q-Hodge filtrations, Habiro cohomology, and ku

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Abstract

Peter Scholze has raised the question whether some variant of the q-de Rham complex is already defined over the *Habiro ring* $\mathcal{H} \coloneqq \lim_{m\in\mathbb{N}} \mathbb{Z}[q]^\wedge_{(q^m-1)}$. Such a variant should then be called *Habiro cohomology*.

In Part I, we'll show that Habiro cohomology exists whenever the q-de Rham complex can be equipped with a *q-Hodge filtration*: a q-deformation of the Hodge filtration, subject to some reasonable conditions. To any such q-Hodge filtration we'll associate a small modification of the q-de Rham complex, which we call the *q-Hodge complex*, and show that it descends canonically to the Habiro ring. This construction recovers and generalises the *Habiro ring of a number field* from [GSWZ24] and is closely related to the q-de Rham–Witt complexes from [Wag24].

While there's no canonical q-Hodge filtration in general, we'll show that it does exist in many cases of interest. For example, for a smooth scheme X over $\mathbb{Z}$, the q-de Rham complex $q\text{-}\Omega_{X/\mathbb{Z}}$ can be equipped with a canonical q-Hodge filtration as soon as one inverts all primes $p \leqslant \dim(X/\mathbb{Z})$.

In Part II we'll explain how another large class of examples arises from homotopy theory: If R is quasi-syntomic and admits a spherical $\mathbb{E}_2$-lift $\mathbb{S}_R$, then the graded pieces of the even filtration on $\mathrm{TC}^-(\mathrm{ku}\otimes\mathbb{S}_R/\mathrm{ku})$ and $\mathrm{TC}^-(\mathrm{KU}\otimes\mathbb{S}_R/\mathrm{KU})$ give rise to a q-Hodge filtration on the (derived) q-de Rham complex of R and the associated q-Hodge complex, respectively. We'll also explain the Habiro descent of the q-Hodge complex in terms of a genuine refinement of the S^1-action on $\mathrm{THH}(\mathrm{KU}\otimes\mathbb{S}_R/\mathrm{KU})$.

In Part III, which is based on joint work with Samuel Meyer [MW24], we'll study a refinement of $\mathrm{THH}/\mathrm{TC}^-$, constructed by Efimov and Scholze as a consequence of Efimov's theorem on the rigidity of localising motives [Efi-Rig]. Using the results from Part II, we'll compute $\pi_*\,\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku}\otimes\mathbb{Q}/\mathrm{ku})$ and $\pi_*\,\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU}\otimes\mathbb{Q}/\mathrm{KU})$.

Contents

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§1. Introduction

This thesis attempts to answer the following question, which was raised by Peter Scholze:

1.1. Question. — *Is it possible to construct a version of q-de Rham cohomology with coefficients in the Habiro ring?*

Before we go into any details, let us give an overview of what these objects are and why Question 1.1 is relevant.

1.2. What's q-de Rham cohomology? — We'll answer this question in detail in §1.1 below. For now, it's enough to know the following: *q-de Rham cohomology* is a cohomology theory for smooth schemes over $\mathbb{Z}$ that was constructed by Bhargav Bhatt and Peter Scholze [Sch17; BS19]. The q-de Rham cohomology of a X a smooth scheme over $\mathbb{Z}$ is the sheaf cohomology

$$\mathrm{H}^*_{q\text{-dR}}(X) \coloneqq \mathrm{H}^*(X, q\text{-}\Omega_{X/\mathbb{Z}})$$

of the *q-de Rham complex* $q\text{-}\Omega_{X/\mathbb{Z}}$. This object, in turn, is a sheaf of complexes of $\mathbb{Z}[\![q-1]\!]$-modules on X, whose reduction modulo $(q-1)$ is the usual de Rham complex of X:

$$q\text{-}\Omega_{X/\mathbb{Z}}/(q-1) \simeq \Omega^*_{X/\mathbb{Z}}\,.$$

In other words, the q-de Rham complex is a *q-deformation* of the usual de Rham complex.

1.3. What's the Habiro ring? — If $\mathbb{Z}[q]^\wedge_{(q^m-1)}$ denotes the completion of the polynomial ring $\mathbb{Z}[q]$ at the ideal (q^m-1), we define the *Habiro ring* as the limit

$$\mathcal{H} \coloneqq \lim_{m\in\mathbb{N}} \mathbb{Z}[q]^\wedge_{(q^m-1)}\,.$$

Here and in the following, $\mathbb{N}$ denotes the set of positive integers, partially ordered by divisibility. An equivalent presentation (and the one originally used in [Hab04]) would be

$$\mathcal{H} \cong \lim_{n\geqslant 0} \mathbb{Z}[q]/(q;q)_n\,,$$

where $(q;q)_n \coloneqq (1-q)(1-q^2)\cdots(1-q^n)$ denotes the *q-Pochhammer symbol.*

We note that for every root of unity ζ there exists a ring morphism $\mathcal{H} \to \mathbb{Z}[\zeta][\![q-\zeta]\!]$; in particular, for $\zeta = 1$ we get a map $\mathcal{H} \to \mathbb{Z}[\![q-1]\!]$. These can be shown to be injective, and so the Habiro ring can be informally thought of as the "subring of $\mathbb{Z}[\![q-1]\!]$ of those power series that admit Taylor expansions around each root of unity".

We can thus informally restate Question 1.1 as follows:

1.1′. Question. — *Given that the de Rham complex admits an interesting deformation around $q = 1$, what can we say around other roots of unity?*

In the following, we'll call a putative answer to Question 1.1 *Habiro cohomology.* Let us now explain why Habiro cohomology should be interesting.

1.4. Motivation from arithmetic geometry. — In arithmetic geometry we're looking for the "one cohomology to rule them all". For p-adic formal schemes and p-adic coefficients, a good candidate for such a theory is Bhatt–Scholze's *prismatic cohomology* [BS19]. But in general, for schemes over $\mathbb{Z}$—which we'll usually call *global*—and arbitrary coefficients, very little is known.

But at least we know this: After completion at any prime p, the q-de Rham complex $(q\text{-}\Omega_{X/\mathbb{Z}})_p^\wedge$ computes prismatic cohomology of the formal scheme $X \times \operatorname{Spf} \mathbb{Z}_p[\zeta_p]$, where ζ_p denotes a primitive p^{th} root of unity. In fact, q-de Rham cohomology is the only known non-trivial case in which the various prismatic cohomologies for all primes p can be combined into one global object!

This makes q-de Rham cohomology interesting. Now a positive answer to Question 1.1 should be even more interesting, since it would contain a lot more information. To explain what form this information takes, it is best to adopt a geometric perspective. This leads us to a brief digression about *stacky approaches* to cohomology theories. The idea behind stacky approaches is the following: To any reasonable cohomology theory $\mathrm{H}^*_?(-)$ and any geometric object X, it should be possible to associate another geometric object $X^?$ (often a stack, hence the name *stacky approach*) in such a way that the cohomology $\mathrm{H}^*_?(X)$ is given by the sheaf cohomology

$$\mathrm{H}^*_?(X) \cong \mathrm{H}^*(X^?, \mathcal{O}) ,$$

where $\mathcal{O}$ denotes the structure sheaf of $X^?$. This reduces studying the algebraic properties of $\mathrm{H}^*_?(X)$ to studying the geometric properties of $X^?$, which often holds much more refined information.

The first instance of a stacky approach is Carlos Simpson's *de Rham stack* X^{dR} [Sim96], whose sheaf cohomology computes de Rham cohomology, as the name suggests. In recent years, this idea has received much attention through the construction of a stacky approach to prismatic cohomology by Drinfeld and Bhatt–Lurie [Dri24; BL22a; BL22b], the construction of a de Rham stack for rigid-analytic varieties by Rodríguez Camargo [RC24b], and the subsequent ongoing efforts to combine both constructions into a stacky approach to prismatic cohomology for rigid-analytic varieties.

Stacky approaches also fit remarkably well with the philosophy that the "one cohomology to rule them all" should be a sheaf on "$X \times \operatorname{Spec} \mathbb{Z}$", where the product is not taken in schemes, but over the "absolute base" (the "field with one element"). This doesn't exist, of course, but we hope that a geometric object playing the role of "$X \times \operatorname{Spec} \mathbb{Z}$" can be constructed nonetheless. Any cohomology theory $\mathrm{H}^*_?(-)$ with a stacky approach should then give rise to a map $X^? \to$ "$X \times \operatorname{Spec} \mathbb{Z}$". Through these maps we can probe the elusive object "$X \times \operatorname{Spec} \mathbb{Z}$" and try to understand its geometry.

It's currently unknown whether q-de Rham cohomology admits a stacky approach $X^{q\text{-dR}}$. But assume that it does, and assume Habiro cohomology as in Question 1.1 not only exists, but also admits a stacky approach $X^{\mathcal{H}}$. Then $X^{\mathcal{H}}$ would be a lot larger than $X^{q\text{-dR}}$, because already the formal spectrum $\operatorname{Spf} \mathcal{H}$ is much larger and much more complicated than $\operatorname{Spf} \mathbb{Z}[\![q-1]\!]$; we attempt to draw a picture in Fig. 2. We expect that $X^{\mathcal{H}}$ would be able to see much more of "$X \times \operatorname{Spec} \mathbb{Z}$" than $X^{q\text{-dR}}$; in particular, $X^{\mathcal{H}}$ should see some interesting geometry that would be invisible to $X^{q\text{-dR}}$. We'll see a concrete instance of this expectation in 1.5 below, and we'll continue the discussion of $X^{q\text{-dR}}$ and $X^{\mathcal{H}}$ (in a slightly different setting) in §1.6.

1.5. Motivation from 3-manifold topology. — The Habiro ring originally comes from 3-manifold topology: Habiro [Hab02] constructs an invariant of knots and homology 3-spheres with values in his ring $\mathcal{H}$.

Through work of Garoufalidis, Scholze, Wheeler, and Zagier [GZ23; GZ24; GSWZ24; GW25], we now have evidence to believe that Habiro's invariant is just the first instance of a much broader theory.[(1.1)] Due to the author's ignorance, we won't discuss their work in detail, but let us at least mention the following construction: Given a number field F and an integer Δ divisible by 6 and by the discriminant of F, the four authors construct a formally étale $\mathcal{H}$-algebra $\mathcal{H}_{\mathcal{O}_F[1/\Delta]}$ (the *Habiro ring of the number field*) and a morphism of abelian groups

$$K_3(F) \longrightarrow \operatorname{Pic}\bigl(\mathcal{H}_{\mathcal{O}_F[1/\Delta]}\bigr)$$

(the *regulator map*); see [GSWZ24, Theorem 2]. Then they show that certain q-hypergeometric series that arise in complex Chern–Simons theory naturally form sections of line bundles in the image of $K_3(F) \to \operatorname{Pic}(\mathcal{H}_{\mathcal{O}_F[1/\Delta]})$.

The Habiro ring of the number field $\mathcal{H}_{\mathcal{O}_F[1/\Delta]}$ is the Habiro cohomology of the scheme $X \coloneqq \operatorname{Spec} \mathcal{O}_F[1/\Delta]$ (see Corollaries 2.13 and 3.13) and the formal spectrum $\operatorname{Spf} \mathcal{H}_{\mathcal{O}_F[1/\Delta]}$ should be—at least in first approximation—the Habiro stack $X^{\mathcal{H}}$ envisioned in 1.4 above. Thus, already in the simplest possible case, where X is étale $\mathbb{Z}$, the Habiro stack $X^{\mathcal{H}}$ exhibits interesting geometry in form of the line bundles above. Moreover, this geometry would be invisible to any q-de Rham stack, since the line bundles in the image of $K_3(F) \to \operatorname{Pic}(\mathcal{H}_{\mathcal{O}_F[1/\Delta]})$ all become trivial after $(q-1)$-completion.

Even more recently, and again motivated by complex Chern–Simons theory (more precisely, the asymptotics of the 3D index), Garoufalidis–Wheeler [GW25] have constructed examples that don't yield sections of line bundles, but cohomology classes in non-zero degree. It's currently still conjectural whether these classes are contained in the Habiro cohomology that we will construct in this thesis, but all the evidence we have is pointing towards this indeed being the case.

All of this is suggests that Habiro cohomology should be at the center of a fruitful connection between 3-manifold topology and arithmetic geometry. For a low dimensional topologist, Habiro cohomology should provide a framework in which generalisations of Habiro's invariant take values. For arithmetic geometers, 3-manifold topology should provide a source of explicit classes in Habiro cohomology. We hope to explore this much further in future work.

We've argued why it should be worthwhile to pursue Question 1.1, from the perspective of arithmetic geometry but also through an unexpected connection to 3-manifold topology. Let us now explain in some detail the contents of this thesis.

§1.1. q-de Rham cohomology

We start with a review of q-de Rham cohomology. For a much more technical introduction, which contains the relevant constructions and proofs, the reader should consult §A.

Throughout the introduction we'll always work over $\mathbb{Z}$ for simplicity. In the main body of the text our base will instead be a Λ-ring A which is *perfectly covered* in the sense defined in 1.50 below.

1.6. q-derivatives. — For a polynomial ring $\mathbb{Z}[x]$, one can define a *q-derivative* (or *Jackson derivative* after [Jac10]) $q\text{-}\partial\colon \mathbb{Z}[x,q] \to \mathbb{Z}[x,q]$ via

$$q\text{-}\partial f(x,q) \coloneqq \frac{f(qx,q) - f(x,q)}{qx - x}\,.$$

(1.1) Perhaps even a *topological quantum field theory.*

For example, $q\text{-}\partial(x^m) = [m]_q x^{m-1}$, where $[m]_q \coloneqq 1 + q + \dots + q^{m-1}$ denotes the *Gaussian* q*-analogue of* m. For a polynomial ring in several variables $\mathbb{Z}[x_1, \dots, x_n]$, one can similarly define *partial* q*-derivatives* $q\text{-}\partial_i$ for $i = 1, \dots, n$ and organise them into a q*-de Rham complex*, as was first done by Aomoto [Aom90].

1.7. The q-de Rham complex. — In [Sch17], Scholze observed that, upon completing at $(q-1)$, this construction can be extended beyond the case of polynomial rings. Define a *framed smooth* $\mathbb{Z}$*-algebra* to be a pair $(S, \square)$ of a smooth algebra S over $\mathbb{Z}$ and an étale map $\square\colon \mathbb{Z}[x_1, \dots, x_n] \to S$. Note that every smooth scheme over $\mathbb{Z}$ admits a Zariski cover by framed smooth $\mathbb{Z}$-algebras. Scholze shows that the partial q-derivatives can be extended to maps

$$q\text{-}\partial_i\colon S[\![q-1]\!] \longrightarrow S[\![q-1]\!]$$

as follows: Let $\gamma_i\colon \mathbb{Z}[x_1, \dots, x_n][\![q-1]\!] \to \mathbb{Z}[x_1, \dots, x_n][\![q-1]\!]$ be morphism of rings that sends $x_i \mapsto q x_i$ and leaves the other variables fixed. Then there exists a unique lift in the following diagram of rings:

$$\begin{array}{ccc} \mathbb{Z}[x_1, \dots, x_n][\![q-1]\!] & \xrightarrow{\gamma_i} & S[\![q-1]\!] \\ {\scriptstyle\square}\downarrow & \overset{\exists !}{\nearrow} & \downarrow \\ S[\![q-1]\!] & \longrightarrow & S \end{array}$$

Indeed, the left vertical arrow is the $(q-1)$-completion of an étale morphism and the right vertical map is a $(q-1)$-complete pro-infinitesimal thickening, so existence and uniqueness follows from the unique lifting property of étale morphisms against infinitesimal thickenings.

This lift will also be denoted γ_i. By construction, $\gamma_i \equiv \mathrm{id} \mod (q-1)$. By lifting against $S[\![q-1]\!]/(q-1)x_i \to S$ instead, which is still a $(q-1)$-complete pro-infinitesimal thickening, we see that even $\gamma_i \equiv \mathrm{id} \mod (q-1)x_i$. This allow us to extend Jackson's q-derivatives to all $f \in S[\![q-1]\!]$ via

$$q\text{-}\partial_i f \coloneqq \frac{\gamma_i(f) - f}{q x_i - x_i}$$

for $i = 1, \dots, n$. Note that $q\text{-}\partial_i$ and $q\text{-}\partial_j$ commute for all i and j. Indeed, this reduces to the same assertion for γ_i and γ_j, which follows once again by an infinitesimal lifting argument. We may thus construct the q*-de Rham complex of* $(S, \square)$ as the Koszul complex of the commuting $\mathbb{Z}[\![q-1]\!]$-module endomorphisms $q\text{-}\partial_1, \dots, q\text{-}\partial_n$:

$$q\text{-}\Omega^*_{S/\mathbb{Z},\square} \coloneqq \left(S[\![q-1]\!] \xrightarrow{q\text{-}\nabla} \Omega^1_{S/\mathbb{Z}}[\![q-1]\!] \xrightarrow{q\text{-}\nabla} \cdots \xrightarrow{q\text{-}\nabla} \Omega^n_{S/\mathbb{Z}}[\![q-1]\!]\right),$$

where $q\text{-}\nabla \coloneqq \sum_{i=1}^n q\text{-}\partial_i \, \mathrm{d}x_i$.

1.8. Coordinate (in-)dependence. — The q-derivatives from 1.6 are extremely sensitive to coordinate transformations such as $x \mapsto x + 1$, and there's no way to make the complex $q\text{-}\Omega^*_{S/\mathbb{Z},\square}$ independent of the choice of coordinates $\square$, not even in the simplest case $S = \mathbb{Z}[x]$. It then comes as a small miracle that $q\text{-}\Omega^*_{S/\mathbb{Z},\square}$, as an object in the derived category $D(\mathbb{Z}[\![q-1]\!])$, is independent of $\square$, and functorial in S. More precisely, we have the following theorem due to Bhatt and Scholze:

1.9. Theorem (Bhatt–Scholze; see Theorem A.1). — *There exists a functor*

$$q\text{-}\Omega_{-/\mathbb{Z}}\colon \mathrm{Sm}_{\mathbb{Z}} \longrightarrow \mathrm{CAlg}\Big(\widehat{\mathcal{D}}_{(q-1)}\big(\mathbb{Z}[\![q-1]\!]\big)\Big)$$

from the category of smooth $\mathbb{Z}$-algebras into the ∞-category $(q-1)$-complete $\mathbb{E}_\infty$-$\mathbb{Z}[\![q-1]\!]$-algebras, satisfying the following properties:

(*a*) $q\text{-}\Omega_{S/\mathbb{Z}}/(q-1) \simeq \Omega^*_{S/\mathbb{Z}}$ *is the usual de Rham complex.*

(*b*) *For all primes p, the p-completion $(q\text{-}\Omega_{S/\mathbb{Z}})^\wedge_p \simeq \mathbb{\Delta}_{\widehat{S}_p[\zeta_p]/\mathbb{Z}_p[\![q-1]\!]}$ agrees with the prismatic cohomology of $\widehat{S}_p[\zeta_p]$ over the q-de Rham prism $(\mathbb{Z}_p[\![q-1]\!], [p]_q)$.*

(*c*) $(q\text{-}\Omega_{S/\mathbb{Z}} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Q})^\wedge_{(q-1)} \simeq (\Omega_{S/\mathbb{Z}} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$ *becomes the trivial q-deformation.*

(*d*) *For every framed smooth $\mathbb{Z}$-algebra $(S, \square)$, the underlying object of $q\text{-}\Omega_{S/\mathbb{Z}}$ in the derived ∞-category of $\mathbb{Z}[\![q-1]\!]$ can be represented as*

$$q\text{-}\Omega_{S/\mathbb{Z}} \simeq q\text{-}\Omega^*_{S/\mathbb{Z},\square}\,.$$

We note that as a consequence of (*a*) (or a combination of (*b*) and (*c*)), $q\text{-}\Omega_{-/\mathbb{Z}}$ satisfies Zariski (even étale) descent and so Theorem 1.9 guarantees the existence of a q-de Rham complex $q\text{-}\Omega_{X/\mathbb{Z}}$ for any smooth scheme X over $\mathbb{Z}$.

A proof of this theorem will be explained in the appendix; see §A. The essential step is to identify the p-completions $(q\text{-}\Omega^*_{S/\mathbb{Z},\square})^\wedge_p$ with prismatic cohomology, which was achieved in [BS19, Theorems 16.18 and 16.22].

§1.2. Can the Hodge filtration be q-deformed?

Given that the de Rham complex admits this canonical q-deformation, it is natural to ask how much additional structure can be q-deformed along. One such piece of structure that features very prominently in the classical theory is the *Hodge filtration* $\mathrm{fil}^\star_{\mathrm{Hdg}}\,\Omega^*_{S/\mathbb{Z}}$, given by

$$\mathrm{fil}^i_{\mathrm{Hdg}}\,\Omega^*_{S/\mathbb{Z}} \coloneqq \left(0 \to \cdots \to 0 \to \Omega^i_{S/\mathbb{Z}} \xrightarrow{\nabla} \Omega^{i+1}_{S/\mathbb{Z}} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^n_{S/\mathbb{Z}}\right).$$

1.10. Question. — *Is there a "q-Hodge filtration" $\mathrm{fil}^\star_{q\text{-Hdg}}\,q\text{-}\Omega_{S/\mathbb{Z}}$, which is a module over the filtered ring $(q-1)^\star\mathbb{Z}[\![q-1]\!]$ in the filtered derived ∞-category $\mathrm{Fil}\,\mathcal{D}(\mathbb{Z})$ and which q-deforms the usual Hodge filtration in the sense that that we have a base change equivalence*

$$\mathrm{fil}^\star_{q\text{-Hdg}}\,q\text{-}\Omega_{S/\mathbb{Z}} \otimes^{\mathrm{L}}_{(q-1)^\star\mathbb{Z}[\![q-1]\!]} \mathbb{Z} \xrightarrow{\simeq} \mathrm{fil}^\star_{\mathrm{Hdg}}\,\Omega^*_{S/\mathbb{Z}}\,?$$

In Part I of this thesis, we study this natural question and find that it has several surprises in store, including a connection to Question 1.1.

(*a*) There is no functorial choice for $\mathrm{fil}^\star_{q\text{-Hdg}}\,q\text{-}\Omega_{S/\mathbb{Z}}$, at least not if one imposes a few natural additional compatibilities (namely, the ones in Definition 1.14 below).

(*b*) A functorial choice for $\mathrm{fil}^\star_{q\text{-Hdg}}\,q\text{-}\Omega_{S/\mathbb{Z}}$ does exist once we invert all primes $p \leqslant \dim(S/\mathbb{Z})$.

(*c*) Whenever $\mathrm{fil}^\star_{q\text{-Hdg}}\,q\text{-}\Omega_{S/\mathbb{Z}}$ does exist, the q-de Rham complex $q\text{-}\Omega_{S/\mathbb{Z}}$ (and in fact, already the *q-Hodge complex* introduced below) descends canonically to the Habiro ring.

We'll discuss these in the order (*a*), (*c*), (*b*). Let us start with why a functorial q-Hodge filtration is impossible. A direct objection is known to the experts and will be reproduced in Lemma 3.3 below; here we'll explain a somewhat indirect objection, which will also introduce a construction that will become very important in the later discussion.

1.11. A coordinate-dependent q-Hodge filtration. — For a framed smooth $\mathbb{Z}$-algebra $(S,\square)$ as in 1.7, there's an obvious guess for what the q-Hodge filtration should be: We could define $\operatorname{fil}^i_{q\text{-Hdg},\square} q\text{-}\Omega^*_{S/\mathbb{Z},\square}$ to be the complex

$$\Big((q-1)^i S[\![q-1]\!] \to (q-1)^{i-1}\Omega^1_{S/\mathbb{Z}}[\![q-1]\!] \to \dotsb \to \Omega^i_{S/\mathbb{Z}}[\![q-1]\!] \to \dotsb \to \Omega^n_{S/\mathbb{Z}}[\![q-1]\!]\Big)\,.$$

The question then becomes: Can this filtration be made functorial as well? As it turns out, this is most likely *not* the case. To formulate a precise objection, let us introduce another construction.

1.12. A coordinate-dependent q-Hodge complex. — Given a framed smooth $\mathbb{Z}$-algebra $(S,\square)$, the *q-Hodge complex of* $(S,\square)$ is the complex

$$q\text{-}\operatorname{Hdg}^*_{S/\mathbb{Z},\square} \coloneqq \Big(S[\![q-1]\!] \xrightarrow{(q-1)\,q\text{-}\nabla} \Omega^1_{S/\mathbb{Z}}[\![q-1]\!] \xrightarrow{(q-1)\,q\text{-}\nabla} \dotsb \xrightarrow{(q-1)\,q\text{-}\nabla} \Omega^n_{S/\mathbb{Z}}[\![q-1]\!]\Big)$$

given by multiplying all the differentials in the de Rham complex by $(q-1)$.

The q-Hodge complex was first studied by Pridham [Pri19], and it was suggested by Peter Scholze to be a more natural object to descend to the Habiro ring than the q-de Rham complex itself. There are several mathematical reason to for this expectation (and it will ultimately be proven right in Theorem 1.16), but let us give a plausibility argument instead: Any cohomology theory that descends to the Habiro ring should "treat all roots of unity equally", and so instead of q-derivatives that send $x^m \mapsto [m]_q x^{m-1}\,\mathrm{d}x$, which gives "special treatment" to $q=1$, we should have differentials that send $x^m \mapsto (q^m-1)x^{m-1}\,\mathrm{d}x$. This precisely gives rise to the q-Hodge complex from 1.12.

Observe that if $\operatorname{fil}^\star_{q\text{-Hdg},\square} q\text{-}\Omega^*_{S/\mathbb{Z},\square}$ could be made functorial, then the same would be true for the q-Hodge complex, as it can also be obtained as

$$q\text{-}\operatorname{Hdg}^*_{S/\mathbb{Z},\square} \cong \operatorname{colim}\Big(\operatorname{fil}^0_{q\text{-Hdg},\square} q\text{-}\Omega^*_{S/\mathbb{Z},\square} \xrightarrow{(q-1)} \operatorname{fil}^1_{q\text{-Hdg},\square} q\text{-}\Omega^*_{S/\mathbb{Z},\square} \xrightarrow{(q-1)} \dotsb\Big)^\wedge_{(q-1)}\,.$$

However, in [Wag24] we've found a strange objection to functoriality of the q-Hodge complex: For all $m\in\mathbb{N}$ and all rings R, we introduce differential-graded $\mathbb{Z}[q]$-algebras $q\text{-}\mathbb{W}_m\Omega^*_{R/\mathbb{Z}}$ that we call *m-truncated q-de Rham–Witt complexes*. The system $(q\text{-}\mathbb{W}_m\Omega^*_{-/\mathbb{Z}})_{m\in\mathbb{N}}$, together with certain *Frobenius* and *Verschiebung operators*

$$F_{m/d}\colon q\text{-}\mathbb{W}_m\Omega^*_{-/\mathbb{Z}} \longrightarrow q\text{-}\mathbb{W}_d\Omega^*_{-/\mathbb{Z}} \quad\text{and}\quad V_{m/d}\colon q\text{-}\mathbb{W}_d\Omega^*_{-/\mathbb{Z}} \longrightarrow q\text{-}\mathbb{W}_m\Omega^*_{-/\mathbb{Z}}$$

for all divisors $d\mid m$, satisfies a similar universal property as the usual de Rham–Witt pro-complex with its Frobenii and Verschiebungen (compare [Wag24, Definitions 3.1 and 3.6] and [LZ04, §1.3]). We then show the following result:

1.13. Theorem (see [Wag24, Theorems 4.27 and 5.1])**.** — *Let $(S,\square)$ be a framed smooth $\mathbb{Z}$-algebra. Then the following is true:*

(a) *For every $m\in\mathbb{N}$, there's an isomorphism of differential-graded $\mathbb{Z}[q]$-algebras*

$$\mathrm{H}^*\big(q\text{-}\operatorname{Hdg}^*_{S/\mathbb{Z},\square}/(q^m-1)\big) \cong \big(q\text{-}\mathbb{W}_m\Omega^*_{S/\mathbb{Z}}\big)^\wedge_{(q-1)}\,,$$

where the differential on the left-hand side is the Bockstein differential. In particular, the cohomology of $q\text{-}\operatorname{Hdg}^*_{S/\mathbb{Z},\square}/(q^m-1)$ is independent of the choice of coordinates $\square$. Moreover, under the isomorphism above, the Frobenius operators $F_{m/d}$ are induced by the canonical projections $q\text{-}\operatorname{Hdg}^*_{S/\mathbb{Z},\square}/(q^m-1) \to q\text{-}\operatorname{Hdg}^*_{S/\mathbb{Z},\square}/(q^d-1)$.

(*b*) $(S,\square)\mapsto q\text{-}\mathrm{Hdg}^*_{S/\mathbb{Z},\square}$ *cannot be extended to a functor* $q\text{-}\mathrm{Hdg}_{-/\mathbb{Z}}\colon \mathrm{Sm}_\mathbb{Z}\to\widehat{\mathcal{D}}_{(q-1)}(\mathbb{Z}[\![q-1]\!])$ *in such a way that the identifications from* (*a*) *also become functorial.*

Theorem 1.13 is by all means a weird result. Part (*a*) promises functoriality and a wealth of extra structure. Moreover, the fact that not $q\text{-}\mathbb{W}_m\Omega^*_{S/\mathbb{Z}}$ itself appears, but only its $(q-1)$-completion, looks encouraging when we're seeking to descend the q-Hodge complex to the Habiro ring. But then part (*b*) shows that functoriality of the q-Hodge complex is impossible, at least not in a way compatible with the extra structure.

As we've argued above, this also raises a serious objection to whether the coordinate-dependent q-Hodge filtration $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg},\square}\,q\text{-}\Omega^*_{S/\mathbb{Z},\square}$ can be made coordinate-independent (and as mentioned before, a more direct no-go is known and will be explained in Lemma 3.3). Unfazed by this, we can introduce the following notion:

1.14. Definition (see Definition 3.2 for a more precise version)**.** — Let S be smooth over $\mathbb{Z}$. A *q-Hodge filtration on* $q\text{-}\Omega_{S/\mathbb{Z}}$ is a module over the filtered ring $(q-1)^\star\mathbb{Z}[\![q-1]\!]$ in the filtered derived category $\mathrm{Fil}\,\mathcal{D}(\mathbb{Z})$ such that:

(*a*) $\mathrm{fil}^0_{q\text{-}\mathrm{Hdg}}\,q\text{-}\Omega_{S/\mathbb{Z}}\simeq q\text{-}\Omega_{S/\mathbb{Z}}$; that is, $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\Omega_{S/\mathbb{Z}}$ is a filtration on $q\text{-}\Omega_{S/\mathbb{Z}}$.

(*b*) $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\Omega_{S/\mathbb{Z}}\otimes^{\mathrm{L}}_{(q-1)^\star\mathbb{Z}[\![q-1]\!]}\mathbb{Z}\simeq\mathrm{fil}^\star_{\mathrm{Hdg}}\,\Omega^*_{S/\mathbb{Z}}$ is the usual Hodge filtration.

(*c*) After rationalisation, $(\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\Omega_{S/\mathbb{Z}}\otimes^{\mathrm{L}}_\mathbb{Z}\mathbb{Q})^\wedge_{(q-1)}\simeq\mathrm{fil}^\star_{(\mathrm{Hdg},q-1)}(\Omega^*_{S/\mathbb{Z}}\otimes^{\mathrm{L}}_\mathbb{Z}\mathbb{Q})[\![q-1]\!]$ becomes the $(q-1)$-completed tensor product of the Hodge filtration on $\Omega^*_{S/\mathbb{Z}}$ and the $(q-1)$-adic filtration on $\mathbb{Q}[\![q-1]\!]$.

(c_p) Similarly, $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}(q\text{-}\Omega_{S/\mathbb{Z}})^\wedge_p[1/p]^\wedge_{(q-1)}\simeq\mathrm{fil}^\star_{(\mathrm{Hdg},q-1)}(\Omega^*_{S/\mathbb{Z}})^\wedge_p[1/p][\![q-1]\!]$ for primes p.

Moreover, to any q-Hodge filtration as above, we associate a *q-Hodge complex*

$$q\text{-}\mathrm{Hdg}_{(S,\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}})/\mathbb{Z}}\coloneqq\operatorname{colim}\Bigl(\mathrm{fil}^0_{q\text{-}\mathrm{Hdg},}\,q\text{-}\Omega_{S/\mathbb{Z}}\xrightarrow{(q-1)}\mathrm{fil}^1_{q\text{-}\mathrm{Hdg}}\,q\text{-}\Omega_{S/\mathbb{Z}}\xrightarrow{(q-1)}\dotsb\Bigr)^\wedge_{(q-1)}.$$

1.15. Remark. — As you'll find, our definition Definition 3.2 in the main text allows for arbitrary *animated rings*, not only smooth $\mathbb{Z}$-algebras, and thus necessarily uses the *derived q-de Rham complex* (see 1.49). We'll explain in Remark 3.6 that it doesn't matter whether we study q-Hodge filtrations on derived or underived q-de Rham complexes.

With this Definition 1.14, we'll give a partial answer to Question 1.1 and simultaneously show an improved version of Theorem 1.13(*a*).

1.16. Theorem (see Theorem 3.11 and Corollary 3.54)**.** — *Let* $\mathrm{Sm}^{q\text{-}\mathrm{Hdg}}_\mathbb{Z}$ *be the category of pairs* $(S,\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\Omega_{S/\mathbb{Z}})$, *where S is smooth over $\mathbb{Z}$ and* $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\Omega_{S/\mathbb{Z}}$ *is a q-Hodge filtration.*

(*a*) *The q-Hodge complex functor* $q\text{-}\mathrm{Hdg}_{-/A}\colon\mathrm{Sm}^{q\text{-}\mathrm{Hdg}}_\mathbb{Z}\to\widehat{\mathcal{D}}_{(q-1)}(\mathbb{Z}[\![q-1]\!])$ *admits a non-trivial factorisation*

$$\begin{array}{ccc} & & \widehat{\mathcal{D}}_{\mathcal{H}}(\mathbb{Z}[q]) \\ & \overset{q\text{-}\mathcal{H}\mathrm{dg}_{-/\mathbb{Z}}}{\nearrow} & \big\downarrow {\scriptstyle(-)^\wedge_{(q-1)}} \\ \mathrm{Sm}^{q\text{-}\mathrm{Hdg}}_\mathbb{Z} & \xrightarrow[q\text{-}\mathrm{Hdg}_{-/\mathbb{Z}}]{} & \widehat{\mathcal{D}}_{(q-1)}(\mathbb{Z}[q]) \end{array}$$

where $\widehat{\mathcal{D}}_{\mathcal{H}}(\mathbb{Z}[q])$ *denotes the full sub-∞-category of Habiro-complete objects in the derived ∞-category of* $\mathbb{Z}[q]$ (*see the appendix,* §B).

(*b*) *For all* $m \in \mathbb{N}$ *there's a natural isomorphism*

$$\mathrm{H}^*\big(q\text{-}\mathcal{H}\mathrm{dg}_{-/A}/(q^m-1)\big) \cong q\text{-}\mathbb{W}_m\Omega^*_{-/A}\,.$$

(*c*) $\mathrm{L}\eta_{(q-1)}\, q\text{-}\mathrm{Hdg}_{-/A} \simeq q\text{-}\Omega_{-/A}$ *is the q-de Rham complex, and so* $\mathrm{L}\eta_{(q-1)}\, q\text{-}\mathcal{H}\mathrm{dg}_{-/A}$ *defines similar factorisation of the functor* $q\text{-}\Omega_{-/A}\colon \mathrm{Sm}_{\mathbb{Z}}^{q\text{-}\mathrm{Hdg}} \to \widehat{\mathcal{D}}_{(q-1)}(\mathbb{Z}[q])$.

The proof of Theorem 1.16 is the technical heart of Part I and will occupy most of §3. At this point, we haven't seen any examples of q-Hodge filtrations except $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg},\square}\, q\text{-}\Omega^*_{S/\mathbb{Z},\square}$ from 1.11 (in this case $q\text{-}\mathcal{H}\mathrm{dg}_{-/A}$ can also be explicitly described as a complex; see Example 3.12). So it's not clear whether there are any non-trivial examples to which Theorem 1.16 applies. Fortunately, it turns out that there are plenty:

1.17. Theorem (see Theorem 4.11)**.** — *A functorial q-Hodge filtration exists as soon as one inverts all primes up to the dimension. More precisely: Let* $\mathrm{Sm}_{\mathbb{Z}[\dim!^{-1}]}$ *be the category of smooth* $\mathbb{Z}$*-algebras* S *such that all primes* $p \leqslant \dim(S/\mathbb{Z})$ *are invertible in* S*. Then the forgetful functor* $\mathrm{Sm}_{\mathbb{Z}}^{q\text{-}\mathrm{Hdg}} \to \mathrm{Sm}_{\mathbb{Z}}$ *admits a partial section*

$$\big(-, \mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\Omega_{-/\mathbb{Z}}\big)\colon \mathrm{Sm}_{\mathbb{Z}[\dim!^{-1}]} \longrightarrow \mathrm{Sm}_{\mathbb{Z}}^{q\text{-}\mathrm{Hdg}}\,.$$

We'll prove this theorem in §4.1, but let us already sketch the construction in the case where $\dim(S/\mathbb{Z}) \leqslant 1$ (so that no primes need to be inverted), as the idea is very simple.

1.18. Canonical q-Hodge filtrations in relative dimension $\leqslant 1$. — Let S be smooth over $\mathbb{Z}$. The most naive idea to equip $q\text{-}\Omega_{S/\mathbb{Z}}$ with a q-Hodge filtration would be to simply take the pullback

$$\begin{array}{ccc} \mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\Omega_{S/\mathbb{Z}} & \longrightarrow & q\text{-}\Omega_{S/\mathbb{Z}} \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{fil}^\star_{\mathrm{Hdg}}\, \Omega_{S/\mathbb{Z}} & \longrightarrow & \Omega_{S/\mathbb{Z}} \end{array}$$

This cannot work, of course, because in this pullback each filtration step $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\Omega_{S/\mathbb{Z}}$ will contain all of $(q-1)\, q\text{-}\Omega_{S/\mathbb{Z}}$. In view of Definition 1.14(*c*) this is only ok for $\star \leqslant 1$.

Now if S has relative dimension $\dim(S/\mathbb{Z}) \leqslant 1$, then $\mathrm{fil}^\star_{\mathrm{Hdg}}\, \Omega^*_{S/\mathbb{Z}}$ is trivial in filtration degrees $\star \geqslant 2$. Consequently any filtration $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\Omega_{S/\mathbb{Z}}$ satisfying Definition 1.14(*b*) will necessarily be given by the $(q-1)$-adic filtration $(q-1)^{\star-1}\, \mathrm{fil}^1_{q\text{-}\mathrm{Hdg}}\, q\text{-}\Omega_{S/A}$ in filtration degrees $\geqslant 1$. We may thus define the first filtration step $\mathrm{fil}^1_{q\text{-}\mathrm{Hdg}}\, q\text{-}\Omega_{S/A}$ using the pullback above and then construct the rest of the filtration $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\Omega_{S/\mathbb{Z}}$ as

$$\Big(q\text{-}\Omega_{S/\mathbb{Z}} \leftarrow \mathrm{fil}^1_{q\text{-}\mathrm{Hdg}}\, q\text{-}\Omega_{S/\mathbb{Z}} \leftarrow (q-1)\, \mathrm{fil}^1_{q\text{-}\mathrm{Hdg}}\, q\text{-}\Omega_{S/\mathbb{Z}} \leftarrow (q-1)^2\, \mathrm{fil}^1_{q\text{-}\mathrm{Hdg}}\, q\text{-}\Omega_{S/\mathbb{Z}} \leftarrow \cdots\Big)\,.$$

One can (and we will) check that this satisfies all expected properties.

§1.3. q-Hodge filtrations from topological Hochschild homology over ku

Another rich source of examples of q-Hodge filtrations, to which Theorem 1.16 (or its fully derived version Theorem 3.11) can be applied, comes from homotopy theory. This will be the content of Part II of this thesis.

To explain how this works, let us first recall how the Hodge filtration on the (derived) de Rham complex is related to Hochschild homology and its cousins.

1.19. Hochschild and negative cyclic homology. — Recall that the *Hochschild homology* of a ring R is defined as

$$\mathrm{HH}(R) \coloneqq R \otimes^{\mathrm{L}}_{R \otimes^{\mathrm{L}}_{\mathbb{Z}} R^{\mathrm{op}}} R\,.$$

Here R is not necessarily commutative and R^{op} denotes the ring with the same underlying abelian group but the opposite multiplication.

Note that the unit circle S^1, which we always identify with the topological group $\mathrm{U}(1)$, acts naturally on $\mathrm{HH}(R)$. In the case where R is commutative, this is easy to explain: $\mathrm{HH}(R)$ is the colimit of the constant R-valued functor $\mathrm{const}\,R\colon S^1 \to \mathrm{CAlg}\,\mathcal{D}(\mathbb{Z})$ from S^1 into the ∞-category of $\mathbb{E}_\infty$-$\mathbb{Z}$-algebras, essentially because pushouts in $\mathrm{CAlg}\,\mathcal{D}(\mathbb{Z})$ are given by (derived) tensor products. So we get an action $S^1 \curvearrowright \mathrm{HH}(R) \simeq \operatorname{colim}_{S^1} R$ via S^1 acting on itself. In the general case, the S^1-action can still be constructed, albeit not quite as easily; see e.g. [NS18, Definition III.2.3]. In any case, we define the *topological negative cyclic homology* of R as the homotopy fixed points

$$\mathrm{HC}^-(R) \coloneqq \mathrm{HH}(R)^{hS^1}\,.$$

It turns out that these constructions are intimately connected to the de Rham complex. The first result in that direction is the celebrated Hochschild–Kostant–Rosenberg theorem [HKR62], which states that

$$\mathrm{H}_*\bigl(\mathrm{HH}(S)\bigr) \cong \Omega^*_{S/\mathbb{Z}}$$

when S is smooth over $\mathbb{Z}$. A much refined version of this result for HC^- has been obtained by Ben Antieau, following previous constructions by Loday [Lod92] for $\mathbb{Q}$-algebras and by Bhatt–Morrow–Scholze [BMS19] in the p-complete case.

1.20. Theorem (Antieau [Ant19])**.** — *For commutative rings R, there exists a motivic filtration $\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{HC}^-(R)$, which is exhaustive and complete if R is quasi-syntomic, and whose associated graded*

$$\Sigma^{-2*}\,\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{HC}^-(R) \simeq \mathrm{fil}^\star_{\mathrm{Hdg}}\,\widehat{\mathrm{dR}}_{R/\mathbb{Z}}$$

is given (up to shift) by the completed Hodge filtration on the derived de Rham complex of R.

1.21. "Motivic filtration = even filtration". — Note that in the definition of Hochschild and negative cyclic homology, one can replace the base $\mathbb{Z}$ by any $\mathbb{E}_\infty$-ring spectrum k and R by any $\mathbb{E}_1$-k-algebra. The resulting constructions are usually called *topological Hochschild/negative cyclic homology relative to k* and denoted

$$\mathrm{THH}(R/k) \quad \text{and} \quad \mathrm{TC}^-(R/k)\,.$$

By an amazing insight due to Hahn–Raksit–Wilson [HRW22], the motivic filtration—and thus a notion of "Hodge-filtered de Rham complex over k"—can also be defined for any $\mathbb{E}_\infty$-ring spectrum k! Their construction works as follows: For any $\mathbb{E}_\infty$-ring spectrum T, they define the *even filtration of T* to be

$$\mathrm{fil}^\star_{\mathrm{ev}}\,T \coloneqq \lim_{T \to E \text{ even}} \tau_{\geqslant 2\star}(E)\,,$$

where the limit is taken over all maps of $\mathbb{E}_\infty$-ring spectra $T \to E$ such that $\pi_*(E)$ vanishes in odd degrees. If T comes equipped with an S^1-action, one can also define an S^1-equivariant version:

$$\mathrm{fil}^\star_{\mathrm{ev},hS^1}\,T^{hS^1} \coloneqq \lim_{\substack{T \to E \text{ even} \\ S^1\text{-equivariant}}} \tau_{\geqslant 2\star}\bigl(E^{hS^1}\bigr)\,,$$

where we now also impose that $T \to E$ is S^1-equivariant. If R is an $\mathbb{E}_\infty$-k-algebra, these constructions can be applied to $T = \mathrm{THH}(R/k)$ to get filtrations

$$\mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}(R/k) \quad \text{and} \quad \mathrm{fil}^\star_{\mathrm{ev},hS^1} \mathrm{THH}(R/k)^{hS^1}$$

on $\mathrm{THH}(R/k)$ and $\mathrm{THH}(R/k)^{hS^1} \simeq \mathrm{TC}^-(R/k)$, respectively. In the case where $k = \mathbb{Z}$ and R is quasi-syntomic, Hahn–Raksit–Wilson [HRW22, Theorem 5.0.2] show that Antieau's motivic filtration agrees with their S^1-equivariant even filtration:

$$\mathrm{fil}^\star_{\mathrm{mot}} \mathrm{HC}^-(R) \simeq \mathrm{fil}^\star_{\mathrm{ev},hS^1} \mathrm{HH}(R)^{hS^1}\,.$$

We'll be interested in the case where k is one of the following $\mathbb{E}_\infty$-ring spectra.

1.22. Complex K-theory spectra. — We define the *periodic complex K-theory spectrum* KU as the spectrum representing periodic complex K-theory, so that $\mathrm{KU}^0(X)$ is the Grothendieck group of $\mathbb{C}$-vector bundles on X whenever X is a compact Hausdorff space. We also define the *connective complex K-theory spectrum* $\mathrm{ku} \coloneqq \tau_{\geqslant 0}(\mathrm{KU})$ as the connective cover of KU. These are well-known to admit $\mathbb{E}_\infty$-structures and their homotopy groups are given by

$$\pi_*(\mathrm{KU}) \cong \mathbb{Z}[\beta^{\pm 1}] \quad \text{and} \quad \pi_*(\mathrm{ku}) \cong \mathbb{Z}[\beta]\,,$$

where the generator $\beta \in \pi_2(\mathrm{KU})$ is called the *Bott element*.

The following unpublished calculation of Arpon Raksit reveals an astonishing connection between the S^1-equivariant even filtration on $\mathrm{TC}^-(-/\mathrm{ku})$ and another construction that we've seen before.

1.23. Theorem (Raksit, unpublished; see Theorem 9.10)**.** — *Let $\mathrm{ku}[x]$ denote the flat polynomial ring over ku. Then the associated graded of the S^1-equivariant even filtration*

$$\Sigma^{-2*} \mathrm{gr}^*_{\mathrm{ev},hS^1} \mathrm{TC}^-\big(\mathrm{ku}[x]/\mathrm{ku}\big) \simeq \mathrm{fil}^\star_{q\text{-}\mathrm{Hdg},\square}\, q\text{-}\Omega^*_{\mathbb{Z}[x]/\mathbb{Z},\square}$$

is given (up to shift) by the q-Hodge filtration from 1.11, applied to $\mathbb{Z}[x]$ equipped with the identical framing $\square\colon \mathbb{Z}[x] \to \mathbb{Z}[x]$.

1.24. Remark. — In fact, Raksit's calculation works more generally for $\mathrm{TC}^-(e[x]/e)$, where $e \coloneqq \tau_{\geqslant}(E)$ is the connective cover of an even-periodic $\mathbb{E}_\infty$-ring spectrum. The resulting 0^{th} graded piece $\mathrm{gr}^0_{\mathrm{ev},hS^1} \mathrm{TC}^-(e[x]/e)$ turns out to be the *F_E-de Rham complex of $\mathbb{Z}[x]$* in the sense of [DM23, Definition 4.3.6], where F_E is the formal group law associated to a complex orientation $t \in \pi_{-2}(E^{hS^1})$. Roughly, the F_E-de Rham complex is a generalisation of the q-de Rham complex of $(\mathbb{Z}[x], \square)$, in which the differentials send $x^m \mapsto \langle m\rangle_E(t) x^{m-1}\,\mathrm{d}x$, where $\langle m\rangle_E(t) \coloneqq [m]_E(t)/t$ denotes the reduced m-series of F_E.

This opens up the exciting possibility of higher chromatic versions of q-de Rham cohomology. We'll include some speculation in that direction in 1.46.

Our main goal in Part II is to show the following generalization of Raksit's result, which also yields a q-de Rham analogue of Antieau's Theorem 1.20.

1.25. Theorem (see Theorem 7.27)**.** — *Let R be quasi-syntomic and $2 \in R^\times$. Suppose that R admits a lift to a connective $\mathbb{E}_2$-ring spectrum $\mathbb{S}_R$ such that $R \simeq \mathbb{S}_R \otimes \mathbb{Z}$. Then there* *exists a q-Hodge filtration $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_{R/\mathbb{Z}}$ on the derived q-de Rham complex of R such that the associated graded of the S^1-equivariant even filtration*[(1.2)]

$$\Sigma^{-2*}\,\mathrm{gr}^*_{\mathrm{ev},hS^1}\,\mathrm{TC}^-(\mathrm{ku}\otimes\mathbb{S}_R/\mathrm{ku})\simeq \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{R/\mathbb{Z}}$$

is given (up to shift) by the completion of this filtration.

1.26. Remark. — Given that ku is a deformation of $\mathbb{Z}\simeq \mathrm{ku}/\beta$ in "homotopical direction", it shouldn't come as a surprise that we see a deformation of $\mathrm{fil}^\star_{\mathrm{Hdg}}\,\widehat{\mathrm{dR}}_{R/\mathbb{Z}}$ appearing. It is surprising, however, that the deformation is always q-de Rham cohomology.

Under the identification above, q corresponds to a canonical class in $\pi_0(\mathrm{ku}^{hS^1})$: If we regard the standard $\mathbb{C}$-representation of S^1 as a $\mathbb{C}$-vector bundle on the classifying space $\mathrm{B}S^1$, then its image under $\mathrm{KU}^0(\mathrm{B}S^1)\cong\pi_0(\mathrm{ku}^{hS^1})$ agrees with the image of q.

1.27. Remark. — Thanks to Burklund's breakthrough on the construction of multiplicative structures on quotients [Bur22], it's easy to construct quasi-syntomic rings R for which a spherical lift $\mathbb{S}_R$ as in Theorem 1.25 exists. We'll discuss several such examples in §9.

The constructions in and the proof of Theorem 1.25 will occupy §§5–7. The key ingredient is a result by Sanath Devalapurkar ([Dev25, Theorem 6.4.1]; see Theorem 7.2), who constructs an equivalence of S^1-equivariant $\mathbb{E}_\infty$-ring spectra

$$\mathrm{THH}\big(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p[\![q-1]\!]\big)^\wedge_p\xrightarrow{\ \simeq\ }\tau_{\geqslant 0}\big(\mathrm{ku}^{tC_p}\big)$$

for all primes $p>2$. As an S^1-equivariant $\mathbb{E}_1$-equivalence, this was shown for all primes in unpublished work of Thomas Nikolaus (see Theorem 7.17). The S^1-equivariant $\mathbb{E}_\infty$-equivalence is also conjectured to be true for $p=2$; if this could be shown, the condition $2\in R^\times$ in Theorem 1.25 could likely be removed.

Devalapurkar's equivalence allows us to relate the p-completion $\mathrm{TC}^-(\mathrm{ku}\otimes\mathbb{S}_R/\mathrm{ku})^\wedge_p$ to $\mathrm{TC}^-(\widehat{R}_p[\zeta_p]/\mathbb{S}_p[\![q-1]\!])^\wedge_p$. The S^1-equivariant even filtration on the latter is known to compute prismatic cohomology (see Proposition A.17), which in the case at hand agrees with the p-completed derived de Rham complex $(q\text{-dR}_{R/\mathbb{Z}})^\wedge_p$ by Theorem 1.9(*b*). This allows us to go from q-de Rham complexes to $\mathrm{TC}^-(-/\mathrm{ku})$, which eventually gives rise to the theorem. See also [Dev25, Corollary 6.4.2] for a closely related observation.

Furthermore, we'll show a version of Theorem 1.25 in which $\mathbb{S}_R$ is allowed to only be $\mathbb{E}_1$. This case will be particularly interesting because it'll allow us to compute q-Hodge filtrations explicitly in Part III. To formulate the $\mathbb{E}_1$-version, let us put ourselves in a p-complete situation, where p is any prime (with $p=2$ allowed) and let us assume that R/p is *semiperfect* in the sense that the Frobenius $(-)^p\colon R/p\to R/p$ is surjective. In combination with R being p-quasi-syntomic, this assumption guarantees that $\mathrm{TC}^-(\mathrm{ku}\otimes\mathbb{S}_R/\mathrm{ku})^\wedge_p$ is even (see the argument in Remark 6.4). So we can consider the double-speed Whitehead filtration $\tau_{\geqslant 2\star}\,\mathrm{TC}^-(\mathrm{ku}\otimes\mathbb{S}_R/\mathrm{ku})^\wedge_p$ as an ad-hoc replacement of the even filtration, which isn't defined in this case as $\mathrm{TC}^-(\mathrm{ku}\otimes\mathbb{S}_R/\mathrm{ku})$ is only an $\mathbb{E}_0$-ring spectrum. The assumption also guarantees that the p-completed derived (q-)de Rham complexes $(q\text{-dR}_{R/\mathbb{Z}})^\wedge_p$ and $(\mathrm{dR}_{R/\mathbb{Z}})^\wedge_p$ are concentrated in degree 0, so that we may regard them as ordinary rings (see Lemma 4.18). The result in the $\mathbb{E}_1$ case is then as follows:

(1.2) Note that in this situation $\mathrm{THH}(\mathrm{ku}\otimes\mathbb{S}_R/\mathrm{ku})$ is only an $\mathbb{E}_1$-ring spectrum, so the Hahn–Raksit–Wilson construction of the even filtration won't apply. Instead, we'll be working with a version of that construction due to Piotr Pstrągowski [Pst23], which also applies to $\mathbb{E}_1$-ring spectra. We'll furthermore use a construction of Raksit [PR; AR24] to get an S^1-equivariant version of Pstrągowski's even filtration. The details are explained in 6.8 and 7.23.

1.28. Theorem (see Theorem 7.18). — *Let R be a p-complete p-torsion free p-quasi-syntomic ring such that R/p is semiperfect. Suppose that R admits a lift to a p-complete connective $\mathbb{E}_1$-ring spectrum $\mathbb{S}_R$ such that $R \simeq \mathbb{S}_R \otimes \mathbb{Z}$. Then there exists a filtration $\mathrm{fil}^\star_{q\text{-Hdg}}(q\text{-}\mathrm{dR}_{R/\mathbb{Z}})^\wedge_p$, which q-deforms the p-completed Hodge filtration, such that*

$$\pi_{2*}\,\mathrm{TC}^-(\mathrm{ku}\otimes\mathbb{S}_R/\mathrm{ku})^\wedge_p \simeq \mathrm{fil}^\star_{q\text{-Hdg}}(q\text{-}\widehat{\mathrm{dR}}_{R/\mathbb{Z}})^\wedge_p$$

is the completion of this filtration. Moreover, $\mathrm{fil}^\star_{q\text{-Hdg}}(q\text{-}\mathrm{dR}_{R/\mathbb{Z}})^\wedge_p$ can explicitly described as the preimage of the combined Hodge and $(q-1)$-adic filtration $\mathrm{fil}^\star_{(\mathrm{Hdg},q-1)}(\mathrm{dR}_{R/\mathbb{Z}})^\wedge_p[1/p]\llbracket q-1\rrbracket$ under the canonical map

$$\left(q\text{-}\mathrm{dR}_{R/\mathbb{Z}}\right)^\wedge_p \longrightarrow \left(\mathrm{dR}_{R/\mathbb{Z}}\right)^\wedge_p\left[\tfrac{1}{p}\right]\llbracket q-1\rrbracket\,.$$

In particular, the filtration $\mathrm{fil}^\star_{q\text{-Hdg}}(q\text{-}\mathrm{dR}_{R/\mathbb{Z}})^\wedge_p$ is independent of the choice of $\mathbb{S}_R$!

1.29. Remark. — It's natural to ask whether the spherical lifts $\mathbb{S}_R$ in Theorems 1.25 and 1.28 can be replaced by lifts ku_R to ku. We don't expect that this works. At the very least Theorem 1.28 cannot work with lifts to ku, since the ring that we use to obtain the contradiction in Lemma 3.3 admits an $\mathbb{E}_1$-lift to ku (e.g. by [HW18]).

We do, however, expect that Theorem 1.28 and the p-complete variant of Theorem 1.25 (see Theorem 7.9) are already true if we replace $\mathbb{S}_R$ with a lift j_R to the j *spectrum*, which is defined as the connective cover $j \coloneqq \tau_{\geqslant 0}(\mathbb{S}_{K(1)})$ of the $K(1)$-local sphere. If we had a chromatic height 2 analogue of the equivalence $\mathrm{THH}(\mathbb{Z}_p)^\wedge_p \simeq \tau_{\geqslant 0}(j^{tC_p})$ from [DR25, Theorem 0.1.4] available, this could be shown along the lines of [DR25, §5].

A lift to j also seems to be the right condition in light of the following: It will be apparent from the construction that the equivalences from Theorem 1.28 and the p-complete variant of Theorem 1.25 (see Theorem 7.9) are $\mathbb{Z}_p^\times$-equivariant with respect to the Adams action on $\mathrm{ku}^\wedge_p$ and a certain Adams action on the q-de Rham complex that we'll explain in A.20. Now a lift j_R is roughly the same as a lift ku_R together with lifts of the Adams operations, so to get the additional $\mathbb{Z}_p^\times$-equivariance, we need at least a lift to j.

§1.4. Habiro cohomology and genuine equivariant homotopy theory

By Theorem 1.25, we can construct examples of q-Hodge filtrations using topological Hochschild/negative cyclic homology over ku. We can then apply Theorem 1.16 (or its fully derived version Theorem 3.11) to these q-Hodge filtrations to see that the associated q-Hodge complexes descend canonically to the Habiro ring. It's a natural question if this Habiro descent can also be expressed in terms of $\mathrm{THH}(-/\mathrm{ku})$. This is indeed the case, as we'll explain in §8.

The homotopical incarnation of the Habiro descent involves *genuine equivariant homotopy theory*. Since this is not part of the standard repertoire of arithmetic geometry, we'll offer the reader a crash course in §8.1. For now, we'll only explain the rough idea in the case of KU.

1.30. Genuine equivariant KU. — Let G be a compact Lie group, which we let act trivially on KU. It would certainly be nice if the homotopy fixed points KU^{hG} were the spectrum that represents the cohomology theory given by $\mathbb{C}$-vector bundles with G-action. More precisely, for every compact Hausdorff space X, we would like for $(\mathrm{KU}^{hG})^0(X)$ to be the Grothendieck group of bundles of G-representations on Y.

Unfortunately, this turns out to be quite false. It fails already in the simplest possible case, where $G = C_m$ is a finite cyclic group of order $m > 1$ and $X \simeq *$ is a point: In this case the Grothendieck group would be the representation ring

$$R_{\mathbb{C}}(C_m) \cong \mathbb{Z}[q]/(q^m - 1)\,, \quad \text{whereas} \quad \left(\mathrm{KU}^{hC_m}\right)^0(*) \cong \mathbb{Z}\llbracket q-1\rrbracket/(q^m-1)$$

holds by the Atiyah–Segal completion theorem [AS69].

Genuine G-equivariant homotopy theory fixes such issues by incorporating G-actions on a more fundamental level into the way we build spectra. To any genuine G-equivariant spectrum Y one can associate a spectrum Y^G of *genuine fixed points*, which behaves more like we would expect. For example, KU can be naturally equipped with a genuine G-equivariant structure for every compact Lie group G, and in the case $G = C_m$ the genuine fixed points really satisfy

$$\pi_0\left(\mathrm{KU}^{C_m}\right) \cong \mathbb{Z}[q]/(q^m - 1)\,,$$

as we'll see 8.33. In general, genuine fixed points often behave like a *decompletion* of homotopy fixed points, just like Question 1.1 asks for a decompletion of q-de Rham cohomology.

1.31. Genuine equivariant structure on THH. — The cyclotomic structure on $\mathrm{THH}(\mathbb{S}_R)$ allows us to upgrade the action of the finite cyclic subgroup $C_m \subseteq S^1$ to a genuine action for all $m \in \mathbb{N}$ (see e.g. [NS18, Theorem II.6.3]). Together with the corresponding structure on KU from 1.30, we obtain an upgrade of the C_m-action on $\mathrm{THH}(\mathrm{KU} \otimes \mathbb{S}_R/\mathrm{KU}) \simeq \mathrm{THH}(\mathbb{S}_R) \otimes \mathrm{KU}$ to a genuine C_m-action.

The genuine C_m-fixed points $\mathrm{THH}(\mathrm{KU}\otimes\mathbb{S}_R/\mathrm{KU})^{C_m}$ will still carry a residual S^1/C_m-action and so we can form

$$\left(\mathrm{THH}(\mathrm{KU} \otimes \mathbb{S}_R/\mathrm{KU})^{C_m}\right)^{h(S^1/C_m)}\,.$$

In 8.47 and 8.59, we'll propose a construction of suitable genuine S^1-equivariant even filtrations $\mathrm{fil}^\star_{\mathrm{ev},S^1}$ on these objects. Afterwards we'll show that these indeed realise the Habiro descent from Theorem 1.25 homotopically:

1.32. Theorem (see Theorem 8.63)**.** — *Let R be quasi-syntomic and $2 \in R^\times$. Suppose that R admits a lift to a connective $\mathbb{E}_2$-ring spectrum*[(1.3)] *$\mathbb{S}_R$ such that $R \simeq \mathbb{S}_R \otimes \mathbb{Z}$ and let $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R/\mathbb{Z}}$ be the q-Hodge filtration from Theorem 1.25. Then the associated q-Hodge complex is*

$$q\text{-}\mathrm{Hdg}_{(R,\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}})/\mathbb{Z}} \simeq \mathrm{gr}^0_{\mathrm{ev},hS^1}\, \mathrm{TC}^-(\mathrm{KU} \otimes \mathbb{S}_R/\mathrm{KU})\,,$$

and its descent to the Habiro ring from Theorem 1.16 is given by

$$q\text{-}\mathcal{H}\mathrm{dg}_{(R,\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}})/\mathbb{Z}} \simeq \lim_{m\in\mathbb{N}} \mathrm{gr}^0_{\mathrm{ev},S^1}\left(\left(\mathrm{THH}(\mathrm{KU} \otimes \mathbb{S}_R/\mathrm{KU})^{C_m}\right)^{h(S^1/C_m)}\right).$$

§1.5. Refined localising invariants and $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q})$

The homotopical theory of q-Hodge filtrations and Habiro cohomology that we develop in Part II has the major drawback that it cannot be functorial in R, since the spherical $\mathbb{E}_2$- (or $\mathbb{E}_1$-)lifts cannot be chosen functorially. However, if R is a $\mathbb{Q}$-algebra, then spherical lifts exist tautologically: We can just take R itself, since $R \simeq R \otimes \mathbb{Z}$ is true for $\mathbb{Q}$-algebras.

(1.3) Again, there will also be an $\mathbb{E}_1$-version under certain additional assumptions on R.

But the results won't be interesting in this case. For instance, the q-Hodge filtration from Theorem 1.25 will just be the combined Hodge and $(q-1)$-adic filtration $\mathrm{fil}^\star_{(\mathrm{Hdg},q-1)}\,\mathrm{dR}_{R/\mathbb{Z}}\llbracket q-1\rrbracket$: This is essentially due to the fact that $\mathrm{ku}\otimes\mathbb{Q}\simeq\mathbb{Q}[\beta]$ is a polynomial algebra on a generator in homotopical degree 2 and it's closely related to Theorem 1.9(c), which tells us that q-de Rham cohomology doesn't contain any new information rationally. In particular, whatever cohomology theory for $\mathbb{Q}$-varieties X we might be able to construct out of the even filtration on $\mathrm{TC}^-(\mathrm{ku}\otimes X/\mathrm{ku})$, the result will always be rational again. Therefore, it will never allow for any comparison map to non-rational cohomology theories, like singular cohomology of $X(\mathbb{C})$ or étale cohomology of $X_{\overline{\mathbb{Q}}}$ with torsion coefficients.

In Part III of this thesis, we'll investigate a refined version of $\mathrm{TC}^-(-/\mathrm{ku})$, due to Efimov and Scholze, which should be able to overcome these issues. Let first explain how this refinement works and what we know about it, then we'll speculate in §1.6 how this should lead to improved versions of q-de Rham/q-Hodge/Habiro cohomology for $\mathbb{Q}$-varieties.

1.33. Rigid symmetric monoidal ∞-categories. — The construction of refined TC^- is based on the following notion due to Gaitsgory–Rozenblyum (see [GR17, Definition I.9.1.2] as well as [Ram24, Corollary 4.57] for a proof that their definition is equivalent to the one we use here). A presentable stable symmetric monoidal[(1.4)] is called *rigid* if the following two conditions are satisfied:

(a) The tensor unit $\mathbb{1}\in\mathcal{E}$ is compact.

(b) $\mathcal{E}$ is generated under colimits by objects of the form $X\simeq\operatorname{colim}(X_1\rightarrow X_2\rightarrow\cdots)$, where each transition map $X_n\rightarrow X_{n+1}$ is *trace-class*. That is, if $X_n^\vee\coloneqq\underline{\mathrm{Hom}}_{\mathcal{E}}(X_n,\mathbb{1})$ denotes the predual of X_n, there exists a morphism $\eta\colon\mathbb{1}\rightarrow X_n^\vee\otimes X_{n+1}$ such that $X_n\rightarrow X_{n+1}$ agrees with the composition
$$X_n\simeq X_n\otimes\mathbb{1}\xrightarrow{\eta}X_n\otimes X_n^\vee\otimes X_{n+1}\xrightarrow{\mathrm{ev}}\mathbb{1}\otimes X_{n+1}\simeq X_{n+1}\,.$$
(see the review in §5.2).

We note that a compactly generated symmetric monoidal presentable stable ∞-category $\mathcal{E}$ is rigid if and only if "compact $\Leftrightarrow$ dualisable" holds in $\mathcal{E}$.

1.34. Rigidity of localising motives. — Let $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$ denote the ∞-category of compactly generated presentable stable ∞-categories and functors that preserve colimits and compact objects. For us, a *localising invariant* is a functor
$$T\colon\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}\longrightarrow\mathcal{D}$$
into a stable ∞-category $\mathcal{D}$ such that T preserves filtered colimits and sends short exact sequences in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$ (that is, sequences $\mathcal{C}'\rightarrow\mathcal{C}\rightarrow\mathcal{C}''$ that are both fibre and cofibre sequences) to cofibre sequences in $\mathcal{D}$.

Blumberg–Gepner–Tabuada [BGT16] defined an ∞-category $\mathrm{Mot}^{\mathrm{loc}}$ of *localising motives* as the target of the universal localising invariant
$$\mathcal{U}^{\mathrm{loc}}\colon\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}\longrightarrow\mathrm{Mot}^{\mathrm{loc}}\,.$$
Any localising invariant T as above then factors uniquely through a functor $\mathrm{Mot}^{\mathrm{loc}}\rightarrow\mathcal{D}$, which we usually (by slight abuse of notation) still denote by T and call a localising invariant.

(1.4) By convention, we'll always assume that the tensor product $-\otimes-$ in a presentable symmetric monoidal ∞-category commutes with colimits in both variables.

A relative variant of this construction was introduced by Efimov [Efi25, Definition 1.20]: For any rigid presentable stable symmetric monoidal ∞-category $\mathcal{E}$, he defines an ∞-category $\mathrm{Mot}^{\mathrm{loc}}_{\mathcal{E}}$ of *localising motives over* $\mathcal{E}$. In [Efi-Rig], Efimov shows the remarkable theorem that the presentable stable symmetric monoidal ∞-category $\mathrm{Mot}^{\mathrm{loc}}_{\mathcal{E}}$ is again rigid.

1.35. Refined localising invariants. — By an observation of Efimov and Scholze, the rigidity of localising motives can be used to construct refined versions of certain localising invariants as follows: Suppose

$$T\colon \mathrm{Mot}^{\mathrm{loc}}_{\mathcal{E}} \longrightarrow \mathcal{D}$$

is a symmetric monoidal localising invariant whose target is *not* rigid. By abstract nonsense, T factors uniquely through the rigidifaction $\mathcal{D}^{\mathrm{rig}}$ of $\mathcal{D}$:

$$\begin{array}{ccc} \mathrm{Mot}^{\mathrm{loc}}_{\mathcal{E}} & \xrightarrow{T} & \mathcal{D} \\ & \underset{T^{\mathrm{ref}}}{\searrow} & \uparrow \\ & & \mathcal{D}^{\mathrm{rig}} \end{array}$$

We then define *refined* T to be the factorisation $T^{\mathrm{ref}}\colon \mathrm{Mot}^{\mathrm{loc}}_{\mathcal{E}} \to \mathcal{D}^{\mathrm{rig}}$.

1.36. Refined THH/TC^{-}. — If k is an $\mathbb{E}_\infty$-ring spectrum, we can apply the above to the case $\mathcal{E} = \mathrm{Mod}_k(\mathrm{Sp})$. Write $\mathrm{Mot}^{\mathrm{loc}}_k \coloneqq \mathrm{Mot}^{\mathrm{loc}}_{\mathrm{Mod}_k(\mathrm{Sp})}$ for short. Then

$$\mathrm{THH}(-/k)\colon \mathrm{Mot}^{\mathrm{loc}}_k \longrightarrow \mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1}$$

is an example of a symmetric monoidal localising invariant with rigid source but non-rigid target. We let $\mathrm{THH}^{\mathrm{ref}}(-/k)$ denote its refinement.

If k is complex orientable and $t \in \pi_{-2}(k^{hS^1})$ is a complex orientation generator, then taking S^1-fixed points induces a symmetric monoidal equivalence

$$(-)^{hS^1}\colon \mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1} \xrightarrow{\simeq} \mathrm{Mod}_{k^{hS^1}}(\mathrm{Sp})_t^\wedge$$

between k-modules with S^1-action and t-complete k^{hS^1}-modules (see Lemma 11.2). Scholze and Efimov then define $\mathrm{TC}^{-,\mathrm{ref}}(-/k)$ to be the composition

$$\mathrm{TC}^{-,\mathrm{ref}}(-/k)\colon \mathrm{Mot}^{\mathrm{loc}}_k \xrightarrow{\mathrm{THH}^{\mathrm{ref}}(-/k)} \left(\mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1}\right)^{\mathrm{rig}} \simeq \left(\mathrm{Mod}_{k^{hS^1}}(\mathrm{Sp})_t^\wedge\right)^{\mathrm{rig}}.$$

In this case, [Efi25, Theorem 4.2] allows us to pin down the rigidification on the right-hand side: It agrees with Efimov's ∞-category $\mathrm{Nuc}(k^{hS^1})$ of *nuclear* k^{hS^1}*-modules*, defined as the full sub-∞-category of $\mathrm{Ind}(\mathrm{Mod}_{k^{hS^1}}(\mathrm{Sp})_t^\wedge)$ generated under colimits by sequential ind-objects of the form "colim"$(M_1 \to M_2 \to \cdots)$ such that each $M_n \to M_{n+1}$ is trace-class.

1.37. Remark. — The refinement procedure from 1.35 is very sensitive to the choice of $\mathcal{E}$. This is a feature, not a bug: It offers a lot of flexibility, even if we stick to THH. For example, if C is a complete non-archimedean algebraically closed field, one can look at the refinement $\mathrm{THH}^{\mathrm{ref}}_{/\mathcal{O}_C}(-;\mathbb{Z}_p)$ of the functor

$$\mathrm{THH}(-;\mathbb{Z}_p)\colon \mathrm{Mot}^{\mathrm{loc}}_{\mathcal{O}_C} \longrightarrow \mathrm{Mod}_{\mathrm{THH}(\mathcal{O}_C;\mathbb{Z}_p)}\big(\mathrm{Sp}^{\mathrm{B}S^1}\big)_p^\wedge.$$

That is, we refine (p-completed) absolute THH, but only accept motives over $\mathcal{O}_C$ as input.(1.5) This is vastly different from $\mathrm{THH}^{\mathrm{ref}}(-;\mathbb{Z}_p)$, where we would allow all localising motives.

Scholze and Efimov [Sch24a] have sketched a computation of $\mathrm{THH}^{\mathrm{ref}}_{/\mathcal{O}_C}(C;\mathbb{Z}_p)$—or rather $\mathrm{TC}^{-,\mathrm{ref}}_{/\mathcal{O}_C}(C;\mathbb{Z}_p)$, which is equivalent by Lemma 11.2—, by reducing the problem to the known computation of $\mathrm{THH}(\mathcal{O}_C/p^\alpha;\mathbb{Z}_p)$ for all $\alpha \geqslant 1$. We'll explain a general version of this reduction in Theorem 10.17 and Example 10.27.

This brings us to the main question that we investigate in Part III of this thesis.

1.38. Question. — *What is* $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q})$*?*

It's clear from the start that the answer to this question is non-trivial: While $\mathrm{THH}(\mathbb{Q}) \simeq \mathbb{Q}$, the p-completions $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q})_p^\wedge$ are non-zero for all primes p. Indeed, this follows from Efimov's and Scholze's result in Remark 1.37, or alternatively from Theorem 1.40 below. The observation $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q})_p^\wedge \not\simeq 0$ is certainly welcome in view of the discussion at the beginning of §1.5

But actually computing $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q})$, or just its p-completions, seems currently out of reach: We'll explain in Example 10.27 how to reduce this to a computation of $\mathrm{THH}(\mathbb{S}/p^\alpha)$ for all sufficiently large α, but computing these spectra seems impossible at the moment.

Scholze and Efimov have suggested that a more approachable goal would be to compute the base change $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q}) \otimes \mathrm{MU} \simeq \mathrm{THH}^{\mathrm{ref}}(\mathrm{MU} \otimes \mathbb{Q}/\mathrm{MU})$ and then to attack the original question—to the extent in which that's possible—via Adams–Novikov descent. While we still don't know how to compute $\mathrm{THH}(\mathbb{S}/p^\alpha)$ after base change to MU, we can compute the answer after base change to ku thanks to Theorems 1.25 and 1.28. This leads to a computation of

$$\mathrm{THH}^{\mathrm{ref}}(\mathrm{ku} \otimes \mathbb{Q}/\mathrm{ku}) \quad \text{and} \quad \mathrm{THH}^{\mathrm{ref}}(\mathrm{KU} \otimes \mathbb{Q}/\mathrm{KU})\,.$$

In §§11–12, which are based on the joint work [MW24] with Samuel Meyer, we explain this computation. First note that it is an equivalent problem to compute $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku} \otimes \mathbb{Q}/\mathrm{ku})$ and $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU} \otimes \mathbb{Q}/\mathrm{KU})$. Via Burklund's results from [Bur22], we construct a certain system of $\mathbb{E}_1$-algebra structures on $\mathbb{S}/m$, where m ranges through a certain coinitial sub-poset $\mathbb{N}^\sharp \subseteq \mathbb{N}$. From these $\mathbb{E}_1$-algebras and Theorem 1.28(1.6) we'll construct pro-systems of completed q-Hodge filtrations and q-Hodge complexes

$$\underset{m\in\mathbb{N}^\sharp}{\text{“}\lim\text{”}}\, \mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}} \quad \text{and} \quad \underset{m\in\mathbb{N}^\sharp}{\text{“}\lim\text{”}}\, q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}\,.$$

We'll show that these are idempotent as pro-algebras. Using the notion of *killing idempotent pro-algebras* that we'll explain in §10.1, we'll then derive a preliminary description of homotopy groups of $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku} \otimes \mathbb{Q}/\mathrm{ku})$ and $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU} \otimes \mathbb{Q}/\mathrm{KU})$:

1.39. Theorem (joint with Meyer [MW24]; see Theorem 11.15)**.** — $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku} \otimes \mathbb{Q}/\mathrm{ku})$ *and* $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU} \otimes \mathbb{Q}/\mathrm{KU})$ *are concentrated in even degrees. Moreover, their even homotopy groups are described as follows:*

(*a*) $\pi_{2*}\,\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku} \otimes \mathbb{Q}/\mathrm{ku}) \cong \mathrm{A}^*_{\mathrm{ku}}$*, where* $\mathrm{A}^*_{\mathrm{ku}}$ *is the idempotent nuclear graded* $\mathbb{Z}[\beta]\llbracket t\rrbracket$*-algebra obtained by killing the idempotent pro-algebra* $\text{“}\lim\text{”}_{m\in\mathbb{N}^\sharp}\, \mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$.

(1.5) Historically, $\mathrm{THH}^{\mathrm{ref}}_{/\mathcal{O}_C}(-;\mathbb{Z}_p)$ is the first refined invariant considered by Scholze and Efimov.

(1.6) Note that Theorem 1.28 doesn't apply in the case $R = \mathbb{Z}/p^\alpha$, since this ring is not torsion free. We'll instead apply it in the case $R = \mathbb{Z}_p\{x\}_\infty/x^\alpha$, where $\mathbb{Z}_p\{x\}_\infty$ denotes the free perfect δ-ring on a generator x, and then use base change along the δ-ring map $\mathbb{Z}_p\{x\}_\infty \to \mathbb{Z}_p$ that sends $x \mapsto p$.

(*b*) $\pi_{2*}\,\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU}\otimes\mathbb{Q}/\mathrm{KU})\cong \mathrm{A}_{\mathrm{KU}}[\beta^{\pm1}]$, *where* A_{KU} *is the idempotent nuclear* $\mathbb{Z}[\![q-1]\!]$-*algebra obtained by killing the idempotent pro-algebra* "$\lim_{m\in\mathbb{N}^\natural}$" $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}$.

In §11.2, we'll compute $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$ and $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}$ explicitly, which enables us to make the above descriptions more concrete. To be able to formulate such a concrete description in geometric terms, we'll replace ku and KU by their p-completions $\mathrm{ku}_p^\wedge$ and $\mathrm{KU}_p^\wedge$ for an arbitrary prime p. Let us first formulate the geometric result for $\mathrm{KU}_p^\wedge$, as it is easier to state. We put

$$\mathrm{A}_{\mathrm{KU},p}\coloneqq \pi_0\,\mathrm{TC}^{-,\mathrm{ref}}\left(\mathrm{KU}_p^\wedge\otimes\mathbb{Q}/\mathrm{KU}_p^\wedge\right),$$

so $\pi_{2*}\,\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU}_p^\wedge\otimes\mathbb{Q}/\mathrm{KU}_p^\wedge)\cong \mathrm{A}_{\mathrm{KU},p}[\beta^{\pm1}]$. Let also $X\coloneqq \operatorname{Spa}\mathbb{Z}_p[\![q-1]\!]\smallsetminus\{p=0,q=1\}$ be the "analytic locus" where p or $q-1$ is invertible. Then $\mathrm{A}_{\mathrm{KU},p}$ has the following description, confirming a conjecture of Scholze and Efimov:

1.40. Theorem (joint with Meyer [MW24])**.** — *Let* $Z\subseteq X$ *denote the union of the closed subsets* $\operatorname{Spa}(\mathbb{F}_p(\!(q-1)\!),\mathbb{F}_p[\![q-1]\!])$ *and* $\operatorname{Spa}(\mathbb{Q}_p(\zeta_{p^n}),\mathbb{Z}_p[\zeta_{p^n}])$ *for all* $n\geqslant 0$. *Let* $Z^\dagger$ *denote the overconvergent neighbourhood of* Z *in* X *and let* $\mathcal{O}(Z^\dagger)$ *be the nuclear* $\mathbb{Z}_p[\![q-1]\!]$-*algebra of overconvergent functions on* Z. *Then*

$$\mathrm{A}_{\mathrm{KU},p}\cong\mathcal{O}(Z^\dagger)\,.$$

In Fig. 1 we show a picture of $Z^\dagger$. It should be reminiscent of Scholze's famous prismatic picture (a nice depiction of which can be found in [HN20, p. 4]), but the rays are "overconvergently blurred" and the "origin" $\{p=0,q=1\}$ has been removed.

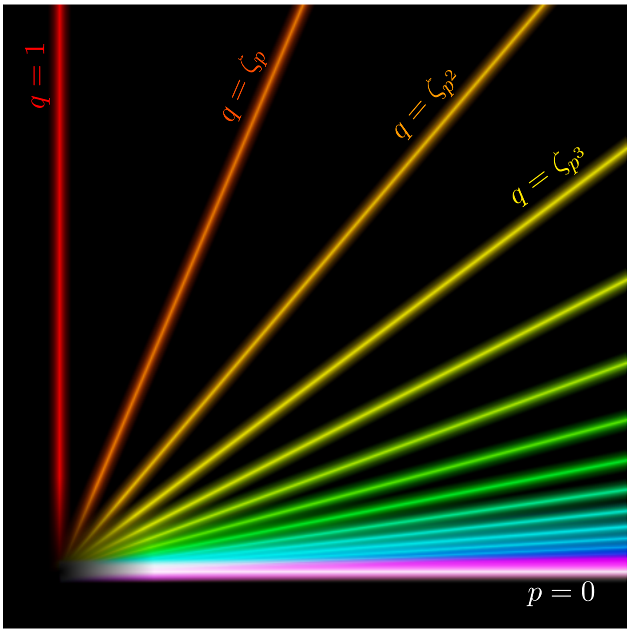


Fig. 1: The overconvergent neighbourhood $Z^\dagger$.

Since $Z^\dagger$ visibly contains the entire infinitesimal neighbourhood of $\{p=0\}$ except for the "origin", we see that $\mathrm{TC}^{-,\mathrm{ref}}((\mathrm{KU}_p^\wedge\otimes\mathbb{Q})/\mathrm{KU}_p^\wedge)_p^\wedge\neq 0$. In particular, it follows that $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q})_p^\wedge\not\simeq 0$, as we've claimed above.

To formulate a similar geometric result for $\mathrm{ku}_p^\wedge$, consider the ungraded ring $\mathbb{Z}[\beta,t]_{(p,t)}^\wedge$ with its (p,t)-adic topology. We wish to encode the graded (p,t)-complete ring $\mathbb{Z}_p[\beta]\llbracket t\rrbracket$ in terms of an action of $\mathbb{G}_m$ on $\operatorname{Spa}\mathbb{Z}[\beta,t]_{(p,t)}^\wedge$, as usual—but we have to be careful: Since we wish that t is a topologically nilpotent element in non-zero graded degree, we can only act by units u "of norm $|u|=1$". More precisely, we have to replace $\mathbb{G}_m$ by the "adic unit circle" $\mathbb{T}\coloneqq \operatorname{Spa}(\mathbb{Z}[u^{\pm 1}],\mathbb{Z}[u^{\pm 1}])$.

With this modification, everything works (as we'll elaborate in §12.2): Declaring β and t to have degree 2 and -2, respectively, determines an action of $\mathbb{T}$ on $\operatorname{Spa}\mathbb{Z}[\beta,t]_{(p,t)}^\wedge$, and we can identify $\mathbb{Z}_p[\beta]\llbracket t\rrbracket$ with the structure sheaf on $(\operatorname{Spa}\mathbb{Z}[\beta,t]_{(p,t)}^\wedge)/\mathbb{T}$, where the quotient is always taken in the derived (or "stacky") sense. We also let $X^*\coloneqq \operatorname{Spa}\mathbb{Z}[\beta,t]_{(p,t)}^\wedge \smallsetminus \{p=0,\beta t=0\}$. Since p and βt are homogeneous, X^* inherits an action of $\mathbb{T}$. Putting

$$\mathrm{A}_{\mathrm{ku},p}^* \coloneqq \pi_{2*}\,\mathrm{TC}^{-,\mathrm{ref}}\left(\mathrm{ku}_p^\wedge\otimes\mathbb{Q}/\mathrm{ku}_p^\wedge\right),$$

we see that $\mathrm{A}_{\mathrm{ku},p}^*$ is a graded $\mathbb{Z}_p[\beta]\llbracket t\rrbracket$-module, hence we can regard it as a quasi-coherent sheaf on $(\operatorname{Spa}\mathbb{Z}[\beta,t]_{(p,t)}^\wedge)/\mathbb{T}$. As we'll see, it is already the pushforward of a sheaf on the open substack $X^*/\mathbb{T}$. This sheaf, which we'll also denote $\mathrm{A}_{\mathrm{ku},p}^*$, can be described as follows:

1.41. Theorem (joint with Meyer [MW24]). — *Let $Z^*\subseteq X^*$ be union of the $\mathbb{T}$-equivariant closed subsets $\{p=0\}$ and $\{[p^n]_{\mathrm{ku}}(t)=0\}$ for all $n\geqslant 0$, where $[p^n]_{\mathrm{ku}}(t)\coloneqq((1+\beta t)^{p^n}-1)/\beta$ denotes the p^n-series of the formal group law of* ku. *Let $Z^{*,\dagger}$ denote the overconvergent neighbourhood of Z^*. Then $Z^{*,\dagger}$ inherits a $\mathbb{T}$-action and*

$$\mathrm{A}_{\mathrm{ku},p}^*\cong \mathcal{O}_{Z^{*,\dagger}/\mathbb{T}}\,.$$

§1.6. Synthesis: Towards a new cohomology theory for $\mathbb{Q}$-varieties

Let us end with an outlook and a bit of speculation. First, it should be possible to adapt the formalism of even filtrations from [HRW22; Pst23] to the nuclear setting to define an even filtration $\mathrm{fil}_{\mathrm{ev},hS^1}^\star\,\mathrm{TC}^{-,\mathrm{ref}}(-/\mathrm{ku})$. We already take some steps towards this goal in §5.

1.42. *q*-de Rham/*q*-Hodge cohomology for $\mathbb{Q}$-varieties. — For a smooth variety X over $\mathbb{Q}$, we can use this even filtration on $\mathrm{TC}^{-,\mathrm{ref}}(-/\mathrm{ku})$ to define cohomology theories $\mathrm{R}\Gamma_{\mathrm{ku}}(X)$ and $\mathrm{R}\Gamma_{\mathrm{KU}}(X)$, the former automatically equipped with a filtration, via

$$\mathrm{fil}^\star\,\mathrm{R}\Gamma_{\mathrm{ku}}(X)\coloneqq\Sigma^{-2*}\,\mathrm{gr}_{\mathrm{ev},hS^1}^*\,\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku}\otimes X/\mathrm{ku})\,,$$
$$\mathrm{R}\Gamma_{\mathrm{KU}}(X)\coloneqq\mathrm{gr}_{\mathrm{ev},hS^1}^0\,\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU}\otimes X/\mathrm{KU})\,.$$

Morally, $\mathrm{fil}^\star\,\mathrm{R}\Gamma_{\mathrm{ku}}(X)$ should be the "q-Hodge-filtered q-de Rham cohomology of X" and $\mathrm{R}\Gamma_{\mathrm{KU}}(X)$ should be the "q-Hodge cohomology of X". But in contrast to the naive notions, which would be rational, $\mathrm{R}\Gamma_{\mathrm{ku}}(X)$ and $\mathrm{R}\Gamma_{\mathrm{KU}}(X)$ will be non-trivial modulo any prime p, as Theorems 1.40 and 1.41 already show.

So it is not out of the question to hope for comparisons to étale cohomology $\mathrm{R}\Gamma_{\text{ét}}(X_{\overline{\mathbb{Q}}},\mathbb{Z}/N)$ with torsion coefficients. We intend to return to this in future work.

1.43. Habiro descent of $\mathrm{R}\Gamma_{\mathrm{KU}}(X)$. — We expect that $\mathrm{R}\Gamma_{\mathrm{KU}}(X)$ can be canonically descended to an object $\mathrm{R}\Gamma_{\mathcal{H}}(X)$ satisfying

$$\mathrm{R}\Gamma_{\mathrm{KU}}(X)\simeq\mathrm{R}\Gamma_{\mathcal{H}}(X)\otimes_{\mathcal{H}}^{\mathrm{L}\blacksquare}\mathbb{Z}\llbracket q-1\rrbracket$$

(here $-\otimes_{\mathcal{H}}^{\mathrm{L}\blacksquare}-$ denotes the *solid tensor product*; see 5.1 below). It should be possible to construct $R\Gamma_{\mathcal{H}}(X)$ via an "analytic" version of Theorem 1.16, but there should also be a construction via genuine equivariant homotopy theory as in Theorem 1.32: Namely, we can regard $\mathrm{THH}(-/\mathrm{KU})$ as a functor

$$\mathrm{THH}(-/\mathrm{KU})\colon \mathrm{Mot}_{\mathrm{KU}}^{\mathrm{loc}} \longrightarrow \mathrm{Mod}_{\mathrm{KU}}(\mathrm{CycnSp})$$

into the ∞-category of KU-modules in cyclonic spectra (see §8.2). Applying the refinement to this functor, we get a version of $\mathrm{THH}^{\mathrm{ref}}(-/\mathrm{KU})$ with values in $\mathrm{Nuc}\,\mathrm{Ind}(\mathrm{Mod}_{\mathrm{KU}}(\mathrm{CycnSp}))$. It should again be possible to construct a version of the even filtration in this ∞-category. Finally, $R\Gamma_{\mathcal{H}}(X)$ should arise as the 0^{th} graded piece of that even filtration on $\mathrm{THH}^{\mathrm{ref}}(\mathrm{KU}\otimes X/\mathrm{KU})$.

The ring of coefficients $\mathcal{A}_{\mathrm{KU}} \coloneqq R\Gamma_{\mathcal{H}}(\operatorname{Spec}\mathbb{Q})$ roughly looks as follows: Generically, it should agree with the Habiro ring, but p-adically, each factor in

$$\widehat{\mathcal{H}}_p \simeq \prod_{(m,p)=1} \mathbb{Z}_p[\zeta_m][\![q-\zeta_m]\!]$$

should be replaced by a base change of Fig. 1. Here's an attempt at a picture:

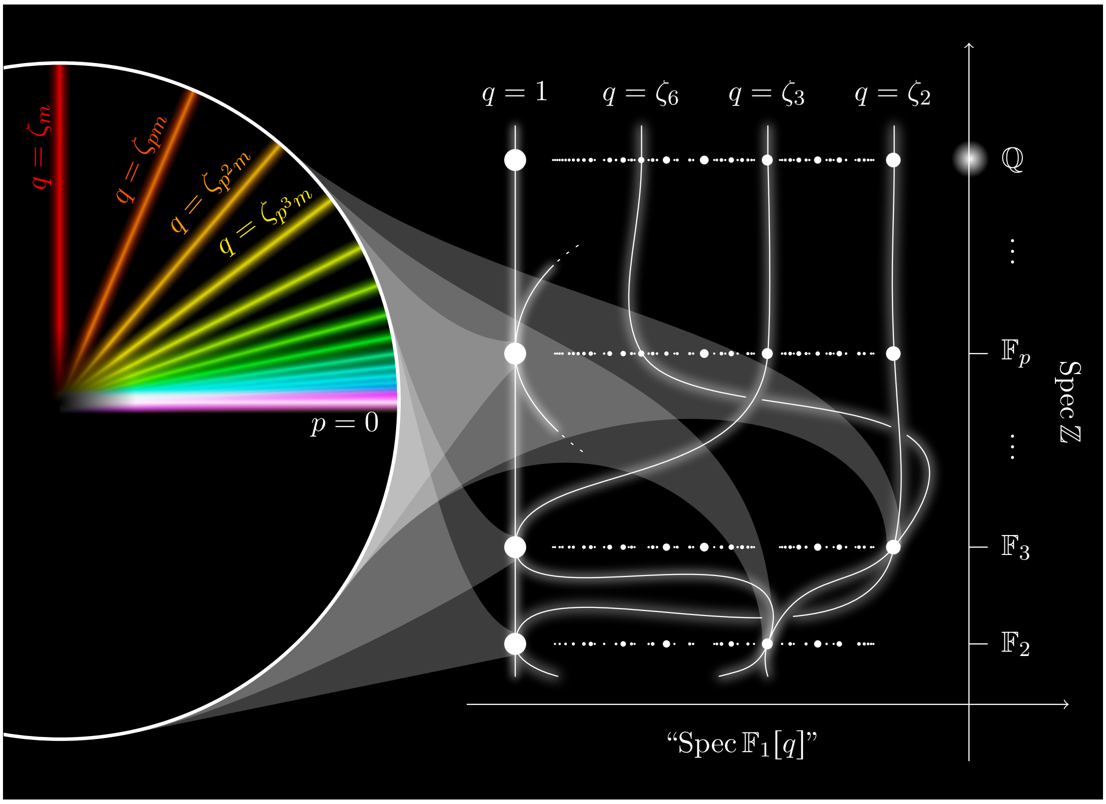


Fig. 2: The analytic spectrum of $\mathcal{A}_{\mathrm{KU}}$. The picture shows how different roots of unity (the "fibres over $\mathbb{F}_1[q]$") collide p-adically. Around each collision, $\operatorname{AnSpec}\mathcal{A}_{\mathrm{KU}}$ should be an overconvergent neighbourhood of the colliding rays, with the collision point removed.

1.44. Habiro stacks. — We furthermore expect that $R\Gamma_{\mathrm{ku}}(-)$, $R\Gamma_{\mathrm{KU}}(-)$, and $R\Gamma_{\mathcal{H}}(-)$ naturally admit *stacky approaches* as in 1.4. In forthcoming work of Devalapurkar–Hahn–Raksit–Yuan [DHRY] (some of which is already contained in [Dev25, §7.1]), it will be explained that the theory of even filtrations can be supplemented by a theory of *even stacks*, which roughly replaces the limit $\mathrm{fil}^\star_{\mathrm{ev}}\, T \simeq \lim_{T\to E \text{ even}} \tau_{\geqslant 2\star}(E)$ by the colimit

$$\operatorname{Spev} T \coloneqq \underset{T\to E \text{ even}}{\operatorname{colim}} \operatorname{Spec} \pi_{2*}(E)/\mathbb{G}_m\,.$$

We hope that an appropriate S^1-equivariant or cyclonic nuclear version of this construction should provide the desired stacky approaches X^{ku}, X^{KU}, and $X^{\mathcal{H}}$.

At this point it is high time to mention that Peter Scholze [Sch25] has proposed another construction of a *Habiro stack* X^{Hab}, which lives over a certain "analytic" version $\mathcal{H}^{\mathrm{an}}$ of the Habiro ring. Funnily enough, $\mathcal{H}^{\mathrm{an}}$ and the base ring $\mathcal{A}_{\mathrm{KU}}$ from 1.43 are almost "complementary" in that q is "close, but not too close" to a root of unity in $\mathcal{H}^{\mathrm{an}}$, whereas q is "overconvergently close" to a root of unity in $\mathcal{A}_{\mathrm{KU}}$. We hope to combine both constructions in future work, each filling in the missing pieces of the other.

Switching back from varieties over $\mathbb{Q}$ to smooth schemes over $\mathbb{Z}$, let us summarise what we now know about Question 1.1.

1.45. The current state of Habiro cohomology. — For a smooth scheme X over $\mathbb{Z}$, we are presently in the comfortable situation where Question 1.1 has not one, but at least three positive answers. Namely:

(*a*) The sheaf cohomology $\mathrm{R}\Gamma(X^{\mathrm{Hab}}, \mathcal{O})$ of Scholze's Habiro stack.

(*b*) The cohomology $\mathrm{R}\Gamma_{\mathcal{H}}(X_{\mathbb{Q}})$ of the generic fibre as sketched in 1.43.

(*c*) If $n \coloneqq \dim(X/\mathbb{Z})$, we can combine Theorems 1.16(*a*) and 1.17 to construct a sheaf of Habiro-descended q-Hodge complexes $q\text{-}\mathcal{H}\mathrm{dg}_{X_{\mathbb{Z}[1/n!]}/\mathbb{Z}}$ on $X_{\mathbb{Z}[1/n!]}$. Its sheaf cohomology $\mathrm{R}\Gamma(X, q\text{-}\mathcal{H}\mathrm{dg}_{X_{\mathbb{Z}[1/n!]}/\mathbb{Z}})$ is another reasonable candidate for the Habiro cohomology of X.

We hope that all three options will turn out to be compatible. For (*a*) and (*b*) we don't know how to do this, but hope to return to this question in future work. For (*b*) and (*c*), the comparison should work by a variant of Theorem 1.32.

By construction, (*a*) has a stacky description, and we've explained in 1.44 why we expect the same for (*b*). For (*c*), this is impossible, due to the rather subtle monoidality properties of the construction that we'll discuss in 4.12. Up to this shortcoming, option (*c*) gives the best integrality properties for primes $p > n$. But also note that both (*a*) and (*b*) contain non-trivial information at primes $p \leqslant n$ that (*c*) can't see.

1.46. Higher chromatic speculation. — Finally, let us point out that the constructions from 1.42 should work just as well if ku and KU are replaced by any $\mathbb{E}_\infty$-ring spectrum. It would also be really interesting to see if "analytic" versions of Raksit's F_E-de Rham complex from Remark 1.24 appear if ku and KU are replaced by e and E.

The long term goal should be to understand $\mathrm{TC}^{-,\mathrm{ref}}(-/\mathrm{MU})$, or better yet, the absolute refined topological Hochschild homology

$$\mathrm{THH}^{\mathrm{ref}}(-)\colon \mathrm{Mot}^{\mathrm{loc}} \longrightarrow \mathrm{Nuc}\,\mathrm{Ind}(\mathrm{CyctSp})$$

and its cyclotomic nuclear even stack (which would need to be defined). In the context of varieties over $\mathbb{Q}$, we should point out that $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q})$ is an $\mathbb{E}_\infty$-algebra over the K-theory spectrum $K(\mathbb{Q})$, which vanishes upon $T(n)$-localisation for $n \geqslant 2$. Due to the delicate nature of the refinement, this doesn't mean that the answer over a higher chromatic base would be trivial, and $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{MU}\otimes X/\mathrm{MU})$ should still contain strictly more information than $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku}\otimes X/\mathrm{ku})$, but that information will necessarily be rather subtle.

§1.7. Notations and conventions

1.47. Conventions on ∞-categories. — We freely use the language of ∞-categories. We denote by Ani and call *anima* what outside of Bonn is known as the ∞-category of spaces.

- **Stable ∞-categories.** We denote by Sp the ∞-category of spectra and by $\mathbb{S} \in \mathrm{Sp}$ the sphere spectrum. For an ordinary ring R, we let $\mathcal{D}(R)$ denote the derived ∞-category of R. We often implicitly regard objects of $\mathcal{D}(R)$ as spectra via the Eilenberg–MacLane functor H, but we'll always suppress this functor in our notation.

  For a stable ∞-category $\mathcal{C}$, we let $\mathrm{Hom}_{\mathcal{C}}(-,-)$ denote the mapping spectra in $\mathcal{C}$. The shift functor and its inverse will always be denoted by Σ and Σ^{-1} (even for $\mathcal{D}(R)$), to avoid confusion with shifts in graded or filtered objects.

- **Symmetric monoidal ∞-categories.** For a symmetric monoidal ∞-category $\mathcal{C}$, we usually denote by $\otimes_{\mathcal{C}}$ and $\mathbb{1}_{\mathcal{C}}$ its tensor product and tensor unit. If no confusion can occur, we simply write $\otimes$ and $\mathbb{1}$ instead. If $\mathcal{C}$ is symmetric monoidal, we let $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$ and $\mathrm{CAlg}(\mathcal{C})$ denote the ∞-categories of $\mathbb{E}_n$-algebras and $\mathbb{E}_\infty$-algebras in $\mathcal{C}$, respectively.

  Whenever we consider a symmetric monoidal ∞-category $\mathcal{C}$ which is stable or presentable, we always implicitly assume that the tensor product commutes in both variables with finite colimits or arbitrary colimits, respectively. In the presentable case, we let $\underline{\mathrm{Hom}}_{\mathcal{C}}(-,-)$ denote the *internal Hom* in $\mathcal{C}$ and $X^\vee \coloneqq \underline{\mathrm{Hom}}_{\mathcal{C}}(X, \mathbb{1})$ the *predual* of an object $X \in \mathcal{C}$.

1.48. Conventions on graded and filtered objects. — For a stable ∞-category $\mathcal{C}$, we let $\mathrm{Gr}(\mathcal{C})$ and $\mathrm{Fil}(\mathrm{Sp})$ denote the ∞-categories of *graded* and (*descendingly*) *filtered objects in* $\mathcal{C}$. The shift in graded or filtered objects is denoted $(-)(1)$. We'll always try to distinguish between *graded/filtered degree* and *homotopical/homological degree.*

- **Descending and ascending filtrations.** Unless specified otherwise, filtrations will be descending by default. An object with a descending filtration is typically denoted

$$\mathrm{fil}^\star X = \left(\cdots \leftarrow \mathrm{fil}^n X \leftarrow \mathrm{fil}^{n+1} X \leftarrow \cdots\right)$$

  and we let $\mathrm{gr}^* X$ denote the *associated graded*, given by $\mathrm{gr}^n X \coloneqq \mathrm{cofib}(\mathrm{fil}^{n+1} X \to \mathrm{fil}^n X)$. We mostly work with filtrations that are constant in degrees $\leqslant 0$ (such as the Hodge filtration and its variants). In this case we'll abusingly write $\mathrm{fil}^\star X = (\mathrm{fil}^0 X \leftarrow \mathrm{fil}^1 X \leftarrow \cdots)$; this should be interpreted as the constant $\mathrm{fil}^0 X$-valued filtration in degrees $\leqslant 0$.

  Sometimes we also consider *ascending* filtrations. Ascendingly filtered objects will be denoted $\mathrm{fil}_\star X = (\cdots \to \mathrm{fil}_n X \to \mathrm{fil}_{n+1} X \to \cdots)$ and the associated graded by $\mathrm{gr}_* X$, where $\mathrm{gr}_n X \coloneqq \mathrm{cofib}(\mathrm{fil}^{n-1} X \to \mathrm{fil}^n X)$.

- **Graded and filtered tensor products.** If $\mathcal{C}$ is presentable stable symmetric monoidal, we'll equip $\mathrm{Gr}(\mathcal{C})$ and $\mathrm{Fil}(\mathcal{C})$ with the Day convolution symmetric monoidal structures. We frequently use the following fact: Let $\mathbb{1}_{\mathrm{Gr}}$ and $\mathbb{1}_{\mathrm{Fil}}$ denote the tensor units in $\mathrm{Gr}(\mathcal{C})$ and $\mathrm{Fil}(\mathcal{C})$, respectively. Then the underlying graded object of $\mathbb{1}_{\mathrm{Fil}}$ can be identified with the graded polynomial ring $\mathbb{1}_{\mathrm{Gr}}[t]$, where t sits in graded degree -1, the forgetful functor $\mathrm{Fil}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$ induces an equivalence

$$\mathrm{Fil}(\mathcal{C}) \xrightarrow{\simeq} \mathrm{Mod}_{\mathbb{1}_{\mathrm{Gr}}[t]}\big(\mathrm{Gr}(\mathcal{C})\big)\,,$$

  and under this equivalence, the associate graded $\mathrm{gr}^*\colon \mathrm{Fil}(\mathcal{C}) \to \mathrm{Gr}(\mathcal{C})$ becomes identified with the base change functor $-\otimes_{\mathbb{1}_{\mathrm{Gr}}[t]} \mathbb{1}_{\mathrm{Gr}}$. See [Rak21, Proposition 3.2.9] for example.

- **Exhaustive and complete filtrations.** We say that a filtered object $\mathrm{fil}^\star X$ is an *exhaustive filtration on* X if $X \simeq \operatorname{colim}_{n\to-\infty} \mathrm{fil}^n X$. We say that $\mathrm{fil}^\star X$ is *complete* if $0 \simeq \lim_{n\to\infty} \mathrm{fil}^n X$. Given an exhaustive filtration $\mathrm{fil}^\star X$ on X, we define its *completion* $\mathrm{fil}^\star \widehat{X}$ via

$$\mathrm{fil}^\star \widehat{X} \coloneqq \lim_{n\to\infty} \operatorname{cofib}\bigl(\mathrm{fil}^{\star+n} X \to \mathrm{fil}^\star X\bigr)\,.$$

This is an exhaustive filtration on $\widehat{X} \coloneqq \lim_{n\to\infty} \operatorname{cofib}(\mathrm{fil}^n X \to X)$. By construction, we have a pullback square

$$\begin{array}{ccc} \mathrm{fil}^\star X & \longrightarrow & \mathrm{fil}^\star \widehat{X} \\ \big\downarrow & \lrcorner & \big\downarrow \\ X & \longrightarrow & \widehat{X} \end{array}$$

We'll often refer to this by saying that *every filtration is the pullback of its completion.*

Let us also remark that under the identification of filtered objects with graded $\mathbb{1}_{\mathrm{Gr}}[t]$-modules, the notions of completeness and completion agree with those of t*-completeness* and t*-completion* that we'll review in 1.49 below. Moreover, this exhibits the pullback square above as a special case of a general fracture square.

1.49. Conventions on derived algebra. — Most algebraic constructions in this thesis will be derived (with one notable exception in §4.2) and we'll use the following terminology:

- **Animated rings.** If k is an ordinary ring, we denote by AniAlg_k the ∞-category of *animated* k*-algebras*. In the case $k = \mathbb{Z}$ we'll write AniRing and say *animated rings* instead. AniAlg_k is the ∞-category freely generated under sifted colimits by the category Poly_k of polynomial k-algebras in finitely many variables; equivalently, AniAlg_k can be described as the (∞-categorical) localisation of the category sCAlg_k of simplicial commutative k-algebras at the weak equivalences.

An animated ring which is concentrated in homological degree 0 (and hence an ordinary ring) will be called *static* ("un-animated"). We'll use the same terminology for spectra concentrated in homotopical degree 0. We don't use the more common term *discrete* to avoid confusion with condensed spectra that are equipped with the discrete ("un-condensed") topology; see 5.1.

- **Animation/nonabelian derived functors.** If $F\colon \mathrm{Poly}_k \to \mathcal{D}$ is any functor into ∞-category $\mathcal{D}$ with all sifted colimits, then F extends uniquely to a sifted colimit preserving functor $\mathrm{L}F\colon \mathrm{AniAlg}_k \to \mathcal{D}$, which we call the *animation* or (*non-abelian*) *derived functor of* F (both names will be used synonymously). The main examples of interest are

$$\Omega^1_{-/k}\colon \mathrm{Poly}_k \longrightarrow \mathcal{D}(k)\,,\quad \Omega^*_{-/k}\colon \mathrm{Poly}_k \longrightarrow \mathcal{D}(k)\,,\quad q\text{-}\Omega_{-/A}\colon \mathrm{Poly}_A \longrightarrow \widehat{\mathcal{D}}_{(q-1)}\bigl(A[\![q-1]\!]\bigr)$$

for a Λ-ring A. The corresponding derived functors will be denoted $\mathrm{L}_{-/k}$ (the *cotangent complex*), $\mathrm{dR}_{-/k}$ (the *derived de Rham complex*), and $q\text{-}\mathrm{dR}_{-/A}$ (the *derived* q*-de Rham complex*), respectively.

- **Derived quotients.** For an $\mathbb{E}_1$-ring spectrum R, a homotopy class $f \in \pi_n(R)$, and a left- or right-R-module M, we denote

$$M/f \coloneqq \operatorname{cofib}(f\colon \Sigma^n M \to M)\,.$$

For several homotopy classes $f_1, \dots, f_r$, we let $M/(f_1, \dots, f_r) \coloneqq (\cdots(M/f_1)/f_2\cdots)/f_r$. Observe that if M is a static module over a static ring R, then $M/(f_1, \dots, f_r)$ agrees with the usual quotient only if $(f_1, \dots, f_r)$ is a Koszul-regular sequence on M, but we'll never use the notation in a case where this is not satisfied.

Similarly, if R^* is a graded $\mathbb{E}_1$-ring spectrum, $f \in \pi_n(R^i)$, and M^* is a left or right-R-module, we put

$$M^*/f \coloneqq \operatorname{cofib}\bigl(f\colon \Sigma^n M(i) \to M\bigr)$$

and define $M^*/(f_1, \dots, f_r)$ analogously. The same notation will also be used in the filtered setting, by regarding filtered objects as graded $\mathbb{1}_{\mathrm{Gr}}[t]$-modules, as explained in 1.47.

- **Completions.** For an $\mathbb{E}_\infty$-ring spectrum R, finitely many homogeneous homotopy classes $f_1, \dots, f_r \in \pi_*(R)$, and an R-module spectrum M, we let

$$\widehat{M}_{(f_1,\dots,f_r)} \coloneqq \lim_{n \geqslant 1} M/(f_1^n, \dots, f_r^n)$$

denote the $(f_1, \dots, f_r)$-*adic completion* of M. Analogous notions will sometimes also be used in the graded or the filtered setting.

Since the completion only depends on the ideal $I = (f_1, \dots, f_r) \subseteq \pi_*(R)$, we often just write $\widehat{M}_I$ (or $(-)_I^\wedge$ for longer arguments). If R is an ordinary ring, this recovers the notion of *derived I-completion*; in particular, all completions in this article will be derived. For the p-completions of $\mathbb{Z}$ and the sphere spectrum $\mathbb{S}$ we omit the hat and just write $\mathbb{Z}_p$ and $\mathbb{S}_p$.

We let $\operatorname{Mod}_R(\mathrm{Sp})_I^\wedge \subseteq \operatorname{Mod}_R(\mathrm{Sp})$, or $\widehat{\mathcal{D}}_I(R) \subseteq \mathcal{D}(R)$ for static rings R, denote the full sub-∞-category spanned by the *I-complete objects*, that is, those M for which $M \simeq \widehat{M}_I$. The following fact will be used countless times: If M is $(f_1, \dots, f_r)$-complete, and the homotopy groups of $M/(f_1, \dots, f_r)$ vanish in some degree d, then also the homotopy groups of M must vanish in degree d.

- **Fracture squares.** We'll frequently use the fact that in the general situation above there are natural pullback squares

$$\begin{array}{ccc} M & \longrightarrow & \widehat{M}_f \\ \downarrow & \lrcorner & \downarrow \\ M\bigl[\tfrac{1}{f}\bigr] & \longrightarrow & \widehat{M}_f\bigl[\tfrac{1}{f}\bigr] \end{array} \qquad \text{and} \qquad \begin{array}{ccc} M & \longrightarrow & \prod_p \widehat{M}_p \\ \downarrow & \lrcorner & \downarrow \\ M \otimes \mathbb{Q} & \longrightarrow & \prod_p \widehat{M}_p \otimes \mathbb{Q} \end{array}$$

where the product on the right is taken over all primes p. In the case where $f = N$ is an integer, we'll refer to the pullback square on the left as the *arithmetic fracture square*; the same terminology will be used for the pullback on the right.

1.50. Perfectly covered Λ-rings. — Throughout the text, A will denote a Λ-ring, which we'll usually assume to be *perfectly covered*. By this we mean that the Adams operations $\psi^m\colon A \to A$ are faithfully flat for all m; or equivalently, that A admits a faithfully flat Λ-ring morphism $A \to A_\infty$ into a perfect Λ-ring. We remark that perfectly covered Λ-rings are p-torsion free for all primes p, because the same is true for perfect Λ-rings.

Part I.
q-Hodge complexes over the Habiro ring

In this part, we show that whenever a q-Hodge complex can be defined, it descends canonically to the Habiro ring. More precisely, for any perfectly covered Λ-ring A, we'll introduce an ∞-category $\mathrm{AniAlg}_A^{q\text{-Hdg}}$ of pairs $(R, \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A})$ consisting of an animated A-algebra R and a filtration on the derived q-de Rham complex $q\text{-}\mathrm{dR}_{R/A}$, which q-deforms the Hodge filtration $\mathrm{fil}^\star_{\mathrm{Hdg}}\, \mathrm{dR}_{R/A}$ and satisfies a few natural compatibilities. To any such pair we associate the *q-Hodge complex*

$$q\text{-}\mathrm{Hdg}_{(R,\mathrm{fil}^\star_{q\text{-Hdg}})/A} \coloneqq \operatorname{colim}\Bigl(\mathrm{fil}^0_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A} \xrightarrow{(q-1)} \mathrm{fil}^1_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A} \xrightarrow{(q-1)} \cdots\Bigr)^\wedge_{(q-1)}\,.$$

We'll then show in Theorem 3.11 that the functor $q\text{-}\mathrm{Hdg}_{-/A}\colon \mathrm{AniAlg}_A^{q\text{-Hdg}} \to \widehat{\mathcal{D}}_{(q-1)}(A[q])$ admits a non-trivial factorisation

$$\begin{array}{ccc} & & \widehat{\mathcal{D}}_{\mathcal{H}}(A[q]) \\ & \overset{q\text{-}\mathcal{H}\mathrm{dg}_{-/A}}{\nearrow} & \big\downarrow (-)^\wedge_{(q-1)} \\ \mathrm{AniAlg}_A^{q\text{-Hdg}} & \xrightarrow[q\text{-}\mathrm{Hdg}_{-/A}]{} & \widehat{\mathcal{D}}_{(q-1)}(A[q]) \end{array}$$

where $\widehat{\mathcal{D}}_{\mathcal{H}}(A[q])$ denotes the derived ∞-category of *Habiro-complete* $A[q]$-modules in the sense of §B. The non-triviality of the factorisation will be measured in a precise way using derived versions of the q-de Rham–Witt complexes from [Wag24].

Overview of Part I. — This part is organised as follows: In §2, we'll construct q-$\mathcal{H}\mathrm{dg}_{-/A}$ in the case where R is étale over A. This is much cleaner than the general case, and we'll recover the construction of the *Habiro ring of a number field* from [GSWZ24]. In the long and technical section §3, we'll construct q-$\mathcal{H}\mathrm{dg}_{-/A}$ in general. In §4, we'll show that even though there's no functorial choice of a q-Hodge filtration—that is, the forgetful functor $\mathrm{AniAlg}_A^{q\text{-Hdg}} \to \mathrm{AniAlg}_A$ provably has no section—such sections exist on surprisingly large full subcategories of AniAlg_A. In particular, there are many examples to which Theorem 3.11 can be applied.

§2. Habiro rings of étale extensions

Fix a perfectly covered Λ-ring A. The goal of this section is to construct a *relative Habiro ring* $\mathcal{H}_{R/A}$ for any étale algebra R over A, and to relate this construction to the theory of q-Witt vectors. In the case where $A = \mathbb{Z}$, our construction $\mathcal{H}_{R/\mathbb{Z}}$ recovers the ring $\mathcal{H}_R$ from [GSWZ24, Definition 1.1].

As we'll see in §3, the construction of $\mathcal{H}_{R/A}$ is a special case of a much more general construction. However, the general case is vastly more technical, so it will be worthwhile to spell out the étale case first.

§2.1. A general descent principle

To construct $\mathcal{H}_{R/A}$, we'll first construct the completions $(\mathcal{H}_{R/A})^\wedge_{\Phi_m(q)}$ for all $m \in \mathbb{N}$ and then "glue them together" using a very general descent principle that we'll explain in this subsection. It will probably seem a little overkill for now, but we'll use the same descent principle again in §3.3 to construct the twisted q-de Rham complexes $q\text{-}\mathrm{dR}^{(m)}_{R/A}$.

2.1. Setup. — Let $\mathcal{I}$ be a site whose underlying category is a partially ordered set. Let $\mathcal{D}$ be a presentable stable symmetric monoidal ∞-category. Suppose that for every $Z \in \mathcal{I}$ we have a full stable sub-∞-category $\mathcal{D}_Z$ satisfying the following conditions:

(a) The inclusion $\mathcal{D}_Z \subseteq \mathcal{D}$ admits a left adjoint $L_Z\colon \mathcal{D} \to \mathcal{D}_Z$.

(b) Whenever $Z_1 \to Z_2$ is a morphism in $\mathcal{I}$, we have $\mathcal{D}_{Z_1} \subseteq \mathcal{D}_{Z_2}$. Note that $L_{Z_1}\colon \mathcal{D}_{Z_2} \to \mathcal{D}_{Z_1}$ is still a left adjoint of this inclusion.

(c) For all $x, y \in \mathcal{D}$ and all $Z \in \mathcal{I}$, the canonical morphism $L_Z(x \otimes y) \to L_Z(L_Z(x) \otimes y)$ is an equivalence in $\mathcal{D}$.

In this case, sending $Z \mapsto \mathcal{D}_Z$ and $(Z_1 \to Z_2) \mapsto (L_{Z_1}\colon \mathcal{D}_{Z_2} \to \mathcal{D}_{Z_1})$ defines a contravariant functor

$$\mathcal{D}_{(-)}\colon \mathcal{I}^{\mathrm{op}} \longrightarrow \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$$

into the ∞-category of presentable stable symmetric monoidal ∞-categories. Indeed, let's ignore the symmetric monoidal structure for the moment and let $\mathcal{D}_\mathcal{I} \subseteq \mathcal{I} \times \mathcal{D}$ be the full sub-∞-category spanned fibrewise by $\mathcal{D}_Z \subseteq \{Z\} \times \mathcal{D}$. By ($b$), $\mathcal{D}_\mathcal{I} \to \mathcal{I}$ is still a cocartesian fibration and so it defines a covariant functor $\mathcal{D}_{(-)}\colon \mathcal{I} \to \mathrm{Cat}_\infty$. By ($a$), this functor factors through $\mathrm{Pr}^{\mathrm{R}}_{\mathrm{st}}$. Using $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}} \simeq (\mathrm{Pr}^{\mathrm{R}}_{\mathrm{st}})^{\mathrm{op}}$ by [L-HTT, Corollary 5.5.3.4], we get the desired functor $\mathcal{D}_{(-)}\colon \mathcal{I}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.

To incorporate the symmetric monoidal structure, let $\mathcal{D}$ be the ∞-operad $\mathcal{D}^\otimes$ associated to the given symmetric monoidal structure on $\mathcal{D}$. By (c) and [L-HA, Proposition 2.2.1.9], for all $Z \in \mathcal{I}$, the inclusion of the full sub-∞-operad $\mathcal{D}^\otimes_Z \subseteq \mathcal{D}^\otimes$ spanned by $\mathcal{D}_Z$ admits a symmetric monoidal left adjoint $L^\otimes_Z\colon \mathcal{D}^\otimes \to \mathcal{D}^\otimes_Z$ which recovers L_Z on underlying ∞-categories. Using this observation, the same argument as above can be repeated with $\mathcal{D}$ replaced by $\mathcal{D}^\otimes$.

2.2. Lemma. — *In the situation of 2.1, assume that covers in $\mathcal{I}$ always have finite refinements and that for any finite covering family $\{Z_i \to Z\}_{i=1,\dots,r}$, the functors $L_{Z_i}\colon \mathcal{D}_Z \to \mathcal{D}_{Z_i}$ are jointly conservative. Then*

$$\mathcal{D}_{(-)}\colon \mathcal{I}^{\mathrm{op}} \longrightarrow \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$$

is a sheaf on $\mathcal{I}$. In particular, $\mathrm{CAlg}(\mathcal{D}_{(-)})\colon \mathcal{I}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L}}$ *is a sheaf as well.*

Proof sketch. Everything can be checked on the level of underlying ∞-categories, so we can disregard the symmetric monoidal structure (but it was still essential to include the symmetric monoidal structure in the construction). By assumption, it's enough to check the sheaf property for a finite cover $\{Z_i \to Z\}_{i=1,\dots,r}$. If $Z_\bullet$ denotes its Čech nerve, we need to show $\mathcal{D}_Z \simeq \lim_\Delta \mathcal{D}_{Z_\bullet}$. Since $\mathcal{I}$ is a partially ordered set, we have $Z_i \times_Z Z_i = Z_i$ for all i. It follows that the cosimplicial limit can be simplified to a limit indexed by the set $\dot{\sqcup}^r \coloneqq \mathcal{P}(\{1,\dots,r\}) \smallsetminus \{\emptyset\}$ of non-empty subsets of $\{1,\dots,r\}$, partially ordered by inclusion. Therefore, we must show

$$\mathcal{D}_Z \simeq \lim_{S \in \dot{\sqcup}^r} \mathcal{D}_{Z_S}\,,$$

where we put $Z_S \coloneqq Z_{i_0} \times_Z \times \dots \times_Z Z_{i_k}$ for every non-empty subset $S = \{i_0,\dots,i_k\} \subseteq \{1,\dots,r\}$. To prove that $\mathcal{D}_Z \to \lim_{S \in \dot{\sqcup}^r} \mathcal{D}_{Z_S}$ is fully faithful, we have to show that

$$\operatorname{Hom}_{\mathcal{D}_Z}(x,y) \longrightarrow \operatorname{Hom}_{\mathcal{D}_{Z_S}}\big(L_{Z_S}(x), L_{Z_S}(y)\big)$$

is an equivalence for all $x,y \in \mathcal{D}_Z$. Rewriting $\operatorname{Hom}_{\mathcal{D}_{Z_S}}(L_{Z_S}(x), L_{Z_S}(y)) \simeq \operatorname{Hom}_{\mathcal{D}}(x, L_{Z_S}(y))$, this reduces to showing that $y \to \lim_{S \in \dot{\sqcup}^r} L_{Z_S}(y)$ is an equivalence. This can be checked after applying the jointly conservative functors $L_{Z_i}\colon \mathcal{D}_Z \to \mathcal{D}_{Z_i}$. After applying L_{Z_i}, each $L_{Z_S}(y) \Rightarrow L_{Z_{S \cup \{i\}}}$ becomes an equivalence. This easily implies $L_{Z_i}(y) \simeq \lim_{S \in \dot{\sqcup}^r} L_{Z_i}(L_{Z_S}(y))$ (for example, by the dual of [L-HA, Lemma 1.2.4.15]). Since L_{Z_i} preserves finite limits, this shows that $y \to \lim_{S \in \dot{\sqcup}^r} L_{Z_S}(y)$ is an equivalence after applying L_{Z_i}, and so fully faithfulness follows. The same argument shows essential surjectivity. ☐

2.3. Remark. — The quintessential example for Lemma 2.2 is the case where R is some ring, $\mathcal{D} \coloneqq \mathcal{D}(R)$ and $\mathcal{I}$ is the partially ordered set of closed subsets $Z \subseteq \operatorname{Spec} R$ with quasi-compact complement. Every such Z is the vanishing set of a finitely generated ideal I and we define $\mathcal{D}_Z \coloneqq \widehat{\mathcal{D}}_I(R)$; note that this only depends on Z, not on the choice of I. The functors $L_Z \coloneqq (-)_I^\wedge$ clearly satisfy the conditions from 2.1, and the condition from Lemma 2.2 is easily checked (see e.g. [Wag24, Lemma 2.4]). Hence the descent from Lemma 2.2 is applicable.

In the case that we're actually interested in, the descent diagram simplifies considerably; in particular, no coherence data needs to be provided!

2.4. Corollary. — *Let $m \in \mathbb{N}$. Suppose we're given the following data:*

(*a*) *For all divisors $d \mid m$, a derived $\Phi_d(q)$-complete $\mathbb{E}_\infty$-$A[q]$-algebra E_d.*

(*b*) *For all divisors $pd \mid m$, where p is a prime, an equivalence of $\mathbb{E}_\infty$-$A[q]$-algebras*

$$h_d\colon (E_{pd})_p^\wedge \xrightarrow{\simeq} (E_d)_p^\wedge\,.$$

Then there exists a unique (q^m-1)-complete $\mathbb{E}_\infty$-$A[q]$-algebra E together with equivalences $E_d \simeq E^\wedge_{\Phi_d(q)}$ for all $d \mid m$ such that h_d becomes identified with the identity on $E^\wedge_{(\Phi_d(q),\Phi_{pd}(q))}$.

Proof. The idea is to apply descent for $R = A[q]$ and the cover $V(q^m-1) = \bigcup_{d \mid m} V(\Phi_d(q))$. The simplifications come from the observation that many intersections are empty; see [Wag24, Lemma 2.1] for example.

For a precise argument, let T be the set of positive divisors of m and let $\dot{\sqcup}^T \coloneqq \mathcal{P}(T) \smallsetminus \{\emptyset\}$ denote the set of non-empty subsets of T, partially ordered by inclusion. For every $S \subseteq T$, put $\widehat{\mathcal{D}}_S \coloneqq \widehat{\mathcal{D}}_{(\Phi_d(q)\ \mid\ d \in S)}(A[q])$. Then Lemma 2.2 implies

$$\widehat{\mathcal{D}}_{(q^m-1)}\big(A[q]\big) \simeq \lim_{S \in \dot{\sqcup}^r} \widehat{\mathcal{D}}_S\,.$$

For every pair (d,p), where $d \mid m$ is a divisor of m and p is a prime such that $p \nmid d$, we let $T_{d,p} \coloneqq \{d, pd, \dots, p^{v_p(m)}d\} \subseteq T(m)$ and write $\dot{\sqcup}^{T_{d,p}} \subseteq \dot{\sqcup}^{T}$ for the corresponding sub-partially ordered set. By [Wag24, Lemma 2.1], we have $\widehat{\mathcal{D}}_S \simeq 0$ if $S \notin \bigcup_{d,p} \dot{\sqcup}^{T_{d,p}}$. By inspection, this means that $\widehat{\mathcal{D}}_{(-)}\colon \dot{\sqcup}^{T} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ is right-Kan extended from $\bigcup_{d,p} \dot{\sqcup}^{T_{d,p}} \subseteq \dot{\sqcup}^{T}$. Furthermore, if $S \subsetneq S'$ are elements of $\dot{\sqcup}^{T_{d,p}}$ such that $|S| \geqslant 2$, then the same result tells us that the corresponding morphism $S \to S'$ is sent to the identity, as both $\widehat{\mathcal{D}}_S$ and $\widehat{\mathcal{D}}_{S'}$ agree with the full sub-∞-category

$$\widehat{\mathcal{D}}_{(p,\Phi_d(q))}\big(A[q]\big) \subseteq \mathcal{D}\big(A[q]\big)\,.$$

Again, by inspection, this means that $\widehat{\mathcal{D}}_{(-)} \bigcup_{d,p} \dot{\sqcup}^{T_{d,p}} \to \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ is right-Kan extended from $P \subseteq \bigcup_{d,p} \dot{\sqcup}^{T_{d,p}}$, where P denotes the sub-partially ordered set spanned by $T_{d,p} \in \bigcup_{d,p} \dot{\sqcup}^{T_{d,p}(m)}$ for all d, p (note that this includes all subsets of the form $\{d\}$, where d is a divisor of m, as $\{d\} = T_{d,\ell}$ if ℓ is any prime not dividing m). In total, this implies $\widehat{\mathcal{D}}_{(q^m-1)}(A[q]) \simeq \lim_{S\in P} \widehat{\mathcal{D}}_S$ and thus

$$\mathrm{CAlg}\Big(\widehat{\mathcal{D}}_{(q^m-1)}\big(A[q]\big)\Big) \simeq \lim_{S\in P} \mathrm{CAlg}\big(\widehat{\mathcal{D}}_S\big)\,.$$

After unravelling of definitions, an object in the limit on the right-hand side is precisely given by the data (*a*) and (*b*). □

2.5. Remark. — In Corollary 2.4, we've glued E from its $\Phi_d(q)$-completions $E^\wedge_{\Phi_d(q)} \simeq E_d$ for all $d \mid m$. But E can also be glued from from the completed localisation $E[1/m]^\wedge_{(q^m-1)}$ and the completions $E^\wedge_{(p,q^m-1)}$ for all primes $p \mid m$ via the usual arithmetic fracture square (see 1.49). For later use, let us explain how to extract the latter from the former: If $m = p^\alpha n$, where n is coprime to p, then

$$E\big[\tfrac{1}{m}\big]^\wedge_{(q^m-1)} \simeq \prod_{d\mid m} E_d\big[\tfrac{1}{m}\big]^\wedge_{\Phi_d(q)} \quad\text{and}\quad E^\wedge_{(p,q^m-1)} \simeq \prod_{d\mid n} (E_{p^i d})^\wedge_p \quad \text{for any } 0 \leqslant i \leqslant \alpha\,.$$

For the equivalence on the left, just observe that the factors in $(q^m-1) = \prod_{d\mid m}\Phi_d(q)$ become coprime as soon as m is invertible. For the equivalence on the right, observe that after p-completion the ℓ-adic gluings for $\ell \neq p$ become vacuous, so the only gluing that happens is along $(E_d)^\wedge_p \simeq (E_{pd})^\wedge_p \simeq \dots \simeq (E_{p^\alpha d})^\wedge_p$ for all $d \mid n$.

2.6. Remark. — Corollary 2.4 remains true if we replace $\mathbb{E}_\infty$-$A[q]$-algebras by derived commutative $A[q]$-algebras in the sense of [Rak21, Example 4.3.1]. The proof is entirely analogous.

§2.2. Habiro rings of étale extensions

In the following, we fix a perfectly covered Λ-ring A as before.

2.7. Relative Habiro rings. — Let R be an étale A-algebra. For all primes p, the p^{th} Adams operation $\psi^p\colon A \to A$ can be uniquely extended to a Frobenius lift $\phi_p\colon \widehat{R}_p \to \widehat{R}_p$. Let us denote by

$$\phi_{p/A}\colon \big(\widehat{R}_p \otimes_{A,\psi^p} A\big)^\wedge_p \xrightarrow{\ \simeq\ } \widehat{R}_p$$

the linearised Frobenius. It is an equivalence as indicated. Indeed, this can be checked modulo p, where it becomes classical; see [Stacks, Tag 0EBS]. We also remark that A being perfectly covered implies that A is p-torsion free (because this is true for the perfect Λ-ring A_∞), and so all p-completions above are static.

For all $m \in \mathbb{N}$, let us now define a (q^m-1)-complete $\mathbb{E}_\infty$-$A[q]$-algebra $\mathcal{H}_{R/A,m}$ via Corollary 2.4: For every $d \mid m$, let $E_d \coloneqq (R \otimes_{A,\psi^d} A)[q]^\wedge_{\Phi_d(q)}$ and for every $pd \mid m$, where p is a prime, let the gluing equivalence h_d be the $A[q]$-linear map induced by $\phi_{p/A}$.

For all $d \mid m$, Corollary 2.4 provides a preferred equivalence $\mathcal{H}_{R/A,d} \simeq (\mathcal{H}_{R/A,m})^\wedge_{(q^d-1)}$. In particular, we get maps $\mathcal{H}_{R/A,m} \to \mathcal{H}_{R/A,d}$. The *Habiro ring of R relative to A* is then defined as the limit[(2.1)]

$$\mathcal{H}_{R/A} \coloneqq \lim_{m \in \mathbb{N}} \mathcal{H}_{R/A,m} \,.$$

2.8. Remark. — If we were to construct $R[q]^\wedge_{(q^m-1)}$ using Corollary 2.4, we would take $E_d \coloneqq R[q]^\wedge_{\Phi_d(q)}$, together with the identity maps on $R[q]^\wedge_{(p,\Phi_d(q))}$ (instead of $\phi_{p/A}$) as gluing equivalences. Thus, there's no reason to expect that $\mathcal{H}_{R/A,m} \simeq R[q]^\wedge_{(q^m-1)}$, unless R itself (rather than only its p-completions) admits Frobenius lifts for all prime factors $p \mid m$. In the case $A = \mathbb{Z}$, a precise obstruction of this kind is shown in [Wag24, Corollary 2.52].

We can now formulate the relation between $\mathcal{H}_{R/A}$ and q-Witt vectors relative to A. To this end, recall from [Wag24, Proposition 2.48] that q-$\mathbb{W}_m(R/A)$ is an étale algebra over q-$\mathbb{W}_m(A/A) \cong A[q]/(q^m-1)$.

2.9. Theorem. — *Let A be a perfectly covered Λ-ring, R an A-algebra, and $m \in \mathbb{N}$. Then*

$$\mathcal{H}_{R/A,m}/(q^m-1) \simeq q\text{-}\mathbb{W}_m(R/A)\,.$$

In fact, $\mathcal{H}_{R/A,m}$ is the unique lift of the étale $A[q]/(q^m-1)$-algebra q-$\mathbb{W}_m(R/A)$ to a (q^m-1)-complete $\mathbb{E}_\infty$-algebra over $A[q]^\wedge_{(q^m-1)}$. In particular, $\mathcal{H}_{R/A,m}$ is an ordinary ring for all $m \in \mathbb{N}$, and the same is true for the relative Habiro ring $\mathcal{H}_{R/A}$.

Proof. Let, temporarily, W denote the unique lift of q-$\mathbb{W}_m(R/A)$ to a (q^m-1)-complete $\mathbb{E}_\infty$-algebra over $A[q]^\wedge_{(q^m-1)}$. If p is prime and $pd \mid m$, then the ghost maps for the usual Witt vectors $\mathbb{W}_m(A/p)$ and $\mathbb{W}_m(R/p)$ satisfy $\mathrm{gh}_{m/d}(x) = \mathrm{gh}_{m/pd}(x)^p$. It follows that the ghost maps for relative q-Witt vectors fit into a commutative diagram

$$\begin{array}{ccccc}
\big(R \otimes_{A,\psi^{pd}} A\big)[q]/\Phi_{pd}(q) & \xleftarrow{\mathrm{gh}_{m/pd}} & q\text{-}\mathbb{W}_m(R/A) & \xrightarrow{\mathrm{gh}_{m/d}} & \big(R \otimes_{A,\psi^d} A\big)[q]/\Phi_d(q) \\
\downarrow & & & & \downarrow \\
\big(R/p \otimes_{A/p,\psi^{pd}} A/p\big)[q]/\Phi_{pd}(q) & & \longrightarrow & & \big(R/p \otimes_{A/p,\psi^d} A/p\big)[q]/\Phi_d(q)
\end{array}$$

where the bottom horizontal map is induced by the relative Frobenius $R/p \otimes_{A/p,(-)^p} A/p \to R/p$. After passing to unique deformations of étale algebras everywhere, we obtain a similar diagram

$$\begin{array}{ccccc}
\big(R \otimes_{A,\psi^{pd}} A\big)[q]^\wedge_{\Phi_{pd}(q)} & \longleftarrow & W & \longrightarrow & \big(R \otimes_{A,\psi^d} A\big)[q]^\wedge_{\Phi_d(q)} \\
\downarrow & & & & \downarrow \\
\big(\widehat{R}_p \otimes_{A,\psi^{pd}} A\big)[q]^\wedge_{(p,\Phi_{pd}(q))} & & \longrightarrow & & \big(\widehat{R}_p \otimes_{A,\psi^d} A\big)[q]^\wedge_{(p,\Phi_d(q))}
\end{array}$$

(2.1) A pedantic remark: To even write down this limit, we need to assemble the maps $\mathcal{H}_{R/A,m} \to \mathcal{H}_{R/A,d}$ into a functor $\mathcal{H}_{R/A,(-)}\colon \mathbb{N} \to \mathrm{CAlg}\,\mathcal{D}(A[q])$, where $\mathbb{N}$ denotes the category of natural numbers partially ordered by divisibility. With a little more effort, this functoriality can be squeezed out of Corollary 2.4. Alternatively, we can take the limit over the sequential subdiagram $\{n!\}_{n\geqslant 1}$, where the existence of maps is enough. Or we could use Theorem 2.9 to realise that we're working with ordinary rings, so there are no higher coherences to check and functoriality can be obtained by hand.

where the bottom horizontal map is induced by $\phi_{p/A}$ from 2.7. By construction of $\mathcal{H}_{R/A,m}$, this yields an $\mathbb{E}_\infty$-$A[q]$-algebra map $W \to \mathcal{H}_{R/A,m}$. As both sides are (q^m-1)-complete, whether this is an equivalence can be checked modulo $\Phi_d(q)$ for all $d \mid m$. By [Wag24, Corollary 2.51] and 2.7,

$$W/\Phi_d(q) \simeq R \otimes_{A,\psi^d} A[q]/\Phi_d(q) \simeq \mathcal{H}_{R/A,m}/\Phi_d\,.$$

As the equivalence on the left is induced via the ghost map $\mathrm{gh}_{m/d}$, it is apparent from our construction that $W/\Phi_d(q) \to \mathcal{H}_{R/A,m}/\Phi_d(q)$ is given by the chain of equivalences above. This finishes the proof that $\mathcal{H}_{R/A,m}$ is the unique deformation of q-$\mathbb{W}_m(R/A)$.

Since $\mathcal{H}_{R/A,m}$ is (q^m-1)-complete and becomes static modulo p, we see that $\mathcal{H}_{R/A,m}$ must be static as well. Therefore it is an ordinary ring. To conclude the same for $\mathcal{H}_{R/A}$, we've seen above that $\mathcal{H}_{R/A}/\Phi_m(q)$ is static for all $m \in \mathbb{N}$. Then Corollary B.4 can be applied. □

2.10. Remark. — By tracing through the proof of Theorem 2.9 and checking on ghost coordinates, we see that the maps $\mathcal{H}_{R/A,m} \to \mathcal{H}_{R/A,d}$ from 2.7 deform the q-Witt vector Frobenii $F_{m/d}\colon q\text{-}\mathbb{W}_m(R/A) \to q\text{-}\mathbb{W}_d(R/A)$. Then the construction of $\mathcal{H}_{R/A}$ is reminiscent of the construction of $\mathbb{A}_{\mathrm{inf}}$ from [BMS18, Lemma 3.2].

In [GSWZ24, Definition 1.1], the Habiro ring of a number field is defined in terms of power series in $q-\zeta$, for ζ ranging through roots of unity. We'll now give a similar hands-on description of $\mathcal{H}_{R/A}$. This will imply that our construction recovers the one from [GSWZ24].

2.11. *p*-adic reexpansions around roots of unity. — In the following, we choose a system of roots of unity $(\zeta_m)_{m\in\mathbb{N}}$ in such a way that

$$\zeta_{mn} = \zeta_m\zeta_n \text{ if } (m,n)=1 \quad \text{and} \quad \zeta_{p^\alpha} = \zeta_{p^{\alpha+1}}^p\,.$$

One possible choice would be $\zeta_m \coloneqq \prod_p \mathrm{e}^{2\pi \mathrm{i}/p^{v_p(m)}}$. The conditions above are also required in [GSWZ24, §1.2] and they ensure $v_p(\zeta_m - \zeta_{mp}) > 0$ whenever p is prime, so that after p-completion, any power series in $(q-\zeta_m)$ can be reexpanded as power series in $(q-\zeta_{pm})$. In other words, there's a canonical zigzag

$$\mathbb{Z}[\zeta_m][\![q-\zeta_m]\!] \longrightarrow \mathbb{Z}_p[\zeta_{pm}][\![q-\zeta_m]\!] \simeq \mathbb{Z}_p[\zeta_{pm},q]^\wedge_{(q-\zeta_m,q-\zeta_{pm})} \longleftarrow \mathbb{Z}[\zeta_{pm}][\![q-\zeta_{pm}]\!]\,,$$

In the situation we're interested in, we get a similar zigzag

$$(R\otimes_{A,\psi^m}A)[\zeta_m][\![q-\zeta_m]\!] \longrightarrow \left(\widehat{R}_p\otimes_{A,\psi^m}A\right)^\wedge_p[\zeta_{pm}][\![q-\zeta_m]\!] \xleftarrow{\phi_{p/A}} (R\otimes_{A,\psi^{pm}}A)[\zeta_{pm}][\![q-\zeta_{pm}]\!]$$

where the map on the right is induced by the relative Frobenius $\phi_{p/A}$ from 2.7, followed by a reexpansion of power series as above. We'll call the map on the left the *canonical map* and the map on the right the *Frobenius*.

2.12. Lemma. — *The ring $\mathcal{H}_{R/A}$ agrees with following equaliser (which can be taken both in $\mathbb{E}_\infty$-$A[q]$-algebras or in ordinary $A[q]$-algebras):*

$$\mathcal{H}_{R/A} \simeq \mathrm{eq}\left(\prod_m (R\otimes_{A,\psi^m}A)[\zeta_m][\![q-\zeta_m]\!] \underset{\phi_{/A}}{\overset{\mathrm{can}}{\rightrightarrows}} \prod_{p,m}\left(\widehat{R}_p\otimes_{A,\psi^m}A\right)^\wedge_p[\zeta_{pm}][\![q-\zeta_m]\!]\right).$$

Here can *and $\phi_{/A}$ are the canonical maps and Frobenius maps described in 2.11.*

Proof. Let, temporarily, E denote the derived equaliser. By construction, $\mathcal{H}_{R/A}$ can be written as a similar equaliser, with $(R \otimes_{A,\psi^m} A)[\zeta_m]⟦q-\zeta_m⟧$ replaced by $(R \otimes_{A,\psi^m} A)[q]^\wedge_{\Phi_m(q)}$. We clearly get a map of underived equalisers, hence a map $\mathcal{H}_{R/A} \to \pi_0(E)$ as $\mathcal{H}_{R/A}$ is static. Since the derived equaliser E is coconnective, this yields a map $\mathcal{H}_{R/A} \to E$ as well. Since both sides are Habiro-complete in the sense of B.1, whether this map is an equivalence can be checked after $(-)^\wedge_\ell$ for all primes ℓ and after $(- \otimes_\mathbb{Z} \mathbb{Q})^\wedge_{\Phi_d(q)}$ for all $d \in \mathbb{N}$.

Proof after ℓ-completion. After $(-)^\wedge_\ell$, all factors in E with $p \neq \ell$ die, and the surviving Frobenii $\phi_{\ell/A}$ become equivalences. Similarly, in $\mathcal{H}_{R/A}$, all p-adic gluings for $p \neq \ell$ vanish, and the ℓ-adic gluings become equivalences. It follows that after ℓ-completion, the map has the form

$$\prod_{(m,\ell)=1} \left(\widehat{R}_\ell \otimes_{A,\psi^m} A\right)^\wedge_\ell [q]^\wedge_{(\ell,\Phi_m(q))} \longrightarrow \prod_{(m,\ell)=1} \left(\widehat{R}_\ell \otimes_{A,\psi^m} A\right)^\wedge_\ell [\zeta_m]⟦q-\zeta_m⟧$$

So it will be enough to show that $\mathbb{Z}_\ell[q]^\wedge_{(\ell,\Phi_m(q))} \to \mathbb{Z}_\ell[\zeta_m]⟦q-\zeta_m⟧$ is an equivalence whenever $(m,\ell) = 1$. This can be checked modulo $(\ell, \Phi_m(q))$. The left-hand side clearly becomes $\mathbb{F}_\ell[q]/\Phi_m(q) \simeq \mathbb{F}_\ell(\zeta_m)$ since the cyclotomic polynomial $\Phi_m(q)$ is irreducible in $\mathbb{F}_\ell[q]$ if $(m,\ell) = 1$. Moreover, $\Phi_m(q)$ has distinct roots in $\overline{\mathbb{F}}_\ell$, and so $\Phi_m(q)/(q-\zeta_m)$ will be a unit in $\mathbb{F}_\ell(\zeta_m)⟦q-\zeta_m⟧$. It follows that $\mathbb{Z}_\ell[\zeta_m]⟦q-\zeta_m⟧/(\ell,\Phi_m(q)) \simeq \mathbb{F}_\ell(\zeta_m)$ as well. This concludes the argument after ℓ-completion.

Proof after $\Phi_d(q)$-completed rationalisation. By 2.7, the $\Phi_d(q)$-completion of $\mathcal{H}_{R/A}$ is $(R \otimes_{A,\psi^d} A)[q]^\wedge_{\Phi_d(q)}$ and so

$$\left(\mathcal{H}_{R/A} \otimes_\mathbb{Z} \mathbb{Q}\right)^\wedge_{\Phi_d(q)} \simeq \left((R \otimes_{A,\psi^d} A) \otimes_\mathbb{Z} \mathbb{Q}\right)[q]^\wedge_{\Phi_d(q)} \simeq \left((R \otimes_{A,\psi^d} A) \otimes_\mathbb{Z} \mathbb{Q}(\zeta_m)\right)⟦q-\zeta_m⟧\,.$$

Here we use that $\mathbb{Q}[q]^\wedge_{\Phi_d(q)} \to \mathbb{Q}(\zeta_m)⟦q-\zeta_m⟧$ is an equivalence. Indeed, this can be checked modulo $\Phi_m(q)$. Since $\Phi_m(q)$ is irreducible and has distinct roots in $\overline{\mathbb{Q}}$, the same argument as above shows that both sides become $\mathbb{Q}(\zeta_m)$ modulo $\Phi_m(q)$, as desired.

Let's compute $\widehat{E}_{\Phi_d(q)}$ next. Since $(R \otimes_{A,\psi^m} A)[\zeta_m]⟦q-\zeta_m⟧$ is $\Phi_m(q)$-complete, it'll vanish upon $\Phi_d(q)$-completion unless m/d is a prime power (possibly with negative exponent). Moreover, if $m/d = p^\alpha$ is a power of p, then the $\Phi_d(q)$-completion of $(R \otimes_{A,\psi^m} A)[\zeta_m]⟦q-\zeta_m⟧$ will also be p-complete, unless $\alpha = 0$. It follows that all surviving Frobenii will become equivalences, except if their source is $(R \otimes_{A,\psi^m} A)[\zeta_d]⟦q-\zeta_d⟧$.

For all primes p, let $\alpha_p \coloneqq v_p(d)$ and write $d = p^{\alpha_p} d_p$. By massaging the limit using our observations so far, we find that $\widehat{E}_{\Phi_d(q)}$ sits inside a pullback diagram

$$\begin{array}{ccc}
\widehat{E}_{\Phi_d(q)} & \longrightarrow & \left(R \otimes_{A,\psi^d} A\right)[\zeta_d]⟦q-\zeta_d⟧ \\
\downarrow & \lrcorner & \downarrow {\scriptstyle (\phi_{p/A}^{\alpha_p})_p} \\
\prod_p \left(\widehat{R}_p \otimes_{A,\psi^{d_p}} A\right)^\wedge_p [\zeta_{d_p}]⟦q-\zeta_{d_p}⟧ & \longrightarrow & \prod_p \left(\widehat{R}_p \otimes_{A,\psi^{d_p}} A\right)^\wedge_p [\zeta_d]⟦q-\zeta_{d_p}⟧
\end{array}$$

Observe that the bottom horizontal arrow is a split injection on underlying $\mathbb{Z}[q]$-modules, because in each factor $\mathbb{Z}[\zeta_{d_p}] \to \mathbb{Z}[\zeta_d]$ is a split injection of abelian groups. However, as we've seen above, $\mathbb{Q}[q]^\wedge_{\Phi_d(q)} \simeq \mathbb{Q}(\zeta_d)⟦q-\zeta_d⟧$ contains ζ_d. Thus the bottom horizontal arrow becomes an equivalence after $(- \otimes_\mathbb{Z} \mathbb{Q})^\wedge_{\Phi_d(q)}$. It follows that

$$(E \otimes_\mathbb{Z} \mathbb{Q})^\wedge_{\Phi_d(q)} \simeq \left((R \otimes_{A,\psi^d} A) \otimes_\mathbb{Z} \mathbb{Q}[\zeta_m]\right)⟦q-\zeta_m⟧\,.$$

Thus $\mathcal{H}_{R/A} \to E$ also becomes an equivalence after $(- \otimes_\mathbb{Z} \mathbb{Q})^\wedge_{\Phi_d(q)}$. $\square$

2.13. Corollary. — *If F is a number field with discriminant Δ and $R \coloneqq \mathcal{O}_F[1/\Delta]$, then $\mathcal{H}_{R/\mathbb{Z}}$ agrees with the Habiro ring $\mathcal{H}_R$ defined in* [*GSWZ24*, *Definition* 1.1].

Proof. This follows immediately from Lemma 2.12. □

§3. Habiro descent for q-Hodge complexes

In this section, we'll show that in those situations where a well-behaved derived q-Hodge complex can be defined, it descends automatically to the Habiro ring, and furthermore a derived analogue of Theorem 1.13(a) holds true.

Throughout this section, we fix a perfectly covered Λ-ring A.

3.1. Convention. — In the following we'll consider filtered modules over the filtered ring $(q^m-1)^\star A[q]$ for various m. For such a filtered module $\operatorname{fil}^\star M$, we always let $\operatorname{fil}^\star M/(q^m-1)$ denote the base change

$$\operatorname{fil}^\star M/(q^m-1) \coloneqq \operatorname{fil}^\star M \otimes^{\mathrm{L}}_{(q^m-1)^\star A[q]} A$$

in filtered objects, or in other words, the quotient by (q^m-1) sitting in filtration degree 1, not filtration degree 0. In particular, the n^{th} filtered piece of the quotient $\operatorname{fil}^\star M/(q^m-1)$ will be

$$\operatorname{cofib}\bigl((q^m-1)\colon \operatorname{fil}^{n-1} M \longrightarrow \operatorname{fil}^n M\bigr)\,.$$

§3.1. q-Hodge filtrations and the q-Hodge complex

Let us start by introducing an appropriate ∞-category of A-algebras equipped with a well-behaved q-deformation of the Hodge filtration. Since Definition 3.2 below is a bit of a mess, let us informally summarise the key points first: In addition to the obvious q-deformation condition (b), we also wish the filtration to be compatible with the rational equivalence

$$\bigl(q\text{-}\mathrm{dR}_{R/A} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Q}\bigr)^\wedge_{(q-1)} \simeq \bigl(\mathrm{dR}_{R/A} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Q}\bigr)[\![q-1]\!]\,,$$

which leads to condition (c). For technical reasons, we also need to require the same for the rationalisations of the p-completed (q-)de Rham complexes, which is why we have to include condition (c_p) below. These conditions need to satisfy some obvious compatibilities; recording those, we end up with the following slightly messy definition:

3.2. Definition (q-Hodge filtrations)**.** — Let R be an animated A-algebra. A *q-Hodge filtration on* $q\text{-}\mathrm{dR}_{R/A}$ is a filtered $(q-1)^\star A[q]$-module

$$\operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}} q\text{-}\mathrm{dR}_{R/A} \simeq \Bigl(\operatorname{fil}^0_{q\text{-}\mathrm{Hdg}} q\text{-}\mathrm{dR}_{R/A} \leftarrow \operatorname{fil}^1_{q\text{-}\mathrm{Hdg}} q\text{-}\mathrm{dR}_{R/A} \leftarrow \operatorname{fil}^2_{q\text{-}\mathrm{Hdg}} q\text{-}\mathrm{dR}_{R/A} \leftarrow \cdots\Bigr)\,,$$

equipped with the following data and compatibilities[(3.1)]:

(a) An equivalence of $A[q]$-modules $q\text{-}\mathrm{dR}_{R/A} \simeq \operatorname{fil}^0_{q\text{-}\mathrm{Hdg}} q\text{-}\mathrm{dR}_{R/A}$. In other words, we require that $\operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}} q\text{-}\mathrm{dR}_{R/A}$ defines a descending filtration on the derived q-de Rham complex.

(b) An equivalence of filtered A-modules

$$c_{(q-1)}\colon \operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}} q\text{-}\mathrm{dR}_{R/A}/(q-1) \xrightarrow{\ \simeq\ } \operatorname{fil}^\star_{\mathrm{Hdg}} \mathrm{dR}_{R/A}\,,$$

which in filtered degrees $\leqslant 0$ agrees with the usual equivalence $q\text{-}\mathrm{dR}_{R/A}/(q-1) \simeq \mathrm{dR}_{R/A}$ under the identification from (a). In other words, the filtration $\operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}} q\text{-}\mathrm{dR}_{R/A}$ has to be a $(q-1)$-deformation of the Hodge filtration.

(3.1) Since we're working with ∞-categories, each compatibility is again a datum that needs to be provided.

(c) An equivalence of filtered $(q-1)^\star(A\otimes\mathbb{Q})[q]$-modules

$$c_{\mathbb{Q}}\colon \bigl(\operatorname{fil}^\star_{q\text{-Hdg}} q\text{-dR}_{R/A}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q}\bigr)^\wedge_{(q-1)} \xrightarrow{\simeq} \operatorname{fil}^\star_{(\mathrm{Hdg},q-1)}\bigl(\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q}\bigr)[\![q-1]\!]\,,$$

where $\operatorname{fil}^\star_{(\mathrm{Hdg},q-1)}$ denotes the $(q-1)$-completed tensor product of the Hodge filtration on $\mathrm{dR}_{R/A}$ and the $(q-1)$-adic filtration on $\mathbb{Q}[\![q-1]\!]$; in the following, we'll often call this the *combined Hodge and $(q-1)$-adic filtration*. In addition, we require that $c_{\mathbb{Q}}$ agrees in filtered degrees $\leqslant 0$ with the usual equivalence $(q\text{-dR}_{R/A}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q})^\wedge_{(q-1)}\simeq(\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q})[\![q-1]\!]$ under the identification from (a), and that $c_{\mathbb{Q}}$ and $c_{(q-1)}$ from (b) fit into a commutative diagram

$$\begin{array}{ccccc}
\operatorname{fil}^\star_{q\text{-Hdg}} q\text{-dR}_{R/A} & \longrightarrow & \operatorname{fil}^\star_{q\text{-Hdg}} q\text{-dR}_{R/A}/(q-1) & \xrightarrow[c_{(q-1)}]{\simeq} & \operatorname{fil}^\star_{\mathrm{Hdg}}\mathrm{dR}_{R/A}\\
\downarrow & & /\!/\!/ & & \downarrow\\
\bigl(\operatorname{fil}^\star_{q\text{-Hdg}} q\text{-dR}_{R/A}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q}\bigr)^\wedge_{(q-1)} & \xrightarrow[c_{\mathbb{Q}}]{\simeq} & \operatorname{fil}^\star_{(\mathrm{Hdg},q-1)}\bigl(\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q}\bigr)[\![q-1]\!] & \longrightarrow & \operatorname{fil}^\star_{\mathrm{Hdg}}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q}
\end{array}$$

which again must agree in filtered degrees $\leqslant 0$ with the corresponding unfiltered diagram under the identification from (a).

(c_p) For every prime p, an equivalence of filtered $(q-1)^\star\widehat{A}_p[1/p][\![q-1]\!]$-modules

$$c_{\mathbb{Q}_p}\colon \operatorname{fil}^\star_{q\text{-Hdg}}\bigl(q\text{-dR}_{R/A}\bigr)^\wedge_p\bigl[\tfrac{1}{p}\bigr]^\wedge_{(q-1)} \xrightarrow{\simeq} \operatorname{fil}^\star_{(\mathrm{Hdg},q-1)}\bigl(\mathrm{dR}_{R/A}\bigr)^\wedge_p\bigl[\tfrac{1}{p}\bigr][\![q-1]\!]\,,$$

which is required to agree in filtered degrees $\leqslant 0$ agrees with the usual equivalence $(q\text{-dR}_{R/A})^\wedge_p[1/p]^\wedge_{(q-1)}\simeq(\mathrm{dR}_{R/A})^\wedge_p[1/p][\![q-1]\!]$ under the identification from (a). In addition, we require that $c_{\mathbb{Q}}$ and $c_{\mathbb{Q}_p}$ are compatible in form of a commutative diagram

$$\begin{array}{ccc}
\bigl(\operatorname{fil}^\star_{q\text{-Hdg}} q\text{-dR}_{R/A}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q}\bigr)^\wedge_{(q-1)} & \xrightarrow[c_{\mathbb{Q}}]{\simeq} & \operatorname{fil}^\star_{(\mathrm{Hdg},q-1)}\bigl(\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q}\bigr)[\![q-1]\!]\\
\downarrow & /\!/\!/ & \downarrow\\
\operatorname{fil}^\star_{q\text{-Hdg}}\bigl(q\text{-dR}_{R/A}\bigr)^\wedge_p\bigl[\tfrac{1}{p}\bigr]^\wedge_{(q-1)} & \xrightarrow[c_{\mathbb{Q}_p}]{\simeq} & \operatorname{fil}^\star_{(\mathrm{Hdg},q-1)}\bigl(\mathrm{dR}_{R/A}\bigr)^\wedge_p\bigl[\tfrac{1}{p}\bigr][\![q-1]\!]
\end{array}$$

which in filtered degrees $\leqslant 0$ must agree with the usual compatibility under the identification from (a), and that $c_{(q-1)}$ and $c_{\mathbb{Q}_p}$ fit into a commutative diagram

$$\begin{array}{ccccc}
\operatorname{fil}^\star_{q\text{-Hdg}}\bigl(q\text{-dR}_{R/A}\bigr)^\wedge_p & \longrightarrow & \operatorname{fil}^\star_{q\text{-Hdg}}\bigl(q\text{-dR}_{R/A}\bigr)^\wedge_p/(q-1) & \xrightarrow[c_{(q-1)}]{\simeq} & \operatorname{fil}^\star_{\mathrm{Hdg}}\bigl(\mathrm{dR}_{R/A}\bigr)^\wedge_p\\
\downarrow & & /\!/\!/ & & \downarrow\\
\operatorname{fil}^\star_{q\text{-Hdg}}\bigl(q\text{-dR}_{R/A}\bigr)^\wedge_p\bigl[\tfrac{1}{p}\bigr]^\wedge_{(q-1)} & \xrightarrow[c_{\mathbb{Q}_p}]{\simeq} & \operatorname{fil}^\star_{(\mathrm{Hdg},q-1)}\bigl(\mathrm{dR}_{R/A}\bigr)^\wedge_p\bigl[\tfrac{1}{p}\bigr][\![q-1]\!] & \longrightarrow & \operatorname{fil}^\star_{\mathrm{Hdg}}\bigl(\mathrm{dR}_{R/A}\bigr)^\wedge_p\bigl[\tfrac{1}{p}\bigr]
\end{array}$$

which must agree in filtered degrees $\leqslant 0$ with the corresponding unfiltered diagram under the identification from (a). Finally, we require that this diagram is compatible with the diagram from (c) under the previous diagram relating $c_{\mathbb{Q}}$ and $c_{\mathbb{Q}_p}$, and that in filtered degrees $\leqslant 0$ this compatibility agrees with the usual compatibility under the identification from (a).

We let $\mathrm{AniAlg}_A^{q\text{-Hdg}}$ denote the ∞-category of pairs $(R,\operatorname{fil}^\star_{q\text{-Hdg}} q\text{-dR}_{R/A})$, where R is an animated A-algebra and $\operatorname{fil}^\star_{q\text{-Hdg}} q\text{-dR}_{R/A}$ is a q-Hodge filtration on $q\text{-dR}_{R/A}$. Formally, the ∞-category $\mathrm{AniAlg}_A^{q\text{-Hdg}}$ can be expressed as an iterated pullback of AniAlg_A and several ∞-categories of filtered modules; this is straightforward, but not very enlightening, so we omit the details.

It is natural to ask whether q-Hodge filtrations can be chosen functorially. Surprisingly, this turns out to be false.

3.3. Lemma. — *If A is not a $\mathbb{Q}$-algebra, then the forgetful functor $\mathrm{AniAlg}_A^{q\text{-Hdg}} \to \mathrm{AniAlg}_A$ is not essentially surjective. In particular, it has no section, not even when restricted to the full subcategory $\mathrm{Sm}_A \subseteq \mathrm{AniAlg}_A$ of smooth A-algebras.*

Proof sketch. As far as the author is aware, this result hasn't been published, but the objection is known among the experts in the field.

Let p be a prime such that $\widehat{A}_p \not\simeq 0$. Let $\widehat{A}_p\{x\}_\infty$ be the free p-complete perfect δ-ring on a generator x. We'll show that the q-de Rham complex of $R \coloneqq \widehat{A}_p\{x\}_\infty/x$ admits no q-Hodge filtration. Suppose it does. Note that $(q\text{-dR}_{R/A})_p^\wedge$ is given by the prismatic envelope

$$\left(q\text{-dR}_{R/A}\right)_p^\wedge \simeq \widehat{A}_p\{x\}_\infty\llbracket q-1\rrbracket\left\{\frac{\phi(x)}{[p]_q}\right\}_{(p,q-1)}^\wedge .$$

In particular, it is static. Since the Hodge filtration $\mathrm{fil}^\star_{\mathrm{Hdg}}(\mathrm{dR}_{R/A})_p^\wedge$ is just the divided power filtration of the PD-envelope $(\mathrm{dR}_{R/A})_p^\wedge \simeq D_{\widehat{A}_p\{x\}_\infty}(x)$, Definition 3.2($b$) implies that $\mathrm{fil}^\star_{q\text{-Hdg}}(q\text{-dR}_{R/A})_p^\wedge$ must also be a descending chain of submodules of $(q\text{-dR}_{R/A})_p^\wedge$. Moreover, we see that $\mathrm{fil}^p_{q\text{-Hdg}}(q\text{-dR}_{R/A})_p^\wedge$ must contain an element $\widetilde{\gamma}_q(x)$ such that $\widetilde{\gamma}_q(x) \equiv x^p/p \mod (q-1)$. Using Definition 3.2($c_p$), we see that $\widetilde{\gamma}_q$ must also be contained in the ideal $(x, q-1)^p$ after completed rationalisation. But it is straightforward to check that the prismatic envelope above doesn't contain any $\widetilde{\gamma}_q(x)$ with these properties (for the details, see Example 4.24 below).

This shows that $\mathrm{AniAlg}_A^{q\text{-Hdg}} \to \mathrm{AniAlg}_A$ is not essentially surjective. Hence it can't have a section, not even over $\mathrm{Sm}_A \subseteq \mathrm{AniAlg}_A$, because we could always animate to extend such a section to all of AniAlg_A. ☐

3.4. Remark. — Despite the general non-existence, it's possible to construct many interesting objects of the ∞-category $\mathrm{AniAlg}_A^{q\text{-Hdg}}$, and the forgetful functor $\mathrm{AniAlg}_A^{q\text{-Hdg}} \to \mathrm{AniAlg}_A$ does admit sections when restricted to certain full subcategories of AniAlg_A. We'll discuss several such examples in §4.

In the remainder of this subsection, we'll study the following objects:

3.5. q-Hodge complexes. — Given a q-Hodge filtration $\mathrm{fil}^\star\, q\text{-dR}_{R/A}$ for R over A, we can construct the *q-Hodge complex* as

$$q\text{-Hdg}_{(R,\mathrm{fil}^\star_{q\text{-Hdg}})/A} \coloneqq \operatorname{colim}\left(\mathrm{fil}^0_{q\text{-Hdg}}\, q\text{-dR}_{R/A} \xrightarrow{(q-1)} \mathrm{fil}^1_{q\text{-Hdg}}\, q\text{-dR}_{R/A} \xrightarrow{(q-1)} \cdots\right)_{(q-1)}^\wedge .$$

If the q-Hodge filtration is clear from the context, we usually just write $q\text{-Hdg}_{R/A}$.

3.6. Remark. — In Definition 1.14 we've seen a variant of Definition 3.2 and 3.5 that only allows for the case where $R = S$ is a smooth A-algebra. Moreover, that variant uses the q-de Rham complex $q\text{-}\Omega_{S/A}$ instead of its derived version $q\text{-dR}_{S/A}$.

Note that $q\text{-}\Omega_{S/A}$ usually *doesn't* agree with the derived q-de Rham complex $q\text{-dR}_{S/A}$, because $\Omega^*_{S/A}$ and $\mathrm{dR}_{S/A}$ usually differ in characteristic 0. But this is not a problem. If we're given a filtration $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\Omega_{S/A}$ that satisfies the obvious analogues of Definition 3.2(a)–(c_p), then its pullback along the canonical map $q\text{-dR}_{S/A} \to q\text{-}\Omega_{S/A}$ yields a filtration $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_{R/A}$ as in Definition 3.2. Indeed, this follows from the fact that $\Omega^*_{S/A} \simeq \widehat{\mathrm{dR}}_{S/A}$ always agrees with the Hodge-completed derived de Rham complex and the fact that any filtration is the pullback of its completion (see 1.48).

Conversely, we'll show in Proposition 3.47 that for any $(S, \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{S/A}) \in \mathrm{AniAlg}_A^{q\text{-Hdg}}$ such that S is smooth over A, we have an equivalence

$$q\text{-}\Omega_{S/A} \simeq q\text{-}\widehat{\mathrm{dR}}_{S/A}$$

of the underived q-de Rham complex and the q-Hodge completed derived q-de Rham complex. Finally, let us remark that in the definition of the q-Hodge complex it doesn't matter whether we use $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A}$ or its completion $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{R/A}$, since every element in $\mathrm{fil}^i_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A}$ becomes divisible by $(q-1)^i$ in $q\text{-}\mathrm{Hdg}_{R/A}$ and the q-Hodge complex is $(q-1)$-complete.

3.7. Proposition. — $\mathrm{AniAlg}_A^{q\text{-Hdg}}$ *admits a canonical symmetric monoidal structure. The tensor product of two objects* $(R_1, \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R_1/A})$ *and* $(R_2, \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R_2/A})$ *is given by*

$$\left(R_1 \otimes^{\mathrm{L}}_A R_2, \left(\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R_1/A} \otimes^{\mathrm{L}}_{(q-1)^\star A[q]} \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R_2/A}\right)^\wedge_{(q-1)}\right),$$

where in the second component we take the derived tensor as filtered modules over the filtered ring $(q-1)^\star A[q]$*. Furthermore, the functor*

$$q\text{-}\mathrm{Hdg}_{-/A} \colon \mathrm{AniAlg}_A^{q\text{-Hdg}} \longrightarrow \widehat{\mathcal{D}}_{(q-1)}\big(A[q]\big)$$

can be equipped with a canonical symmetric monoidal structure.

To prove Proposition 3.7, let us first construct a filtration on $q\text{-}\mathrm{Hdg}_{-/A}/(q-1)$.

3.8. The conjugate filtration. — Let $(R, \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A})$ be an object in $\mathrm{AniAlg}_A^{q\text{-Hdg}}$. Let's consider the localisation of the filtered $(q-1)^\star A[q]$-module $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A}$ at $(q-1)$:

$$\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A}\left[\tfrac{1}{q-1}\right] \simeq \operatorname{colim}\left(\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A} \xrightarrow{(q-1)} \mathrm{fil}^{\star+1}_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A} \xrightarrow{(q-1)} \dotsb\right).$$

Upon completing the filtration, this filtered object becomes the $(q-1)$-adic filtration on the q-Hodge complex $q\text{-}\mathrm{Hdg}_{R/A}$.

Before taking the colimit, the diagram above can be regarded as a bifiltered object, with one ascending ("horizontal") filtration, given by the steps in the colimit, and one descending ("vertical") filtration, given by the filtrations on each step $\mathrm{fil}^{\star+n}_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A}$. If we pass to the associated graded in the vertical direction, we obtain

$$q\text{-}\mathrm{Hdg}_{R/A}/(q-1) \simeq \operatorname{colim}\left(\mathrm{gr}^0_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A} \xrightarrow{(q-1)} \mathrm{gr}^1_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A} \xrightarrow{(q-1)} \dotsb\right).$$

This representation as a colimit defines an exhaustive ascending filtration on $q\text{-}\mathrm{Hdg}_{R/A}/(q-1)$, which we define to be the *conjugate filtration* $\mathrm{fil}_\star^{\mathrm{conj}}(q\text{-}\mathrm{Hdg}_{R/A}/(q-1))$.

3.9. Lemma. — *The associated graded of the conjugate filtration* $\mathrm{fil}_\star^{\mathrm{conj}}\, q\text{-}\mathrm{Hdg}_{R/A}/(q-1)$ *is given by*

$$\mathrm{gr}_*^{\mathrm{conj}}\big(q\text{-}\mathrm{Hdg}_{R/A}/(q-1)\big) \simeq \Sigma^{-*}\mathrm{dR}^*_{R/A} \simeq \mathrm{gr}^*_{\mathrm{Hdg}}\, \mathrm{dR}_{R/A}\,.$$

Proof. To avoid ambiguous notation, let us identify the filtered ring $(q-1)^\star A[q]$ with the graded ring $A[\beta, t]$, where $|\beta| = 1$, $|t| = -1$, and $\beta t = q - 1$.[3.2] The filtered structure on $A[\beta, t]$ comes from the $A[t]$-module structure (see 1.48), so t can be regarded as the filtration parameter and β can be regarded as the element "$(q-1)$ sitting in degree 1". If we regard $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A}$ as a graded $A[\beta, t]$-module, then

$$\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A}/\beta \simeq \mathrm{fil}^\star_{\mathrm{Hdg}}\, \mathrm{dR}_{R/A} \quad \text{and} \quad \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A}/t \simeq \mathrm{gr}^*_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A}$$

as graded $A[t]$- or $A[\beta]$-modules, respectively. The first equivalence follows from Definition 3.2(*b*), the second follows because modding out t is the same as taking the associated graded (see 1.48). Hence also

$$\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A}/(\beta, t) \simeq \mathrm{gr}^*_{\mathrm{Hdg}}\, \mathrm{dR}_{R/A}$$

as filtered A-modules. Finally, by construction, we can identify $q\text{-}\mathrm{Hdg}_{R/A}/(q-1)$ with $(\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A} \otimes^{\mathrm{L}}_{A[\beta]} A[\beta^{\pm 1}])_0/(\beta t)$, where $(-)_0$ denotes the restriction of a graded object to its degree-0 part. Then the desired assertion follows from Lemma 3.10 below. □

3.10. Lemma. — *Let M^* be a graded module over the graded ring $A[\beta, t]$, where $|\beta| = 1$, $|t| = -1$. Then $(M^* \otimes^{\mathrm{L}}_{A[\beta]} A[\beta^{\pm 1}])_0/(\beta t)$ admits a canonical exhaustive ascending filtration whose associated graded is $M^*/(\beta, t)$.*

Proof. We formally get $(M^* \otimes^{\mathrm{L}}_{A[\beta]} A[\beta^{\pm 1}])_0/(\beta t) \simeq (M^*/t \otimes^{\mathrm{L}}_{A[\beta]} A[\beta^{\pm 1}])_0$. Let $\beta^{-\star} A[\beta]$ denote the ascendingly filtered graded ring

$$\beta^{-\star} A[\beta] \coloneqq \left(\cdots \xrightarrow{\beta} A[\beta](1) \xrightarrow{\beta} A[\beta](0) \xrightarrow{\beta} A[\beta](-1) \xrightarrow{\beta} \cdots \right),$$

where $A[\beta](i)$ denotes the shift of the graded object $A[\beta]$ by i (to account for the fact that multiplication by β shifts degrees). The colimit of this filtration is $\operatorname{colim} \beta^{-\star} A[\beta] \simeq A[\beta^{\pm 1}]$. Hence $(M^*/t \otimes^{\mathrm{L}}_{A[\beta]} \beta^{-\star} A[\beta])_0$ defines an exhaustive ascending filtration on $(M^*/t \otimes^{\mathrm{L}}_{A[\beta]} A[\beta^{\pm 1}])_0$ (by inspection, this is also precisely how the conjugate filtration from 3.8 arises). Since the associated graded of $\beta^{-\star} A[\beta]$ is $\bigoplus_{i \in \mathbb{Z}} A(-i)$, the associated graded of the filtration we've just constructed is indeed

$$\left(\bigoplus_{i \in \mathbb{Z}} M^*/t \otimes^{\mathrm{L}}_{A[\beta]} A(-i) \right)_0 \simeq \left(\bigoplus_{i \in \mathbb{Z}} M^*/(\beta, t)(-i) \right)_0 \simeq M^*/(\beta, t)\,. \qquad \square$$

Proof of Proposition 3.7. $\mathrm{AniAlg}_A^{q\text{-Hdg}}$ can be written as an iterated pullback of symmetric monoidal ∞-categories along symmetric monoidal functors, so there's a canonical way to equip it with a symmetric monoidal structure itself. The forgetful functors

$$\mathrm{AniAlg}_A^{q\text{-Hdg}} \longrightarrow \mathrm{AniAlg}_A \quad \text{and} \quad \mathrm{AniAlg}_A^{q\text{-Hdg}} \longrightarrow \mathrm{Mod}_{(q-1)^\star A[q]}\big(\mathrm{Fil}\,\mathcal{D}(A)\big)^\wedge_{(q-1)}$$

will then be symmetric monoidal, which shows the formula for tensor products.

To construct a symmetric monoidal structure on $q\text{-}\mathrm{Hdg}_{-/A}$, we use 3.8. Since localising is symmetric monoidal and passing to the 0^{th} filtration step is lax symmetric monoidal, we get

[3.2] In Remark 7.4 we'll recognise $(q-1)^\star \mathbb{Z}[\![q-1]\!] \cong \mathbb{Z}[\beta][\![t]\!] \cong \pi_{2*}(\mathrm{ku}^{hS^1})$, where $\beta \in \pi_2(\mathrm{ku})$ is the Bott element and $t \in \pi_{-2}(\mathrm{ku}^{hS^1})$ is a suitable complex orientation.

a lax symmetric monoidal structure on $q\text{-Hdg}_{-/A}$. Strict symmetric monoidality can then be checked modulo $(q-1)$ because the values of $q\text{-Hdg}_{-/A}$ are $(q-1)$-complete.

From the proof of Lemma 3.10 above, it is clear that $\operatorname{fil}_\star^{\mathrm{conj}}(q\text{-Hdg}_{-/A}/(q-1))$ can be equipped with a lax symmetric monoidal structure compatible with the one on $q\text{-Hdg}_{-/A}/(q-1)$ (modding out t or β as well as $-\otimes^{\mathrm{L}}_{A[\beta]}\beta^{-\star}A[\beta]$ are symmetric monoidal and $(-)_0$ is lax symmetric monoidal). Furthermore the equivalence

$$\operatorname{gr}_*^{\mathrm{conj}}\bigl(q\text{-Hdg}_{-/A}/(q-1)\bigr)\simeq \operatorname{gr}^*_{\mathrm{Hdg}}\mathrm{dR}_{-/A}$$

is an equivalence of lax symmetric monoidal functors. Strict symmetric monoidality of $\operatorname{fil}_\star^{\mathrm{conj}}(q\text{-Hdg}_{-/A}/(q-1))$ can now be checked on the associated graded, so we win since it's well-known that $\operatorname{gr}^*_{\mathrm{Hdg}}\mathrm{dR}_{-/A}$ is symmetric monoidal. $\square$

§3.2. The main result

We can now state the general Habiro descent result. We let $q\text{-}\mathbb{W}_m\Omega^*_{-/A}$ denote the m-truncated q-de Rham Witt complex from [Wag24, Definition 3.12] and $q\text{-}\mathbb{W}_m\mathrm{dR}_{-/A}\colon \mathrm{AniAlg}_A\to\mathcal{D}(A[q])$ its non-abelian derived functor.

3.11. Theorem. — *Let A be a perfectly covered Λ-ring and $\mathrm{AniAlg}_A^{q\text{-Hdg}}$ be the ∞-category of animated A-algebras equipped with a q-Hodge filtration on their q-de Rham complex.*

(*a*) *Let $\widehat{\mathcal{D}}_{\mathcal{H}}(A[q])\subseteq\mathcal{D}(A[q])$ denote the full sub-∞-category of Habiro-complete objects (in the sense of B.1). Then the q-Hodge complex functor admits a symmetric monoidal factorisation*

$$\begin{array}{ccc} & & \widehat{\mathcal{D}}_{\mathcal{H}}\bigl(A[q]\bigr)\\ & \overset{q\text{-}\mathcal{H}\mathrm{dg}_{-/A}}{\nearrow} & \big\downarrow (-)^\wedge_{(q-1)}\\ \mathrm{AniAlg}_A^{q\text{-Hdg}} & \xrightarrow[q\text{-Hdg}_{-/A}]{} & \widehat{\mathcal{D}}_{(q-1)}\bigl(A[q]\bigr)\end{array}$$

(*b*) *For all $m\in\mathbb{N}$, the quotient $q\text{-}\mathcal{H}\mathrm{dg}_{-/A}/(q^m-1)$ admits an exhaustive ascending filtration $\operatorname{fil}_\star^{q\text{-}\mathbb{W}_m\Omega}(q\text{-}\mathcal{H}\mathrm{dg}_{-/A}/(q^m-1))$ with associated graded*

$$\operatorname{gr}_*^{q\text{-}\mathbb{W}_m\Omega}\bigl(q\text{-}\mathcal{H}\mathrm{dg}_{-/A}/(q^m-1)\bigr)\simeq\Sigma^{-*}\,q\text{-}\mathbb{W}_m\mathrm{dR}^*_{-/A}\,.$$

Furthermore, $\operatorname{fil}_\star^{q\text{-}\mathbb{W}_m\Omega}(q\text{-}\mathcal{H}\mathrm{dg}_{-/A}/(q^m-1))$ can be equipped with a canonical lax symmetric monoidal structure compatible with the one on $q\text{-}\mathcal{H}\mathrm{dg}_{-/A}/(q^m-1)$, and the equivalence above is an equivalence of lax symmetric monoidal functors.

3.12. Example. — If S is a smooth over A and $\square\colon A[x_1,\dots,x_n]\to S$ is an étale framing, then we can define a filtration on the coordinate-dependent q-de Rham complex $q\text{-}\Omega^*_{S/A,\square}$ via

$$\operatorname{fil}^n_{q\text{-Hdg},\square}\,q\text{-}\Omega^*_{S/A,\square}\coloneqq(q-1)^{\max\{n-*,0\}}\,q\text{-}\Omega^*_{S/A,\square}\,.$$

(compare the construction in 1.11). As explained in Remark 3.6, we can take the pullback along $q\text{-dR}_{S/A}\to q\text{-}\Omega^*_{S/A,\square}$ to get a filtration $\operatorname{fil}^\star_{q\text{-Hdg},\square}\,q\text{-dR}_{S/A}$ on the derived q-de Rham complex. It's straightforward to equip it with the additional structure from Definition 3.2(*a*)–(*c_p*): Just construct everything on the level of complexes and then take the pullback.

Therefore, the pair $(S, \operatorname{fil}^\star_{q\text{-Hdg},\square} q\text{-dR}_{S/A})$ determines an $\mathbb{E}_0$-algebra in $\mathrm{AniAlg}_A^{q\text{-Hdg}}$. We'll explain in Remark 9.14 that it can be refined to an $\mathbb{E}_\infty$-algebra. The derived q-Hodge complex associated to $(S, \operatorname{fil}^\star_{q\text{-Hdg},\square} q\text{-dR}_{S/A})$ is the coordinate-dependent q-Hodge complex $q\text{-Hdg}^*_{S/A,\square}$. Indeed, as we've seen in Remark 3.6, in the definition of $q\text{-Hdg}_{-/A}$ it doesn't matter whether we work with the q-Hodge filtration on $q\text{-dR}_{-/A}$ or its completion. Since $\operatorname{fil}^\star_{q\text{-Hdg},\square} q\text{-}\Omega^*_{S/A,\square}$ is already complete, it's automatically the completion of its pullback $\operatorname{fil}^\star_{q\text{-Hdg},\square} q\text{-dR}_{S/A}$. We conclude that the corresponding derived q-Hodge complex is

$$\operatorname{colim}\left(\operatorname{fil}^0_{q\text{-Hdg},\square} q\text{-}\Omega^*_{S/A,\square} \xrightarrow{(q-1)} \operatorname{fil}^1_{q\text{-Hdg},\square} q\text{-}\Omega^*_{S/A,\square} \xrightarrow{(q-1)} \cdots\right) \cong q\text{-Hdg}^*_{S/A,\square}\,,$$

as claimed.

In this case, Theorem 3.11(a) shows that $q\text{-Hdg}^*_{S/A,\square}$ descends to an $\mathbb{E}_\infty$-algebra $q\text{-}\mathcal{H}\mathrm{dg}_{S/A,\square}$ in $\widehat{\mathcal{D}}_{\mathcal{H}}(A[q])$. As we'll see in Corollary 3.31 below, $\Sigma^{-n}\, q\text{-}\mathbb{W}_m\mathrm{dR}^n_{S/A} \simeq q\text{-}\mathbb{W}_m\Omega^n_{S/A}$ holds for all n. Thus, Theorem 3.11(b) shows

$$\mathrm{H}^*\left(q\text{-}\mathcal{H}\mathrm{dg}_{S/A,\square}/(q^m-1)\right) \cong q\text{-}\mathbb{W}_m\Omega^*_{S/A}$$

as graded $A[q]/(q^m-1)$-modules. With a little more effort (see Corollary 3.54 below), we can even get an equivalence as differential-graded $A[q]/(q^m-1)$-algebras, so we obtain an improved version of [Wag24, Theorem 4.27].[(3.3)]

In fact, $q\text{-}\mathcal{H}\mathrm{dg}_{S/A,\square}$ can be described as an explicit complex; this was first presented in [Sch25, Lecture 4]. To this end, equip $A[x_1,\dots,x_n]$ with the *toric* Λ-A-*algebra structure* in which the Adams operations are given by $\psi^m(x_i) = x_i^m$ and consider the relative Habiro ring $\mathcal{H}_{S/A[x_1,\dots,x_n]}$. For $i = 1,\dots,n$ let γ_i be the $A[q]$-algebra endomorphism of $A[x_1,\dots,x_n,q]$ given by $\gamma_i(x_i) = qx_i$ and $\gamma_i(x_j) = x_j$ for $j \neq i$. We wish to extend γ_i to an automorphism of $\mathcal{H}_{S/A[x_1,\dots,x_n]}$. To do so, we'll extend γ_i to each of the factors of the equaliser in Lemma 2.12. Fix $m \in \mathbb{N}$ and put $S^{(m)} \coloneqq (S \otimes_{A[x_1,\dots,x_n],\psi^m} A[x_1,\dots,x_n])[\zeta_m]$. Consider the diagram

$$\begin{array}{ccc} A[x_1,\dots,x_n,\zeta_m]\llbracket q-\zeta_m\rrbracket & \xrightarrow{\gamma_i} & S^{(m)}\llbracket q-\zeta_m\rrbracket \\ \square\big\downarrow & \overset{\gamma_i^{(m)}}{\dashrightarrow} & \big\downarrow \\ S^{(m)}\llbracket q-\zeta_m\rrbracket & \xrightarrow{\overline{\gamma}_i^{(m)}} & S^{(m)} \end{array}$$

where $\overline{\gamma}_i^{(m)}$ is given by the identity on the tensor factor S, $\overline{\gamma}_i^{(m)}(x_i) = \zeta_m x_i$, and $\overline{\gamma}_i^{(m)}(x_j) = x_j$ for $j \neq i$. By the infinitesimal lifting property of formally étale morphisms, there exists a unique dashed arrow $\gamma_i^{(m)}$ making the diagram commutative. Then $\left(\gamma_i^{(m)}\right)_{m\in\mathbb{N}}$ defines the desired automorphism γ_i of $\mathcal{H}_{S/A[x_1,\dots,x_n]}$ via Lemma 2.12. It's also straightforward to check that $\gamma_i \equiv \mathrm{id} \mod x_i$.

Letting $q\text{-}\widetilde{\partial}_i \coloneqq (\gamma_i - \mathrm{id})/x_i$ and $q\text{-}\widetilde{\nabla} \coloneqq \sum_i q\text{-}\widetilde{\partial}_i\, \mathrm{d}x_i$, the Koszul complex of the commuting endomorphisms $q\text{-}\widetilde{\partial}_i$,

$$\left(\mathcal{H}_{S/A[x_1,\dots,x_n]} \xrightarrow{q\text{-}\widetilde{\nabla}} \bigoplus_i \mathcal{H}_{S/A[x_1,\dots,x_n]}\,\mathrm{d}x_i \xrightarrow{q\text{-}\widetilde{\nabla}} \cdots \xrightarrow{q\text{-}\widetilde{\nabla}} \mathcal{H}_{S/A[x_1,\dots,x_n]}\,\mathrm{d}x_1\cdots\mathrm{d}x_n\right),$$

is an explicit complex representing $q\text{-}\mathcal{H}\mathrm{dg}_{S/A,\square}$. This can be shown by unravelling the proof of Theorem 3.11 (which is less horrible than it sounds).

(3.3) But this theorem is being used in the proof, so we don't get a new proof.

Example 3.12 covers in particular the case of étale A-algebras. In this special case, we recover a familiar construction.

3.13. Corollary. — *If R is étale over A, then q-$\mathcal{H}\mathrm{dg}_{R/A}$ is the relative Habiro ring $\mathcal{H}_{R/A}$ constructed in 2.7.*

Proof. This is clear from the explicit presentation in Example 3.12, but it can also be shown without having to unravel the proof of Theorem 3.11.

We'll see in Corollary 3.31 that $q\text{-}\mathbb{W}_m\mathrm{dR}^n_{R/A} \simeq \Sigma^{-n}\, q\text{-}\mathbb{W}_m\Omega^n_{R/A}$ holds for all $n \geqslant 0$ whenever R is smooth over A. If R is étale, then combining this observation with Theorem 3.11(a) and [Wag24, Proposition 3.31] shows

$$q\text{-}\mathcal{H}\mathrm{dg}_{R/A}/(q^m-1) \simeq q\text{-}\mathbb{W}_m(R/A) \simeq \mathcal{H}_{R/A}/(q^m-1)\,.$$

By uniqueness of deformations of étale extensions, these automatically lift to a unique equivalence of $\mathbb{E}_\infty$-$\mathcal{H}$-algebras $(q\text{-}\mathcal{H}\mathrm{dg}_{R/A})^\wedge_{(q^m-1)} \simeq \mathcal{H}_{R/A,m}$; furthermore, uniqueness also ensures that these equivalences are compatible for varying m. It follows that $q\text{-}\mathcal{H}\mathrm{dg}_{R/A} \simeq \mathcal{H}_{R/A}$, as desired. ☐

The proof of Theorem 3.11 has many ingredients and will occupy §§3.3–3.6. Before we get lost in the technicalities, let us already outline the main argument and point out where the missing pieces will be provided.

Proof outline of Theorem 3.11. In §3.3 we'll introduce *twisted q-de Rham complexes* for all $m \in \mathbb{N}$. These are (q^m-1)-complete $\mathbb{E}_\infty$-$A[q]$-algebras $q\text{-}\mathrm{dR}^{(m)}_{R/A}$ satisfying

$$q\text{-}\mathrm{dR}^{(m)}_{R/A}/(q^m-1) \simeq q\text{-}\mathbb{W}_m\mathrm{dR}_{R/A}$$

(see Proposition 3.19[3.4]). By animating the stupid filtration $q\text{-}\mathbb{W}_m\Omega^{\geqslant n,*}_{-/A}$, we obtain a filtration $\mathrm{fil}^\star_{\mathcal{H}\mathrm{dg}_m}\, q\text{-}\mathbb{W}_m\mathrm{dR}_{-/A}$ on $q\text{-}\mathbb{W}_m\mathrm{dR}_{-/A}$. For $m=1$, this is the Hodge filtration on $\mathrm{dR}_{-/A}$; for higher m, it should be thought of as a q-Witt vector analogue of the Hodge filtration. By construction,

$$\mathrm{gr}^n_{\mathcal{H}\mathrm{dg}_m}\, q\text{-}\mathbb{W}_m\mathrm{dR}_{-/A} \simeq q\text{-}\mathbb{W}_m\mathrm{dR}^n_{-/A}\,.$$

In §3.5, specifically Proposition 3.39, we'll show that given a q-Hodge filtration on $q\text{-}\mathrm{dR}_{R/A}$, we can construct a filtration $\mathrm{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_m}\, q\text{-}\mathrm{dR}^{(m)}_{R/A}$ satisfying

$$\mathrm{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_m}\, q\text{-}\mathrm{dR}^{(m)}_{R/A}/(q^m-1) \simeq \mathrm{fil}^\star_{\mathcal{H}\mathrm{dg}_m}\, q\text{-}\mathbb{W}_m\mathrm{dR}_{R/A}\,,$$

where (q^m-1) sits in filtration degree 1. We'll also verify that $\mathrm{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_m}\, q\text{-}\mathrm{dR}^{(m)}_{R/A}$ is lax symmetric monoidal in $(R, \mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R/A})$ and the equivalence above is an equivalence of lax symmetric monoidal functors $\mathrm{AniAlg}^{q\text{-}\mathrm{Hdg}}_A \to \mathrm{Mod}_{(q^m-1)^\star A[q]}(\mathrm{Fil}\,\mathcal{D}(A))$.

With this construction, we'll build the desired Habiro descent of $q\text{-}\mathrm{Hdg}_{R/A}$ in §3.6 by mimicking the definition of the q-Hodge complex in 3.5. For all $m \in \mathbb{N}$, we define

$$q\text{-}\mathcal{H}\mathrm{dg}_{R/A,m} \coloneqq \operatorname{colim}\Big(\mathrm{fil}^0_{q\text{-}\mathcal{H}\mathrm{dg}_m}\, q\text{-}\mathrm{dR}^{(m)}_{R/A} \xrightarrow{(q^m-1)} \mathrm{fil}^1_{q\text{-}\mathcal{H}\mathrm{dg}_m}\, q\text{-}\mathrm{dR}^{(m)}_{R/A} \xrightarrow{(q^m-1)} \dots\Big)^\wedge_{(q^m-1)}\,.$$

[3.4] Informally, just as the q-de Rham complex is a q-deformation of $\mathrm{dR}_{R/A} \simeq q\text{-}\mathbb{W}_1\mathrm{dR}_{R/A}$, the twisted q-de Rham complexes are q^m-deformations of $q\text{-}\mathbb{W}_m\mathrm{dR}_{R/A}$.

In Proposition 3.43, we'll show $(q\text{-}\mathcal{H}\mathrm{dg}_{R/A,m})^\wedge_{(q^d-1)} \simeq q\text{-}\mathcal{H}\mathrm{dg}_{R/A,d}$ whenever $d \mid m$. It follows that $q\text{-}\mathcal{H}\mathrm{dg}_{R/A} \coloneqq \lim_{m\in\mathbb{N}} q\text{-}\mathcal{H}\mathrm{dg}_{R/A,m}$ determines a Habiro descent of $q\text{-}\mathrm{Hdg}_{R/A}$, thus proving Theorem 3.11(a), except for the symmetric monoidality statement. As in the proof of Proposition 3.7, it's formal to construct a lax symmetric monoidal structure on $q\text{-}\mathcal{H}\mathrm{dg}_{-/A}$ which reduces to the one on $q\text{-}\mathrm{Hdg}_{-/A}$ after $(q-1)$-completion; see 3.45 for the details. Strict symmetric monoidality will then be checked in Lemma 3.46, finishing the proof of Theorem 3.11(a).

To show Theorem 3.11(b), we will mimic the arguments for the conjugate filtration (and in fact, for $m=1$, the desired filtration on $q\text{-}\mathcal{H}\mathrm{dg}_{R/A}/(q-1) \simeq q\text{-}\mathrm{Hdg}_{R/A}/(q-1)$ *is* the conjugate filtration). By the same argument as in 3.8, we obtain

$$q\text{-}\mathcal{H}\mathrm{dg}_{R/A,m}/(q^m-1) \simeq \operatorname{colim}\left(\mathrm{gr}^0_{q\text{-}\mathcal{H}\mathrm{dg}_m} q\text{-}\mathrm{dR}^{(m)}_{R/A} \xrightarrow{(q^m-1)} \mathrm{gr}^1_{q\text{-}\mathcal{H}\mathrm{dg}_m} q\text{-}\mathrm{dR}^{(m)}_{R/A} \xrightarrow{(q^m-1)} \dots\right).$$

The colimit defines an exhaustive ascending filtration on $q\text{-}\mathcal{H}\mathrm{dg}_{R/A,m}/(q^m-1)$, which we take to be our definition of $\mathrm{fil}^{q\text{-}\mathbb{W}_m\Omega}_\star(q\text{-}\mathcal{H}\mathrm{dg}_{R/A,m}/(q^m-1))$. The associated graded of this filtration can be determined by via Lemma 3.10 (for this we identify the filtered ring $(q^m-1)^\star A[q]$ with the graded ring $A[q,\beta,t]/(\beta t-(q^m-1))$, where $|q|=0$, $|\beta|=1$, and $|t|=-1$): We obtain

$$\mathrm{gr}^{q\text{-}\mathbb{W}_m\Omega}_*\big(q\text{-}\mathcal{H}\mathrm{dg}_{R/A,m}/(q^m-1)\big) \simeq \Sigma^{-*} q\text{-}\mathbb{W}_m\mathrm{dR}^*_{R/A} \simeq \mathrm{gr}^*_{\mathcal{H}\mathrm{dg}_m} q\text{-}\mathbb{W}_m\mathrm{dR}_{R/A}\,.$$

As in the proof of Proposition 3.7, the lax symmetric monoidality statements are formal, and so the proof of Theorem 3.11(b) is finished. □

§3.3. Deformations of q-de Rham–Witt complexes

We fix a perfectly covered Λ-ring A as before. We let ψ^m denote its Adams operations, which we extend to a map $\psi^m\colon A[q]\to A[q]$ via $\psi^m(q)\coloneqq q^m$. We'll also frequently use the Berthelot–Ogus décalage functor $\mathrm{L}\eta_{[m]_q}$ (see [BMS18, §6] or [Stacks, Tag 0F7N]).

In this subsection, we'll study *twisted q-de Rham complexes*: For S smooth over A, these are certain (q^m-1)-complete $\mathbb{E}_\infty$-$A[q]$-algebras $q\text{-}\Omega^{(m)}_{S/A}$, refining the $(q-1)$-complete $\mathbb{E}_\infty$-$A[q]$-algebras $\mathrm{L}\eta_{[m]_q} q\text{-}\Omega_{S/A}$ for all $m\in\mathbb{N}$. The rationale behind our notation and the name *twisted q-de Rham complexes* is as follows: If the global q-de Rham complex would admit Adams operations ψ^m inducing equivalences

$$\psi^m\colon \left(q\text{-}\Omega_{S/A}\otimes^{\mathrm{L}}_{A[q],\psi^m} A[q]\right)^\wedge_{(q-1)} \xrightarrow{\simeq} \mathrm{L}\eta_{[m]_q} q\text{-}\Omega_{S/A}\,,$$

then the corresponding twisted q-de Rham complex could simply be constructed as the (q^m-1)-completion of the "Adams-twist" $q\text{-}\Omega_{S/A}\otimes^{\mathrm{L}}_{A[q],\psi^m} A[q]$.[(3.5)] However, such global Adams operations don't exist in general (this already fails if S is étale over A, as Λ-structures usually don't extend along étale maps). The best we have is, for every prime p, a Frobenius ϕ_p on the p-completion $(q\text{-}\Omega_{S/A})^\wedge_p$. Still, these p-adic Frobenii are enough to construct $q\text{-}\Omega^{(m)}_{S/A}$.

3.14. Twisted q-de Rham complexes — Let S be a smooth A-algebra. We'll construct a (q^m-1)-complete $\mathbb{E}_\infty$-$A[q]$-algebra using Corollary 2.4. In the notation of that corollary, take

$$E_d \coloneqq \left(\mathrm{L}\eta_{[m/d]_q} q\text{-}\Omega_{S/A}\otimes^{\mathrm{L}}_{A[q],\psi^d} A[q]\right)^\wedge_{\Phi_d(q)}.$$

(3.5) If ψ^m is finite (for example, this holds if $A=\mathbb{Z}$ or more generally if A is a polynomial ring), then $q\text{-}\Omega_{S/A}\otimes^{\mathrm{L}}_{A[q],\psi^m} A[q]$ is already (q^m-1)-complete.

We must also provide p-adic gluing equivalences. For p a prime such that $pd \mid m$, the required gluing equivalence $(E_{pd})_p^\wedge \simeq (E_d)_p^\wedge$ should be of the form

$$\Bigl(\mathrm{L}\eta_{[m/pd]_q}\, q\text{-}\Omega_{S/A} \otimes^{\mathrm{L}}_{A[q],\psi^{pd}} A[q]\Bigr)^\wedge_{(p,\Phi_{pd}(q))} \xrightarrow{\simeq} \Bigl(\mathrm{L}\eta_{[m/d]_q}\, q\text{-}\Omega_{S/A} \otimes^{\mathrm{L}}_{A[q],\psi^{d}} A[q]\Bigr)^\wedge_{(p,\Phi_{d}(q))}\,.$$

To construct this, we may replace $\mathrm{L}\eta_{[m/pd]_q}$ and $\mathrm{L}\eta_{[m/d]_q}$ by $\mathrm{L}\eta_{[p^\alpha]_q}$ and $\mathrm{L}\eta_{[p^{\alpha+1}]_q}$, where $\alpha \coloneqq v_p(m/pd)$, because the factor $[m/pd]_q/[p^\alpha]_q$ will be invertible on either side. It will thus be enough to construct an equivalence

$$\Bigl(\mathrm{L}\eta_{[p^\alpha]_q}\, q\text{-}\Omega_{S/A} \otimes^{\mathrm{L}}_{A[q],\psi^{p}} A[q]\Bigr)^\wedge_{(p,q-1)} \xrightarrow{\simeq} \Bigl(\mathrm{L}\eta_{[p^{\alpha+1}]_q}\, q\text{-}\Omega_{S/A}\Bigr)^\wedge_{(p,q-1)}$$

Now $(\mathrm{L}\eta_{[p^\alpha]_q}\, q\text{-}\Omega_{S/A})_p^\wedge \simeq \mathrm{L}\eta_{[p^\alpha]_q}(q\text{-}\Omega_{S/A})_p^\wedge$. Indeed, $q\text{-}\Omega_{S/A}$ is $(q-1)$-complete, so p-completion agrees with $[p^\alpha]_q$-completion, which always commutes with $\mathrm{L}\eta_{[p^\alpha]_q}$ (see [BMS18, Lemma 6.20]). Thus, we may replace $q\text{-}\Omega_{S/A}$ by its p-completion on the left-hand side; the same argument applies to the right-hand side as well.

Finally, if (B,J) denotes the prism $(\widehat{A}_p[\![q-1]\!],[p]_q)$ and $T \coloneqq \widehat{S}_p[\zeta_p]$, then $(q\text{-}\Omega_{S/A})_p^\wedge \simeq \Delta_{T/B}$, and so the desired gluing equivalence can be constructed using the general fact that the relative Frobenius induces an equivalence (see [BS19, Theorem 15.3])

$$\phi_{/B}\colon \Delta_{T/B} \mathbin{\widehat{\otimes}}^{\mathrm{L}}_{B,\phi_B} B \xrightarrow{\simeq} \mathrm{L}\eta_J \Delta_{T/B}\,.$$

According to Corollary 2.4, we can glue the E_d for all $d \mid m$ to a (q^m-1)-complete $\mathbb{E}_\infty$-$A[q]$-algebra $q\text{-}\Omega^{(m)}_{S/A}$. This is the m^{th} *twisted q-de Rham complex of S over A*. Via animation, we can then define a functor

$$q\text{-}\mathrm{dR}^{(m)}_{-/A}\colon \mathrm{AniAlg}_A \longrightarrow \mathrm{CAlg}\Bigl(\widehat{\mathcal{D}}_{(q^m-1)}\bigl(A[q]\bigr)\Bigr)\,,$$

which agrees with $q\text{-}\Omega^{(m)}_{-/A}$ on polynomial-A-algebras (but not on all smooth A-algebras, due to the usual issues in characteristic 0).

The arithmetic fracture square for $q\text{-}\Omega^{(m)}_{S/A}$ (in the sense of 1.49) can be read off from the construction.

3.15. Lemma. — *Fix $m \in \mathbb{N}$ and $N \neq 0$ divisible by m. For any prime $p \mid N$ and any divisor $d \mid m$ write $m = p^{v_p(m)} m_p$ and $d = p^{v_p(d)} d_p$, where m_p and d_p are coprime to p. Let also*

$$\phi_{p/A[q]}\colon q\text{-}\Omega_{S/A} \otimes^{\mathrm{L}}_{A[q],\psi^p} A[q] \longrightarrow \bigl(q\text{-}\Omega_{S/A}\bigr)^\wedge_p$$

denote the relative Frobenius coming from the identification with prismatic cohomology. Then we have a functorial pullback square

$$\begin{array}{ccc} q\text{-}\Omega^{(m)}_{S/A} & \longrightarrow & \displaystyle\prod_{p\mid N}\prod_{d_p\mid m_p}\Bigl(q\text{-}\Omega_{S/A}\otimes^{\mathrm{L}}_{A[q],\psi^{p^{v_p(m)}d_p}} A[q]\Bigr)^\wedge_{(p,\Phi_{d_p}(q))} \\ \Big\downarrow & \lrcorner & \Big\downarrow \Bigl(\phi_{p/A[q]}^{v_p(m/d)}\Bigr)_{p\mid N,\, d\mid m} \\ \displaystyle\prod_{d\mid m}\Bigl(q\text{-}\Omega_{S/A}\otimes^{\mathrm{L}}_{A[q],\psi^{d}} A\bigl[\tfrac{1}{N},q\bigr]\Bigr)^\wedge_{\Phi_d(q)} & \longrightarrow & \displaystyle\prod_{p\mid N}\prod_{d\mid m}\Bigl(q\text{-}\Omega_{S/A}\otimes^{\mathrm{L}}_{A[q],\psi^{d}} A[q]\Bigr)^\wedge_p\bigl[\tfrac{1}{p}\bigr]^\wedge_{\Phi_d(q)} \end{array}$$

Proof. Using Remark 2.5, the desired pullback square can be identified with the (q^m-1)-completed arithmetic fracture square

$$\begin{array}{ccc} q\text{-}\Omega^{(m)}_{S/A} & \longrightarrow & \prod_{p\mid N}\big(q\text{-}\Omega^{(m)}_{S/A}\big)^\wedge_p \\ \downarrow & \lrcorner & \downarrow \\ q\text{-}\Omega^{(m)}_{S/A}\big[\tfrac{1}{N}\big]^\wedge_{(q^m-1)} & \longrightarrow & \prod_{p\mid N}\big(q\text{-}\Omega^{(m)}_{S/A}\big)^\wedge_p\big[\tfrac{1}{p}\big]^\wedge_{(q^m-1)} \end{array}$$

Here we also use that $[m/d]_q$ is mapped to a unit under $\psi^d\colon A[q]\to A[1/N,q]^\wedge_{\Phi_d(q)}$, so we may ignore $\mathrm{L}\eta_{[m/d]_q}$ for any $d\mid m$ in the bottom left corner. Similarly, we may ignore any $\mathrm{L}\eta_{[m/p^{v_p(m)}d]_q}$ in the top or bottom right corner. ☐

3.16. Transition maps. — Whenever $n\mid m$, there's a map of $\mathbb{E}_\infty$-$A[q]$-algebras

$$\big(q\text{-}\Omega^{(m)}_{S/A}\big)^\wedge_{(q^n-1)}\longrightarrow q\text{-}\Omega^{(n)}_{S/A}\,,$$

functorial in S. To construct this map, we once again appeal to the gluing procedure of Corollary 2.4. On Φ_d-completions, where $d\mid n$, the desired map is induced by the symmetric monoidal natural transformation $\mathrm{L}\eta_{[m/d]_q}\Rightarrow\mathrm{L}\eta_{[n/d]_q}$. It's straightforward to check that this is compatible with the p-adic gluings from 3.14. Alternatively, we can use the pullback square from Lemma 3.15: On the bottom part of the diagram, $\big(q\text{-}\Omega^{(m)}_{S/A}\big)^\wedge_{(q^n-1)}\to q\text{-}\Omega^{(n)}_{S/A}$ is induced by projection to those factors where $d\mid n$. In the top right corner, we also need to apply the relative Frobenius $\phi^{v_p(m/n)}_{p/A[q]}$ in any factor where $d_p\mid n_p$.

These maps can be assembled into a functor $q\text{-}\Omega^{(-)}_{S/A}\colon\mathbb{N}\to\mathrm{CAlg}(\widehat{\mathcal{D}}_{\mathcal{H}}(A[q]))$, where $\mathbb{N}$ denotes the category of natural numbers partially ordered by divisibility. Furthermore, this functor is itself functorial in S. We'll refrain from spelling out the argument, as it would just add one more layer of technicalities. To construct the Habiro descent eventually, we only need the individual maps, not the whole functor with all its higher coherences, since any $\lim_{m\in\mathbb{N}}$ can be replaced by the limit over the sequential subdiagram given by $\{n!\}_{n\geqslant 1}$.

3.17. Remark. — The maps $\big(q\text{-}\Omega^{(m)}_{S/A}\big)^\wedge_{(q^n-1)}\to q\text{-}\Omega^{(n)}_{S/A}$ are usually quite far from being equivalences, as can be seen from the discrepancy between $\mathrm{L}\eta_{[m/d]_q}$ and $\mathrm{L}\eta_{[n/d]_q}$. Thus, we can form the limit

$$\lim_{m\in\mathbb{N}} q\text{-}\Omega^{(m)}_{S/A}\,,$$

but it will usually be a pathological object (unless S is étale over A, in which case we recover 2.7). In particular, it won't be a Habiro descent of $q\text{-}\Omega_{S/A}$.

3.18. Remark. — To get $\big(q\text{-}\Omega^{(m)}_{S/A}\big)^\wedge_{(q^n-1)}\to q\text{-}\Omega^{(n)}_{S/A}$ closer to being an equivalence, a natural idea goes as follows: The Berthelot–Ogus décalage functors $\mathrm{L}\eta_{[m/d]_q}$ and $\mathrm{L}\eta_{[n/d]_q}$ come equipped with canonical filtrations (see [BMS19, Proposition 5.8]). If these filtrations would glue to give filtrations on $q\text{-}\Omega^{(m)}_{S/A}$ and $q\text{-}\Omega^{(n)}_{S/A}$, we could modify $q\text{-}\Omega^{(m)}_{S/A}$ and $q\text{-}\Omega^{(n)}_{S/A}$ by "making elements in each filtration degree i divisible by $[m]_q^i$ and $[n]_q^i$, respectively". It is then reasonable to hope that the map between the modifications is an equivalence after (q^n-1)-completion, so that in the limit we get a Habiro descent of $q\text{-}\Omega_{S/A}$.

However, the filtrations on $\mathrm{L}\eta_{[m/d]_q}$ *do not* glue. To make the idea work, we need the additional datum of a q-Hodge filtration on $q\text{-}\Omega_{S/A}$; and, we won't get a Habiro descent of $q\text{-}\Omega_{S/A}$, but of $q\text{-}\mathrm{Hdg}_{S/A}$. This is precisely how we'll prove Theorem 3.11. See the outline at the end of §3.2. Also see §3.7 for a discussion of Habiro descent for $q\text{-}\Omega_{S/A}$.

Let us now explain the relationship between $q\text{-}\Omega_{S/A}^{(m)}$ and the q-de Rham–Witt complexes. To this end, recall from [Wag24, Proposition 3.17] that we have a map of graded $A[q]/(q^m-1)$-algebras $F_{m/d}\colon q\text{-}\mathbb{W}_m\Omega_{S/A}^* \to q\text{-}\mathbb{W}_d\Omega_{S/A}^*$ for all divisors $d \mid m$ (the *Frobenius* on q-de Rham–Witt complexes). This satisfies $\mathrm{d}\circ F_{m/d} = (m/d)\circ F_{m/d}$. Therefore, if $\widetilde{F}_{m/d}$ is given by $(m/d)^n F_{m/d}$ in degree n, then

$$\widetilde{F}_{m/d}\colon q\text{-}\mathbb{W}_m\Omega_{S/A}^* \longrightarrow q\text{-}\mathbb{W}_d\Omega_{S/A}^*$$

is a map of differential-graded $A[q]/(q^m-1)$-algebras.

3.19. Proposition. — *Let A be a perfectly covered Λ-ring and let S be a smooth A-algebra. There's a functorial equivalence of $\mathbb{E}_\infty$-$A[q]/(q^m-1)$-algebras*

$$q\text{-}\Omega_{S/A}^{(m)}/(q^m-1) \xrightarrow{\simeq} q\text{-}\mathbb{W}_m\Omega_{S/A}\,.$$

Under this identification, the map $q\text{-}\Omega_{S/A}^{(m)}/(q^m-1) \to q\text{-}\Omega_{S/A}^{(d)}/(q^d-1)$ induced by 3.16 agrees with the map $\widetilde{F}_{m/d}$ above.

Proof sketch. By [Wag24, Corollary 4.37], for any $N \neq 0$ divisible by m the arithmetic fracture square for $q\text{-}\mathbb{W}_m\Omega_{S/A}$ has the the following form:

$$\begin{array}{ccc}
q\text{-}\mathbb{W}_m\Omega_{S/A} & \longrightarrow & \displaystyle\prod_{p\mid N}\prod_{d_p\mid m_p}\Big(\Omega_{S/A}\otimes^{\mathrm{L}}_{A,\psi^{p^{v_p(m)}d_p}} A[q]\Big)_p^\wedge/\Phi_{d_p}(q^{v_p(m)}) \\
\Big\downarrow {\scriptstyle (\mathrm{gh}_{m/d})_{d\mid m}} & \lrcorner & \Big\downarrow {\scriptstyle \left(\phi_{p/A}^{v_p(m/d)}\right)_{p\mid N,\, d\mid m}} \\
\displaystyle\prod_{d\mid m}\Big(\Omega_{S/A}\otimes^{\mathrm{L}}_{A,\psi^d} A\big[\tfrac{1}{N},q\big]\Big)/\Phi_d(q) & \longrightarrow & \displaystyle\prod_{p\mid N}\prod_{d\mid m}\Big(\Omega_{S/A}\otimes^{\mathrm{L}}_{A,\psi^d} A[q]\Big)_p^\wedge\big[\tfrac{1}{p}\big]/\Phi_d(q)
\end{array}$$

This agrees with the reduction modulo (q^m-1) of the arithmetic fracture square from Lemma 3.15. Here we note that upon reduction modulo (q^m-1), every occurence of the q-de Rham complex $q\text{-}\Omega_{S/A}$ in Lemma 3.15 can be replaced by $\Omega_{S/A}$. For example, for the d^{th} factor in the bottom left corner, reduction modulo (q^m-1) is the same as reduction modulo $\Phi_d(q)$, as (q^m-1) and $\Phi_d(q)$ only differ by a unit in $A[1/N,q]_{\Phi_d(q)}^\wedge$. Now $(q-1)$ maps to 0 under $\psi^d\colon A[q]\to A[1/N,q]/\Phi_d(q)$, so indeed $q\text{-}\Omega_{S/A}$ can be replaced by $\Omega_{S/A}$ in that corner. Similar arguments apply to the other corners.

This yields the desired equivalence $q\text{-}\Omega_{S/A}^{(m)}/(q^m-1)\simeq q\text{-}\mathbb{W}_m\Omega_{S/A}$. It's straightforward to check that this equivalence doesn't depend on the choice of N (compare 3.38 below).

The additional assertion about $q\text{-}\Omega_{S/A}^{(m)}/(q^m-1)\to q\text{-}\Omega_{S/A}^{(d)}/(q^d-1)$ follows similarly by a comparison of arithmetic fracture squares (where we may now choose the same N). The only non-trivial step is to check that under the equivalence $(q\text{-}\mathbb{W}_{p^\alpha}\Omega_{S/A})_p^\wedge \simeq (\Omega_{S/A}\otimes^{\mathrm{L}}_{A,\psi^{p^\alpha}} A[q]/(q^{p^\alpha}-1))_p^\wedge$ the maps $\widetilde{F}_p$ and $\phi_{p/A}$ get identified. This is explained in [Wag24, Corollary 4.38]. $\square$

§3.4. The Nygaard filtration on q-de Rham–Witt complexes

In this subsection, we'll study an auxiliary filtration on q-de Rham–Witt complexes. Throughout §3.4, we fix a prime p. We'll also write ϕ instead of ψ^p for the p^{th} Adams operation of the Λ-ring A. We extend ϕ to $A[q]$ via $\phi(q) \coloneqq q^p$.

Let's first recall the Nygaard filtration on q-de Rham cohomology.

3.20. The Nygaard filtration on q-de Rham cohomology. — Let S be a smooth A-algebra. By Lemma 3.15, for all $\alpha \geqslant 0$,

$$\left(q\text{-}\Omega_{S/A}^{(p^\alpha)}\right)_p^\wedge \simeq \left(q\text{-}\Omega_{S/A} \otimes_{A[q],\phi^\alpha}^{\mathrm{L}} A[q]\right)_{(p,q-1)}^\wedge$$

agrees with the α-fold Frobenius-twist of $(q\text{-}\Omega_{S/A})_p^\wedge$. Since q-de Rham cohomology is a special case of prismatic cohomology, the general theory of Nygaard filtrations [BS19, §15] provides a filtration $\operatorname{fil}_{\mathcal{N}}^\star\big(q\text{-}\Omega_{S/A}^{(p)}\big)_p^\wedge$: It is the preimage of the filtered décalage filtration on $\mathrm{L}\eta_{\Phi_p(q)}(q\text{-}\Omega_{S/A})_p^\wedge$ under the relative Frobenius

$$\phi_{/A[q]}\colon \left(q\text{-}\Omega_{S/A}^{(p)}\right)_p^\wedge \xrightarrow{\simeq} \mathrm{L}\eta_{\Phi_p(q)}\left(q\text{-}\Omega_{S/A}\right)_p^\wedge .$$

Via pullback along $\phi^{\alpha-1}\colon A[q] \to A[q]$, we also get Nygaard filtrations $\operatorname{fil}_{\mathcal{N}}^\star\big(q\text{-}\Omega_{S/A}^{(p^\alpha)}\big)_p^\wedge$ for all $\alpha \geqslant 2$. By construction, these Nygaard filtrations are canonically filtered $\mathbb{E}_\infty$-algebras over the filtered ring $\Phi_{p^\alpha}(q)^\star A[q]$, hence over $(q^{p^\alpha}-1)^\star A[q]$ as well. By Proposition 3.19, we also have an equivalence

$$\left(q\text{-}\Omega_{S/A}^{(p^\alpha)}\right)_p^\wedge/(q^{p^\alpha}-1) \simeq \left(q\text{-}\mathbb{W}_{p^\alpha}\Omega_{S/A}\right)_p^\wedge .$$

Our goal in this subsection is to identify the image of the Nygaard filtration under this equivalence with an explicit filtration on the complex $q\text{-}\mathbb{W}_{p^\alpha}\Omega_{S/A}^*$.

3.21. The Nygaard filtration on q-de Rham–Witt complexes. — Let S be smooth over A. The *the Nygaard filtration* is the filtration $\operatorname{fil}_{\mathcal{N}}^\star q\text{-}\mathbb{W}_m\Omega_{S/A}^*$ whose n^{th} term is the subcomplex $\operatorname{fil}_{\mathcal{N}}^n q\text{-}\mathbb{W}_{p^\alpha}\Omega_{S/A}^* \subseteq q\text{-}\mathbb{W}_{p^\alpha}\Omega_{S/A}^*$ given by

$$\left(p^{n-1}V_p\left(q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega_{S/A}^0\right) \to \dots \to p^0 V_p\left(q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega_{S/A}^{n-1}\right) \to q\text{-}\mathbb{W}_{p^\alpha}\Omega_{S/A}^n \to \dotsb\right).$$

3.22. Proposition. — *For smooth A-algebras S, there exists a unique functorial equivalence of filtered $\mathbb{E}_\infty$-$A[q]/(q^{p^\alpha}-1)$-algebras*

$$\operatorname{fil}_{\mathcal{N}}^\star\left(q\text{-}\Omega_{S/A}^{(p^\alpha)}\right)_p^\wedge/(q^{p^\alpha}-1) \xrightarrow{\simeq} \operatorname{fil}_{\mathcal{N}}^\star\left(q\text{-}\mathbb{W}_{p^\alpha}\Omega_{S/A}\right)_p^\wedge$$

(*the quotient on the left-hand side is taken in accordance with Convention* 3.1) *which in degree* 0 *recovers the equivalence* $\big(q\text{-}\Omega_{S/A}^{(p^\alpha)}\big)_p^\wedge/(q^{p^\alpha}-1) \simeq (q\text{-}\mathbb{W}_{p^\alpha}\Omega_{S/A})_p^\wedge$ *from* 3.20.

The proof of Proposition 3.22 requires several preliminary lemmas.

3.23. Lemma. — *Let S be smooth over A. For all $n \geqslant 0$, the Frobenius $\widetilde{F}_p$, when restricted to $\operatorname{fil}_{\mathcal{N}}^n q\text{-}\mathbb{W}_{p^\alpha}\Omega_{S/A}^*$, is divisible by p^n. The divided Frobenius $p^{-n}\widetilde{F}_p$ induces a map*

$$p^{-n}\widetilde{F}_p\colon \operatorname{gr}_{\mathcal{N}}^n q\text{-}\mathbb{W}_{p^\alpha}\Omega_{S/A}^* \longrightarrow \tau^{\leqslant n}\left(q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega_{S/A}^*/p\right)$$

which is surjective in degree n and an isomorphism in all other degrees.

Proof. It follows directly from the construction that $\widetilde{F}_p$ is divisible by p^n on $\operatorname{fil}_{\mathcal{N}}^n q\text{-}\mathbb{W}_{p^\alpha}\Omega_{S/A}^*$. The Verschiebung $V_p\colon q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega_{S/A}^* \to q\text{-}\mathbb{W}_{p^\alpha}\Omega_{S/A}^*$ satisfies $F_p \circ V_p = p$ and $q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega_{S/A}^*$ is degree-wise p-torsion free by [Wag24, Proposition 4.1], hence V_p must be injective. It follows that

$$p^{-n}\widetilde{F}_p\colon \operatorname{gr}_{\mathcal{N}}^n q\text{-}\mathbb{W}_{p^\alpha}\Omega_{S/A}^* \longrightarrow q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega_{S/A}^*/p$$

is an isomorphism in degrees $\leqslant n-1$. Also $\operatorname{gr}_{\mathcal{N}}^n q\text{-}\mathbb{W}_{p^\alpha}\Omega_{S/A}^*$ vanishes in degrees $\geqslant n+1$. In degree n, the map above is given by

$$F_p\colon q\text{-}\mathbb{W}_{p^\alpha}\Omega_{S/A}^n/V_p \longrightarrow q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega_{S/A}^n/p\,.$$

Since $\mathrm{d}\circ F_p = p(F_p\circ \mathrm{d})$, this map lands in $\ker(\mathrm{d}\colon q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega_{S/A}^n/p \to q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega_{S/A}^{n+1}/p)$, and so $\widetilde{F}_p/p^n$ indeed factors through $\tau^{\leqslant n}(q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega_{S/A}^*/p)$.

To finish the proof, we must show that F_p maps surjectively onto this kernel. First suppose that $S = P$ is a polynomial A-algebra. If $\xi \in q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega_{P/A}^n$ satisfies $\mathrm{d}\xi \equiv 0 \mod p$, then [Wag24, 4.3($d_\alpha$)] shows that there exist ω and η satisfying $\xi = F_p(\omega) + p\eta$, proving the desired surjectivity in the polynomial case. If S admits an étale map $\square\colon P \to S$, then surjectivity follows via base change along the étale map $q\text{-}\mathbb{W}_{p^\alpha}(P/A) \to q\text{-}\mathbb{W}_{p^\alpha}(S/A)$. Here we use [Wag24, Propositions 2.48 and 3.31] as well as the observation that

$$\mathrm{d}\colon q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega_{S/A}^n/p \longrightarrow q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega_{S/A}^{n+1}/p$$

is a map of $q\text{-}\mathbb{W}_{p^\alpha}(S/A)$-modules, as $\mathrm{d}\circ F_p \equiv 0 \mod p$. For general S, we find a Zariski cover $S \to S'$ such that S' admits an étale map from a polynomial A-algebra. Then we can again argue via base change along the étale cover $q\text{-}\mathbb{W}_{p^\alpha}(S/A) \to q\text{-}\mathbb{W}_{p^\alpha}(S'/A)$. $\square$

3.24. Lemma. — *Let S be smooth over A. There exists canonical isomorphisms*

$$\begin{aligned}\Omega_{S/A}^n \otimes_{A,\phi^\alpha} A[\zeta_{p^\alpha}] &\cong \ker\Big(F_p\colon q\text{-}\mathbb{W}_{p^\alpha}\Omega_{S/A}^n \to q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega_{S/A}^n\Big)\\ &\cong \ker\Big(F_p\colon q\text{-}\mathbb{W}_{p^\alpha}\Omega_{S/A}^n/V_p \to q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega_{S/A}^n/p\Big)\end{aligned}$$

Proof. We prove the second isomorphism first. For injectivity, suppose $\omega \in q\text{-}\mathbb{W}_{p^\alpha}\Omega_{S/A}^n$ satisfies $F_p(\omega) = 0$, but is also contained in the image of V_p, say, $\omega = V_p(\eta)$. Then $0 = F_p(\omega) = p\eta$ implies $\eta = 0$ by p-torsion freeness, hence $\omega = 0$. For surjectivity, suppose $\omega \in q\text{-}\mathbb{W}_{p^\alpha}\Omega_{S/A}^n$ satisfies $F_p(\omega) = p\eta$ for some η. Then $\omega - V_p(\eta)$ is contained in the kernel of F_p. This proves the second isomorphism.

To show the first isomorphism, consider the ghost map

$$\mathrm{gh}_1\colon q\text{-}\mathbb{W}_{p^\alpha}\Omega_{S/A}^n \longrightarrow \Omega_{S/A}^n \otimes_{A,\phi^\alpha} A[\zeta_{p^\alpha}]\,.$$

We claim that gh_1 maps the kernel of F_p isomorphically onto $(\zeta_p - 1)(\Omega_{S/A}^n \otimes_{A,\phi^\alpha} A[\zeta_{p^\alpha}])$, which would provide the desired isomorphism, as $(\zeta_p - 1)$ is a non-zerodivisor. We only need to show this claim in the case where $S = P$ is a polynomial A-algebra; the general case will follow by the same base change arguments as in the proof of Lemma 3.23 above.

To show injectivity, recall from [Wag24, Lemma 4.5] that gh_1 is surjective with kernel $\operatorname{im} V_p + \operatorname{im} \mathrm{d}V_p$. Thus, suppose $\omega \in q\text{-}\mathbb{W}_{p^\alpha}\Omega_{P/A}^n$ is contained both the kernel of F_p and of gh_1, then we may write $\omega = V_p(\eta_0) + \mathrm{d}V_p(\eta_1)$. Using $F_p \circ \mathrm{d} \circ V_p = \mathrm{d}$, we get $0 = F_p(\omega) = p\eta_0 + \mathrm{d}\eta_1$. In particular, $\mathrm{d}\eta_1 \equiv 0 \mod p$. By [Wag24, 4.3($d_\alpha$)], η_1 can be written as $\eta_1 = F_p(\xi_0) + p\xi_1$, so that $\mathrm{d}\eta_1 = pF_p(\mathrm{d}\xi_0) + p\,\mathrm{d}\xi_1$. Now $p\eta_0 = -\,\mathrm{d}\eta_1$ and p-torsion freeness imply $\eta_0 = -F_p(\mathrm{d}\xi_0) - \mathrm{d}\xi_1$. Thus

$$\omega = V_p(\eta_0) + \mathrm{d}V_p(\eta_1) = -V_pF_p(\mathrm{d}\xi_0) - V_p(\mathrm{d}\xi_1) + \mathrm{d}V_pF_p(\xi_0) + p\,\mathrm{d}V_p(\xi_1)\,.$$

Using $V_p \circ F_p = \Phi_{p^\alpha}$ and $V_p \circ \mathrm{d} = p(\mathrm{d} \circ V_p)$, we conclude $\omega = 0$. This proves injectivity.

Let us now show that the image is precisely $(\zeta_p - 1)(\Omega^n_{P/A} \otimes_{A,\phi^\alpha} A[\zeta_{p^\alpha}])$. By p-torsion freeness, it's enough to check this after p-completion and after inverting p. Once we invert p, the q-de Rham–Witt complexes $q\text{-}\mathbb{W}_{p^\alpha}\Omega^*_{P/A}$ and $q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega^*_{P/A}$ split into products of base-changed de Rham complexes by [Wag24, Corollary 3.34] and the assertion is clear.

So let us see what happens after p-completion. First observe that we can replace $(\ker F_p)^\wedge_p$ by the kernel of $F_p\colon (q\text{-}\mathbb{W}_{p^\alpha}\Omega^*_{P/A})^\wedge_p \to (q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega^*_{P/A})^\wedge_p$. Indeed, to show that the image is contained in $(\zeta_p - 1)(\Omega^n_{P/A} \otimes_{A,\phi^\alpha} A[\zeta_{p^\alpha}])^\wedge_p$, this is certainly sufficient. To see that all of $(\zeta_p - 1)(\Omega^n_{P/A} \otimes_{A,\phi^\alpha} A[\zeta_{p^\alpha}])^\wedge_p$ is hit, we may use base change [Wag24, Lemma 3.16] and reduce to the case where $A = \mathbb{Z}$. In this case we're dealing with finitely generated modules over a noetherian ring [Wag24, Corollary 2.39 and Proposition 3.12(*a*)], so p-completion commutes with kernels.

In any case, we can now use [Wag24, Theorem 4.27] to identify the q-de Rham–Witt Frobenius $F_p\colon (q\text{-}\mathbb{W}_{p^\alpha}\Omega^*_{P/A})^\wedge_p \to (q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega^*_{P/A})^\wedge_p$ with

$$\mathrm{H}^*\big((q\text{-}\mathrm{Hdg}^*_{P/A,\square})^\wedge_p/(q^{p^\alpha} - 1)\big) \longrightarrow \mathrm{H}^*\big((q\text{-}\mathrm{Hdg}^*_{P/A,\square})^\wedge_p/(q^{p^{\alpha-1}} - 1)\big)\,,$$

where the framing $\square$ can be any choice of coordinates of the polynomial ring P. Also note that we can ignore the $(q-1)$-completion in the cited theorem, because everything is p-completed but also $(q^{p^\alpha} - 1)$-torsion. In [Wag24, 4.28–4.30] we construct a direct summand $(q\text{-}\mathrm{Hdg}^{*,0}_{P/A,\square})^\wedge_p \subseteq (q\text{-}\mathrm{Hdg}^*_{P/A,\square})^\wedge_p$ that fits into a commutative diagram

$$\begin{array}{ccccc}
 & & \mathrm{H}^*\big((q\text{-}\mathrm{Hdg}^{*,0}_{P/A,\square})^\wedge_p/(q^{p^\alpha}-1)\big) & \longrightarrow & \mathrm{H}^*\big((q\text{-}\mathrm{Hdg}^{*,0}_{P/A,\square})^\wedge_p/(q^{p^{\alpha-1}}-1)\big) \\
 & \swarrow^{\mathrm{gh}_1} & \downarrow{\cong} & & \downarrow{\cong} \\
\big(\Omega^*_{P/A} \otimes_{A,\phi^\alpha} A[\zeta_{p^\alpha}]\big)^\wedge_p & \twoheadleftarrow & \big(\Omega^*_{P/A} \otimes_{A,\phi^\alpha} A[q]/(q^{p^\alpha}-1)\big)^\wedge_p & \twoheadrightarrow & \big(\Omega^*_{P/A} \otimes_{A,\phi^\alpha} A[q]/(q^{p^{\alpha-1}}-1)\big)^\wedge_p
\end{array}$$

It is also checked there that the complementary direct summand is sent to 0 under gh_1. It follows that the image of $\ker F_p$ under gh_1 is the image of $(q^{p^{\alpha-1}} - 1)(\Omega^*_{P/A} \otimes_{A,\phi^\alpha} A[q]/(q^{p^\alpha} - 1))^\wedge_p$ in $(\Omega^*_{P/A} \otimes_{A,\phi^\alpha} A[\zeta_{p^\alpha}])^\wedge_p$, which is indeed exactly $(\zeta_p - 1)(\Omega^n_{P/A} \otimes_{A,\phi^\alpha} A[\zeta_{p^\alpha}])^\wedge_p$ in degree n. This finishes the proof. ◻

3.25. Corollary. — *Let R be an animated A-algebra and let $q\text{-}\mathbb{W}_{p^\alpha}\mathrm{dR}_{-/A}$ denote the (p-completed) animations of the q-de Rham–Witt complex functors. For all $n \geqslant 0$ and all $\alpha \geqslant 0$, there exists a functorial divided Frobenius*

$$p^{-n}\widetilde{F}_p\colon \mathrm{gr}^n_{\mathcal{N}}\, q\text{-}\mathbb{W}_{p^\alpha}\mathrm{dR}_{R/A} \longrightarrow \mathrm{fil}^{\mathrm{conj}}_n\big(\mathrm{dR}_{R/A}/p\big) \otimes^{\mathrm{L}}_{A,\phi^{\alpha-1}} A[q]/\big(q^{p^{\alpha-1}} - 1\big)\,.$$

with fibre given by $\mathrm{fib}(p^{-n}\widetilde{F}_p) \simeq \Sigma^{-n}\mathrm{dR}^n_{R/A} \otimes^{\mathrm{L}}_{A,\phi^\alpha} A[\zeta_{p^\alpha}]$. Here $\mathrm{fil}^{\mathrm{conj}}_\star(\mathrm{dR}_{R/A}/p)$ denotes the conjugate filtration on the derived de Rham complex, i.e. the animation of $\tau^{\leqslant \star}(\Omega_{-/A}/p)$.

Proof. For S smooth over A, Lemmas 3.23 and 3.24 provide a short exact sequence of complexes

$$0 \longrightarrow \Omega^n_{S/A}[-n] \otimes_{A,\phi^\alpha} A[\zeta_{p^\alpha}] \longrightarrow \mathrm{gr}^n_{\mathcal{N}}\, q\text{-}\mathbb{W}_{p^\alpha}\Omega_{S/A} \xrightarrow{p^{-n}\widetilde{F}_p} \tau^{\leqslant n}\big(q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega_{S/A}/p\big) \longrightarrow 0\,.$$

Using $q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega_{S/A}/p \simeq \Omega_{S/A}/p \otimes^{\mathrm{L}}_{A,\phi^{\alpha-1}} A[q]/(q^{p^{\alpha-1}}-1)$ by [Wag24, Proposition 4.2], this provides the desired cofibre sequence. The case of general R follows by passing to animations. $\square$

3.26. Corollary. — *For all animated A-algebras R and all $\alpha \geqslant 0$, let us denote the animated Nygaard filtration by $\mathrm{fil}^n_{\mathcal{N}}\, q\text{-}\mathbb{W}_{p^\alpha}\mathrm{dR}_{R/A}$. Then:*

(a) $\mathrm{fil}^n_{\mathcal{N}}(q\text{-}\mathbb{W}_{p^\alpha}\mathrm{dR}_{R/A})^\wedge_p$ *satisfies quasi-syntomic descent in R.*

(b) *If R is smooth over A, then $\mathrm{fil}^n_{\mathcal{N}}(q\text{-}\mathbb{W}_{p^\alpha}\mathrm{dR}_{R/A})^\wedge_p$ agrees with its un-animated variant $\mathrm{fil}^n_{\mathcal{N}}(q\text{-}\mathbb{W}_{p^\alpha}\Omega_{R/A})^\wedge_p$.*

Proof. It's clear that $(q\text{-}\mathbb{W}_{p^\alpha}\mathrm{dR}_{R/A})^\wedge_p \simeq (\mathrm{dR}_{R/A} \otimes^{\mathrm{L}}_{A,\phi^\alpha} A[q]/(q^{p^\alpha}-1))^\wedge_p$ satisfies quasi-syntomic descent and agrees with its un-animated variant when R is smooth over A. To prove (*a*) and (*b*), it will thus be enough to show that $\mathrm{gr}^n_{\mathcal{N}}(q\text{-}\mathbb{W}_{p^\alpha}\mathrm{dR}_{R/A})^\wedge_p$ satisfies quasi-syntomic descent for all $n \geqslant 0$ and agrees with $\mathrm{gr}^n_{\mathcal{N}}(q\text{-}\mathbb{W}_{p^\alpha}\Omega_{R/A})^\wedge_p$ when R is smooth. Both assertions follow from Corollary 3.25. $\square$

Next we construct an analog of the fibre sequence from Corollary 3.25 for the other Nygaard filtration $\mathrm{fil}^\star_{\mathcal{N}}\big(q\text{-}\Omega^{(p^\alpha)}_{S/A}\big)^\wedge_p/(q^{p^\alpha}-1)$. After that we'll prove Proposition 3.22 by carefully comparing these fibre sequences.

3.27. Lemma. — *Let R be an animated A-algebra. For brevity, let us write*

$$\mathrm{fil}^\star_{\mathcal{N},q\text{-}\Omega} \coloneqq \mathrm{fil}^\star_{\mathcal{N}}\big(q\text{-}\mathrm{dR}^{(p^\alpha)}_{R/A}\big)^\wedge_p/(q^{p^\alpha}-1)$$

and let $\mathrm{gr}^*_{\mathcal{N},q\text{-}\Omega}$ denote the associated graded of this filtered object. Let also $\phi_{/A}$ denote the relative Frobenius on $(\mathrm{dR}_{R/A})^\wedge_p$. Then for all $n \geqslant 0$ there are canonical maps

$$p^{-n}\phi_{/A}\colon \mathrm{gr}^n_{\mathcal{N},q\text{-}\Omega} \longrightarrow \mathrm{fil}^{\mathrm{conj}}_n(\mathrm{dR}_{R/A}/p) \otimes^{\mathrm{L}}_{A,\phi^{\alpha-1}} A[q]/\big(q^{p^{\alpha-1}}-1\big)$$

with fibre $\mathrm{fib}(p^{-n}\phi_{/A}) \simeq \Sigma^{-n}(\mathrm{dR}^n_{R/A} \otimes^{\mathrm{L}}_{A,\phi^\alpha} A[\zeta_{p^\alpha}])^\wedge_p$.

Proof. By definition of the Nygaard filtration, the Frobenius on q-de Rham cohomology is divisible by $\Phi_{p^\alpha}(q)^n$ on $\mathrm{fil}^n_{\mathcal{N}}\big(q\text{-}\mathrm{dR}^{(p^\alpha)}_{R/A}\big)^\wedge_p$. Therefore, for all $n \geqslant 0$ there's a commutative diagram

$$\begin{array}{ccc}
\mathrm{gr}^{n-1}_{\mathcal{N}}\big(q\text{-}\mathrm{dR}^{(p^\alpha)}_{R/A}\big)^\wedge_p & \xrightarrow{(q^{p^\alpha}-1)} & \mathrm{gr}^n_{\mathcal{N}}\big(q\text{-}\mathrm{dR}^{(p^\alpha)}_{R/A}\big)^\wedge_p \\
{\scriptstyle \Phi_{p^\alpha}(q)^{-(n-1)}\phi_{/A[q]}}\downarrow{\scriptstyle\simeq} & & {\scriptstyle\simeq}\downarrow{\scriptstyle \Phi_{p^\alpha}(q)^{-n}\phi_{/A[q]}} \\
\mathrm{fil}^{\mathrm{conj}}_{n-1}\big(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}/\Phi_{p^\alpha}(q)\big) & \xrightarrow{(q^{p^{\alpha-1}}-1)} & \mathrm{fil}^{\mathrm{conj}}_n\big(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}/\Phi_{p^\alpha}(q)\big)
\end{array}$$

The vertical arrows are equivalences by [BS19, Theorem 15.2(2)] (plus quasi-syntomic descent and passing to animations to allow for arbitrary animated A-algebras R).

Now $\mathrm{gr}^n_{\mathcal{N},q\text{-}\Omega}$ is the cofibre of the top horizontal arrow and thus also the cofibre of the bottom horizontal arrow; we wish to compute the latter. To this end, note that

$$\mathrm{fil}^{\mathrm{conj}}_n\big(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}/\Phi_{p^\alpha}(q)\big)/\big(q^{p^{\alpha-1}}-1\big) \simeq \mathrm{fil}^{\mathrm{conj}}_n\big(\mathrm{dR}_{R/A}/p\big) \otimes^{\mathrm{L}}_{A,\phi^{\alpha-1}} A[q]/\big(q^{p^{\alpha-1}}-1\big)\,.$$

Indeed, without the Frobenius-twists, $\mathrm{fil}^{\mathrm{conj}}_n(q\text{-}\mathrm{dR}_{R/A}/\Phi_p(q)) \otimes^{\mathrm{L}}_{A[\![q-1]\!]} A \simeq \mathrm{fil}^{\mathrm{conj}}_n(\mathrm{dR}_{R/A}/p)$ follows from the base change result in [BS19, Theorem 15.2(3)] plus quasi-syntomic descent, using that $-\otimes^{\mathrm{L}}_{A[\![q-1]\!]} A$ commutes with all limits. To incorporate the Frobenius twists, just take the base change along $\phi^{\alpha-1}$.

As a consequence, we obtain the desired canonical map

$$p^{-n}\phi_{/A}\colon \mathrm{gr}^n_{\mathcal{N},q\text{-}\Omega} \longrightarrow \mathrm{fil}^{\mathrm{conj}}_n\big(\mathrm{dR}_{R/A}/p\big)\otimes^{\mathrm{L}}_{A,\phi^{\alpha-1}} A[q]/\big(q^{p^{\alpha-1}}-1\big)\,.$$

By the diagram above and the Hodge–Tate comparison for prismatic cohomology (see [BS19, Construction 7.6]) the fibre is indeed $\mathrm{gr}^{\mathrm{conj}}_n(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}/\Phi_{p^\alpha}(q)) \simeq \Sigma^{-n}(\mathrm{dR}^n_{R/A}\otimes^{\mathrm{L}}_{A,\phi^\alpha} A[\zeta_{p^\alpha}])^\wedge_p$, as desired. $\square$

3.28. Remark. — By contemplating the bottom row of the diagram in the proof above, we find that $\mathrm{fib}(p^{-n}\phi_{/A}) \to \mathrm{gr}^n_{\mathcal{N},q\text{-}\Omega}$ sits inside the following diagram for all $n \geqslant 0$:

$$\begin{array}{ccccc}
 & & \mathrm{fil}^n_{\mathcal{N}}\big(q\text{-}\mathrm{dR}^{(p^\alpha)}_{R/A}\big)^\wedge_p & \xrightarrow{\;\Phi_{p^\alpha}(q)^{-n}\phi_{/A[q]}\;} & \\
 & & \downarrow & & \searrow \\
\mathrm{gr}^{\mathrm{conj}}_n\big(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}/\Phi_{p^\alpha}(q)\big) & \longrightarrow & \mathrm{gr}^n_{\mathcal{N},q\text{-}\Omega} & & \mathrm{fil}^{\mathrm{conj}}_n\big(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}/\Phi_{p^\alpha}(q)\big) \\
 & \searrow^{(q^{p^{\alpha-1}}-1)} & \downarrow & \swarrow & \\
 & & \mathrm{gr}^{\mathrm{conj}}_n\big(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}/\Phi_{p^\alpha}(q)\big) & &
\end{array}$$

Proof of Proposition 3.22. Thanks to Corollary 3.26, we can tackle the question using quasi-syntomic descent. Let R be a p-complete quasi-syntomic A-algebra which is *large* in the sense of [BS19, Definition 15.1], i.e. there exists a surjection $\widehat{A}_p\langle x_i^{1/p^\infty} \mid i\in I\rangle \twoheadrightarrow R$ for some set I. Let $\mathrm{fil}^\star_{\mathcal{N},q\text{-}\Omega}$ and $\mathrm{fil}^\star_{\mathcal{N},q\text{-}\mathbb{W}}$ denote the two filtrations on $(\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A,\phi^\alpha} A[q]/(q^{p^\alpha}-1))^\wedge_p$ given by

$$\mathrm{fil}^\star_{\mathcal{N},q\text{-}\Omega} \coloneqq \mathrm{fil}^\star_{\mathcal{N}}\big(q\text{-}\mathrm{dR}^{(p^\alpha)}_{R/A}\big)^\wedge_p/(q^{p^\alpha}-1) \quad\text{and}\quad \mathrm{fil}^\star_{\mathcal{N},q\text{-}\mathbb{W}} \coloneqq \mathrm{fil}^\star_{\mathcal{N}}\big(q\text{-}\mathbb{W}_{p^\alpha}\mathrm{dR}_{R/A}\big)^\wedge_p\,.$$

Our assumptions on R ensure that $(\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A,\phi^\alpha} A[q]/(q^{p^\alpha}-1))^\wedge_p$ is static and that $\mathrm{fil}^\star_{\mathcal{N},q\text{-}\Omega}$ is a descending filtrations by ordinary ideals. So once we've shown $\mathrm{fil}^\star_{\mathcal{N},q\text{-}\Omega} = \mathrm{fil}^\star_{\mathcal{N},q\text{-}\mathbb{W}}$ as ideals, the comparison will automatically be functorial in R (of the given form) and an equivalence of filtered $\mathbb{E}_\infty$-$A[q]$-algebras. Moreover, uniqueness will also be clear. Via quasi-syntomic descent we can then recover the smooth case.

To prove the proposition for R, we show using induction on n that $\mathrm{fil}^n_{\mathcal{N},q\text{-}\Omega} = \mathrm{fil}^n_{\mathcal{N},q\text{-}\mathbb{W}}$ as ideals in the ring $(\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A,\phi^\alpha} A[q]/(q^{p^\alpha}-1)))^\wedge_p$. The case $n=0$ is clear. So assume we know $\mathrm{fil}^n_{\mathcal{N},q\text{-}\Omega} = \mathrm{fil}^n_{\mathcal{N},q\text{-}\mathbb{W}} \eqqcolon \mathrm{fil}^n_{\mathcal{N}}$ for some $n \geqslant 0$. Let

$$K \coloneqq \mathrm{fib}\Big(p^{-n}\phi_{/A}\colon \mathrm{fil}^n_{\mathcal{N}} \to \mathrm{fil}^{\mathrm{conj}}_n\big(\mathrm{dR}_{R/A}/p\big)\otimes^{\mathrm{L}}_{A,\phi^{\alpha-1}} A[q]/\big(q^{p^{\alpha-1}}-1\big)\Big)\,.$$

Via $\mathrm{fil}^n_{\mathcal{N}} = \mathrm{fil}^n_{\mathcal{N},q\text{-}\Omega}$ we know that $p^{-n}\phi_{/A}$ is surjective and so K is static. According to Corollary 3.25 we have an equivalence

$$\mathrm{cofib}\big(\mathrm{fil}^{n+1}_{\mathcal{N},q\text{-}\mathbb{W}} \to K\big) \simeq \Sigma^{-n}\big(\mathrm{dR}^n_{R/A}\otimes^{\mathrm{L}}_{A,\phi^\alpha} A[\zeta_{p^\alpha}]\big)^\wedge_p\,.$$

Moreover, this equivalence can be explicitly described as follows: Consider the ghost map gh_1 for $q\text{-}\mathbb{W}_{p^\alpha}\mathrm{dR}_{R/A}$, which by [Wag24, Proposition 4.2] just corresponds to the canonical projection

$$\big(\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A,\phi^\alpha} A[q]/(q^{p^\alpha}-1)\big)^\wedge_p \longrightarrow \big(\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A,\phi^\alpha} A[\zeta_{p^\alpha}]\big)^\wedge_p$$

sending $q \mapsto \zeta_{p^\alpha}$. When restricted to $\mathrm{fil}_{\mathcal{N}}^n = \mathrm{fil}_{\mathcal{N},q\text{-}\mathbb{W}}^n$, this lands in $(\mathrm{fil}_{\mathrm{Hdg}}^n \mathrm{dR}_{R/A} \otimes_{A,\phi^\alpha}^{\mathrm{L}} A[\zeta_{p^\alpha}])_p^\wedge$. Indeed, for smooth A-algebras this follows directly from 3.21, as the image of V_p dies under gh_1; the general case follows via animation. By tracing through the proof of Lemma 3.24, we now see that the diagram

$$
\begin{array}{ccccc}
\operatorname{cofib}(\mathrm{fil}_{\mathcal{N},q\text{-}\mathbb{W}}^{n+1} \to K) & \longleftarrow & K & \longrightarrow & \mathrm{fil}_{\mathcal{N}}^n \\
\downarrow{\scriptstyle\simeq} & & & & \downarrow \\
\mathrm{gr}_{\mathrm{Hdg}}^n\big(\mathrm{dR}_{R/A} \otimes_{A,\phi^\alpha}^{\mathrm{L}} A[\zeta_{p^\alpha}]\big)_p^\wedge & & & & \mathrm{fil}_{\mathrm{Hdg}}^n\big(\mathrm{dR}_{R/A} \otimes_{A,\phi^\alpha}^{\mathrm{L}} A[\zeta_{p^\alpha}]\big)_p^\wedge \\
& \searrow^{(\zeta_p - 1)} & & & \downarrow \\
& & & & \mathrm{gr}_{\mathrm{Hdg}}^n\big(\mathrm{dR}_{R/A} \otimes_{A,\phi^\alpha}^{\mathrm{L}} A[\zeta_{p^\alpha}]\big)_p^\wedge
\end{array}
$$

commutes. Thus K is mapped into the submodule $(\zeta_p - 1)(\mathrm{gr}_{\mathrm{Hdg}}^n \mathrm{dR}_{R/A} \otimes_{A,\phi^\alpha}^{\mathrm{L}} A[\zeta_{p^\alpha}])_p^\wedge$ and $\mathrm{fil}_{\mathcal{N},q\text{-}\mathbb{W}}^{n+1}$ is the fibre of this map.

According to Lemma 3.27 and the left half of the diagram from Remark 3.28, for $\mathrm{fil}_{\mathcal{N},q\text{-}\Omega}^{n+1}$ we have a similar diagram:

$$
\begin{array}{ccccc}
\operatorname{cofib}(\mathrm{fil}_{\mathcal{N},q\text{-}\Omega}^{n+1} \to K) & \longleftarrow & K & \longrightarrow & \mathrm{fil}_{\mathcal{N}}^n \\
\downarrow{\scriptstyle\simeq} & & & & \downarrow \\
\mathrm{gr}_n^{\mathrm{conj}}\big(q\text{-}\mathrm{dR}_{R/A}^{(p^{\alpha-1})}/\Phi_{p^\alpha}(q)\big) & & & & \mathrm{gr}_{\mathcal{N},q\text{-}\Omega}^n \\
& \searrow^{(q^{p^{\alpha-1}} - 1)} & & & \downarrow \\
& & & & \mathrm{gr}_n^{\mathrm{conj}}\big(q\text{-}\mathrm{dR}_{R/A}^{(p^{\alpha-1})}/\Phi_{p^\alpha}(q)\big)
\end{array}
$$

Note that $(q^{p^{\alpha-1}} - 1)$ is sent to $(\zeta_p - 1)$ under $q \mapsto \zeta_{p^\alpha}$. Therefore, to show $\mathrm{fil}_{\mathcal{N},q\text{-}\Omega}^{n+1} = \mathrm{fil}_{\mathcal{N},q\text{-}\mathbb{W}}^{n+1}$ and thus to finish the induction, it will be enough to show that the following diagram commutes; here we also use the right half of the diagram from Remark 3.28:

$$
\begin{array}{ccc}
\mathrm{fil}_{\mathcal{N}}^n\big(q\text{-}\mathrm{dR}_{R/A}^{(p^\alpha)}\big)_p^\wedge & \longrightarrow & \mathrm{fil}_{\mathcal{N}}^n \\
\downarrow{\scriptstyle \Phi_{p^\alpha}(q)^{-n}\phi_{/A[q]}} & & \downarrow \\
\mathrm{fil}_n^{\mathrm{conj}}\big(q\text{-}\mathrm{dR}_{R/A}^{(p^{\alpha-1})}/\Phi_{p^\alpha}(q)\big) & & \mathrm{fil}_{\mathrm{Hdg}}^n\big(\mathrm{dR}_{R/A} \otimes_{A,\phi^\alpha}^{\mathrm{L}} A[\zeta_{p^\alpha}]\big)_p^\wedge \\
\downarrow & & \downarrow \\
\mathrm{gr}_n^{\mathrm{conj}}\big(q\text{-}\mathrm{dR}_{R/A}^{(p^{\alpha-1})}/\Phi_{p^\alpha}(q)\big) & \xrightarrow[\text{Hodge–Tate comparison}]{\simeq} & \mathrm{gr}_{\mathrm{Hdg}}^n\big(\mathrm{dR}_{R/A} \otimes_{A,\phi^\alpha}^{\mathrm{L}} A[\zeta_{p^\alpha}]\big)_p^\wedge
\end{array}
$$

To show commutativity, let us first get rid of $(\alpha - 1)$ Frobenius-twists (thus reducing to $\alpha = 1$), as these Frobenius-twists just amount to a pullback. Moreover, commutativity can be checked after the faithfully flat base change along the map $A \to A_\infty$ into the colimit perfection of the perfectly covered Λ-ring A. Since everything is p-complete, working relative to A_∞ is the same as working absolutely, so we can reduce to the case $A = \mathbb{Z}$. We can then use the method from [BS19, §12]. Let us first check commutativity in the single case $R = \mathbb{Z}_p\langle x^{1/p^\infty}\rangle/x$.

In this case, everything is explicit: First off, $(q\text{-dR}_{R/\mathbb{Z}})_p^\wedge$ is the ring

$$\mathbb{Z}_p[\![q-1]\!]\langle x^{1/p^\infty}\rangle\left\{\frac{x^p}{\Phi_p(q)}\right\}_{(p,q-1)}^\wedge \simeq \left(\bigoplus_{i\in\mathbb{N}[1/p]}\mathbb{Z}_p[\![q-1]\!]\cdot\frac{x^i}{[\lfloor i\rfloor]_q!}\right)_{(p,q-1)}^\wedge .$$

The graded piece $\operatorname{gr}^n_{\mathrm{Hdg}}(\mathrm{dR}_{R/\mathbb{Z}}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Z}[\zeta_p])_p^\wedge$ is generated by the divided power $x^n/n!$, which is the image of the q-divided power $x^n/[n]_q!\in q\text{-dR}_{R/\mathbb{Z}}$. We have

$$\Phi_p(q)^{-n}\phi\left(\frac{x^n}{[n]_q!}\right)=\frac{x^{pn}}{[n]_{q^p}!\cdot\Phi_p(q)^n}\equiv\frac{\left(x^p/\Phi_p(q)\right)^n}{n!}\mod\Phi_p(q)$$

By [BS19, Lemma 12.6], $[n]_{q^p}!\cdot\Phi_p(q)^n$ is a unit multiple of $[pn]_q!$. This shows that $\phi(x^n/[n]_q!)$ is divisible by $\Phi_p(q)^n$ and so the image of $x^n/[n]_q!$ under $(q\text{-dR}_{R/\mathbb{Z}})_p^\wedge\to(q\text{-dR}^{(p)}_{R/\mathbb{Z}})_p^\wedge$ lies in Nygaard filtration degree n. The proof of [BS19, Lemma 12.7] also explains that the graded algebra $\operatorname{gr}^{\mathrm{conj}}_*(q\text{-dR}_{R/\mathbb{Z}}/\Phi_p(q))$ is generated by divided powers of $x^p/\Phi_p(q)$ and that these generators induce the Hodge-Tate comparison. As we've seen above, said divided powers are precisely the images of $x^n/[n]_q!$, so we obtain commutativity in our special case.

The method from [BS19, §12] then shows commutativity in general: First consider the case $R=\mathbb{Z}_p\langle x_1^{1/p^\infty},\dots,x_n^{1/p^\infty}\rangle/(x_1,\dots,x_n)$. This follows from the special case above by multiplicativity. Next consider the case $R=R'/(f_1,\dots,f_r)$, where R' is a perfectoid ring and $(f_1,\dots,f_r)$ is a p-completely regular sequence. If each f_i admits compatible p-power roots, we can reduce to the previous special case via base change. In general, by Andre's lemma [BS19, Theorem 7.14], we find a p-completely faithfully flat cover $R'\to R''$ such that R'' is perfectoid again and each f_i admits compatible p-power roots in R'', so we can conclude via descent.

Now assume R is p-completely smooth over $\mathbb{Z}_p$. In this case we can choose a surjection $\mathbb{Z}_p\langle x_1,\dots,x_n\rangle\twoheadrightarrow R$ and put

$$R_\infty\coloneqq\left(\mathbb{Z}_p\langle x_1^{1/p^\infty},\dots,x_n^{1/p^\infty}\rangle\otimes_{\mathbb{Z}_p\langle x_1,\dots,x_n\rangle}R\right)_p^\wedge .$$

Using descent for $R\to R_\infty$, we only need to check the assertion for each term in the Čech nerve $(R_\infty^{\otimes_R\bullet})_p^\wedge$. These terms are Zariski-locally of the form considered in the previous paragraph and so the smooth case follows. Finally, the case of arbitrary R follows by passing to animations. ☐

The same slightly convoluted method of proof can be used to show the following technical lemma, which we'll need below.

3.29. Lemma. — *The equivalence $(q\text{-dR}_{R/A})_p^\wedge[1/p]_{(q-1)}^\wedge\simeq(\mathrm{dR}_{R/A})_p^\wedge[1/p][\![q-1]\!]$ upgrades uniquely to an equivalence of filtered $\mathbb{E}_\infty$-$A[1/p,q]$-algebras*

$$\operatorname{fil}^\star_{\mathcal{N}}\left(q\text{-dR}^{(p)}_{R/A}\right)_p^\wedge\left[\tfrac{1}{p}\right]_{\Phi_p(q)}^\wedge\xrightarrow{\simeq}\operatorname{fil}^\star_{(\mathrm{Hdg},\Phi_p(q))}\left(\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A,\phi}A\right)_p^\wedge\left[\tfrac{1}{p},q\right]_{\Phi_p(q)}^\wedge,$$

where $\operatorname{fil}^\star_{(\mathrm{Hdg},\Phi_p(q))}$ denotes the combined Hodge and $\Phi_p(q)$-adic filtration.

Proof. Let us first construct the map. It's enough to do this in the case where R a p-complete quasi-syntomic A-algebra which is *large* in the sense that there exists a surjection $\widehat{A}_p\langle x_i^{1/p^\infty}\mid i\in I\rangle\twoheadrightarrow R$ for some set I. Via quasi-syntomic descent, we can then recover the case where R is smooth over A, and the general case follows via animation.

If R is as above, then $(q\text{-dR}_{R/A})_p^\wedge[1/p]_{(q-1)}^\wedge \simeq (\mathrm{dR}_{R/A})_p^\wedge[1/p]\llbracket q-1\rrbracket$ are static and so are the filtrations on them. So we have to compare two descending filtrations of a ring by ideals. It follows at once that the comparison, if it exists, must be unique, and it will automatically be compatible with the filtered $\mathbb{E}_\infty$-$A[1/p,q]$-algebra structures. Moreover, to compare the two filtrations by ideals, we may base change along the faithfully flat map $A \to A_\infty$.[(3.6)] Since working relative to the perfect δ-ring A_∞ is equivalent to working absolutely, we may thus assume $A = \mathbb{Z}$. Then we use the method from [BS19, §12] as in the previous proof of Proposition 3.22.

So we only have to check the single case $R = \mathbb{Z}_p\langle x^{1/p^\infty}\rangle/x$. In this case, $(q\text{-dR}_{R/\mathbb{Z}}^{(p)})_p^\wedge$ is given by a completed direct sum

$$(q\text{-dR}_{R/\mathbb{Z}}^{(p)})_p^\wedge \simeq \left(\bigoplus_{i\in\mathbb{N}[1/p]} \mathbb{Z}_p\llbracket q-1\rrbracket \cdot \frac{x^i}{[\lfloor i\rfloor]_{q^p}!}\right)_{(p,\Phi_p(q))}^\wedge .$$

By definition, $\mathrm{fil}_{\mathcal{N}}^n(q\text{-dR}_{R/\mathbb{Z}}^{(p)})_p^\wedge$ consists of those elements whose Frobenius becomes divisible by $\Phi_p(q)^n$. By inspection, these are precisely

$$\mathrm{fil}_{\mathcal{N}}^n\big(q\text{-dR}_{R/\mathbb{Z}}^{(p)}\big)_p^\wedge \simeq \left(\bigoplus_{i\in\mathbb{N}[1/p]} \Phi_p(q)^{\max\{n-\lfloor i\rfloor,0\}}\mathbb{Z}_p\llbracket q-1\rrbracket \cdot \frac{x^i}{[\lfloor i\rfloor]_{q^p}!}\right)_{(p,\Phi_p(q))}^\wedge .$$

After $(-)[1/p]_{\Phi_p(q)}^\wedge$, this becomes the ideal $(x,\Phi_p(q))^n$, which is the n^{th} step in the combined Hodge and $\Phi_p(q)$-adic filtration on $(\mathrm{dR}_{R/\mathbb{Z}})_p^\wedge[1/p,q]_{\Phi_p(q)}^\wedge$. This finishes the discussion of the special case and thus the construction of the comparison between the two filtrations.

To show that we get an equivalence, let A be arbitrary again and let R be any animated A-algebra. We'll show that both sides agree if we reduce them modulo $\Phi_p(q)$, where $\Phi_p(q)$ sits in filtration degree 1. Since both sides also agree in filtration degree 0, it will follow inductively that they agree everywhere. By construction,

$$\mathrm{fil}^\star_{(\mathrm{Hdg},\Phi_p(q))}\big(\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A,\phi}A\big)_p^\wedge\big[\tfrac{1}{p},q\big]_{\Phi_p(q)}^\wedge/\Phi_p(q) \simeq \mathrm{fil}^\star_{\mathrm{Hdg}}\big(\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A,\phi}A[\zeta_p]\big)_p^\wedge\big[\tfrac{1}{p}\big]$$

is just a base change of the Hodge filtration. So let's see what happens on the left-hand side. Since $(q-1)$ becomes invertible after $(-)[1/p]_{\Phi_p(q)}^\wedge$, we may as well reduce modulo (q^p-1), again sitting in filtration degree 1. Then Proposition 3.22 shows

$$\mathrm{fil}^\star_{\mathcal{N}}\big(q\text{-dR}_{R/A}^{(p)}\big)_p^\wedge\big[\tfrac{1}{p}\big]_{\Phi_p(q)}^\wedge/(q^p-1) \simeq \mathrm{fil}^\star_{\mathcal{N}}\big(q\text{-}\mathbb{W}_p\mathrm{dR}_{R/A}\big)_p^\wedge\big[\tfrac{1}{p}\big]_{\Phi_p(q)}^\wedge .$$

We claim that the right-hand side is equivalent to $(\mathrm{fil}^\star_{\mathrm{Hdg}}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A,\phi}A[\zeta_p])_p^\wedge[1/p]$ via the ghost map gh_1. This may be checked in the case where R is smooth over A, as then the general case follows via animation. As we've seen above, $(-)[1/p]_{\Phi_p(q)}^\wedge$ forces $(q-1)$ to be invertible, and so all the images of V_p in 3.21 die because they're all $(q-1)$-torsion. It follows that for R smooth over A, the ghost map

$$\mathrm{gh}_1\colon \mathrm{fil}^\star_{\mathcal{N}}\big(q\text{-}\mathbb{W}_p\Omega^*_{R/A}\big)_p^\wedge\big[\tfrac{1}{p}\big]_{\Phi_p(q)}^\wedge \xrightarrow{\simeq} \mathrm{fil}^\star_{\mathrm{Hdg}}\big(\Omega^*_{R/A}\otimes_{A,\phi}A[\zeta_p]\big)_p^\wedge\big[\tfrac{1}{p}\big]$$

is already an isomorphism on the level of complexes and so we're done. □

(3.6) Recall from Remark A.7 that for every fixed $n \geqslant 0$ there exists an N such that the canonical map $(q\text{-dR}_{R/A})_p^\wedge \to (\mathrm{dR}_{R/A})_p^\wedge[1/p]\llbracket q-1\rrbracket/(q-1)^n$ already factors through $p^{-N}(\mathrm{dR}_{R/A})_p^\wedge\llbracket q-1\rrbracket/(q-1)^n$. The existence of a map

$$\mathrm{fil}^\star_{\mathcal{N}}\big(q\text{-dR}_{R/A}^{(p)}\big)_p^\wedge \longrightarrow \mathrm{fil}^\star_{(\mathrm{Hdg},\Phi_p(q))}\big(\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A,\phi}A\big)_p^\wedge\big[\tfrac{1}{p},q\big]_{\Phi_p(q)}^\wedge$$

boils down to an inclusion of ideals. Using the observation above, this inclusion can be checked modulo powers of p and $\Phi_p(q)$, and so we can use base change along $A \to A_\infty$ without having to worry about completion issues.

§3.5. The twisted q-Hodge filtration

For smooth A-algebras S, let $\operatorname{fil}^\star_{\mathcal{H}\mathrm{dg}_m} q\text{-}\mathbb{W}_m\Omega^*_{S/A}$ denote the stupid filtration given by $q\text{-}\mathbb{W}_m\Omega^{\geqslant n,*}_{S/A}$ in degree n. In general, let

$$\operatorname{fil}^\star_{\mathcal{H}\mathrm{dg}_m} q\text{-}\mathbb{W}_m\mathrm{dR}_{-/A}\colon \mathrm{AniAlg}_A \longrightarrow \mathrm{CAlg}\Big(\mathrm{Fil}\,\mathcal{D}\big(A[q]/(q^m-1)\big)\Big)$$

be the animation of this functor. In this subsection, we'll show that once $q\text{-}\mathrm{dR}_{R/A}$ is equipped with a q-Hodge filtration, the filtration $\operatorname{fil}^\star_{\mathcal{H}\mathrm{dg}_m} q\text{-}\mathbb{W}_m\mathrm{dR}_{R/A}$ admits a canonical q^m-deformation

$$\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_m} q\text{-}\mathrm{dR}^{(m)}_{R/A}\,.$$

This will eventually allow us to prove Theorem 3.11 in §3.6 below. Let us first explain how $\operatorname{fil}^\star_{\mathcal{H}\mathrm{dg}_m} q\text{-}\mathbb{W}_m\mathrm{dR}_{R/A}$ is related to the Nygaard filtration from §3.4.

3.30. Lemma. — *For all smooth A-algebras S, all primes p and all $\alpha \geqslant 1$ the diagram*

$$\begin{array}{ccc} \operatorname{fil}^n_{\mathcal{H}\mathrm{dg}_{p^\alpha}} q\text{-}\mathbb{W}_{p^\alpha}\Omega^*_{S/A} & \longrightarrow & \operatorname{fil}^n_{\mathcal{N}} q\text{-}\mathbb{W}_{p^\alpha}\Omega^*_{S/A} \\ \big\downarrow & \lrcorner & \big\downarrow {\scriptstyle p^{-n}\widetilde{F}_p} \\ \operatorname{fil}^n_{\mathcal{H}\mathrm{dg}_{p^{\alpha-1}}} q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega^*_{S/A} & \longrightarrow & q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega^*_{S/A} \end{array}$$

becomes a pullback in $\mathcal{D}(A[q])$ for all $n \geqslant 0$.

Proof. It's enough to check that the induced map on horizontal cofibres is an equivalence. Since $\operatorname{fil}^n_{\mathcal{H}\mathrm{dg}_{p^\alpha}} q\text{-}\mathbb{W}_{p^\alpha}\Omega^*_{S/A} \to \operatorname{fil}^n_{\mathcal{N}} q\text{-}\mathbb{W}_{p^\alpha}\Omega^*_{S/A}$ is injective, the cofibre agrees with the cokernel, which is given by

$$\Big(p^{n-1}V_p\big(q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega^0_{S/A}\big) \to \cdots \to p^0V_p\big(q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega^{n-1}_{S/A}\big) \to 0 \to 0 \to \cdots\Big)\,.$$

Under $p^{-n}\widetilde{F}_p$, this complex is mapped isomorphically onto

$$\Big(q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega^0_{S/A} \to \cdots \to q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega^{n-1}_{S/A} \to 0 \to 0 \to \cdots\Big)\,,$$

which is the cokernel (and the cofibre) of $\operatorname{fil}^n_{\mathcal{H}\mathrm{dg}_{p^{\alpha-1}}} q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega^*_{S/A} \to q\text{-}\mathbb{W}_{p^{\alpha-1}}\Omega^*_{S/A}$ □

3.31. Corollary. — *If S is smooth over A, then $q\text{-}\mathbb{W}_m\mathrm{dR}^n_{S/A} \simeq \Sigma^{-n}\, q\text{-}\mathbb{W}_m\Omega^n_{S/A}$ for all $m \in \mathbb{N}$ and all degrees $n \geqslant 0$.*

Proof. It's enough to show this rationally and after p-completion for all primes p. Rationally, [Wag24, Corollary 3.34] shows

$$q\text{-}\mathbb{W}_m\Omega^n_{S/A} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \prod_{d\mid m}\Big(\Omega^n_{S/A} \otimes_{A,\psi^d} (A\otimes\mathbb{Q})[\zeta_d]\Big)$$

and it's well-known that the values of $\Omega^n_{-/A}$ on smooth A-algebras don't change under animation. After p-completion, [Wag24, Lemma 4.36] allows us to restrict to the case where $m = p^\alpha$ is a prime power. Since $q\text{-}\mathbb{W}_{p^\alpha}\mathrm{dR}^n_{S/A} \simeq \operatorname{gr}^n_{\mathcal{H}\mathrm{dg}_{p^\alpha}} q\text{-}\mathbb{W}_{p^\alpha}\mathrm{dR}_{S/A}$, it will be enough to show that the filtration $\operatorname{fil}^\star_{\mathcal{H}\mathrm{dg}_{p^\alpha}} q\text{-}\mathbb{W}_{p^\alpha}\Omega^*_{S/A}$ is unchanged under animation. This follows via induction on α from Lemma 3.30 and Corollary 3.26(*b*). □

We now set out to construct the desired q^m-deformation of $\operatorname{fil}^\star_{\mathcal{H}\mathrm{dg}_m} q\text{-}\mathbb{W}_m\mathrm{dR}_{R/A}$.

3.32. The twisted q-Hodge filtration (p-adically). — Let's first construct the filtration for prime powers $m = p^\alpha$ and after p-completion. We'll use a recursive definition. For $\alpha = 0$, $q\text{-}\mathrm{dR}^{(p^0)}_{R/A} \simeq q\text{-}\mathrm{dR}_{R/A}$ is just the q-de Rham complex and we choose

$$\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_{p^0}}\bigl(q\text{-}\mathrm{dR}^{(p^0)}_{R/A}\bigr)^\wedge_p \coloneqq \operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}}\bigl(q\text{-}\mathrm{dR}_{R/A}\bigr)^\wedge_p$$

to be the given q-Hodge filtration. For $\alpha \geqslant 1$, we consider the "rescaling" of the filtration $\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_{p^{\alpha-1}}}\bigl(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}\bigr)^\wedge_p$ by Φ_{p^α}, that is,

$$\Phi_{p^\alpha}(q)^\star \operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_{p^{\alpha-1}}} \coloneqq \left(\operatorname{fil}^0_{q\text{-}\mathcal{H}\mathrm{dg}_{p^{\alpha-1}}} \xleftarrow{\Phi_{p^\alpha}(q)} \operatorname{fil}^1_{q\text{-}\mathcal{H}\mathrm{dg}_{p^{\alpha-1}}} \xleftarrow{\Phi_{p^\alpha}(q)} \cdots\right).$$

We also equip $\bigl(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}\bigr)^\wedge_p$ with its $\Phi_{p^\alpha}(q)$-adic filtration. Then we define $\operatorname{fil}^\star_{\mathcal{H}\mathrm{dg}_{p^\alpha}}\bigl(q\text{-}\mathrm{dR}^{(p^\alpha)}_{R/A}\bigr)^\wedge_p$ as the following pullback of filtered objects:

$$\begin{array}{ccc}
\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_{p^\alpha}}\bigl(q\text{-}\mathrm{dR}^{(p^\alpha)}_{R/A}\bigr)^\wedge_p & \longrightarrow & \operatorname{fil}^\star_{\mathcal{N}}\bigl(q\text{-}\mathrm{dR}^{(p^\alpha)}_{R/A}\bigr)^\wedge_p \\
\big\downarrow & \lrcorner & \big\downarrow {\scriptstyle \phi_{p/A[q]}} \\
\Phi_{p^\alpha}(q)^\star \operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_{p^{\alpha-1}}}\bigl(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}\bigr)^\wedge_p & \longrightarrow & \Phi_{p^\alpha}(q)^\star\bigl(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}\bigr)^\wedge_p
\end{array}$$

Using this pullback diagram, we can also inductively equip $\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_{p^\alpha}}\bigl(q\text{-}\mathrm{dR}^{(p^\alpha)}_{R/A}\bigr)^\wedge_p$ with the structure of a filtered module over the filtered ring $(q^{p^\alpha}-1)^\star A[q]$.

3.33. Remark. — If we reduce the pullback diagram above modulo $(q^{p^\alpha}-1)$ (where we invoke Convention 3.1 as usual), we obtain the pullback diagram from Lemma 3.30. Indeed, this follows via induction on α, using Proposition 3.22. It follows that

$$\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_{p^\alpha}}\bigl(q\text{-}\mathrm{dR}^{(p^\alpha)}_{R/A}\bigr)^\wedge_p/(q^{p^\alpha}-1) \simeq \operatorname{fil}_{\mathcal{H}\mathrm{dg}_{p^\alpha}}(q\text{-}\mathbb{W}_{p^\alpha}\mathrm{dR}_{R/A})^\wedge_p\,.$$

3.34. Lax symmetric monoidal structure I. — The functor

$$\operatorname{fil}^\star_{\mathcal{H}\mathrm{dg}_{p^\alpha}}\bigl(q\text{-}\mathrm{dR}^{(p^\alpha)}_{-/A}\bigr)^\wedge_p \colon \mathrm{AniAlg}^{q\text{-}\mathrm{Hdg}}_A \longrightarrow \mathrm{Mod}_{(q^{p^\alpha}-1)^\star A[q]}\Bigl(\mathrm{Fil}\,\mathcal{D}\bigl(A[q]\bigr)\Bigr)^\wedge_{(p,q-1)}$$

comes equipped with a canonical lax symmetric monoidal structure. This follows from the recursive construction. For $\alpha = 0$, Proposition 3.7 even provides a symmetric monoidal structure. For $\alpha \geqslant 1$, we must equip the legs of the pullback in 3.32 with the structure of symmetric monoidal transformations. This is not hard. First, the Frobenius

$$\phi_{/A[q]} \colon \operatorname{fil}^\star_{\mathcal{N}}\bigl(q\text{-}\mathrm{dR}^{(p^\alpha)}_{R/A}\bigr)^\wedge_p \longrightarrow \Phi_{p^\alpha}(q)^\star\bigl(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}\bigr)^\wedge_p$$

becomes a symmetric monoidal transformation by quasi-syntomic descent from the case where R is a p-complete quasi-syntomic A-algebra with a surjection $\widehat{A}_p\langle x_i^{1/p^\infty} \mid i \in I\rangle \twoheadrightarrow R$. In this case, we're dealing with filtrations of rings by ideals, so symmetric monoidality is automatic.

Second, the functor that "rescales" a filtration by $\Phi_{p^\alpha}(q)$ as in 3.32 is lax symmetric monoidal. Indeed, if we regard our filtered objects as graded modules over $\mathbb{Z}[q,t]$, with the filtration parameter t in graded degree -1, then rescaling corresponds to restriction along the $\mathbb{Z}[q]$-linear map $\mathbb{Z}[q,t] \to \mathbb{Z}[q,t]$ that sends $t \mapsto \Phi_{p^\alpha}(q)t$. This is lax symmetric monoidal.

3.35. Lax symmetric monoidal structure II. — It follows from the construction in 3.32 that we have a canonical map

$$\operatorname{fil}^\star_{\mathcal{H}\mathrm{dg}_{p^\alpha}}\left(q\text{-}\mathrm{dR}^{(p^\alpha)}_{R/A}\right)^\wedge_p \longrightarrow \operatorname{fil}^\star_{\mathcal{H}\mathrm{dg}_{p^{\alpha-1}}}\left(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}\right)^\wedge_p$$

compatible with the relative Frobenius $\phi_{p/A[q]}\colon \left(q\text{-}\mathrm{dR}^{(p^\alpha)}_{R/A}\right)^\wedge_p \to \left(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}\right)^\wedge_p$, because the "rescaling" by $\Phi_{p^\alpha}(q)$ of any filtration in non-negative degrees has a canonical map back to the original filtration. Moreover, by the discussion in 3.34, the map above can be canonically equipped with the structure of a symmetric monoidal transformation.

3.36. Lemma. — *For all primes p and all $\alpha \geqslant 0$, there exists a canonical equivalence of filtered $(q^{p^\alpha}-1)^\star A[q]$-modules*

$$\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_{p^\alpha}}\left(q\text{-}\mathrm{dR}^{(p^\alpha)}_{R/A}\right)^\wedge_p\left[\tfrac{1}{p}\right]^\wedge_{\Phi_{p^\alpha}(q)} \xrightarrow{\ \simeq\ } \operatorname{fil}^\star_{(\mathrm{Hdg},\Phi_{p^\alpha}(q))}\left(\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A,\psi^{p^\alpha}}A\right)^\wedge_p\left[\tfrac{1}{p},q\right]^\wedge_{\Phi_{p^\alpha}}\,,$$

where $\operatorname{fil}^\star_{(\mathrm{Hdg},\Phi_{p^\alpha}(q))}$ denotes the combined Hodge and $\Phi_{p^\alpha}(q)$-adic filtration. This equivalence is compatible with $(q\text{-}\mathrm{dR}_{R/A})^\wedge_p[1/p]^\wedge_{(q-1)} \simeq (\mathrm{dR}_{R/A})^\wedge_p[1/p]\llbracket q-1\rrbracket$.

Proof. For $\alpha=0$, this is the condition from Definition 3.2(c_p). So let $\alpha \geqslant 1$. After applying $(-)[1/p]^\wedge_{\Phi_{p^\alpha}(q)}$, the polynomial $(q^{p^{\alpha-1}}-1)$ becomes invertible, and so the filtered $(q^{p^{\alpha-1}}-1)^\star A[q]$-module

$$\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_{p^{\alpha-1}}}\left(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}\right)^\wedge_p\left[\tfrac{1}{p}\right]^\wedge_{\Phi_{p^\alpha}(q)}$$

must be the constant filtration on $\left(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}\right)^\wedge_p[1/p]^\wedge_{\Phi_{p^\alpha}(q)}$. Consequently, after applying $(-)[1/p]^\wedge_{\Phi_{p^\alpha}(q)}$ the bottom horizontal arrow in the pullback diagram from 3.32 becomes an equivalence and thus the top horizontal arrow becomes an equivalence too. The desired assertion then follows via base change from Lemma 3.29. ☐

3.37. Lemma. — *For all primes p, all $\alpha \geqslant 1$, and all $0 \leqslant i \leqslant \alpha-1$, the canonical map from 3.35 induces an equivalence of filtered $(q^{p^\alpha}-1)^\star A[q]$-modules*

$$\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_{p^\alpha}}\left(q\text{-}\mathrm{dR}^{(p^\alpha)}_{R/A}\right)^\wedge_p\left[\tfrac{1}{p}\right]^\wedge_{\Phi_{p^i}(q)} \xrightarrow{\ \simeq\ } \operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_{p^{\alpha-1}}}\left(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}\right)^\wedge_p\left[\tfrac{1}{p}\right]^\wedge_{\Phi_{p^i}(q)}\,.$$

Proof. After $(-)[1/p]^\wedge_{\Phi_{p^i}(q)}$, the polynomial $\Phi_{p^\alpha}(q)$ becomes invertible. Consequently, the "rescaling" of filtrations in 3.32 has no effect anymore. Moreover, it follows that the filtered $\Phi_{p^\alpha}(q)^\star A[q]$-module

$$\operatorname{fil}^\star_{\mathcal{N}}\left(q\text{-}\mathrm{dR}^{(p^\alpha)}_{R/A}\right)^\wedge_p\left[\tfrac{1}{p}\right]^\wedge_{\Phi_{p^i}(q)}$$

must be the constant filtration on $\left(q\text{-}\mathrm{dR}^{(p^\alpha)}_{R/A}\right)^\wedge_p\left[\tfrac{1}{p}\right]^\wedge_{\Phi_{p^i}(q)}$. Thus, after applying $(-)[1/p]^\wedge_{\Phi_{p^i}(q)}$, the pullback from 3.32 collapses to the desired equivalence. ☐

Let us finally construct the filtration $\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_m} q\text{-}\mathrm{dR}^{(m)}_{R/A}$ in general.

3.38. The twisted q-Hodge filtration (globally) — Choose $N \neq 0$ divisible by m (we'll argue below that the choice of N doesn't matter). For every divisor $d \mid m$ and every prime $p \mid N$, write $m = p^{v_p(m)}m_p$ and $d = p^{v_p(d)}d_p$, where m_p and d_p are coprime to p. Using the animated version of Lemma 3.15, we obtain a pullback diagram

$$\begin{array}{ccc}
q\text{-}\mathrm{dR}^{(m)}_{R/A} & \longrightarrow & \displaystyle\prod_{p\mid N}\prod_{d_p\mid m_p}\Bigl(q\text{-}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A[q],\psi^{p^{v_p(m)}d_p}}A[q]\Bigr)^\wedge_{(p,\Phi_{d_p}(q))}\\
\big\downarrow & \lrcorner & \big\downarrow\Bigl(\phi^{v_p(m/d)}_{p/A[q]}\Bigr)_{p\mid N,\, d\mid m}\\
\displaystyle\prod_{d\mid m}\Bigl(q\text{-}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A[q],\psi^d}A\bigl[\tfrac{1}{N},q\bigr]\Bigr)^\wedge_{\Phi_d(q)} & \longrightarrow & \displaystyle\prod_{p\mid N}\prod_{d\mid m}\Bigl(q\text{-}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A[q],\psi^d}A[q]\Bigr)^\wedge_p\bigl[\tfrac{1}{p}\bigr]^\wedge_{\Phi_d(q)}
\end{array}$$

To construct $\mathrm{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_m}\, q\text{-}\mathrm{dR}^{(m)}_{R/A}$, we'll equip each factor of the pullback above with a filtration and then check that these filtrations are compatible.

(a) On the factor $(q\text{-}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A[q],\psi^d}A[1/N,q])^\wedge_{\Phi_d(q)}$ for any $d\mid m$, we put the base-changed q-Hodge filtration

$$\Bigl(\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A[q],\psi^d}A\bigl[\tfrac{1}{N},q\bigr]\Bigr)^\wedge_{\Phi_d(q)}\,.$$

(b) On the factor $(q\text{-}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A[q],\psi^d}A[q])^\wedge_p[1/p,q]^\wedge_{\Phi_d(q)}$ for any prime $p\mid N$ and any $d\mid m$, we put again the base-changed q-Hodge filtration.

(c) On the factor $(q\text{-}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A[q],\psi^{p^{v_p(m)}d_p}}A[q])^\wedge_{(p,\Phi_{d_p}(q))}$ for any prime $p\mid N$ and any $d_p\mid m_p$, we put the base-changed filtration

$$\Bigl(\mathrm{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_{p^{v_p(m)}}}\bigl(q\text{-}\mathrm{dR}^{(p^{v_p(m)})}_{R/A}\bigr)^\wedge_p\otimes^{\mathrm{L}}_{A[q],\psi^{d_p}}A[q]\Bigr)^\wedge_{(p,\Phi_{d_p}(q))}\,.$$

Moreover, each of these filtrations is canonically a module over the filtered ring $(q^m-1)^\star A[q]$. It's clear that (a) and (b) are compatible as filtered $(q^m-1)^\star A[q]$-modules. To check that (c) and (b) are compatible, we may reduce via base change to the case where $m=p^\alpha$ is a power of p. From Lemmas 3.36 and 3.37 and our assumptions on $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}$ we deduce that both filtrations can be identified with the combined Hodge and $\Phi_{p^\alpha}(q)$-adic filtration

$$\mathrm{fil}^\star_{(\mathrm{Hdg},\Phi_{p^\alpha}(q))}\bigl(\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A,\psi^{p^\alpha}}A\bigr)^\wedge_p\bigl[\tfrac{1}{p},q\bigr]^\wedge_{\Phi_{p^\alpha}(q)}\,,$$

which yields the desired compatibility.

Let us now argue that the choice of N is irrelevant. Suppose $N\mid N'$. Then the pullback diagrams for N' is obtained from the pullback square for N by replacing the bottom left corner $\prod_{d\mid m}(q\text{-}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A[q],\psi^d}A[1/N,q])^\wedge_{\Phi_d(q)}$ by the pullback square

$$\begin{array}{ccc}
\displaystyle\prod_{d\mid m}\Bigl(q\text{-}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A[q],\psi^d}A\bigl[\tfrac{1}{N},q\bigr]\Bigr)^\wedge_{\Phi_d(q)} & \longrightarrow & \displaystyle\prod_{\ell}\prod_{d\mid m}\Bigl(q\text{-}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A[q],\psi^d}A\bigl[\tfrac{1}{N},q\bigr]\Bigr)^\wedge_{(\ell,\Phi_d(q))}\\
\big\downarrow & \lrcorner & \big\downarrow\\
\displaystyle\prod_{d\mid m}\Bigl(q\text{-}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A[q],\psi^d}A\bigl[\tfrac{1}{N'},q\bigr]\Bigr)^\wedge_{\Phi_d(q)} & \longrightarrow & \displaystyle\prod_{\ell}\prod_{d\mid m}\Bigl(q\text{-}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A[q],\psi^d}A\bigl[\tfrac{1}{N},q\bigr]\Bigr)^\wedge_\ell\bigl[\tfrac{1}{\ell}\bigr]^\wedge_{\Phi_d(q)}
\end{array}$$

where the product is taken over all primes ℓ such that $\ell\mid N'$ but $\ell\nmid N$. Note that for any such prime we also have $\ell\nmid m$, so each $v_\ell(m/d)=0$ and so each iterated Frobenius $\phi^{v_\ell(m/d)}_{\ell/A[q]}$

is the identity. Moreover, we see that the filtrations we put on the different factors is always $\operatorname{fil}^\star_{q\text{-Hdg}}$, base changed along ψ^d. It follows that the filtrations constructed using N and using N' must indeed agree, as claimed. To get a canonical construction, we can let N vary through a totally ordered initial sub-poset of $\mathbb{N}$ (like $\{n!\}_{n\geqslant m}$) and then take the limit. This finishes the construction of $\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_m} q\text{-}\mathrm{dR}^{(m)}_{R/A}$.

The construction is clearly functorial. Using 3.34 and 3.35 we also see that the functor

$$\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_m} q\text{-}\mathrm{dR}^{(m)}_{-/A}\colon \mathrm{AniAlg}^{q\text{-Hdg}}_A \longrightarrow \mathrm{Mod}_{(q^m-1)^\star A[q]}\Big(\operatorname{Fil}\mathcal{D}\big(A[q]\big)\Big)^\wedge_{(q^m-1)}$$

comes equipped with a canonical lax symmetric monoidal structure.

3.39. Proposition. — *For all $m\in\mathbb{N}$, the equivalence $q\text{-}\mathrm{dR}^{(m)}_{R/A}/(q^m-1)\simeq q\text{-}\mathbb{W}_m\mathrm{dR}_{R/A}$ from the animated version of Proposition 3.19 upgrades canonically to an equivalence of filtered $A[q]/(q^m-1)$-modules*

$$\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_m} q\text{-}\mathrm{dR}^{(m)}_{R/A}/(q^m-1)\xrightarrow{\ \simeq\ }\operatorname{fil}^\star_{\mathcal{H}\mathrm{dg}_m} q\text{-}\mathbb{W}_m\mathrm{dR}_{R/A}$$

(*the quotient on the left-hand side is taken in accordance with Convention 3.1*).

Proof sketch. We analyse the effect of $(-)/(q^m-1)$ on each of the factors in 3.38. For the factors in 3.38(c), note that (q^m-1) and $\Phi_{d_p}(q^{p^{v_p(m)}})$ will only differ by a unit upon $(p,\Phi_{d_p}(q))$-adic completion. Then the argument in Remark 3.33 plus base change shows that after modding out (q^m-1) we get

$$\Big(\operatorname{fil}^\star_{\mathcal{H}\mathrm{dg}_{p^{v_p(m)}}} q\text{-}\mathbb{W}_{p^{v_p(m)}}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A[q],\psi^{d_p}} A[q]\Big)^\wedge_p/\Phi_{d_p}\big(q^{p^{v_p(m)}}\big)$$

It follows from [Wag24, Lemma 4.36] that $\operatorname{fil}^\star_{\mathcal{H}\mathrm{dg}_m}(q\text{-}\mathbb{W}_m\mathrm{dR}_{R/A})^\wedge_p$ is indeed a product of factors of this form.

For the factors in 3.38(a), note that (q^m-1) and $\Phi_d(q)$ will only differ by a unit after $(-)[1/N]^\wedge_{\Phi_d(q)}$. By construction, the q-Hodge filtration becomes the Hodge filtration modulo $(q-1)$. Thus, after base change along $\psi^d\colon A[q]\to A[q]$, we get

$$\Big(\operatorname{fil}^\star_{q\text{-Hdg}} q\text{-}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A[q],\psi^d} A\big[\tfrac{1}{N},q\big]\Big)^\wedge_{\Phi_d(q)}/\Phi_d(q)\simeq \operatorname{fil}^\star_{\mathrm{Hdg}}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A,\psi^d} A\big[\tfrac{1}{N},\zeta_d\big]\,.$$

It follows from [Wag24, Corollary 3.34] that $\operatorname{fil}^\star_{\mathcal{H}\mathrm{dg}_m} q\text{-}\mathbb{W}_m\mathrm{dR}_{R/A}[1/N]$ is indeed a product of factors of this form. The same argument applies for the factors in 3.38(b). □

3.40. Remark. — It follows from the proof that the equivalence in Proposition 3.39 is, in fact, an equivalence of lax symmetric monoidal functors $\mathrm{AniAlg}^{q\text{-Hdg}}_A\to\operatorname{Fil}\mathcal{D}(A[q]/(q^m-1))$. Thus, if $(R,\operatorname{fil}^\star_{q\text{-Hdg}} q\text{-}\mathrm{dR}_{R/A})$ admits the structure of an $\mathbb{E}_n$-algebra in $\mathrm{AniAlg}^{q\text{-Hdg}}_A$ for any $0\leqslant n\leqslant\infty$, then the equivalence in Proposition 3.39 will be one of filtered $\mathbb{E}_n$-$A[q]/(q^m-1)$-algebras.

3.41. Transition maps. — Whenever $n\mid m$, there's a canonical map of filtered objects

$$\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_m} q\text{-}\mathrm{dR}^{(m)}_{R/A}\longrightarrow \operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_n} q\text{-}\mathrm{dR}^{(n)}_{R/A}\,.$$

To construct this, we look at the factors of the pullback from 3.38 (we're allowed to use the same N for both m and n). For the factors from 3.38(a) and (b), we simply project to those where $d \mid n$. For the factors from 3.38(c), we first project to those where $d_p \mid n_p$ and then we use the maps from 3.35, base changed along $\psi^{d_p}\colon A[q] \to A[q]$.

It's clear from the construction that these maps $\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_m} q\text{-}\mathrm{dR}^{(m)}_{R/A} \to \operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_n} q\text{-}\mathrm{dR}^{(n)}_{R/A}$ assemble canonically into a symmetric monoidal transformation of lax symmetric monoidal functors. With some more effort, one can also make these transformations functorial in $n, m \in \mathbb{N}$, where $\mathbb{N}$ denotes the category of natural numbers partially ordered by divisibility. For our purposes, the existence of the individual maps is enough, as any $\lim_{m\in\mathbb{N}}$ can be replaced by the limit over the sequential subdiagram given by $\{n!\}_{n\geqslant 1}$. We will therefore not spell out the construction of this additional functoriality.

§3.6. Habiro descent for q-Hodge complexes

In this subsection, we'll finish the proof of Theorem 3.11, following the outline that we have explained at the end of §3.2.

3.42. The (q^m-1)-complete descent. — For all $m \in \mathbb{N}$, we consider the colimit

$$q\text{-}\mathcal{H}\mathrm{dg}_{R/A,m} \coloneqq \operatorname{colim}\Bigl(\operatorname{fil}^0_{q\text{-}\mathcal{H}\mathrm{dg}_m} q\text{-}\mathrm{dR}^{(m)}_{R/A} \xrightarrow{(q^m-1)} \operatorname{fil}^1_{q\text{-}\mathcal{H}\mathrm{dg}_m} q\text{-}\mathrm{dR}^{(m)}_{R/A} \xrightarrow{(q^m-1)} \dots\Bigr)^\wedge_{(q^m-1)}\,.$$

In the following, we'll informally write

$$q\text{-}\mathcal{H}\mathrm{dg}_{R/A,m} \simeq q\text{-}\mathrm{dR}^{(m)}_{R/A}\left[\frac{\operatorname{fil}^i_{q\text{-}\mathcal{H}\mathrm{dg}_m}}{(q^m-1)^i}\;\middle|\; i \geqslant 1\right]^\wedge_{(q^m-1)}$$

and we'll say that $q\text{-}\mathcal{H}\mathrm{dg}_{R/A,m}$ *is given by adjoining* $(q^m-1)^{-\star}\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_m}$ *to* $q\text{-}\mathrm{dR}^{(m)}_{R/A}$. We'll also use similar notation and terminology for related filtrations such as the Nygaard filtration or the combined Hodge and $\Phi_d(q)$-adic filtration for some $d \mid m$.

3.43. Proposition. — *Let $m \in \mathbb{N}$. For all divisors $n \mid m$, the map from 3.41 induces an equivalence*

$$\bigl(q\text{-}\mathcal{H}\mathrm{dg}_{R/A,m}\bigr)^\wedge_{(q^n-1)} \xrightarrow{\simeq} q\text{-}\mathcal{H}\mathrm{dg}_{R/A,n}\,.$$

In particular, $q\text{-}\mathcal{H}\mathrm{dg}_{R/A,m}$ is a descent of $q\text{-}\mathrm{Hdg}_{R/A}$ along $\mathbb{Z}[q]^\wedge_{(q^m-1)} \to \mathbb{Z}[\![q-1]\!]$.

Proof sketch. Again, we look at the different factors from 3.38. Let's start with those from 3.38(a) for some $d \mid m$. If $d \nmid n$, then $\Phi_d(q)$ and (q^n-1) are coprime in $\mathbb{Q}[q]$ and so the factor will die after (q^n-1)-completion. Therefore in $(q\text{-}\mathcal{H}\mathrm{dg}_{R/A,m})^\wedge_{(q^n-1)}$ only those factors where $d \mid n$ will survive. These are precisely the factors that are also used in the construction of of $q\text{-}\mathcal{H}\mathrm{dg}_{R/A,n}$. Moreover, if $d \mid n$ then both (q^m-1) and (q^n-1) are unit multiples of $\Phi_d(q)$ in $\mathbb{Q}[q]^\wedge_{\Phi_d(q)}$, so it doesn't matter whether we adjoin $(q^m-1)^{-\star}\operatorname{fil}^\star_{(\mathrm{Hdg},\Phi_d(q))}$ or $(q^n-1)^{-\star}\operatorname{fil}^\star_{(\mathrm{Hdg},\Phi_d(q))}$. It follows that on the factors from 3.38(a) we get indeed an equivalence. The same argument applies to the factors from 3.38(b).

It remains to show that we also get an equivalence on the factors from 3.38(c). So let's consider such a factor for some prime p and some $d_p \mid m_p$. Using induction, we may assume that m and n differ only by a single prime factor. If that prime is different from p, then (q^m-1) and (q^n-1) will differ by a unit after $(p, \Phi_{d_p}(q))$-completion and we can argue as above. So assume $n = m/p$. Via base change along $\psi^{d_p}\colon A[q] \rightarrow A[q]$, we may reduce to the case where $m = p^\alpha$ is a prime power and $n = p^{\alpha-1}$. From 3.32 we obtain a pullback diagram

$$\begin{array}{ccc} \left(q\text{-dR}_{R/A}^{(p^\alpha)}\right)\left[\dfrac{\mathrm{fil}^i_{q\text{-}\mathcal{H}\mathrm{dg}_{p^\alpha}}}{(q^{p^\alpha}-1)^i}\ \middle|\ i\geqslant 0\right]^\wedge_{(p,q^{p^{\alpha-1}}-1)} & \longrightarrow & \left(q\text{-dR}_{R/A}^{(p^\alpha)}\right)\left[\dfrac{\mathrm{fil}^i_{\mathcal{N}}}{(q^{p^\alpha}-1)^i}\ \middle|\ i\geqslant 0\right]^\wedge_{(p,q^{p^{\alpha-1}}-1)} \\ \downarrow & \lrcorner & \downarrow \\ \left(q\text{-dR}_{R/A}^{(p^{\alpha-1})}\right)\left[\dfrac{\mathrm{fil}^i_{q\text{-}\mathcal{H}\mathrm{dg}_{p^{\alpha-1}}}}{(q^{p^{\alpha-1}}-1)^i}\ \middle|\ i\geqslant 0\right]^\wedge_{(p,q^{p^{\alpha-1}}-1)} & \longrightarrow & \left(q\text{-dR}_{R/A}^{(p^{\alpha-1})}\right)\left[\dfrac{1}{(q^{p^{\alpha-1}}-1)}\right]^\wedge_{(p,q^{p^{\alpha-1}}-1)} \end{array}$$

To finish the proof, we must show that the left vertical arrow is an equivalence. Since the diagram is a pullback, it will be enough to show that the right vertical arrow is an equivalence.(3.7)

This is now purely an assertion about the Nygaard filtration. Via base change, we may reduce to the case $\alpha = 1$. This case will be shown in Lemma 3.44 below. □

3.44. Lemma. — *The relative Frobenius* $\phi_{p/A[q]}\colon \left(q\text{-dR}_{R/A}^{(p)}\right)^\wedge_p \rightarrow (q\text{-dR}_{R/A})^\wedge_p$ *induces functorial equivalences*

$$\left(q\text{-dR}_{R/A}^{(p)}\right)^\wedge_p\left[\frac{\mathrm{fil}^i_{\mathcal{N}}}{\Phi_p(q)^i}\ \middle|\ i\geqslant 0\right]^\wedge_{(p,q-1)} \xrightarrow{\simeq} (q\text{-dR}_{R/A})^\wedge_p\,,$$
$$\left(q\text{-dR}_{R/A}^{(p)}\right)^\wedge_p\left[\frac{\mathrm{fil}^i_{\mathcal{N}}}{(q^p-1)^i}\ \middle|\ i\geqslant 0\right]^\wedge_{(p,q-1)} \xrightarrow{\simeq} (q\text{-dR}_{R/A})^\wedge_p\left[\frac{1}{(q-1)}\right]^\wedge_{(p,q-1)} \simeq 0\,.$$

Proof. We start with the first equivalence. Since both sides are $\Phi_p(q)$-complete, it will be enough to show the equivalence modulo $\Phi_p(q)$. The same argument as in 3.8 shows

$$\left(q\text{-dR}_{R/A}^{(p)}\right)^\wedge_p\left[\frac{\mathrm{fil}^i_{\mathcal{N}}}{\Phi_p(q)^i}\ \middle|\ i\geqslant 0\right]\Big/\Phi_p(q) \simeq \operatorname{colim}\left(\mathrm{gr}^0_{\mathcal{N}} \xrightarrow{\Phi_p(q)} \mathrm{gr}^1_{\mathcal{N}} \xrightarrow{\Phi_p(q)} \cdots\right).$$

The divided Frobenius $\Phi_p(q)^{-i}\phi_{p/A[q]}$ maps $\mathrm{gr}^i_{\mathcal{N}}$ isomorphically onto $\mathrm{fil}_i^{\mathrm{conj}}(q\text{-dR}_{R/A}/\Phi_p(q))$ (by [BS19, Theorem 15.2] plus quasi-syntomic descent and animation to cover all animated A-algebras R). Since the conjugate filtration is exhaustive, this shows the first of the two claimed equivalences.

For the second equivalence, note that the inclusion of the diagonal into any $\mathbb{Z}_{\geqslant 0}\times\mathbb{Z}_{\geqslant 0}$-shaped diagram is coinitial. Therefore, we can write

$$\left(q\text{-dR}_{R/A}^{(p)}\right)^\wedge_p\left[\frac{\mathrm{fil}^i_{\mathcal{N}}}{(q^p-1)^i}\ \middle|\ i\geqslant 0\right] \simeq \operatorname{colim}\begin{pmatrix} \mathrm{fil}^0_{\mathcal{N}} & \xrightarrow{\Phi_p(q)} & \mathrm{fil}^1_{\mathcal{N}} & \xrightarrow{\Phi_p(q)} & \cdots \\ \downarrow{\scriptstyle (q-1)} & & \downarrow{\scriptstyle (q-1)} & & \\ \mathrm{fil}^0_{\mathcal{N}} & \xrightarrow{\Phi_p(q)} & \mathrm{fil}^1_{\mathcal{N}} & \xrightarrow{\Phi_p(q)} & \cdots \\ \downarrow{\scriptstyle (q-1)} & & \downarrow{\scriptstyle (q-1)} & & \\ \vdots & & \vdots & & \ddots \end{pmatrix}$$

(3.7) Also note that the bottom right corner vanishes, so it will follow that the top right corner vanishes as well. But this will be irrelevant for our argument.

By the first equivalence, the $(p, q-1)$-completed colimit of every row in this diagram is $(q\text{-dR}_{R/A})_p^\wedge$. If we then take the colimit in the vertical direction, the second equivalence follows and we're done $\square$

3.45. The Habiro descent — Let $(R, \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_{R/A})$ be an object in $\mathrm{AniAlg}_A^{q\text{-Hdg}}$. We define the *Habiro–Hodge complex of R over A* to be

$$q\text{-}\mathcal{H}\mathrm{dg}_{R/A} \coloneqq \lim_{m\in\mathbb{N}} q\text{-}\mathcal{H}\mathrm{dg}_{R/A,m}\,.$$

The same argument as in the proof of Proposition 3.7 allows us to equip $q\text{-}\mathcal{H}\mathrm{dg}_{-/A,m}$ with a lax symmetric monoidal structure for all $m \in \mathbb{N}$; thanks to 3.41, the equivalences from Proposition 3.43 will be compatible with this lax symmetric monoidal structure. It follows that there's a diagram of lax symmetric monoidal functors

$$\begin{array}{ccc} & & \widehat{\mathcal{D}}_{\mathcal{H}}\bigl(A[q]\bigr) \\ & \overset{q\text{-}\mathcal{H}\mathrm{dg}_{-/A}}{\nearrow} & \big\downarrow (-)^\wedge_{(q-1)} \\ \mathrm{AniAlg}_A^{q\text{-Hdg}} & \xrightarrow[q\text{-Hdg}_{-/A}]{} & \widehat{\mathcal{D}}_{(q-1)}\bigl(A[q]\bigr) \end{array}$$

3.46. Lemma. — *The lax symmetric monoidal functor $q\text{-}\mathcal{H}\mathrm{dg}_{-/A}\colon \mathrm{AniAlg}_A^{q\text{-Hdg}} \to \widehat{\mathcal{D}}_{\mathcal{H}}(A[q])$ is, in fact, symmetric monoidal.*

Proof. It will be enough to show that for all $m \in \mathbb{N}$ the functor

$$q\text{-}\mathcal{H}\mathrm{dg}_{-/A}/(q^m-1)\Bigl[\bigl\{(q^d-1)^{-1}\bigr\}_{d\mid m,\, d\neq m}\Bigr]$$

is symmetric monoidal. The same argument as in the proof of Proposition 3.7 allows us to equip the filtration $\mathrm{fil}_\star^{q\text{-}\mathbb{W}_m\Omega}(q\text{-}\mathcal{H}\mathrm{dg}_{-/A}/(q^m-1))$ from Theorem 3.11(*b*) with a lax symmetric monoidal structure. Symmetric monoidality can then be checked on the associated graded

$$\mathrm{gr}_*^{q\text{-}\mathbb{W}_m\Omega}\bigl(q\text{-}\mathcal{H}\mathrm{dg}_{-/A}/(q^m-1)\bigr) \simeq \Sigma^{-*}\, q\text{-}\mathbb{W}_m\mathrm{dR}^*_{-/A} \simeq \mathrm{gr}^*_{\mathcal{H}\mathrm{dg}_m}\, q\text{-}\mathbb{W}_m\mathrm{dR}_{-/A}\,.$$

Thus, it would be enough to show that $\mathrm{fil}^\star_{\mathcal{H}\mathrm{dg}_m}\, q\text{-}\mathbb{W}_m\mathrm{dR}_{-/A}$ is symmetric monoidal. This is not true on the nose. However, once we invert (q^d-1) for all divisors $d \mid m$, $d \neq m$, we claim that the first ghost map

$$\mathrm{gh}_1\colon \mathrm{fil}^\star_{\mathcal{H}\mathrm{dg}_m}\, q\text{-}\mathbb{W}_m\mathrm{dR}_{-/A} \xrightarrow{\simeq} \mathrm{fil}^\star_{\mathrm{Hdg}}\, \mathrm{dR}_{-/A} \otimes^{\mathrm{L}}_{A,\psi^m} A[\zeta_m]$$

becomes an equivalence. If we can show this, we're done, since the Hodge filtration $\mathrm{fil}^\star_{\mathrm{Hdg}}\, \mathrm{dR}_{-/A}$ is symmetric monoidal.

To prove this claim, observe that for any ordinary R-algebra A and any $d \mid m$, $d \neq m$, the q-de Rham–Witt complex $q\text{-}\mathbb{W}_d\Omega^*_{R/A}$ is (q^d-1)-torsion and so it dies after inverting (q^d-1). With this observation, a simple comparison of universal properties (compare the argument in [Wag24, Lemma 4.5]) shows that

$$\mathrm{gh}_1\colon q\text{-}\mathbb{W}_m\Omega^*_{R/A}\Bigl[\bigl\{(q^d-1)^{-1}\bigr\}_{d\mid m,\, d\neq m}\Bigr] \xrightarrow{\cong} \Omega^*_{R/A} \otimes_{A,\psi^m} A\Bigl[\zeta_m, \bigl\{(\zeta_m^d-1)^{-1}\bigr\}_{d\mid m,\, d\neq m}\Bigr]$$

is an isomorphism of complexes. In particular, it induces an isomorphism on stupid filtrations. By passing to animations, the above claim about gh_1 follows and so we're done. $\square$

At this point, we've assembled all the ingredients to carry out the proof of Theorem 3.11 as outlined at the end of §3.2, and so the proof is finally finished.

§3.7. Habiro descent for q-de Rham complexes

In this short subsection, we discuss to what extent the q-de Rham complex $q\text{-}\mathrm{dR}_{R/A}$ can or cannot be descended to the Habiro ring. Let's start with the case that works.

3.47. Proposition. — *Let $(S, \mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{S/A}) \in \mathrm{AniAlg}_A^{q\text{-}\mathrm{Hdg}}$ be an object such that S is a smooth A-algebra. Then:*

(*a*) $q\text{-}\Omega_{S/A} \simeq q\text{-}\widehat{\mathrm{dR}}_{S/A}$ *is the completion of* $q\text{-}\mathrm{dR}_{S/A}$ *at the q-Hodge filtration* $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}$.

(*b*) *We have* $\mathrm{L}\eta_{(q-1)}\, q\text{-}\mathrm{Hdg}_{S/A} \simeq q\text{-}\Omega_{R/A}$. *In particular,* $\mathrm{L}\eta_{(q-1)}\, q\text{-}\mathcal{H}\mathrm{dg}_{S/A}$ *is a Habiro descent of* $q\text{-}\Omega_{S/A}$.

Proof. We construct the equivalence $q\text{-}\Omega_{S/A} \simeq q\text{-}\widehat{\mathrm{dR}}_{S/A}$ using an arithmetic fracture square. Let us first construct an equivalence after p-completion for any prime p. Note that the canonical maps $q\text{-}\mathrm{dR}_{S/A} \to q\text{-}\Omega_{S/A}$ and $q\text{-}\mathrm{dR}_{S/A} \to q\text{-}\widehat{\mathrm{dR}}_{S/A}$ become equivalences after p-completion for any prime p. Indeed, this can be checked modulo $(q-1)$, where we recover the well-known fact $(\Omega_{S/A})^\wedge_p \simeq (\mathrm{dR}_{S/A})^\wedge_p \simeq (\widehat{\mathrm{dR}}_{S/A})^\wedge_p$. So we obtain the desired equivalence $(q\text{-}\Omega_{S/A})^\wedge_p \simeq (q\text{-}\widehat{\mathrm{dR}}_{S/A})^\wedge_p$.

Let us now construct the equivalence rationally. We know that $\mathrm{dR}_{S/A} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Q} \to \Omega_{S/A} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Q}$ identifies the right-hand side with the completion of the left-hand side at the Hodge filtration. Consequently, $(\mathrm{dR}_{S/A} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]^\wedge_{(\mathrm{Hdg}, q-1)} \simeq (\Omega_{S/A} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$, which yields the desired equivalence rationally. The data from Definition 3.2(c_p) ensures that the p-complete and rational equivalences glue, which finishes the proof of (a).

To prove (b), first observe that the natural map $q\text{-}\mathrm{dR}_{S/A} \to q\text{-}\mathrm{Hdg}_{S/A}$ factors through the completion at the q-Hodge filtration, because each filtration step $\mathrm{fil}^i_{q\text{-}\mathrm{Hdg}}$ becomes divisible by $(q-1)^i$ in $q\text{-}\mathrm{Hdg}_{S/A}$ and $q\text{-}\mathrm{Hdg}_{S/A}$ is $(q-1)$-complete. Now consider the map of filtered objects

$$\begin{array}{ccccccccc}
\cdots = & q\text{-}\widehat{\mathrm{dR}}_{S/A} & = & q\text{-}\widehat{\mathrm{dR}}_{S/A} & \xleftarrow{\simeq} & \mathrm{fil}^0_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{S/A} & \longleftarrow & \mathrm{fil}^1_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{S/A} & \longleftarrow \cdots \\
 & \downarrow{\scriptstyle (q-1)^2} & & \downarrow{\scriptstyle (q-1)} & & \downarrow & & \downarrow & \\
\cdots \longleftarrow & q\text{-}\mathrm{Hdg}_{S/A} & \xleftarrow{(q-1)} & q\text{-}\mathrm{Hdg}_{S/A} & \xleftarrow{(q-1)} & q\text{-}\mathrm{Hdg}_{S/A} & \xleftarrow{(q-1)} & q\text{-}\mathrm{Hdg}_{S/A} & \longleftarrow \cdots
\end{array}$$

We claim that the top row is the connective cover of the bottom row in the Beilinson t-structure. If we can prove this, then [BMS19, Proposition 5.8] will show $q\text{-}\widehat{\mathrm{dR}}_{S/A} \simeq \mathrm{L}\eta_{(q-1)}\, q\text{-}\mathrm{Hdg}_{S/A}$, hence $q\text{-}\Omega_{S/A} \simeq \mathrm{L}\eta_{(q-1)}\, q\text{-}\mathrm{Hdg}_{S/A}$ by (a), as desired. Since $\mathrm{L}\eta_{(q-1)}$ commutes with $(q-1)$-completion [BMS18, Lemma 6.20], we also deduce that $\mathrm{L}\eta_{(q-1)}\, q\text{-}\mathcal{H}\mathrm{dg}_{S/A}$ is indeed a Habiro descent of $q\text{-}\Omega_{S/A}$.

To show the claim, let us first verify that the top row is indeed connective in the Beilinson t-structure. We must show that $\mathrm{gr}^n_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{S/A}$ is concentrated in cohomological degrees $\leqslant n$ for all n. If $n < 0$, this is clear as then $\mathrm{gr}^n_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{S/A} \simeq 0$. If $n \geqslant 0$, we have a finite-length filtration

$$0 \longrightarrow \mathrm{gr}^0_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{S/A} \xrightarrow{(q-1)} \mathrm{gr}^1_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{S/A} \xrightarrow{(q-1)} \cdots \xrightarrow{(q-1)} \mathrm{gr}^n_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{S/A}\,.$$

The i^{th} graded piece of this filtration is $\Sigma^{-i}\Omega^i_{S/A}$ by Lemma 3.9, which is concentrated in cohomological degree i. Hence $\mathrm{gr}^n_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{S/A}$ is indeed concentrated in cohomological degrees $\leqslant n$ and so the top row is Beilinson-connective.

Moreover, the argument shows that $\operatorname{gr}^n_{q\text{-Hdg}} q\text{-dR}_{S/A} \to q\text{-Hdg}_{S/A}/(q-1)$ induces an equivalence $\operatorname{gr}^n_{q\text{-Hdg}} q\text{-dR}_{S/A} \simeq \tau^{\leqslant n}(q\text{-Hdg}_{S/A}/(q-1))$. By [BMS19, Theorem 5.4(2)], this shows that the map from the top row to the Beilinson-connective cover of the bottom row is an equivalence on associated gradeds. As both filtered objects are complete, we're done. □

Let us now discuss what probably doesn't work.

3.48. Remark. — For an arbitrary object $(R, \operatorname{fil}^\star_{q\text{-Hdg}} q\text{-dR}_{R/A})$ in $\operatorname{AniAlg}_A^{q\text{-Hdg}}$, without a smoothness assumption on R, we don't know how to construct a Habiro descent of $q\text{-dR}_{R/A}$. A naive guess would be

$$\lim_{m\in\mathbb{N}} q\text{-dR}^{(m)}_{R/A}\left[\frac{\operatorname{fil}^i_{q\text{-}\mathcal{H}\mathrm{dg}_m}}{[m]_q^i}\ \middle|\ i\geqslant 0\right]^\wedge_{(q^m-1)},$$

but this object doesn't exist, since $\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_m} q\text{-dR}^{(m)}_{R/A}$ is only a filtered module over the filtered ring $(q^m-1)^\star A[q]$, but not necessarily over $[m]_q^\star A[q]$.

3.49. Remark. — We also don't expect that the Habiro descent of $q\text{-}\Omega_{S/A}$ in Proposition 3.47 can be constructed without the datum of a q-Hodge filtration $\operatorname{fil}^\star_{q\text{-Hdg}} q\text{-dR}_{S/A}$, let alone functorially in S. While it seems hard to get any definite no-go theorem, let us at least explain why the most natural attempt doesn't work.

In Remark 3.18, we've explained an attempt to construct a filtration $\operatorname{fil}^\star_{\mathrm{L}\eta} q\text{-}\Omega^{(m)}_{S/A}$: Each $\mathrm{L}\eta_{[m/d]_q}$ carries a natural filtration via [BMS19, Proposition 5.8]. If these filtrations could be glued to give the desired $\operatorname{fil}^\star_{\mathrm{L}\eta}$, we could attempt to construct a Habiro descent of $q\text{-}\Omega_{S/A}$ via

$$\lim_{m\in\mathbb{N}} q\text{-}\Omega^{(m)}_{S/A}\left[\frac{\operatorname{fil}^i_{\mathrm{L}\eta}}{[m]_q^i}\ \middle|\ i\geqslant 0\right]^\wedge_{(q^m-1)}.$$

However, the filtrations on $\mathrm{L}\eta_{[m/d]_q}$ *do not* glue. This can already be seen in the case $m = p$. In this case we have a pullback diagram

$$\begin{array}{ccc} q\text{-}\Omega^{(p)}_{S/A} & \longrightarrow & \left(q\text{-}\Omega_{S/A}\otimes^{\mathrm{L}}_{A[q],\psi^p} A[q]\right)^\wedge_{[p]_q} \\ \downarrow & \lrcorner & \downarrow \phi_{p/A[q]} \\ \mathrm{L}\eta_{[p]_q}\, q\text{-}\Omega_{S/A} & \longrightarrow & \mathrm{L}\eta_{[p]_q}\left(q\text{-}\Omega_{S/A}\right)^\wedge_p \end{array}$$

The filtration on $\mathrm{L}\eta_{[1]_q}\simeq \mathrm{id}$ is trivial. But the trivial filtration on $(q\text{-}\Omega_{S/A}\otimes^{\mathrm{L}}_{A[q],\psi^p} A[q])^\wedge_{[p]_q}$ will not be compatible with the natural filtration on $\mathrm{L}\eta_{[p]_q}(q\text{-}\Omega_{S/A})^\wedge_p$, so gluing fails.

To make the gluing work, we should instead equip $(q\text{-}\Omega_{S/A}\otimes^{\mathrm{L}}_{A[q],\psi^p} A[q])^\wedge_{[p]_q}$ with a global version of the Nygaard filtration. But such a global Nygaard filtration likely doesn't exist. To see this, let's attempt to construct it via an arithmetic fracture square. On the p-completion $(q\text{-}\Omega_{S/A}\otimes^{\mathrm{L}}_{A[q],\psi^p} A[q])^\wedge_{(p,[p]_q)}$, we put the usual Nygaard filtration. In view of Lemma 3.29, on the rationalisation we should put the combined Hodge and $[p]_q$-adic filtration. But then on the ℓ-completion $(q\text{-}\Omega_{S/A}\otimes^{\mathrm{L}}_{A[q],\psi^p} A[q])^\wedge_{(\ell,[p]_q)}$ for any prime $\ell\neq p$, we would need to put a filtration that becomes the combined Hodge and $[p]_q$-adic filtration after $(-)[1/\ell]^\wedge_{[p]_q}$.

It is entirely unclear (at least to the author) how to construct such a filtration, unless we're already given a q-Hodge filtration $\operatorname{fil}^\star_{q\text{-Hdg}} q\text{-dR}_{S/A}$. This explains the need for the additional datum of $\operatorname{fil}^\star_{q\text{-Hdg}} q\text{-dR}_{S/A}$.

§3.8. Habiro descent in derived commutative algebras

Raksit [Rak21] has introduced ∞-categories of *derived commutative algebras*, along with filtered, graded, and differential-graded variants. The filtrations $\operatorname{fil}^\star_{\mathcal{H}\mathrm{dg}_m} q\text{-}\mathbb{W}_m\mathrm{dR}_{R/A}$ admit canonical filtered derived commutative $A[q]/(q^m-1)$-algebra structures (if R is a polynomial A-algebra, these structures can be constructed on the level of complexes, then one can pass to animations) and the $\mathbb{E}_\infty$-structure on the derived q-de Rham complex $q\text{-}\mathrm{dR}_{R/A}$ can be canonically enhanced to a derived commutative $A[q]$-algebra structure (see A.13).

In this subsection, we sketch how Theorem 3.11 can be made compatible with these derived commutative structures. As a warm-up, let us instead consider $\mathbb{E}_n$-monoidal structures for some $0 \leqslant n \leqslant \infty$.

3.50. $\mathbb{E}_n$-monoidal upgrade. — Let $(R, \operatorname{fil}^\star_{q\text{-Hdg}} q\text{-}\mathrm{dR}_{R/A}) \in \mathrm{AniAlg}_A^{q\text{-Hdg}}$. Suppose that the q-Hodge filtration $\operatorname{fil}^\star_{q\text{-Hdg}} q\text{-}\mathrm{dR}_{R/A}$ can be equipped with the structure of an $\mathbb{E}_n$-algebra in filtered $(q-1)^\star A[q]$-modules, compatible with the $\mathbb{E}_\infty$-$A[q]$-algebra structure on $q\text{-}\mathrm{dR}_{R/A}$. Suppose furthermore that the data from Definition 3.2(a)–(c_p) can be made compatible with this $\mathbb{E}_n$-structure. Then $(R, \operatorname{fil}^\star_{q\text{-Hdg}} q\text{-}\mathrm{dR}_{R/A})$ becomes an $\mathbb{E}_n$-algebra in $\mathrm{AniAlg}_A^{q\text{-Hdg}}$.

By the symmetric monoidality statement in Theorem 3.11(a), we can conclude that the Habiro–Hodge complex $q\text{-}\mathcal{H}\mathrm{dg}_{R/A}$ becomes an $\mathbb{E}_n$-algebra in $\widehat{\mathcal{D}}_{\mathcal{H}}(A[q])$. Similarly, the lax symmetric monoidality statements in Theorem 3.11(b) show that $\operatorname{fil}_\star^{q\text{-}\mathbb{W}_m\Omega}(q\text{-}\mathcal{H}\mathrm{dg}_{R/A}/(q^m-1))$ becomes a filtered $\mathbb{E}_n$-algebra and the identification of its associated graded

$$\operatorname{gr}_*^{q\text{-}\mathbb{W}_m\Omega}\bigl(q\text{-}\mathcal{H}\mathrm{dg}_{R/A}/(q^m-1)\bigr) \simeq \Sigma^{-*} q\text{-}\mathbb{W}_m\mathrm{dR}^*_{R/A} \simeq \operatorname{gr}^*_{\mathcal{H}\mathrm{dg}_m} q\text{-}\mathbb{W}_m\mathrm{dR}_{R/A}$$

becomes a graded $\mathbb{E}_n$-monoidal equivalence.

3.51. Derived commutative upgrade I. — Similar to 3.50, suppose that $\operatorname{fil}^\star_{q\text{-Hdg}} q\text{-}\mathrm{dR}_{R/A}$ can be equipped with the structure of a *filtered derived commutative algebra* over $(q-1)^\star A[q]$, that is, an element in the slice ∞-category $(\operatorname{Fil}\mathrm{DAlg}_{A[q]})_{(q-1)^\star A[q]/}$, where $\operatorname{Fil}\mathrm{DAlg}_{A[q]}$ is Raksit's ∞-category of filtered derived commutative $A[q]$-algebras [Rak21, Definition 4.3.4]. Suppose furthermore that this derived commutative structure is compatible with the derived commutative $A[q]$-algebra structure on $q\text{-}\mathrm{dR}_{R/A}$ (see A.13) and that the data from Definition 3.2(a)–(c_p) can be made compatible with the filtered derived commutative algebra structures everywhere.

For example, this can be done in the special cases from Example 3.12 above and Construction 4.28 below. In the former case, we'll verify this in Remark 9.14, in the latter case see Remark 4.31.

3.52. Lemma. — *In the situation of 3.51, $q\text{-}\mathcal{H}\mathrm{dg}_{R/A}$ admits a canonical derived commutative $A[q]$-algebra structure. Furthermore, for all $m \in \mathbb{N}$, $\operatorname{fil}_\star^{q\text{-}\mathbb{W}_m\Omega}(q\text{-}\mathcal{H}\mathrm{dg}_{R/A}/(q^m-1))$ admits a filtered derived commutative $A[q]/(q^m-1)$-algebra structure, compatible with the derived commutative $A[q]/(q^m-1)$-algebra structure on $q\text{-}\mathcal{H}\mathrm{dg}_{R/A}/(q^m-1)$, and the equivalence*

$$\operatorname{gr}_*^{q\text{-}\mathbb{W}_m\Omega}\bigl(q\text{-}\mathcal{H}\mathrm{dg}_{R/A}/(q^m-1)\bigr) \simeq \Sigma^{-*} q\text{-}\mathbb{W}_m\mathrm{dR}^*_{R/A} \simeq \operatorname{gr}^*_{\mathcal{H}\mathrm{dg}_m} q\text{-}\mathbb{W}_m\mathrm{dR}_{R/A}$$

from Theorem 3.11(b) is an equivalence of graded derived commutative $A[q]/(q^m-1)$-algebras.

Proof sketch. First note that our results about the Nygaard filtration, specifically Proposition 3.22 and Lemma 3.29, also hold true as equivalences of filtered derived commutative algebras, since the proofs work in this setting as well. By tracing through 3.32–3.41, we now see that each $\mathrm{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_m} q\text{-}\mathrm{dR}^{(m)}_{R/A}$ acquires a filtered derived commutative algebra structure over $(q^m-1)^\star A[q]$, and that the transition maps in 3.41 are compatible with these structures.

The construction from 3.42 produces a canonical derived commutative algebra structure on $\mathcal{H}_{R/A,m}$, because we can view the construction as a filtered localisation followed by restriction to filtered degree 0; compare 3.8. It is then clear from 3.45 that $q\text{-}\mathcal{H}\mathrm{dg}_{R/A}$ acquires a derived commutative $A[q]$-algebra structure. Moreover, since the filtration $\mathrm{fil}^{q\text{-}\mathbb{W}_m\Omega}_\star(q\text{-}\mathcal{H}\mathrm{dg}_{R/A}/(q^m-1))$ and the identification of its associated graded were constructed in a completely way (see the proofs of Lemmas 3.9 and 3.10), they will also work on the level of derived commutative algebras. The only input this needs is that Proposition 3.39 holds as an equivalence of filtered derived commutative $A[q]/(q^m-1)$-algebras, which is again apparent from the constructions. □

But there's one more piece of structure.

3.53. Derived commutative upgrade II. — Since $q\text{-}\mathbb{W}_m\Omega^*_{-/A}$ is a functor with values in commutative differential-graded $A[q]/(q^m-1)$-algebras, we see that its animation $\Sigma^{-*}q\text{-}\mathbb{W}_m\mathrm{dR}^*_{-/A} \simeq \mathrm{gr}^*_{\mathcal{H}\mathrm{dg}_m} q\text{-}\mathbb{W}_m\mathrm{dR}_{-/A}$ upgrades to a functor with values in Raksit's ∞-category $\mathrm{DG}_-\mathrm{DAlg}_{A[q]/(q^m-1)}$ of derived differential-graded $A[q]/(q^m-1)$-algebras [Rak21, Definition 5.1.10].

By transfer of structure, the associated graded $\mathrm{gr}^{q\text{-}\mathbb{W}_m\Omega}_*(q\text{-}\mathcal{H}\mathrm{dg}_{R/A}/(q^m-1))$ becomes an element in $\mathrm{DG}_-\mathrm{DAlg}_{A[q]/(q^m-1)}$ as well. Via the following corollary, we can figure out what the differentials are, at least in the case where R is smooth over A.

3.54. Corollary. — *Let $(S, \mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}} q\text{-}\mathrm{dR}_{S/A}) \in \mathrm{AniAlg}^{q\text{-}\mathrm{Hdg}}_A$ be an object such that S is smooth over A. Then:*

(*a*) *$\mathrm{fil}^{q\text{-}\mathbb{W}_m\Omega}_\star(q\text{-}\mathcal{H}\mathrm{dg}_{S/A}/(q^m-1))$ is the Whitehead filtration $\tau_{\geqslant\star}(q\text{-}\mathcal{H}\mathrm{dg}_{S/A}/(q^m-1))$.*

(*b*) *The equivalence from Theorem 3.11(b) becomes an isomorphism of graded $A[q]/(q^m-1)$-modules*
$$\mathrm{H}^*\big(q\text{-}\mathcal{H}\mathrm{dg}_{S/A}/(q^m-1)\big) \cong q\text{-}\mathbb{W}_m\Omega^*_{S/A}$$
(and an isomorphism of graded $A[q]/(q^m-1)$-algebras as soon as $(S, \mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}} q\text{-}\mathrm{dR}_{S/A})$ is at least an $\mathbb{E}_1$-algebra in $\mathrm{AniAlg}^{q\text{-}\mathrm{Hdg}}_A$).

(*c*) *Under the isomorphism from (b), the canonical differential on $q\text{-}\mathbb{W}_m\Omega^*_{S/A}$ corresponds to the Bockstein differential on $\mathrm{H}^*(q\text{-}\mathcal{H}\mathrm{dg}_{S/A}/(q^m-1))$.*

Proof. We've seen in Corollary 3.31 that $q\text{-}\mathbb{W}_m\Omega^n_{S/A} \simeq q\text{-}\mathbb{W}_m\mathrm{dR}^n_{S/A}$ for all n. It follows that each graded piece $\mathrm{gr}^{q\text{-}\mathbb{W}_m\Omega}_n(q\text{-}\mathcal{H}\mathrm{dg}_{S/A}/(q^m-1))$ is concentrated in cohomological degree n. Since $\mathrm{fil}^{q\text{-}\mathbb{W}_m\Omega}_\star(q\text{-}\mathcal{H}\mathrm{dg}_{S/A}/(q^m-1))$ is bounded below and thus complete, it has to be the Whitehead filtration. This shows (*a*) as well as the graded $A[q]/(q^m-1)$-module isomorphism from (*b*). The isomorphism as graded $A[q]/(q^m-1)$-algebras follows from 3.50.

It remains to show (*c*). Similar to the proof of Lemma 3.9, let us identify the filtered ring $(q^m-1)^\star A[q]$ with the graded ring $A[q,\beta,t_m]/(\beta t_m-(q^m-1))$, where $|q|=0$, $|\beta|=1$, and $|t_m|=-1$.(3.8) The filtered structure comes from the $A[t_m]$-module structure. In particular, modding out t_m is the same as passing to the associated graded. Let us also regard the

(3.8) In 8.33, we'll recognise $\mathbb{Z}[q,\beta,t_m]/(\beta t_m-(q^m-1)) \cong \pi_*(\mathrm{ku}^{C_m})$.

filtrations $\mathrm{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_m} q\text{-}\mathrm{dR}^{(m)}_{S/A}$ and $\mathrm{fil}^\star_{\mathcal{H}\mathrm{dg}_m} q\text{-}\mathbb{W}_m\mathrm{dR}_{S/A}$ as graded $A[q,\beta,t_m]/(\beta t_m-(q^m-1))$-modules $\mathrm{fil}^*_{q\text{-}\mathcal{H}\mathrm{dg}}$ and $\mathrm{fil}^*_{\mathcal{H}\mathrm{dg}}$. Finally, let us denote by $\beta^{-\star}A[\beta]$ the ascendingly filtered graded ring

$$\beta^{-\star}A[\beta]\coloneqq\Bigl(\dotsb\xrightarrow{\beta}A[\beta](1)\xrightarrow{\beta}A[\beta](0)\xrightarrow{\beta}A[\beta](-1)\xrightarrow{\beta}\dotsb\Bigr)\,,$$

As explained in the proof of Lemma 3.10, the filtration $\mathrm{fil}^{q\text{-}\mathbb{W}_m\Omega}_\star(q\text{-}\mathcal{H}\mathrm{dg}_{S/A}/(q^m-1))$ can be written as follows:

$$\mathrm{fil}^{q\text{-}\mathbb{W}_m\Omega}_\star\bigl(q\text{-}\mathcal{H}\mathrm{dg}_{S/A}/(q^m-1)\bigr)\simeq\bigl(\mathrm{fil}^*_{q\text{-}\mathcal{H}\mathrm{dg}}/t_m\otimes^{\mathrm{L}}_{A[\beta]}\beta^{-\star}A[\beta]\bigr)_0$$

(note that $*$ on the right-hand side refers to the graded degree whereas $\star$ corresponds to the filtration degree). Now consider the Bockstein cofibre sequence for $\mathrm{fil}^*_{q\text{-}\mathcal{H}\mathrm{dg}}/t_m$. It fits into a commutative diagram of graded $A[q,\beta,t_m]/(\beta t_m-(q^m-1))$-modules

$$\begin{array}{ccccc}\mathrm{fil}^*_{q\text{-}\mathcal{H}\mathrm{dg}}(-1)/t_m & \xrightarrow{t_m} & \mathrm{fil}^*_{q\text{-}\mathcal{H}\mathrm{dg}}/t_m^2 & \longrightarrow & \mathrm{fil}^*_{q\text{-}\mathcal{H}\mathrm{dg}}/t_m\\ {\scriptstyle\beta}\big\uparrow & \nearrow{\scriptstyle (q^m-1)} & & & \\ \mathrm{fil}^*_{q\text{-}\mathcal{H}\mathrm{dg}}/t_m & & & & \end{array}$$

If we apply $(-\otimes^{\mathrm{L}}_{A[\beta]}A[\beta^{\pm1}])_0$ to this diagram, the left vertical arrow becomes an equivalence and so the cofibre sequence from the top row will become equivalent to the Bockstein cofibre sequence

$$q\text{-}\mathcal{H}\mathrm{dg}_{R/A}/(q^m-1)\xrightarrow{(q^m-1)}q\text{-}\mathcal{H}\mathrm{dg}_{R/A}/(q^m-1)^2\longrightarrow q\text{-}\mathcal{H}\mathrm{dg}_{R/A}/(q^m-1)\,.$$

If we apply $(-\otimes^{\mathrm{L}}_{A[\beta]}\beta^{-\star}A[\beta])_0$ to the top row, we get a filtration on this cofibre sequence. By (*a*), this filtration will be of the form

$$\tau^{\leqslant\star+1}\bigl(q\text{-}\mathcal{H}\mathrm{dg}_{S/A}/(q^m-1)\bigr)\longrightarrow\bigl(\mathrm{fil}_{q\text{-}\mathcal{H}\mathrm{dg}}/t_m^2\otimes^{\mathrm{L}}_{A[\beta]}\beta^{-\star}A[\beta]\bigr)_0\longrightarrow\tau^{\leqslant\star}\bigl(q\text{-}\mathcal{H}\mathrm{dg}_{S/A}/(q^m-1)\bigr)$$

where $\bigl(\mathrm{fil}_{q\text{-}\mathcal{H}\mathrm{dg}}/t_m^2\otimes^{\mathrm{L}}_{A[\beta]}\beta^{-\star}A[\beta]\bigr)_0$ is an ascending filtration on $q\text{-}\mathcal{H}\mathrm{dg}_{S/A}/(q^m-1)^2$ that lies between $\tau^{\leqslant\star}$ and $\tau^{\leqslant\star+1}$. After passing to associated gradeds, the connecting morphism will then necessarily be the usual Bockstein differential

$$\mathrm{H}^*\bigl(q\text{-}\mathcal{H}\mathrm{dg}_{S/A}/(q^m-1)\bigr)\longrightarrow\mathrm{H}^{*+1}\bigl(q\text{-}\mathcal{H}\mathrm{dg}_{S/A}/(q^m-1)\bigr)\,.$$

On the other hand, the associated graded of $\beta^{-\star}A[\beta]$ is given by $\bigoplus_{i\in\mathbb{Z}}A(-i)$. If we apply $(-\otimes^{\mathrm{L}}_{A[\beta]}\bigoplus_{i\in\mathbb{Z}}A(-i))_0$ to the top row of the diagram, we get the Bockstein cofibre sequence

$$\mathrm{fil}^*_{\mathcal{H}\mathrm{dg}}(-1)/t_m\xrightarrow{t_m}\mathrm{fil}^*_{\mathcal{H}\mathrm{dg}}/t_m^2\longrightarrow\mathrm{fil}^*_{\mathcal{H}\mathrm{dg}}/t_m\,,$$

because $\mathrm{fil}^*_{\mathcal{H}\mathrm{dg}}\simeq\mathrm{fil}^*_{q\text{-}\mathcal{H}\mathrm{dg}}/\beta$ by Proposition 3.39. Since $\mathrm{fil}^\star_{\mathcal{H}\mathrm{dg}}\,q\text{-}\mathbb{W}_m\mathrm{dR}_{R/A}$ is the stupid filtration on the complex $q\text{-}\mathbb{W}_m\Omega^*_{R/A}$, the differential of $q\text{-}\mathbb{W}_m\Omega^*_{R/A}$ agrees with the connecting morphism for the Bockstein cofibre sequence of $\mathrm{fil}^*_{\mathcal{H}\mathrm{dg}}/t_m$. This finishes the proof of (*c*). $\square$

§4. Functorial q-Hodge filtrations

Fix a perfectly covered Λ-ring A. We've seen in Lemma 3.3 that it's impossible to get a functorial q-Hodge filtration for all animated A-algebras, or even just for smooth A-algebras. Despite this general no-go result, we'll see in this section that functorial q-Hodge filtrations exist for fairly large full subcategories of AniAlg_A.

A further ample source of examples comes from homotopy theory and will be discussed at length in Part II.

§4.1. Functorial q-Hodge filtrations away from small primes

In this subsection, we'll give an elementary construction of a functorial q-Hodge filtration on certain smooth A-algebras. In the introduction (see 1.18), we've already explained the idea in the case of relative dimension $\leqslant 1$. The general case follows the same simple idea.

4.1. Canonical q-Hodge filtrations I. — Let S be smooth of arbitrary dimension over A and let n be a positive integer such that all primes $p \leqslant n$ are invertible in S. This assumption ensures that the canonical map $q\text{-}\Omega_{S/A} \to \Omega_{S/A}$ factors through an $\mathbb{E}_\infty$-$A⟦q-1⟧$-algebra map

$$q\text{-}\Omega_{S/A} \longrightarrow \Omega_{S/A}⟦q-1⟧/(q-1)^n\,.$$

Indeed, by construction of the global q-de Rham complex (see Construction A.12), it's enough to check this after completion at any prime p. In general, $(q\text{-}\Omega_{S/A})_p^\wedge \to (\Omega_{S/A})_p^\wedge$ factors through $(q\text{-}\Omega_{S/A})_p^\wedge \to (\Omega_{S/A})_p^\wedge⟦q-1⟧/(q-1)^{p-1}$ by Lemma A.6. For primes $p > n$, this does what we want. For $p \leqslant n$, our assumption on S ensures that $(q\text{-}\Omega_{S/A})_p^\wedge$ vanishes, so this case is fine too.

Let us now equip $\Omega_{S/A}⟦q-1⟧/(q-1)^n$ with the following filtration: We first define $\mathrm{fil}^\star_{(\mathrm{Hdg},q-1)}\Omega_{S/A}⟦q-1⟧ \coloneqq (\mathrm{fil}^\star_{\mathrm{Hdg}}\Omega_{S/A} \otimes^{\mathrm{L}}_{\mathbb{Z}} (q-1)^\star\mathbb{Z}⟦q-1⟧)^\wedge_{(q-1)}$ to be the combined Hodge and $(q-1)$-adic filtration, as usual. We then let $\mathrm{fil}^\star_{(\mathrm{Hdg},q-1)}\Omega_{S/A}⟦q-1⟧/(q-1)^n$ denotes its reduction modulo $(q-1)^n$, which we regard as an element in filtration degree n.(4.1) We may then form the following pullback of filtered objects in degrees $\leqslant n$:

$$\begin{array}{ccc}\mathrm{fil}^{\star\leqslant n}_{q\text{-}\mathrm{Hdg},n}\,q\text{-}\Omega_{S/A} & \longrightarrow & q\text{-}\Omega_{S/A}\\ \downarrow & \lrcorner & \downarrow\\ \mathrm{fil}^{\star\leqslant n}_{(\mathrm{Hdg},q-1)}\Omega_{S/A}⟦q-1⟧/(q-1)^n & \longrightarrow & \Omega_{S/A}⟦q-1⟧/(q-1)^n\end{array}$$

Here $\mathrm{fil}^{\star\leqslant n}_{(\mathrm{Hdg},q-1)}\Omega_{S/A}⟦q-1⟧/(q-1)^n$ denotes the restriction of the to degrees $\star \leqslant n$; more precisely, we apply the truncation functor τ_n^* from Lemma 4.2 below.

We then wish to extend $\mathrm{fil}^{\star\leqslant n}_{q\text{-}\mathrm{Hdg},n}\,q\text{-}\Omega_{S/A}$ to degrees $\star \geqslant n+1$. Intuitively, this should be done via the $(q-1)$-adic filtration $(q-1)^{\star-n}\,\mathrm{fil}^n_{q\text{-}\mathrm{Hdg},n}\,q\text{-}\Omega_{S/A}$ as in 1.18. To do this formally and make the resulting filtered $(q-1)^\star A⟦q-1⟧$-module structure apparent, we need to show a technical lemma.

4.2. Lemma. — *Let $\mathrm{Fil}^{\geqslant 0}\mathcal{D}(\mathbb{Z})$ denote the full sub-∞-categories of filtered objects that are constant in filtration degrees $\star \leqslant 0$. Let $\mathrm{Fil}^{[0,n]}\mathcal{D}(\mathbb{Z}) \subseteq \mathrm{Fil}^{\geqslant 0}\mathcal{D}(\mathbb{Z})$ denote the full sub-∞-category of filtered objects that also vanish in filtration degree $\star \geqslant n+1$.*

(4.1) Said differently, we wish to equip $\mathbb{Z}⟦q-1⟧/(q-1)^n$ with the finite filtration given by $(q-1)^i\mathbb{Z}⟦q-1⟧/(q-1)^n$ in degree i. This is *not* the $(q-1)$-adic filtration in our sense, since the latter would be $\mathbb{Z}⟦q-1⟧/(q-1)^n$ in every degree, with transition maps given by multiplication by $(q-1)$.

(*a*) *The inclusion* $\operatorname{Fil}^{[0,n]}\mathcal{D}(\mathbb{Z})\rightarrow\operatorname{Fil}^{\geqslant 0}\mathcal{D}(\mathbb{Z})$ *has a left adjoint* τ_n^**, which on objects is given by replacing all filtration degrees* $\star\geqslant n+1$ *by* 0*. Moreover, if* $\operatorname{fil}^\star M$ *and* $\operatorname{fil}^\star N$ *are filtered* $\mathbb{Z}$*-modules, the canonical map*

$$\tau_n^*\Bigl(\operatorname{fil}^\star M\otimes_{\mathbb{Z}}^{\mathrm{L}}\tau_n^*(\operatorname{fil}^\star N)\Bigr)\xrightarrow{\simeq}\tau_n^*\Bigl(\operatorname{fil}^\star M\otimes_{\mathbb{Z}}^{\mathrm{L}}\operatorname{fil}^\star N\Bigr)$$

is an equivalence. Consequently there's a canonical way to equip $\operatorname{Fil}^{[0,n]}\mathcal{D}(\mathbb{Z})$ *and* $\tau_n^*\colon\operatorname{Fil}^{\geqslant 0}\mathcal{D}(\mathbb{Z})\rightarrow\operatorname{Fil}^{[0,n]}\mathcal{D}(\mathbb{Z})$ *with symmetric monoidal structures.*

(*b*) *For any filtered* $\mathbb{E}_\infty$*-algebra* $T\in\operatorname{CAlg}(\operatorname{Fil}^{\geqslant 0}\mathcal{D}(\mathbb{Z}))$*, the induced symmetric monoidal functor*

$$\tau_n^*\colon\operatorname{Mod}_T\bigl(\operatorname{Fil}^{\geqslant 0}\mathcal{D}(\mathbb{Z})\bigr)\longrightarrow\operatorname{Mod}_{\tau_n^*T}\bigl(\operatorname{Fil}^{[0,n]}\mathcal{D}(\mathbb{Z})\bigr)$$

admits an oplax symmetric monoidal left adjoint

$$\tau_{n,!}^T\colon\operatorname{Mod}_{\tau_n^*T}\bigl(\operatorname{Fil}^{[0,n]}\mathcal{D}(\mathbb{Z})\bigr)\longrightarrow\operatorname{Mod}_T\bigl(\operatorname{Fil}^{\geqslant 0}\mathcal{D}(\mathbb{Z})\bigr)$$

(*if* T *is clear from the context, we'll often just write* $\tau_{n,!}$).

(*c*) *Let* $T_1\rightarrow T_2$ *be any map in* $\operatorname{CAlg}(\operatorname{Fil}^{\geqslant 0}\mathcal{D}(\mathbb{Z}))$ *and let* $\operatorname{fil}^\star M\in\operatorname{Mod}_{\tau_n^*T}(\operatorname{Fil}^{[0,n]}\mathcal{D}(\mathbb{Z}))$*. Then there's a natural equivalence*

$$\tau_{n,!}^{T_2}\bigl(\operatorname{fil}^\star M\otimes_{\tau_n^*T_1}\tau_n^*T_2\bigr)\xrightarrow{\simeq}\tau_{n,!}^{T_1}(\operatorname{fil}^\star M)\otimes_{T_1}T_2\,.$$

Proof. We start with (*a*). It's straightforward to see that τ_n^* exists and is given as claimed. To show the equivalence, since τ_n^* and the inclusion preserve colimits, it will be enough to check the case where $\operatorname{fil}^\star M\simeq\mathbb{Z}(i)$ and $\operatorname{fil}^\star N\simeq\mathbb{Z}(j)$, where $i,j\geqslant 0$. If $j\leqslant n$, then $\tau_n^*\mathbb{Z}(j)\rightarrow\mathbb{Z}(j)$ is an equivalence and the claim is clear. If $j\geqslant n+1$, then we must check that $\tau_n^*\mathbb{Z}(i+j)\rightarrow\tau_n^*\mathbb{Z}(i+n)$ is an equivalence. This is clear as both sides are just $\mathbb{Z}(n)$. The final claim in (*a*) is general abstract nonsense about symmetric monoidal structures on localisations (see [L-HA, Proposition 2.2.1.9] for example).

Let us now prove (*b*) and (*c*) simultaneously. For any map $T_1\rightarrow T_2$ in $\operatorname{CAlg}(\operatorname{Fil}^{\geqslant 0}\mathcal{D}(\mathbb{Z}_p))$, the diagram

$$\begin{array}{ccc}\operatorname{Mod}_{T_2}\bigl(\operatorname{Fil}^{\geqslant 0}\mathcal{D}(\mathbb{Z})\bigr) & \xrightarrow{\tau_n^*} & \operatorname{Mod}_{\tau_n^*T_2}\bigl(\operatorname{Fil}^{[0,n]}\mathcal{D}(\mathbb{Z})\bigr)\\ \downarrow & & \downarrow\\ \operatorname{Mod}_{T_1}\bigl(\operatorname{Fil}^{\geqslant 0}\mathcal{D}(\mathbb{Z})\bigr) & \xrightarrow{\tau_n^*} & \operatorname{Mod}_{\tau_n^*T_1}\bigl(\operatorname{Fil}^{[0,n]}\mathcal{D}(\mathbb{Z})\bigr)\end{array}$$

commutes. In the special case where $T_1=\mathbb{Z}$ is the filtered tensor unit and $T_2=T$, this allows us to show that $\tau_n^*\colon\operatorname{Mod}_T(\operatorname{Fil}^{\geqslant 0}\mathcal{D}(\mathbb{Z}))\rightarrow\operatorname{Mod}_{\tau^*T}(\operatorname{Fil}^{[0,n]}\mathcal{D}(\mathbb{Z}))$ preserves all limits and colimits. Therefore the claimed left adjoint $\tau_{n,!}^T$ exists by Lurie's adjoint functor theorem. By abstract nonsense, $\tau_{n,!}^T$ will automatically acquire an oplax symmetric monoidal structure. This shows (*b*). By passing to left adjoints in the diagram above, we immediately obtain (*c*). ☐

4.3. Canonical q-Hodge filtrations II. — We resume the discussion from 4.1. As we know now, the pullback defining $\operatorname{fil}_{q\text{-Hdg},n}^{\star\leqslant n}q\text{-}\Omega_{S/A}$ can be taken in $\operatorname{Mod}_{\tau_n^*((q-1)^\star A[\![q-1]\!])}(\operatorname{Fil}^{[0,n]}\mathcal{D}(\mathbb{Z}))$. Applying the functor $\tau_{n,!}$ from Lemma 4.2(*b*), we obtain a filtered $(q-1)^\star A[\![q-1]\!]$-module

$$\operatorname{fil}_{q\text{-Hdg},n}^\star q\text{-}\Omega_{S/A}\coloneqq\tau_{n,!}\Bigl(\operatorname{fil}_{q\text{-Hdg},n}^{\star\leqslant n}q\text{-}\Omega_{S/A}\Bigr)_{(q-1)}^\wedge\,.$$

We can also take the pullback along $q\text{-}\mathrm{dR}_{S/A}\rightarrow q\text{-}\Omega_{S/A}$ to construct $\operatorname{fil}_{q\text{-Hdg},n}^\star q\text{-}\mathrm{dR}_{S/A}$ (in order to be in line with Definition 3.2).

4.4. Remark. — If S satisfies the assumptions of 4.3 and is additionally equipped with an étale framing $\square\colon A[x_1,\dots,x_n]\to S$, then there exists an equivalence of filtered $(q-1)^\star A[\![q-1]\!]$-modules

$$\mathrm{fil}^\star_{q\text{-Hdg},n}\, q\text{-}\Omega_{S/A}\xrightarrow{\ \simeq\ }\mathrm{fil}^\star_{q\text{-Hdg},\square}\, q\text{-}\Omega^*_{S/A,\square}$$

between the q-Hodge filtration from 4.3 and the one from Example 3.12. Indeed, we observe $\mathrm{fil}^{\star\leqslant n}_{q\text{-Hdg},n}\, q\text{-}\Omega_{S/A}\simeq\tau_n^*(\mathrm{fil}^\star_{q\text{-Hdg},\square}\, q\text{-}\Omega^*_{S/A,\square})$, since both sides fit into the same pullback diagram by construction. Since $\tau_{n,!}$ was defined as a the left adjoint of τ_n^*, we obtain the map above. To see that it is an equivalence, we may reduce modulo $(q-1)$, where we get the identity on $\mathrm{fil}^\star_{\mathrm{Hdg}}\,\Omega_{S/A}$ by inspection and Lemma 4.6 below.

4.5. Remark. — Here's another way to do the construction from 4.1 and 4.3. Fix a prime p. Recall that Bhatt–Lurie [BL22a, Construction 4.8.3] have defined a *$\tilde{p}$-de Rham complex* $\tilde{p}\text{-}\Omega_{\widehat{S}_p/\widehat{A}_p}$. Explicitly, it is the homotopy-fixed points of the action of μ_{p-1} on $(q\text{-}\Omega_{S/A})^\wedge_p$, where we let $\mu_{p-1}\subseteq\mathbb{Z}_p^\times$ act via the Adams operations from A.20.

We can then define $\mathrm{fil}^{\star\leqslant n}_{\tilde{p}\text{-Hdg},n}\,\tilde{p}\text{-}\Omega_{\widehat{S}_p/\widehat{A}_p}$ as the pullback of the Hodge filtration along the canonical map $\tilde{p}\text{-}\Omega_{\widehat{S}_p/\widehat{A}_p}\to(\Omega_{S/A})^\wedge_p$ (no combined Hodge and $(q-1)$-adic filtration is needed here), extend via $\tau_{n,!}$, and then finally base change to $(q-1)^\star\mathbb{Z}_p[\![q-1]\!]$ to define a p-completed q-Hodge filtration $\mathrm{fil}^\star_{q\text{-Hdg},n}(q\text{-}\Omega_{S/A})^\wedge_p$.

These filtrations for all p can be glued with the combined Hodge and $(q-1)$-adic filtration on $(\Omega_{S/A}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q})[\![q-1]\!]$ to get the same filtration $\mathrm{fil}^\star_{q\text{-Hdg},n}\, q\text{-}\Omega_{S/A}$ as in 4.3. We prefer the construction in 4.3, since spelling out the gluing argument is a bit of a pain.

4.6. Lemma. — *With notation as in 4.1, assume additionally that* $\dim(S/A)\leqslant n$. *Then* $\mathrm{fil}^\star_{q\text{-Hdg},n}\, q\text{-dR}_{S/A}$ *can naturally be equipped with the structure of a q-Hodge filtration as in Definition 3.2.*

Proof. In the following, we'll regard $(q-1)$ as sitting in filtration degree 1, as per Convention 3.1. We first compute

$$\begin{aligned}\mathrm{fil}^\star_{q\text{-Hdg},n}\, q\text{-}\Omega_{S/A}/(q-1)&\simeq\tau^{\mathbb{Z}}_{n,!}\Big(\mathrm{fil}^{\star\leqslant n}_{q\text{-Hdg},n}\, q\text{-}\Omega_{S/A}\otimes^{\mathrm{L}}_{\tau_n^*((q-1)^\star\mathbb{Z}[\![q-1]\!])}\mathbb{Z}\Big)\\&\simeq\tau^{\mathbb{Z}}_{n,!}\tau_n^*(\mathrm{fil}^\star_{\mathrm{Hdg}}\,\Omega_{S/A})\\&\simeq\mathrm{fil}^\star_{\mathrm{Hdg}}\,\Omega_{S/A}\,.\end{aligned}$$

In the first equivalence we apply Lemma 4.2(c) to $(q-1)^\star\mathbb{Z}[\![q-1]\!]\to\mathbb{Z}$. The second equivalence follows by construction. To see the third equivalence, first observe that the Hodge filtration $\mathrm{fil}^\star_{\mathrm{Hdg}}\,\Omega_{S/A}$ is already contained in $\mathrm{Fil}^{[0,n]}\,\mathcal{D}(\mathbb{Z})$ because we assume $\dim(S/A)\leqslant n$. Since the right adjoint of $\tau_n^*\colon\mathrm{Fil}^{\geqslant 0}\,\mathcal{D}(\mathbb{Z})\to\mathrm{Fil}^{[0,n]}\,\mathcal{D}(\mathbb{Z})$ is fully faithful, so is the left adjoint $\tau^{\mathbb{Z}}_{n,!}$, which yields the third equivalence. Similarly,

$$\begin{aligned}\big(\mathrm{fil}^\star_{q\text{-Hdg},n}\, q\text{-}\Omega_{S/A}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q}\big)^\wedge_{(q-1)}&\simeq\tau_{n,!}\Big(\big(\mathrm{fil}^{\star\leqslant n}_{q\text{-Hdg},n}\, q\text{-}\Omega_{S/A}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q}\big)^\wedge_{(q-1)}\Big)\\&\simeq\tau_{n,!}\Big(\big(\mathrm{fil}^\star_{\mathrm{Hdg}}\,\Omega_{S/A}\otimes^{\mathrm{L}}_{\mathbb{Z}}\tau_n^*\big((q-1)^\star\mathbb{Q}[\![q-1]\!]\big)\big)^\wedge_{(q-1)}\Big)\\&\simeq\mathrm{fil}^\star_{(\mathrm{Hdg},q-1)}\big(\Omega_{S/A}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q}\big)[\![q-1]\!]\,.\end{aligned}$$

The first equivalence is Lemma 4.2(c) applied to $\mathbb{Z}\to\mathbb{Q}$. For the second equivalence, we apply $(-\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q})^\wedge_{(q-1)}$ to the pullback defining $\mathrm{fil}^\star_{q\text{-Hdg},n}\, q\text{-}\Omega_{S/A}$ in 4.3 and use the fact that

$(q\text{-}\Omega_{S/A} \otimes_{\mathbb{Z}}^{\mathrm{L}} \mathbb{Q})_{(q-1)}^{\wedge} \simeq (\Omega_{S/A} \otimes_{\mathbb{Z}}^{\mathrm{L}} \mathbb{Q})[\![q-1]\!]$. The third equivalence is Lemma 4.2(c) applied to $\mathbb{Z} \to (q-1)^{\star}\mathbb{Q}[\![q-1]\!]$.

In a completely analogus way, we obtain natural equivalences

$$\mathrm{fil}^{\star}_{q\text{-Hdg},n}\big(q\text{-}\Omega_{S/A}\big)_p^{\wedge}\big[\tfrac{1}{p}\big]_{(q-1)}^{\wedge} \xrightarrow{\simeq} \mathrm{fil}^{\star}_{(\mathrm{Hdg},q-1)}\big(\Omega_{S/A}\big)_p^{\wedge}\big[\tfrac{1}{p}\big][\![q-1]\!]$$

for all primes p. Via pullback along $q\text{-}\mathrm{dR}_{S/A} \to q\text{-}\Omega_{S/A}$, we obtain analogous equivalences for $\mathrm{fil}^{\star}_{q\text{-Hdg},n}\, q\text{-}\mathrm{dR}_{S/A}$. The required compatibilities from Definition 3.2 can all be induced from those for $q\text{-}\mathrm{dR}_{S/A}$, and so $\mathrm{fil}^{\star}_{q\text{-Hdg},n}\, q\text{-}\mathrm{dR}_{S/A}$ can indeed be equipped with the structure of a q-Hodge filtration. □

4.7. Lemma. — *With assumptions as in Lemma 4.6, $\mathrm{fil}^{\star}_{q\text{-Hdg},n}\, q\text{-}\Omega_{S/A}$ is automatically the completion of $\mathrm{fil}^{\star}_{q\text{-Hdg},n}\, q\text{-}\mathrm{dR}_{S/A}$.*

Proof. By Proposition 3.47, $q\text{-}\Omega_{S/A}$ is automatically the completion of $q\text{-}\mathrm{dR}_{S/A}$ at the filtration $\mathrm{fil}^{\star}_{q\text{-Hdg},n}\, q\text{-}\mathrm{dR}_{S/A}$. Since $\mathrm{fil}^{\star}_{q\text{-Hdg},n}\, q\text{-}\mathrm{dR}_{S/A}$ is defined as the pullback of $\mathrm{fil}^{\star}_{q\text{-Hdg},n}\, q\text{-}\Omega_{S/A}$ along $q\text{-}\mathrm{dR}_{S/A} \to q\text{-}\Omega_{S/A}$, the desired assertion follows. □

We will now make the construction from 4.3 functorial.

4.8. Functoriality across dimensions. — For all non-negative integers n and d let $\mathrm{Sm}^{\leqslant d}_{A[n!^{-1}]}$ be the category of all smooth A-algebras S of relative dimension $\dim(S/A) \leqslant d$ such that all primes $p \leqslant n$ are invertible in S. Then 4.3 and Lemma 4.6 provide us with a functor

$$\big(-, \mathrm{fil}^{\star}_{q\text{-Hdg},n}\, q\text{-}\mathrm{dR}_{-/A}\big)\colon \mathrm{Sm}^{\leqslant n}_{A[n!^{-1}]} \longrightarrow \mathrm{AniAlg}_A^{q\text{-Hdg}}\,.$$

We let $\mathrm{Sm}^{\leqslant n}_{A[\dim!^{-1}]} \subseteq \mathrm{Sm}_A$ be the full subcategory spanned by $\bigcup_{d \leqslant n} \mathrm{Sm}^{\leqslant d}_{A[d!^{-1}]}$ and we put

$$\mathrm{Sm}_{A[\dim!^{-1}]} \coloneqq \bigcup_{n \geqslant 0} \mathrm{Sm}^{\leqslant n}_{A[\dim!^{-1}]}\,.$$

Our goal is to show that the functors above for varying n combine into a single functor defined on all of $\mathrm{Sm}_{A[\dim!^{-1}]}$. This will be achieved by the technical Lemmas 4.9 and 4.10 below.

4.9. Lemma. — *For all $n \geqslant 0$, the following diagram is a pushout of ∞-categories:*

$$\begin{array}{ccc} \mathrm{Sm}^{\leqslant n}_{A[(n+1)!^{-1}]} & \longrightarrow & \mathrm{Sm}^{\leqslant n+1}_{A[(n+1)!^{-1}]} \\ \downarrow & \ulcorner & \downarrow \\ \mathrm{Sm}^{\leqslant n}_{A[\dim!^{-1}]} & \longrightarrow & \mathrm{Sm}^{\leqslant n+1}_{A[\dim!^{-1}]} \end{array}$$

Proof. Let $\mathcal{P}$ denote the pushout. Since the diagram above commutes, we get a functor $\mathcal{P} \to \mathrm{Sm}^{\leqslant n+1}_{A[\dim!^{-1}]}$. This functor is clearly essentially surjective. To show that it is fully faithful, we must show that

$$\mathrm{Hom}_{\mathcal{P}}(S_1, S_2) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{Sm}^{\leqslant n+1}_{A[\dim!^{-1}]}}(S_1, S_2)$$

is an equivalence for all $S_1, S_2 \in \mathcal{P}$. We may assume without loss of generality that S_1 and S_2 are the images of objects in $\mathrm{Sm}^{\leqslant n}_{A[\dim!^{-1}]}$ or $\mathrm{Sm}^{\leqslant n+1}_{A[(n+1)!^{-1}]}$.

By an observation of Maxime Ramzi [Ram], fully faithful functors are preserved under pushouts. Since both legs of our pushout are fully faithful, the claimed equivalence is clear if S_1 and S_2 come from the same cofactor. It remains to deal with the following two cases.

Case 1: $S_1 \in \mathrm{Sm}_{A[\dim!^{-1}]}^{\leqslant n}$ and $S_2 \in \mathrm{Sm}_{A[(n+1)!^{-1}]}^{\leqslant n+1}$. Observe that the fully faithful functor

$$\mathrm{Sm}_{A[(n+1)!^{-1}]}^{\leqslant n} \longrightarrow \mathrm{Sm}_{A[\dim!^{-1}]}^{\leqslant n}$$

has a left adjoint given by localisation at $(n+1)!$. It follows formally that $\mathrm{Sm}_{A[\dim!^{-1}]}^{\leqslant n+1} \rightarrow \mathcal{P}$ also has a left adjoint and that the diagram of left adjoints is still a pushout (and in particular commutative). Indeed, we can simply define unit and counit by taking the pushout of the original unit and counit; the triangle identities will automatically be satisfied. Therefore, we can replace S_1 by $S_1[(n+1)!^{-1}]$ and thus reduce to the case where S_1 and S_2 come from the same cofactor.

Case 2: $S_1 \in \mathrm{Sm}_{A[(n+1)!^{-1}]}^{\leqslant n+1}$ and $S_2 \in \mathrm{Sm}_{A[\dim!^{-1}]}^{\leqslant n}$. We may additionally assume that $(n+1)!$ is not invertible in S_2; otherwise we would be in a case already covered. But then

$$\mathrm{Hom}_{\mathrm{Sm}_{A[\dim!^{-1}]}^{\leqslant n+1}}(S_1, S_2) \simeq \emptyset$$

and so the map in question must be an equivalence, since only $\emptyset$ maps to $\emptyset$. $\square$

4.10. Lemma. — *For all $n \geqslant 0$, in the ∞-category of functors $\mathrm{Sm}_{A[(n+1)!^{-1}]}^{\leqslant n} \rightarrow \mathrm{AniAlg}_A^{q\text{-Hdg}}$, there exists a natural equivalence*

$$\left(-, \mathrm{fil}_{q\text{-Hdg},n}^{\star}\, q\text{-dR}_{-/A}\right) \simeq \left(-, \mathrm{fil}_{q\text{-Hdg},n+1}^{\star}\, q\text{-dR}_{-/A}\right).$$

Proof. First observe that every morphism in $\mathrm{Fil}^{\geqslant 0}\mathcal{D}(\mathbb{Z})$ that is sent to an equivalence by $\tau_{n+1}^{*}\colon \mathrm{Fil}^{\geqslant 0}\mathcal{D}(\mathbb{Z}) \rightarrow \mathrm{Fil}^{[0,n]}\mathcal{D}(\mathbb{Z})$ is also sent to an equivalence by τ_n^{*}. Since τ_{n+1}^{*} is a symmetric monoidal localisation, there exists a unique (up to contractible choice) symmetric monoidal functor $\tau_{n,n+1}^{*}$ such that

$$\begin{array}{ccc} \mathrm{Fil}^{\geqslant 0}\mathcal{D}(\mathbb{Z}) & \xrightarrow{\tau_n^*} & \mathrm{Fil}^{[0,n]}\mathcal{D}(\mathbb{Z}) \\ {\scriptstyle \tau_{n+1}^*}\downarrow & \nearrow{\scriptstyle \tau_{n,n+1}^*} & \\ \mathrm{Fil}^{[0,n+1]}\mathcal{D}(\mathbb{Z}) & & \end{array}$$

commutes. Moreover, arguing as in Lemma 4.2(*b*), we see that for any filtered $\mathbb{E}_\infty$-algebra $T \in \mathrm{CAlg}(\mathrm{Fil}^{\geqslant 0}\mathcal{D}(\mathbb{Z}))$, the induced symmetric monoidal functor

$$\tau_{n,n+1}^{*}\colon \mathrm{Mod}_{\tau_{n+1}^{*}T}\left(\mathrm{Fil}^{[0,n+1]}\mathcal{D}(\mathbb{Z})\right) \longrightarrow \mathrm{Mod}_{\tau_n^{*}T}\left(\mathrm{Fil}^{[0,n]}\mathcal{D}(\mathbb{Z})\right)$$

admits an oplax symmetric monoidal left adjoint

$$\tau_{n,n+1,!}\colon \mathrm{Mod}_{\tau_n^{*}T}\left(\mathrm{Fil}^{[0,n]}\mathcal{D}(\mathbb{Z})\right) \longrightarrow \mathrm{Mod}_{\tau_{n+1}^{*}T}\left(\mathrm{Fil}^{[0,n+1]}\mathcal{D}(\mathbb{Z})\right).$$

Let us now apply this in the case where $T = (q-1)^{\star}A[\![q-1]\!]$. Let S be a smooth A-algebra such that $\dim(S/A) \leqslant n$ and all primes $p \leqslant n+1$ are invertible in S. Plugging the canonical projection $\mathrm{fil}_{(\mathrm{Hdg},q-1)}^{\star}(\Omega_{S/A}[\![q-1]\!]/(q-1)^{n+1}) \rightarrow \mathrm{fil}_{(\mathrm{Hdg},q-1)}^{\star}(\Omega_{S/A}[\![q-1]\!]/(q-1)^{n})$ into the pullback from 4.1, we obtain a morphism

$$\tau_{n,n+1}^{*}\,\mathrm{fil}_{q\text{-Hdg},n+1}^{\star\leqslant n+1}\, q\text{-}\Omega_{S/A} \longrightarrow \mathrm{fil}_{q\text{-Hdg},n}^{\star\leqslant n}\, q\text{-}\Omega_{S/A}.$$

Applying $\tau_{n,!}(-)^\wedge_{(q-1)}$ on both sides and using the counit $\tau_{n,n+1,!}\circ\tau_{n,n+1}^*\Rightarrow\mathrm{id}$, we obtain a canonical zigzag

$$\mathrm{fil}^\star_{q\text{-Hdg},n+1}\,q\text{-}\Omega_{S/A}\xleftarrow{\simeq}\tau_{n,!}\left(\tau^*_{n,n+1}\,\mathrm{fil}^{\star\leqslant n+1}_{q\text{-Hdg},n+1}\,q\text{-}\Omega_{S/A}\right)^\wedge_{(q-1)}\xrightarrow{\simeq}\mathrm{fil}^\star_{q\text{-Hdg},n}\,q\text{-}\Omega_{S/A}$$

It is now straightforward to check that both morphisms are equivalences. Indeed, everything is filtered $(q-1)$-complete, so we may check this after reduction modulo $(q-1)$. For the outer two terms, the reduction is $\mathrm{fil}^\star_{\mathrm{Hdg}}\,\Omega_{S/A}$ by the calculation in the proof of Lemma 4.6. An analogous calculation shows that the inner term also becomes $\mathrm{fil}^\star_{\mathrm{Hdg}}\,\Omega_{S/A}$ and that the morphisms become the identity.

The zigzag above provides a functorial equivalence $\mathrm{fil}^\star_{q\text{-Hdg},n+1}\,q\text{-}\Omega_{-/A}\simeq\mathrm{fil}^\star_{q\text{-Hdg},n}\,q\text{-}\Omega_{-/A}$. Taking the pullback along $q\text{-}\mathrm{dR}_{S/A}\to q\text{-}\Omega_{S/A}$, we get what we want. □

In total we've shown:

4.11. Theorem. — *Let A be a perfectly covered Λ-ring and let S be a smooth A-algebra such that all primes $p\leqslant\dim(S/A)$ are invertible in S. Then $q\text{-}\mathrm{dR}_{S/A}$ admits a canonical q-Hodge filtration. More precisely, there exists a functor*

$$\left(-,\mathrm{fil}^\star_{q\text{-Hdg}}\,q\text{-}\mathrm{dR}_{-/A}\right)\colon \mathrm{Sm}_{A[\dim!^{-1}]}\longrightarrow \mathrm{AniAlg}_A^{q\text{-Hdg}}$$

which is a partial section of the forgetful functor $\mathrm{AniAlg}_A^{q\text{-Hdg}}\to\mathrm{AniAlg}_A$.

Proof. This is the quintessence of 4.1–4.10. □

4.12. Monoidality. — We wish to study to what extent the q-Hodge filtrations from 4.3 can be equipped with multiplicative structures. To this end, it would be nice to equip the functor from Theorem 4.11 with a symmetric monoidal structure. This is made complicated by the following issue:

(!) $\mathrm{Sm}_{A[\dim!^{-1}]}$ *is not closed under tensor products in* Sm_A *and we don't see a way of equipping it with a symmetric monoidal structure.*

To address this problem, let $\mathrm{Sm}_A^\otimes\to\mathrm{Fin}_*$ be the ∞-operad associated with the symmetric monoidal structure on Sm_A. We define a sub-∞-operad $\mathrm{Sm}^\otimes_{A[\dim!^{-1}]}\subseteq\mathrm{Sm}_A^\otimes$ as follows:

(*a*) An object $(S_1,\dots,S_i)\in\mathrm{Sm}_A^i$ in the fibre over $\langle i\rangle\in\mathrm{Fin}_*$ is contained in $\mathrm{Sm}_{A[\dim!^{-1}]}$ if and only $S_1,\dots,S_i$ are all contained in $\mathrm{Sm}_{A[\dim!^{-1}]}$.

(*b*) A morphism $(S_1,\dots,S_i)\to(S_1',\dots,S_{i'}')$ over $\alpha\colon\langle i\rangle\to\langle i'\rangle$ is contained in $\mathrm{Sm}_{A[\dim!^{-1}]}$ if and only if both source and target satisfy the condition from (*a*) and the target of a cocartesian lift of α with source $(S_1,\dots,S_i)$ also satisfies the condition from (*a*). Equivalently, we only retain those morphisms that factor through a cocartesian lift of their image in Fin_*.

Let us immediately warn the reader that $\mathrm{Sm}^\otimes_{A[\dim!^{-1}]}$ is *not* the full sub-∞-operad of $\mathrm{Sm}_A^\otimes$ spanned by the full subcategory $\mathrm{Sm}_{A[\dim!^{-1}]}\subseteq\mathrm{Sm}_A$, precisely because the condition from (*b*) yields a *non-full* sub-∞-operad.

Below we'll sketch how to make the functor from Theorem 4.11 into a functor of ∞-operads (this wouldn't work if we had used the full sub-∞-operad spanned by $\mathrm{Sm}_{A[\dim!^{-1}]}$). Let us discuss what kind of multiplicative structures this induces on $\mathrm{fil}^\star_{q\text{-Hdg}}\,q\text{-}\mathrm{dR}_{S/A}$. In general, any multiplicative structure on S as an object in $\mathrm{Sm}_{A[\dim!^{-1}]}$ will induce the same kind of multiplicative structure on $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{S/A}$. For arbitrary $S \in \mathrm{Sm}_{A[\dim!^{-1}]}$ there's nothing we can say. But as soon as all primes $p \leqslant 2\dim(S/A)$ are invertible in S, the multiplication map $S \otimes_A S \to S$ is a morphism in $\mathrm{Sm}_{A[\dim!^{-1}]}$, and so S will have an $\mathbb{A}_2$-structure in $\mathrm{Sm}^{\otimes}_{A[\dim!^{-1}]}$; that is, a homotopy-unital multiplication. If for some $r \geqslant 3$ all primes $p \leqslant r\dim(S/A)$ are invertible in S, then the multiplication will be $\mathbb{A}_r$; that is, coherently associative for up to r factors. A similar analysis works for commutativity.

We'll now sketch how to make the functor from Theorem 4.11 into a functor of ∞-operads. Let us temporarily fix $n \geqslant 0$.

4.13. Lemma. — *Let $\mathrm{Mod}_{\tau_n^*((q-1)^\star A[\![q-1]\!])}(\mathrm{Fil}^{[0,n]}\,\mathcal{D}(\mathbb{Z}))$ be as in 4.3 and equip the full sub-∞-category of $(q-1)$-complete objects $\mathrm{Mod}_{\tau_n^*((q-1)^\star A[\![q-1]\!])}(\mathrm{Fil}^{[0,n]}\,\mathcal{D}(\mathbb{Z}))^\wedge_{(q-1)}$ with the $(q-1)$-completed tensor product. Then the functor*

$$\mathrm{fil}^\star_{q\text{-Hdg},n}\, q\text{-}\mathrm{dR}_{-/A}\colon \mathrm{Sm}_{A[n!^{-1}]} \longrightarrow \mathrm{Mod}_{\tau_n^*((q-1)^\star A[\![q-1]\!])}\big(\mathrm{Fil}^{[0,n]}\,\mathcal{D}(\mathbb{Z})\big)^\wedge_{(q-1)}$$

from 4.3 can be equipped with a symmetric monoidal structure.

Proof sketch. From the construction it's straightforward to get a lax symmetric monoidal structure. Whether it is symmetric monoidal can be checked modulo $(q-1)$, where we reduce to the fact that $\tau_n^*(\mathrm{fil}^\star_{\mathrm{Hdg}}\,\mathrm{dR}_{-/A})$ is symmetric monoidal. □

4.14. Lax vs. oplax symmetric monoidal functors. — For every symmetric monoidal ∞-category with associated cocartesian fibration $\mathcal{C}^\otimes \to \mathrm{Fin}_*$, let $(\mathcal{C}^\otimes)^\vee \to \mathrm{Fin}_*^{\mathrm{op}}$ denote the dual cartesian fibration. Lax symmetric monoidal functors $\mathcal{C} \to \mathcal{D}$ are then encoded as functors $\mathcal{C}^\otimes \to \mathcal{D}^\otimes$ in $\mathrm{Cat}_{\infty/\mathrm{Fin}_*}$ that preserve cocartesian lifts of inert morphisms, whereas oplax symmetric monoidal functors are encoded as functors $(\mathcal{C}^\otimes)^\vee \to (\mathcal{D}^\otimes)^\vee$ in $\mathrm{Cat}_{\infty/\mathrm{Fin}_*^{\mathrm{op}}}$ that preserve cartesian lifts of inert morphisms.

In general, the dual cartesian fibration $(\mathcal{C}^\otimes)^\vee \to \mathrm{Fin}_*$ has a very nice description in terms of span ∞-categories. This is due to Barwick–Glasman–Nardin; see [BGN18, 1.2]. We will now apply this to the oplax symmetric monoidal structure on

$$\big(-, \mathrm{fil}^\star_{q\text{-Hdg},n}\, q\text{-}\mathrm{dR}_{-/A}\big)\colon \mathrm{Sm}_{A[n!^{-1}]} \longrightarrow \mathrm{AniAlg}_A^{q\text{-Hdg}}$$

that we obtain by composing the symmetric monoidal functor from Lemma 4.13 with the oplax symmetric monoidal functor $\tau_{n,!}(-)^\wedge_{(q-1)}$.

4.15. Lemma. — *If $\varphi\colon (S_1', \dots, S_{i'}') \to (S_1, \dots, S_i)$ is a cartesian morphism in $(\mathrm{Sm}^{\otimes}_{A[n!^{-1}]})^\vee$ such that $S_1', \dots, S_{i'}'$ are all of relative dimension $\leqslant n$ over A, then φ is sent to a cartesian morphism under*

$$\big(\mathrm{Sm}^{\otimes}_{A[n!^{-1}]}\big)^\vee \longrightarrow \big(\mathrm{AniAlg}_A^{q\text{-Hdg},\otimes}\big)^\vee\,.$$

Proof sketch. This essentially reduces to the observation that whenever a tensor product of smooth $A[n!^{-1}]$-algebras $S_1 \otimes_A \dots \otimes_A S_i$ has relative dimension $\leqslant n$ over A, the q-Hodge filtration $\mathrm{fil}^\star_{q\text{-Hdg},n}\, q\text{-}\mathrm{dR}_{S_1 \otimes_A \dots \otimes_A S_i/A}$ will agree with

$$\Big(\mathrm{fil}^\star_{q\text{-Hdg},n}\, q\text{-}\mathrm{dR}_{S_1/A} \otimes^{\mathrm{L}}_{(q-1)^\star A[\![q-1]\!]} \dotsb \otimes^{\mathrm{L}}_{(q-1)^\star A[\![q-1]\!]} \mathrm{fil}^\star_{q\text{-Hdg},n}\, q\text{-}\mathrm{dR}_{S_i/A}\Big)^\wedge_{(q-1)}\,.$$

Indeed, this can be checked modulo $(q-1)$, where the desired claim follows using symmetric monoidality of $\mathrm{fil}^\star_{\mathrm{Hdg}}\,\mathrm{dR}_{-/A}$. □

4.16. Corollary. — *The functor from Theorem 4.11 underlies a functor of ∞-operads*

$$\mathrm{Sm}_{A[\dim!^{-1}]}^{\otimes} \longrightarrow \mathrm{AniAlg}_A^{q\text{-Hdg},\otimes}\,,$$

which preserves all cocartesian lifts that exist in the source.

Proof sketch. As in 4.12, we can define a sub-∞-operad $\mathrm{Sm}_{A[n!^{-1}]}^{\leqslant n,\otimes} \subseteq \mathrm{Sm}_{A[n!^{-1}]}^{\otimes}$ given by those objects whose entries are of dimension $\leqslant n$ and those morphisms that factor through a cocartesian lift of their image in Fin_*. Analogously, we can define $(\mathrm{Sm}_{A[n!^{-1}]}^{\leqslant n,\otimes})^\vee \subseteq (\mathrm{Sm}_{A[n!^{-1}]}^{\otimes})^\vee$ given by those objects whose entries are of dimension $\leqslant n$ and those morphisms that factor through a cartesian lift of their image in $\mathrm{Fin}_*^{\mathrm{op}}$.

The dualising construction from [BGN18, 1.2] can not only be applied to cartesian fibrations, but also to $(\mathrm{Sm}_{A[n!^{-1}]}^{\leqslant n,\otimes})^\vee$, and it is straightforward to check that we get back $\mathrm{Sm}_{A[n!^{-1}]}^{\leqslant n,\otimes}$ in this case. Moreover, by Lemma 4.15, the functor

$$\left(\mathrm{Sm}_{A[n!^{-1}]}^{\leqslant,\otimes}\right)^\vee \longrightarrow \left(\mathrm{AniAlg}_A^{q\text{-Hdg},\otimes}\right)^\vee$$

preserves all cartesian lifts that exist in the source. We may thus dualise via [BGN18, 1.2] to obtain a functor

$$\mathrm{Sm}_{A[n!^{-1}]}^{\leqslant,\otimes} \longrightarrow \mathrm{AniAlg}_A^{q\text{-Hdg},\otimes}\,.$$

Now the ∞-operad $\mathrm{Sm}_{A[\dim!^{-1}]}^{\otimes}$ is built from $\mathrm{Sm}_{A[n!^{-1}]}^{\leqslant,\otimes}$ for all $n \geqslant 0$ via a sequence of pushouts as in Lemma 4.9. Combining this with a straightforward analogue of Lemma 4.10, we can inductively construct the desired map of ∞-operads. $\square$

§4.2. Functorial q-Hodge filtrations for certain quasi-regular quotients

In this subsection, we'll explain another elementary construction of functorial q-Hodge filtrations. To this end, let us first fix a prime p and work in a p-complete setting (at the end of this subsection, we'll get back to the global case). Throughout this subsection, all (q-)de Rham complexes or cotangent complexes relative to p-complete rings will be implicitly p-completed.

4.17. Rings of interest. — Temporarily, A will not be a perfectly covered Λ-ring, but a *p-completely perfectly covered δ-ring*, by which we mean a p-complete δ-ring for which the map $A \to A_\infty$ into its p-completed colimit perfection is p-completely faithfully flat. Equivalently, the Frobenius $\phi\colon A \to A$ is p-completely flat (as being faithful is automatic). Since perfect δ-rings are p-torsion free, it follows that A must be p-torsion free too.

Throughout, we will consider *p-quasi-lci algebras over A*: These are p-complete rings R for which the cotangent complex $\mathrm{L}_{R/A}$ (which, by our convention above, we always take to be implicitly p-completed) has p-complete Tor-amplitude over R concentrated in degree $[0,1]$. Additionally, we'll usually assume that R/p is *relatively semiperfect over A*: That is, the relative Frobenius $R/p \otimes_{A,\phi} A \twoheadrightarrow R/p$ is surjective. This forces $\Omega^1_{R/A}/p$ to vanish, so $\mathrm{L}_{R/A}$ will have p-complete Tor-amplitude over R concentrated in degree 1.

An important special case are A-algebras of *perfect-regular presentation*: These are the quotients $R \cong B/J$, where B is a p-complete relatively perfect δ-A-algebra, by which we mean that the relative Frobenius $\phi_{B/A}\colon (B \otimes_{A,\phi} A)_p^\wedge \to B$ is an isomorphism, and $J \subseteq B$ is an ideal generated by a Koszul-regular sequence. We'll sometimes refer to B/J as a *perfect-regular presentation of R*.

The reason for restricting to rings R as above is the following lemma.

4.18. Lemma. — *Let R be a p-torsion free A-algebra such that $\mathrm{L}_{R/A}$ has p-complete* Tor-*amplitude over R concentrated in degree* 1.

(*a*) *The de Rham complex $\mathrm{dR}_{R/A}$, its Hodge-completion $\widehat{\mathrm{dR}}_{R/A}$, every degree in the completed Hodge filtration $\mathrm{fil}^\star_{\mathrm{Hdg}}\widehat{\mathrm{dR}}_{R/A}$, and the q-de Rham complex $q\text{-}\mathrm{dR}_{R/A}$ are all static and p-torsion free.*

(*b*) *The un-completed Hodge filtration $\mathrm{fil}^\star_{\mathrm{Hdg}}\mathrm{dR}_{R/A}$ is static in every degree if and only if R/p is relatively semiperfect over A.*

Proof. To show that every degree in the completed Hodge filtration is static and p-torsion free, just observe that the same is true for the associated graded $\mathrm{gr}^*_{\mathrm{Hdg}}\widehat{\mathrm{dR}}_{R/A}\simeq\Sigma^{-*}\bigwedge^*\mathrm{L}_{R/A}$, because our assumption on R guarantees that $\Sigma^{-1}\mathrm{L}_{R/A}$ is a p-completely flat module over the p-torsion free ring R. To show that the $(q-1)$-complete object $q\text{-}\mathrm{dR}_{R/A}$ is static and p-torsion free, it will be enough to show the same for $q\text{-}\mathrm{dR}_{R/A}/(q-1)\simeq\mathrm{dR}_{R/A}$. Now all assertions about $\mathrm{dR}_{R/A}$ and its Hodge filtration can be checked after base change along the p-completely faithfully flat map $A\to A_\infty$.

So let us put $R_\infty\coloneqq(R\otimes_A A_\infty)^\wedge_p$ and consider $\mathrm{dR}_{R_\infty/A_\infty}$ and let $\overline{R}_\infty\coloneqq R_\infty/p$. Since A_∞ is a perfect δ-ring, $\mathrm{L}_{A_\infty/\mathbb{Z}_p}\simeq 0$, so we may as well consider $\mathrm{dR}_{R_\infty/\mathbb{Z}_p}$. To see that $\mathrm{dR}_{R_\infty/\mathbb{Z}_p}$ is static and p-torsion free, it suffices to check that its modulo p reduction $\mathrm{dR}_{R_\infty/\mathbb{Z}_p}/p\simeq\mathrm{dR}_{\overline{R}_\infty/\mathbb{F}_p}$ is static. The latter admits an ascending exhaustive filtration, the *conjugate filtration*, whose associated graded $\Sigma^{-*}\bigwedge^*\mathrm{L}_{\overline{R}_\infty/\mathbb{F}_p}\simeq\Sigma^{-*}\bigwedge^*\mathrm{L}_{R_\infty/\mathbb{Z}_p}/p$ is static in every degree since $\Sigma^{-1}\mathrm{L}_{R_\infty/\mathbb{Z}_p}$ is p-completely flat over the p-torsion free ring R_∞. This shows that $\mathrm{dR}_{\overline{R}_\infty/\mathbb{F}_p}$ is indeed static and we've finished the proof of (*a*).

For (*b*), we've already seen that $\mathrm{dR}_{R_\infty/\mathbb{Z}_p}$ and the associated graded of the Hodge filtration are static and p-torsion free in every degree. Hence $\mathrm{fil}^\star_{\mathrm{Hdg}}\mathrm{dR}_{R_\infty/\mathbb{Z}_p}$ is degree-wise static if and only if it consists of sub-modules of $\mathrm{dR}_{R_\infty/\mathbb{Z}_p}$, which must be p-torsion free too. Thus $\mathrm{fil}^\star_{\mathrm{Hdg}}\mathrm{dR}_{R_\infty/\mathbb{Z}_p}$ is degree-wise static if and only if the same is true for $\mathrm{fil}^\star_{\mathrm{Hdg}}\mathrm{dR}_{R_\infty/\mathbb{Z}_p}/p\simeq\mathrm{fil}^\star_{\mathrm{Hdg}}\mathrm{dR}_{\overline{R}_\infty/\mathbb{F}_p}$. In the case where $\overline{R}_\infty$ is semiperfect, this holds by [BMS19, Proposition 8.14]. Conversely, assume $\mathrm{fil}^\star_{\mathrm{Hdg}}\mathrm{dR}_{\overline{R}_\infty/\mathbb{F}_p}$ is degree-wise static. If $\mathrm{fil}^\star_{\mathcal{N}}\mathrm{WdR}_{\overline{R}_\infty/\mathbb{F}_p}$ denotes the Nygaard filtration on the derived de Rham–Witt complex, then

$$\mathrm{fil}^n_{\mathcal{N}}\mathrm{WdR}_{\overline{R}_\infty/\mathbb{F}_p}/p\,\mathrm{fil}^{n-1}_{\mathcal{N}}\mathrm{WdR}_{\overline{R}_\infty/\mathbb{F}_p}\simeq\mathrm{fil}^n_{\mathrm{Hdg}}\mathrm{dR}_{\overline{R}_\infty/\mathbb{F}_p}$$

holds for all n by deriving [BMS19, Lemma 8.3]. Inductively it follows that $\mathrm{WdR}_{R_\infty/\mathbb{Z}_p}$ and each step in its Nygaard filtration must be static too. By definition, $\mathrm{fil}^n_{\mathcal{N}}\mathrm{WdR}_{\overline{R}_\infty/\mathbb{F}_p}$ is the fibre of

$$\mathrm{WdR}_{\overline{R}_\infty/\mathbb{F}_p}\xrightarrow{\phi}\mathrm{WdR}_{\overline{R}_\infty/\mathbb{F}_p}\longrightarrow\mathrm{WdR}_{\overline{R}_\infty/\mathbb{F}_p}/p^n\,,$$

so this composition must be surjective for all n. Then $\phi\colon\mathrm{WdR}_{\overline{R}_\infty/\mathbb{F}_p}\to\mathrm{WdR}_{\overline{R}_\infty/\mathbb{F}_p}$ must be surjective as well. Since $\mathrm{WdR}_{\overline{R}_\infty/\mathbb{F}_p}/p\simeq\mathrm{dR}_{\overline{R}_\infty/\mathbb{F}_p}\to\overline{R}_\infty$ is surjective by our assumption that $\mathrm{fil}^1_{\mathrm{Hdg}}\mathrm{dR}_{\overline{R}_\infty/\mathbb{F}_p}$ is static, we conclude that the Frobenius on $\overline{R}_\infty$ must be surjective too. □

4.19. Remark. — In the case where $R\cong B/J$ is of perfect-regular presentation over A, everything can be made explicit: $\mathrm{dR}_{R/A}\simeq D_B(J)$ is the (p-completed) PD-envelope of J, the Hodge filtration is just the PD-filtration, and the q-de Rham complex $q\text{-}\mathrm{dR}_{R/A}$ is the corresponding q-PD-envelope in the sense of [BS19, Lemma 16.10].

4.20. Remark. — There exist p-complete $\mathbb{Z}_p$-algebras whose cotangent complex has p-complete Tor-amplitude concentrated in degree 1, but whose reduction modulo p is not semiperfect. For example, if $p \geqslant 3$, the $\mathbb{F}_p$-algebra constructed in [Gul21] can be lifted in a straightforward way to a p-complete $\mathbb{Z}_p$-algebra with this property.

Let us now define a q-Hodge filtration for rings R as in Lemma 4.18.

4.21. Construction. — Suppose R is a p-torsion free quasi-lci A-algebra such that R/p is relatively semiperfect over A. By Lemma A.4, after rationalisation, $\mathrm{dR}_{R/A}$ and $q\text{-}\mathrm{dR}_{R/A}$ are related via a functorial equivalence

$$q\text{-}\mathrm{dR}_{R/A}\big[\tfrac{1}{p}\big]^\wedge_{(q-1)} \simeq \mathrm{dR}_{R/A}\big[\tfrac{1}{p}\big]\llbracket q-1\rrbracket\,.$$

By Lemma 4.18, both sides are static rings. Let us equip the right-hand side with the combined Hodge and $(q-1)$-adic filtration $\mathrm{fil}^\star_{(\mathrm{Hdg},q-1)}\mathrm{dR}_{R/A}[1/p]\llbracket q-1\rrbracket$ as in Definition 3.2(c_p). This is a descending filtration by ideals.

We now construct $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A}$ as the 1-categorical (!) preimage of this filtration under $q\text{-}\mathrm{dR}_{R/A}\rightarrow \mathrm{dR}_{R/A}[1/p]\llbracket q-1\rrbracket$; in other words, as the pullback

$$\begin{array}{ccc} \mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A} & \longrightarrow & \mathrm{fil}^\star_{(\mathrm{Hdg},q-1)}\mathrm{dR}_{R/A}\big[\tfrac{1}{p}\big]\llbracket q-1\rrbracket \\ \downarrow & \lrcorner & \downarrow \\ q\text{-}\mathrm{dR}_{R/A} & \longrightarrow & \mathrm{dR}_{R/A}\big[\tfrac{1}{p}\big]\llbracket q-1\rrbracket \end{array}$$

taken in the 1-category of filtered $(q-1)^\star A\llbracket q-1\rrbracket$-modules. We remark that $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A}$ will be a descending filtration of ideals in the static ring $q\text{-}\mathrm{dR}_{R/A}$, hence it's automatically a filtered $\mathbb{E}_\infty$-algebra over $(q-1)^\star A\llbracket q-1\rrbracket$.

Let us also remark that the canonical projection $q\text{-}\mathrm{dR}_{R/A}\rightarrow \mathrm{dR}_{R/A}$ induces a (necessarily unique) filtered map

$$\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A}\longrightarrow \mathrm{fil}^\star_{\mathrm{Hdg}}\mathrm{dR}_{R/A}\,.$$

Indeed, to see this, we must check that $\mathrm{fil}^\star_{\mathrm{Hdg}}\mathrm{dR}_{R/A}$ is the preimage of $\mathrm{fil}^\star_{\mathrm{Hdg}}\mathrm{dR}_{R/A}[1/p]$ under $\mathrm{dR}_{R/A}\rightarrow \mathrm{dR}_{R/A}[1/p]$. Since any filtration is the preimage of its completion, we may further replace the Hodge filtration $\mathrm{fil}^\star_{\mathrm{Hdg}}\mathrm{dR}_{R/A}[1/p]$ by its completion $\mathrm{fil}^\star_{\mathrm{Hdg}}\mathrm{dR}_{R/A}[1/p]^\wedge_{\mathrm{Hdg}}$. To check that $\mathrm{fil}^\star_{\mathrm{Hdg}}\mathrm{dR}_{R/A}$ is the preimage, it will thus be enough to check that the map on associated gradeds is injective. Now

$$\Sigma^{-n}\bigwedge^n \mathrm{L}_{R/A}\rightarrow \Sigma^{-n}\bigwedge^n \mathrm{L}_{R/A}\big[\tfrac{1}{p}\big]$$

will be injective for all $n\geqslant 0$, because $\Sigma^{-n}\bigwedge^n \mathrm{L}_{R/A}$ is a p-completely flat module over the p-torsion free ring R and thus p-torsion free itself.

In general, the q-Hodge filtration from Construction 4.21 will be nonsense. But it does behave as desired in the following cases:

4.22. Theorem. — *Let A be a p-completely perfectly covered δ-ring and let R be a p-torsion free quasi-lci A-algebra such that R/p is relatively semiperfect over A. Suppose that one of the following two additional assumptions is satisfied:*

(a) *There exists a perfect-regular presentation $R\cong B/J$, where the ideal $J\subseteq B$ is generated by a Koszul-regular sequence of higher powers, that is, a Koszul-regular sequence $(x_1^{\alpha_1},\dots,x_r^{\alpha_r})$ with $\alpha_i\geqslant 2$ for all i.*

(*b*) *The ring* $R_\infty \coloneqq (R\otimes_A A_\infty)_p^\wedge$ *admits a lift to a p-complete connective $\mathbb{E}_1$-ring spectrum* $\mathbb{S}_{R_\infty}$ *satisfying* $R_\infty \simeq \mathbb{S}_{R_\infty}\otimes_{\mathbb{S}_p}\mathbb{Z}_p$.

Then $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A}$ *is a q-deformation of* $\mathrm{fil}^\star_{\mathrm{Hdg}}\,\mathrm{dR}_{R/A}$ *in the sense that the canonical map from Construction 4.21 induces an equivalence*

$$\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A}/(q-1) \xrightarrow{\simeq} \mathrm{fil}^\star_{\mathrm{Hdg}}\,\mathrm{dR}_{R/A}\,.$$

Here we take the quotient in filtered $(q-1)^\star A[\![q-1]\!]$-modules, with $(q-1)$ regarded as an element in filtration degree 1.

4.23. Remark. — For primes $p>2$, Theorem 4.22(*b*) implies (*a*). Indeed, if we put $B_\infty \coloneqq (B\otimes_A A_\infty)_p^\wedge$, then B_∞ is a perfect δ-ring and so it lifts uniquely to a connective p-complete $\mathbb{E}_\infty$-ring spectrum. We can then use Burklund's theorem on $\mathbb{E}_n$-structures on quotients [Bur22, Theorem 1.5] to construct an $\mathbb{E}_1$-structure on

$$\mathbb{S}_{R_\infty} \coloneqq \mathbb{S}_{B_\infty}/\big(x_1^{\alpha_1},\dots,x_r^{\alpha_r}\big)\,.$$

More precisely, since $p>2$, each $\mathbb{S}_{B_\infty}/x_i$ admits a right-unital multiplication (the relevant obstruction $Q_1(x_i)$ is 2-torsion), and so Burklund's result provides $\mathbb{E}_1$-structures on $\mathbb{S}_{B_\infty}/x_i^{\alpha_i}$ in $\mathrm{Mod}_{\mathbb{S}_{B_\infty}}(\mathrm{Sp})$, of which we can take the tensor product.

For $p=2$, $\mathbb{S}_{B_\infty}/x_i^2$ still admits a right-unital multiplication (see [Bur22, Remark 5.5]) and so the same argument shows that Theorem 4.22(*b*) implies (*a*) if all α_i are even and $\geqslant 4$. It is somewhat surprising that Theorem 4.22(*a*) is true without this additional restriction at $p=2$.

Before we prove Theorem 4.22, let us discuss two examples.

4.24. Example. — Let $A\coloneqq \mathbb{Z}_p\{x\}_p^\wedge$ be the free p-complete δ-ring on a generator x and let $R\coloneqq \mathbb{Z}_p\{x\}_p^\wedge/x^\alpha$ for some $\alpha\geqslant 1$. Then Theorem 4.22(*a*) will apply as soon as $\alpha\geqslant 2$, but not for $\alpha=1$. So let's see what goes wrong for $\alpha=1$ and how higher powers (or divine intervention?) fix the issue.

In the case at hand, $\mathrm{dR}_{R/A}$ and $q\text{-}\mathrm{dR}_{R/A}$ are the usual PD-envelope and the q-PD envelope

$$D_\alpha \coloneqq \mathbb{Z}_p\{x\}\left\{\frac{\phi(x^\alpha)}{p}\right\}_p^\wedge \quad\text{and}\quad q\text{-}D_\alpha \coloneqq \mathbb{Z}_p\{x\}[\![q-1]\!]\left\{\frac{\phi(x^\alpha)}{[p]_q}\right\}_{(p,q-1)}^\wedge,$$

respectively. If the q-Hodge filtration were to be a q-deformation of the Hodge filtration, then $\mathrm{fil}^p_{q\text{-Hdg}}\, q\text{-}D_\alpha$ would need to contain a lift $\widetilde{\gamma}_q(x^\alpha)$ of the divided power $\gamma(x^\alpha)\coloneqq x^{\alpha p}/p\in \mathrm{fil}^p_{\mathrm{Hdg}}\, D_\alpha$. Certainly, $q\text{-}D_\alpha$ itself contains such a lift; namely, the q-divided power

$$\gamma_q(x^\alpha)\coloneqq \frac{\phi(x^\alpha)}{[p]_q}-\delta(x^\alpha)\,.$$

The problem is that $\gamma_q(x^\alpha)$ is usually not contained in $\mathrm{fil}^p_{q\text{-Hdg}}\, q\text{-}D_\alpha$. So for the q-Hodge filtration to be a q-deformation of the Hodge filtration, it must be possible to modify $\gamma_q(x^\alpha)$ by elements from $(q-1)\,q\text{-}D_\alpha$ to get an element in $\mathrm{fil}^p_{q\text{-Hdg}}\, q\text{-}D_\alpha$. As we'll see momentarily, this is impossible for $\alpha=1$, but it works for $\alpha\geqslant 2$.

By definition, $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A}$ is the preimage of the combined Hodge and $(q-1)$-adic filtration on $D_\alpha[1/p][\![q-1]\!]$. Since every filtration is the preimage of its completion, we may replace the latter by its completion, which is the $(x^\alpha,q-1)$-adic filtration on $\mathbb{Q}_p\langle\delta(x),\delta^2(x),\dots\rangle[\![x,q-1]\!]$.

So our task is to modify $\gamma_q(x^\alpha)$ by elements from the $(q-1)\,q\text{-}D_\alpha$ such that the result is contained in the ideal $(x^\alpha, q-1)^p \subseteq \mathbb{Q}_p\langle\delta(x), \delta^2(x), \dots\rangle[\![x, q-1]\!]$.

Write $[p]_q = pu + (q-1)^{p-1}$, where $u \equiv 1 \mod q-1$. In particular, u is a unit in $q\text{-}D_\alpha$. In $\mathbb{Q}_p\langle\delta(x), \delta^2(x), \dots\rangle[\![x, q-1]\!]$, we can rewrite $\gamma_q(x^\alpha)$ as

$$\frac{x^{\alpha p}}{[p]_q} + \left(\frac{p}{[p]_q} - 1\right)\delta(x^\alpha) = \frac{x^{\alpha p}}{[p]_q} + \left((u^{-1}-1) - u^{-2}\frac{(q-1)^{p-1}}{p} + \mathrm{O}\big((q-1)^p\big)\right)\delta(x^\alpha)\,.$$

Here $\mathrm{O}((q-1)^p)$ denotes "error terms" which are divisible by $(q-1)^p$. Observe that these error terms are contained in $(x^\alpha, q-1)^p$, so we can safely ignore them. Also $x^{\alpha p}/[p]_q$ is clearly contained in $(x^\alpha, q-1)^p$. The term $(u^{-1}-1)\delta(x^\alpha)$ is contained in $(q-1)\,q\text{-}D_\alpha$, so we can just kill it. This leaves the term $u^{-2}(q-1)^{p-1}\delta(x^\alpha)/p$.

If $\alpha = 1$, there's nothing we can do: No modification by elements from $(q-1)\,q\text{-}D_\alpha$ will ever get rid of a non-integral multiple of $\delta(x)$, as $\delta(x)$ is a polynomial variable in $\mathbb{Z}_p\{x\}$. This shows that for $\alpha = 1$, the q-Hodge filtration on $q\text{-}D_\alpha$ is *not* a q-deformation of the Hodge filtration. For $\alpha = 2$, however, we have $\delta(x^2) = 2x^p\delta(x) + p\delta(x)^2$. Now the term $2x^p\delta(x)u^{-2}(q-1)^{p-1}/p$ is contained in $(x^2, q-1)^p$ and so

$$\widetilde{\gamma}_q(x^\alpha) \coloneqq \gamma_q(x^\alpha) - (u^{-1}-1)\delta(x^2) + u^{-2}(q-1)^{p-1}\delta(x)^2$$

is contained in $\mathrm{fil}^p_{q\text{-Hdg}}\, q\text{-}D_\alpha$ and satisfies $\widetilde{\gamma}_q(x^\alpha) \equiv x^{2p}/p \mod q-1$, as desired. For $\alpha \geqslant 3$, we can similarly decompose $\delta(x^\alpha)$ into a multiple of $x^{p(\alpha-1)}$ and a multiple of p.

This explains what goes wrong at $\alpha = 1$ and how the objection is resolved for $\alpha \geqslant 2$. In the latter case, it is possible to continue the analysis above and construct for all $n \geqslant 1$ a lift of the divided power $x^{\alpha n}/n!$ that lies in in $\mathrm{fil}^n_{q\text{-Hdg}}\, q\text{-dR}_{R/A}$. This will be explained in §11.2 and leads to an elementary proof of Theorem 4.22(*a*).

4.25. Example. — An example for Theorem 4.22(*b*) that is not covered by Theorem 4.22(*a*) is the case $A \cong \mathbb{Z}_p[x]_p^\wedge$, with δ-structure defined by $\delta(x) = 0$, and $R \cong A/(x-1) \cong \mathbb{Z}_p$. Then A lifts to the p-complete $\mathbb{E}_\infty$-ring spectrum $\mathbb{S}_p[x]_p^\wedge$ and $A \to R$ lifts to an $\mathbb{E}_\infty$-map $\mathbb{S}_p[x]_p^\wedge \to \mathbb{S}_p$. Base changing along $\mathbb{S}_p[x]_p^\wedge \to \mathbb{S}_p[x^{1/p^\infty}]_p^\wedge$ yields a lift of R_∞, even as an $\mathbb{E}_\infty$-ring. In this case, $q\text{-dR}_{R/A}$ is the q-PD envelope

$$q\text{-}D \coloneqq \mathbb{Z}_p[x][\![q-1]\!]\left\{\frac{x^p-1}{[p]_q}\right\}^\wedge_{(p,q-1)}\,.$$

It can be shown that this ring contains elements of the form $(x-1)(x-q)\cdots(x-q^{n-1})/[n]_q!$ for all $n \geqslant 1$ (see Lemma 9.11 for an argument). After completed rationalisation, these elements are visibly contained in the ideal $(x-1, q-1)^n$. Hence they belong to $\mathrm{fil}^n_{q\text{-Hdg}}\, q\text{-dR}_{R/A}$ and lift the usual divided powers.

Let us now prove Theorem 4.22, albeit large parts of the argument will be postponed to later sections. We start with the observation that only surjectivity is critical.

4.26. Lemma. — *Let R be a p-torsion free p-quasi-lci A-algebra such that R/p is relatively semiperfect over A. Then the canonical map from Construction 4.21 induces a degree-wise injection*

$$\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_{R/A}/(q-1) \longhookrightarrow \mathrm{fil}^\star_{\mathrm{Hdg}}\, \mathrm{dR}_{R/A}\,.$$

Proof. We need to check

$$(q-1)\,\mathrm{fil}^{n-1}_{q\text{-Hdg}}\,q\text{-dR}_{R/A} = \mathrm{fil}^{n}_{q\text{-Hdg}}\,q\text{-dR}_{R/A} \cap (q-1)\,q\text{-dR}_{R/A}$$

for all n. This immediately reduces to the analogous assertion for the combined Hodge and $(q-1)$-adic filtration on $\mathrm{dR}_{R/A}[1/p]\llbracket q-1\rrbracket$, which is straightforward to check. □

The q-Hodge filtration from Construction 4.21 enjoys a general flat base change property. This will allow us to reduce the proof of Theorem 4.22 to the case where A is perfect.

4.27. Lemma. — *Let R be a p-torsion free p-quasi-lci A-algebra such that R/p is relatively semiperfect over A. Let $A \to A'$ be a p-completely flat morphism of δ-rings, where A' is also p-completely perfectly covered, and put $R' \coloneqq (R \otimes_A A')^\wedge_p$. Then the canonical map*

$$\left(\mathrm{fil}^{\star}_{q\text{-Hdg}}\,q\text{-dR}_{R/A} \otimes^{\mathrm{L}}_A A'\right)^\wedge_{(p,q-1)} \xrightarrow{\simeq} \mathrm{fil}^{\star}_{q\text{-Hdg}}\,q\text{-dR}_{R'/A'}$$

is an equivalence.

Proof. This is not completely automatic since we have to be careful with completions. Fix n. By Remark A.7, the canonical map $q\text{-dR}_{R/A} \to (\mathrm{dR}_{R/A} \otimes_{\mathbb{Z}} \mathbb{Q})\llbracket q-1\rrbracket/(q-1)^n$ already factors through $p^{-N}\mathrm{dR}_{R/A}\llbracket q-1\rrbracket/(q-1)^n$ for sufficiently large N. Since $\mathrm{fil}^n_{q\text{-Hdg}}\,q\text{-dR}_{R/A}$ contains $(q-1)^n\,q\text{-dR}_{R/A}$, we can also express it as a pullback of $A\llbracket q-1\rrbracket$-modules

$$\begin{array}{ccc}
\mathrm{fil}^n_{q\text{-Hdg}}\,q\text{-dR}_{R/A} & \longrightarrow & q\text{-dR}_{R/A} \\
\downarrow & \lrcorner & \downarrow \\
p^{-N}\,\mathrm{fil}^n_{(\mathrm{Hdg},q-1)}\,\mathrm{dR}_{R/A}\llbracket q-1\rrbracket/(q-1)^n & \longrightarrow & p^{-N}\mathrm{dR}_{R/A}\llbracket q-1\rrbracket/(q-1)^n
\end{array}$$

(here the combined Hodge and $(q-1)$-adic filtration $\mathrm{fil}^{\star}_{(\mathrm{Hdg},q-1)}\,\mathrm{dR}_{R/A}\llbracket q-1\rrbracket/(q-1)^n$ is constructed as in 4.1 above).

It will be enough to show that the pullback is preserved $(-\otimes^{\mathrm{L}}_A A')^\wedge_{(p,q-1)}$. To this end, let P denote the derived pullback (that is, the pullback taken in the derived ∞-category $\mathcal{D}(A\llbracket q-1\rrbracket)$) and recall that derived tensor products preserve derived pullbacks. It is then enough to check that $(\mathrm{H}_{-1}(P) \otimes^{\mathrm{L}}_A A')^\wedge_{(p,q-1)}$ is static. We claim that $\mathrm{H}_{-1}(P)$ is $(q-1)^n$-torsion and p^m-torsion for sufficiently large m. Believing this for the moment, p-complete flatness of $A \to A'$ guarantees that $\mathrm{H}_{-1}(P) \otimes^{\mathrm{L}}_A A'$ is static. Since it is also p^m- and $(q-1)^n$-torsion, the completion doesn't change anything and we're done.

To prove the claim, observe that the cokernel of $q\text{-dR}_{R/A} \to p^{-N}\mathrm{dR}_{R/A}$ must clearly be p^N-torsion. Hence the cokernel of the right vertical map

$$q\text{-dR}_{R/A} \longrightarrow p^{-N}\mathrm{dR}_{R/A}\llbracket q-1\rrbracket/(q-1)^n$$

is p^{nN}-torsion and also $(q-1)^n$-torsion. Since $\mathrm{H}_{-1}(P)$ is a quotient of that cokernel (explicitly the quotient by the bottom left corner of the pullback diagram), we conclude that $\mathrm{H}_{-1}(P)$ is p^{nN}-torsion and $(q-1)^n$-torsion too, as desired. □

Proof of Theorem 4.22. By Lemma 4.26, we only need to check surjectivity. By Lemma 4.27, we can check this after the p-completely faithfully flat base change $A \to A_\infty$ and thus assume that A is perfect. Since in this case $q\text{-dR}_{R/A} \simeq q\text{-dR}_{R/\mathbb{Z}_p}$, we may further reduce to the case $A = \mathbb{Z}_p$.

Then part (b) is a special case of Theorem 7.18 below. As we've explained in Remark 4.23, this also proves part (a) if $p > 2$. In §11.2, we'll give an elementary proof of (a), which also covers the case $p = 2$. □

To finish this subsection, we'll extract a global construction from the above. From now on, we cancel the assumptions from 4.17 and return to our usual notation, where A is a perfectly covered Λ-ring.

4.28. Construction. — Let R be an A-algebra such that for all primes p, R is p-torsion free, the p-completion $\widehat{R}_p$ is p-quasi-lci over $\widehat{A}_p$, and R/p is relatively semiperfect over $\widehat{A}_p$. We construct $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_{R/A}$ as the pullback

$$\begin{array}{ccc} \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_{R/A} & \longrightarrow & \prod_p \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_{\widehat{R}_p/\widehat{A}_p} \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{fil}^\star_{(\mathrm{Hdg},q-1)}\big(\mathrm{dR}_{R/A} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Q}\big)[\![q-1]\!] & \longrightarrow & \mathrm{fil}^\star_{(\mathrm{Hdg},q-1)}\Big(\prod_p \mathrm{dR}_{\widehat{R}_p/\widehat{A}_p} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Q}\Big)[\![q-1]\!] \end{array}$$

taken in the ∞-category filtered $\mathbb{E}_\infty$-algebras over $(q-1)^\star A[\![q-1]\!]$. To see that the right vertical map in the pullback exists, observe that we're dealing with two filtrations by submodules, so there's only a set-level condition to check, which follows directly from the definition of $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_{\widehat{S}_p/\widehat{A}_p}$.

4.29. Theorem. — *Let A be a perfectly covered Λ-ring and let $\mathrm{QReg}^{q\text{-Hdg}}_A$ be the category of all A-algebras R such that for all primes p, R is p-torsion free, the p-completion $\widehat{R}_p$ is p-quasi-lci over $\widehat{A}_p$, R/p is relatively semiperfect over $\widehat{A}_p$, and the canonical morphism from Construction 4.21 induces an equivalence*

$$\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_{\widehat{R}_p/\widehat{A}_p}/(q-1) \xrightarrow{\simeq} \mathrm{fil}^\star_{\mathrm{Hdg}}\, \mathrm{dR}_{\widehat{R}_p/\widehat{A}_p}\,.$$

Then Construction 4.28 determines a functor

$$\big(-, \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_{-/A}\big)\colon \mathrm{QReg}^{q\text{-Hdg}}_A \longrightarrow \mathrm{CAlg}\big(\mathrm{AniAlg}^{q\text{-Hdg}}_A\big)\,,$$

which is a partial section of the forgetful functor $\mathrm{CAlg}(\mathrm{AniAlg}^{q\text{-Hdg}}_A) \to \mathrm{AniAlg}_A$.

Proof sketch. Let us construct the required data from Definition 3.2. In degree 0, the pullback square from Construction 4.28 becomes the one from Construction A.12, which provides the datum from Definition 3.2(a). If we reduce the pullback from Construction 4.28 modulo $(q-1)$, we'll get the arithmetic fracture square for $\mathrm{fil}^\star_{\mathrm{Hdg}}\, \mathrm{dR}_{R/A}$ by our assumptions on R. This provides the data from Definition 3.2(b). Similarly, if we apply $(-\otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Q})^\wedge_{(q-1)}$ or $(-)^\wedge_p[1/p]^\wedge_{(q-1)}$ to the pullback, we get the data from Definition 3.2(c) and (c_p).

So $(R, \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_{R/A})$ can be made into an object of $\mathrm{AniAlg}^{q\text{-Hdg}}_A$. Since all constructions above can also be done on the level of filtered $\mathbb{E}_\infty$-algebras, it immediately upgrades to an object of $\mathrm{CAlg}(\mathrm{AniAlg}^{q\text{-Hdg}}_A)$. Finally, all steps of the construction can easily be made functorial in R. To this end, one writes $\mathrm{CAlg}(\mathrm{AniAlg}_A^{q\text{-Hdg}})$ as an iterated pullback of $\mathrm{CAlg}(-)$ of various symmetric monoidal ∞-categories of filtered objects. We know how to make $\mathrm{fil}^\star_{(\mathrm{Hdg},q-1)}(\mathrm{dR}_{R/A} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$ functorial; in the other factors of the iterated pullback, the objects in question will be 1-categorical in nature, so all functorialities and compatibilities can easily be constructed by hand. $\square$

4.30. Remark. — Thanks to Theorem 4.22, it's easy to write down objects of $\mathrm{QReg}_A^{q\text{-Hdg}}$. For example, it contains the category $\mathrm{QReg}_A^{\not{\sim}}$ of A-algebras R which are p-torsion free for all primes p and can be written in the form $R \cong B/J$, where B is a relatively perfect Λ-A-algebra (by which we mean that the relative Adams operations $\psi^m_{B/A}\colon B \otimes_{A,\psi^m} A \to B$ are isomorphisms) and $J \subseteq B$ is an ideal generated by a Koszul-regular sequence of higher powers, that is, a Koszul-regular sequence $(x_1^{\alpha_1}, \dots, x_r^{\alpha_r})$ with $\alpha_i \geqslant 2$ for all i.

4.31. Remark. — We can not only equip $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_{R/A}$ with a filtered $\mathbb{E}_\infty$-algebra structure, but even with the structure of a filtered derived commutative $(q-1)^\star A[\![q-1]\!]$-algebra as in 3.51, and the various compatibilities all respect this structure.

4.32. Monoidality. — Similar to 4.12, the functor from Theorem 4.29 can be equipped with an ∞-operad structure. To this end, let

$$\mathrm{QReg}_A^{q\text{-Hdg},\otimes} \subseteq \mathrm{AniAlg}_A^{\otimes}$$

be the non-full sub-∞-operad spanned by those objects whose entries are all contained in $\mathrm{QReg}_A^{q\text{-Hdg}}$ and those morphisms that factor through a cocartesian lift of its image in Fin_* (compare the construction of $\mathrm{Sm}^{\otimes}_{A[\dim!^{-1}]}$ in 4.12).

Note that $\mathrm{QReg}_A^{q\text{-Hdg},\otimes} \to \mathrm{Fin}_*$ is not a cocartesian fibration, because $\mathrm{QReg}_A^{q\text{-Hdg}}$ is not closed under tensor products in AniAlg_A. The problem is that $R_1 \otimes^{\mathrm{L}}_A R_2$ might not be static or not p-torsion free for some prime p. As we'll see momentarily, this is the only obstruction.

4.33. Lemma. — *Let $R_1, R_2 \in \mathrm{QReg}_A^{q\text{-Hdg}}$ and put $R \coloneqq R_1 \otimes^{\mathrm{L}}_A R_2$.*

(*a*) *If R is static and p-torsion free for all primes p, then also $R \in \mathrm{QReg}_A^{q\text{-Hdg}}$.*

(*b*) *In the situation from (a) the canonical map*

$$\left(\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_{R_1/A} \otimes^{\mathrm{L}}_{(q-1)^\star A[\![q-1]\!]} \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_{R_2/A}\right)^\wedge_{(q-1)} \xrightarrow{\simeq} \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_{R/A}$$

is an equivalence of filtered $\mathbb{E}_\infty$-algebras over $(q-1)^\star A[\![q-1]\!]$.

Proof. Let p be any prime. Using $\mathrm{L}_{R/A} \simeq (\mathrm{L}_{R_1/A} \otimes^{\mathrm{L}}_A R_2) \oplus (R_1 \otimes^{\mathrm{L}}_A \mathrm{L}_{R_2/A})$, it's clear that $\widehat{R}_p$ is again p-quasi-lci over $\widehat{A}_p$. Similarly, R/p will still be relatively semiperfect over $\widehat{A}_p$. To show $R \in \mathrm{QReg}_A^{q\text{-Hdg}}$, it remains to verify that

$$\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_{\widehat{R}_p/\widehat{A}_p}/(q-1) \xrightarrow{\simeq} \mathrm{fil}^\star_{\mathrm{Hdg}}\, \mathrm{dR}_{\widehat{R}_p/\widehat{A}_p}\,.$$

is an equivalence. By Lemma 4.26, only surjectivity needs to be checked. But since we have $\mathrm{fil}^\star_{\mathrm{Hdg}}\, \mathrm{dR}_{\widehat{R}_p/\widehat{A}_p} \simeq (\mathrm{fil}^\star_{\mathrm{Hdg}}\, \mathrm{dR}_{\widehat{R}_{1,p}/\widehat{A}_p} \otimes^{\mathrm{L}}_A \mathrm{fil}^\star_{\mathrm{Hdg}}\, \mathrm{dR}_{\widehat{R}_{2,p}/\widehat{A}_p})^\wedge_p$, surjectivity for R follows from the analogous assertions for R_1 and R_2. This shows (a).

To show (b), we can reduce both sides modulo $(q-1)$ and then once again reduce to the well-known fact that $\mathrm{fil}^\star_{\mathrm{Hdg}}\, \mathrm{dR}_{-/A}$ is symmetric monoidal. $\square$

4.34. Corollary. — *The functor from Theorem 4.29 underlies a functor of ∞-operads*

$$\mathrm{QReg}_A^{q\text{-}\mathrm{Hdg},\otimes} \longrightarrow \mathrm{CAlg}\big(\mathrm{AniAlg}_A^{q\text{-}\mathrm{Hdg}}\big)^{\otimes}\,,$$

which preserves all cocartesian lifts that exist in the source. In particular, when we restrict to the full subcategory $\mathrm{QReg}_A^{q\text{-}\mathrm{Hdg},\flat} \subseteq \mathrm{QReg}_A^{q\text{-}\mathrm{Hdg}}$ spanned by those R that are flat over A, the functor from Theorem 4.29 is symmetric monoidal.

Proof sketch. To construct the functor of ∞-operads, we repeat the argument from the proof of Theorem 4.29: Write $\mathrm{CAlg}(\mathrm{AniAlg}_A^{q\text{-}\mathrm{Hdg}})^{\otimes}$ as an iterated pullback of $\mathrm{CAlg}(-)^{\otimes}$ of various symmetric monoidal ∞-categories of filtered objects. For $\mathrm{fil}^{\star}_{(\mathrm{Hdg},q-1)}(\mathrm{dR}_{-/A} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$ we know what to do, for all other factors of the iterated pullbacks the objects in question are 1-categorical in nature, so everything can be constructed by hand.

That all existing cocartesian lifts are preserved boils down to Lemma 4.33(b). Finally, if $R_1, R_2 \in \mathrm{QReg}_A^{q\text{-}\mathrm{Hdg}}$ are flat over A, then $R_1 \otimes^{\mathrm{L}}_A R_2$ will be static and p-torsion free for all p, so Lemma 4.33(a) implies that the full sub-∞-operad of $\mathrm{QReg}_A^{q\text{-}\mathrm{Hdg},\otimes}$ spanned by $\mathrm{QReg}_A^{q\text{-}\mathrm{Hdg},\flat}$ will be a cocartesian fibration. Since our map preserves all cocartesian lifts, we deduce that we indeed get a symmetric monoidal functor

$$\big(-, \mathrm{fil}^{\star}_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{-/A}\big)\colon \mathrm{QReg}_A^{q\text{-}\mathrm{Hdg},\flat} \longrightarrow \mathrm{CAlg}\big(\mathrm{AniAlg}_A^{q\text{-}\mathrm{Hdg}}\big)\,. \qquad \square$$

Part II.
q-de Rham cohomology and topological Hochschild homology over ku

In this part, we'll prove a version of Antieau's Theorem 1.20 over the connective complex K-theory spectrum ku. We'll show in Theorem 7.27 that for any quasi-syntomic ring R (with $2 \in R^\times$), if R admits a lift to an $\mathbb{E}_2$-ring spectrum $\mathbb{S}_R$ such that $R \simeq \mathbb{S}_R \otimes \mathbb{Z}$, then the derived q-de Rham complex $q\text{-dR}_{R/\mathbb{Z}}$ can be equipped with a q-Hodge filtration $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_{R/\mathbb{Z}}$ in the sense of Definition 3.2 in such a way that the associated graded of the even filtration

$$\Sigma^{-2*} \mathrm{gr}^*_{\mathrm{ev},hS^1} \mathrm{TC}^-(\mathrm{ku} \otimes \mathbb{S}_R/\mathrm{ku}) \simeq \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{R/\mathbb{Z}}$$

is the completion of this filtration (up to shift). We'll also show a version if $\mathbb{S}_R$ is only $\mathbb{E}_1$ under additional hypotheses.

This provides another large supply of examples to which the Habiro descent from Theorem 3.11 can be applied. For these examples the Habiro descent can also be obtained homotopically: First, the q-Hodge complex arises as $q\text{-Hdg}_{R/\mathbb{Z}} \simeq \mathrm{gr}^0_{\mathrm{ev},hS^1} \mathrm{TC}^-(\mathrm{KU} \otimes \mathbb{S}_R/\mathrm{KU})$. To get its descent to the Habiro ring, we use the cyclotomic structure on $\mathrm{THH}(\mathbb{S}_R)$ and the usual genuine equivariant structure on KU to see that for all $m \in \mathbb{N}$ the action of the cyclic subgroup $C_m \subseteq S^1$ on $\mathrm{THH}(\mathrm{KU} \otimes \mathbb{S}_R/\mathrm{KU}) \simeq \mathrm{THH}(\mathbb{S}_R) \otimes \mathrm{KU}$ can be refined to a genuine action. We'll then construct appropriate even filtrations on $(\mathrm{THH}(\mathrm{KU} \otimes \mathbb{S}_R/\mathrm{KU})^{C_m})^{hS^1/C_m}$ and recover the Habiro descent $q\text{-}\mathcal{H}\mathrm{dg}_{R/\mathbb{Z}}$ in Theorem 8.63 via

$$q\text{-}\mathcal{H}\mathrm{dg}_{R/\mathbb{Z}} \simeq \lim_{m \in \mathbb{N}} \mathrm{gr}^0_{\mathrm{ev},S^1} \left(\mathrm{THH}(\mathrm{KU} \otimes \mathbb{S}_R/\mathrm{KU})^{C_m}\right)^{h(S^1/C_m)}.$$

Overview of Part II. — This part is organised as follows: In §5 we'll develop a version of the even filtration in the solid condensed setting. This makes it much easier to compare, for example, even filtrations on $\mathrm{THH}(-/\mathrm{ku})^\wedge_p$ and $\mathrm{THH}(-/\mathbb{Q}[\beta])$, but it may also be of independent interest. In §6, we'll apply this construction to $\mathrm{THH}(-/\mathrm{ku})$ and show that it satisfies all expected properties. In §7 we'll explain the connection to q-de Rham cohomology. In §8 we'll show how the Habiro descent from Theorem 3.11 can also be recovered in this framework. Finally, we'll use the short section §9 to discuss several examples, including the one mentioned below.

Relation to work of Devalapurkar and Raksit. — It will become apparent to the reader that the results in §7 are very closely connected to work of Sanath Devalapurkar and Arpon Raksit. In fact, it was Raksit who first computed in unpublished work that

$$\Sigma^{-2*} \mathrm{gr}^*_{\mathrm{ev},hS^1} \mathrm{TC}^-\left(\mathrm{ku}[x]/\mathrm{ku}\right) \simeq \mathrm{fil}^\star_{q\text{-Hdg},\square}\, q\text{-}\Omega^*_{\mathbb{Z}[x]/\mathbb{Z},\square}$$

is the explicit q-Hodge filtration from 1.11. This amazing observation has motivated much of our work in Parts II and III, and we'll give it a proof in Theorem 9.10.

Our proof of Theorem 7.27 crucially uses the equivalence $\mathrm{THH}(\mathbb{Z}_p)^\wedge_p \simeq \tau_{\geqslant 0}(j^{tC_p})$ of Devalapurkar and Raksit ([DR25, Theorem 0.1.4]; reproduced as Theorem 7.13 below) as well as its variant $\mathrm{THH}(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p[\![q-1]\!])^\wedge_p \simeq \tau_{\geqslant 0}(\mathrm{ku}^{tC_p})$ from Devalapurkar's thesis ([Dev25, Theorem 6.4.1]; reproduced as Theorem 7.2 below). That these results could be used to prove a result like Theorem 7.27 has already been anticipated in Devalapurkar's thesis; see e.g. the discussion after [Dev25, Corollary 6.4.2].

§5. The solid even filtration

In this section we'll sketch how to adapt Pstrągowski's perfect even filtration [Pst23] to $\mathbb{E}_1$-algebras in solid condensed spectra. This facilitates many p-completion arguments later on. However, as we'll see, not all of the nice properties of the perfect even filtration carry over to the solid condensed case. But in the cases we need—and probably most cases of interest in general—it works as expected. It would be desirable to develop a more complete (and perhaps less naive) theory of the perfect even filtration in the condensed setting.

Before we begin, let us briefly recall the solid condensed setting. There are no properly published sources yet, so we have to refer the reader to the recordings of [CS24] and the unfinished notes [RC24a].

5.1. Solid condensed recollections. — Let $\mathrm{Cond}(\mathrm{Sp})$ denote the ∞-category of (*light*) *condensed spectra*, that is, hypersheaves of spectra on the site of light profinite sets as defined by Clausen and Scholze [CS24]. The evaluation at the point $(-)(*)\colon \mathrm{Cond}(\mathrm{Sp}) \to \mathrm{Sp}$ admits a fully faithful symmetric monoidal left adjoint $\underline{(-)}\colon \mathrm{Sp} \to \mathrm{Cond}(\mathrm{Sp})$, sending a spectrum X to the *discrete* condensed spectrum $\underline{X}$.

One can develop a theory of *solid condensed spectra* along the lines of [CS24, Lectures 5–6]. Let $\mathrm{Null} \coloneqq \mathrm{cofib}(\mathbb{S}[\{\infty\}] \to \mathbb{S}[\mathbb{N}\cup\{\infty\}])$ be the free condensed spectrum on a null sequence. Let $\sigma\colon \mathrm{Null} \to \mathrm{Null}$ be the endomorphism induced by the shift map $(-)+1\colon \mathbb{N}\cup\{\infty\} \to \mathbb{N}\cup\{\infty\}$. Recall that a condensed spectrum M is called *solid* if

$$1-\sigma^*\colon \underline{\mathrm{Hom}}_{\mathbb{S}}(\mathrm{Null}, M) \xrightarrow{\simeq} \underline{\mathrm{Hom}}_{\mathbb{S}}(\mathrm{Null}, M)$$

is an equivalence, where $\underline{\mathrm{Hom}}_{\mathbb{S}}$ denotes the internal Hom in $\mathrm{Cond}(\mathrm{Sp})$. We let $\mathrm{Sp}_{\blacksquare} \subseteq \mathrm{Cond}(\mathrm{Sp})$ denote the full sub-∞-category of solid condensed spectra. Then $\mathrm{Sp}_{\blacksquare}$ is closed under all limits and colimits. This implies that the inclusion $\mathrm{Sp}_{\blacksquare} \subseteq \mathrm{Cond}(\mathrm{Sp})$ admits a left adjoint $(-)^{\blacksquare}\colon \mathrm{Cond}(\mathrm{Sp}) \to \mathrm{Sp}_{\blacksquare}$. It satisfies $(M\otimes N)^{\blacksquare} \simeq (M^{\blacksquare}\otimes N)^{\blacksquare}$, which allows us to endow $\mathrm{Sp}_{\blacksquare}$ with a symmetric monoidal structure, called the *solid tensor product*, via $M\otimes^{\blacksquare} N \coloneqq (M\otimes N)^{\blacksquare}$.

5.2. Solid condensed spectra and p-completions. — If X is a p-complete spectrum, then $\underline{X}$ is usually *not* p-complete in $\mathrm{Cond}(\mathrm{Sp})$ because $\underline{(-)}$ doesn't commute with limits. After passing to p-completions, we still get an adjunction on p-complete objects $\underline{(-)}_p^\wedge\colon \mathrm{Sp}_p^\wedge \rightleftarrows \mathrm{Cond}(\mathrm{Sp})_p^\wedge : (-)(*)$ and the left adjoint is still fully faithful because the unit is still an equivalence.

It's straightforward to check that any discrete condensed spectrum is solid. By closure under limits it follows that $\underline{(-)}_p^\wedge\colon \mathrm{Sp}_p^\wedge \to \mathrm{Cond}(\mathrm{Sp})_p^\wedge$ takes values in $\mathrm{Sp}_{\blacksquare}$. The solid tensor product has the magical property that if M and N are p-complete and bounded below solid condensed spectra, then $M\otimes^{\blacksquare} N$ is again p-complete; see [CS24, Lecture 6] or [Bos23, Proposition A.3]. In particular, the fully faithful embedding $\underline{(-)}_p^\wedge\colon \mathrm{Sp}_p^\wedge \to \mathrm{Sp}_{\blacksquare}$ is symmetric monoidal when restricted to bounded below objects.

§5.1. Definitions and basic properties

In the following we let R be an $\mathbb{E}_1$-algebra in the symmetric monoidal ∞-category of solid condensed spectra $\mathrm{Sp}_{\blacksquare}$ and we let

$$-\otimes_R^{\blacksquare} -\colon \mathrm{RMod}_R(\mathrm{Sp}_{\blacksquare}) \times \mathrm{LMod}_R(\mathrm{Sp}_{\blacksquare}) \longrightarrow \mathrm{Sp}_{\blacksquare}$$

denote the relative tensor product over R. We start setting up the theory in a completely analogous way to [Pst23, §§2–3].

5.3. Solid perfect even modules. — We let $\mathrm{Null}_R := R \otimes^{\blacksquare} \mathrm{Null}^{\blacksquare}$, where we define $\mathrm{Null} := \mathrm{cofib}(\mathbb{S}[\{\infty\}] \to \mathbb{S}[\mathbb{N} \cup \{\infty\}])$ to be the free condensed spectrum on a null sequence as in 5.1. It can be shown that the solidification $\mathrm{Null}^{\blacksquare}$ agrees with $\prod_{\mathbb{N}} \mathbb{S}$ and defines a compact generator of $\mathrm{Sp}_{\blacksquare}$, so that Null_R is a compact generator of $\mathrm{LMod}_R(\mathrm{Sp}_{\blacksquare})$.

We say that an R-module Q is *solid perfect even* if it is contained in the smallest sub-∞-category

$$\mathrm{Perf}_{\mathrm{ev}}(R_{\blacksquare}) \subseteq \mathrm{LMod}_R(\mathrm{Sp}_{\blacksquare})$$

which contains $\Sigma^{2n}\mathrm{Null}_R$ for all $n \in \mathbb{Z}$ and is closed under extensions and retracts.

Note that $R[S]^{\blacksquare}$ is solid perfect even for all light condensed sets S. Also note that in contrast to the uncondensed situation, it is no longer true that $\mathrm{Perf}_{\mathrm{ev}}(R_{\blacksquare})$ is closed under duals. Already for $R = \mathbb{S}$ we have $\mathrm{Hom}_{\mathbb{S}}(\mathrm{Null}_{\mathbb{S}}, \mathbb{S}) \simeq \bigoplus_{\mathbb{N}} \mathbb{S}$, which is not solid perfect even. This is ultimately the reason why the solid theory is not quite as nice.

5.4. The solid even filtration. — Equip $\mathrm{Perf}_{\mathrm{ev}}(R_{\blacksquare})$ with a Grothendieck topology in which covers are maps $P \to Q$ whose fibre is again solid perfect even. Every left-R-module M defines a $\mathrm{Sp}_{\blacksquare}$-valued sheaf on the additive site $\mathrm{Perf}_{\mathrm{ev}}(R_{\blacksquare})$ via

$$\underline{\mathrm{Hom}}_R(-, M)\colon \mathrm{Perf}_{\mathrm{ev}}(R_{\blacksquare})^{\mathrm{op}} \longrightarrow \mathrm{Sp}_{\blacksquare}\,.$$

We can form its truncations $\tau_{\geqslant 2n} \underline{\mathrm{Hom}}_R(-, M)$ in the sheaf ∞-category $\mathrm{Sh}(\mathrm{Perf}_{\mathrm{ev}}(R_{\blacksquare}), \mathrm{Sp}_{\blacksquare})$ and then define the *solid even filtration of M* as the sections

$$\mathrm{fil}^{\star}_{\mathrm{ev}\,/R} M := \Gamma_{\mathrm{Perf}_{\mathrm{ev}}(R_{\blacksquare})}\big(R, \tau_{\geqslant 2\star} \underline{\mathrm{Hom}}_R(-, M)\big)\,.$$

If R is clear from the context, we'll often just write $\mathrm{fil}^{\star}_{\mathrm{ev}} M$. In particular, if we write $\mathrm{fil}^{\star}_{\mathrm{ev}} R$, it is understood that we take the solid even filtration of R over itself.

For any half-integer weight w, we also define the *even sheaf of weight w*, denoted $\mathcal{F}_M(w)$, as the sheafification of the presheaf of solid abelian groups $\pi_{2w} \underline{\mathrm{Hom}}_R(-, M)\colon \mathrm{Perf}_{\mathrm{ev}}(R_{\blacksquare})^{\mathrm{op}} \to \mathrm{Ab}_{\blacksquare}$. For $w = 0$ we just write $\mathcal{F}_M := \mathcal{F}_M(0)$. We call M *solid homologically even* if $\mathcal{F}_M(w) \cong 0$ for all proper half-integers $w \in \frac{1}{2} + \mathbb{Z}$.

The results from [Pst23, §2] can be carried over verbatim to the solid setting. In particular, it's still true that an R-module E, whose condensed homotopy groups $\pi_*(E)$ are concentrated in even degrees, will be homogically even and its solid even filtration will be the double-speed Whitehead filtration $\mathrm{fil}^{\star}_{\mathrm{ev}\,/R} E \simeq \tau_{\geqslant 2\star}(E)$.

5.5. Monoidality of the solid even filtration. — The arguments from [Pst23, §3] can mostly be adapted to the solid situation, but we need some enriched ∞-category to do so.

Let us first set up the enriched setting. We use the formalism from [Hei23]. The ∞-category $\mathrm{LMod}_R(\mathrm{Sp}_{\blacksquare})$ is naturally a module over $\mathrm{Sp}_{\blacksquare}$ in Pr^{L} and so it will be enriched in the sense of [Hei23]. Explicitly, for left R-modules M and N, the mapping spectrum $\mathrm{Hom}_R(M, N)$ comes with a natural condensed structure $\underline{\mathrm{Hom}}_R(M, N)$ which will be solid if N is (we've already used this in 5.4). Restricting the module structure, we see that $\mathrm{LMod}_R(\mathrm{Sp}_{\blacksquare})$ is also a module over the connective part $\mathrm{Sp}_{\blacksquare, \geqslant 0}$ in Pr^{L}, which yields an enrichment given by $\tau_{\geqslant 0} \underline{\mathrm{Hom}}_R(M, N)$. The full sub-$\infty$-category $\mathrm{Perf}_{\mathrm{ev}}(R_{\blacksquare}) \subseteq \mathrm{LMod}_R(\mathrm{Sp}_{\blacksquare})$ inherits an enrichment over $\mathrm{Sp}_{\blacksquare, \geqslant 0}$. There is an established notion of an *enriched presheaf ∞-category* $\mathrm{PSh}^{\mathrm{Sp}_{\blacksquare, \geqslant 0}}(\mathrm{Perf}_{\mathrm{ev}}(R_{\blacksquare}), \mathrm{Sp}_{\blacksquare, \geqslant 0})$ with an enriched Yoneda embedding; see [Hin20; Hei25]. By considering enriched presheaves which are additive and local with respect to all covering sieves, we can also define an enriched version of additive sheaves. To avoid cumbersome notation, we'll drop the superscript and just write

$\mathrm{Sh}_\Sigma(\mathrm{Perf}_{\mathrm{ev}}(R_\blacksquare), \mathrm{Sp}_{\blacksquare,\geqslant 0})$ and $\mathrm{Sh}_\Sigma(\mathrm{Perf}_{\mathrm{ev}}(R_\blacksquare), \mathrm{Sp}_\blacksquare)$ in the following, implicitly assuming that all sheaves are enriched over $\mathrm{Sp}_{\blacksquare,\geqslant 0}$.

Let us now explain how to adapt [Pst23, §3] to turn the solid even filtration into a lax symmetric monoidal functor

$$\mathrm{fil}^\star_{\mathrm{ev}\,/-}(-)\colon \mathrm{LMod}(\mathrm{Sp}_\blacksquare) \longrightarrow \mathrm{LMod}(\mathrm{Fil}\,\mathrm{Sp}_\blacksquare)\,.$$

Let $\mathcal{U}^{\geqslant 0}$ and $\mathcal{U}$ denote the cocartesian unstraightenings of the functors lax symmetric monoidal functors $R \mapsto \mathrm{Sh}_\Sigma(\mathrm{Perf}_{\mathrm{ev}}(R_\blacksquare), \mathrm{Sp}_{\blacksquare,\geqslant 0})$ and $\mathrm{Sh}_\Sigma(\mathrm{Perf}_{\mathrm{ev}}(R_\blacksquare), \mathrm{Sp}_\blacksquare)$. The ∞-category of enriched (pre)sheaves satisfies a similar universal property as usual; see [Hei23, Theorem 5.1]. As in [Pst23, Construction 3.8], we obtain a symmetric monoidal natural transformation between the lax symmetric monoidal functors $R \mapsto \mathrm{Sh}_\Sigma(\mathrm{Perf}_{\mathrm{ev}}(R_\blacksquare), \mathrm{Sp}_{\blacksquare,\geqslant 0})$ and $R \mapsto \mathrm{LMod}_R(\mathrm{Sp}_\blacksquare)$. Applying unstraightening, we obtain a diagram

$$\begin{array}{ccc} \mathcal{U}^{\geqslant 0} & \xrightarrow{\quad F \quad} & \mathrm{LMod}(\mathrm{Sp}_\blacksquare) \\ & \searrow \quad \swarrow & \\ & \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}_\blacksquare) & \end{array}$$

where the vertical arrows are cocartesian fibrations and the top horizontal arrow F is symmetric monoidal.

The functor F admits a fibre-wise right adjoint: In the fibre over R, the right adjoint is given by the restricted enriched Yoneda embedding $\mathrm{LMod}_R(\mathrm{Sp}_\blacksquare) \to \mathrm{Sh}_\Sigma(\mathrm{Perf}_{\mathrm{ev}}(R_\blacksquare), \mathrm{Sp}_{\blacksquare,\geqslant 0})$ sending $M \mapsto \tau_{\geqslant 0}\,\underline{\mathrm{Hom}}_R(-, M)$. Since our sheaves take values in $\mathrm{Sp}_{\blacksquare,\geqslant 0}$, the truncation can be performed section-wise and no sheafification is necessary. By [L-HA, Corollary 7.3.2.7], the fibre-wise right adjoints assemble into a lax symmetric monoidal right adjoint $G\colon \mathrm{LMod}(\mathrm{Sp}_\blacksquare) \to \mathcal{U}^{\geqslant 0}$. We'll now study the composition

$$\mathrm{LMod}(\mathrm{Sp}_\blacksquare) \xrightarrow{G} \mathcal{U}^{\geqslant 0} \longrightarrow \mathcal{U}\,.$$

In the fibre over R, this composition is given by sending $M \mapsto \nu_R(M) \coloneqq \tau_{\geqslant 0}\,\underline{\mathrm{Hom}}_R(-, M)$, where now the truncation is performed in $\mathrm{Sh}_\Sigma(\mathrm{Perf}_{\mathrm{ev}}(R_\blacksquare), \mathrm{Sp}_\blacksquare)$.

Another application of the universal property [Hei23, Theorem 5.1] allows us to extend the lax symmetric monoidal functor $\tau_{\geqslant -2\star}\,\underline{\mathrm{Hom}}_{\mathbb{S}}(-, \mathbb{S})\colon \mathbb{Z} \to \mathrm{Sh}_\Sigma(\mathrm{Perf}_{\mathrm{ev}}(\mathbb{S}_\blacksquare), \mathrm{Sp}_\blacksquare)$ to a lax symmetric monoidal functor

$$\mathrm{Fil}\,\mathrm{Sp}_\blacksquare \longrightarrow \mathrm{Sh}_\Sigma\big(\mathrm{Perf}_{\mathrm{ev}}(\mathbb{S}_\blacksquare), \mathrm{Sp}_\blacksquare\big)$$

As in [Pst23, Construction 3.20], for any $R \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}_\blacksquare)$, $\mathrm{Sh}_\Sigma(\mathrm{Perf}_{\mathrm{ev}}(R_\blacksquare), \mathrm{Sp}_\blacksquare)$ is a module over $\mathrm{Sh}_\Sigma(\mathrm{Perf}_{\mathrm{ev}}(\mathbb{S}_\blacksquare), \mathrm{Sp}_\blacksquare)$ and thus over $\mathrm{Fil}\,\mathrm{Sp}_\blacksquare$. Therefore, if X and Y are $\mathrm{Sp}_\blacksquare$-valued sheaves on $\mathrm{Perf}_{\mathrm{ev}}(R_\blacksquare)$, we can define a filtered solid condensed mapping spectrum $\underline{\mathrm{Hom}}^\star(X, Y)$. Using the enriched Yoneda lemma of [Hin20], we can argue as in [Pst23, Lemma 3.23] to show

$$\underline{\mathrm{Hom}}^\star\big(\nu_R(R), \nu_R(M)\big) \simeq \mathrm{fil}^\star_{\mathrm{ev}\,/R}\, M\,.$$

Now consider the functor $R \mapsto \mathrm{Sh}_\Sigma(\mathrm{Perf}_{\mathrm{ev}}(R_\blacksquare), \mathrm{Sp}_\blacksquare)$. As in [Pst23, Construction 3.27] we can refine it to a lax symmetric monoidal functor $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}_\blacksquare) \to \mathrm{Alg}_{\mathbb{E}_0}(\mathrm{Mod}_{\mathrm{Fil}\,\mathrm{Sp}_\blacksquare}(\mathrm{Pr}^{\mathrm{L}}))$.

We don't know if this functor factors through the image of the fully faithful embedding $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Fil}\,\mathrm{Sp}_\blacksquare) \hookrightarrow \mathrm{Alg}_{\mathbb{E}_0}(\mathrm{Mod}_{\mathrm{Fil}\,\mathrm{Sp}_\blacksquare}(\mathrm{Pr}^{\mathrm{L}}))$, as it does in the uncondensed setting.[(5.1)] But this fully faithful embedding has a right adjoint by [L-HA, Theorem 4.8.5.11], which sends a $\mathrm{Fil}\,\mathrm{Sp}_\blacksquare$-module $\mathcal{M}$ with a distinguished object $X \in \mathcal{M}$ to $\underline{\mathrm{End}}^\star(X) \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Fil}\,\mathrm{Sp}_\blacksquare)$. Composing with this right adjoint allows us to turn $R \mapsto \mathrm{fil}^\star_{\mathrm{ev}\,/R}\,R$ into a lax symmetric monoidal functor

$$\mathrm{fil}^\star_{\mathrm{ev}\,/-}(-)\colon \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}_\blacksquare) \longrightarrow \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Fil}\,\mathrm{Sp}_\blacksquare)$$

and provides a symmetric monoidal natural transformation from $R \mapsto \mathrm{Sh}_\Sigma(\mathrm{Perf}_{\mathrm{ev}}(R_\blacksquare), \mathrm{Sp}_\blacksquare)$ to $R \mapsto \mathrm{Mod}_{\mathrm{fil}^\star_{\mathrm{ev}\,/R}R}(\mathrm{Fil}\,\mathrm{Sp}_\blacksquare)$. The unstraightening of the latter functor is the the pullback of $\mathrm{LMod}(\mathrm{Fil}\,\mathrm{Sp}_\blacksquare) \to \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Fil}\,\mathrm{Sp}_\blacksquare)$ along $\mathrm{fil}_{\mathrm{ev}\,/-}(-)^\star$. We obtain a diagram

$$\begin{array}{ccc} \mathcal{U} & \longrightarrow & \mathrm{LMod}(\mathrm{Fil}\,\mathrm{Sp}_\blacksquare) \\ \downarrow & & \downarrow \\ \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}_\blacksquare) & \xrightarrow{\mathrm{fil}^\star_{\mathrm{ev}\,/-}(-)} & \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Fil}\,\mathrm{Sp}_\blacksquare) \end{array}$$

with lax symmetric monoidal horizontal arrows. We can now finally define a functorial lax symmetric monoidal solid even filtration as the composite

$$\mathrm{fil}_{\mathrm{ev}\,/-}(-)\colon \mathrm{LMod}(\mathrm{Sp}_\blacksquare) \xrightarrow{G} \mathcal{U}^{\geqslant 0} \longrightarrow \mathcal{U} \longrightarrow \mathrm{LMod}(\mathrm{Fil}\,\mathrm{Sp}_\blacksquare)\,.$$

5.6. Calculus of solid evenness. — Deviating from [Pst23, Definition 4.4], let us call a left-R-module M *solid ind-perfect even* if it can be written as a filtered colimit of solid perfect evens, and *solid even flat* if $\pi_*(E \otimes^\blacksquare_R M)$ is concentrated in even degrees for any right-R-module E such that $\pi_*(E)$ is concentrated in even degrees. In the uncondensed setting these notions are equivalent by the "even Lazard theorem" [Pst23, Theorem 4.14]. In the solid setting it is still true that solid ind-perfect even modules are solid even flat (as we'll see). However, we don't know if the converse is true. Similarly, we don't know if [Pst23, Theorem 4.16] still works. In §5.2, we'll discuss what the problem is, and in §5.3 we'll see how to fix this, at least under certain additional assumptions.

Despite these problems, the formalism of *π_*-even envelopes* can entirely be carried over to the solid setting: Any left-R-module M admits a map $M \to E$ such that:

(*a*) $\mathrm{cofib}(M \to E)$ is ind-solid perfect even.

(*b*) $\pi_*(E)$ is concentrated in even degrees.

(*c*) for any other map $M \to F$ into a left-R-module F such that $\pi_*(F)$ is even, a dashed arrow can be found to make the following diagram commutative:

$$\begin{array}{ccccc} & & M & & \\ & \swarrow & & \searrow & \\ E & & \dashrightarrow & & F \end{array}$$

[(5.1)] In particular, we don't know if the analogue of [Pst23, Proposition 3.26] is true, i.e. whether

$$\underline{\mathrm{Hom}}^\star(\nu_R(R), -)\colon \mathrm{Sh}_\Sigma(\mathrm{Perf}_{\mathrm{ev}}(R_\blacksquare), \mathrm{Sp}_\blacksquare) \longrightarrow \mathrm{LMod}_{\underline{\mathrm{End}}^\star(\nu_R(R))}(\mathrm{Fil}\,\mathrm{Sp}_\blacksquare)$$

is an equivalence. The problem is that the even filtration $\mathrm{fil}^\star_{\mathrm{ev}\,/R}(M)$ only knows about the values of the sheaf $\tau_{\geqslant 2\star}\,\mathrm{Hom}_R(-, M)$ on R (plus even shifts, extensions, and retracts thereof), but not about the value on Null_R.

The proof is the same as in the uncondensed setting, except that we have to consider maps $\Sigma^n \mathrm{Null}_R \rightarrow M$ from odd suspensions of Null_R.

5.7. Comparison with the uncondensed theory. — Let R be a discrete solid condensed ring and let M be a discrete left-R-module. Let $\mathrm{fil}^\star_{\text{P-ev}} M$ be Pstrągowski's perfect even filtation, regarded as a filtered discrete solid spectrum. Since Pstrągowski's category $\mathrm{Perf}_{\mathrm{ev}}(R)$ is a full sub-∞-category of $\mathrm{Perf}_{\mathrm{ev}}(R_\blacksquare)$, we get a canonical comparison map

$$\mathrm{fil}^\star_{\text{P-ev}} M \longrightarrow \mathrm{fil}^\star_{\mathrm{ev}} M\,.$$

As a consequence of the fact that π_*-even envelopes still work, we obtain:

5.8. Corollary. — *With assumptions as in 5.7, let M be homologically even as a left-R-module. If M is solid homologically even as well, then the comparison map*

$$\mathrm{fil}^\star_{\text{P-ev}} M \xrightarrow{\simeq} \mathrm{fil}^\star_{\mathrm{ev}} M$$

is an equivalence. In particular, this applies if $M = R$.

Proof. It's straightforward to check that the construction of a π_*-even envelopes of M as a discrete left-R-module in [Pst23, Proposition 4.11] also yields a π_*-even envelope as a solid condensed left-R-module.[(5.2)] Assuming homological evenness, both $\mathrm{gr}^*_{\text{P-ev}} M$ and $\mathrm{gr}^*_{\mathrm{ev}} M$ can be computed by repeatedly taking π_*-even envelopes, as explained in [Pst23, §5]. It follows that $\mathrm{fil}^\star_{\text{P-ev}} M \rightarrow \mathrm{fil}^\star_{\mathrm{ev}} M$ is an equivalence on associated gradeds. Since both filtrations are exhaustive, we conclude. □

5.9. Remark. — We believe that in the context of Corollary 5.8 it's automatically true that M is solid homologically even.

§5.2. Recollections on trace-class morphisms and nuclear objects

In contrast to the mostly smooth sailing of 5.3–5.8, it's not so clear how to transport Pstrągowski's discussion of even flatness—in particular, the powerful results [Pst23, Theorems 4.14 and 4.16]—to the solid setting. The main problem is the following: In the proofs, Pstrągowski repeatedly uses the trick that a map $P \rightarrow Q$ of perfect even R-modules can be equivalently described by a map $\mathbb{S} \rightarrow P^\vee \otimes_R Q$. This doesn't work anymore in the solid setting, since most solid perfect even R-modules are *not* dualisable, the quintessential example being Null_R.

This is not the first time that such a problem occurs in solid condensed mathematics. The usual way to deal with these issues (which will also work in our case) is to replace dualisable objects by the weaker notions of *trace-class morphisms* and *nuclear objects* that we'll review in this subsection.

5.10. Trace-class morphisms. — Let $\mathcal{C}$ be a presentable symmetric monoidal[(5.3)] ∞-category. Let R be an $\mathbb{E}_1$-algebra in $\mathcal{C}$. By Lurie's adjoint functor theorem, for all left-R-modules M and N there exists an object $\underline{\mathrm{Hom}}_R(M,N) \in \mathcal{C}$ characterised by

$$\mathrm{Hom}_{\mathcal{C}}\big(-, \underline{\mathrm{Hom}}_R(M,N)\big) \simeq \mathrm{Hom}_R(M \otimes -, N)\,.$$

[(5.2)] Implicitly, we use that discrete condensed abelian groups have vanishing higher cohomology on any light profinite set; see [CS24, Lecture 4].

[(5.3)] By convention, this includes the assumption that $- \otimes -$ commutes with colimits in both variables, so the adjoint functor theorem is applicable.

We remark that $\underline{\mathrm{Hom}}_R(M, R)$ is naturally a right-R-module. A morphism $\varphi\colon M \to N$ of left-R-modules is called *trace-class* if there exists a morphism $\eta\colon \mathbb{1}_{\mathcal{C}} \to \underline{\mathrm{Hom}}_R(M, R) \otimes_R N$, such that φ is the composition

$$M \simeq M \otimes \mathbb{1}_{\mathcal{C}} \xrightarrow{\eta} M \otimes \underline{\mathrm{Hom}}_R(M, R) \otimes_R N \xrightarrow{\mathrm{ev}_M} R \otimes_R N \simeq N\,.$$

We often call η the *classifier of* φ.

Trace-class morphism have a number of nice properties. We'll often use the properties from [CS22, Lemma 8.2] as well as the following lemma.

5.11. Lemma. — *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor between presentable symmetric monoidal ∞-categories. Let $R \in \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$. By abuse of notation, we'll denote both $\underline{\mathrm{Hom}}_R(-, R)$ and $\underline{\mathrm{Hom}}_{F(R)}(-, F(R))$ by $(-)^\vee$.*

(*a*) *There exists a natural transformation $F((-)^\vee) \Rightarrow F(-)^\vee$.*

(*b*) *If $M \to N$ is a trace-class morphism in $\mathrm{LMod}_R(\mathcal{C})$, then $N^\vee \to M^\vee$ is trace-class in $\mathrm{RMod}_R(\mathcal{C})$ and $F(M) \to F(N)$ is trace-class in $\mathrm{LMod}_{F(R)}(\mathcal{D})$.*

(*c*) *The commutative square in $\mathrm{RMod}_{F(R)}(\mathcal{D})$ formed by the morphisms from (a) and (b)*

$$\begin{array}{ccc} F(N^\vee) & \longrightarrow & F(M^\vee) \\ \big\downarrow & \nearrow & \big\downarrow \\ F(N)^\vee & \longrightarrow & F(M)^\vee \end{array}$$

admits a canonical diagonal map $F(N)^\vee \to F(M^\vee)$ that makes both triangles commute.

Proof. The natural transformation from (a) is adjoint to $F((-)^\vee) \otimes_{F(R)} F(-) \Rightarrow F(R)$, which is in turn given by applying F to the evaluation $(-)^\vee \otimes_R (-) \Rightarrow R$.

Now let $M \to N$ be trace-class in $\mathrm{LMod}_R(\mathcal{C})$ with classifier $\mathbb{1}_{\mathcal{C}} \to M^\vee \otimes_R N$. If we apply F to the classifier and compose with the morphism $F(M^\vee) \to F(M)^\vee$ from (a), we obtain a morphism $\mathbb{1}_{\mathcal{D}} \to F(M^\vee) \otimes_{F(R)} F(N) \to F(M)^\vee \otimes_{F(R)} F(N)$, which serves as a classifier for $F(M) \to F(N)$. If we compose instead with $N \to N^{\vee\vee}$, we obtain $\mathbb{1}_{\mathcal{C}} \to M^\vee \otimes_R N \to M^\vee \otimes_R N^{\vee\vee}$, which serves as a classifier for $N^\vee \to M^\vee$ being trace-class. This shows (b). To show (c), we construct the diagonal map $F(N)^\vee \to F(M^\vee)$ as follows:

$$F(N)^\vee \longrightarrow F(M^\vee \otimes_R N) \otimes_{\mathcal{D}} F(N)^\vee \simeq F(M^\vee) \otimes_{F(R)} F(N) \otimes_{\mathcal{D}} F(N)^\vee \longrightarrow F(M^\vee)\,.$$

Here we use the classifier $\mathbb{1}_{\mathcal{C}} \to M^\vee \otimes_R N$ and the evaluation map for $F(N)$. ☐

5.12. Nuclear objects — In addition to the assumptions from 5.10, let us now assume that $\mathcal{C}$ is stable, compactly generated, and $\mathbb{1}_{\mathcal{C}}$ is compact.

(*a*) A left-R-module M is called *nuclear* if every morphism $P \to M$ from a compact left-R-module P is trace-class.

(*b*) We call a left-R-module M *basic nuclear* if M can be written as a sequential colimit $M \simeq \mathrm{colim}(M_0 \to M_1 \to \cdots)$ such that each transition map $M_n \to M_{n+1}$ is trace-class.

We let $\mathrm{Nuc}(\mathrm{LMod}_R(\mathcal{C})) \subseteq \mathrm{LMod}_R(\mathcal{C})$ denote the full sub-∞-category spanned by the nuclear left-R-modules.

5.13. Theorem. — *Let $\mathcal{C}$ be a presentable stable symmetric monoidal ∞-category such that $\mathcal{C}$ is compactly generated and the tensor unit $1_{\mathcal{C}} \in \mathcal{C}$ is compact. Let $R \in \mathrm{Alg}_{\mathbb{E}_1}(\mathcal{C})$*

(*a*) $\mathrm{Nuc}(\mathrm{LMod}_R(\mathcal{C})) \subseteq \mathrm{LMod}_R(\mathcal{C})$ *closed under shifts and colimits. Moreover, if M is a nuclear left-R-module and $X \in \mathrm{Nuc}(\mathcal{C})$, then $M \otimes X \in \mathrm{Nuc}(\mathrm{LMod}_R(\mathcal{C}))$.*

(*b*) $\mathrm{Nuc}(\mathrm{LMod}_R(\mathcal{C}))$ *is ω_1-compactly generated and the ω_1-compact objects are precisely the basic nuclears.*

(*c*) *If $R \to S$ is a map of $\mathbb{E}_1$-algebras in $\mathcal{C}$, then $S \otimes_R -\colon \mathrm{LMod}_R(\mathcal{C}) \to \mathrm{LMod}_S(\mathcal{C})$ preserves the full sub-∞-categories of nuclear objects.*

(*d*) *Suppose that for all compact left-R-modules P and all compact $C \in \mathcal{C}$ the tensor product $P \otimes C$ is still compact as a left-R-module. If P is compact and M is nuclear, the natural map*

$$\underline{\mathrm{Hom}}_R(P, R) \otimes_R M \xrightarrow{\simeq} \underline{\mathrm{Hom}}_R(P, M)$$

is an equivalence. Furthermore, if $R \to S$ is a map of $\mathbb{E}_1$-algebras in $\mathcal{C}$ such that S is nuclear as a left-R-module, then the forgetful functor $\mathrm{LMod}_S(\mathcal{C}) \to \mathrm{LMod}_R(\mathcal{C})$ preserves the full sub-∞-categories of nuclear objects.

Proof sketch. For parts (*a*) and (*b*), the case $R \simeq 1_{\mathcal{C}}$ is covered in [CS22, Theorem 8.6]; the arguments given therein apply verbatim for general R as well. For (*c*), it's straightforward to check that $S \otimes_R -$ preserves trace-class maps, hence basic nuclear objects and thus all nuclear objects by (*b*).

For (*d*), the assumption implies that every compact left-R-module is also *internally compact* in the sense that $\underline{\mathrm{Hom}}_R(P, -)$ preserves filtered colimits. We may thus reduce to the case where M is basic nuclear. Write M as a sequential colimit $M \simeq \mathrm{colim}(M_0 \to M_1 \to \cdots)$ with trace-class transition maps. If $\eta\colon 1_{\mathcal{C}} \to \underline{\mathrm{Hom}}_R(M_n, R) \otimes_R M_{n+1}$ is a classifier for $M_n \to M_{n+1}$ and $c\colon \underline{\mathrm{Hom}}_R(P, M_n) \otimes \underline{\mathrm{Hom}}_R(M_n, R) \to \underline{\mathrm{Hom}}_R(P, R)$ is the canonical composition map, we get a commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}_R(P, M_n) & \longrightarrow & \underline{\mathrm{Hom}}_R(P, M_{n+1}) \\ \downarrow{\scriptstyle\eta} & & \uparrow \\ \underline{\mathrm{Hom}}_R(P, M_n) \otimes \underline{\mathrm{Hom}}_R(M_n, R) \otimes_R M_{n+1} & \xrightarrow{c} & \underline{\mathrm{Hom}}_R(P, R) \otimes_R M_{n+1} \end{array}$$

Using these diagrams for all n we see that $\mathrm{colim}\, \underline{\mathrm{Hom}}_R(P, R) \otimes_R M_n \to \mathrm{colim}\, \underline{\mathrm{Hom}}_R(P, M_n)$ has an inverse. It follows that $\underline{\mathrm{Hom}}_R(P, R) \otimes_R M \simeq \underline{\mathrm{Hom}}_R(P, M)$, as desired.

Now let N be a nuclear left-S-module and let $P \to N$ be a map from a compact left-R-module. Then $S \otimes_R P \to N$ is trace-class, because it factors through $S \otimes_R P \to S \otimes_R N$ and $S \otimes_R -$ preserves trace-class morphisms. If $\eta\colon 1_{\mathcal{C}} \to \underline{\mathrm{Hom}}_S(S \otimes_R P, S) \otimes_S N$ is a classifier, we note $\underline{\mathrm{Hom}}_S(S \otimes_R P, S) \simeq \underline{\mathrm{Hom}}_R(P, S) \simeq \underline{\mathrm{Hom}}_R(P, R) \otimes_R S$ by our assumption that S is nuclear. Thus $\underline{\mathrm{Hom}}_S(S \otimes_R P, S) \otimes_S N \simeq \underline{\mathrm{Hom}}_R(P, R) \otimes_R N$ and so η is also a classifier witnessing $P \to N$ being trace-class. This shows that the forgetful functor $\mathrm{LMod}_S(\mathcal{C}) \to \mathrm{LMod}_R(\mathcal{C})$ preserves the full sub-∞-categories of nuclear objects. □

5.14. Remark. — If $\mathcal{C}_0$ is a small stable symmetric monoidal ∞-category, then Theorem 5.13 can be applied to $\mathrm{Ind}(\mathcal{C}_0)$. Since every trace-class map in $\mathrm{Ind}(\mathcal{C}_0)$ factors through a compact object by [CS22, Lemma 8.4], we see that the basic nuclear objects in $\mathrm{Ind}(\mathcal{C}_0)$ are of the form "colim"$(X_1 \to X_2 \to \cdots)$, where each $X_n \to X_{n+1}$ is trace-class in $\mathcal{C}_0$.

If $\mathcal{C}$ is a presentable stable symmetric monoidal ∞-category (hence $\mathcal{C}$ is large unless $\mathcal{C} \simeq 0$), one can still make sense of $\operatorname{Nuc}\operatorname{Ind}(\mathcal{C})$ without running into set-theoretic problems. Indeed, if κ is a sufficiently large regular cardinal such that $\mathcal{C}$ is κ-compactly generated and $\mathbb{1}$ is κ-compact, the same argument as in [CS22, Lemma 8.4] shows that every trace-class morphism in $\mathcal{C}$ factors through a κ-compact object. Then every basic nuclear object is equivalent to one in which each X_n is κ-compact and so the basic nuclear objects in form an essentially small ∞-category. We may then define $\operatorname{Nuc}\operatorname{Ind}(\mathcal{C})$ as $\operatorname{Ind}_{\omega_1}(-)$ of the ∞-category of basic nuclear objects.

§5.3. Solid even flatness in the nuclear case

In this subsection we explain that the analogues of [Pst23, Theorems 4.14 and 4.16] are still true under certain additional nuclearity assumptions.

5.15. Assumptions on *R*. — From now on let us assume that R satisfies the following condition:

(R) *$\underline{\operatorname{Hom}}_R(\operatorname{Null}_R, R)$ is nuclear and solid ind-perfect even both as a left-R-module and as a right-R-module.*

Here we use that $\operatorname{Null}_R \simeq \prod_{\mathbb{N}} \mathbb{S} \otimes^{\blacksquare} R$ is naturally a bimodule over R. Also note that Assumption (R) implies that that $\underline{\operatorname{Hom}}_R(P, R)$ is nuclear and solid ind-perfect even for any solid perfect even left- or right-R-module P.

5.16. Lemma. — *Let R° be a discrete condensed $\mathbb{E}_1$-ring spectrum and let M° be any discrete condensed left-R°-module.*

(a) *Assumption 5.15(R) is satisfied for $R = R^\circ$. Moreover, M° is nuclear as a left-R°-module.*

(b) *Assumption 5.15(R) is satisfied for $R = (R^\circ)_p^\wedge$. Moreover, if R° is connective, then $(M^\circ)_p^\wedge$ is nuclear over $(R^\circ)_p^\wedge$.*

(c) *Assumption 5.15(R) is satisfied for $R = (R^\circ)_p^\wedge[1/p]$. Moreover, if R° is connective, then $(M^\circ)_p^\wedge[1/p]$ is nuclear over $(R^\circ)_p^\wedge[1/p]$.*

Proof. In the following, we won't specify whether we're working with left- or right-R-modules, since the arguments will be valid in either case. For arbitrary solid $\mathbb{E}_1$-algebras R, we have $\underline{\operatorname{Hom}}_R(\operatorname{Null}_R, R) \simeq \underline{\operatorname{Hom}}_{\mathbb{S}}(\prod_{\mathbb{N}} \mathbb{S}, R)$. If $R = R^\circ$ is discrete, then $\underline{\operatorname{Hom}}_{\mathbb{S}}(\prod_{\mathbb{N}} \mathbb{S}, R) \simeq \bigoplus_{\mathbb{N}} R^\circ$, which is solid ind-perfect even. Since R is nuclear over itself and nuclear objects are closed under shifts and colimits, it follows that every discrete R-module is nuclear. This shows (a).

If $R = (R^\circ)_p^\wedge$, then the same argument shows $\underline{\operatorname{Hom}}_{\mathbb{S}}(\prod_{\mathbb{N}} \mathbb{S}, R) \simeq (\bigoplus_{\mathbb{N}} R^\circ)_p^\wedge$. To show the solid ind-perfect evenness condition, write

$$\left(\bigoplus_{\mathbb{N}} R^\circ\right)_p^\wedge \simeq \operatorname*{colim}_{\substack{f\colon \mathbb{N}\to\mathbb{N},\\ f(n)\to\infty}} \prod_{\mathbb{N}} p^{f(n)} R\,,$$

where the colimit is taken over all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that $f(n) \to \infty$ as $n \to \infty$. We claim that whenever $g \leqslant f$ is growing so slowly that $f(n) - g(n) \to \infty$, the transition map $\prod_{\mathbb{N}} p^{f(n)} R \to \prod_{\mathbb{N}} p^{g(n)} R$ is trace-class and factors through Null_R. This will show that every map from a compact left-R-module to $(\bigoplus_{\mathbb{N}} R^\circ)_p^\wedge$ is trace-class and factors through Null_R, so that $(\bigoplus_{\mathbb{N}} R^\circ)_p^\wedge$ is nuclear and solid ind-perfect even by the solid analogue of [Pst23, Proposition 4.3].

To show the claim, we may as well assume $g = 0$ and show that $(p^{f(n)})_{n\in\mathbb{N}}\colon \prod_{\mathbb{N}} R \to \prod_{\mathbb{N}} R$ is trace-class and factors through Null_R. Let e_n denote the n^{th} basis vector in the standard basis of $\bigoplus_{\mathbb{N}} R^\circ$. Then $\sum p^{f(n)}(e_n \otimes e_n)$ is a well-defined π_0-class in $(\bigoplus_{\mathbb{N}} \tau_{\geqslant 0}(R^\circ))_p^\wedge \otimes^{\blacksquare}_{\tau_{\geqslant 0}(R)} \prod_{\mathbb{N}} \tau_{\geqslant 0}(R)$, since the solid tensor product of connective p-complete objects will be p-complete again. The image of this π_0-class in $(\bigoplus_{\mathbb{N}} R^\circ)_p^\wedge \otimes^{\blacksquare}_R \prod_{\mathbb{N}} R$ defines a morphism

$$\mathbb{S} \longrightarrow \underline{\operatorname{Hom}}_R(\operatorname{Null}_R, R) \otimes^{\blacksquare}_R \prod_{\mathbb{N}} R\,,$$

which classifies a trace-class map $\operatorname{Null}_R \to \prod_{\mathbb{N}} R$. By inspection, this is a factorisation of $(p^{f(n)})_{n\in\mathbb{N}}\colon \prod_{\mathbb{N}} R \to \prod_{\mathbb{N}} R$, as desired.

This argument shows, in particular, that the p-completion of any countable direct sum of copies of R° is nuclear over R. We deduce the same for arbitrary direct sums, as p-completion commutes with ω_1-filtered colimits. Now suppose R° is connective. First consider the case where M° is bounded below. Let M be the p-completion of M°. Define a sequence of left-R-modules $M_0, M_1, \dots$ as follows: $M_0 \coloneqq M$; for $n \geqslant 0$, we choose a map $\bigoplus \Sigma^n R^\circ \to M_n$ that is surjective on π_n and then define $M_{n+1} \coloneqq \operatorname{cofib}(\bigoplus \Sigma^n R^\circ \to M_n)_p^\wedge$. Then $M \simeq \operatorname{colim} \operatorname{fib}(M \to M_n)$; note that the colimit doesn't need to be p-completed, since each term is p-complete and in each homotopical degree the colimit stabilises after finitely many steps. Thus, it will be enough to check that each $\operatorname{fib}(M \to M_n)$ is nuclear, which follows from our observation that p-completions of arbitrary direct sums of copies of R° are nuclear. This shows that $(M^\circ)_p^\wedge$ is nuclear in the bounded below case. For general M°, note that $(M^\circ)_p^\wedge$ and $(\tau_{\geqslant -n} M^\circ)_p^\wedge$ agree in homotopical degrees $\geqslant -n+1$. It follows that $(M^\circ)_p^\wedge \simeq \operatorname{colim}_{n\geqslant 0}(\tau_{\geqslant -n} M^\circ)_p^\wedge$. By the bounded below case, this is a (non-p-completed) colimit of nuclear objects and so $(M^\circ)_p^\wedge$ must be nuclear too. This finishes the proof of (*b*).

If $R = (R^\circ)_p^\wedge[1/p]$, then $\underline{\operatorname{Hom}}_{\mathbb{S}}(\prod_{\mathbb{N}} \mathbb{S}, R) \simeq (\bigoplus_{\mathbb{N}} R^\circ)_p^\wedge[1/p]$ by compactness of $\prod_{\mathbb{N}} \mathbb{S}$. The desired assertions then follow from (*b*) using base change for nuclear modules (Theorem 5.13(*c*)). This shows (*c*). □

Under Assumption 5.15(R), we can show the following weaker analogue of the "even Lazard theorem" [Pst23, Theorem 4.14].

5.17. Lemma. — *Let R be a solid condensed $\mathbb{E}_1$-ring spectrum and let M be a left-R-module.*

(*a*) *If M is solid ind-perfect even, then M is solid even flat.*

(*b*) *Let M be solid even flat. If R satisfies Assumption 5.15(R) and M is nuclear, then is solid ind-perfect even.*

Proof. For (*a*), we only need to check that Null_R is solid even flat. This follows from the fact that $\operatorname{Null}_{\mathbb{Z}} \simeq \prod_{\mathbb{N}} \mathbb{Z}$ is flat for the solid tensor product on $\mathrm{Ab}_{\blacksquare}$ by [CS24, Lecture 6].

For (*b*), let $\varphi\colon P \to M$ be a map from a compact left-R-module. By the solid analogue of [Pst23, Proposition 4.3], it will be enough to show that φ factors through a solid perfect even. Since M is nuclear, φ will be trace-class, with classifier $\eta\colon \mathbb{S} \to \underline{\operatorname{Hom}}_R(P, R) \otimes^{\blacksquare}_R M$. As in the proof of [Pst23, Theorem 4.14], let us choose a map $\underline{\operatorname{Hom}}_R(P, R) \to E$ whose suspension is a π_*-even envelope in right-R-modules. Then $\pi_*(E \otimes^{\blacksquare}_R M)$ is concentrated in odd degrees, hence the composite

$$\mathbb{S} \longrightarrow \underline{\operatorname{Hom}}_R(P, R) \otimes^{\blacksquare}_R M \longrightarrow E \otimes^{\blacksquare}_R M$$

must vanish.[(5.4)] It follows that the classifier η lifts to a map $\eta'\colon \mathbb{S} \to \Sigma^{-1} C \otimes^{\blacksquare}_R M$, where $C \simeq \operatorname{cofib}(\underline{\operatorname{Hom}}_R(P, R) \to E)$. By definition of π_*-even envelopes, $\Sigma^{-1}C$ is solid ind-perfect

[(5.4)] This argument still works with condensed homotopy groups since any cover of the one-point set $*$ in the site of light profinite sets is split.

even as a right-R-module. Writing $\Sigma^{-1}C$ as a filtered colimit of solid perfect evens and using that $\mathbb{S}$ is compact, we obtain a further factorisation

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{\ \eta''\ } & Q \otimes_R^{\blacksquare} M \\ & \underset{\eta}{\searrow} & \downarrow \\ & & \underline{\mathrm{Hom}}_R(P, R) \otimes_R^{\blacksquare} M \end{array}$$

where Q is solid perfect even. Assumption 5.15(R) guarantees that $\underline{\mathrm{Hom}}_R(P, R)$ is nuclear, hence the composition $Q \to \Sigma^{-1}C \to \underline{\mathrm{Hom}}_R(P, R)$ is trace-class as a map of right-R-modules. Choose a classifier $\vartheta\colon \mathbb{S} \to \underline{\mathrm{Hom}}_R(P, R) \otimes_R^{\blacksquare} \underline{\mathrm{Hom}}_R(Q, R)$. We see that the original map $\varphi\colon P \to M$ is given by tensoring P with η'' and ϑ and then applying the evaluation maps $\mathrm{ev}_Q\colon \underline{\mathrm{Hom}}_R(Q, R) \otimes^{\blacksquare} Q \to R$ and $\mathrm{ev}_P\colon P \otimes^{\blacksquare} \underline{\mathrm{Hom}}_R(P, R) \to R$. This can be done in any order, hence φ also agrees with the composition

$$P \xrightarrow{\vartheta} P \otimes^{\blacksquare} P^\vee \otimes_R^{\blacksquare} \underline{\mathrm{Hom}}_R(Q, R) \xrightarrow{\mathrm{ev}_P} \underline{\mathrm{Hom}}_R(Q, R) \xrightarrow{\eta''} \underline{\mathrm{Hom}}_R(Q, R) \otimes^{\blacksquare} Q \otimes_R^{\blacksquare} M \xrightarrow{\mathrm{ev}_M} M\,,$$

where we wrote $P^\vee \coloneqq \underline{\mathrm{Hom}}_R(P, R)$ for short. We conclude that φ factors through $\underline{\mathrm{Hom}}_R(Q, R)$. Again by Assumption 5.15(R), $\underline{\mathrm{Hom}}_R(Q, R)$ is a filtered colimit of solid perfect even left-R-modules. Since P is compact, we conclude that $\varphi\colon P \to M$ factors through a solid perfect even left-R-module, as desired. $\square$

We can also show the following weaker analogue of [Pst23, Theorem 4.16].

5.18. Lemma. — *Let R be a solid condensed $\mathbb{E}_1$-ring spectrum and let M be a left-R-module.*

(a) *M is solid homologically even if and only if every map $P \to \Sigma M$, where P is solid perfect even, factors through a map $P \to \Sigma Q$, where Q is solid perfect even.*

(b) *Suppose M is solid homologically even. If E is a solid even flat right-R-module such that $\pi_*(E)$ is even, then any map $\mathbb{S} \to E \otimes_R^{\blacksquare} \Sigma M$ vanishes.*

(c) *Suppose R satisfies Assumption 5.15(R) and M is nuclear. Suppose furthermore that for any solid ind-perfect even right-R-module E such that $\pi_*(E)$ is even, any morphism $\mathbb{S} \to E \otimes_R^{\blacksquare} \Sigma M$ vanishes. Then M is solid homologically even. In particular, this applies if M is nuclear and solid even flat.*

Proof. For part (a), the proof of [Pst23, Theorem 4.16(2)] can be copied verbatim. For (b), let $\eta\colon \mathbb{S} \to E \otimes_R^{\blacksquare} \Sigma M$ be any map. Let $M \to F$ be a π_*-even envelope and let $C \coloneqq \mathrm{cofib}(M \to F)$. Since E is solid even flat, $\pi_*(E \otimes_R^{\blacksquare} \Sigma F)$ is concentrated in odd degrees and so the composite

$$\mathbb{S} \longrightarrow E \otimes_R^{\blacksquare} \Sigma M \longrightarrow E \otimes_R^{\blacksquare} \Sigma F$$

must vanish. Choosing a null-homotopy, we see that η factors through a map $\eta'\colon \mathbb{S} \to E \otimes_R^{\blacksquare} C$. By assumption, C is solid ind-perfect even. Since $\mathbb{S}$ is compact, η' factors through another map $\eta''\colon \mathbb{S} \to E \otimes_R^{\blacksquare} P$, where P is solid perfect even. Since M is solid homologically even, (a) shows that the composite $P \to C \to \Sigma M$ factors through ΣQ, where Q is solid perfect even. Now Q is solid even flat by Lemma 5.17(a) and so $\pi_*(E \otimes_R^{\blacksquare} \Sigma Q)$ is concentrated in odd degrees. Thus any map $\mathbb{S} \to E \otimes_R^{\blacksquare} \Sigma Q$ vanishes. Composing with $\Sigma Q \to \Sigma M$, we find that our original map $\mathbb{S} \to E \otimes_R^{\blacksquare} \Sigma M$ must vanish as well, as desired.

Let us now show (c). Let $P \to \Sigma M$ be any map from a solid perfect even. Since M is assumed to be nuclear, any such map is trace-class. Choose a classifier $\eta\colon \mathbb{S} \to \underline{\mathrm{Hom}}_R(P,R) \otimes^{\blacksquare}_R \Sigma M$ as well as a π_*-even envelope $\underline{\mathrm{Hom}}_R(P,R) \to E$ in right-R-modules. By Assumption 5.15(R), $\underline{\mathrm{Hom}}_P(P,R)$ is solid ind-perfect even, hence the same is true for any π_*-even envelope. Our assumption then implies that any map $\mathbb{S} \to E \otimes^{\blacksquare}_R \Sigma M$ vanishes. It follows that η factors through a map $\eta'\colon \mathbb{S} \to \Sigma^{-1} C \otimes^{\blacksquare}_R \Sigma M$, where $C \coloneqq \mathrm{cofib}(\underline{\mathrm{Hom}}_R(P,R) \to E)$. By assumption, C is solid ind-perfect even; since $\mathbb{S}$ is compact, we find a solid perfect even right-R-module Q and a commutative diagram

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{\ \eta''\ } & \Sigma^{-1} Q \otimes^{\blacksquare}_R \Sigma M \\ & {\searrow}^{\eta} & \downarrow \\ & & \underline{\mathrm{Hom}}_R(P,R) \otimes^{\blacksquare}_R \Sigma M \end{array}$$

By Assumption 5.15(R), $\underline{\mathrm{Hom}}_R(P,R)$ is nuclear as a right-R-module and so the composition $\Sigma^{-1} Q \to \Sigma^{-1} C \to \underline{\mathrm{Hom}}_R(P,R)$ is trace-class. Arguing as in the proof of Lemma 5.17(b), we find that our original map $P \to \Sigma M$ factors through $\underline{\mathrm{Hom}}_R(\Sigma^{-1} Q, R)$. By Assumption 5.15($R$) again, $\underline{\mathrm{Hom}}_R(Q,R)$ is solid ind-perfect even. Writing $\underline{\mathrm{Hom}}_R(\Sigma^{-1} Q, R) \simeq \Sigma\, \underline{\mathrm{Hom}}_R(Q,R)$ as a filtered colimit of suspensions of solid perfect even left R-modules and using that P is compact, we deduce that $P \to \Sigma M$ factors through the suspension of a solid perfect even left-R-module, as desired.

For the "in particular", just observe that M being solid even flat implies that $\pi_*(E \otimes^{\blacksquare}_R \Sigma M)$ is concentrated in odd degrees and so indeed any map $\mathbb{S} \to E \otimes^{\blacksquare}_R \Sigma M$ vanishes. $\square$

§5.4. Solid faithfully flat descent in the nuclear case

In this subsection we'll show a flat descent result for the solid even filtration. We start with the definition of faithful flatness; it is slightly more restrictive than [Pst23, Definition 6.15], but we expect that this doesn't cause any problems in practice.

5.19. Definition. — A map $R \to S$ of solid condensed $\mathbb{E}_1$-algebras is called *solid faithfully even flat* if S and $\mathrm{cofib}(R \to S)$ are solid even flat both as left- and as right-R-modules.

5.20. Theorem. — *Let $R \to S$ be a solid faithfully even flat map of solid condensed $\mathbb{E}_1$-algebras such that R satisfies Assumption 5.15(R) and S is nuclear as a left-R-module. We denote the Čech nerve of $R \to S$ by $R \to S^\bullet$. Then for every nuclear solid homologically even left-R-module M, the canonical map*

$$\mathrm{fil}^\star_{\mathrm{ev}\,/R} M \longrightarrow \lim_{\Delta} \mathrm{fil}^\star_{\mathrm{ev}\,/R}(S^\bullet \otimes^{\blacksquare}_R M)$$

is an equivalence up to completing the filtrations on either side.

Proof. Put $C \coloneqq \mathrm{cofib}(R \to S)$ for short. First observe that $S \otimes_R M$ and $C \otimes_R M$ are again nuclear by Theorem 5.13(c) and (d). If E is any π_*-even and solid even flat right-R-module, then $E \otimes^{\blacksquare}_R S$ is π_*-even and solid even flat since S is solid even flat both as as a left- and as a right-R-module. Using that M is solid homologically even, we find that any map $\mathbb{S} \to E \otimes^{\blacksquare}_R S \otimes^{\blacksquare}_R \Sigma M$ vanishes by Lemma 5.18(b). Since $S \otimes^{\blacksquare}_R M$ is nuclear, we conclude that it must be solid homologically even by Lemma 5.18(c). The same argument applies to $C \otimes^{\blacksquare}_R M$.

Therefore we get a short exact sequence $0 \to \mathcal{F}_M \to \mathcal{F}_{S \otimes_R^{\blacksquare} M} \to \mathcal{F}_{C \otimes_R^{\blacksquare} M} \to 0$. Arguing as in the proof of [Pst23, Theorem 6.26], we conclude that the Moore complex

$$0 \longrightarrow \mathcal{F}_M \longrightarrow \mathcal{F}_{S \otimes_R^{\blacksquare} M} \longrightarrow \mathcal{F}_{S \otimes_R^{\blacksquare} S \otimes_R^{\blacksquare} M} \longrightarrow \cdots$$

is exact. Replacing M by an even suspension, we deduce the same for $\mathcal{F}_{(-)}(w)$ for every integral weight $w \in \mathbb{Z}$. For proper half-integral weights $w \in \frac{1}{2} + \mathbb{Z}$ this is true as well for trivial reasons, since our argument above shows that all terms in $S^\bullet \otimes_R^{\blacksquare} M$ are homologically even. We can thus apply the solid analogue of [Pst23, Proposition 5.5]. $\square$

We also need the following variant of faithfully flat descent.

5.21. Theorem. — *Let R_0 be a solid condensed $\mathbb{E}_\infty$-algebra and let S_0 be an $\mathbb{E}_1$-algebra in R_0-modules such that $R_0 \to S_0$ is solid faithfully even flat and S_0 is nuclear over R_0. We denote the Čech nerve of $R_0 \to S_0$ by $R_0 \to S_0^\bullet$. Let $R_0 \to R$ be another map of solid condensed $\mathbb{E}_1$-algebras such that R satisfies Assumption 5.15(R). Then for every solid homologically flat R_0-module M_0, the canonical map*

$$\operatorname{fil}^\star_{\mathrm{ev}/R}(R \otimes_{R_0}^{\blacksquare} M_0) \longrightarrow \lim_{\Delta} \operatorname{fil}^\star_{\mathrm{ev}/R}(R \otimes_{R_0}^{\blacksquare} M_0 \otimes_{R_0}^{\blacksquare} S_0^\bullet)$$

is an equivalence up to completing the filtrations on both sides.

Proof. This doesn't follow from Theorem 5.20 since we can't produce an $\mathbb{E}_1$-structure on $R \otimes_{R_0}^{\blacksquare} S_0$. But the argument can be adapted in a straightforward way.

Let $C_0 \coloneqq \operatorname{cofib}(R_0 \to S_0)$. A combination of Theorem 5.13(c) and (d) shows again that $R \otimes_{R_0}^{\blacksquare} M_0 \otimes_{R_0}^{\blacksquare} S_0$ and $R \otimes_{R_0}^{\blacksquare} M_0 \otimes_{R_0}^{\blacksquare} C_0$ are nuclear over R. Moreover, both are solid even flat as left-R-modules, hence solid homologically even by Lemma 5.18(c). It follows that

$$0 \to \mathcal{F}_{R \otimes_{R_0}^{\blacksquare} M_0} \longrightarrow \mathcal{F}_{R \otimes_{R_0}^{\blacksquare} M_0 \otimes_{R_0}^{\blacksquare} S_0} \longrightarrow \mathcal{F}_{R \otimes_{R_0}^{\blacksquare} M_0 \otimes_{R_0}^{\blacksquare} C_0} \longrightarrow 0$$

is a short exact sequence. Since the cosimplicial R_0-module $M_0 \otimes_{R_0}^{\blacksquare} S_0 \otimes_{R_0}^{\blacksquare} S_0^\bullet$ is split, we can still use an analogous argument as in the proof of [Pst23, Theorem 6.26] to conclude that the Moore complex

$$0 \longrightarrow \mathcal{F}_{R \otimes_{R_0}^{\blacksquare} M_0} \longrightarrow \mathcal{F}_{R \otimes_{R_0}^{\blacksquare} M_0 \otimes_{R_0}^{\blacksquare} S_0} \longrightarrow \mathcal{F}_{R \otimes_{R_0}^{\blacksquare} M_0 \otimes_{R_0}^{\blacksquare} S_0 \otimes_{R_0}^{\blacksquare} S_0} \longrightarrow \cdots$$

is exact. The same follows for $\mathcal{F}_{(-)}(w)$ for every half-integral weight w: If $w \in \mathbb{Z}$, replace M_0 by an even suspension, otherwise exactness holds for trivial reasons as the whole complex vanishes by solid homological evenness. We can thus apply the solid analogue of [Pst23, Proposition 5.5] again to finish the proof. $\square$

5.22. Remark. — Note that $M = R$ satisfies the nuclearity and homological evenness assumption in Theorem 5.20. Similarly, $M_0 = R$ satisfies the assumptions in Theorem 5.21. So in either case we get a way of computing $\operatorname{fil}^\star_{\mathrm{ev}/R} R$ via descent, provided R satisfies Assumption 5.15(R).

§6. The solid even filtration for THH

The purpose of this section is to construct and study an appropriate even filtration on $\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku}_A)$, where ku_A and ku_R denote certain lifts to ku of rings A and R (subject to strong additional assumptions to be specified below). In the subsequent section §7 we'll show that the associated graded of this even filtration is closely related to the q-de Rham complex $q\text{-}\mathrm{dR}_{R/A}$.

Throughout §6 and §§7.1–7.3, we fix a prime p as well as rings A and R satisfying the following assumptions:

6.1. Assumptions on *A*. — We let A be a p-complete and *p-completely perfectly covered* δ-ring as in 4.17. We assume that A is equipped with the following additional structure:

(^{tC_p}) *A has a lift to a p-complete connective $\mathbb{E}_\infty$-ring spectrum $\mathbb{S}_A$ such that $\mathbb{S}_A \otimes_{\mathbb{S}_p} \mathbb{Z}_p \simeq A$ and such that the Tate-valued Frobenius*

$$\phi_{tC_p}\colon \mathbb{S}_A \longrightarrow \mathbb{S}_A^{tC_p}$$

agrees with the δ-ring Frobenius $\phi\colon A \to A$ on π_0. Furthermore, ϕ_{tC_p} must be equipped with an S^1-equivariant structure as a map of $\mathbb{E}_\infty$-ring spectra, where $\mathbb{S}_A$ receives the trivial S^1-action and $\mathbb{S}_A^{tC_p}$ the induced $S^1 \simeq S^1/C_p$-action.

The S^1-equivariant structure in (^{tC_p}) ensures that $\mathbb{S}_A$ is a *p-cyclotomic base*: By the universal property of THH, the augmentation $\mathrm{THH}(\mathbb{S}_A) \to \mathbb{S}_A$ becomes a map of $\mathbb{E}_\infty$-algebras in cyclotomic spectra in a unique way, where the p-cyclotomic Frobenius on $\mathbb{S}_A$ is ϕ_{tC_p} with its chosen S^1-equivariant structure. In particular, $\mathrm{THH}(-/\mathbb{S}_A) \simeq \mathrm{THH}(-) \otimes_{\mathrm{THH}(\mathbb{S}_A)} \mathbb{S}_A$ carries a p-cyclotomic structure. We also put $\mathrm{ku}_A \coloneqq (\mathrm{ku} \otimes \mathbb{S}_A)_p^\wedge$.

6.2. Assumptions on *R*. — We let R be a p-complete A-algebra of bounded p^∞-torsion. We assume that R is *p-quasi-lci* over A in the sense that the cotangent complex $\mathrm{L}_{R/A}$ has p-complete Tor-amplitude in homological degrees $[0,1]$ over R. In addition, one of the following two conditions must be satisfied:

($\mathbb{E}_2$) *R has a lift to a p-complete connective $\mathbb{E}_2$-algebra $\mathbb{S}_R \in \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Mod}_{\mathbb{S}_A}(\mathrm{Sp}))$ such that $\mathbb{S}_R \otimes_{\mathbb{S}_p} \mathbb{Z}_p \simeq R$.*

($\mathbb{E}_1$) *R is p-torsion free and has a p-quasi-syntomic cover $R \to R_\infty$ such that:*

- (*a*) *R_∞/p is relatively semiperfect over A in the sense that its relative Frobenius over the δ-ring A is a surjection $R_\infty/p \otimes_{A,\phi} A \twoheadrightarrow R_\infty/p$.*
- (*b*) *If $R_\infty^\bullet$ denotes the p-completed Čech nerve of $R \to R_\infty$, then the augmented cosimplicial diagram $R \to R_\infty^\bullet$ has a lift to an augmented cosimplicial diagram $\mathbb{S}_R \to \mathbb{S}_{R_\infty^\bullet}$ in $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}_{\mathbb{S}_A}(\mathrm{Sp}))$, which is p-complete and connective in every degree.*

We put $\mathrm{ku}_R \coloneqq (\mathrm{ku} \otimes \mathbb{S}_R)_p^\wedge$ and, in case ($\mathbb{E}_1$), $\mathrm{ku}_{R_\infty^\bullet} \coloneqq (\mathrm{ku} \otimes \mathbb{S}_{R_\infty^\bullet})_p^\wedge$.

6.3. Remark. — Even though the assumptions in 6.1 and 6.2 seem quite restrictive, they allow for many interesting examples, as we'll see in §9.1.

6.4. Remark. — Let us motivate the rather artificial condition 6.2($\mathbb{E}_1$). If our lifts are only $\mathbb{E}_1$, there's no even filtration on $\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku}_A)_p^\wedge$. However, if $\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku}_A)_p^\wedge$ happens to be an even spectrum, then we can still consider its double-speed Whitehead filtration $\tau_{\geqslant 2\star}\,\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku}_A)_p^\wedge$. This case turns out to be quite interesting: As we'll see in §7.3, the

q-deformation of the Hodge filtration that we get in this case is independent of the choice of the $\mathbb{E}_1$-lift $\mathbb{S}_R$! This is the reason why we don't content ourselves with the $\mathbb{E}_2$-case.

More generally, given a resolution $\mathbb{S}_R \to \mathbb{S}_{R_\infty^\bullet}$ as in 6.2($\mathbb{E}_1$), then $\mathrm{TC}^-(\mathrm{ku}_{R_\infty^\bullet}/\mathrm{ku}_A)_p^\wedge$ is even in every cosimplicial degree, so we can use it to define an ad-hoc replacement of the even filtration. Indeed, evenness can be checked modulo β, so we only need to check that $\mathrm{HC}^-(R_\infty^\bullet/A)_p^\wedge$ is even. By assumption, R_∞/p is relatively semiperfect over A, hence the same is true for $R_\infty^\bullet/p$ in every cosimplicial degree. Then the desired evenness follows from Lemma 4.18(a) and [BMS19, Theorem 1.17].

6.5. Remark. — Throughout §6, we won't use that the lifts ku_A and ku_R come from spherical lifts $\mathbb{S}_A$ and $\mathbb{S}_R$, nor will we use the structure of a p-cyclotomic base on $\mathbb{S}_A$. But for the comparison with q-de Rham cohomology in §7, these assumptions will become relevant.

§6.1. Solid THH

Throughout §§6–7, we'll work in the world of *solid condensed spectra* (see 5.1). In many cases, it makes no difference whether we work solidly or p-completely; for the most part, the reader not familiar with the solid theory may safely replace each "$\blacksquare$" by a p-completion. But working solidly has the advantage that that THH will automatically be p-complete (Lemma 6.7). This simplifies the p-completed descent for the even filtration (Lemma 6.12) and it makes it much easier to deal with rationalisations, as not having to p-complete allows us to appeal directly to the fact that $\mathrm{ku}_p^\wedge \otimes \mathbb{Q} \simeq \mathbb{Q}_p[\beta]$.

6.6. Convention. — For readability we'll adopt the following abusive convention: If X is a p-complete spectrum, we'll identify X with the solid condensed spectrum $\underline{X}_p^\wedge$, otherwise we identify X with the discrete solid condensed spectrum $\underline{X}$. In particular, we'll regard ku as a discrete condensed spectrum, but ku_R and ku_A as a p-complete ones.

For any $\mathbb{E}_\infty$-algebra k in $\mathrm{Sp}_\blacksquare$, the module ∞-category $\mathrm{Mod}_k(\mathrm{Sp}_\blacksquare)$ is symmetric monoidal for the solid tensor product $-\otimes_k^\blacksquare -$. We can then consider topological Hochschild homology inside $\mathrm{Mod}_k(\mathrm{Sp}_\blacksquare)$. This yields a functor

$$\mathrm{THH}_\blacksquare(-/k)\colon \mathrm{Alg}_{\mathbb{E}_1}\left(\mathrm{Mod}_k(\mathrm{Sp}_\blacksquare)\right) \longrightarrow \mathrm{Mod}_k(\mathrm{Sp}_\blacksquare)^{\mathrm{B}S^1}\,.$$

We also let $\mathrm{TC}^-_\blacksquare(-/k) \coloneqq \mathrm{THH}_\blacksquare(-/k)^{hS^1}$ and $\mathrm{TP}_\blacksquare(-/k) \coloneqq \mathrm{THH}_\blacksquare(-/k)^{tS^1}$, where the fixed points and Tate construction are taken inside $\mathrm{Mod}_k(\mathrm{Sp}_\blacksquare)^{\mathrm{B}S^1}$.

6.7. Lemma. — *Let k° be a discrete connective $\mathbb{E}_\infty$-ring spectrum and let T° be a discrete connective $\mathbb{E}_1$-algebra in k°-modules. Let $k \coloneqq (k^\circ)_p^\wedge$ and $T \coloneqq (T^\circ)_p^\wedge$. Then solid condensed spectrum $\mathrm{THH}_\blacksquare(T/k)$ is the p-completion of the discrete spectrum $\mathrm{THH}(T^\circ/k^\circ)$.*

Proof. By the magical property of the solid tensor product,

$$\mathrm{THH}_\blacksquare(T/k) \simeq T \otimes^\blacksquare_{T^{\mathrm{op}}\otimes_k^\blacksquare T} T$$

is again p-complete. Hence we get a map $\mathrm{THH}(T^\circ/k^\circ)_p^\wedge \to \mathrm{THH}_\blacksquare(T/k)$. Whether this map is an equivalence can be checked modulo p^5. By Burklund's result [Bur22, Theorem 1.2], the quotient $k/p^5 \simeq k \otimes^\blacksquare \mathbb{S}/p^5$ admits an $\mathbb{E}_2$-k-algebra structure, and so we may regard $T/p^5 \simeq T \otimes_k^\blacksquare k/p^5$ as an $\mathbb{E}_1$-algebra in the $\mathbb{E}_1$-monoidal ∞-category $\mathrm{RMod}_{k/p^5}(\mathrm{Sp}_\blacksquare)$. Since $k/p^5 \simeq k^\circ \otimes \mathbb{S}/p^5$ and $T/p^5 \simeq T^\circ \otimes \mathbb{S}/p^5$ are discrete and the inclusion of discrete objects into all solid condensed spectra preserves tensor products, we obtain

$$\mathrm{THH}(T^\circ/k^\circ)_p^\wedge/p^5 \simeq (T/p^5) \otimes^\blacksquare_{(T/p^5)^{\mathrm{op}}\otimes^\blacksquare_{k/p^5}(T/p^5)} (T/p^5) \simeq \mathrm{THH}_\blacksquare(T/k)/p^5\,. \qquad \square$$

§6.2. The solid even filtration via even resolutions

Let us now construct the desired even filtrations. We'll use the adaptation of Pstrągowski's *perfect even filtration* to the solid setting that we've sketched in §5.

Throughout this subsection, we'll fix a connective even $\mathbb{E}_\infty$-ring spectrum k such that $\pi_{2*}(k)$ is p-torsion free. The example of interest is of course $k = \mathrm{ku}$, but we'll later apply the same results in other cases as well (e.g. for $\mathrm{ku} \otimes \mathbb{Q}$ or the geometric fixed points $\mathrm{ku}^{\Phi C_m}$), so the additional generality will be worthwhile. We put $k_A \coloneqq k \otimes^{\blacksquare} \mathbb{S}_A$, $k_R \coloneqq k \otimes^{\blacksquare} \mathbb{S}_R$, and in case 6.2($\mathbb{E}_1$) also $k_{R^\bullet_\infty} \coloneqq k \otimes^{\blacksquare} \mathbb{S}_{R^\bullet_\infty}$, where we regard k, $\mathbb{S}_A$, and $\mathbb{S}_R$ as solid condensed spectra per Convention 6.6. Note that these are all even by our assumptions on k, A, and R, but they are not necessarily p-complete; in the case $k = \mathrm{ku}$ however, p-completeness is satisfied.

6.8. Even filtrations. — If we are in situation 6.2($\mathbb{E}_2$), then $\mathrm{THH}_{\blacksquare}(k_R/k_A)$ is an $\mathbb{E}_1$-algebra and so we can define

$$\mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}_{\blacksquare}(k_R/k_A)$$

to be its solid even filtration as a module over itself. For $k = \mathrm{ku}$, we'll see in Corollary 6.24 below that $\mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A)$ is the p-completion of Pstrągowski's perfect even filtration on the discrete $\mathbb{E}_1$-ring spectrum $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)$. For $k = \mathbb{Z}$, we'll see in Corollary 6.21, that $\mathrm{fil}^\star_{\mathrm{ev}} \mathrm{HH}_{\blacksquare}(R/A)$ agrees with the Hahn–Raksit–Wilson/HKR filtration on $\mathrm{HH}(R/A)^\wedge_p$.

In situation 6.2($\mathbb{E}_1$), $\mathrm{THH}_{\blacksquare}(k_R/k_A)$ doesn't have any multiplicative structure; instead, we use the following ad-hoc definition as discussed in Remark 6.4:

$$\mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}_{\blacksquare}(k_R/k_A) \coloneqq \lim_{\Delta} \tau_{\geqslant 2\star} \mathrm{THH}_{\blacksquare}(k_{R^\bullet_\infty}/k_A)\,.$$

To define filtrations on $\mathrm{TC}^-_{\blacksquare}(k_R/k_A)$ and $\mathrm{TP}_{\blacksquare}(k_R/k_A)$ in either situation, we use a construction due to Pstrągowski and Raksit that will appear in forthcoming work [PR] and has already been used in [AR24]. Let $\mathbb{S}_{\mathrm{ev}} \coloneqq \mathrm{fil}^\star_{\mathrm{ev}} \mathbb{S}$ and $\mathbb{T}_{\mathrm{ev}} \coloneqq \mathrm{fil}^\star_{\mathrm{ev}} \mathbb{S}[S^1]$ denote the even filtrations of $\mathbb{S}$ and $\mathbb{S}[S^1]$, respectively.(6.1) Following [AR24, Definition 2.11], we define the ∞-category of *synthetic solid condensed spectra* to be $\mathrm{SynSp}_{\blacksquare} \coloneqq \mathrm{Mod}_{\mathbb{S}_{\mathrm{ev}}}(\mathrm{Fil}\,\mathrm{Sp}_{\blacksquare})$. Then $\mathbb{T}_{\mathrm{ev}}$ is a bicommutative bialgebra in $\mathrm{SynSp}_{\blacksquare}$ and we can equip $\mathrm{Mod}_{\mathbb{T}_{\mathrm{ev}}}(\mathrm{SynSp}_{\blacksquare})$ with the symmetric monoidal structure coming from the coalgebra structure on $\mathbb{T}_{\mathrm{ev}}$. By monoidality of the even filtration, $\mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}_{\blacksquare}(k_R/k_A)$ is an object in $\mathrm{Mod}_{\mathbb{T}_{\mathrm{ev}}}(\mathrm{SynSp}_{\blacksquare})$ (in case 6.2($\mathbb{E}_2$) it is even an $\mathbb{E}_1$-algebra). We can then finally define the desired filtrations as

$$\begin{aligned}\mathrm{fil}^\star_{\mathrm{ev},hS^1} \mathrm{TC}^-_{\blacksquare}(k_R/k_A) &\coloneqq \big(\mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}_{\blacksquare}(k_R/k_A)\big)^{h\mathbb{T}_{\mathrm{ev}}}\,,\\ \mathrm{fil}^\star_{\mathrm{ev},tS^1} \mathrm{TP}_{\blacksquare}(k_R/k_A) &\coloneqq \big(\mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}_{\blacksquare}(k_R/k_A)\big)^{t\mathbb{T}_{\mathrm{ev}}}\,,\end{aligned}$$

where the fixed points and Tate constructions $(-)^{h\mathbb{T}_{\mathrm{ev}}}$ and $(-)^{t\mathbb{T}_{\mathrm{ev}}}$ with respect to $\mathbb{T}_{\mathrm{ev}}$ are defined as in [AR24, §2.3].(6.2)

(6.1) It doesn't matter whether they are defined in à la Hahn–Raksit–Wilson or à la Pstrągowski or in the solid setting. Indeed, by [Pst23, Theorem 7.5], the Hahn–Raksit–Wilson filtration is the completion of Pstrągowski's filtration in either case (to apply this result, we use that $\mathbb{S}[S^1] \to \mathbb{S}$ and $\mathbb{S} \to \mathrm{MU}$ are eff by [AR24, Corollary 2.36] and [HRW22, Proposition 2.2.20]). But the filtrations are also exhaustive: For Pstrągowski's, this is always the case, for the Hahn–Raksit–Wilson filtration of connective $\mathbb{E}_\infty$-rings it is an unpublished result of Burklund and Krause. Finally, the comparison with the solid version is Corollary 5.8.

(6.2) To avoid confusion with the *genuine* fixed points that will appear later, we deviate from the notation in [AR24] and write $(-)^{h\mathbb{T}_{\mathrm{ev}}}$ instead of $(-)^{\mathbb{T}_{\mathrm{ev}}}$.

In situation 6.2($\mathbb{E}_1$), the ad-hoc even filtration being given as a cosimplicial limit gives us good control over it. We'll now show a similar description in situation 6.2($\mathbb{E}_2$).

6.9. Even resolutions. — Assume we're in situation 6.2($\mathbb{E}_2$). Let $P \coloneqq \mathbb{Z}[x_i \mid i \in I]$ be a polynomial ring with a surjection $P \twoheadrightarrow R$. Since $\mathbb{S}_P \coloneqq \mathbb{S}[x_i \mid i \in I]$ is the free $\mathbb{E}_1$-ring on commuting generators x_i, we get an $\mathbb{E}_1$-map $\mathbb{S}_P \to \mathrm{ku}_R$. It is a folklore result that $\mathbb{S}_P$ admits an even cell decomposition as an $\mathbb{E}_2$-ring; see Lemma C.1 for a proof. Since k_R is even, the map $\mathbb{S}_P \to k_R$ can be upgraded to an $\mathbb{E}_2$-map.

Now let $\mathbb{Z} \to P^\bullet$ denote the Čech nerve of $\mathbb{Z} \to P$ and define $\mathbb{S} \to \mathbb{S}_{P^\bullet}$ similarly. We also let $\mathbb{Z}_p \to \widehat{P}_p^\bullet$ and $\mathbb{S}_p \to \mathbb{S}_{\widehat{P}_p}$ denote the p-completed Čech nerves. The Čech nerve of the augmentation $\mathrm{THH}_\blacksquare(\mathbb{S}_{\widehat{P}_p}) \to \mathbb{S}_{\widehat{P}_p}$ is the cosimplicial diagram $\mathrm{THH}_\blacksquare(\mathbb{S}_{\widehat{P}_p}/\mathbb{S}_{\widehat{P}_p^\bullet})$. If we base change this diagram along the $\mathbb{E}_1$-map $\mathrm{THH}_\blacksquare(\mathbb{S}_{\widehat{P}_p}) \to \mathrm{THH}_\blacksquare(\mathrm{ku}_R/\mathrm{ku}_A)$, we get an augmented cosimplicial diagram of left-$\mathrm{THH}_\blacksquare(k_R/k_A)$-modules

$$\mathrm{THH}_\blacksquare(k_R/k_A) \longrightarrow \mathrm{THH}_\blacksquare\big(k_R/k_A \otimes^\blacksquare \mathbb{S}_{\widehat{P}_p^\bullet}\big)\,.$$

In the case $k = \mathbb{Z}$, this becomes the descent diagram $\mathrm{HH}_\blacksquare(R/A) \to \mathrm{HH}_\blacksquare(R/A \otimes^\blacksquare_{\mathbb{Z}_p} \widehat{P}_p^\bullet)$.

6.10. Remark. — Instead of the resolution from 6.9, we could also use the following: Let $\mathbb{S}_{P_\infty} \coloneqq \mathbb{S}[x_i^{1/p^\infty} \mid i \in I]$, let $\mathbb{S}_P \to \mathbb{S}_{P_\infty^\bullet}$ be the Čech nerve of $\mathbb{S}_P \to \mathbb{S}_{P_\infty}$ and define

$$k_{R_\infty^\bullet} \coloneqq \big(k_R \otimes_{\mathbb{S}_P} \mathbb{S}_{P_\infty^\bullet}\big)_p^\wedge\,.$$

In this way we get resolutions of the same form in both cases 6.2($\mathbb{E}_1$) and ($\mathbb{E}_2$). Most arguments below would work for this resolution as well, but the one from 6.9 is more convenient for Corollary 6.24 and for the global case in §7.4.

6.11. Proposition. — *Assume we are in situation* 6.2($\mathbb{E}_2$)*. Then the cosimplicial resolution from* 6.9 *induces a canonical equivalence*

$$\mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}_\blacksquare(k_R/k_A) \xrightarrow{\ \simeq\ } \lim_{\Delta} \tau_{\geqslant 2\star} \mathrm{THH}_\blacksquare\big(k_R/k_A \otimes^\blacksquare \mathbb{S}_{\widehat{P}_p^\bullet}\big)\,.$$

To prove Proposition 6.11, we'll send two technical lemmas in advance.

6.12. Lemma. — *The augmentation maps* $\mathrm{THH}_\blacksquare(\mathbb{S}_P) \to \mathbb{S}_P$ *and* $\mathrm{THH}_\blacksquare(\mathbb{S}_{\widehat{P}_p}) \to \mathbb{S}_{\widehat{P}_p}$ *are solid faithfully even flat in the sense of Definition* 5.19*. Moreover,* $\mathbb{S}_P$ *is nuclear as a* $\mathrm{THH}_\blacksquare(\mathbb{S}_P)$*-module and* $\mathbb{S}_{\widehat{P}_p}$ *is nuclear as a* $\mathrm{THH}_\blacksquare(\mathbb{S}_{\widehat{P}_p})$*-module.*

Proof. The nuclearity assumptions follow from Lemma 5.16. We only show solid faithful even flatness for $\mathrm{THH}_\blacksquare(\mathbb{S}_{\widehat{P}_p}) \to \mathbb{S}_{\widehat{P}_p}$; the argument for $\mathrm{THH}_\blacksquare(\mathbb{S}_P) \to \mathbb{S}_P$ is similar (but easier). Let E be a π_*-even module over $\mathrm{THH}_\blacksquare(\mathbb{S}_{\widehat{P}_p})$. We have a convergent spectral sequence

$$\mathrm{E}^2 = \mathrm{H}_*\Big(\pi_*(E) \otimes^{\mathrm{L}\blacksquare}_{\pi_* \mathrm{THH}_\blacksquare(\mathbb{S}_{\widehat{P}_p})} \pi_*(\mathbb{S}_{\widehat{P}_p})\Big) \Longrightarrow \pi_*\Big(E \otimes^\blacksquare_{\mathrm{THH}_\blacksquare(\mathbb{S}_{\widehat{P}_p})} \mathbb{S}_{\widehat{P}_p}\Big)\,.$$

To show that the right-hand side is even, so that $\mathbb{S}_{\widehat{P}_p}$ will be solid even flat as a $\mathrm{THH}_\blacksquare(\mathbb{S}_{\widehat{P}_p})$-module, it will be enough to show that the E^2-page is concentrated in even bidegrees. The calculation in the proof of [HRW22, Proposition 4.2.4] shows that

$$\pi_* \mathrm{THH}_\blacksquare(\mathbb{S}_{\widehat{P}_p}) \cong \pi_*(\mathbb{S}_{\widehat{P}_p}) \otimes^\blacksquare_{\mathbb{Z}_p} \Lambda^*_{\mathbb{Z}_p}(\mathrm{d}x_i \mid i \in I)_p^\wedge$$

is a graded p-completed exterior algebra over $\pi_*(\mathbb{S}_{\widehat{P}_p})$ on generators $\mathrm{d}x_i$ in bidegree $(1,0)$. Since $\pi_*(E)$ is concentrated in even degrees, each $\mathrm{d}x_i$ must act by 0, and so

$$\pi_*(E)\otimes^{\mathrm{L}\blacksquare}_{\pi_*\mathrm{THH}_\blacksquare(\mathbb{S}_{\widehat{P}_p})}\pi_*(\mathbb{S}_{\widehat{P}_p})\simeq\pi_*(E)\otimes^{\mathrm{L}\blacksquare}_{\mathbb{Z}_p}\Gamma^*_{\mathbb{Z}_p}(\sigma^2x_i\mid i\in I)^\wedge_p\,,$$

where $\Gamma^*_{\mathbb{Z}_p}(\sigma^2x_i\mid i\in I)^\wedge_p$ denotes a p-completed divided power algebra on generators in bidegree $(2,0)$. Thus, to show that the E^2-page is concentrated in even bidegrees, we only need to check that any p-completed direct sum $(\bigoplus_J\mathbb{Z}_p)^\wedge_p$ is solid even flat over $\mathbb{Z}_p$. For finite direct sums this is obvious, for countable direct sums we can use the argument from the proof of Lemma 5.16, and for uncountable direct sums we can reduce to the countable case since p-completion commutes with ω_1-filtered colimits. This finishes the proof of evenness of the E^2-page, so that $\mathbb{S}_{\widehat{P}_p}$ is indeed solid even flat over $\mathrm{THH}_\blacksquare(\mathbb{S}_{\widehat{P}_p})$.

Since the unit component $\mathbb{Z}_p\to\Gamma^*_{\mathbb{Z}_p}(\sigma^2x_i\mid i\in I)^\wedge_p$ is a direct summand, we see that the condensed homotopy groups

$$\pi_*\Big(E\otimes^\blacksquare_{\mathrm{THH}_\blacksquare(\mathbb{S}_{\widehat{P}_p})}\operatorname{cofib}\big(\mathrm{THH}_\blacksquare(\mathbb{S}_{\widehat{P}_p})\to\mathbb{S}_{\widehat{P}_p}\big)\Big)$$

are also computed by a spectral sequence with E^2-page concentrated in even bidegrees. This shows that $\operatorname{cofib}(\mathrm{THH}_\blacksquare(\mathbb{S}_{\widehat{P}_p})\to\mathbb{S}_{\widehat{P}_p})$ is also solid even flat over $\mathrm{THH}_\blacksquare(\mathbb{S}_{\widehat{P}_p})$ and we're done. ☐

6.13. Lemma. — *There exists a natural convergent spectral sequence*

$$\mathrm{E}^2_{r,s}=\mathrm{H}_r\big(\mathrm{HH}_\blacksquare(R/A)\otimes^{\mathrm{L}\blacksquare}_{\mathbb{Z}}\pi_{2s}(k)\big)\Longrightarrow\pi_{r+s}\,\mathrm{THH}_\blacksquare(k_R/k_A)\,.$$

Proof. The argument is the same as in [HRW22, Proposition 4.2.4] except for different grading conventions. Consider the filtered spectrum $\mathrm{THH}_\blacksquare(\tau_{\geqslant\star}(k_R)/\tau_{\geqslant\star}(k_A))$. This is an exhaustive and complete (due to increasing connectivity) filtration on $\mathrm{THH}_\blacksquare(k_R/k_A)$ and so it determines a convergent spectral sequence.

It remains to check that the E^2-page has the desired form. The associated graded of the filtered spectrum above is $\mathrm{THH}_\blacksquare(\Sigma^*\pi_*(k_R)/\Sigma^*\pi_*(k_A))$. Since $\pi_*(k_A)$ and $\pi_*(k_R)$ are concentrated in even graded degrees and $\mathbb{Z}$-linear, the shearing functor Σ^* is symmetric monoidal and commutes with THH. The associated graded can thus be rewritten as $\Sigma^*\,\mathrm{HH}_\blacksquare(\pi_*(k_R)/\pi_*(k_A))\simeq\Sigma^*\,\mathrm{HH}_\blacksquare(R/A)\otimes^{\mathrm{L}\blacksquare}_{\mathbb{Z}}\pi_*(k)$. This yields the desired E^2-page. ☐

Proof of Proposition 6.11. Using the spectral sequence from Lemma 6.13 (applied to $\mathbb{S}_A\otimes^\blacksquare\mathbb{S}_{\widehat{P}_p^\bullet}$ instead of $\mathbb{S}_A$) and our asssumption that $A\otimes^\blacksquare_{\mathbb{Z}}\widehat{P}_p\twoheadrightarrow R$ is p-quasi-lci and surjective, we see that $\mathrm{THH}_\blacksquare(k_R/k_A\otimes^\blacksquare\mathbb{S}_{\widehat{P}_p^\bullet})$ is even. It follows by the solid analogue of [Pst23, Lemma 2.36] that the solid even filtration (taken in left modules over $\mathrm{THH}_\blacksquare(k_R/k_A)$) is the double speed Whitehead filtration

$$\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}_\blacksquare\big(k_R/k_A\otimes^\blacksquare\mathbb{S}_{\widehat{P}_p^\bullet}\big)\simeq\tau_{\geqslant2\star}\,\mathrm{THH}_\blacksquare\big(k_R/k_A\otimes^\blacksquare\mathbb{S}_{\widehat{P}_p^\bullet}\big)\,.$$

Using the flat descent result from Theorem 5.21, which applies thanks to Lemmas 6.12 and 5.16(b), we find that

$$\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}_\blacksquare(k_R/k_A)\longrightarrow\lim_{\Delta}\tau_{\geqslant2\star}\,\mathrm{THH}_\blacksquare\big(k_R/k_A\otimes^\blacksquare\mathbb{S}_{\widehat{P}_p^\bullet}\big)$$

becomes an equivalence upon completion of the filtrations. Since the left-hand side is exhaustive whereas the right-hand side is complete, to finish the proof of the THH case, it will be enough to check that the right-hand side is also exhaustive.

In other words, we must show $\mathrm{THH}_\blacksquare(k_R/k_A) \simeq \lim_\Delta \mathrm{THH}_\blacksquare(k_R/k_A \otimes^\blacksquare \mathbb{S}_{\widehat{P}_p^\bullet})$. By the same argument as in [BMS19, Corollary 3.4(2)], it's enough to show instead

$$\mathrm{THH}_\blacksquare(k_R/k_A) \otimes_k^\blacksquare \tau_{\leqslant 2s}k \xrightarrow{\simeq} \lim_\Delta\Big(\mathrm{THH}_\blacksquare\big(k_R/k_A \otimes^\blacksquare \mathbb{S}_{\widehat{P}_p^\bullet}\big) \otimes_k^\blacksquare \tau_{\leqslant 2s}k\Big)$$

for all $s \geqslant 0$. This can be checked on associated gradeds in s. So we must show that $\mathrm{HH}_\blacksquare(R/A) \otimes_\mathbb{Z}^\blacksquare \pi_{2s}(k) \simeq \lim_\Delta(\mathrm{HH}_\blacksquare(R/A \otimes_\mathbb{Z}^\blacksquare \widehat{P}_p^\bullet) \otimes_\mathbb{Z}^\blacksquare \pi_{2s}(k))$ for all $s \geqslant 0$. By our assumptions on R and A, the HKR filtrations $\mathrm{fil}^\star_{\mathrm{HKR}} \mathrm{HH}_\blacksquare(R/A)$ and $\mathrm{fil}^\star_{\mathrm{HKR}} \mathrm{HH}_\blacksquare(R/A \otimes_\mathbb{Z}^\blacksquare \widehat{P}_p^\bullet)$ increase in connectivity as $\star \to \infty$. They are therefore still complete after $- \otimes_\mathbb{Z}^\blacksquare \pi_{2s}(k)$. So we may also pass to the associated graded of the HKR filtration. It remains to show that

$$\bigwedge^n \mathrm{L}_{R/A} \otimes_\mathbb{Z}^{\mathrm{L}\blacksquare} \pi_{2s}(k) \longrightarrow \lim_\Delta\Big(\bigwedge^n \mathrm{L}_{R/A\otimes_\mathbb{Z} P^\bullet} \otimes_\mathbb{Z}^{\mathrm{L}\blacksquare} \pi_{2s}(k)\Big)$$

is an equivalence for all $n, s \geqslant 0$ (here the cotangent complexes are implicitly p-completed). By descent for the cotangent complex, this would be true without $- \otimes_\mathbb{Z}^{\mathrm{L}\blacksquare} \pi_{2s}(k)$ on either side, so we must check that $- \otimes_\mathbb{Z}^{\mathrm{L}\blacksquare} \pi_{2s}(k)$ commutes with the cosimplicial limit. Since R is p-quasi-lci over A and $P \twoheadrightarrow R$ is surjective, each $\bigwedge^n \mathrm{L}_{R/A\otimes_\mathbb{Z} P^\bullet}$ is concentrated in homological degree n. Writing $\bigwedge^n \mathrm{L}_{R/A\otimes_\mathbb{Z} P^i} \simeq \Sigma^n L_i$, it follows that the cosimplicial limit $\lim_\Delta \bigwedge^n \mathrm{L}_{R/A\otimes_\mathbb{Z} P^\bullet}$ is given by the unnormalised Moore complex $L_* \simeq (\cdots \leftarrow L_1 \leftarrow L_0)$, sitting in homological degrees $(-\infty, n]$. Now since $\pi_{2s}(k)$ is p-torsion free and discrete by our assumptions on k, we see that $L_i \otimes_\mathbb{Z}^{\mathrm{L}\blacksquare} \pi_{2s}(k) \simeq L_i \otimes_\mathbb{Z}^\blacksquare \pi_{2s}(k)$ is static. It follows that

$$L_* \otimes_\mathbb{Z}^{\mathrm{L}\blacksquare} \pi_{2s}(k) \simeq \Big(\cdots \leftarrow \big(L_1 \otimes_\mathbb{Z}^\blacksquare \pi_{2s}(k)\big) \leftarrow \big(L_0 \otimes_\mathbb{Z}^\blacksquare \pi_{2s}(k)\big)\Big)\,.$$

So in this case it is indeed true that $- \otimes_\mathbb{Z}^{\mathrm{L}\blacksquare} \pi_{2s}(k)$ commutes with the cosimplicial limit. This finishes the proof. □

6.14. Corollary. — *In both situations 6.2($\mathbb{E}_1$) and 6.2($\mathbb{E}_2$), $\mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}_\blacksquare(k_R/k_A)$ is an exhaustive complete filtration on $\mathrm{THH}_\blacksquare(k_R/k_A)$.*

Proof. In case 6.2($\mathbb{E}_1$) completeness is clear and exhaustiveness follows from the same argument as in the proof of Proposition 6.11 above. In case 6.2($\mathbb{E}_2$) exhaustiveness is automatic and completeness follows from Proposition 6.11. □

6.15. Corollary. — *Put $(\tau_{\leqslant 2s}k)_A \coloneqq (\mathbb{S}_A \otimes \tau_{\leqslant 2s}k)_p^\wedge$ and $(\tau_{\leqslant 2s}k)_R \coloneqq (\mathbb{S}_R \otimes \tau_{\leqslant 2s}k)_p^\wedge$ for all $s \geqslant 0$. In both situations 6.2($\mathbb{E}_1$) and 6.2($\mathbb{E}_2$), consider the bifiltered object given by*

$$\mathrm{fil}^s \mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}_\blacksquare(k_R/k_A) \coloneqq \mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}_\blacksquare\big((\tau_{\leqslant 2*}k)_R/(\tau_{\leqslant 2s}k)_A\big)\,.$$

(a) *We have $\mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}_\blacksquare(k_R/k_A) \simeq \lim_{s\geqslant 0} \mathrm{fil}^s \mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}_\blacksquare(k_R/k_A)$.*

(b) *If $\mathrm{fil}^\star_{\mathrm{HKR}}$ denotes the usual HKR filtration, then for all $s \geqslant 0$,*

$$\mathrm{gr}^s \mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}_\blacksquare(k_R/k_A) \simeq \big(\mathrm{fil}^{\star-s}_{\mathrm{HKR}} \mathrm{HH}_\blacksquare(R/A)\big) \otimes_\mathbb{Z}^{\mathrm{L}\blacksquare} \Sigma^{2s+1}\pi_{2s}(k)\,.$$

Proof. We explain the argument in the context of 6.2($\mathbb{E}_1$). The other case is analogous, using the cosimplicial resolution from Proposition 6.11 instead. Put $(\tau_{\leqslant 2s}k)_{R_\infty^\bullet} \coloneqq (\mathbb{S}_{R_\infty^\bullet} \otimes \tau_{\leqslant 2s}k)_p^\wedge$ and consider the cosimplicial bifiltered object

$$\mathrm{fil}^s \tau_{\geqslant 2\star} \mathrm{THH}_\blacksquare(k_{R_\infty^\bullet}/k_A) \coloneqq \tau_{\geqslant 2\star} \mathrm{THH}_\blacksquare\big((\tau_{\leqslant 2s}k)_{R_\infty^\bullet}/(\tau_{\leqslant 2s}k)_A\big)\,.$$

Then clearly $\tau_{\geqslant 2\star}\operatorname{THH}_{\blacksquare}(k_{R^\bullet_\infty}/k_A) \simeq \lim_{s\geqslant 0}\operatorname{fil}^s \tau_{\geqslant 2\star}\operatorname{THH}_{\blacksquare}(k_{R^\bullet_\infty}/k_A)$. Applying $\lim_{\Delta}$ on both sides already shows (*a*). To prove (*b*), observe that the functor $\tau_{\geqslant 2\star}(-)$ is non-exact in general, but nevertheless it preserves the cofibre sequence

$$\operatorname{HH}_{\blacksquare}(R^\bullet_\infty/A)\otimes^{\blacksquare}_{\mathbb{Z}}\Sigma^{2s}\pi_{2s}(k)\longrightarrow \operatorname{THH}_{\blacksquare}(k_{R^\bullet_\infty}/k_A)\otimes^{\blacksquare}_{k}\tau_{\leqslant 2s}k\longrightarrow \operatorname{THH}_{\blacksquare}(k_{R^\bullet_\infty}/k_A)\otimes^{\blacksquare}_{k}\tau_{\leqslant 2(s-1)}k\,.$$

Indeed, consider the spectral sequence[(6.3)] from Lemma 6.13 with k replaced by $\tau_{\leqslant 2s}(k)$ or $\tau_{\leqslant 2(s-1)}(k)$. Our assumptions on $R^\bullet_\infty$ guarantee that both E^2-pages are concentrated in even bidegrees and so the spectral sequences collapse. A closer examination of the induced map on E^2-pages then shows that $\tau_{\geqslant 2\star}(-)$ indeed preserves the cofibre sequence above.

Using this observation, we conclude that the graded pieces of $\operatorname{fil}^s\tau_{\geqslant 2\star}\operatorname{THH}_{\blacksquare}(k_{R^\bullet_\infty}/k_A)$ are given by[(6.4)]

$$\operatorname{gr}^s\tau_{\geqslant 2\star}\operatorname{THH}_{\blacksquare}(k_{R^\bullet_\infty}/k_A)\simeq \Sigma\tau_{\geqslant 2\star}\Big(\operatorname{HH}_{\blacksquare}(R^\bullet_\infty/A)\otimes^{\mathrm{L}\blacksquare}_{\mathbb{Z}}\Sigma^{2s}\pi_{2s}(k)\Big)\,.$$

The right-hand side agrees with $\tau_{\geqslant 2(\star-s)}\operatorname{HH}_{\blacksquare}(R^\bullet_\infty/A)\otimes^{\mathrm{L}\blacksquare}_{\mathbb{Z}}\Sigma^{2s+1}\pi_{2s}(k)$ since $\pi_{2s}(k)$ was assumed to be discrete and p-torsion free. Now the HKR filtration can be computed as the cosimplicial limit $\operatorname{fil}^\star_{\mathrm{HKR}}\operatorname{HH}_{\blacksquare}(R/A)\simeq \lim_{\Delta}\tau_{\geqslant 2\star}\operatorname{HH}_{\blacksquare}(R^\bullet_\infty/A)$. Thus, to prove (*b*), it remains to check that $-\otimes^{\mathrm{L}\blacksquare}_{\mathbb{Z}}\pi_{2s}(k)$ commutes with the cosimplicial limit. Since the HKR filtration stays complete after $-\otimes^{\mathrm{L}\blacksquare}_{\mathbb{Z}}\pi_{2s}(k)$ (due to increasing connectivity), we may pass to the associated graded. This reduces us to an assertion that was checked in the proof of Proposition 6.11 above. ☐

6.16. Corollary. — *In situation 6.2($\mathbb{E}_1$), the given cosimplicial resolution induces equivalences*

$$\operatorname{fil}^\star_{\mathrm{ev},hS^1}\operatorname{TC}^-_{\blacksquare}(k_R/k_A)\xrightarrow{\ \simeq\ }\lim_{\Delta}\tau_{\geqslant 2\star}\operatorname{TC}^-_{\blacksquare}(k_{R^\bullet_\infty}/k_A)\,,$$
$$\operatorname{fil}^\star_{\mathrm{ev},tS^1}\operatorname{TP}_{\blacksquare}(k_R/k_A)\xrightarrow{\ \simeq\ }\lim_{\Delta}\tau_{\geqslant 2\star}\operatorname{TP}_{\blacksquare}(k_{R^\bullet_\infty}/k_A)\,.$$

If we are in situation 6.2($\mathbb{E}_2$), the cosimplicial resolution from 6.9 induces equivalences

$$\operatorname{fil}^\star_{\mathrm{ev},hS^1}\operatorname{TC}^-_{\blacksquare}(k_R/k_A)\xrightarrow{\ \simeq\ }\lim_{\Delta}\tau_{\geqslant 2\star}\operatorname{TC}^-_{\blacksquare}\big(k_R/k_A\otimes^{\blacksquare}\mathbb{S}_{\widehat{P}^\bullet_p}\big)\,,$$
$$\operatorname{fil}^\star_{\mathrm{ev},tS^1}\operatorname{TP}_{\blacksquare}(k_R/k_A)\xrightarrow{\ \simeq\ }\lim_{\Delta}\tau_{\geqslant 2\star}\operatorname{TP}_{\blacksquare}\big(k_R/k_A\otimes^{\blacksquare}\mathbb{S}_{\widehat{P}^\bullet_p}\big)\,.$$

Proof. To see the assertion for TC^- in both cases, just observe that $(-)^{h\mathbb{T}_{\mathrm{ev}}}$ commutes with the cosimplicial limit and that $(\tau_{\geqslant 2\star}\operatorname{THH}_{\blacksquare}(-))^{h\mathbb{T}_{\mathrm{ev}}}\simeq\tau_{\geqslant 2\star}\operatorname{TC}^-_{\blacksquare}(-)$ holds in this case by [AR24, Lemma 2.75(vi)]. To show the same for TP, we need to commute $(-)_{h\mathbb{T}_{\mathrm{ev}}}\simeq\mathbb{S}_{\mathrm{ev}}\otimes^{\blacksquare}_{\mathbb{T}_{\mathrm{ev}}}-$ past the cosimplicial limit.

Let us explain how to do this in case 6.2($\mathbb{E}_1$); the other case is analogous. We use the bifiltration from Corollary 6.15. By Corollary 6.15(*b*), $\operatorname{cofib}(\operatorname{fil}^s\operatorname{fil}^\star_{\mathrm{ev}}\to\operatorname{fil}^\star_{\mathrm{ev}})$ is $\star+s$-connective. Using Corollary 6.15(*a*) follows that $(\operatorname{fil}^\star_{\mathrm{ev}})_{h\mathbb{T}_{\mathrm{ev}}}\simeq(\lim_{s\geqslant 0}\operatorname{fil}^s\operatorname{fil}^\star_{\mathrm{ev}})_{h\mathbb{T}_{\mathrm{ev}}}\simeq\lim_{s\geqslant 0}(\operatorname{fil}^s\operatorname{fil}^\star_{\mathrm{ev}})_{h\mathbb{T}_{\mathrm{ev}}}$. So we may pass to the associated graded in s-direction and thus, using Corollary 6.15(*b*) again, it will be enough to check

$$\Big(\operatorname{fil}^\star_{\mathrm{HKR}}\operatorname{HH}_{\blacksquare}(R/A)\otimes^{\mathrm{L}\blacksquare}_{\mathbb{Z}}\pi_{2s}(k)\Big)_{h\mathbb{T}_{\mathrm{ev}}}\xrightarrow{\ \simeq\ }\lim_{\Delta}\Big(\big(\tau_{\geqslant 2\star}\operatorname{HH}_{\blacksquare}(R^\bullet_\infty/A)\otimes^{\mathrm{L}\blacksquare}_{\mathbb{Z}}\pi_{2s}(k)\big)_{h\mathbb{T}_{\mathrm{ev}}}\Big)\,.$$

(6.3) In the construction of the spectral sequence in Lemma 6.13 we used the Postnikov filtration $\tau_{\geqslant\star}k$, while here we're working with the double speed Whitehead filtration $\tau_{\leqslant 2\star}k$. We could have used the Postnikov filtration as well to construct a similar spectral sequence as in Lemma 6.13. But we still use the one from Lemma 6.13.
(6.4) Note that gr^s is defined as a cofibre, not a fibre. Hence the extra Σ.

Now both sides are $\mathbb{Z}$-linear. By [AR24, Proposition 2.54], the construction $(-)_{h\mathbb{T}_{\mathrm{ev}}}$ agrees with the orbits with respect to Raksit's filtered circle [Rak21, Notation 6.3.2]. Combining this observation with [BMS19, Corollary 3.4(1)] (plus an easy argument as in the proof of Proposition 6.11 to deal with the extra $-\otimes_{\mathbb{Z}}^{\mathrm{L}\blacksquare}\pi_{2s}(k)$), we conclude that both sides are exhaustive filtrations on $(\mathrm{HH}_\blacksquare(R/A)\otimes_{\mathbb{Z}}^{\mathrm{L}\blacksquare}\pi_{2s}(k))_{hS^1}$.

The equivalence can now be checked on associated gradeds. By [Rak21, Proposition 6.3.3], the n^{th} graded piece of $(\mathrm{fil}_{\mathrm{HKR}}^\star \mathrm{HH}_\blacksquare(R/A)\otimes_{\mathbb{Z}}^{\mathrm{L}\blacksquare}\pi_{2s}(k))_{h\mathbb{T}_{\mathrm{ev}}}$ will be an iterated extension of $\mathrm{gr}_{\mathrm{HKR}}^i \mathrm{HH}_\blacksquare(R/A)\otimes_{\mathbb{Z}}^{\mathrm{L}\blacksquare}\pi_{2s}(k)$ for $i=0,1,\dots,n$. A similar argument applies on the right-hand side. So we can finally deduce the desired equivalence from Proposition 6.11. $\square$

§6.3. Base change

We continue to fix a k as specified at the beginning of §6.2. As a consequence of Proposition 6.11, we show that the even filtrations constructed in 6.8 satisfy all expected base change properties.

6.17. Corollary. — *Let $k\to l$ be any map of $\mathbb{E}_\infty$-ring spectra where l is also connective, even, and p-torsion free in every homotopical degree. Let $l_A\coloneqq l\otimes^\blacksquare \mathbb{S}_A$ and $l_R\coloneqq l\otimes^\blacksquare\mathbb{S}_R$. Let furthermore $k_{\mathrm{ev}}\coloneqq\tau_{\geqslant 2\star}k$ and $l_{\mathrm{ev}}\coloneqq\tau_{\geqslant 2\star}l$. Then the canonical base change morphism is an equivalence*

$$\mathrm{fil}_{\mathrm{ev}}^\star \mathrm{THH}_\blacksquare(k_R/k_A)\otimes_{k_{\mathrm{ev}}}^\blacksquare l_{\mathrm{ev}}\xrightarrow{\simeq}\mathrm{fil}_{\mathrm{ev}}^\star\mathrm{THH}_\blacksquare(l_R/l_A)\,.$$

Proof. Using Corollary 6.14, we see that both sides are exhaustive filtrations on $\mathrm{THH}_\blacksquare(l_R/l_A)$. It is thus enough to check the equivalence on associated gradeds. Let us now assume we're in case 6.2($\mathbb{E}_2$); the 6.2($\mathbb{E}_2$) is analogous using the resolution from Proposition 6.11. Using the spectral sequence from Lemma 6.13, we see that the cosimplicial graded object $\pi_{2*}\mathrm{THH}_\blacksquare(k_{R_\infty^\bullet}/k_A)$ has a finite filtration[(6.5)] in every graded degree-wise finite filtration whose associated graded satisfies

$$\mathrm{gr}^*\pi_{2(\star+*)}\mathrm{THH}_\blacksquare(k_{R_\infty^\bullet}/k_A)\simeq\pi_{2\star}\mathrm{HH}_\blacksquare(R_\infty^\bullet/A)\otimes_{\mathbb{Z}}^{\mathrm{L}\blacksquare}\pi_{2*}(k)$$

as cosimplicial bigraded objects. Applying $\lim_\Delta$ (which commutes with $-\otimes_{\mathbb{Z}}^{\mathrm{L}\blacksquare}\pi_{2*}(k)$ by the argument in the proof of Proposition 6.11), we find that $\mathrm{gr}_{\mathrm{ev}}^*\mathrm{THH}_\blacksquare(k_R/k_A)$ has a finite filtration in every graded degree in such a way that the associated graded satisfies

$$\mathrm{gr}^*\mathrm{gr}_{\mathrm{ev}}^{\star+*}\mathrm{THH}_\blacksquare(k_R/k_A)\simeq\mathrm{gr}_{\mathrm{HKR}}^\star\mathrm{HH}_\blacksquare(R/A)\otimes_{\mathbb{Z}}^\blacksquare\Sigma^{2*}\pi_{2*}(k)$$

as bigraded objects. This equivalence is compatible with $\mathrm{gr}^* k_{\mathrm{ev}}\simeq\Sigma^{2*}\pi_{2*}(k)$, since the latter can be obtained from the spectral sequence for $\mathrm{THH}_\blacksquare(k/k)$. Using the same for l, the desired equivalence now follows from the trivial observation

$$\left(\mathrm{gr}_{\mathrm{HKR}}^\star\mathrm{HH}_\blacksquare(R/A)\otimes_{\mathbb{Z}}^\blacksquare\Sigma^{2*}\pi_{2*}(k)\right)\otimes_{\Sigma^{2*}\pi_{2*}(k)}^\blacksquare\Sigma^{2*}\pi_{2*}(l)\simeq\mathrm{gr}_{\mathrm{HKR}}^\star\mathrm{HH}_\blacksquare(R/A)\otimes_{\mathbb{Z}}^\blacksquare\Sigma^{2*}\pi_{2*}(l)\,,$$

so we're done. $\square$

6.18. Corollary. — *Let $k\to l$ be as in Corollary 6.17 and put $k_{\mathrm{ev}}^{hS^1}\coloneqq\tau_{\geqslant 2\star}(k^{hS^1})$ as well as $l_{\mathrm{ev}}^{hS^1}\coloneqq\tau_{\geqslant 2\star}(l^{hS^1})$. Let also $t\in\pi_{-2}(k^{hS^1})$ be a complex orientation of k. We regard t as sitting in homotopical degree -2 and filtration degree -1 of $k_{\mathrm{ev}}^{hS^1}$. Then the canonical base change morphism is an equivalence*

$$\left(\mathrm{fil}_{\mathrm{ev},hS^1}^\star\mathrm{TC}_\blacksquare^-(k_R/k_A)\otimes_{k_{\mathrm{ev}}^{hS^1}}^\blacksquare l_{\mathrm{ev}}^{hS^1}\right)_t^\wedge\xrightarrow{\simeq}\mathrm{fil}_{\mathrm{ev},hS^1}^\star\mathrm{TC}_\blacksquare^-(l_R/l_A)\,.$$

(6.5) This is *not* the filtration from Corollary 6.15.

Proof. Using Corollary 6.16, we see that both sides are t-complete. Upon reduction modulo t, we get the equivalence from Corollary 6.17. □

A similar base change equivalence exists for $\operatorname{fil}_{\mathrm{ev},tS^1}\operatorname{TP}_{\blacksquare}(k_R/k_A)$, but one has to be a little careful about completions. One way to formulate the result would be via Corollary 6.18 combined with the following:

6.19. Corollary. — *Let $k_{\mathrm{ev}}^{tS^1} \coloneqq \tau_{\geqslant 2\star}(k^{tS^1})$. We have a canonical equivalence*

$$\operatorname{fil}^{\star}_{\mathrm{ev},hS^1}\operatorname{TC}^-_{\blacksquare}(k_R/k_A)\otimes^{\blacksquare}_{k_{\mathrm{ev}}^{hS^1}} k_{\mathrm{ev}}^{tS^1} \xrightarrow{\simeq} \operatorname{fil}^{\star}_{\mathrm{ev},tS^1}\operatorname{TP}_{\blacksquare}(k_R/k_A)\,.$$

Proof. Using Corollary 6.16, we see that both sides are exhaustive filtrations on $\operatorname{TP}_{\blacksquare}(k_R/k_A)$. It is thus enough to check the equivalence on associated gradeds. Using Corollary 6.16, we find that

$$\operatorname{gr}^{*}_{\mathrm{ev},hS^1}\operatorname{TC}^-_{\blacksquare}(k_R/k_A) \to \operatorname{gr}^{*}_{\mathrm{ev},tS^1}\operatorname{TP}_{\blacksquare}(k_R/k_A)$$

is an equivalence in negative graded degrees and that the right-hand side is periodic. Since $-\otimes^{\blacksquare}_{\operatorname{gr}^* k_{\mathrm{ev}}^{hS^1}}\operatorname{gr}^* k_{\mathrm{ev}}^{tS^1}$ will also make the left-hand side periodic, we're done. □

§6.4. Comparison of even filtrations

As another consequence of Proposition 6.11, we can show that the even filtrations from 6.8 agree with the those defined by [BMS19; HRW22; Pst23].

6.20. Even filtrations on ordinary Hochschild homology. — In the case $k=\mathbb{Z}$, the constructions in 6.8 yield filtrations

$$\operatorname{fil}^{\star}_{\mathrm{ev}}\operatorname{HH}_{\blacksquare}(R/A)\,,\quad \operatorname{fil}^{\star}_{\mathrm{ev},hS^1}\operatorname{HC}^-_{\blacksquare}(R/A)\,,\quad\text{and}\quad \operatorname{fil}^{\star}_{\mathrm{ev},tS^1}\operatorname{HP}_{\blacksquare}(R/A)\,.$$

But $\operatorname{HH}_{\blacksquare}(R/A)\simeq \operatorname{HH}(R/A)_p^{\wedge}$ is a p-complete $\mathbb{E}_\infty$-ring spectrum and so we can also consider the Hahn–Raksit–Wilson even filtrations

$$\operatorname{fil}^{\star}_{\text{HRW-ev}}\operatorname{HH}(R/A)_p^{\wedge}\,,\quad \operatorname{fil}^{\star}_{\text{HRW-ev},hS^1}\operatorname{HC}^-(R/A)_p^{\wedge}\,,\quad\text{and}\quad \operatorname{fil}^{\star}_{\text{HRW-ev},tS^1}\operatorname{HP}(R/A)_p^{\wedge}\,.$$

These can be regarded as filtrations on $\operatorname{HH}_{\blacksquare}(R/A)$, $\operatorname{HC}^-_{\blacksquare}(R/A)$, and $\operatorname{HP}_{\blacksquare}(R/A)$ in a natural way. For HH, we simply regard p-complete spectra as solid condensed spectra per Convention 6.6 and use Lemma 6.7. For HC^- and HP, we must be a little more careful: If $\operatorname{HH}(R/A)\to E$ is an S^1-equivariant $\mathbb{E}_\infty$-map into an even p-complete ring spectrum with bounded p^∞-torsion, we regard E^{hS^1} as a solid condensed spectrum by performing both the p-completion and the homotopy fixed points $(-)^{hS^1}$ in $\mathrm{Sp}_{\blacksquare}$. We then regard

$$\operatorname{fil}^{\star}_{\text{HRW-ev},hS^1}\operatorname{HC}^-(R/A)_p^{\wedge}\simeq \lim_{\operatorname{HH}(R/A)\to E}\tau_{\geqslant 2\star}\big(E^{hS^1}\big)\,;$$

as a solid condensed spectrum by also performing the limit in $\mathrm{Sp}_{\blacksquare}$. In the same way we can regard $\operatorname{fil}^{\star}_{\text{HRW-ev},tS^1}\operatorname{HP}(R/A)_p^{\wedge}$ as a filtered solid condensed spectrum.

If E is even, then the perfect even filtration of E is the double-speed Whitehead filtration $\tau_{\geqslant 2\star}(E)$ by [Pst23, Lemma 2.36] and its solid analogue. Moreover, $(\tau_{\geqslant 2\star}(E))^{h\mathbb{T}_{\mathrm{ev}}}\simeq \tau_{\geqslant 2\star}(E^{hS^1})$ by [AR24, Lemma 2.75(vi)] and similarly $(\tau_{\geqslant 2\star}(E))^{t\mathbb{T}_{\mathrm{ev}}}\simeq \tau_{\geqslant 2\star}(E^{tS^1})$. It follows that there's a canonical map $\operatorname{fil}^{\star}_{\mathrm{ev}}\to \operatorname{fil}^{\star}_{\text{HRW-ev}}$ in each case.

6.21. Corollary. — *Via the comparison maps constructed in 6.20 above, the filtrations*

$$\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{HH}_\blacksquare(R/A)\,,\quad \mathrm{fil}^\star_{\mathrm{ev},hS^1}\,\mathrm{HC}^-_\blacksquare(R/A)\,,\quad \textit{and}\quad \mathrm{fil}^\star_{\mathrm{ev},tS^1}\,\mathrm{HP}_\blacksquare(R/A)\,,$$

agree with the Hahn–Raksit–Wilson/HKR even filtrations

$$\mathrm{fil}_{\text{HRW-ev}}\,\mathrm{HH}(R/A)^\wedge_p\,,\quad \mathrm{fil}_{\text{HRW-ev},hS^1}\,\mathrm{HC}^-(R/A)^\wedge_p\,,\quad \textit{and}\quad \mathrm{fil}_{\text{HRW-ev},tS^1}\,\mathrm{HP}(R/A)^\wedge_p\,.$$

Proof. The solid even filtration $\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{HH}_\blacksquare(R/A)$ can be computed by a certain cosimplicial resolution (in case 6.2($\mathbb{E}_1$) by definition, in case 6.2($\mathbb{E}_2$) by Proposition 6.11). The same resolutions also compute the even filtration of Hahn–Raksit–Wilson. The same argument also works for $\mathrm{HC}^-_\blacksquare$ and $\mathrm{HP}_\blacksquare$ thanks to Corollary 6.16. □

6.22. Remark. — For later use, let us point out the following consequence: Using Corollary 6.18 for $\mathrm{ku}\to\mathrm{ku}\otimes\mathbb{Q}\simeq\mathbb{Q}[\beta]$ and $\mathbb{Z}\to\mathbb{Q}[\beta]$, we deduce that

$$\left(\mathrm{fil}^\star_{\mathrm{ev},hS^1}\,\mathrm{TC}^-_\blacksquare(\mathrm{ku}_R/\mathrm{ku}_A)\otimes^\blacksquare\mathbb{Q}\right)^\wedge_t\simeq\left(\mathrm{fil}^\star_{\mathrm{ev},hS^1}\,\mathrm{HC}^-_\blacksquare(R/A)\otimes^\blacksquare_{\mathbb{Z}^{hS^1}_{\mathrm{ev}}}\mathbb{Q}[\beta]^{hS^1}_{\mathrm{ev}}\right)^\wedge_t\,.$$

Moreover, the filtration on the right-hand side is the usual Hahn–Raksit–Wilson/HKR even filtration. This will give us good control over the constructions in §7 after rationalisation.

The filtration on $\mathrm{TC}^-(S/\mathbb{S}_A[\![q-1]\!])[1/u]^\wedge_{(p,q-1)}$ from Proposition A.17, whose associated graded computes prismatic/q-de Rham cohomology, is also recovered by the solid even filtration.

6.23. Corollary. — *If S is any p-complete p-quasi-lci $A[\zeta_p]$-algebra of bounded p^∞-torsion, then there's a canonical filtered $\mathbb{E}_\infty$-equivalence*

$$\left(\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}_\blacksquare\big(S/\mathbb{S}_A[\![q-1]\!]\big)\big[\tfrac{1}{u}\big]^\wedge_p\right)^{h\mathbb{T}_{\mathrm{ev}}}\xrightarrow{\simeq}\mathrm{fil}^\star_{\text{HRW-ev},hS^1}\left(\mathrm{TC}^-\big(S/\mathbb{S}_A[\![q-1]\!]\big)\big[\tfrac{1}{u}\big]^\wedge_{(p,q-1)}\right)$$

(*where the right-hand side is regarded as a filtered solid condensed spectrum in the way described in 6.20 above*).

Proof. Let us first construct the canonical map in question. For every S^1-equivariant $\mathbb{E}_\infty$-map $\mathrm{THH}(S/\mathbb{S}_A[\![q-1]\!])[1/u]\to E$ into a p-complete even ring spectrum, we get a canonical filtered $\mathbb{E}_\infty$-map

$$\left(\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}_\blacksquare\big(S/\mathbb{S}_A[\![q-1]\!]\big)\big[\tfrac{1}{u}\big]^\wedge_p\right)^{h\mathbb{T}_{\mathrm{ev}}}\longrightarrow(\tau_{\geqslant 2\star}E)^{h\mathbb{T}_{\mathrm{ev}}}\simeq\tau_{\geqslant 2\star}\big(E^{hS^1}\big)$$

using [AR24, Lemma 2.75(vi)]. This induces the desired comparison map. To prove that we get an equivalence, we can use the same arguments as before: Choose a polynomial ring $P=\mathbb{Z}[x_i\mid i\in I]$ with a surjection $P\twoheadrightarrow S$ and then show that both sides are computed by the cosimplicial resolution $\tau_{\geqslant 2\star}\,\mathrm{TC}^-_\blacksquare(S/(\mathbb{S}_A\otimes^\blacksquare\mathbb{S}_{\widehat{P}^\bullet_p})[\![q-1]\!])[1/u]^\wedge_{(p,q-1)}$. □

Finally, we show that in the case $k=\mathrm{ku}$ our solid even filtration on $\mathrm{THH}_\blacksquare(\mathrm{ku}_R/\mathrm{ku}_A)$ agrees with the p-completion of Pstrągowski's perfect even filtration $\mathrm{fil}^\star_{\text{P-ev}}\,\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)$. This won't be needed in the rest of the text, but it is perhaps a nice sanity check.

6.24. Corollary. — *The canonical map induced by 5.7 is an equivalence*

$$\big(\mathrm{fil}^\star_{\text{P-ev}}\,\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)\big)^\wedge_p\xrightarrow{\simeq}\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}_\blacksquare(\mathrm{ku}_R/\mathrm{ku}_A)\,.$$

Proof. Let $T \coloneqq \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)$ for short. Since $\mathrm{THH}(\mathbb{S}_P) \to \mathbb{S}_P$ is eff, we can compute $\mathrm{fil}^\star_{\text{P-ev}} T$ using descent; more precisely, using the uncondensed version of Theorem 5.21. We find that

$$\mathrm{fil}^\star_{\text{P-ev}/T} T \longrightarrow \lim_{\Delta} \mathrm{fil}^\star_{\text{P-ev}/T}\big(T \otimes_{\mathrm{THH}(\mathbb{S}_P)} \mathrm{THH}(\mathbb{S}_P/\mathbb{S}_{P^\bullet})\big)$$

is an equivalence up to completing the filtrations on both sides. Let us now study the right-hand side. Fix some cosimplicial degree i and put $M \coloneqq \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A \otimes \mathbb{S}_{P^i})$ for short. We claim that there is a canonical equivalence

$$(\mathrm{fil}^\star_{\text{P-ev}} M)^\wedge_p \xrightarrow{\simeq} \mathrm{fil}^\star_{\text{P-ev}} \widehat{M}_p \simeq \tau_{\geqslant 2\star}(\widehat{M}_p)\,.$$

If we can show this, we're done. Indeed, by comparison with the resolution from Proposition 6.11, we find that $\big(\mathrm{fil}^\star_{\text{P-ev}} \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)\big)^\wedge_p \to \mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}_\blacksquare(\mathrm{ku}_R/\mathrm{ku}_A)$ is an equivalence up to completion. But the filtrations on both sides are exhaustive and the right-hand side is complete by Proposition 6.11 again, and so the map must be an equivalence.

To show the claim, first observe that the homotopy groups of $\widehat{M}_p/\beta \simeq \mathrm{HH}(R/A \otimes_{\mathbb{Z}} P^i)^\wedge_p$ are concentrated in even degrees and p-completely flat over R, where the R-module structure on $\pi_*(\widehat{M}_p/\beta)$ comes from the left-T-module structure on M. We would like to show that the same conclusion is true for $\pi_*(\mathrm{Hom}_T(Q, \widehat{M}_p)/\beta)$ for any perfect even T-module Q; however, the seemingly obvious argument doesn't quite work, since T is only $\mathbb{E}_1$ and so there's no left-T-module structure on $\mathrm{Hom}_T(-,-)$.

To fix this, observe that $T \otimes_{\mathrm{THH}(\mathbb{S}_P)} \mathbb{S}_P$ has a right-$\mathbb{S}_P$-module structure commuting with the left-T-module structure. Restricting to $\pi_0(\mathbb{S}_P) \cong P$, we get a right homotopy action of P on $T \otimes_{\mathrm{THH}(\mathbb{S}_P)} \mathbb{S}_P$. Since $\pi_0 \mathrm{THH}(\mathbb{S}_P) \cong P$ as well, this action agrees with the right action of P on T via $P \twoheadrightarrow R \cong \pi_0(T)$. In particular, the right homotopy action by P factors through R. An analogous right homotopy action of R can be constructed on $M \simeq T \otimes_{\mathrm{THH}(\mathbb{S}_P)} \mathbb{S}_P^{\otimes_{\mathrm{THH}(\mathbb{S}_P)}(i+1)}$, by picking our favourite tensor factor.

This explains how $\pi_* \mathrm{Hom}_T(-, \widehat{M}_p)$ can be equipped with an R-module structure. With this R-module structure, it is still true that the homotopy groups $\pi_*(\widehat{M}_p/\beta)$ are concentrated in even degrees and are p-completely flat R-modules, because $\mathrm{HH}(R/A \otimes_{\mathbb{Z}} P^i)$ is commutative. This allows us to deduce that the homotopy groups $\pi_*(\mathrm{Hom}_T(Q, \widehat{M}_p)/\beta)$ are also concetrated in even degrees and p-completely flat over R for any perfect even left T-module Q. Since M is bounded below, we deduce that also $\mathrm{Hom}_T(Q, \widehat{M}_p)$ is even and its homotopy groups are p-completely flat R-modules. In particular, this is true for $\widehat{M}_p$ itself. By [BMS19, Lemma 4.7], the p^∞-torsion in $\pi_{2*} \mathrm{Hom}_T(Q, \widehat{M}_p)$ is therefore bounded. In fact, there's a uniform bound N that works for all Q, since we can use the same bound as for R.

Let us use this to analyse the canonical map

$$\mathrm{gr}^*_{\text{P-ev}} \widehat{M}_p \longrightarrow \lim_{\alpha \geqslant 0} \mathrm{gr}^*_{\text{P-ev}}(\widehat{M}_p/p^\alpha)\,.$$

By definition, $(\mathrm{gr}^*_{\text{P-ev}} \widehat{M}_p)/p^\alpha$ is given by the sections over T of the sheafification of the spectra-valued presheaf $\Sigma^{2*}(\pi_{2*} \mathrm{Hom}_T(-, \widehat{M}_p))/p^\alpha$ on the perfect even site $\mathrm{Perf}_{\mathrm{ev}}(T)$. In homotopical degree $2*$, this presheaf agrees with $\Sigma^{2*}\pi_{2*} \mathrm{Hom}_T(-, \widehat{M}_p/p^\alpha)$, but in homotopical degree $2*+1$ it has an extra torsion component. However, if we go from $\alpha + N$ to α, then the transition map will vanish on the torsion component, because N is a uniform bound for the p^∞-torsion. Thus, in the limit we get an equivalence $\lim_{\alpha \geqslant 0}(\mathrm{gr}^*_{\text{P-ev}} \widehat{M}_p)/p^\alpha \simeq \lim_{\alpha \geqslant 0} \mathrm{gr}^*_{\text{P-ev}}(\widehat{M}_p/p^\alpha)$. The left-hand side agrees with $\pi_{2*}(\widehat{M}_p)$ since $\widehat{M}_p$ is already even and p-complete. We conclude that

$$\tau_{\geqslant 2\star}(\widehat{M}_p) \simeq \operatorname{fil}^\star_{\text{P-ev}} \widehat{M}_p \longrightarrow \lim_{\alpha\geqslant 0} \operatorname{fil}^\star_{\text{P-ev}}(\widehat{M}_p/p^\alpha)$$

is an equivalence up to completion of the filtration on the right-hand side.

Since $\operatorname{THH}(\mathbb{S}_P) \to \mathbb{S}_P$ is eff, M will be even flat, hence homologically even over T. Thus [Pst23, Remark 2.35] shows $\operatorname{fil}^\star_{\text{P-ev}} M \simeq \operatorname{fil}^{\star-1/2}_{\text{P-ev}} M$. By definition, $(\operatorname{fil}^{\star-1/2}_{\text{P-ev}} M)/p^\alpha$ is given by the sections over T of the sheafification of the spectra-valued presheaf

$$\operatorname{cofib}\bigl(p^\alpha\colon \tau_{\geqslant 2\star-1} \operatorname{Hom}_T(-,M) \longrightarrow \tau_{\geqslant 2\star-1} \operatorname{Hom}_T(-,M)\bigr)$$

on $\operatorname{Perf}_{\text{ev}}(T)$. In homotopical degrees $\geqslant 2\star$, this presheaf agrees with $\tau_{\geqslant 2\star} \operatorname{Hom}_T(-,M/p^\alpha)$, but in homotopical degree $2\star-1$ there might be an additional component that injects into $\Sigma^{2\star-1}\pi_{2\star-1} \operatorname{Hom}_T(-,M/p^\alpha)$. However, the transition maps from $\alpha+N$ to α will vanish on this additional component by our uniform p^∞-torsion bound, so in the limit we get an equivalence

$$(\operatorname{fil}^\star_{\text{P-ev}} M)^\wedge_p \simeq \lim_{\alpha\geqslant 0}(\operatorname{fil}^\star_{\text{P-ev}} M)/p^\alpha \xrightarrow{\simeq} \lim_{\alpha\geqslant 0} \operatorname{fil}^\star_{\text{P-ev}}(M/p^\alpha)\,.$$

At this point we've shown that $(\operatorname{fil}^\star_{\text{P-ev}} M)^\wedge_p \to \tau_{\geqslant 2\star}(\widehat{M}_p)$ is an equivalence up to completion. But both sides are already complete: The right-hand side by inspection, the left-hand side by [Pst23, Theorem 8.3(2)]. So we're done. □

6.25. Remark. — The argument can be adapted to any even ring spectrum k such that $\pi_*(k)$ is a graded polynomial ring over $\mathbb{Z}$ with finitely many generators in each given degree. In particular, it works for $k = \mathrm{MU}$. We don't know to what extent Corollary 6.24 is true in complete generality. At the very least, one would need some finiteness assumption on k; otherwise k_A and k_R won't be p-complete in general.

§7. q-de Rham cohomology and TC^- over ku

In this section we'll finally formulate and prove the precise relationship between the even filtration on $\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku}_A)$ and the q-de Rham complex $q\text{-}\mathrm{dR}_{R/A}$.

7.1. Convention — To avoid excessive use of completions, throughout §7, all (q-)de Rham complexes and cotangent complexes relative to a p-complete ring will be implicitly p-completed.

§7.1. The p-complete comparison (case $p > 2$)

We fix a prime $p > 2$. We'll also continue to fix rings A and R satisfying the assumptions from 6.1 and 6.2.

Our main tool will be a striking result of Devalapurkar. To formulate this result, let us regard $\mathbb{Z}_p[\zeta_p]$ as a $\mathbb{S}_p[\![q-1]\!]$-algebra via $q \mapsto \zeta_p$. We let S^1 act on $\mathrm{THH}(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p[\![q-1]\!])_p^\wedge$ in the usual way and let $\mathbb{Z}_p^\times$ act via A.20. We let S^1 act on ku^{tC_p} via the residual $S^1 \simeq S^1/C_p$-action and let $\mathbb{Z}_p^\times$ act via the Adams operations on $\mathrm{ku}_p^\wedge$.

7.2. Theorem (Devalapurkar [Dev25, Theorem 6.4.1]). — *For primes $p > 2$, there exists an $S^1 \times \mathbb{Z}_p^\times$-equivariant equivalence of $\mathbb{E}_\infty$-ring spectra*

$$\mathrm{THH}\big(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p[\![q-1]\!]\big)_p^\wedge \xrightarrow{\simeq} \tau_{\geqslant 0}\big(\mathrm{ku}^{tC_p}\big)\,.$$

Moreover, this equivalence fits into a commutative diagram of S^1-equivariant $\mathbb{E}_\infty$-algebras

$$\begin{array}{ccc}
\mathrm{THH}\big(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p[\![q-1]\!]\big)_p^\wedge & \xrightarrow{\simeq} & \tau_{\geqslant 0}(\mathrm{ku}^{tC_p}) \\
\downarrow & & \downarrow \\
\mathrm{THH}(\mathbb{F}_p) & \xrightarrow{\simeq} & \tau_{\geqslant 0}(\mathbb{Z}_p^{tC_p})
\end{array}$$

where the bottom row is the equivalence from [NS18, Corollary IV.4.13].

7.3. Remark. — Theorem 7.2 was conjectured for all p by Lurie and Nikolaus. By an unpublished result of Nikolaus, Theorem 7.2 is true as an S^1-equivariant $\mathbb{E}_1$-equivalence for all p (see Theorem 7.17 below). As far as the author is aware, constructing an S^1-equivariant $\mathbb{E}_\infty$-equivalence case $p = 2$ is still open.

7.4. Remark. — If we also let $q \in \pi_0(\mathrm{ku}^{hS^1}) \cong \mathrm{ku}^0(\mathrm{B}S^1)$ denote the class corresponding to the standard representation of S^1 on $\mathbb{C}$, then the map from Theorem 7.2 sends $q \mapsto q$.

Moreover, there's a unique complex orientation $t \in \pi_{-2}(\mathrm{ku}^{hS^1})$ satisfying $q - 1 = \beta t$. In the following, we'll frequently use $\pi_*(\mathrm{ku}^{hS^1}) \cong \mathbb{Z}[\beta][\![t]\!]$, and we'll identify this graded $\mathbb{Z}[t]$-algebra with the filtered ring $(q-1)^\star\mathbb{Z}[\![q-1]\!]$, where $(q-1)$ in degree 1 corresponds to β.

7.5. The comparison map I. — We import the equivalence from Theorem 7.2 into the solid world via 6.6. Using this equivalence, we can construct an S^1-equivariant map of solid condensed spectra as follows:

$$\begin{array}{ccc}
\big(\mathrm{THH}_\blacksquare(\mathbb{S}_R/\mathbb{S}_A) \otimes^\blacksquare_{\mathbb{S}_A,\phi_{tC_p}} \mathbb{S}_A\big) \otimes^\blacksquare \mathrm{THH}_\blacksquare\big(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p[\![q-1]\!]\big) & \longrightarrow & \mathrm{THH}_\blacksquare(\mathbb{S}_R/\mathbb{S}_A)^{tC_p} \otimes^\blacksquare \mathrm{ku}^{tC_p} \\
\simeq\downarrow & & \downarrow \\
\mathrm{THH}_\blacksquare\Big((R \otimes^{\mathrm{L}}_{A,\phi} A)_p^\wedge[\zeta_p]/\mathbb{S}_A[\![q-1]\!]\Big) & \dashrightarrow & \mathrm{THH}_\blacksquare(\mathrm{ku}_R/\mathrm{ku}_A)^{tC_p}
\end{array}$$

The map in the top row is given by $\phi_{p/\mathbb{S}_A} \otimes^{\blacksquare}$ (7.2), where $\phi_{p/\mathbb{S}_A}$ denotes the relative cyclotomic Frobenius on $\mathrm{THH}(-/\mathbb{S}_A)$. The right vertical arrow comes from lax symmetric monoidality of $(-)^{tC_p}$. The left vertical arrow is an equivalence since THH is symmetric monoidal. So the dashed bottom horizontal arrow exists.

Now $\mathrm{THH}_{\blacksquare}(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p\llbracket q-1\rrbracket) \to \mathrm{ku}^{tC_p}$ sends the generator $u \in \pi_2$ to a unit. Indeed, this can be checked modulo $(q-1) = \beta t$, so we reduce to the same question for $\mathrm{THH}(\mathbb{F}_p) \to \mathbb{Z}_p^{tC_p}$. Under the equivalence $\mathbb{Z}_p^{tC_p} \simeq \mathrm{THH}(\mathbb{F}_p)^{tC_p}$, this map becomes the cyclotomic Frobenius for $\mathrm{THH}(\mathbb{F}_p)$, which is well-known to send u to a unit. The diagram above thus induces an S^1-equivariant map

$$\psi_R\colon \mathrm{THH}_{\blacksquare}\big(R^{(p)}[\zeta_p]/\mathbb{S}_A\llbracket q-1\rrbracket\big)\big[\tfrac{1}{u}\big] \longrightarrow \mathrm{THH}_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A)^{tC_p}\,,$$

where $R^{(p)} \coloneqq (R \otimes^{\mathrm{L}}_{A,\phi} A)^\wedge_p$ as in A.19. From ψ_R, we can now construct a filtered map

$$\psi_R^\star\colon \mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{TC}^-_{\blacksquare}\big(R^{(p)}[\zeta_p]/\mathbb{S}_A\llbracket q-1\rrbracket\big)\big[\tfrac{1}{u}\big]^\wedge_{(p,q-1)} \longrightarrow \mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{TP}_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A)\,,$$

where the filtration on the left-hand side agrees with the Bhatt–Morrow–Scholze filtration, the Hahn–Raksit–Wilson, and the Pstrągowski–Raksit even filtration. To construct $\psi_R^\star$, we have to distinguish the two cases:

($\mathbb{E}_1$) In situation 6.2($\mathbb{E}_1$), we construct $\psi_R^\star$ as the limit

$$\lim_{\Delta} \tau_{\geqslant 2\star}\,\mathrm{TC}^-_{\blacksquare}\big((R_\infty^\bullet)^{(p)}[\zeta_p]/\mathbb{S}_A\llbracket q-1\rrbracket\big)\big[\tfrac{1}{u}\big]^\wedge_{(p,q-1)} \xrightarrow{(7.5)} \lim_{\Delta} \tau_{\geqslant 2\star}\,\mathrm{TP}_{\blacksquare}(\mathrm{ku}_{R_\infty^\bullet}/\mathrm{ku}_A)$$

The left-hand side is $\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{TC}^-_{\blacksquare}(R^{(p)}[\zeta_p]/\mathbb{S}_A\llbracket q-1\rrbracket)[1/u]^\wedge_{(p,q-1)}$ by quasi-syntomic descent for the Bhatt–Morrow–Scholze even filtration and the right-hand side is $\mathrm{fil}_{\mathrm{ev}}\,\mathrm{TP}_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A)$ by definition.

($\mathbb{E}_2$) In situation 6.2($\mathbb{E}_2$), we construct $\psi_R^\star$ by applying $(\mathrm{fil}^\star_{\mathrm{ev}}(-))^{h(\mathbb{T}/C_p)_{\mathrm{ev}}}$ to the map from 7.5 and composing with a certain canonical map

$$\big(\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A)^{tC_p}\big)^{h(\mathbb{T}/C_p)_{\mathrm{ev}}} \longrightarrow \mathrm{fil}^\star_{\mathrm{ev},tS^1}\,\mathrm{TP}_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A)\,,$$

that will be constructed in 7.7 below.

7.6. Even filtrations and the Tate construction. — To construct such a map, let more generally T be a complex orientable solid $\mathbb{E}_1$-ring spectrum and let M be an S^1-equivariant left-T-module such that M^{hC_p} is solid homologically even over T^{hC_p}. Let $T^{hS^1}_{\mathrm{ev}} \coloneqq \mathrm{fil}^\star_{\mathrm{ev}}\,T^{hS^1}$ and $T^{tS^1}_{\mathrm{ev}} \coloneqq \mathrm{fil}^\star_{\mathrm{ev}}\,T^{tS^1}$. First observe that we have an equivalence

$$T^{tS^1}_{\mathrm{ev}} \otimes^{\blacksquare}_{T^{hS^1}_{\mathrm{ev}}} \mathrm{fil}^\star_{\mathrm{ev}\,/T^{hC_p}}\,M^{hC_p} \xrightarrow{\simeq} \mathrm{fil}^\star_{\mathrm{ev}\,/T^{tC_p}}\,M^{tC_p}$$

Indeed, choose a complex orientation $t \in \pi_{-2}(T^{hS^1})$. It's well-known that $T^{tS^1} \simeq T^{hS^1}[t^{-1}]$ and $M^{tC_p} \simeq M^{hC_p}[t^{-1}]$. In particular, we see that both sides above are exhaustive filtrations on M^{tC_p}, and so it's enough to check the equivalence on graded pieces. Since t sits in even degree -2, if we take any π_*-even envelope over T^{hS^1} or T^{hC_p} and invert t, we get a π_*-even envelope over T^{tS^1} or T^{tC_p}, respectively. Since the associated graded of the even filtration can be computed by successively taking π_*-even envelopes (see [Pst23, §5]; the solid analogue is discussed in 5.6), the claimed equivalence follows.

Now let $(-)^{hC_{p,\mathrm{ev}}}$ and $(-)^{tC_{p,\mathrm{ev}}}$ denote the synthetic fixed point and Tate constructions from [AR24, Definition 2.61]. We have canonical maps

$$\begin{aligned} \mathrm{fil}^\star_{\mathrm{ev}\,/T^{hC_p}} M^{hC_p} &\longrightarrow \left(\mathrm{fil}^\star_{\mathrm{ev}\,/T} M\right)^{hC_{p,\mathrm{ev}}}\,, \\ T_{\mathrm{ev}}^{tS^1} \otimes^{\blacksquare}_{T_{\mathrm{ev}}^{hS^1}} \left(\mathrm{fil}^\star_{\mathrm{ev}\,/T} M\right)^{hC_{p,\mathrm{ev}}} &\longrightarrow \left(\mathrm{fil}^\star_{\mathrm{ev}\,/T} M\right)^{tC_{p,\mathrm{ev}}}\,. \end{aligned}$$

Composing these with the equivalence above, we get a canonical map

$$\mathrm{fil}^\star_{\mathrm{ev}\,/T^{tC_p}} M^{tC_p} \longrightarrow \left(\mathrm{fil}^\star_{\mathrm{ev}\,/T} M\right)^{tC_{p,\mathrm{ev}}}\,.$$

7.7. The comparison map II. — To construct the map that we need in 7.5($\mathbb{E}_2$), we apply $(-)^{h(\mathbb{T}/C_p)_{\mathrm{ev}}}$ to the general construction from 7.6, where $(-)^{h(\mathbb{T}/C_p)_{\mathrm{ev}}}$ denotes fixed points in the sense of [AR24, §2.3] with respect to the even filtration on $\mathbb{S}[S^1/C_p]$. It then remains to check that the canonical map

$$\mathrm{fil}^\star_{\mathrm{ev}} \mathrm{TP}_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A) \xrightarrow{\ \simeq\ } \left(\left(\mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A)\right)^{tC_{p,\mathrm{ev}}}\right)^{h(\mathbb{T}/C_p)_{\mathrm{ev}}}$$

is an equivalence. To see this, we'll use the cosimplicial resolution from Proposition 6.11. A similar argument as in the proof of Corollary 6.16 can be used to verify that $(-)^{tC_{p,\mathrm{ev}}}$ commutes with the cosimplicial limit. We can thus reduce to the case where $\mathrm{THH}_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A)$ is already even. The desired result then follows from [AR24, Lemma 2.75(vi)], its analogue for $(-)_{hC_{p,\mathrm{ev}}}$, and the classical fact that $(-)^{tS^1} \simeq ((-)^{tC_p})^{h(S^1/C_p)}$ holds on bounded below p-complete spectra by [NS18, Lemma II.4.2].

7.8. The q-Hodge filtration. — We can pass to the 0^{th} graded piece of our filtered comparison map $\psi_R^\star$ and use Proposition A.17 to obtain a map

$$\psi_R^0\colon q\text{-}\mathrm{dR}_{R/A} \longrightarrow \mathrm{gr}^0_{\mathrm{ev},tS^1} \mathrm{TP}_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A) \simeq \mathrm{gr}^0_{\mathrm{ev},tS^1} \mathrm{TC}^-_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A)\,.$$

Now $\mathrm{gr}^*_{\mathrm{ev},hS^1} \mathrm{TC}^-_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A)$ is a graded module over $\mathrm{gr}^*_{\mathrm{ev},hS^1}(\mathrm{ku}^{hS^1}) \simeq \Sigma^{2*}\pi_{2*}(\mathrm{ku}^{hS^1})$. Hence the double shearing $\Sigma^{-2*} \mathrm{gr}^*_{\mathrm{ev},hS^1} \mathrm{TC}^-_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A)$ is a graded module over $\mathbb{Z}_p[\beta]\llbracket t\rrbracket$, with $|\beta| = 2$, $|t| = -2$.[(7.1)] We can regard t as a filtration parameter, so that the graded $\mathbb{Z}_p[\beta]\llbracket t\rrbracket$-module $\Sigma^{-2*} \mathrm{gr}^*_{\mathrm{ev},hS^1} \mathrm{TC}^-_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A)$ defines a filtration on $\mathrm{gr}^0_{\mathrm{ev},hS^1} \mathrm{TC}^-_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A)$. We define the *$q$-Hodge filtration* as the pullback

$$\begin{array}{ccc} \mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R/A} & \longrightarrow & \Sigma^{-2*} \mathrm{gr}^*_{\mathrm{ev},hS^1} \mathrm{TC}^-_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A) \\ \big\downarrow & \lrcorner & \big\downarrow \\ q\text{-}\mathrm{dR}_{R/A} & \xrightarrow{\ \psi_R^0\ } & \mathrm{gr}^0_{\mathrm{ev},hS^1} \mathrm{TC}^-_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A) \end{array}$$

The name *q-Hodge filtration* is justified by the fact that $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{R/A}$ is indeed a q-deformation of the Hodge filtration on $\mathrm{dR}_{R/A}$. This is part of the main result of this subsection, which we can now formulate and prove. Here we identify the graded $\mathbb{Z}[t]$-algebra $\mathbb{Z}_p[\beta]\llbracket t\rrbracket$ with the $(q-1)$-adic filtration $(q-1)^\star\mathbb{Z}_p\llbracket q-1\rrbracket$ as explained in Remark 7.4.

(7.1) Also note that since everything is $\mathbb{Z}$-linear, the double shearing functor Σ^{2*} is symmetric monoidal.

7.9. Theorem. — *Let $p > 2$ be a prime and let A and R satisfy the assumptions from 6.1 and 6.2. Then the map ψ_R^0 from 7.7 induces an equivalence of graded $\mathbb{Z}_p[\beta][\![t]\!]$-modules*

$$\mathrm{fil}_{q\text{-Hdg}}^{\star}\, q\text{-}\widehat{\mathrm{dR}}_{R/A} \xrightarrow{\ \simeq\ } \Sigma^{-2*}\,\mathrm{gr}_{\mathrm{ev},hS^1}^{*}\,\mathrm{TC}_{\blacksquare}^{-}(\mathrm{ku}_R/\mathrm{ku}_A)\,,$$

where the left-hand side denotes the completion of the q-Hodge filtration $\mathrm{fil}_{q\text{-Hdg}}^{*}\, q\text{-}\mathrm{dR}_{R/A}$ from 7.8. Moreover, modulo β and after rationalisation, we get equivalences

$$\begin{aligned}\mathrm{fil}_{q\text{-Hdg}}^{\star}\, q\text{-}\mathrm{dR}_{R/A} \otimes_{\mathbb{Z}_p[\beta][\![t]\!]}^{\mathrm{L}} \mathbb{Z}_p[\![t]\!] &\xrightarrow{\ \simeq\ } \mathrm{fil}_{\mathrm{Hdg}}^{\star}\,\mathrm{dR}_{R/A}\,,\\ \mathrm{fil}_{q\text{-Hdg}}^{\star}\, q\text{-}\mathrm{dR}_{R/A}\big[\tfrac{1}{p}\big]_{(q-1)}^{\wedge} &\xrightarrow{\ \simeq\ } \mathrm{fil}_{(\mathrm{Hdg},q-1)}^{\star}\,\mathrm{dR}_{R/A}\big[\tfrac{1}{p}\big][\![q-1]\!]\end{aligned}$$

with the usual Hodge filtration and the combined Hodge and $(q-1)$-adic filtration, respectively.

7.10. Remark. — In case 6.2($\mathbb{E}_2$), all equivalences in Theorem 7.9 are canonically $\mathbb{E}_1$-monoidal. In fact, if $\mathbb{S}_R$ can be equipped with an $\mathbb{E}_n$-algebra structure in $\mathbb{S}_A$-modules for any $2 \leqslant n \leqslant \infty$, then all equivalences will be canonically $\mathbb{E}_{n-1}$-monoidal. To see this, observe that for any $T \in \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Mod}_{\mathbb{S}_A}(\mathrm{Sp}_{\blacksquare}))$, we can use the same construction as in 7.5 to produce an S^1-equivariant map

$$\mathrm{THH}_{\blacksquare}\big((T \otimes_{\mathbb{S}_A,\phi_{tC_p}}^{\blacksquare} \mathbb{S}_A) \otimes^{\blacksquare} \mathbb{Z}_p[\zeta_p]/\mathbb{S}_A[\![q-1]\!]\big)\big[\tfrac{1}{u}\big] \longrightarrow \mathrm{THH}_{\blacksquare}(\mathrm{ku} \otimes^{\blacksquare} T/\mathrm{ku}_A)^{tC_p}\,;$$

these maps assemble into a symmetric monoidal transformation of symmetric monoidal functors $\mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Mod}_{\mathbb{S}_A}(\mathrm{Sp}_{\blacksquare})) \to \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}_{\blacksquare}^{\mathrm{B}S^1})$. If $\mathbb{S}_R$ admits an $\mathbb{E}_n$-algebra structure in $\mathbb{S}_A$-modules, then $\mathbb{S}_R \in \mathrm{Alg}_{\mathbb{E}_{n-2}}(\mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Mod}_{\mathbb{S}_A}(\mathrm{Sp}_{\blacksquare})))$ and so ψ_R is S^1-equivariantly $\mathbb{E}_{n-2}$ as a map in $\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp}_{\blacksquare})$, hence S^1-equivariantly $\mathbb{E}_{n-1}$ as a map in $\mathrm{Sp}_{\blacksquare}$. The other parts of the construction clearly preserve $\mathbb{E}_{n-1}$-monoidality.

If we are in case 6.2($\mathbb{E}_1$), then a priori we only get $\mathbb{E}_0$-monoidal structures. However, we can a posteriori upgrade everything from $\mathbb{E}_0$ to $\mathbb{E}_\infty$ by applying Theorem 7.18 below to the given resolution $R \to R_\infty^{\bullet}$.

The main step in the proof of Theorem 7.9 is to describe ψ_R^0 modulo $(q-1)$.

7.11. Lemma. — *The reduction modulo $(q-1) = \beta t$ of the map ψ_R^0 from 7.7 agrees with the canonical Hodge completion map*

$$\mathrm{dR}_{R/A} \longrightarrow \widehat{\mathrm{dR}}_{R/A} \simeq \mathrm{gr}_{\mathrm{ev},tS^1}^{0}\,\mathrm{HP}_{\blacksquare}(R/A)\,.$$

Proof (*initial reduction*). In the following, we'll assume we're in case 6.2($\mathbb{E}_2$). In case 6.2($\mathbb{E}_1$), we repeat the arguments below instead for each term in the cosimplicial resolution $R_\infty^{\bullet}$, with the even filtration replaced by $\tau_{\geqslant 2\star}$.

Put $\overline{R} \coloneqq R \otimes_{\mathbb{Z}_p}^{\mathrm{L}} \mathbb{F}_p$ and $\overline{R}^{(p)} \coloneqq \overline{R} \otimes_{A,\phi}^{\mathrm{L}} A$ for short. If we reduce the diagram from 7.5 modulo $(q-1) = \beta t$, we obtain the following commutative diagram:

$$\begin{array}{ccc}\big(\mathrm{THH}_{\blacksquare}(\mathbb{S}_R/\mathbb{S}_A) \otimes_{\mathbb{S}_A,\phi_{tC_p}}^{\blacksquare} \mathbb{S}_A\big) \otimes^{\blacksquare} \mathrm{THH}(\mathbb{F}_p) & \longrightarrow & \mathrm{THH}_{\blacksquare}(\mathbb{S}_R/\mathbb{S}_A)^{tC_p} \otimes^{\blacksquare} \mathbb{Z}_p^{tC_p}\\ \simeq\big\downarrow & & \big\downarrow\\ \mathrm{THH}_{\blacksquare}\big(\overline{R}^{(p)}/\mathbb{S}_A\big) & \dashrightarrow & \mathrm{HH}_{\blacksquare}(R/A)^{tC_p}\end{array}$$

The top row is induced by the equivalence $\mathrm{THH}(\mathbb{F}_p) \simeq \tau_{\geqslant 0}(\mathbb{Z}_p^{tC_p})$ from [NS18, Corollary IV.4.13] and the relative cyclotomic Frobenius $\phi_{p/\mathbb{S}_A}$ for $\mathrm{THH}(-/\mathbb{S}_A)$. After passing to homotopy S^1-fixed points, the bottom row of this diagram factors induces a map

$$\overline{\psi}_R^{hS^1}\colon \mathrm{TC}^-_{\blacksquare}\bigl(\overline{R}^{(p)}/\mathbb{S}_A\bigr)\bigl[\tfrac{1}{u}\bigr]_p^\wedge \longrightarrow \mathrm{HP}_{\blacksquare}(R/A)\,.$$

The key observation is now that the map $\overline{\psi}_R^{hS^1}$ can be constructed without the choice of a spherical lift $\mathbb{S}_R$. Let us interrupt the proof for the moment and discuss how this works. $\square$

7.12. Constructing $\overline{\psi}_R^{hS^1}$ without a spherical lift. — Let us first assume that $A \cong \mathrm{W}(k)$ is the ring of Witt vectors over a perfect field of characteristic p. In this case, Petrov and Vologodsky [PV23] construct an equivalence $\mathrm{TP}_{\blacksquare}(\overline{R}/\mathbb{S}_A) \simeq \mathrm{HP}_{\blacksquare}(R/A)$ without choosing any spherical lift $\mathbb{S}_R$. We claim that this equivalence holds, in fact, for arbitrary A, and that the composition with the relative cyclotomic Frobenius

$$\phi_{p/\mathbb{S}_A}^{hS^1}\colon \mathrm{TC}^-_{\blacksquare}\bigl(\overline{R}^{(p)}/\mathbb{S}_A\bigr)\bigl[\tfrac{1}{u}\bigr]_p^\wedge \longrightarrow \mathrm{TP}(\overline{R}/\mathbb{S}_A)$$

agrees with the map $\overline{\psi}_R^{hS^1}$. Both of these claims follow from work of Devalapurkar and Raksit [DR25]: They give a new proof of the equivalence $\mathrm{TP}_{\blacksquare}(\overline{R}/\mathbb{S}_A) \simeq \mathrm{HP}_{\blacksquare}(R/A)$, which works for arbitrary A, and from their proof it will be apparent that the maps indeed coincide. The new proof is based on the following result:

7.13. Theorem (Devalapurkar–Raksit [DR25])**.** — *Let $j \coloneqq \tau_{\geqslant 0}(\mathbb{S}_{K(1)})$ be the connective cover of the $K(1)$-local sphere.*

(a) There is an equivalence $\mathrm{THH}(\mathbb{Z}_p)_p^\wedge \simeq \tau_{\geqslant 0}(j^{tC_p})$ as well as a commutative diagram

$$\begin{array}{ccc} j & \longrightarrow & \mathrm{THH}(\mathbb{Z}_p)_p^\wedge \\ \big\downarrow & {\scriptstyle ///}\;\swarrow & \big\downarrow \\ \mathbb{Z}_p & \longrightarrow & \mathrm{THH}(\mathbb{F}_p) \end{array}$$

of S^1-equivariant (in fact, cyclotomic) $\mathbb{E}_\infty$-rings. Moreover, there exists a dashed diagonal arrow that makes the upper left but not the lower right triangle commute S^1-equivariantly.

(b) The horizontal maps $j \rightarrow \mathrm{THH}(\mathbb{Z}_p)_p^\wedge$ and $\mathbb{Z}_p \rightarrow \mathrm{THH}(\mathbb{F}_p)$ are S^1-nilpotent, that is, for any spectrum X with S^1-action the maps $X \otimes j \rightarrow X \otimes \mathrm{THH}(\mathbb{Z}_p)_p^\wedge$ and $X \otimes \mathbb{Z}_p \rightarrow X \otimes \mathrm{THH}(\mathbb{F}_p)$ become equivalences upon $(-)^{tS^1}$.

The new proof of the equivalence $\mathrm{TP}_{\blacksquare}(\overline{R}/\mathbb{S}_A) \simeq \mathrm{HP}_{\blacksquare}(R/A)$ in [DR25, §5] then proceeds as follows: By Theorem 7.13(*a*) we have an S^1-equivariant commutative diagram

$$\begin{array}{ccc} \mathrm{THH}_{\blacksquare}(R/\mathbb{S}_A) \otimes_j^{\blacksquare} \mathbb{Z}_p & \xrightarrow{(\simeq)^{tS^1}} & \mathrm{THH}_{\blacksquare}(R/\mathbb{S}_A) \otimes_{\mathrm{THH}_{\blacksquare}(\mathbb{Z}_p)}^{\blacksquare} \mathbb{Z}_p \\ {\scriptstyle (\simeq)^{tS^1}}\big\downarrow & & \big\downarrow \\ \mathrm{THH}_{\blacksquare}(R/\mathbb{S}_A) \otimes_j^{\blacksquare} \mathrm{THH}(\mathbb{F}_p) & \xrightarrow{(\simeq)^{tS^1}} & \mathrm{THH}_{\blacksquare}(R/\mathbb{S}_A) \otimes_{\mathrm{THH}_{\blacksquare}(\mathbb{Z}_p)} \mathrm{THH}(\mathbb{F}_p) \end{array}$$

By Theorem 7.13(*b*), the horizontal arrows and the left vertical arrow become equivalences after applying $(-)^{tS^1}$.[(7.2)] Hence after $(-)^{tS^1}$ the dashed vertical arrow exists and it induces the desired equivalence $\mathrm{HP}_{\blacksquare}(R/A) \simeq \mathrm{TP}_{\blacksquare}(\overline{R}/\mathbb{S}_A)$.

Using $\mathrm{THH}_{\blacksquare}(R/\mathbb{S}_A) \simeq \mathrm{THH}_{\blacksquare}(\mathbb{S}_R/\mathbb{S}_A) \otimes^{\blacksquare} \mathrm{THH}_{\blacksquare}(\mathbb{Z}_p)$, it is also apparent that the composition of this equivalence with the relative cyclotomic Frobenius $\phi_{p/\mathbb{S}_A}^{hS^1}$ agrees with the map $\overline{\psi}_R^{hS^1}$, as we've claimed above.

Proof of Lemma 7.11 (end of proof). The proof can now be finished as follows: Let S be a p-torsion free p-complete p-quasi-lci A-algebra, put $\overline{S} \coloneqq S/p$ and $\overline{S}^{(p)} \coloneqq \overline{S} \otimes^{\mathrm{L}}_{A,\phi} A$. Via quasi-syntomic descent as in the proof of Proposition A.17, we can define a Bhatt–Morrow–Scholze-style even filtration $\mathrm{fil}^\star_{\text{BMS-ev},hS^1} \mathrm{TC}^-_{\blacksquare}(\overline{S}^{(p)}/\mathbb{S}_A)[1/u]_p^\wedge$ together with a map

$$\overline{\psi}_S^\star \colon \mathrm{fil}^\star_{\text{BMS-ev},hS^1} \mathrm{TC}^-_{\blacksquare}\left(\overline{S}^{(p)}/\mathbb{S}_A\right)\left[\tfrac{1}{u}\right]_p^\wedge \longrightarrow \mathrm{fil}^\star_{\text{BMS-ev},tS^1} \mathrm{HP}_{\blacksquare}(S/A)\,;$$

to construct this map, we use 7.12 above. By passing to animations, we can also cover the case $S = R$.[(7.3)] A comparison with prismatic cohomology as in the proof of Proposition A.17 shows that the 0^{th} graded piece of $\overline{\psi}_S^\star$ has the form

$$\overline{\psi}_S^0 \colon \mathbb{\Delta}_{\overline{S}^{(p)}/A} \simeq \mathrm{dR}_{S/A} \longrightarrow \widehat{\mathrm{dR}}_{S/A}\,;$$

here we also use the crystalline comparison for prismatic cohomology [BS19, Theorem 5.2] and the fact that the de Rham cohomology of S agrees with the crystalline cohomology of its reduction $\overline{S}$. If we can show that $\overline{\psi}_S^0$ is the canonical Hodge completion map, then we'll be done, because from the comparison results in Corollaries 6.21 and 6.23 it's clear that in the case $S = R$ the map $\overline{\psi}_R^\star$ agrees with the reduction of ψ_R^0 modulo $(q-1)$.

To show that $\overline{\psi}_S^0$ has the desired form, we can now use quasi-syntomic descent. In particular, we may reduce to a situation where S/p is relatively semiperfect over A (i.e. the relative Frobenius $S/p \otimes_{A,\phi} A \twoheadrightarrow S/p$ is surjective). Then everything is even, hence both sides of $\overline{\psi}_S^\star$ are double speed Whitehead filtrations on even spectra and $\overline{\psi}_S^0$ is a map between two static condensed rings. Whether this map is the correct one can be checked on the level of sets and hence after any p-completely faithfully flat base change. Let A_∞ denote the p-completed colimit perfection of A. By our assumption 6.1, $A \to A_\infty$ is p-completely faithfully flat, and it can be lifted to an $\mathbb{E}_\infty$-map $\mathbb{S}_A \to \mathbb{S}_{A_\infty}$ (see Lemma A.15 for example). Via base change along this map, we may reduce to the case where A is perfect. Then S/p is semiperfect on the nose and so $\mathrm{A}_{\mathrm{inf}} \coloneqq \mathrm{W}(S^\flat) \twoheadrightarrow S$ is surjective.

Now everything becomes rather explicit: Let $J \coloneqq \ker(\mathrm{A}_{\mathrm{inf}} \to R)$ and let $\mathrm{A}_{\mathrm{crys}} \coloneqq D_{\mathrm{A}_{\mathrm{inf}}}(J)$ denote the p-completed PD-envelope of J. It's well-known[(7.4)] that

$$\mathrm{dR}_{S/A} \simeq \mathrm{dR}_{R/\mathrm{A}_{\mathrm{inf}}} \simeq \mathrm{A}_{\mathrm{crys}}\,.$$

Since the un-p-completed PD-envelope $\mathrm{A}_{\mathrm{crys}}^\circ$ of $J \subseteq \mathrm{A}_{\mathrm{inf}}$ is contained in $\mathrm{A}_{\mathrm{inf}}[1/p]$, the Hodge completion map $\mathrm{A}_{\mathrm{crys}} \to \widehat{\mathrm{A}}_{\mathrm{crys}}$ is uniquely characterised by the following two properties:

(7.2) The functor $(-)^{tS^1}$ factors through a certain category, denoted $\widehat{\mathrm{Mod}}^t_{W[S^1]}$ by [PV23] and $(\mathrm{Mod}_j^{tS^1})^\wedge_{(p,v_1)}$ by [Dev25]; the S^1-nilpotence property from Theorem 7.13(*b*) ensures that $j \to \mathrm{THH}(\mathbb{Z}_p)_p^\wedge$ and $\mathbb{Z}_p \to \mathrm{THH}(\mathbb{F}_p)$ become equivalences in that category.

(7.3) Observe that $\overline{R}^{(p)}$ might only be an animated ring.

(7.4) Indeed, the first equivalence follows from the fact that A and $\mathrm{A}_{\mathrm{inf}}$ being are perfect δ-rings. For the second, note that $\mathrm{dR}_{R/\mathrm{A}_{\mathrm{inf}}}$ is p-torsion free and contains divided powers for all $x \in J$, as can be seen from $\mathrm{dR}_{\mathbb{Z}/\mathbb{Z}[x]} \to \mathrm{dR}_{R/\mathrm{A}_{\mathrm{inf}}}$. Hence there's a map $\mathrm{A}_{\mathrm{crys}} \to \mathrm{dR}_{R/\mathrm{A}_{\mathrm{inf}}}$, and this map is an equivalence modulo p by [BMS19, Proposition 8.12].

(*a*) *It is a map of* $\mathrm{A}_{\mathrm{inf}}$*-modules.*

(*b*) *It is continuous with respect to the natural topologies on either side.*

It's clear from the construction that $\overline{\psi}_S^0$ satisfies (*b*) since it is a map of condensed rings. To see (*a*), just observe that in the construction of $\overline{\psi}_S^0$, instead of working with $\mathrm{THH}_{\blacksquare}(-/\mathbb{S}_A)$, we could have worked with $\mathrm{THH}_{\blacksquare}(-/\mathbb{S}_{\mathrm{A}_{\mathrm{inf}}})$, where $\mathbb{S}_{\mathrm{A}_{\mathrm{inf}}}$ denotes the unique lift of the perfect δ-ring $\mathrm{A}_{\mathrm{inf}}$ to a p-complete connective $\mathbb{E}_\infty$-ring spectrum. □

Next let us describe ψ_R^0 after rationalisation.

7.14. Lemma. — *The rationalisation of the map* ψ_R^0 *from 7.7 fits into a commutative diagram*

$$\begin{array}{ccc} q\text{-}\mathrm{dR}_{R/A}\big[\tfrac{1}{p}\big]^\wedge_{(q-1)} & \xrightarrow{\psi^0_{R,\mathbb{Q}_p}} & \mathrm{gr}^0_{\mathrm{ev},hS^1}\,\mathrm{TC}^-_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A)\big[\tfrac{1}{p}\big]^\wedge_{(q-1)} \\ \simeq\big\downarrow & & \big\downarrow\simeq \\ \mathrm{dR}_{R/A}\big[\tfrac{1}{p}\big]\llbracket q-1\rrbracket & \longrightarrow & \mathrm{dR}_{R/A}\big[\tfrac{1}{p}\big]^\wedge_{\mathrm{Hdg}}\llbracket q-1\rrbracket \end{array}$$

where the left vertical arrow is the usual equivalence for rationalised q-de Rham cohomology, the right vertical arrow is obtained via Remark 6.22, and the bottom arrow is the natural Hodge completion map.

Proof. The following argument was suggested by Peter Scholze (any errors are due to the author). Observe that the usual rationalisation equivalence $q\text{-}\mathrm{dR}_{R/A}[1/p]^\wedge_{(q-1)} \simeq \mathrm{dR}_{R/A}[1/p]\llbracket q-1\rrbracket$ is $\mathbb{Z}_p^\times$-equivariant, where the action on the left-hand side is the one discussed in A.20 and on the right-hand side $u\in\mathbb{Z}_p^\times$ acts via $q\mapsto q^u$. Since the equivalence from Theorem 7.2 is also $\mathbb{Z}_p^\times$-equivariant, we obtain a $\mathbb{Z}_p^\times$-equivariant map

$$\mathrm{dR}_{R/A}\big[\tfrac{1}{p}\big]\llbracket q-1\rrbracket \longrightarrow \mathrm{dR}_{R/A}\big[\tfrac{1}{p}\big]^\wedge_{\mathrm{Hdg}}\llbracket q-1\rrbracket\,,$$

which we must show to agree with the natural Hodge completion map. In general, if $M\in\mathcal{D}(\mathbb{Q}_p)$ is equipped with the trivial action of $\mathbb{Z}_p^\times$, there's a functorial equivalence

$$M \xrightarrow{\simeq} M\llbracket q-1\rrbracket^{h\mathbb{Z}_p^\times}\otimes^{\mathrm{L}}_{\mathbb{Z}_p^{h\mathbb{Z}_p^\times}}\mathbb{Z}_p\,.$$

Indeed, the fixed points $M\llbracket q-1\rrbracket^{h\mathbb{Z}_p^\times}$ would be $M\oplus\Sigma^{-1}M$; to kill the shifted copy of M, we take the tensor product along $\mathbb{Z}_p^{h\mathbb{Z}_p^\times}\to\mathbb{Z}_p$.

Applying this in the situation at hand, we get a map $\mathrm{dR}_{R/A}[1/p]\to\mathrm{dR}_{R/A}[1/p]^\wedge_{\mathrm{Hdg}}$. By comparison with the reduction modulo $(q-1)$ and using Lemma 7.11, we see that this map must be the canonical Hodge completion map. By applying $(-\otimes^{\mathrm{L}}_{\mathbb{Q}_p}\mathbb{Q}_p\llbracket q-1\rrbracket)^\wedge_{(q-1)}$ to this map, we deduce that the original map must have been the natural Hodge completion as well. □

Proof of Theorem 7.9. By definition of the filtration $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A}$ (see 7.8), the base change $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{\mathbb{Z}_p[\beta]\llbracket t\rrbracket}\mathbb{Z}_p\llbracket t\rrbracket$ is the pullback of the filtered module $\Sigma^{-2\star}\,\mathrm{gr}^\star_{\mathrm{ev},hS^1}\,\mathrm{HC}^-_{\blacksquare}(R/A)$ along $\overline{\psi}^0_R\colon \mathrm{dR}_{R/A}\to\mathrm{gr}^0_{\mathrm{ev},hS^1}\,\mathrm{HC}^-_{\blacksquare}(R/A)$. The rationalisation $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A}[1/p]^\wedge_{(q-1)}$ can be described analogously. Using Lemmas 7.11 and 7.14 as well as the fact that any filtration is the pullback of its completion (see 1.48), we deduce that

$$\begin{aligned} \mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{\mathbb{Z}_p[\beta]\llbracket t\rrbracket}\mathbb{Z}_p\llbracket t\rrbracket &\xrightarrow{\simeq} \mathrm{fil}^\star_{\mathrm{Hdg}}\,\mathrm{dR}_{R/A}\,,\\ \mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A}\big[\tfrac{1}{p}\big]^\wedge_{(q-1)} &\xrightarrow{\simeq} \mathrm{fil}^\star_{(\mathrm{Hdg},q-1)}\,\mathrm{dR}_{R/A}\big[\tfrac{1}{p}\big]\llbracket q-1\rrbracket \end{aligned}$$

are indeed equivalences. Finally, whether

$$\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{R/A} \xrightarrow{\simeq} \Sigma^{-2*}\,\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{TC}^-_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A)$$

is an equivalence can be checked modulo β. By the base change result that we've already shown, this follows from $\mathrm{fil}^\star_{\mathrm{Hdg}}\,\widehat{\mathrm{dR}}_{R/A} \simeq \Sigma^{-2*}\,\mathrm{gr}^*_{\mathrm{ev}}\,\mathrm{HC}^-_{\blacksquare}(R/A)$. □

§7.2. The p-complete comparison (case $p = 2$)

In this subsection, we'll discuss how much of §7.1 can be salvaged in the case $p = 2$. We expect that Theorem 7.9 is still true for $p = 2$, but our proof fails at several places. Here are the two main issues:

(!) *The S^1-equivariant $\mathbb{E}_\infty$-equivalence $\mathrm{THH}(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p[\![q-1]\!]) \simeq \tau_{\geqslant 0}(\mathrm{ku}^{tC_p})$ from Theorem 7.2 is still conjectural for $p = 2$.*

(!!) *Theorem 7.13 is provably false for $p = 2$.*

The objection in the second issue is essentially the discrepancy between Nygaard and divided power completion at $p = 2$; see [DR25, Remark 0.5.3] for example. The goal of this subsection is to show that both issues only affect the case 6.2($\mathbb{E}_2$).

7.15. Theorem. — *If R satisfies the assumptions from 6.2($\mathbb{E}_1$), then the conclusions of Theorem 7.9 are true in the case $p = 2$ as well.*

7.16. Remark. — Note that a priori $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A}$ will only be a graded $\mathbb{E}_0$-algebra over $\mathbb{Z}_p[\beta][\![t]\!]$. A posteriori, we get an $\mathbb{E}_\infty$-structure by applying Theorem 7.18 below to the given cosimplicial resolution $R \to R^\bullet_\infty$.

To show Theorem 7.15, let us first address the less serious issue (!) above.

7.17. Theorem (Nikolaus, unpublished). — *For all primes p there exists an S^1-equivariant equivalence of $\mathbb{E}_1$-ring spectra*

$$\mathrm{THH}\big(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p[\![q-1]\!]\big)^\wedge_p \xrightarrow{\simeq} \tau_{\geqslant 0}\big(\mathrm{ku}^{tC_p}\big)\,,$$

compatible with $\mathrm{THH}(\mathbb{F}_p) \simeq \tau_{\geqslant 0}(\mathbb{Z}_p^{tC_p})$. For $p > 2$, this equivalence agrees with the underlying S^1-equivariant $\mathbb{E}_1$-equivalence of Theorem 7.2.

Proof. We thank Sanath Devalapurkar for explaining the following argument to us; any errors are our own responsibility. Let us first construct an S^1-equivariant $\mathbb{E}_\infty$-map $\mathbb{S}[\![q-1]\!] \to \mathrm{ku}^{tC_p}$, where the left-hand side receives the trivial S^1 action and the right-hand side the residual $S^1 \simeq S^1/C_p$-action. It's enough to construct an S^1-equivariant $\mathbb{E}_\infty$-map $\mathbb{S}[\![q-1]\!] \to \mathrm{ku}^{hC_p}$, or equivalently, an $\mathbb{E}_\infty$-map $\mathbb{S}[\![q-1]\!] \to (\mathrm{ku}^{hC_p})^{h(S^1/C_p)} \simeq \mathrm{ku}^{hS^1}$. But the element $q \in \pi_0(\mathrm{ku}^{hS^1})$ is is detected by an $\mathbb{E}_\infty$-map $\mathbb{S}[q] \to \mathrm{ku}^{hS^1}$; see Corollary D.2. This factors over the $(q-1)$-completion $\mathbb{S}[q] \to \mathbb{S}[\![q-1]\!]$ and so we obtain the desired map.

Now let us construct an $\mathbb{E}_2$-$\mathbb{S}_p[\![q-1]\!]$-algebra map $\mathbb{Z}_p[\zeta_p] \to \mathrm{ku}^{tC_p}$. To this end, observe that $\mathbb{Z}_p[\zeta_p]$ is the free $(q-1)$-complete $\mathbb{E}_2$-$\mathbb{S}_p[\![q-1]\!]$-algebra satisfying $[p]_q = 0$. Indeed, since $[p]_q = 0$ holds in $\mathbb{Z}_p[\zeta_p]$, it certainly receives an $\mathbb{E}_2$-$\mathbb{S}_p[\![q-1]\!]$-map from the free guy. Whether this map is an equivalence can be checked modulo $(q-1)$, where it reduces to the classical fact that $\mathbb{F}_p$ is the free $\mathbb{E}_2$-algebra satisfying $p = 0$. Since $[p]_q = 0$ holds in $\pi_*(\mathrm{ku}^{tC_p}) \cong \pi_*(\mathrm{ku}^{tS^1})/[p]_q$ and any nullhomotopy witnessing this must be unique by evenness, we get our desired $\mathbb{E}_2$-$\mathbb{S}_p[\![q-1]\!]$-algebra map $\mathbb{Z}_p[\zeta_p] \to \mathrm{ku}^{tC_p}$. It induces S^1-equivariant $\mathbb{E}_1$-$\mathbb{S}_p[\![q-1]\!]$-algebra maps

$$\mathrm{THH}\big(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p[\![q-1]\!]\big)_p^\wedge \longrightarrow \mathrm{THH}\big(\mathrm{ku}^{tC_p}/\mathbb{S}[\![q-1]\!]\big)_p^\wedge \longrightarrow \mathrm{ku}^{tC_p}\,,$$

where the arrow on the right comes from the universal property of $\mathrm{THH}(-/\mathbb{S}[\![q-1]\!])$ on $\mathbb{E}_\infty$-$\mathbb{S}[\![q-1]\!]$-algebras.[7.5] Since the left-hand side is connective, the above composition factors through an S^1-equivariant $\mathbb{E}_1$-$\mathbb{S}_p[\![q-1]\!]$-algebra map $\mathrm{THH}(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p[\![q-1]\!])_p^\wedge \to \tau_{\geqslant 0}(\mathrm{ku}^{tC_p})$.

We wish to show that this map is an equivalence. This can be checked modulo $(q-1)$, so it will be enough to prove that modulo $(q-1)$ we obtain the equivalence $\mathrm{THH}(\mathbb{F}_p) \simeq \tau_{\geqslant 0}(\mathbb{Z}_p^{tC_p})$ from [NS18, Corollary IV.4.13]. To this end, observe that by the universal properties of $\mathbb{Z}_p[\zeta_p]$ and $\mathbb{F}_p$ as free $\mathbb{E}_2$-algebras, the $\mathbb{E}_\infty$-map $\mathrm{ku}^{tC_p} \to \mathbb{Z}_p^{tC_p}$ fits into a commutative diagram of $\mathbb{E}_2$-algebras

$$\begin{array}{ccc} \mathbb{Z}_p[\zeta_p] & \longrightarrow & \mathrm{ku}^{tC_p} \\ \big\downarrow & & \big\downarrow \\ \mathbb{F}_p & \longrightarrow & \mathbb{Z}_p^{tC_p} \end{array}$$

which on the level of underlying spectra exhibits the bottom row as the mod-$(q-1)$-reduction of the top row. Using the same recipe as above, the bottom row induces an S^1-equivariant maps of $\mathbb{E}_1$-algebras

$$\mathrm{THH}(\mathbb{F}_p) \longrightarrow \mathrm{THH}\big(\mathbb{Z}_p^{tC_p}\big) \longrightarrow \mathbb{Z}_p^{tC_p}$$

After passing to connective covers, we get an S^1-equivariant $\mathbb{E}_1$-map $\mathrm{THH}(\mathbb{F}_p) \to \tau_{\geqslant 0}(\mathbb{Z}_p^{tC_p})$. We claim that this map necessarily agrees with the underlying $\mathbb{E}_1$-map of the S^1-equivariant $\mathbb{E}_\infty$-equivalence $\mathrm{THH}(\mathbb{F}_p) \simeq \tau_{\geqslant 0}(\mathbb{Z}^{tC_p})$ from [NS18, Corollary IV.4.13]. Indeed, by the universal property of THH for $\mathbb{E}_\infty$-ring spectra, this equivalence must also be given by a composition as above, where the first arrow is given by the non-equivariant $\mathbb{E}_\infty$-map $\mathbb{F}_p \to \mathbb{Z}_p^{tC_p}$ induced by the equivalence. But $\mathbb{F}_p$ is the free $\mathbb{E}_2$-algebra with $p=0$. Since $\mathbb{Z}_p^{tC_p}$ is even, any nullhomotopy witnessing $p=0$ is unique, and so there's a unique $\mathbb{E}_2$-map $\mathbb{F}_p \to \mathbb{Z}_p^{tC_p}$. This shows that the S^1-equivariant $\mathbb{E}_1$-map $\mathrm{THH}(\mathbb{F}_p) \to \tau_{\geqslant 0}(\mathbb{Z}_p^{tC_p})$ agrees with the equivalence $\mathrm{THH}(\mathbb{F}_p) \simeq \tau_{\geqslant 0}(\mathbb{Z}_p^{tC_p})$ and concludes the proof that $\mathrm{THH}(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p[\![q-1]\!])_p^\wedge \to \tau_{\geqslant 0}(\mathrm{ku}^{tC_p})$ is an equivalence.

To show that for $p>2$ this equivalence agrees with the underlying S^1-equivariant $\mathbb{E}_1$-equivalence of Theorem 7.2, we can use the same argument as above, noting that the $\mathbb{E}_2$-$\mathbb{S}[\![q-1]\!]$-algebra map $\mathbb{Z}_p[\zeta_p] \to \mathrm{ku}^{tC_p}$ is unique. □

We can now show Theorem 7.15.

Proof sketch of Theorem 7.15. Let us indicate how to modify the arguments in order to avoid those that don't work for $p=2$. To construct the comparison map ψ_R^0 as an $\mathbb{E}_0$-map, we don't need the full strength of Theorem 7.2, so Theorem 7.17 will suffice. In the proof of Lemma 7.11, we don't need quasi-syntomic descent (and in particular, we don't need Theorem 7.13, so we circumvent the more serious issue (!!) above), since the given resolution $R \to R_\infty^\bullet$ places us already in a relatively semiperfect situation.

[7.5] In particular, this map $\mathrm{THH}(\mathrm{ku}^{tC_p}/\mathbb{S}_p[\![q-1]\!])_p^\wedge \to \mathrm{ku}^{tC_p}$ is *not* the usual augmentation, as the augmentation would only be S^1-equivariant for the trivial S^1-action on ku^{tC_p}.

It remains to explain how to adapt the proof of Lemma 7.14. We don't know if the $\mathbb{Z}_p^\times$-equivariance argument still works, but fortunately, we can replace it by a simple argument similar to the proof of Lemma 7.11. In the given resolution, $R_\infty^\bullet/p$ is already relatively semiperfect over A and so $\mathrm{TC}^-_{\blacksquare}(\mathrm{ku}_{R_\infty^\bullet}\otimes^{\blacksquare}\mathbb{Q}_p/\mathrm{ku}_A\otimes^{\blacksquare}\mathbb{Q}_p)$ is already even. This reduces the question whether $\psi^0_{R,\mathbb{Q}_p}$ is the correct map to a question that can be checked on underlying sets. In particular, we can base change again to a situation where A is already perfect, so that $R_\infty^\bullet/p$ is semiperfect on the nose. If we put $\mathrm{A}_{\mathrm{inf}}^\bullet \coloneqq \mathrm{W}((R_\infty^\bullet)^\flat)$, $J^\bullet \coloneqq \ker(\mathrm{A}_{\mathrm{inf}}^\bullet \to R_\infty^\bullet)$, and let $\mathrm{A}_{\mathrm{crys}}^\bullet$ denote the p-completed PD-envelope of $J^\bullet$, then

$$q\text{-}\mathrm{dR}_{R_\infty^\bullet/A}\left[\tfrac{1}{p}\right]^\wedge_{(q-1)} \simeq \mathrm{dR}_{R_\infty^\bullet/A}\left[\tfrac{1}{p}\right]\llbracket q-1\rrbracket \simeq \mathrm{A}_{\mathrm{crys}}^\bullet\left[\tfrac{1}{p}\right]\llbracket q-1\rrbracket\,.$$

So to prove Lemma 7.14 in this particular case, we must check whether a certain map $\mathrm{A}_{\mathrm{crys}}^\bullet[1/p]\llbracket q-1\rrbracket \to \mathrm{A}_{\mathrm{crys}}^\bullet[1/p]^\wedge_{\mathrm{Hdg}}\llbracket q-1\rrbracket$ agrees with the canonical Hodge completion map. As in the proof of Lemma 7.11, the Hodge completion map is uniquely determined by:

(a) *It is a map of $\mathrm{A}_{\mathrm{inf}}^\bullet\llbracket q-1\rrbracket$-modules.*

(b) *It is continuous with respect to the natural topologies on either side.*

Condition (b) is again clear from our condensed setup, whereas (a) follows by working over $\mathbb{S}_{\mathrm{A}_{\mathrm{inf}}^\bullet}$ rather than $\mathbb{S}_A$. This finishes the proof. ☐

§7.3. The case of quasi-regular quotients

Let us continue to fix a prime p (with $p=2$ allowed) and keep Convention 7.1. Let A be a δ-ring as in 6.1 and suppose that R is an A-algebra satisfying 6.2($\mathbb{E}_1$) for the identical cover $\mathrm{id}\colon R\to R$. In other words, R is a p-quasi-lci A-algebra with a lift to a p-complete connective $\mathbb{E}_1$-algebra $\mathbb{S}_R \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}_{\mathbb{S}_A}(\mathrm{Sp}))$ such that R/p is relatively semiperfect over A.

These assumptions ensure that $q\text{-}\mathrm{dR}_{R/A}$ and $\mathrm{dR}_{R/A}$ are static rings and that the Hodge filtration $\mathrm{fil}^\star_{\mathrm{Hdg}}\,\mathrm{dR}_{R/A}$ is a descending filtration by ideals (see Lemma 4.18(b)). As it turns out, the q-Hodge filtration from 7.8 has a very explicit description in this case.

7.18. Theorem. — *Under the assumptions above, the q-Hodge filtration $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A}$ is the descending filtration by ideals given by the* (1*-categorical*) *preimage of the combined Hodge- and* $(q-1)$*-adic filtration under the rationalisation map $q\text{-}\mathrm{dR}_{R/A}\to \mathrm{dR}_{R/A}[1/p]\llbracket q-1\rrbracket$. In other words, there's a pullback*

$$\begin{array}{ccc} \mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A} & \longrightarrow & \mathrm{fil}^\star_{(\mathrm{Hdg},q-1)}\,\mathrm{dR}_{R/A}\left[\tfrac{1}{p}\right]\llbracket q-1\rrbracket \\ \big\downarrow & \lrcorner & \big\downarrow \\ q\text{-}\mathrm{dR}_{R/A} & \longrightarrow & \mathrm{dR}_{R/A}\left[\tfrac{1}{p}\right]\llbracket q-1\rrbracket \end{array}$$

in the 1*-category of filtered $(q-1)^\star A\llbracket q-1\rrbracket$-modules. In particular, $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A}$ is independent of the choice of the spherical $\mathbb{E}_1$-lift $\mathbb{S}_R$, and canonically a filtered $\mathbb{E}_\infty$-algebra over the filtered ring $(q-1)^\star A\llbracket q-1\rrbracket$.*

Proof. That $q\text{-}\mathrm{dR}_{R/A}$ is static and $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A}$ is a descending filtration by subgroups follows from the corresponding assertions for $\mathrm{dR}_{R/A}$ and $\mathrm{fil}^\star_{\mathrm{Hdg}}\,\mathrm{dR}_{R/A}$, using $q\text{-}\mathrm{dR}_{R/A}/(q-1)\simeq \mathrm{dR}_{R/A}$ and $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A}/\beta \simeq \mathrm{fil}^\star_{\mathrm{Hdg}}\,\mathrm{dR}_{R/A}$ by Theorems 7.9 and 7.15.

To show the description as a preimage, we first note that $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{R/A}$ is the preimage of its completion under $q\text{-}\mathrm{dR}_{R/A} \to q\text{-}\widehat{\mathrm{dR}}_{R/A}$ and likewise for $\mathrm{fil}^\star_{(\mathrm{Hdg},q-1)}\,\mathrm{dR}_{R/A}[1/p]\llbracket q-1\rrbracket$. Thus, it remains to show that the filtration on $\pi_0\,\mathrm{TC}^-_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A)$ induced by the homotopy fixed point spectral sequence is the preimage of the analogous filtration on $\pi_0\,\mathrm{TC}_{\blacksquare}(\mathrm{ku}_R \otimes^{\blacksquare} \mathbb{Q}_p/\mathrm{ku}_A \otimes^{\blacksquare} \mathbb{Q}_p)$ under the rationalisation map

$$\pi_0\,\mathrm{TC}^-_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A) \longrightarrow \pi_0\,\mathrm{TC}_{\blacksquare}(\mathrm{ku}_R \otimes^{\blacksquare} \mathbb{Q}_p/\mathrm{ku}_A \otimes^{\blacksquare} \mathbb{Q}_p)\,.$$

As both filtrations are complete, it will be enough to show that the map on associated gradeds is injective. That is, we must show $\pi_{2*}\,\mathrm{THH}_{\blacksquare}(\mathrm{ku}_R/\mathrm{ku}_A) \to \pi_{2*}\,\mathrm{THH}_{\blacksquare}(\mathrm{ku}_R \otimes^{\blacksquare} \mathbb{Q}_p/\mathrm{ku}_A \otimes^{\blacksquare} \mathbb{Q}_p)$ is injective. This can be checked modulo β, so we've reduced the problem to checking injectivity of $\pi_{2*}\,\mathrm{HH}_{\blacksquare}(R/A) \to \pi_{2*}\,\mathrm{HH}_{\blacksquare}(R \otimes^{\blacksquare} \mathbb{Q}_p/A \otimes^{\blacksquare} \mathbb{Q}_p)$. By the HKR theorem, we must show that

$$\Sigma^{-n} \bigwedge^n \mathrm{L}_{R/A} \longrightarrow \Sigma^{-n} \bigwedge^n \mathrm{L}_{R/A} \otimes^{\blacksquare} \mathbb{Q}_p$$

is injective for all n. Our assumptions guarantee that $\Sigma^{-1}\mathrm{L}_{R/A}$ is a p-completely flat module over the p-torsion free ring R and so each $\Sigma^{-n}\bigwedge^n \mathrm{L}_{R/A}$ will be a p-torsion free R-module. ☐

§7.4. The global case

In this subsection we'll sketch a global analogue of the p-complete comparison between $q\text{-}\mathrm{dR}_{R/A}$ and $\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku}_A)$ from §7.1. So let us no longer fix a prime p and update our assumptions on A and R accordingly.

7.19. New assumptions on A and R. — From now on, A and R must satisfy the following:

(A) *We assume that A is a perfectly covered Λ-ring* (*in the sense defined in* 1.50) *such that for all primes p the p-completion $\widehat{A}_p$ satisfies* 6.1(^{tC_p}), *with $\mathbb{S}_{\widehat{A}_p}$ denoting the p-complete spherical lift.*

(R) *We assume that R is a quasi-lci A-algebra in the sense that the cotangent complex $\mathrm{L}_{R/A}$ has* Tor*-amplitude in homogical degrees* $[0,1]$ *over R. In addition, for every prime p, the ring R must have bounded p^∞-torsion and its p-completion $\widehat{R}_p$ must satisfy one of the conditions* 6.2($\mathbb{E}_2$) *or* ($\mathbb{E}_1$) (*but not necessarily the same for every p*). *We let $\mathbb{S}_{\widehat{R}_p}$ denote the p-complete spherical lift of $\widehat{R}_p$.*

We note that the p-complete lifts $\mathbb{S}_{\widehat{A}_p}$ and $\mathbb{S}_{\widehat{R}_p}$ for all primes p can be glued with $A\otimes\mathbb{Q}$ and $R\otimes\mathbb{Q}$ to a connective $\mathbb{E}_\infty$-ring spectrum $\mathbb{S}_A$ and a connective $\mathbb{E}_1$-algebra $\mathbb{S}_R \in \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Mod}_{\mathbb{S}_A}(\mathrm{Sp}))$ satisfying

$$\mathbb{S}_A \otimes \mathbb{Z} \simeq A \quad \text{and} \quad \mathbb{S}_R \otimes \mathbb{Z} \simeq R\,.$$

By construction, $\mathbb{S}_A$ acquires the structure of a cyclotomic base. If 6.2($\mathbb{E}_2$) was chosen for every p, then $\mathbb{S}_R$ will be an $\mathbb{E}_2$-algebra in $\mathbb{S}_A$-modules. We also let $\mathrm{ku}_{\widehat{A}_p} \coloneqq (\mathrm{ku}\otimes\mathbb{S}_{\widehat{A}_p})^\wedge_p$ and $\mathrm{ku}_A \coloneqq \mathrm{ku}\otimes\mathbb{S}_A$ and define $\mathrm{ku}_{\widehat{R}_p}$ and ku_R analogously.

7.20. Remark. — Despite the restrictive hypotheses, there are many examples of such A and R, as we'll see in §9.1.

To carry out our global constructions, we'll proceed by gluing the p-complete constructions from §7.1 with the rational case. For the gluing to work, we'll the notion of *profinite completion* and the fact that it interacts well with the solid tensor product; see the review in B.8.

7.21. Profinite even filtrations. — Let $\widehat{A}$ and $\widehat{R}$ denote the profinite completions of A and R. Let k be any connective even $\mathbb{E}_\infty$-ring spectrum such that $\pi_*(k)$ is p-torsion free for all primes p (the most relevant case is of course $k = \mathrm{ku}$, but we'll also need $k = \mathrm{ku} \otimes \mathbb{Q}$ and later $k = \mathrm{ku}^{\Phi C_m}$). Let $k_{\widehat{A}} \coloneqq k \otimes^{\blacksquare} \prod_p \mathbb{S}_{\widehat{A}_p}$ and $k_{\widehat{R}} \coloneqq k \otimes^{\blacksquare} \prod_p \mathbb{S}_{\widehat{R}_p}$. We wish to construct an appropriate even filtration

$$\mathrm{fil}^{\star}_{\mathrm{ev}} \mathrm{THH}_{\blacksquare}\bigl(k_{\widehat{R}}/k_{\widehat{A}}\bigr)\,.$$

Once we have this, we can also construct versions for $\mathrm{TC}^-_{\blacksquare}$ and $\mathrm{TP}_{\blacksquare}$ via

$$\begin{aligned}\mathrm{fil}^{\star}_{\mathrm{ev},hS^1} \mathrm{TC}^-_{\blacksquare}\bigl(k_{\widehat{R}}/k_{\widehat{A}}\bigr) &\coloneqq \bigl(\mathrm{fil}^{\star}_{\mathrm{ev}} \mathrm{THH}_{\blacksquare}(k_{\widehat{R}}/k_{\widehat{A}})\bigr)^{h\mathbb{T}_{\mathrm{ev}}}\,,\\ \mathrm{fil}^{\star}_{\mathrm{ev},tS^1} \mathrm{TP}_{\blacksquare}\bigl(k_{\widehat{R}}/k_{\widehat{A}}\bigr) &\coloneqq \bigl(\mathrm{fil}^{\star}_{\mathrm{ev}} \mathrm{THH}_{\blacksquare}(k_{\widehat{R}}/k_{\widehat{A}})\bigr)^{t\mathbb{T}_{\mathrm{ev}}}\end{aligned}$$

Before we discuss the construction in general, let us start with two special cases:

($\mathbb{E}_1$) If we chose condition 6.2($\mathbb{E}_1$) for all primes p, and $\mathbb{S}_{\widehat{R}_p} \to \mathbb{S}_{\widehat{R}^\bullet_{p,\infty}}$ are the given cosimplicial resolutions, we put $k_{\widehat{R}^\bullet_\infty} \coloneqq k \otimes^{\blacksquare} \prod_p \mathbb{S}_{\widehat{R}^\bullet_{p,\infty}}$ and define our filtration via

$$\mathrm{fil}^{\star}_{\mathrm{ev}} \mathrm{THH}_{\blacksquare}\bigl(k_{\widehat{R}}/k_{\widehat{A}}\bigr) \coloneqq \lim_{\Delta} \tau_{\geqslant 2\star} \mathrm{THH}_{\blacksquare}\bigl(k_{\widehat{R}^\bullet_\infty}/k_{\widehat{A}}\bigr)\,.$$

($\mathbb{E}_2$) If instead 6.2($\mathbb{E}_2$) was chosen for all primes p, so that $k_{\widehat{R}}$ is an $\mathbb{E}_2$-algebra in $k_{\widehat{A}}$-modules, we simply define $\mathrm{fil}^{\star}_{\mathrm{ev}} \mathrm{THH}_{\blacksquare}(k_{\widehat{R}}/k_{\widehat{A}})$ to be the solid even filtration of $\mathrm{THH}_{\blacksquare}(k_{\widehat{R}}/k_{\widehat{A}})$ as a left module over itself.

In general, let P_1 and P_2 be the set of primes where we choose 6.2($\mathbb{E}_1$) and 6.2($\mathbb{E}_2$), respectively. Let $k_{\widehat{R},\mathbb{E}_1} \coloneqq \prod_{p\in P_1} k_{\widehat{R}_p}$ and $k_{\widehat{R},\mathbb{E}_2} \coloneqq \prod_{p\in P_2} k_{\widehat{R}_p}$. Then

$$\mathrm{THH}_{\blacksquare}\bigl(k_{\widehat{R}}/k_{\widehat{A}}\bigr) \simeq \mathrm{THH}_{\blacksquare}\bigl(k_{\widehat{R},\mathbb{E}_1}/k_{\widehat{A}}\bigr) \times \mathrm{THH}_{\blacksquare}\bigl(k_{\widehat{R},\mathbb{E}_2}/k_{\widehat{A}}\bigr)$$

and we can apply the constructions from ($\mathbb{E}_1$) and ($\mathbb{E}_2$) to the two factors separately.

The results from §§6.2–6.4 can all be adapted to the profinite case in a straightforward way and the proofs can be copied verbatim. For example, in case ($\mathbb{E}_2$), let $P \coloneqq \mathbb{Z}[x_i \mid i \in I]$ be a polynomial ring with a surjection $P \twoheadrightarrow R$ and let $\widehat{P}$ be its profinite completion. Let $\mathbb{S}_P \coloneqq \mathbb{S}[x_i \mid i \in I]$ and let $\mathbb{S}_{\widehat{P}}$ be its profinite completion. Finally, let $\mathbb{S} \to \mathbb{S}_{\widehat{P}^\bullet}$ denote the profinitely completed Čech nerve of $\mathbb{S} \to \mathbb{S}_{\widehat{P}}$. Then

$$\mathrm{fil}^{\star}_{\mathrm{ev}} \mathrm{THH}_{\blacksquare}\bigl(k_{\widehat{R}}/k_{\widehat{A}}\bigr) \xrightarrow{\simeq} \lim_{\Delta} \tau_{\geqslant 2\star} \mathrm{THH}_{\blacksquare}\bigl(k_{\widehat{R}}/k_{\widehat{A}} \otimes^{\blacksquare} \mathbb{S}_{\widehat{P}^\bullet}\bigr)\,.$$

To show this, we can simply copy the proof of Proposition 6.11. The key points are that $\mathrm{THH}_{\blacksquare}(\mathbb{S}_{\widehat{P}}) \to \mathbb{S}_{\widehat{P}}$ is still solid faithfully even flat, which can be shown by the same argument as in Lemma 6.12, and that $\mathrm{HH}_{\blacksquare}(\widehat{R}/\widehat{A} \otimes^{\blacksquare}_{\mathbb{Z}} \widehat{P}^\bullet)$ is still even.

7.22. Lemma. — *For $k = \mathrm{ku}$, we have canonical equivalences*

$$\begin{aligned}\mathrm{fil}^{\star}_{\mathrm{ev}} \mathrm{THH}_{\blacksquare}\bigl(\mathrm{ku}_{\widehat{R}}/\mathrm{ku}_{\widehat{A}}\bigr) &\xrightarrow{\simeq} \prod_p \mathrm{fil}^{\star}_{\mathrm{ev}} \mathrm{THH}_{\blacksquare}\bigl(\mathrm{ku}_{\widehat{R}_p}/\mathrm{ku}_{\widehat{A}_p}\bigr)\,,\\ \mathrm{fil}^{\star}_{\mathrm{ev},hS^1} \mathrm{TC}^-_{\blacksquare}\bigl(\mathrm{ku}_{\widehat{R}}/\mathrm{ku}_{\widehat{A}}\bigr) &\xrightarrow{\simeq} \prod_p \mathrm{fil}^{\star}_{\mathrm{ev},hS^1} \mathrm{TC}^-_{\blacksquare}\bigl(\mathrm{ku}_{\widehat{R}_p}/\mathrm{ku}_{\widehat{A}_p}\bigr)\,,\\ \mathrm{fil}^{\star}_{\mathrm{ev},tS^1} \mathrm{TP}_{\blacksquare}\bigl(\mathrm{ku}_{\widehat{R}}/\mathrm{ku}_{\widehat{A}}\bigr) &\xrightarrow{\simeq} \prod_p \mathrm{fil}^{\star}_{\mathrm{ev},tS^1} \mathrm{TP}_{\blacksquare}\bigl(\mathrm{ku}_{\widehat{R}_p}/\mathrm{ku}_{\widehat{A}_p}\bigr)\,.\end{aligned}$$

Proof. Let us first show the assertion for $\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}_\blacksquare$. Note that $\mathrm{ku}_{\widehat{A}} \simeq \prod_p \mathrm{ku}_{\widehat{A}_p} \simeq (\mathrm{ku}_A)^\wedge$ is the profinite completion of ku_A and likewise for $\mathrm{ku}_{\widehat{R}}$. Using Lemma 6.7 and its profinite analogue, we see

$$\mathrm{THH}_\blacksquare\big(\mathrm{ku}_{\widehat{R}}/\mathrm{ku}_{\widehat{A}}\big) \simeq \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^\wedge \simeq \prod_p \mathrm{THH}_\blacksquare\big(\mathrm{ku}_{\widehat{R}_p}/\mathrm{ku}_{\widehat{A}_p}\big)\,.$$

Applying the same observation to the cosimplicial resolutions $\mathrm{THH}_\blacksquare(\mathrm{ku}_{\widehat{R}^\bullet_\infty}/\mathrm{ku}_{\widehat{A}})$ (in the special case 7.21($\mathbb{E}_1$)) or $\mathrm{THH}_\blacksquare(\mathrm{ku}_{\widehat{R}}/\mathrm{ku}_{\widehat{A}} \otimes^\blacksquare \mathbb{S}_{\widehat{P}^\bullet})$ (in the special case 7.21($\mathbb{E}_1$)) or a mixture thereof (in the general case), we get the desired equivalence for $\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}_\blacksquare$.

The equivalence for $\mathrm{fil}^\star_{\mathrm{ev},hS^1}\,\mathrm{TC}^-_\blacksquare$ immediately follows. For $\mathrm{fil}^\star_{\mathrm{ev},tS^1}\,\mathrm{TP}_\blacksquare$, we must explain why $(-)_{h\mathbb{T}_{\mathrm{ev}}} \simeq \mathbb{S}_{\mathrm{ev}} \otimes^\blacksquare_{\mathbb{T}_{\mathrm{ev}}} -$ commutes with the infinite product $\prod_p$. By arguing as in the proof of Corollary 6.16 (or just reduction modulo β), we can reduce this to showing that $(-)_{hS^1}$ commutes with the infinite product in $\prod_p \mathrm{fil}^\star_{\mathrm{HKR}}\,\mathrm{HH}_\blacksquare(\widehat{R}_p/\widehat{A}_p)$. Since the HKR filtration increases in connectivity, it's enough to show the same for each graded piece $\prod_p \mathrm{gr}^n_{\mathrm{HKR}}\,\mathrm{HH}_\blacksquare(\widehat{R}_p/\widehat{A}_p)$. Since R was assumed to be quasi-lci over A, each graded piece is concentrated in a finite range of degrees. Thus, in any given homotopical degree, only finitely many cells of $\mathbb{CP}^\infty \simeq BS^1$ will contribute to $(-)_{hS^1}$, so it commutes with the infinite product. □

Finally, we can put everything together.

7.23. Global even filtrations. — Since ku_A and ku_R are discrete, $\mathrm{THH}_\blacksquare$ agrees with the usual THH. We can thus equip $\mathrm{THH}(\mathrm{ku}_R \otimes \mathbb{Q}/\mathrm{ku}_A \otimes \mathbb{Q})$ with the solid even filtration, which agrees with Pstrągowski's perfect even filtration by Corollary 5.8, and with the Hahn–Raksit–Wilson filtration by [Pst23, Theorem 7.5] and our assumption that R is quasi-lci over A. We can now define an even filtration on $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)$ via the pullback diagram

$$\begin{array}{ccc}
\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A) & \longrightarrow & \mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}_\blacksquare\big(\mathrm{ku}_{\widehat{R}}/\mathrm{ku}_{\widehat{A}}\big) \\
\downarrow & \lrcorner & \downarrow \\
\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}(\mathrm{ku}_R \otimes \mathbb{Q}/\mathrm{ku}_A \otimes \mathbb{Q}) & \longrightarrow & \mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}_\blacksquare\big(\mathrm{ku}_{\widehat{R}} \otimes^\blacksquare \mathbb{Q}/\mathrm{ku}_{\widehat{A}} \otimes^\blacksquare \mathbb{Q}\big)
\end{array}$$

where the right vertical map is given by 7.21 applied to $k = \mathrm{ku}$ and $k = \mathrm{ku} \otimes \mathbb{Q}$.

We must explain where the bottom horizontal map comes from. It's straightforward to check that $\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}(\mathrm{ku}_R \otimes \mathbb{Q}/\mathrm{ku}_A \otimes \mathbb{Q}) \simeq \mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{HH}(R/A) \otimes \mathbb{Q}[\beta]_{\mathrm{ev}}$. Moreover, since the base change result from Corollary 6.17 is still true in the profinite situation (see the discussion in 7.21), we can use base change for $\mathbb{Z} \to \mathbb{Q}[\beta] \simeq \mathrm{ku} \otimes \mathbb{Q}$ to get

$$\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}_\blacksquare\big(\mathrm{ku}_{\widehat{R}} \otimes^\blacksquare \mathbb{Q}/\mathrm{ku}_{\widehat{A}} \otimes^\blacksquare \mathbb{Q}\big) \simeq \mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{HH}_\blacksquare(\widehat{R}/\widehat{A}) \otimes^\blacksquare \mathbb{Q}[\beta]_{\mathrm{ev}}\,.$$

Moreover, the profinite analogue of Corollary 6.21 shows that $\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{HH}_\blacksquare(\widehat{R}/\widehat{A})$ agrees with $\prod_p \mathrm{fil}^\star_{\mathrm{HRW\text{-}ev}}\,\mathrm{HH}(R/A)^\wedge_p$. We then have a canonical map $\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{HH}(R/A) \to \mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{HH}_\blacksquare(\widehat{R}/\widehat{A})$, which provides us with the desired bottom horizontal map in the diagram above.

Once we have constructed $\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)$, we can also construct filtrations on TC^- and TP in the usual manner:

$$\begin{aligned}
\mathrm{fil}^\star_{\mathrm{ev},hS^1}\,\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku}_A) &\coloneqq \big(\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)\big)^{h\mathbb{T}_{\mathrm{ev}}}\,,\\
\mathrm{fil}^\star_{\mathrm{ev},tS^1}\,\mathrm{TP}(\mathrm{ku}_R/\mathrm{ku}_A) &\coloneqq \big(\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)\big)^{t\mathbb{T}_{\mathrm{ev}}}\,.
\end{aligned}$$

Here's a sanity check:

7.24. Lemma. — *Suppose we chose condition* 6.2($\mathbb{E}_2$) *for all primes p, so that ku_R is an $\mathbb{E}_2$-algebra in ku_A-modules. Then $\mathrm{fil}_{\mathrm{ev}}^\star \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)$ agrees with the solid perfect even filtration on the solid $\mathbb{E}_1$-ring $\mathrm{THH}_\blacksquare(\mathrm{ku}_R/\mathrm{ku}_A)$, and also with Pstrągowski's perfect even filtration $\mathrm{fil}_{\text{P-ev}}^\star \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)$.*

Proof sketch. The solid even filtration agrees with Pstrągowski's construction by Corollary 5.8. To show that both agree with the pullback $\mathrm{fil}_{\mathrm{ev}}^\star \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)$ from 7.23, we verify that all even filtrations in sight can be computed by cosimplicial resolutions as in Proposition 6.11. To show this, the proof of said proposition can be adapted in a straightforward way. The key points are that $\mathrm{THH}_\blacksquare(\mathbb{S}_P) \to \mathbb{S}_P$ is still solid faithfully even flat by Lemma 6.12 and that $\mathrm{HH}(R/A \otimes_{\mathbb{Z}} P^\bullet)$ is still even. □

We're now ready to construct the global comparison with q-de Rham cohomology. Due to the problems at $p = 2$ that we've discussed at the end of §7.1, we need a small addendum to the assumptions from 7.19(R).

7.19*a*. New assumptions on A and R. — From now on we'll assume that R satisfies not only 7.19(R) but also:

(R_2) *The 2-adic completion $\widehat{R}_2$ satisfies* 6.2($\mathbb{E}_1$).

We note that this is true, in particular, if 2 is invertible in R.

7.25. The global comparison map. — Let us denote $q\text{-}\mathrm{dR}_{\widehat{R}/\widehat{A}} \coloneqq \prod_p q\text{-}\mathrm{dR}_{\widehat{R}_p/\widehat{A}_p}$ and $\mathrm{dR}_{\widehat{R}/\widehat{A}} \coloneqq \prod_p \mathrm{dR}_{\widehat{R}_p/\widehat{A}_p}$ for short. Then the global q-de Rham complex sits inside a pullback

$$\begin{array}{ccc} q\text{-}\mathrm{dR}_{R/A} & \longrightarrow & q\text{-}\mathrm{dR}_{\widehat{R}/\widehat{A}} \\ \downarrow & \lrcorner & \downarrow \\ \left(\mathrm{dR}_{R/A} \otimes_{\mathbb{Z}}^{\mathrm{L}} \mathbb{Q}\right)[\![q-1]\!] & \longrightarrow & \left(\mathrm{dR}_{\widehat{R}/\widehat{A}} \otimes_{\mathbb{Z}}^{\mathrm{L}} \mathbb{Q}\right)[\![q-1]\!] \end{array}$$

(see Construction A.12). We claim that this diagram maps canonically to the pullback square

$$\begin{array}{ccc} \mathrm{gr}^0_{\mathrm{ev},hS^1} \mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku}_A) & \longrightarrow & \mathrm{gr}^0_{\mathrm{ev},hS^1} \mathrm{TC}^-_\blacksquare\left(\mathrm{ku}_{\widehat{R}}/\mathrm{ku}_{\widehat{A}}\right) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{gr}^0_{\mathrm{ev},hS^1} \mathrm{TC}^-(\mathrm{ku}_R \otimes \mathbb{Q}/\mathrm{ku}_A \otimes \mathbb{Q}) & \longrightarrow & \mathrm{gr}^0_{\mathrm{ev},hS^1} \mathrm{TC}^-_\blacksquare\left(\mathrm{ku}_{\widehat{R}} \otimes^\blacksquare \mathbb{Q}/\mathrm{ku}_{\widehat{A}} \otimes^\blacksquare \mathbb{Q}\right) \end{array}$$

coming from 7.23. To construct this map of pullback squares, we need:

(*a*) A map $q\text{-}\mathrm{dR}_{\widehat{R}/\widehat{A}} \to \mathrm{gr}^0_{\mathrm{ev},hS^1} \mathrm{TC}^-_\blacksquare(\mathrm{ku}_{\widehat{R}}/\mathrm{ku}_{\widehat{A}})$. This we get by taking the product of the maps $\psi^0_{\widehat{R}_p}$ from 7.7 for all primes p.

(*b*) A map $(\mathrm{dR}_{R/A} \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!] \to \mathrm{gr}^0_{\mathrm{ev}} \mathrm{TC}^-(\mathrm{ku}_R \otimes \mathbb{Q}/\mathrm{ku}_A \otimes \mathbb{Q})$. Since $\mathrm{ku}_A \otimes \mathbb{Q} \simeq A \otimes \mathbb{Q}[\beta]$ and $\mathrm{ku}_R \otimes \mathbb{Q} \simeq R \otimes \mathbb{Q}[\beta]$, we get

$$\mathrm{TC}^-(\mathrm{ku}_R \otimes \mathbb{Q}/\mathrm{ku}_A \otimes \mathbb{Q}) \simeq \mathrm{HC}^-\left(R \otimes \mathbb{Q}[\beta]/A \otimes \mathbb{Q}[\beta]\right).$$

A standard computation identifies $\mathrm{gr}^0_{\mathrm{ev}}$ with the Hodge completion $(\mathrm{dR}_{R/A} \otimes \mathbb{Q})^\wedge_{\mathrm{Hdg}}[\![q-1]\!]$, so we can choose our desired map to be the Hodge completion map.

(c) A map $(\mathrm{dR}_{\widehat{R}/\widehat{A}}\otimes_{\mathbb{Z}}\mathbb{Q})\llbracket q-1\rrbracket\rightarrow \mathrm{gr}^0_{\mathrm{ev},hS^1}\mathrm{TC}^-_{\blacksquare}(\mathrm{ku}_{\widehat{R}}\otimes^{\blacksquare}\mathbb{Q}/\mathrm{ku}_{\widehat{A}}\otimes^{\blacksquare}\mathbb{Q})$. This works as in (b) above.

Clearly (b) and (c) are compatible; compatibility of (a) and (c) will be checked in Lemma 7.29 below. So we get our map of pullback squares and thus a map

$$\psi_R^0\colon q\text{-}\mathrm{dR}_{R/A}\longrightarrow \mathrm{gr}^0_{\mathrm{ev},hS^1}\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku}_A)\,.$$

7.26. The global q-Hodge filtration. — As in the p-complete case 7.8, we identify $\Sigma^{-2*}\mathrm{gr}^*_{\mathrm{ev},hS^1}(\mathrm{ku}^{hS^1})\simeq\mathbb{Z}[\beta]\llbracket t\rrbracket$ with the filtered ring $(q-1)^\star\mathbb{Z}\llbracket q-1\rrbracket$, where t is the filtration parameter and β corresponds to $(q-1)$ in filtration degree 1. We then define the *q-Hodge filtration* as the pullback

$$\begin{array}{ccc}\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A} & \longrightarrow & \Sigma^{-2*}\mathrm{gr}^*_{\mathrm{ev},hS^1}\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku}_A)\\ \downarrow & \lrcorner & \downarrow\\ q\text{-}\mathrm{dR}_{R/A} & \xrightarrow{\ \psi_R^0\ } & \mathrm{gr}^0_{\mathrm{ev},hS^1}\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku}_A)\end{array}$$

As the name suggests, $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A}$ is indeed a q-Hodge filtration in the sense of Definition 3.2.

7.27. Theorem. — *Suppose A and R satisfy the assumptions from 7.19 along with the addendum (R_2). Then the map ψ_R^0 from 7.25 induces an equivalence of graded $\mathbb{Z}[\beta]\llbracket t\rrbracket$-modules*

$$\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{R/A}\xrightarrow{\ \simeq\ }\Sigma^{-2*}\mathrm{gr}^*_{\mathrm{ev},hS^1}\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku}_A)\,,$$

where the left-hand side denotes the completion of the q-Hodge filtration $\mathrm{fil}^*_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A}$ from 7.26. Moreover, modulo β and after rationalisation, we get equivalences

$$\begin{aligned}\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{\mathbb{Z}[\beta]\llbracket t\rrbracket}\mathbb{Z}\llbracket t\rrbracket &\xrightarrow{\ \simeq\ }\mathrm{fil}^\star_{\mathrm{Hdg}}\,\mathrm{dR}_{R/A}\,,\\ \mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\big(q\text{-}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q}\big)^\wedge_{(q-1)} &\xrightarrow{\ \simeq\ }\mathrm{fil}^\star_{(\mathrm{Hdg},q-1)}\big(\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q}\big)\llbracket q-1\rrbracket\end{aligned}$$

with the usual Hodge filtration and the combined Hodge and $(q-1)$-adic filtration, respectively. Via these equivalences, $(R,\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A})$ becomes canonically an object in $\mathrm{AniAlg}_A^{q\text{-}\mathrm{Hdg}}$.

7.28. Remark. — Fix $2\leqslant n\leqslant\infty$. If for every prime p either 6.2($\mathbb{E}_1$) was chosen or $\mathbb{S}_{\widehat{R}_p}$ admits an $\mathbb{E}_n$-algebra structure in $\mathbb{S}_{\widehat{A}_p}$-modules, then all equivalences in Theorem 7.27 are canonically $\mathbb{E}_{n-1}$-monoidal. Indeed, for those primes where $\mathbb{S}_{\widehat{R}_p}$ is $\mathbb{E}_n$, we get $\mathbb{E}_{n-1}$-monoidality by carefully tracing through all constructions. For the other primes use Theorem 7.18. It follows that $(R,\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A})$ is canonically an $\mathbb{E}_{n-1}$-algebra in $\mathrm{AniAlg}_A^{q\text{-}\mathrm{Hdg}}$ (compare 3.50).

7.29. Lemma. — *The maps from 7.25(a) and (c) fit into a commutative diagram*

$$\begin{array}{ccc}\big(q\text{-}\mathrm{dR}_{\widehat{R}/\widehat{A}}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q}\big)^\wedge_{(q-1)} & \xrightarrow{\ 7.25(a)\ } & \Big(\mathrm{gr}^0_{\mathrm{ev},hS^1}\mathrm{TC}^-_{\blacksquare}(\mathrm{ku}_{\widehat{R}}/\mathrm{ku}_{\widehat{A}})\otimes^{\blacksquare}\mathbb{Q}\Big)^\wedge_{(q-1)}\\ \simeq\downarrow & & \downarrow\simeq\\ \big(\mathrm{dR}_{\widehat{R}/\widehat{A}}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q}\big)\llbracket q-1\rrbracket & \xrightarrow{\ 7.25(c)\ } & \mathrm{gr}^0_{\mathrm{ev},hS^1}\mathrm{HC}^-_{\blacksquare}\Big(\widehat{R}\otimes^{\blacksquare}\mathbb{Q}[\beta]/\widehat{A}\otimes^{\blacksquare}\mathbb{Q}[\beta]\Big)\end{array}$$

where the left vertical arrow is the usual equivalence for rationalised q-de Rham cohomology and the right vertical arrow is obtained as explained in 7.23.

Proof. In the following, we'll assume that 2 is invertible in R. To treat the general case, we can just split off the factor $p=2$ from $\big(\prod_p q\text{-}\mathrm{dR}_{\widehat{R}_p/\widehat{A}_p}\otimes_{\mathbb{Z}}^{\mathrm{L}}\mathbb{Q}\big)^\wedge_{(q-1)}$ and use Lemma 7.14.[(7.6)]

We'll use an adaptation of the argument from the proof of Lemma 7.14. Observe that all maps in question are equivariant with respect to the Adams action of $\widehat{\mathbb{Z}}^\times\coloneqq\prod_p\mathbb{Z}_p^\times$, so the problem boils down to checking that a certain $\widehat{\mathbb{Z}}^\times$-equivariant map

$$\big(\mathrm{dR}_{\widehat{R}/\widehat{A}}\otimes_{\mathbb{Z}}^{\mathrm{L}}\mathbb{Q}\big)[\![q-1]\!]\longrightarrow\big(\mathrm{dR}_{\widehat{R}/\widehat{A}}\otimes_{\mathbb{Z}}^{\mathrm{L}}\mathbb{Q}\big)^\wedge_{\mathrm{Hdg}}[\![q-1]\!]$$

is the canonical Hodge completion map.

To see this, consider the element $\psi\coloneqq(\zeta_{p-1}(1+p))_p\in\prod_p\mathbb{Z}_p^\times$, where $\zeta_{p-1}\in\mathbb{Z}_p^\times$ denotes any primitive $(p-1)^{\mathrm{st}}$ root of unity. We claim that for any $M\in\mathcal{D}(\mathbb{Z})$, equipped with the trivial action of $\widehat{\mathbb{Z}}^\times$, one has a functorial equivalence

$$\big(\widehat{M}\otimes_{\mathbb{Z}}^{\mathrm{L}}\mathbb{Q}\big)[\![q-1]\!]^{\psi=1}\simeq\big(\widehat{M}\otimes_{\mathbb{Z}}^{\mathrm{L}}\mathbb{Q}\big)\oplus\Sigma^{-1}\big(\widehat{M}\otimes_{\mathbb{Z}}^{\mathrm{L}}\mathbb{Q}\big)$$

To show the claim, it'll be enough to show $\mathrm{H}_{-1}\big(\widehat{\mathbb{Z}}[\![q-1]\!]^{\psi=1}/(q-1)^n\big)\simeq\widehat{\mathbb{Z}}\oplus(\text{torsion group})$ for every n. This H_{-1} agrees with π_{-1} of the spectrum

$$\prod_p\Big(\big(\mathrm{ku}_p^\wedge\big)^{\mathrm{B}S^1}\Big)^{\psi=1}/t^n\simeq\prod_p\Big(\big(\mathrm{ku}_p^\wedge\big)^{\psi=1}\Big)^{\mathbb{CP}^n}$$

The homotopy groups of $(\mathrm{ku}_p^\wedge)^{\psi=1}$ are $\mathbb{Z}_p$ in degrees $\{-1,0\}$ and torsion groups in degrees $\geqslant 2p-3$. Since $\mathbb{CP}^n$ has a finite even cell decomposition, the torsion groups in positive degrees will only contribute to $\pi_{-1}\big(\prod_p((\mathrm{ku}_p^\wedge)^{\psi=1})^{\mathbb{CP}^n}\big)$ for finitely many primes, and so the result will indeed be of the form $\widehat{\mathbb{Z}}\oplus(\text{torsion group})$. This proves the claim.

To deduce that our map above must be the canonical Hodge completion, we apply $(-)^{\psi=1}\otimes^{\mathrm{L}}_{\mathbb{Z}^{\psi=1}}\mathbb{Z}$ to get a map $\mathrm{dR}_{\widehat{R}/\widehat{A}}\otimes_{\mathbb{Z}}^{\mathrm{L}}\mathbb{Q}\to(\mathrm{dR}_{\widehat{R}/\widehat{A}}\otimes_{\mathbb{Z}}^{\mathrm{L}}\mathbb{Q})^\wedge_{\mathrm{Hdg}}$. By comparison with the reduction modulo $(q-1)$ and Lemma 7.11 (applied for all primes p), we know that this map must be the canonical Hodge completion. By applying $(-\otimes_{\mathbb{Q}}^{\mathrm{L}}\mathbb{Q}[\![q-1]\!])^\wedge_{(q-1)}$ to this map, we deduce that our original map must be the Hodge completion as well. □

Proof sketch of Theorem 7.27. Using Corollary 6.21, we see that the base change of our even filtration $\mathrm{fil}^\star_{\mathrm{ev},hS^1}\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku}_A)$ along $\mathrm{ku}_{\mathrm{ev}}^{hS^1}\to\mathbb{Z}_{\mathrm{ev}}^{hS^1}$ is the Hahn–Raksit–Wilson even filtration on $\mathrm{HC}^-(R/A)$. Moreover, it's clear from the construction in 7.25 and Lemma 7.11 that the induced map

$$\overline{\psi}_R^0\colon\mathrm{dR}_{R/A}\longrightarrow\widehat{\mathrm{dR}}_{R/A}\simeq\mathrm{gr}^0_{\mathrm{HRW\text{-}ev},hS^1}\mathrm{HC}^-(R/A)$$

is the canonical Hodge completion map. Similarly, by the construction in 7.25(b), the rationalisation

$$\psi_{R,\mathbb{Q}}^0\colon\big(\mathrm{dR}_{R/A}\otimes_{\mathbb{Z}}^{\mathrm{L}}\mathbb{Q}\big)[\![q-1]\!]\longrightarrow\mathrm{gr}^0_{\mathrm{ev},hS^1}\mathrm{TC}^-(\mathrm{ku}_R\otimes\mathbb{Q}/\mathrm{ku}_A\otimes\mathbb{Q})$$

gets identified with the canonical Hodge completion map. With these two observations, the proof of Theorem 7.9 can be copied verbatim to show everything but the last claim.

To give $(R,\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A})$ the structure of an object in $\mathrm{AniAlg}_A^{q\text{-}\mathrm{Hdg}}$, the equivalences from Definition 3.2(b) and (c) have already been constructed; the compatibility between them follows by comparing the even filtrations on $\mathrm{TC}^-(\mathrm{ku}_R\otimes\mathbb{Q}/\mathrm{ku}_A\otimes\mathbb{Q})\simeq\mathrm{HC}^-(R\otimes\mathbb{Q}[\beta]/A\otimes\mathbb{Q}[\beta])$ and $\mathrm{HC}^-(R\otimes\mathbb{Q}/A\otimes\mathbb{Q})$. For Definition 3.2($c_p$), we use Theorem 7.9; the compatibilities come for free via the adelic gluing constructions in 7.23 and 7.25. □

(7.6) Recall that Lemma 7.14 still works for $p=2$ as long as 6.2($\mathbb{E}_1$) was chosen; see the argument in the proof of Theorem 7.15.

§8. Habiro descent via genuine equivariant homotopy theory

We've seen in Theorem 7.27 that the even filtration on $\mathrm{TC}^-(\mathrm{ku}_R/\mathrm{ku}_A)$ gives rise to a q-Hodge filtration $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_{R/A}$ in the sense of Definition 3.2. In particular, this provides many examples to which Theorem 3.11 can be applied.

The goal of this section is to show that, in the situation at hand, the Habiro descent from Theorem 3.11 can also be obtained homotopically. As a straightforward corollary of Theorem 7.27, one checks that the q-Hodge complex associated to $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_{R/A}$ agrees with

$$q\text{-Hdg}_{R/A} \simeq \mathrm{gr}^0_{\mathrm{ev},hS^1} \mathrm{TC}^-(\mathrm{KU}_R/\mathrm{KU}_A)\,,$$

where we put $\mathrm{KU}_A \coloneqq \mathrm{KU} \otimes \mathbb{S}_A$ and $\mathrm{KU}_R \coloneqq \mathrm{KU} \otimes \mathbb{S}_R$. To get the Habiro descent, we'll show that for every $m \in \mathbb{N}$ the action of the cyclic subgroup $C_m \subseteq S^1$ on $\mathrm{THH}(\mathrm{KU}_R/\mathrm{KU}_A)$ can be made *genuine*. We'll then construct an even filtration on $(\mathrm{THH}(\mathrm{KU}_R/\mathrm{KU}_A)^{C_m})^{h(S^1/C_m)}$. The Habiro descent $q\text{-}\mathcal{H}\mathrm{dg}_{R/A}$ will finally be recovered as the 0^{th} graded piece

$$q\text{-}\mathcal{H}\mathrm{dg}_{R/A} \simeq \lim_{m\in\mathbb{N}} \mathrm{gr}^0_{\mathrm{ev},S^1}\big(\mathrm{THH}(\mathrm{KU}_R/\mathrm{KU}_A)^{C_m}\big)^{h(S^1/C_m)}$$

This section is organised as follows: In §§8.1–8.3 we review genuine equivariant homotopy theory, its special case of *cyclonic spectra*, and the genuine equivariant structure on ku. In §8.4, we finally construct the desired even filtrations in the cyclonic setting and prove that they give rise to the same Habiro descent as in Theorem 3.11.

§8.1. Recollections on genuine equivariant homotopy theory

In this subsection, we briefly review theory of genuine equivariant spectra. We'll follow the model-independent treatment of [GM23, Appendix C] and the lecture notes [Hau24].

8.1. Genuine equivariant anima. — Let G be a compact Lie group (of relevance to us will only be the case of S^1 and its finite cyclic subgroups $C_m \subseteq S^1$). We let Orb_G denote the category whose objects are quotient spaces G/H, where $H \subseteq G$ is a closed subgroup, and whose morphisms are G-equivariant maps. Orb_G is canonically topologically enriched; through this enrichment we view it as an ∞-category.

We define the ∞-category of *G-anima* (or *G-spaces*) as well as its pointed variant as

$$\mathrm{Ani}^G \coloneqq \mathrm{PSh}(\mathrm{Orb}_G) \quad \text{and} \quad \mathrm{Ani}^G_* \coloneqq \mathrm{PSh}(\mathrm{Orb}_G)_*\,,$$

where $\mathrm{PSh}(-) \coloneqq \mathrm{Fun}((-)^{\mathrm{op}}, \mathrm{Ani})$ and $\mathrm{PSh}(-)_* \coloneqq \mathrm{Fun}((-)^{\mathrm{op}}, \mathrm{Ani}_*)$ denote the presheaf ∞-category and its pointed variant. The pointwise product or smash product induces symmetric monoidal structures on Ani^G and Ani^G_* and thus turns them into objects in $\mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}})$. We denote the evaluation at G/H by $(-)^H\colon \mathrm{Ani}^G \to \mathrm{Ani}$ and likewise for Ani^G_*. By construction, these functors are symmetric monoidal.

8.2. Genuine equivariant spectra. — For every finite-dimensional real G-representation V, we have a topologically enriched functor $\mathrm{Orb}_G^{\mathrm{op}} \to \mathrm{Top}_*$ sending $G/H \mapsto S^{V^H}$, where S^{V^H} denotes the 1-point compactification of the vector space V^H. This functor defines a pointed G-anima $S^V \in \mathrm{Ani}^G_*$, which we call the *representation sphere of V*. We finally define the ∞-category of *genuine G-equivariant spectra*

$$\mathrm{Sp}^G \coloneqq \mathrm{Ani}^G_*\Big[\big\{(S^V)^{\otimes -1}\big\}_V\Big]$$

to be the initial Ani_*^G-algebra in Pr^{L} in which all representation spheres S^V become $\otimes$-invertible. Explicitly, Sp^G can be written as a colimit in Pr^{L} of a diagram whose objects are copies of Ani_*^G and whose transition maps are of the form $S^V \wedge -\colon \mathrm{Ani}_*^G \to \mathrm{Ani}_*^G$, where V ranges through finite-dimensional G-representations; see [GM23, §C.1]. By construction, Sp^G comes with a symmetric monoidal functor

$$\Sigma_G^\infty\colon \mathrm{Ani}_*^G \longrightarrow \mathrm{Sp}^G$$

in Pr^{L}, which thus admits a lax monoidal right adjoint $\Omega_G^\infty\colon \mathrm{Sp}^G \to \mathrm{Ani}_*^G$.

We let $\Sigma^V\colon \mathrm{Sp}^G \to \mathrm{Sp}^G$ denote the functor $\Sigma_G^\infty S^V \otimes -$. By construction, this functor is an equivalence, and we let Σ^{-V} denote its inverse. If $(-)_+\colon \mathrm{Ani}^G \to \mathrm{Ani}_*^G$ denotes the left adjoint of the forgetful functor, we also define

$$\mathbb{S}_G[-]\colon \mathrm{Ani}^G \xrightarrow{(-)_+} \mathrm{Ani}_*^G \xrightarrow{\Sigma_G^\infty} \mathrm{Sp}^G$$

and we let $\mathbb{S}_G \coloneqq \mathbb{S}_G[*]$ be the *genuine G-equivariant sphere spectrum.*

The ∞-category Ani_*^G is compactly generated, with a set of compact generators given by $(G/H)_+$ for all closed subgroups $H \subseteq G$. The transition maps $S^V \wedge -$ preserve compact objects and $\mathrm{Pr}_\omega^{\mathrm{L}} \to \mathrm{Pr}^{\mathrm{L}}$ preserves colimits. It follows that Sp^G is compactly generated, with a set of compact generators given by $\Sigma^{-V}\mathbb{S}_G[G/H]$ for all representation spheres and all closed subgroups $H \subseteq G$. In fact, we can do slightly better; see Lemma 8.9 below.

8.3. Pullback functors. — Given any morphism $\varphi\colon G \to K$ of compact Lie groups, we can define a functor $\mathrm{Orb}_G \to \mathrm{Orb}_K$ by sending $G/H \mapsto K/\varphi(H)$. By precomposition, we obtain a symmetric monoidal functor $\varphi^*\colon \mathrm{Ani}_*^K \to \mathrm{Ani}_*^G$ in Pr^{L}, which sends representation spheres to representation spheres and therefore determines a unique symmetric monoidal colimit-preserving functor

$$\varphi^*\colon \mathrm{Sp}^K \longrightarrow \mathrm{Sp}^G\,.$$

8.4. Lemma. — *For every morphism $\varphi\colon G \to K$ of compact Lie groups, the following diagrams commute:*

$$\begin{array}{ccc} \mathrm{Ani}_*^K & \xrightarrow{\varphi^*} & \mathrm{Ani}_*^G \\ \Sigma_K^\infty \big\downarrow & & \big\downarrow \Sigma_G^\infty \\ \mathrm{Sp}^K & \xrightarrow{\varphi^*} & \mathrm{Sp}^G \end{array} \quad \textit{and} \quad \begin{array}{ccc} \mathrm{Ani}_*^K & \xrightarrow{\varphi^*} & \mathrm{Ani}_*^G \\ \Omega_K^\infty \big\uparrow & & \big\uparrow \Omega_G^\infty \\ \mathrm{Sp}^K & \xrightarrow{\varphi^*} & \mathrm{Sp}^G \end{array}$$

Proof sketch. The diagram on the left commutes by construction. To see that the diagram on the right commutes as well, rewrite the colimits defining Sp^G and Sp^K as limits in Pr^{R}. It's then enough to check that $\varphi^*\colon \mathrm{Ani}_*^K \to \mathrm{Ani}_*^G$ intertwines the right adjoints of $S^V \wedge -$ and $S^{\varphi^*(V)} \wedge -$ for any finite-dimensional K-representation V. Since $\varphi^*\colon \mathrm{Ani}_*^K \to \mathrm{Ani}_*^G$ has a left adjoint $\varphi_!$, given by left Kan extension, we may pass to left adjoints and show the equivalent assertion $\varphi_!(S^{\varphi^*(V)} \wedge -) \simeq S^V \wedge \varphi_!(-)$. Now in general, for any functor $\varphi\colon \mathcal{C} \to \mathcal{D}$ of small ∞-categories, the adjunction $\varphi_!\colon \mathrm{PSh}(\mathcal{C})_* \rightleftarrows \mathrm{PSh}(\mathcal{D})_* \colon \varphi^*$ satisfies the "projection formula" $\varphi_!(\varphi^*(Y) \wedge X) \simeq Y \wedge \varphi_!(X)$ by abstract nonsense. $\square$

8.5. Lemma. — *Let $i\colon H \hookrightarrow G$ be the inclusion of a closed subgroup. Then $i^*\colon \mathrm{Sp}^G \to \mathrm{Sp}^H$ preserves all limits and and thus admits a left adjoint $i_!\colon \mathrm{Sp}^H \to \mathrm{Sp}^G$.*(8.1) *If we also let*

(8.1) The functor $i_!$ is usually denoted Ind_H^G and called *induction.*

$i_! \colon \mathrm{Ani}_*^H \to \mathrm{Ani}_*^G$ *denote the left Kan extension functor, then the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{Ani}_*^H & \xrightarrow{i_!} & \mathrm{Ani}_*^G \\ {\scriptstyle \Sigma_H^\infty}\downarrow & & \downarrow{\scriptstyle \Sigma_G^\infty} \\ \mathrm{Sp}^H & \xrightarrow{i_!} & \mathrm{Sp}^G \end{array}$$

In particular, $i_! \mathbb{S}_H \simeq \mathbb{S}_G[G/H]$.

Proof sketch. To form Sp^H, it's enough to invert all representation spheres of the form $S^{i^*(V)}$ in Ani_*^H, where V is a finite-dimensional G-representation. Thus, we can obtain Sp^G and Sp^H by colimit diagrams of the same shape in Pr^{L}. Treating them as limit diagrams in Pr^{R} and noting that the transition maps still commute with $i^* \colon \mathrm{Ani}_*^G \to \mathrm{Ani}_*^H$ (see the argument in the proof of Lemma 8.4) shows that i^* indeed preserves limits. Commutativity of the diagram follows from the right diagram in Lemma 8.4 by passing to left adjoints. □

8.6. Borel-complete spectra. — The full sub-∞-category spanned by $G/\{1\} \in \mathrm{Orb}_G^{\mathrm{op}}$ defines a functor $\mathrm{B}G \to \mathrm{Orb}_G^{\mathrm{op}}$. Via precomposition we get a symmetric monoidal functor $\mathrm{Ani}_*^G \to \mathrm{Ani}_*^{\mathrm{B}G}$. Since all representation spheres $S^V \in \mathrm{Ani}_*^G$ become $\otimes$-invertible under $\Sigma^\infty \colon \mathrm{Ani}_*^{\mathrm{B}G} \to \mathrm{Sp}^{\mathrm{B}G}$, we can use the universal property of Sp^G to obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Ani}_*^G & \longrightarrow & \mathrm{Ani}_*^{\mathrm{B}G} \\ {\scriptstyle \Sigma_G^\infty}\downarrow & & \downarrow{\scriptstyle \Sigma^\infty} \\ \mathrm{Sp}^G & \xrightarrow{U_G} & \mathrm{Sp}^{\mathrm{B}G} \end{array}$$

of symmetric monoidal functors in Pr^{L}. For a genuine G-equivariant spectrum X, we think of $U_G(X)$ as the underlying spectrum with its non-genuine G-action, and we'll often suppress U_G in the notation. Genuine G-equivariant spectra in the image of the right adjoint

$$B_G \colon \mathrm{Sp}^{\mathrm{B}G} \longrightarrow \mathrm{Sp}^G$$

will be called *Borel-complete* and we call the functor $B_G \circ U_G$ *Borel completion.*

8.7. Lemma. — *The functor* $B_G \colon \mathrm{Sp}^{\mathrm{B}G} \to \mathrm{Sp}^G$ *is fully faithful.*

Proof. As in Lemma 8.5, one shows that U_G also preserves limits and hence admits a left adjoint L. It will be enough to show that the unit $u \colon \mathrm{id} \Rightarrow U_G \circ L$ is an equivalence. Since both U_G and L preserve all colimits, we only need to check that u is an equivalence on the generator $\mathbb{S}[G]$ of $\mathrm{Sp}^{\mathrm{B}G}$.

To see this, note that the forgetful functor $\mathrm{Sp}^{\mathrm{B}G} \to \mathrm{Sp}$ is conservative. Moreover, it's clear from the construction that $\mathrm{Sp}^G \to \mathrm{Sp}^{\mathrm{B}G} \to \mathrm{Sp}$ equals $e^* \colon \mathrm{Sp}^G \to \mathrm{Sp}$, where $e \colon \{1\} \hookrightarrow G$ is the inclusion of the identity element. Since $\mathbb{S}[G]$ is the image of $\mathbb{S}$ under the left adjoint of $\mathrm{Sp}^{\mathrm{B}G} \to \mathrm{Sp}$, it will thus be enough to check that $\mathbb{S} \to e^* e_! \mathbb{S}$ is an equivalence. Using the commutative diagram of Lemma 8.5, this reduces to checking that $S^0 \to e_! e^* S^0$ is an equivalence in Ani_*, which is clear since Kan extension along a fully faithful functor is fully faithful. □

8.8. Genuine fixed points. — For every morphism $\varphi \colon G \to K$ of compact Lie groups, the right adjoint $\varphi_* \colon \mathrm{Sp}^G \to \mathrm{Sp}^K$ of φ^* is lax symmetric monoidal and still preserves colimits.

Indeed, since φ_* is an exact functor between compactly generated stable ∞-categories, it will be enough to check that φ^* preserves compact objects, which is clear from the description of compact generators in 8.2. In the case where φ is the projection $\pi_G\colon G \to \{1\}$ to the trivial group, we also denote $\pi_{G,*}$ by

$$(-)^G\colon \mathrm{Sp}^G \longrightarrow \mathrm{Sp}$$

and call this the *genuine G-fixed points*. We have $(-)^G \simeq \mathrm{Hom}_{\mathrm{Sp}}(\mathbb{S}, (-)^G) \simeq \mathrm{Hom}_{\mathrm{Sp}^G}(\mathbb{S}_G, -)$ by adjunction, and so $(-)^G$ is represented by $\mathbb{S}_G$.

If $X \in \mathrm{Sp}^G$ and $i\colon H \hookrightarrow G$ is the inclusion of a closed subgroup, we'll usually write X^H instead of $(i^*X)^H$ for brevity. It follows formally that $(-)^H\colon \mathrm{Sp}^G \to \mathrm{Sp}$ is represented by $i_!\mathbb{S}_H \simeq \mathbb{S}_G[G/H]$.

8.9. Lemma. — *The ∞-category Sp^G is compactly generated, with a set of compact generators given by $\Sigma^{-n}\mathbb{S}_G[G/H]$ for all $n \geqslant 0$ and all closed subgroups $H \subseteq G$.*

Proof. The following argument is taken from [Hau24, Proposition 2.7]. We use induction on $(\dim G, |\pi_0(G)|)$, ordered lexicographically. Suppose a genuine G-equivariant spectrum X satisfies $\mathrm{Hom}_{\mathrm{Sp}^G}(\Sigma^{-n}\mathbb{S}_G[G/H], X) \simeq 0$ for all closed subgroups H and all $n \geqslant 0$. If $i\colon H \hookrightarrow G$ is the inclusion of any such H, then for any closed subgroup $K \subseteq H$ we have

$$0 \simeq \mathrm{Hom}_{\mathrm{Sp}^G}\bigl(\mathbb{S}_G[G/K], X\bigr) \simeq \mathrm{Hom}_{\mathrm{Sp}^G}\bigl(i_!\mathbb{S}_H[H/K], X\bigr) \simeq \mathrm{Hom}_{\mathrm{Sp}^H}\bigl(\mathbb{S}_H[H/K], i^*(X)\bigr)$$

and therefore $i^*(X) \simeq 0$ by the inductive hypothesis. As a consequence, we see that $\mathrm{Hom}_{\mathrm{Sp}^G}(\Sigma^{-V}\mathbb{S}_G[G/H], X) \simeq 0$ for all proper closed subgroups $H \subsetneq G$ and all finite-dimensional G-representations V.

It remains to show $\mathrm{Hom}_{\mathrm{Sp}^G}(\Sigma^{-V}\mathbb{S}_G, X) \simeq 0$ for all V. Let $j\colon \mathrm{Orb}_{<G} \hookrightarrow \mathrm{Orb}_G$ denote the inclusion of the full sub-∞-category spanned by all objects except the terminal object G/G. Let $\mathrm{Ani}_*^{<G} \coloneqq \mathrm{PSh}(\mathrm{Orb}_{<G})_*$. A straightforward application of the Kan extension formula shows that the left Kan extension functor $j_!\colon \mathrm{Ani}_*^{<G} \to \mathrm{Ani}_*^G$ is fully faithful, with essential image given by those pointed genuine G-equivariant anima Y that satisfy $Y^G \simeq *$ (i.e. those presheaves that vanish on $G/G \in \mathrm{Orb}_G$). Since $\mathrm{cofib}(S^{V^G} \to S^V)$ is of this form, it can be written as a colimit of $(G/H)_+$ for proper closed subgroups $H \subsetneq G$. It follows that $\mathrm{cofib}(\Sigma^{V^G}X \to \Sigma^V X) \simeq 0$, since it can be written as a colimit of terms of the form

$$\mathbb{S}_G[G/H] \otimes X \simeq i_!\mathbb{S}_H \otimes X \simeq i_!\bigl(\mathbb{S}_H \otimes i^*(X)\bigr) \simeq 0\,.$$

By our assumption on X, we also have $\mathrm{Hom}_{\mathrm{Sp}^G}(\mathbb{S}_G, \Sigma^{V^G}X) \simeq \mathrm{Hom}_{\mathrm{Sp}^G}(\Sigma^{-n}\mathbb{S}_G, X) \simeq 0$, where $n \coloneqq \dim V^G$. We conclude $0 \simeq \mathrm{Hom}_{\mathrm{Sp}^G}(\mathbb{S}_G, \Sigma^V X) \simeq \mathrm{Hom}_{\mathrm{Sp}^G}(\Sigma^{-V}\mathbb{S}_G, X)$, as desired. □

8.10. Lemma. — *If G is finite, then the compact objects $\mathbb{S}_G[G/H] \in \mathrm{Sp}^G$ are self-dual for all subgroups $H \subseteq G$. In particular, Sp^G is a rigid symmetric monoidal ∞-category.*

Proof sketch. We need to construct a coevaluation $\eta\colon \mathbb{S}_G \to \mathbb{S}_G[G/H] \otimes \mathbb{S}_G[G/H]$ and an evaluation $\varepsilon\colon \mathbb{S}_G[G/H] \otimes \mathbb{S}_G[G/H] \to \mathbb{S}_G$ satisfying the triangle identities. To construct ε, we simply apply Σ_G^∞ to the map $(G/H \times G/H)_+ \to S^0$ that sends the diagonal to the non-basepoint and everything else to the basepoint.

Let us now construct η. Let $V \coloneqq \mathbb{R}[G/H]$. Equip V with an inner product in such a way that $\{\sigma\}_{\sigma \in G/H}$ is an orthonormal basis. Consider the "diagonal map" $V \to V \times (G/H)$, whose σ^{th} component is given by $\mathbb{R}[G/H] \to \mathbb{R}\sigma \to \mathbb{R}\sigma \times \{\sigma\}$, where the first map is the orthogonal projection. This map is G-equivariant and proper, so it induces a G-equivariant map of one-point compactifications, which takes the form $S^V \to S^V \wedge (G/H)_+$. Applying $\Sigma_G^{\infty - V}$, we obtain a map $\mathrm{tr}_H^G \colon \mathbb{S}_G \to \mathbb{S}_G[G/H]$, called the *transfer*. Let also $\Delta\colon G/H \to G/H \times G/H$ denote the diagonal. We finally define η as the composite

$$\eta\colon \mathbb{S}_G \xrightarrow{\mathrm{tr}_H^G} \mathbb{S}_G[G/H] \xrightarrow{\mathbb{S}_G[\Delta]} \mathbb{S}_G[G/H] \otimes \mathbb{S}_G[G/H]\,.$$

The triangle identities can already be verified on the level of genuine G-equivariant anima. □

8.11. Remark. — For arbitrary compact Lie groups G, it is still true that $\mathbb{S}_G[G/H]$ are dualisable, so that Sp^G is still rigid. See e.g. [Hau24, §2.3].

8.12. Genuine vs. homotopy fixed points. — By abuse of notation, let us denote the composition ofthe functor $U_G\colon \mathrm{Sp}^G \to \mathrm{Sp}^{\mathrm{B}G}$ from 8.6 with the homotopy fixed point functor $(-)^{hG}\colon \mathrm{Sp}^{\mathrm{B}G} \to \mathrm{Sp}$ also by $(-)^{hG}$. For any $X \in \mathrm{Sp}^G$ we have

$$\big(B_G U_G(X)\big)^G \simeq \mathrm{Hom}_{\mathrm{Sp}^G}\big(\mathbb{S}_G, B_G U_G(X)\big) \simeq \mathrm{Hom}_{\mathrm{Sp}^{\mathrm{B}G}}\big(\mathbb{S}, U_G(X)\big) \simeq X^{hG}\,.$$

Thus, the natural transformation $\mathrm{id} \Rightarrow U_G \circ B_G$ (Borel-completion) induces a symmetric monoidal transformation of lax symmetric monoidal functors

$$(-)^G \Longrightarrow (-)^{hG}$$

In general, this is far from being an equivalence; in fact, the goal of this whole section is to explain how the Habiro descent of the q-Hodge complex is accounted for by the failure of $\mathrm{THH}(\mathrm{KU}_R/\mathrm{KU}_A)^{C_m} \to \mathrm{THH}(\mathrm{KU}_R/\mathrm{KU}_A)^{hC_m}$ to be an equivalence.

8.13. Geometric fixed points. — The functor $\Sigma^\infty \circ (-)^G\colon \mathrm{Ani}_*^G \to \mathrm{Sp}$ is symmetric monoidal and inverts all representation spheres. Therefore it induces a symmetric monoidal functor

$$(-)^{\Phi G}\colon \mathrm{Sp}^G \longrightarrow \mathrm{Sp}$$

in Pr^{L}, called *geometric fixed points.*[(8.2)] There always exists a natural transformation

$$(-)^G \Longrightarrow (-)^{\Phi G}\,.$$

One way to construct this would be as the following composite (see [Hau24, §2.2]):

$$X^G \simeq \mathrm{Hom}_{\mathrm{Sp}^G}(\mathbb{S}_G, X) \longrightarrow \mathrm{Hom}_{\mathrm{Sp}}\big(\mathbb{S}_G^{\Phi G}, X^{\Phi G}\big) \simeq \mathrm{Hom}_{\mathrm{Sp}}\big(\mathbb{S}, X^{\Phi G}\big) \simeq X^{\Phi G}\,.$$

Just as for genuine fixed points, for every closed subgroup $H \subseteq G$, we also consider the functor $(-)^{\Phi H}\colon \mathrm{Sp}^G \to \mathrm{Sp}$, suppressing the pullback $\mathrm{Sp}^G \to \mathrm{Sp}^H$ in the notation.

8.14. Lemma. — *The family of functors* $\{(-)^H\}_{H \subseteq G}$ *in* $\mathrm{Fun}(\mathrm{Sp}^G, \mathrm{Sp})$ *is jointly conservative. The same is true for* $\{(-)^{\Phi H}\}_{H \subseteq G}$.

Proof. Both assertions are classical; see e.g. [Sch18, Proposition 3.3.10] for the case of geometric fixed points. We'll give a proof by abstract nonsense, following [Hau24, §§2.2–2.3].

For genuine fixed points, joint conservativity follows immediately from Lemma 8.9. For geometric fixed points, assume $X \in \mathrm{Sp}^G$ satisfies $X^{\Phi H} \simeq 0$ for all H. We wish to show

(8.2) Geometric fixed points are usually denoted Φ^G. We chose $(-)^{\Phi G}$ to be in line with $(-)^G$, $(-)^{hG}$, and $(-)^{tG}$.

$X \simeq 0$. Arguing by induction over $(\dim G, |\pi_0(G)|)$, we may assume $i^*(X) \simeq 0$ for all inclusions $i\colon H \hookrightarrow G$ of proper closed subgroups. As in the proof of Lemma 8.9, this implies $\mathbb{S}_G[G/H] \otimes X \simeq 0$ for all such H.

As in the proof of Lemma 8.9, let now $j\colon \mathrm{Orb}_{<G} \hookrightarrow \mathrm{Orb}_G$ denote the inclusion of the full sub-∞-category spanned by all objects except G/G and put $\mathrm{Ani}^{<G} \coloneqq \mathrm{PSh}(\mathrm{Orb}_{<G})$. Let $s\colon \{G/G\} \hookrightarrow \mathrm{Orb}_G$ denote the complementary inclusion. Let $j_!\colon \mathrm{Ani}^{<G} \rightleftarrows \mathrm{Ani}^G : j^*$ be the adjunction given by left Kan extension/restriction along j and let $s^*\colon \mathrm{Ani}_*^G \rightleftarrows \mathrm{Ani}_* : s_*$ be the adjunction given by restriction/right Kan extension along s. We denote $\mathrm{E}\mathcal{P}_G \coloneqq j_!j^*(*)$ and $\widetilde{\mathrm{E}}\mathcal{P}_G \coloneqq s_*s^*S^0$ (in the classical setup this has intrinsic meaning; for us it's just notation). Then the Kan extension formula shows that

$$(\mathrm{E}\mathcal{P}_G)^H \simeq \begin{cases} \emptyset & \text{if } H = G \\ * & \text{if } H \subsetneq G \end{cases} \quad \text{and} \quad (\widetilde{\mathrm{E}}\mathcal{P}_G)^H \simeq \begin{cases} S^0 & \text{if } H = G \\ * & \text{if } H \subsetneq G \end{cases}.$$

Thus the canonical sequence $(\mathrm{E}\mathcal{P}_G)_+ \to S^0 \to \widetilde{\mathrm{E}}\mathcal{P}_G$ induced by the universal property of Kan extension is a cofibre sequence in Ani_*^G. It follows that $\mathbb{S}_G[\mathrm{E}\mathcal{P}_G] \to \mathbb{S}_G \to \Sigma_G^\infty(\widetilde{\mathrm{E}}\mathcal{P}_G)$ is a cofibre sequence in Sp^G, respectively. We have $\mathbb{S}_G[\mathrm{E}\mathcal{P}_G] \otimes X \simeq 0$ as $\mathbb{S}_G[\mathrm{E}\mathcal{P}_G]$ is contained in the full sub-∞-category generated under colimits by $\mathbb{S}_G[G/H]$ for proper closed subgroups $H \subsetneq G$. It will thus be enough to show $\Sigma_G^\infty(\widetilde{\mathrm{E}}\mathcal{P}_G) \otimes X \simeq 0$. Since $(-)^{\Phi H}$ is symmetric monoidal, we still have $(\Sigma_G^\infty(\widetilde{\mathrm{E}}\mathcal{P}_G) \otimes X)^{\Phi H} \simeq 0$ and so the inductive hypothesis shows $(\Sigma_G^\infty(\widetilde{\mathrm{E}}\mathcal{P}_G) \otimes X)^H \simeq 0$ for all proper closed subgroups $H \subsetneq G$. It remains to show $(\Sigma_G^\infty(\widetilde{\mathrm{E}}\mathcal{P}_G) \otimes X)^G \simeq 0$, which follows from the assumption $X^{\Phi G} \simeq 0$ using Lemma 8.15 below. □

8.15. Lemma. — *With notation as above, for any $X \in \mathrm{Sp}^G$ there is a functorial equivalence*

$$\left(\Sigma_G^\infty(\widetilde{\mathrm{E}}\mathcal{P}_G) \otimes X\right)^G \xrightarrow{\simeq} X^{\Phi G}.$$

Proof. Let us first construct the functorial map. With notation as in the proof of Lemma 8.14 above, we have $\mathbb{S}_G[\mathrm{E}\mathcal{P}_G]^{\Phi G} \simeq \mathbb{S}[(\mathrm{E}\mathcal{P}_G)^G] \simeq 0$. Thus, if we apply the natural transformation $(-)^G \Rightarrow (-)^{\Phi G}$ to the cofibre sequence $\mathbb{S}_G[\mathrm{E}\mathcal{P}_G] \otimes X \to \mathbb{S}_G \otimes X \to \Sigma_G^\infty(\widetilde{\mathrm{E}}\mathcal{P}_G) \otimes X$, it will induce the desired map.

Let us now verify that this map is an equivalence. Since $(-)^G$ and $(-)^{\Phi G}$ preserve colimits, it's enough to check the case $X \simeq \mathbb{S}_G[G/H]$. For proper subgroups $H \subsetneq G$ we have $(G/H)^G \simeq *$ and so $\mathbb{S}_G[G/H]^{\Phi G} \simeq 0$ as well as

$$\Sigma_G^\infty(\widetilde{\mathrm{E}}\mathcal{P}_G) \otimes \mathbb{S}_G[G/H] \simeq \Sigma_G^\infty\left(\widetilde{\mathrm{E}}\mathcal{P}_G \wedge (G/H)_+\right) \simeq \Sigma_G^\infty(*) \simeq 0.$$

It remains to show that $\Sigma_G^\infty(\widetilde{\mathrm{E}}\mathcal{P}_G)^G \to \mathbb{S}_G^{\Phi G} \simeq \mathbb{S}$ is an equivalence. This can be checked on underlying anima. Using the definition of Sp^G as a colimit, we see

$$\Omega^\infty\left(\Sigma_G^\infty(\widetilde{\mathrm{E}}\mathcal{P}_G)^G\right) \simeq \Omega^\infty \operatorname{Hom}_{\mathrm{Sp}^G}\left(\mathbb{S}_G, \Sigma_G^\infty(\widetilde{\mathrm{E}}\mathcal{P}_G)\right) \simeq \operatorname*{colim}_{V \subseteq \mathcal{U}} \operatorname{Map}_{\mathrm{Ani}_*^G}\left(S^V, S^V \wedge \widetilde{\mathrm{E}}\mathcal{P}_G\right),$$

where $\mathcal{U}$ is a complete G-universe, that is, a direct sum of countably many copies of each irreducible G-representation, and V ranges through all finite-dimensional subrepresentations of $\mathcal{U}$. Now recall that $\widetilde{\mathrm{E}}\mathcal{P}_G \simeq s_*s^*S^0$. Using the Kan extension formula, it's straightforward to check $S^V \wedge s_*s^*S^0 \simeq s_*S^{V^G}$ and so the colimit above can be rewritten as desired:

$$\operatorname*{colim}_{V \subseteq \mathcal{U}} \operatorname{Map}_{\mathrm{Ani}_*^G}\left(S^V, s_*S^{V^G}\right) \simeq \operatorname*{colim}_{V \subseteq \mathcal{U}} \operatorname{Map}_{\mathrm{Ani}_*}\left(S^{V^G}, S^{V^G}\right) \simeq \Omega^\infty\mathbb{S}.$$ □

Using a similar argument, we can also show the following assertion:

8.16. Lemma. — *Let G be finite. For a genuine G-equivariant spectrum X, the following are equivalent:*

(*a*) *For all subgroups $H \subseteq G$, the genuine fixed points X^H are bounded below.*

(*b*) *For all subgroups $H \subseteq G$, the geometric fixed points $X^{\Phi H}$ are bounded below.*

Proof. Via induction on $|G|$, it will be enough to show under the hypothesis that X^H is bounded below for all proper subgroups $H \subsetneq G$, the genuine fixed points X^G are bounded below if and only if the geometric fixed points $X^{\Phi G}$ are bounded below. Using Lemma 8.15 and the proof of Lemma 8.14, we find a cofibre sequence $(\mathbb{S}_G[\mathrm{E}\mathcal{P}_G] \otimes X)^G \to X^G \to X^{\Phi G}$. Moreover, $\mathbb{S}_G[\mathrm{E}\mathcal{P}_G]$ can be written as a colimit of $\mathbb{S}_G[G/H]$ for proper subgroups $H \subsetneq G$. Thus, it will be enough to show that each $(\mathbb{S}_G[G/H] \otimes X)^G$ is bounded below (here we use finiteness of G to ensure that there are only finitely many H). This follows from

$$\big(\mathbb{S}_G[G/H] \otimes X\big)^G \simeq \mathrm{Hom}_{\mathrm{Sp}^G}\big(\mathbb{S}_G, \mathbb{S}_G[G/H] \otimes X\big) \simeq \mathrm{Hom}_{\mathrm{Sp}^G}\big(\mathbb{S}_G[G/H], X\big) \simeq X^H\,,$$

where we use self-duality of $\mathbb{S}_G[G/H]$ (Lemma 8.10) □

8.17. Inflation maps. — Given any morphism $\varphi\colon G \to K$ of compact Lie groups, one has a symmetric monoidal natural transformation of lax symmetric monoidal functors

$$\inf_\varphi\colon (-)^K \Longrightarrow \big(\varphi^*(-)\big)^G$$

Indeed, from 8.8 we see that $(-)^G \simeq (-)^K \circ \varphi_*$ and then the desired natural transformation arises by postcomposing the unit transformation $\mathrm{id} \Rightarrow \varphi_* \circ \varphi^*$ with $(-)^K$.

If φ is injective, the transformation above is called *restriction* and denoted res^K_G. We're instead interested in the case where φ is surjective, where it is customary to call these maps *inflations*. In the surjective case, there's also a symmetric monoidal inflation

$$\inf_\varphi\colon (-)^{\Phi K} \Longrightarrow \big(\varphi^*(-)\big)^{\Phi G}\,.$$

Indeed, on the level of genuine equivariant pointed anima, the pullback $\varphi^*\colon \mathrm{Ani}^K_* \to \mathrm{Ani}^G_*$ satisfies $(-)^K \simeq (\varphi^*(-))^G$ (this needs surjectivity, so that evaluation at $K/K \in \mathrm{Orb}^{\mathrm{op}}_K$ agrees with evaluation at $K/\varphi(G)$) and then the desired inflation transformation is induced by the universal property of Sp^K as an Ani^K_*-algebra in Pr^{L}. It's straightforward to check that for all $X \in \mathrm{Sp}^K$ the diagram

$$\begin{array}{ccc} X^K & \xrightarrow{\inf_\varphi} & (\varphi^*X)^G \\ \downarrow & & \downarrow \\ X^{\Phi K} & \xrightarrow{\inf_\varphi} & (\varphi^*X)^{\Phi G} \end{array}$$

commutes functorially in X, where the horizontal maps are the inflations and the vertical maps are the ones from 8.13.

8.18. Residual actions. — Let $i\colon N \hookrightarrow G$ be the inclusion of a normal subgroup, let $\pi\colon G \to G/N$ denotes the canonical projection and let $e\colon \{1\} \hookrightarrow G/N$ the inclusion of the identity element. Then the diagram

$$\begin{array}{ccc} \mathrm{Sp}^G & \xrightarrow{\pi_*} & \mathrm{Sp}^{G/N} \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle e^*} \\ \mathrm{Sp}^N & \xrightarrow{(-)^N} & \mathrm{Sp} \end{array}$$

commutes. Indeed, commutativity can be checked after passing to left adjoints, and then it follows from $\pi^* e_! \mathbb{S} \simeq \pi^* \mathbb{S}_{G/N}[G/N] \simeq \mathbb{S}_G[G/N] \simeq i_! \mathbb{S}_N$, using the diagram from 8.8 to compute the values of $i_!$ and $e_!$.

In particular, for any $X \in \mathrm{Sp}^G$, the genuine fixed points X^N can be equipped with a *residual genuine G/N-action*. In a similar way, one can equip $X^{\Phi N}$ with a residual genuine G/N-action, and it can be checked that $X^N \to X^{\Phi N}$ is genuine G/N-equivariant.

8.19. Lemma. — *With the residual actions from 8.18, for all $X \in \mathrm{Sp}^G$ we have canonical equivalences*

$$X^G \simeq (X^N)^{G/N} \quad \textit{and} \quad X^{\Phi G} \simeq (X^{\Phi N})^{\Phi(G/N)}\,.$$

Proof. If $\pi_G \colon G \to \{1\}$ and $\pi_{G/N} \colon G/N \to \{1\}$ denote the canonical projections, then clearly $\pi_G^* \simeq \pi^* \circ \pi_{G/N}^*$. Since adjoints compose, the equivalence for genuine fixed points follows. To see the equivalence for geometric fixed points, it's enough to check the case $X \simeq \mathbb{S}_G[Y]$ for Y a genuine G-equivariant anima; this case follows from $Y^G \simeq (Y^N)^{G/N}$. □

§8.2. The ∞-category of cyclonic spectra

After reviewing the general framework of genuine equivariant homotopy theory, from now on we'll restrict to the following special case:

8.20. Cyclonic spectra. — In the following, we'll consider spectra with an S^1-action that is genuine with respect to all finite cyclic subgroups $C_m \subseteq S^1$. These were introduced under the name *cyclonic spectra* by Barwick and Glasman [BG16].

While the original construction uses spectral Mackey functors, we'll follow [AMR17, Notation 2.3(3)] and construct ∞-category of cyclonic spectra as the full stable sub-∞-category $\mathrm{CycnSp} \subseteq \mathrm{Sp}^{S^1}$ generated under colimits by $\Sigma^{-n}\mathbb{S}_{S^1}[S^1/C_m]$ for all finite cyclic subgroups $C_m \subseteq S^1$ and all $n \geqslant 0$.

8.21. Lemma. — *The family of functors $\{(-)^{C_m}\}_{m\in\mathbb{N}}$ in $\mathrm{Fun}(\mathrm{CycnSp}, \mathrm{Sp})$ is jointly conservative. The same is true for $\{(-)^{\Phi C_m}\}_{m\in\mathbb{N}}$.*

Proof. For genuine fixed points this follows since $\{\Sigma^{-n}\mathbb{S}_{S^1}[S^1/C_m]\}_{m\in\mathbb{N},n\geqslant 0}$ is a system of generators for CycnSp by construction. The assertion about geometric fixed points then follows from Lemma 8.14. □

8.22. Lemma. — *The fully faithful inclusion $j_! \colon \mathrm{CycnSp} \hookrightarrow \mathrm{Sp}^{S^1}$ admits a right adjoint $j^* \colon \mathrm{Sp}^{S^1} \to \mathrm{CycnSp}$ with the following properties:*

(*a*) *j^* still preserves all colimits.*

(*b*) *The counit transformation $c \colon j_! \circ j^* \Rightarrow \mathrm{id}$ is an equivalence after applying $(-)^{C_m}$ or $(-)^{\Phi C_m}$ for any finite cyclic subgroup $C_m \subseteq S^1$.*

(*c*) *For all* $X, Y \in \mathrm{Sp}^{S^1}$ *the canonical map*

$$j^*(X \otimes j_! j^* Y) \xrightarrow{\simeq} j^*(X \otimes Y)$$

is an equivalence. Thus, there's a canonical way to equip CycnSp *and* $j^* \colon \mathrm{Sp}^{S^1} \to \mathrm{CycnSp}$ *with symmetric monoidal structures.*

Proof. The right adjoint j^* exists since $j_!$ preserves all colimits. Since CycnSp is compactly generated and $j_!$ preserves compact objects, j^* preserves filtered colimits and thus all colimits by exactness, proving (*a*). By construction,

$$\mathrm{Hom}_{\mathrm{Sp}^{S^1}}\left(\mathbb{S}_{S^1}[S^1/C_n], j_! j^* X\right) \simeq \mathrm{Hom}_{\mathrm{Sp}^{S^1}}\left(\mathbb{S}_{S^1}[S^1/C_m], X\right)$$

and so $(j_! j^* X)^{C_m} \to X^{C_m}$ is indeed an equivalence. Since this is true for all divisors $d \mid m$, Lemma 8.14 shows that $(j_! j^* X)^{\Phi C_m} \to X^{\Phi C_m}$ is an equivalence as well. This shows (*b*).

Whether $j^*(X \otimes j_! j^* Y) \to j^*(X \otimes Y)$ is an equivalence can be checked on geometric fixed points by Lemma 8.21. But after applying $(j_!(-))^{\Phi C_m}$, both sides become $X^{\Phi C_m} \otimes Y^{\Phi C_m}$ by (*b*) and symmetric monoidality of $(-)^{\Phi C_m}$. This shows the first claim in (*c*); the second claim is general abstract nonsense about localisations of symmetric monoidal ∞-categories (see [L-HA, Proposition 2.2.1.9] for example). □

In the following, we'll usually suppress $j_!$ and j^* in the notation.

8.23. Lemma. — *For* $m, n \in \mathbb{N}$, *let us identify* $C_{mn}/C_m \cong C_n$, $C_{mn}/C_n \cong C_m$. *For all cyclonic spectra* X, *the residual actions from* 8.18 *satisfy the following functorial identites:*

(*a*) $(X^{C_m})^{C_n} \simeq X^{C_{mn}}$, $(X^{\Phi C_m})^{\Phi C_n} \simeq X^{\Phi C_{mn}}$, *and* $(X^{hC_m})^{hC_n} \simeq X^{hC_{mn}}$.

(*b*) *If* m *and* n *are coprime, then* $(X^{C_m})^{\Phi C_n} \simeq (X^{\Phi C_n})^{C_m}$.

Proof. The first two assertions from (*a*) are special cases of Lemma 8.19, the third assertion is classical. For (*b*), let us first note that $\mathrm{Orb}_{C_{mn}} \simeq \mathrm{Orb}_{C_m} \times \mathrm{Orb}_{C_n}$, which easily implies $\mathrm{Sp}_{C_{mn}} \simeq \mathrm{Sp}_{C_m} \otimes \mathrm{Sp}_{C_n}$ for the Lurie tensor product. By construction of geometric fixed points it's clear that $(-)^{\Phi C_n} \colon \mathrm{Sp}_{C_m} \otimes \mathrm{Sp}_{C_n} \to \mathrm{Sp}_{C_m}$ is given by applying $(-)^{\Phi C_n} \colon \mathrm{Sp}_{C_n} \to \mathrm{Sp}$ in the second tensor factor. If we can show a similar assertion for $(-)^{C_m}$, we'll be done.

To this end, let $\pi \colon C_{mn} \to C_n$ and $\pi_{C_m} \colon C_m \to \{1\}$ denote the canonical projections. It is again clear from the construction that $\pi^* \colon \mathrm{Sp}_{C_n} \to \mathrm{Sp}_{C_m} \otimes \mathrm{Sp}_{C_n}$ is given by applying $\pi_{C_m}^* \colon \mathrm{Sp} \to \mathrm{Sp}_{C_m}$ in the first tensor factor. Its right adjoint π_* must then also be given by applying the right adjoint $\pi_{C_m,*}$ (which is also a functor in Pr^{L}) in the first tensor factor, because we can just apply $- \otimes \mathrm{Sp}_{C_n}$ to the unit, the counit, and the triangle identities. □

Nikolaus–Scholze [NS18, Theorem II.6.9] showed that on bounded below objects, the structure of a *cyclotomic spectrum* is equivalent to a "naive" notion, in which one only asks for S^1-equivariant maps $X \to X^{tC_p}$. We'll now show a similar result in the cyclonic case. This is based on the following well-known fact (see e.g. [HM97] or [NS18, Lemma II.4.5]):

8.24. Lemma. — *There's a pullback square of symmetric monoidal transformations between lax symmetric monoidal functors in* $\mathrm{Fun}(\mathrm{Sp}^{C_p}, \mathrm{Sp})$

$$\begin{array}{ccc} (-)^{C_p} & \xRightarrow{(8.13)} & (-)^{\Phi C_p} \\ {\scriptstyle (8.12)}\Big\Downarrow & \lrcorner & \Big\Downarrow \\ (-)^{hC_p} & \Longrightarrow & (-)^{tC_p} \end{array}$$

Proof. If we regard the orbit C_p/C_1 as a genuine C_p-equivariant anima via the Yoneda embedding, we find $(C_p/C_1)^{C_p} \simeq \emptyset$ and $(C_p/C_1)^{C_1} \simeq C_p$. By direct inspection, it follows that $\mathrm{E}\mathcal{P}_{C_p} \simeq (C_p/C_1)_{hC_p}$, where C_p acts on C_p/C_1 in the obvious way.

For every genuine C_p-equivariant spectrum X, we've seen in Lemma 8.15 that the fibre of $X^{C_p} \to X^{\Phi C_p}$ is given by $(\mathbb{S}_{C_p}[\mathrm{E}\mathcal{P}_{C_p}] \otimes X)^{C_p}$. Using that $(-)^{C_p}$ preserves all colimits and $\mathbb{S}_{C_p}[C_p]$ is self-dual by Lemma 8.10, we find

$$\left(\mathbb{S}_{C_p}[\mathrm{E}\mathcal{P}_{C_p}] \otimes X\right)^{C_p} \simeq \left(\mathbb{S}_{C_p}[C_p]_{hC_p} \otimes X\right)^{C_p} \simeq \left((\mathbb{S}_{C_p}[C_p] \otimes X)^{C_p}\right)_{hC_p} \simeq X_{hC_p}\,.$$

In the case where X is Borel-complete, it's straightforward to check that the induced map $X_{hC_p} \to X^{C_p} \simeq X^{hC_p}$ is the norm map and so $X^{\Phi C_p} \simeq X^{tC_p}$ for Borel-complete X. In general, composing the Borel completion transformation $\mathrm{id} \Rightarrow B_{C_p} U_{C_p}$ with the natural trasformation $(-)^{C_p} \Rightarrow (-)^{\Phi C_p}$, we obtain the desired commutative square. It is a pullback square since the row-wise fibres are given by $(-)_{hC_p}$, as we've just verified. Symmetric monoidality is also clear from the construction. □

8.25. Naive cyclonic spectra — Informally, a *naive cyclonic spectrum* should consist of a collection of spectra $(Y_m)_{m\in\mathbb{N}}$, each Y_m equipped with an (S^1/C_m)-action, together with (S^1/C_{pm})-equivariant maps $\phi_{p,m}\colon Y_{pm} \to Y_m^{tC_p}$ for all m and all primes p. The intuition is that $Y_m \simeq X^{\Phi C_m}$ records the geometric fixed points of some cyclonic spectrum X. To see obtain the maps $\phi_{p,m}\colon X^{\Phi C_{pm}} \to (X^{\Phi C_m})^{tC_p}$ in this case, we plug $X^{\Phi C_m}$ into the natural transformation $(-)^{\Phi C_p} \Rightarrow (-)^{tC_p}$; by naturality, the map $\phi_{p,m}$ that we obtain is (non-genuinely) (S^1/C_{pm})-equivariant.

Formally, we define the ∞-category of *naive cyclonic spectra* to be the lax equaliser (in the sense of [NS18, Definition II.1.4])

$$\mathrm{CycnSp}^{\mathrm{naiv}} \coloneqq \mathrm{LEq}\left(\prod_{m\in\mathbb{N}} \mathrm{Sp}^{\mathrm{B}(S^1/C_m)} \underset{((-)^{tC_p})_{p,m}}{\overset{\mathrm{can}}{\rightrightarrows}} \prod_{p}\prod_{m\in\mathbb{N}} \mathrm{Sp}^{\mathrm{B}(S^1/C_{pm})}\right),$$

where p runs through all primes, the top functor is given by $(Y_m)_m \mapsto (Y_{pm})_{p,m}$, and the bottom functor is given by $(Y_m)_m \mapsto (Y_m^{tC_p})_{p,m}$. By the universal property of lax equalisers there is a functor

$$(-)^{\Phi C}\colon \mathrm{CycnSp} \longrightarrow \mathrm{CycnSp}^{\mathrm{naiv}}$$

which sends $X \mapsto (X^{\Phi C_m})_{m\in\mathbb{N}}$, equipped with the canonical maps $\phi_{p,m}\colon X^{\Phi C_{pm}} \to (X^{\Phi C_m})^{tC_p}$ described above. Using Lemma 8.27 below, we can also equip $\mathrm{CycnSp}^{\mathrm{naiv}}$ with a symmetric monoidal structure in such a way that $(-)^{\Phi C}$ is symmetric monoidal.

Let us also call a cyclonic spectrum X *bounded below* if each X^{C_m} is bounded below (not necessarily with a uniform bound for all m); equivalently by Lemma 8.16, all $X^{\Phi C_m}$ are bounded below. Similarly, a naive cyclonic spectrum $Y = ((Y_m)_m, (\phi_{p,m})_{p,m})$ will be called *bounded below* if each Y_m is bounded below (not necessarily with a uniform bound). We denote by CycnSp_+ and $\mathrm{CycnSp}_+^{\mathrm{naiv}}$ the respective full sub-∞-categories of bounded below objects.

8.26. Proposition. — *When restricted to the respective full sub-∞-categories of bounded below objects, the functor $(-)^{\Phi C}$ becomes a symmetric monoidal equivalence*

$$(-)^{\Phi C}\colon \mathrm{CycnSp}_+ \xrightarrow{\simeq} \mathrm{CycnSp}_+^{\mathrm{naiv}}\,.$$

To prove Proposition 8.26, let us first construct the desired symmetric monoidal structure.

8.27. Lemma. — *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor and let $G\colon \mathcal{C} \to \mathcal{D}$ be a lax symmetric monoidal functor of symmetric monoidal ∞-categories. Let $F^{\otimes}$ and $G^{\otimes}$ denote the corresponding functors between the ∞-operads $\mathcal{C}^{\otimes} \to \mathrm{Fin}_*$ and $\mathcal{D}^{\otimes} \to \mathrm{Fin}_*$ and define*

$$\mathrm{LEq}(F,G)^{\otimes} \coloneqq \mathrm{LEq}(F^{\otimes}, G^{\otimes}) \times_{\mathrm{LEq}(\mathrm{id}_{\mathrm{Fin}_*}, \mathrm{id}_{\mathrm{Fin}_*})} \mathrm{Fin}_* \,.$$

(*a*) $\mathrm{LEq}(F,G)^{\otimes} \to \mathrm{Fin}_*$ *is an ∞-operad associated to a symmetric monoidal structure on the ∞-category* $\mathrm{LEq}(F,G)$ *and* $\mathrm{LEq}(F,G) \to \mathcal{C}$ *is symmetric monoidal.*

(*b*) *If $\mathcal{C}$ and $\mathcal{D}$ are presentably symmetric monoidal, F preserves colimits, and G is accessible, then* $\mathrm{LEq}(F,G)$ *is again presentably monoidal.*

Proof sketch. Let $\langle i\rangle \in \mathrm{Fin}_*$. Using $\mathrm{LEq}(\mathrm{id}_{\{\langle i\rangle\}}, \mathrm{id}_{\{\langle i\rangle\}}) \simeq *$, the fact that lax equalisers commute with pullbacks, and the fact that the fibres over $F^{\otimes}$ and $G^{\otimes}$ over $\langle i\rangle$ are $F^i\colon \mathcal{C}^i \to \mathcal{D}^i$ and $G^i\colon \mathcal{C}^i \to \mathcal{D}^i$ respectively, we find that the fibre of $\mathrm{LEq}(F,G)^{\otimes} \to \mathrm{Fin}_*$ over $\langle i\rangle \in \mathrm{Fin}_*$ is of the desired form:

$$\mathrm{LEq}(F^{\otimes}, G^{\otimes}) \times_{\mathrm{LEq}(\mathrm{id}_{\mathrm{Fin}_*}, \mathrm{id}_{\mathrm{Fin}_*})} \mathrm{LEq}\big(\mathrm{id}_{\{\langle i\rangle\}}, \mathrm{id}_{\{\langle i\rangle\}}\big) \simeq \mathrm{LEq}(F^i, G^i) \simeq \mathrm{LEq}(F,G)^i \,.$$

Let us next check that $\mathrm{LEq}(F,G)^{\otimes} \to \mathrm{Fin}_*$ is a cocartesian fibration. For simplicity, we'll only describe locally cocartesian lifts of the unique active morphism $f_2\colon \langle 2\rangle \to \langle 1\rangle$; it will be obvious how to perform the construction in general, as will be the fact that the locally cocartesian lifts compose, so that we obtain a cocartesian fibration by the dual of [L-HTT, Proposition 2.4.2.8]. So suppose we're given $((x_1,\varphi_1),(x_2,\varphi_2)) \in \mathrm{LEq}(F,G)^2$, where $\varphi_1\colon F(x_1) \to G(x_1)$ and $\varphi_2\colon F(x_2) \to G(x_2)$. Let φ denote the composite

$$\varphi\colon F(x_1 \otimes_{\mathcal{C}} x_2) \simeq F(x_1) \otimes_{\mathcal{D}} F(x_2) \xrightarrow{\varphi_1 \otimes \varphi_2} G(x_1) \otimes_{\mathcal{D}} G(x_2) \longrightarrow G(x_1 \otimes_{\mathcal{C}} x_2)\,,$$

where we use strict and lax symmetric monoidality of F and G, respectively. Now let $\mu\colon (x_1,x_2) \to x_1 \otimes_{\mathcal{C}} x_2$ be a locally cocartesian lift of f_2 along $\mathcal{C}^{\otimes} \to \mathrm{Fin}_*$. Moreover, let $\mu_F \simeq F^{\otimes}(\mu)\colon (F(x_1),F(x_2)) \to F(x_1) \otimes_{\mathcal{D}} F(x_2)$ and $\mu_G\colon (G(x_1),G(x_2)) \to G(x_1) \otimes_{\mathcal{D}} G(x_2)$ be locally cocartesian lifts of f_2 along $\mathcal{D}^{\otimes} \to \mathrm{Fin}_*$. We have $\varphi \circ \mu_F \simeq \mu_G \circ (\varphi_1,\varphi_2)$ by construction of φ, and so we obtain a morphism $((x_1,\varphi_1),(x_2,\varphi_2)) \to (x_1 \otimes_{\mathcal{C}} x_2, \varphi)$ in $\mathrm{LEq}(F,G)^{\otimes}$. Using the formula for mapping anima in lax equalisers from [NS18, Proposition II.1.5(ii)] and the general criterion from the dual of [L-HTT, Proposition 2.4.4.3], it's straightforward to verify that this morphism is indeed a locally cocartesian lift of f_2, as desired.

Therefore, $\mathrm{LEq}(F,G)^{\otimes} \to \mathrm{Fin}_*$ is indeed a cocartesian fibration. From the description of cocartesian lifts above, it's clear that $\mathrm{LEq}(F,G)^{\otimes} \to \mathcal{C}^{\otimes}$ preserves cocartesian lifts, hence $\mathrm{LEq}(F,G) \to \mathcal{C}$ is indeed symmetric monoidal. This finishes the proof sketch of (*a*).

For (*b*), we must check that $\mathrm{LEq}(F,G)$ is presentable and that the tensor product preserves colimits in either variable. Both assertions follow from [NS18, Proposition II.1.5(iv)–(v)]. □

Let us now commence with the proof of Proposition 8.26. The main ingredient is a formula that allows to compute genuine fixed points for finite cyclic groups in terms of homotopy fixed points, geometric fixed points, and the Tate construction.

8.28. Lemma. — *Let X be a cyclonic spectrum and let $m \in \mathbb{N}$. If the geometric fixed points $X^{\Phi C_d}$ are bounded below for all divisors $d \mid m$, then the following canonical (S^1/C_m)-equivariant map is an equivalence:*

$$X^{C_m} \xrightarrow{\simeq} \mathrm{eq}\left(\prod_{d \mid m} (X^{\Phi C_d})^{hC_{m/d}} \underset{\phi}{\overset{\mathrm{can}}{\rightrightarrows}} \prod_{p} \prod_{pd \mid m} \big((X^{\Phi C_d})^{tC_p}\big)^{hC_{m/pd}}\right).$$

Here the second product is taken over all primes p. The two maps can *and ϕ in the equaliser are given as follows:*

$$\begin{aligned}\left(X^{\Phi C_d}\right)^{hC_{m/d}} &\simeq \left(\left(X^{\Phi C_d}\right)^{hC_p}\right)^{hC_{m/pd}} \longrightarrow \left(\left(X^{\Phi C_d}\right)^{tC_p}\right)^{hC_{m/pd}}\,,\\ \left(X^{\Phi C_{pd}}\right)^{hC_{m/pd}} &\simeq \left(\left(X^{\Phi C_d}\right)^{\Phi C_p}\right)^{hC_{m/pd}} \longrightarrow \left(\left(X^{\Phi C_d}\right)^{tC_p}\right)^{hC_{m/pd}}\,,\end{aligned}$$

using the natural transformations $(-)^{hC_p} \Rightarrow (-)^{tC_p}$ and $(-)^{\Phi C_p} \Rightarrow (-)^{tC_p}$, respectively.

Proof. We use induction on m. If $m = p^\alpha$ is a prime power, the assertion is [NS18, Corollary II.4.7]. Now let m be arbitrary. We may assume that all but one prime factors of m act invertibly on X, because an arbitrary X can be written as a finite Čech limit of such objects (also the assumption that all $X^{\Phi C_d}$ are bounded below is preserved under any localisation). Write $m = p^\alpha m_p$, where p is the not necessarily invertible prime and m_p is coprime to p. Using the inductive hypothesis and the fact that the Tate construction $(-)^{tC_\ell}$ vanishes on $\mathbb{S}[1/\ell]$-modules, we find

$$X^{C_{m_p}} \simeq \prod_{d_p \mid m_p} \left(X^{\Phi C_{d_p}}\right)^{hC_{m_p/d_p}}\,.$$

Also observe that all homotopy fixed points $(-)^{hC_{m_p/d_p}}$ in this formula can be computed as finite limits, as $\mathrm{B}C_{m_p/d_p}$ has a finite cell structure once m_p is invertible. An argument as in Lemma 8.23(*b*) then allows us to deduce that the formula above is also true as genuine C_{p^α}-equivariant spectra and that the homotopy fixed points $(-)^{hC_{m_p/d_p}}$ commute with the geometric fixed points $(-)^{\Phi C_{p^i}}$. With these observations, the formula for $X^{C_m} \simeq (X^{C_{m_p}})^{C_{p^\alpha}}$ becomes precisely the desired equaliser. □

With a similar argument, one can show the following technical lemma.

8.29. Lemma. — *Let $Y = ((Y_m)_m, (\phi_{p,m})_{p,m})$ be a naive cyclotomic spectrum. Then Y is bounded below if and only if for all $m \in \mathbb{N}$ the following equaliser is bounded below:*

$$\mathrm{eq}\left(\prod_{d \mid m} Y_d^{hC_{m/d}} \overset{\mathrm{can}}{\underset{\phi}{\rightrightarrows}} \prod_p \prod_{pd \mid m} \left(Y_d^{tC_p}\right)^{hC_{m/pd}}\right).$$

Proof. We only prove the "only if" part, the "if" will follow from Proposition 8.26 (and won't be used in the proof). So let Y be bounded below. We may once again assume that all but one prime factors of m act invertibly on Y, since the property of being bounded below is preserved under finite Čech limits. So write $m = p^\alpha m_p$, where p is the not necessarily invertible prime and m_p is coprime to p. Since the Tate constructions $(-)^{tC_\ell}$ vanish for all primes $p \neq \ell$, the equaliser simplifies to

$$\mathrm{eq}\left(\prod_{d \mid m} Y_d^{hC_{m/d}} \overset{\mathrm{can}}{\underset{\phi_p}{\rightrightarrows}} \prod_{pd \mid m} \left(Y_d^{tC_p}\right)^{hC_{m/pd}}\right)$$

Let $pd \mid m$ and write $d = p^i d_p$, where $i \leqslant \alpha - 1$ and d_p is coprime to p. Using the Tate fixed point lemma [NS18, Lemma II.4.1], we find

$$\mathrm{fib}\left(Y_d^{hC_{m/d}} \to \left(Y_d^{tC_p}\right)^{hC_{m/pd}}\right) \simeq \left((Y_d)_{hC_{p^{\alpha-i}}}\right)^{hC_{m/p^\alpha d_p}}\,.$$

Since $(-)_{hC_{p^{\alpha-i}}}$ preserves bounded below objects and $(-)^{hC_{m/p^\alpha d_p}}$ can be written as a finite limit in our situation, we deduce that the fibre is bounded below. An easy induction shows that the equaliser in question must be bounded below as well. □

Proof of Proposition 8.26. Let us first show that $(-)^{\Phi C}\colon \mathrm{CycnSp}_+ \to \mathrm{CycnSp}_+^{\mathrm{naiv}}$ is fully faithful. For any $m \in \mathbb{N}$, we have

$$\mathbb{S}_{S^1}[S^1/C_m]^{\Phi C_d} \simeq \begin{cases} \mathbb{S}[S^1/C_{m/d}] & \text{if } d \mid m \\ 0 & \text{else} \end{cases}.$$

By unravelling the general formula for mapping anima/spectra in lax equalisers [NS18, Proposition II.1.5(ii)], we find that $\mathrm{Hom}_{\mathrm{CycnSp}^{\mathrm{naiv}}}(\mathbb{S}_{S^1}[S^1/C_m]^{\Phi C}, X^{\Phi C})$ is given by the equaliser from Lemma 8.28 for all cyclonic spectra X. If X is bounded below, it follows that

$$\mathrm{Hom}_{\mathrm{CycnSp}^{\mathrm{naiv}}}\left(\mathbb{S}_{S^1}[S^1/C_m]^{\Phi C}, X^{\Phi C}\right) \simeq X^{C_m} \simeq \mathrm{Hom}_{\mathrm{CycnSp}}\left(\mathbb{S}_{S^1}[S^1/C_m], X\right),$$

as desired. Since CycnSp is generated under colimits by shifts of $\mathbb{S}_{S^1}[S^1/C_m]$ for all $m \in \mathbb{N}$, we deduce that $(-)^{\Phi C}\colon \mathrm{CycnSp}_+ \to \mathrm{CycnSp}_+^{\mathrm{naiv}}$ is indeed fully faithful.

Using [NS18, Proposition II.1.5(iv)–(v)], we see that $(-)^{\Phi C}\colon \mathrm{CycnSp} \to \mathrm{CycnSp}^{\mathrm{naiv}}$ is a colimit-preserving functor between presentable ∞-categories and so it admits a right adjoint $R\colon \mathrm{CycnSp}^{\mathrm{naiv}} \to \mathrm{CycnSp}$. We note that R restricts to a functor $R\colon \mathrm{CycnSp}_+^{\mathrm{naiv}} \to \mathrm{CycnSp}_+$. Indeed, an analogous computation as above shows that

$$R(Y)^{C_m} \simeq \mathrm{Hom}_{\mathrm{CycnSp}^{\mathrm{naiv}}}\left(\mathbb{S}_{S^1}[S^1/C_m]^{\Phi C}, Y\right) \simeq \mathrm{eq}\left(\prod_{d \mid m} Y_d^{hC_{m/d}} \underset{\phi}{\overset{\mathrm{can}}{\rightrightarrows}} \prod_p \prod_{pd \mid m} (Y_d^{tC_p})^{hC_{m/pd}}\right)$$

for all $Y \in \mathrm{CycnSp}^{\mathrm{naiv}}$. Thus, if Y is bounded below, Lemma 8.29 shows that $R(Y)$ will be bounded below as well.

The same calculation shows that R is conservative. Indeed, if $Y \to Y'$ is a morphism of naive cyclonic spectra such that $R(Y) \to R(Y')$ is an equivalence, then the induced morphisms on the equalisers from Lemma 8.29 are equivalences for all $m \in \mathbb{N}$. Arguing inductively, this implies that $Y_m \to Y'_m$ must be an equivalence for all $m \in \mathbb{N}$ and so $Y \to Y'$ is indeed an equivalence as well.

In general, if the left adjoint in any adjunction is fully faithful and the right adjoint is conservative, the adjunction is a pair of inverse equivalences. This finishes the proof. □

8.30. Remark. — Ayala–Mazel-Gee–Rozenblyum derive another "naive" description of cyclonic spectra in [AMR17, Corollary 0.4]. In contrast to Proposition 8.26, which is only valid in the bounded below case, their result covers all cyclonic spectra. This comes at a cost of additional coherence data. The moral reason why, in the bounded below case, we can get away with only the maps $X^{\Phi C_{pm}} \to (X^{\Phi C_m})^{tC_p}$, with no coherence data to be specified, is the following: For X bounded below, the composition maps for the *proper* Tate construction are equivalences

$$X^{\tau C_{mn}} \xrightarrow{\simeq} (X^{\tau C_m})^{\tau C_n},$$

and unless m and n are powers of the same prime, both sides vanish. This determines all coherence data uniquely. We expect that by formalising this observation, one can deduce Proposition 8.26 from [AMR17, Corollary 0.4], but we have not attempted to do so.

8.31. Cyclonic vs. cyclotomic spectra. — Let CyctSp denote the ∞-category of cyclotomic spectra and let $\mathrm{CyctSp}^{\mathrm{naiv}}$ denote its naive variant introduced by Nikolaus–Scholze [NS18, Definition II.1.6(i)]. We have a symmetric monoidal functor

$$\mathrm{CyctSp}^{\mathrm{naiv}} \longrightarrow \mathrm{CycnSp}^{\mathrm{naiv}}$$

sending a cyclotomic spectrum X to the constant family $((X)_m, (\phi_{p,m})_{p,m})$ in which each $\phi_{p,m}$ is given by the cyclotomic Frobenius $X \to X^{tC_p}$. This functor is not fully faithful (this will become useful in 8.43 below).

One can also construct a functor CyctSp $\to$ CycnSp on the non-naive ∞-categories (see [AMR17, §2.5] for example) which agrees with the functor above on bounded below objects.

§8.3. Genuine equivariant ku

In this subsection we'll equip ku with the structure of a cyclonic spectrum and compute its genuine and geometric fixed points ku^{C_m} and $\mathrm{ku}^{\Phi C_m}$ for all m.

8.32. Cyclonic ku. — Recall that Schwede [Sch18, Construction 6.3.9] constructs a model $\mathrm{ku}_{\mathrm{gl}}$ of ku as an *ultracommutative global*[(8.3)] *ring spectrum*. Throwing away most of the structure, this yields an $\mathbb{E}_\infty$-algebra $\mathrm{ku}_{S^1} \in \mathrm{CAlg}(\mathrm{Sp}_{S^1})$ with underlying non-equivariant $\mathbb{E}_\infty$-algebra ku. We still have a Bott map $\beta\colon \Sigma^2\mathbb{S}_{S^1} \to \mathrm{ku}_{S^1}$ (in fact, β already exists for $\mathrm{ku}_{\mathrm{gl}}$) and we define $\mathrm{KU}_{S^1} \coloneqq \mathrm{ku}_{S^1}[\beta^{-1}]$. In the following we'll often abusingly drop the index and just write ku or KU for the genuine S^1-equivariant versions. We also note that by restriction, ku and KU define $\mathbb{E}_\infty$-algebras in cyclonic spectra.

8.33. Genuine fixed points of ku. — Let q denote the standard representation of S^1 on $\mathbb{C}$ via rotations, so that the complex representation rings of S^1 and C_m are given by $\mathrm{RU}(S^1) \cong \mathbb{Z}[q^{\pm 1}]$ and $\mathrm{RU}(C_m) \cong \mathbb{Z}[q]/(q^m-1)$. Via the canonical map $\mathrm{RU}(S^1) \to \pi_0(\mathrm{ku}^{S^1})$, we can regard q as a class in $\pi_0(\mathrm{ku}^{S^1})$, compatible with Remark 7.4. It's a well-known fact that q is a *strict* element, that is, it is detected by an $\mathbb{E}_\infty$-algebra map $\mathbb{S}[q] \to \mathrm{ku}^{S^1}$. See Corollary D.2 for a proof.

For the finite groups C_m the analogous maps $\mathrm{RU}(C_m) \to \pi_0(\mathrm{ku}^{C_m})$ are isomorphisms [Sch18, Theorem 6.3.33] and so, by equivariant Bott periodicity,

$$\pi_*(\mathrm{ku}^{C_m}) \cong \mathbb{Z}[\beta, q]/(q^m-1) \quad \text{and} \quad \pi_*(\mathrm{KU}^{C_m}) \cong \mathbb{Z}[\beta^{\pm 1}, q]/(q^m-1)\,.$$

In particular, $\mathrm{ku}^{C_m} \simeq \tau_{\geqslant 0}(\mathrm{KU}^{C_m})$. Using the homotopy fixed point spectral sequence, we can also compute the homotopy fixed points of the residual (S^1/C_m)-action:

$$\pi_*\big((\mathrm{ku}^{C_m})^{h(S^1/C_m)}\big) \cong \mathbb{Z}[\beta, q][\![t_m]\!]/\big(\beta t_m - (q^m-1)\big)\,,$$

where $|t_m| = -2$. The canonical map $(\mathrm{ku}^{C_m})^{h(S^1/C_m)} \to \mathrm{ku}^{hS^1}$ sends $t_m \mapsto [m]_q t$. In particular, on π_0 this map recovers the $(q-1)$-completion $\mathbb{Z}[q]^\wedge_{(q^m-1)} \to \mathbb{Z}[\![q-1]\!]$, and $t_m = [m]_{\mathrm{ku}}(t)$ agrees with the m-series of the formal group law of ku.

8.34. Inflation maps for ku. — Consider the inflation maps from 8.17 in the special case where φ is the n^{th} power map $(-)^n\colon S^1 \to S^1$ for some $n \geqslant 1$. We have $\varphi^*\mathrm{ku}_{S^1} \simeq \mathrm{ku}_{S^1}$, since the genuine S^1-equivariant structure comes from a global spectrum $\mathrm{ku}_{\mathrm{gl}}$, where all actions are trivial (compare [Sch18, §4.1]). Since $(-)^n$ maps the subgroups C_{mn} to C_m, we get inflations

$$\mathrm{inf}_n\colon \mathrm{ku}^{C_m} \longrightarrow \mathrm{ku}^{C_{mn}} \quad \text{and} \quad \mathrm{inf}_n\colon \mathrm{ku}^{\Phi C_m} \longrightarrow \mathrm{ku}^{\Phi C_{mn}}\,.$$

These are maps of $\mathbb{E}_\infty$-algebras in Sp_{S^1} for the residual genuine $S^1 \simeq S^1/C_m$-equivariant structure on the left-hand sides and the residual $S^1 \simeq S^1/C_{mn}$-equivariant structure on the right-hand sides. A straightforward check shows $\mathrm{inf}_n(q) = q^n$ and $\mathrm{inf}_n(\beta) = \beta$ (compare D.3).

(8.3) "Global" in the sense of global homotopy theory, not in the sense of §7.4. Very roughly, it means to have compatible trivial actions by all compact Lie groups. "Ultracommutative" refers to the fact that Schwede's model admits a strictly commutative multiplication on the point-set level.

8.35. Corollary. — *For all m and n, the inflation map induces an S^1-equivariant equivalence of $\mathbb{E}_\infty$-algebras*

$$\inf_n\colon \mathrm{ku}^{C_m}\otimes_{\mathbb{S}[q],\psi^n}\mathbb{S}[q]\longrightarrow \mathrm{ku}^{C_{mn}}\,,$$

where $\psi^n\colon \mathbb{S}[q]\rightarrow\mathbb{S}[q]$ is given by $\psi^n(q)\coloneqq q^n$.

Proof. This can be checked on homotopy groups, where it follows from 8.33 and 8.34. ☐

8.36. Remark. — The notation $\psi^n\colon \mathbb{S}[q]\rightarrow\mathbb{S}[q]$ is chosen to be compatible with the Adams operations on the Λ-ring $\mathbb{Z}[q]$. One can also construct equivariant Adams operations on ku (see D.5), but these *do not* coincide with $\inf_n$.

8.37. Geometric fixed points of ku. — To prove our Habiro descent result, it will be crucial to know the geometric fixed points $\mathrm{ku}^{\Phi C_m}$ as well, at least after inverting m and after p-completion for any prime $p\mid m$. This will be our goal for the rest of this subsection. Our strategy will be to compute the geometric fixed points inductively using Lemma 8.28. To apply said lemma, observe that we already know that each $\mathrm{ku}^{\Phi C_m}$ is bounded below thanks to Lemma 8.16.

For KU, the geometric fixed points can essentially already be found in the literature (even though the author could only find the precise result in the case where m is a prime power): We have an equivalence of S^1-equivariant $\mathbb{E}_\infty$-ring spectra

$$\mathrm{KU}^{C_m}\Big[\big\{(q^d-1)^{-1}\big\}_{d\mid m,\, d\neq m}\Big]\xrightarrow{\simeq}\mathrm{KU}^{\Phi C_m}\,.$$

One way to prove this is via the corresponding statement for equivariant MU [Sin01, Proposition 4.6] and the equivariant Conner–Floyd theorem [Cos87]. The result can also be deduced from Proposition 8.42 below.

8.38. Lemma. — *The canonical map $\mathrm{ku}[1/m]^{C_m}\rightarrow\mathrm{ku}[1/m]^{\Phi C_m}$ induces an equivalence of S^1-equivariant $\mathbb{E}_\infty$-ring spectra*

$$\big(\mathrm{ku}\big[\tfrac{1}{m}\big]^{C_m}\big)^\wedge_{\Phi_m(q)}\xrightarrow{\simeq}\mathrm{ku}\big[\tfrac{1}{m}\big]^{\Phi C_m}\,.$$

In particular, $\pi_*(\mathrm{ku}[1/m]^{\Phi C_m})\cong\mathbb{Z}[1/m,\beta,q]/\Phi_m(q)$.

Proof. Since we already know that $\mathrm{ku}^{\Phi C_d}$ is bounded below for all $d\mid m$, we can apply the formula from Lemma 8.28 to $\mathrm{ku}[1/m]$. Because we've inverted m, all Tate constructions will vanish, and the formula becomes an equivalence

$$\mathrm{ku}\big[\tfrac{1}{m}\big]^{C_m}\simeq\prod_{d\mid m}\mathrm{ku}\big[\tfrac{1}{m}\big]^{\Phi C_d}\,.$$

The claim then follows via induction on m and Corollary 8.35. ☐

8.39. Lemma. — *Let $m=p^\alpha m_p$, where p is a prime and m_p is coprime to p. The inflation map induces an S^1-equivariant equivalence of $\mathbb{E}_\infty$-ring spectra*

$$\inf_{m_p}\colon\Big(\big(\mathrm{ku}^{\Phi C_{p^\alpha}}\big)^\wedge_p\otimes_{\mathbb{S}[q],\psi^{m_p}}\mathbb{S}[q]\Big)^\wedge_{\Phi_m(q)}\xrightarrow{\simeq}\big(\mathrm{ku}^{\Phi C_m}\big)^\wedge_p\,.$$

Proof. Note that m_p is invertible on $(\mathrm{ku}^{\Phi C_{p^\alpha}})_p^\wedge$. The same argument as in the proof of Lemma 8.38 shows that the canonical map

$$\big((\mathrm{ku}^{\Phi C_{p^\alpha}})_p^\wedge\big)^{C_{m_p}} \simeq \big((\mathrm{ku}^{\Phi C_{p^\alpha}})^{C_{m_p}}\big)_p^\wedge \longrightarrow \big((\mathrm{ku}^{\Phi C_{p^\alpha}})^{\Phi C_{m_p}}\big)_p^\wedge \simeq \big(\mathrm{ku}^{\Phi C_m}\big)_p^\wedge$$

exhibits the target as the $(p, \Phi_m(q))$-completion of the source. It remains to show that inflation induces an equivalence $(\mathrm{ku}^{\Phi C_{p^\alpha}})_p^\wedge \otimes_{\mathbb{S}[q],\psi^{m_p}} \mathbb{S}[q] \simeq ((\mathrm{ku}^{\Phi C_{p^\alpha}})^{C_{m_p}})_p^\wedge$. As both sides are p-complete, this can be checked modulo p. Moreover, note that with geometric fixed points replaced by genuine fixed points, this would follow from Corollary 8.35. In fact, applying Corollary 8.35 to $\mathrm{ku}^{C_{p^i}}$ for all $i \leqslant \alpha$, we deduce that

$$\inf_{m_p}\colon \mathrm{ku}/p \otimes_{\mathbb{S}[q],\psi^{m_p}} \mathbb{S}[q] \xrightarrow{\ \simeq\ } (\mathrm{ku}/p)^{C_{m_p}}$$

is an equivalence of genuine C_{p^α}-equivariant spectra, as it induces equivalences on genuine fixed points for all subgroups (see 8.8). Then it must induce equivalences on geometric fixed points as well, which proves what we want. $\square$

8.40. Lemma. — *For all primes p and all $\alpha \geqslant 1$, the following assertions are true.*

(*a*) *The canonical map $\mathrm{ku}^{\Phi C_{p^\alpha}} \to (\mathrm{ku}^{\Phi C_{p^{\alpha-1}}})^{tC_p}$ induces an S^1-equivariant equivalence of $\mathbb{E}_\infty$-ring spectra*

$$\big(\mathrm{ku}^{\Phi C_{p^\alpha}}\big)_p^\wedge \xrightarrow{\ \simeq\ } \tau_{\geqslant 0}\big((\mathrm{ku}^{\Phi C_{p^{\alpha-1}}})^{tC_p}\big)\,.$$

(*b*) *On homotopy groups, we have $\pi_*((\mathrm{ku}^{\Phi C_{p^\alpha}})_p^\wedge) \cong \mathbb{Z}_p[u_{p^\alpha}, q]/\Phi_{p^\alpha}(q)$ where $|u_{p^\alpha}| = 2$, and*

$$\pi_*\Big(\big((\mathrm{ku}^{\Phi C_{p^\alpha}})_p^\wedge\big)^{h(S^1/C_{p^\alpha})}\Big) \cong \mathbb{Z}_p[u_{p^\alpha}, q][\![t_{p^\alpha}]\!]/\big(u_{p^\alpha} t_{p^\alpha} - \Phi_{p^\alpha}(q)\big)\,.$$

With notation as in 8.33, the canonical map $(\mathrm{ku}^{C_{p^\alpha}})^{h(S^1/C_{p^\alpha})} \to ((\mathrm{ku}^{\Phi C_{p^\alpha}})_p^\wedge)^{h(S^1/C_{p^\alpha})}$ sends $q \mapsto q$, $t_{p^\alpha} \mapsto t_{p^\alpha}$, and $\beta \mapsto (q^{p^{\alpha-1}} - 1)u_{p^\alpha}$.

(*c*) *The inflation map induces an equivalence of S^1-equivariant $\mathbb{E}_\infty$-ring spectra*

$$\inf_{p^{\alpha-1}}\colon \Big(\mathrm{ku}^{\Phi C_p} \otimes_{\mathbb{S}[q],\psi^{p^{\alpha-1}}} \mathbb{S}[q]\Big)_p^\wedge \xrightarrow{\ \simeq\ } \big(\mathrm{ku}^{\Phi C_{p^\alpha}}\big)_p^\wedge\,.$$

Proof. We show all three assertions at once using induction on α. In general, using Lemma 8.28, or more directly the iterated pullback diagram from [NS18, Corollary II.4.7], we obtain a pullback square

$$\begin{array}{ccc} \mathrm{ku}^{C_{p^\alpha}} & \longrightarrow & \mathrm{ku}^{\Phi C_{p^\alpha}} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \big(\mathrm{ku}^{C_{p^{\alpha-1}}}\big)^{hC_p} & \longrightarrow & \big(\mathrm{ku}^{\Phi C_{p^{\alpha-1}}}\big)^{tC_p} \end{array}$$

For $\alpha = 1$, we see that $\mathrm{ku}^{C_p} \to \mathrm{ku}^{hC_p}$ induces an equivalence $(\mathrm{ku}^{C_p})_p^\wedge \simeq \tau_{\geqslant 0}(\mathrm{ku}^{hC_p})_p^\wedge$, and $(\mathrm{ku}^{hC_p})_p^\wedge \to \mathrm{ku}^{tC_p}$ is an equivalence in homotopical degrees $\leqslant -1$. From the pullback square we deduce that $(\mathrm{ku}^{\Phi C_p})_p^\wedge \simeq \tau_{\geqslant 0}(\mathrm{ku}^{tC_p})$, proving (*a*). Assertion (*b*) for $\alpha = 1$ is then a standard calculation; see [DR25, Proposition 3.3.1] for example. Assertion (*c*) is tautological for $\alpha = 1$. For the inductive step, let $\alpha \geqslant 2$. We claim that

$$\begin{aligned} \pi_*\big(\mathrm{ku}^{C_{p^\alpha}}\big) &\cong \pi_*\big(\mathrm{ku}^{C_p}\big) \otimes_{\mathbb{Z}[q],\psi^{p^{\alpha-1}}} \mathbb{Z}[q]\,, \\ \pi_*\big((\mathrm{ku}^{C_{p^{\alpha-1}}})^{hC_p}\big) &\cong \pi_*\big(\mathrm{ku}^{hC_p}\big) \otimes_{\mathbb{Z}[q],\psi^{p^{\alpha-1}}} \mathbb{Z}[q]\,, \\ \pi_*\big((\mathrm{ku}^{\Phi C_{p^{\alpha-1}}})^{tC_p}\big) &\cong \pi_*\big(\mathrm{ku}^{tC_p}\big) \otimes_{\mathbb{Z}[q],\psi^{p^{\alpha-1}}} \mathbb{Z}[q] \end{aligned}$$

Indeed, the first two isomorphism follow from Corollary 8.35 and the third one from (*b*) for $\mathrm{ku}^{\Phi C_{p^{\alpha-1}}}$, which we already know by induction. Then (*b*) and (*c*) follow immediately from the pullback square above. Moreover, we see that the vertical map $\mathrm{ku}^{C_{p^\alpha}} \to (\mathrm{ku}^{C_{p^{\alpha-1}}})^{hC_p}$ induces an equivalence $(\mathrm{ku}^{C_{p^\alpha}})_p^\wedge \simeq \tau_{\geqslant 0}((\mathrm{ku}^{C_{p^{\alpha-1}}})^{hC_p})_p^\wedge$, and that after p-completion the horizontal map $((\mathrm{ku}^{C_{p^{\alpha-1}}})^{hC_p})_p^\wedge \to (\mathrm{ku}^{\Phi C_{p^{\alpha-1}}})^{tC_p}$ is an equivalence in homotopical degrees $\leqslant -1$. As in the case $\alpha = 1$, this implies (*a*). □

8.41. Remark. — Let $p > 2$, so that $\mathrm{THH}(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p[\![q-1]\!])_p^\wedge \simeq \tau_{\geqslant 0}(\mathrm{ku}^{tC_p})$ holds as S^1-equivariant $\mathbb{E}_\infty$-ring spectra by Theorem 7.2. As a consequence of Lemma 8.40(*a*), we get an equivalence

$$\mathrm{THH}\big(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p[\![q-1]\!]\big)_p^\wedge \simeq \big(\mathrm{ku}^{\Phi C_p}\big)_p^\wedge$$

of S^1-equivariant $\mathbb{E}_\infty$-ring spectra.

But we can say even more. Devalapurkar shows in [Dev25, Theorem 6.4.1] that the equivalence $\mathrm{THH}(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p[\![q-1]\!])_p^\wedge \simeq \tau_{\geqslant 0}(\mathrm{ku}^{tC_p})$ holds as *cyclotomic* $\mathbb{E}_\infty$-ring spectra, where $\tau_{\geqslant 0}(\mathrm{ku}^{tC_p})$ is equipped with the cyclotomic structure induced from the trivial cyclotomic structure on ku (see [DR25, Construction 1.1.3]). Since the inflation maps for ku are similarly induced via the trivial S^1-action on the global ultracommutative ring spectrum $\mathrm{ku}_{\mathrm{gl}}$, we see that the cyclotomic Frobenius

$$\phi_p\colon \tau_{\geqslant 0}\big(\mathrm{ku}^{tC_p}\big) \longrightarrow \big(\tau_{\geqslant 0}(\mathrm{ku}^{tC_p})\big)^{tC_p}$$

agrees, up to passing to the connective cover in the target, with $\mathrm{inf}_p\colon (\mathrm{ku}^{\Phi C_p})_p^\wedge \to (\mathrm{ku}^{\Phi C_{p^2}})_p^\wedge$, as maps of S^1-equivariant $\mathbb{E}_\infty$-ring spectra. Therefore we obtain a commutative diagram

$$\begin{array}{ccc}
\big(\mathrm{ku}^{\Phi C_p}\big)_p^\wedge \otimes_{\mathbb{S}[q],\psi^p} \mathbb{S}[q] & \xrightarrow{\quad\mathrm{inf}_p\quad} & \big(\mathrm{ku}^{\Phi C_{p^2}}\big)_p^\wedge \\
{\scriptstyle\simeq}\downarrow & & \downarrow \\
\mathrm{THH}\big(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p[\![q-1]\!]\big)_p^\wedge \otimes_{\mathbb{S}[q],\psi^p} \mathbb{S}[q] & \xrightarrow{\phi_{p/\mathbb{S}[q]}} & \mathrm{THH}\big(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p[\![q-1]\!]\big)^{tC_p}
\end{array}$$

of S^1-equivariant $\mathbb{E}_\infty$-ring spectra.

For our purposes, the description of $\mathrm{ku}^{\Phi C_m}$ that we get from Lemmas 8.38–8.40 would be enough, but for the sake of completeness, let us deduce a complete computation of $\pi_*(\mathrm{ku}^{\Phi C_m})$.

8.42. Proposition. — *Let $m \in \mathbb{N}$. For all divisors $d \mid m$ let $[d]_{\mathrm{ku}}(t) = \beta^{-1}(q^d - 1)$ denote the d-series of the formal group law of* ku. *Then*

$$\pi_*\big(\mathrm{ku}^{\Phi C_m}\big) \cong \mathbb{Z}[\beta, t]/[m]_{\mathrm{ku}}(t)\Big[\big\{[d]_{\mathrm{ku}}(t)^{-1}\big\}_{d\mid m, d\neq m}\Big]^{\geqslant 0}\,,$$

where $(-)^{\geqslant 0}$ on the right-hand side denotes the restriction to non-negative graded degrees.

Proof sketch. We use the arithmetic fracture square (see 1.49)

$$\begin{array}{ccc}
\mathrm{ku}^{\Phi C_m} & \longrightarrow & \prod_{p\mid m} \big(\mathrm{ku}^{\Phi C_m}\big)_p^\wedge \\
\downarrow & \lrcorner & \downarrow \\
\mathrm{ku}\big[\tfrac{1}{m}\big]^{\Phi C_m} & \longrightarrow & \prod_{p\mid m} \big(\mathrm{ku}^{\Phi C_m}\big)_p^\wedge\big[\tfrac{1}{p}\big]
\end{array}$$

Using Lemmas 8.38–8.40, one readily checks that the right vertical and bottom horizontal maps are jointly surjective on π_*. Therefore, we also get a pullback on π_*. It is then straightforward to construct a map $\mathbb{Z}[\beta,t]/[m]_{\mathrm{ku}}(t)\big[\{[d]_{\mathrm{ku}}(t)^{-1}\}_{d\mid m, d\neq m}\big]_{\geqslant 0} \to \pi_*(\mathrm{ku}^{\Phi C_m})$. Whether this map is an equivalence can be checked after localising m and after p-completion for all $p \mid m$, which is again straightforward via Lemmas 8.38–8.40. $\square$

§8.4. Cyclonic even filtrations and Habiro descent of q-Hodge complexes

Let A and R be rings that satisfy the assumptions from 7.19 and assume that $2 \in R^\times$ (so that the addendum (R_2) is automatically satisfied as well). In this subsection, we'll finally explain how to obtain the Habiro descent $q\text{-}\mathcal{H}\mathrm{dg}_{R/A}$ of the q-Hodge complex from a cyclonic structure on $\mathrm{THH}(\mathrm{KU}_R/\mathrm{KU}_A)$.

To this end, let us first discuss how to equip $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A) \simeq \mathrm{THH}(\mathbb{S}_R/\mathbb{S}_A) \otimes \mathrm{ku}$ with a suitable cyclonic structure. At first, one would expect that the cyclonic structure on $\mathrm{THH}(\mathbb{S}_R/\mathbb{S}_A)$ coming from its cyclotomic structure via 8.31 would do the job. But it doesn't! For example, the constructions in §3 are all $A[q]$-linear. But $\mathrm{THH}(\mathbb{S}_R/\mathbb{S}_A)^{\Phi C_p} \to \mathrm{THH}(\mathbb{S}_R/\mathbb{S}_A)^{tC_p}$, which by definition agrees with the cyclotomic Frobenius, is not $\mathbb{S}_A$-linear; instead, it is semilinear over the Tate-valued Frobenius $\phi_{tC_p}\colon \mathbb{S}_A \to \mathbb{S}_A^{tC_p}$. It is thus unclear how one would construct an $A[q]$-linear structure on the associated graded of some even filtration on $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^{C_p}$.

8.43. Cyclonic structure on $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)$. — To fix this, we need to modify the cyclonic structure on $\mathrm{THH}(\mathbb{S}_R/\mathbb{S}_A)$. This requires yet another assumption on A.

(A_2) *Let $\mathbb{S}_A^{\mathrm{cyct}}$ and $\mathbb{S}_A^{\mathrm{triv}}$ denote the cyclonic structures on $\mathbb{S}_A$ given by the cyclotomic structure from 7.19(A) and the trivial cyclotomic structure, respectively. Then we must assume that there exists a map*

$$\mathbb{S}_A^{\mathrm{cyct}} \longrightarrow \mathbb{S}_A^{\mathrm{triv}}$$

of $\mathbb{E}_\infty$-algebras in CycnSp[(8.4)] *whose underlying map of S^1-equivariant $\mathbb{E}_\infty$-algebras is the identity on $\mathbb{S}_A$, equipped with the trivial action.*

Now let $\mathrm{THH}(\mathbb{S}_R/\mathbb{S}_A)^{\mathrm{cyct}}$ denotes the cyclonic structure on $\mathrm{THH}(\mathbb{S}_R/\mathbb{S}_A)$ coming from the usual cyclotomic structure. Assuming (A_2), we can instead consider the following cyclonic structure:

$$\mathrm{THH}(\mathbb{S}_R/\mathbb{S}_A)^{\mathrm{cyct}} \otimes_{\mathbb{S}_A^{\mathrm{cyct}}} \mathbb{S}_A^{\mathrm{triv}}\,.$$

We'll then regard $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A) \simeq \mathrm{THH}(\mathbb{S}_R/\mathbb{S}_A) \otimes \mathrm{ku}$ as a cyclonic spectrum in the apparent way, using the above cyclonic structure on $\mathrm{THH}(\mathbb{S}_R/\mathbb{S}_A)$ as well as the cyclonic structure on ku from 8.32. As we'll see, this has the desired properties.

Let us unravel Assumption (A_2). Since both $\mathbb{S}_A^{\mathrm{cyct}}$ and $\mathbb{S}_A^{\mathrm{triv}}$ are cyclotomic spectra, we have $(\mathbb{S}_A^{\mathrm{cyct}})^{\Phi C_m} \simeq \mathbb{S}_A$ and $(\mathbb{S}_A^{\mathrm{triv}})^{\Phi C_m} \simeq \mathbb{S}_A$ for all m, identifying the residual (S^1/C_m)-action with the trivial S^1-action on $\mathbb{S}_A$. In particular, after taking $(-)^{\Phi C_m}$, a map $\mathbb{S}_A^{\mathrm{cyct}} \to \mathbb{S}_A^{\mathrm{triv}}$ induces S^1-equivariant $\mathbb{E}_\infty$-maps $\psi^m\colon \mathbb{S}_A \to \mathbb{S}_A$ that fit into commutative diagrams

$$\begin{array}{ccc} \mathbb{S}_A & \xrightarrow{\ \psi^{pm}\ } & \mathbb{S}_A \\ {\scriptstyle \phi_p}\big\downarrow & & \big\downarrow \\ \mathbb{S}_A^{tC_p} & \xrightarrow{(\psi^m)^{tC_p}} & \mathbb{S}_A^{tC_p} \end{array}$$

[(8.4)]Beware that there may be more maps as cyclonic $\mathbb{E}_\infty$-algebras than as cyclotomic $\mathbb{E}_\infty$-algebras.

for all $m \in \mathbb{N}$ and all primes p. It follows inductively that $\psi^m\colon \mathbb{S}_A \to \mathbb{S}_A$ must be a lift of the Λ-ring Adams operation $\psi^m\colon A \to A$.

8.44. Lemma. — *The data of S^1-equivariant $\mathbb{E}_\infty$-maps $\psi^m\colon \mathbb{S}_A \to \mathbb{S}_A$ together with commutative diagrams as above uniquely determines a map $\mathbb{S}_A^{\mathrm{cyct}} \to \mathbb{S}_A^{\mathrm{triv}}$ of cyclonic $\mathbb{E}_\infty$-algebras.*

Proof. By Proposition 8.26 we may equivalently construct $\mathbb{S}_A^{\mathrm{cyct}} \to \mathbb{S}_A^{\mathrm{triv}}$ as a map of $\mathbb{E}_\infty$-algebras in naive cyclonic spectra. It's clear from the construction in Lemma 8.27 that

$$\mathrm{CAlg}\big(\mathrm{CycnSp}^{\mathrm{naiv}}\big) \simeq \mathrm{LEq}\left(\prod_{m\in\mathbb{N}} \mathrm{CAlg}\big(\mathrm{Sp}^{\mathrm{B}(S^1/C_m)}\big) \underset{((-)^{tC_p})_{p,m}}{\overset{\mathrm{can}}{\rightrightarrows}} \prod_p \prod_{m\in\mathbb{N}} \mathrm{CAlg}\big(\mathrm{Sp}^{\mathrm{B}(S^1/C_{pm})}\big)\right),$$

and so the given data indeed uniquely determines such a map. $\square$

We'll verify in §9.1 that in all examples we can construct, Assumption (A_2) is satisfied as well. This concludes our discussion of the cyclonic structure on $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)$. For convenience, let us also introduce the following notation.

8.45. Definition. — For all $m \in \mathbb{N}$, the m^{th} *topological cyclonic homology of* ku_R *over* ku_A is the spectrum

$$\mathrm{TC}^{-(m)}(\mathrm{ku}_R/\mathrm{ku}_A) \coloneqq \big(\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^{C_m}\big)^{h(S^1/C_m)}\,.$$

8.46. Cyclonic even filtrations in general. — Let T be a cyclonic $\mathbb{E}_1$-algebra and let M be a cyclonic left T-module. Suppose that T and M are bounded below and that for all $m \in \mathbb{N}$ the geometric fixed points $T^{\Phi C_m}$ are complex orientable (but we don't require any genuine equivariant or cyclonic complex orientation). In this situation, we expect that the correct filtration to put on $M^{\Phi C_m}$ is simply the non-equivariant even perfect filtration $\mathrm{fil}^\star_{\mathrm{ev}} M^{\Phi C_m} \coloneqq \mathrm{fil}^\star_{\mathrm{P\text{-}ev}/T^{\Phi C_m}} M^{\Phi C_m}$ of $M^{\Phi C_m}$ as a left module over $T^{\Phi C_m}$. Moreover, the genuine C_m-fixed points should be equipped with the filtration

$$\mathrm{fil}^\star_{\mathrm{ev}\,/T,C_m} M^{C_m} \coloneqq \mathrm{eq}\left(\prod_{d\mid m}\big(\mathrm{fil}^\star_{\mathrm{ev}} M^{\Phi C_d}\big)^{hC_{m/d,\mathrm{ev}}} \underset{\phi}{\overset{\mathrm{can}}{\rightrightarrows}} \prod_p \prod_{pd\mid m} \big((\mathrm{fil}^\star_{\mathrm{ev}} M^{\Phi C_d})^{tC_{p,\mathrm{ev}}}\big)^{hC_{m/pd,\mathrm{ev}}}\right).$$

Here $(-)^{hC_{m/d,\mathrm{ev}}}$, $(-)^{tC_{p,\mathrm{ev}}}$, and $(-)^{hC_{m/pd,\mathrm{ev}}}$ refer to the filtered fixed points and Tate construction defined [AR24, §2.3].(8.5) The map can in the equaliser is induced by the natural transformation $(-)^{hC_{p,\mathrm{ev}}} \Rightarrow (-)^{tC_{p,\mathrm{ev}}}$ and the map ϕ is induced by the canonical maps

$$\mathrm{fil}^\star_{\mathrm{P\text{-}ev}/T^{\Phi C_{pd}}}\big((M^{\Phi C_d})^{tC_p}\big) \longrightarrow \mathrm{fil}^\star_{\mathrm{P\text{-}ev}/(T^{\Phi C_d})^{tC_p}}\big((M^{\Phi C_d})^{tC_p}\big) \longrightarrow \big(\mathrm{fil}^\star_{\mathrm{P\text{-}ev}/T^{\Phi C_d}} M^{\Phi C_d}\big)^{tC_{p,\mathrm{ev}}}$$

using the construction from 7.6. To apply this construction, we need the additional assumption that $(M^{\Phi C_m})^{hC_p}$ is homologically even over $(T^{\Phi C_m})^{hC_p}$; this is certainly satisfied in the case $M = T$ that is relevant for us.

A genuine equivariant version of the even filtration is currently in the works; for example, the author has been informed of (independent) work in progress by Jeremy Hahn and Lucas Piessevaux. We have little doubt that in the foreseeable future, an intrinsically defined genuine equivariant even filtration will be available and we expect that for M as above (maybe subject to some extra assumptions), the true even filtration will agree with our formula.

(8.5) This needs the residual S^1-actions, so as stated the formula above only applies in the cyclonic setting but not in the genuine C_m-equivariant setting.

8.47. Cyclonic even filtrations on $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)$. — Put $R^{(m)} \coloneqq R \otimes^{\mathrm{L}}_{A,\psi^m} A$. Note that $R^{(m)}$ is static, since the Adams operation ψ^m is flat in any perfectly covered Λ-ring. Moreover, $R^{(m)}$ satisfies the assumptions from 7.19(R); in particular, it admits a spherical lift given by $\mathbb{S}_{R^{(m)}} \coloneqq \mathbb{S}_R \otimes_{\mathbb{S}_A,\psi^m} \mathbb{S}_A$, where $\psi^m \colon \mathbb{S}_A \to \mathbb{S}_A$ is the lift of the Λ-ring Adams operation from 8.43. We may thus define $\mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}(\mathrm{ku}_{R^{(m)}}/\mathrm{ku}_A)$ via 7.23. Via base change along the inflation $\mathrm{inf}_m \colon \mathrm{ku} \to \mathrm{ku}^{\Phi C_m}$, we may then equip the geometric fixed points $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^{\Phi C_m}$ with the filtration

$$\mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^{\Phi C_m} \coloneqq \mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}(\mathrm{ku}_{R^{(m)}}/\mathrm{ku}_A) \otimes_{\mathrm{ku}_{\mathrm{ev}}} \mathrm{ku}_{\mathrm{ev}}^{\Phi C_m}\,,$$

where $\mathrm{ku}_{\mathrm{ev}}^{\Phi C_m} \coloneqq \tau_{\geqslant 2\star}(\mathrm{ku}^{\Phi C_m})$ denotes the double-speed Whitehead filtration. We'll check in Lemma 8.48 below that this agrees with the usual perfect even filtration on $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^{\Phi C_m}$, as long as the latter is defined. Next, we construct the filtration on genuine fixed points

$$\mathrm{fil}^\star_{\mathrm{ev},C_m} \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^{C_m}$$

via the formula in 8.46. Finally, using the notation introduced in Definition 8.45, we define

$$\mathrm{fil}^\star_{\mathrm{ev},S^1} \mathrm{TC}^{-(m)}(\mathrm{ku}_R/\mathrm{ku}_A) \coloneqq \left(\mathrm{fil}^\star_{\mathrm{ev},C_m} \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^{C_m}\right)^{h(\mathbb{T}/C_m)_{\mathrm{ev}}}\,,$$

where $(-)^{h(\mathbb{T}/C_m)_{\mathrm{ev}}}$ denotes fixed points in the sense of [AR24, §2.3] with respect to the even filtration on $\mathbb{S}[S^1/C_m]$.

Here are two sanity checks:

8.48. Lemma. — *Suppose we chose condition 6.2($\mathbb{E}_2$) for all primes p, so that ku_R is an $\mathbb{E}_2$-algebra in ku_A-modules. Then $\mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^{\Phi C_m}$ agrees with Pstrągowski's perfect even filtration on the $\mathbb{E}_1$-ring $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^{\Phi C_m}$.*

Proof sketch. We know from Lemma 7.24 that $\mathrm{fil}_{\mathrm{ev}} \mathrm{THH}(\mathrm{ku}_{R^{(m)}}/\mathrm{ku}_A)$ agrees with Pstrągowski's perfect even filtration. It will thus be enough to show that the canonical base change map

$$\mathrm{fil}^\star_{\mathrm{P\text{-}ev}} \mathrm{THH}(\mathrm{ku}_{R^{(m)}}/\mathrm{ku}_A) \otimes_{\mathrm{ku}_{\mathrm{ev}}} \mathrm{ku}_{\mathrm{ev}}^{\Phi C_m} \longrightarrow \mathrm{fil}^\star_{\mathrm{P\text{-}ev}} \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^{\Phi C_m}$$

is an equivalence. It's enough to check this on associated gradeds as both sides are exhaustive filtrations on $\mathrm{THH}(\mathrm{ku}_{R^{(m)}}/\mathrm{ku}_A) \otimes_{\mathrm{ku}} \mathrm{ku}^{\Phi C_m} \simeq \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^{\Phi C_m}$. Now on associated gradeds (and in fact, one the nose) both sides can be computed by a cosimplicial resolution as in Proposition 6.11, because $\mathrm{THH}(\mathbb{S}_P) \to \mathbb{S}_P$ is faithfully even flat. We can then use a similar argument as in Corollary 6.17 to show the desired base change equivalence. Here we use that $\mathrm{ku}^{\Phi C_m}$ is even with p-torsion free homotopy groups for all primes p by Proposition 8.42. □

8.49. Lemma. — *For all $m \in \mathbb{N}$,*

$$\mathrm{fil}^\star_{\mathrm{ev},C_m} \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^{C_m} \quad \text{and} \quad \mathrm{fil}^\star_{\mathrm{ev},S^1} \mathrm{TC}^{-(m)}(\mathrm{ku}_R/\mathrm{ku}_A)$$

are complete exhaustive filtrations on $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^{C_m}$ and $\mathrm{TC}^{-(m)}(\mathrm{ku}_R/\mathrm{ku}_A)$, respectively.

Proof sketch. For completeness, apply [AR24, Lemma 2.75(iv)] to each of the constituents of the equaliser from 8.46. The only non-obvious thing to check is that $(\mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^{\Phi C_d})^{tC_{p,ev}}$ is complete, which follows from an argument as in 7.7. For exhaustiveness, apply [AR24, Lemma 2.75(iv)] to each of the constituents in the equaliser from 8.46. To see that this lemma applies, one can use Corollary 6.15. □

We can now formulate the main result of §8.

8.50. Return of the twisted q-Hodge filtration. — We can plug the q-Hodge filtration $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_{R/A}$ from Theorem 7.27 into construction 3.38 to obtain the *twisted q-Hodge filtration*

$$\mathrm{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_m}\, q\text{-dR}^{(m)}_{R/A}\,.$$

We will relate this to $\mathrm{gr}^*_{\mathrm{ev},S^1}\,\mathrm{TC}^{-(m)}(\mathrm{ku}_R/\mathrm{ku}_A)$. To this end, we must explain how the latter acquires a filtered structure.

Observe that $\mathrm{fil}^\star_{\mathrm{ev},S^1}\,\mathrm{TC}^{-(m)}(\mathrm{ku}/\mathrm{ku}) \simeq \tau_{\geqslant 2\star}((\mathrm{ku}^{C_m})^{h(S^1/C_m)})$. This computation is not completely trivial, but it can be done in the same way as Theorem 8.51 below.[8.6] As a consequence, we see that in general,

$$\Sigma^{-2*}\,\mathrm{gr}^*_{\mathrm{ev},S^1}\,\mathrm{TC}^{-(m)}(\mathrm{ku}_R/\mathrm{ku}_A)$$

is a module over the graded ring $\mathbb{Z}[\beta,q]\llbracket t_m\rrbracket/(\beta t_m-(q^m-1)) \cong \pi_{2*}((\mathrm{ku}^{C_m})^{h(S^1/C_m)})$ (see 8.33). Regarding t_m as the filtration parameter, this graded ring can be identified with the (q^m-1)-adic filtration $(q^m-1)^\star\mathbb{Z}[q]^\wedge_{(q^m-1)}$.

8.51. Theorem. — *Let $m\in\mathbb{N}$. Suppose A and R satisfy the assumptions from 7.19 along with the addenda $2\in R^\times$ and 8.43(A_2). Then there exists a canonical equivalence of filtered $\mathbb{Z}[\beta,q]\llbracket t_m\rrbracket/(\beta t_m-(q^m-1))$-modules*

$$\mathrm{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_m}\, q\text{-}\widehat{\mathrm{dR}}^{(m)}_{R/A} \xrightarrow{\simeq} \Sigma^{-2*}\,\mathrm{gr}^*_{\mathrm{ev},S^1}\,\mathrm{TC}^{-(m)}(\mathrm{ku}_R/\mathrm{ku}_A)\,,$$

where the left-hand side denotes the completion of the twisted q-Hodge filtration $\mathrm{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_m}\, q\text{-dR}^{(m)}_{R/A}$ from 8.50 and the right-hand side is defined in 8.47.

To show Theorem 8.51, we'll decompose $\Sigma^{-2*}\,\mathrm{gr}^*_{\mathrm{ev},S^1}\,\mathrm{TC}^{-(m)}(\mathrm{ku}_R/\mathrm{ku}_A)$ into a fracture square and match it up with 3.38.

8.52. Fracture squares for even filtrations. — Let N be a positive integer. We construct an even filtration

$$\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}\big(\mathrm{ku}_R\big[\tfrac{1}{N}\big]/\mathrm{ku}_A\big[\tfrac{1}{N}\big]\big)$$

as in 7.23, except that we replace every occurence of ku by a $\mathrm{ku}[1/N]$. Moreover, for any prime p we let

$$\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}_\blacksquare\big(\mathrm{ku}_{\widehat{R}_p}/\mathrm{ku}_{\widehat{A}_p}\big)\,,\quad\text{and}\quad \mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}_\blacksquare\big(\mathrm{ku}_{\widehat{R}_p}\big[\tfrac{1}{p}\big]/\mathrm{ku}_{\widehat{A}_p}\big[\tfrac{1}{p}\big]\big)$$

be the even filtrations given by applying 6.8 for $k=\mathrm{ku}$ and $k=\mathrm{ku}[1/p]$, respectively. By construction, we then have a pullback square

$$\begin{array}{ccc}\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A) & \longrightarrow & \prod_{p\mid N}\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}_\blacksquare\big(\mathrm{ku}_{\widehat{R}_p}/\mathrm{ku}_{\widehat{A}_p}\big)\\ \downarrow & \lrcorner & \downarrow\\ \mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}\big(\mathrm{ku}_R\big[\tfrac{1}{N}\big]/\mathrm{ku}_A\big[\tfrac{1}{N}\big]\big) & \longrightarrow & \prod_{p\mid N}\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}_\blacksquare\big(\mathrm{ku}_{\widehat{R}_p}\big[\tfrac{1}{p}\big]/\mathrm{ku}_{\widehat{A}_p}\big[\tfrac{1}{p}\big]\big)\end{array}$$

[8.6] In fact, it is almost a special case of that theorem, except that $2\notin\mathbb{Z}^\times$. Even so, to formulate the theorem properly, we need this special case first.

A similar fracture square exists for the geometric C_m-fixed points. To this end, replace R by $R^{(m)}$ in the above construction and apply the base change $-\otimes_{\mathrm{ku}_{\mathrm{ev}}}\mathrm{ku}_{\mathrm{ev}}^{\Phi C_m}$ to obtain

$$\mathrm{fil}_{\mathrm{ev}}^{\star}\operatorname{THH}\big(\mathrm{ku}_R\big[\tfrac{1}{N}\big]/\mathrm{ku}_A\big[\tfrac{1}{N}\big]\big)^{\Phi C_m},$$
$$\mathrm{fil}_{\mathrm{ev}}^{\star}\operatorname{THH}_{\blacksquare}\big(\mathrm{ku}_{\widehat{R}_p}/\mathrm{ku}_{\widehat{A}_p}\big)^{\Phi C_m}\quad\text{and}\quad \mathrm{fil}_{\mathrm{ev}}^{\star}\operatorname{THH}_{\blacksquare}\big(\mathrm{ku}_{\widehat{R}_p}\big[\tfrac{1}{p}\big]/\mathrm{ku}_{\widehat{A}_p}\big[\tfrac{1}{p}\big]\big)^{\Phi C_m}.$$

These fit into a pullback square

$$\begin{array}{ccc}
\mathrm{fil}_{\mathrm{ev}}^{\star}\operatorname{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^{\Phi C_m} & \longrightarrow & \prod_{p\mid N}\mathrm{fil}_{\mathrm{ev}}^{\star}\operatorname{THH}_{\blacksquare}\big(\mathrm{ku}_{\widehat{R}_p}/\mathrm{ku}_{\widehat{A}_p}\big)^{\Phi C_m}\\
\downarrow & \lrcorner & \downarrow\\
\mathrm{fil}_{\mathrm{ev}}^{\star}\operatorname{THH}\big(\mathrm{ku}_R\big[\tfrac{1}{N}\big]/\mathrm{ku}_A\big[\tfrac{1}{N}\big]\big)^{\Phi C_m} & \longrightarrow & \prod_{p\mid N}\mathrm{fil}_{\mathrm{ev}}^{\star}\operatorname{THH}_{\blacksquare}\big(\mathrm{ku}_{\widehat{R}_p}\big[\tfrac{1}{p}\big]/\mathrm{ku}_{\widehat{A}_p}\big[\tfrac{1}{p}\big]\big)^{\Phi C_m}
\end{array}$$

We also note that if we define the *∞-category of cyclonic solid condensed spectra* as the Lurie tensor product $\mathrm{CycnSp}\otimes\mathrm{Sp}_{\blacksquare}$, then $\operatorname{THH}_{\blacksquare}(\mathrm{ku}_{\widehat{R}_p}/\mathrm{ku}_{\widehat{A}_p})$ and $\operatorname{THH}_{\blacksquare}(\mathrm{ku}_{\widehat{R}_p}[1/p]/\mathrm{ku}_{\widehat{A}_p}[1/p])$ can be equipped with cyclonic solid condensed structures as in 8.43 and so the expressions $\operatorname{THH}_{\blacksquare}(\mathrm{ku}_{\widehat{R}_p}/\mathrm{ku}_{\widehat{A}_p})^{\Phi C_m}$ and $\operatorname{THH}_{\blacksquare}(\mathrm{ku}_{\widehat{R}_p}[1/p]/\mathrm{ku}_{\widehat{A}_p}[1/p])^{\Phi C_m}$ make sense. Finally, the constructions from 8.47 can also be applied in this setting, and so we obtain

$$\mathrm{fil}_{\mathrm{ev},S^1}^{\star}\operatorname{TC}^{-(m)}\big(\mathrm{ku}_R\big[\tfrac{1}{N}\big]/\mathrm{ku}_A\big[\tfrac{1}{N}\big]\big),$$
$$\mathrm{fil}_{\mathrm{ev},S^1}^{\star}\operatorname{TC}_{\blacksquare}^{-(m)}\big(\mathrm{ku}_{\widehat{R}_p}/\mathrm{ku}_{\widehat{A}_p}\big)\quad\text{and}\quad \mathrm{fil}_{\mathrm{ev},S^1}^{\star}\operatorname{TC}_{\blacksquare}^{-(m)}\big(\mathrm{ku}_{\widehat{R}_p}\big[\tfrac{1}{p}\big]/\mathrm{ku}_{\widehat{A}_p}\big[\tfrac{1}{p}\big]\big),$$

which fit into a pullback square

$$\begin{array}{ccc}
\mathrm{fil}_{\mathrm{ev},S^1}^{\star}\operatorname{TC}^{-(m)}(\mathrm{ku}_R/\mathrm{ku}_A) & \longrightarrow & \prod_{p\mid N}\mathrm{fil}_{\mathrm{ev},S^1}^{\star}\operatorname{TC}_{\blacksquare}^{-(m)}\big(\mathrm{ku}_{\widehat{R}_p}/\mathrm{ku}_{\widehat{A}_p}\big)\\
\downarrow & \lrcorner & \downarrow\\
\mathrm{fil}_{\mathrm{ev},S^1}^{\star}\operatorname{TC}^{-(m)}\big(\mathrm{ku}_R\big[\tfrac{1}{N}\big]/\mathrm{ku}_A\big[\tfrac{1}{N}\big]\big) & \longrightarrow & \prod_{p\mid N}\mathrm{fil}_{\mathrm{ev},S^1}^{\star}\operatorname{TC}_{\blacksquare}^{-(m)}\big(\mathrm{ku}_{\widehat{R}_p}\big[\tfrac{1}{p}\big]/\mathrm{ku}_{\widehat{A}_p}\big[\tfrac{1}{p}\big]\big)
\end{array}$$

We will now analyse this pullback. Let us begin with the part where N is invertible.

8.53. Lemma. — *Suppose N is divisible by m. Then the inflation map $\inf_m\colon \mathrm{ku}\to\mathrm{ku}^{\Phi C_m}$ induces a filtered S^1-equivariant (or more precisely, $\mathbb{T}_{\mathrm{ev}}$-module) equivalence*

$$\Big(\mathrm{fil}_{\mathrm{ev}}^{\star}\operatorname{THH}\big(\mathrm{ku}_R^{(m)}\big[\tfrac{1}{N}\big]/\mathrm{ku}_A\big[\tfrac{1}{N}\big]\big)\otimes_{\mathbb{S}[q],\psi^m}\mathbb{S}[q]\Big)^{\wedge}_{\Phi_m(q)}\xrightarrow{\ \simeq\ }\mathrm{fil}_{\mathrm{ev}}^{\star}\operatorname{THH}\big(\mathrm{ku}_R\big[\tfrac{1}{N}\big]/\mathrm{ku}_A\big[\tfrac{1}{N}\big]\big)^{\Phi C_m}.$$

Proof. Observe that the $\Phi_m(q)$-adic completion is just the projection to the m^{th} factor in the decomposition

$$\mathbb{S}\big[\tfrac{1}{N},q\big]/(q^m-1)\simeq\prod_{d\mid m}\mathbb{S}\big[\tfrac{1}{N},q\big]/\Phi_d(q).$$

The claim then follows from Lemma 8.38 and the definition of $\mathrm{fil}_{\mathrm{ev}}^{\star}\operatorname{THH}(\mathrm{ku}_R[1/N]/\mathrm{ku}_A[1/N])$ and $\mathrm{fil}_{\mathrm{ev}}^{\star}\operatorname{THH}_{\blacksquare}(\mathrm{ku}_{\widehat{R}_p}[1/p]/\mathrm{ku}_{\widehat{A}_p}[1/p])$ as base changes along $-\otimes_{\mathrm{ku}_{\mathrm{ev}}}\mathrm{ku}_{\mathrm{ev}}^{\Phi C_m}$. $\square$

Let us now analyse the p-adic part.

8.54. Lemma. — *For all primes p, the inflation map $\inf_m\colon \mathrm{ku}\rightarrow \mathrm{ku}^{\Phi C_m}$ induces a filtered S^1-equivariant (or more precisely, $\mathbb{T}_{\mathrm{ev}}$-module) equivalence*

$$\Bigl(\mathrm{fil}_{\mathrm{ev}}^\star \operatorname{THH}_\blacksquare\bigl(\mathrm{ku}\widehat{R}_p^{(m)}\bigl[\tfrac{1}{p}\bigr]/\mathrm{ku}\widehat{A}_p\bigl[\tfrac{1}{p}\bigr]\bigr)\otimes_{\mathbb{S}[q],\psi^m}\mathbb{S}[q]\Bigr)_{\Phi_m(q)}^\wedge \xrightarrow{\simeq} \mathrm{fil}_{\mathrm{ev}}^\star \operatorname{THH}_\blacksquare\bigl(\mathrm{ku}\widehat{R}_p\bigl[\tfrac{1}{p}\bigr]/\mathrm{ku}\widehat{A}_p\bigl[\tfrac{1}{p}\bigr]\bigr)^{\Phi C_m}\,.$$

Proof. Analogous to Lemma 8.53. □

8.55. Lemma. — *Write $m=p^\alpha m_p$, where p is a prime and m_p is coprime to p. Then the inflation map $\inf_{m_p}\colon \mathrm{ku}^{\Phi C_{p^\alpha}}\rightarrow \mathrm{ku}^{\Phi C_m}$ induces a filtered S^1-equivariant (or more precisely, $\mathbb{T}_{\mathrm{ev}}$-module) equivalence*

$$\Bigl(\mathrm{fil}_{\mathrm{ev}}^\star \operatorname{THH}_\blacksquare\bigl(\mathrm{ku}\widehat{R}_p^{(m_p)}/\mathrm{ku}\widehat{A}_p\bigr)^{\Phi C_{p^\alpha}}\otimes_{\mathbb{S}[q],\psi^{m_p}}\mathbb{S}[q]\Bigr)_{\Phi_m(q)}^\wedge \xrightarrow{\simeq} \mathrm{fil}_{\mathrm{ev}}^\star \operatorname{THH}_\blacksquare\bigl(\mathrm{ku}\widehat{R}_p/\mathrm{ku}\widehat{A}_p\bigr)^{\Phi C_m}\,.$$

Proof. As in the proof of Lemma 8.53, observe that the $\Phi_m(q)$-adic completion, which agrees with $\Phi_{m_p}(q)$-adic completion as everything is already p-complete, is just a projection to the m_p^{th} factor in the product decomposition

$$\bigl(\mathbb{S}[q]/(q^m-1)\bigr)_p^\wedge\simeq\prod_{d_p\mid m_p}\bigl(\mathbb{S}_p[q]/(q^m-1)\bigr)_{(p,\Phi_{d_p}(q))}^\wedge\,.$$

The claim then follows from the constructions and Lemma 8.39. □

8.56. Lemma. — *In the case $m=p^\alpha$, where $p>2$ is a prime and $\alpha\geqslant 1$, we have a canonical equivalence of filtered $\mathbb{Z}_p[u_{p^\alpha},q]\llbracket t_{p^\alpha}\rrbracket/(u_{p^\alpha}t_{p^\alpha}-\Phi_{p^\alpha}(q))$-modules*

$$\mathrm{fil}_{\mathcal{N}}^\star\bigl(q\text{-}\mathrm{dR}_{R/A}^{(p^\alpha)}\bigr)_{(p,\mathcal{N})}^\wedge\xrightarrow{\simeq}\Sigma^{-2*}\operatorname{gr}^\star\Bigl(\bigl(\mathrm{fil}_{\mathrm{ev}}^* \operatorname{THH}_\blacksquare\bigl(\mathrm{ku}\widehat{R}_p/\mathrm{ku}\widehat{A}_p\bigr)^{\Phi C_{p^\alpha}}\bigr)^{h(\mathbb{T}/C_{p^\alpha})_{\mathrm{ev}}}\Bigr)\,.$$

Proof. We'll explain the case $\alpha=1$; the general case will follow from an analogous argument using Lemma 8.40(*c*). Let $\widehat{R}_p^{(p)}$, $\mathbb{S}_{\widehat{R}_p^{(p)}}$, and $\mathrm{ku}_{\widehat{R}_p^{(p)}}$ denote the p-completions of $R^{(p)}$, $\mathbb{S}_{R^{(p)}}$, and $\mathrm{ku}_{R^{(p)}}$, respectively. By Remark 8.41, $(\mathrm{ku}^{\Phi C_p})_p^\wedge\simeq\operatorname{THH}_\blacksquare(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p\llbracket q-1\rrbracket)$, and so we get S^1-equivariant equivalences

$$\operatorname{THH}_\blacksquare\bigl(\mathrm{ku}\widehat{R}_p/\mathrm{ku}\widehat{A}_p\bigr)^{\Phi C_p}\simeq\operatorname{THH}_\blacksquare\bigl(\mathbb{S}\widehat{R}_p^{(p)}/\mathbb{S}\widehat{A}_p\bigr)\otimes^\blacksquare\mathrm{ku}^{\Phi C_p}\simeq\operatorname{THH}_\blacksquare\bigl(\widehat{R}_p^{(p)}[\zeta_p]/\mathbb{S}\widehat{A}_p\llbracket q-1\rrbracket\bigr)\,.$$

This also induces an equivalence of S^1-equivariant even filtrations

$$\Bigl(\mathrm{fil}_{\mathrm{ev}}^\star \operatorname{THH}_\blacksquare\bigl(\mathrm{ku}\widehat{R}_p/\mathrm{ku}\widehat{A}_p\bigr)^{\Phi C_p}\Bigr)^{h(\mathbb{T}/C_p)_{\mathrm{ev}}}\simeq\mathrm{fil}_{\mathrm{HRW\text{-}ev},hS^1}^\star\operatorname{TC}^-\bigl(\widehat{R}_p^{(p)}[\zeta_p]/\mathbb{S}\widehat{A}_p\llbracket q-1\rrbracket\bigr)_p^\wedge\,.$$

Indeed, depending on whether we are in case 6.2($\mathbb{E}_1$) or ($\mathbb{E}_2$), the given resolution $\widehat{R}_p\rightarrow\widehat{R}_{p,\infty}^\bullet$ or the resolution from Proposition 6.11 will also compute the Hahn–Raksit–Wilson even filtration. By Proposition A.17 and A.19, the associated graded

$$\Sigma^{-2*}\operatorname{gr}_{\mathrm{HRW\text{-}ev},hS^1}^*\operatorname{TC}^-\bigl(\widehat{R}_p^{(p)}[\zeta_p]/\mathbb{S}\widehat{A}_p\llbracket q-1\rrbracket\bigr)_p^\wedge\simeq\mathrm{fil}_{\mathcal{N}}^\star\bigl(q\text{-}\mathrm{dR}_{R/A}^{(p)}\bigr)_{(p,\mathcal{N})}^\wedge$$

is the completion of the Nygaard filtration on $\bigl(q\text{-}\mathrm{dR}_{R/A}^{(p)}\bigr)_p^\wedge$, as desired. □

8.57. Lemma. — *In the case $m = p^\alpha$, where $p > 2$ is a prime and $\alpha \geqslant 1$, we have a canonical equivalence of filtered $\mathbb{Z}_p[\beta, q][\![t_{p^\alpha}]\!]/(\beta t_{p^\alpha} - (q^{p^\alpha} - 1))$-modules*

$$\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_{p^\alpha}} \big(q\text{-}\widehat{\mathrm{dR}}^{(p^\alpha)}_{R/A}\big)^\wedge_p \xrightarrow{\simeq} \Sigma^{-2*} \operatorname{gr}^*_{\mathrm{ev},S^1} \mathrm{TC}^{-(p^\alpha)}_{\blacksquare}\big(\mathrm{ku}\widehat{R}_p/\mathrm{ku}\widehat{A}_p\big)\,.$$

Proof. We use induction on α. Unravelling the equaliser from 8.46 in the case $m = p^\alpha$ provides us with a pullback diagram

$$\begin{array}{ccc} \operatorname{fil}^\star_{\mathrm{ev},S^1} \mathrm{TC}^{-(p^\alpha)}_{\blacksquare}\big(\mathrm{ku}\widehat{R}_p/\mathrm{ku}\widehat{A}_p\big) & \longrightarrow & \Big(\big(\operatorname{fil}^\star_{\mathrm{ev}} \mathrm{THH}_{\blacksquare}\big(\mathrm{ku}\widehat{R}_p/\mathrm{ku}\widehat{A}_p\big)^{\Phi C_{p^\alpha}}\Big)^{h(\mathbb{T}/C_{p^\alpha})_{\mathrm{ev}}} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \operatorname{fil}^\star_{\mathrm{ev},S^1} \mathrm{TC}^{-(p^{\alpha-1})}_{\blacksquare}\big(\mathrm{ku}\widehat{R}_p/\mathrm{ku}\widehat{A}_p\big) & \longrightarrow & \Big(\Big(\big(\operatorname{fil}^\star_{\mathrm{ev}} \mathrm{THH}_{\blacksquare}\big(\mathrm{ku}\widehat{R}_p/\mathrm{ku}\widehat{A}_p\big)^{\Phi C_{p^{\alpha-1}}}\Big)^{tC_{p,\mathrm{ev}}}\Big)^{h(\mathbb{T}/C_{p^\alpha})_{\mathrm{ev}}} \end{array}$$

Let us first consider the case $\alpha = 1$. In this case the bottom left corner of the diagram above is just $\operatorname{fil}^\star_{\mathrm{ev},hS^1} \mathrm{TC}^-_{\blacksquare}(\mathrm{ku}\widehat{R}_p/\mathrm{ku}\widehat{A}_p)$, whose associated graded is $\operatorname{fil}^*_{q\text{-}\mathrm{Hdg}}(q\text{-}\widehat{\mathrm{dR}}_{R/A})^\wedge_p$ by Theorem 7.9. The argument in 7.7 shows that the bottom right corner can be identified with $\operatorname{fil}^\star_{\mathrm{ev},tS^1} \mathrm{TP}_{\blacksquare}(\mathrm{ku}\widehat{R}_p/\mathrm{ku}\widehat{A}_p)$, whose associated graded is $(q\text{-}\widehat{\mathrm{dR}}_{R/A})^\wedge_p$ in every degree. The associated graded of the top right corner has been computed in Lemma 8.56. We conclude that the associated graded of the pullback diagram above will be of the form

$$\begin{array}{ccc} \Sigma^{-2*} \operatorname{gr}^*_{\mathrm{ev},S^1} \mathrm{TC}^{-(p)}_{\blacksquare}\big(\mathrm{ku}\widehat{R}_p/\mathrm{ku}\widehat{A}_p\big) & \longrightarrow & \operatorname{fil}^\star_{\mathcal{N}}\big(q\text{-}\mathrm{dR}^{(p)}_{R/A}\big)^\wedge_{(p,\mathcal{N})} \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \phi_{p/A[q]}} \\ \operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}}\big(q\text{-}\widehat{\mathrm{dR}}_{R/A}\big)^\wedge_p & \longrightarrow & \big(q\text{-}\widehat{\mathrm{dR}}_{R/A}\big)^\wedge_p \end{array}$$

By A.18 and the construction of the comparison map in 7.5–7.7, we see that the right vertical map is indeed the relative Frobenius $\phi_{p/A[q]}$ on q-de Rham cohomology.

The filtered structure on $\operatorname{fil}^\star_{\mathcal{N}}(q\text{-}\mathrm{dR}^{(p)}_{R/A})^\wedge_{(p,\mathcal{N})}$ comes from the structure as a graded module over $\mathbb{Z}_p[u_p, q][\![t]\!]/(u_p t_p - \Phi_p(q))$, whereas the filtered structure on $\operatorname{fil}^*_{q\text{-}\mathrm{Hdg}}(q\text{-}\mathrm{dR}_{R/A})^\wedge_p$ and the constant filtration on $(q\text{-}\mathrm{dR}_{R/A})^\wedge_p$ are presented as graded $\mathbb{Z}[\beta][\![t]\!]$-modules. Changing the filtration parameter from t to $t_p = \Phi_p(q)t$ has the effect of "rescaling" filtrations by $\Phi_p(q)$ as in 3.32. The resulting diagram almost looks like the completion of the defining pullback of $\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_p}(q\text{-}\mathrm{dR}^{(p)}_{R/A})^\wedge_p$, except for the following subtlety: The rescaled filtrations

$$\Phi_p(q)^\star \operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}}(q\text{-}\mathrm{dR}_{R/A})^\wedge_p \quad \text{and} \quad \Phi_p(q)^\star(q\text{-}\mathrm{dR}_{R/A})^\wedge_p$$

are already complete, so $\Phi_p(q)^\star \operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}}(q\text{-}\widehat{\mathrm{dR}}_{R/A})^\wedge_p$ and $\Phi_p(q)^\star(q\text{-}\widehat{\mathrm{dR}}_{R/A})^\wedge_p$ are *not* the completions of these filtrations. To see that the pullback above still yields the completion of $\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_p}(q\text{-}\mathrm{dR}^{(p)}_{R/A})^\wedge_p$, just observe that the pullback

$$\begin{array}{ccc} \operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}}\big(q\text{-}\mathrm{dR}_{R/A}\big)^\wedge_p & \longrightarrow & \big(q\text{-}\mathrm{dR}_{R/A}\big)^\wedge_p \\ \big\downarrow & \lrcorner & \big\downarrow \\ \operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}}\big(q\text{-}\widehat{\mathrm{dR}}_{R/A}\big)^\wedge_p & \longrightarrow & \big(q\text{-}\widehat{\mathrm{dR}}_{R/A}\big)^\wedge_p \end{array}$$

stays a pullback after rescaling everything by $\Phi_p(q)$. This is clear since rescaling preserves all limits. This concludes the proof in the case $\alpha = 1$.

Now let $\alpha \geqslant 2$. Using a similar argument as in 7.7, we see that the associated graded of $((\operatorname{fil}^\star_{\mathrm{ev}} \mathrm{THH}_\blacksquare(\mathrm{ku}\widehat{R}_p/\mathrm{ku}\widehat{A}_p)^{\Phi C_{p^{\alpha-1}}})^{tC_{p,\mathrm{ev}}})^{h(\mathbb{T}/C_{p^\alpha})_{\mathrm{ev}}}$ is given by $\big(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}\big)^\wedge_{(p,\mathcal{N})}$ in every degree. Thus, the associated graded of the pullback diagram from the beginning of the proof will take the form

$$\begin{array}{ccc} \Sigma^{-2*}\operatorname{gr}^*_{\mathrm{ev},S^1}\mathrm{TC}^{-(p^\alpha)}_\blacksquare\big(\mathrm{ku}\widehat{R}_p/\mathrm{ku}\widehat{A}_p\big) & \longrightarrow & \operatorname{fil}^\star_{\mathcal{N}}\big(q\text{-}\mathrm{dR}^{(p^\alpha)}_{R/A}\big)^\wedge_{(p,\mathcal{N})} \\ \big\downarrow & \lrcorner & \big\downarrow \phi_{p/A[q]} \\ \operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_{p^{\alpha-1}}}\big(q\text{-}\widehat{\mathrm{dR}}^{(p^{\alpha-1})}_{R/A}\big)^\wedge_p & \longrightarrow & \big(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}\big)^\wedge_{(p,\mathcal{N})} \end{array}$$

Again, changing the filtration parameter from $t_{p^{\alpha-1}}$ to t_{p^α} introduces a "rescaling" by $\Phi_{p^\alpha}(q)$ in the bottom row. The resulting diagram looks almost like the completion of the defining pullback of $\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_{p^\alpha}}\big(q\text{-}\mathrm{dR}^{(p^\alpha)}_{R/A}\big)^\wedge_p$, except that again the rescaled filtrations are already complete. To fix this and to finish the proof, it will be enough to check that the diagram

$$\begin{array}{ccccc} \operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_{p^{\alpha-1}}}\big(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}\big)^\wedge_p & \longrightarrow & \operatorname{fil}^\star_{\mathcal{N}}\big(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}\big)^\wedge_p & \longrightarrow & \big(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}\big)^\wedge_p \\ \big\downarrow & \lrcorner & \big\downarrow & \lrcorner & \big\downarrow \\ \operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_{p^{\alpha-1}}}\big(q\text{-}\widehat{\mathrm{dR}}^{(p^{\alpha-1})}_{R/A}\big)^\wedge_p & \longrightarrow & \operatorname{fil}^\star_{\mathcal{N}}\big(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}\big)^\wedge_{(p,\mathcal{N})} & \longrightarrow & \big(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}\big)^\wedge_{(p,\mathcal{N})} \end{array}$$

consists of two pullback squares (so that we still get a pullback after rescaling the outer rectangle by $\Phi_{p^\alpha}(q)$). Now the right square is a pullback since every filtration is the pullback of its completion. To see that the left square is a pullback, we observe that in the definition of $\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_{p^{\alpha-1}}}\big(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}\big)^\wedge_p$ the only occuring non-complete filtration is $\operatorname{fil}^\star_{\mathcal{N}}\big(q\text{-}\mathrm{dR}^{(p^{\alpha-1})}_{R/A}\big)^\wedge_p$, as the other two filtrations are rescaled by $\Phi_{p^{\alpha-1}}(q)$ and thus automatically complete. □

Proof sketch of Theorem 8.51. We analyse the factors of the last fracture square from 8.52 in the case where N is divisible by m and check that they match up with those from 3.38.

(*a*) Once we invert N, all filtered Tate constructions $(-)^{tC_{p,\mathrm{ev}}}$ for $p \mid m$ will vanish, using that the non-filtered Tate construction $(-)^{tC_p}$ vanishes on $\mathbb{S}[1/p]$-modules plus an argument as in 7.7. So the equaliser from 8.46 will just be a product. Together with Lemma 8.53, we conclude that $\operatorname{fil}^\star_{\mathrm{ev},S^1}\mathrm{TC}^{-(m)}(\mathrm{ku}_R[1/N]/\mathrm{ku}_A[1/N])$ is the product

$$\prod_{d\mid m}\Big(\operatorname{fil}^\star_{\mathrm{ev}}\mathrm{TC}^-\big(\mathrm{ku}_R\big[\tfrac{1}{N}\big]/\mathrm{ku}_A\big[\tfrac{1}{N}\big]\big)\otimes_{\mathbb{S}[q],\psi^d}\mathbb{S}[q]\Big)^\wedge_{\Phi_d(q)}$$

and therefore $\Sigma^{-2*}\operatorname{gr}^*_{\mathrm{ev},S^1}\mathrm{TC}^{-(m)}(\mathrm{ku}_R[1/N]/\mathrm{ku}_A[1/N])$ is the completion of the filtered $\mathbb{Z}[\beta,q]\llbracket t_m\rrbracket/(\beta t_m-(q^m-1))$-module

$$\prod_{d\mid m}\Big(\operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A[q],\psi^d}A\big[\tfrac{1}{N},q\big]\Big)^\wedge_{\Phi_d(q)}\,.$$

(*b*) A similar analysis as in (*a*) shows that $\Sigma^{-2*}\operatorname{gr}^*_{\mathrm{ev},S^1}\mathrm{TC}^{-(m)}(\mathrm{ku}\widehat{R}_p[1/p]/\mathrm{ku}\widehat{A}_p[1/p])$ is the completion of the filtered $\mathbb{Z}[\beta,q]\llbracket t_m\rrbracket/(\beta t_m-(q^m-1))$-module

$$\prod_{d\mid m}\Big(\operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\mathrm{dR}_{R/A}\otimes^{\mathrm{L}}_{A[q],\psi^d}A[q]\Big)^\wedge_p\big[\tfrac{1}{p}\big]^\wedge_{\Phi_d(q)}\,.$$

(*c*) After p-completion for any $p \mid N$, we observe as in (*a*) that all filtered Tate constructions $(-)^{tC_{\ell,ev}}$ vanish for $\ell \neq p$. Simplifying the equaliser accordingly and using Lemma 8.55, we find that $\operatorname{fil}^\star_{\mathrm{ev},S^1} \mathrm{TC}^{-(m)}(\mathrm{ku}\widehat{_{R_p}}/\mathrm{ku}\widehat{_{A_p}})$ is given by the product

$$\prod_{d_p \mid m_p} \left(\operatorname{fil}^\star_{\mathrm{ev},S^1} \mathrm{TC}^{-(p^\alpha)}\left(\mathrm{ku}\widehat{_{R_p^{(d_p)}}}/\mathrm{ku}\widehat{_{A_p}}\right) \otimes_{\mathbb{S}[q],\psi^{d_p}} \mathbb{S}[q]\right)^\wedge_{\Phi_{d_p}(q)},$$

where we put $m = p^\alpha m_p$ with m_p coprime to p. Using Lemma 8.56, we deduce that the sheared associated graded $\Sigma^{-2*} \operatorname{gr}^*_{\mathrm{ev},S^1} \mathrm{TC}^{-(m)}(\mathrm{ku}\widehat{_{R_p}}/\mathrm{ku}\widehat{_{A_p}})$ is the completion of the filtered $\mathbb{Z}[\beta, q][\![t_m]\!]/(\beta t_m - (q^m - 1))$-module

$$\prod_{d_p \mid m_p} \left(\operatorname{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_{p^\alpha}} \left(q\text{-}\mathrm{dR}^{(p^\alpha)}_{R/A}\right)^\wedge_p \otimes^{\mathrm{L}}_{A[q],\psi^{d_p}} A[q]\right)^\wedge_{(p,\Phi_{d_p}(q))}$$

Evidently, (*a*)–(*c*) above match up with 3.38(*a*)–(*c*). It's straightforward to check (using Lemma 7.14) that also the maps between them match up. This proves what we want. □

As a consequence we obtain a "TR-style" description of derived q-de Rham–Witt complexes. The question whether such a description exists was first raised by Johannes Anschütz in the author's Master's thesis defense.

8.58. Corollary. — *The associated graded of the even filtration* $\operatorname{fil}^\star_{\mathrm{ev},C_m} \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^{C_m}$ *is given by*

$$\Sigma^{-2*} \operatorname{gr}^*_{\mathrm{ev},C_m} \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^{C_m} \simeq q\text{-}\mathbb{W}_m\mathrm{dR}^*_{R/A}.$$

Proof sketch. This follows from Theorem 8.51 and Proposition 3.39. □

Finally, let us explain how to recover the Habiro-Hodge complex $q\text{-}\mathcal{H}\mathrm{dg}_{R/A}$.

8.59. Cyclonic even filtrations on THH($\mathrm{KU}_R/\mathrm{KU}_A$). — Put $\mathrm{KU}_A \coloneqq \mathrm{KU} \otimes \mathbb{S}_A$ and $\mathrm{KU}_R \coloneqq \mathrm{KU} \otimes \mathbb{S}_R$. We equip KU with its cyclonic structure from 8.32 and

$$\mathrm{THH}(\mathrm{KU}_R/\mathrm{KU}_A) \simeq \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A) \otimes_{\mathrm{ku}} \mathrm{KU}$$

with the base change of the cyclonic structure from 8.43. We also let

$$\operatorname{fil}^\star_{\mathrm{ev},C_m} \mathrm{THH}(\mathrm{KU}_R/\mathrm{KU}_A)^{C_m} \coloneqq \operatorname{fil}^\star_{\mathrm{ev},C_m} \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^{C_m} \otimes_{\mathrm{ku}^{C_m}_{\mathrm{ev}}} \mathrm{KU}^{C_m}_{\mathrm{ev}},$$

where $\mathrm{ku}^{C_m}_{\mathrm{ev}} \coloneqq \tau_{\geqslant 2\star}(\mathrm{ku}^{C_m})$ and $\mathrm{KU}^{C_m}_{\mathrm{ev}} \coloneqq \tau_{\geqslant 2\star}(\mathrm{KU}^{C_m})$. Observe that $- \otimes_{\mathrm{ku}^{C_m}_{\mathrm{ev}}} \mathrm{KU}^{C_m}_{\mathrm{ev}}$ can be regarded as a localisation at the element β sitting in homotopical degree 2 and filtration degree 1. Finally, we construct

$$\operatorname{fil}^\star_{\mathrm{ev},S^1} \mathrm{TC}^{-(m)}(\mathrm{KU}_R/\mathrm{KU}_A) \coloneqq \left(\operatorname{fil}^\star_{\mathrm{ev},C_m} \mathrm{THH}(\mathrm{KU}_R/\mathrm{KU}_A)^{C_m}\right)^{h(\mathbb{T}/C_m)_{\mathrm{ev}}}.$$

8.60. Remark. — If we believe that our construction of $\operatorname{fil}^\star_{\mathrm{ev},C_m} \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^{C_m}$ is the "correct" filtration to put on $\mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^{C_m}$ (see the discussion in 8.46), then the construction from 8.59 provides the correct even filtration for $\mathrm{THH}(\mathrm{KU}_R/\mathrm{KU}_A)^{C_m}$, since taking even filtrations should commute with filtered colimits.

8.61. Lemma. — *For all $m \in \mathbb{N}$, the filtered objects*

$$\operatorname{fil}^{\star}_{\mathrm{ev},C_m} \mathrm{THH}(\mathrm{KU}_R/\mathrm{KU}_A)^{C_m} \quad \textit{and} \quad \operatorname{fil}^{\star}_{\mathrm{ev},S^1} \mathrm{TC}^{-(m)}(\mathrm{KU}_R/\mathrm{KU}_A)$$

are complete and exhaustive filtrations on $\mathrm{THH}(\mathrm{KU}_R/\mathrm{KU}_A)^{C_m}$ *and* $\mathrm{TC}^{-(m)}(\mathrm{KU}_R/\mathrm{KU}_A)$, *respectively.*

Proof sketch. Observe that inverting the element β in homotopical degree 2 and filtration degree 1 preserves the assumptions of [AR24, Lemma 2.75(iv)]. We can thus use the same argument as in Lemma 8.49. ☐

8.62. Remark. — In the general setup of 8.46, we have canonical maps

$$\operatorname{fil}^{\star}_{\mathrm{ev}\,/T,C_m} M^{C_m} \longrightarrow \left(\operatorname{fil}^{\star}_{\mathrm{ev}\,/T,C_n} M^{C_n}\right)^{hC_{m/n,\mathrm{ev}}}$$

whenever $n \mid m$. Indeed, upon applying $(-)^{hC_{m/n,\mathrm{ev}}}$, the equaliser diagram for $\operatorname{fil}^{\star}_{\mathrm{ev}\,/T,C_n} M^{C_n}$ becomes a subdiagram of that for $\operatorname{fil}^{\star}_{\mathrm{ev}\,/T,C_m} M^{C_m}$. As a consequence, we get canonical maps

$$\operatorname{fil}^{\star}_{\mathrm{ev},S^1} \mathrm{TC}^{-(m)}(\mathrm{KU}_R/\mathrm{KU}_A) \longrightarrow \operatorname{fil}^{\star}_{\mathrm{ev},S^1} \mathrm{TC}^{-(n)}(\mathrm{KU}_R/\mathrm{KU}_A)\,.$$

and similarly for ku. It's possible to construct these maps coherently, that is, assemble them into functor $\mathbb{N} \to \mathrm{SynSp}$. Since we're only interested in the limit, the individual maps will suffice, as we can always restrict to the sequential subposet $\{n!\}_{n\geqslant 1} \subseteq \mathbb{N}$.

8.63. Theorem. — *Let $m \in \mathbb{N}$. Suppose A and R satisfy the assumptions from 7.19 along with the addenda $2 \in R^\times$ and 8.43(A_2). Then there exists a canonical $\mathbb{Z}[\beta^{\pm 1}]$-linear equivalence*

$$q\text{-}\mathcal{H}\mathrm{dg}_{R/A}[\beta^{\pm 1}] \xrightarrow{\simeq} \operatorname{gr}^*\Bigl(\lim_{m\in\mathbb{N}} \operatorname{fil}^{\star}_{\mathrm{ev},S^1} \mathrm{TC}^{-(m)}(\mathrm{KU}_R/\mathrm{KU}_A)\Bigr)\,.$$

Proof. Let us first verify that

$$\Bigl(\bigl(\operatorname{fil}^{\star}_{\mathrm{ev},S^1} \mathrm{TC}^{-(m)}(\mathrm{ku}_R/\mathrm{ku}_A)\bigr)[\beta^{-1}]\Bigr)^\wedge_{t_m} \xrightarrow{\simeq} \operatorname{fil}^{\star}_{\mathrm{ev},S^1} \mathrm{TC}^{-(m)}(\mathrm{KU}_R/\mathrm{KU}_A)\,,$$

where β sits in homotopical degree 2 and filtration degree 1, whereas t_m sits in homotopical degree -2 and filtration degree -1 of $\tau_{\geqslant 2\star}\bigl((\mathrm{ku}^{C_m})^{h(S^1/C_m)}\bigr)$. Indeed, we can identify the t_m-adic filtration on $(-)^{h(\mathbb{T}/C_m)_{\mathrm{ev}}}$ with the filtration coming from the *CW filtration* on $\mathrm{ku}[S^1/C_m]_{\mathrm{ev}}$ in the sense of [AR24, Construction 2.52]. This shows that both sides above are t_m-complete, so the map exists, and after reduction modulo t_m we recover the defining equivalence $\operatorname{fil}^{\star}_{\mathrm{ev},C_m} \mathrm{THH}(\mathrm{ku}_R/\mathrm{ku}_A)^{C_m}[\beta^{-1}] \simeq \operatorname{fil}^{\star}_{\mathrm{ev},C_m} \mathrm{THH}(\mathrm{KU}_R/\mathrm{KU}_A)^{C_m}$, so also the map above is an equivalence.

As a consequence of this observation and Theorem 8.51, we obtain that the filtration $\operatorname{fil}^{\star}_{\mathrm{ev},S^1} \mathrm{TC}^{-(m)}(\mathrm{KU}_R/\mathrm{KU}_A)$ is periodic and each graded piece is equivalent to

$$\operatorname{gr}^0_{\mathrm{ev},S^1} \mathrm{TC}^{-(m)}(\mathrm{KU}_R/\mathrm{KU}_A) \simeq q\text{-}\widehat{\mathrm{dR}}^{(m)}_{R/A}\left[\frac{\operatorname{fil}^i_{q\text{-}\mathcal{H}\mathrm{dg}_m}}{(q^m-1)^i}\;\middle|\; i \geqslant 1\right]^\wedge_{(q^m-1)}\,,$$

where we use the notation from 3.42. Also observe that since we complete at (q^m-1) anyway, it doesn't matter whether we use $q\text{-}\widehat{\mathrm{dR}}^{(m)}_{R/A}$ or $q\text{-}\mathrm{dR}^{(m)}_{R/A}$ in this formula, so the right-hand side agrees with $q\text{-}\mathcal{H}\mathrm{dg}_{R/A,m}$.

By tracing through the constructions it's straightforward to check that the associated graded of $\mathrm{fil}^\star_{\mathrm{ev},S^1}\mathrm{TC}^{-(m)}(\mathrm{ku}_R/\mathrm{ku}_A)\rightarrow\mathrm{fil}^\star_{\mathrm{ev},S^1}\mathrm{TC}^{-(n)}(\mathrm{ku}_R/\mathrm{ku}_A)$ from Remark 8.62 is the completion of the transition map

$$\mathrm{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_m}\, q\text{-}\mathrm{dR}^{(m)}_{R/A}\longrightarrow\mathrm{fil}^\star_{q\text{-}\mathcal{H}\mathrm{dg}_n}\, q\text{-}\mathrm{dR}^{(n)}_{R/A}$$

from 3.41. Thus, the associated graded of $\lim_{m\in\mathbb{N}}\mathrm{fil}^\star_{\mathrm{ev},S^1}\mathrm{TC}^{-(m)}(\mathrm{KU}_R/\mathrm{KU}_A)$ is indeed given by $\lim_{m\in\mathbb{N}} q\text{-}\mathcal{H}\mathrm{dg}_{R/A,m}[\beta^{\pm1}]\simeq q\text{-}\mathcal{H}\mathrm{dg}_{R/A}[\beta^{\pm1}]$. □

§9. Examples

§9.1. Examples of spherical lifts

The assumptions of our main results—Theorems 7.27, 8.51, and 8.63—seem quite restrictive at first. In this subsection we'll show that there are nevertheless many nontrivial examples to which the theorems apply. We'll start with examples of Λ-rings A that satisfy the assumptions from 7.19(A).

9.1. Example. — If $A = \mathbb{Z}[x_i \mid i \in I]$ is a polynomial ring equipped with the *toric Λ-structure* in which $\psi^m(x_i) = x_i^m$ for all m, then the assumptions from 6.1 are satisfied. Indeed, we can choose $\mathbb{S}_A \simeq \mathbb{S}[x_i \mid i \in I]$ to be flat spherical polynomial ring. As explained in [BMS19, Proposition 11.3], this is a cyclotomic basis and for every prime p the Tate-valued Frobenius satisfies $\phi_{tC_p}(x_i) = x_i^p = \psi^p(x_i)$.

9.2. Example. — If A is a perfect Λ-ring, then the assumptions from 6.1 are also satisfied: For every prime p, the spherical Witt vector ring $\mathbb{S}_{\mathrm{W}(A/p)}$ from [L-Ell$_{\mathrm{II}}$, Example 5.2.7] yields a p-complete lift of A. These can be glued with $A \otimes \mathbb{Q}$ in a canonical way to yield $\mathbb{S}_A$. To construct the structure of a cyclotomic base and check 6.1(tC_p) for all primes p, we must equip the Tate-valued Frobenius

$$\phi_{tC_p} \colon \mathbb{S}_A \longrightarrow \mathbb{S}_A^{tC_p}$$

with an S^1-equivariant structure, where $\mathbb{S}_A$ receives the trivial action and $\mathbb{S}_A^{tC_p}$ the residual $S^1/C_p \simeq S^1$-action. Equivalently, we must factor ϕ_{tC_p} through an $\mathbb{E}_\infty$-map

$$\mathbb{S}_A \longrightarrow \left(\mathbb{S}_A^{tC_p}\right)^{h(S^1/C_p)} \simeq \left(\mathbb{S}_A^{tS^1}\right)_p^\wedge .$$

By the universal property of spherical Witt vectors, for all $m \in \mathbb{N}$ and all primes p the Adams operation $\psi^m \colon A \to A$ lifts to an $\mathbb{E}_\infty$-map $\psi^m \colon \mathbb{S}_{\mathrm{W}(A/p)} \to \mathbb{S}_{\mathrm{W}(A/p)}$. These can be glued with the rationalisation to obtain an $\mathbb{E}_\infty$-map $\psi^m \colon \mathbb{S}_A \to \mathbb{S}_A$. From the trivial S^1-action we also obtain a map $\mathbb{S}_A \to \mathbb{S}_A^{hS^1}$ that splits the usual limit projection. The desired factorisation of ϕ_{tC_p} is then given by

$$\mathbb{S}_A \xrightarrow{\psi^p} \mathbb{S}_A \longrightarrow \mathbb{S}_A^{hS^1} \longrightarrow \left(\mathbb{S}_A^{tS^1}\right)_p^\wedge \longrightarrow \mathbb{S}_A^{tC_p} .$$

To see that the composition is really ϕ_{tC_p}, we use the universal property of spherical Witt vectors again: It's enough to check that the map on $\pi_0(-)/p$ is the Frobenius on A/p, which is clear from the construction.

9.3. Example. — We can also combine Examples 9.1 and 9.2 and consider A to be a polynomial ring over a perfect Λ-ring, or even a localisation of such a ring, as long as it still carries a Λ-structure.

The examples where A is a polynomial ring (over a perfect Λ-ring) are the most relevant for us, since they are expected to show up in the connection with the work of Garoufalidis–Scholze–Wheeler–Zagier ([GSWZ24], but the relative case was only discussed in [Sch24b]). Nevertheless, there are examples that are not of this form, such as the following.

9.4. Example. — Recall that the polynomial ring $\mathbb{Z}[y]$ admits one more Λ-structure besides the toric one ([Cla94]; see also [Man16]). This other Λ-structure is called the *Chebyshev Λ-structure*, since $\psi^m(y)$ is given by the Chebyshev polynomial $T_m(y)$. If $\mathbb{Z}[x^{\pm 1}]$ is equipped with the toric Λ-structure, then the Chebyshev Λ-structure on $\mathbb{Z}[y]$ can be identified with the fixed points of the C_2-action on $\mathbb{Z}[x^{\pm 1}]$ that sends $x \mapsto x^{-1}$. Under this identification we have $y = x + x^{-1}$.

We'll show that $A = \mathbb{Z}[\frac{1}{2}, y]$ still satisfies 7.19(A). Indeed, as soon as 2 is invertible, the homotopy fixed points $\mathbb{S}[\frac{1}{2}, y] \coloneqq \mathbb{S}[\frac{1}{2}, x^{\pm 1}]^{hC_2}$ define the desired $\mathbb{E}_\infty$-lift. To verify that 6.1(^{tC_p}) is satisfied for all primes p, there's nothing to do for $p = 2$, as then $\mathbb{S}[\frac{1}{2}, y]^{tC_2} \simeq 0$. For $p \neq 2$, $(-)^{tC_p}$ and $(-)^{hC_2}$ commute (see [KN17, Lemma 9.3] for example) and so 6.1(^{tC_p}) follows from the corresponding assertions for $\mathbb{S}[\frac{1}{2}, x^{\pm 1}]$ by applying $(-)^{hC_2}$. The same argument shows that the addendum from 8.43(A_2) is satisfied as well.

9.5. Remark. — Recall that a cyclotomic spectrum X has *Frobenius lifts* in the sense of [KN17, Definition 8.2] if for each prime p the cyclotomic Frobenius $\phi_p\colon X \to X^{tC_p}$ factors S^1-equivariantly through a map $\psi_p\colon X \to X^{hC_p}$ such that the ψ_p commute for different primes.

In each of Examples 9.1–9.4 it's clear that $\mathbb{S}_A$ admits Frobenius lifts as a cyclotomic $\mathbb{E}_\infty$-algebra. Using Lemma 8.44, this implies that Assumption 8.43(A_2) is satisfied. Indeed, since the S^1-action is trivial, we may equivalently regard $\psi_p\colon \mathbb{S}_A \to \mathbb{S}_A^{hC_p}$ as an S^1-equivariant $\mathbb{E}_\infty$-algebra map $\psi^p\colon \mathbb{S}_A \to \mathbb{S}_A$. The commutativity datum simply provides homotopies $\psi^p \circ \psi^\ell \simeq \psi^\ell \circ \psi^p$ for all $p \neq \ell$. Inductively defining $\psi^1 \coloneqq \mathrm{id}$, $\psi^{pm} \coloneqq \psi^m \circ \psi^p$, we obtain the necessary commutative diagrams

$$\begin{array}{ccc} \mathbb{S}_A & \xrightarrow{\psi^{pm}} & \mathbb{S}_A \\ {\scriptstyle\phi_p}\downarrow & & \downarrow \\ \mathbb{S}_A^{tC_p} & \xrightarrow{(\psi^m)^{tC_p}} & \mathbb{S}_A^{tC_p} \end{array}$$

and thus the desired map $\mathbb{S}_A^{\mathrm{cyct}} \to \mathbb{S}_A^{\mathrm{triv}}$.

9.6. Non-example. — In the case where $A = \mathbb{Z}\{x\}_\Lambda$ is a free Λ-ring, it's not known whether a spherical lift $\mathbb{S}_A$ as in 6.1 exist.[(9.1)]

Let us now give several examples of A-algebras R that satisfy the assumptions of 7.19(R).

9.7. Example. — Suppose that S is a smooth A-algebra equipped with an étale map $\square\colon A[x_1, \dots, x_n] \to S$. By [L-HA, Theorem 7.5.4.3], $\square$ lifts uniquely to an étale map $\mathbb{S}_A[x_1, \dots, x_n] \to \mathbb{S}_{S,\square}$ of $\mathbb{E}_\infty$-ring spectra. Then $R = S$ satisfies the assumptions of 7.19(R), choosing 6.2($\mathbb{E}_2$) for every prime p. We'll continue to study this example in §9.2 below.

9.8. Example. — In the setting from Example 9.7, suppose that $(y_1, \dots, y_r)$ is a regular sequence in S. By Burklund's theorem about multiplicative structures on quotients [Bur22, Theorem 1.5] (see also the argument in Remark 4.23), the spectrum

$$\mathbb{S}_R \coloneqq \mathbb{S}_{S,\square}/\big(y_1^{\alpha_1}, \dots, y_r^{\alpha_r}\big)$$

admits an $\mathbb{E}_2$-structure in $\mathbb{S}_A$-modules (even in $\mathbb{S}_{S,\square}$-modules) if all α_i are even and $\geqslant 6$. If 2 is invertible in S, it's already enough to have all $\alpha_i \geqslant 3$, with no evenness assumption. In either case, we see that $R = S/(y_1^{\alpha_1}, \dots, y_r^{\alpha_r})$ satisfies the assumptions of 7.19(R), choosing 6.2($\mathbb{E}_2$) for every prime p.

(9.1) In fact, it is a conjecture of Thomas Nikolaus that such a spherical lift *doesn't* exist.

If we only assume that all α_i are even and $\geqslant 4$, or 2 is invertible in S and all $\alpha_i \geqslant 2$, then $\mathbb{S}_R$ still admits an $\mathbb{E}_1$-structure in $\mathbb{S}_{S,\square}$-modules. Provided that R is p-torsion free, condition 6.2($\mathbb{E}_1$) is satisfied for every prime p. Indeed, if we put

$$\widehat{R}_{p,\infty} \coloneqq \left(\widehat{A}_p\langle x_1^{1/p^\infty}, \dots, x_n^{1/p^\infty}\rangle \otimes_{\widehat{A}_p\langle x_1,\dots,x_n\rangle} \widehat{R}_p\right)_p^\wedge .$$

then the p-completed Čech nerve of $\widehat{R}_p \to \widehat{R}_{p,\infty}$ admits a spherical $\mathbb{E}_1$-lift, given by the p-completed base change along $\mathbb{S}_A[x_1,\dots,x_n] \to \mathbb{S}_R$ of the Čech nerve of the $\mathbb{E}_\infty$-algebra map $\mathbb{S}_A[x_1,\dots,x_n] \to \mathbb{S}_A[x_1^{1/p^\infty},\dots,x_n^{1/p^\infty}]$.

9.9. Example. — The easiest way for 6.2($\mathbb{E}_1$) to be satisfied is the case where R/p is already relatively semiperfect over A, so that we can take the trivial descent diagram for the identity on $\widehat{R}_p$. Then the only condition is for $\widehat{R}_p$ to admit an $\mathbb{E}_1$-lift $\mathbb{S}_{\widehat{R}_p}$ in $\mathbb{S}_A$-modules.

Thanks to Burklund's result again, it's easy to write down rings for which this is satisfied for all primes p. Here's one possible construction: Let B be a relatively perfect Λ-A-algebra such that $A \to B$ is quasi-lci.[(9.2)] For example, we could take $B = A[x^{1/n} \mid n \geqslant 1]$ with the toric Λ-structure or $B = A \otimes_{\mathbb{Z}} \mathbb{Z}\{x\}_{\Lambda,\mathrm{perf}}$, the free Λ-A-algebra on a perfect generator. Let B' be an étale B-algebra and let $(y_1,\dots,y_r)$ be a regular sequence in B'. Then $R \cong B'/(y_1^{\alpha_1},\dots,y_r^{\alpha_r})$ satisfies 6.2($\mathbb{E}_1$) if all α_i are even and $\geqslant 4$. If 2 is invertible in R, it's already enough to have all $\alpha_i \geqslant 2$ with no evenness assumption.

Indeed, since each p-completions $\widehat{B}'_p$ is all p-completely formally étale over A, it lifts uniquely to a p-complete connective $\mathbb{E}_\infty$-$\mathbb{S}_A$-algebra $\mathbb{S}_{\widehat{B}'_p}$. Our assumptions on the α_i ensure that [Bur22, Theorem 1.5] applies, so that

$$\mathbb{S}_{\widehat{R}_p} \coloneqq \mathbb{S}_{\widehat{B}'_p}/(y_1^{\alpha_1},\dots,y_r^{\alpha_r})$$

admits an $\mathbb{E}_1$-structure in $\mathbb{S}_A$-modules (even in $\mathbb{S}_{\widehat{R}_p}$-modules), as desired.

§9.2. The case of a framed smooth algebra

In the situation of Example 9.7, the q-deformation of the Hodge filtration that we see has a very nice explicit description. This result is due to Arpon Raksit; in fact, his result is what motivated our investigation. To formulate the result, recall that in the situation at hand, the (underived) q-de Rham complex $q\text{-}\Omega_{S/A}$ can be represented by an explicit complex

$$q\text{-}\Omega^*_{S/A,\square} = \left(S[\![q-1]\!] \xrightarrow{q\text{-}\nabla} \Omega^1_{S/A}[\![q-1]\!] \xrightarrow{q\text{-}\nabla} \cdots \xrightarrow{q\text{-}\nabla} \Omega^n_{S/A}[\![q-1]\!]\right).$$

9.10. Theorem (Raksit, unpublished)**.** — *Let $(S,\square)$ be a framed smooth A-algebra as in Example 9.7 and put $\mathrm{ku}_{S,\square} \coloneqq \mathrm{ku} \otimes \mathbb{S}_{S,\square}$. For all integers i we let $\mathrm{fil}^i_{q\text{-Hdg},\square}\, q\text{-}\Omega^*_{S/A,\square}$ denote the subcomplex*

$$\left((q-1)^i S[\![q-1]\!] \to (q-1)^{i-1}\Omega^1_{S/A}[\![q-1]\!] \to \cdots \to \Omega^i_{S/A}[\![q-1]\!] \to \cdots \to \Omega^n_{S/A}[\![q-1]\!]\right).$$

of the coordinate-dependent q-de Rham complex $q\text{-}\Omega^*_{S/A,\square}$ (which we regard as sitting in homotopical degrees $[-n,0]$). Then

$$\Sigma^{-2*}\,\mathrm{gr}^i_{\mathrm{ev}}\,\mathrm{TC}^-(\mathrm{ku}_{S,\square}/\mathrm{ku}_A) \simeq \mathrm{fil}^\star_{q\text{-Hdg},\square}\, q\text{-}\Omega^*_{S/A,\square}\,.$$

(9.2) For every prime p, the relatively perfect map of δ-rings $\widehat{A}_p \to \widehat{B}_p$ will automatically be p-quasi-lci, so $A \to B$ being quasi-lci is a rational condition.

While Raksit's original proof uses geometric arguments, we'll give a more algebraic proof of Theorem 9.10. We first need a general fact about q-divided powers.

9.11. Lemma. — *Fix a prime p. Consider $\mathbb{Z}_p[x,y,q]$, equipped with the toric δ-structure, and let*

$$q\text{-}D \coloneqq \mathbb{Z}_p[x,y,q]\left\{\frac{\phi(x-y)}{[p]_q}\right\}^\wedge_{(p,q-1)}.$$

Then q-D is the $(p,q-1)$-completion of the subalgebra of $\mathbb{Q}_p[x,y]\llbracket q-1\rrbracket$ generated by $\mathbb{Z}_p[x,y,q]$ as well as elements $(q-1)^d\widetilde{\gamma}_q^d(x-y)$ for all $d\geqslant 1$, where we put

$$\widetilde{\gamma}_q^d(x-y) \coloneqq \frac{(x-y)(x-qy)\cdots(x-q^{d-1}y)}{(q;q)_d}.$$

Proof. It will be enough to show that q-D contains $(q-1)^d\widetilde{\gamma}_q^d(x-y)$ for all $d\geqslant 1$, as then the fact that these are generators as well as the claimed description of q-D can be checked modulo $(q-1)$.

First observe that $(p,q-1)$ is a regular sequence in q-D. Indeed, q-$D/(q-1)$, where the quotient is taken in the derived sense as usual, is the PD-envelope of $(x-y)\subseteq\mathbb{Z}_p[x,y]$, which is a p-torsion free ring. It follows that $(p,(q;q)_d)$ is a regular sequence for all $d\geqslant 1$. Indeed, up to factors that are invertible in q-D, the Pochhammer symbol is a product of factors of the form $(1-q^{p^\alpha})$, and $(1-q^{p^\alpha})\equiv(1-q)^{p^\alpha} \mod p$. In particular, each $(q;q)_d$ is a non-zerodivisor in q-D.

If we equip $\mathbb{Z}_p[x,y,q]$ with the toric Λ-structure, then the Adams operations ψ^ℓ for $\ell\neq p$ are δ-ring maps. Using the universal property it is then straightforward to check that the ψ^ℓ extend to q-D, hence q-D carries a Λ-$\mathbb{Z}_p[x,y,q]$-structure extending the given δ-structure. This Λ-structure extends then uniquely to the localisation $q\text{-}D[(q;q)_d^{-1}\mid d\geqslant 1]$. In the localisation, we have

$$\lambda^d\left(\frac{x-y}{q-1}\right) = \widetilde{\gamma}_q^d(x-y);$$

see [Pri19, Lemma 1.3]. So we must show $(q-1)^d\lambda^d\left(\frac{x-y}{q-1}\right)\in q\text{-}D$. To this end, first observe that $(q-1)\psi^d\left(\frac{x-y}{q-1}\right)\in q\text{-}D$ for all $d\geqslant 1$. Indeed, it's enough to check this if $d=p^\alpha$ is a power of p. So we must check that $x^{p^\alpha}-y^{p^\alpha}$ is divisible by $[p^\alpha]_q$ in q-D. Since q-D is $(p,q-1)$-completely flat over $\mathbb{Z}_p\llbracket q-1\rrbracket$ by [BS19, Lemma 16.10] and thus flat on the nose over $\mathbb{Z}[q]$, it will be enough to check that $x^{p^\alpha}-y^{p^\alpha}$ is divisible by each cyclotomic polynomial in the factorisation $[p^\alpha]_q=\Phi_p(q)\Phi_{p^2}(q)\cdots\Phi_{p^\alpha}(q)$. Since $x^{p^i}-y^{p^i}$ divides $x^{p^\alpha}-y^{p^\alpha}$ for $i\leqslant\alpha$, it suffices to show that $x^{p^\alpha}-y^{p^\alpha}$ is divisible by $\Phi_{p^\alpha}(q)$, which follows by applying $\phi^{\alpha-1}$ to $\phi(x-y)/[p]_q$.

Now let us put $\lambda_t(-)\coloneqq\sum_{d\geqslant 0}\lambda^d(-)t^n$ and $\psi_t(-)\coloneqq\sum_{d\geqslant 1}\psi^d(-)t^d$, where t is a formal variable. Our observation above shows that $\psi_{(q-1)t}\left(\frac{x-y}{q-1}\right)$ has coefficients in q-D. From the general Λ-ring formula $\psi_t=-t\frac{\mathrm{d}}{\mathrm{d}t}\log\lambda_{-t}$ we deduce that $\lambda_{(q-1)t}\left(\frac{x-y}{q-1}\right)$ has coefficients in $q\text{-}D[p^{-1}]$. Since $(p,(q;q)_d)$ is a regular sequence in q-D, the we get

$$q\text{-}D\big[p^{-1}\big]\cap q\text{-}D\big[(q;q)_d^{-1}\big]=q\text{-}D,$$

where the intersection is taken in $q\text{-}D[p^{-1},(q;q)_d^{-1}]$ (and on the level of sets—nothing derived is happening). This shows $(q-1)^d\lambda^d\left(\frac{x-y}{q-1}\right)\in q\text{-}D$, as desired. □

9.12. A cosimplicial resolution. — To show Theorem 9.10, we'll compute the even filtration via an explicit resolution. To this end, let us fix the following notation:

(*a*) Let $P \coloneqq A[x_1, \dots, x_n]$ and $\mathbb{S}_P \coloneqq \mathbb{S}_A[x_1, \dots, x_n]$. Let $A \to P^\bullet$ and $\mathbb{S}_A \to \mathbb{S}_{P^\bullet}$ denote the Čech nerves of $A \to P$ and $\mathbb{S}_A \to \mathbb{S}_P$ and put $\mathrm{ku}_{P^\bullet} \coloneqq \mathrm{ku} \otimes \mathbb{S}_{P^\bullet}$.

(*b*) Let $x_i^{(r)} \in P^\bullet \cong P^{\otimes_A (\bullet+1)}$ denote the element $1 \otimes \dots \otimes 1 \otimes x_i \otimes 1 \otimes \dots \otimes 1$ coming from the r^{th} tensor factor for any $1 \leqslant r \leqslant \bullet + 1$.

(*c*) Let $q\text{-}D^\bullet$ denote the $(q-1)$-completion of the sub-algebra of $(S^{\otimes_A (\bullet+1)} \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$ generated by $S^{\otimes_A (\bullet+1)}[\![q-1]\!]$ as well as the elements $(q-1)^d \widetilde{\gamma}_q^d(x_i^{(r)} - x_i^{(s)})$ for all integers $d \geqslant 1$, all tensor factors $1 \leqslant r, s \leqslant \bullet + 1$, and all indices $1 \leqslant i \leqslant n$.

(*d*) Let $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}D^\bullet$ be the descending filtration of ideals generated by $(q-1)$ in filtration degree 1 and the elements $(q-1)^d \widetilde{\gamma}_q^d(x_i^{(r)} - x_i^{(s)})$ in filtration degree d, and let $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{D}^\bullet$ denote the completion of this filtration.

9.13. Lemma. — *With notation as above, there exists a canonical isomorphism of graded* $\mathbb{Z}[\beta][\![t]\!] \cong (q-1)^\star \mathbb{Z}[\![q-1]\!]$*-modules*

$$\pi_{2*} \operatorname{TC}^-(\mathrm{ku}_{S,\square}/\mathrm{ku}_{P^\bullet}) \cong \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{D}^\bullet\,.$$

Proof. We know from Theorem 7.27 that $\pi_{2*} \operatorname{TC}^-(\mathrm{ku}_{S,\square}/\mathrm{ku}_{P^\bullet})$ is the completion of a filtration $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{S/P^\bullet}$. Consider the arithmetic fracture square for the completed filtration:

$$\begin{array}{ccc}
\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{S/P^\bullet} & \longrightarrow & \prod_p \mathrm{fil}^\star_{q\text{-Hdg}} \left(q\text{-}\widehat{\mathrm{dR}}_{S/P^\bullet}\right)_p^\wedge \\
\downarrow & \lrcorner & \downarrow \\
\mathrm{fil}^\star_{(\mathrm{Hdg},q-1)} \left(\mathrm{dR}_{S/P^\bullet} \otimes_{\mathbb{Z}} \mathbb{Q}\right)^\wedge_{\mathrm{Hdg}}[\![q-1]\!] & \longrightarrow & \mathrm{fil}^\star_{(\mathrm{Hdg},q-1)} \left(\prod_p \left(\mathrm{dR}_{S/P^\bullet}\right)_p^\wedge \otimes_{\mathbb{Z}} \mathbb{Q}\right)^\wedge_{\mathrm{Hdg}}[\![q-1]\!]
\end{array}$$

Observe that all corners of this pullback square are static in every filtration degree. Indeed, this can easily be checked modulo $(q-1)$. More precisely, if we identify the $(q-1)$-adic filtration $(q-1)^\star \mathbb{Z}[\![q-1]\!]$ with the graded ring $\mathbb{Z}[\beta][\![t]\!]$ as in 7.26, then everything is β-complete; modulo β, we're then reduced to checking that $\mathrm{fil}^\star_{\mathrm{Hdg}}\, \widehat{\mathrm{dR}}_{S/P^\bullet}$ as well as its p-completions and its Hodge-completed rationalisation are static, which is standard.

We conclude that this diagram is also a pullback of filtered abelian groups, which will make it easy to construct a map $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{D}^\bullet \to \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{S/P^\bullet}$. To this end, let us now analyse the factors of the pullback. Let us start with the p-completed q-de Rham complex $(q\text{-}\mathrm{dR}_{S/P^\bullet})_p^\wedge$. Since δ-structures extend uniquely along p-completely étale maps, the toric δ-A-algebra structure on $\widehat{P}_p^\bullet$ extends uniquely to a δ-A-algebra structure on $(S^{\otimes_A (\bullet+1)})_p^\wedge$. Then $(q\text{-}\mathrm{dR}_{S/P^\bullet})_p^\wedge$ is the q-PD-envelope in the sense of [BS19, Lemma 16.10] of the $(p, q-1)$-completely regular ideal

$$\widehat{J}_p^\bullet \coloneqq \ker\Bigl(\bigl(S^{\otimes_A (\bullet+1)}\bigr)_p^\wedge \twoheadrightarrow \widehat{S}_p\Bigr)$$

Using Lemma 9.11 we see that $(q\text{-}\mathrm{dR}_{S/P^\bullet})_p^\wedge$ contains all the elements $(q-1)^d \widetilde{\gamma}_q^d(x_i^{(r)} - x_i^{(s)})$. By Theorem 7.18, for any fixed d, these elements are contained in $\mathrm{fil}^d_{q\text{-Hdg}}(q\text{-}\mathrm{dR}_{S/P^\bullet})_p^\wedge$.

The rational factor is similar: Since $P \to S$ is étale, the Hodge-completed de Rham complex satisfies $\widehat{\mathrm{dR}}_{S/P^\bullet} \simeq \widehat{\mathrm{dR}}_{S/S^{\otimes_A (\bullet+1)}}$, and so $(\mathrm{dR}_{S/P^\bullet} \otimes_{\mathbb{Z}} \mathbb{Q})^\wedge_{\mathrm{Hdg}}[\![q-1]\!]$ is the $(J_{\mathbb{Q}}^\bullet, q-1)$-adic completion of $(S^{\otimes_A (\bullet+1)} \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$, where

$$J_{\mathbb{Q}}^\bullet \coloneqq \ker\Bigl(\bigl(S^{\otimes_A (\bullet+1)} \otimes_{\mathbb{Z}} \mathbb{Q}\bigr) \twoheadrightarrow (S \otimes_{\mathbb{Z}} \mathbb{Q})\Bigr)\,.$$

Since $x_i^{(r)}-x_i^{(s)}$ is an element of $J_{\mathbb{Q}}^\bullet$, it's also clear that $\tilde{\gamma}_q^d\big(x_i^{(r)}-x_i^{(s)}\big)$ is contained in $\operatorname{fil}^\star_{(\mathrm{Hdg},q-1)}(\mathrm{dR}_{S/P^\bullet}\otimes_{\mathbb{Z}}\mathbb{Q})^\wedge_{\mathrm{Hdg}}[\![q-1]\!]$. Using the pullback above we get a filtered map

$$\operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{D}^\bullet \longrightarrow \operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{S/P^\bullet}\,.$$

Reducing modulo $(q-1)$, or more precisely modulo β, we see that this map is an isomorphism, which finishes the proof. ☐

Proof of Theorem 9.10. The even filtration in question can be computed via the cosimplicial resolution

$$\operatorname{fil}^\star_{\mathrm{ev}}\mathrm{TC}^-(\mathrm{ku}_{S,\square}/\mathrm{ku}_A)\simeq \lim_{\Delta}\tau_{\geqslant 2\star}\mathrm{TC}^-(\mathrm{ku}_{S,\square}/\mathrm{ku}_{P^\bullet})\,.$$

Using Lemma 9.13, it remains to show that the totalisation of the cosimplicial filtered ring $\operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{D}^\bullet$ is quasi-isomorphic to the filtered complex $\operatorname{fil}^\star_{q\text{-}\mathrm{Hdg},\square}\, q\text{-}\Omega^*_{S/A,\square}$. We'll show this using a similar argument as in the proof of [BS19, Theorem 16.22].

To this end, first observe that the q-divided powers from Lemma 9.11 interact with the q-derivatives as follows:

$$q\text{-}\partial_x\big(\tilde{\gamma}_q^d(x-y)\big)=\tilde{\gamma}_q^{d-1}(x-y)\quad\text{and}\quad q\text{-}\partial_y\big(\tilde{\gamma}_q^d(x-y)\big)=-\tilde{\gamma}_q^{d-1}(x-qy)\,.$$

It follows that the q-derivatives extend to $q\text{-}\widehat{D}^\bullet$. We can then consider the filtered cosimplicial filtered complex $\operatorname{fil}^\star q\text{-}M^{\bullet,*}$ given by

$$\Big(\operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{D}^\bullet \xrightarrow{q\text{-}\nabla} \operatorname{fil}^{\star-1}_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{D}^\bullet\otimes_{P^\bullet}\Omega^1_{P^\bullet/A}\xrightarrow{q\text{-}\nabla}\operatorname{fil}^{\star-2}_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{D}^\bullet\otimes_{P^\bullet}\Omega^2_{P^\bullet/A}\xrightarrow{q\text{-}\nabla}\cdots\Big)\,.$$

Then each column $\operatorname{fil}^\star q\text{-}M^{i,*}$ is quasi-isomorphic to $\operatorname{fil}^\star q\text{-}M^{0,*}$; indeed, this can be checked modulo $(q-1)$, and then it follows from the Poincaré lemma for the completed Hodge-filtered de Rham complex. On the other hand the rows $\operatorname{fil}^\star q\text{-}M^{\bullet,j}$ for $j>0$ are acyclic; this can be seen e.g. by [Stacks, Tag 07L7] applied to the cosimplicial filtered ring $\operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{D}^\bullet$. It follows formally that the 0^{th} column $\operatorname{fil}^\star q\text{-}M^{0,*}$ is quasi-isomorphic to the totalisation of the 0^{th} row $\operatorname{fil}^\star q\text{-}M^{\bullet,0}$, which is exactly what we wanted to show. ☐

9.14. Remark. — As a consequence of Theorem 9.10, the filtered complex $\operatorname{fil}^i_{q\text{-}\mathrm{Hdg},\square}\, q\text{-}\Omega^*_{S/A,\square}$ can be promoted to a filtered $\mathbb{E}_\infty$-algebra over the filtered ring $(q-1)^\star A[\![q-1]\!]$. In fact, we even get the structure of a filtered derived commutative algebra, as we desired in 3.51.

§9.3. The Habiro ring of a number field, homotopically

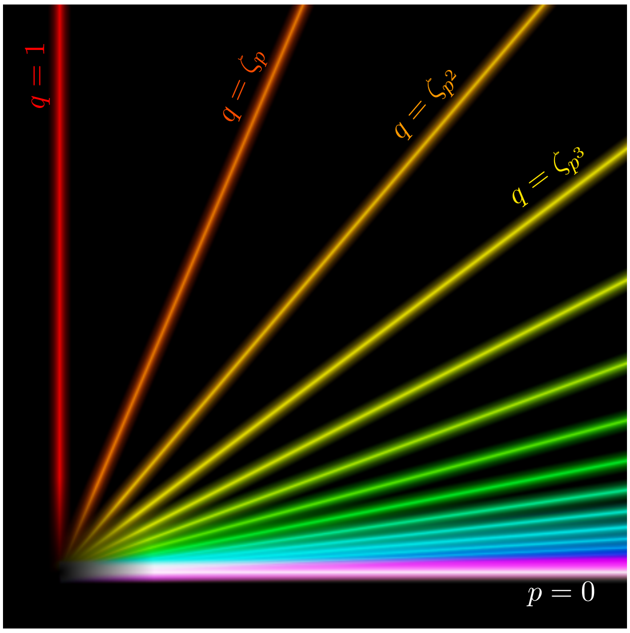
As a final example, let us give a homotopical description of the Habiro ring of a number field from [GSWZ24, Definition 1.1].

9.15. Corollary. — *Let F be a number field and let Δ be divisible by 6 and by the discriminant of F. Let $\mathbb{S}_{\mathcal{O}_F[1/\Delta]}$ denote the unique lift of $\mathcal{O}_F[1/\Delta]$ to an étale extension of $\mathbb{S}$. Then*

$$\mathcal{H}_{\mathcal{O}_F[1/\Delta]}\cong\pi_0\Big(\lim_{m\in\mathbb{N}}\big(\mathrm{THH}(\mathrm{KU}\otimes\mathbb{S}_{\mathcal{O}_F[1/\Delta]}/\mathrm{KU})^{C_m}\big)^{h(S^1/C_m)}\Big)\,.$$

Proof. By Corollary 3.13, $q\text{-}\mathcal{H}\mathrm{dg}_{\mathcal{O}_F[1/\Delta]/\mathbb{Z}}\simeq\mathcal{H}_{\mathcal{O}_F[1/\Delta]}$. In particular, the Habiro–Hodge complex must be static. By Theorem 8.63, the filtration $\lim_{m\in\mathbb{N}}\operatorname{fil}^\star_{\mathrm{ev},S^1}\mathrm{TC}^{-(m)}(\mathrm{KU}\otimes\mathbb{S}_{\mathcal{O}_F[1/\Delta]}/\mathrm{KU})$ must be the double-speed Whitehead filtration $\tau_{\geqslant 2\star}$ and the result follows. ☐

Part III.
q-Hodge complexes and refined THH/TC$^-$



This part is based on joint work with Samuel Meyer [MW24]. We'll discuss the construction of refined localising invariants due to Efimov and Scholze, and we'll explain a recipe how to compute them in certain cases (see Theorem 10.17), using the notion of *killing an idempotent pro-algebra*. We'll then apply this recipe to compute

$$\pi_* \operatorname{TC}^{-,\mathrm{ref}}(k \otimes \mathbb{Q}/k) \quad \text{for } k \in \left\{\mathrm{ku}, \mathrm{KU}, \mathrm{ku}_p^\wedge, \mathrm{KU}_p^\wedge\right\}$$

in Theorem 11.15.

The main input is a complete computation of the homotopy groups $\pi_* \operatorname{TC}^-((\mathrm{ku}\otimes\mathbb{S}/p^\alpha)/\mathrm{ku})$ for $\alpha \geqslant 2$ (the case $p = 2$ needs α even and $\geqslant 4$ instead), where $\mathbb{S}/p^\alpha$ is equipped with a Burklund-style $\mathbb{E}_1$-structure. To perform this computation, we use the relation between q-de Rham cohomology and $\operatorname{TC}^-(-/\mathrm{ku})$, particularly Theorem 7.18 as well as an explicit description of the canonical q-Hodge filtration $\operatorname{fil}^\star_{q\text{-Hdg}}\, q\text{-dR}_{(\mathbb{Z}_p\{x\}_\infty/x^\alpha)/\mathbb{Z}_p}$ from Construction 4.21. This leads to proofs of Theorems 1.40 and 1.41 as well as to an elementary proof of Theorem 4.22(*a*).

Overview of Part III. — This part is organised as follows: In §10, we'll discuss the construction of refined localising invariants and the recipe for computation. In §11, we apply this recipe to describe the homotopy groups $\pi_* \operatorname{TC}^{-,\mathrm{ref}}(\mathrm{ku}\otimes\mathbb{Q})$ and $\pi_* \operatorname{TC}^{-,\mathrm{ref}}(\mathrm{KU}\otimes\mathbb{Q}/\mathrm{KU})$. In §12, we study overconvergent neighbourhoods in analytic stacks and then derive the simpler descriptions of $\pi_* \operatorname{TC}^{-,\mathrm{ref}}(\mathrm{ku}_p^\wedge\otimes\mathbb{Q}/\mathrm{ku}_p^\wedge)$ and $\pi_* \operatorname{TC}^{-,\mathrm{ref}}(\mathrm{KU}_p^\wedge\otimes\mathbb{Q}/\mathrm{KU}_p^\wedge)$ from Theorems 1.40 and 1.41

§10. Refined localising invariants and how to compute them

In this section we'll present Efimov–Scholze's construction of refined localising invariants and we'll explain a method for computing them in the case of certain "open submotives" of "smooth and proper" rigid symmetric monoidal ∞-categories over some base (these notions will be made precise below). As a consequence, we'll get a recipe for computing $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q})$, which we'll carry out (after base change to ku) in §§11–12, but the method would apply just as well to other cases like $\mathrm{THH}^{\mathrm{ref}}(\mathrm{L}_n^f\mathbb{S}_{(p)}/\mathbb{S}_{(p)})$ or $\mathrm{THH}^{\mathrm{ref}}(\mathbb{S}[x])$.

§10.1. Killing (pro-)algebra objects

In this subsection we review the general formalism for passing to the "open complement" of an algebra object. We'll follow [CS24, Lecture 13]. Throughout, let's fix a presentable stable symmetric monoidal ∞-category $\mathcal{C}$.

10.1. Killing algebras. — Let $A \in \mathcal{C}$ be an object equipped maps $\mu\colon A \otimes A \to A$ and $\mathbb{1} \to A$ such that μ is left-unital (or right-unital; this doesn't matter). We let $\mathcal{C}^A \subseteq \mathcal{C}$ be the full sub-∞-category spanned by those $U \in \mathcal{C}$ for which

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(A, U) \simeq 0\,,$$

where $\underline{\mathrm{Hom}}_{\mathcal{C}}$ denotes the internal Hom of $\mathcal{C}$, as usual.

Clearly $\mathcal{C}^A$ is closed under limits in $\mathcal{C}$. If κ is a sufficiently large cardinal such that $S \otimes A$ are κ-compact for all S in a set of generators for $\mathcal{C}$, then $\mathcal{C}^A$ is also closed under κ-filtered colimits. By the ∞-categorical reflection theorem [RS22], it follows that the inclusion $\mathcal{C}^A \to \mathcal{C}$ admits a left adjoint $j^*\colon \mathcal{C} \to \mathcal{C}^A$. Since $\mathcal{C}^A$ is also clearly closed under $\underline{\mathrm{Hom}}_{\mathcal{C}}(Y, -)$ for any $Y \in \mathcal{C}$, we see that

$$j^*(X \otimes Y) \xrightarrow{\simeq} j^*\bigl(j^*(X) \otimes Y\bigr)$$

is an equivalence for all $X, Y \in \mathcal{C}$. By abstract nonsense about symmetric monoidal localisations (see [L-HA, Proposition 2.2.1.9]), it follows that $\mathcal{C}^A$ and $j^*\colon \mathcal{C} \to \mathcal{C}^A$ can be equipped with canonical symmetric monoidal structures and the inclusion $\mathcal{C}^A \to \mathcal{C}$ with a lax symmetric monoidal structure. In particular, $j^*(\mathbb{1})$ is an $\mathbb{E}_\infty$-algebra in $\mathcal{C}$. We'll often say that $j^*(\mathbb{1})$ *is obtained from* $\mathbb{1}$ *by killing* A.

Our first goal is now to give a formula for j^* in certain cases.

10.2. Lemma. — *Let $\mathcal{I} \coloneqq \mathrm{fib}(\mathbb{1} \to A)$. Then for every $X \in \mathcal{C}$ the canonical map*

$$\eta_X\colon X \simeq \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathbb{1}, X) \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{I}, X)$$

becomes an equivalence upon applying $\mathrm{Hom}_{\mathcal{C}}(-, U)$ for any $U \in \mathcal{C}^A$.

Proof. It's enough to show $\mathrm{Hom}_{\mathcal{C}}(\mathrm{fib}(\eta_X), U) \simeq 0$. Note that the fibre $\mathrm{fib}(\eta_X) \simeq \underline{\mathrm{Hom}}_{\mathcal{C}}(A, X)$ is a weak A-module in the sense that there exists a unital multiplication map

$$A \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(A, X) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(A, X)\,.$$

In particular, $\underline{\mathrm{Hom}}_{\mathcal{C}}(A, X)$ is a retract of $A \otimes \underline{\mathrm{Hom}}_{\mathcal{C}}(A, X)$ and so it suffices to show that $\mathrm{Hom}_{\mathcal{C}}(-, U)$ vanishes on the latter. Now $\mathrm{Hom}_{\mathcal{C}}(A \otimes Y, U) \simeq \mathrm{Hom}_{\mathcal{C}}(Y, \underline{\mathrm{Hom}}_{\mathcal{C}}(A, U)) \simeq 0$ holds for all $Y \in \mathcal{C}$, so we conclude. $\square$

10.3. Proposition. — *With notation as above, suppose that one of the following two conditions is satisfied:*

(*a*) *For all $X \in \mathcal{C}$, we recursively put $X_0 \coloneqq X$ and $X_{n+1} \coloneqq \underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{I}, X_n)$. Then the diagram*

$$X \xrightarrow{\eta_X} X_1 \xrightarrow{\eta_{X_1}} X_2 \xrightarrow{\eta_{X_2}} \cdots$$

stabilises at some finite stage (for example, this is satisfied if A is idempotent—then the colimit always stabilises after the first step).

(*b*) *The functor $\underline{\operatorname{Hom}}_{\mathcal{C}}(A, -)$ commutes with sequential colimits (for example, this is satisfied if A is dualisable in $\mathcal{C}$).*

Then $j^*(X)$ is the colimit of the diagram from (a) for all $X \in \mathcal{C}$.

Proof. Let us denote the colimit of the diagram from (*a*) by X_∞. Then Lemma 10.2 ensures that $\operatorname{Hom}_{\mathcal{C}}(X_\infty, U) \to \operatorname{Hom}_{\mathcal{C}}(X, U)$ is an equivalence for all $U \in \mathcal{C}^A$, so we only need to check $X_\infty \in \mathcal{C}^A$; that is, $\underline{\operatorname{Hom}}_{\mathcal{C}}(A, X_\infty) \simeq 0$. Equivalently, $\eta_{X_\infty} \colon X_\infty \to \underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{I}, X_\infty)$ needs to be an equivalence. But either of the two assumptions above makes sure that $\underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{I}, -)$ commutes with the colimit defining X_∞ and so η_{X_∞} is an equivalence by construction. □

We'll now explain a variant of the construction above in a pro-/ind-setting.

10.4. Killing pro-algebras — We keep $\mathcal{C}$ a presentable symmetric monoidal stable ∞-category. The tensor product on $\mathcal{C}$ extends to symmetric monoidal structures on $\operatorname{Pro}(\mathcal{C})$ and $\operatorname{Ind}(\mathcal{C})$.[10.1] Observe that $\underline{\operatorname{Hom}}_{\mathcal{C}}$ can also be extended to a functor

$$\operatorname{Pro}(\mathcal{C})^{\mathrm{op}} \otimes \operatorname{Ind}(\mathcal{C}) \simeq \operatorname{Ind}(\mathcal{C}^{\mathrm{op}}) \otimes \operatorname{Ind}(\mathcal{C}) \xrightarrow{\operatorname{Ind}(\underline{\operatorname{Hom}}_{\mathcal{C}})} \operatorname{Ind}(\mathcal{C})\,,$$

which, by abuse of notation, we still denote $\underline{\operatorname{Hom}}_{\mathcal{C}}$. Explicitly,

$$\underline{\operatorname{Hom}}_{\mathcal{C}}\Bigl(\text{“}\lim_{j \in J}\text{”}\, Y_j, \text{“}\operatorname*{colim}_{k \in K}\text{”}\, Z_k\Bigr) \simeq \text{“}\operatorname*{colim}_{(j,k) \in J^{\mathrm{op}} \times K}\text{”}\, \underline{\operatorname{Hom}}_{\mathcal{C}}(Y_j, Z_k)\,.$$

Let now $A \coloneqq \text{“}\lim_{i \in I}\text{”}\, A_i \in \operatorname{Pro}(\mathcal{C})$ be a pro-object equipped with maps $\mu \colon A \otimes A \to A$ and $\mathbb{1} \to A$ such that μ is left-unital. We let $\operatorname{Ind}(\mathcal{C})^A \subseteq \operatorname{Ind}(\mathcal{C})$ denote the full sub-∞-category spanned by those ind-objects for which

$$\underline{\operatorname{Hom}}_{\mathcal{C}}(A, M) \simeq 0\,.$$

Our goal is again to describe a left adjoint $j^* \colon \operatorname{Ind}(\mathcal{C})^A \to \operatorname{Ind}(\mathcal{C})$ of the inclusion. To this end, let $\mathcal{I} \coloneqq \operatorname{fib}(\mathbb{1} \to A)$ and consider the canonical maps $\eta_X \colon X \simeq \underline{\operatorname{Hom}}_{\mathcal{C}}(\mathbb{1}, X) \to \underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{I}, X)$ for all $X \in \operatorname{Ind}(\mathcal{C})$, as in Lemma 10.2.

10.5. Lemma. — *The inclusion of $\operatorname{Ind}(\mathcal{C})^A$ admits a left adjoint $j^* \colon \operatorname{Ind}(\mathcal{C}) \to \operatorname{Ind}(\mathcal{C})^A$, which can be explicitly described as follows: For $X \in \mathcal{C}$ we recursively put $X_0 \coloneqq X$ and $X_{n+1} \coloneqq \underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{I}, X_n)$. Then*

$$j^*(X) \simeq \operatorname{colim}\Bigl(X \xrightarrow{\eta_X} X_1 \xrightarrow{\eta_{X_1}} X_2 \xrightarrow{\eta_{X_2}} \cdots\Bigr)\,.$$

[10.1] We'll ignore the set-theoretic difficulties that arise with applying $\operatorname{Pro}(-)$ and $\operatorname{Ind}(-)$ to large ∞-categories. In all cases of interest, we can safely replace $\mathcal{C}$ by its κ-compact objects $\mathcal{C}^\kappa \subseteq \mathcal{C}$ for some large enough regular cardinal κ (usually $\kappa = \omega_1$ is enough).

Proof. Since $\underline{\mathrm{Hom}}_{\mathcal{C}}(A,-)\colon \mathrm{Ind}(\mathcal{C})\to\mathrm{Ind}(\mathcal{C})$ preserves filtered colimits, we can argue as in the proof of Proposition 10.3 to see that $j^*(X)\in\mathrm{Ind}(\mathcal{C})^A$. It remains to show that the canonical morphism $X\to j^*(X)$ induces equivalences

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}\bigl(j^*(X),U\bigr)\xrightarrow{\simeq}\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(X,U)$$

for all $U\in\mathrm{Ind}(\mathcal{C})^A$. It will be enough to show the same for η_X, or equivalently, that $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(\underline{\mathrm{Hom}}_{\mathcal{C}}(A,X),U)\simeq 0$. To this end, let $M\in\mathrm{Ind}(\mathcal{C})$ be any object for which the natural transformation $\underline{\mathrm{Hom}}_{\mathcal{C}}(A,-)\Rightarrow\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathbb{1},-)\simeq(-)$ admits a section.(10.2) Via such a section $M\to\underline{\mathrm{Hom}}_{\mathcal{C}}(A,M)$, the identity on $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(M,U)$ factors through

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}\bigl(\underline{\mathrm{Hom}}_{\mathcal{C}}(A,M),\underline{\mathrm{Hom}}_{\mathcal{C}}(A,U)\bigr)\simeq 0\,,$$

and so $\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(M,U)\simeq 0$. Since such a section exists for $M=\underline{\mathrm{Hom}}_{\mathcal{C}}(A,X)$, we conclude. □

10.6. Killing idempotent pro-algebras. — Suppose that A is *idempotent* in $\mathrm{Pro}(\mathcal{C})$, that is, $\mathbb{1}\to A$ induces an equivalence

$$A\simeq\mathbb{1}\otimes A\xrightarrow{\simeq}A\otimes A\,.$$

Let us spell out how $j^*(\mathbb{1})$ looks like in this case: We write $A=\text{“}\lim\text{”}A_i$ and denote by $(-)^\vee:=\underline{\mathrm{Hom}}_{\mathcal{C}}(-,\mathbb{1})$ the predual in $\mathcal{C}$. Then Lemma 10.5 implies that there is a cofibre sequence

$$\text{“}\operatorname*{colim}_{i\in I^{\mathrm{op}}}\text{”}\,A_i^\vee\longrightarrow\mathbb{1}\longrightarrow j^*(\mathbb{1})\,.$$

For idempotent A, we check in Lemma 10.7 below that $j^*\colon\mathrm{Ind}(\mathcal{C})\to\mathrm{Ind}(\mathcal{C})^A$ can be equipped with a symmetric monoidal structure (we don't know if this works in general—the argument from 10.1 doesn't seem to work anymore). As a consequence, $j^*(\mathbb{1})$ will be an $\mathbb{E}_\infty$-algebra in $\mathrm{Ind}(\mathcal{C})$. We'll say that $j^*(\mathbb{1})$ *is obtained from* $\mathbb{1}$ *by killing the idempotent pro-algebra* A.

10.7. Lemma. — *Suppose that A is an idempotent pro-object. Then for all $X,Y\in\mathrm{Ind}(\mathcal{C})$, the canonical morphism*

$$j^*(X\otimes Y)\xrightarrow{\simeq}j^*\bigl(j^*(X)\otimes Y\bigr)$$

is an equivalence. In particular, there's a canonical way to equip $j^*\colon\mathrm{Ind}(\mathcal{C})\to\mathrm{Ind}(\mathcal{C})^A$ with a symmetric monoidal structure.

Proof. By Lemma 10.5 and idempotence of A, $j^*(X)\simeq\mathrm{cofib}(\underline{\mathrm{Hom}}_{\mathcal{C}}(A,X)\to X)$. Thus, to show the first assertion, we may equivalently show that the canonical morphism

$$\underline{\mathrm{Hom}}_{\mathcal{C}}\bigl(A,\underline{\mathrm{Hom}}_{\mathcal{C}}(A,X)\otimes Y\bigr)\xrightarrow{\simeq}\underline{\mathrm{Hom}}_{\mathcal{C}}(A,X)\otimes Y$$

induced by $\mathbb{1}\to A$ is an equivalence. To see this, first observe that this morphism has a left inverse given by

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(A,X)\otimes Y\simeq\underline{\mathrm{Hom}}_{\mathcal{C}}\bigl(A,\underline{\mathrm{Hom}}_{\mathcal{C}}(A,X)\bigr)\otimes Y\longrightarrow\underline{\mathrm{Hom}}_{\mathcal{C}}\bigl(A,\underline{\mathrm{Hom}}_{\mathcal{C}}(A,X)\otimes Y\bigr)$$

using idempotence of A and $Y\simeq\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathbb{1},Y)$. Now, in general, let $M\in\mathrm{Ind}(\mathcal{C})$ be an ind-object for which $\underline{\mathrm{Hom}}_{\mathcal{C}}(A,M)\to M$ has a left inverse. We can then exhibit $\underline{\mathrm{Hom}}_{\mathcal{C}}(A,M)\to M$

(10.2)Intuitively, the condition should be that M admits a unital multiplication $A\otimes M\to M$, but this doesn't make sense in our setting. So we replace this by the condition that $\underline{\mathrm{Hom}}_{\mathcal{C}}(A,M)\to M$ admits a section.

as a retract of $\underline{\operatorname{Hom}}_{\mathcal{C}}(A, \underline{\operatorname{Hom}}_{\mathcal{C}}(A, M)) \to \underline{\operatorname{Hom}}_{\mathcal{C}}(A, M)$. But the latter is an equivalence by pro-idempotence of A, so already $\underline{\operatorname{Hom}}_{\mathcal{C}}(A, M) \to M$ must be an equivalence.

This finishes the proof that $j^*(X \otimes Y) \to j^*(j^*(X) \otimes Y)$ is an equivalence. By abstract nonsense about symmetric monoidal structures on localisations (see [L-HA, Proposition 2.2.1.9]), it follows that j^* can be canonically equipped with a symmetric monoidal structure. □

10.8. Remark. — In general, $j^*(\mathbb{1})$ is not an idempotent $\mathbb{E}_\infty$-algebra in $\operatorname{Ind}(\mathcal{C})$; it is idempotent if and only if $A^\vee :=$ "$\operatorname{colim}_{i \in I^{\mathrm{op}}}$" $A_i^\vee$ is an *ind-idempotent coalgebra* in the sense that $A^\vee \to \mathbb{1}$ induces an equivalence $A^\vee \otimes A^\vee \simeq A^\vee$ in $\operatorname{Ind}(\mathcal{C})$.

In the following lemma we'll study a special situation in which this is the case. This uses the notions of *trace-class maps* and *nuclear objects*; see the review in §5.2.

10.9. Lemma. — *Let $A =$ "$\lim_{i \in I}$" A_i be an idempotent pro-object whose transition maps are eventually trace-class in the sense that for all $i \in I$ there exists an object $j \to i$ such that $A_j \to A_i$ is trace-class. Let $A^\vee :=$ "$\operatorname{colim}_{i \in I}$" $A_i^\vee$. Then the canonical map*

$$X \otimes A^\vee \xrightarrow{\simeq} \underline{\operatorname{Hom}}_{\mathcal{C}}(A, X)$$

is an equivalence for all $X \in \operatorname{Ind}(\mathcal{C})$. In particular, this implies:

(*a*) *$A^\vee$ is an idempotent coalgebra in $\operatorname{Ind}(\mathcal{C})$ with eventually trace-class transition maps.*

(*b*) *$j^*(\mathbb{1})$ is an idempotent nuclear $\mathbb{E}_\infty$-algebra in $\operatorname{Ind}(\mathcal{C})$, $\operatorname{Ind}(\mathcal{C})^A \subseteq \operatorname{Ind}(\mathcal{C})$ is precisely the full sub-∞-category of $j^*(\mathbb{1})$-modules, and $- \otimes j^*(\mathbb{1}) \simeq j^*(-)$.*

(*c*) *If $F \colon \mathcal{C} \to \mathcal{D}$ is any symmetric monoidal functor of presentable symmetric monoidal ∞-categories, then $F(j^*(\mathbb{1}))$ is obtained by killing the idempotent pro-algebra $F(A)$.*

Proof sketch. We can construct an inverse of $X \otimes A^\vee \to \underline{\operatorname{Hom}}_{\mathcal{C}}(A, X)$ as follows: Fix some $i \in I$, choose $j \to i$ such that $A_j \to A_i$ is trace-class and let $\mathbb{1} \to A_i \otimes A_j^\vee$ be the corresponding classifier. Then consider the composition

$$\underline{\operatorname{Hom}}_{\mathcal{C}}(A_i, X) \longrightarrow \underline{\operatorname{Hom}}_{\mathcal{C}}(A_i, X) \otimes A_i \otimes A_j^\vee \longrightarrow X \otimes A_j^\vee\,.$$

In the first map, we tensor $\underline{\operatorname{Hom}}_{\mathcal{C}}(A_i, X)$ with the classifier above. In the second map we use the evaluation $\underline{\operatorname{Hom}}_{\mathcal{C}}(A_i, X) \otimes A_i \to X$. It's straightforward but a little tedious to check that

$$X \otimes A_i^\vee \longrightarrow \underline{\operatorname{Hom}}_{\mathcal{C}}(A_i, X) \longrightarrow X \otimes A_j^\vee$$
$$\underline{\operatorname{Hom}}_{\mathcal{C}}(A_i, X) \longrightarrow X \otimes A_j^\vee \longrightarrow \underline{\operatorname{Hom}}_{\mathcal{C}}(A_j, X)$$

agree with the transition maps in the ind-objects $X \otimes A^\vee$ and $\underline{\operatorname{Hom}}_{\mathcal{C}}(A, X)$, respectively; we'll omit the argument.

Proving that these maps assemble into an inverse map $X \otimes A^\vee \to \underline{\operatorname{Hom}}_{\mathcal{C}}(A, X)$ requires a non-trivial argument, since we're working in an ∞-category, but there's an easier way to show that $X \otimes A^\vee \to \underline{\operatorname{Hom}}_{\mathcal{C}}(A, X)$ is an equivalence: Equivalences are detected by $\pi_0 \operatorname{Hom}_{\operatorname{Ind}(\mathcal{C})}(Z, -)$, where Z ranges through all compact objects of $\operatorname{Ind}(\mathcal{C})$; now any morphism from a compact object factors through $X \otimes A_i^\vee$ or $\underline{\operatorname{Hom}}_{\mathcal{C}}(A_i, X)$ for some $i \in I$, and so the observations above will be enough.

To show (*a*), plug in $X \simeq A^\vee$: We obtain $A^\vee \otimes A^\vee \simeq \underline{\operatorname{Hom}}_{\mathcal{C}}(A, A^\vee) \simeq (A \otimes A)^\vee$. This proves idempotence as a coalgebra, because $(A \otimes A)^\vee \simeq A^\vee$ follows by dualising $A \simeq A \otimes A$. If

$j \to i$ is large enough so that $A_j \to A_i$ is trace-class, then the dual transition map $A_i^\vee \to A_j^\vee$ is again trace-class by Lemma 5.11(*b*). This shows (*a*).

For (*b*), since we've shown that $A^\vee$ is an idempotent coalgebra in $\operatorname{Ind}(\mathcal{C})$, it follows that $j^*(\mathbb{1})$ is an idempotent algebra. Also $A^\vee$ is a nuclear object in $\operatorname{Ind}(\mathcal{C})$, since every map $Z \to A^\vee$ from a compact object factors through a trace-class morphism and is therefore trace-class itself. Since $\mathbb{1}$ is nuclear too, it follows that $j^*(\mathbb{1})$ is nuclear. $X \otimes j^*(\mathbb{1}) \simeq j^*(X)$ follows immediately from the above equivalence $X \otimes A^\vee \simeq \underline{\operatorname{Hom}}_{\mathcal{C}}(A, X)$. Since the inclusion $\operatorname{Ind}(\mathcal{C})^A \to \operatorname{Ind}(\mathcal{C})$ is lax monoidal by Lemma 10.7, it factors through a functor

$$\operatorname{Ind}(\mathcal{C})^A \to \operatorname{Mod}_{j^*(\mathbb{1})}\bigl(\operatorname{Ind}(\mathcal{C})\bigr)\,.$$

Since $j^*(\mathbb{1})$ is idempotent, $\operatorname{Mod}_{j^*(\mathbb{1})}(\operatorname{Ind}(\mathcal{C})) \subseteq \operatorname{Ind}(\mathcal{C})$ is the full sub-∞-category spanned by the objects of the form $X \otimes j^*(\mathbb{1})$. Hence we also get an inclusion $\operatorname{Ind}(\mathcal{C})^A \subseteq \operatorname{Mod}_{j^*(\mathbb{1})}(\operatorname{Ind}(\mathcal{C}))$. On the other hand, every object of the form $X \otimes j^*(\mathbb{1}) \simeq j^*(X)$ is contained in $\operatorname{Ind}(\mathcal{C})^A$. This finishes the proof of (*b*).

To show (*c*), we only need "$\operatorname{colim}_{i\in I}$" $F(A_i^\vee) \simeq$ "$\operatorname{colim}_{i\in I}$" $F(A_i)^\vee$. If $A_j \to A_i$ is trace-class, Lemma 5.11(*c*) provides a map $F(A_i)^\vee \to F(A_j^\vee)$ in the reverse direction. By a formal argument as above, this is enough to show the desired equivalence. □

§10.2. Generalities on refined localising invariants

Throughout this subsection and the next, we fix the following notation: Let $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ denote the ∞-category of presentable stable ∞-categories and colimit-preserving functors. For a regular cardinal κ, we also denote by $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\kappa} \subseteq \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ the non-full sub-∞-category spanned by the κ-compactly generated presentable stable ∞-categories and those colimit-preserving functors that also preserve κ-compact objects (equivalently, the right adjoint preserves κ-filtered colimits). We equip these ∞-categories with the Lurie tensor product and we let $\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}} \subseteq \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ denote the non-full sub-∞-category spanned by the dualisable objects and those functors whose right adjoint still preserves all colimits.

We also let $\mathcal{E} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}})$ be a rigid presentable stable symmetric monoidal ∞-category in the sense of 1.33. We denote

$$\mathrm{Pr}^{\mathrm{L}}_{\mathcal{E}} \coloneqq \operatorname{Mod}_{\mathcal{E}}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}) \quad \text{and} \quad \mathrm{Pr}^{\mathrm{L}}_{\mathcal{E},\kappa} \coloneqq \operatorname{Mod}_{\mathcal{E}}(\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\kappa})\,,$$

the latter assuming that $\mathcal{E}$ is κ-compactly generated. If $\mathcal{E} \simeq \operatorname{Mod}_k(\mathrm{Sp})$ is the ∞-category of modules over some $\mathbb{E}_\infty$-ring spectrum k, we'll usually abbreviate these as $\mathrm{Pr}^{\mathrm{L}}_k$ and $\mathrm{Pr}^{\mathrm{L}}_{k,\kappa}$, respectively.

10.10. Localising motives over $\mathcal{E}$. — We define the *∞-category of dualisable $\mathcal{E}$-modules* as the module ∞-category $\mathrm{Cat}^{\mathrm{dual}}_{\mathcal{E}} \coloneqq \operatorname{Mod}_{\mathcal{E}}(\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}})$.(10.3) Following Efimov [Efi25, Definition 1.20], we let the *∞-category* $\mathrm{Mot}^{\mathrm{loc}}_{\mathcal{E}}$ *of localising motives over* $\mathcal{E}$ be the recipient of the universal localising invariant on dualisable $\mathcal{E}$-modules.

In the case where $\mathcal{E} \simeq \operatorname{Mod}_k(\mathrm{Sp})$ is the ∞-category of modules over some $\mathbb{E}_\infty$-ring spectrum k, we'll write $\mathrm{Mot}^{\mathrm{loc}}_k$ instead; this agrees with the ∞-category of localising motives over k defined by Blumberg–Gepner–Tabuada [BGT16].

(10.3) $\mathrm{Cat}^{\mathrm{dual}}_{\mathcal{E}}$ can be defined without assuming that $\mathcal{E}$ is rigid, but usually it won't agree with $\operatorname{Mod}_{\mathcal{E}}(\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}})$. See [Efi25, §1.3].

10.11. Refined localising invariants (Efimov–Scholze)**.** — A deep theorem of Efimov [Efi-Rig] states that $\mathrm{Mot}_{\mathcal{E}}^{\mathrm{loc}}$ is rigid itself. This has the following curious consequence, as first observed by Efimov and Scholze: Suppose T is a localising invariant over $\mathcal{E}$, that is, a colimit-preserving functor

$$T\colon \mathrm{Mot}_{\mathcal{E}}^{\mathrm{loc}} \longrightarrow \mathcal{D}$$

into a presentable stable ∞-category $\mathcal{D}$. If T can be equipped with a symmetric monoidal structure, then Efimov's rigidity result implies that there's a unique symmetric monoidal factorisation

$$\begin{array}{ccc} \mathrm{Mot}_{\mathcal{E}}^{\mathrm{loc}} & \xrightarrow{\;T\;} & \mathcal{D} \\ & {\scriptstyle T^{\mathrm{ref}}} \searrow & \uparrow \\ & & \mathcal{D}^{\mathrm{rig}} \end{array}$$

This factorisation $T^{\mathrm{ref}}\colon \mathrm{Mot}_{\mathcal{E}}^{\mathrm{loc}} \to \mathcal{D}^{\mathrm{rig}}$ is the *refinement of* T defined by Efimov–Scholze.

Here $\mathcal{D}^{\mathrm{rig}}$ denotes the *rigidification of* $\mathcal{D}$ in the sense of [Ram24, Construction 4.75]; see also [Efi25, Proposition 1.23]. We recall from these references that $\mathcal{D}^{\mathrm{rig}}$ can be described as the full sub-∞-category of $\mathrm{Ind}(\mathcal{D})$(10.4) generated under colimits by ind-objects of the form "$\mathrm{colim}_{i\in\mathbb{Q}}$" x_i, where all transition maps $x_i \to x_j$ for rational numbers $i<j$ are trace-class. If $\mathcal{D}$ is *locally rigid* and its tensor unit is ω_1-compact, then it suffices to consider $\mathbb{Z}_{\geqslant 0}$-indexed ind-objects instead of $\mathbb{Q}$-indexed ones. In other words, in this case

$$\mathcal{D}^{\mathrm{rig}} \xrightarrow{\;\simeq\;} \mathrm{Nuc}\,\mathrm{Ind}(\mathcal{D})$$

is an equivalence. See [Efi25, Theorem 4.2] for a proof.

10.12. Lemma. — *Let $M_{(-)}\colon \mathbb{Q} \to \mathrm{Mot}_{\mathcal{E}}^{\mathrm{loc}}$ be a diagram such that $M_i \to M_j$ is trace-class for all rational numbers $i<j$. Then*

$$T^{\mathrm{ref}}\Bigl(\operatorname*{colim}_{i\in\mathbb{Q}} M_i\Bigr) \simeq \text{“}\operatorname*{colim}_{i\in\mathbb{Q}}\text{”}\, T(M_i)\,.$$

If $\mathcal{D}$ is locally rigid and its tensor unit is ω_1-compact, then the same is true for $\mathbb{Z}_{\geqslant 0}$-indexed diagrams with trace-class transition maps.

Proof. This is almost tautological: Since $\mathrm{Mot}_{\mathcal{E}}^{\mathrm{loc}}$ is rigid, $(\mathrm{Mot}_{\mathcal{E}}^{\mathrm{loc}})^{\mathrm{rig}} \to \mathrm{Mot}_{\mathcal{E}}^{\mathrm{loc}}$ is an equivalence. Since the ind-object "$\mathrm{colim}_{i\in\mathbb{Q}}$" M_i is a preimage of M under this equivalence, the first claim follows. The second claim is completely analogous, since the additional assumptions imply $\mathcal{D}^{\mathrm{rig}} \simeq \mathrm{Nuc}\,\mathrm{Ind}(\mathcal{D})$, as we've seen in 10.11. □

10.13. Why computing T^{ref} is hard. — In general, we're faced with at least two difficult problems:

(!) *For an arbitrary motive $M \in \mathrm{Mot}_{\mathcal{E}}^{\mathrm{loc}}$, it can be very hard to decompose M into pieces for which resolutions as in Lemma 10.12 exist.*

(!!) *Even if such resolutions can be found, computing $T(M_i)$ (and the transition maps between them) can still be a very hard problem.*

(10.4) The set-theoretic difficulties here can be fixed as in Remark 5.14.

In §10.3, we'll explain how to solve problem (!) in many cases of interest, which will include $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q})$, $\mathrm{THH}^{\mathrm{ref}}(\mathrm{L}_n^f\mathbb{S}_{(p)}/\mathbb{S}_{(p)})$ and $\mathrm{THH}^{\mathrm{ref}}(\mathbb{S}[x])$. The entirety of §§11–12 below will then be spent on problem (!!) for $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q})$, and we will only be able to obtain an answer after base change to ku.

But before we dive into the difficult calculations, let us discuss another easy case. To this end, recall from [Efi25, Definition 1.48] that a dualisable $\mathcal{E}$-module category $\mathcal{X}$ is called *smooth* if the coevaluation $\mathrm{Sp} \to \mathcal{X}^\vee \otimes_{\mathcal{E}} \mathcal{X}$ preserves compact objects, and *proper* if the evaluation $\mathcal{X} \otimes \mathcal{X}^\vee \to \mathcal{E}$ preserves compact objects. Here $\mathcal{X}^\vee$ denotes the dual of $\mathcal{X}$ as an $\mathcal{E}$-module.

10.14. Lemma. — *Let $\mathcal{X}$ be a dualisable $\mathcal{E}$-module.*

(*a*) *$\mathcal{X}$ is smooth and proper in the sense above if and only if $\mathcal{X}$ is dualisable in $\mathrm{Cat}_{\mathcal{E}}^{\mathrm{dual}}$.*(10.5)

(*b*) *If this is the case, then $T^{\mathrm{ref}}(\mathcal{X}) \simeq T(\mathcal{X})$.*

Proof sketch. Assume first that $\mathcal{X}$ is smooth and proper. We'll only explain why the coevaluation and the evaluation over $\mathcal{E}$, i.e. $\mathcal{E} \to \mathcal{X}^\vee \otimes_{\mathcal{E}} \mathcal{X}$ and $\mathcal{X} \otimes_{\mathcal{E}} \mathcal{X}^\vee \to \mathcal{E}$, are functors in $\mathrm{Cat}_{\mathcal{E}}^{\mathrm{dual}}$; the triangle identities are then straightforward to verify. Since $\mathrm{Sp} \to \mathcal{X}^\vee \otimes_{\mathcal{E}} \mathcal{X}$ is strongly continuous by smoothness, the same will be true for the composition

$$\mathcal{E} \longrightarrow \mathcal{E} \otimes (\mathcal{X}^\vee \otimes_{\mathcal{E}} \mathcal{X}) \longrightarrow \mathcal{X}^\vee \otimes_{\mathcal{E}} \mathcal{X}$$

by [Efi25, Proposition 1.12(ii)]. So the coevaluation is a functor in $\mathrm{Cat}_{\mathcal{E}}^{\mathrm{dual}}$. Moreover, we have $\mathcal{X}^\vee \simeq \underline{\mathrm{Hom}}_{\mathcal{E}}^{\mathrm{dual}}(\mathcal{X}, \mathcal{E})$ by [Efi25, Proposition 3.4(iii)]. Since $\mathcal{E}$ was assumed symmetric monoidal, $\mathrm{Cat}_{\mathcal{E}}^{\mathrm{dual}}$ admits an internal Hom, which necessarily lifts $\underline{\mathrm{Hom}}_{\mathcal{E}}^{\mathrm{dual}}$. Hence we get an evaluation $\mathcal{X} \otimes_{\mathcal{E}} \mathcal{X}^\vee \to \mathcal{E}$ in $\mathcal{E}$ as well.

Now assume that $\mathcal{X}$ is dualisable in $\mathrm{Cat}_{\mathcal{E}}^{\mathrm{dual}}$. Then $\mathcal{E} \to \mathcal{X}^\vee \otimes_{\mathcal{E}} \mathcal{X}$ is strongly continuous, hence it sends the tensor unit (which is compact as $\mathcal{E}$ is rigid) to a compact object. Then the same must be true for $\mathrm{Sp} \to \mathcal{X}^\vee \otimes_{\mathcal{E}} \mathcal{X}$, proving smoothness. For properness, we already know that $\mathcal{X} \otimes_{\mathcal{E}} \mathcal{X}^\vee \to \mathcal{E}$ is strongly continuous, so it remains to show the same for $\mathcal{X} \otimes \mathcal{X}^\vee \to \mathcal{X} \otimes_{\mathcal{E}} \mathcal{X}^\vee$. To this end, write

$$\mathcal{X} \otimes_{\mathcal{E}} \mathcal{X}^\vee \simeq (\mathcal{X} \otimes \mathcal{X}^\vee) \otimes_{\mathcal{E} \otimes \mathcal{E}} \mathcal{E}$$

and use that $\mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$ is strongly continuous by rigidity. This finishes the proof of (*a*). Part (*b*) is an immediate consequence of this and Lemma 10.12, applied to the constant $\mathcal{X}$-valued diagram, which has trace-class transition maps since the identity on any dualisable object is trace-class. □

10.15. Corollary. — *Let $\mathcal{E} \to \mathcal{X}$ be a strongly continuous symmetric monoidal functor into another rigid symmetric monoidal presentable stable ∞-category. If $\mathcal{X}$ is smooth and proper as an $\mathcal{E}$-module, then the forgetful functor $\mathrm{Cat}_{\mathcal{X}}^{\mathrm{dual}} \to \mathrm{Cat}_{\mathcal{E}}^{\mathrm{dual}}$ preserves trace-class morphisms.*

Proof. By Lemma 10.14(*a*) and the general fact that $\mathcal{X}^\vee \simeq \mathcal{X}$ (see [GR17, 1.9.2.1] or [Efi25, Proposition 1.3]), we see that $\mathcal{X}$ is a self-dual $\mathbb{E}_\infty$-algebra in $\mathrm{Cat}_{\mathcal{E}}^{\mathrm{dual}}$. The assertion then becomes purely abstract nonsense: For $\mathcal{X}$-modules $\mathcal{M}$ and $\mathcal{N}$, the diagram

$$\begin{array}{ccccc}
\underline{\mathrm{Hom}}_{\mathcal{X}}^{\mathrm{dual}}(\mathcal{M}, \mathcal{X}) \otimes_{\mathcal{X}} \mathcal{N} & \longrightarrow & \underline{\mathrm{Hom}}_{\mathcal{X}}^{\mathrm{dual}}(\mathcal{M}, \mathcal{X}) \otimes_{\mathcal{X}} (\mathcal{X} \otimes_{\mathcal{E}} \mathcal{N}) & \xrightarrow{\simeq} & \underline{\mathrm{Hom}}_{\mathcal{E}}^{\mathrm{dual}}(\mathcal{M}, \mathcal{E}) \otimes_{\mathcal{E}} \mathcal{N} \\
\downarrow & & \downarrow & & \downarrow \\
\underline{\mathrm{Hom}}_{\mathcal{X}}^{\mathrm{dual}}(\mathcal{M}, \mathcal{N}) & \longrightarrow & \underline{\mathrm{Hom}}_{\mathcal{X}}^{\mathrm{dual}}(\mathcal{M}, \mathcal{X} \otimes_{\mathcal{E}} \mathcal{N}) & \xrightarrow{\simeq} & \underline{\mathrm{Hom}}_{\mathcal{E}}^{\mathrm{dual}}(\mathcal{M}, \mathcal{N})
\end{array}$$

(10.5) Note that being dualisable in $\mathrm{Cat}_{\mathcal{E}}^{\mathrm{dual}}$ is much stronger than being a dualisable $\mathcal{E}$-module.

commutes, where the horizontal arrows in the left square are given by the unit $\mathcal{N} \to \mathcal{X} \otimes_{\mathcal{E}} \mathcal{N}$ of the "wrong way" adjunction between the forgetful functor and $\mathcal{X} \otimes_{\mathcal{E}} -\colon \mathrm{Cat}_{\mathcal{E}}^{\mathrm{dual}} \to \mathrm{Cat}_{\mathcal{X}}^{\mathrm{dual}}$. □

§10.3. A recipe for computation

We continue to fix the notation from §10.2 as well as a symmetric monoidal localising invariant

$$T\colon \mathrm{Mot}_{\mathcal{E}}^{\mathrm{loc}} \longrightarrow \mathcal{D}\,.$$

From now on, we'll additionally assume that $\mathcal{D}$ is locally rigid and its tensor unit is ω_1-compact, so that $\mathcal{D}^{\mathrm{rig}} \simeq \mathrm{Nuc}\,\mathrm{Ind}(\mathcal{D})$ by [Efi25, Theorem 4.2].

Our goal in this subsection is to explain a method to compute certain values of the refinement T^{ref}. This method is a more or less straightforward abstract reformulation of the method that Efimov uses in his computations (see e.g. [Efi24, Talk 6]).

10.16. Motives of interest. — Let $\mathcal{E} \to \mathcal{X}$ be a strongly continuous symmetric monoidal functor into another rigid symmetric monoidal presentable stable ∞-category. Assume that $\mathcal{X}$ is smooth and proper as an $\mathcal{E}$-module. We wish to compute $T^{\mathrm{ref}}(\mathcal{U})$ for localisations $\mathcal{U} \subseteq \mathcal{X}$ that arise as in 10.1. That is, there is some object $V_0 \in \mathcal{X}$ with a left-unital multiplication such that $\mathcal{U}$ is the full sub-∞-category spanned by those $X \in \mathcal{X}$ for which $\underline{\mathrm{Hom}}_{\mathcal{X}}(V_0, X) \simeq 0$. Let us additionally assume that the following is satisfied:

(*V*) *There exists a tower of $\mathbb{E}_1$-algebras in $\mathcal{X}$,*

$$V_0 \longleftarrow V_1 \longleftarrow V_2 \longleftarrow \cdots\,,$$

such that each V_r is dualisable in $\mathcal{X}$ and contained in the thick tensor ideal (that is, the smallest full sub-∞-category closed under finite limits and colimits, retracts, and $-\otimes X$ for all $X \in \mathcal{X}$) generated by V_0. Moreover, we assume that for all $r \geqslant 0$, the induced map $V_{r+1} \otimes V_r \to V_r \otimes V_r$ factors through the multiplication

$$V_{r+1} \otimes V_r \xrightarrow{\mu} V_r$$

as a map of V_{r+1}-V_r-bimodules.

The main example to keep in mind is the following: Suppose we're given maps $v_i\colon \mathcal{I}_i \to \mathbb{1}_{\mathcal{X}}$ for $i = 0, 1, \dots, n$, where each $\mathcal{I}_i$ is dualisable in $\mathcal{X}$. Then we can define V_r as the iterated cofibre

$$V_r \coloneqq \mathbb{1}_{\mathcal{X}}/\bigl(v_0^{\alpha_{r,0}}, \dots, v_n^{\alpha_{r,n}}\bigr)$$

for some entry-wise increasing sequence of $(n+1)$-tuples $\alpha_r = (\alpha_{r,1}, \dots, \alpha_{r,n})$ and equip the tower $\{V_r\}_{r\geqslant 0}$ with Burklund-style $\mathbb{E}_1$-structures. We'll discuss in §10.4 why this satisfies (*V*) and how this allows us to recover many examples of interest, such as $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q})$, $\mathrm{THH}^{\mathrm{ref}}(\mathbb{S}[x])$, and $\mathrm{THH}^{\mathrm{ref}}(\mathrm{L}_n^f\mathbb{S}_{(p)}/\mathbb{S}_{(p)})$ (note that the last example doesn't quite fit this situation, which will cause us some pain).

10.17. Theorem. — *Let $\mathcal{E}$ be rigid and let $T\colon \mathrm{Mot}_{\mathcal{E}}^{\mathrm{loc}} \to \mathcal{D}$ be a localising invariant such that $\mathcal{D}$ is locally rigid and its tensor unit is ω_1-compact. Let $\mathcal{X}$ and $\mathcal{U}$ be as in 10.16.*

(*a*) *The pro-object "$\lim_{r\geqslant 0}$" $T(\mathrm{RMod}_{V_r}(\mathcal{X}))$ is idempotent over $T(\mathcal{X})$ and its transition maps are trace-class.*

(*b*) *$T^{\mathrm{ref}}(\mathcal{U})$ is obtained from $T(\mathcal{X})$ by killing this idempotent pro-algebra. In particular, $T^{\mathrm{ref}}(\mathcal{U})$ sits inside the following cofibre sequence in $\mathcal{D}^{\mathrm{rig}} \simeq \operatorname{Nuc}\operatorname{Ind}(\mathcal{D})$:*

$$\text{“}\operatorname*{colim}_{r\geqslant 0}\text{”}\, T\big(\mathrm{RMod}_{V_r}(\mathcal{X})\big)^{\vee} \longrightarrow T(\mathcal{X}) \longrightarrow T^{\mathrm{ref}}(\mathcal{U})\,.$$

We start the proof of Theorem 10.17 with a few easy observations about the “closed complement” of $\mathcal{U}$ in $\mathcal{X}$.

10.18. Lemma. — *Let $\mathcal{X}$ and $\mathcal{U}$ be as in 10.16.*

(*a*) *The inclusion $\mathcal{U} \to \mathcal{X}$ admits a left adjoint $j^*\colon \mathcal{X} \to \mathcal{U}$, which can be canonically equipped with a symmetric monoidal structure.*

(*b*) *If $\mathcal{V} \subseteq \mathcal{X}$ denotes the kernel of j^*, then $\mathcal{V}$ is a tensor ideal and closed under colimits, finite limits, and retracts in $\mathcal{X}$. If S runs through a set of generators of $\mathcal{X}$, then $V_0 \otimes S$ forms a set of generators of $\mathcal{V}$.*

(*c*) *For all $r \geqslant 0$, the $\mathbb{E}_1$-algebra V_r is a compact object of $\mathcal{X}$, and every left- or right-module over V_r is contained in $\mathcal{V}$.*

Proof. Part (*a*) follows immediately from 10.1. Since j^* is symmetric monoidal and preserves all colimits, its kernel $\mathcal{V}$ must be a tensor ideal and closed under colimits, finite limits, and retracts. Now let $V \in \mathcal{V}$ be an object such that

$$0 \simeq \mathrm{Hom}_{\mathcal{X}}(V_0 \otimes S, V) \simeq \mathrm{Hom}_{\mathcal{X}}\big(S, \underline{\mathrm{Hom}}_{\mathcal{X}}(V_0, V)\big)$$

for all S. Since S runs through a set of generators of $\mathcal{X}$, this implies $\underline{\mathrm{Hom}}_{\mathcal{X}}(V_0, V) \simeq 0$. Hence also $V \in \mathcal{U}$ and so $V \simeq j^*(V) \simeq 0$. This finishes the proof of (*b*).

To show (*c*), observe that any $X \in \mathcal{X}$ is dualisable if and only if it is compact (because in a rigid presentable symmetric monoidal ∞-category $\mathrm{id}_X\colon X \to X$ is trace-class if and only if it is compact; see [Ram24, Corollary 4.52] or [Efi25, Proposition 1.7]). Hence V_r is compact for all $r \geqslant 0$. To show that any left- or right-V_r-module is contained in $\mathcal{V}$, it suffices to show the same for induced modules (i.e. those of the form $V_r \otimes X$), since every module is a colimit of induced ones. By the thick tensor ideal condition in 10.16(*V*), we can furthermore reduce to objects of the form $V_0 \otimes X$. Now if $U \in \mathcal{U}$, then

$$\mathrm{Hom}_{\mathcal{X}}(V_0 \otimes X, U) \simeq \mathrm{Hom}_{\mathcal{X}}\big(X, \underline{\mathrm{Hom}}_{\mathcal{X}}(V_0, U)\big) \simeq 0\,,$$

proving $j^*(V_0 \otimes X) \simeq 0$, as desired. □

10.19. Lemma. — *For every $r \geqslant 0$, the base change functor*

$$-\otimes_{V_{r+1}} V_r\colon \mathrm{RMod}_{V_{r+1}}(\mathcal{X}) \longrightarrow \mathrm{RMod}_{V_r}(\mathcal{X})$$

is a trace-class morphism in $\mathrm{Cat}^{\mathrm{dual}}_{\mathcal{X}}$, hence also in $\mathrm{Cat}^{\mathrm{dual}}_{\mathcal{E}}$.

Proof. The additional assertion will follow immediately from Corollary 10.15 once we've shown the rest. Writing $\mathrm{RMod}_{V_r}(\mathcal{X}) \simeq \mathrm{RMod}_{V_r}(\mathrm{Ind}(\mathcal{X}^\omega)) \otimes_{\mathrm{Ind}(\mathcal{X}^\omega)} \mathcal{X}$, we may reduce to the case where $\mathcal{X}$ is compactly generated, as $-\otimes_{\mathrm{Ind}(\mathcal{X}^\omega)} \mathcal{X}$ preserves trace-class morphisms by Lemma 5.11(*b*). In the compactly generated case, we'll even show that $-\otimes_{V_{r+1}} V_r$ is trace-class in $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{X},\omega}$.

Recall from [L-HA, Remark 4.8.4.8] that $\mathrm{RMod}_{V_{r+1}}(\mathcal{X})$ is dualisable in $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{X}}$ with dual $\mathrm{LMod}_{V_{r+1}}(\mathcal{X})$. Therefore, the base change functor is always trace-class in $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{X}}$. The witnessing functor $\mathcal{X} \to \mathrm{LMod}_{V_{r+1}}(\mathcal{X}) \otimes_{\mathcal{X}} \mathrm{RMod}_{V_r}(\mathcal{X}) \simeq \mathrm{LMod}_{V_{r+1}\otimes V_r^{\mathrm{op}}}(\mathcal{X})$ is the classifier of V_r as a left module over $V_{r+1} \otimes V_r^{\mathrm{op}}$, or equivalently, a V_{r+1}-$V-r$-bimodule. If we work in $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{X},\omega}$ instead, then $\mathrm{RMod}_{V_{r+1}}(\mathcal{X})$ will no longer be dualisable, but we can still form the predual

$$\underline{\mathrm{Hom}}_{\mathrm{Pr}^{\mathrm{L}}_{\mathcal{X},\omega}}\big(\mathrm{RMod}_{V_{r+1}}(\mathcal{X}), \mathcal{X}\big) \simeq \mathrm{Ind}\big(\mathrm{Fun}_{\mathcal{X}^\omega}(\mathrm{RMod}_{V_{r+1}}(\mathcal{X})^\omega, \mathcal{X}^\omega)\big) \simeq \mathrm{Ind}\big(\mathrm{LMod}_{V_{r+1}}(\mathcal{X}^\omega)\big)\,,$$

where we've used [L-HA, Theorem 4.8.4.1] and the fact that $V_{r+1} \in \mathcal{X}^\omega$ by Lemma 10.18(c). Using [L-HA, Theorem 4.8.4.6], we still have a functor

$$\mathcal{X} \longrightarrow \mathrm{Ind}\big(\mathrm{LMod}_{V_{r+1}}(\mathcal{X}^\omega)\big) \otimes_{\mathcal{X}} \mathrm{RMod}_{V_r}(\mathcal{X}) \simeq \mathrm{RMod}_{V_r}\big(\mathrm{Ind}\big(\mathrm{LMod}_{V_{r+1}}(\mathcal{X}^\omega)\big)\big)$$

in $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{X}}$ that classifies V_r has a right V_r-module in $\mathrm{Ind}(\mathrm{LMod}_{V_{r+1}}(\mathcal{X}))$. For the desired trace-class property to hold, this functor needs to be contained in $\mathrm{Pr}^{\mathrm{L}}_{\mathcal{X},\omega}$. That is, we need V_r to be a compact object in $\mathrm{RMod}_{V_r}(\mathrm{Ind}(\mathrm{LMod}_{V_{r+1}}(\mathcal{X}^\omega)))$.

To this end, recall our assumption 10.16(V) that $V_{r+1} \otimes V_r \to V_r \otimes V_r$ factors through the multiplication $V_{r+1} \otimes V_r \to V_r$ as a map of V_{r+1}-V_r-bimodules. Consequently, V_r is a retract of $V_r \otimes V_r$ in $\mathrm{RMod}_{V_r}(\mathrm{Ind}(\mathrm{LMod}_{V_{r+1}}(\mathcal{X}^\omega)))$. This is enough to show compactness. Indeed, the object $V_r \in \mathrm{Ind}(\mathrm{LMod}_{V_{r+1}}(\mathcal{X}^\omega))$ is compact(10.6) and so the induced right-V_r-module $V_r \otimes V_r$ must be compact. □

10.20. Remark. — As a consequence of the proof of Lemma 10.19 and Lemma 5.11(b), we see that the functors

$$\mathrm{Ind}\,\mathrm{LMod}_{V_r}(\mathcal{X}^\omega) \otimes_{\mathrm{Ind}(\mathcal{X}^\omega)} \mathcal{X} \longrightarrow \mathrm{Ind}\,\mathrm{LMod}_{V_{r+1}}(\mathcal{X}^\omega) \otimes_{\mathrm{Ind}(\mathcal{X}^\omega)} \mathcal{X}\,.$$

induced by the forgetful functors $\mathrm{LMod}_{V_r}(\mathcal{X}^\omega) \to \mathrm{LMod}_{V_{r+1}}(\mathcal{X}^\omega)$ are also trace-class in $\mathrm{Cat}^{\mathrm{dual}}_{\mathcal{X}}$, hence in $\mathrm{Cat}^{\mathrm{dual}}_{\mathcal{E}}$ by Corollary 10.15.

The reader familiar with some of Efimov's computations of refined invariants will have already seen $\mathrm{Ind}\,\mathrm{LMod}_{V_r}(\mathcal{X}^\omega) \otimes_{\mathrm{Ind}(\mathcal{X}^\omega)} \mathcal{X}$, albeit in disguise: For example, it is the abstract analogue of $\mathcal{D}^b_{\mathrm{coh}}(\mathbb{Q}[x]/x^n)$ in Efimov's computation of $\mathrm{HC}^{-,\mathrm{ref}}(\mathbb{Q}[x^{\pm 1}]/\mathbb{Q}[x])$ (see e.g. [Efi24, Talk 6]). Also note that the forgetful functors $\mathrm{LMod}_{V_r}(\mathcal{X}^\omega) \to \mathcal{X}^\omega$ will land in $\mathcal{V}$ by Lemma 10.18(c) and so we get functors

$$\mathrm{Ind}\,\mathrm{LMod}_{V_r}(\mathcal{X}^\omega) \otimes_{\mathrm{Ind}(\mathcal{X}^\omega)} \mathcal{X} \longrightarrow \mathcal{V}\,.$$

for all $r \geqslant 0$. These are compatible with the functors above.

10.21. Lemma. — *With notation as above, the functors from Remark 10.20 induce an equivalence of $\mathcal{X}$-linear presentable ∞-categories*

$$\operatorname*{colim}_{r\geqslant 0}\big(\mathrm{Ind}\,\mathrm{LMod}_{V_r}(\mathcal{X}^\omega) \otimes_{\mathrm{Ind}(\mathcal{X}^\omega)} \mathcal{X}\big) \xrightarrow{\simeq} \mathcal{V}\,.$$

Here the colimit on the left-hand side is taken in $\mathrm{Cat}^{\mathrm{dual}}_{\mathcal{X}}$, or equivalently, in $\mathrm{Cat}^{\mathrm{dual}}_{\mathcal{E}}$ or $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$.

Proof. We'll prove this under the assumption that $\mathcal{X}$ is compactly generated; to reduce to this special case, apply Lemma 10.22 below for $\mathrm{Ind}(\mathcal{X}^\omega) \to \mathcal{X}$. Since $\mathcal{X}$ is rigid, compact objects are closed under tensor products, since they coincide with the dualisable objects. By Lemma 10.18(b), this implies that $\mathcal{V}$ is again compactly generated. By construction,

(10.6) By contrast, V_r is usually *not* compact in $\mathrm{LMod}_{V_{r+1}}(\mathcal{X})$.

$\operatorname{Ind}\operatorname{LMod}_{V_r}(\mathcal{X}^\omega) \to \mathcal{X}$ preserves compact objects, hence the same is true if we restrict the codomain to $\mathcal{V}$. Using that $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega} \to \mathrm{Pr}^{\mathrm{L}}_{\mathrm{st}}$ preserves all colimits, we deduce that

$$L\colon \operatorname*{colim}_{r\geqslant 0} \operatorname{Ind}\operatorname{LMod}_{V_r}(\mathcal{X}^\omega) \longrightarrow \mathcal{V}$$

is a functor in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{st},\omega}$. In particular, whether L is fully faithful can be checked on compact objects. So let M and N be compact.

Writing $\operatorname{colim}_{r\geqslant 0}\operatorname{Ind}(\operatorname{LMod}_{V_r}(\mathcal{X}^\omega)) \simeq \operatorname{Ind}(\operatorname{colim}_{r\geqslant 0}\operatorname{LMod}_{V_r}(\mathcal{X}^\omega))$, we may assume that M and N are V_r-modules for some r. We must then show that

$$\operatorname*{colim}_{s\geqslant r} \operatorname{Hom}_{V_s}(M,N) \xrightarrow{\ \simeq\ } \operatorname{Hom}_{\mathcal{X}}(M,N)\,.$$

is an equivalence. To this end, let us rewrite this map as

$$\operatorname*{colim}_{s\geqslant r} \operatorname{Hom}_{V_r}\bigl((V_r\otimes_{V_s} V_r)\otimes_{V_r} M, N\bigr) \longrightarrow \operatorname{Hom}_{V_r}\bigl((V_r\otimes V_r)\otimes_{V_r} M, N\bigr)\,.$$

For all $s\geqslant r$, consider $V_r\otimes V_r$ as a right-V_{s+1}-module via the right action on the first tensor factor and as a left-V_{s+1}-module via the left action on the second tensor factor. In total, we've produced a right-$(V_{s+1}\otimes V_{s+1}^{\mathrm{op}})$-module structure on $V_r\otimes V_r$. Since $V_r\otimes V_r$ is already a right-$(V_s\otimes V_s^{\mathrm{op}})$-module via the same construction, the identity on $V_r\otimes V_r$ factors through $(V_r\otimes V_r)\otimes_{V_{s+1}\otimes V_{s+1}^{\mathrm{op}}} V_s\otimes V_s^{\mathrm{op}}$. By Assumption 10.16($V$), $V_{s+1}\otimes V_{s+1}\to V_s\otimes V_s$ factors through V_{s+1} as a map of V_{s+1}-V_{s+1}-bimodules, or equivalently, as a map of left-$V_{s+1}\otimes V_{s+1}^{\mathrm{op}}$-modules. This shows that the identity on $V_r\otimes V_r$ factors through

$$(V_r\otimes V_r)\otimes_{V_{s+1}\otimes V_{s+1}^{\mathrm{op}}} V_{s+1} \simeq V_r\otimes_{V_{s+1}} V_r\,.$$

This factorisation works as V_r-V_r-bimodules, since we haven't touched the "outer" V_r-V_r-bimodule structure anywhere and have only worked with the "inner" bimodule structures. Thus, the colimit diagram above can be intertwined with the constant $\operatorname{Hom}_{V_r}((V_r\otimes V_r)\otimes_{V_r} M, N)$-valued diagram, which proves that we get the desired equivalence.

Hence L is fully faithful. Once we know this, essential surjectivity follows immediately from Lemma 10.18(b), so we win. □

10.22. Lemma. — *Let $\mathcal{X}\to\mathcal{X}'$ be a symmetric monoidal colimit-preserving functor into another rigid presentable stable symmetric monoidal ∞-category $\mathcal{X}'$. Let V_0' denote the image of V_0, let $\mathcal{U}' \coloneqq (\mathcal{X}')^{V_0'}\subseteq\mathcal{X}'$ and let $\mathcal{V}'$ be the kernel of the left adjoint $\mathcal{X}'\to\mathcal{U}'$ of the inclusion. Then the induced functor*

$$\mathcal{V}\otimes_{\mathcal{X}}\mathcal{X}' \xrightarrow{\ \simeq\ } \mathcal{V}'$$

is an equivalence of ∞-categories.

Proof. It's enough to show this in the case where $\mathcal{X}$ is compactly generated, since the general case will follow by considering $\operatorname{Ind}(\mathcal{X}^\omega)\to\mathcal{X}\to\mathcal{X}'$. By Lemma 10.18($b$), $\mathcal{V}$ is a tensor ideal and so the inclusion $\mathcal{V}\to\mathcal{X}$ is $\mathcal{X}$-linear. Note that its right adjoint is again $\mathcal{X}$-linear. Indeed, the right adjoint is given by $\operatorname{fib}(X\to j^*(X))$ for all $X\in\mathcal{X}$, so we must show that $j^*(X)\otimes Y\to j^*(X\otimes Y)$ is an equivalence for all $Y\in\mathcal{X}$. Since we assume $\mathcal{X}$ to be compactly generated, it suffices to show this in the case $Y\in\mathcal{X}^\omega$, as both sides commute with filtered colimits. But then Y is dualisable as $\mathcal{X}$ is rigid. Since $\mathcal{U}'$ is stable under tensoring with dualisable objects, we obtain $j^*(X)\otimes Y\simeq j^*(j^*(X)\otimes Y)\simeq j^*(X\otimes Y)$ from 10.1, as desired.

It follows that $\mathcal{V} \otimes_{\mathcal{X}} \mathcal{X} \to \mathcal{X}'$ is fully faithful, since we can now just base change the fact that the unit is an equivalence. Its essential image is clearly contained in $\mathcal{V}'$, and it's clear from Lemma 10.18(*b*) that $\mathcal{V}_\omega \otimes_{\mathcal{X}} \mathcal{X}' \to \mathcal{V}'$ is essentially surjective. □

Proof of Theorem 10.17. By Lemma 10.19 and Lemma 5.11(*b*) applied to the symmetric monoidal functor $T\colon \mathrm{Cat}_{\mathcal{X}}^{\mathrm{dual}} \simeq \mathrm{Mod}_{\mathcal{X}}(\mathrm{Cat}_{\mathcal{E}}^{\mathrm{dual}}) \to \mathrm{Mod}_{T(\mathcal{X})}(\mathcal{D})$, the transition maps of the pro-object "$\lim_{r\geqslant 0}$" $T(\mathrm{RMod}_{V_r}(\mathcal{X}))$ are trace-class morphisms in $\mathrm{Mod}_{T(\mathcal{X})}(\mathcal{D})$. To prove (*a*), it will thus be enough to check that the dual ind-object is an idempotent coalgebra.

To see this, write $\mathrm{RMod}_{V_r}(\mathcal{X}) \simeq \mathrm{RMod}_{V_r}(\mathrm{Ind}(\mathcal{X}^\omega)) \otimes_{\mathrm{Ind}(\mathcal{X}^\omega)} \mathcal{X}$. We've seen in the proof of Lemma 10.19 that the predual of $\mathrm{RMod}_{V_r}(\mathrm{Ind}(\mathcal{X}^\omega))$ in $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ind}(\mathcal{X}^\omega),\omega}$ is $\mathrm{Ind}\,\mathrm{LMod}_{V_r}(\mathcal{X}^\omega)$. Now consider the diagram of symmetric monoidal functors

$$\begin{array}{ccccccc} \mathrm{Pr}^{\mathrm{L}}_{\mathrm{Ind}(\mathcal{X}^\omega),\omega} & \xrightarrow{-\otimes_{\mathrm{Ind}(\mathcal{X}^\omega)}\mathcal{X}} & \mathrm{Cat}_{\mathcal{X}}^{\mathrm{dual}} & \longrightarrow & \mathrm{Mod}_{\mathcal{X}}(\mathrm{Mot}_{\mathcal{E}}^{\mathrm{loc}}) & \xrightarrow{T} & \mathrm{Mod}_{T(\mathcal{X})}(\mathcal{D}) \\ \downarrow & & & & \uparrow & & \\ \mathrm{Cat}^{\mathrm{dual}}_{\mathrm{Ind}(\mathcal{X}^\omega)} & & \longrightarrow & & \mathrm{Mot}^{\mathrm{loc}}_{\mathrm{Ind}(\mathcal{X}^\omega)} & & \end{array}$$

In general, none of them preserves preduals, but once we pass to "$\mathrm{colim}_{r\geqslant 0}$" this isn't a problem anymore by Lemma 5.11(*c*). Thus, it will be enough to check that the image of "$\mathrm{colim}_{r\geqslant 0}$" $\mathrm{Ind}\,\mathrm{LMod}_{V_r}(\mathcal{X}^\omega)$ is idempotent in $\mathrm{Ind}(\mathrm{Mot}^{\mathrm{loc}}_{\mathrm{Ind}(\mathcal{X}^\omega)})$.

For ease of notation, let us now replace $\mathcal{X}$ by $\mathrm{Ind}(\mathcal{X}^\omega)$, thereby assuming that $\mathcal{X}$ is compactly generated. Since "$\mathrm{colim}_{r\geqslant 0}$" $\mathrm{Ind}\,\mathrm{LMod}_{V_r}(\mathcal{X}^\omega)$ has trace-class transition maps and $\mathrm{Nuc}\,\mathrm{Ind}(\mathrm{Mot}^{\mathrm{loc}}_{\mathcal{X}}) \simeq \mathrm{Mot}^{\mathrm{loc}}_{\mathcal{X}}$ by Efimov's rigidity theorem, it will be enough to show that $\mathrm{colim}_{r\geqslant 0}\,\mathrm{Ind}\,\mathrm{LMod}_{V_r}(\mathcal{X}^\omega) \simeq \mathcal{V}$ is idempotent in $\mathrm{Mot}^{\mathrm{loc}}_{\mathcal{X}}$. We claim that $\mathcal{V}$ is already idempotent in $\mathrm{Cat}^{\mathrm{dual}}_{\mathcal{X}}$. To see this, just observe that the same argument as in Lemma 10.21 also proves that

$$\operatorname*{colim}_{r\geqslant 0}\,\mathrm{Ind}\,\mathrm{LMod}_{V_r\otimes V_r}(\mathcal{X}^\omega) \xrightarrow{\simeq} \mathcal{V}$$

is an equivalence of ∞-categories. This finishes the proof of (*a*).

Let us now show (*b*). In the following, we'll use several times (and in a somewhat confusing way) that $\mathrm{Nuc}\,\mathrm{Ind}(\mathrm{Mod}_{\mathcal{X}}(\mathrm{Mot}^{\mathrm{loc}}_{\mathcal{E}})) \simeq \mathrm{Mod}_{\mathcal{X}}(\mathrm{Mot}^{\mathrm{loc}}_{\mathcal{E}})$ by Efimov's rigidity theorem.

The proof of (*a*) shows that "$\lim_{r\geqslant 0}$" $\mathrm{RMod}_{V_r}(\mathcal{X})$ is idempotent in $\mathrm{Pro}(\mathrm{Mod}_{\mathcal{X}}(\mathrm{Mot}^{\mathrm{loc}}_{\mathcal{E}}))$, its dual ind-object has nuclear transition map, and the dual ind-object is sent to $\mathcal{V}$ under $\mathrm{Nuc}\,\mathrm{Ind}(\mathrm{Mod}_{\mathcal{X}}(\mathrm{Mot}^{\mathrm{loc}}_{\mathcal{E}})) \simeq \mathrm{Mod}_{\mathcal{X}}(\mathrm{Mot}^{\mathrm{loc}}_{\mathcal{E}})$. Since $\mathcal{V} \to \mathcal{X} \to \mathcal{U}$ becomes a cofibre sequence in $\mathrm{Mot}_{\mathcal{X}}(\mathrm{Mot}^{\mathrm{loc}}_{\mathcal{E}})$, it follows that the preimage of $\mathcal{U}$ under $\mathrm{Nuc}\,\mathrm{Ind}(\mathrm{Mod}_{\mathcal{X}}(\mathrm{Mot}^{\mathrm{loc}}_{\mathcal{E}})) \simeq \mathrm{Mod}_{\mathcal{X}}(\mathrm{Mot}^{\mathrm{loc}}_{\mathcal{E}})$ is obtained from $\mathcal{X}$ by killing the pro-idempotent "$\lim_{r\geqslant 0}$" $\mathrm{RMod}_{V_r}(\mathcal{X})$. This is necessarily also true as $\mathbb{E}_\infty$-$\mathcal{X}$-algebras, since the $\mathbb{E}_\infty$-structure will be idempotent over $\mathcal{X}$ by Lemma 10.9(*b*) and thus unique. Since any symmetric monoidal functor preserves killing idempotent pro-algebras with trace-class transition maps by Lemma 10.9(*c*), the statement of (*b*) follows. □

§10.4. Burklund's $\mathbb{E}_1$-structures and square-zero extensions

In this subsection we show that tensor products of two Burklund-style $\mathbb{E}_1$-structures on quotients are often trivial square zero algebras. We then use this technical result to make Theorem 10.17 applicable in many cases of interest.

For the abstract setup, let $\mathcal{C}$ be a presentable stable $\mathbb{E}_2$-monoidal ∞-category and $v\colon \mathcal{I} \to \mathbb{1}$ be a morphism in $\mathcal{C}$ such that $\mathbb{1}/v$ admits a right-unital multiplication. Fix $\alpha_0 \geqslant 3$, so that

$\mathbb{1}/v^{\alpha_0}$ admits a preferred $\mathbb{E}_2$-algebra structure by [Bur22, Theorem 1.5]. The same theorem shows that $\mathbb{1}/v^{\alpha}$ admits a preferred $\mathbb{E}_1$-algebra structure for all $\alpha \geqslant 2$. Via base change, we get an $\mathbb{E}_1$-structure on $\mathbb{1}/v^{\alpha_0} \otimes \mathbb{1}/v^{\alpha}$ in the $\mathbb{E}_1$-monoidal stable ∞-category $\mathrm{LMod}_{\mathbb{1}/v^{\alpha_0}}(\mathcal{C})$.

10.23. Proposition. — *With notation and assumptions as above, suppose additionally that $\mathcal{C}$ is rigid, $\mathcal{I}$ is dualisable in $\mathcal{C}$, and $\alpha \geqslant \alpha_0 + 3$.*

(*a*) *If we equip $\mathbb{1}/v^{\alpha_0} \oplus \Sigma(\mathcal{I}^{\otimes \alpha}/v^{\alpha_0})$ with the trivial square-zero $\mathbb{E}_1$-structure over $\mathbb{1}/v^{\alpha_0}$, then the equivalence of left $\mathbb{1}/v^{\alpha_0}$-modules*

$$\mathbb{1}/v^{\alpha_0} \otimes \mathbb{1}/v^{\alpha} \simeq \mathbb{1}/v^{\alpha_0} \oplus \Sigma(\mathcal{I}^{\otimes \alpha}/v^{\alpha_0})$$

lifts canonically to an equivalence of $\mathbb{E}_1$-algebras in $\mathrm{LMod}_{\mathbb{1}/v^{\alpha_0}}(\mathcal{C})$. Under this identification, the multiplication $\mathbb{1}/v^{\alpha_0} \otimes \mathbb{1}/v^{\alpha} \to \mathbb{1}/v^{\alpha_0}$ becomes the augmentation map $\mathbb{1}/v^{\alpha_0} \oplus \Sigma(\mathcal{I}^{\otimes \alpha}/v^{\alpha_0}) \to \mathbb{1}/v^{\alpha_0}$.

(*b*) *For all $\alpha' \geqslant \alpha \geqslant \alpha_0 + 3$, the map $\mathbb{1}/v^{\alpha_0} \otimes \mathbb{1}/v^{\alpha'} \to \mathbb{1}/v^{\alpha_0} \otimes \mathbb{1}/v^{\alpha}$ agrees with the map of trivial square-zero extensions induced by $v^{\alpha'-\alpha}\colon \mathcal{I}^{\otimes \alpha'}/v^{\alpha_0} \to \mathcal{I}^{\otimes \alpha}/v^{\alpha_0}$, as maps of $\mathbb{E}_1$-algebras in $\mathrm{LMod}_{\mathbb{1}/v^{\alpha_0}}(\mathcal{C})$.*

10.24. Remark. — The bound $\alpha \geqslant \alpha_0 + 3$ doesn't seem optimal and the author suspects that Proposition 10.23 might already be true for $\alpha \geqslant \alpha_0$. It also seems reasonable that the result should be true for any compatible $\mathbb{E}_1$-structures on $\mathbb{1}/v^{\alpha_0}$ and $\mathbb{1}/v^{\alpha}$, but we don't know how to show this.

10.25. Remark. — Since the bounds $\alpha_0 \geqslant 3$ and $\alpha \geqslant \alpha_0 + 3$ ensure that the $\mathbb{E}_1$-algebra structures on $\mathbb{1}/v^{\alpha_0}$ and $\mathbb{1}/v^{\alpha}$ refine to $\mathbb{E}_2$-algebra structures, the multplication map in Proposition 10.23(*a*) is canonically a map of $\mathbb{E}_1$-algebras. The identification with the augmentation $\mathbb{1}/v^{\alpha_0} \oplus \Sigma(\mathcal{I}^{\otimes \alpha}/v^{\alpha_0}) \to \mathbb{1}/v^{\alpha_0}$ also holds as $\mathbb{E}_1$-algebra maps (as we'll see in the proof).

Proof of Proposition 10.23. Recall [Bur22, Constructions 4.7 and 4.8]: Let $\widetilde{\mathcal{C}} \coloneqq \mathrm{Def}(\mathcal{C}, \mathcal{Q})$ be the deformation of $\mathcal{C}$ that Burklund uses. The specific construction is irrelevant for the purpose of this proof; the reader only needs to know that $\widetilde{\mathcal{C}}$ is a presentable stable $\mathbb{E}_2$-monoidal ∞-category and comes with $\mathbb{E}_2$-monoidal functors $\nu\colon \mathcal{C} \to \widetilde{\mathcal{C}}$ (which is non-exact) and $(-)^{\tau=1}\colon \widetilde{\mathcal{C}} \to \mathcal{C}$ (which preserves colimits and is therefore exact) such that $\nu(-)^{\tau=1} \simeq \mathrm{id}_{\mathcal{C}}$. Let furthermore $\widetilde{\mathbb{1}} \coloneqq \nu(\mathbb{1})$ denote the tensor unit of $\widetilde{\mathcal{C}}$ and let $\widetilde{\mathcal{I}} \coloneqq \Sigma^{-1}\nu(\Sigma\mathcal{I})$. Even though ν is non-exact, $\nu(\mathbb{1}) \to \nu(\mathbb{1}/v) \to \nu(\Sigma\mathcal{I})$ is still a cofibre sequence in $\widetilde{\mathcal{C}}$ and so $\nu(v)\colon \nu(\mathcal{I}) \to \widetilde{\mathbb{1}}$ factors through a map

$$\widetilde{v}\colon \widetilde{\mathcal{I}} \longrightarrow \widetilde{\mathbb{1}}\,.$$

Then $\widetilde{v}$ is a deformation of v in the sense that $\widetilde{v}^{\tau=1} \simeq v$.(10.7) It will thus be enough to show the assertions with v replaced by $\widetilde{v}\colon \widetilde{\mathcal{I}} \to \widetilde{\mathbb{1}}$.

Burklund constructs $\mathbb{E}_1$-structures on $\widetilde{\mathbb{1}}/\widetilde{v}^{\alpha}$ for $\alpha \geqslant 2$ using the obstruction theory from [Bur22, Proposition 2.4] in $\widetilde{\mathcal{C}}$. The reason to replace $\mathcal{C}$ and v by their deformations $\widetilde{\mathcal{C}}$ and $\widetilde{v}$ is that for the deformed versions all obstructions vanish (because the obstruction group vanishes), and the witnessing nullhomotopies are unique (because the next homotopy group also vanishes).

The base-changed $\mathbb{E}_1$-structure on $\widetilde{\mathbb{1}}/\widetilde{v}^{\alpha_0} \otimes \widetilde{\mathbb{1}}/\widetilde{v}^{\alpha}$ is then obtained via Burklund's obstruction theory in the $\mathbb{E}_1$-monoidal(10.8) presentable stable ∞-category $\mathrm{LMod}_{\widetilde{\mathbb{1}}/\widetilde{v}^{\alpha_0}}(\widetilde{\mathcal{C}})$. The main step to

(10.7) Note that $\widetilde{v}$ is usually *not* the trivial deformation $\nu(v)$, as the canonical map $\nu(\mathcal{I}) \to \widetilde{\mathcal{I}}$ is usually not an equivalence. This is crucial to make Burklund's construction work.

(10.8) Burklund's paper assumes an $\mathbb{E}_2$-monoidal structure, but for the purpose of [Bur22, §2] only an $\mathbb{E}_1$-monoidal structure is necessary.

prove both (a) and (b) is to show that in this case too all obstructions vanish and the witnessing nullhomotopies are unique. More precisely, we'll show that for all $k \geqslant 2$ and all $\alpha' \geqslant \alpha \geqslant \alpha_0 + 3$,

$$\pi_i \operatorname{Hom}_{\mathrm{LMod}_{\widetilde{\mathbb{1}}/\widetilde{v}^{\alpha_0}}(\widetilde{\mathcal{C}})}\left(\Sigma^{-3}\left(\Sigma^2(\widetilde{\mathcal{I}}/\widetilde{v}^{\alpha_0})^{\otimes\alpha'}\right)^{\otimes k}, \widetilde{\mathbb{1}}/\widetilde{v}^{\alpha_0} \otimes \widetilde{\mathbb{1}}/\widetilde{v}^{\alpha}\right) \cong 0 \quad \text{for } i \in \{0,1\}\,.$$

To show this, we use that $\widetilde{\mathbb{1}}/\widetilde{v}^{\alpha_0} \otimes - \colon \widetilde{\mathcal{C}} \to \mathrm{LMod}_{\widetilde{\mathbb{1}}/\widetilde{v}^{\alpha_0}}(\widetilde{\mathcal{C}})$ is left adjoint to the forgetful functor, that $\widetilde{\mathbb{1}}/\widetilde{v}^{\alpha_0} \otimes \widetilde{\mathbb{1}}/\widetilde{v}^{\alpha} \simeq \widetilde{\mathbb{1}}/\widetilde{v}^{\alpha_0} \oplus \Sigma(\widetilde{\mathcal{I}}^{\otimes\alpha}/\widetilde{v}^{\alpha_0})$ as left-$\widetilde{\mathbb{1}}/\widetilde{v}^{\alpha_0}$-modules, and that $\widetilde{\mathcal{I}}$ is still dualisable, with dual $\widetilde{\mathcal{I}}^\vee \simeq \Sigma\nu(\Sigma^{-1}\mathcal{I}^\vee)$. The left-hand side above can then be rewritten as follows:

$$\begin{aligned}
&\pi_i \operatorname{Hom}_{\widetilde{\mathcal{C}}}\left(\Sigma^{2k-3}\widetilde{\mathcal{I}}^{\otimes\alpha' k}, \widetilde{\mathbb{1}}/\widetilde{v}^{\alpha_0} \oplus \Sigma(\widetilde{\mathcal{I}}^{\otimes\alpha}/\widetilde{v}^{\alpha_0})\right)\\
&\qquad\cong \pi_i \operatorname{Hom}_{\widetilde{\mathcal{C}}}\left(\Sigma^{2k-3}\widetilde{\mathcal{I}}^{\otimes\alpha' k}, \widetilde{\mathbb{1}}/\widetilde{v}^{\alpha_0}\right) \oplus \pi_i \operatorname{Hom}_{\widetilde{\mathcal{C}}}\left(\Sigma^{2k-2}\widetilde{\mathcal{I}}^{\otimes\alpha' k} \otimes (\widetilde{\mathcal{I}}^\vee)^{\otimes\alpha'}, \widetilde{\mathbb{1}}/\widetilde{v}^{\alpha_0}\right)\\
&\qquad\cong \pi_i \operatorname{Hom}_{\widetilde{\mathcal{C}}}\left(\Sigma^{-\alpha' k+2k-3}\nu(X), \widetilde{\mathbb{1}}/\widetilde{v}^{\alpha_0}\right) \oplus \pi_i \operatorname{Hom}_{\widetilde{\mathcal{C}}}\left(\Sigma^{-\alpha' k+\alpha+2k-2}\nu(Y), \widetilde{\mathbb{1}}/\widetilde{v}^{\alpha_0}\right),
\end{aligned}$$

where $X \simeq (\Sigma\mathcal{I})^{\otimes\alpha' k}$ and $Y \simeq (\Sigma\mathcal{I})^{\otimes\alpha' k} \otimes (\Sigma^{-1}\mathcal{I}^\vee)^{\otimes\alpha}$. According to [Bur22, Lemma 4.8] (which is applicable thanks to our rigidity assumption on $\mathcal{C}$), both summands on the right-hand side vanish for $i \in \{0,1\}$ as soon as $\alpha' k - \alpha - 2k + 1 \geqslant \alpha_0$. Under our assumptions $\alpha' \geqslant \alpha \geqslant \alpha_0 + 3$ and $k \geqslant 2$, we can estimate

$$\alpha' k - \alpha - 2k + 1 \geqslant (\alpha_0 + 3)(k-1) - 2k + 1 = (k-1)\alpha_0 + k - 2 \geqslant \alpha_0\,,$$

as desired. This shows that indeed all obstructions vanish (because the obstruction group π_0 vanishes) and the witnessing nullhomotopies are unique (because π_1 also vanishes).

Now (b) as well as the first part of (a) immediately follow. Indeed, in the case $\alpha' = \alpha$, the vanishing result above combined with [Bur22, Remark 2.5] shows that the $\mathbb{E}_1$-structure on $\widetilde{\mathbb{1}}/\widetilde{v}^{\alpha_0} \otimes \widetilde{\mathbb{1}}/\widetilde{v}^{\alpha}$ is unique, so it has to be the trivial square zero structure. For general $\alpha' \geqslant \alpha$, the same argument shows that the $\mathbb{E}_1$-map $\widetilde{\mathbb{1}}/\widetilde{v}^{\alpha_0} \otimes \widetilde{\mathbb{1}}/\widetilde{v}^{\alpha'} \to \widetilde{\mathbb{1}}/\widetilde{v}^{\alpha_0} \otimes \widetilde{\mathbb{1}}/\widetilde{v}^{\alpha}$ is unique, proving (b). To show the second part of (a), observe that, with notation as above, we must also have

$$\pi_i \operatorname{Hom}_{\widetilde{\mathcal{C}}}\left(\Sigma^{-\alpha' k+2k-3}\nu(X), \widetilde{\mathbb{1}}/\widetilde{v}^{\alpha_0}\right) \cong 0 \quad \text{for } i \in \{0,1\}\,.$$

This precisely ensures that $\widetilde{\mathbb{1}}/\widetilde{v}^{\alpha_0} \otimes \widetilde{\mathbb{1}}/\widetilde{v}^{\alpha} \to \widetilde{\mathbb{1}}/\widetilde{v}^{\alpha_0}$ is unique as well, and so it has to be the augmentation map. □

10.26. Corollary. — *If $\mathcal{I}$ is dualisable, $\alpha \geqslant \alpha_0 + 3$, and $\alpha' \geqslant \alpha + \alpha_0$, then*

$$\mathbb{1}/v^{\alpha_0} \otimes \mathbb{1}/v^{\alpha'} \longrightarrow \mathbb{1}/v^{\alpha_0} \otimes \mathbb{1}/v^{\alpha}$$

factors through the tensor unit $\mathbb{1}/v^{\alpha_0}$ as a map of $\mathbb{E}_1$-algebras in $\mathrm{LMod}_{\mathbb{1}/v^{\alpha_0}}(\mathcal{C})$.

Proof. By Proposition 10.23(b), it's enough to check that $v^{\alpha'-\alpha} \colon \mathcal{I}^{\otimes\alpha'}/v^{\alpha_0} \to \mathcal{I}^{\otimes\alpha}/v^{\alpha_0}$ is zero in $\mathrm{LMod}_{\mathbb{1}/v^{\alpha_0}}(\mathcal{C})$ for $\alpha' \geqslant \alpha + \alpha_0$. This reduces to $v^{\alpha_0} \colon \mathcal{I}^{\otimes\alpha_0}/v^{\alpha_0} \to \mathbb{1}/v^{\alpha_0}$ being zero in $\mathrm{LMod}_{\mathbb{1}/v^{\alpha_0}}(\mathcal{C})$. Since $\mathbb{1}/v^{\alpha_0} \otimes - \colon \mathcal{C} \to \mathrm{LMod}_{\mathbb{1}/v^{\alpha_0}}(\mathcal{C})$ is left adjoint to the forgetful functor, this is equivalent to $v^{\alpha_0} \colon \mathcal{I}^{\otimes\alpha_0} \to \mathbb{1}/v^{\alpha_0}$ being zero in $\mathcal{C}$, which is true by construction. □

Thanks to Corollary 10.26, it is now easy to construct examples where Assumption 10.16(V) is satisfied and thus Theorem 10.17 is applicable.

10.27. Example. — Let m be a positive integer that is either coprime to 2 or divisible by 4. Then $\mathbb{S}/m$ admits a right-unital multiplication and so Burklund's construction applied to $m\colon \mathbb{S} \to \mathbb{S}$ provides a tower of $\mathbb{E}_1$-algebras

$$\mathbb{S}/m^2 \longleftarrow \mathbb{S}/m^3 \longleftarrow \mathbb{S}/m^4 \longleftarrow \cdots .$$

Up to passing to an appropriate subtower, this satisfies Assumption 10.16(V). Indeed, dualisability and the thick tensor ideal condition are clear and the factorisation condition follows from Corollary 10.26 above.

Thus, for any $\mathbb{E}_\infty$-ring spectrum k, Theorem 10.17 shows that $\mathrm{THH}^{\mathrm{ref}}(k[1/m]/k)$ is obtained from $\mathrm{THH}(k/k) \simeq k$ by killing the idempotent pro-algebra "$\lim_{\alpha\geqslant 1}$" $\mathrm{THH}((k \otimes \mathbb{S}/m^\alpha)/k)$. In particular, there's a cofibre sequence

$$\text{“}\operatorname*{colim}_{\alpha\geqslant 1}\text{”}\, \mathrm{THH}\big((k \otimes \mathbb{S}/m^\alpha)/k\big)^\vee \longrightarrow k \longrightarrow \mathrm{THH}^{\mathrm{ref}}\big(k\big[\tfrac{1}{m}\big]/k\big) .$$

in $\mathrm{Nuc}\,\mathrm{Ind}(\mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1})$. Since $\mathrm{THH}^{\mathrm{ref}}(-/k)$ commutes with filtered colimits, this also allows us to compute $\mathrm{THH}^{\mathrm{ref}}(k \otimes \mathbb{Q}/k) \simeq \operatorname{colim}_{m\in\mathbb{N}} \mathrm{THH}^{\mathrm{ref}}(k[1/m]/k)$.

10.28. Example. — If k is any $\mathbb{E}_\infty$-ring spectrum, we can compute $\mathrm{THH}^{\mathrm{ref}}(k[x]/k)$ as follows: Let $\mathbb{P}^1_k$ denote the flat projective line over k, which is smooth and proper over k. We can construct a tower of $\mathbb{E}_1$-algebras

$$k[x^{-1}]/x^{-1} \longleftarrow k[x^{-1}]/x^{-2} \longleftarrow k[x^{-1}]/x^{-3} \longleftarrow \cdots$$

either by hand (construct $k[x^{-1}]$ as a graded $\mathbb{E}_\infty$-k-algebra with x^{-1} in graded degree -1, then truncate the grading) or by applying Burklund's construction to $\mathcal{O}_{\mathbb{P}^1_k}(-1) \to \mathcal{O}_{\mathbb{P}^1_k}$ (this will only give the tower from the second step onwards, but this is no problem). In either case, Assumption 10.16(V) will be satisfied and so Theorem 10.17 provides a cofibre sequence

$$\text{“}\operatorname*{colim}_{\alpha\geqslant 1}\text{”}\, \mathrm{THH}\big((k[x^{-1}]/x^{-\alpha})/k\big)^\vee \longrightarrow \mathrm{THH}(\mathbb{P}^1_k/k) \longrightarrow \mathrm{THH}^{\mathrm{ref}}\big(k[x]/k\big)$$

in $\mathrm{Nuc}\,\mathrm{Ind}(\mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1})$.

As a final example, let us explain how Theorem 10.17 applies to $\mathrm{THH}^{\mathrm{ref}}(\mathrm{L}_n^f\mathbb{S}_{(p)}/\mathbb{S}_{(p)})$, where L_n^f denotes telescopic localisation to chromatic height $\leqslant n$. First we need a technical lemma:

10.29. Lemma. — *Let $m \geqslant 2$ and $n \geqslant 0$. Let $V' \to V$ be a map of $\mathbb{E}_{m+1}$-algebras whose underlying spectra are of type n. Let $v\colon \Sigma^N V \to V$ be a v_n-self map of V and $v'\colon \Sigma^{N'}V' \to V'$ a v_n-self map of V'.*

(*a*) *Up to replacing v' by a suitable power, the induced map $v' \otimes_{V'} V\colon \Sigma^{N'}V \to V$ can be chosen to be a power of v.*

(*b*) *Suppose v is the fourth power of another v_n-self map of V, so that V/v admits a right-unital multiplication in $\mathrm{LMod}_V(\mathrm{Sp}_{(p)})$. Furthermore, assume that v' is as in (a) and V'/v' admits a right-unital multiplication in $\mathrm{LMod}_{V'}(\mathrm{Sp})$. Then the canonical left-V-module map*

$$V'/v'^{m+1} \otimes_{V'} V \longrightarrow V/v^{m+1} .$$

can be upgraded to an $\mathbb{E}_m$-algebra map in $\mathrm{LMod}_V(\mathrm{Sp})$, where we equip V/v^{m+1} and V'/v'^{m+1} with Burklund's $\mathbb{E}_m$-structures in $\mathrm{LMod}_V(\mathrm{Sp})$ and $\mathrm{LMod}_{V'}(\mathrm{Sp})$, respectively.

Proof sketch. Part (*a*) follows immediately from asymptotic uniqueness of v_n-self maps (see [L-Ch, Lemma 27.10] for example).

To show (*b*), let us denote $V/v'^{m+1} \coloneqq V'/v'^{m+1} \otimes_{V'} V$ for short. First note that the claim is not completely automatic, since the $\mathbb{E}_m$-structures on V/v^{m+1} and V/v'^{m+1} are constructed via different deformation categories. More precisely, let $\mathcal{Q}$ and $\mathcal{Q}'$ be the classes of morphisms in $\mathrm{LMod}_V(\mathrm{Sp})^\omega$ that become split epimorphisms upon $- \otimes_V V/v$ or $- \otimes_V V/v'$, respectively. Then the $\mathbb{E}_m$-structure on V/v^{m+1} is constructed via $\mathrm{Def}(\mathrm{LMod}_V(\mathrm{Sp}); \mathcal{Q})$, whereas for V/v'^{m+1} we use $\mathrm{Def}(\mathrm{LMod}_V(\mathrm{Sp}); \mathcal{Q}')$.

Our assumptions on v and v' imply that $V/v' \to V/v$ can be turned into an $\mathbb{E}_1$-map in $\mathrm{LMod}_V(\mathrm{Sp})$. This need not be compatible with the $\mathbb{E}_1$-structures on V/v^{m+1} or V/v'^{m+1}, but it is enough to ensure $\mathcal{Q}' \subseteq \mathcal{Q}$, because any morphism that becomes split after $- \otimes_V V/v'$ will also become split after $(- \otimes_V V/v') \otimes_{V/v'} V/v \simeq - \otimes_V V/v$. Sheafification then induces a strongly continuous $\mathbb{E}_{m+1}$-monoidal functor $\mathrm{Def}(\mathrm{LMod}_V(\mathrm{Sp}); \mathcal{Q}') \to \mathrm{Def}(\mathrm{LMod}_V(\mathrm{Sp}); \mathcal{Q})$ which fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Def}\big(\mathrm{LMod}_V(\mathrm{Sp}); \mathcal{Q}'\big) & \longrightarrow & \mathrm{Def}\big(\mathrm{LMod}_V(\mathrm{Sp}); \mathcal{Q}\big) \\ {\scriptstyle \nu'}\big\uparrow & \nearrow{\scriptstyle \nu} & \\ \mathrm{LMod}_V(\mathrm{Sp}) & & \end{array}$$

where ν and ν' denote the respective Yoneda embeddings.

Let us now denote deformations in $\mathrm{Def}(\mathrm{LMod}_V(\mathrm{Sp}); \mathcal{Q})$ by $(\widetilde{-})$ as in the proof of Proposition 10.23. Via the functor above and [Bur22, Proposition 2.4], we can write $\widetilde{V}/\widetilde{v}'^{m+1}$ as an iterated pushout of $\mathbb{E}_m$-algebras in $\mathrm{Def}(\mathrm{LMod}_V(\mathrm{Sp}); \mathcal{Q})$. This yields a sequence of obstructions to constructing an $\mathbb{E}_m$-algebra map $\widetilde{V}/\widetilde{v}'^{m+1} \to \widetilde{V}/\widetilde{v}^{m+1}$. Since the functor $\mathrm{Def}(\mathrm{LMod}_V(\mathrm{Sp}); \mathcal{Q}') \to \mathrm{Def}(\mathrm{LMod}_V(\mathrm{Sp}); \mathcal{Q})$ intertwines ν' and ν, the obstructions are still of the form that automatically vanishes. $\square$

10.30. Example. — For all $m \geqslant 2$ and $n \geqslant 0$ let us construct a tower of $\mathbb{E}_m$-algebras

$$V(n)_0 \longleftarrow V(n)_1 \longleftarrow V(n)_2 \longrightarrow \cdots$$

of the form $V(n)_r \simeq \mathbb{S}/(p^{\alpha_{r,0}}, v_1^{\alpha_{r,1}}, \dots, v_n^{\alpha_{r,n}})$, such that Assumption 10.16(V) is satisfied. Note that the dualisability condition in 10.16(V) is trivial and the thick tensor ideal condition is automatic by the thick subcategory theorem (see [L-Ch, Theorem 26.8] for example). So we only have to construct the tower and verify the factorisation condition.

We use induction on n. Suppose we've already constructed a tower of $\mathbb{E}_{m+1}$-algebras $(V(n-1)_r)_{r\geqslant 0}$ with the desired properties. We'll write $V_r \coloneqq V(n-1)_r$ for brevity. Using Lemma 10.29 for $V_{r+1} \to V_r$, we can inductively construct v_n-self maps $v_{n,r}\colon \Sigma^{N_r} V_r \to Vr$ such that each of them is the fourth power of another v_n-self map and the quotients

$$\overline{V}_r \coloneqq V_r/v_{n,r}^{2^r(m+1)}$$

fit into a tower of $\mathbb{E}_m$-algebras. Note that this would already work with $V_r/v_{n,r}^{m+1}$; the extra factor in the exponent will only be used for the factorisation condition.

As in the proof of Lemma 10.21, consider the right-$V_{r+1} \otimes V_r^{\mathrm{op}}$-module structure on $\overline{V}_{r+1} \otimes \overline{V}_r$ given by its "inner" bimodule structure. Since $V_{r+1} \otimes V_r \to V_r \otimes V_r$ factors through V_r by the inductive hypothesis, we see that $\overline{V}_{r+1} \otimes \overline{V}_r \to \overline{V}_r \otimes \overline{V}_r$ factors through

$$\big(\overline{V}_{r+1} \otimes \overline{V}_r\big) \otimes_{V_{r+1} \otimes V_r^{\mathrm{op}}} V_r \simeq \big(\overline{V}_{r+1} \otimes_{V_{r+1}} V_r\big) \otimes_{V_r} \overline{V}_r$$

as a map of $\overline{V}_{r+1}$-$\overline{V}_r$-bimodules. If we now consider the composition $\overline{V}_{r+2} \otimes \overline{V}_r \to \overline{V}_r \otimes \overline{V}_r$, we see that it factors through

$$V_r/v_{n,r+1}^{2^{r+2}(m+1)} \otimes_{V_r} \overline{V}_r \longrightarrow V_r/v_{n,r+1}^{2^{r+1}(m+1)} \otimes_{V_r} \overline{V}_r\,.$$

This, in turn, factors through $\overline{V}_r$ as a map of $\mathbb{E}_1$-algebras in $\mathrm{RMod}_{\overline{V}_r}(\mathrm{Sp})$. Indeed, this follows from Corollary 10.26 via base change along $V_r/v_{n,r+1}^{2^r(m+1)} \to V_r/v_{n,r}^{2^r(m+1)} \simeq \overline{V}_r$. So we get the desired factorisation for $\overline{V}_{r+2} \otimes \overline{V}_r \to \overline{V}_r \otimes \overline{V}_r$. Thus, if we put $V(n)_r \coloneqq \overline{V}_{2r}$, we get a tower of the desired form.

With these disgusting technicalities out of the way, we can finally apply Theorem 10.17: We deduce that $\mathrm{THH}^{\mathrm{ref}}(\mathrm{L}_n^f\mathbb{S}_{(p)}/\mathbb{S}_{(p)})$ is obtained from $\mathbb{S}_{(p)}$ by killing an idempotent pro-algebra of the form "$\lim_{r\geqslant 0}$" $\mathrm{THH}(\mathbb{S}/(p^{\alpha_{r,0}}, v_1^{\alpha_{r,1}}, \dots, v_n^{\alpha_{r,n}}))$. In particular, we get a cofibre sequence

$$\text{“}\operatorname*{colim}_{r\geqslant 0}\text{”}\, \mathrm{THH}\big(\mathbb{S}/(p^{\alpha_{r,0}}, v_1^{\alpha_{r,1}}, \dots, v_n^{\alpha_{r,n}})\big)^\vee \longrightarrow \mathbb{S}_{(p)} \longrightarrow \mathrm{THH}^{\mathrm{ref}}\big(\mathrm{L}_n^f\mathbb{S}_{(p)}/\mathbb{S}_{(p)}\big)$$

in $\mathrm{Nuc}\,\mathrm{Ind}(\mathrm{Sp}_{(p)}^{\mathrm{B}S^1})$.

§11. Refined THH and TC$^-$ over ku

We've seen in Example 10.27 that to compute $\mathrm{THH}^{\mathrm{ref}}(\mathbb{Q})$, one essentially has to compute an ind-object of the form "$\operatorname{colim}_{\alpha\geqslant 2}$" $\mathrm{THH}(\mathbb{S}/p^\alpha)^\vee$ for all primes p. This seems currently out of reach. However, after base change to ku, we can get some control over $\mathrm{THH}((\mathrm{ku}\otimes\mathbb{S}/p^\alpha)/\mathrm{ku})$ thanks to the results from Part II, and so $\mathrm{THH}^{\mathrm{ref}}(\mathrm{ku}\otimes\mathbb{Q}/\mathrm{ku})$ is approachable.

In this section we study $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku}\otimes\mathbb{Q}/\mathrm{ku})$ and $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU}\otimes\mathbb{Q}/\mathrm{KU})$, which contain the same information as $\mathrm{THH}^{\mathrm{ref}}(\mathrm{ku}\otimes\mathbb{Q}/\mathrm{ku})$ and $\mathrm{THH}^{\mathrm{ref}}(\mathrm{KU}\otimes\mathbb{Q}/\mathrm{KU})$ by Lemma 11.2 below. In §11.1, we compute the homotopy groups

$$\mathrm{A}_{\mathrm{ku}}^* \coloneqq \pi_{2*}\,\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku}\otimes\mathbb{Q}/\mathrm{ku}) \quad\text{and}\quad \mathrm{A}_{\mathrm{KU}} \coloneqq \pi_0\,\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU}\otimes\mathbb{Q}/\mathrm{KU})$$

in terms of certain q-Hodge filtrations $\mathrm{fil}^\star_{q\text{-Hdg}}\,q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ and the associated q-Hodge complexes $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ that we get from the chosen $\mathbb{E}_1$-structures on $\mathbb{S}/p^\alpha$. In §11.2 we'll explain how to describe these objects explicitly. These explicit descriptions will then be used in §12 to finish the proof of Theorems 1.40 and 1.41.

11.1. Convention — Throughout §§11–12, all (q-)de Rham complexes and q-Hodge complexes relative to a p-complete ring will be implicitly p-completed.

§11.1. q-Hodge filtrations and $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku}\otimes\mathbb{Q}/\mathrm{ku})$

We begin by showing that for complex orientable ring spectra k, $\mathrm{THH}^{\mathrm{ref}}(k\otimes\mathbb{Q}/k)$ with its S^1-action contains the same information as $\mathrm{TC}^{-,\mathrm{ref}}(k\otimes\mathbb{Q}/k)$.

11.2. Lemma. — *Let k be a complex orientable $\mathbb{E}_\infty$-ring spectrum, equipped with trivial S^1-action, and let $t\in\pi_{-2}(k^{hS^1})$ be any complex orientation. Then taking S^1-fixed points defines a symmetric monoidal equivalence*

$$(-)^{hS^1}\colon \mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1} \xrightarrow{\ \simeq\ } \mathrm{Mod}_{k^{hS^1}}(\mathrm{Sp})_t^\wedge\,,$$

where $\mathrm{Mod}_{k^{hS^1}}(\mathrm{Sp})_t^\wedge$ denotes ∞-category of t-complete k^{hS^1}-module spectra, which we equip with the t-completed tensor product $-\mathbin{\widehat{\otimes}}_{k^{hS^1}}-$.

Proof. By construction $(-)^{hS^1}$ is lax symmetric monoidal. To see that it is strictly symmetric monoidal, we must check whether $M^{hS^1}\mathbin{\widehat{\otimes}}_{k^{hS^1}}N^{hS^1}\to(M\otimes_k N)^{hS^1}$ is an equivalence. As both sides are t-complete, this can be checked modulo t, where it follows from [HRW22, Lemma 2.2.10] for example.

By definition, $(-)^{hS^1}\colon \mathrm{Sp}^{\mathrm{B}S^1}\to\mathrm{Sp}$ has a left adjoint, given by the symmetric monoidal functor $\mathrm{const}\colon\mathrm{Sp}\to\mathrm{Sp}^{\mathrm{B}S^1}$, which sends a spectrum X to itself equipped with the trivial S^1-action. By general nonsense about how symmetric monoidal adjunctions pass to module categories, we see that $(-)^{hS^1}\colon\mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1}\simeq\mathrm{Mod}_k(\mathrm{Sp}^{\mathrm{B}S^1})\to\mathrm{Mod}_{k^{hS^1}}(\mathrm{Sp})$ admits a left adjoint L, which is given as the composition

$$L\colon \mathrm{Mod}_{k^{hS^1}}(\mathrm{Sp})\xrightarrow{\mathrm{const}}\mathrm{Mod}_{k^{hS^1}}(\mathrm{Sp}^{\mathrm{B}S^1})\xrightarrow{-\otimes_{k^{hS^1}}k}\mathrm{Mod}_k\big(\mathrm{Sp}^{\mathrm{B}S^1}\big)\simeq\mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1}\,.$$

In particular, on underlying k-modules, L is simply given by $(-)/t$. Since $(-)/t$ is conservative on t-complete k^{hS^1}-modules, it follows that $L\colon\mathrm{Mod}_{k^{hS^1}}(\mathrm{Sp})_t^\wedge\to\mathrm{Mod}_k(\mathrm{Sp})^{\mathrm{B}S^1}$ must be conservative too. Furthermore, the counit $c\colon L((-)^{hS^1})\Rightarrow\mathrm{id}$ is an equivalence, as follows from

[HRW22, Lemma 2.2.10] again. Thus $(-)^{hS^1}$ must be fully faithful. We conclude using the standard fact that an adjunction in which the right adjoint is fully faithful and the left adjoint is conservative must be a pair of inverse equivalences. □

We'll now set out to compute $\pi_* \mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku}\otimes\mathbb{Q}/\mathrm{ku})$ and $\pi_* \mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU}\otimes\mathbb{Q}/\mathrm{KU})$.

11.3. Outline of the computation. — For convenience, let's call a positive integer m *high-powered* if its prime factorisation $m=\prod_p p^{\alpha_p}$ has the following property: For all primes $p>2$ either $\alpha_p=0$ or $\alpha_p\geqslant 2$ and for $p=2$ either $\alpha_2=0$ or α_2 is even and $\geqslant 4$. We let $\mathbb{N}^\sharp$ denote the set of high-powered positive integers, partially ordered by divisibility.

Since $\mathbb{S}/4$ and $\mathbb{S}/p$ admit right-unital multiplications, we can use Burklund's general construction [Bur22, Theorem 1.5][11.1] to construct $\mathbb{E}_1$-structures on

$$\mathbb{S}/m\simeq\prod_p \mathbb{S}/p^{\alpha_p}$$

for every high-powered m. These assemble into a functor $\mathbb{S}/-\colon \mathbb{N}^\sharp\longrightarrow \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Sp})$. In the following we'll write $\mathrm{ku}/m\coloneqq \mathrm{ku}\otimes\mathbb{S}/m$ and $\mathrm{KU}/m\coloneqq\mathrm{KU}\otimes\mathbb{S}/m$, where it is understood that the $\mathbb{E}_1$-structure is always base changed from the one on $\mathbb{S}/m$ above. By Example 10.27 and Lemma 11.2, we get a cofibre sequence

$$\underset{m\in(\mathbb{N}^\sharp)^{\mathrm{op}}}{\text{“colim”}}\mathrm{TC}^-\big((\mathrm{ku}/m)/\mathrm{ku}\big)^\vee\longrightarrow \mathrm{ku}^{hS^1}\longrightarrow \mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku}\otimes\mathbb{Q}/\mathrm{ku})$$

(where now $(-)^\vee\coloneqq \mathrm{Hom}_{\mathrm{ku}^{hS^1}}(-,\mathrm{ku}^{hS^1})$ denotes the dual in ku^{hS^1}-modules) and a similar one for KU. To compute the pro-object on the left, we'll proceed in three steps:

(*a*) We compute $\pi_*\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})$ and $\pi_*\mathrm{TC}^-((\mathrm{KU}/m)/\mathrm{KU})$ using Theorem 7.27. This will be the content of Corollary 11.7.

(*b*) We compute $\pi_*\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})^\vee$ and $\pi_*\mathrm{TC}^-((\mathrm{KU}/m)/\mathrm{KU})^\vee$, essentially showing that in this case taking duals commutes with π_* in a derived way. This will be achieved in Corollary 11.11.

(*c*) We show that pro-idempotence and the transition maps being trace-class passes to homotopy groups in this case. This will be the content of Corollaries 11.13 and 11.14.

This leads to a preliminary description of the homotopy rings $\pi_*\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku}\otimes\mathbb{Q}/\mathrm{ku})$ and $\pi_*\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU}\otimes\mathbb{Q}/\mathrm{KU})$ in Theorem 11.15.

We begin with step (*a*).

11.4. Reduction to the *p*-torsion free case. — Decomposing $m=\prod_p p^{\alpha_p}$ into prime powers, we have

$$\mathrm{TC}^-\big((\mathrm{ku}/m)/\mathrm{ku}\big)\simeq\prod_p\mathrm{TC}^-\big((\mathrm{ku}/p^{\alpha_p})/\mathrm{ku}\big)\,,$$

so we may reduce to the case where $m=p^\alpha$ is a high-powered prime power. Let us remark that $\mathrm{TC}^-((\mathrm{ku}/p^\alpha)/\mathrm{ku})$ is automatically p-complete. Indeed, it is (β,t)-complete and $\mathrm{TC}^-((\mathrm{ku}/p^\alpha)/\mathrm{ku})/(\beta,t)\simeq\mathrm{HH}((\mathbb{Z}/p^\alpha)/\mathbb{Z})$ is p^α-torsion, hence p-complete.

[11.1] We could also use [Bur22, Theorem 3.2] to get another tower of $\mathbb{E}_1$-algebras $\mathbb{S}/8\leftarrow\mathbb{S}/16\leftarrow\mathbb{S}/32\leftarrow\cdots$. This one is potentially different from ours (as different deformation categories are used in the construction). It will become apparent in 11.4 why we made that choice.

To compute $\mathrm{TC}^-((\mathrm{ku}/p^\alpha)/\mathrm{ku})$, we lift to a p-torsion free case. Let $\mathbb{Z}_p\{x\}_\infty$ be the free p-complete perfect δ-ring on a generator x and let $\mathbb{S}_{\mathbb{Z}_p\{x\}_\infty}$ be its unique lift to a p-complete connective $\mathbb{E}_\infty$-ring spectrum (see Example 9.2). By [Bur22, Theorem 1.5], we get a tower of $\mathbb{E}_1$-algebras in $\mathbb{S}_{\mathbb{Z}_p\{x\}_\infty}$-modules

$$\mathbb{S}_{\mathbb{Z}_p\{x\}_\infty}/x^2 \longleftarrow \mathbb{S}_{\mathbb{Z}_p\{x\}_\infty}/x^3 \longleftarrow \mathbb{S}_{\mathbb{Z}_p\{x\}_\infty}/x^4 \longleftarrow \cdots$$

for $p > 2$; the case $p = 2$ needs powers of x^2 instead. The map of perfect δ-rings $\mathbb{Z}_p\{x\}_\infty \to \mathbb{Z}_p$ sending $x \mapsto p$ lifts uniquely to an $\mathbb{E}_\infty$-map $\mathbb{S}_{\mathbb{Z}_p\{x\}_\infty} \to \mathbb{S}_p$. If we base change the tower above along this map, we get the tower of $\mathbb{E}_1$-algebras $(\mathbb{S}/p^\alpha)$ from 11.3. Indeed, this follows from the uniqueness statement in [Bur22, Theorem 1.5].(11.2)

Now put $\mathrm{ku}_{\mathbb{Z}_p\{x\}_\infty} \coloneqq (\mathrm{ku} \otimes \mathbb{S}_{\mathbb{Z}_p\{x\}_\infty})_p^\wedge$. Then $\mathrm{THH}(-/\mathrm{ku}_{\mathbb{Z}_p\{x\}_\infty})_p^\wedge \simeq \mathrm{THH}(-/\mathrm{ku})_p^\wedge$ holds by the same argument as in [BMS19, Proposition 11.7] and so we get a base change equivalence

$$\Bigl(\mathrm{TC}^-\bigl((\mathrm{ku}_{\mathbb{Z}_p\{x\}_\infty}/x^\alpha)/\mathrm{ku}\bigr) \otimes_{\mathrm{ku}_{\mathbb{Z}_p\{x\}_\infty}} \mathrm{ku}_p^\wedge\Bigr)_{(p,t)}^\wedge \xrightarrow{\simeq} \mathrm{TC}^-\bigl((\mathrm{ku}/p^\alpha)/\mathrm{ku}\bigr)\,.$$

11.5. A q-Hodge filtration for $\mathbb{Z}/m$. — We can apply Theorem 7.18 to $\mathbb{Z}_p\{x\}_\infty/x^\alpha$ with its spherical $\mathbb{E}_1$-lift $\mathbb{S}_{\mathbb{Z}_p\{x\}_\infty}/x^\alpha$ to obtain a q-Hodge filtration $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{(\mathbb{Z}_p\{x\}_\infty/x^\alpha)/\mathbb{Z}_p}$, which doesn't depend on the choice of spherical lift (only its existence). We then construct a filtration on $q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ as the base change

$$\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} \coloneqq \Bigl(\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{(\mathbb{Z}_p\{x\}_\infty/x^\alpha)/\mathbb{Z}_p} \otimes^{\mathrm{L}}_{\mathbb{Z}_p\{x\}_\infty} \mathbb{Z}_p\Bigr)_{(p,q-1)}^\wedge\,.$$

For a general high-powered positive integer $m \in \mathbb{N}^{\not\downarrow}$ with prime factorisation $m = \prod_p p^{\alpha_p}$, we put

$$\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{(\mathbb{Z}/m)/\mathbb{Z}} \coloneqq \prod_p \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^{\alpha_p})/\mathbb{Z}_p}\,,$$

and denote its completion by $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$. We regard these filtrations as filtered modules over $(q-1)^\star\mathbb{Z}[\![q-1]\!]$, which we identify with $\mathbb{Z}[\beta][\![t]\!]$ as in 7.26. We can also form the associated q-Hodge complex $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}$ as in 3.5.

11.6. Lemma. — *As the notation suggests, $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{(\mathbb{Z}/m)/\mathbb{Z}}$ is indeed a q-Hodge filtration in the sense of Definition 3.2. Moreover, $q\text{-}\mathrm{dR}_{(\mathbb{Z}/m)/\mathbb{Z}}$ and $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}$ are static $(q-1)$-torsion free rings and the q-Hodge filtration is a descending filtration by ideals.*

Proof. For any prime p, $q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^{\alpha_p})/\mathbb{Z}}$ vanishes after $(-)[1/p]_{(q-1)}^\wedge$, as $(\mathbb{Z}/p^{\alpha_p})[1/p] \cong 0$, and thus it also vanishes after $(-)_p^\wedge[1/p]_{(q-1)}^\wedge$, as any module over the trivial ring is trivial. It follows that $q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^{\alpha_p})/\mathbb{Z}}$ is already p-complete and thus agrees with $q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^{\alpha_p})/\mathbb{Z}_p}$.

With this observation, Definition 3.2(a) is straightforward to verify. Definition 3.2(b) follows via base change from $\mathbb{Z}_p\{x\}_\infty/x^{\alpha_p}$. Definition 3.2($c$) and ($c_p$) are vacuous, since the rationalisations vanish. Therefore, $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{(\mathbb{Z}/m)/\mathbb{Z}}$ is indeed a q-Hodge filtration in the sense of Definition 3.2.

To verify that $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{(\mathbb{Z}/m)/\mathbb{Z}}$ is degree-wise static and $(q-1)$-torsion free, just observe that its reduction modulo $(q-1)$ is $\mathrm{fil}^\star_{\mathrm{Hdg}}\, \mathrm{dR}_{(\mathbb{Z}/m)/\mathbb{Z}}$, which is degree-wise static. Via base change from $\mathbb{Z}_p\{x\}_\infty/x^{\alpha_p}$ it's then clear that $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\mathrm{dR}_{(\mathbb{Z}/m)/\mathbb{Z}}$ must be a descending filtration by ideals. By construction, this implies that $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}$ is a static and $(q-1)$-torsion free ring, as claimed. □

(11.2) Burklund only shows that the objects in the tower are unique and therefore satisfy base change. But the same argument shows that the transition maps too are unique, so they satisfy base change as well.

The upshot of 11.4–11.6 is the following.

11.7. Corollary. — *Let $m \in \mathbb{N}^{\sharp}$ be a high-powered positive integer. Then the spectra $\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})$ and $\mathrm{TC}^-((\mathrm{KU}/m)/\mathrm{KU})$ are concentrated in even degrees and we have*

$$\pi_{2*}\,\mathrm{TC}^-\big((\mathrm{ku}/m)/\mathrm{ku}\big) \cong \mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}\,,$$
$$\pi_{2*}\,\mathrm{TC}^-\big((\mathrm{KU}/m)/\mathrm{KU}\big) \cong q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}[\beta^{\pm 1}]\,.$$

Proof. It's enough to check evenness modulo t, so we may pass from TC^- to THH. Since $\mathrm{THH}((\mathrm{ku}/m)/\mathrm{ku})$ is connective, we may further pass to $\mathrm{THH}((\mathrm{ku}/m)/\mathrm{ku})/\beta \simeq \mathrm{HH}((\mathbb{Z}/m)/\mathbb{Z})$, which is indeed even. This shows evenness for $\mathrm{THH}((\mathrm{ku}/m)/\mathrm{ku})$ and then the same follows for $\mathrm{THH}((\mathrm{ku}/m)/\mathrm{ku})[1/\beta] \simeq \mathrm{THH}((\mathrm{KU}/m)/\mathrm{KU})$.

By decomposing m into prime factors as in 11.4 and using the base change equivalence, we get a map

$$\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}} \longrightarrow \pi_{2\star}\,\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})\,.$$

Whether this is an equivalence can be checked modulo β, where we recover the well-known fact that the even homotopy groups of $\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})/\beta \simeq \mathrm{HC}^-((\mathbb{Z}/m)/\mathbb{Z})$ are the completed Hodge filtration $\mathrm{fil}^\star_{\mathrm{Hdg}}\,\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$. The claim that the even homotopy groups of

$$\mathrm{TC}^-\big((\mathrm{KU}/m)/\mathrm{KU}\big) \simeq \mathrm{TC}^-\big((\mathrm{ku}/m)/\mathrm{ku}\big)\big[\tfrac{1}{\beta}\big]^\wedge_t$$

are given by $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}[\beta^{\pm1}]$ follows formally. □

This finishes step (*a*) of our plan in 11.3. Before we move onwards to step (*b*), let us make two remarks.

11.8. Remark. — Let $m = \prod_p p^{\alpha_p}$ be an integer such that for all primes $p > 2$ either $\alpha_p = 0$ or $\alpha_p \geqslant 3$ and for $p = 2$ either $\alpha_2 = 0$ or α_2 is even and $\geqslant 6$. Then the $\mathbb{E}_1$-structure on $\mathbb{S}/m$ can be upgraded to an $\mathbb{E}_2$-structure. We can thus apply Theorem 7.27 to obtain another q-Hodge filtration on $q\text{-}\mathrm{dR}_{(\mathbb{Z}/m)/\mathbb{Z}}$. This necessarily agrees with the one from 11.5.

Indeed, this can be reduced to a similar claim for $\mathbb{Z}_p\{x\}_\infty/x^{\alpha_p}$, noting that the $\mathbb{E}_1$-structure on $\mathbb{S}_{\mathbb{Z}_p\{x\}_\infty}/x^{\alpha_p}$ also admits an $\mathbb{E}_2$-upgrade, compatible with the one on $\mathbb{S}/p^{\alpha_p}$. The assertion then follows by observing that the solid even filtration on the already even $\mathbb{E}_1$-ring spectrum $\mathrm{TC}^-_{\blacksquare}((\mathrm{ku}_{\mathbb{Z}_p\{x\}_\infty}/x^{\alpha_p})/\mathrm{ku})$ necessarily agrees with the double-speed Whitehead filtration $\tau_{\geqslant 2\star}$.

11.9. Remark. — We don't know if $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\mathrm{dR}_{(\mathbb{Z}/m)/\mathbb{Z}}$ is canonical: The results from §4.2 don't apply, as $\mathbb{Z}/m$ is not torsion free. However, in the case where $m = p^\alpha$ is a prime power, we have the following weak form of canonicity:

($*$) *Let $A \to \mathbb{Z}_p$ be any map from a p-completely perfectly covered δ-ring and $R \to \mathbb{Z}/p^\alpha$ be an A-algebra map, where R is as in Theorem 4.22(a). Then the induced map*

$$q\text{-}\mathrm{dR}_{R/A} \longrightarrow q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$$

is compatible with q-Hodge filtrations. Moreover, the q-Hodge filtration on $q\text{-}\mathrm{dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ is the smallest multiplicative filtration with this property.

To prove this, let $R \cong B/J$ be a perfect-regular presentation as in Theorem 4.22(*a*), where J is generated by a regular sequence of higher powers $(x_1^{\alpha_1}, \dots, x_r^{\alpha_r})$. Using $q\text{-}\mathrm{dR}_{R/A} \simeq q\text{-}\mathrm{dR}_{R/B}$ and the base change assertion from Lemma 4.27 applied to $\mathbb{Z}_p\{x_1, \dots, x_r\} \to B$, we can reduce to the case $A = \mathbb{Z}_p\{x_1, \dots, x_r\}$, $R = \mathbb{Z}_p\{x_1, \dots, x_r\}/(x_1^{\alpha_1}, \dots, x_r^{\alpha_r})$; note that we still get a δ-ring map $\mathbb{Z}_p\{x_1, \dots, x_r\} \to \mathbb{Z}_p$ by lifting the images of the x_i in $\mathbb{Z}/p^\alpha$ arbitrarily. By symmetric monoidality (Lemma 4.33(*b*)), we can further reduce to the case $r = 1$.

In this case suppose $\mathbb{Z}_p\{x_1\} \to \mathbb{Z}_p$ sends $x_1 \mapsto ap^N$, where $(a, p) = 1$. In order to have a map $\mathbb{Z}_p\{x_1\}/x_1^{\alpha_1} \to \mathbb{Z}/p^\alpha$, we must have $\alpha_1 N \geqslant \alpha$. Then the map $\mathbb{Z}_p\{x_1\} \to \mathbb{Z}_p\{x\}$ sending $x_1 \mapsto ax^N$ induces a map $\mathbb{Z}_p\{x_1\}/x_1^{\alpha_1} \to \mathbb{Z}_p\{x\}/x^\alpha$ and so the desired compatibility of q-Hodge filtrations follows from functoriality of Construction 4.21. The minimality claim is clear since $\operatorname{fil}^\star_{q\text{-Hdg}} q\text{-dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ is base changed from $\operatorname{fil}^\star_{q\text{-Hdg}} q\text{-dR}_{(\mathbb{Z}_p\{x\}_\infty/x^\alpha)/\mathbb{Z}_p}$.

We'll now commence step 11.3(*b*). We start with a general fact (which is usually formulated as a spectral sequence).

11.10. Lemma. — *Let k be an even $\mathbb{E}_1$-ring spectrum and let M, N be even left-k-modules. Then the mapping spectrum $\operatorname{Hom}_k(M, N)$ admits a complete exhaustive descending filtration with graded pieces*

$$\operatorname{gr}^* \operatorname{Hom}_k(M, N) \simeq \Sigma^{2*} \mathrm{R}\underline{\operatorname{Hom}}_{\pi_{2*}(k)}\bigl(\pi_{2*}(M), \pi_{2*}(N)\bigr)\,.$$

Here $\Sigma^{2*}\colon \operatorname{Gr}(\mathrm{Sp}) \to \operatorname{Gr}(\mathrm{Sp})$ is the "double shearing" functor and $\mathrm{R}\underline{\operatorname{Hom}}_{\pi_{2*}(k)}$ denotes the derived internal Hom in graded $\pi_{2*}(k)$-modules.

Proof. In the usual adjunction $\operatorname{colim}\colon \operatorname{Fil}(\mathrm{Sp}) \rightleftarrows \mathrm{Sp} :\!\operatorname{const}$, the left adjoint is symmetric monoidal and the right adjoint is lax symmetric monoidal. Furthermore, $\operatorname{colim} \tau_{\geqslant 2\star}(k) \simeq k$. It follows formally that $\operatorname{colim}\colon \operatorname{LMod}_{\tau_{\geqslant 2\star}(k)}(\operatorname{Fil}(\mathrm{Sp})) \rightleftarrows \operatorname{LMod}_k(\mathrm{Sp}) :\!\operatorname{const}$ is an adjunction as well and so $\operatorname{Hom}_k(M, N) \simeq \operatorname{Hom}_{\tau_{\geqslant 2\star}(k)}(\tau_{\geqslant 2\star}(M), \operatorname{const} N)$. Hence we may define the desired filration via

$$\operatorname{fil}^n \operatorname{Hom}_k(M, N) \coloneqq \operatorname{Hom}_{\tau_{\geqslant 2\star}(k)}\bigl(\tau_{\geqslant 2\star}(M), \tau_{\geqslant 2(\star+n)}(N)\bigr)\,.$$

This filtration is clearly complete since we may pull $0 \simeq \lim_{n\to\infty} \tau_{\geqslant 2(\star+n)}(N)$ out of the Hom. To show that the filtration is exhaustive, we need to check that $\operatorname{const} N \simeq \operatorname{colim}_{n\to-\infty} \tau_{\geqslant 2(\star+n)}(N)$ can similarly be pulled out of the Hom. To this end, recall that $\operatorname{Fil}(\mathrm{Sp})$ can be equipped with the *double Postnikov t-structure* in which objects in the image of $\tau_{\geqslant 2\star}(-)$ are connective and connective objects are closed under tensor products (see [Rak21, Construction 3.3.6] for example and double everything). Then $\operatorname{Mod}_{\tau_{\geqslant 2\star}(k)}(\operatorname{Fil}(\mathrm{Sp}))$ inherits a t-structure in which $\tau_{\geqslant 2\star}(M)$ is connective and the cofibres of $\tau_{\geqslant 2(\star+n)}(N) \to \operatorname{const} N$ get more and more coconnective as $n \to -\infty$. This shows that the colimit can be pulled out.

It remains to determine the associated graded. By construction, the n^{th} graded piece is given by $\operatorname{gr}^n \operatorname{Hom}_k(M, N) \simeq \operatorname{Hom}_{\tau_{\geqslant 2\star}(k)}(\tau_{\geqslant 2*}(M), \Sigma^{2(\star+n)}\pi_{2(\star+n)}(N))$. To simplify this further, let $\mathbb{S}_{\mathrm{Gr}}$ and $\mathbb{S}_{\mathrm{Fil}}$ denote the tensor units in graded and filtered spectra, respectively. By abuse of notation, we identify $\mathbb{S}_{\mathrm{Fil}}$ with its underlying graded spectrum. As remarked in 1.48, we have $\operatorname{Fil}(\mathrm{Sp}) \simeq \operatorname{Mod}_{\mathbb{S}_{\mathrm{Fil}}}(\operatorname{Gr}(\mathrm{Sp}))$; this identifies passing to the associated graded with the base change functor $-\otimes_{\mathbb{S}_{\mathrm{Fil}}} \mathbb{S}_{\mathrm{Gr}}$. Since the $\mathbb{S}_{\mathrm{Fil}}$-module structure on $\Sigma^{2(\star+n)}\pi_{2(\star+n)}(N)$ already factors through $\mathbb{S}_{\mathrm{Fil}} \to \mathbb{S}_{\mathrm{Gr}}$, we obtain

$$\begin{aligned}\operatorname{Hom}_{\tau_{\geqslant 2\star}(k)}\bigl(\tau_{\geqslant 2*}(M), \Sigma^{2(\star+n)}\pi_{2(\star+n)}(N)\bigr) &\simeq \operatorname{Hom}_{\Sigma^{2*}\pi_{2*}(k)}\bigl(\Sigma^{2*}\pi_{2*}(M), \Sigma^{2(*+n)}\pi_{2(*+n)}(N)\bigr)\\ &\simeq \Sigma^{2n} \operatorname{Hom}_{\pi_{2*}(k)}\bigl(\pi_{2*}(M), \pi_{2*}(N)(-n)\bigr)\,.\end{aligned}$$

The first step is the usual base change equivalence for $\tau_{\geqslant 2\star}(k) \to \tau_{\geqslant 2\star}(k) \otimes_{\mathbb{S}_{\mathrm{Fil}}} \mathbb{S}_{\mathrm{Gr}} \simeq \Sigma^{2*}\pi_{2*}(k)$, the second step uses that the shearing functor $\Sigma^{2*}\colon \operatorname{Gr}(\mathrm{Sp}) \to \operatorname{Gr}(\mathrm{Sp})$ is an $\mathbb{E}_1$-monoidal equivalence (even $\mathbb{E}_2$-monoidal, see [DHL+23, Proposition 3.10], but we don't need that). Now the right-hand side is precisely the n^{th} graded piece of $\mathrm{R}\underline{\mathrm{Hom}}_{\pi_{2*}(k)}(\pi_{2*}(M),\pi_{2*}(N))$ and so we're done. □

We'll apply this now in the case $k\simeq \mathrm{ku}^{hS^1}$, so that $\pi_{2*}(k)\cong\mathbb{Z}[\beta][\![t]\!]$. We also let $\underline{\mathrm{Ext}}^i_{\mathbb{Z}[\beta][\![t]\!]}$ denote the graded $\mathbb{Z}[\beta][\![t]\!]$-module $\mathrm{H}_{-i}\,\mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[\beta][\![t]\!]}$ for all $i\geqslant 0$.

11.11. Corollary. — *Let $m\in\mathbb{N}^\sharp$ be a high-powered positive integer. Then the spectra $\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})^\vee$ and $\mathrm{TC}^-((\mathrm{KU}/m)/\mathrm{KU})^\vee$ are concentrated in odd degrees and we have*

$$\pi_{-(2*+1)}\,\mathrm{TC}^-\big((\mathrm{ku}/m)/\mathrm{ku}\big)^\vee\cong\underline{\mathrm{Ext}}^1_{\mathbb{Z}[\beta][\![t]\!]}\Big(\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}},\mathbb{Z}[\beta][\![t]\!]\Big)$$
$$\pi_{-(2*+1)}\,\mathrm{TC}^-\big((\mathrm{KU}/m)/\mathrm{KU}\big)^\vee\cong\mathrm{Ext}^1_{\mathbb{Z}[\![q-1]\!]}\Big(q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}},\mathbb{Z}[\![q-1]\!]\Big)[\beta^{\pm1}]\,.$$

Proof. According to Corollary 11.7 and Lemma 11.10, the spectrum $\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})^\vee$ admits a complete exhaustive filtration with associated graded $\Sigma^{2*}(\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}})^\vee$, where now the dual is taken in graded $\mathbb{Z}[\beta][\![t]\!]$-modules. It'll be enough to show that this dual is concentrated in homological degree -1 (which precisely accounts for the $\underline{\mathrm{Ext}}^1_{\mathbb{Z}[\![q-1]\!][\beta^{\pm1}]}$-terms). Since $\mathbb{Z}[\beta][\![t]\!]$ is (β,t)-complete as a graded object, the same is true for any dual in graded $\mathbb{Z}[\beta][\![t]\!]$-modules, and so it'll be enough that

$$\mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}[\beta][\![t]\!]}\Big(\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}},\mathbb{Z}[\beta][\![t]\!]\Big)/(\beta,t)\simeq\mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}}\Big(\mathrm{gr}^*_{\mathrm{Hdg}}\,\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}},\mathbb{Z}\Big)$$

is concentraded in homological degree -1. Since $\mathrm{gr}^n_{\mathrm{Hdg}}\,\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}\simeq\Sigma^{-n}\bigwedge^n\mathrm{L}_{(\mathbb{Z}/m)/\mathbb{Z}}\simeq\mathbb{Z}/m$, the n^{th} graded piece of the right-hand side is precisely $\mathrm{RHom}_{\mathbb{Z}}(\mathbb{Z}/m,\mathbb{Z})$, which is indeed concentrated in homological degree -1. This finishes the proof for $\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})^\vee$.

The proof for $\mathrm{TC}^-((\mathrm{KU}/m)/\mathrm{KU})^\vee$ is analogous, except that we need a different argument to show that the dual $(q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}})^\vee$ in $\mathbb{Z}[\![q-1]\!]$-modules is concentrated in homological degree -1. By $(q-1)$-completeness, it'll be enough to check the same for $\mathrm{RHom}_{\mathbb{Z}}(q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}/(q-1),\mathbb{Z})$. By 3.8 we see that $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}/(q-1)$ admits an exhaustive ascending filtration with associated graded given by $\mathrm{gr}^*_{\mathrm{Hdg}}\,\mathrm{dR}_{(\mathbb{Z}/m)/\mathbb{Z}}$. It follows that $\mathrm{RHom}_{\mathbb{Z}}(q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}/(q-1),\mathbb{Z})$ admits a descending filtration with associated graded $\mathrm{R}\underline{\mathrm{Hom}}_{\mathbb{Z}}(\mathrm{gr}^*_{\mathrm{Hdg}}\,\mathrm{dR}_{(\mathbb{Z}/m)/\mathbb{Z}},\mathbb{Z})$. This is indeed concentrated in homological degree -1 as we've seen above, so we're done. □

This finishes step (*b*) in our plan from 11.3. We continue with step (*c*). Note that neither pro-idempotence of "$\lim_{m\in\mathbb{N}^\sharp}$" $\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})$ nor the fact that its transition maps become eventually trace-class are automatically preserved under passing to homotopy groups. The problem is that $\pi_*(-)$—or really passing to the associated graded of the Whitehead filtration $\tau_{\geqslant\star}$—is not a symmetric monoidal functor.

As we'll see, in our situation, passing to the associated graded of the *double-speed* Whitehead filtration $\tau_{\geqslant 2\star}$ behaves as if it were symmetric monoidal, which fixes all issues. Our starting point is the following general fact, which is quite similar to Lemma 11.10 (and is also usually formulated as a spectral sequence).

11.12. Lemma. — *Let k be an even $\mathbb{E}_\infty$-ring spectrum, let $t\in\pi_{2*}(k)$ be a homogeneous element, and let M, N be even k-modules. Then the t-completed tensor product $M\mathbin{\widehat{\otimes}}_k N$ admits a complete exhaustive descending filtration with graded pieces*

$$\mathrm{gr}^*(M\mathbin{\widehat{\otimes}}_k N)\simeq\Sigma^{2*}\Big(\pi_{2*}(M)\mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\pi_{2*}(k)}\pi_{2*}(N)\Big)\,.$$

Here $-\mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\pi_{2*}(k)}-$ *denotes the graded t-completed derived tensor product over $\pi_{2*}(k)$.*

Proof. The filtered spectrum $\tau_{\geqslant 2\star}(M)\otimes_{\tau_{\geqslant 2\star}(k)}\tau_{\geqslant 2\star}(N)$ defines a filtration on $M\otimes_k N$. This filtration is exhaustive, since $\operatorname{colim}\colon \mathrm{Fil}(\mathrm{Sp})\to\mathrm{Sp}$ is symmetric monoidal, and complete, since $\tau_{\geqslant 2\star}(M)\otimes_{\tau_{\geqslant 2\star}(k)}\tau_{\geqslant 2\star}(N)$ is a connective object in the double Postnikov t-structure (see the proof of Lemma 11.10).

Now consider the t-adically completed tensor product $\tau_{\geqslant 2\star}(M)\mathbin{\widehat{\otimes}}_{\tau_{\geqslant 2\star}(k)}\tau_{\geqslant 2\star}(N)$, where t in the filtration degree corresponding to its homotopical degree. This now defines a filtration on $M\mathbin{\widehat{\otimes}}_k N$, which is clearly still complete. It is also still exhaustive. Indeed, for all n, the cofibre of $(\tau_{\geqslant 2\star}(M)\otimes_{\tau_{\geqslant 2\star}(k)}\tau_{\geqslant 2\star}(N))_{-n}\to M\otimes N$ is $(2n+1)$-coconnective. Upon t-adic completion, the coconnectivity can go down by at most 1, and so we see that the cofibre of $(\tau_{\geqslant 2\star}(M)\mathbin{\widehat{\otimes}}_{\tau_{\geqslant 2\star}(k)}\tau_{\geqslant 2\star}(N))_{-n}\to M\mathbin{\widehat{\otimes}} N$ will still be $2n$-coconnective. This ensures exhaustiveness.

Passing to the associated graded is symmetric and commutes with t-adic completion (in the filtered and graded setting, respectively). Moreover, the double shearing functor Σ^{2*} is $\mathbb{E}_1$-monoidal (even $\mathbb{E}_2$, but we won't need that). Hence

$$\mathrm{gr}^*(M\mathbin{\widehat{\otimes}}_k N)\simeq \Sigma^{2*}\pi_{2*}(M)\mathbin{\widehat{\otimes}}_{\Sigma^{2*}\pi_{2*}(k)}\Sigma^{2*}\pi_{2*}(N)\simeq\Sigma^{2*}\Bigl(\pi_{2*}(M)\mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\pi_{2*}(k)}\pi_{2*}(N)\Bigr)\,. \qquad\square$$

11.13. Corollary. — *“$\lim_{m\in\mathbb{N}^\sharp}$” $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$ and “$\lim_{m\in\mathbb{N}^\sharp}$” $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}$ are idempotent pro-algebras, respectively, in the derived ∞-categories of t-complete graded $\mathbb{Z}[\beta]\llbracket t\rrbracket$-modules and of $(q-1)$-complete $\mathbb{Z}\llbracket q-1\rrbracket$-modules.*

Proof. Throughout the proof, $\widehat{\otimes}$ will denote a t-completed tensor product. We also put $\mathrm{fil}^\star\,q\text{-}\widehat{\mathrm{dR}}_m\coloneqq\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$ and $A\coloneqq$ “$\lim_{m\in\mathbb{N}^\sharp}$” $\mathrm{fil}^\star\,q\text{-}\widehat{\mathrm{dR}}_m$ for short.

Since each $\mathrm{fil}^\star\,q\text{-}\widehat{\mathrm{dR}}_m$ is a graded $\mathbb{Z}[\beta]\llbracket t\rrbracket$-algebra, we get a unit map $\mathbb{Z}[\beta]\llbracket t\rrbracket\to A$ and a multiplication $A\mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathbb{Z}[\beta]\llbracket t\rrbracket}A\to A$ such that the composition

$$A\simeq\mathbb{Z}[\beta]\llbracket t\rrbracket\mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathbb{Z}[\beta]\llbracket t\rrbracket}A\longrightarrow A\mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathbb{Z}[\beta]\llbracket t\rrbracket}A\longrightarrow A$$

is the identity. For the other composition, let $m_1,m_2\in\mathbb{N}^\sharp$ and consider the t-completed tensor product

$$\mathrm{TC}^-\bigl((\mathrm{ku}/m_1\otimes_{\mathrm{ku}}\mathrm{ku}/m_2)/\mathrm{ku}\bigr)\simeq\mathrm{TC}^-\bigl((\mathrm{ku}/m_1)/\mathrm{ku}\bigr)\mathbin{\widehat{\otimes}}_{\mathrm{ku}^{hS^1}}\mathrm{TC}^-\bigl((\mathrm{ku}/m_2)/\mathrm{ku}\bigr)\,.$$

By Lemma 11.12, this has a complete exhaustive filtration with graded pieces given by $\Sigma^{2*}(\mathrm{fil}^\star\,q\text{-}\widehat{\mathrm{dR}}_{m_1}\mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathbb{Z}[\beta]\llbracket t\rrbracket}\mathrm{fil}^\star\,q\text{-}\widehat{\mathrm{dR}}_{m_2})$. Observe that this graded completed tensor product is concentrated in homological degrees $[0,1]$. Indeed, this can be checked modulo (β,t). Then $\mathrm{fil}^\star\,q\text{-}\widehat{\mathrm{dR}}_{m_i}/(\beta,t)\simeq\mathrm{gr}^*_{\mathrm{Hdg}}\,\mathrm{dR}_{(\mathbb{Z}/m_i)/\mathbb{Z}_p}$ is given by $\mathbb{Z}/m_i$ in every graded degree for $i=1,2$, and $\mathbb{Z}/m_1\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Z}/m_2$ is indeed concentrated in homological degrees $[0,1]$. It follows that the filtration on $\mathrm{TC}^-((\mathrm{ku}/p^{m_1}\otimes_{\mathrm{ku}}\mathrm{ku}/p^{m_2})/\mathrm{ku})$ must be the double speed Whitehad filtration $\tau_{\geqslant 2\star}$.

By Corollary 10.26, $\mathrm{TC}^-((\mathrm{ku}/m^3\otimes_{\mathrm{ku}}\mathrm{ku}/m)/\mathrm{ku})\to\mathrm{TC}^-((\mathrm{ku}/m^2\otimes_{\mathrm{ku}}\mathrm{ku}/m)/\mathrm{ku})$ factors through the even spectrum $\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})$. By passing to the associated graded of the double speed Whitehead filtration, we see that

$$\mathrm{fil}^\star\,q\text{-}\widehat{\mathrm{dR}}_{m^3}\mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathbb{Z}[\beta]\llbracket t\rrbracket}\mathrm{fil}^\star\,q\text{-}\widehat{\mathrm{dR}}_m\longrightarrow\mathrm{fil}^\star\,q\text{-}\widehat{\mathrm{dR}}_{m^2}\mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathbb{Z}[\beta]\llbracket t\rrbracket}\mathrm{fil}^\star\,q\text{-}\widehat{\mathrm{dR}}_{m^2}$$

factors through $\mathrm{fil}^\star\,q\text{-}\widehat{\mathrm{dR}}_m$. This finishes the proof that $A=$ “$\lim_{m\in\mathbb{N}^\sharp}$” $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$ is an idempotent pro-algebra.

The argument for "$\lim_{m\in\mathbb{N}^\natural}$" $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}$ is analogous, except that we work with KU instead of ku, and to show that $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m_1)/\mathbb{Z}}\mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathbb{Z}[\![q-1]\!]} q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m_2)/\mathbb{Z}}$ is concentrated in homological degrees $[0,1]$, we need a slightly different argument: First, we can reduce modulo $(q-1)$. The conjugate filtration from 3.8 gives an ascending filtration on $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m_i)/\mathbb{Z}}/(q-1)$ for $i=1,2$, whose graded pieces are copies of $\mathbb{Z}/m_i$. Moreover, $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m_i)/\mathbb{Z}}/(q-1)$ is an $\mathbb{Z}/m_i$-algebra, since $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m_i)/\mathbb{Z}}$ contains an element of the form $m_i/(q-1)$. Thus, abstractly, $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m_i)/\mathbb{Z}}/(q-1)\simeq\bigoplus_{\mathbb{N}}\mathbb{Z}/m_i$. So we're done since $\mathbb{Z}/m_1\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Z}/m_2$ is concentrated in homological degrees $[0,1]$. ☐

11.14. Corollary. — *"$\lim_{m\in\mathbb{N}^\natural}$" $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$ and "$\lim_{m\in\mathbb{N}^\natural}$" $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}$ are equivalent to pro-objects with trace-class transition maps.*

Proof. Throughout the proof, $\widehat{\otimes}$ will denote a t-completed tensor product. Using Corollary 10.26 and unravelling the proof of Lemma 10.19, we find that that for every high-powered m, $\mathrm{TC}^-((\mathrm{ku}/m^3)/\mathrm{ku})\to\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})$ is trace-class in t-complete ku^{hS^1}-modules. Hence it must be induced by a map

$$\eta\colon \mathrm{ku}^{hS^1}\longrightarrow \mathrm{TC}^-\big((\mathrm{ku}/m^3)/\mathrm{ku}\big)^\vee\mathbin{\widehat{\otimes}}_{\mathrm{ku}^{hS^1}}\mathrm{TC}^-\big((\mathrm{ku}/m)/\mathrm{ku}\big)$$

By Lemma 11.12 (applied to the shift $\Sigma\,\mathrm{TC}^-((\mathrm{ku}/m^3)/\mathrm{ku})^\vee$ to get an even spectrum, then we shift back afterwards), the right-hand side has a complete exhaustive filtration with graded pieces $(\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m^3)/\mathbb{Z}})^\vee\mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathbb{Z}[\beta][\![t]\!]}\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$. As in the proof of Corollary 11.13, one easily checks that this graded completed tensor product is concentrated in homological degrees $[-1,0]$. It follows that the filtration must be given by $\tau_{\geqslant 2\star-1}(-)$. Thus, by considering $\tau_{\geqslant 2\star-1}(\eta)$ and then passing to associated gradeds, we obtain a morphism

$$\mathbb{Z}[\beta][\![t]\!]\longrightarrow\big(\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m^3)/\mathbb{Z}}\big)^\vee\mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathbb{Z}[\beta][\![t]\!]}\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}\,.$$

which witnesses that the morphism $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m^3)/\mathbb{Z}_p}\to\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$ is indeed trace-class, as desired.

The argument for $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m^3)/\mathbb{Z}}\to q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}$ being trace-class is analogous, except that we use KU instead of ku. Moreover, we need a different argument to show that $(q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m^3)/\mathbb{Z}})^\vee\mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathbb{Z}[\![q-1]\!]}q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}$ is concentrated in homological degrees $[-1,0]$: First, we can reduce modulo $(q-1)$. As we've seen in the proof of Corollary 11.13, on underlying abelian groups we get an equivalence $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}/(q-1)\simeq\bigoplus_{\mathbb{N}}\mathbb{Z}/m$. An analogous conclusion holds for $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m^3)/\mathbb{Z}}/(q-1)$. Thus, the tensor product modulo $(q-1)$ becomes $\Sigma^{-1}\prod_{\mathbb{N}}\mathbb{Z}/m^3\otimes^{\mathrm{L}}_{\mathbb{Z}}\bigoplus_{\mathbb{N}}\mathbb{Z}/m$, which is clearly concentrated in homological degrees $[-1,0]$. ☐

This finishes step 11.3(c) and we arrive at the result of our computation.

11.15. Theorem. — *$\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku}\otimes\mathbb{Q}/\mathrm{ku})$ and $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU}\otimes\mathbb{Q}/\mathrm{KU})$ are concentrated in even degrees. Furthermore, their even homotopy groups are given as follows:*

(a) *$\pi_{2*}\,\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku}\otimes\mathbb{Q}/\mathrm{ku})\cong\mathrm{A}^*_{\mathrm{ku}}$, where $\mathrm{A}^*_{\mathrm{ku}}$ is obtained by killing the idempotent pro-graded $\mathbb{Z}[\beta][\![t]\!]$-algebra "$\lim_{m\in\mathbb{N}^\natural}$" $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$. In particular, there's a short exact sequence*

$$0\longrightarrow\mathbb{Z}[\beta][\![t]\!]\longrightarrow\mathrm{A}^*_{\mathrm{ku}}\longrightarrow\underset{m\in(\mathbb{N}^\natural)^{\mathrm{op}}}{\text{“colim”}}\,\underline{\mathrm{Ext}}^1_{\mathbb{Z}[\beta][\![t]\!]}\Big(\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}},\mathbb{Z}[\beta][\![t]\!]\Big)\longrightarrow 0\,,$$

and $\mathrm{A}^*_{\mathrm{ku}}$ is an idempotent nuclear graded $\mathbb{Z}[\beta][\![t]\!]$-algebra.

(*b*) $\pi_{2*}\,\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{KU}\otimes\mathbb{Q}/\mathrm{KU})\cong \mathrm{A}_{\mathrm{KU}}[\beta^{\pm1}]$, *where* A_{KU} *is obtained by killing the idempotent pro-*$\mathbb{Z}[\![q-1]\!]$*-algebra* "$\lim_{m\in\mathbb{N}^\sharp}$" $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}$. *In particular, there's a short exact sequence*

$$0\longrightarrow \mathbb{Z}[\![q-1]\!]\longrightarrow \mathrm{A}_{\mathrm{KU}}\longrightarrow \underset{m\in(\mathbb{N}^\sharp)^{\mathrm{op}}}{\text{“colim”}}\operatorname{Ext}^1_{\mathbb{Z}[\![q-1]\!]}\Big(q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}},\mathbb{Z}[\![q-1]\!]\Big)\longrightarrow 0\,,$$

and A_{KU} *is an idempotent nuclear* $\mathbb{Z}[\![q-1]\!]$*-algebra.*

Proof. We use the cofibre sequence of 11.3. To compute $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku}\otimes\mathbb{Q}/\mathrm{ku})$, we must study the cofibres of $\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})^\vee\to\mathrm{ku}^{hS^1}$ for high-powered integers $m\in\mathbb{N}^\sharp$. Put

$$\begin{aligned}\mathrm{fil}^\star\, q\text{-}\overline{\mathrm{dR}}_m &\coloneqq \operatorname{cofib}\Big(\mathbb{Z}[\beta][\![t]\!]\to \mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}\Big)\,,\\ \overline{\mathrm{TC}}{}^-_m &\coloneqq \operatorname{cofib}\Big(\mathrm{ku}^{hS^1}\to \mathrm{TC}^-\big((\mathrm{ku}/m)/\mathrm{ku}\big)\Big)\,.\end{aligned}$$

Since ku^{hS^1} and $\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})$ are even spectra, the sequence of double speed Whitehead filtrations $\tau_{\geqslant 2\star}(\mathrm{ku}^{hS^1})\to\tau_{\geqslant 2\star}\,\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})\to\tau_{\geqslant 2\star}\overline{\mathrm{TC}}{}^-_m$ is still a cofibre sequence in filtered spectra. Applying the construction from the proof of Lemma 11.10, we get complete exhaustive filtrations on the duals of ku^{hS^1}, $\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})$, and $\overline{\mathrm{TC}}{}^-_m$ in such a way that they fit into a cofibre sequence $\mathrm{fil}^\star(\overline{\mathrm{TC}}{}^-_m)^\vee\to\mathrm{fil}^\star\,\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})^\vee\to\mathrm{fil}^\star(\mathrm{ku}^{hS^1})^\vee$. After passing to associated gradeds, we get a cofibre sequence of graded $\Sigma^{2*}\mathbb{Z}[\beta][\![t]\!]$-modules

$$\mathrm{gr}^*(\overline{\mathrm{TC}}{}^-_m)^\vee\longrightarrow\Sigma^{2*}\Big(\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}\Big)^\vee\longrightarrow\Sigma^{2*}\mathbb{Z}[\beta][\![t]\!]^\vee\,,$$

where $\Sigma^{2*}\colon\mathrm{Gr}(\mathrm{Sp})\to\mathrm{Gr}(\mathrm{Sp})$ denotes the "double shearing" functor. It's clear from the construction that the morphism on the right must really be given by $\Sigma^{2*}(-)^\vee$ applied to the unit map $\mathbb{Z}[\beta][\![t]\!]\to\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$. It follows that $\mathrm{gr}^*(\overline{\mathrm{TC}}{}^-_m)^\vee\simeq\Sigma^{2*}(\mathrm{fil}^\star\,q\text{-}\overline{\mathrm{dR}}_m)^\vee$. Observe that $(\mathrm{fil}^\star\,q\text{-}\overline{\mathrm{dR}}{}^*_m)^\vee$ sits in homological degree -1. Indeed, this can be checked modulo (β,t). Then $\mathrm{fil}^\star\,q\text{-}\overline{\mathrm{dR}}_m/(\beta,t)\simeq\operatorname{cofib}(\mathbb{Z}\to\mathrm{gr}^*_{\mathrm{Hdg}}\,\mathrm{dR}_{(\mathbb{Z}/m)/\mathbb{Z}})$ is given by $\Sigma\mathbb{Z}$ in graded degree 0 and $\mathbb{Z}/m$ in every other graded degree, so it's straightforward to see that its graded dual over $\mathbb{Z}$ sits indeed in homological degree -1.

Thus, $\mathrm{fil}^\star(\overline{\mathrm{TC}}{}^-_m)^\vee$ must be the double speed Whitehead filtration, $(\overline{\mathrm{TC}}{}^-_m)^\vee$ is concentrated in odd degrees, and $\pi_{2*-1}((\overline{\mathrm{TC}}{}^-_m)^\vee)\cong\mathrm{H}_{-1}(\mathrm{fil}^\star(q\text{-}\overline{\mathrm{dR}}_m)^\vee)$ as a graded $\mathbb{Z}[\beta][\![t]\!]$-modules. Combining this with Corollary 11.11, we see that the long exact homotopy sequence of the rotated cofibre sequence $(\mathrm{ku}^{hS^1})^\vee\to\Sigma(\overline{\mathrm{TC}}{}^-_m)^\vee\to\Sigma\,\mathrm{TC}^-((\mathrm{ku}/m)/\mathrm{ku})^\vee$ breaks up into a short exact sequence of graded $\mathbb{Z}[\beta][\![t]\!]$-modules of the following form:

$$0\longrightarrow\mathbb{Z}[\beta][\![t]\!]\longrightarrow\mathrm{H}_{-1}\big(\mathrm{fil}^\star(q\text{-}\overline{\mathrm{dR}}_m)^\vee\big)\longrightarrow\underline{\operatorname{Ext}}^1_{\mathbb{Z}[\beta][\![t]\!]}\Big(\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}},\mathbb{Z}[\beta][\![t]\!]\Big)\longrightarrow 0\,.$$

Since $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku}\otimes\mathbb{Q}/\mathrm{ku})\simeq$ "$\mathrm{colim}_{m\in(\mathbb{N}^\sharp)^{\mathrm{op}}}$" $\Sigma(\overline{\mathrm{TC}}{}^-_m)$ by the cofibre sequence from 11.3, it follows at once that $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku}\otimes\mathbb{Q}/\mathrm{ku})$ is concentrated in even degrees and that $\mathrm{A}^*_{\mathrm{ku}}$ fits into the desired short exact sequence. Furthermore, it's clear from our considerations above that

$$\Big(\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}\Big)^\vee\simeq\Sigma^{-1}\,\underline{\operatorname{Ext}}_{\mathbb{Z}[\beta][\![t]\!]}\Big(\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}},\mathbb{Z}[\beta][\![t]\!]\Big)\longrightarrow\mathbb{Z}[\beta][\![t]\!]\,,$$

induced by the short exact sequence, is given by dualising the canonical unit morphism $\mathbb{Z}[\beta][\![t]\!]\to\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$. Then the underlying graded ind-$\mathbb{Z}[\beta][\![t]\!]$-module of $\mathrm{A}^*_{\mathrm{ku}}$ must really be given by killing the pro-idempotent "$\lim_{m\in\mathbb{N}^\sharp}$" $\mathrm{fil}^\star_{q\text{-}\mathrm{Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/m)/\mathbb{Z}}$. Idempotence and nuclearity of $\mathrm{A}_{\mathrm{ku}}^*$ follow from Lemma 10.9(*b*) and Corollary 11.14. Since idempotents admit a unique $\mathbb{E}_\infty$-algebra structure, it follows that the desired description of $\mathrm{A}_{\mathrm{ku}}^*$ also holds as a nuclear ind-$\mathbb{Z}[\beta]\llbracket t\rrbracket$-algebra. This finishes the proof of (*a*).

The proof of (*b*) is analogous; the only difference is that we need a different argument why $\operatorname{cofib}(\mathbb{Z}\llbracket q-1\rrbracket \to q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}})^\vee$ is concentrated in homological degree -1. This can be checked modulo $(q-1)$. We've seen in the proof of Corollary 11.13 that $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}/(q-1)$ is a $\mathbb{Z}/m$-algebra and, abstractly, $q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}/(q-1) \simeq \bigoplus_{\mathbb{N}} \mathbb{Z}/m$. We can choose this decomposition in such a way that one of those summands corresponds to the unit $\mathbb{Z}/m \to q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}/(q-1)$. It follows that

$$\operatorname{cofib}\bigl(\mathbb{Z} \to q\text{-}\mathrm{Hdg}_{(\mathbb{Z}/m)/\mathbb{Z}}/(q-1)\bigr)^\vee \simeq \Biggl(\Sigma\mathbb{Z} \oplus \bigoplus_{\mathbb{N}\smallsetminus\{1\}} \mathbb{Z}/m\Biggr)^\vee \simeq \Sigma^{-1}\mathbb{Z} \oplus \Sigma^{-1} \prod_{\mathbb{N}\smallsetminus\{1\}} \mathbb{Z}/m$$

is indeed concentrated in homological degree -1 and we're done. □

§11.2. Explicit q-Hodge filtrations

In this subsection, we'll give an explicit description of the q-Hodge filtration $\operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}} q\text{-}\mathrm{dR}_{(\mathbb{Z}/m)/\mathbb{Z}}$. This will be used in §12 to prove Theorems 1.40 and 1.41, but it also leads to an elementary proof of Theorem 4.22(*a*), which also covers the remaining cases for $p = 2$.

By construction, it will be enough to describe the q-Hodge filtration in the case where $m = p^\alpha$ is a prime power. In this case, the filtration is obtained via base change from $q\text{-}\mathrm{dR}_{(\mathbb{Z}_p\{x\}_\infty/x^\alpha)/\mathbb{Z}_p}$. Using $q\text{-}\mathrm{dR}_{(\mathbb{Z}_p\{x\}_\infty/x^\alpha)/\mathbb{Z}_p} \simeq q\text{-}\mathrm{dR}_{(\mathbb{Z}_p\{x\}_\infty/x^\alpha)/\mathbb{Z}_p\{x\}_\infty}$ and Lemma 4.27, we can further reduce the problem to describing the filtration from Construction 4.21 on the derived q-de Rham complex

$$q\text{-}\mathrm{dR}_{(\mathbb{Z}_p\{x\}/x^\alpha)/\mathbb{Z}_p\{x\}} \simeq \mathbb{Z}_p\{x\}\llbracket q-1\rrbracket\left\{\frac{\phi(x^\alpha)}{[p]_q}\right\}^\wedge_{(p,q-1)}.$$

Let us denote this ring by $q\text{-}D_\alpha$ for short and let $D_\alpha \coloneqq \mathrm{dR}_{(\mathbb{Z}_p\{x\}/x^\alpha)/\mathbb{Z}_p\{x\}}$. Then D_α is the p-completed PD-envelope of $(x^\alpha) \subseteq \mathbb{Z}_p\{x\}$ and $q\text{-}D_\alpha/(q-1) \simeq D_\alpha$. The filtration $\operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}} q\text{-}D_\alpha$ from Construction 4.21 is then given as the (1-categorical) preimage of the $(x^\alpha, q-1)$-adic filtration on $D_\alpha[1/p]^\wedge_{\mathrm{Hdg}}\llbracket q-1\rrbracket$.

11.16. Lifts of divided powers. — Let $\gamma(-) \coloneqq (-)^p/p$ denote the divided power operation and let $\gamma^{(n)}(-)$ denote its n-fold iteration. To get an explicit description of the filtration $\operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}} q\text{-}D_\alpha$, we must find elements $\widetilde{\gamma}_q^{(n)}(x^\alpha) \in \operatorname{fil}^{p^n}_{q\text{-}\mathrm{Hdg}} q\text{-}D_\alpha$ for all $n \geqslant 0$ such that the following two conditions hold:

(*a*) *We have* $\widetilde{\gamma}_q^{(n)}(x^\alpha) \equiv \gamma^{(n)}(x^\alpha) \mod (q-1)$.

(*b*) *The image of* $\widetilde{\gamma}_q^{(n)}(x^\alpha)$ *in* $D_\alpha[1/p]^\wedge_{\mathrm{Hdg}}\llbracket q-1\rrbracket$ *is contained in the ideal* $(x^\alpha, q-1)^{p^n}$.

Indeed, if $\operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}} q\text{-}D_\alpha/(q-1) \cong \operatorname{fil}^\star_{\mathrm{Hdg}} D_\alpha$ (which we know for $p > 2$ as well as for $p = 2$ in the case where α is even and $\geqslant 4$; in the remaining cases we wish to show it), then such elements must exist. Conversely, if such elements exist, then $\operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}} q\text{-}D_\alpha/(q-1) \to \operatorname{fil}^\star_{\mathrm{Hdg}} D_\alpha$ must be surjective, thus an isomorphism by Lemma 4.26, and so $\operatorname{fil}^\star_{q\text{-}\mathrm{Hdg}} q\text{-}D_\alpha$ must be generated as a $(p, q-1)$-complete filtered $q\text{-}D_\alpha$-algebra by $(q-1)$ in filtration degree 1 and the elements $\widetilde{\gamma}_q^{(n)}(x^\alpha)$ in filtration degree p^n for all $n \geqslant 0$.

In Example 4.24, we've got a first taste why describing such $\widetilde{\gamma}_q^{(n)}(x^\alpha)$ is not an easy task. The following technical lemma due to Samuel Meyer shows existence of these lifts along with some structural information about them, and we'll even see an explicit recursive construction in the proof. Moreover, all of this works for all $\alpha \geqslant 2$ without any restrictions in the case $p = 2$.

11.17. Lemma (Meyer). — *For all primes p, there is a sequence $(\Gamma_n)_{n\geqslant 0}$ of polynomials in $\mathbb{Z}_p\{x\}[q]$ with the following properties:*

(a) $\Gamma_n \equiv x^{p^n} \mod (q-1)^{p-1}$ *and* $\Gamma_n \in (x^p, (q-1)^{p-1})^{p^{n-1}}$.

(b) $\Gamma_n \in ((\phi^i(x), \Phi_{p^i}(q))^p, \Phi_{p^i}(q)^{p-1})^{p^{n-1-i}}$ *for all* $1 \leqslant i \leqslant n-1$.

(c) $\Gamma_n \in (\phi^n(x), \Phi_{p^n}(q))$.

(d) $(\Gamma_n)^\alpha \in \prod_{i=1}^n \Phi_{p^i}(q)^{p^{n-i}} \cdot q\text{-}D_\alpha$ *for all* $\alpha \geqslant 2$.

In particular, for all $\alpha \geqslant 2$, $(\Gamma_n)^\alpha$ is contained in the ideal $(x^\alpha, q-1)^{p^n}$, and

$$\widetilde{\gamma}_q^{(n)}(x^\alpha) \coloneqq \frac{(\Gamma_n)^\alpha}{\prod_{i=1}^n \Phi_{p^i}(q)^{p^{n-i}}} \in \mathrm{fil}^{p^n}_{q\text{-Hdg}}\, q\text{-}D_\alpha$$

is a lift of the n-fold iterated divided power $\gamma^{(n)}(x^\alpha)$ and contained in the $(p^n)^{th}$ step of the q-Hodge filtration on q-D_α.

Proof. We'll do a proof by induction. For the base case of the induction, $n = 0$, let $\Gamma_0 \coloneqq x$. All of the statements are trivial in this case.

For the induction step, we first want to construct the element Γ_n. For this, let P_n, Q_n be some polynomials in $\mathbb{Z}[q]$ such that $p = P_n(q)(q-1)^{(p-1)p^{n-1}} + Q_n(q)\Phi_{p^n}(q)$. Note that such polynomials always exist, since $\Phi_{p^n}(1) = p$ and $\Phi_{p^n}(q) \equiv (q-1)^{(p-1)p^{n-1}} \mod p$, so

$$\frac{\Phi_{p^n}(q) - (q-1)^{(p-1)p^{n-1}}}{p}$$

is a unit modulo $(q-1)^{(p-1)p^{n-1}}$. Now define

$$\Gamma_n \coloneqq (\Gamma_{n-1})^p + P_n(q)(q-1)^{(p-1)p^{n-1}}\delta(\Gamma_{n-1}) = \phi(\Gamma_{n-1}) - Q_n(q)\Phi_{p^n}(q)\delta(\Gamma_{n-1}).$$

Statement (*a*) follows trivially. For (*b*) and (*c*), by Lemma 11.18 below it's enough to check that $p \cdot \Gamma_n$ is contained in these ideals. We have

$$\begin{aligned} p \cdot \Gamma_n &= p \cdot (\Gamma_{n-1})^p + P_n(q)(q-1)^{(p-1)p^{n-1}}\big(\phi(\Gamma_{n-1}) - (\Gamma_{n-1})^p\big) \\ &= p \cdot \phi(\Gamma_{n-1}) - Q_n(q)\Phi_{p^n}(q)\big(\phi(\Gamma_{n-1}) - (\Gamma_{n-1})^p\big). \end{aligned}$$

Now $(\Gamma_{n-1})^p$ and $\phi(\Gamma_{n-1})$ are contained in each one of the ideals from (*b*). Indeed, for $(\Gamma_{n-1})^p$, this follows from statements (*b*) and (*c*) of the induction hypothesis, and for $\phi(\Gamma_{n-1})$ this follows similarly from (*a*) and (*b*). Therefore, the first of the two equations above shows that $p \cdot \Gamma_n$ is contained in each of the ideals from (*b*). Similarly, using statement (*c*) of the induction hypothesis, we get $\phi(\Gamma_{n-1}) \in (\phi^n(x), \Phi_{p^n}(q))$ and so the second of the equations above shows that $p \cdot \Gamma_n$ is contained in this ideal as well. This finishes the induction step for (*b*) and (*c*).

It remains to show statement (*d*). By [BS19, Lemma 16.10], q-D_α is $(p, q-1)$-completely flat over $\mathbb{Z}_p[\![q-1]\!]$ and thus flat on the nose over $\mathbb{Z}[q]$. Therefore

$$\prod_{i=1}^n \Phi_{p^i}(q)^{p^{n-i}} \cdot q\text{-}D_\alpha = \bigcap_{i=1}^n \Phi_{p^i}(q)^{p^{n-i}} \cdot q\text{-}D_\alpha\,.$$

To show that $(\Gamma_n)^\alpha \in \Phi_{p^i}(q)^{p^{n-i}} \cdot q\text{-}D_\alpha$ for $1 \leqslant i \leqslant n-1$, by the already proven statement (*b*), it's enough to show the same for any element in the ideal $((\phi^i(x), \Phi_{p^i}(q))^p, \Phi_{p^i}(q)^{p-1})^{\alpha p^{n-1-i}}$. So consider a monomial of the form

$$\left(\phi^i(x)^j \Phi_{p^i}(q)^k\right)^\ell \Phi_{p^i}(q)^{(p-1)m}\,,$$

where $j+k=p$ and $\ell+m=\alpha p^{n-1-i}$. By construction, $\phi(x)^\alpha$ becomes divisible by $\Phi_p(q)$ in $q\text{-}D_\alpha$ and so $\phi^i(x)^\alpha \in \Phi_{p^i}(q)\cdot q\text{-}D_\alpha$. Hence $\phi^i(x)^{j\ell}$ is divisible by $\Phi_{p^i}(q)^{\lfloor j\ell/\alpha\rfloor}$. It will therefore be enough to show

$$\left\lfloor \frac{j\ell}{\alpha}\right\rfloor + k\ell + (p-1)m \geqslant p^{n-i}\,.$$

This is straightforward: For $\ell = 0$, the inequality follows from $\alpha(p-1)\geqslant p$ as $\alpha \geqslant 2$. In general, if we replace (j,k) by $(j-1,k+1)$, the left-hand side changes by at least $\ell - \lfloor \ell/\alpha\rfloor - 1$; for $\ell \geqslant 1$ and $\alpha \geqslant 2$ this term is always nonnegative. Therefore we may assume $j=p$, $k=0$, and we must show $\lfloor p\ell/\alpha\rfloor + (p-1)m \geqslant p^{n-i}$. If $p=2$ and $\alpha=2$, this becomes the equality $\ell+m=2^{n-i}$ and so the inequality is sharp in this case. If $p\geqslant 3$ or $\alpha \geqslant 3$, we have $(p-1)-\lfloor p/\alpha\rfloor - 1 \geqslant 0$ and so by the same argument as before we may assume $\ell = \alpha p^{n-1-i}$, $m=0$. The the desired inequality follows from $\alpha(p-1)\geqslant p$ again.

A similar but easier argument shows that every element in $(\phi^n(x), \Phi_{p^n}(q))^\alpha$ becomes divisible by $\Phi_{p^n}(q)$ in $q\text{-}D_\alpha$ and we have an inclusion of ideals $(x^p, (q-1)^{p-1})^{\alpha p^{n-1}} \subseteq (x^\alpha, q-1)^{p^n}$ in $\mathbb{Z}_p\{x\}[q]$. This finishes the proof of (*d*) and shows $(\Gamma_n)^\alpha \in (x^\alpha, q-1)^{p^n}$. Hence $\widetilde{\gamma}_q^{(n)}(x^\alpha)$ is really contained in the $(p^n)^{\text{th}}$ step of the q-Hodge filtration and it lifts $\gamma^{(n)}(x^\alpha)$ by (*a*). ☐

11.18. Lemma. — *If $J \subseteq \mathbb{Z}_p\{x\}[q]$ is any of the ideals in Lemma 11.17(b) or (c), then $\mathbb{Z}_p\{x\}[q]/J$ is p-torsion free.*

Proof. Consider the map $\psi_i\colon \mathbb{Z}_p\{x\}[q] \to \mathbb{Z}_p\{x\}[q]$ given by the i-fold iterated Frobenius $\phi^i\colon \mathbb{Z}_p\{x\}\to\mathbb{Z}_p\{x\}$ and $q \mapsto \Phi_{p^i}(q)$. If we replace $\phi^i(x)$ and $\Phi_{p^i}(q)$ in the definition of J by x and q, respectively, we obtain an ideal $J_0 \subseteq \mathbb{Z}_p\{x\}[q]$ such that

$$\mathbb{Z}_p\{x\}/J \cong \mathbb{Z}_p\{x\}/J_0 \otimes_{\mathbb{Z}_p\{x\}[q],\psi_i} \mathbb{Z}_p\{x\}[q]\,.$$

Now ϕ^i is flat by [BS19, Lemma 2.11] and $q\mapsto \Phi_{p^i}(q)$ is finite free, as the polynomial $\Phi_{p^i}(q)$ is monic. So ψ_i is flat and it suffices to show that $\mathbb{Z}_p\{x\}[q]/J_0$ is p-torsion free. But $\mathbb{Z}_p\{x\}[q]$ is a free module over $\mathbb{Z}_p$ with basis given by monomials in $x, \delta(x), \delta^2(x), \dots$ and q. By construction, J_0 is a free submodule on a subset of that basis. It follows that $\mathbb{Z}_p\{x\}[q]/J_0$ is free over $\mathbb{Z}_p$, hence p-torsion free. ☐

Finally, as a simple corollary of Lemma 11.17, we get an elementary proof of Theorem 4.22(*a*).

Proof of Theorem 4.22(a). We already know from Lemma 4.26 that

$$\left(\operatorname{fil}^\star_{q\text{-Hdg}} q\text{-dR}_{R/A}\right)/(q-1) \longrightarrow \operatorname{fil}^\star_{\text{Hdg}} \operatorname{dR}_{R/A}$$

is degree-wise injective, so it suffices to show surjectivity. It'll be enough to show that for each of the generators of $J=(x_1^{\alpha_1},\dots,x_r^{\alpha_r})$ and all $n\geqslant 0$, the n-fold iterated divided power $\gamma^{(n)}(x_i^{\alpha_i})$ admits a lift which lies in the $(p^n)^{\text{th}}$ step of the q-Hodge filtration. Thus, it's enough to treat the case $A=\mathbb{Z}_p\{x\}$ and $R=\mathbb{Z}_p\{x\}/x^\alpha$ for all $\alpha\geqslant 2$. In this case the desired lifts have been constructed in Lemma 11.17. ☐

§12. Algebras of overconvergent functions

In this section we prove Theorems 1.40 and 1.41. In §§12.1–12.2 we'll review Clausen's and Scholze's approach to adic spaces via solid analytic rings [CS24, Lecture 10] and study algebras of overconvergent functions as well as gradings in this setup. In §12.3, we'll then combine this with our explicit computation of the q-Hodge filtration on $q\text{-dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ from §11.2 to finish the proof of Theorems 1.40 and 1.41.

§12.1. Adic spaces as analytic stacks

In the following, we'll use the formalism of analytic stacks from [CS24]. Recall the notion of *solid condensed spectra* from 5.1. We let $\mathcal{D}(\mathbb{Z}_\blacksquare) \simeq \operatorname{Mod}_\mathbb{Z}(\operatorname{Sp}_\blacksquare)$ denote the derived ∞-category of solid abelian groups. Let also $\operatorname{Null}_\mathbb{Z} \coloneqq \operatorname{Null} \otimes \mathbb{Z} \simeq \operatorname{cofib}(\mathbb{Z}[\{\infty\}] \to \mathbb{Z}[\mathbb{N} \cup \{\infty\}])$ denote the free condensed abelian group on a nullsequence and let $\sigma\colon \operatorname{Null}_\mathbb{Z} \to \operatorname{Null}_\mathbb{Z}$ denote the shift endomorphism.

12.1. Huber pairs à la Clausen–Scholze. — Recall that to any Huber pair (R, R^+) one can associate an *analytic ring* $(R, R^+)_\blacksquare$ in the sense of [CS24, Lecture 1] as follows: First consider R as a condensed ring via its given topology. For $f \in R(*)$ and $M \in \operatorname{Mod}_R(\mathcal{D}(\mathbb{Z}_\blacksquare))$ we say that M is *f-solid* if

$$1 - f\sigma^*\colon \operatorname{R}\underline{\operatorname{Hom}}_\mathbb{Z}(\operatorname{Null}_\mathbb{Z}, M) \xrightarrow{\simeq} \operatorname{R}\underline{\operatorname{Hom}}_\mathbb{Z}(\operatorname{Null}_\mathbb{Z}, M)$$

is an equivalence. The inclusion of the full sub-∞-category of f-solid R-modules admits a left adjoint $(-)^{f\blacksquare}$, called *f-solidification*. The underlying animated condensed ring of $(R, R^+)_\blacksquare$ is then defined as

$$(R, R^+)^\triangleright_\blacksquare \coloneqq \operatorname*{colim}_{\{f_1, \dots, f_r\} \subseteq R^+} R^{f_1\blacksquare, \dots, f_r\blacksquare},$$

where the colimit is taken over all finite subsets of R^+, and $\mathcal{D}((R, R^+)_\blacksquare) \subseteq \operatorname{Mod}_R(\mathcal{D}(\mathbb{Z}_\blacksquare))$ is the full sub-∞-category of solid condensed R-modules that are f-solid for all $f \in R^+ \subseteq R(*)$. In the following, we'll always work with Huber pairs for which $(R, R^+)^\triangleright_\blacksquare$ is just R itself.

The classical notion of *affinoid open subsets* fits naturally into this formalism. Suppose we're given $f_1, \dots, f_r \in R(*)$ generating an open ideal as well as another element $g \in R(*)$, so that $U \coloneqq \bigl\{x \in \operatorname{Spa}(R, R^+) \bigm| |f_1|_x, \dots, |f_r|_x \leqslant |g|_x \neq 0\bigr\}$ defines a rational open subset. We can define an analytic ring $\mathcal{O}(U_\blacksquare)$ as follows: The underlying animated condensed ring is the solidification

$$\mathcal{O}(U) \coloneqq R\bigl[\tfrac{1}{g}\bigr]^{(f_1/g)\blacksquare, \dots, (f_r/g)\blacksquare}$$

and we let $\mathcal{D}(U_\blacksquare) \coloneqq \mathcal{D}(\mathcal{O}(U_\blacksquare)) \subseteq \operatorname{Mod}_{R[1/g]}(\mathcal{D}((R, R^+)_\blacksquare))$ be the full sub-∞-category spanned by those $R[1/g]$-modules in $\mathcal{D}((R, R^+)_\blacksquare)$ that are also (f_i/g)-solid for $i = 1, \dots, r$. If $\mathcal{O}(U)$ is static and quasi-separated, it agrees with the Huber ring from the classical theory of adic spaces. In practice, this will almost always be the case.

12.2. Adic spaces à la Clausen–Scholze. — Clausen and Scholze associate to any Tate[(12.1)] adic space X an *analytic stack* $X_\blacksquare \to \operatorname{AnSpec} \mathbb{Z}_\blacksquare$. If $X = \operatorname{Spa}(R, R^+)$ is Tate affinoid, we simply put $X_\blacksquare \coloneqq \operatorname{AnSpec}(R, R^+)_\blacksquare$. If $U \subseteq \operatorname{Spa}(R, R^+)$ is an open subset of a Tate affinoid

(12.1) To avoid confusion with analytic stacks, we'll call an adic space *Tate* rather than *analytic* if, locally, there exists a topologically nilpotent unit. The restriction to Tate adic spaces makes sure that open immersions go to open immersions (see Lemma 12.3 below); analytic stacks can be associated to any adic space.

adic space, choose a cover $V \coloneqq \coprod_{i\in I} V_i \to U$ by rational open subsets and form the Čech nerve $V_\bullet \coloneqq \check{C}_\bullet(V \to X)$. Every V_n is a disjoint union of affinoid adic spaces, hence $V_{n,\blacksquare}$ is already defined. Then we can put $U_\blacksquare \coloneqq \operatorname{colim}_{[n]\in\Delta^{\mathrm{op}}} V_{n,\blacksquare}$. Finally, if X is an arbitrary Tate adic space, choose a cover $W \coloneqq \coprod_{j\in J} W_j \to X$ by affinoids and form the Čech nerve $W_\bullet \coloneqq \check{C}_\bullet(W \to X)$. Each W_m is a disjoint union of open subsets of Tate affinoid adic spaces, so $W_{m,\blacksquare}$ is already defined, and we put $X_\blacksquare \coloneqq \operatorname{colim}_{[m]\in\Delta^{\mathrm{op}}} W_{m,\blacksquare}$.

It can be shown that these constructions are well-defined and independent of the choices involved. We'll omit the verification, but let us at least mention the crucial input.

12.3. Lemma. — *Let (R, R^+) be a Huber pair and let $X_\blacksquare \coloneqq \operatorname{AnSpec}(R, R^+)_\blacksquare$ be the associated affine analytic stack.*

(*a*) *If $U, U' \subseteq \operatorname{Spa}(R, R^+)$ are rational open subsets, then*

$$\operatorname{AnSpec}\mathcal{O}(U_\blacksquare) \times_{\operatorname{AnSpec}(R,R^+)_\blacksquare} \operatorname{AnSpec}\mathcal{O}(U'_\blacksquare) \simeq \operatorname{AnSpec}\mathcal{O}\big((U\cap U')_\blacksquare\big)\,.$$

(*b*) *If R is Tate and $U \subseteq \operatorname{Spa}(R, R^+)$ is a rational open subset, then $j\colon U_\blacksquare \to X_\blacksquare$ is an open immersion of affine analytic stacks in the sense of [CS24, Lecture 16]. That is, j^* admits a fully faithful left adjoint $j_!$ satisfying the projection formula.*

(*c*) *If R is Tate and $\coprod_{i=1}^n U_i \to \operatorname{Spa}(R, R^+)$ is a cover by rational open subsets, then $\coprod_{i=1}^n U_{i,\blacksquare} \to X_\blacksquare$ is a !-cover of affine analytic stacks.*

12.4. Remark. — The Tate condition in Lemma 12.3(*b*) and (*c*) is crucial and it is the reason why we restrict to the Tate case when we describe adic spaces in terms of analytic stacks. Without this assumption, (*b*) will be wrong. For example, if R is a discrete ring, any Zariski-open also determines a rational open of $\operatorname{Spa}(R, R)$, but in this case j^* almost never preserves limits, so it can't have a left adjoint $j_!$.

Proof sketch of Lemma 12.3. Suppose U and U' are given by $|f_1|, \dots, |f_r| \leqslant |g| \neq 0$ and $|f'_1|, \dots, |f'_s| \leqslant |g'| \neq 0$, respectively. Using the description of pushouts from [CS24, Lecture 11], it's clear that $\mathcal{O}(U_\blacksquare) \otimes^{\mathrm{L}}_{(R,R^+)_\blacksquare} \mathcal{O}(U'_\blacksquare)$ is the solidification of $R[1/(gg')]$ at the elements f_i/g and f'_j/g' for $i = 1, \dots, r$, $j = 1, \dots, s$. But that's precisely $\mathcal{O}((U \cap U')_\blacksquare)$, proving (*a*).

For (*b*), assume U is given by $|f_1|, \dots, |f_r| \leqslant |g| \neq 0$. Since R is assumed to be Tate, the open ideal generated by $f_1, \dots, f_r$ must be all of R. Hence g will aready be invertible in $R[T_1, \dots, T_r]/(gT_i - f_i \mid i = 1, \dots, r)$ and this quotient is automatically a derived quotient as well. It follows that the functor $j^*\colon \mathcal{D}(X_\blacksquare) \to \mathcal{D}(U_\blacksquare)$ can also be written as

$$(-)[T_1, \dots, T_r]^{T_1\blacksquare, \dots, T_r\blacksquare}/(gT_i - f_i \mid i = 1, \dots, r)\,.$$

By [CS24, Lecture 7], the functor $(-)[T]^{T\blacksquare}$ of adjoining a variable and then solidifiying it can be explicitly described as $\mathrm{R}\underline{\operatorname{Hom}}_{\mathbb{Z}}(\mathbb{Z}(\!(T^{-1})\!)/\mathbb{Z}[T], -)$ and so $j^*(-) \simeq \mathrm{R}\underline{\operatorname{Hom}}_R(Q, -)$, where

$$Q \coloneqq \left(\bigotimes_{i=1}^r \Sigma^{-1}\mathbb{Z}(\!(T_i^{-1})\!)/\mathbb{Z}[T_i] \otimes^{\mathrm{L}\blacksquare}_{\mathbb{Z}} R\right)/(gT_i - f_i \mid i = 1, \dots, r)\,.$$

It follows immediately that j^* admits a left adjoint $j_!(-) \simeq Q \otimes^{\mathrm{L}}_{(R,R^+)_\blacksquare} -$. It remains to check the projection formula

$$j_!(M) \otimes^{\mathrm{L}}_{(R,R^+)_\blacksquare} N \simeq j_!\big(M \otimes^{\mathrm{L}}_{\mathcal{O}(U_\blacksquare)} j^*(N)\big)\,.$$

By the same argument as above, Q is already an $R[1/g]$-module and the functor j^* is insensitive to inverting g. Therefore, it's enough to check the projection formula in the case where N is an $R[1/g]$-module. When restricting to $R[1/g]$-modules, j^* is just given by successively killing the idempotent algebras $\mathbb{Z}(\!(T_i^{-1})\!) \otimes^{\mathrm{L}\blacksquare}_{\mathbb{Z}[T_i], T_i\mapsto f_i/g} R[1/g]$ for $i = 1, \dots, r$. Now for killing an idempotent it's completely formal to see that the left adjoint indeed satisfies the projection formula. This finishes the proof of (*b*).

To show (*c*), since we already know that each $j_i\colon U_{i,\blacksquare} \to X_\blacksquare$ is an open immersion, we can use the criterion from [CS24, Lecture 18] to verify that $\coprod_{i=1}^n U_{i,\blacksquare} \to X_\blacksquare$ is indeed a !-cover. That is, if $A_i \coloneqq \operatorname{cofib}(j_{i,!}\mathcal{O}(U_i) \to R)$, we need to show $A_1 \otimes^{\mathrm{L}}_{(R,R^+)_\blacksquare} \cdots \otimes^{\mathrm{L}}_{(R,R^+)_\blacksquare} A_n \simeq 0$. Using [Hub94, Lemma 2.6] and an inductive argument as in [CS19, Lemma 10.3], this can be reduced to the special case where $n = 2$ and $U_1 = \{x \in X \mid 1 \leqslant |f|_x\}$, $U_2 = \{x \in X \mid |f|_x \leqslant 1\}$ for some $f \in R$. This is now a straightforward calculation. □

12.5. Remark. — Let $U \subseteq X$ be an open inclusion of Tate adic spaces and let $j\colon U_\blacksquare \to X_\blacksquare$ be the corresponding map of analytic stacks. In the following, if its clear that we're working in $\mathcal{D}(X_\blacksquare)$, we often abuse notation and write $\mathcal{O}_U$ instead of $j_*\mathcal{O}_{U_\blacksquare}$ for the pushforward of the structure sheaf of $U_\blacksquare$. We also use $-\otimes^{\mathrm{L}}_{\mathcal{O}_{X_\blacksquare}} \mathcal{O}_{U_\blacksquare}$ to denote the functor $j_*j^*\colon \mathcal{D}(X_\blacksquare) \to \mathcal{D}(X_\blacksquare)$.

Let us point out that $-\otimes^{\mathrm{L}}_{\mathcal{O}_{X_\blacksquare}} \mathcal{O}_{U_\blacksquare}$ is *not* just the tensor product with $\mathcal{O}_U$ in the symmetric monoidal ∞-category $\mathcal{D}(X_\blacksquare)$. We can already see the difference if $X = \operatorname{Spa}(R, R^+)$ and $U \subseteq X$ is a rational open given by $|f_1|, \dots, |f_r| \leqslant |g| \neq 0$: In this case,

$$-\otimes^{\mathrm{L}}_{\mathcal{O}_{X_\blacksquare}} \mathcal{O}_{U_\blacksquare} \simeq \left(-\otimes^{\mathrm{L}}_{\mathcal{O}_{X_\blacksquare}} \mathcal{O}_U\right)^{(f_1/g)_\blacksquare, \dots, (f_r/g)_\blacksquare}.$$

In particular, even though $\mathcal{O}_U \otimes^{\mathrm{L}}_{\mathcal{O}_{X_\blacksquare}} \mathcal{O}_{U_\blacksquare} \simeq \mathcal{O}_U$ (see Lemma 12.3(*a*) and Lemma 12.10(*b*) below), it's rarely true that $\mathcal{O}_U$ is idempotent in $\mathcal{D}(X_\blacksquare)$.

Thus, there's a priori no reason to expect that sheaves of overconvergent functions $\mathcal{O}_{Z^\dagger}$ would be idempotent. In the following, we'll investigate why idempotence is satisfied in the situation of Theorems 1.40 and 1.41. Let's start by introducing a notion of open immersions for analytic stacks that need not be affine.

12.6. Open immersions of analytic stacks. — We call a map of analytic stacks $j\colon U \to X$ a *naive open immersion* if j is a !-able monomorphism and $j^* \simeq j^!$. Since j is a monomorphism, $U \times_X U \simeq U$. Combining this with proper base change, we get $j^*j_! \simeq \mathrm{id}_{\mathcal{D}(U)}$ and so $j_!$ is fully faithful. Then the right adjoint j_* of j^* must be fully faithful as well.

Using the projection formula and $j^*j_! \simeq \mathrm{id}_{\mathcal{D}(U)}$, we see that $j_!\mathcal{O}_U \to \mathcal{O}_X$ exhibits $j_!\mathcal{O}_U$ as an idempotent coalgebra in $\mathcal{D}(X)$. Then $\operatorname{cofib}(j_!\mathcal{O}_U \to \mathcal{O}_X)$ must be an idempotent algebra. In this way, we can associate to any naive open immersion an idempotent algebra in $\mathcal{D}(X)$, which we call the *complementary idempotent determined by* U and denote $\mathcal{O}_{X\smallsetminus U}$. It's straightforward to check that the forgetful functor $i_*\colon \mathrm{Mod}_{\mathcal{O}_{X\smallsetminus U}}(\mathcal{D}(X)) \to \mathcal{D}(X)$, which is fully faithful by idempotence, fits into a recollement

$$\mathrm{Mod}_{\mathcal{O}_{X\smallsetminus U}}\big(\mathcal{D}(X)\big) \;\begin{smallmatrix}\xleftarrow{\;i^*\;}\\ \xrightarrow{\;i_*\;}\\ \xleftarrow[\;i^!\;]{}\end{smallmatrix}\; \mathcal{D}(X) \;\begin{smallmatrix}\xleftarrow{\;j_!\;}\\ \xrightarrow{\;j^*\;}\\ \xleftarrow[\;j_*\;]{}\end{smallmatrix}\; \mathcal{D}(U)$$

and so $j_*\mathcal{O}_U$ is obtained from $\mathcal{O}_X$ by killing the idempotent algebra $\mathcal{O}_{X\smallsetminus U}$. As long as it's clear that we're working in $\mathcal{D}(X)$, we often abuse notation and just write $\mathcal{O}_U$ instead of $j_*\mathcal{O}_X$.

12.7. Remark. — Every open immersion of affine analytic stacks in the sense of [CS24, Lecture 16] is also a naive open immersion.

12.8. Remark. — If $A \in \mathcal{D}(X)$ is an idempotent algebra, we can define an analytic substack $U_A \subseteq X$ by declaring that a map $f\colon Y \to X$ factors through U_A if and only if $f^*\colon \mathcal{D}(X) \to \mathcal{D}(Y)$ factors through the localisation $\mathcal{D}(X)/\operatorname{Mod}_A(\mathcal{D}(X))$, or equivalently, if and only if $f^*(A) \simeq 0$. However, it's *not* true that the constructions $U \mapsto \mathcal{O}_{X \smallsetminus U}$ and $A \mapsto U_A$ are inverses; it's not even clear why $\mathcal{D}(U_A)$ would coincide with $\mathcal{D}(X)/\operatorname{Mod}_A(\mathcal{D}(X))$.

It's not obvious what conditions should be put on U and A to make these constructions mutually inverse (moreover, whatever the condition, it should be satisfied for open immersions of affine analytic stacks). This explains why we call the notion from 12.6 *naive*: An *honest* open immersion of analytic stacks should be a naive open immersion for which the idempotent algebra $\mathcal{O}_{X \smallsetminus U}$ meets the putative condition. In the following, we'll work with the naive notion, since it is enough for our purposes.

12.9. Lemma. — *Let $U' \to U \to X$ be naive open immersions of analytic stacks. Suppose that U contains the closure of U' in the sense that there exists another naive open immersion $j\colon V \to X$ such that $U' \times_X V \simeq \emptyset$ and $\mathcal{O}_{X \smallsetminus V} \otimes^{\mathrm{L}}_{\mathcal{O}_X} \mathcal{O}_{X \smallsetminus U} \simeq 0$. Then $\mathcal{O}_U \otimes^{\mathrm{L}}_{\mathcal{O}_X} \mathcal{O}_{U'} \simeq \mathcal{O}_{U'}$. Moreover, the map $\mathcal{O}_U \to \mathcal{O}_{U'}$ is trace-class in $\mathcal{D}(X)$ and factors through $\mathcal{O}_{X \smallsetminus V}$.*

Proof. The condition $U' \times_X V \simeq \emptyset$ implies that $\mathcal{O}_{U'}$ is in the kernel of the pullback functor $j^*\colon \mathcal{D}(X) \to \mathcal{D}(V)$ and so $\mathcal{O}_{U'}$ is an algebra over the idempotent $A \coloneqq \mathcal{O}_{X \smallsetminus V}$ by 12.6. We also know that $\mathcal{O}_U$ is obtained from $\mathcal{O}_X$ by killing the idempotent $B \coloneqq \mathcal{O}_{X \smallsetminus U}$. Hence $\mathcal{O}_U \simeq \operatorname{cofib}(B^\vee \to \mathcal{O}_X)$. Since $B^\vee$ is a B-module, $\mathcal{O}_{U'}$ is an A-module, and $A \otimes B \simeq 0$, we get $B^\vee \otimes^{\mathrm{L}}_{\mathcal{O}_X} \mathcal{O}_{U'} \simeq 0$, hence indeed $\mathcal{O}_U \otimes^{\mathrm{L}}_{\mathcal{O}_X} \mathcal{O}_{U'} \simeq \mathcal{O}_{U'}$.

Since the double dual $B^{\vee\vee}$ is still a B-module, the same argument shows $\mathcal{O}_U^\vee \otimes^{\mathrm{L}}_{\mathcal{O}_X} \mathcal{O}_{U'} \simeq \mathcal{O}_{U'}$. Hence $\mathcal{O}_U \to \mathcal{O}_{U'}$ is trace-class, with classifier given by the unit $\mathcal{O}_X \to \mathcal{O}_{U'}$. We've already seen that $\mathcal{O}_{U'}$ is an A-algebra. The condition $A \otimes B \simeq 0$ also implies $\mathrm{R}\underline{\operatorname{Hom}}_X(B, A) \simeq 0$, since $\mathrm{R}\underline{\operatorname{Hom}}_X(B, A)$ is both an A-module and a B-module. It follows that A is contained in the image of $j_*\colon \mathcal{D}(U) \to \mathcal{D}(X)$ and hence A is an $\mathcal{O}_U$-algebra. This shows that $\mathcal{O}_U \to \mathcal{O}_{U'}$ factors through A. $\square$

12.10. Lemma. — *Let X be a Tate adic space with associated analytic stack $X_\blacksquare \to \operatorname{AnSpec} \mathbb{Z}_\blacksquare$, and let $U, U' \subseteq X$ be open subsets.*

(*a*) *The map $j\colon U_\blacksquare \to X_\blacksquare$ is a naive open immersion of analytic stacks. Moreover, an arbitary map $f\colon Y \to X_\blacksquare$ of analytic stacks factors through $U_\blacksquare$ if and only if $f^*(\mathcal{O}_{X \smallsetminus U}) \simeq 0$.*

(*b*) *We have $U_\blacksquare \times_{X_\blacksquare} U'_\blacksquare \simeq (U \cap U')_\blacksquare$. In particular, $\mathcal{O}_U \otimes^{\mathrm{L}}_{\mathcal{O}_{X_\blacksquare}} \mathcal{O}_{U'_\blacksquare} \simeq \mathcal{O}_{U \cap U'}$ and vice versa if U and U' are exchanged.*

(*c*) *If $\overline{U}' \subseteq U$, then $U_\blacksquare$ contains the closure of $U'_\blacksquare$ in the sense of Lemma 12.9.*

Proof sketch. Let's start with (*b*). In the case where U and U' are affinoid, $U_\blacksquare \times_{X_\blacksquare} U'_\blacksquare \simeq (U \cap U')_\blacksquare$ follows essentially by the construction of $X_\blacksquare$ in 12.2, because we can choose both U and U' to be part of an affinoid cover of X (and to prove that said construction is independent of the choice of cover, we need Lemma 12.3(*a*)). To show the general case, just cover U and U' by affinoid open subsets.

Let's show (*a*) next. Let's first consider the case where $X = \operatorname{Spa}(R, R^+)$ is affinoid and $U \subseteq X$ is a rational open. We've already seen in Lemma 12.3(*b*) that $j\colon U_\blacksquare \to X_\blacksquare$ is a naive open immersion. Suppose $f\colon Y \to X_\blacksquare$ is a map of analytic stacks such that $f^*(\mathcal{O}_{X\smallsetminus U}) \simeq 0$. If $Y \simeq \operatorname{AnSpec} S$ is affine, then the map of analytic rings $(R, R^+)_\blacksquare \to S$ factors through $\mathcal{O}(U_\blacksquare)$ if and only if $f^*\colon \mathcal{D}((R, R^+)_\blacksquare) \to \mathcal{D}(S)$ factors through $\mathcal{D}(U_\blacksquare)$. Since $f^*(\mathcal{O}_{X\smallsetminus U}) \simeq 0$, this is satisfied in our case. This proves the claim in the case where $Y \simeq \operatorname{AnSpec} S$ is affine. In particular, $U_\blacksquare \times_{X_\blacksquare} \operatorname{AnSpec} S \simeq \operatorname{AnSpec} S$. For the general case, write Y as a colimit of affines to see $U_\blacksquare \times_{X_\blacksquare} Y \simeq Y$. Then $f\colon Y \to X_\blacksquare$ clearly factors through $U_\blacksquare$.

Now let U and X be arbitrary. Proving that $j\colon U_\blacksquare \to X_\blacksquare$ is a naive open immersion formally reduces to the special case considered above; we omit the argument. Now let $f\colon Y \to X_\blacksquare$ be a map of analytic stacks such that $f^*(\mathcal{O}_{X\smallsetminus U}) \simeq 0$. Whether f factors through $U_\blacksquare$ can be checked locally on $X_\blacksquare$. By (b), if $\operatorname{Spa}(R, R^+) \to X$ is an affinoid open supset, then $U_\blacksquare \times_{X_\blacksquare} \operatorname{AnSpec}(R, R^+)_\blacksquare \simeq (U \cap \operatorname{Spa}(R, R^+))_\blacksquare$, so we can reduce to the case where X is affinoid. As above, we may also assume that $Y \simeq \operatorname{AnSpec} S$ is affine. Let $\coprod_{i\in I} U_i \to U$ be a cover by rational open subsets. Then

$$\mathcal{O}_{X\smallsetminus U} \simeq \operatorname*{colim}_{\{i_1,\dots,i_n\}\subseteq I} \left(\mathcal{O}_{X\smallsetminus U_{i_1}} \otimes^{\mathrm{L}}_{\mathcal{O}_{X_\blacksquare}} \cdots \otimes^{\mathrm{L}}_{\mathcal{O}_{X_\blacksquare}} \mathcal{O}_{X\smallsetminus U_{i_n}}\right),$$

where the colimit is taken over all finite subsets of I. Since the colimit is filtered and $f^*(\mathcal{O}_{X\smallsetminus U})$ is detected by the single condition $1 = 0$, there exists a finite subset $\{i_1, \dots, i_n\} \subseteq I$ such that already $f^*(\mathcal{O}_{X\smallsetminus U_{i_1}}) \otimes^{\mathrm{L}}_S \cdots \otimes^{\mathrm{L}}_S f^*(\mathcal{O}_{X\smallsetminus U_{i_n}}) \simeq 0$ in $\mathcal{D}(S)$. By the criterion from [CS24, Lecture 18], it follows that $\coprod_{j=1}^n U_{i_j,\blacksquare} \times_{X_\blacksquare} \operatorname{AnSpec} S \to \operatorname{AnSpec} S$ is a !-cover. We may therefore replace S by the constituents of this cover, and for each of them it's clear that they factor through $U_\blacksquare$. This finishes the proof of (a).

Part (c) is a formal consequence: If $V \coloneqq X \smallsetminus \overline{U}'$, then $V_\blacksquare \to X_\blacksquare$ is a naive open immersion by (a), $U_\blacksquare \times_{X_\blacksquare} V_\blacksquare \simeq \emptyset$ follows from (b), and if $A \coloneqq \mathcal{O}_{X\smallsetminus U} \otimes^{\mathrm{L}}_{\mathcal{O}_{X_\blacksquare}} \mathcal{O}_{X\smallsetminus V}$, then it's formal to see that $\operatorname{Mod}_A(\mathcal{D}(X_\blacksquare))$ is the kernel of the pullback functor $\mathcal{D}(X_\blacksquare) \to \mathcal{D}(U_\blacksquare) \times_{\mathcal{D}((U\cap V)_\blacksquare)} \mathcal{D}(V_\blacksquare)$. But this functor is an equivalence as $U \cup V = X$, and so $A \simeq 0$. ☐

We can finally show the desired criterion for idempotence.

12.11. Definition. — If X is a Tate adic space and $Z \subseteq X$ is a closed subset, the *overconvergent neighbourhood of Z* is the analytic stack

$$Z^\dagger \coloneqq \lim_{U\supseteq Z} U_\blacksquare\,,$$

where the limit is taken over all open neighbourhoods of Z. If it's clear that we're working in $\mathcal{D}(X_\blacksquare)$, we often abuse notation and denote by $\mathcal{O}_{Z^\dagger} \coloneqq \operatorname{colim}_{U\supseteq Z} \mathcal{O}_U \in \mathcal{D}(X_\blacksquare)$ the *sheaf of overconvergent functions on Z*. This is in favorable situations, but not always, the pushforward of the structure sheaf of $Z^\dagger$; see Theorem 12.12(b) below.

12.12. Theorem. — *Let X be a quasi-compact quasi-separated Tate adic space and let $Z \subseteq X$ be a closed subset such that for all points $z \in Z$ and all generalisations $z' \rightsquigarrow z$ also $z' \in Z$.*

(a) *The ind-object*

$$\text{“}\operatorname*{colim}_{U\supseteq Z}\text{”}\, \mathcal{O}_U \in \operatorname{Ind}\mathcal{D}(X_\blacksquare)$$

is idempotent, nuclear, and obtained by killing the pro-idempotent “$\lim$”$_{Z\cap\overline{W}=\emptyset}\, \mathcal{O}_W$, where the limit is taken over all open subsets $W \subseteq X$ such that $Z \cap \overline{W} = \emptyset$. In particular, $\mathcal{O}_{Z^\dagger} \in \mathcal{D}(X_\blacksquare)$ is idempotent and nuclear.

(*b*) *If for every affinoid open* $j\colon \operatorname{Spa}(R,R^+)\to X$ *the pullback* $j^*(\mathcal{O}_{Z^\dagger})\in\mathcal{D}((R,R^+)_\blacksquare)$ *is connective*[(12.2)], *then pushforward along* $Z^\dagger\to X_\blacksquare$ *induces a symmetric monoidal equivalence* $\mathcal{D}(Z^\dagger)\simeq\operatorname{Mod}_{\mathcal{O}_{Z^\dagger}}(\mathcal{D}(X_\blacksquare))$. *In particular, in this case* $\mathcal{O}_{Z^\dagger}$ *is really the pushforward of the structure sheaf of* $Z^\dagger$.

To prove Theorem 12.12, we send a lemma in advance.

12.13. Lemma. — *Let* X *be a spectral space and let* $Y,Z\subseteq X$ *be closed subsets such that for* $z\in Z$ *and* $y\in Y$ *there never exists a common generalisation* $z\leftsquigarrow x\rightsquigarrow y$ *(in particular* $Z\cap Y=\emptyset$*). Then there exist open neighbourhoods* $U\supseteq Z$ *and* $V\supseteq Y$ *such that* $U\cap V=\emptyset$.

Proof. Fix $z\in Z$. By [Stacks, Tag 0906], $y\in Y$ there exist open neighbourhoods $U_y\ni z$ and $V_y\ni y$ such that $U_y\cap V_y=\emptyset$. By compactness of Y, there exist finitely many $y_1,\dots,y_n\in Y$ such that $Y\subseteq V_z\coloneqq V_{y_1}\cup\dots\cup V_{y_n}$. Let also $U_z\coloneqq U_{y_1}\cap\cdots U_{y_n}$, so that $U_z\cap V_z=\emptyset$. By compactness of Z, there exist finitely many $z_1,\dots,z_m\in Z$ such that $Z\subseteq U\coloneqq U_{z_1}\cup\dots\cup U_{z_m}$. Putting $V\coloneqq V_{z_1}\cap\dots\cap V_{z_m}$, we have constructed U and V with the required properties. □

Proof of Theorem 12.12. First observe that Lemma 12.13 can be applied to any closed subset $Y\subseteq X$ such that $Z\cap Y=\emptyset$. Indeed, for any common generalisation $z\leftsquigarrow x\rightsquigarrow y$, we would have $x\in Z$, as Z is closed under generalisations, but then $y\in Z$, as Z is also closed under specialisations.

It follows that in the ind-object "$\operatorname{colim}_{U\supseteq Z}$" $\mathcal{O}_U$ we can restrict to open neighbouhoods of the form $U=X\smallsetminus\overline{W}$ for some open subset W such that $Z\cap\overline{W}=\emptyset$. Indeed, for arbitrary U, apply Lemma 12.13 to Z and $X\smallsetminus U$ to get an open neighbourhood $W\supseteq(X\smallsetminus U)$ such that $Z\cap\overline{W}=\emptyset$. Then $(X\smallsetminus\overline{W})\subseteq U$, as desired.

Let $\mathcal{O}_{\overline{W}}\coloneqq\mathcal{O}_{X\smallsetminus(X\smallsetminus\overline{W})}\in\mathcal{D}(X_\blacksquare)$ be the complementary idempotent determined by the open subset $X\smallsetminus\overline{W}$. Since each $\mathcal{O}_U$ is obtained by killing the idempotent $\mathcal{O}_{X\smallsetminus U}$, our observation implies that "$\operatorname{colim}_{U\supseteq Z}$" $\mathcal{O}_U$ is obtained by killing the pro-idempotent "$\lim_{Z\cap\overline{W}=\emptyset}$" $\mathcal{O}_{\overline{W}}$. For all such W, applying Lemma 12.13 to Z and $\overline{W}$ provides another open neighbourhood $W'\supseteq\overline{W}$ such that still $Z\cap\overline{W'}=\emptyset$. By Lemma 12.9 and Lemma 12.10(*c*), $\mathcal{O}_{W'}\to\mathcal{O}_W$ is trace-class and factors through $\mathcal{O}_{\overline{W}}$. It follows that "$\lim_{Z\cap\overline{W}=\emptyset}$" $\mathcal{O}_W\simeq$ "$\lim_{Z\cap\overline{W}=\emptyset}$" $\mathcal{O}_{\overline{W}}$ and that the condition of Lemma 10.9 is satisfied, so that "$\operatorname{colim}_{U\supseteq Z}$" $\mathcal{O}_U$ is indeed idempotent and nuclear in $\operatorname{Ind}\mathcal{D}(X_\blacksquare)$. Since $\operatorname{colim}\colon\operatorname{Ind}\mathcal{D}(X_\blacksquare)\to\mathcal{D}(X_\blacksquare)$ preserves idempotents and nuclear objects, it follows that $\mathcal{O}_{Z^\dagger}\in\mathcal{D}(X_\blacksquare)$ is idempotent and nuclear as well. This finishes the proof of (*a*).

For (*b*), note that $Z^\dagger$ is clearly compatible with base change and so is $\mathcal{O}_{Z^\dagger}$ by (*a*) and Lemma 10.9(*c*). We may therefore assume that $X=\operatorname{Spa}(R,R^+)$ is affinoid and $\mathcal{O}_{Z^\dagger}$ is connective. Then $\mathcal{O}_{Z^\dagger}$ can be turned into an analytic ring using the induced analytic ring structure from $(R,R^+)_\blacksquare$. It follows that a map $f\colon\operatorname{AnSpec}S\to\operatorname{AnSpec}(R,R^+)_\blacksquare$ factors through $\mathcal{O}_{Z^\dagger}$ if and only if $S\simeq f^*(\mathcal{O}_{Z^\dagger})$. By Lemma 10.9(*b*), we have $\mathcal{O}_{Z^\dagger}\otimes^{\mathrm{L}}_{(R,R^+)_\blacksquare}\mathcal{O}_W\simeq 0$ for all open W such that $Z\cap\overline{W}=\emptyset$. Thus $S\simeq f^*(\mathcal{O}_{Z^\dagger})$ implies $f^*(\mathcal{O}_W)\simeq 0$ for all such W. By sandwiching open and closed subsets, we get $f^*(\mathcal{O}_{X\smallsetminus U})\simeq 0$ for all open neighbourhoods $U\supseteq Z$. By Lemma 12.10(*a*), this implies that f factors through $Z^\dagger\simeq\lim_{U\supseteq Z}U_\blacksquare$.

Conversely, if f factors through $Z^\dagger$, then $f^*(\mathcal{O}_{X\smallsetminus U})\simeq 0$ for all U and thus $f^*(\mathcal{O}_W)\simeq 0$ for all W as above, using the same sandwiching argument. It follows that S is a module over the nuclear idempotent ind-algebra obtained by killing "$\lim_{Z\cap\overline{W}=\emptyset}$" $f^*(\mathcal{O}_W)$ in $\mathcal{D}(S)$. By

(12.2) Following discussions with Ben Antieau and Peter Scholze, we believe that connectivity can be replaced by the much weaker condition that $\operatorname{Mod}_{j^*(\mathcal{O}_{Z^\dagger})}(\mathcal{D}(R))$ is left-complete, using an adaptation of [MM24, Proposition 2.16].

Lemma 10.9(c), this is "$\mathrm{colim}''_{U\supseteq Z} f^*(\mathcal{O}_U)$. Then S is also a module over the honest colimit $\mathrm{colim}_{U\supseteq Z} f^*(\mathcal{O}_U) \simeq f^*(\mathcal{O}_{Z^\dagger})$, proving $S \simeq f^*(\mathcal{O}_{Z^\dagger})$.

In conclusion, this argument shows that $Z^\dagger \simeq \mathrm{AnSpec}\,\mathcal{O}_{Z^\dagger}$ is an affine analytic stack and so $\mathcal{D}(Z^\dagger) \simeq \mathrm{Mod}_{\mathcal{O}_{Z^\dagger}}(\mathcal{D}((R,R^+)_\blacksquare))$ follows by construction, as we've put the induced analytic ring structure on $\mathcal{O}_{Z^\dagger}$. ☐

This implies idempotence and nuclearity in the situation of Theorem 1.40.

12.14. Corollary. — *Let $X \coloneqq \mathrm{Spa}\,\mathbb{Z}_p[\![q-1]\!] \smallsetminus \{p=0, q=1\}$ and let $Z \subseteq X$ be the union of the closed subsets $\mathrm{Spa}(\mathbb{F}_p(\!(q-1)\!), \mathbb{F}_p[\![q-1]\!])$ and $\mathrm{Spa}(\mathbb{Q}_p(\zeta_{p^n}), \mathbb{Z}_p[\zeta_{p^n}])$ for all $n \geqslant 0$.*

(a) *Z is closed and closed under generalisations.*

(b) *For $n, r, s \geqslant 1$ such that $(p-1)p^n > s$, let $W_{n,r,s} \subseteq X$ be the rational open subset determined by $|p^r| \leqslant |q^{p^n} - 1| \neq 0$, $|(q-1)^s| \leqslant |p| \neq 0$. Then $\mathcal{O}_{Z^\dagger}$ is idempotent, nuclear, and the colimit of the idempotent nuclear ind-algebra obtained by killing the idempotent pro-algebra "$\lim''_{n,r,s} \mathcal{O}_{W_{n,r,s}}$.*

Proof. Let $x \in X \smallsetminus Z$. Then $|p|_x \neq 0$, hence $|(q-1)^s|_x \leqslant |p|_x$ for $s \gg 0$. Choose such an s. Moreover, $|q^{p^n} - 1|_x \neq 0$ holds for all $n \geqslant 0$. Choose n such that $(p-1)p^n > s$ and choose $r \gg 0$ such that $|p^r|_x \leqslant |q^{p^n} - 1|_x$. Then $x \in W_{n,r,s}$. If we can show $Z \cap \overline{W}_{n,r,s} = \emptyset$, both ($a$) and ($b$) will follow. Indeed, this will imply that $X \smallsetminus Z$ is open and closed under specialisations, proving (a). Moreover, $X \smallsetminus Z = \bigcup_{n,r,s} W_{n,r,s}$ and so for any open subset W such that $Z \cap \overline{W} = \emptyset$ we must have $W_{n,r,s} \supseteq \overline{W}$ for sufficiently large n, r, and s by quasi-compactness of $\overline{W}$. Hence (b) follows from Theorem 12.12(a).

To show $Z \cap \overline{W}_{n,r,s} = \emptyset$, let $w \in W_{n,r,s}$. Since $(p-1)p^n > s$, we get $|(q-1)^{(p-1)p^{i-1}}|_w < |p|_x$ for all $i > n$ and so $|\Phi_{p^i}(q)|_w = |p|_w$, where $\Phi_{p^i}(q)$ denotes the $(p^i)^{\mathrm{th}}$ cyclotomic polynomial. Thus $0 < |p^{r+i-n}|_w \leqslant |q^{p^i} - 1|_w$ for $i > n$. In particular, $w \notin Z$. Even better: If U_i denotes the rational open subset determined by $|q^{p^i} - 1| \leqslant |p^{r+i-n+1}| \neq 0$ and V denotes the rational open subset determined by $|p| \leqslant |(q-1)^{s+1}| \neq 0$, then the open set $\bigcup_{i \geqslant n} U_i \cup V$ contains Z and doesn't intersect $W_{n,r,s}$, so indeed $Z \cap \overline{W}_{n,r,s} = \emptyset$. ☐

§12.2. Graded adic spaces

To deduce idempotence and nuclearity in the situation of Theorem 1.41, let us describe how to encode gradings in terms of actions of the analytic stack

$$\mathbb{T} \coloneqq \mathrm{AnSpec}\,\mathbb{Z}[u^{\pm 1}]_\blacksquare\,,$$

where $\mathbb{Z}[u^{\pm 1}]_\blacksquare$ is obtained from $\mathbb{Z}[u^{\pm 1}]$ by solidifying both u and u^{-1}. Equivalently, $\mathbb{Z}[u^{\pm 1}]_\blacksquare$ is the analytic ring associated to the discrete Huber pair $(\mathbb{Z}[u^{\pm 1}], \mathbb{Z}[u^{\pm 1}])$.

12.15. Graded adic spaces via $\mathbb{T}$-actions. — Classically, the grading on $\mathbb{Z}[\beta, t]$ in which β and t receive degree 2 and -2, respectively, is encoded by an action of $\mathbb{G}_m \coloneqq \mathrm{Spec}\,\mathbb{Z}[u^{\pm 1}]$ on $\mathrm{Spec}\,\mathbb{Z}[\beta, t]$. The action map $\mathrm{Spec}\,\mathbb{Z}[\beta, t] \times \mathbb{G}_m \to \mathrm{Spec}\,\mathbb{Z}[\beta, t]$ corresponds to the ring map $\Delta\colon \mathbb{Z}[\beta, t] \to \mathbb{Z}[\beta, t] \otimes_{\mathbb{Z}} \mathbb{Z}[u^{\pm 1}]$ given by $\Delta(\beta) \coloneqq u^2\beta$, $\Delta(t) \coloneqq u^{-2}t$.

In our situation, we're forced to work with the adic spectrum $\overline{X}^* \coloneqq \mathrm{Spa}\,\mathbb{Z}[\beta, t]^\wedge_{(p,t)}$ instead. But in the map Δ we can't just replace $\mathbb{Z}[\beta, t]$ by its (p, t)-completion, since the tensor product $\mathbb{Z}[\beta, t]^\wedge_{(p,t)} \otimes_{\mathbb{Z}} \mathbb{Z}[u^{\pm 1}]$ won't be (p, t)-complete anymore.

To fix this, consider $\pi\colon \mathbb{T} \to \operatorname{AnSpec}\mathbb{Z}_\blacksquare$ and let $-\otimes^{\mathrm{L}}_{\mathbb{Z}_\blacksquare}\mathbb{Z}[u^{\pm 1}]_\blacksquare$ denote the pullback $\pi^*\colon \mathcal{D}(\mathbb{Z}_\blacksquare) \to \mathcal{D}(\mathbb{T})$. By [CS24, Lecture 7], the process of adjoining a variable and then solidifying it preserves limits, and so

$$\mathbb{Z}[\beta, t]^\wedge_{(p,t)} \otimes^{\mathrm{L}}_{\mathbb{Z}_\blacksquare} \mathbb{Z}[u^{\pm 1}]_\blacksquare \simeq \mathbb{Z}[\beta, t, u^{\pm 1}]^\wedge_{(p,t)}\,.$$

Thus, if we put $\overline{X}^*_\blacksquare \coloneqq \operatorname{AnSpec}(\mathbb{Z}[\beta,t]^\wedge_{(p,t)}, \mathbb{Z}[\beta,t]^\wedge_{(p,t)})_\blacksquare$, we do get an action $\overline{X}^*_\blacksquare \times \mathbb{T} \to \overline{X}^*_\blacksquare$ simply by (p,t)-completing the map Δ above. Here and in the following, all products are taken in the ∞-category $\mathrm{AnStk}_{\mathbb{Z}_\blacksquare}$ of analytic stacks over $\mathbb{Z}_\blacksquare$. We let $\mathbb{T}^\bullet\colon \Delta^{\mathrm{op}} \to \mathrm{AnStk}_{\mathbb{Z}_\blacksquare}$ denote the simplicial analytic stack corresponding to the underlying $\mathbb{E}_1$-structure of the $\mathbb{E}_\infty$-group object $\mathbb{T}$, and we let $\overline{X}^*_\blacksquare \times \mathbb{T}^\bullet\colon \Delta^{\mathrm{op}} \to \mathrm{AnStk}_{\mathbb{Z}_\blacksquare}$ denote the simplicial analytic stack corresponding to the $\mathbb{T}$-action on $\overline{X}^*_\blacksquare$. Finally, let

$$\mathrm{B}\mathbb{T} \coloneqq \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \mathbb{T}^n \quad\text{and}\quad \overline{X}^*_\blacksquare/\mathbb{T} \coloneqq \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} \overline{X}^*_\blacksquare \times \mathbb{T}^n\,.$$

12.16. Lemma. — *Let $\mathcal{O}_{\overline{X}^*/\mathbb{T}} \in \mathcal{D}(\mathrm{B}\mathbb{T})$ denote the pushforward of the structure sheaf of $\overline{X}^*_\blacksquare/\mathbb{T}$. Then pushforward along $\overline{X}^*_\blacksquare/\mathbb{T} \to \mathrm{B}\mathbb{T}$ induces a symmetric monoidal equivalence of ∞-categories*

$$\mathcal{D}(\overline{X}^*_\blacksquare/\mathbb{T}) \simeq \mathrm{Mod}_{\mathcal{O}_{\overline{X}^*/\mathbb{T}}}\big(\mathcal{D}(\mathrm{B}\mathbb{T})\big)\,.$$

Proof. The same argument as in 12.15 shows $\overline{X}^*_\blacksquare \times \mathbb{T}^n \simeq \operatorname{AnSpec}(\mathbb{Z}[\beta, t, u_1^{\pm 1}, \dots, u_n^{\pm 1}]^\wedge_{(p,t)})_\blacksquare$. By definition, $\mathcal{D}(\mathrm{B}\mathbb{T}) \simeq \lim_{[n]\in\Delta}\mathcal{D}(\mathbb{T}^n)$ and $\mathcal{D}(\overline{X}^*_\blacksquare/\mathbb{T}) \simeq \lim_{[n]\in\Delta}\mathcal{D}(\overline{X}^*_\blacksquare \times \mathbb{T}^n)$, where the cosimplicial limits are taken along the pullback functors. Observe that the pushforward functors $\pi_*\colon \mathcal{D}(\overline{X}^*_\blacksquare \times \mathbb{T}^n) \to \mathcal{D}(\mathbb{T}^n)$ commute with these pullbacks. Indeed, if we would take the limit along the !-pullbacks, this would follow from proper base change (by passing to right adjoints). Since $\mathbb{Z} \to \mathbb{Z}[u^{\pm 1}]$ is smooth of relative dimension 1 and $\Omega^1_{\mathbb{Z}[u^{\pm 1}]/\mathbb{Z}} \cong \mathbb{Z}[u^{\pm 1}]\,\mathrm{d}u$ is a free module of rank 1, we get $\pi^! \simeq \Sigma^{-1}\pi^*$ by [CS19, Theorem 11.6], and so commutativity for the $*$-pullbacks follows.

Therefore $\mathcal{O}_{\overline{X}^*/\mathbb{T}} \in \mathcal{D}(\mathrm{B}\mathbb{T})$ is given by the degree-wise pushforwards of the structure sheaves $\mathcal{O}_{\overline{X}^*_\blacksquare \times \mathbb{T}^n}$, that is, by $\mathbb{Z}[\beta, t, u_1^{\pm 1}, \dots, u_n^{\pm 1}]^\wedge_{(p,t)} \in \mathcal{D}(\mathbb{T}^n)$ for all $[n] \in \Delta$. In every degree, the pushforward induces an equivalence

$$\mathcal{D}(\overline{X}^*_\blacksquare \times \mathbb{T}^n) \xrightarrow{\ \simeq\ } \mathrm{Mod}_{\mathbb{Z}[\beta,t,u_1^{\pm 1},\dots,u_n^{\pm 1}]^\wedge_{(p,t)}}\big(\mathcal{D}(\mathbb{T}^n)\big)\,.$$

Using this observation, $\mathcal{D}(\overline{X}^*_\blacksquare/\mathbb{T}) \simeq \mathrm{Mod}_{\mathcal{O}_{\overline{X}^*/\mathbb{T}}}(\mathcal{D}(\mathrm{B}\mathbb{T}))$ is completely formal. □

12.17. Graded objects and sheaves on $\mathrm{B}\mathbb{T}$. — Let $\mathbb{G}_{m,\mathbb{Z}_\blacksquare} \coloneqq \mathbb{G}_m \times \operatorname{AnSpec}\mathbb{Z}_\blacksquare$. By adapting the usual proof, it's straightforward to show that

$$\mathcal{D}(\mathrm{B}\mathbb{G}_{m,\mathbb{Z}_\blacksquare}) \simeq \operatorname{Gr}\mathcal{D}(\mathbb{Z}_\blacksquare)$$

is the ∞-category of graded solid condensed abelian groups. Since we have a map of analytic stacks $c\colon \mathrm{B}\mathbb{T} \to \mathrm{B}\mathbb{G}_{m,\mathbb{Z}_\blacksquare}$, we get a pullback functor $c^*\colon \operatorname{Gr}\mathcal{D}(\mathbb{Z}_\blacksquare) \to \mathcal{D}(\mathrm{B}\mathbb{T})$. In this way, we can associate to any graded solid condensed $\mathbb{Z}$-module a quasi-coherent sheaf on $\mathrm{B}\mathbb{T}$.

We don't know if c^* is fully faithful (it probably isn't), but at least it's fully faithful when restricted to the full sub-∞-category $\operatorname{Gr}\mathcal{D}(\mathbb{Z}) \subseteq \operatorname{Gr}\mathcal{D}(\mathbb{Z}_\blacksquare)$ spanned by the discrete graded $\mathbb{Z}$-modules. Indeed, for discrete objects, solidification doesn't do anything, and so for all $[n] \in \Delta$ the functor $\mathcal{D}(\mathbb{G}^n_{m,\mathbb{Z}_\blacksquare}) \to \mathcal{D}(\mathbb{T}^n)$, given by solidifying $u_i^{\pm 1}$ for $i = 1, \dots, n$, is fully faithful when restricted to discrete objects.

The following lemma takes this one step further and allows us to regard the graded $\mathbb{Z}_p[\beta][\![t]\!]$-modules $\operatorname{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ as sheaves on $\mathcal{D}(\mathrm{B}\mathbb{T})$ without loss of information.

12.18. Lemma. — *Let $\mathbb{Z}_p[\beta][\![t]\!] \in \operatorname{Gr}\mathcal{D}(\mathbb{Z})$ denote the graded (p,t)-completion of the discrete graded ring $\mathbb{Z}[\beta,t]$ and equip $\operatorname{Mod}_{\mathbb{Z}_p[\beta][\![t]\!]}(\operatorname{Gr}\mathcal{D}(\mathbb{Z}))^\wedge_{(p,t)}$ with the (p,t)-completed graded tensor product. Then c^* induces a fully faithful lax symmetric monoidal functor*

$$\operatorname{Mod}_{\mathbb{Z}_p[\beta][\![t]\!]}\big(\operatorname{Gr}\mathcal{D}(\mathbb{Z})\big)^\wedge_{(p,t)} \longrightarrow \operatorname{Mod}_{\mathcal{O}_{\overline{X}^*/\mathbb{T}}}\big(\mathcal{D}(\mathrm{B}\mathbb{T})\big)\,,$$

which is symmetric monoidal when restricted to the full sub-∞-category spanned by those objects in $\operatorname{Mod}_{\mathbb{Z}_p[\beta][\![t]\!]}(\operatorname{Gr}\mathcal{D}(\mathbb{Z}))^\wedge_{(p,t)}$ that are uniformly bounded below in every graded degree.[12.3]

Proof. To construct the desired functor, we compose c^* with (p,t)-completion to obtain

$$\operatorname{Mod}_{\mathbb{Z}_p[\beta][\![t]\!]}\big(\operatorname{Gr}\mathcal{D}(\mathbb{Z}_\blacksquare)\big) \xrightarrow{c^*} \operatorname{Mod}_{c^*(\mathbb{Z}_p[\beta][\![t]\!])}\big(\mathcal{D}(\mathrm{B}\mathbb{T})\big) \xrightarrow{(-)^\wedge_{(p,t)}} \operatorname{Mod}_{\mathcal{O}_{\overline{X}^*/\mathbb{T}}}\big(\mathcal{D}(\mathrm{B}\mathbb{T})\big)\,.$$

The functor c^* is symmetric monoidal and $(-)^\wedge_{(p,t)}$ is lax symmetric monoidal. Hence the composition is lax symmetric monoidal. Moreover, it is symmetric monoidal when restricted to graded $\mathbb{Z}_p[\beta][\![t]\!]$-modules that are uniformly bounded below in every graded degree. Indeed, the image of such objects in $\operatorname{Mod}_{\mathcal{O}_{\overline{X}^*/\mathbb{T}}}(\mathcal{D}(\mathrm{B}\mathbb{T})) \simeq \lim_{[n]\in\Delta}\mathcal{D}(\overline{X}^*_\blacksquare \times \mathbb{T}^n)$ will be bounded below and (p,t)-complete in every cosimplicial degree, because the pullback functors along which the limit is taken preserve bounded below and (p,t)-complete objects (the latter because they preserve limits; see the argument in 12.15). So we can reduce to the fact that the solid tensor product in $\mathcal{D}(\overline{X}^*_\blacksquare \times \mathbb{T}^n)$ preserves bounded below (p,t)-complete objects.

Clearly $(-)^\wedge_{(p,t)} \circ c^*$ factors through $\operatorname{Mod}_{\mathbb{Z}_p[\beta][\![t]\!]}(\operatorname{Gr}\mathcal{D}(\mathbb{Z}_\blacksquare))^\wedge_{(p,t)}$. By restricting to the full sub-∞-category $\operatorname{Mod}_{\mathbb{Z}_p[\beta][\![t]\!]}(\operatorname{Gr}\mathcal{D}(\mathbb{Z}))^\wedge_{(p,t)}$, we get the desired functor

$$\operatorname{Mod}_{\mathbb{Z}_p[\beta][\![t]\!]}\big(\operatorname{Gr}\mathcal{D}(\mathbb{Z})\big)^\wedge_{(p,t)} \longrightarrow \operatorname{Mod}_{\mathcal{O}_{\overline{X}^*/\mathbb{T}}}\big(\mathcal{D}(\mathrm{B}\mathbb{T})\big)\,.$$

We've already seen that this functor is symmetric monoidal on uniformly bounded below objects. Fully faithfulness can be checked modulo (p,t), so it'll be enough to check that $\operatorname{Mod}_{\mathbb{F}_p[\beta]}(\operatorname{Gr}\mathcal{D}(\mathbb{Z})) \to \operatorname{Mod}_{c^*(\mathbb{F}_p[\beta])}(\mathcal{D}(\mathrm{B}\mathbb{T}))$ is fully faithful. This follows from the fact that $c^*\colon \operatorname{Gr}\mathcal{D}(\mathbb{Z}) \to \mathcal{D}(\mathrm{B}\mathbb{T})$ is fully faithful, as we've seen in 12.17. ☐

12.19. Lemma. — *Let $X^* \subseteq \overline{X}^*$ be the subset $\operatorname{Spa}\mathbb{Z}[\beta,t]^\wedge_{(p,t)} \smallsetminus \{p=0, \beta t=0\}$. Then X^* is a Tate adic space and its associated analytic stack $X^*_\blacksquare$ can be written as the following pushout:*

$$\begin{array}{ccc}
\operatorname{AnSpec}\Big(\mathbb{Z}[\beta,t]^\wedge_{(p,t)}\big[\tfrac{1}{p\beta t}\big], \mathbb{Z}[\beta,t]^\wedge_{(p,t)}\Big)_\blacksquare & \longrightarrow & \operatorname{AnSpec}\Big(\mathbb{Z}[\beta,t]^\wedge_{(p,t)}\big[\tfrac{1}{\beta t}\big], \mathbb{Z}[\beta,t]^\wedge_{(p,t)}\Big)_\blacksquare \\
\big\downarrow & \ulcorner & \big\downarrow \\
\operatorname{AnSpec}\Big(\mathbb{Z}[\beta,t]^\wedge_{(p,t)}\big[\tfrac{1}{p}\big], \mathbb{Z}[\beta,t]^\wedge_{(p,t)}\Big)_\blacksquare & \longrightarrow & X^*_\blacksquare
\end{array}$$

Moreover, the $\mathbb{T}$-action on $\overline{X}^*_\blacksquare$ restricts to an action on $X^*_\blacksquare$, and if $\mathcal{O}_{X^*/\mathbb{T}} \in \mathcal{D}(\mathrm{B}\mathbb{T})$ denotes the pushforward of the structure sheaf of $X^*_\blacksquare/\mathbb{T}$, then pushforward along $X^*_\blacksquare/\mathbb{T} \to \mathrm{B}\mathbb{T}$ induces a symmetric monoidal equivalence

$$\mathcal{D}(X^*_\blacksquare/\mathbb{T}) \simeq \operatorname{Mod}_{\mathcal{O}_{X^*/\mathbb{T}}}\big(\mathcal{D}(\mathrm{B}\mathbb{T})\big)\,.$$

[12.3] By contrast, the graded solid tensor product on $\operatorname{Gr}\mathcal{D}(\mathbb{Z}_\blacksquare)$ does *not* preserve p-complete objects, not even if they're uniformly bounded below, because being p-complete is not preserved under infinite direct sums.

Proof. By 12.2, $X^*_\blacksquare$ is glued together from rational open subsets of $\overline{X}^*$. For example, one can take $U_1 = \{x \in \overline{X}^* \mid |\beta t|_x \leqslant |p|_x \neq 0\}$ and $U_2 = \{x \in \overline{X}^* \mid |p|_x \leqslant |\beta t|_x \neq 0\}$ and then

$$X^*_\blacksquare \simeq U_{1,\blacksquare} \sqcup_{(U_1 \cap U_2)_\blacksquare} U_{2,\blacksquare}\,.$$

To show the desired pushout, it's enough that $Y_{1,\blacksquare} \coloneqq \operatorname{AnSpec}(\mathbb{Z}[\beta,t]^\wedge_{(p,t)}[1/p], \mathbb{Z}[\beta,t]^\wedge_{(p,t)})_\blacksquare$ and $Y_{2,\blacksquare} \coloneqq \operatorname{AnSpec}(\mathbb{Z}[\beta,t]^\wedge_{(p,t)}[1/(\beta t)], \mathbb{Z}[\beta,t]^\wedge_{(p,t)})_\blacksquare$ form a !-cover after pullback to $U_{1,\blacksquare}$ and $U_{2,\blacksquare}$. This is clear, as $Y_{1,\blacksquare} \times_{\overline{X}^*_\blacksquare} U_{1,\blacksquare} \simeq U_{1,\blacksquare}$ and similarly $Y_{2,\blacksquare} \times_{\overline{X}^*_\blacksquare} U_{2,\blacksquare} \simeq U_{2,\blacksquare}$.

To see that the $\mathbb{T}$-action on $\overline{X}^*_\blacksquare$ restricts to an action on $X^*_\blacksquare$, just observe that p and βt are homogeneous elements. The pushout above implies that the pushforward $\mathcal{O}_{X^*} \in \mathcal{D}(\mathbb{Z}_\blacksquare)$ of the structure sheaf of $X^*_\blacksquare$ is given by

$$\mathcal{O}_{X^*} \simeq \mathbb{Z}[\beta,t]^\wedge_{(p,t)}\left[\tfrac{1}{p}\right] \times_{\mathbb{Z}[\beta,t]^\wedge_{(p,t)}\left[\frac{1}{p\beta t}\right]} \mathbb{Z}[\beta,t]^\wedge_{(p,t)}\left[\tfrac{1}{\beta t}\right],$$

the pullback being taken in the derived sense. Now $\mathcal{D}(X^*_\blacksquare \times \mathbb{T}^n) \simeq \operatorname{Mod}_{\mathcal{O}_{X^* \times \mathbb{T}^n}}(\mathcal{D}(\mathbb{T}^n))$ holds for all $[n] \in \Delta$, since the same is true for $Y_{1,\blacksquare}$, $Y_{2,\blacksquare}$, and $Y_{1,\blacksquare} \times_{X^*_\blacksquare} Y_{2,\blacksquare}$. This finally implies $\mathcal{D}(X^*_\blacksquare/\mathbb{T}) \simeq \operatorname{Mod}_{\mathcal{O}_{X^*/\mathbb{T}}}(\mathcal{D}(\mathrm{B}\mathbb{T}))$, as desired. □

We can finally show idempotence and nuclearity in the situation of Theorem 1.41.

12.20. Corollary. — *Let $Z^* \subseteq X^*$ be union of the closed subsets $\{p = 0\}$ and $\{[p^n]_{\mathrm{ku}}(t) = 0\}$ for all $n \geqslant 0$, where $[p^n]_{\mathrm{ku}}(t) \coloneqq ((1+\beta t)^{p^n} - 1)/\beta$ denotes the p^n-series of the formal group law of* ku.

(*a*) *Z^* is closed and closed under generalisations. Moreover, the $\mathbb{T}$-action on $X^*_\blacksquare$ restricts to an action on the overconvergent neighbourhood $Z^{*,\dagger}$ of Z^*.*

(*b*) *For $n, r, s \geqslant 1$ such that $(p-1)p^n > s$, let $W^*_{n,r,s} \subseteq X^*$ be the rational open subset determined by $|p^r| \leqslant |[p^n]_{\mathrm{ku}}(t)| \neq 0$, $|(\beta t)^s| \leqslant |p| \neq 0$. Then $\mathcal{O}_{Z^{*,\dagger}/\mathbb{T}} \in \mathcal{D}(X^*_\blacksquare/\mathbb{T})$ is idempotent, nuclear, and the colimit of the ind-algebra obtained by killing the idempotent pro-algebra "$\lim$"$_{n,r,s}\, \mathcal{O}_{W^*_{n,r,s}/\mathbb{T}}$.*

Proof. The proof of Corollary 12.14 can be carried over to show that $Z^* \cap \overline{W}^*_{n,r,s} = \emptyset$ and $X^* \smallsetminus Z^* = \bigcup_{n,r,s} W^*_{n,r,s}$. Hence Z^* is closed and closed under generalisations. Moreover, the $\mathbb{T}$-equivariant open subsets $X^* \smallsetminus \overline{W}^*_{n,r,s}$ are coinitial among all open neighbourhoods of Z^*, because for an arbitrary $U \supseteq Z^*$, the complement $X^* \smallsetminus U$ is quasi-compact and thus contained in some $W^*_{n,r,s}$. Since the $W^*_{n,r,s}$ are $\mathbb{T}$-equivariant, as they're defined by homogeneous elements, we see that $Z^{*,\dagger}$ acquires a $\mathbb{T}$-action. This finishes the proof of (*a*).

For part (*b*), Theorem 12.12 shows that $\mathcal{O}_{Z^{*,\dagger}}$ is the colimit of the idempotent nuclear ind-algebra obtained by killing "$\lim$"$_{n,r,s}\, \mathcal{O}_{W^*_{n,r,s}}$. Since $Z^{*,\dagger} \times \mathbb{T}^n \simeq \lim_{U^* \supseteq Z^*}(U^*_\blacksquare \times \mathbb{T})$, where the limit is taken over all $\mathbb{T}$-equivariant open neighbourhoods, and since killing pro-idempotents is compatible with base change in the nuclear case by Lemma 10.9(*c*), we get that $\mathcal{O}_{Z^{*,\dagger} \times \mathbb{T}^n}$ is similarly given by killing "$\lim$"$_{n,r,s}\, \mathcal{O}_{W^*_{n,r,s} \times \mathbb{T}^n}$ in $\mathcal{D}(X^*_\blacksquare \times \mathbb{T}^n)$. Now let $A \in \mathcal{D}(X^*_\blacksquare/\mathbb{T})$ be the colimit of the ind-algebra given by killing "$\lim$"$_{n,r,s}\, \mathcal{O}_{W^*_{n,r,s}/\mathbb{T}}$. Then Lemma 12.9 shows that all sufficiently large transition maps in this pro-object are trace-class again. Hence A is idempotent, nuclear, and the base change result from Lemma 10.9(*c*) shows that the pullbacks of A to $X^*_\blacksquare \times \mathbb{T}^n$ agree with $\mathcal{O}_{Z^{*,\dagger} \times \mathbb{T}^n}$ for all $[n] \in \Delta$. This implies $\mathcal{O}_{Z^{*,\dagger}/\mathbb{T}} \simeq A$, as both of the maps

$$\mathcal{O}_{Z^{*,\dagger}/\mathbb{T}} \longrightarrow \mathcal{O}_{Z^{*,\dagger}/\mathbb{T}} \otimes^{\mathrm{L}}_{\mathcal{O}_{X^*_\blacksquare/\mathbb{T}}} A \longleftarrow A$$

become equivalences after pullback to $X^*_\blacksquare \times \mathbb{T}^n$ for all $[n] \in \Delta$. □

§12.3. Proof of Theorems 1.40 and 1.41

In this final subsection, we'll give a completely explicit description of the homotopy groups of

$$\mathrm{TC}^{-,\mathrm{ref}}\big(\mathrm{ku}_p^\wedge\otimes\mathbb{Q}/\mathrm{ku}_p^\wedge\big)\quad\text{and}\quad\mathrm{TC}^{-,\mathrm{ref}}\big(\mathrm{KU}_p^\wedge\otimes\mathbb{Q}/\mathrm{KU}_p^\wedge\big)\,.$$

By Example 10.27 and Lemma 11.2, these objects are obtained from $(\mathrm{ku}_p^\wedge)^{hS^1}$ and $(\mathrm{KU}_p^\wedge)^{hS^1}$, respectively, by killing the idempotent pro-algebras[(12.4)]

$$\text{“}\lim_{\alpha\geqslant 2}\text{”}\,\mathrm{TC}^-\big((\mathrm{ku}/p^\alpha)/\mathrm{ku}\big)\quad\text{and}\quad\text{“}\lim_{\alpha\geqslant 2}\text{”}\,\mathrm{TC}^-\big((\mathrm{KU}/p^\alpha)/\mathrm{KU}\big)\,.$$

The arguments from §11.1, particularly Corollaries 11.13, 11.14, and the proof of Theorem 11.15, show that $\mathrm{TC}^{-,\mathrm{ref}}$ is concentrated in even degrees in both cases, and the even homotopy groups are given by

$$\pi_{2*}\,\mathrm{TC}^{-,\mathrm{ref}}\big(\mathrm{ku}_p^\wedge\otimes\mathbb{Q}/\mathrm{ku}_p^\wedge\big)\cong \mathrm{A}_{\mathrm{ku},p}^*\,,\quad \pi_{2*}\,\mathrm{TC}^{-,\mathrm{ref}}\big(\mathrm{KU}_p^\wedge\otimes\mathbb{Q}/\mathrm{KU}_p^\wedge\big)\cong \mathrm{A}_{\mathrm{KU},p}[\beta^{\pm1}]\,,$$

where $\mathrm{A}_{\mathrm{ku},p}^*$ is obtained by killing the idempotent pro-algebra “$\lim_{\alpha\geqslant 2}$” $\mathrm{fil}^\star_{q\text{-Hdg}}\,q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ in graded (p,t)-complete $\mathbb{Z}_p[\beta]\llbracket t\rrbracket$-modules and $\mathrm{A}_{\mathrm{KU},p}$ is obtained by killing the idempotent pro-algebra “$\lim_{\alpha\geqslant 2}$” $q\text{-Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ in $(p,q-1)$-complete $\mathbb{Z}_p\llbracket q-1\rrbracket$-modules. Moreover, we already know that $\mathrm{A}_{\mathrm{ku},p}^*$ and $\mathrm{A}_{\mathrm{KU},p}$ are idempotent nuclear ind-objects.

Our goal is to identify $\mathrm{A}_{\mathrm{ku},p}^*$ and $\mathrm{A}_{\mathrm{KU},p}$ with the structure sheaves of the analytic stacks $Z^{*,\dagger}/\mathbb{T}$ and $Z^\dagger$, respectively (see Corollaries 12.14 and 12.20). To this end, let us first discuss how to transport $\mathrm{A}_{\mathrm{ku},p}^*$ and $\mathrm{A}_{\mathrm{KU},p}$ into the solid condensed world.

12.21. Nuclear modules à la Efimov and à la Clausen–Scholze. — Let R be a ring and $I\subseteq R$ a finitely generated homogeneous ideal. Efimov defines an ∞-category of *nuclear* $\widehat{R}_I$*-modules*, which (along many equivalent characterisations) can be described as

$$\mathrm{Nuc}(\widehat{R}_I)\simeq \mathrm{Nuc}\,\mathrm{Ind}\big(\widehat{\mathcal{D}}_I(R)\big);$$

see [Efi25, Corollary 4.4] (also recall that $\mathrm{Nuc}\,\mathrm{Ind}(-)$ is set-theoretically ok thanks to Remark 5.14). Let $\widehat{R}_{I,\blacksquare}\coloneqq(\widehat{R}_I,\widehat{R}_I)_\blacksquare$ be the analytic ring associated to the Huber pair $(\widehat{R}_I,\widehat{R}_I)$ (see 12.1). Then we can also consider the ∞-category $\mathrm{Nuc}(\mathcal{D}(\widehat{R}_{I,\blacksquare}))$ of nuclear $\widehat{R}_{I,\blacksquare}$-modules.[(12.5)] Efimov [Efi25, Corollary 7.6] constructs a fully faithful strongly continuous symmetric monoidal functor

$$\mathrm{Nuc}\,\mathcal{D}(\widehat{R}_{I,\blacksquare})\longrightarrow \mathrm{Nuc}(\widehat{R}_I)\,,$$

which is an equivalence on bounded objects.

12.22. $\mathrm{A}_{\mathrm{KU},p}$ and $\mathrm{A}_{\mathrm{ku},p}^*$ as sheaves on analytic stacks. — Applying Efimov's result above for $R=\mathbb{Z}[q]$ and $I=(p,q-1)$, we see that the bounded object $\mathrm{A}_{\mathrm{KU},p}$ is in the essential image of $\mathrm{Nuc}(\mathcal{D}(\mathbb{Z}_p\llbracket q-1\rrbracket_\blacksquare))$. Its preimage can be explicitly described: As usual (compare 5.2), we can regard each $q\text{-Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ as a $(p,q-1)$-complete[(12.6)] solid condensed $\mathbb{Z}_p\llbracket q-1\rrbracket$-module

(12.4)In the case $p=2$, the pro-systems need to be indexed by α even and $\geqslant 4$, but we'll ignore this since it makes no difference

(12.5)In fact, for *any* Huber pair $(\widehat{R}_I,R^+)$ the nuclear objects $\mathrm{Nuc}(\mathcal{D}((\widehat{R}_I,R^+)_\blacksquare))$ will be independent of the choice of R^+. See [AM24, Example 3.34] for example.

(12.6)Observe that $q\text{-Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ is automatically p-complete, since it is $(q-1)$-complete and contains an element of the form $p^\alpha/(q-1)$ by construction.

by $(p, q-1)$-completing the associated discrete condensed abelian group. The pro-algebra "$\lim_{\alpha\geqslant 2}$" $q\text{-Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ is still idempotent in $\operatorname{Pro}\mathcal{D}(\mathbb{Z}_p[\![q-1]\!]_\blacksquare)$ and has eventually trace-class transition maps. Thus, by killing it, we get an idempotent nuclear algebra in $\operatorname{Ind}\mathcal{D}(\mathbb{Z}_p[\![q-1]\!]_\blacksquare)$. Its colimit is the preimage of $\mathrm{A}_{\mathrm{KU},p}$.

In a similar way, via Lemma 12.18, we can regard "$\lim_{\alpha\geqslant 2}$" $\operatorname{fil}^\star_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ as an idempotent pro-algebra in $\operatorname{Mod}_{\mathcal{O}_{\overline{X}^*_\blacksquare/\mathbb{T}}}(\mathcal{D}(\mathrm{B}\mathbb{T}))$. By killing it and taking the colimit of the result idempotent nuclear ind-algebra, we can regard $\mathrm{A}^*_{\mathrm{ku},p}$ as an object in $\operatorname{Nuc}\operatorname{Mod}_{\mathcal{O}_{\overline{X}^*_\blacksquare/\mathbb{T}}}(\mathcal{D}(\mathrm{B}\mathbb{T}))$

The following lemma shows that $\mathrm{A}^*_{\mathrm{ku},p}$ and $\mathrm{A}_{\mathrm{KU},p}$ are already sheaves on $X^*_\blacksquare/\mathbb{T}$ and $X_\blacksquare$, where we put $X^* \coloneqq \overline{X}^* \smallsetminus \{p=0, \beta t=0\}$ and $X \coloneqq \operatorname{Spa}\mathbb{Z}_p[\![q-1]\!] \smallsetminus \{p=0, q=1\}$ as before.

12.23. Lemma. — *$\mathrm{A}^*_{\mathrm{ku},p}$ vanishes after (p,β)-completion and after (p,t)-completion. $\mathrm{A}_{\mathrm{KU},p}$ vanishes after $(p,q-1)$-completion. In particular, $\mathrm{A}^*_{\mathrm{ku},p}$ and $\mathrm{A}_{\mathrm{KU},p}$ are already contained in the full sub-∞-categories $\mathcal{D}(X^*_\blacksquare/\mathbb{T}) \simeq \operatorname{Mod}_{\mathcal{O}_{X^*/\mathbb{T}}}(\mathcal{D}(\mathrm{B}\mathbb{T}))$ and $\mathcal{D}(X_\blacksquare) \simeq \operatorname{Mod}_{\mathcal{O}_X}(\mathcal{D}(\mathbb{Z}_\blacksquare))$.*

Proof. By Nakayama's lemma it's enough to show $\mathrm{A}^*_{\mathrm{ku},p}/(p,\beta) \simeq 0$ and $\mathrm{A}^*_{\mathrm{ku},p}/(p,t) \simeq 0$. Since $\mathrm{A}_{\mathrm{KU},p}[\beta^{\pm 1}]$ is a $\mathrm{A}^*_{\mathrm{ku},p}$-algebra, this will also show $\mathrm{A}_{\mathrm{KU},p}/(p,q-1) \simeq 0$. Since $\mathrm{A}^*_{\mathrm{ku},p}/t$ is concentrated in nonnegative graded degrees, it is automatically β-complete, so it's already enough to show $\mathrm{A}^*_{\mathrm{ku},p}/(p,\beta) \simeq 0$. Now $\mathrm{ku} \to \mathrm{ku}/(p,\beta) \simeq \mathbb{F}_p$ is a map of $\mathbb{E}_\infty$-ring spectra, and it's clear from Example 10.27 and Lemma 11.2 that $\mathrm{TC}^{-,\mathrm{ref}}(-\otimes\mathbb{Q}/-)$ satisfies base change along $\mathbb{E}_\infty$-maps. So $\mathrm{TC}^{-,\mathrm{ref}}(\mathrm{ku}\otimes\mathbb{Q}/\mathrm{ku})/(p,\beta) \simeq \mathrm{TC}^{-,\mathrm{ref}}(\mathbb{F}_p\otimes\mathbb{Q}/\mathbb{F}_p) \simeq 0$.

It follows that $(\mathrm{A}^*_{\mathrm{ku},p})^\wedge_{(p,\beta t)} \simeq 0$. Using the pullback square from Lemma 12.19, we get

$$\mathrm{A}^*_{\mathrm{ku},p} \simeq \mathrm{A}^*_{\mathrm{ku},p} \otimes^{\mathrm{L}}_{\mathcal{O}_{\overline{X}^\star_\blacksquare/\mathbb{T}}} \mathcal{O}_{X^*/\mathbb{T}}$$

and so $\mathrm{A}^*_{\mathrm{ku},p}$ is indeed a $\mathcal{O}_{X^*/\mathbb{T}}$-module. The argument for $\mathrm{A}_{\mathrm{KU},p}$ is analogous. □

To finish the proof, we analyse the pro-systems "$\lim_{n,r,s}$" $\mathcal{O}_{W_{n,r,s}}$ and "$\lim_{n,r,s}$" $\mathcal{O}_{W^*_{n,r,s}/\mathbb{T}}$ from Corollaries 12.14 and 12.20.

12.24. Lemma. — *For every fixed $\alpha \geqslant 2$ and all sufficiently large n, r, s, there exist maps*

$$\mathcal{O}_{W^*_{n,r,s}/\mathbb{T}} \longrightarrow \operatorname{fil}^\star_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} \otimes^{\mathrm{L}}_{\mathcal{O}_{\overline{X}^*_\blacksquare/\mathbb{T}}} \mathcal{O}_{X^*/\mathbb{T}}\,,$$
$$\mathcal{O}_{W_{n,r,s}} \longrightarrow q\text{-Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} \otimes^{\mathrm{L}}_{\mathbb{Z}_p[\![q-1]\!]_\blacksquare} \mathcal{O}_X$$

in $\mathcal{D}(X^*_\blacksquare/\mathbb{T})$ and $\mathcal{D}(X_\blacksquare)$, respectively.

Proof. By construction, the q-de Rham complex $q\text{-dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ contains elements of the form $\phi^i(\phi(p^\alpha)/\Phi_p(q)) = p^\alpha/\Phi_{p^{i+1}}(q)$ for all $i \geqslant 0$, and we have $p^\alpha \in \operatorname{fil}^1_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$. When we regard $\operatorname{fil}^\star_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ as a graded $\mathbb{Z}_p[\beta][\![t]\!]$-module, this precisely means that p^α is divisible by t. Hence we have elements of the form

$$\frac{p^{(n+1)\alpha}}{[p^n]_{\mathrm{ku}}(t)} = \frac{p^\alpha}{t}\cdot\frac{\phi(p^\alpha)}{\Phi_p(q)}\cdots\frac{\phi^n(p^\alpha)}{\Phi_{p^n}(q)} \in \operatorname{fil}^\star_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$$

for all $n \geqslant 0$. Similarly, there exist elements of the form $(\beta t)^N/p$ in $\operatorname{fil}^\star_{q\text{-Hdg}} q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ for sufficiently large N. Indeed, the ring $q\text{-dR}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ is $(p,\Phi_p(q))$-complete and contains an element of the form $p^\alpha/\Phi_p(q)$. Applying the nilpotence criterion from [BCM20, Proposition 2.5], we see that $\Phi_p(q)$ is nilpotent in $\mathrm{Fil}^*_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}/p$. Then $(q-1)^{p-1}$ must be nilpotent as well, and so $(q-1)^N$ must be divisible by p in $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ for $N \gg 0$.

In particular, as soon as we invert βt in $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}/p$, we see that p will be invertible as well, and so

$$\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} \otimes^{\mathrm{L}}_{\mathcal{O}_{\overline{X}^*_\blacksquare/\mathbb{T}}} \mathcal{O}_{X^*/\mathbb{T}} \simeq \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}\left[\tfrac{1}{p}\right].$$

Moreover, as soon as p is invertible, $[p^n]_{\mathrm{ku}}(t)$ will be invertible for all $n \geqslant 0$. Choosing $s > N$, we see that $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ contains an element of the form $(\beta t)^s/p$ which is topologically nilpotent, hence automatically solid. Moreover, for $(p-1)p^n > s$ and $r > (n+1)\alpha$, we get an element of the form $p^r/[p^n]_{\mathrm{ku}}(t)$, which is again topologically nilpotent and thus solid. Thus, for such n, r, and s, a map $\mathcal{O}_{W^*_{n,r,s}/\mathbb{T}} \to \mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}[1/p]$ exists. The argument in the q-Hodge case is analogous. □

12.25. Remark. — As a consequence of Theorem 3.11(*b*), $q\text{-Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}/(q^{p^n}-1)$ is an algebra over the p-typical Witt vectors $\mathbb{W}_{p^n}(\mathbb{Z}/p^\alpha)$. Since this ring is $p^{\alpha+n}$-torsion, we already have elements of the form $p^{\alpha+n}/(q^{p^n}-1)$ in $q\text{-Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}$ for all $n \geqslant 0$.

12.26. Lemma. — *For all fixed n, r, s such that $(p-1)p^n > s$ and all sufficiently large $\alpha \geqslant 2$, there exist canonical maps*

$$\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} \otimes^{\mathrm{L}}_{\mathcal{O}_{\overline{X}^*_\blacksquare/\mathbb{T}}} \mathcal{O}_{X^*/\mathbb{T}} \longrightarrow \mathcal{O}_{W^*_{n,r,s}/\mathbb{T}},$$
$$q\text{-Hdg}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p} \otimes^{\mathrm{L}}_{\mathbb{Z}_p[\![q-1]\!]_\blacksquare} \mathcal{O}_X \longrightarrow \mathcal{O}_{W_{n,r,s}}$$

in $\mathcal{D}(X^*_\blacksquare/\mathbb{T})$ and $\mathcal{D}(X_\blacksquare)$, respectively.

Proof. Let $q\text{-}D_\alpha \coloneqq q\text{-dR}_{(\mathbb{Z}_p\{x\}/x^\alpha)/\mathbb{Z}_p\{x\}}$ as in §11.2 and let $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{D}_\alpha$ denote its completed q-Hodge filtration. It follows from 11.16 that $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{D}_\alpha$ is generated as a (p,t)-complete graded $\mathbb{Z}_p[\beta][\![t]\!]$-algebra by lifts of the iterated divided powers $\gamma^{(d)}(x^\alpha)$ sitting in filtration degree $2p^d$. Thanks to Lemma 11.17, we know that these lifts can be chosen to be of the form

$$\frac{(\Gamma_d)^\alpha}{t^{p^d}\prod_{i=1}^d \Phi_{p^i}(q)^{p^{d-i}}}$$

for $\Gamma_d \in (x^p, (q-1)^{p-1})^{p^{d-1}}$. The extra t^{p^d} in the denominator accomodates for the fact that this element must sit in degree $2p^d$. Note that the denominators all become invertible in $\mathcal{O}_{W^*_{n,r,s}/\mathbb{T}}$, but that's not enough to obtain the desired map: We must send the generators to *solid* elements, to ensure that the map extends over the (p,t)-completion.

By construction, $(q-1)^s/p$ and $p^r/[p^n]_{\mathrm{ku}}(t)$ are solid. In particular, $p^r/(t\Phi_{p^i}(q))$ is solid for all $i = 1, \dots, n$. For $i > n$, we have $(p-1)p^{i-1} > s$ by assumption. Hence $(q-1)^{(p-1)p^{i-1}}/p$ is topologically nilpotent in $\mathcal{O}_{W^*_{n,r,s}/\mathbb{T}}$. It follows that $\Phi_{p^i}(q) = p(1+w)$, where w is topologically nilpotent, and so $p^r/\Phi_{p^i}(q)$ is solid in $\mathcal{O}_{W^*_{n,r,s}/\mathbb{T}}$ for $i > n$. Therefore the elements $p^{2r}/(t\Phi_{p^i}(q))$ are solid for all $i \geqslant 1$.

By choosing α large enough, we can ensure that for every monomial $x^{pi}(q-1)^{(p-1)j}$ in the ideal $(x^p, (q-1)^{p-1})^{\alpha p^{d-1}}$ we have $pi \geqslant 2rp^d$ or $(p-1)j \geqslant sp^d$. Now $(\Gamma_d)^\alpha$ is a $\mathbb{Z}_p\{x\}[q]$-linear combination of such terms. It follows that the δ-ring map $\mathbb{Z}_p\{x\} \to \mathbb{Z}_p$ sending $x \mapsto p$ can really be extended to a map $\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{D}_\alpha \to \mathcal{O}_{W^*_{n,r,s}/\mathbb{T}}$ of graded solid condensed $\mathbb{Z}_p[\beta][\![t]\!]$-algebras.

Via (p,t)-completed base change along $\mathbb{Z}_p\{x\}\to\mathbb{Z}_p$ and extension of scalars to $\mathcal{O}_{X^*/\mathbb{T}}$, this yields the desired map

$$\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}\otimes^{\mathrm{L}}_{\mathcal{O}_{\overline{X}^*_\blacksquare/\mathbb{T}}}\mathcal{O}_{X^*/\mathbb{T}}\longrightarrow\mathcal{O}_{W^*_{n,r,s}/\mathbb{T}}$$

The argument in the q-Hodge case is analogous. □

Proof of Theorems 1.40 and 1.41. By Lemma 12.23 and Lemma 10.9(*c*), we see that $\mathrm{A}^*_{\mathrm{ku},p}$ is the colimit of the idempotent nuclear ind-algebra given by killing the pro-idempotent

$$\text{“}\lim_{\alpha\geqslant 2}\text{”}\,\mathrm{fil}^\star_{q\text{-Hdg}}\, q\text{-}\widehat{\mathrm{dR}}_{(\mathbb{Z}/p^\alpha)/\mathbb{Z}_p}\otimes^{\mathrm{L}}_{\mathcal{O}_{\overline{X}^*_\blacksquare/\mathbb{T}}}\mathcal{O}_{X^*/\mathbb{T}}$$

in $\mathcal{D}(X^*_\blacksquare/\mathbb{T})$. By Lemmas 12.24 and 12.26, we see that this pro-system is equivalent to $\text{“}\lim\text{”}_{n,r,s}\,\mathcal{O}_{W^*_{n,r,s}/\mathbb{T}}$, which proves $\mathrm{A}^*_{\mathrm{ku},p}\simeq\mathcal{O}_{Z^{*,\dagger}/\mathbb{T}}$. The argument for $\mathrm{A}_{\mathrm{KU},p}\simeq\mathcal{O}_{Z^\dagger}$ is completely analogous. □

12.27. Remark. — An obvious adaptation of Theorem 11.15 shows that $\mathrm{A}_{\mathrm{KU},p}$ and $\mathrm{A}^*_{\mathrm{ku},p}$ are connective. Therefore the condition from Theorem 12.12(*b*) is satisfied and so $\mathcal{O}_{Z^\dagger}$ and $\mathcal{O}_{Z^{*,\dagger}/\mathbb{T}}$ are really the pushforwards of the respective structure sheaves.

Appendix A. The q-de Rham complex

Let p be a prime. In [BS19, §16], Bhatt and Scholze construct a functorial $(p, q-1)$-complete q-de Rham complex relative to any q-PD pair (D, I). This verifies Scholze's conjecture [Sch17, Conjecture 3.1] after p-completion, but leaves open the global case. There are (at least) two strategies to tackle the global case:

(*a*) One can glue the global q-de Rham complex from its p-completions and its rationalisation using an arithmetic fracture square.

(*b*) Following Kedlaya [Ked21, §29], one can construct the global q-de Rham complex as the cohomology of a global q-crystalline site.

Strategy (*a*) is what Bhatt and Scholze originally had in mind, but they never published the argument. It is essentially straightforward, but not entirely trivial. Since all our global constructions proceed similarly by gluing p-completions and rationalisations, it will be worthwhile to fill in the missing details of strategy (*a*). Our goal is to show the following theorem.

A.1. Theorem. — *Let A be a Λ-ring that is p-torsion free for all primes p. Then there exists a functor*

$$q\text{-}\Omega_{-/A}\colon \mathrm{Sm}_A \longrightarrow \mathrm{CAlg}\Big(\widehat{\mathcal{D}}_{(q-1)}\big(A\llbracket q-1\rrbracket\big)\Big)$$

from the ∞-category of smooth A-algebras into the ∞-category of $(q-1)$-complete $\mathbb{E}_\infty$-algebras over $A\llbracket q-1\rrbracket$, satisfying the following properties:

(*a*) *$q\text{-}\Omega_{-/A}/(q-1) \simeq \Omega_{-/A}$ agrees with the usual de Rham complex functor.*

(*b*) *For all primes p, the p-completion*

$$(q\text{-}\Omega_{-/A})_p^\wedge \simeq {\mathbbl{\Delta}}_{(-)^{(p)}[\zeta_p]/\widehat{A}_p\llbracket q-1\rrbracket}$$

agrees with prismatic cohomology relative to the q-de Rham prism $(\widehat{A}_p\llbracket q-1\rrbracket, [p]_q)$. Here we denote the p-adic Frobenius twist by $(-)^{(p)} \coloneqq (- \otimes_{A,\psi^p} A)_p^\wedge$.

(*c*) *$(q\text{-}\Omega_{-/A} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Q})^\wedge_{(q-1)} \simeq (\Omega_{-/A} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Q})\llbracket q-1\rrbracket$ becomes the trivial q-deformation.*

(*d*) *For every framed smooth A-algebra $(S, \square)$, the underlying object of $q\text{-}\Omega_{S/A}$ in the derived ∞-category of $A\llbracket q-1\rrbracket$ can be represented as*

$$q\text{-}\Omega_{S/A} \simeq q\text{-}\Omega^*_{S/A,\square}\,,$$

where $q\text{-}\Omega^*_{S/A,\square}$ denotes the coordinate-dependent q-de Rham complex as in 1.7.

Moreover, if $A \to A'$ is a map of Λ-rings such that A' is also p-torsion free for all primes p, there's a canonical base change equivalence

$$\big(q\text{-}\Omega_{-/A} \otimes^{\mathrm{L}}_A A'\big)^\wedge_{(q-1)} \xrightarrow{\simeq} q\text{-}\Omega_{(-\otimes_A A')/A'}\,.$$

Modulo $(q-1)$ this reduces to the usual base change equivalence of the de Rham complex.

A.2. Remark. — It will be apparent from our proof of Theorem A.1 (and we'll give a precise argument in A.13) that the q-de Rham complex functor lifts canonically to a functor

$$q\text{-}\Omega_{-/A}\colon \mathrm{Sm}_A \longrightarrow \big(\mathrm{DAlg}_{A\llbracket q-1\rrbracket}\big)^\wedge_{(q-1)}$$

into $(q-1)$-complete objects of the *the ∞-category of derived commutative $A\llbracket q-1\rrbracket$-algebras* $\mathrm{DAlg}_{A\llbracket q-1\rrbracket}$ as defined in [Rak21, Definition 4.2.22].

A.3. Convention. — Throughout §A, to increase readability and avoid excessive use of completions, all (q-)de Rham complexes or cotangent complexes relative to a p-complete ring will be implicitly p-completed.

§A.1. Rationalised q-crystalline cohomology

Fix a prime p. Then $(\widehat{A}_p[\![q-1]\!],(q-1))$ is a q-PD pair as in [BS19, Definition 16.1] and so we can use q-crystalline cohomology to construct a functorial $(p,q-1)$-complete q-de Rham complex $q\text{-}\Omega_{S/\widehat{A}_p}$ for every p-completely smooth $\widehat{A}_p$-algebra S. We let $q\text{-}\mathrm{dR}_{-/\widehat{A}_p}$ denote its non-abelian derived functor (or *animation*), which is now defined for all p-complete animated $\widehat{A}_p$-algebras. Observe that animation leaves the values on p-completely smooth $\widehat{A}_p$-algebras unchanged, as can be seen modulo $(p,q-1)$, where it reduces to a well-known fact about derived de Rham cohomology in characteristic p.

Our first goal is to show that after rationalisation derived q-de Rham cohomology is just a base change of derived de Rham cohomology relative to $\widehat{A}_p$. In coordinates, such an equivalence was already constructed in [Sch17, Lemma 4.1] (see A.8 for a review), but here we need a different argument: We want a coordinate-independent equivalence, so we have to work with the definition of the q-de Rham complex via q-crystalline cohomology.

A.4. Lemma. — *For all p-complete animated $\widehat{A}_p$-algebras R there is a functorial equivalence of $\mathbb{E}_\infty$-$(\widehat{A}_p\otimes_{\mathbb{Z}}\mathbb{Q})[\![q-1]\!]$-algebras*

$$\left(q\text{-}\mathrm{dR}_{R/\widehat{A}_p}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q}\right)^\wedge_{(q-1)}\simeq\left(\mathrm{dR}_{R/\widehat{A}_p}\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q}\right)[\![q-1]\!]\,.$$

Proof. By passing to non-abelian derived functors, it's enough to construct such a functorial equivalence for p-completely smooth $\widehat{A}_p$-algebras S. In this case, we can identify derived (q-)de Rham and (q-)crystalline cohomology:

$$q\text{-}\mathrm{dR}_{S/\widehat{A}_p}\simeq\mathrm{R}\Gamma_{q\text{-crys}}\left(S/\widehat{A}_p[\![q-1]\!]\right)\quad\text{and}\quad\mathrm{dR}_{S/\widehat{A}_p}\simeq\mathrm{R}\Gamma_{\mathrm{crys}}\left(S/\widehat{A}_p\right).$$

To construct the desired identification between q-crystalline and crystalline cohomology after rationalisation, let $P\twoheadrightarrow S$ be a surjection from a p-completely ind-smooth δ-$\widehat{A}_p$-algebra. Extend the δ-structure on P to $P[\![q-1]\!]$ via $\delta(q)\coloneqq 0$. Let J be the kernel of $P\twoheadrightarrow S$ and let $D\coloneqq D_P(J)$ be its p-completed PD-envelope. Finally, let q-D denote the corresponding q-PD-envelope as defined in [BS19, Lemma 16.10]. It will be enough to construct a functorial equivalence

$$\left(q\text{-}D\otimes_{\mathbb{Z}}\mathbb{Q}\right)^\wedge_{(q-1)}\simeq\left(D\otimes_{\mathbb{Z}}\mathbb{Q}\right)[\![q-1]\!]\,.$$

If D° denotes the un-p-completed PD-envelope of J, then $P\rightarrow q\text{-}D\rightarrow(q\text{-}D\otimes_{\mathbb{Z}}\mathbb{Q})^\wedge_{(q-1)}$ uniquely factors through $D^\circ\rightarrow(q\text{-}D\otimes_{\mathbb{Z}}\mathbb{Q})^\wedge_{(q-1)}$. The tricky part is to show that this map extends over the p-completion. Since D° is p-torsion free, its p-completion agrees with $D^\circ[\![t]\!]/(t-p)$. By Lemma A.6 below, for every fixed $n\geqslant 0$, every p-power series in D° converges in the p-adic topology on $(q\text{-}D\otimes_{\mathbb{Z}}\mathbb{Q})/(q-1)^n$, so we indeed get our desired extension $D\rightarrow(q\text{-}D\otimes_{\mathbb{Z}}\mathbb{Q})^\wedge_{(q-1)}$.

Extending further, we get a map $(D\otimes_{\mathbb{Z}}\mathbb{Q})[\![q-1]\!]\rightarrow(q\text{-}D\otimes_{\mathbb{Z}}\mathbb{Q})^\wedge_{(q-1)}$ of the desired form. Whether this is an equivalence can be checked modulo $(q-1)$ by the derived Nakayama lemma. Then the base change property from [BS19, Lemma 16.10(3)] finishes the proof—up to verifying convergence for p-power series in D°. □

To complete the proof of Lemma A.4, we need to prove two technical lemmas about (q-)divided powers. Let's fix the following notation: According to [BS19, Lemmas 2.15 and 2.17], we may uniquely extend the δ-structure from $q\text{-}D$ to $(q\text{-}D\otimes_{\mathbb{Z}}\mathbb{Q})^\wedge_{(q-1)}$. We still let ϕ and δ denote the extended Frobenius and δ-map. Furthermore, we denote by

$$\gamma(x)=\frac{x^p}{p}\quad\text{and}\quad\gamma_q(x)=\frac{\phi(x)}{[p]_q}-\delta(x)$$

the maps defining a PD-structure and a q-PD structure, respectively. Note that $\gamma(x)$ and $\gamma_q(x)$ make sense for all $x\in(q\text{-}D\otimes_{\mathbb{Z}}\mathbb{Q})^\wedge_{(q-1)}$ since p and $[p]_q$ are invertible.

A.5. Lemma. — *With notation as above, the following is true for the self-maps δ and γ_q of $(q\text{-}D\otimes_{\mathbb{Z}}\mathbb{Q})^\wedge_{(q-1)}$:*

(*a*) *For all $n\geqslant 1$ and all $\alpha\geqslant 1$, the map δ sends $(q-1)^n\,q\text{-}D$ into itself, and $p^{-\alpha}(q-1)^n\,q\text{-}D$ into $p^{-(p\alpha+1)}(q-1)^n\,q\text{-}D$.*

(*b*) *For all $n\geqslant 1$ and all $\alpha\geqslant 1$, the map γ_q sends $(q-1)^n\,q\text{-}D$ into $(q-1)^{n+1}\,q\text{-}D$, and $p^{-\alpha}(q-1)^n\,q\text{-}D$ into $p^{-(p\alpha+1)}(q-1)^{n+1}\,q\text{-}D$.*

Proof. Let's prove (*a*) first. Let $x=p^{-\alpha}(q-1)^ny$ for some $y\in q\text{-}D$. Since $q\text{-}D$ is flat over $\mathbb{Z}_p[\![q-1]\!]$ and thus is p-torsion free, we can compute

$$\delta(x)=\frac{\phi(x)-x^p}{p}=\frac{(q^p-1)^n\phi(y)}{p^{\alpha+1}}-\frac{(q-1)^{pn}y^p}{p^{p\alpha+1}}\,.$$

As q^p-1 is divisible by $q-1$, the right-hand side lies in $p^{-(p\alpha+1)}(q-1)^n\,q\text{-}D$. If $\alpha=0$, then the right-hand side must also be contained in $q\text{-}D$. But $q\text{-}D\cap p^{-1}(q-1)^n\,q\text{-}D=(q-1)^n\,q\text{-}D$ by flatness again. This proves both parts of (*a*). Now for (*b*), we first compute

$$\gamma_q(q-1)=\frac{\phi(q-1)}{[p]_q}-\delta(q-1)=-(q-1)^2\sum_{i=2}^{p-1}\frac{1}{p}\binom{p}{i}(q-1)^{i-2}\,.$$

Hence $\gamma_q(q-1)$ is divisible by $(q-1)^2$. In the following, we'll repeatedly use the relation $\gamma_q(xy)=\phi(y)\gamma_q(x)-x^p\delta(y)$ from [BS19, Remark 16.6] repeatedly. First off, it shows that

$$\gamma_q\big((q-1)^nx\big)=\phi\big((q-1)^{n-1}x\big)\gamma_q(q-1)-(q-1)^p\delta\big((q-1)^{n-1}x\big)\,.$$

It follows from (*a*) that $\delta((q-1)^{n-1}x)$ and $\phi((q-1)^{n-1}x)$ are divisible by $(q-1)^{n-1}$. Hence $\gamma_q((q-1)^nx)$ is indeed divisible by $(q-1)^{n+1}$. Moreover, we obtain

$$\gamma_q\big(p^{-\alpha}(q-1)^nx\big)=\phi(p^{-\alpha})\gamma_q\big((q-1)^nx\big)-(q-1)^{np}x^p\delta(p^{-\alpha})\,.$$

Now $\phi(p^{-\alpha})=p^{-\alpha}$ and $\delta(p^{-\alpha})$ is contained in $p^{-(p\alpha+1)}\,q\text{-}D$, hence $\gamma_q(p^{-\alpha}(q-1)^nx)$ is contained in $p^{-(p\alpha+1)}(q-1)^n\,q\text{-}D$. This finishes the proof of (*b*). □

A.6. Lemma. — *Let $x\in J$. For every $n\geqslant 1$, there are elements $y_0,\dots,y_n\in q\text{-}D$ such that y_0 admits q-divided powers in $q\text{-}D$ and*

$$\gamma^{(n)}(x)=y_0+\sum_{i=1}^{n}p^{-2(p^{i-1}+\dots+p+1)}(q-1)^{(p-2)+i}y_i$$

holds in $q\text{-}D\otimes_{\mathbb{Z}}\mathbb{Q}$, where $\gamma^{(n)}=\gamma\circ\dots\circ\gamma$ denotes the n-fold iteration of γ.

Proof. We use induction on n. For $n = 1$, we compute

$$\gamma(x) = \frac{x^p}{p} = \gamma_q(x) + \frac{[p]_q - p}{p}\big(\gamma_q(x) + \delta(x)\big)\,.$$

Note that x admits q-divided powers in q-D since we assume $x \in J$. Then $\gamma_q(x)$ admits q-divided powers again by [BS19, Lemma 16.7]. Moreover, writing $[p]_q = pu + (q-1)^{p-1}$, we find that $([p]_q - p)/p = (u-1) + p^{-1}(q-1)^{p-1}$. Then $(u-1)(\gamma_q(x) + \delta(x))$ admits q-divided powers since $u \equiv 1 \mod (q-1)$. This settles the case $n = 1$. We also remark that the above equation for $\gamma(x)$ remains true without the assumption $x \in J$ as long as the expression $\gamma_q(x)$ makes sense.

Now assume $\gamma^{(n)}$ can be written as above. We put $z_i = p^{-2(p^{i-1}+\dots+p+1)}(q-1)^{(p-2)+i}y_i$ for short, so that $\gamma^n(x) = y_0 + z_1 + \dots + z_n$. Recall the relations

$$\gamma_q(a+b) = \gamma_q(a) + \gamma_q(b) + \sum_{i=1}^{p-1}\frac{1}{p}\binom{p}{i}a^ib^{p-i}\,,\ \ \delta(a+b) = \delta(a) + \delta(b) - \sum_{i=1}^{p-1}\frac{1}{p}\binom{p}{i}a^ib^{p-i}\,.$$

The first relation implies that $\gamma_q(y_0 + z_1 + \dots + z_n)$ is equal to $\gamma_q(y_0) + \gamma_q(z_1) + \dots + \gamma_q(z_n)$ plus a linear combination of terms of the form $y_0^{\alpha_0}z_1^{\alpha_1}\cdots z_n^{\alpha_n}$ with $0 \leqslant \alpha_i < p$ and $\alpha_0 + \dots + \alpha_n = p$. Now $\gamma_q(y_0)$ admits q-divided powers again. Moreover, Lemma A.5(*b*) makes sure that each $\gamma_q(z_i)$ is contained in $p^{-2(p^i+\dots+p+1)}(q-1)^{(p-2)+i+1}\,q\text{-}D$. It remains to consider monomials $y_0^{\alpha_0}z_1^{\alpha_1}\cdots z_n^{\alpha_n}$. Put $m \coloneqq \max\{i \mid \alpha_i \neq 0\}$. If $\alpha_0 = p-1$, then all other α_i must vanish except $\alpha_m = 1$. In this case, the monomial is contained in $p^{-2(p^{m-1}+\dots+p+1)}(q-1)^{(p-2)+m}\,q\text{-}D$. If $\alpha_0 < p-1$, we get at least one more factor $(q-1)$ and the monomial $y_0^{\alpha_0}z_1^{\alpha_1}\cdots z_n^{\alpha_n}$ is contained in $p^{-2(p^m+\dots+p+1)}(q-1)^{(p-2)+m+1}\,q\text{-}D$.

A similar analysis, using the second of the above relations as well as Lemma A.5(*a*), shows that $(u-1)\delta(y_0 + z_1 + \dots + z_n)$ and $p^{-1}(q-1)^{p-1}\delta(y_0 + z_1 + \dots + z_n)$ can be decomposed into a bunch of terms, each of which is either a multiple of $(q-1)$ in q-D, so that it admits q-divided powers, or contained in $p^{-2(p^i+\dots+p+1)}(q-1)^{i+1}\,q\text{-}D$ for some $1 \leqslant i \leqslant n+1$. We conclude that

$$\gamma^{(n+1)}(x) = \gamma_q\big(\gamma^{(n)}(x)\big) + \frac{[p]_q - p}{p}\Big(\gamma_q\big(\gamma^{(n)}(x)\big) + \delta\big(\gamma^{(n)}(x)\big)\Big)$$

can be written in the desired form. ☐

The following remark is irrelevant for our proof of Theorem A.1, but it is occasionally useful for technical arguments.

A.7. Remark. — There's also an analogue of Lemma A.6 with the roles of D and q-D reversed. For every $x \in J$ and $n \geqslant 1$, there's an infinite sequence $y_0, y_1, \dots, \in D$ such that y_0 admits divided powers and

$$\gamma_q^{(n)}(x) = y_0 + \sum_{i\geqslant 1} p^{-2(p^{i-1}+\dots+1)}(q-1)^{(p-2)+i}y_i$$

holds in $(D \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$. The proof is very similar to Lemma A.6: We write

$$\gamma_q(x) = \left(\gamma(x) + \frac{[p]_q - p}{p}\delta(x)\right)\frac{p}{[p]_q}$$

and $[p]_q = pu + (q-1)^{p-1}$. Then we use induction on $n \geqslant 1$. For the inductive step, we first check that the operations $\gamma(-)$, $(u-1)\delta(-)$ and $p^{-1}(q-1)^{p-1}\delta(-)$ all preserve expressions of the desired form. Then we observe that u is a unit in $\mathbb{Z}_p[\![q-1]\!]$ and so multiplication by $p/[p]_q = u^{-1}\sum_{i\geqslant 0} p^{-i}u^{-i}(q-1)^{(p-1)i}$ also preserves expressions of the desired form.

A.8. The equivalence on q-de Rham complexes. — Suppose we're given a p-completely smooth $\widehat{A}_p$-algebra S together with a p-completely étale framing $\square\colon \widehat{A}_p\langle T_1,\dots,T_d\rangle \rightarrow S$. In this case, the q-crystalline cohomology can be computed as a q-de Rham complex

$$\mathrm{R}\Gamma_{q\text{-crys}}\big(S/\widehat{A}_p[\![q-1]\!]\big) \simeq q\text{-}\Omega^*_{S/\widehat{A}_p,\square}$$

by [BS19, Theorem 16.22]. Similarly, it's well-known that the crystalline cohomology is given by the ordinary de Rham complex $\Omega^*_{S/\widehat{A}_p}$ (recall that according to Convention A.3, all (q-)de Rham complexes of the p-complete ring S will implicitly be p-completed). In this case, an explicit isomorphism of complexes

$$\big(q\text{-}\Omega^*_{S/\widehat{A}_p,\square}\otimes_{\mathbb{Z}}\mathbb{Q}\big)^\wedge_{(q-1)} \xrightarrow{\cong} \big(\Omega^*_{S/\widehat{A}_p}\otimes_{\mathbb{Z}}\mathbb{Q}\big)[\![q-1]\!]$$

can be constructed as explained in [Sch17, Lemma 4.1]: One first observes that, after rationalisation, the partial q-derivatives $q\text{-}\partial_i$ can be computed in terms of the usual partial derivative ∂_i via the formula

$$q\text{-}\partial_i = \left(\frac{\log(q)}{q-1} + \sum_{n\geqslant 2}\frac{\log(q)^n}{n!(q-1)}(\partial_i T_i)^{(n-1)}\right)\partial_i\,;$$

see [BMS18, Lemma 12.4]. Here $\log(q)$ refers to the usual Taylor series for the logarithm around $q=1$. Noticing that the first factor is an invertible automorphism, one can then appeal to the following general fact: If M is an abelian group together with commuting endomorphisms $g_1,\dots,g_d$ and commuting automorphisms $h_1,\dots,h_d$ such that h_i commutes with g_j for $i\neq j$ one always has an isomorphism $\mathrm{Kos}^*(M,(g_1,\dots,g_d)) \cong \mathrm{Kos}^*(M,(h_1g_1,\dots,h_dg_d))$ of Koszul complexes.(A.1)

We would like to show that this explicit isomorphism is compatible with the one constructed in Lemma A.4. To this end, let's put ourselves in a slightly more general situation: Instead of a p-completely étale framing $\square$ as above, let's assume we're given a surjection $P \twoheadrightarrow S$ from a p-completely ind-smooth $\widehat{A}_p$-algebra P, which is in turn equipped with a p-completely ind-étale framing $\square\colon \widehat{A}_p\langle x_i \mid i\in I\rangle \rightarrow P$ for some (possible infinite) set I. Then $\widehat{A}_p\langle x_i \mid i\in I\rangle$ carries a δ-$\widehat{A}_p$-algebra structure characterised by $\delta(x_i)=0$ for all $i\in I$. By [BS19, Lemma 2.18], this extends uniquely to a δ-$\widehat{A}_p$-algebra structure on P. If J denotes the kernel of $P\twoheadrightarrow S$, we can form the usual PD-envelope $D \coloneqq D_P(J)^\wedge_p$ and the q-PD-envelope q-D as before. Furthermore, we let $\breve{\Omega}^*_{D/\widehat{A}_p}$ and $q\text{-}\breve{\Omega}^*_{q\text{-}D/\widehat{A}_p,\square}$ denote the usual PD-de Rham complex and the q-PD-de Rham complex from [BS19, Construction 16.20], respectively (both are implicitly p-completed).

A.9. Lemma. — *With notation as above, there is again an explicit isomorphism of complexes*

$$\big(q\text{-}\breve{\Omega}^*_{q\text{-}D/\widehat{A}_p,\square}\otimes_{\mathbb{Z}}\mathbb{Q}\big)^\wedge_{(q-1)} \xrightarrow{\cong} \big(\breve{\Omega}^*_{D/\widehat{A}_p}\otimes_{\mathbb{Z}}\mathbb{Q}\big)[\![q-1]\!]\,.$$

Proof. This follows from the same recipe as in A.8, provided we can show that the formula for $q\text{-}\partial_i$ in terms of ∂_i remains true under the identification $(q\text{-}D\otimes_{\mathbb{Z}}\mathbb{Q})^\wedge_{(q-1)} \cong (D\otimes_{\mathbb{Z}}\mathbb{Q})[\![q-1]\!]$

(A.1) We don't require h_i to commute with g_i (and it's not true in the case at hand).

from the proof of Lemma A.4. But for every fixed n, the images of the diagonal maps in the diagram

$$\begin{array}{ccc} & (P\otimes_{\mathbb{Z}}\mathbb{Q})[\![q-1]\!] & \\ \swarrow & & \searrow \\ (q\text{-}D\otimes_{\mathbb{Z}}\mathbb{Q})/(q-1)^n & \xrightarrow{\ \cong\ } & (D\otimes_{\mathbb{Z}}\mathbb{Q})[\![q-1]\!]/(q-1)^n \end{array}$$

are dense for the p-adic topology and for elements of $(P\otimes_{\mathbb{Z}}\mathbb{Q})[\![q-1]\!]$ the formula is clear. □

A.10. Lemma. — *With notation as above, the following diagram commutes:*

$$\begin{array}{ccc} \Big(\mathrm{R}\Gamma_{q\text{-crys}}\big(S/\widehat{A}_p[\![q-1]\!]\big)\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q}\Big)^\wedge_{(q-1)} & \xrightarrow[\text{(A.4)}]{\simeq} & \Big(\mathrm{R}\Gamma_{\mathrm{crys}}\big(S/\widehat{A}_p\big)\otimes^{\mathrm{L}}_{\mathbb{Z}}\mathbb{Q}\Big)[\![q-1]\!] \\ \simeq\downarrow & & \downarrow\simeq \\ \big(q\text{-}\breve{\Omega}^*_{q\text{-}D/\widehat{A}_p,\square}\otimes_{\mathbb{Z}}\mathbb{Q}\big)^\wedge_{(q-1)} & \xrightarrow[\text{(A.9)}]{\cong} & \big(\breve{\Omega}^*_{D/\widehat{A}_p,\square}\otimes_{\mathbb{Z}}\mathbb{Q}\big)[\![q-1]\!] \end{array}$$

Here the left vertical arrow is the quasi-isomorphism from [BS19, Theorem 16.22] and the right vertical arrow is the usual quasi-isomorphism between crystalline cohomology and PD-de Rham complexes.

Proof. Let $P^\bullet$ be the degreewise p-completed Čech nerve of $\widehat{A}_p\rightarrow P$ and let $J^\bullet\subseteq P^\bullet$ be the kernel of the augmentation $P^\bullet\twoheadrightarrow S$. Let $D^\bullet\coloneqq D_{P^\bullet}(J^\bullet)^\wedge_p$ be the PD-envelope and let q-$D^\bullet$ be the corresponding q-PD-envelope. Finally, form the cosimplicial complexes

$$M^{\bullet,*}\coloneqq\breve{\Omega}^*_{D^\bullet/\widehat{A}_p}\quad\text{and}\quad q\text{-}M^{\bullet,*}\coloneqq q\text{-}\breve{\Omega}^*_{q\text{-}D^\bullet/\widehat{A}_p,\square}\,.$$

In the proof of [BS19, Theorem 16.22] it's shown that the totalisation $\mathrm{Tot}(q\text{-}M^{\bullet,*})$ of $q\text{-}M^{\bullet,*}$ is quasi-isomorphic to the 0^{th} column $q\text{-}M^{0,*}\cong q\text{-}\breve{\Omega}^*_{q\text{-}D/\widehat{A}_p,\square}$, but also to the totalisation of the 0^{th} row $\mathrm{Tot}(q\text{-}M^{\bullet,0})\cong\mathrm{Tot}(q\text{-}D^\bullet)$. This provides the desired quasi-isomorphism

$$q\text{-}\breve{\Omega}^*_{q\text{-}D/\widehat{A}_p,\square}\simeq\mathrm{Tot}(q\text{-}M^{\bullet,*})\simeq\mathrm{Tot}(q\text{-}D^\bullet)\simeq\mathrm{R}\Gamma_{q\text{-crys}}\big(S/\widehat{A}[\![q-1]\!]\big)\,.$$

In the exact same way, the quasi-isomorphism $\breve{\Omega}^*_{D/\widehat{A}_p}\simeq\mathrm{R}\Gamma_{\mathrm{crys}}(S/\widehat{A}_p)$ is constructed using the cosimplicial complex $M^{\bullet,*}$ in [Stacks, Tag 07LG]. Applying Lemma A.9 column-wise gives an isomorphism of cosimplicial complexes $(q\text{-}M^{\bullet,*}\otimes_{\mathbb{Z}}\mathbb{Q})^\wedge_{(q-1)}\cong(M^{\bullet,*}\otimes_{\mathbb{Z}}\mathbb{Q})[\![q-1]\!]$. On 0^{th} columns, this is the isomorphism from Lemma A.9, whereas on 0^{th} rows it is the isomorphism from Lemma A.4. This proves commutativity of the diagram. □

§A.2. The global q-de Rham complex

From now on, we no longer work in a p-complete setting, but we keep Convention A.3.

A.11. Doing §A.1 for all primes at once. — Fix n and put $N_n\coloneqq\prod_{\ell\leqslant n}\ell^{2(\ell^{n-1}+\dots+\ell+1)}$, where the product is taken over all primes $\ell\leqslant n$. Now fix an arbitrary prime p and let P, D, and $q\text{-}D$ be as in §A.1. We've verified that the map $P \to q\text{-}D \to q\text{-}D/(q-1)^n \otimes_{\mathbb{Z}} \mathbb{Q}$ admits a unique continuous extension

$$\begin{array}{ccc} P & \longrightarrow & q\text{-}D/(q-1)^n \otimes_{\mathbb{Z}} \mathbb{Q} \\ \downarrow & \nearrow & \\ D & & \end{array}$$

But in fact, Lemma A.6 shows that this extension already factors through $N_n^{-1}\, q\text{-}D/(q-1)^n$, no matter how our implicit prime p is chosen. This observation allows us to construct canonical maps $\mathrm{dR}_{\widehat{R}_p/\widehat{A}_p} \to N_n^{-1}\, q\text{-}\mathrm{dR}_{\widehat{R}_p/\widehat{A}_p}/(q-1)^n$ for all animated rings R and all $n \geqslant 0$. Taking the product over all p and the limit over all n allows us to construct a map

$$\Bigl(\prod_p q\text{-}\mathrm{dR}_{\widehat{R}_p/\widehat{A}_p} \otimes_{\mathbb{Z}}^{\mathrm{L}} \mathbb{Q}\Bigr)^{\wedge}_{(q-1)} \xleftarrow{\simeq} \Bigl(\prod_p \mathrm{dR}_{\widehat{R}_p/\widehat{A}_p} \otimes_{\mathbb{Z}}^{\mathrm{L}} \mathbb{Q}\Bigr)[\![q-1]\!]\,.$$

compatible with the one from Lemma A.4. This map is an equivalence as indicated, as one immediately checks modulo $q-1$.

A.12. Construction. — For all smooth A-algebras S, we construct the *q-de Rham complex of S over A* as the pullback

$$\begin{array}{ccc} q\text{-}\Omega_{S/A} & \longrightarrow & \prod_p q\text{-}\Omega_{\widehat{S}_p/\widehat{A}_p} \\ \downarrow & \lrcorner & \downarrow \\ (\Omega_{S/A} \otimes_{\mathbb{Z}}^{\mathrm{L}} \mathbb{Q})[\![q-1]\!] & \longrightarrow & \Bigl(\prod_p \Omega_{\widehat{S}_p/\widehat{A}_p} \otimes_{\mathbb{Z}}^{\mathrm{L}} \mathbb{Q}\Bigr)[\![q-1]\!] \end{array}$$

Here the right vertical map is the one constructed in A.11 above.

Proof of Theorem A.1. We've constructed $q\text{-}\Omega_{S/A}$ in Construction A.12. Functoriality is clear since all constituents of the pullback are functorial and so are the arrows between them. Modulo $(q-1)$, the pullback reduces to the usual arithmetic fracture square for $\Omega_{R/A}$, proving (*a*). By construction, $(q\text{-}\Omega_{S/A})_p^{\wedge} \simeq q\text{-}\Omega_{\widehat{S}_p/\widehat{A}_p}$, and so (*b*) follows from [BS19, Theorem 16.18]. Part (*c*) follows again from the construction.

For (*d*), suppose S is equipped with an étale framing $\square\colon A[x_1,\dots,x_d] \to S$. The same argument as in A.8 provides an isomorphism $(q\text{-}\Omega^*_{S/A,\square} \otimes_{\mathbb{Z}} \mathbb{Q})^{\wedge}_{(q-1)} \cong (\Omega^*_{S/A} \otimes_{\mathbb{Z}} \mathbb{Q})[\![q-1]\!]$. The compatibility check from Lemma A.10 now allows us to identify the pullback square for $q\text{-}\Omega_{S/A}$ with the usual arithmetic fracture square for the complex $q\text{-}\Omega^*_{S/A,\square}$, completed at $(q-1)$. This shows $q\text{-}\Omega_{S/A} \simeq q\text{-}\Omega^*_{S/A,\square}$, as desired.

For the additional assertion, it's clear from the construction that a base change morphism

$$\bigl(q\text{-}\Omega_{-/A} \otimes_A^{\mathrm{L}} A'\bigr)^{\wedge}_{(q-1} \longrightarrow q\text{-}\Omega_{(-\otimes_A A')/A'}$$

exists and that it reduces modulo $(q-1)$ to the usual base change equivalence for the de Rham complex. In particular, it must be an equivalence as well. This finishes the proof. $\square$

A.13. Upgrade to derived commutative $A[\![q-1]\!]$-algebras. — Let us explain how to lift the q-de Rham complex to a functor

$$q\text{-}\Omega_{-/A}\colon \mathrm{Sm}_A \longrightarrow \left(\mathrm{DAlg}_{A[\![q-1]\!]}\right)^{\wedge}_{(q-1)}$$

into the ∞-category of $(q-1)$-complete derived commutative $A[\![q-1]\!]$-algebras. The key observation is that all limits and colimits in derived commutative $A[\![q-1]\!]$-algebras can be computed on the level of underlying $\mathbb{E}_\infty$-$A[\![q-1]\!]$-algebras by [Rak21, Proposition 4.2.27]. Thus, by compatibility with pullbacks, it'll be enough to lift the three components of the pullback from Construction A.12 to derived commutative $A[\![q-1]\!]$-algebras. By compatibility with cosimplicial limits, it'll be enough to construct functorial cosimplicial realisations of $\Omega_{S/A}$, $\Omega_{\widehat{S}_p/\widehat{A}_p}$, and q-$\Omega_{\widehat{S}_p/\widehat{A}_p}$.

For the latter two, the comparison with (q-)crystalline cohomology easily provides such realisations. But the same trick works just as well for $\Omega_{S/A}$: Let $P \twoheadrightarrow S$ be any surjection from an ind-smooth-A-algebra (which can be chosen functorially; for example, take $P \coloneqq A[\{T_s\}_{s\in S}]$), form the Čech nerve $P^\bullet$ of $A \to P$, let $J^\bullet \subseteq P^\bullet$ be the kernel of the augmentation $P^\bullet \twoheadrightarrow S$, and let $D^\bullet \coloneqq D_{P^\bullet}(J^\bullet)$ be its PD-envelope. Then $\Omega_{S/A} \simeq \operatorname{Tot} D_{P^\bullet}(J^\bullet)$ holds by a straightforward adaptation of the proof of [BS19, Theorem 16.22]: Namely, one considers the cosimplicial complex

$$M^{\bullet,*} \coloneqq \breve{\Omega}^*_{D^\bullet/A}$$

and checks that each column $M^{i,*}$ is quasi-isomorphic to $M^{0,*}$ (this is the Poincaré lemma) and that each row $M^{\bullet,j}$ for $j > 0$ is nullhomotopic (e.g. by [Stacks, Tag 07L7] applied to the cosimplicial ring $D^\bullet$).

In fact, this argument can be used to show something even better: Since the de Rham complex $\Omega^*_{S/A}$ and its PD-variants $\breve{\Omega}^*_{D_{P^\bullet}(J^\bullet)/A}$ are commutative differential-graded A-algebras, they define elements in Raksit's ∞-category $\mathrm{DG}_-\,\mathrm{DAlg}_A$ [Rak21, Definition 5.1.10], which gives another construction of a derived commutative algebra structure on $\Omega_{S/A}$. But the argument above shows that $\Omega_{S/A} \simeq \operatorname{Tot} D_{P^\bullet}(J^\bullet)$ holds true as derived commutative A-algebras.

A.14. Derived global q-de Rham complexes. — We let q-$\mathrm{dR}_{-/A}$ denote the animation of q-$\Omega_{-/A}$. For all animated A-algebras R, we call q-$\mathrm{dR}_{R/A}$ the *derived q-de Rham complex of R over A*. By construction, it sits inside a pullback square

$$\begin{array}{ccc} q\text{-}\mathrm{dR}_{R/A} & \longrightarrow & \prod_p q\text{-}\mathrm{dR}_{\widehat{R}_p/\widehat{A}_p} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \left(\mathrm{dR}_{R/A} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Q}\right)[\![q-1]\!] & \longrightarrow & \left(\prod_p \mathrm{dR}_{\widehat{R}_p/\widehat{A}_p} \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Q}\right)[\![q-1]\!] \end{array}$$

where the right vertical map again comes from A.11. It's still true that q-$\mathrm{dR}_{-/A}/(q-1) \simeq \mathrm{dR}_{-/A}$ and that q-$\mathrm{dR}_{-/A}$ lifts canonically to $(q-1)$-complete derived commutative $A[\![q-1]\!]$-algebras (this follows immediately from compatibility with colimits as explained in A.13).

However, in contrast to the p-complete situation, it's no longer true that the values on smooth A-algebras remain unchanged under animation (only the values on polynomial algebras do). In fact, this already fails for the derived de Rham complex in characteristic 0. If q-$\mathrm{dR}_{R/A}$ can be equipped with a q-deformation of the Hodge filtration, this problem can be fixed by considering the q-Hodge-completed derived q-de Rham complex q-$\widehat{\mathrm{dR}}_{R/A}$.

§A.3. The q-de Rham complex via TC^-

In [BMS19, §11] and [BS19, §15.2], it is explained how prismatic cohomology relative to a Breuil–Kisin prism $(\mathrm{W}(k)[\![z]\!], E(z))$ can be understood in terms of $\mathrm{TC}^-(-/\mathbb{S}[z])_p^\wedge$. In this subsection, we'll show how the p-complete q-de Rham complex can be understood in a completely analogous way.

For this to work, we assume that A satisfies the conditions from 6.1, that is, A is a p-complete and p-completely covered δ-ring with a flat spherical lift $\mathbb{S}_A$ which admits the structure of a p-cyclotomic base. We also put Convention 7.1 into effect again.

A.15. Lemma. — *The p-completed colimit-perfection A_∞ of A admits a unique lift to a p-complete connective $\mathbb{E}_\infty$-ring spectrum $\mathbb{S}_{A_\infty}$ and $A \to A_\infty$ can be lifted to an $\mathbb{E}_\infty$-map $\mathbb{S}_A \to \mathbb{S}_{A_\infty}$.*

Proof. Since A_∞ is a perfect δ-ring, the lift $\mathbb{S}_{A_\infty}$ exists uniquely; it is given by the spherical Witt vectors $\mathbb{S}_{\mathrm{W}(A_\infty^\flat)}$ from Example [L-Ell$_{\mathrm{II}}$, 5.2.7].

To construct the map $\mathbb{S}_A \to \mathbb{S}_{A_\infty}$, first observe that the canonical map $\mathbb{S}_A \to \mathbb{S}_A^{tC_p}$ is an equivalence. Indeed, we can choose a two-term resolution $0 \to \bigoplus_I \mathbb{Z}_p \to \bigoplus_J \mathbb{Z}_p \to A \to 0$ and lift it to a cofibre sequence $\bigoplus_I \mathbb{S}_p \to \bigoplus_J \mathbb{S}_p \to \mathbb{S}_A$ of spectra. By the Segal conjecture, $(\bigoplus_I \mathbb{S}_p)^{tC_p} \simeq (\bigoplus_I \mathbb{S}_p)_p^\wedge$ and likewise for J, so the same will be true for $\mathbb{S}_A$. We can then form the sequential colimit

$$\operatorname{colim}\left(\mathbb{S}_A \xrightarrow{\phi_{tC_p}} \mathbb{S}_A^{tC_p} \simeq \mathbb{S}_A \xrightarrow{\phi_{tC_p}} \cdots\right)_p^\wedge .$$

By our assumptions on A, the Tate-valued Frobenius ϕ_{tC_p} agrees with ϕ on π_0, and so this colimit is a p-complete connective $\mathbb{E}_\infty$-lift of A_∞. By uniqueness, it must agree with $\mathbb{S}_{A_\infty}$, and so we get our desired map $\mathbb{S}_A \to \mathbb{S}_{A_\infty}$. □

A.16. Lemma. — *There are generators u and v in π_2 and π_{-2} of $\mathrm{TC}^-(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p[\![q-1]\!])_p^\wedge$ such that*

$$\pi_* \mathrm{TC}^-\big(\mathbb{Z}_p[\zeta_p]/\mathbb{S}_p[\![q-1]\!]\big)_p^\wedge \simeq \mathbb{Z}_p[\![q-1]\!][u,v]/\big(uv-[p]_q\big) .$$

Proof. This can be shown in the same way as [BMS19, Proposition 11.10], using base change along $\mathbb{S}[\![q-1]\!] \to \mathbb{S}[\![q^{1/p^\infty}-1]\!]$. □

A.17. Proposition. — *Let S be a p-complete p-quasi-lci $A[\zeta_p]$-algebra of bounded p^∞-torsion. Then there is an equivalence of graded $\mathbb{E}_\infty$-$\mathbb{Z}_p[\![q-1]\!][u,v]/(uv-[p]_q)$-algebras*

$$\Sigma^{-2*}\operatorname{gr}^*_{\mathrm{HRW\text{-}ev},hS^1}\mathrm{TC}^-\big(S/\mathbb{S}_A[\![q-1]\!]\big)_p^\wedge \simeq \operatorname{fil}^*_{\mathcal{N}}\widehat{\Delta}^{(p)}_{S/A[\![q-1]\!]} ,$$

where $\operatorname{gr}^*_{\mathrm{HRW\text{-}ev},hS^1}$ denotes the associated graded of the p-complete S^1-equivariant Hahn–Raksit–Wilson even filtration and $(-)^{(p)}$ (instead of $(-)^{(1)}$) denotes the Frobenius twist of prismatic cohomology. Moreover, after inverting u, we get an equivalence of graded $\mathbb{E}_\infty$-$\mathbb{Z}_p[u^{\pm1}][\![q-1]\!]$-algebras

$$\Sigma^{-2*}\operatorname{gr}^*_{\mathrm{HRW\text{-}ev},hS^1}\Big(\mathrm{TC}^-\big(S/\mathbb{S}_A[\![q-1]\!]\big)\big[\tfrac{1}{u}\big]^\wedge_{(p,q-1)}\Big) \simeq \Delta_{S/A[\![q-1]\!]}[u^{\pm1}] ,$$

where now $\operatorname{gr}^*_{\mathrm{HRW\text{-}ev}}$ refers to the p-complete S^1-equivariant Hahn–Raksit–Wilson even filtration on $\mathrm{THH}(S/\mathbb{S}_A[\![q-1]\!])[1/u]_p^\wedge$.

Proof sketch. First observe that $S_\infty \coloneqq (S \otimes^{\mathrm{L}}_{A[\![q-1]\!]} A_\infty[\![q^{1/p^\infty}-1]\!])_p^\wedge$ will be static and of bounded p^∞-torsion, as $\phi\colon A \to A$ is p-completely flat. Moreover, S_∞ will be p-quasi-lci over $A_\infty[\![q^{1/p^\infty}-1]\!]$, hence over $\mathbb{Z}_p$, as the cotangent complex $\mathrm{L}_{A_\infty[\![q^{1/p^\infty}-1]\!]/\mathbb{Z}_p}$ vanishes after p-completion. Thus S_∞ is p-quasi-syntomic.

If S is *large* in the sense that there exists a surjection $A\langle x_i^{1/p^\infty} \mid i \in I\rangle \twoheadrightarrow S$, then $\mathrm{TC}^-(S/\mathbb{S}_A[\![q-1]\!])_p^\wedge$ will be even. Indeed, evenness can be checked after base change along $\mathbb{S}_A[\![q-1]\!] \to \mathbb{S}_{A_\infty}[\![q^{1/p^\infty}-1]\!]$. By an analogous argument as in [BMS19, Proposition 11.7],

$$\mathrm{THH}\bigl(\mathbb{S}_{A_\infty}[\![q^{1/p^\infty}-1]\!]\bigr) \to \mathbb{S}_{A_\infty}[\![q^{1/p^\infty}-1]\!]$$

is an equivalence after p-completion. This reduces the assertion to $\mathrm{TC}^-(S_\infty)_p^\wedge$ being even, which is shown in [BMS19, Theorem 7.2].

Via quasi-syntomic descent from the large case, we can now construct a filtration on $\mathrm{TC}^-(S/\mathbb{S}_A[\![q-1]\!])_p^\wedge$. Arguing as in [BMS19, §11.2] and [BS19, §15.2], we find that the associated graded of this filtration yields the completion of the Nygaard filtration on the Frobenius-twisted prismatic cohomology relative to the q-de Rham prism $(A[\![q-1]\!], [p]_q)$. To see that the filtration agrees with the p-complete S^1-equivariant Hahn–Raksit–Wilson even filtration, we argue as in the proof of [HRW22, Theorem 5.0.3]. Choose a surjection from a polynomial ring $\mathbb{Z}[x_i \mid i \in I] \twoheadrightarrow S$. Both filtrations satisfy descent along the p-completely eff map $\mathrm{THH}(\mathbb{S}[x_i \mid i \in I]) \to \mathrm{THH}(\mathbb{S}[x_i^{1/p^\infty} \mid i \in I])$. By descent, it will then be enough to check that the filtrations agree when S is large, which is clear by evenness.

After inverting u, the argument is analogous: As in [BMS19, §11.3], we use quasi-syntomic descent again to construct a filtration

$$\mathrm{fil}^\star_{\text{BMS-ev}}\Bigl(\mathrm{TC}^-\bigl(S/\mathbb{S}_A[\![q-1]\!]\bigr)\bigl[\tfrac{1}{u}\bigr]^\wedge_{(p,q-1)}\Bigr)$$

and check via descent along $\mathrm{THH}(\mathbb{S}[x_i \mid i \in I]) \to \mathrm{THH}(\mathbb{S}[x_i^{1/p^\infty} \mid i \in I])$ that this filtration is really the Hahn–Raksit–Wilson even filtration. To see $\mathrm{gr}^*_{\text{HRW-ev},hS^1} \simeq \mathbb{\Delta}_{S/A[\![q-1]\!]}[u^{\pm 1}]$, observe that inverting the degree 2 class u amounts to adjoining $[p]_q^{-i}\,\mathrm{fil}^i_{\mathcal{N}}$ for all $i \geqslant 0$ in the sense of 3.42; we must then show that the relative Frobenius induces an equivalence

$$\phi_{/A[\![q-1]\!]}\colon \widehat{\mathbb{\Delta}}^{(p)}_{S/A[\![q-1]\!]}\left[\frac{\mathrm{fil}^i_{\mathcal{N}}}{[p]_q^i}\;\middle|\; i \geqslant 0\right]^\wedge_{(p,q-1)} \xrightarrow{\ \simeq\ } \mathbb{\Delta}_{S/A[\![q-1]\!]}\,.$$

This is a general fact about the Nygaard filtration on prismatic cohomology; it follows, for example, from [BS19, Theorem 15.2(2)] via quasi-syntomic descent. See also Lemma 3.44. ☐

A.18. Frobenii. — The same argument as in [BMS19, Proposition 11.10] shows that the p-cyclotomic Frobenius

$$\phi_p^{hS^1}\colon \mathrm{TC}^-\bigl(\mathbb{Z}_p[\zeta_p]/\mathbb{S}[\![q-1]\!]\bigr)_p^\wedge \longrightarrow \mathrm{TP}\bigl(\mathbb{Z}_p[\zeta_p]/\mathbb{S}[\![q-1]\!]\bigr)_p^\wedge$$

inverts the generator u in degree 2. Moreover, the p-cyclotomic Frobenius on $\mathrm{THH}(-/\mathbb{S}_{A,p}[\![q-1]\!])$ is semilinear with respect to the Tate-valued Frobenius $\phi_{tC_p}\colon \mathbb{S}_{A,p}[q] \to \mathbb{S}_{A,p}[q]$, which on π_0 is given by $\phi\colon A \to A$ and $q \mapsto q^p$. It follows that the p-cyclotomic Frobenius induces a map

$$\Bigl(\mathrm{TC}^-\bigl(S/\mathbb{S}_A[\![q-1]\!]\bigr)\bigl[\tfrac{1}{u}\bigr] \otimes_{\mathbb{S}_A[q],\phi_{tC_p}} \mathbb{S}_A[q]\Bigr)^\wedge_{(p,q-1)} \longrightarrow \mathrm{TP}\bigl(S/\mathbb{S}_A[\![q-1]\!]\bigr)^\wedge_p\,.$$

On $\mathrm{gr}^0_{\mathrm{HRW\text{-}ev}}$, this map agrees with the Nygaard completion $\Delta^{(p)}_{S/A[\![q-1]\!]} \to \widehat{\Delta}^{(p)}_{S/A[\![q-1]\!]}$, as the proof of Proposition A.17 shows. The relative Frobenius on prismatic cohomology,

$$\phi_{/A[\![q-1]\!]}\colon \widehat{\Delta}^{(p)}_{S/A[\![q-1]\!]} \longrightarrow \Delta_{S/A[\![q-1]\!]}\,,$$

can then be identified as the composition of $\mathrm{gr}^0_{\mathrm{HRW\text{-}ev}}\,\mathrm{TP} \simeq \mathrm{gr}^0_{\mathrm{HRW\text{-}ev}}\,\mathrm{TC}^-$ with

$$\mathrm{gr}^0_{\mathrm{HRW\text{-}ev}}\,\mathrm{TC}^-\big(S/\mathbb{S}_A[\![q-1]\!]\big)^\wedge_p \longrightarrow \mathrm{gr}^0_{\mathrm{HRW\text{-}ev}}\Big(\mathrm{TC}^-\big(S/\mathbb{S}_A[\![q-1]\!]\big)\big[\tfrac{1}{u}\big]^\wedge_{(p,q-1)}\Big)\,.$$

A.19. Recovering q-de Rham cohomology. — Let R be a p-torsion free p-quasi-lci A-algebra and let $R^{(p)} \coloneqq (R \otimes^{\mathrm{L}}_{A,\phi} A)^\wedge_p$. Then [BS19, Theorem 16.18] shows

$$q\text{-}\mathrm{dR}_{R/A} \simeq \Delta_{R^{(p)}[\zeta_p]/A[\![q-1]\!]}\,.$$

Therefore Proposition A.17 and A.18 contain q-de Rham cohomology (which is implicitly p-completed per Convention A.3) as a special case.

A.20. The Adams action. — In [BL22a, §3.8], Bhatt–Lurie describe an action of $\mathbb{Z}_p^\times$ on the q-de Rham prism $(A[\![q-1]\!], [p]_q)$, where $u \in \mathbb{Z}_p^\times$ acts by sending $q \mapsto q^u$. Here q^u denotes the convergent power series

$$q^u \coloneqq \sum_{n\geqslant 0} \binom{u}{n}(q-1)^n\,.$$

By functoriality of prismatic cohomology, the action on the prism induces an action of $\mathbb{Z}_p^\times$ on $q\text{-}\mathrm{dR}_{R/A}$, which is precisely the action predicted in [Sch17, Conjecture 6.2].

Under the identification

$$q\text{-}\mathrm{dR}_{R/A} \simeq \mathrm{gr}^0_{\mathrm{HRW\text{-}ev}}\Big(\mathrm{TC}^-\big(R^{(p)}[\zeta_p]/\mathbb{S}_A[\![q-1]\!]\big)\big[\tfrac{1}{u}\big]^\wedge_{(p,q-1)}\Big)\,,$$

this action comes from an action of $\mathbb{Z}_p^\times$ on $\mathbb{S}_A[\![q-1]\!]$. Indeed, following [DR25, Notation 3.3.3], we can write

$$\mathbb{S}_A[\![q-1]\!] \simeq \lim_{\alpha\geqslant 0} \mathbb{S}_A[q]/\big(q^{p^\alpha}-1\big) \simeq \lim_{\alpha\geqslant 0} \mathbb{S}_A[\mathbb{Z}/p^\alpha]$$

and then let $\mathbb{Z}_p^\times$ act on $\mathbb{Z}/p^\alpha$ via multiplication (this is another way of making precise what q^u is supposed to mean). To see that this induces the same action on $(q\text{-}\mathrm{dR}_{R/A})^\wedge_p$ as above, we can use quasi-syntomic descent as in the proof of Proposition A.17 to reduce to an even situation, where the claim is straightforward to verify.

We call this action the *Adams action*, since it turns out to agree with the action of $\mathbb{Z}_p^\times$ on $\mathrm{ku}^\wedge_p$ via Adams operations (see §7.1).

Appendix B. Habiro-completion

In this appendix we'll study the *Habiro completion functor* $(-)^\wedge_{\mathcal{H}} \coloneqq \lim_{m\in\mathbb{N}}(-)^\wedge_{(q^m-1)}$ and show that it behaves for all practical purposes like completion at a finitely generated ideal. We'll also study Habiro completion in the setting of solid condensed mathematics.

In the following, we'll use the notion of *killing an idempotent algebra*, which is nicely reviewed in [CS24, Lecture 13].

B.1. Habiro-complete spectra. — Following Manin [Man10, §0.2], let us denote the localisation $\mathbb{Z}\big[q^{\pm 1}, \{(q^m-1)^{-1}\}_{m\in\mathbb{N}}\big]$ by $\mathcal{R}$ and let $\mathbb{S}_{\mathcal{R}} \coloneqq \mathbb{S}\big[q^{\pm 1}, \{(q^m-1)^{-1}\}_{m\in\mathbb{N}}\big]$ be its obvious spherical lift. Then $\mathbb{S}_{\mathcal{R}}$ is an idempotent algebra over $\mathbb{S}[q^{\pm 1}]$ and we define the *∞-category of Habiro-complete spectra*

$$\mathrm{Mod}_{\mathbb{S}_{\mathcal{H}}}(\mathrm{Sp})^\wedge_{\mathcal{H}} \subseteq \mathrm{Mod}_{\mathbb{S}[q^{\pm 1}]}(\mathrm{Sp})$$

to be the full sub-∞-category obtained by killing the idempotent $\mathbb{S}_{\mathcal{R}}$. That is, $\mathrm{Mod}_{\mathbb{S}_{\mathcal{H}}}(\mathrm{Sp})^\wedge_{\mathcal{H}}$ consists of those $M \in \mathrm{Mod}_{\mathbb{S}[q^{\pm 1}]}(\mathrm{Sp})$ such that $\mathrm{Hom}_{\mathbb{S}[q^{\pm 1}]}(\mathbb{S}_{\mathcal{R}}, M) \simeq 0$.

It'll be apparent from Lemma B.2 below that the inclusion $\mathrm{Mod}_{\mathbb{S}_{\mathcal{H}}}(\mathrm{Sp})^\wedge_{\mathcal{H}} \subseteq \mathrm{Mod}_{\mathbb{S}[q^{\pm 1}]}(\mathrm{Sp})$ has a left adjoint $(-)^\wedge_{\mathcal{H}} \coloneqq \lim_{m\in\mathbb{N}}(-)^\wedge_{(q^m-1)}$ which we call *Habiro-completion*. When applied to the tensor unit, we obtain the *spherical Habiro ring*

$$\mathbb{S}_{\mathcal{H}} \coloneqq \lim_{m\in\mathbb{N}} \mathbb{S}[q]^\wedge_{(q^m-1)}\,.$$

Note that q is already a unit in $\mathbb{S}_{\mathcal{H}}$, so it doesn't matter whether we complete $\mathbb{S}[q]$ or $\mathbb{S}[q^{\pm 1}]$. We let $-\mathbin{\widehat{\otimes}}_{\mathbb{S}_{\mathcal{H}}}-$ denote the Habiro-completed tensor product in $\mathrm{Mod}_{\mathbb{S}_{\mathcal{H}}}(\mathrm{Sp})^\wedge_{\mathcal{H}}$. We also let $\widehat{\mathcal{D}}(\mathcal{H}) \subseteq \mathcal{D}(\mathbb{Z}[q^{\pm 1}])$ denote the full sub-∞-category of Habiro-complete objects and denote its completed tensor product by $-\mathbin{\widehat{\otimes}}^{\mathrm{L}}_{\mathcal{H}}-$.

B.2. Lemma. — *For a $\mathbb{S}[q^{\pm 1}]$-module spectrum M, the following conditions are equivalent.*

(*a*) *M is Habiro-complete.*

(*b*) $\mathrm{Hom}_{\mathbb{S}[q^{\pm 1}]}(\mathbb{S}_{\mathcal{R}}, M) \simeq 0$.

(*c*) *The canonical $\mathbb{S}[q^{\pm 1}]$-module morphism*

$$M \longrightarrow \lim_{n\geqslant 1} M/(q;q)_n \simeq \lim_{m\in\mathbb{N}} M^\wedge_{(q^m-1)}$$

is an equivalence. Here $(a;q)_n \coloneqq (1-a)(1-aq)\cdots(1-aq^{n-1})$ denotes the q-Pochhammer symbol, as usual.

(*d*) *All homotopy groups $\pi_n(M)$, $n\in\mathbb{Z}$, are Habiro-complete.*

Proof. The proof is analogous to [Stacks, Tag 091P]. Equivalence of (*a*) and (*b*) follows by definition of what it means to kill the idempotent $\mathbb{S}_{\mathcal{R}}$. Condition (*b*) is equivalent to $M \simeq \mathrm{Hom}_{\mathbb{S}[q^{\pm 1}]}(\mathrm{fib}(\mathbb{S}[q^{\pm 1}] \to \mathbb{S}_{\mathcal{R}}), M)$. Writing

$$\mathrm{fib}\big(\mathbb{S}[q^{\pm 1}] \to \mathbb{S}_{\mathcal{R}}\big) \simeq \Sigma^{-1}\operatorname*{colim}\left(\mathbb{S}[q^{\pm 1}]/(q;q)_1 \xrightarrow{(1-q^2)} \mathbb{S}[q^{\pm 1}]/(q;q)_2 \xrightarrow{(1-q^3)} \cdots\right)$$

we see that this condition is equivalent to $M \simeq \lim_{n\geqslant 1} M/(q;q)_n$, thus (*b*) $\Leftrightarrow$ (*c*). Finally, to show (*a*) $\Leftrightarrow$ (*d*), consider the Postikov filtration $\tau_{\geqslant\star}(M)$. This allows us to define a descending filtration on $\mathrm{Hom}_{\mathbb{S}[q^{\pm 1}]}(\mathbb{S}_{\mathcal{R}}, M)$ via

$$\mathrm{fil}^\star \mathrm{Hom}_{\mathbb{S}[q^{\pm 1}]}(\mathbb{S}_{\mathcal{R}}, M) \coloneqq \mathrm{Hom}_{\mathbb{S}[q^{\pm 1}]}\big(\mathbb{S}_{\mathcal{R}}, \tau_{\geqslant\star}(M)\big)\,.$$

This filtration is complete, because $0 \simeq \lim_{n\to\infty} \tau_{\geqslant n}(M)$ can be pulled into $\operatorname{Hom}_{\mathbb{S}[q^{\pm 1}]}(\mathbb{S}_{\mathcal{R}}, -)$. To show that the filtration is exhaustive, we need to check that $M \simeq \operatorname{colim}_{n\to-\infty} \tau_{\geqslant n}(M)$ can similarly be pulled into $\operatorname{Hom}_{\mathbb{S}[q^{\pm 1}]}(\mathbb{S}_{\mathcal{R}}, -)$. This works because $\mathbb{S}_{\mathcal{R}}$ is connective, whereas the cofibres $\operatorname{cofib}(\tau_{\geqslant n}(M) \to M) \simeq \tau_{\leqslant n-1}(M)$ become more and more coconnective as $n \to -\infty$.

Since each $\pi_n(M)$ is already a $\mathbb{Z}[q^{\pm 1}]$-module, the associated graded of this filtration is given by

$$\operatorname{gr}^n \operatorname{Hom}_{\mathbb{S}[q^{\pm 1}]}(\mathbb{S}_{\mathcal{R}}, M) \simeq \operatorname{Hom}_{\mathbb{S}[q^{\pm 1}]}\bigl(\mathbb{S}_{\mathcal{R}}, \Sigma^n \pi_n(M)\bigr) \simeq \Sigma^n \operatorname{RHom}_{\mathbb{Z}[q^{\pm 1}]}\bigl(\mathcal{R}, \pi_n(M)\bigr)\,.$$

Now $\mathcal{R}$ has a two-term resolution by free $\mathbb{Z}[q^{\pm 1}]$-modules. For example, take

$$0 \longrightarrow \bigoplus_{i\geqslant 0} \mathbb{Z}[q^{\pm 1}] \longrightarrow \bigoplus_{i\geqslant 0} \mathbb{Z}[q^{\pm 1}] \longrightarrow \mathcal{R} \longrightarrow 0\,,$$

where the first arrow sends $(a_i)_{i\geqslant 0} \mapsto (a_i - (q;q)_i a_{i-1})_{i\geqslant 0}$ (with $a_{-1} \coloneqq 0$) and the second arrow sends $(a_i)_{i\geqslant 0} \mapsto \sum_{i\geqslant 0} a_i/(q;q)_i$. It follows that $\Sigma^n \operatorname{RHom}_{\mathbb{Z}[q^{\pm 1}]}(\mathcal{R}, \pi_n(M))$ is concentrated in homological degrees $[n-1, n]$. Combined with the fact that the filtration is complete and exaustive[(B.1)], we obtain short exact sequences

$$0 \longrightarrow \operatorname{Ext}^1_{\mathbb{Z}[q^{\pm 1}]}\bigl(\mathcal{R}, \pi_{n+1}(M)\bigr) \longrightarrow \pi_n \operatorname{Hom}_{\mathbb{S}[q^{\pm 1}]}(\mathbb{S}_{\mathcal{R}}, M) \longrightarrow \operatorname{Hom}_{\mathbb{Z}[q^{\pm 1}]}\bigl(\mathcal{R}, \pi_n(M)\bigr) \longrightarrow 0$$

for all $n \in \mathbb{Z}$. Therefore, $\operatorname{Hom}_{\mathbb{S}[q^{\pm 1}]}(\mathbb{S}_{\mathcal{R}}, M)$ vanishes if and only if $\operatorname{RHom}_{\mathbb{Z}[q^{\pm 1}]}(\mathcal{R}, \pi_n(M))$ vanishes for all $n \in \mathbb{Z}$, which proves that M is Habiro-complete if and only if each $\pi_n(M)$ is. ☐

We have the following "derived Nakayama lemma".

B.3. Lemma. — *Let M be a Habiro-complete spectrum. If $M/\Phi_m(q) \simeq 0$ for all $m \in \mathbb{N}$, then $M \simeq 0$. If M is an ordinary $\mathbb{Z}[q^{\pm 1}]$-module, the same conclusion is already true if the quotients are taken in the underived sense.*

Proof. By the usual derived Nayama lemma, if $M/\Phi_m(q) \simeq 0$, then $M^\wedge_{\Phi_m(q)} \simeq 0$, hence $M^\wedge_{(q^m-1)} \simeq 0$. By Lemma B.2(*c*), this implies $M \simeq 0$. Now suppose M is an ordinary $\mathbb{Z}[q^{\pm 1}]$-module such that the underived quotients $M/\Phi_m(q)$ vanish for all $m \in \mathbb{N}$. We argue as in [Stacks, Tag 09B9]. The assumption implies that multiplication by $(q;q)_n$ is surjective on M for all $n \geqslant 1$. It follows that the underived limit of

$$\left(M \xleftarrow{(q;q)_1} M \xleftarrow{(q;q)_2} M \xleftarrow{(q;q)_3} \cdots\right)$$

is non-zero. Then the derived limit is non-zero as well, which forces $\operatorname{Hom}_{\mathbb{S}[q^{\pm 1}]}(\mathbb{S}_{\mathcal{R}}, M) \not\simeq 0$, so M is not Habiro-complete. ☐

B.4. Corollary. — *Let M be a Habiro-complete spectrum and fix $n \in \mathbb{Z}$. If $\pi_n(M/\Phi_m(q)) \cong 0$ for all $m \in \mathbb{N}$, then already $\pi_n(M) \cong 0$.*

Proof. The underived quotient $\pi_n(M)/\Phi_m(q)$ is a sub-$\mathbb{Z}[q^{\pm 1}]$-module of $\pi_n(M/\Phi_m(q))$, so if $\pi_n(M/\Phi_m(q))$ vanishes, then the underived quotient $\pi_n(M)/\Phi_m(q)$ vanishes as well. If this is happens for all $m \in \mathbb{N}$, Lemma B.3 implies $\pi_n(M) \cong 0$, because $\pi_n(M)$ is Habiro-complete by Lemma B.2(*d*). ☐

(B.1) Alternatively, observe that the spectral sequence associated to the filtered spectrum $\operatorname{fil}^\star \operatorname{Hom}_{\mathbb{S}[q^{\pm 1}]}(\mathbb{S}_{\mathcal{R}}, M)$ collapses on the E^2-page.

B.5. Remark. — In Lemma B.3 and Corollary B.4, we could equally well replace $\{\Phi_m(q)\}_{m\in\mathbb{N}}$ by $\{(q^m-1)\}_{m\in\mathbb{N}}$, or $\{(q;q)_n\}_{n\geqslant 1}$, or any set of polynomials in which each $\Phi_m(q)$ occurs as a factor at least once.

Let us now study Habiro-completeness in the setting of solid condensed mathematics (see the brief review in 5.1).

B.6. Habiro-complete solid condensed spectra. — We can also define Habiro-complete objects and Habiro completion inside $\mathrm{Mod}_{\mathbb{S}[q^{\pm1}]}(\mathrm{Sp}_\blacksquare)$. To every ordinary Habiro-complete spectrum M, we can associate a Habiro-complete solid condensed spectrum by taking the condensed Habiro-completion of the associated discrete condensed spectrum $\underline{M}$. By abuse of notation, this Habiro-complete solid condensed spectrum will be denoted M again, and then "$M\mapsto M$" defines a fully faithful functor

$$\mathrm{Mod}_{\mathbb{S}_\mathcal{H}}(\mathrm{Sp})_\mathcal{H}^\wedge \longrightarrow \mathrm{Mod}_{\mathbb{S}_\mathcal{H}}(\mathrm{Sp}_\blacksquare)\,.$$

B.7. Lemma. — *The solidified tensor product $-\otimes_{\mathbb{S}_\mathcal{H}}^\blacksquare -$ preserves bounded below Habiro-complete objects. In particular, the fully faithful functor $\mathrm{Mod}_{\mathbb{S}_\mathcal{H}}(\mathrm{Sp})_\mathcal{H}^\wedge\rightarrow\mathrm{Mod}_{\mathbb{S}_\mathcal{H}}(\mathrm{Sp}_\blacksquare)$ from B.6 is symmetric monoidal when restricted to bounded below objects.*

Proof sketch. The proof is analogous to the proof that the solid tensor product preserves bounded below p-complete objects (see [CS24, Lecture 6] or [Bos23, Proposition A.3]), but let us still sketch the argument.

First we claim that $\mathbb{S}_\mathcal{H}$ is idempotent in $\mathrm{Mod}_{\mathbb{S}[q^{\pm1}]}(\mathrm{Sp}_\blacksquare)$. Indeed, each stage of the limit $\mathbb{S}_\mathcal{H}\simeq\lim_{n\geqslant1}\mathbb{S}[q^{\pm1}]/(q;q)_n$ is a finite direct sum of copies of $\mathbb{S}$. Limits of this form interact well with the solid tensor product (as $\prod_\mathbb{N}\mathbb{S}\otimes^\blacksquare\prod_\mathbb{N}\mathbb{S}\simeq\prod_{\mathbb{N}\times\mathbb{N}}\mathbb{S}$) and we obtain

$$\mathbb{S}_\mathcal{H}\otimes^\blacksquare\mathbb{S}_\mathcal{H}\simeq\lim_{m,n\geqslant1}\left(\mathbb{S}[q_1^{\pm1}]/(q_1;q_1)_m\otimes^\blacksquare\mathbb{S}[q_2^{\pm1}]/(q_2;q_2)_n\right)\simeq\lim_{m\in\mathbb{N}}\mathbb{S}[q_1,q_2]^\wedge_{(q_1^m-1,q_2^m-1)}\,.$$

Taking the solidified tensor product over $\mathbb{S}[q^{\pm1}]$ instead amounts to identifying q_1 and q_2, which implies $\mathbb{S}_\mathcal{H}\otimes^\blacksquare_{\mathbb{S}[q^{\pm1}]}\mathbb{S}_\mathcal{H}\simeq\mathbb{S}_\mathcal{H}$, as desired. A similar argument shows $\prod_\mathbb{N}\mathbb{S}\otimes^\blacksquare\mathbb{S}_\mathcal{H}\simeq\prod_\mathbb{N}\mathbb{S}_\mathcal{H}$, so $\mathrm{Mod}_{\mathbb{S}_\mathcal{H}}(\mathrm{Sp}_\blacksquare)$ is compactly generated by shifts of $\prod_\mathbb{N}\mathbb{S}_\mathcal{H}$.

Now let M and N be bounded below and Habiro-complete. We wish to show that $M\otimes^\blacksquare_{\mathbb{S}_\mathcal{H}}N$ is Habiro-complete again. Using that Habiro-completion is a countable limit and thus commutes with ω_1-filtered colimits, we can reduce to the case where M and N are the Habiro-completions of countable direct sums of the form $\bigoplus_{n\in\mathbb{N}}\prod_{I_n}\mathbb{S}_\mathcal{H}$, where each I_n is countable as well. For ease of notation, let us assume $|I_n|=1$ for all n; the argument in the general case is exactly the same. The Habiro completion of $\bigoplus_{n\in\mathbb{N}}\mathbb{S}_\mathcal{H}$ can be written as

$$\left(\bigoplus_{n\in\mathbb{N}}\mathbb{S}_\mathcal{H}\right)^\wedge_\mathcal{H}\simeq\operatorname*{colim}_{\substack{f\colon\mathbb{N}\to\mathbb{N},\\ f(n)\to\infty}}\prod_{n\in\mathbb{N}}(q;q)_{f(n)}\mathbb{S}_\mathcal{H}\,,$$

where the colimit is taken over all functions $f\colon\mathbb{N}\to\mathbb{N}$ such that $f(n)\to\infty$ as $n\to\infty$. It follows that

$$M\otimes^\blacksquare_{\mathbb{S}_\mathcal{H}}N\simeq\operatorname*{colim}_{\substack{f,\,g\colon\mathbb{N}\to\mathbb{N},\\ f(n),\,g(n)\to\infty}}\prod_{(m,n)\in\mathbb{N}\times\mathbb{N}}(q;q)_{f(m)}(q;q)_{g(n)}\mathbb{S}_\mathcal{H}\,.$$

Observe that $(q;q)_{f(m)}(q;q)_{g(n)}$ divides $(q;q)_{f(m)+g(n)}$, because q-binomial coefficients are polynomials in $\mathbb{Z}[q]$. Moreover, for every $h\colon\mathbb{N}\times\mathbb{N}\to\mathbb{N}$ such that $h(m,n)\to\infty$ as $m+n\to\infty$

there exist $f, g\colon \mathbb{N} \to \mathbb{N}$ such that $f(n), g(n) \to \infty$ and $h(m,n) \geqslant f(m) + g(n)$ for all m, n. By the same argument as for p-completions, it follows that the colimit above can be rewritten as

$$M \otimes_{\mathbb{S}_{\mathcal{H}}}^{\blacksquare} N \simeq \operatorname*{colim}_{\substack{h\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N},\\ h(m,n)\to\infty}} \prod_{(m,n)\in\mathbb{N}\times\mathbb{N}} (q;q)_{h(m,n)}\mathbb{S}_{\mathcal{H}} \simeq \left(\bigoplus_{m\in\mathbb{N}} \mathbb{S}_{\mathcal{H}} \otimes_{\mathbb{S}_{\mathcal{H}}}^{\blacksquare} \bigoplus_{n\in\mathbb{N}} \mathbb{S}_{\mathcal{H}}\right)_{\mathcal{H}}^{\wedge}. \qquad \square$$

Finally, let us mention that the arguments in the proofs above are quite robust and also work for similar notions of completion, such as the following classical case:

B.8. Profinite completion. — The ∞-category of *profinitely complete spectra* $\mathrm{Sp}^\wedge \subseteq \mathrm{Sp}$ is obtained by killing the idempotent $\mathbb{Q}$. For a spectrum M, the following are then equivalent:

(a) M is profinitely complete.

(b) $\mathrm{Hom}_{\mathrm{Sp}}(\mathbb{Q}, M) \simeq 0$.

(c) The canonical map $M \to \lim_{m\in\mathbb{N}} M/m \simeq \prod_p \widehat{M}_p$ is an equivalence.

(d) All homototopy groups $\pi_n(M)$, $n \in \mathbb{Z}$, are profinitely complete.

To show equivalence, one can just copy the arguments from Lemma B.2 and replace each occurrence of the q-Pochhammer symbol $(q;q)_n$ by $n!$.

In the same way, one can also define profinite completeness for solid condensed spectra and show that the solid tensor product of bounded below profinitely complete solid condensed spectra will again be profinitely complete.

Appendix C. Even $\mathbb{E}_2$-cell structures on flat polynomial rings

In this appendix we show the following technical result.

C.1. Lemma. — *Let $\mathbb{S}[x_i \mid i \in I]$ be the flat graded polynomial ring on generators x_i in graded degree 1 and homotopical degree 0. As a graded $\mathbb{E}_2$-ring, $\mathbb{S}[x_i \mid i \in I]$ admits a cell decomposition with all cells in even homotopical degree.*

C.2. Remark. — For polynomial rings in one variable this is shown in [ABM23, Proposition 3.11]. We believe the argument given there can be adapted to several variables as well. The authors of that paper also remark that an alternative proof of the one-variable case is given in the second (but not in the final) arXiv version of [HW22]; we'll follow the proof given therein.

Proof of Lemma C.1. To avoid issues with double duals of infinite direct sums, we work in the ∞-category of graded solid condensed spectra $\mathrm{Gr}(\mathrm{Sp}_\blacksquare)$. Usual graded spectra embed fully faithfully as the full sub-∞-category of graded discrete solid condensed spectra. We let

$$\mathbb{D}^{(2)} \coloneqq \underline{\mathrm{Hom}}_{\mathrm{Gr}(\mathrm{Sp}_\blacksquare)}\big(\mathrm{Bar}^{(2)}(-), \mathbb{S}\big) \colon \mathrm{Alg}_{\mathbb{E}_2}\big(\mathrm{Gr}(\mathrm{Sp}_\blacksquare)\big) \longrightarrow \mathrm{Alg}_{\mathbb{E}_2}\big(\mathrm{Gr}(\mathrm{Sp}_\blacksquare)\big)$$

denote the $\mathbb{E}_2$-Koszul duality functor.

Let us first compute $D \coloneqq \mathbb{D}^{(2)}(\mathbb{S}[x_i \mid i \in I])$. A standard computation shows that the double Bar construction $\mathrm{Bar}^{(2)}(\mathbb{S}[x_i])$ is given by $\bigoplus_{n\geqslant 0} \Sigma^{2n}\mathbb{S}(n)$ as a graded spectrum. Thus, if $I_n \coloneqq \mathrm{Sym}^n I$ denotes the n^{th} symmetric power of I as a set, then

$$D \simeq \bigoplus_{n\geqslant 0} \Sigma^{-2n} \prod_{I_n} \mathbb{S}(-n)\,.$$

If $D_{\geqslant -n}$ denotes the restriction of D to graded degrees $\geqslant -n$, then D is the limit of the tower of square-zero extensions $\cdots \to D_{\geqslant -2} \to D_{\geqslant -1} \to D_{\geqslant 0}$. For all $n \geqslant 1$, the square-zero extension $D_{\geqslant -n} \to D_{\geqslant -(n-1)}$ is determined by a pullback diagram

$$\begin{array}{ccc} D_{\geqslant -n} & \longrightarrow & \mathbb{S} \\ \downarrow & \lrcorner & \downarrow \\ D_{\geqslant -(n-1)} & \longrightarrow & \mathbb{S} \oplus \Sigma^{-2n+1} \prod_{I_n} \mathbb{S}(-n) \end{array}$$

After applying the Koszul duality functor, this becomes a pushout diagram

$$\begin{array}{ccc} \mathrm{Free}_{\mathbb{E}_2}\Big(\Sigma^{2n+1} \bigoplus_{I_n} \mathbb{S}(n)\Big) & \longrightarrow & \mathbb{D}^{(2)}\big(D_{\geqslant -(n-1)}\big) \\ \downarrow & \ulcorner & \downarrow \\ \mathbb{S} & \longrightarrow & \mathbb{D}^{(2)}\big(D_{\geqslant -n}\big) \end{array}$$

Here we use $\mathrm{Hom}_{\mathrm{Sp}_\blacksquare}(\prod_{I_n} \mathbb{S}, \mathbb{S}) \simeq \bigoplus_{I_n} \mathbb{S}$; this is the advantage of working in solid condensed spectra. Taking the colimit, we see that $\mathbb{D}^{(2)}(D)$ has an $\mathbb{E}_2$-cell decomposition with cells in even homotopical degrees. Once again using that we're working in the solid condensed world, we find $\mathbb{D}^{(2)}(D) \simeq \mathbb{S}[x_i \mid i \in I]$ and so we're done. ☐

Appendix D. On the equivariant Snaith theorem

For abelian compact Lie groups, Spitzweck and Østvær [SØ10] show a genuine equivariant form of Snaith's theorem. However, the equivalence they construct is only one of homotopy ring spectra. In this short appendix, we explain how to make their equivalence $\mathbb{E}_\infty$-algebras. We'll restrict to S^1 for simplicity, but the argument would work for any abelian compact Lie group.

D.1. Construction. — In [Sch18, (2.3.20)] Schwede introduces an orthogonal space $\mathbb{P}^{\mathbb{C}}$ that sends an inner product space V to the infinite projective space $\mathbb{P}(\operatorname{Sym}^*_{\mathbb{C}} V_{\mathbb{C}})$. We can construct a morphism of orthogonal spaces

$$c\colon \mathbb{P}^{\mathbb{C}} \longrightarrow \Omega^\bullet \mathrm{ku}_{\mathrm{gl}}$$

using a similar construction as in [Sch18, Construction 6.3.24]: Namely, for any inner product space V, the required map $c(V)\colon \mathbb{P}(\operatorname{Sym}^*_{\mathbb{C}} V_{\mathbb{C}}) \to \operatorname{Map}_*(S^V, \mathrm{ku}_{\mathrm{gl}}(V))$ is adjoint to the tautological map $\mathbb{P}(\operatorname{Sym}^*_{\mathbb{C}} V_{\mathbb{C}}) \wedge S^V \to \mathrm{ku}_{\mathrm{gl}}(V)$ that sends $(L, v) \mapsto [L; v]$ for any line $L \subseteq \operatorname{Sym}^*_{\mathbb{C}} V_{\mathbb{C}}$ and any point $v \in S^V$.

Schwede equips $\mathbb{P}^{\mathbb{C}}$ with an ultracommutative monoid structure by sending a pair of lines $(L_1 \subseteq \operatorname{Sym}^*_{\mathbb{C}} V_{\mathbb{C}}, L_2 \subseteq \operatorname{Sym}^*_{\mathbb{C}} W_{\mathbb{C}})$ to $L_1 \otimes_{\mathbb{C}} L_2 \subseteq \operatorname{Sym}^*_{\mathbb{C}} V_{\mathbb{C}} \otimes_{\mathbb{C}} \operatorname{Sym}^*_{\mathbb{C}} W_{\mathbb{C}} \cong \operatorname{Sym}^*_{\mathbb{C}}(V \oplus W)_{\mathbb{C}}$. It's clear from the construction that c is multiplicative. Thus, by adjunction, it induces a map of ultracommutative global ring spectra

$$\mathbb{S}_{\mathrm{gl}}[\mathbb{P}^{\mathbb{C}}] \longrightarrow \mathrm{ku}_{\mathrm{gl}}\,.$$

Before we continue, let us deduce that the element $q \in \pi_0(\mathrm{ku}^{S^1})$ is *strict*.

D.2. Corollary. — *Let $q \in \pi_0(\mathrm{ku}^{S^1})$ be the image of the standard representation of S^1 under $\mathrm{RU}(S^1) \to \pi_0(\mathrm{ku}^{S^1})$. Then q is detected by an $\mathbb{E}_\infty$-algebra map*

$$\mathbb{S}_{S^1}[q] \longrightarrow \mathrm{ku}_{S^1}$$

in Sp_{S^1}. In particular, q is a strict element in $(\mathrm{ku}^{C_m})^{h(S^1/C_m)}$ for all m.

Proof. By [Sch18, Proposition 4.1.8] (plus a simple argument to get rid of the telescope), the restriction of $\mathbb{S}_{\mathrm{gl}}[\mathbb{P}^{\mathbb{C}}]$ to a genuine S^1-equivariant ring spectrum is given by $\mathbb{S}_{S^1}[\mathbb{P}^{\mathbb{C}}]$, where $\mathcal{U}$ is any complete complex S^1-universe, that is, a direct sum of countably many copies of each irreducible complex S^1-representation. Choosing any copy of the standard representation q inside $\mathcal{U}$, we get a $\mathbb{C}$-algebra map $\mathbb{C} \oplus q \oplus q^2 \oplus \dotsb \to \operatorname{Sym}^* \mathcal{U}$, which induces an S^1-equivariant monoid map $\{1, q, q^2, \dots\} \simeq \mathbb{P}(\mathbb{C}) \sqcup \mathbb{P}(q) \sqcup \mathbb{P}(q^2) \sqcup \dotsb \to \mathbb{P}(\operatorname{Sym}^* \mathcal{U})$ and thus the desired map of $\mathbb{E}_\infty$-algebras in Sp_{S^1}

$$\mathbb{S}_{S^1}[q] \longrightarrow \mathbb{S}_{S^1}[\mathbb{P}^{\mathbb{C}}] \longrightarrow \mathrm{ku}_{S^1}\,. \qquad \square$$

D.3. The Bott element. — Let $\mathcal{U}$ be a complete complex S^1-universe as in the proof above. Let ε denote any copy of the trivial representation inside $\mathcal{U}$. The inclusion $\mathbb{C} \oplus \varepsilon \subseteq \operatorname{Sym}^*_{\mathbb{C}} \mathcal{U}$, where $\mathbb{C}$ denotes the unit component of the symmetric algebra, defines a map of genuine S^1-equivariant spectra $\xi\colon \mathbb{S}_{S^1}[\mathbb{P}(\mathbb{C} \oplus \varepsilon)] \to \mathbb{S}_{S^1}[\mathbb{P}(\operatorname{Sym}^*_{\mathbb{C}} \mathcal{U})]$. When we restrict to $\mathbb{S}_{S^1} \simeq \mathbb{S}_{S^1}[\mathbb{P}(\mathbb{C})]$ in source and target, ξ is canonically the identity, and so we can construct the *Bott map* as the factorisation

$$\begin{array}{ccc} \mathbb{S}_{S^1} \oplus \Sigma^2 \mathbb{S}_{S^1} & \longrightarrow & \Sigma^2 \mathbb{S}_{S^1} \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\beta} \\ \mathbb{S}_{S^1}\big[\mathbb{P}(\mathbb{C} \oplus \varepsilon)\big] & \xrightarrow{1-\xi} & \mathbb{S}_{S^1}\big[\mathbb{P}(\operatorname{Sym}^*_{\mathbb{C}} \mathcal{U})\big] \end{array}$$

It's clear from the construction that the $\mathbb{E}_\infty$-map $\mathbb{S}_{S^1}[\mathbb{P}(\mathrm{Sym}^*_{\mathbb{C}}\mathcal{U})] \to \mathrm{ku}_{S^1}$, that was constructed in the proof of Corollary D.2, sends $\beta \mapsto \beta$.

We also note that if ε' is another copy of the trivial representation inside $\mathcal{U}$, then the map $\mathbb{S}_{S^1}[\mathbb{P}(\varepsilon \oplus \varepsilon')] \to \mathbb{S}_{S^1}[\mathbb{P}(\mathrm{Sym}^*_{\mathbb{C}}\mathcal{U})]$ is homotopic to β. Indeed, already the inclusions of $\mathbb{P}(\mathbb{C} \oplus \varepsilon)$ and $\mathbb{P}(\varepsilon \oplus \varepsilon')$ into $\mathbb{P}(\mathbb{C} \oplus \varepsilon \oplus \varepsilon')$ are S^1-equivariantly homotopic. It follows that β already factors through the map $\mathbb{S}_{S^1}[\mathbb{P}(\mathcal{U})] \to \mathbb{S}_{S^1}[\mathbb{P}(\mathrm{Sym}^*_{\mathbb{C}}\mathcal{U})]$ induced by $\mathcal{U} \cong \mathrm{Sym}^1_{\mathbb{C}}\mathcal{U} \subseteq \mathrm{Sym}^*_{\mathbb{C}}\mathcal{U}$. Finally, recall that Spitzweck and Østvær construct a homotopy ring spectrum structure on $\mathbb{S}_{S^1}[\mathbb{P}(\mathcal{U})]$, so that we can consider the localisation $\mathbb{S}_{S^1}[\mathbb{P}(\mathcal{U})][\beta^{-1}]$.

D.4. Lemma. — *The induced map of $\mathbb{E}_\infty$-algebras in Sp_{S^1}*

$$\mathbb{S}_{S^1}\big[\mathbb{P}(\mathrm{Sym}^*_{\mathbb{C}}\mathcal{U})\big][\beta^{-1}] \xrightarrow{\simeq} \mathrm{KU}_{S^1}$$

is an equivalence. Moreover, its precomposition with $\mathbb{S}_{S^1}[\mathbb{P}(\mathcal{U})][\beta^{-1}] \to \mathbb{S}_{S^1}[\mathbb{P}(\mathrm{Sym}^*_{\mathbb{C}}\mathcal{U})][\beta^{-1}]$ is the equivalence constructed in [SØ10].

Proof. Since $\mathbb{S}_{S^1}[\mathbb{P}(\mathcal{U})] \to \mathbb{S}_{S^1}[\mathbb{P}(\mathrm{Sym}^*_{\mathbb{C}}\mathcal{U})]$ is an equivalence as both $\mathcal{U}$ and $\mathrm{Sym}^*_{\mathbb{C}}\mathcal{U}$ are complete complex S^1-universes, it will be enough to show the second statement.

To this end, let $\mathrm{Gr}^{\mathbb{C}}$ be the orthogonal space from [Sch18, Example 2.3.16] that sends an inner product space V to $\coprod_{i \geqslant 0} \mathrm{Gr}_i^{\mathbb{C}}(V_{\mathbb{C}})$, where $\mathrm{Gr}_i^{\mathbb{C}}$ denotes the Grassmannian of i-dimensional complex subspaces. Let $\mathrm{Gr}_1^{\mathbb{C}} \to \mathrm{Gr}^{\mathbb{C}}$ be the component where $i = 1$. Using [Sch18, Proposition 4.1.8] (plus a simple argument to get rid of the telescope), we see that $\mathbb{S}_{S^1}[\mathbb{P}(\mathcal{U})]$ is the restriction of the global spectrum $\mathbb{S}_{\mathrm{gl}}[\mathrm{Gr}_1^{\mathbb{C}}]$ to a genuine S^1-equivariant spectrum. By unravelling the proof of Corollary D.2, we immediately see that the diagram

$$\begin{array}{ccc} \mathbb{S}_{\mathrm{gl}}[\mathrm{Gr}_1^{\mathbb{C}}] & \longrightarrow & \mathbb{S}_{\mathrm{gl}}[\mathbb{P}^{\mathbb{C}}] \\ \downarrow & & \downarrow \\ \mathbb{S}_{\mathrm{gl}}[\mathrm{Gr}^{\mathbb{C}}] & \longrightarrow & \mathrm{ku}_{\mathrm{gl}} \end{array}$$

commutes, where the bottom map is the adjoint of [Sch18, Construction 6.3.24]. By another straightforward unravelling, the composition $\mathbb{S}_{\mathrm{gl}}[\mathrm{Gr}_1^{\mathbb{C}}] \to \mathbb{S}_{\mathrm{gl}}[\mathrm{Gr}^{\mathbb{C}}] \to \mathrm{ku}_{\mathrm{gl}}$ restricts to the map $\mathbb{S}_{S^1}[\mathbb{P}(\mathcal{U})] \to \mathrm{ku}_{S^1}$ constructed in [SØ10]. □

D.5. Equivariant Adams operations — Let ρ_n denote the n^{th} power map $(-)^n\colon S^1 \to S^1$. Writing the monoid operation multiplicatively, we also consider the monoid endomorphism $(-)^n\colon \mathbb{P}(\mathrm{Sym}^*_{\mathbb{C}}\mathcal{U}) \to \mathbb{P}(\mathrm{Sym}^*_{\mathbb{C}}\mathcal{U})$. This is equivariant over ρ_n and therefore induces an endomorphism

$$\psi^n\colon \rho_n^*\mathbb{S}_{S^1}\big[\mathbb{P}(\mathrm{Sym}^*_{\mathbb{C}}\mathcal{U})\big] \longrightarrow \mathbb{S}_{S^1}\big[\mathbb{P}(\mathrm{Sym}^*_{\mathbb{C}}\mathcal{U})\big]$$

of $\mathbb{E}_\infty$-algebras in S^1-equivariant spectra. Clearly $\psi^n(q) = q^n$. Moreover, $\psi^n(\beta) = n\beta$ holds S^1-equivariantly. Indeed, to see this, let $\mathcal{U}_{\mathrm{triv}} \subseteq \mathcal{U}$ be the direct summand consisting of all copies of the trivial S^1-representation. Then the usual non-equivariant argument can be applied to $\mathbb{S}_{S^1}[\mathbb{P}(\mathrm{Sym}^*_{\mathbb{C}}\mathcal{U}_{\mathrm{triv}})]$. Inverting β and passing to connected covers, we obtain maps

$$\psi^n\colon \mathrm{KU}_{S^1} \longrightarrow \mathrm{KU}_{S^1}\big[\tfrac{1}{n}\big] \quad \text{and} \quad \psi^n\colon \mathrm{ku}_{S^1} \longrightarrow \mathrm{ku}_{S^1}\big[\tfrac{1}{n}\big]$$

of $\mathbb{E}_\infty$-algebras in Sp_{S^1}. Here we also use $\rho_n^*\mathrm{ku}_{S^1} \simeq \mathrm{ku}_{S^1}$ and $\rho_n^*\mathrm{ku}_{S^1} \simeq \mathrm{ku}_{S^1}$, since we've modelled ku by an ultracommutative global ring spectrum $\mathrm{ku}_{\mathrm{gl}}$, where everything acts trivially.

References

[ABM23] Christian Ausoni, Haldun Özgür Bayındır, and Tasos Moulinos, *Adjunction of roots, algebraic K-theory and chromatic redshift*, 2023. arXiv: 2211.16929 [math.AT].

[AM24] Johannes Anschütz and Lucas Mann, *Descent for solid quasi-coherent sheaves on perfectoid spaces*, 2024. arXiv: 2403.01951 [math.AG].

[AMR17] David Ayala, Aaron Mazel-Gee, and Nick Rozenblyum, *A naive approach to genuine G-spectra and cyclotomic spectra*, 2017. arXiv: 1710.06416 [math.AT].

[Ant19] Benjamin Antieau, *Periodic cyclic homology and derived de Rham cohomology*, Annals of K-Theory, vol. 4, no. 3, pp. 505–519, 2019.

[Aom90] Kazuhiko Aomoto, *q-analogue of de Rham cohomology associated with Jackson integrals. I*, Japan Academy. Proceedings. Series A. Mathematical Sciences, vol. 66, no. 7, pp. 161–164, 1990.

[AR24] Benjamin Antieau and Noah Riggenbach, *Cyclotomic synthetic spectra*, 2024. arXiv: 2411.19929 [math.KT].

[AS69] Michael F. Atiyah and Graeme B. Segal, *Equivariant K-theory and completion*, Journal of Differential Geometry, vol. 3, pp. 1–18, 1969.

[BCM20] Bhargav Bhatt, Dustin Clausen, and Akhil Mathew, *Remarks on $K(1)$-local K-theory*, Selecta Mathematica. New Series, vol. 26, no. 3, pp. 1–16 (paper no. 39), 2020.

[BG16] Clark Barwick and Saul Glasman, *Cyclonic spectra, cyclotomic spectra, and a conjecture of Kaledin*, 2016. arXiv: 1602.02163 [math.AT].

[BGN18] Clark Barwick, Saul Glasman, and Denis Nardin, *Dualizing cartesian and cocartesian fibrations*, Theory and Applications of Categories, vol. 33, pp. 67–94 (paper no. 4), 2018.

[BGT16] Andrew J. Blumberg, David Gepner, and Gonçalo Tabuada, *K-theory of endomorphisms via noncommutative motives*, Transactions of the American Mathematical Society, vol. 368, no. 2, pp. 1435–1465, 2016.

[BL22a] Bhargav Bhatt and Jacob Lurie, *Absolute prismatic cohomology*, 2022. arXiv: 2201.06120 [math.AG].

[BL22b] ———, *The prismatization of p-adic formal schemes*, 2022. arXiv: 2201.06124 [math.AG].

[BMS18] Bhargav Bhatt, Matthew Morrow, and Peter Scholze, *Integral p-adic Hodge theory*, Publications Mathématiques de l'IHÉS, vol. 128, pp. 219–397, 2018.

[BMS19] ———, *Topological Hochschild homology and integral p-adic Hodge theory*, Publications Mathématiques. Institut de Hautes Études Scientifiques, vol. 129, pp. 199–310, 2019.

[Bos23] Guido Bosco, *Rational p-adic Hodge theory for rigid-analytic varieties*, 2023. arXiv: 2306.06100 [math.AG].

[BS19] Bhargav Bhatt and Peter Scholze, *Prisms and prismatic cohomology*, 2019. arXiv: 1905.08229 [math.AG].

[Bur22] Robert Burklund, *Multiplicative structures on Moore spectra*, 2022. arXiv: 2203.14787 [math.AT].

[Cla94] F.J.-B.J. Clauwens, *Commuting polynomials and λ-ring structures on $\mathbb{Z}[x]$*, Journal of Pure and Applied Algebra, vol. 95, no. 3, pp. 261–269, 1994.

[Cos87] Steven R. Costenoble, *The equivariant Conner-Floyd isomorphism*, Transactions of the American Mathematical Society, vol. 304, no. 2, pp. 801–818, 1987.

[CS19] Dustin Clausen and Peter Scholze, *Condensed Mathematics*, lecture course held at University of Bonn, 2019. Available at https://people.mpim-bonn.mpg.de/scholze/Condensed.pdf.

[CS22] ———, *Condensed Mathematics and Complex Geometry*, lecture course held at IHÉS/University of Bonn, 2022. Lecture notes available at people.mpim-bonn.mpg.de/scholze/Complex.pdf.

[CS24] ———, *Analytic Stacks*, lecture course held at IHÉS/MPIM Bonn, Winter term 2023/24. Recordings available at https://www.youtube.com/playlist?list=PLx5f8IelFRgGmu6gmL-Kf_R1_6Mm7juZO.

[Dev25] Sanath K. Devalapurkar, *Spherochromatism in representation theory and arithmetic geometry*, Ph.D. Thesis, Harvard University, 2025. Available at https://sanathdevalapurkar.github.io/files/thesis.pdf.

[DHL+23] Sanath K. Devalapurkar *et al.*, *Examples of disk algebras*, 2023. arXiv: 2302.11702 [math.AT].

[DHRY] Sanath K. Devalapurkar *et al.*, *Prismatization of commutative ring spectra*, in preparation.

[DM23] Sanath K. Devalapurkar and Max L. Misterka, *Generalized n-series and de Rham complexes*, 2023. arXiv: 2304.04739 [math.CO].

[DR25] Sanath K. Devalapurkar and Arpon Raksit, $\mathrm{THH}(\mathbb{Z})$ *and the image of J*, 2025. arXiv: 2505.02218 [math.AT].

[Dri24] Vladimir Drinfeld, *Prismatization*, Selecta Mathematica. New Series, vol. 30, no. 3, pp. 1–150 (paper no. 49), 2024.

[Efi-Rig] Alexander I. Efimov, *Rigidity of the category of localizing motives*, in preparation.

[Efi24] ———, *Dualizable categories and localizing motives*, Masterclass at University of Copenhagen, 2024. Notes by Claudius Heyer available at https://www.math.ku.dk/english/calendar/events/masterclass-continuous-k-theory/Efimov1.pdf.

[Efi25] ———, *Localizing invariants of inverse limits*, 2025. arXiv: 2502.04123 [math.KT].

[GM23] David Gepner and Lennart Meier, *On equivariant topological modular forms*, Compositio Mathematica, vol. 159, no. 12, pp. 2638–2693, 2023.

[GR17] Dennis Gaitsgory and Nick Rozenblyum, *A study in derived algebraic geometry, Vol. I: Correspondences and duality*, Mathematical Surveys and Monographs, vol. 221, pp. xl+533. American Mathematical Society, Providence, RI, 2017.

[GSWZ24] Stavros Garoufalidis *et al.*, *The Habiro ring of a number field*, 2024. arXiv: 2412.04241 [math.NT].

[Gul21] Daniel Gulotta (user dgulotta), *Vanishing of Kahler differentials vs. surjective Frobenius?* MathOverflow post (version: 2021-08-11). Available at https://mathoverflow.net/q/401491.

[GW25] Stavros Garoufalidis and Campbell Wheeler, *Explicit classes in habiro cohomology*, 2025. arXiv: 2505.19885 [math.AG].

[GZ23] Stavros Garoufalidis and Don Zagier, *Knots and their related q-series*, SIGMA. Symmetry, Integrability and Geometry. Methods and Applications, vol. 19, pp. 1–39 (paper no. 082), 2023.

[GZ24] ———, *Knots, perturbative series and quantum modularity*, SIGMA. Symmetry, Integrability and Geometry. Methods and Applications, vol. 20, pp. 1–87 (paper no. 055), 2024.

[Hab02] Kazuo Habiro, *On the quantum $\mathfrak{sl}_2$ invariants of knots and integral homology spheres*, Invariants of knots and 3-manifolds, Geometry & Topology Monographs, vol. 4, pp. 55–68, Geometry & Topology Publications, Coventry, 2002.

[Hab04] ———, *Cyclotomic completions of polynomial rings*, Kyoto University. Research Institute for Mathematical Sciences. Publications, vol. 40, no. 4, pp. 1127–1146, 2004.

[Hau24] Markus Hausmann, *Equivariant homotopy at east 2024*, extended lecture notes, 2024. Available at https://www.math.ru.nl/~sagave/east2024/EAST_Equivariant_homotopy.pdf.

[Hei23] Hadrian Heine, *An equivalence between enriched ∞-categories and ∞-categories with weak action*, Advances in Mathematics, vol. 417, paper no. 108941, 2023.

[Hei25] ———, *An equivalence between two models of ∞-categories of enriched presheaves*, Applied Categorical Structures, vol. 33, no. 1, paper no. 2, 2025.

[Hin20] Vladimir Hinich, *Yoneda lemma for enriched ∞-categories*, Advances in Mathematics, vol. 367, paper no. 107129, 2020.

[HKR62] Gerhard Hochschild, Bertram Kostant, and Alex Rosenberg, *Differential forms on regular affine algebras*, Transactions of the American Mathematical Society, vol. 102, pp. 383–408, 1962.

[HM97] Lars Hesselholt and Ib Madsen, *On the K-theory of finite algebras over Witt vectors of perfect fields*, Topology. An International Journal of Mathematics, vol. 36, no. 1, pp. 29–101, 1997.

[HN20] Lars Hesselholt and Thomas Nikolaus, *Topological cyclic homology*, Handbook of homotopy theory, CRC Press/Chapman Hall Handb. Math. Ser., CRC Press, Boca Raton, FL, 2020, pp. 619–656.

[HRW22] Jeremy Hahn, Arpon Raksit, and Dylan Wilson, *A motivic filtration on the topological cyclic homology of commutative ring spectra*, 2022. arXiv: 2206.11208 [math.KT].

[Hub94] Roland Huber, *A generalization of formal schemes and rigid analytic varieties*, Mathematische Zeitschrift, vol. 217, no. 4, pp. 513–551, 1994.

[HW18] Jeremy Hahn and Dylan Wilson, *Quotients of even rings*, 2018. arXiv: 1809.04723 [math.AT].

[HW22] ———, *Redshift and multiplication for truncated Brown–Peterson spectra*, Annals of Mathematics. Second Series, vol. 196, no. 3, pp. 1277–1351, 2022.

[Jac10] Frank H. Jackson, *q-Difference Equations*, American Journal of Mathematics, vol. 32, no. 4, pp. 305–314, 1910.

[Ked21] Kiran S. Kedlaya, *Prismatic Cohomology*, graduate topics course held at UC San Diego, 2021. Lecture notes available at https://kskedlaya.org/prismatic.

[KN17] Achim Krause and Thomas Nikolaus, *Lectures on topological Hochschild homology and cyclotomic spectra*, lecture notes, 2017. Available at https://www.uni-muenster.de/IVV5WS/WebHop/user/nikolaus/Papers/Lectures.pdf.

[L-Ch] Jacob Lurie, *Chromatic homotopy theory* (*Math 252x*), lecture course held at Harvard University, spring 2010. Notes available at https://www.math.ias.edu/~lurie/252x.html.

[L-Ell$_{\mathrm{II}}$] ———, *Elliptic Cohomology II: Orientations*, 2018. Available at https://www.math.ias.edu/~lurie/papers/Elliptic-II.pdf.

[L-HA] ———, *Higher algebra*, 2017. Available at http://people.math.harvard.edu/~lurie/papers/HA.pdf.

[L-HTT] ———, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, pp. xviii+925. Princeton University Press, Princeton, NJ, 2009.

[Lod92] Jean-Louis Loday, *Cyclic homology*, Grundlehren der mathematischen Wissenschaften, vol. 301, pp. xviii+454. Springer-Verlag, Berlin, 1992, Appendix E by María O. Ronco.

[LZ04] Andreas Langer and Thomas Zink, *De Rham-Witt cohomology for a proper and smooth morphism*, Journal de l'Institut de Mathématiques de Jussieu, vol. 3, no. 2, pp. 231–314, 2004.

[Man10] Yuri I. Manin, *Cyclotomy and analytic geometry over* $\mathbb{F}_1$, Quanta of maths, Clay Mathematical Proceedings, vol. 11, American Mathematical Society, Providence, RI, 2010, pp. 385–408.

[Man16] ———, *Local zeta factors and geometries under* $\operatorname{Spec}\mathbb{Z}$, Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya, vol. 80, no. 4, pp. 123–130, 2016.

[MM24] Akhil Mathew and Shubhodip Mondal, *Affine stacks and derived rings*, 2024. Available at https://personal.math.ubc.ca/~smondal/papers/affinestacks.pdf.

[MW24] Samuel Meyer and Ferdinand Wagner, *Derived q-Hodge complexes and refined* TC^-, 2024. arXiv: 2410.23115 [math.AT].

[NS18] Thomas Nikolaus and Peter Scholze, *On topological cyclic homology*, Acta Mathematica, vol. 221, no. 2, pp. 203–409, 2018.

[PR] Piotr Pstrągowski and Arpon Raksit, *Prismatic cohomology of $\mathbb{E}_2$-ring spectra*, in preparation.

[Pri19] Jonathan P. Pridham, *On q-de Rham cohomology via Λ-rings*, Mathematische Annalen, vol. 375, no. 1-2, pp. 425–452, 2019.

[Pst23] Piotr Pstrągowski, *Perfect even modules and the even filtration*, 2023. arXiv: 2304.04685 [math.AT].

[PV23] Alexander Petrov and Vadim Vologodsky, *On the periodic topological cyclic homology of DG categories in characteristic p*, 2023. arXiv: 1912.03246 [math.AG].

[Rak21] Arpon Raksit, *The Filtered Circle*. ProQuest LLC, Ann Arbor, MI, 2021, Ph.D. Thesis, Stanford University.

[Ram] Maxime Ramzi, *Fun with pushouts*. Available at https://sites.google.com/view/maxime-ramzi-en/notes/fun-with-pushouts.

[Ram24] ———, *Locally rigid ∞-categories*, 2024. arXiv: 2410.21524 [math.CT].

[RC24a] Juan Esteban Rodríguez Camargo, *Seminar on solid geometry*, Columbia University, 2024. Notes available at https://sites.google.com/view/jerodriguezcamargo/courses-and-seminars.

[RC24b] ———, *The analytic de Rham stack in rigid geometry*, 2024. arXiv: 2401.07738 [math.AG].

[RS22] Shaul Ragimov and Tomer Schlank, *The ∞-categorical reflection theorem and applications*, 2022. arXiv: 2207.09244 [math.AT].

[Sch17] Peter Scholze, *Canonical q-deformations in arithmetic geometry*, Annales de la Faculté des Sciences de Toulouse Série 6, vol. 26, no. 5, pp. 1163–1192, 2017.

[Sch18] Stefan Schwede, *Global homotopy theory*, New Mathematical Monographs, vol. 34, xviii +828. Cambridge University Press, Cambridge, 2018.

[Sch24a] Peter Scholze, *Talk 1: Refined* TC^- *over perfectoid rings*, Workshop on Dualisable Categories & Continuous K-Theory, 2024. Recording available at https://archive.mpim-bonn.mpg.de/id/eprint/5092/.

[Sch24b] ———, *The Habiro ring of a number field*, lecture course held at MPIM Bonn, Winter term 2024/25. Recordings available at https://archive.mpim-bonn.mpg.de/id/eprint/5132/.

[Sch25] ———, *Habiro Cohomology*, lecture course held at MPIM Bonn, Summer term 2025. Recordings available at https://archive.mpim-bonn.mpg.de/id/eprint/5155/.

[Sim96] Carlos Simpson, *Homotopy over the complex numbers and generalized de Rham cohomology*, Moduli of vector bundles (Sanda, 1994; Kyoto, 1994), Lecture Notes in Pure and Appl. Math., vol. 179, Dekker, New York, 1996, pp. 229–263.

[Sin01] Dev P. Sinha, *Computations of complex equivariant bordism rings*, American Journal of Mathematics, vol. 123, no. 4, pp. 577–605, 2001.

[SØ10] Markus Spitzweck and Paul Arne Østvær, *A Bott inverted model for equivariant unitary topological K-theory*, Mathematica Scandinavica, vol. 106, no. 2, pp. 211–216, 2010.

[Stacks] The Stacks Project Authors, *The Stacks Project*, 2021. Available at http://stacks.math.columbia.edu.

[Wag24] Ferdinand Wagner, *q-Witt vectors and q-Hodge complexes*, 2024. arXiv: 2410.23078 [math.NT].