Essays in Bargaining, Auctions and Payments

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To my friends, with love

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Contents

List of Figures

List of Tables

Introduction

This dissertation explores strategic interactions in economic models that involve bargaining, auctions, and payment systems, using a game-theoretic approach. Each chapter focuses on a different context where asymmetric information, strategic behavior, and equilibrium dynamics play a crucial role in shaping outcomes.

Chapter 1 investigates the role of second-order beliefs in a reputational bargaining model involving two agents, A and B. Both agents can be either irrational (refusing to concede and sticking to their initial offer) or rational. B can take one of two rational forms: omniscient, who is certain of A's rationality, or ignorant, who is uncertain. In typical reputational bargaining, agents make an offer at the beginning and adhere to it throughout the negotiation. However, we allow B to propose a 'fair' 50-50 split of the surplus, which reveals B's rationality and could serve as a potential signal for the omniscient type. Using a hybrid discrete-continuous time framework proposed by Abreu and Pearce (2007), we examine how reputation effects can arise even when one agent (omniscient) is fully aware of the other's true nature and decides whether to reveal or withhold this information. Our analysis reveals multiple equilibria, including scenarios where no fair offers are made, as rational players strategically avoid disclosing their rationality to preserve their advantage. If B's irrational demand exceeds a fair division of the surplus, this scenario is the unique equilibrium. Conversely, when the demand is less than 50%, an equilibrium with a fair offer can occur. Every equilibrium of this type is characterized by a period t in which the fair deal is offered with positive probability exclusively at t.

In Chapter 2 we analyze a model of competing sealed-bid first-price and second-price auctions where bidders have unit demand and can bid on multiple auctions simultaneously. We show that there is no symmetric pure equilibrium with strategies that are increasing in the lowest type, unlike in standard auction games. However, for a two-player game a symmetric mixed-strategy equilibrium exists, and bidders place bids on all available auctions with probability one. This holds true for any mixed equilibrium and for any number of bidders. We then solve the case of two auctions and two bidders. Analyzing the case of binary type space, we are able to identify mixed strategy equilibria and analyze the consequences of discrete bid spaces.

Chapter 3 investigates the acceptance and usage of card payments, as well as the transactions' demand for cash through a game-theoretic model. Buyers can search for shops that accept cards, creating competition among merchants: even though card payments are costly because of fees, sellers might be willing to accept them to attract more customers. The economy features *no-acceptance* and *full-acceptance equilibria*, as well as *imperfect acceptance* ones that resemble the prevailing situation in most countries. After studying the existence, uniqueness, and stability properties of our equilibria, we analyze how the equilibrium responds to changes in search frictions, consumers' tastes, and the opportunity cost of holding cash. We bring the model to the data by solving an augmented version of the problem which features a dynamic cash management problem for buyers, and we calibrate it using data from ECB payment diaries. We use our calibrated model to compare the partial and general equilibrium implications of a policy that makes card payments cheaper for buyers, showing that when taking into account the optimal response by sellers, such policies may generate unintended consequences and lower card usage in equilibrium.

References

Abreu, D., and D. Pearce. 2007. "Bargaining, Reputation, and Equilibrium Selection in Repeated Games with Contracts." *Econometrica* 75: 653–710. [1]

Chapter 1

Reputational Bargaining with an Omniscient Type

1.1 Introduction

When two agents negotiate over the division of some surplus, they may each attempt to convince the other that they are a tough party to bargain with, refusing to accept anything less than their demands. In other words, they try to build a reputation for being firm. As noted by Milgrom and Roberts (1982), for reputation effects to emerge, it is not necessary for one agent to be uncertain about the nature of their opponent. In fact, it is enough for the opponent to believe that they are building a reputation.

Consider two agents, A (she) and B (he), negotiating over a surplus. Suppose that B is certain, based on information gathered prior to the negotiation, that A is not as tough as she appears. If A is unsure about the information B possesses or what he actually believes, she may still attempt to build a reputation by delaying the agreement. Moreover, would B attempt to convince A that he is omniscient, even if doing so would expose his rationality?

In this first chapter, we not only study the reputation effects that arise from second-order beliefs, but also examine the incentives for a player, who is certain of the other's rationality, to reveal his information. We analyze a bargaining game in a hybrid discrete-continuous time framework, as proposed in Abreu and Pearce (2007). In this model, two agents, A and B, make initial demands and stick to them until one concedes to the other. Both players can either be irrational (or stubborn)[1] or perfectly rational. An irrational player holds firm to their demand and never concedes, while a rational player can concede at any point. While A is described by just these two types, B has a more complex structure. In fact, if B is rational, his type is further determined by the information he possesses. The first type, called the ig-

1. We will also refer to this type as the behavioral type.

norant type, holds a nondegenerate prior belief about *A*'s true nature, meaning he is uncertain whether *A* is perfectly rational. The second type, the omniscient type, *knows* that *A* is rational. Importantly, *A* does not know whether *B* is irrational, rational ignorant, or rational omniscient. At the beginning of the game, agents *A* and *B* can offer to the other a behavioral contract such that their demands are incompatible. Both can concede to the other at any point in time, yet, at any discrete period, we allow *B* to propose a fair 50-50 split of the surplus. Such an offer would expose *B*'s rationality, but it could also signal to the rational type of *A* that *B* is omniscient and prepared to engage in a war of attrition to prove he has information about her rationality. In our model, we assume that once *A* believes with certainty that *B* is omniscient, she immediately concedes to the fair offer. This assumption allows us to avoid the complexities of a war of attrition between two fully rational agents negotiating over the surplus.

To analyze this game, we first examine the continuation game in which *B* has already revealed his rationality. This helps us understand the consequences of offering a fair deal during the bargaining process. We refer to this continuation game as the one behavioral type model, as only *A* can be irrational in this scenario. We find that there are two classes of equilibria. In the first, there is a continuum of degenerate equilibria (all outcome-equivalent), where *B* concedes immediately, regardless of his rational type. We interpret this as *B*'s belief (confirmed in equilibrium) that *A*'s rational type is so stubborn that any informational advantage is irrelevant, leading *A* to never concede. In the second class, there is a unique nondegenerate equilibrium where the omniscient type of *B* does not concede. Here, *A* attempts to convince *B* that she is a tough negotiator who will not back down, while *B* tries to convince *A* that he knows she is only pretending to be tough—that is, that he is omniscient.

Once we have understood the consequences of the fair offer, we return to the original game (referred to as the two behavioral types model). First, we find that an equilibrium without any fair offers is always possible, regardless of the parameters. This equilibrium can be sustained by having the agents play the degenerate equilibrium from the one behavioral type game. In this case, the omniscient type prefers not to reveal his rationality, as doing so would put him at a disadvantage, with *A* never conceding. Naturally, this discourages the ignorant type from proposing a fair deal as well. Moreover, when *B*'s irrational demand exceeds the fair offer (i.e., more than 50%), this is the unique equilibrium. In this scenario, the omniscient type is unwilling to reveal his information if it means proposing a deal that leaves him with less than the irrational demand. We demonstrate that this outcome holds even when the war of attrition following the fair offer is shorter. On the other hand, when *B*'s irrational offer is less than 50%, an equilibrium with a fair contract becomes possible. We prove that, in equilibrium, there is exactly one period in which this offer can be made with positive probability. During this period, the omniscient type proposes the fair contract with probability 1, while the ignorant type mixes between proposing the fair contract and sticking with the irrational demand. An example of an equilibrium for certain parameter values is provided at the end.

The rest of the paper is organized as follows. Section 1.2 presents the literature review. Sections 1.3 and 1.4 analyze the models with one and two behavioral types, respectively, where the main results of the paper and an equilibrium example are presented. Finally, Section 1.5 concludes. In the Appendix, we provide a brief summary of the key results from Abreu and Gul (2000), which we use in subsequent sections, a discussion of equilibrium consistency, and a model of the informational structure based on the Appendix of Milgrom and Roberts (1982) to illustrate the derivation of the type structure we employ. At the end of the Appendix, we include the proofs.

1.2 Literature review

The work on reputation started with Kreps and Wilson (1982) and Milgrom and Roberts (1982). Both papers analyze a model where an incumbent threatens potential entrants to start a price war so that new firms are discouraged from entering the market. Even if a price war is not immediately convenient to the incumbent, it is shown that building a reputation for being aggressive has long-term advantages. These concepts are then developed in the bargaining framework by Abreu and Gul (2000) (AG in the following sections). The authors study a model where two agents can assume at the beginning of the bargaining process one out of many different 'unfair' postures, that is, an infinite path of high demands. This model predicts that the game ends before some fixed time (and therefore the negotiation does not proceed forever) and that some party concedes to the unfair demand of the other. Many papers on reputational bargaining then followed. For example, Abreu and Pearce (2007) builds a similar model where players in each period can also take some action that affects the utility. Moreover, they allow behavioral non-stationary strategies. More recently, in Fanning (2016), the author explains how 'deadline effects' can be caused by reputational effects when the deadline is stochastic.

This work is closely related to two recent contributions. The first one is Wolitzky (2012). In this study, the author explores the effect of reputational bargaining when the agents *know* that the other is rational but may be committed to a certain behavioral posture with some small probability. The solution concept used in that work is *minmax*. In contrast, our work employs the classic *sequential equilibrium* solution.

The second paper is Zhao (2023). In this study, the author considers a model where the rational agent may have either an optimistic or pessimistic view regarding the other's rationality. An optimistic (rational) agent assigns a high probability to the other's rationality, while a pessimistic agent assigns a small probability to the same event. The game in that paper follows the structure of a classic war of attrition.

In our model, we allow the agent who may possess full information to signal their level of information, deviating from the model presented in Zhao (2023).

This paper also contributes to the literature on war of attrition with more than two actions. Two recent examples are Leeuwen, Offerman, and Ven (2020) and Hörner and Sahuguet (2011). The first paper study a war of attrition where the players have the possibility to fight. The players are uncertain about the strength of their opponent and the fight resolves the conflicts. This differs from our model as the additional action starts a new war of attrition. The second paper analyzes a war of attrition with alternate moves, and players make arbitrary payments. Their opponent can either match the payment or concede.

A recent contribution to this literature and the literature on reputational bargaining is Ekmekci and Zhang (2024), where the agents have the opportunity to end the conflict through an external resolution whose outcome depends on on the strength of their claims.

1.3 The one behavioral type model

We first analyze a game in which player B is commonly known to be rational. Therefore, A knows B is rational, B knows that A knows B is rational and so on. B can only offer the contract $\left(1/2, 1/2\right)$, that we will call 'fair' throughout the paper. Consequently, in this game, B can only be either ignorant (lacking knowledge of A's type) or omniscient (knowing that A is rational).

This game features two distinct classes of equilibria. In the first class, there is a continuum of equilibria where B concedes immediately, even if he is omniscient. In this scenario, A is perceived as too stubborn even when rational, and thus, B's knowledge of her rationality does not help him achieve a higher payoff. In the second class, there is a unique equilibrium characterized by a war of attrition between A and B. Here, A attempts to convince B that she is a behavioral type (without knowing if this is possible, as she is uncertain whether B is ignorant), while B tries to convince A that he is omniscient and therefore knows she is pretending to be irrational.

To simplify the analysis and avoid detailing the complexities of a war of attrition that occurs after A is convinced that B is omniscient, we assume that as soon as A is certain of B's omniscience, she accepts the fair deal of $\left(1/2, 1/2\right)$.

This simplified game is then used to address the larger game where B can also be irrational and has the choice to either imitate the behavioral type or reveal his rationality by offering $\left(1/2, 1/2\right)$, thereby attempting to convince A that he possesses information about her rationality. By using backward induction, we solve this game to understand the consequences of offering (or not offering) a fair contract in the middle of the standard war of attrition and determine whether B will ever choose to reveal his rationality.

1.3.1 The bargaining game

The bargaining protocol is defined in continuous time, on the interval $[0, +\infty)$. The agents have to split a surplus of 1. If they never reach an agreement, both get a payoff of 0. At $t = 0$, A and B make the offers $(a, 1-a)$ and $(1-b, b)$ respectively, where the first element of the vector is the quantity of the surplus for player A and the second element is the quantity for player B. If the offers are compatible, i.e., $a+b < 1$, the game ends, and an even randomization decides the contract to be implemented. When $a+b = 1$, this share of the split is enforced. If $a+b > 1$, instead, the game continues. Payoffs are then exponentially discounted according to the rate δ, which is symmetric across the agents. The behavioral type θ_b^A is restricted to the offer $(\gamma, 1-\gamma)$ where $\gamma > 1/2$, which is what we call the 'unfair' offer. On the other hand, since B is known to be rational, we restrict his strategy to either offering a 'fair' contract $(1/2, 1/2)$ or a split which is compatible with A's unfair offer. Without loss of generality, we can restrict the action space of B to $\{(1/2, 1/2), (\gamma, 1-\gamma)\}$. This structure resembles a classic war of attrition, where 'waiting' corresponds to $(1/2, 1/2)$ for player B and $(\gamma, 1-\gamma)$ for player A and 'conceding' corresponds to $(\gamma, 1-\gamma)$ for B and $(1/2, 1/2)$ for A. Whenever a player waits at some t and the other concedes, the game ends and each one gets the split assigned. Therefore, what determines the outcome of the game is the time t at which a player switches from waiting to conceding.

Player A can have type $\theta^A \in \{\theta_r^A, \theta_b^A\}$, where θ_r^A is referred to as the rational type, while θ_b^A is the behavioral type. The latter one can only offer the unfair contract $(\gamma, 1-\gamma)$ and never concedes to worse contract. Player B, on the other hand, can have type $\theta^B \in \{\theta_i^B, \theta_o^B\}$. We refer to the first one as the ignorant type and to the second one as the omniscient type. Throughout the game, the agents have beliefs about the type of the other player. Let $\mu_i^B(\tau)$ be the probability that type θ_i^B assigns to the event $\theta^A = \theta_b^A$, which is a function of the period that the game has reached. Observe $\mu_i^B(0) = z$. In the same fashion, let μ_r^A be the probability that θ_r^A assigns to the event $\theta^B = \theta_i^B$. Therefore, $\mu_r^A(0) = q$. These beliefs are updated through Bayes rule. Then, consider also the following assumption.

Assumption 1. *Whenever* $\mu_r^A(\tau) = 0$, θ_r^A *accepts the fair contract* $(1/2, 1/2)$ *at* τ.

This assumption makes sure that whenever we reach a complete information game (i.e., A's rationality becomes common knowledge), the players do not start a new war of attrition. In fact, we aim to capture the types' behavior *before* full rationality is revealed. Observe that θ_r^A may believe with probability 1 that $\theta^B = \theta_o^B$ even when $\theta^B = \theta_i^B$. In this case, we do not have a complete information game, yet we want to ignore all the strategic interactions that happen after θ_r^A is sure of the other's type. Whenever θ_r^A puts probability 1 to $\theta^B = \theta_o^B$, she gives up the behavioral posture and accepts B's fair split. This can be interpreted as a new continuation game in which there is common knowledge of rationality and both players have an expected payoff of (1/2, 1/2) from the bargain they are going to start. Even if $\theta^B = \theta_i^B$, as soon as A quits the irrational behavior, θ_i^B *knows* that A is rational.

Now, we provide the formal definition of strategy. In the spirit of Laraki, Solan, and Vieille (2005), we define strategies in the following way. Let $\Delta(\overline{\mathbb{R}}_+)$ be the set of probability measure over $\overline{\mathbb{R}}+$, the positive extended real numbers. We topologize it with the the topology of weak convergence.

Definition 1.1. A strategy is a function $\sigma : \mathbb{R}_+ \longrightarrow \Delta(\overline{\mathbb{R}}_+)$, that satisfies the following properties.

(*i*) Properness: σ_t assigns probability 1 to $[t, +\infty) \cup \{+\infty\}$;

(*ii*) Conditioning requirement: For all $t \in [0, \tau)$ and borel set $\mathscr{B} \subseteq [\tau, +\infty) \cup \{+\infty\}$, we have

$$\sigma_t(\mathscr{B}) = (1 - \sigma_t([t, \tau)))\sigma_\tau(\mathscr{B}).$$

The first property guarantees that σ_t is a concession strategy of the continuation game t. The second property says that the measure's distribution must be computed through Bayes rule when possible. Therefore, σ_t represent the plan of action that a type of player B wants to play in continuation game t.

We define the utilities of A and B for their rational types. Each utility function depends on the (potentially mixed) strategy of the other player and the time t at which the player decides to concede. We assume that when both concede at the same time, the contract implemented is randomized. Denote with σ_r^A, σ_i^B, and σ_o^B the strategies of type θ_r^A, θ_i^B, and θ_o^B respectively. Then, consider continuation game τ. When player A with type θ_r^A, concedes at $t \geq \tau$, has continuation utility[2]

$$u_r^A((\sigma_i^B, \sigma_o^B), t|\tau) = \mu_r^A(\tau) \cdot \left[\int_0^t \gamma e^{-\delta x} d\sigma_{i,\tau}^B(x) + \left(\sigma_{i,\tau}^B(t)\frac{1}{2}\left(\gamma + 1/2\right) + \sigma_{i,\tau}^B((t, +\infty])\frac{1}{2}\right)e^{-\delta t}\right] +$$
$$(1 - \mu_r^A(\tau)) \cdot \left[\int_0^t \gamma e^{-\delta x} d\sigma_{o,\tau}^B(x) + \left(\sigma_{o,\tau}^B(t)\frac{1}{2}\left(\gamma + 1/2\right) + \sigma_{o,\tau}^B((t, +\infty])\frac{1}{2}\right)e^{-\delta t}\right],$$

That is, with probability $\mu_r^A(\tau)$ player B is of type θ_i^B and therefore is playing according to σ_i^B. If θ_i^B stops before t, then θ_r^A receives γ. If θ_i^B stops exactly at t, then a random contract is enforced. Finally, if θ_i^B decided to stop after t, the rational type θ_r^A receives a fair split of 1/2. With probability $1 - \mu_r^A(\tau)$ player B is omniscient and therefore plays the strategy σ_o^B.

2. Throughout the paper, we assume

$$\int_0^t f(x)dF(x) = \lim_{\tau \uparrow t} \int_0^\tau f(x)dF(x).$$

That is, the integral does not include the mass point at t. This holds in case F is a CDF or a measure.

Player B with type θ_i^B concedes at time $t \geq \tau$ gets utility

$$u_i^B(\sigma_r^A, t|\tau) =$$
$$(1-\mu_i^B(\tau)) \cdot \left[\int_0^t \frac{1}{2} e^{-\delta x} d\sigma_{r,\tau}^A(x) + \left(\sigma_{r,\tau}^A(t)\frac{1}{2}\left(3/2-\gamma\right) + \sigma_{r,\tau}^A((t,+\infty])(1-\gamma)\right)e^{-\delta t}\right]+$$
$$\mu_i^B(\tau)\cdot(1-\gamma)e^{-\delta t},$$

Observe that with probability $\mu_i^A(\tau)$, the other player is behavioral and therefore will never concede. Hence θ_i^B receives the share $1-\gamma$ at the time he decided to stop.

Finally, when type θ_o^B concedes at time t he gets utility

$$u_o^B(\sigma_r^A, t|\tau) = \int_0^t \frac{1}{2} e^{-\delta x} d\sigma_{r,\tau}^A(x) + \left(\sigma_{r,\tau}^A(t)\frac{1}{2}\left(3/2-\gamma\right) + \sigma_{r,\tau}^A((t,+\infty])(1-\gamma)\right)e^{-\delta t}.$$

In what follows, we use the solution concept of sequential equilibrium. For completeness, we list the properties of a sequential equilibrium of this game.

Definition 1.2. A sequential equilibrium of the bargaining game is a vector of strategies $\left(\sigma_i^B, \sigma_o^B, \sigma_r^A\right)$ and beliefs (μ_i^B, μ_r^A) such that

(1) σ_i^B maximizes θ_i^B's expected utility in any continuation game $t \in [0, +\infty)$ given beliefs μ_i^B and σ_r^A;
(2) σ_o^B maximizes θ_o^B's expected utility in any continuation game $t \in [0, +\infty)$ given σ_r^A;
(3) σ_r^A maximizes θ_r^A's expected utility in any continuation game $t \in [0, +\infty)$ given beliefs μ_r^A and σ_i^B and σ_o^B;
(4) $\left((\sigma_i^B, \sigma_o^B, \sigma_r^A), (\mu_i^B, \mu_r^A)\right)$ is a consistent assessment.

The discussion and definition of consistent assessment is in the Appendix. The next Proposition is key for finding the equilibria of the game. It provides the relation between the incentives of the ignorant and the omniscient types, showing how the informational advantage of θ_o^B over θ_i^B manifests. Then, the next Corollary states an important property of the equilibrium.

Proposition 1.3. *Suppose θ_r^A plays the strategy σ_r^A. Consider continuation game τ and let $t \in [\tau, \infty)$ and $\hat{t} > t$. Then*

(i) θ_i^B weakly prefers concession at $\hat{t} \Rightarrow \theta_o^B$ strictly prefers concession at $\hat{t}$.
(ii) θ_o^B weakly prefers concession at $t \Rightarrow \theta_i^B$ strictly prefers concession at t.

Corollary 1.4. *In any sequential equilibrium, θ_o^B plays a pure strategy in every continuation game.*

Proof. Suppose otherwise, i.e., θ_o^B plays a mixed strategy in the continuation game $t \geq 0$. Then, by the indifference of θ_o^B and Proposition 1.3, the type θ_i^B strictly prefers to concede in the support of θ_o^B's strategy. Then, whenever θ_r^A observes waiting at any time of the support, he updates his beliefs and assigns probability 1 to the event $\theta^B = \theta_o^B$ and concedes immediately. Then, at the minimum time in the support, θ_i^B can wait, and in case $\theta^A = \theta_r^A$, he gets his best payoff $\frac{1}{2}$; in case $\theta^A = \theta_b^A$, he immediately observes waiting and can concede. Therefore θ_i^B has a profitable deviation, a contradiction. Therefore θ_o^B plays a pure strategy in every continuation game $t \geq 0$. □

1.3.2 Degenerate equilibria

The game exhibits a continuum of degenerate equilibria, wherein player B readily concedes. In the subsequent analysis, we aim to characterize this collection. Initially, we establish a specific degenerate equilibrium and subsequently demonstrate the process of generating additional equilibria from it. Finally, we establish that the equilibria we characterize are the only degenerate ones, ruling out the existence of any others.

The equilibrium candidate is the following:

- θ_i^B and θ_o^B concede with probability 1 at any $t \geq 0$;
- θ_r^A chooses the strategy σ_r^A such that $\sigma_{r,\tau}^A([a,b]) = 0$ for all $[a,b] \subseteq [\tau, +\infty)$[3].

Consequently, every type of player B always concedes. Conversely, the behavior of θ_r^A resembles that of the irrational type θ_b^A. Our task is to identify consistent beliefs that can support these strategies as sequentially rational choices. Consider θ_r^A first. Observe $\sigma_{r,\tau}^A(0) = 0$ is optimal given B's strategy as long as $\mu_r^A(\tau) > 0$. Therefore, suppose we reached time $\tau > 0$. Let θ_r^A's beliefs μ_r^A be such that $\mu_r^A(\tau) > 0$. That is, upon reaching time τ, the type θ_r^A does not assign probability 1 to the omniscient type. Then, θ_r^A knows that B concedes with probability 1 at τ. Hence, at τ, θ_r^A is indifferent among all distributions F_r^A such that

$$\sigma_{r,\tau}^A(\tau) = 0. \tag{1.1}$$

This condition is necessary and sufficient for the sequential rationality of F_r^A. Therefore, the strategy proposed is sequentially rational together with μ_r^A. We show in the appendix that these strategies can be sustained by consistent beliefs.

Now consider player B. By Proposition 1.3, it is sufficient to show that θ_o^B weakly prefers to concede at every $t \geq 0$. In fact, if this is the case, then θ_i^B strictly prefers to adopt the same strategy. Suppose we are in continuation game τ. Since θ_r^A never concedes in this continuation game ($\sigma_{r,\tau}([\tau, +\infty)) = 0$), θ_o^B is indifferent among

3. Therefore, the probability measure $\sigma_{r,0}^A$ assigns probability 1 to $\{+\infty\}$

all those strategies σ_o^B such that $\sigma_{o,\tau}^B(\tau) = 1$. In fact, for any deviation, θ_r^A would still not assign probability 1 to the event $\theta^B = \theta_o^B$, and would still consider the equilibrium strategies where B immediately concedes. Since θ_o^B optimally concedes, so does θ_i^B by Proposition 1.3. Therefore, the candidate equilibrium is a degenerate sequential equilibrium.

Note, however, that there is a continuum of degenerate equilibria. In fact, θ_r^A is indifferent among all the strategies $\overline{\sigma}_r^A$ such that $\overline{\sigma}_{r,\tau}^A(\tau) = 0$. Hence, any strategy $\overline{\sigma}_r^A$ with no jumps and $\overline{\sigma}_{r,0}^A(0) = 0$ is still optimal for θ_r^A. Therefore, any strategy $\overline{\sigma}_r^A$ that makes θ_o^B weakly better off by conceding immediately (in any continuation game) can still be part of a degenerate equilibrium. In fact, in this case, by Proposition 1.3, θ_i^B is strictly better off by conceding immediately.

Now we are going to characterize the entire set of degenerate equilibria. Consider the following condition

$$\begin{aligned} &\forall \tau \geq 0,\ \forall t' > \tau, \\ &\int_\tau^{t'} \frac{1}{2} e^{-\delta x} d\sigma_{r,\tau}^A(x) + \left[\sigma_{r,\tau}^A(t')\left(3/2 - \gamma\right) + \sigma_{r,\tau}^A([t', +\infty])(1-\gamma)\right] e^{-\delta t'} \\ &\leq (1-\gamma) e^{-\delta \tau}. \end{aligned} \tag{1.2}$$

This condition states that for every continuation game $t \geq 0$, the type θ_o^B weakly prefers to concede immediately than to wait any time after t. Note that the payoff is conditioned on the event that we have reached continuation game t. Observe that when this condition is satisfied, θ_o^B concedes immediately with probability 1 in any continuation game. Define the set:

$$DE := \{\sigma_r^A | \forall \tau \geq 0\ \sigma_{r,\tau}^A(\tau) = 0 \wedge (1.2)\}.$$

Let $\hat{\sigma}_i^B$ and $\hat{\sigma}_o^B$ be such that $\hat{\sigma}_{i,\tau}^B(\tau) = \hat{\sigma}_{o,\tau}^B(\tau) = 1$ for all $\tau \geq 0$. Then, we get the following result.

Proposition 1.5. *The set of all strategies sustainable as degenerate equilibria of the bargaining game is*

$$\{(\sigma_i^B, \sigma_o^B, \sigma_r^A) | \sigma_i^B = \hat{\sigma}_i^B \wedge \sigma_o^B = \hat{\sigma}_o^B \wedge \sigma_r^A \in DE\}.$$

1.3.3 Nondegenerate equilibrium

In a nondegenerate equilibrium, no player concedes with a probability of 1 at time $t = 0$. We will demonstrate the existence of a unique nondegenerate equilibrium. In this equilibrium, players randomize over a finite support interval $[0, T^0]$. However, as stated in Corollary 1.4, the omniscient type cannot play a mixed strategy equilibrium. Hence, the ignorant type engages in mixing. According to Proposition 1.3, the omniscient type does not concede within the interval $[0, T^0]$, thereby exhibiting behavior akin to an irrational type. This allows us to analyze the game in a manner similar to the standard AG model, where each player has only one behavioral contract available. Unlike the standard model, the strategy of the omniscient type is endogenous in our context. Since player B is limited to being either rational or irrational in the standard version, the strategy of the irrational player is exogenously determined. In contrast, Proposition 1.3 informs us that θ_o^B adopts the same strategy, but this choice arises from a strategic decision. We define only strategies σ_0, and derive the equilibrium with their equivalent cdf. Conditional distribution are derived through Bayes rule until the last period of possible concession.

First, define $F^A(t) = (1-\mu_i^B(t)) \cdot F_r^A(t)$ and $F^B(t) = \mu_r^A(t) F_i^B(t)$, where F_r^A and F_i^B are CDFs representing $\sigma_{r,0}^A$ and $\sigma_{i,0}^B$ respectively. The conditional measures $\sigma_{r,\tau}^A$ and $\sigma_{i,\tau}^B$ can be represented by $F_r^A/(1-F_r^A(\tau))$ and $F_i^B/(1-F_i^B(\tau))$. Clearly, $\sigma_{o,\tau}^B([\tau,+\infty)) = 0$ for each τ. Then, let

$$\lambda^A = \frac{(1-\gamma)\delta}{1/2-(1-\gamma)} \quad \text{and} \quad \lambda^B = \frac{1/2\delta}{1/2-(1-\gamma)}.$$

Define $T^A = -\log(z)/\lambda^A$ and $T^B = -\log(1-q)/\lambda^B$. As in AG, we have $T^0 = \min\{T^A, T^B\}$. Then, we have that

$$F^A(t) = 1-c^A e^{-\lambda^A t} \quad \text{and} \quad F^B(t) = 1-c^B e^{-\lambda^B t},$$

where $c^i = e^{-\lambda^i(T^i-T^0)}$. Observe that if $T^i = T^0$, then player i never concedes at $t=0$.
By Proposition 1 in AG[4], $(F^B/q, F_o^B, F^A/(1-z))$, where $F_o^B(t) = 0$ for all $t \in [0, T^0]$, constitutes the unique nondegenerate equilibrium. Conditional distributions for $\tau \in [0, T^0]$ are computed through Bayes' rule.

Now that we have characterized the entire set of equilibria in this game (later referred to as the game after the signal, or GAS), we are ready to apply these findings to the larger game where B can also be an irrational type. This allows us to study the incentives for θ_i^B and θ_o^B to reveal their rationality and initiate a new war of attrition. As we will see, whenever A and B are expected to play a degenerate equilibrium, θ_o^B lacks the incentive to offer the fair contract. In this case, he understands that doing so would weaken his position, as A would anticipate an immediate concession (or an early one in case B deviates). Furthermore, when B's behavioral demand is high, he is disincentivized to reveal his rationality, as he stands to gain a better deal by continuing to pretend to be irrational.

4. A summary of Abreu and Gul (2000) Proposition 1 can be found in the appendix.

1.4 The two behavioral types model

In this section, we enlarge B's type space to include a behavioral type θ_b^B, having then two behavioral types, one for each player. We therefore distinguish between γ_A and γ_B, the behavioral demand of players A and B respectively. The type θ_b^B has all the features of θ_b^A. Therefore, throughout the game he offers a split of the surplus corresponding to $(1-\gamma_B, \gamma_B)$, where $\gamma_A + \gamma_B > 1$, and does not accept anything less. Whenever player A offers $(x, 1-x)$ with $1-x \geq \gamma_B$, θ_b^B accepts immediately. A's types remain unchanged, while B's new type space is $\Theta_1^B := \{\theta_b^B, \theta_o^B, \theta_i^B\}$, where θ_o^B and θ_i^B are as described in the previous sections. We let q_b, q_o, q_i be the prior probabilities of B being irrational, omniscient and ignorant, respectively. A is irrational with probability $z \in (0,1)$ as before. Now, define $\mathcal{Q} := \{(q_o, q_i, z) \in (0,1)^3 | q_o + q_i \in (0,1)\}$. This space of initial beliefs will be helpful when we characterize the equilibrium set. Clearly, $q_b = 1 - q_o - q_i$.

In this new version of the model, the ignorant type θ_i^B decides whether to imitate the behavioral type θ_b^B or try to signal the information he possesses by pretending to know that his opponent is rational. Type θ_o^B has the same dilemma, with the difference that he actually knows about A's rationality. Hence, contrary to the classic war of attrition we want to give one of the players, B, the possibility of using more than two actions (wait and concede). Therefore, suppose that B has offered $(1-\gamma_B, \gamma_B)$ at the beginning of the bargaining procedure. In this continuation game, B can either reject A's offer (waiting), accept A's offer (concede) or offer the fair contract $(1/2, 1/2)$. The last action reveals B's rationality, putting him in a difficult position. Yet, at the same time, it signals A that B is ready to start a new war of attrition with his rationality exposed, which can be taken by A as a sign of strength (omniscient type).

In order to model these choices, we follow Abreu and Pearce (2007) and Abreu, Pearce, and Stacchetti (2015). They formulate a new hybrid model with both discrete and continuous time features. As they explain, this allows for easier calculations in the war of attrition, avoiding openness problems when a new offer is made and there is no "first" time to accept it. Therefore, we allow the players to concede at any period, while B can change his offers only at integer times. For every $t \in \mathbb{N}$, we split the date into three subdates, $(t, -1)$, $(t, 0)$ and $(t, +1)$. At $(t, -1)$, A has her last opportunity to accept the pending offer of her opponent. B can also accept $1-\gamma_A$ at this dates. At $(t, 0)$, B can offer the fair contract $(1/2, 1/2)$ in case he has not done it before. Finally, at $(t, +1)$ A and B can accept the standing offer of their opponent. There is no discounting among subdates of the same period. A and B can concede at any $t \in \mathbb{R}_{++} \setminus \mathbb{N}$ as well. Date $t = 0$ is split into $(0, 0)$ and $(0, +1)$. At $(0, 0)$ the players choose an initial contract. For simplicity and without loss of generality A is restricted to $(\gamma_A, 1-\gamma_A)$ while B can offer $(1-\gamma_B, \gamma_B)$ and $(1/2, 1/2)$ only. At $(0, +1)$ both can accept their opponent's contract. If we extend any non-integer number to two dimensions, writing $(t,+1)$ for $t \in \mathbb{R}_{++} \backslash \mathbb{N}$, we can put a complete order on our periods. We put the lexicographic order, where $(t,k) \geq (t',k')$ in one of these two cases: $t > t'$ OR $t = t'$ and $k \geq k'$. Hence, the end of a period and the beginning of a new period are separated. This implies that a player can condition his or her choice of the action at the beginning of a period on the events happened at a previous period.

We denote the continuation game at $(n,0)$, $n \in \mathbb{N}_0$ after the fair contract has been offered as the *game after the signal* (GAS).

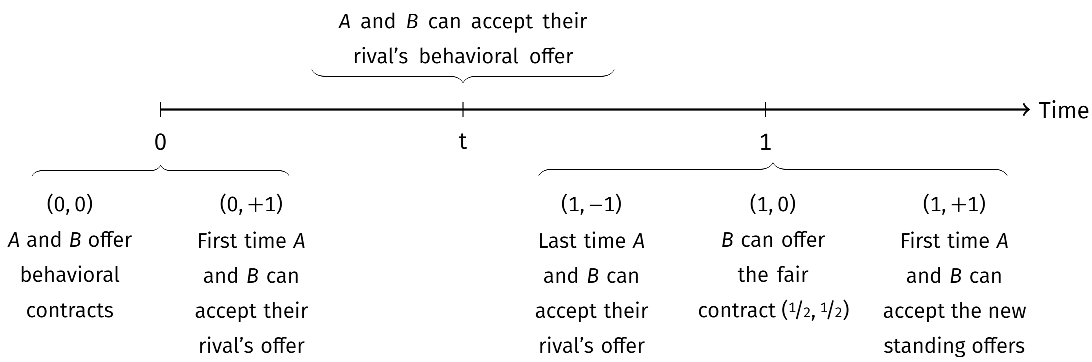

Figure 1.1. Timeline representation

In Figure 1.1, we illustrate a game scenario unfolding over the time interval from $(0,0)$ to $(1,+1)$. In this depiction, players A and B commence by proposing their behavioral contracts. Subsequently, at $(0,+1)$, they are presented with their initial opportunity to accept the behavioral contracts proposed by their respective opponents. Throughout any point in time $t \in (0,1)$, they retain the option to accept these standing offers. This opportunity persists until period $(0,-1)$, marking the final chance to accept before B can alter his offer. Following this, B can present the fair contract at $(1,0)$. Should B choose to do so, A can accept this new offer at $(1,+1)$. At this date, B also has the option to accept $(\gamma_A, 1-\gamma_A)$. If B has not proposed the fair contract by $(1,0)$, the players can start their concession again from $(1,+1)$.

We turn to the description of the mixed strategies. First, note that a pure strategy for player B in this game can be one of three things:

(1) The offer of the behavioral contract at $(0,0)$ and a concession time (t,k);

(2) The offer of the behavioral contract at $(0,0)$, a time $n \in \mathbb{N}$ for the fair offer contract and a concession time $(t,k) > (n,0)$;

(3) The offer of the fair contract at $(0,0)$ and a concession time (t,k)[5].

5. Player B can decide to never concede. In this case, $t = +\infty$ and the choice of k would have no meaning.

In the first case, B decides to not offer the fair contract, while in the second and third case he offers $(1/2, 1/2)$ before conceding to A's behavioral demand. This is a heuristic description of available pure strategies. In fact, the players have to specify an action for any possible continuation game [6]. For this reason, we turn directly to the description of mixed strategies as done in the previous section. In order to introduce the modeling of mixed strategies, we make use of the following notation. Let $n \in \mathbb{N}$ and $t \in (n-1, n)$. Then, concession in the interval $[t, n]$ means concession from t up until $(n, -1)$. On the other hand, when $t \in (n, n+1)$, concession in $[n, t]$ means concession from $(n, +1)$ to t. Finally, concession in $[n-1, n]$ is from $(n-1, +1)$ to $(n, -1)$.

We require A and B to play strategies σ^A and σ^B described in Definition 1.1 after B offers the fair contract. Therefore, suppose B changes offer from $(1-\gamma_B, \gamma_B)$ to $(1/2, 1/2)$ at $(\tau^*, 0)$, $\tau^* \in \mathbb{N}$, henceforth revealing his rationality. Then, A and B play strategies σ^A and σ^B that satisfy Definition 1.1 with the difference that their domain and codomain are now $[\tau^*, +\infty)$ and $\Delta([\tau^*, +\infty))$. Condition (i) is left unchanged while (ii) becomes:

- *Conditioning requirement: For all $t \in [\tau^*, \tau)$ and Borel set $\mathscr{B} \subseteq [\tau, +\infty) \cup \{+\infty\}$, we have*
$$\sigma_t(\mathscr{B}) \;=\; (1-\sigma_t([t,\tau)))\sigma_\tau(\mathscr{B}).$$

Therefore, for each $n \in \mathbb{N}_0$, A and B need to specify strategies $\sigma^A[n]$, $\sigma^B[n]$ that satisfy the new version of Definition 1.1.

Player B at the beginning of the game chooses whether to offer $(1/2, 1/2)$ at some point or concede to A's demand first. Suppose B concedes first. Then, he selects (t_0, k) such that $t_0 \in \mathbb{R}_+$ and $k \in \{-1, +1\}$ such that $k = +1$ for $t_0 \in \mathbb{R}_+ \backslash \mathbb{N}$. If B offers $(1/2, 1/2)$ first, instead, he chooses $t_1 \in \mathbb{N}_0$ and offers the fair contract at $(t_1, 0)$. Hence, we define a function $Y^B : \mathbb{N}_0 \to [0,1]$ that assigns to each $n \in \mathbb{N}_0$ the probability that B offers $(1/2, 1/2)$ at $(n, 0)$. We denote with $Y_i^B(n)$ and $Y_o^B(n)$ the probabilities assigned by θ_i^B and θ_o^B respectively. Finally, we let $X^B = (X_n^B)_{n \in \mathbb{N}_0}$ be a sequence of measures, such that $X_n^B : \mathbb{B}([n, n+1]) \to [0,1]$ and

$$\sum_{n=0}^{+\infty} X_n^B([n, n+1]) + \sum_{n=0}^{+\infty} Y^B(n) \;\leq\; 1 - q_b. \tag{1.3}$$

Each X_n^B describes the concession of B to A's demand in the interval $[n, n+1]$, hence from $(n, +1)$ to $(n+1, -1)$ [7]. The sigma-algebra $\mathbb{B}$ imposed on each interval

6. Note, however, that as long as the fair contract is not offered, there are no unexpected events for θ_r^A. In fact, any continuation game with standing offer $(1-\gamma_B, \gamma_B)$ has positive probability of being reached, since B can be behavioral. This is different from the model with one behavioral type as B is commonly known as rational. For example, in its degenerate equilibrium, B waiting at $t = 0$ is an unexpected event, and strategies that describe the actions after this event must be specified.

7. The probability of concession in a set $S_n \cup S_m$ (both measurable sets) where $S_n \subseteq [n, n+1]$, $S_m \subseteq [m, m+1]$ with $n \neq m$ can be calculated by $X_n^B(S_n) + X_m^B(S_m)$.

is the Borel sigma-algebra derived from the relative Euclidean topology. We write X_i^B and X_o^B to distinguish the strategies used by θ_i^B and θ_o^B respectively. X^A is similarly defined.

Summarizing, B's strategy includes:

- A sequence of measures $X^B = (X_n^B)_{n\in\mathbb{N}_0}$;
- A function $Y^B : \mathbb{N}_0 \to [0,1]$;
- A sequence $(\sigma^B[n])_{n\in\mathbb{N}_0}$ such that $\forall n \in \mathbb{N}_0$ the modified version of Definition 1.1 is satisfied,

and (1.3) holds. A's strategy, on the other hand, includes

- A sequence of measures $X^A = (X_n^A)_{n\in\mathbb{N}_0}$;
- A sequence $(\sigma^A[n])_{n\in\mathbb{N}_0}$ such that $\forall n \in \mathbb{N}_0$ the modified version of Definition 1.1 is satisfied,

and

$$\sum_{n=0}^{+\infty} X_n^A([n, n+1]) \;\leq\; 1 - z.$$

Beliefs are expressed as follows: the belief of B's ignorant type θ_i^B at period (t,k) is denoted by $\mu_i^B((t,k);\theta_b^A)$ and represents the probability that θ_i^B assigns to the event $\theta^A = \theta_b^A$. Beliefs also depend on the history up to period (t,k), but we omit this from the notation for simplicity. When necessary, we specify the history preceding (t,k). The same applies to θ_r^A's beliefs. Clearly, in the game after the signal beliefs coincide with the beliefs of the one behavioral type model. Therefore, for each n we have

$$\sigma^A[n] \;=\; (1 - \mu_i^B((n,0);\theta_b^A))\sigma_r^A[n],$$

and

$$\sigma^B[n] \;=\; \mu_r^A((n,0);\theta_i^B)\sigma_i^B[n] + \mu_r^A((n,0);\theta_o^B)\sigma_o^B[n],$$

where $\sigma_i^B[n]$, $\sigma_o^B[n]$, $\sigma_r^A[n]$ are the strategies used by types θ_i^B, θ_o^B and θ_r^A respectively and beliefs are computed considering the fair offer happening at $(n,0)$.

Before we give the definition of equilibrium, we provide a brief description of the utilities. Utilities are written at time 0, the beginning of the bargaining game. Consider θ_i^B and suppose he chooses to offer the fair contract at t_1, before conceding to A's contract, and concedes at t_2 in the game after the signal. His utility is then

$$\begin{aligned} U_i^B((X^A,(\sigma^A[n])_{n\in\mathbb{N}_0}),t_1,t_2) \;&=\; \sum_{n=0}^{t_1-1}\int_n^{n+1} \gamma_B e^{-\delta t} dX_n^A(t) \\ &\quad + \left(1 - \sum_{n=0}^{t_1-1} X_n^A([n,n+1])\right) u_i^B(\sigma_r^A[t_1], t_2 | t_1) e^{-\delta t_1}. \end{aligned}$$

The term in the first line refers to the event in which A concedes to $(1-\gamma_B,\gamma_B)$ before t_1. In the second term, $1-\sum_{n=0}^{t_1-1} X_n^A([n,n+1])$ is the probability that A does not concede before the behavioral demand at t_1. Then, this probability is multiplied by $u_i^B(\sigma_r^A[t_1],t_2|t_1)e^{-\delta t_1}$, the expected utility of θ_i^B in the GAS. The omniscient type knows that A is rational, therefore if he follows the previous example of strategies, he gets

$$\begin{aligned} U_o^B((X^A,(\sigma^A[n])_{n\in\mathbb{N}_0}),t_1,t_2) &= \frac{1}{1-z}\sum_{n=0}^{t_1-1}\int_n^{n+1}\gamma_B e^{-\delta t} dX_n^A(t) \\ &\quad + \left(1-\frac{1}{1-z}\sum_{n=0}^{t_1-1} X_n^A([n,n+1])\right) u_o^B(\sigma_r^A[t_1],t_2|t_1)e^{-\delta t_1}. \end{aligned}$$

Finally, suppose θ_r^A chooses $(t_0,+1)$ such that $t_0\in(n^*,n^*+1)$ for some $n^*\in\mathbb{N}_0$ and concession $(t_2^n)_{n=0}^{n^*}$. Each t_2^n represents θ_r^A's concession in case B offers $(1/2,1/2)$ at n. Let $\Sigma^B=(X^B,Y^B,(\sigma^B[n])_{n\in\mathbb{N}_0})$. Her utility is then,

$$\begin{aligned} U_r^A(\Sigma^B,(t_0,+1),(t_2^n)_{n=0}^{n^*}) &= \sum_{n=0}^{n^*-1}\int_n^{n+1}\gamma_A e^{-\delta t} dX_n^B(t) + \int_{n^*}^{t_0}\gamma_A e^{-\delta t} dX_{n^*}^B(t) \\ &\quad + \sum_{n=0}^{n^*} Y^B(n) u_r^A((\sigma_i^B[n],\sigma_o^B[n]),t_2^n|n) \\ &\quad + \left(1-\sum_{n=0}^{n^*-1} X_n^B([n,n+1]) - X_{n^*}^B([n^*,t_0]) - \sum_{n=0}^{n^*} Y^B(n)\right)(1-\gamma_B)e^{-\delta t_0} \end{aligned}$$

The first line captures the events in which B concedes to A's demand before offering the fair contract. The second line includes the probability that B offers $(1/2,1/2)$ before A concedes. In the third line we have the probability that B neither offers the fair contract nor concedes to A before t_0. In this event θ_r^A's profit is $1-\gamma_B$.

Remark. *We can express the utility of θ_i^B in relation to the utility of θ_o^B. Consider the previous case as an example, that is, θ_i^B is offering the fair contract at t_1 and conceding in the GAS at t_2. Observe that at t_1, beliefs are not anymore z and $1-z$. Since A has not conceded in case the players arrive to time t_1, θ_i^B believes A is irrational with probability $z/\left(1-\sum_{n=0}^{t_1-1} X_n^A([n,n+1])\right)$. For ease of notation, let $\sum_{n=0}^{t_1-1} X_n^A([n,n+1])=X$. We have*

$$u_i^B(\cdot) = \left(1-\frac{z}{1-X}\right)u_o^B(\cdot) + \frac{z}{1-X}e^{-\delta t_2+\delta t_1}(1-\gamma_A).$$

We discounted by $e^{+\delta t_1}$ the term $e^{-\delta t_2}(1-\gamma_A)$ because the GAS is shifted from $t=0$ to $t=t_1$. Hence,

$$u_i^B(\cdot)e^{-\delta t_1} = \left(1-\frac{z}{1-X}\right)u_o^B(\cdot)e^{-\delta t_1} + \frac{z}{1-X}e^{-\delta t_2}(1-\gamma_A).$$

Finally,

$$\begin{aligned}
U_i^B((X^A,(\sigma^A[n])_{n\in\mathbb{N}_0}),t_1,t_2) &= \sum_{n=0}^{t_1-1}\int_n^{n+1}\gamma_B e^{-\delta t}dX_n^A(t) + (1-X)\,u_i^B(\cdot)e^{-\delta t_1} \\
&= (1-z)\left(\frac{1}{1-z}\sum_{n=0}^{t_1-1}\int_n^{n+1}\gamma_B e^{-\delta t}dX_n^A(t)\right) \\
&+ (1-X)\left(\left(1-\frac{z}{1-X}\right)u_o^B(\cdot)e^{-\delta t_1}+\frac{z}{1-X}e^{-\delta t_2}(1-\gamma_A)\right) \\
&= (1-z)\left(\frac{1}{1-z}\sum_{n=0}^{t_1-1}\int_n^{n+1}\gamma_B e^{-\delta t}dX_n^A(t)\right) \\
&+ (1-z)\left(1-\frac{X}{1-z}\right)u_o^B(\cdot)e^{-\delta t_1}+ze^{-\delta t_2}(1-\gamma_A) \\
&= (1-z)\left(\frac{1}{1-z}\sum_{n=0}^{t_1-1}\int_n^{n+1}\gamma_B e^{-\delta t}dX_n^A(t)+\left(1-\frac{X}{1-z}\right)u_o^B(\cdot)e^{-\delta t_1}\right) \\
&+ ze^{-\delta t_2}(1-\gamma_A) \\
&= (1-z)U_o^B((X^A,(\sigma^A[n])_{n\in\mathbb{N}_0}),t_1,t_2)+ze^{-\delta t_2}(1-\gamma_A).
\end{aligned}$$

Therefore, with probability $1-z$, θ_i^B *has* θ_o^B*'s payoff, with probability* z *he will get* $1-\gamma_A$ *at time* $t=t_2$. *This resembles the GAS payoff of the previous section.*

We need a last step before the equilibrium definition. We now solve a game where the last possible chance of sending the signal has not been taken, and therefore the players continue a standard reputational bargaining with behavioral offers.

Game with No Signal Suppose $\tau^*\in\mathbb{N}_0$ is the last period where B can offer $(1/2,1/2)$, and assume that instead he offers again $(1-\gamma_B,\gamma_B)$. We are left with an AG game where the first period is shifted from 0 to τ^*. Therefore, the solution is unique and can be recovered from the AG results. We also know from the previous section that θ_o^B mimics θ_b^B and therefore θ_i^B is the only type of B that concedes. Hence, let $T_N^A=-\log(\mu_i^B((\tau^*,0;\theta_b^A)))/\lambda_N^A+\tau^*$, $T_N^B=-\log(\mu_r^A((\tau^*,0);\theta_o^B,\theta_b^B))/\lambda_N^B+\tau^*$, where

$$\lambda_N^A=\frac{(1-\gamma_A)\delta}{\gamma_A+\gamma_B-1}\quad\text{and}\quad\lambda_N^B=\frac{(1-\gamma_B)\delta}{\gamma_A+\gamma_B-1}.$$

Thus, B's concession distribution from τ^*, $(X_n^B)_{n\geq\tau^*}$ can be represented by the cdf $F_N^B(t)=1-c^Be^{-\lambda_N^B(t-\tau^*)}$. A's concession is instead distributed according to $F_N^A(t)=1-c^Ae^{-\lambda_N^A(t-\tau^*)}$, where $c^i=e^{-\lambda_N^i(T_N^i-T_N^0)}$ and $T_N^0=\min\{T_N^A,T_N^B\}$. We refer this continuation game as *game with no signal* (GNS).

We are ready to provide the equilibrium definition. Observe that we exploit Definition 1.2 to specify the sequential rationality imposed on the equilibrium at $t\in\mathbb{N}_0$ after $(1/2,1/2)$ has been offered.

Definition 1.6. Let $\boldsymbol{q} \in \mathcal{Q}$. A sequential equilibrium given $\boldsymbol{q}$ of the full bargaining game is a vector of strategies $(\Sigma_i^B, \Sigma_o^B, \Sigma_r^A)$, where $\Sigma_s^B = (X_s^B, Y_s^B, (\sigma_s^B[n])_{n\in\mathbb{N}_0})$, $s \in \{i, o\}$, and $\Sigma_r^A = (X_r^A, (\sigma_r^A[n])_{n\in\mathbb{N}_0})$, beliefs (μ_i^B, μ_r^A), such that

(1) Σ_i^B maximizes θ_i^B's expected utility in any continuation game (t, k) given beliefs μ_i^B and Σ_r^A;
(2) Σ_o^B maximizes θ_o^B's expected utility in any continuation game (t, k) given Σ_r^A;
(3) Σ_r^A maximizes θ_r^A's expected utility in any continuation game (t, k) given beliefs μ_r^A and Σ_i^B and Σ_o^B;
(4) For each $n \in \mathbb{N}_0$, $(\sigma_i^B[n], \sigma_o^B[n], \sigma_r^A[n])$ is GAS degenerate or nondegenerate equilibrium;
(5) A and B play the unique equilibrium in GNS;
(6) $((\Sigma_i^B, \Sigma_o^B, \Sigma_r^A), (\mu_i^B, \mu_r^A))$ is a consistent assessment.

We also make another assumption on the equilibrium behavior of B. This assumption states that in case any B's type decides to offer $(1/2, 1/2)$ at $(n, 0)$, implying $\mu_r^A((n, 0); \theta_i^B) = 1$, then we force this type to concede at $(n, -1)$. These two actions are equivalent. In fact, when $\mu_r^A((n, 0); \theta_i^B) = 1$, B optimally concedes at $(n, +1)$. Since there is no discounting between $(n, -1)$ and $(n, +1)$, we get the equivalence.

Assumption 2. *In any equilibrium, there exists no $n \in \mathbb{N}_0$ such that $(1/2, 1/2)$ offered at $(n, 0) \Rightarrow \mu_r^A((n, 0); \theta_i^B) = 1$.*

Assumption 3. *Suppose $\exists n \in \mathbb{N}_0$ such that $X_n^B([n, n+1]) > 0$ or $Y^B(n) > 0$. Then, there is strictly positive probability that continuation game $(n, 0)$ is reached.*

In the next Proposition, we show that B can offer the fair contract only for a limited amount periods in any sequential equilibrium. For this, we use the following notation. Let $T_S^0(n)$ be the time at which the game is certain to end before it, with a probability of 1, provided that A and B play the strategy profile $(\sigma^A[n], \sigma^B[n])$ in the GAS. In the next results, we define $T_N^0(n)$ in the same way for the GNS.

Proposition 1.7. *In any sequential equilibrium, both holds:*

(1) $supp(Y_i^B) = supp(Y_o^B)$;
(2) $|supp(Y^B)| < +\infty$.

This Proposition tells us that either there is no event in which $(1/2, 1/2)$ is offered, or there exists a last time τ' where the fair contract can be offered. Therefore for all $\tau > \tau'$, $Y^B(\tau) = 0$.

1.4.1 Equilibrium with no signal

We now find an equilibrium in the game where the fair contract is not offered. Consequently, the opportunity for B to signal their rationality is never utilized. This outcome must be optimal in equilibrium, therefore we explore one way to achieve this through the use of degenerate equilibria. As anticipated, when A and B play a degenerate equilibrium, θ_o^B does not offer the fair contract, and thus θ_i^B refrains from it as well. We show that in this case, the equilibrium is unique and possesses the properties of the AG solution. The only difference is that θ_o^B does not concede to A 's irrational demands due to its maximization problem, making θ_o^B's behavior endogenous, unlike θ_b^B. In the next section, we demonstrate that an equilibrium with no signal is the unique possible equilibrium when B's behavioral demand is high.

Lemma 1.8. *Let $n \in \mathbb{N}_0$. In equilibrium, if $(\sigma^A[n], \sigma^B[n])$ represents the degenerate equilibrium of GAS, then $Y^B(n) = 0$ whenever $\mu_r^A((n,0);\theta_o^B) < 1$.*

Proof. Clear since θ_o^B obtains a higher payoff by waiting instead of offering $(1/2, 1/2)$ at n. Hence, $Y_o^B(n) = 0$. From Proposition 1.7, $Y_i^B(n) = 0$ and so $Y^B(n) = 0$. □

Therefore, if we let A and B play the degenerate equilibrium for each $n \in \mathbb{N}_0$, then $\sum_{n=0}^{+\infty} Y^B(n) = 0$. We show that this can be part of an equilibrium[8]. First, consider the following result.

Lemma 1.9. *Suppose $Y^B(n) = 0$ for each $n \in \mathbb{N}_0$, and take $\hat{t}, t \in \mathbb{R}_+$ such that $\hat{t} > t$ Then,*

(i) θ_i^B weakly prefers concession at $\hat{t} \Rightarrow \theta_o^B$ strictly prefers concession at $\hat{t}$.
(ii) θ_o^B weakly prefers concession at $t \Rightarrow \theta_i^B$ strictly prefers concession at t.

Proof. Apply the same steps of the proof of Proposition 1.3, substituting γ with γ_A and $\frac{1}{2}$ with γ_B and $1-\gamma_B$. □

This implies that, under the assumption that $Y^B(n) = 0$ for each $n \in \mathbb{N}_0$, in equilibrium, θ_o^B never concedes as long as θ_r^A has not conceded first. To see this, suppose θ_r^A does not concede at time t with probability 1, but assume, for the sake of contradiction, that θ_o^B concedes at t. According to the previous Lemma, θ_i^B strictly prefers to concede no later than t with probability 1.

Now, let $\tau^* = \sup\{\bigcup_{n=0}^{+\infty} \operatorname{supp}(X_{n,o}^B)\}$. Note that θ_r^A concedes with probability 1 no later than τ^*. For any $\varepsilon > 0$, there exists some $t' \in [\tau^{-}\varepsilon, \tau^*]$ such that t' is in the support of θ_o^B's concession strategy. However, if θ_o^B does not concede by τ^*, then θ_r^A will accept B's behavioral demand.

Conceding at t' gives θ_o^B a payoff of $(1-\gamma_A)e^{-\delta t'}$, whereas waiting to concede after τ^* yields a payoff of at least $\gamma_B e^{-\delta\tau^*}$. Since by assumption $\gamma_B > 1-\gamma_A$, there exists an $\varepsilon > 0$ such that waiting is strictly better than conceding at t'. Thus, t' cannot

8. Note however that in case θ_r^A updates her beliefs to $\mu_r^A((n,0);\theta_o^B) = 1$ after the fair contract is offered at $(n,0)$, then by Assumption 1 θ_r^A concedes immediately. Note, however, that if $\mu_r^A((n,0);\theta_o^B) = 1$ in equilibrium and this is optimal for θ_o^B, then it is optimal for θ_i^B too, and so $Y_i^B(n) > 0$, implying $\mu_r^A((n,0);\theta_o^B) < 1$, a contradiction. Therefore, in equilibrium, $\mu_r^A((n,0);\theta_o^B) < 1$.

be in the support of θ_o^B's equilibrium strategy, leading to a contradiction. Therefore, θ_o^B does not concede as long as θ_r^A has not done so.

As in the previous section, we have that θ_o^B assumes the posture of the behavioral types, who never concede in the war of attrition. Therefore, we can recover the equilibrium from AG. The types θ_i^B and θ_r^A randomize over some support $[0, T^0]$, with rates

$$\lambda^A = \frac{(1-\gamma_A)\delta}{\gamma_A + \gamma_B - 1} \quad \text{and} \quad \lambda^B = \frac{(1-\gamma_B)\delta}{\gamma_A + \gamma_B - 1}.$$

The final period T^0 is computed as $\min\{T^A, T^B\}$, where

$$T^A = -\frac{\log(z)}{\lambda^A} \quad \text{and} \quad T^B = -\frac{\log(q_o + q_b)}{\lambda^B}.$$

Clearly, if $T^k > T^j$, for $k \neq j$, player k concedes with positive probability at $(0, +1)$ to compensate for his or her reputation reaching 1 later than the opponent. In this equilibrium, it is necessary that after the unexpected event of the fair contract offer at $(n, 0)$, θ_r^A does not update beliefs with $\mu_r^A((n, 0); \theta_o^B) = 1$. In fact, in this case A would immediately concede and for some parameters θ_o^B prefers this deviation. If, for example, θ_r^A assumes that θ_o^B and θ_i^B made the mistake with the same probability, beliefs are not degenerate and θ_o^B does not want to deviate. Therefore, in the game with two behavioral types, there always exists an equilibrium where the fair contract is not used. The intuition is that the fair contract is perceived by A and B as a signal for weakness, and therefore it is not used. In fact, in case it is offered, both B's types prefer to concede immediately, as A would be too stubborn in that continuation game. This behavior and beliefs resemble second-order optimism in Friedenberg (2019). In this paper, the author shows that under the assumptions of rationality and common strong belief of rationality, two agents who bargain over a surplus can delay their agreement because any Pareto improved and earlier offer from one agent may make the other player too optimistic, letting her believe she can obtain even more from a longer negotiation.

1.4.2 Signaling equilibrium

Now we turn to the analysis of equilibria that feature signaling, that is, equilibria where player B offer the fair contract with positive probability. As anticipated, these equilibria not always exist, and their existence depend on the parameter γ_B.

For the next proposition, let $F_S^A(\cdot; \tau^*)$ be the cdf representing A's concession strategy after B has offered the fair contract at $(\tau^*, 0)$. Then, we denote with $Y_s^B(\tau^*|\tau^*)$ the probability that type θ_s^B, $s \in \{i, o\}$, offers the fair contract at $(\tau^*, 0)$ conditioning on the event the game arrives at $(\tau^*, 0)$. We get the following result.

Lemma 1.10. *Let* $\tau^* = \max supp(Y^B)$*. Then, for* $\tau^* > 0$*,* $Y_i^B(\tau^*|\tau^*) \in (0, 1)$ *and* $Y_o^B(\tau^*|\tau^*) \in (0, 1]$.

Hence, if τ^* is the last period where the fair contract is offered with positive probability, at $\tau^*, 0)$, θ_i^B mixes between the offer $(1/2, 1/2)$ and the offer $(1-\gamma_B, \gamma_B)$. This implies he has to be indifferent between the two.

Theorem 1.11. *In any equilibrium we have* $\sum_{n=0}^{+\infty} Y^B(n) = 0$ *for all* $\boldsymbol{q} \in \mathcal{Q}$ *whenever* $\gamma_B > 1/2$.

Therefore, in any equilibrium where $\gamma_B > 1/2$, the omniscient player B opts to conceal their information. This choice arises because if B reveals information by offering the fair contract, player A might still suspect that B is bluffing. At this point, the best outcome he can achieve is the split $(1/2, 1/2)$. Conversely, by continuing to offer the behavioral contract, θ_o^B can secure γ_B. Consequently, the omniscient type θ_o^B cannot prevent the ignorant type θ_i^B from also offering the fair contract with positive probability, making it difficult for A to distinguish between the two.

As established in Lemma 1.9, in equilibrium, θ_o^B does not concede until θ_r^A has done so. Thus, the omniscient type behaves like a behavioral type, leading to a unique equilibrium, as described in the preceding section. The AG model applies, with the distinction that the probability of B behaving like an irrational type is $q_o + q_b$.

Having resolved the case where $\gamma_B > 1/2$, we now focus on the scenario where $\gamma_B < 1/2$ for the remainder of the paper. Here, the dynamics differ significantly. A fair contract offer not only signals the potential possession of information but also allows B to secure a better deal if A concedes. However, this advantage comes at a cost: B's reputation for being omniscient grows more slowly than the reputation for being behavioral (or omniscient) in the war of attrition before the fair contract was offered.

As stated in the previous proof, concession distributions in the GAS and GNS can be represented by CDFs. Therefore, let τ_1^* be the last period for the fair contract offer. For player $m \in A, B$, denote the concession distribution in the GAS by $F_S^m(\cdot;\tau_1^*)$ and the concession distribution in the GNS by $F_N^m(\cdot;\tau_1^*)$.

Lemma 1.12. *Let* $\tau_1^*, \tau_0^* \in \mathbb{N}$ *be such that* τ_1^* *is the last period and* τ_0^* *is the second to last period in which B offers* $(1/2, 1/2)$ *with positive probability. Then, A does not concede with positive probability to either contract at* $(\tau_1^*, +1)$. *Moreover, in the event B does not offer* $(1/2, 1/2)$ *at* τ_0^**, A and B concede over* (τ_0^*, τ_1^*) *with rates* λ_N^A *and* λ_N^B*, respectively. If* τ_0^* *cannot be defined, set* $\tau_0^* = 0$.

Lemma 1.13. *Let* $\tau_0^* \in \mathbb{N}$ *be the second to last period in which B offers* $(1/2, 1/2)$ *with positive probability. Then, A does not concede with positive probability to either contract at* $(\tau_0^*, +1)$.

We proceed by establishing a theorem crucial for grasping the dynamics of signaling equilibria. This theorem states that B can only manifest his rationality through signaling in a single period $t \in \mathbb{N}_0$. Consequently, if B foregoes the opportunity to propose the fair contract at t, his offer remains unchanged throughout the game. To gain insight into why this holds true in any signaling equilibrium, consider the following scenario. Suppose B is randomizing his fair contract offer between τ_0^* and τ_1^*. Then θ_o^B is indifferent, and the same holds for θ_i^B. Recall that in equilibrium, θ_i^B can optimally choose θ_o^B's posture, so he can concede in the last period $T_S^0(\cdot)$ in the GAS. Hence, by our previous remark, we know that θ_i^B's utility is the utility of θ_o^B with probability $1-z$, while with probability z he gets the worst possible outcome, that is $1-\gamma_A$ on the very last period $T_S^0(\cdot)$. Therefore, since with probability $1-z$ he is indifferent between τ_0^* and τ_1^* (by θ_o^B's indifference), we know that θ_i^B must be indifferent even in the event that A is irrational, which happens with probability z. Since this worst case scenario depends only on $T_S^0(\cdot)$, indifference necessitates $T_S^0(\tau_0^*) = T_S^0(\tau_1^*)$. Hence, even if B postpones offering the fair contract until τ_1^*, the GAS still concludes at $T_S^0(\tau_0^*)$. From Lemma 1.12 and Lemma 1.13, we know that A does not concede to the fair contract with positive probability at $(\tau_0^*, +1)$ and $(\tau_1^*, +1)$, implying that $T_S^0(\cdot)$ depends on A's reputation and concession rate at both τ_0^* and τ_1^*. However, to maintain uniform deadlines, it must be that if B refrains from proposing the fair contract at τ_0^*, A's reputation grows at a rate of λ_S^A from τ_0^* to τ_1^*. This guarantees that even in the absence of the fair contract at τ_0^*, A's reputation progresses as it actually happened, ensuring that when the players reach τ_1^*, the absence of $(1/2, 1/2)$ at τ_0^* is inconsequential due to A's reputation evolving at rate λ_S^A, thereby maintaining identical deadlines $T_S^0(\tau_0^*)$ and $T_S^0(\tau_1^*)$. Nonetheless, as hinted by Lemma 1.12, if B refrains from offering $(1/2, 1/2)$ at τ_0^*, A's reputation progresses at a rate of λ_N^A. Given that $\lambda_N^A \neq \lambda_S^A$, randomization between τ_0^* and τ_1^* is untenable.

Theorem 1.14. *In any signaling sequential equilibrium,* $|supp(Y^B)| = 1$.

Now, we know that in any separating sequential equilibrium, the fair contract can be offered, with positive probability, on a single period τ^* only. By Lemma 1.12 we also know that θ_r^A does not concede at $(\tau^*, +1)$, no matter the contract offered by B at $(\tau^*, 0)$. Therefore, in the GAS at $(\tau^*, 0)$ we have

$$T_S^0(\tau^*) = T_S^A(\tau^*) = -\frac{\mu_i^B((\tau^*, 0); \theta_b^A)}{\lambda_S^A} + \tau^*$$

and

$$T_N^0(\tau^*) = T_N^A(\tau^*) = -\frac{\mu_i^B((\tau^*, 0); \theta_b^A)}{\lambda_N^A} + \tau^*.$$

Since

$$\lambda_N^A = \frac{(1-\gamma_A)\delta}{\gamma_A + \gamma_B - 1} > \frac{(1-\gamma_A)\delta}{\gamma_A + 1/2 - 1} = \lambda_S^A,$$

we get

$$T_S^0(\tau^*) > T_N^0(\tau^*) \quad \forall \tau^*.$$

From this, the next Corollary follows.

Corollary 1.15. *In any separating sequential equilibrium,* $Y_o^B(\tau^*) = 1$.

Proof. Recall $Y_i^B(\tau^*|\tau^*) \in (0,1)$ by Lemma 1.10, and so $Y_i^B(\tau^*) \in (0,1)$. That is, θ_i^B offers the fair contract at $(\tau^*, 0)$ with a probability strictly less than 1. Moreover, from Theorem 1.14, $Y_i^B(n) = 0\ \forall n \neq \tau^*$. Hence, θ_i^B is indifferent between offering $(1/2, 1/2)$ at $(\tau^*, 0)$ and conceding at $T_S^0(\tau^*)$ or concession to A's behavioral demand at $T_N^0(\tau^*)$. That is, in equilibrium,

$$U_i^B(\Sigma^A, (\tau^*, T_S^0(\tau^*))) \ = \ U_i^B(\Sigma^A, T_N^0(\tau^*)).$$

Therefore,

$$(1-z)U_o^B(\Sigma^A, (\tau^*, T_S^0(\tau^*))) + z(1-\gamma_A)e^{-\delta T_S^0(\tau^*)}$$

$$=$$

$$(1-z)U_o^B(\Sigma^A, T_N^0(\tau^*)) + z(1-\gamma_A)e^{-\delta T_N^0(\tau^*)}$$

Since $T_S^0(\tau^*) > T_N^0(\tau^*)$, $z(1-\gamma_A)e^{-\delta T_N^0(\tau^*)} > z(1-\gamma_A)e^{-\delta T_S^0(\tau^*)}$, and so

$$U_o^B(\Sigma^A, (\tau^*, T_S^0(\tau^*))) \ > \ U_o^B(\Sigma^A, T_N^0(\tau^*)).$$

Since $T_N^0(\tau^*)$ is θ_o^B's optimal action in the GNS, θ_o^B strictly prefers to offer the fair contract at τ^*, and so $Y_o^B(\tau^*|\tau^*) = 1$. Now, since τ_o^* of Lemma 1.12 is equal to 0, we have that B concedes at rate λ_N^B from 0 to τ^*. Yet, since concession from θ_i^B is necessary in equilibrium (Proposition 1.3 can be applied to the GNS as well), we have that θ_i^B is indifferent in any concession in the interval $[0, \tau^*]$. Therefore, θ_o^B strictly prefers to wait until $(\tau^*, 0)$ at least. Hence,

$$\sum_{n=0}^{\tau^*-1} X_{n,o}^B([n, n+1]) \ = \ 0,$$

and so $Y_o^B(\tau^*) = 1$. □

We summarize now the properties of any separating equilibrium we have found. For $\gamma_B > 1/2$, there exists a unique equilibrium in which the fair contract is never offered. For $\gamma_B < 1/2$, instead, we get the following.

(1) $\exists! \tau^* \in \mathbb{N}_0$ such that $Y^B(\tau^*) > 0$. For all $n \neq \tau^*$, $Y^B(n) = 0$;
(2) $Y_i^B(\tau^*|\tau^*) \in (0,1)$ and $Y_o^B(\tau^*) = 1$;
(3) A and B concede with rates λ_N^A and λ_N^B respectively in the interval $[0, \tau^*]$;
(4) At $(\tau^*, +1)$ player A does not concede with positive probability, no matter B's standing offer.

Note that these condition are necessary for any separating equilibrium, therefore we need to make sure that they do not create a contradiction. In particular, we need to check the payoff indifference conditions for the rational types. Hence, we show how to make sure that all four condition holds in equilibrium so that no rational type has profitable deviations.

(1) For the first condition, we need that both θ_i^B and θ_o^B are not willing to offer the fair contract at $n < \tau^*$. This is trivial when $\tau^* = 0$, so suppose $\tau^* > 0$ and take $n \in \mathbb{N}_0$ such that $n < \tau^*$. From Definition 1.6.4, we have that $(\sigma_i^B[n], \sigma_o^B[n], \sigma_r^A[n])$ is a GAS degenerate or nondegenerate equilibrium. Hence, we can assume that for all $n < \tau^*$, $(\sigma_i^B[n], \sigma_o^B[n], \sigma_r^A[n])$ is a GAS degenerate equilibrium. In this case, any offer of the fair contract before τ^* implies a payoff of $(1 - \gamma_A)e^{-\delta n}$ for θ_i^B and θ_o^B. Clearly, θ_i^B is indifferent between the deviation and the equilibrium strategy, and therefore θ_o^B strictly prefers to not offer it at $(n, 0)$. Hence, $Y^B(n) = 0$ can be part of the equilibrium. For what regards its consistency, it is enough that any deviation at n is sustained by A's beliefs that put equal probability of mistake by θ_i^B and θ_o^B (clearly, these are not the unique beliefs that can sustain it).

(2) By the previous point 4., we have that A does not concede at $(\tau^*, +1)$, no matter the contract. By Lemma 1.8, we have that $(\sigma_i^B[\tau^*], \sigma_o^B[\tau^*], \sigma_r^A[\tau^*])$ is the GAS nondegenerate equilibrium. Hence, θ_i^B's payoff in case of fair contract offer is $(1 - \gamma_A)e^{-\delta\tau^*}$. In case the fair contract is not offered, A and B play the unique GNS equilibrium, and since A does not concede at $(\tau^*, +1)$, we get that θ_i^B's expected payoff of offering $(1 - \gamma_B, \gamma_B)$ is $(1 - \gamma_A)e^{-\delta\tau^*}$. Therefore, θ_i^B is indifferent and we can have $Y_i^B(\tau^*|\tau^*) \in (0, 1)$. In the proof of Corollary 1.15 we have shown that θ_i^B's indifference at $(\tau^*, 0)$ implies θ_o^B strictly prefers to offer the fair contract at $(\tau^*, 0)$. Therefore, we have $Y_i^B(\tau^*|\tau^*) = 1$, and since θ_o^B does not concede in $[0, \tau^*]$ (proof of Corollary 1.15) we get $Y_o^B(\tau^*) = 1$.

(3) Clear from Lemma 1.12.

(4) Clear from Lemma 1.12.

We need one last condition for the signaling equilibrium. We have that $(\tau^*, -1)$ is in θ_r^A concession support, and we know that at $(\tau^*, 0)$ θ_r^A's jump, depending on the contract offer. This may cause a jump in θ_r^A's payoff. Moreover, the contract offered from B may change, and this is another source of jump in θ_r^A's expected utility. Therefore, in order for θ_r^A to be indifferent between concession at $(\tau^*, -1)$ and any period after τ^*, we need to calibrate B's concessions at $(\tau^*, +1)$ and the probability of fair contract offer $Y^B(\tau^*)$. So, for ease of notation let $\overline{U}_r^A$ the equilibrium utility of θ_r^A of concession after $(\tau^*, +1)$, and $\underline{U}_r^A$ the equilibrium utility of concession at $(\tau^*, -1)$. We have

$$\begin{aligned}\overline{U}_r^A &= \sum_{n=0}^{\tau^*-1}\int_n^{n+1}\gamma_A e^{-\delta t}dX_n^B(t)\\ &+ Y^B(\tau^*)[F_S^B(\tau^*;\tau^*)\gamma_A + (1-F_S^B(\tau^*;\tau^*))1/2]e^{-\delta\tau^*}\\ &+\left(1-\sum_{n=0}^{\tau^*-1}X_n^B([n,n+1]) - Y^B(\tau^*)\right)[F_N^B(\tau^*;\tau^*)\gamma_A + (1-F_N^B(\tau^*;\tau^*))(1-\gamma_B)]e^{-\delta\tau^*},\end{aligned}$$

and

$$\underline{U}_r^A = \sum_{n=0}^{\tau^*-1}\int_n^{n+1}\gamma_A e^{-\delta t}dX_n^B(t) + \left(1-\sum_{n=0}^{\tau^*-1}X_n^B([n,n+1])\right)(1-\gamma_B)e^{-\delta\tau^*}.$$

Hence, $\overline{U}_r^A = \underline{U}_r^A$ implies

$$\begin{aligned}&Y^B(\tau^*)[F_S^B(\tau^*;\tau^*)\gamma_A + (1-F_S^B(\tau^*;\tau^*))1/2]e^{-\delta\tau^*}\\ &+\left(1-\sum_{n=0}^{\tau^*-1}X_n^B([n,n+1]) - Y^B(\tau^*)\right)[F_N^B(\tau^*;\tau^*)\gamma_A + (1-F_N^B(\tau^*;\tau^*))(1-\gamma_B)]e^{-\delta\tau^*}\\ &=\\ &\left(1-\sum_{n=0}^{\tau^*-1}X_n^B([n,n+1])\right)(1-\gamma_B)e^{-\delta\tau^*}.\end{aligned}$$

This equation can be rewritten as

$$\begin{aligned}&\frac{Y^B(\tau^*)}{1-\sum_{n=0}^{\tau^*-1}X_n^B([n,n+1])}[F_S^B(\tau^*;\tau^*)\gamma_A + (1-F_S^B(\tau^*;\tau^*))1/2]\\ &+\left(1-\frac{Y^B(\tau^*)}{1-\sum_{n=0}^{\tau^*-1}X_n^B([n,n+1])}\right)[F_N^B(\tau^*;\tau^*)\gamma_A + (1-F_N^B(\tau^*;\tau^*))(1-\gamma_B)]\\ &=\\ &(1-\gamma_B),\end{aligned}$$

and so,

$$\begin{aligned}&Y^B(\tau^*|\tau^*)F_S^B(\tau^*;\tau^*)\gamma_A + (1-F_S^B(\tau^*;\tau^*))1/2]\\ &+\left(1-Y^B(\tau^*|\tau^*)\right)[F_N^B(\tau^*;\tau^*)\gamma_A + (1-F_N^B(\tau^*;\tau^*))(1-\gamma_B)]\\ &=\\ &(1-\gamma_B).\end{aligned}$$

Note that this is the indifference condition of θ_r^A at continuation game $(\tau^*,-1)$ for immediate concession and any concession afterwards. Therefore, $Y^B(\tau^*|\tau^*)$, $F_S^B(\tau^*;\tau^*)$ and $F_N^B(\tau^*;\tau^*)$ must satisfy this equation. Note that $F_S^B(\tau^*;\tau^*)$ and $F_N^B(\tau^*;\tau^*)$ depends on θ_r^A's beliefs, and these beliefs depend on $Y^B(\tau^*;\tau^*)$. Therefore, in order to conclude, we give an example of a separating equilibrium, showing the relation among these variables. The probabilities of concession at $(0,0)$ are going to be fundamental as the allow for beliefs manipulation. Moreover, recall that in equilibrium $T_S^0(\tau) = T_S^A(\tau^*)$ and $T_N^0(\tau^*) = T_N^A(\tau^*)$, which add two additional constraint on the equilibrium, that is, $T_S^A(\tau^*) \leq T_S^B(\tau^*)$ and $T_N^A(\tau^*) \leq T_N^B(\tau^*)$. These constraint are dependent on beliefs as well and so they are dependent also on the probabilities of concession at $(0,0)$. The following Proposition and remark are useful for the computation of $Y^B(\tau^*|\tau^*)$ and θ_r^A's indifference condition at $(\tau^*,-1)$.

Proposition 1.16. *We have*

$$Y^B(\tau^*|\tau^*) \;=\; \mu_r^A((\tau^*,-1);\theta_o^B) + \mu_r^A((\tau^*,-1);\theta_i^B)Y_i^B(\tau^*|\tau^*).$$

Remark. *Let* $x = Y_i^B(\tau^*|\tau^*)$*,* $\mu_r^A((\tau^*,-1);\theta_i^B) = Q_i$ *and* $\mu_r^A((\tau^*,-1);\theta_o^B) = Q_o$ *for ease of notation. Then, when* $(1/2, 1/2)$ *is offered,* θ_r^A*'s belief is:*

$$\mu_r^A((\tau^*,0);\theta_o^B) \;=\; \frac{Q_o}{Q_o + Q_i x},$$

while if $(1-\gamma_B,\gamma_B)$ *is offered, her belief is:*

$$\mu_r^A((\tau^*,0);\theta_b^B) \;=\; \frac{Q_b}{Q_b + Q_i(1-x)}.$$

Moreover, note

$$\begin{aligned} F_S^B(\tau^*;\tau^*) \;&=\; 1 - e^{-\lambda_S^B(T_S^B - T_S^A)} \\ &=\; 1 - e^{-\lambda_S^B\left(-\log\left(\frac{Q_o}{Q_o+Q_i x}\right)\frac{1}{\lambda_S^B} + \log(\mu_i^B((\tau^*,0);\theta_b^A))\frac{1}{\lambda_S^A}\right)} \\ &=\; 1 - \frac{Q_o}{Q_o + Q_i x}\mu_i^B((\tau^*,0);\theta_b^A)^{-\frac{1/2}{1-\gamma_A}}. \end{aligned}$$

In the same fashion, we can prove

$$F_N^B(\tau^*|\tau^*) \;=\; 1 - \frac{Q_b}{Q_b + Q_b(1-x)}\mu_i^B((\tau^*,0);\theta_b^A)^{-\frac{1-\gamma_B}{1-\gamma_A}}.$$

Let $Z = \mu_i^B((\tau^*,0);\theta_b^A)$*.* θ_r^A*'s indifference condition is*

$$\begin{aligned} &(Q_o + Q_i x)\left[\left(1 - \frac{Q_o}{Q_o + Q_i x}Z^{-\frac{1/2}{1-\gamma_A}}\right)\gamma_A + \frac{Q_o}{Q_o + Q_i x}Z^{-\frac{1/2}{1-\gamma_A}}\frac{1}{2}\right] \\ &+ (\overbrace{1 - Q_i}^{Q_b+Q_i} - Q_i x)\left[\left(1 - \frac{Q_b}{Q_b + Q_i(1-x)}Z^{-\frac{1-\gamma_B}{1-\gamma_A}}\right)\gamma_A + \frac{Q_b}{Q_b + Q_i(1-x)}Z^{-\frac{1-\gamma_B}{1-\gamma_A}}(1-\gamma_B)\right] \\ &= \\ &(1-\gamma_B) \end{aligned}$$

The LHS of this equation can be rewritten as:

$$\left(Q_o + Q_i x - Q_o Z^{-\frac{1/2}{1-\gamma_A}}\right)\gamma_A + Q_o Z^{-\frac{1/2}{1-\gamma_A}}\frac{1}{2} + \left(Q_b + Q_i(1-x) - Q_b Z^{-\frac{1-\gamma_B}{1-\gamma_A}}\right)\gamma_A + Q_b Z^{-\frac{1-\gamma_B}{1-\gamma_A}}(1-\gamma_B)$$

$$=$$

$$\left(Q_o - Q_o Z^{-\frac{1/2}{1-\gamma_A}}\right)\gamma_A + Q_o Z^{-\frac{1/2}{1-\gamma_A}}\frac{1}{2} + \left(Q_b + Q_i - Q_b Z^{-\frac{1-\gamma_B}{1-\gamma_A}}\right)\gamma_A + Q_b Z^{-\frac{1-\gamma_B}{1-\gamma_A}}(1-\gamma_B)$$

Hence, the indifference condition is independent from x, i.e., $Y_i^B(\tau^*|\tau^*)$, as Q_o and Q_i are beliefs computed at $(\tau^*,-1)$. The only role that $Y_i^B(\tau^*|\tau^*)$ plays is in making sure that the constraints $T_S^A(\tau^*) \leq T_S^B(\tau^*)$ and $T_N^A(\tau^*) \leq T_N^B(\tau^*)$ are satisfied. This generates a continuum of separating equilibria.

Consistency

We have not specified how consistency is obtained so far. Clearly, consistency in the GAS is taken from the results in the section of the model with one behavioral type. Instead, the consistency before the fair contract is not offered is obtained through Bayes' rule as long as there are no surprise events. Note that the only surprise event in which the war of attrition does not end is an unexpected offer of the fair contract, for some $n \neq \tau^*$. For simplicity, consider $n < \tau^*$. In this case, as previously described, we can just assume that A believes that the mistake was made with equal probability by θ_i^B and θ_o^B. Here, as the probability of mistake goes to zero, beliefs are

$$\mu_r^A((n,0);\theta_i^B) = \frac{\mu_r^A((n,-1);\theta_i^B)}{\mu_r^A((n,-1);\theta_i^B) + \mu_r^A((n,-1);\theta_o^B)}$$

and

$$\mu_r^A((n,0);\theta_o^B) = \frac{\mu_r^A((n,-1);\theta_o^B)}{\mu_r^A((n,-1);\theta_i^B) + \mu_r^A((n,-1);\theta_o^B)}.$$

Both are strictly less than 1. Then, we can easily sustain the equilibrium by letting A and B to play the nondegenerate GAS equilibrium.

Example 1.17. Consider the following set of parameters.

Table 1.1. Example table with parameters and values

Parameter	Value
γ_A	0.8
γ_B	0.4
δ	1
q_o	$0.2e^{-3}$
q_b	$0.1e^{-3}$
z	0.2

Clearly, we have $q_i = 1 - q_o - q_b$. Note that $\gamma_A + \gamma_B > 1$, so that behavioral demands are incompatible. We let $\tau^* = 1$. That is, in equilibrium, B offers the fair contract with positive probability only at period 1. The concessions rate are the following:

$$\lambda_S^A = \frac{(1-\gamma_A)\delta}{\gamma_A + 1/2 - 1} = \frac{2}{3} \qquad \lambda_S^B = \frac{1/2\delta}{\gamma_A + 1/2 - 1} = \frac{5}{3}$$
$$\lambda_N^A = \frac{(1-\gamma_A)\delta}{\gamma_A + \gamma_B - 1} = 1 \qquad \lambda_N^B = \frac{(1-\gamma_B)\delta}{\gamma_A + \gamma_B - 1} = 3$$

By Lemma 1.12, we know that A and B concede with rates λ_N^A and λ_N^B respectively. Therefore, we have

$$Q_o + Q_b = \mu_r^A((\tau^*, -1); \theta_b^B, \theta_o^B) = (0.3e^{-3})e^{\lambda_N^B} = 0.3.$$

Note that this implies $Q_b = \mu_r^A((1,-1);\theta_b^B) = 0.1$ and $Q_o = \mu_r^A((1,-1);\theta_o^B) = 0.2$. Now, let Z^* be the value that satisfies the previous indifference condition. We can derive $F^A(0;0)$, i.e., the probability of concession from A at time $(0,0)$ that makes her reputation jump from z to $Z^* e^{-\lambda_N^A}$. Note that given the parameters given and $Q_o = 0.2$, $Q_b = 0.1$ we have $Z^* \approx 0.7$, which gives $Z^* e^{-\lambda_N^A} \approx 0.26$. Hence, we compute θ_r^A's probability of concession at $(0,+1)$ that allows this shift in θ_i^B's beliefs. Denote y the probability of θ_r^A's concession at $(0,+1)$.

$$Z^* e^{-1} = \frac{z}{z + (1-z)(1-y)} \Rightarrow y = \frac{1}{4}(5 - e/Z^*) \approx 0.28.$$

Therefore, for a reputation jump from $z = 0.2$ to $Z^* e^{-1} \approx 0.26$ type θ_r^A needs to concede with approximate probability of 0.28. Concession at $(0,+1)$ from A implies no concession from B at the same date. Finally, we need to check that the constraints $T_S^A(\tau^*) \le T_S^B(\tau^*)$ and $T_N^A(\tau^*) \le T_N^B(\tau^*)$ are satisfied. Therefore,

$$T_S^A(1) = -\frac{\log(Z^*)}{2/3} + 1 \approx 1.52$$
$$T_S^B(1) = -\log\left(\frac{Q_o}{Q_o + Q_i x}\right)\frac{3}{5} + 1 = -\log\left(\frac{2}{2+7x}\right)\frac{3}{5} + 1$$
$$T_N^A(1) = -\frac{-\log(Z^*)}{1} + 1 \approx 1.34$$
$$T_N^B(1) = -\log\left(\frac{Q_b}{Q_b + Q_i(1-x)}\right)\frac{1}{3} + 1 = -\log\left(\frac{1}{8-7x}\right)\frac{1}{3} + 1.$$

From these, we get

$$x = Y_i^B(1|1) \in \left(\frac{2(Z^*)^{-5/2} - 2}{7}, \frac{8 - (Z^*)^{-3}}{7}\right) \approx (0.06, 0.73).$$

Figure 1.2 describes the unfolding of the equilibrium graphically.

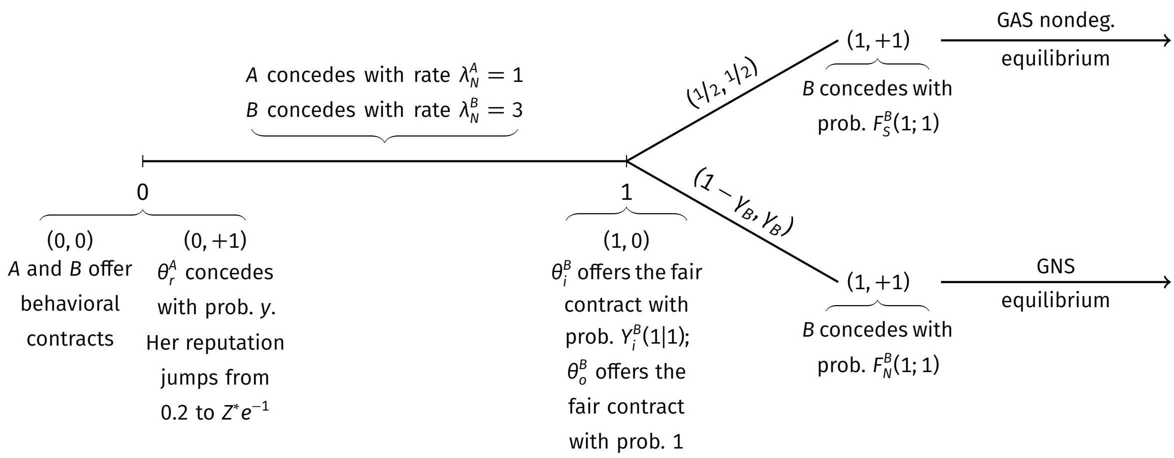

Figure 1.2. Example of a Signaling Equilibrium

1.5 Conclusion

In conclusion, this paper explores the role of reputation in bargaining scenarios where an agent may be aware of the other's rationality. Hence, we examined how reputation effects can arise not only from direct uncertainty about an opponent's type, but also from second-order beliefs, where one agent attempts to appear tough despite the opponent's awareness of their true nature. The other main ingredient of the model is the possibility of the informed type to signal his knowledge by a fair split 50-50 of the surplus. By analyzing a bargaining game with both discrete and continuous time dynamics, we focused on the strategic choices player *B*, and his respective types—rational, ignorant, or omniscient.

In the one behavioral type model, our results highlight two main classes of equilibria. In one, *B* concedes immediately regardless of his type, due to the belief that *A* is so stubborn that any informational advantage is useless. In the other, the omniscient type of *B* does not concede, leading to a situation where both agents engage in signaling games to convince the other of their toughness or knowledge.

We then linked this analysis to the original two-behavioral-type model, demonstrating that an equilibrium without fair offers is always possible, especially when *B*'s irrational demand exceeds the fair offer. However, when *B*'s irrational offer is less than 50%, an equilibrium with a fair contract becomes viable, with a specific period in which this contract can be proposed.

Appendix 1.A AG Proposition 1 summary

We present here a summary of AG results of Proposition 1. In their initial model, there are two players A and B, and each of them is one of two possible types: rational and behavioral. Player i is irrational with probability z_i, and demand a fixed amount γ_i, where $\gamma_A + \gamma_B > 1$. Each player is allowed to concede to the other's demand, or wait. Therefore, since each player has only one rational type, strategies can be described by the cdf F_i.

Players discount payoffs exponentially at the rate of δ_i. Utility functions u_i are written in the same way of θ_i^B's utility in the model with one behavioral type. Then, this result follows:

(1) Let $\tau^i = \inf\{t \geq 0 | F_i(t) = \lim_{t' \to \infty} F_i(t')\}$. Then $\tau^A = \tau^B$;
(2) If $\lim_{t' \to t^-} F_i(t') \neq F_i(t)$, then $\lim_{t' \to t^-} F_j(t') = F_j(t)$, for $j \neq i$;
(3) If F_i is continuous at t, then u_i is continuous at (concession at) t;
(4) There is no $t_1, t_2 \in \mathbb{R}_+$ such that $0 \leq t_1 < t_2 \leq \tau_i$ such that F_A and F_B are constant over (t_1, t_2);
(5) F_i is strictly increasing over $(0, \tau^i)$;
(6) F_i is continuous for $t > 0$.

From these properties, AG proves that both players concede at constant rate that makes the opponent indifferent, for every $t > 0$, between immediate concession and waiting. In order to calculate this rate, denote it first by λ_i for player i. Then, in order to make j indifferent, the cost and the benefit of waiting must be the same. The cost of not conceding at t, instead of some $t + dt$, is $\delta_j(1 - \gamma_i)dt$, that is, the lost of the interest of i's offer, $1 - \gamma_i$. The benefit of waiting until $t + dt$ is instead the probability that i concedes in the interval $(t, t + dt)$ times the gain j gets from i's acceptance. Therefore, the benefit is $\lambda_i(\gamma_j - (1 - \gamma_i))dt$. Hence, the rate λ_i of concession that makes j indifferent is

$$\lambda_i = \frac{\delta_j(1 - \gamma_i)}{\gamma_i + \gamma_j - 1}.$$

Hence, we have that $F_i(t) = 1 - (1 - F_i(0))e^{-\lambda_i t}$. From the previous results, we have that $\tau^A = \tau^B$. For this to happen, we need that both players reach reputation[9] 1 at the same time, at some period T^0. Since in general $\lambda_i \neq \lambda_j$, one of the player concedes with positive probability at time $t = 0$ in order to boost her reputation and make sure she reaches reputation 1 at the same time of her opponent. Note that i's reputation reaches 1 at T^i if

$$T^i = -\frac{\log(z_i)}{\lambda_i}.$$

9. The other player must believe with probability 1 that she is irrational.

Therefore, let $T^0 = \min\{T^A, T^B\}$. In case $T^0 = T^j$, then i has to concede at time zero with strictly positive probability $F_i(0)$. We have

$$F_i(0) \;=\; 1 - e^{-\lambda_i(T^i - T^0)}.$$

Appendix 1.B Consistency in the model with one behavioral type

First, we define the definition of convergence of sequence of strategies.

Definition 1.18. A sequence of strategies $(\sigma^k)_{k\in\mathbb{N}}$ is said to converge to the strategy σ if and only if

$$\forall t \in \mathbb{R}, \quad \sigma_t^k \xrightarrow{w} \sigma_t.$$

Definition 1.19. Let $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$ be the vectors collecting the strategies and beliefs of all the players. We say that $(\boldsymbol{\sigma}, \boldsymbol{\mu})$ is a consistent assessment if and only if there exists a sequence of completely mixed strategies $(\boldsymbol{\sigma}^k)_{k\in\mathbb{N}}$ converging to $\boldsymbol{\sigma}$ and a sequence of beliefs $(\boldsymbol{\mu}^k)_{k\in\mathbb{N}}$ converging to $\boldsymbol{\mu}$ in Euclidean space with the property that for each k, $\boldsymbol{\mu}^k$ is derived from $\boldsymbol{\sigma}^k$ using Bayes' rule.

When we write completely mixed, we mean that σ_τ assigns positive probability to each $(a,b) \subseteq [\tau, +\infty)$. Therefore, a sequence $(\sigma^k)_{k\in\mathbb{N}}$ converges to some strategy σ if and only if every continuation game measure of the sequence weakly converges to the limit continuation game measure.
Recall that weakly convergence is necessary and sufficient for convergence in distribution. Hence, let F^k and F be the cdf associated with σ_t^k and σ_t respectively. We have that

$$\sigma_t^k \xrightarrow{w} \sigma_t \iff F_k(x) \;\to\; F(x),\ x \text{ continuity point of } F.$$

Therefore, even though the definition are described using measures $\sigma(t)$, we prove the statements using their respective cdf $F(\cdot|t)$, i.e., conditional distributions. Moreover, we use $F(\cdot)$ to denote $F(\cdot|0)$. Observe there is no ambiguity as we can always derive a cdf from a measure and vice versa. Conditional utilities are derived using continuation game beliefs $\mu(\cdot|\tau)$ and conditional distribution $F(\cdot|\tau)$.

In section 1.3.2, we proposed the following candidate equilibrium:

- $F_i^B(t|\tau) = F_o^B(t|\tau) = \begin{cases} 1 & \text{if } t \geq \tau \\ 0 & \text{otherwise} \end{cases}$
- $F_r^A(t) = 0$ for all $t \geq 0$.

We already showed sequential rationality for all the types. Now we prove consistency of F_i^B and F_o^B (as F_r^A can be trivially be proven to be consistent since μ_i^B is always updated through Bayes' rule).

We now prove consistency. That is, there exists a sequence of completely mixed strategy such that it converges to the equilibrium assessment and beliefs are always derived from Bayes' rule in the sequence. We start with F_r^A and beliefs μ_i^B. Observe that in equilibrium, we must have $\mu_i^B(t) = z$ for each t, as the rational type never concedes. Consider a sequence of completely mixed strategies $\sigma_{r,k}^A$, that assigns probability $1/k$ over the interval $[0, +\infty)$ and $1 - 1/k$ to $\{+\infty\}$. When θ_i^B observe waiting at time t, he updates beliefs $\mu_{i,k}^B(t) = \frac{(1-z)(1-F_{r,k}^A(t))}{(1-z)(1-F_{r,k}^A(t))+z}$, where $F_{r,k}^A$ is the cdf that describes how $\sigma_{r,k}^A$ distributes the mass $1/k$ over $[0, +\infty)$. Since $F_{r,k}^A(t) \to 0$ for each $t \geq 0$, we have that $\mu_{i,k}^B \to \mu_i^B$. Hence, F_r^A and μ_i^B satisfy the consistency requirement.

Now consider F_i^B, F_o^B and μ_r^A. In order to sustain the equilibrium, we set $\mu_r^A(t) = q$ for each $t \geq 0$. Observe that these beliefs allow F_r^A to satisfy the sequential rationality requirement. Consider the sequence of completely mixed strategies $\sigma_{i,k}^B$, $\sigma_{o,k}^B$ such that $\sigma_{i,k}^B = \sigma_{o,k}^B$ for all k, and both have cdf $F_k(t) = 1 - e^{-kt}$. Then, the conditional distribution upon reaching continuation game τ is

$$\begin{aligned} F_k(t|\tau) &= \frac{F(t) - F(\tau)}{1 - F(\tau)} \\ &= \frac{(1 - e^{-kt}) - (1 - e^{-k\tau})}{1 - (1 - e^{-k\tau})} \\ &= 1 - e^{-k(t-\tau)}. \end{aligned}$$

Then, for each continuity point $t > \tau$, we have that $F_k(t|\tau) \to 1 = F_i^B(t|\tau) = F_o^B(t|\tau)$, as $k \to +\infty$. Therefore, this strategy converges to the candidate equilibrium strategy. Moreover, since θ_i^B and θ_o^B use the same strategy for each k, θ_r^A have constant beliefs in every continuation game τ, that is, $\mu_r^A(\tau) = Pr(\theta^B = \theta_i^B|\tau) = q$ for each $\tau \geq 0$. Hence, $((\sigma_{i,k}^B, \sigma_{o,k}^B), \mu_{r,k}^A)) \to ((\sigma_i^B, \sigma_o^B), \mu_r^A)$ where $\mu_{r,k}^A(\tau) = q$ for each k and τ and (σ_i^B, σ_o^B) is the candidate strategy for B. Thus, $F_i^B(\cdot|\tau)$, F_o^B and μ_r^A satisfy the consistency requirement for each τ. Therefore, $\big((F_i^B(\cdot|\tau), F_o^B(\cdot|\tau), F_r^A), (\mu_i^B, \mu_r^A)\big)_{\tau \geq 0}$ is a consistent assessment and therefore it is a sequential equilibrium.

Appendix 1.C The information structure

We propose an information structure that could generate the type space introduced in the model with one behavioral type. From this, we can easily construct a larger information structure capable of generating the model with two behavioral types. We do this in the spirit of Milgrom and Roberts (1982), who propose in their appendix an information structure that could induce reputation effects even when one of the agents knows the other is rational. As they emphasize, the key factor is that the agent attempting to build a reputation is unaware that the other knows about her rationality.

We start by constructing an Aumann model of incomplete information and then we derive the corresponding Harsanyi type space. [10] There are two players, A and B, who bargain over some surplus. There is a set of states of nature $\mathscr{S} = \{s_1, s_2\}$. In s_1, A is irrational (or behavioral), while in s_2 is rational. B is known to be rational by both players and this constitutes common knowledge. We then construct a set of states of the world $\Omega := \{a, b, c\}$. The function that associates each state of the world to the states of nature is $\mathfrak{s} : \Omega \longrightarrow \mathscr{S}$, such that

$$\begin{aligned}\mathfrak{s}(a) &= s_1 \\ \mathfrak{s}(b) &= \mathfrak{s}(c) = s_2.\end{aligned}$$

Hence, in the first state of the world, A is irrational, while in the other two states, he is rational. The players' information sets are the following:

$$\begin{aligned}\mathscr{F}_A &:= \{\{a\}, \{b, c\}\} \\ \mathscr{F}_B &:= \{\{a, b\}, \{c\}\}.\end{aligned}$$

Observe that A can only distinguish the set of states in which he is rational or not, while B can either have complete information (in $\omega = c$) or be completely ignorant (state $\omega \in \{a, b\}$).

In this framework, A's rationality cannot be common knowledge in any of the states. In fact, in state $\omega = b$ player B cannot distinguish the rational type from the behavioral type. In state $\omega = c$, B knows that player A is rational but the latter does not possess this information. As noted in Appendix B of Milgrom and Roberts (1982), reputation effects can emerge even when both players are rational and know that the other is rational. What is key, is the absence of its common knowledge. In this model, common knowledge of rationality fails because player A never knows whether B possesses information about the true state of nature. This missing link will generate reputation strategies from the rational player A. From this information structure, we can derive the usual Harsanyi types. Consider player A. Associate to the first partition element, $\{a\}$, the type θ_b^A, which corresponds to his behavioral type. Then, associate with his second element, $\{b, c\}$, the type θ_r^A, the rational type. Apply the same process to player B. We obtain θ_i^B, the ignorant type, corresponding to the partition element $\{a, b\}$, and θ_o^B, the omniscient type, for $\{c\}$. Note that Harsanyi players' types are not independent. In fact, when $\theta^B = \theta_o^B$, B assigns probability 1 to the event (hence, knows) $\theta^A = \theta_r^A$. Therefore, with this type structure, we get the following joint mass distribution of types:

- $p(\theta_b^A, \theta_i^B) = Pr(\omega = a) = p_a$;
- $p(\theta_b^A, \theta_o^B) = 0$;

10. For a reference, see Maschler, Solan, and Zamir (2013), Chapter 9.

- $p(\theta_r^A, \theta_i^B) = Pr(\omega = b) = p_b$;
- $p(\theta_r^A, \theta_o^B) = Pr(\omega = c) = p_c$.

Therefore, for example, when B is of type θ_i^B, he has the following beliefs:

$$z := Pr(\theta_b^A|\theta_i^B) = \frac{p_a}{p_a + p_b} \quad Pr(\theta_r^A|\theta_i^B) = \frac{p_b}{p_a + p_b} = 1 - z.$$

Note we defined with z the probability that A is irrational when B is ignorant. We also define $q := Pr(\theta_i^B|\theta_r^A)$.

Appendix 1.D Proofs

Proof of Proposition 1.3

Proof. We only consider the case of indifference between concession at t and t'. The case of strict preference easily follows.

(i) By assumption,

$$(1-\mu_i^B(\tau)) \cdot \left[\int_0^t \frac{1}{2} e^{-\delta x} d\sigma_{r,\tau}^A(x) + \left(\sigma_{r,\tau}^A(t)\frac{1}{2}\left(3/2-\gamma\right) + \sigma_{r,\tau}^A((t,+\infty])(1-\gamma)\right)e^{-\delta t}\right] + \mu_i^B(\tau)\cdot(1-\gamma)e^{-\delta t}$$

$$=$$

$$(1-\mu_i^B(\tau)) \cdot \left[\int_0^{\hat{t}} \frac{1}{2} e^{-\delta x} d\sigma_{r,\tau}^A(x) + \left(\sigma_{r,\tau}^A(\hat{t})\frac{1}{2}\left(3/2-\gamma\right) + \sigma_{r,\tau}^A((\hat{t},+\infty])(1-\gamma)\right)e^{-\delta \hat{t}}\right] + \mu_i^B(\tau)\cdot(1-\gamma)e^{-\delta \hat{t}}.$$

Observe that $\mu_i^B(\tau)(1-\gamma)e^{-\delta t} > \mu_i^B(\tau)(1-\gamma)e^{-\delta \hat{t}}$, hence

$$\int_0^{\hat{t}} \frac{1}{2} e^{-\delta x} d\sigma_{r,\tau}^A(x) + \left(\sigma_{r,\tau}^A(\hat{t})\frac{1}{2}\left(3/2-\gamma\right) + \sigma_{r,\tau}^A((\hat{t},+\infty])(1-\gamma)\right)e^{-\delta \hat{t}} \quad (1.D.1)$$

$$>$$

$$\int_0^{t} \frac{1}{2} e^{-\delta x} d\sigma_{r,\tau}^A(x) + \left(\sigma_{r,\tau}^A(t)\frac{1}{2}\left(3/2-\gamma\right) + \sigma_{r,\tau}^A((t,+\infty])(1-\gamma)\right)e^{-\delta t}, \quad (1.D.2)$$

where (1.D.1) and (1.D.2) are θ_o^B's payoffs when he concedes at t' and t respectively. Therefore, θ_o^B prefers to concede at t'.

(*ii*) Following the same lines, we assume

$$\int_0^{\hat{t}} \frac{1}{2} e^{-\delta x} d\sigma^A_{r,\tau}(x) + \left(\sigma^A_{r,\tau}(\hat{t}) \frac{1}{2} \left(3/2 - \gamma\right) + \sigma^A_{r,\tau}((\hat{t}, +\infty])(1-\gamma)\right) e^{-\delta \hat{t}}$$

$$=$$

$$\int_0^{t} \frac{1}{2} e^{-\delta x} d\sigma^A_{r,\tau}(x) + \left(\sigma^A_{r,\tau}(t) \frac{1}{2} \left(3/2 - \gamma\right) + \sigma^A_{r,\tau}((t, +\infty])(1-\gamma)\right) e^{-\delta t}.$$

Then,

$$(1-\mu^B_i(\tau)) \cdot \left[\int_0^{t} \frac{1}{2} e^{-\delta x} d\sigma^A_{r,\tau}(x) + \left(\sigma^A_{r,\tau}(t) \frac{1}{2} \left(3/2 - \gamma\right) + \sigma^A_{r,\tau}((t, +\infty])(1-\gamma)\right) e^{-\delta t}\right] + \mu^B_i(\tau) \cdot (1-\gamma) e^{-\delta t}$$

$$>$$

$$(1-\mu^B_i(\tau)) \cdot \left[\int_0^{\hat{t}} \frac{1}{2} e^{-\delta x} d\sigma^A_{r,\tau}(x) + \left(\sigma^A_{r,\tau}(\hat{t}) \frac{1}{2} \left(3/2 - \gamma\right) + \sigma^A_{r,\tau}((\hat{t}, +\infty])(1-\gamma)\right) e^{-\delta \hat{t}}\right] + \mu^B_i(\tau) \cdot (1-\gamma) e^{-\delta \hat{t}}.$$

Therefore θ^B_i strictly prefers to concede at t.

□

Proof of Proposition 1.5

Proof. First, we show that whenever $F^A_r \notin DE$, then no vector (F^B_i, F^B_o, F^A_r) can be sustained as a degenerate sequential equilibrium. Consider the first condition of the set DE, and suppose otherwise, i.e., $F^A_r(0) > 0$. Then, B (each type) has an incentive to wait an $\varepsilon > 0$ at $t = 0$, and hence $F^B_i(0) = F^B_o(0) = 0$. Since we assume the equilibrium is degenerate, it must be that $F^A_r(0) = 1$. Yet, if θ^A_r wait an $\varepsilon > 0$, then θ^B_i assigns probability 1 to the event $\theta^A = \theta^A_b$ and concedes immediately. Hence, θ^A_r strictly prefers to concede at $t = \varepsilon$ than at $t = 0$, a contradiction. Hence $F^A_r(0) = 0$. Now consider the second condition, that is, F^A_r does not admits a jump at some t. On the contrary, suppose there exists $t > 0$ such that $\Delta(F^A_r(t)) > 0$. Therefore θ^A_r and so A concedes with positive probability at time t. But then, each type of B does not concede in the interval $[t-\varepsilon, t]$ for some $\varepsilon > 0$. Hence, F^B_i and F^B_o are constant over the interval $[t-\varepsilon, t]$. In this case, θ^A_r either prefers to concede at $t-\varepsilon$ or strictly after t. This implies that F^A_r is constant too on the interval $[t-\varepsilon, t+\sigma]$ for some $\sigma > 0$.

But this is a contradiction, as $\Delta(F_r^A(t)) > 0$. Thus, in any degenerate equilibrium, F_r^A does not admit jumps.

Consider then *DE* final condition (1.2). Suppose it is violated for some $t \geq 0$, some $t' > t$ such that $t' = t + \varepsilon$. Observe that in any equilibrium, the set of optimal ε is such that there exists ε' where $t + \varepsilon' = t^A_{r,max}$, where $t^A_{r,max} = \inf\{t | F_r^A(t) = \lim_{\tau \to +\infty} F_r^A(\tau)\}$. In fact, suppose otherwise, i.e., in a sequential equilibrium the type θ_o^B wants to concede at $t + \varepsilon < t^A_{r,max}$ after they reached continuation game t, and they strictly prefer this choice to concession at $t^A_{r,max}$. By Proposition 1.3, θ_i^B concedes no later than $t + \varepsilon$. Therefore, θ_r^A knows that B concedes before $t + \varepsilon$ with probability 1. Then there exists $\sigma > 0$ such that θ_r^A waits in the interval $[t + \varepsilon - \sigma, t + \varepsilon]$. But if this is the case, θ_o^B either concedes before $t + \varepsilon - \sigma$ or strictly after $t + \varepsilon$, a contradiction. We can conclude that when the players reach continuation game t, θ_o^B's strategy support includes $t^A_{r,max}$. Let t^* be the infimum of the set of t such that θ_o^B can optimally concede at $t^A_{r,max}$ when players reach continuation game t. Suppose first $t^* > 0$. At t^*, either θ_o^B is indifferent between concession at t^* and concession at $t^A_{r,max}$ or he strictly prefers to concede at $t^A_{r,max}$. Suppose he is indifferent. Then, by Proposition 1.3 type θ_i^B strictly prefers immediate concession. By continuity, there exists $\sigma > 0$ such that θ_i^B strictly prefers to concede at $\tau \in [t^* + \sigma, t^A_{r,max})$ than at $t^A_{r,max}$. Again, by continuity and definition of t^*, θ_o^B strictly prefers to concession at $t^A_{r,max}$ than concession at τ in the continuation game starting at $t^* + \sigma$. Hence, at τ, B is playing a separating strategy. Then θ_i^B can profitably deviate imitating θ_o^B at τ, a contradiction.

Now suppose $t^* = 0$. θ_o^B cannot be indifferent between concession at t^* and $t^A_{r,max}$ by the same argument. Yet, if θ_o^B strictly prefers to concede at $t^A_{r,max}$, then $F_o^B(0) = 0$, and then they are not playing a degenerate sequential equilibrium. Hence, in a degenerate equilibrium, condition (1.2) is satisfied.

Since in any degenerate equilibrium $F_r^A \in DE$, we have that θ_o^B weakly prefers to concede at every t. Therefore, by Proposition 1.3, θ_i^B strictly concedes in every continuation game, hence $F_i^B = \hat{F}_i^B$. Now we show that when θ_o^B is indifferent, we cannot have an equilibrium in which he does not concede immediately with probability 1. Recall by Corollary 1.4 that θ_o^B cannot play a mixed strategy in any sequential equilibrium. Therefore, suppose that θ_o^B plays $F_o^B(t) = 0$ for some continuation game $t > 0$. Then, since by the previous argument $F_i^B(t) = \hat{F}_i^B(t) = 1$, B is playing a separating strategy. But then θ_i^B has the incentive to imitate θ_o^B, so that in the event $\theta^A = \theta_r^A$ he gets the best contract. Moreover, in the event $\theta^A = \theta_b^A$ he observes waiting and can then concede. Hence, θ_i^B has a profitable deviation. Therefore, in any degenerate equilibrium $F_o^B = \hat{F}_o^B$. □

Proof of Proposition 1.7

Proof. First, we claim that $supp(Y_i^B) = supp(Y_o^B)$. By contradiction, we have two cases:

(1) $\exists n_i \in supp(Y_i^B)$ such that $Y_o^B(n_i) = 0$;

(2) $\exists n_o \in supp(Y_o^B)$ such that $Y_i^B(n_o) = 0$.

The first case can be easily excluded. Since $Y_o^B(n_i) = 0$, we have

$$\langle (1/2, 1/2) \text{ offered at } (n_i, 0)\rangle \Rightarrow \mu_r^A((n_i, 0); \theta_i^B) = 1,$$

which contradicts Assumption 2. Therefore, consider case 2., and check continuation game $(n_o, -1)$. First, we claim that $\mu_r^A((n_o, -1); \theta_i^B) > 0$, that is, θ_i^B does not concede with probability 1 before $(n_o, -1)$. In order to prove it, note that by Assumption 3, since $Y^B(n_o) > 0$, it must be the case that $\bigcup_{n=n_o}^{+\infty} X_n^A([n, n+1]) > 0$. Therefore, in case θ_i^B concedes with probability 1 no later than some $t < n_o$, we have that $\mu_i^B((t,k); \theta_r^A) > 0$. But then, if θ_i^B deviates by waiting at (t, k), θ_r^A has beliefs $\mu_r^A((t,k); \theta_i^B) = 0$. Now, observe that $Y^B(n) = 0$ for $n > n_o$ since the fair contract is accepted at $(n_o, +1)$ by Assumption 1. Hence, if $(1/2, 1/2)$ is not offered at $(n_o, 0)$, θ_r^A accepts $(1-\gamma_B, \gamma_B)$ at $(n_o, +1)$ (since no rational types in the following war of attrition believes she is irrational with positive probability). Hence, we must have $1/2 \geq \gamma_B$ since θ_o^B offers the fair contract with positive probability. But then, θ_r^A is better off by accepting $(1-\gamma_B, \gamma_B)$ at (t, k), as she is certain, in that continuation game, that she cannot get more than $1-\gamma_B$. This is a contradiction, and so θ_i^B does not concede before $(n_o, -1)$ with probability 1.

Since $Y_o^B(n_o) > 0$, we can assume that θ_o^B has not offered $(1/2, 1/2)$ yet. Then,

$$\mu_r^A((n_o, -1); \theta_o^B) \in (0, 1).$$

When $(1/2, 1/2)$ is offered at $(n_o, 0)$, we have $\mu_r^A((n_o, 0), \theta_o^B) = 1$. By Assumption 1 θ_r^A immediately concedes. Hence, the action that offers $(1/2, 1/2)$ at $(n_o, 0)$ and concedes to $(\gamma_A, 1-\gamma_A)$ at $n_o + \varepsilon$ is not a profitable deviation for θ_i^B for any $\varepsilon > 0$, since $Y_i^B(n_o) = 0$. This implies that θ_i^B can obtain at least the same payoff through his strategy. Note that in the event $\theta^A = \theta_b^A$, the former action is strictly dominant for some $\varepsilon > 0$ to any other strategy that does not offer $(1/2, 1/2)$. But then, θ_i^B's action dominates the other in the event $\theta^A = \theta_r^A$. Yet, any strategy s that provides θ_i^B a payoff of $\tilde{u}$ in the event A is rational, can be replicated by θ_o^B. Hence, θ_o^B can obtain $\tilde{u}$ with probability 1. Since $Y_o^B(n_o) > 0$, s cannot provide a strictly higher payoff $\tilde{u}$ to θ_o^B than offering $(1/2, 1/2)$. But this is a contradiction, since $\tilde{u}$ is strictly higher than the utility of the fair contract offer at $(n_o, 0)$ in the event $\theta^A = \theta_r^A$. Therefore, we can conclude $supp(Y_i^B) = supp(Y_o^B)$.

Now, suppose $|supp(Y^B)| = +\infty$. Take $t' \in supp(Y_i^B)$. Then $t' \in supp(Y_o^B)$. There exists a sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n > t'$ and $t_n \in supp(Y^B)$ for each $n \in \mathbb{N}$, and moreover $t_n \to +\infty$. Since θ_o^B is indifferent between t' and any t_n, and θ_o^B does not concede in GAS, we have

$$U_o^B(\Sigma^A, t', T_S^0(t')) = U_o^B(\Sigma^A, t_n, T_S^0(t_n)). \tag{1.D.3}$$

Now, since θ_i^B is indifferent, and waiting until $T_S^0(\cdot)$ is optimal in any GAS, we have

$$U_i^B(\Sigma^A, t', T_S^0(t')) = U_i^B(\Sigma^A, t_n, T_S^0(t_n)),$$

which implies

$$(1-z)U_o^B(\Sigma^A, t', T_S^0(t')) + ze^{-\delta T_S^B(t')}(1-\gamma_A) = (1-z)U_o^B(\Sigma^A, t_n, T_S^0(t_n)) + ze^{-\delta T_S^B(t_n)}(1-\gamma_A).$$

By equation (1.D.3), we get $ze^{-\delta T_S^0(t')}(1-\gamma_A) = ze^{-\delta T_S^0(t_n)}(1-\gamma_A)$, and so $T_S^0(t') = T_S^0(t_n)$. Yet, since $t_n \leq T_S^0(t_n)$ for each n, and $t_n \to +\infty$, $T_S^0(t_n) \to +\infty$. Therefore, $\exists n' \in \mathbb{N}$ such that $T_S^0(t') < T_S^0(t_{n'})$. Hence, indifference of θ_o^B implies θ_i^B strictly prefers $(t', 0)$ over $(t_{n'}, 0)$, a contradiction since $n' \in supp(Y_i^B)$ by definition. Therefore, $|supp(Y^B)| < +\infty$. □

Proof of Lemma 1.10

Proof. First, suppose $Y_i^B(\tau^*|\tau^*) = 0$. Then $Y_i^B(\tau^*) = 0$. By Proposition 1.7, $Y_o^B(\tau^*) = 0$, and so $\tau^* \notin supp(Y^B)$, a contradiction. Therefore, $Y_i^B(\tau^*|\tau^*) > 0$.
Now assume $Y_i^B(\tau^*|\tau^*) = 1$. We compare this action with a deviation in which θ_i^B offers the behavioral contract at $(\tau^*, 0)$ and concedes to $\tau^* + \varepsilon$ for some arbitrary $\varepsilon > 0$ in case A does not accept the contract at $(\tau^*, +1)$. In case $Y_o^B(\tau^*|\tau^*) = 1$, then the behavioral offer at $(\tau^*, 0)$ implies beliefs $\mu_r^A((\tau^*, 0); \theta_b^B) = 1$, while in case $Y_o^B(\tau^*|\tau^*) < 1$, beliefs are $\mu_r^A((\tau^*, 0); \theta_o^B, \theta_b^B) = 1$. Since $Y_o^B(n) = 0$ for all $n > \tau^*$, τ_r^A accepts the behavioral contract at $(\tau^*, +1)$ since θ_o^B does not concede in that continuation game. Therefore, θ_i^B can concede at any $\tau^* + \varepsilon$, $\varepsilon > 0$ in case A does not concede at $(\tau^*, +1)$, since in this event A is behavioral. Since ε is arbitrary, we obtain that θ_i^B's payoff from this deviation is

$$\mu_i^B((\tau^*, -1); \theta_b^A)(1-\gamma_A)e^{-\delta\tau^*} + (1-\mu_i^B((\tau^*, -1); \theta_b^A))\gamma_B e^{-\delta\tau^*}.$$

The payoff of the ignorant type in case of no deviation is

$$\frac{1}{2}F_S^A(\tau^*; \tau^*)e^{-\delta\tau^*} + (1-\gamma_A)(1-F_S^A(\tau^*; \tau^*))e^{-\delta\tau^*}.$$

In equilibrium, we must have

$$\frac{1}{2}F_S^A(\tau^*; \tau^*) + (1-\gamma_A)(1-F_S^A(\tau^*; \tau^*)) \geq \mu_i^B((\tau^*, 0); \theta_b^A)(1-\gamma_A) + (1-\mu_i^B((\tau^*, 0); \theta_b^A))\gamma_B.$$

Clearly, $F_S^A(\tau^*; \tau^*) < 1 - \mu_i^B((\tau^*, 0); \theta_b^A)$. Therefore, if $\gamma_B > 1/2$, θ_i^B has a profitable deviation. Hence, assume $\gamma_B < 1/2$. Then, at $(\tau^*, -1)$, θ_r^A is certain to receive the fair offer contract on the next period $(\tau^*, 0)$ in case B is rational. Otherwise, she receives again the behavioral offer. Since $1-\gamma_B > 1/2$ by assumption and θ_b^B, θ_o^B do not accept her behavioral offer $(\gamma_A, 1-\gamma_A)$ at $(\tau^*, +1)$, θ_r^A is better off by accepting $(1-\gamma_B, \gamma_B)$ at $(\tau^*, -1)$ than waiting $(\tau^*, 0)$. But then, $(\tau^*, 0)$ cannot be reached with positive probability, a contradiction to Assumption 3. Therefore, $Y_i^B(\tau^*|\tau^*) < 1$.
The proof of $Y_o^B(\tau^*|\tau^*) > 0$ is clear by the same argument that proves $Y_i^B(\tau^*|\tau^*) > 0$. □

Proof of Theorem 1.11

Proof. Let τ^* be the last period at which B offers the fair contract. We can analyze the game as such B can offer the fair contract at $\tau' = 0$ only. In fact, the two continuation games are equivalent, except for the discount factor and beliefs. In the second continuation game A and B are splitting a surplus of 1 instead of $e^{-\delta\tau^*}$. Hence, we consider the second continuation game only as it is strategically equivalent to the first one.

We proceed by contradiction, therefore, suppose that one of B's rational type send the signal with positive probability. Then, we can only have an equilibrium where θ_i^B mixes and θ_o^B offers $(1/2, 1/2)$ with positive probability at $\tau^* = 0$, by Lemma 1.10.

Since $(0,0)$ is the last period for the fair contract offer, the concession distributions following either $(1/2, 1/2)$ or $(1-\gamma_B, \gamma_B)$ can be represented by CDFs. Therefore, call F_S^m the concession cdf for player $m \in \{A, B\}$ in the GAS, and denote with F_N^m the concession cdf for player $m \in \{A, B\}$ in the GNS. From AG Proposition 1, we know these two functions are exponential distributions and hence differentiable. Denote their densities with f_S^m and f_N^m.

First, suppose $T_S^A \le T_S^B$. Then, θ_i^B's payoff in the continuation equilibrium after he offered $(1/2, 1/2)$ is $1-\gamma_A$. In the continuation equilibrium after $(1-\gamma_B, \gamma_B)$ his expected utility is $F_N^A(0)\gamma_B + (1-F_N^A(0))(1-\gamma_A)$. Hence, in order to make θ_i^B indifferent between $(1/2, 1/2)$ and $(1-\gamma_B, \gamma_B)$, we need $F_N^A(0) = 0$, and therefore $T_N^A \le T_N^B$. Now, consider the type θ_o^B. His expected payoff after the offer $(1/2, 1/2)$ is $\int_0^{T_S^A} 1/2 e^{-\delta t}\frac{f_S^A(t)}{1-z}dt$. The offer $(1-\gamma_B, \gamma_B)$ provides, instead, $\int_0^{T_N^A} \gamma_B e^{-\delta t}\frac{f_N^A(t)}{1-z}dt$. Since in the candidate equilibrium θ_o^B offers $(1/2, 1/2)$ with positive probability, we must have

$$\int_0^{T_S^A} 1/2 e^{-\delta t}\frac{f_S^A(t)}{1-z}dt \ \ge\ \int_0^{T_N^A} \gamma_B e^{-\delta t}\frac{f_N^A(t)}{1-z}dt.$$

$$\Longleftrightarrow$$

$$(1-\gamma_A)(1-e^{-(\delta+\lambda_S^A)T_S^A}) \ \ge\ (1-\gamma_A)(1-e^{-(\delta+\lambda_N^A)T_N^A})$$

and so, we require

$$\frac{\delta+\lambda_N^A}{\delta+\lambda_S^A} \le \frac{T_S^A}{T_N^A} = \frac{\lambda_N^A}{\lambda_S^A}.$$

This inequality is satisfied if and only if $\lambda_S^A \le \lambda_N^A$, but this is true if and only if $\gamma_B \le 1/2$, a contradiction. Hence, we consider the case $T_S^A > T_S^B$.

Type θ_i^B's payoff of offering $(1/2, 1/2)$ is $F_S^A(0)1/2 + (1-F_S^A(0))(1-\gamma_A)$, with $F_S^A(0) > 0$. As stated above, the payoff from $(1-\gamma_B, \gamma_B)$ is $F_N^A(0)\gamma_B + (1-F_N^A(0))(1-\gamma_A)$, so $F_N(0) > 0$ which implies $T_N^B < T_N^A$. Therefore, $T_S^0 = T_S^B$ and $T_N^0 = T_N^B$. Assume first that $T_S^0 < T_N^0$, i.e., $T_S^B < T_N^B$.

Recall that waiting until the end of the continuation game is always optimal for θ_i^B in a nondegenerate equilibrium, independently from the contract offered at $\tau = 0$. Therefore, θ_i^B indifference can be rewritten as

$$(1-z)\left[\frac{F_S^A(0)}{1-z}\frac{1}{2} + \int_0^{T_S^B} \frac{1}{2}e^{-\delta t}\frac{f_S^A(t)}{1-z}dt\right] + ze^{-\delta T_S^B}(1-\gamma^A)$$

$$=$$

$$(1-z)\left[\frac{F_N^A(0)}{1-z}\gamma_B + \int_0^{T_N^B} \gamma_B e^{-\delta t}\frac{f_N^A(t)}{1-z}dt\right] + ze^{-\delta T_N^B}(1-\gamma^A).$$

Since $T_S^B < T_N^B$, we have $ze^{-\delta T_S^B}(1-\gamma_A) > ze^{-\delta T_N^B}(1-\gamma_A)$, therefore

$$\frac{F_S^A(0)}{1-z}\frac{1}{2} + \int_0^{T_S^B} \frac{1}{2}e^{-\delta t}\frac{f_S^A(t)}{1-z}dt \; < \; \frac{F_N^A(0)}{1-z}\gamma_B + \int_0^{T_N^B} \gamma_B e^{-\delta t}\frac{f_N^A(t)}{1-z}dt,$$

and so θ_o^B does not offer $(1/2, 1/2)$ in equilibrium, a contradiction.

Next, assume $T_S^B \geq T_N^B$. Type θ_i^B's payoff indifference in equilibrium is

$$F_S^A(0)\frac{1}{2} + (1-F_S^A(0))(1-\gamma_A) \; = \; F_N^A(0)\gamma_B + (1-F_N^A(0))(1-\gamma_A).$$

Since $\gamma_B > 1/2$, we need $F_S^A(0) > F_N^A(0)$, hence $c_S^A < c_N^A$, which implies

$$\begin{aligned} e^{-\lambda_S^A(T_S^A - T_S^B)} \; < \; e^{-\lambda_N^A(T_N^A - T_N^B)} &\Rightarrow \lambda_S^A(T_S^A - T_S^B) \; > \; \lambda_N^A(T_N^A - T_N^B) \\ &\Rightarrow -\log(z) - \lambda_S^A T_S^B \; > \; -\log(z) - \lambda_N^A T_N^B \\ &\Rightarrow \lambda_S^A T_S^B \; < \; \lambda_N^A T_N^B. \end{aligned}$$

Since $\gamma_B > 1/2$, $\lambda_S^A > \lambda_N^A$, and so $T_S^B < T_N^B$, contradiction.

Therefore, in any equilibrium, for each vector of parameters $\boldsymbol{q}$, we have $Y^B(n) = 0$ for each n. □

Proof of Lemma 1.12

Proof. Assume first that A concedes with a positive probability at $(\tau_1^*, +1)$ following the offer $(1/2, 1/2)$. It's important to note that since τ_1^* marks the final opportunity for the fair contract to be offered, A concedes with a non-zero probability even if B doesn't propose the fair deal (otherwise θ_i^B would strictly prefer to offer the fair contract over the behavioral). Consequently, there exists a time $t < \tau_1^*$ at which the rational player B strictly prefers proposing $(1/2, 1/2)$ over conceding at any time within the interval $(t, \tau_1^*]$. This observation implies that A refrains from conceding during this interval as well[11]. Let t' be the last time before B prefers to wait, i.e., $t' := \sup\{\bigcup_{n=\tau_0^*}^{\tau_1^*-1} supp(X_n^B)\}$. Given that A's strategy includes conceding to both $(1/2, 1/2)$ and $(1-\gamma_B, \gamma_B)$, her expected payoff at the continuation game t' is $[1/2Y^B(\tau_1^*|\tau_1^*) + (1-\gamma_B)(1-Y^B(\tau_1^*|\tau_1^*))]e^{-\delta\tau_1^*}$. However, conceding at t' yields $(1-\gamma_B)e^{-\delta t'}$, which is evidently strictly higher, leading to a contradiction.

Now, let's assume that A doesn't concede with a positive probability at τ_1^* for $(1/2, 1/2)$. Consequently, A also refrains from conceding for $(1-\gamma_B, \gamma_B)$. Suppose there exists $t' < \tau_1^*$ such that A and B do not concede in $(t', \tau_1^*]$. Since A does not concede at $(\tau_1^*, +1)$, θ_i^B's payoff in the continuation game t' is $(1-\gamma_A)e^{-\delta\tau_1^*}$. Yet, concession at t' provides $(1-\gamma_A)e^{-\delta t'}$, a contradiction. Hence, there exists $t \in (\tau_0^*, \tau_1^*)$ such that A and B concede with positive density in $[t, \tau_1^*]$. Observe that if there exists an interval $(t_1, t_2) \subseteq [\tau_0^*, t)$ with no concessions, then θ_r^A and θ_i^B strictly prefer concession at t_1 over any $\tau > t_2$, again leading to a contradiction. Therefore, A and B concede with everywhere positive probability in $[\tau_0^*, \tau_1^*]$. In fact, consider the following. Define the function $X_A^* : [\tau_0^*, \tau_1^*] \to [0, 1]$ where:

$$X_A^0 = \sum_{n=0}^{\tau_0^*-1} X_n^A([n, n+1]),$$

for $m \in [\tau_0^*, \tau_1^*]$, $m \in \mathbb{N}$,

$$X_A^*(t) = X_A^0 + \sum_{n=\tau_0^*}^{m-1} X_n^A([n, n+1]) + X_{m+1}^A(\{m\}),$$

and for $t \in (m, m+1)$,

$$X_A^*(t) = X_A^0 + \sum_{n=\tau_0^*}^{m-1} X_n^A([n, n+1]) + X_{m+1}^A([m, t])$$

Observe that X_A^* is weakly increasing in $[\tau_0^*, \tau_1^*]$. This function represents the cumulative distribution of concession of player A in the interval $[\tau_0^*, \tau_1^*]$, with no distinction between $(t, +1)$ and $(t, -1)$ for $t \in \mathbb{N}$. In fact, since $Y^B(t) = 0$, we can treat the two subdates as the same date. We define X_B^* in the same manner. Type θ_i^B's utility of concession at $t \in [\tau_0^*, \tau_1^*]$ can be written as

$$\begin{aligned} U_i^B((X^A, (\sigma^A[n])_{n\in\mathbb{N}_0}), t) = & \sum_{n=0}^{\tau_0^*-1} \int_n^{n+1} \gamma_B e^{-\delta z} dX_n^A(z) + \int_{\tau_0^*}^{t} \gamma_B e^{-\delta z} dX_A^*(z) \\ & + (1 - X_A^*(t))(1-\gamma_A)e^{-\delta t}. \end{aligned}$$

We now describe the properties of X_A^* and X_B^*. We follow AG Proposition 1, in particular the points $(b) - (f)$.

11. See Abreu and Gul (2000), Proposition 1

(*i*) *If X_A^* jumps at $t \in (\tau_0^*, \tau_1^*]$, X_B^* does not jump at t.*
In fact, B can just wait an instant after t. If $t = \tau_1^*$, B can wait until $(\tau_1^*, +1)$.

(*ii*) *If X_A^* is continuous at $t \in (\tau_0^*, \tau_1^*)$, then U_i^B is continuous at t. If X_B^* is continuous at $t \in (\tau_0^*, \tau_1^*)$ then U_r^A is continuous at t.*
These properties stem directly from definitions.

(*iii*) *There is no interval (t', t'') with $\tau_0 \leq t' < t'' \leq \tau_1^*$ such that X_A^* and X_B^* are both constant in (t', t'').*
First we claim that A does not concede at $(\tau_1^*, -1)$ with positive probability, i.e., $\lim_{\tau \to \tau_1^*} X_A^*(\tau) = X_A^*(\tau_1^*)$. If A concedes at $(\tau_1^*, -1)$ with positive probability, $\exists \varepsilon > 0$ such that B does not concede in $[\tau_1^* - \varepsilon, \tau_1^*]$ and prefers instead concession at $(\tau_1^*, +1)$. If this is the case, A's concession at $\tau_1^* - \varepsilon$ provides $(1-\gamma_B)e^{-\delta(\tau_1^* - \varepsilon)}$ (in the continuation game $\tau_1^* - \varepsilon$), while concession at $(\tau_1^*, -1)$ gives A $(1-\gamma_B)e^{-\delta\tau_1^*}$ (in the same continuation game) since B does not concede in $[\tau_1^* - \varepsilon, \tau_1^*]$, a contradiction.
Now assume there is a time interval (t', t''), as described in the statement. Let t^* be the supremum of t'' for which $\exists t \in [\tau_0^*, \tau_1^*)$ such that (t', t^*) has the property stated above. We first argue that $t^* < \tau_1^*$. Assume otherwise, i.e., $t^* = \tau_1^*$. Then by assumption A does not concede in $(t', \tau_1^*]$ (since A does not concede at $(\tau_1^*, -1)$ too). Moreover, A does not concede with positive probability to the fair contract at $(\tau_1^*, +1)$. Therefore, at continuation game $t \in (t', \tau_1^*)$, θ_i^B's utility of immediate concession is $(1-\gamma_A)e^{-\delta t}$, while the offer of the fair contract at $(\tau_1^*, 0)$ provides him with $(1-\gamma_A)e^{-\delta\tau_1^*}$ (in continuation game t). Hence concession at t is a profitable deviation, a contradiction. Therefore $t^* < \tau_1^*$. The remaining part of the proof follows AG Proposition 1 closely, and we include it for completeness.

Fix $t \in (t', t^*)$. Observe that for both players (in particular for types θ_i^B and θ_r^A) $\exists \varepsilon_t > 0$ such that concession at t is strictly better than any concession in $(t^* - \varepsilon_t, t^*)$. Furthermore, by (*i*) and (*ii*) there exists one type between θ_i^B and θ_r^A for which their utility is continuous at t^*. Hence, since $t^* < \tau_1^*$, $\exists \eta > 0$ such that concession at t^* is still strictly better than concession in $(t^*, t^* + \eta)$ for this type. But then X_K^* is constant in $(t^*, t^* + \eta)$, where K is the player whose type has continuous utility at t^*. Yet, if X_K^* is constant in $(t^*, t^* + \eta)$, by optimality X_j^* is constant in $(t^*, t^* + \eta)$ too for $j \neq k$, a contradiction to the definition of t^*.

(*iv*) *For $t' < t'' < \tau_1^*$, $X_K^*(t') < X_K^*(t'')$, $K \in \{A, B\}$.*
If X_K^* is constant in (t', t''), by optimality X_J^* is constant in (t', t''), but this contradicts (*iii*).

(*v*) *X_K^* is continuous in (τ_0^*, τ_1^*).*
A jump at $t \in (\tau_0^*, \tau_1^*)$ implies the opponent waits in some $(t - \varepsilon, t)$, a contradiction to (*iv*).

Since X_A^* and X_B^* are strictly increasing, A and B randomize over the entire interval (τ_0^*, τ_1^*). Clearly, θ_o^B strictly prefers to wait and hence does not concede in (τ_0^*, τ_1^*).

Therefore, U_r^A and U_i^B are constant through (τ_0^*, τ_1^*) and so these utilities are differentiable. From θ_i^B's utility, we get

$$\gamma_B e^{-\delta t} x_A^*(t) - \delta(1 - X_A^*(t))(1-\gamma_A)e^{-\delta t} - x_A^*(t)(1-\gamma_A)e^{-\delta t} = 0,$$

where x_A^* is the derivative of X_A^*. Hence, for all $t \in (\tau_0^*, \tau_1^*)$ we get

$$\frac{x_A^*(t)}{1 - X_A^*(t)} = \frac{(1-\gamma_A)\delta}{\gamma_A + \gamma_B - 1} = \lambda_N^A.$$

Note that $x_A^*(t)/(1 - X_A^*(t))$ represents A's rate of concession at t. Then, we get that A concedes with a constant rate of λ_N^A in the interval $[\tau_0^*, \tau_1^*]$. Through the same calculations, we conclude that B must concede with a constant rate of λ_N^B in the same interval. Note that in order to find the concession rates in $[\tau_0^*, \tau_1^*]$ we never used the fact that a fair contract could be offered at $(\tau_0^*, 0)$ with positive probability. Therefore, in case that is not true, we can have $\tau_0^* = 0$. □

Proof of Lemma 1.13

Proof. Suppose first that A concedes with positive probability x_0 to $(1/2, 1/2)$ at τ_0^*. Then, there exists $t < \tau_0^*$ such that B does not concede in the interval (t, τ_0^*). This implies that neither A concedes in the same interval. Now, if B does not concede in the continuation game τ_0^* after offering again $(1-\gamma_B, \gamma_B)$, then A concedes to $(1-\gamma_B, \gamma_B)$ no later than t. In fact, concession at t provides a payoff of $(1-\gamma_B)e^{-\delta t}$, while any other action that moves at τ_0^* or after gives $[1/2Y^B(\tau_0^*|\tau_0^*) + (1-\gamma_B)(1 - Y^B(\tau_0^*|\tau_0^*))]e^{-\delta\tau_0^*}$, which is strictly lower than the previous one. Therefore, when $x_0 > 0$, it must be that B concedes with positive probability at τ_0^* after offering again $(1-\gamma_B, \gamma_B)$. Now, since θ_i^B and θ_o^B randomize the offer of the fair contract between τ_0^* and τ_1^*, we have that it is optimal for θ_i^B to offer $(1-\gamma_B, \gamma_B)$ at τ_0^*. Moreover, he is conceding with positive probability to $(\gamma_A, 1-\gamma_A)$ in that event. Hence, θ_i^B's indifference condition implies

$$(1-\gamma_A)e^{-\delta\tau_0^*} = [1/2x_0 + (1-x_0)(1-\gamma_A)]e^{-\delta\tau_0^*}.$$

Yet, this equation cannot be true for $x_0 > 0$. Therefore, we have to assume $x_0 = 0$. Suppose again that A is conceding to $(1-\gamma_B, \gamma_B)$ at τ_0^* with positive probability. Then, θ_i^B's payoff from offering $(1-\gamma_B, \gamma_B)$ is strictly greater than the payoff from the offer of the fair contract, and so θ_i^B does not offer $(1/2, 1/2)$ at τ_0^*, a contradiction.

Therefore, A cannot concede with positive probability to either contract at τ_0^*. □

Proof of Theorem 1.14

Proof. By contradiction, assume $|supp(Y^B)| \geq 2$. From Proposition 1.7 we know $|supp(Y^B)| < +\infty$. Therefore, $\exists \tau_0^*, \tau_1^* \in \mathbb{N}_0$ such that $\tau_0^* < \tau_1^*$ and

$$\sum_{n=\tau_0^*+1}^{\tau_1^*-1} Y^B(n) + \sum_{n=\tau_1^*+1}^{\infty} Y^B(n) \;=\; 0,$$

so that τ_0^* and τ_1^* are as described in Lemma 1.12 and Lemma 1.13. From Proposition 1.7 we know that $supp(Y_i^B) = supp(Y_o^B)$, so that both θ_o^B and θ_i^B randomize between τ_0^* and τ_1^* (and possibly, between not offering the fair contract at all). Now, since θ_o^B randomizes, we need

$$U_o^B(\Sigma^A, \tau_0^*, T_S^0(\tau_0^*)) \;=\; U_o^B(\Sigma^A, \tau_1^*, T_S^0(\tau_1^*))$$

in equilibrium. Recall that θ_o^B waits until the end of GAS $T_S^0(\cdot)$ in equilibrium. Since $T_S^0(\cdot)$ is in θ_i^B's support as well, and θ_i^B randomizes between τ_0^* and τ_1^*, we need

$$U_i^B(\Sigma^A, \tau_0^*, T_S^0(\tau_0^*)) \;=\; U_i^B(\Sigma^A, \tau_1^*, T_S^0(\tau_1^*)).$$

Therefore, we have

$$(1-z)U_o^B(\Sigma^A, \tau_0^*, T_S^0(\tau_0^*)) + z(1-\gamma_A)e^{-\delta T_S^0(\tau_0^*)} \;=\; (1-z)U_o^B(\Sigma^A, \tau_1^*, T_S^0(\tau_1^*)) + z(1-\gamma_A)e^{-\delta T_S^0(\tau_1^*)}.$$

By θ_o^B's indifference, we get

$$T_S^0(\tau_0^*) \;=\; T_S^0(\tau_1^*).$$

From Lemma 1.12 and Lemma 1.13 we know that A does not concede with positive probability at $(\tau_0^*, +1)$ and $(\tau_1^*, +1)$ to the fair contract. Therefore, $T_S^0(\tau_0^*)$ and $T_S^0(\tau_1^*)$ depend on λ_S^A, $\mu_i^B((\tau_0^*, 0); \theta_b^A)$ and $\mu_i^B((\tau_1^*, 0); \theta_b^A)$. Since the GAS which starts at $n \in \mathbb{N}$ has the same concession rates of a game started at 0, with the difference that the share of the pie shrinks (with no impact on the concession probability), we have that

$$T_S^0(\tau_0^*) \;=\; -\frac{\log(\mu_i^B((\tau_0^*, 0); \theta_b^A))}{\lambda_S^A} + \tau_0^*$$

and

$$T_S^0(\tau_1^*) \;=\; -\frac{\log(\mu_i^B((\tau_1^*, 0); \theta_b^A))}{\lambda_S^A} + \tau_1^*.$$

Hence, the GAS at τ_0^* is equivalent to a game started at 0 shifted by τ_0^* periods. The same holds for τ_1^*. We have

$$-\frac{\log(\mu_i^B((\tau_0^*, 0); \theta_b^A))}{\lambda_S^A} + \tau_0^* \;=\; -\frac{\log(\mu_i^B((\tau_1^*, 0); \theta_b^A))}{\lambda_S^A} + \tau_1^*,$$

which implies

$$\mu_i^B((\tau_0^*, 0); \theta_b^A)e^{-\tau_0^*\lambda_S^A} \;=\; \mu_i^B((\tau_1^*, 0); \theta_b^A)e^{-\tau_1^*\lambda_S^A},$$

and so

$$\mu_i^B((\tau_1^*,0);\theta_b^A) \;=\; \mu_i^B((\tau_0^*,0);\theta_b^A)e^{\lambda_S^A(\tau_1^*-\tau_0^*)}.$$

Yet, by Lemma 1.12, we know that A concedes at rate λ_N^A in the time interval $[\tau_0^*,\tau_1^*]$. Therefore, A's reputation grows at rate λ_N^A. Hence, in equilibrium, we get

$$\mu_i^B((\tau_1^*,0);\theta_b^A) \;=\; \mu_i^B((\tau_0^*,0);\theta_b^A)e^{\lambda_N^A(\tau_1^*-\tau_0^*)}.$$

Since $\gamma_B < 1/2$, $\lambda_N^A \neq \lambda_S^A$, and so A's reputation grows at a rate that, given θ_o^B's indifference, does not make θ_i^B indifferent between τ_0^* and τ_1^*. This is a contradiction since $\tau_0^*,\tau_1^* \in supp(Y_i^B)$. □

Proof of Proposition 1.16

Proof. First, note

$$Y^B(\tau^*) \;=\; q_o + q_i Y_i^B(\tau^*),$$

as $Y_o^B(\tau^*) = 1$. Therefore,

$$Y^B(\tau^*|\tau^*) \;=\; \frac{Y^B(\tau^*)}{1-X} \;=\; \frac{q_o}{1-X} + \frac{q_i Y_i^B(\tau^*)}{1-X},$$

where $X = \sum_{n=0}^{\tau^*-1} X_n^B([n,n+1])$. Observe that $q_o/(1-X) = \mu_r^A((\tau^*,-1);\theta_o^B)$. Therefore, we are left to prove that $\mu_r^A((\tau^*,-1);\theta_i^B)Y_i^B(\tau^*|\tau^*) = q_i Y_i^B(\tau^*)/(1-X)$. Note that

$$\mu_r^A((\tau^*,-1);\theta_i^B) \;=\; 1-(\mu_r^A((\tau^*,-1);\theta_b^B)+\mu_r^A((\tau^*,-1);\theta_o^B)) \;=\; 1-\frac{q_b+q_o}{1-X}.$$

Hence,

$$\begin{aligned}
\mu_r^A((\tau^*,-1);\theta_i^B)Y_i^B(\tau^*|\tau^*) \;&=\; \mu_r^A((\tau^*,-1);\theta_i^B)\frac{Y_i^B(\tau^*)}{1-\frac{1}{q_i}X}\\
&=\; q_i\mu_r^A((\tau^*,-1);\theta_i^B)\frac{Y_i^B(\tau^*)}{q_i-X}\\
&=\; q_i\left(1-\frac{q_b+q_o}{1-X}\right)\frac{Y_i^B(\tau^*)}{1-(q_b+q_o)-X}\\
&=\; q_i\left(\frac{1-X-(q_b+q_o)}{1-X}\right)\frac{Y_i^B(\tau^*)}{1-(q_b+q_o)-X}\\
&=\; q_i\frac{Y_i^B(\tau^*)}{1-X}.
\end{aligned}$$

□

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Chapter 2

Simultaneous Bidding in Sealed-Bid Auctions

2.1 Introduction

In this second chapter, we study the bidding behavior in a model of competing and simultaneous sealed-bid auctions. We make use of second-price auctions and first-price auctions. Buyers are symmetric, have unit demand, and can participate and bid on all the auctions with no entry costs. The structure of the game is as follows. Several sellers hold simultaneous auctions and sell homogeneous goods (one each) to a group of buyers. Each buyer perceives the goods as perfect substitutes: they are interested in acquiring just one unit. The game has incomplete information, where valuations are independent and ex-ante symmetric. After the buyers have decided on their bids, the auctions are solved. This framework raises the following trade-off: bidding on multiple auctions increases the probability of winning at least one object, but concurrently it increases the sum of the expected prices, as the bidder may win more than one object, which is undesired. Yet, winning two goods is better than zero if the prices paid are low.

Recall that agents bid their type in the classic equilibrium of a second-price auction. Here, if bidders find it optimal to bid on multiple auctions, they reduce their bids to offset the higher sum of expected prices and hence would not reveal their type. Solving these games is not trivial: Cai and Dimitriou (2014) show that these games are at least PP-hard[1]. Tipically, we aim to identify symmetric and pure equilibria. The first part of Theorem 2.1 shows that no such equilibrium exists for a large class of 'regular' strategies. The regularity condition we impose that bidders, on at least an auction, use a bidding function that is increasing from the lowest type to any arbitrary higher type. Theorem 2.1 also states that in any pure equilibrium of

1. A PP-hard problem can be solved by probabilistic polynomial time. NP problems are included in this class.

a game with excess demand (the number of auctions is lower than the number of buyers), at least one bidder will place bids in more than one auction. Consequently, the assumption that bidders can participate in only a single auction is not without loss of generality, even when bidders have unit demand.

We believe that symmetric behavior is a plausible property of the equilibrium of a symmetric auction game, even if its existence is not trivial. Theorem 2.3 shows that a symmetric equilibrium exists in the two-player case if we allow for mixed strategies. This existence result relies on Reny (1999). The author shows that under some technical conditions of the strategy space (compactness and Hausdorff) and payoffs (better-reply security), any discontinuous symmetric game possesses a symmetric mixed strategy equilibrium. By showing that the game satisfies these properties, we prove existence. Furthermore, it turns out that in any of these equilibria all the bidders bid on all the auctions with probability 1. We are not able to prove existence in general. In fact, better-reply security relies on the fact that for each $\varepsilon > 0$ the players can always play strategies that avoid ties and lose no more than ε expected utility. When the game has more than two players, we cannot guarantees this. Therefore, the game may not have enough continuity to get the existence result. In Appendix 2.*B* we provide an example that shows why ties cannot be easily excluded as in standard auction games.

In Appendix 2.*A*, we propose a brief analysis of non-trivial, pure asymmetric equilibria with increasing strategies[2]. We provide some examples and find their respective equilibria. As anticipated in the discussion of Theorem 2.1, in equilibrium some of the players bid on multiple auctions.

Building upon Szentes (2007), we characterize symmetric mixed-strategy equilibria in a game with two auctions and two bidders. We consider two kinds of binary type space. In the first one, the low type has a valuation of 0 for the object. This type does not participate in any auctions. Therefore, the only incomplete information for a bidder with high type is whether the other player wants to participate. Then, we consider the case where the low type has a strictly positive valuation, so that both types want to bid positive amounts and so the high type faces stronger competition. Solving a particular functional equation, we find a closed-form expression of a symmetric mixed strategy equilibrium in both cases. In the case of lowest type has zero valuation for the objects, the high types randomize their bid over two decreasing lines. These lines are cut in half by the 45-degree line, and one line is strictly above the other. Moreover, the distribution of bids is the same for both auctions. On the other hand, when the lowest type has positive valuation for the goods, we find a

2. An increasing strategy can be defined in multiple ways in the concurrent auctions setting. As we try to be as general as possible, we posit a prerequisite condition, stipulating that the strategy must exhibit strict monotonicity from the lowest type to a higher type in at least one auction. This definition accommodates functions with diminishing segments as well. The crucial element is the presence of at least one auction where the function strictly increase from the lowest to a higher type.

continuum of equilibria where each type randomizes over a decreasing line, and the support of the low type is strictly below the support of the high type.

We also study the case of discrete bids, showing how richer bid spaces are responsible for higher probabilities of positive bids on all the auctions.

The rest of the paper is organized as follows. In section 2, we present the literature review. In section 3 we describe the model and set the basics of the game. Section 4 discusses equilibrium existence for a general number of auctions and bidders. Here the main two theorems are presented. In section 5, we find symmetric equilibria in the specific case of two auctions and two bidders and discuss the consequences of discrete bids. Finally, we present the conclusions at the end of the paper.

2.2 Literature Review

Many works on competing auctions assume that buyers (who desire to acquire just one unit of the good) can participate in one auction only and allow them to randomize their participation decision (McAfee (1993), Peters and Severinov (1997), Delnoij and De Jaegher (2020)). Peters and Severinov (2006) consider instead simultaneous English auctions in which bidders can bid on multiple auctions. When there are no bidding cost and no fixed ending time for the auctions, the authors find that the strategy that bids on the auction with the lowest standing bid is a Bayesian equilibrium. Anwar, McMillan, and Zheng (2006) perform an empirical analysis using evidence from eBay. They find that bidders bid across multiple auctions; their strategy is coherent with what is suggested by Peters and Severinov (2006). We follow this approach in a framework of sealed-bid auctions. Gerding et al. (2008b) have also addressed this problem by categorizing bidders into two groups: local and global. A local bidder is a bidder who can bid on one auction only, while a global bidder can bid on multiple auctions at the same time[3]. In their model, there is only one global bidder, and the mechanism is a second-price auction. They study the behavior of the global bidder and prove that no matter the number of local bidders and available auctions, she wants to place a bid on all the auctions. We can interpret this result in the following way: in a model with only global bidders, there exists no equilibrium in which all want to bid on one auction only. Our work diverts from this path, considering games with global bidders only. At the end of their paper, Gerding et al. (2008b) analyze the game with three global bidders and no local bidders. They approach the problem with numerical simulation. Their algorithm oscillates among two states and hence it does not converge. In Appendix 2.*A*, we solve this specific

3. The authors provide many reasons why a bidder should be local. For example, she may have bounded rationality, and therefore be unable to compute the optimal strategy when considering multiple bids. The bidder may also have a budget constraint: Gerding et al. (2008a) prove in their paper that a bidder may prefer to concentrate her resources on one auction when budget-constrained. Alternatively, the bidder may be unaware of the other auctions (unlikely in the case of online auctions).

game analytically, and we find a (non-trivial) equilibrium when the distribution of types is uniform, the same distribution the authors assume in their example.

Our paper also builds on the work of Szentes (2007), who analyzed the scenario of two auctions and two symmetric bidders. Szentes examined both perfect complements and perfect substitutes goods, assuming that bidders have only one type, which implies complete information. The auction mechanism studied was a first-price auction. However, in a separate paper, Szentes (2005) provided methods to convert first-price auction equilibria into second-price auction equilibria. We extend Szentes' results by introducing an additional bidder type into the model, thus incorporating incomplete information.

2.3 The Model

We consider an auction game G, in which $K \geq 2$ independent sealed-bid auctions selling the same good are held simultaneously. In this game, N bidders can participate in any number of auctions. The mechanism can either be a first-price auction, $G = G^{FPA}$, or a second-price auction, $G = G^{SPA}$. When not specified, G can be either of them. Bidders are assumed ex-ante symmetric and with unit demand. In the game's first stage, Nature specifies a type for each bidder from the set $\Theta \subseteq [0, 1]$ according to some distribution F (assumed to be atomless when types are continuous). Types are independent across players. Once they know their type $\theta \in \Theta$, each bidder selects a bid for each auction from the bid space $\mathscr{B}$. Since there are K simultaneous sealed-bid auctions, the action space is $\mathscr{A} = \mathscr{B}^K$. We assume that $0 \in \mathscr{B}$ and consider a bid of 0 on auction j equivalent to the decision of not participating in that auction.

Since goods are homogeneous and bidders have unit demand, the object of interest is the probability of winning at least one good. We denote it with Q. Q is a function of the player's vector of bids $b \in \mathscr{A}$ and of the other players' strategies. A pure strategy in this game is a function $\beta = (\beta^1, \dots, \beta^K)$ where, for all k, $\beta^k : \Theta \longrightarrow \mathscr{B}$ is a measurable function. Each β^k assigns to each type θ a bid $\beta^k(\theta)$ on auction k. Therefore, for each $\theta \in \Theta$, $\beta(\theta)$ gives the vector of bids of the player.

Preferences are assumed to be linear. Hence, the interim expected payoff of bidder i when her type is θ_i, she bids $\beta_i(\theta) = (\beta_i^1(\theta_i), \dots, \beta_i^K(\theta_i))$ and each $j \neq i$ uses strategy β_j is

$$u_i(\theta_i, \beta_i, \beta_{-i}) \; = \; \theta_i Q_i(\beta_i(\theta_i), \beta_{-i}) - \sum_{k=1}^{K} E[P_k | \beta_i^k(\theta), \beta_{-i}^k],$$

where $E[P_k | \beta_i^k(\theta), \beta_{-i}^k]$ denotes the expected price of auction k paid by i given i's bid and strategies β_{-i}^k. Clearly, the expected price depends on the format FPA or SPA. We compute the expected payoff of a mixed strategy in the obvious way.

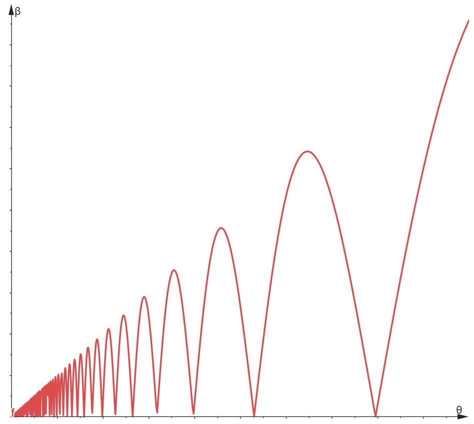

Figure 2.1. Bidding function k with $\theta^k = 0$

It is important to note that while the expected gain $\theta_i Q_i$ is influenced by the strategies used in all the auctions, each expected price is independent of the bidding strategies on the other auctions.

2.4 Equilibrium Existence

In this section, we let $\Theta = \mathscr{B} = [0, 1]$. In symmetric auction games we usually search for symmetric equilibria. We investigate the nature of pure strategy equilibria first. A symmetric pure strategy equilibrium in this game is a strategy β^* such that $\forall i \in \{1, \dots, N\}$, $\forall \theta_i \in \Theta$, $\beta^*(\theta_i)$ maximizes $u_i(\theta_i, \beta_i, \beta^*_{-i})$. We know that, in any symmetric equilibrium, $\beta^k(\theta) \leq \theta$ for each θ and auction format. Now, for each auction k, define

$$\theta^k := \inf\{\theta \in [0, 1] : \exists \tilde{\theta} < \theta \text{ s.t } \beta^k(\tilde{\theta}) \geq \beta^k(\theta)\}.$$

Note that when $\theta^k > 0$, the bidding function β^k is strictly increasing in the interval $[0, \theta^k]$. In fact, consider $\theta < \theta' < \theta^k$. Then, by definition of θ^k we have that $\beta^k(\theta) < \beta^k(\theta')$. Moreover, observe β^k is allowed to have decreasing parts in the interval $[\theta^k, 1]$. An example of a bidding function β^k with $\theta^k = 0$ and $\beta^k(\theta) \leq \theta$ is when it exhibits infinite oscillations (decreasing branch followed by an increasing branch) as θ gets close to 0. Figure 2.4 depicts an example of a bidding function of this kind. Clearly, θ^k is a function of β^k and of β in general. When the set where we take the infimum is empty, let $\theta^k = +\infty$. In the next Theorem, we show that the existence of an auction k with $\theta^k > 0$ is enough to exclude the possibility of a symmetric pure equilibrium. Moreover, we show that whenever there are more buyers than sellers (so that there is excess of demand), there cannot be an equilibrium where the bidders split and bid on a single auction each. Therefore, if they coordinate and enter one auction only, they have an incentive to deviate and bid across multiple auctions.

Theorem 2.1. *Consider strategy β and suppose there exists $k \in \{1,\dots,K\}$ such that $\theta^k > 0$. Then, β cannot be a symmetric pure strategy equilibrium. Moreover, whenever $N > K$ there are no equilibria in pure strategies where all the bidders bid on one auction only.*

Before the proof, we want to report Gerding et al. (2008b) result about participation on multiple auctions, adapting it to the current setting.

Theorem (Gerding et al. (2008b)) let $G = G^{SPA}$. Suppose that $\forall i \in \{1,\dots,N-1\}$ there exists $k \in \{1,\dots,K\}$ such that for all $\theta_i \in [0,1]$, $\beta_i^k(\theta_i) = \theta_i$ and for $j \neq k$, $\beta_i^j(\theta_i) = 0$. Then, β_N is optimal for player N only if $\beta_N^k(\theta_N) > 0$ for all $\theta_N > 0$, and for all $k \in \{1,\dots,K\}$.

We are now ready to prove Theorem 2.1.

Proof. First, observe $\forall \theta' \in (0,1]$ we cannot have more than one auction k where β^k is strictly increasing in $[0,\theta']$. If there are two or more, these auctions have perfectly correlated allocations, and in this case, a profitable deviation is to bid on only one of these auctions. In fact, not that if k and m are two auctions such that β^k and β^m are strictly increasing in $[0,\theta']$, then for $\theta_i \in (0,\theta']$,

$$Q_i(\beta_i^1(\theta_i),\dots,\beta_i^k(\theta_i),\dots,\beta_i^m(\theta_i),\dots,\beta^K(\theta_i),\beta_{-i}) = Q_i(\beta_i^1(\theta_i),\dots,\beta_i^k(\theta_i),\dots,0,\dots,\beta^K(\theta_i),\beta_{-i}).$$

Therefore, bidding 0 on auction m leaves the expected allocation unchanged and reduces the expected price of auction m to 0.

Hence, suppose there exists a unique $k \in \{1,\dots,K\}$ such that $\theta^k > 0$. Take $\theta' \leq \theta^k$. Now, consider auction $m \neq k$. Let

$$\theta^* := \inf\{\theta \in [0,\theta'] : \exists \tilde{\theta} < \theta \text{ s.t } \beta^m(\tilde{\theta}) > \beta^m(\theta)\}.$$

The set on which we take the infimum, can either be empty or non-empty. Assume it is empty first. Then, $\forall \theta \in [0,\theta']$, and $\forall \hat{\theta} \in (\theta,\theta']$, $\beta^m(\hat{\theta}) \geq \beta^m(\theta)$. Hence, β^m is weakly increasing in $[0,\theta']$. First, we exclude $\beta^m(\theta) = 0$ for all $\theta \in [0,\delta)$ for any $\delta < \theta'$. If so, take $\theta < \delta$, and observe that $\beta^k(\theta) = 0$ and $\beta^m(\theta) > 0$ is a profitable deviation (when $G = G^{FPA}$, we consider small $\beta^m(\theta)$). Therefore, $\beta^m(\theta) > 0$ for all $\theta \in (0,\delta]$ for some $\delta > 0$. But then, since β^m is weakly increasing and positive for positive types, winning auction m implies winning auction k. Therefore, $\beta^m(\theta) = 0$ is a profitable deviation for these types, a contradiction.

Now, assume $\{\theta \in [0,\theta'] : \exists \tilde{\theta} < \theta \text{ s.t } \beta^m(\tilde{\theta}) > \beta^m(\theta)\}$ is non-empty. We have two different cases, $\theta^* = 0$ and $\theta^* > 0$. Suppose first $\theta^m = 0$. We first claim that there exists $\theta \in [0,\theta']$ such that $\beta^m(\theta) \geq \beta^m(\tilde{\theta})$ for all $\tilde{\theta} < \theta$. Suppose not. Then, for all $\theta \in (0,\theta']$ there exits $\underline{\theta} < \theta$ such that $\beta^m(\theta) < \beta^m(\underline{\theta})$. Now, construct a sequence of types in the following way.

(1) Let $\theta_1 \in (0, \theta']$. Then take $\theta_2 < \theta_1$ be such that $\beta^m(\theta_2) > \beta^m(\theta_1)$.

(2) Let $\theta_n \in (0, \theta_{n-1})$ be such that $\beta^m(\theta_n) > \beta^m(\theta_{n-1})$.

(3) Let $\theta_n \to 0$. Observe this is possible by assumption.

Now, take n^* such that $\theta_{n^*} < \beta^m(\theta_1)$. Such n^* exists since $\theta_n \to 0$. Observe that by construction we have $\beta^m(\theta_{n^*}) > \beta^m(\theta_1) > \theta_{n^*}$, a contradiction. Therefore, there exits $\theta \in (0, \theta']$ such that for all $\tilde{\theta} < \theta$, $\beta^m(\theta) \geq \beta^m(\tilde{\theta})$. But then, type θ wins auction m only if she wins auction k. Hence, θ has a profitable deviation by bidding on k.

Hence, let $\theta^* > 0$. Observe that when this is the case, then β^m is weakly increasing in $[0, \theta^*]$ (it cannot be strictly increasing by assumption). By the previous argument, these types win auction m only if they win auction k, so they have a profitable deviation.

The second result for $G = G^{SPA}$ stems from Gerding et al. (2008b) in the following way. By contradiction, suppose that in equilibrium all the bidders $i \in \{1, \dots, N-1\}$ bid on one auction only. Then, player N has the incentive to bid on all the auctions. Hence, in a pure equilibrium, at least one bidder bids on multiple auctions.

For $G = G^{FPA}$ consider the following. Suppose each bidder place her bid on a single auction. Clearly, there exists $i \in \{1, \dots, N\}$ such that $Q_i < 1$, as $N > K$. Moreover, in a FPA, we have that in equilibrium $\beta_j^k(\theta_j) \leq \theta_j$ for all bidders j and auction k. Now, suppose i bids on auction m, and consider a deviation that bids the same amount on auction m and $\varepsilon > 0$ on auction k. Then, since the auctions are independent, the new probability of winning at least one object is $Q_i' = Q_i^m + Q_i^k - Q_i^m Q_i^k$ (we suppressed the arguments for readability). Note that $Q_i^k \geq F^n(\varepsilon)$, where n is the number of bidders on auction k. Moreover, note that the expected price on auction k is $Q_i^k \varepsilon$. Hence, the new expected utility increases by the amount

$$\theta_i Q_i^k (1 - Q_i^m) - Q_i^k \varepsilon \;=\; Q_i^k (\theta_i (1 - Q_i^m) - \varepsilon). \tag{2.1}$$

Since $Q_i^k > 0$ for $\varepsilon > 0$ and $Q_i = Q_i^m < 1$ by assumption, $\exists \varepsilon > 0$ such that (2.1) is strictly positive. Therefore, i can deviate and this is a contradiction. □

This Theorem tells us that we cannot have any symmetric pure strategy equilibrium unless for each auction k we have $\theta^k = 0$. This excludes any equilibrium for which β^k is increasing in an interval $[0, \theta']$ for any θ' and any auction k. An example of a strategy β with $\theta^k = 0$ for each k, is a function such that for all k, β^k has infinite oscillation as θ gets close to 0 (Figure 2.4). Our conjecture is, in case an equilibrium like this exists, that oscillations work as a coordination device. Clearly, $\beta^k \neq \beta^m$ (on sets with positive measure) for each pair of auctions k and m, otherwise these auctions are perfectly correlated in terms of allocation and bidders prefer to bid on a single auction. Hence, each auction will have a different strategy. Now, to have an intuition of our conjecture, suppose there are only two auctions. We think that every type will bid 'high' on an auction and bid 'low' on the other one.

High and low are calibrated to minimize the probability of winning both auctions under the constraint of monotonic allocation (a higher type has higher probability of winning at least one object, i.e., a higher Q). This attempt of coordination could generate oscillations. Another example of $\theta^k = 0$ is a function β^k that is weakly increasing from 0 to some θ and exhibits constant values in intervals that become progressively smaller as θ approaches zero. As shown in the Appendix for the proof of Theorem 2.3, we cannot exclude the possibility of ties in equilibrium for a general number of bidders. In fact, we believe that ties allow the players to hedge against the risk of winning too many objects.

Another natural candidate for the equilibrium is the case in which the bidders can coordinate their entry to reduce competition. For example, if there are $N = 4$ bidders and $K = 2$ auctions, naive intuition may expect the participation of two bidders in the first auction and the two other bidders in the second auction. Our second statement says that whenever $N > K$, this kind of coordination fails. In fact, when agents split and participate on one auction only, independent types imply that the allocation on one auction is independent from the allocation on another auction. As Gerding et al. (2008b) suggests, since $Q < 1$, bidders always "demand" for more probability of winning. As they can also control their expected price through their bid they always have incentive to bid on multiple auctions. This may not be achieved if auctions are highly correlated. An intuition of this result is also provided by the next example.

Example 2.2. Let $G = G^{SPA}$. Suppose there are $N = 2n + 1$ bidders, where $n \in \mathbb{N}$ and $K = 2$. They independently draw their type from the uniform distribution over $[0, 1]$. There are $2n$ local bidders and one global bidder. n local bidders bid on auction 1, and the other n bid on auction 2. Local bidders play either $\beta(\theta) = (\theta, 0)$ or $\beta(\theta) = (0, \theta)$ depending on the auction in which they participate. Suppose the global bidder places a bid of θ (her true type) on the first auction. Her interim utility is then

$$u(\theta, (\theta, 0), \beta_{-i}) \;=\; \theta^{n+1} - \frac{n}{n+1}\theta^{n+1}.$$

Now, consider placing a bid $b \in \mathbb{R}_+$ on auction 2 as well. Then

$$u(\theta, (\theta, b), \beta_{-i}) \;=\; \theta(\theta^n + b^n - \theta^n b^n) - \frac{n}{n+1}\theta^{n+1} - \frac{n}{n+1}b^{n+1}.$$

Therefore, the expected gain is

$$\begin{aligned}
&u(\theta, (\theta, b), \beta_{-i}) - u(\theta, (\theta, 0), \beta_{-i}) \\
&= \theta(\theta^n + b^n - \theta^n b^n) - \frac{n}{n+1}\theta^{n+1} - \frac{n}{n+1}b^{n+1} - \left(\theta^{n+1} - \frac{n}{n+1}\theta^{n+1}\right) \\
&= \theta b^n - \theta^{n+1} b^n - \frac{n}{n+1}b^{n+1} \\
&= b^n\left(\theta - \theta^{n+1} - \frac{n}{n+1}b\right).
\end{aligned}$$

Observe that the sign of the last expression depends on

$$\theta(1-\theta^n) - \frac{n}{n+1}b \gtreqless 0. \tag{2.2}$$

Choose $b \in \left(0, \frac{n+1}{n}\theta(1-\theta^n)\right)$. Expression (2.2) becomes strictly positive, and then the agent gains from bidding b.

Theorem 2.1 highlights two fundamental incentives within the game. First, when auctions are highly correlated, bidders tend to bid on fewer auctions. Conversely, when the auctions are independent and each bidder initially bids on a single auction, they wish to deviate by bidding on all of them. Therefore, any equilibrium lies between these two extremes: the auctions will be neither entirely independent nor perfectly correlated.

It is not trivial to obtain symmetric behavior in this game. The following result states symmetric behavior is possible in equilibrium with mixed strategies when $N = 2$. Proving the existence of such solutions can be problematic. Auctions present discontinuity in the payoffs. Therefore, we cannot apply classical results in fixed point theory. Reny (1999) provided Nash equilibrium existence results for a large class of discontinuous games. His main Theorem gives sufficient conditions for the existence of pure strategy equilibria that generalizes the mixed strategy equilibrium existence in the previous literature (e.g., Nash (1950), Glicksberg (1952)). Moreover, he provides additional conditions which are sufficient for the existence of symmetric equilibria. Proving that G possesses all the sufficient conditions requires many technical steps. Thus, we leave the proof of the following Theorem in the Appendix.

Theorem 2.3. *Let $N = 2$. The game G possesses a symmetric equilibrium in mixed strategies.*

In this section we have seen that pure, regular strategies and symmetric behavior cannot be achieved at the same time. Therefore in the next section we analyze the game G focusing on symmetric strategies. As proved in the previous theorem, we can obtain such an equilibrium when $N = 2$ allowing the bidders to use mixed strategies. The reason why we cannot extend the proof to any $N \geq 2$ is related to the fact that we cannot grant that the bidders do not strictly prefer ties in equilibrium. This blocks us from generating enough continuity in the game to use Reny (1999) results. We discuss this in Appendix 2.*B*. The analysis of (asymmetric) pure strategy equilibria is left in the Appendix 2.*A*, where we find the equilibria in several different examples and show their properties.

2.5 Symmetric Equilibria

2.5.1 Discrete bids

We start the section with a simple example of a game with discrete bids, two players, two auctions and two types. This game allows us to show the role of discrete bids on the equilibrium. Let $G = G^{SPA}$, $\Theta = \{0, 1\}$, $N = 2$ and $\mathscr{B} = \{0, 1/2, 1\}$, and set $Pr(\theta = 0) = 1/2$. We denote with $\sigma(\theta)$ the probability distribution over bids when the player's type is θ. For example, $\sigma(1) = [0.5(x,y), 0.5(y,x)]$ means that type $\theta = 1$ plays (x,y) (i.e., bids x on auction 1 and y on auction 2) and (y,x) with probability of 0.5 each.

Proposition 2.4. *The game G^{SPA} has only two symmetric mixed equilibria σ_1, σ_2, where*

$$\sigma_1(\theta) = \begin{cases} [1(0,0)] & \text{if } \theta = 0 \\ [1/6(1,0), 1/6(0,1), 2/3(1/2,1/2)] & \text{if } \theta = 1. \end{cases}$$

and

$$\sigma_2(\theta) = \begin{cases} [1(0,0)] & \text{if } \theta = 0 \\ [1/2(1,1/2), 1/2(1/2,1)] & \text{if } \theta = 1. \end{cases}$$

In the first equilibrium, bidders have a positive probability of bidding on one auction only (recall 0 bids are equivalent to non-participation). It is natural to ask whether this kind of equilibrium exists because of the low cardinality of the action space. In the next Proposition we show that this is indeed the case. The probability of bidding on one auction decreases as the action space becomes richer. Therefore, consider the sequence of games G_n similar to the previous one, where the type space is $\Theta = \{0, 1\}$ and the bid space is $\mathscr{B}_n = \{0 = x_0, x_1, \dots, x_n, x_{n+1} = 1\}$ such that the points in the set are equidistant. We get the following proposition.

Proposition 2.5. *Consider the sequence of games G_n^{SPA} and let p_n be the probability that a player bids on one auction only in a symmetric mixed equilibrium of G_n^{SPA}. Then, $p_n \to 0$ as $n \to \infty$.*

Proof. We drop the subscript n for readability. Suppose both bidders play the action $(1, 0)$ with probability p by both players in a symmetric equilibrium. Next, consider player i and the deviation $(1, x_1)$. We check the net gain of player i of moving mass p from $(1, 0)$ to $(1, x_1)$. This deviation benefits i in the event in which the other bidder plays $(1, 0)$ and she loses the tie. The benefit is 1, and the probability that this event happens is $\frac{1}{4}p$ (0.5 for $\theta = 1$ of the other player and 0.5 for losing the tie). Therefore the benefit is at least $\frac{1}{4}p$. Next, observe that playing $(1, x_1)$ over $(1, 0)$ can be detrimental to i in the case in which she is already winning auction 1. Yet, if the other player plays $(1,0)$, i does not pay any additional price. Hence, the maximum loss is $\frac{1}{2}(1-p)x_1$. Thus, a necessary condition for the optimality of $(1,0)$ is

$$\frac{1}{4}p \le x_1\frac{1}{2}(1-p) \Leftrightarrow p \le \frac{x_1}{1/2 + x_1}.$$

Then, as $x_1 \to 0$, we must have that $p \to 0$. □

Therefore, a richer type space induces the players to bid more frequently on all the auctions. In what follows, we prove that this fact holds in the limit too, where the bid space is the continuum $[0,1]$. Moreover, it holds for any number of auctions and bidders.

Consider again $G \in \{G^{FPA}, G^{SPA}\}$, $\Theta = \mathscr{B} = [0,1]$.

Proposition 2.6. *In any symmetric mixed strategy equilibrium of G, the players bid on all the available auctions with probability* 1.

This Proposition is true for all $N \ge 2$. Yet, as previously discussed, we cannot prove the existence of a symmetric mixed strategy equilibrium when the number of players is more than 2. The intuition of the previous Proposition is pretty straightforward: by symmetry, if one bidder puts positive probability on the strategy that bids on, say, auction 1 only, then everyone assigns the same probability to the same action. Hence, there is a strictly positive probability that another auction, for example, auction 2, will be left with no participants. Therefore, the strategy that bids the same amount on auction 1 and a small amount on auction 2 makes the player strictly better off, as the amount on the second auction can be arbitrarily small[4]. We start with the following Lemma, which is the core of the proof.

Lemma 2.7. *Suppose $N = 2$, $K = 2$, and types are uniformly distributed over* $[0,1]$. *Let $b > 0$. Then, in any symmetric mixed strategy equilibrium players do not put positive probability on the strategies* $(b,0)$ *and* $(0,b)$.

Proof. Suppose otherwise, that is, both players put $q_1 > 0$ on $(\beta^1(\theta), 0)$, or $q_2 > 0$ on $(0, \beta^2(\theta))$, or both. Without loss of generality, we assume both. Consider bidder $i \in \{1,2\}$. We claim $\exists \varepsilon > 0$ such that $(\beta^1(\theta_i), \varepsilon)$ (where θ_i is i's type) is a profitable deviation against $(\beta^1(\theta_i), 0)$. To do so, we compare the interim payoff provided by $(\beta^1(\theta_i), \varepsilon)$ and $(\beta^1(\theta_i), 0)$ in all the relevant events. We use interim payoffs as all the statements in the following steps hold for all $\theta > 0$.

With probability $q_1 > 0$ bidder $j \ne i$ plays $(\beta^1(\theta_j), 0)$. In equilibrium, $\beta^1(\theta') > \beta^1(\theta'')$ for $\theta' > \theta''$ (this is trivially true when the player decides to bid on a single auction). In this event, the action $(\beta^1(\theta_i), \varepsilon)$ makes i win at least one object.

4. The intuition is similar to the result in war of attrition in continuous time where at most one bidder can concede with positive probability. In fact, if both do, one of the player can wait an ε at time 0 and get a strictly higher payoff. See Abreu and Gul (2000), Hendricks, Weiss, and Wilson (1988).

Moreover, the expected prices of $(\beta^1(\theta_i), 0)$ and $(\beta^1(\theta_i), \varepsilon)$ are the same if we condition on j playing $(\beta^1(\theta_j), 0)$ in G^{SPA}, and are arbitrarily close in G^{FPA}. Therefore, $(\beta^1(\theta_i), \varepsilon)$ increases payoff by $\theta_i(1-\theta_i)$ with probability q_1.

Consider next j playing $(0, \beta^2(\theta_j))$. This event happens with a probability of $q_2 > 0$. Here, both $(\beta^1(\theta_i), 0)$ and $(\beta^1(\theta_i), \varepsilon)$ grant i the object. Yet, the second bid increases the expected price of auction 2 by a maximum of ε.

Finally, with probability q_3 bidder j is playing some bid (β', β''). Then, $(\theta_i, 0)$ and (θ_i, ε) provide the same probability of winning and expected price on auction 1. Winning auction 2 may be correlated with the event of winning auction 1. We consider the worst-case scenario, in which bidding on auction 2 does not increase the probability of winning at least one object. Therefore, $(\beta^1(\theta_i), \varepsilon)$ increases the expected price on auction 2 by no more than ε. In the worst-case scenario the difference in the payoff is at least

$$q_1\theta_i(1-\theta_i) + q_2(-\varepsilon) + q_3(-\varepsilon).$$

Hence, $\exists\varepsilon > 0$ such that $(\beta^1(\theta_i), \varepsilon)$ is strictly better than $(\beta^1(\theta_i), 0)$ (note that if $\varepsilon > 0$ is a profitable deviation for θ_i, then it is a profitable deviation for all $\tilde{\theta}_i > \theta_i$). □

Now, we are able to prove the Proposition.

Proof. Whenever $q_1 > 0$, there is a strictly positive probability that the other player is giving up auction 2. Hence, the bidder can just put a small amount in that auction and win the object for free. We can extend this result to any number of bidders. In fact, suppose there are $n+1 \in \mathbb{N}$ bidders. Consider again $q_1 > 0$ and $q_2 > 0$. There is still q_1^n probability of increasing the payoff by $\theta_i(1-\theta_i)^n$. Since in the other cases the expected costs can be controlled by the player via $\varepsilon > 0$, $(\beta^1(\theta_i), \varepsilon)$ is still a profitable deviation against $(\beta^1(\theta_i), 0)$. Again, the same holds for $q_2 > 0$. Finally, we can allow for any number of auctions. By the same reasoning as the last part, strategies of the kind $(0, \ldots, \beta^k(\theta_i), \ldots, 0)$ are dominated. When bidders put a positive probability on a strategy that bids on multiple auctions but not all of them, we can still apply the same logic. In fact, suppose the equilibrium strategy puts probability $q > 0$ on $(\beta^1(\theta_i), \beta^2(\theta_i), \ldots, 0, \ldots, \beta^K(\theta_i))$. Hence, there is a probability q^n that one auction has no bidders. Therefore, there exists $\varepsilon > 0$ such that $(\beta^1(\theta_i), \beta^2(\theta_i), \ldots, \varepsilon, \ldots, \beta^K(\theta_i))$ dominates the strategy in the support.

Note that the uniform distribution does not play a role. Hence, we can substitute it with any atomless distribution. □

2.5.2 Continuous bids

We have described the property that holds in any symmetric mixed equilibrium, and so we now describe and analyze a particular case. We find symmetric mixed strategy equilibria of a game G in which $K = 2$ and $N = 2$. Theorem 2.3 proved that this game always have a symmetric mixed strategy equilibrium for this number of bidders, and the Theorem could be easily extended to the discrete types case. This problem was previously considered by Szentes (2007) with type space $\Theta = \{1\}$. Therefore, he assumes complete information in his model. We extend his results. First, we consider the case $\Theta = \{0, 1\}$ and $\mathscr{B} = [0, 1]$. We assume $G = G^{FPA}$ to find a closed form solution, as this ensures an easier payoff structure. Then, we transform the equilibrium into an equilibrium of the second-price auction game G^{SPA} through a modified version of a technique provided in Szentes (2005). We leave the details in the Appendix. Finally, we modify the type space to $\Theta = \{a, 1\}$, $a > 0$. In this case, the competition for the high type is tighter as the low type is interested in the object. On both cases, $Pr(\theta = 1) = 1/2$.

Since the game has perfect recall, we can describe strategies in behavioral form, as in Proposition 2.4. Therefore, a symmetric equilibrium is a set of strategies such that each type randomizes over the square $[0, 1] \times [0, 1]$ and no profitable deviations are possible. Szentes (2007) proves that, in any symmetric mixed equilibrium with atomless strategies, agents randomize over two decreasing lines that lie in the space $\mathscr{A} = [0, 1] \times [0, 1]$. The reason the support includes decreasing lines only is intuitive: whenever a bidder increases the bid on one auction, say, auction 1, the marginal value of the object sold in auction 2 will decrease, making it optimal to place a lower bid on this auction. We now consider the type space $\Theta = \{0, 1\}$ as in the previous example and seek an explicit solution to the game. With incomplete information and $\Theta = \{0, 1\}$, randomization can occur along a single decreasing line. However, following Szentes' approach, we aim to find an equilibrium with a two-line support, as this equilibrium resembles the one found in the case of $\Theta = \{a, 1\}$. Before delving into the technical analysis, we provide an intuitive example to illustrate why the support must consist of either one or two decreasing lines.

So, take two points P_2 and P_1 in $[0, 1]^2$ such that $P_2 >> P_1$. These corresponds to two different vectors of bids. We assume that P_2 has higher bids on auction 1 and auction 2, as in Figure 2.2. Now consider alternative bids D_1 and D_2 as in the picture. Now compare the randomization $\frac{1}{2}P_1 + \frac{1}{2}P_2$ against $\frac{1}{2}D_1 + \frac{1}{2}D_2$. Note that both provide the same payoff if we condition on the event in which the other bidder plays outside of the red square. Instead, in case the opponent plays inside the square, P_1 wins no object while P_2 wins both; D_1 and D_2 win exactly one object each. Therefore, in order for P_1 and P_2 to be in the equilibrium support, the players cannot put positive mass inside the square, otherwise $\frac{1}{2}D_1 + \frac{1}{2}D_2$ is a deviation.

This exclude bidimensional supports, increasing lines or more than two decreasing lines, as in Figure 2.3.

Therefore, we are left with the possibility of two or one decreasing line. We proceed with the former. Let g_1 and g_2 be two decreasing functions. The union of their graphs is going to be the support of the equilibrium strategy of the high type as, clearly, $\sigma(0) = [1(0, 0)]$. The distribution of this strategy is determined by two other functions that we call G_1 and G_2. Finally, these four functions are related in

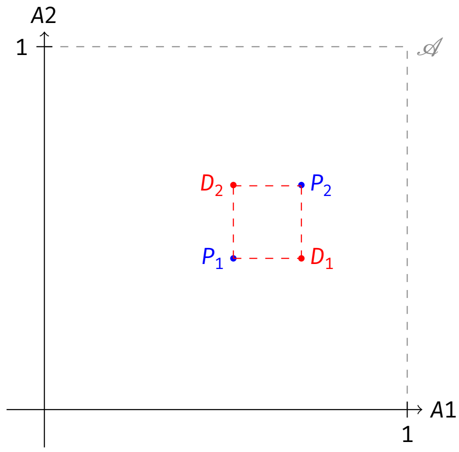

Figure 2.2. P_1 and P_2 against D_1 and D_2

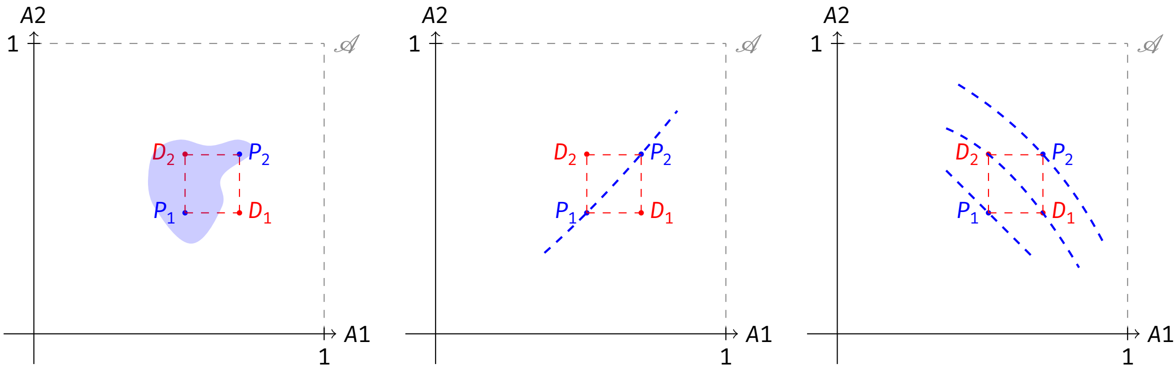

Figure 2.3. Examples of supports that cannot emerge in equilibrium

the following way:

$$G_1(g_1(x)) = \frac{1}{2} - G_1(x)$$
$$G_2(g_2(x)) = \frac{1}{2} - G_2(x),$$

where $G_1(x) = \frac{x}{1-x}(1 + G_2(x))$ and $G_2(0) = 0$, $G_2(1/4) = 1/2$. Then, we have the following.

Lemma 2.8. *The curves g_1 and g_2 are both symmetric with respect to the objects, strictly decreasing and for all $x \in (0, 1/4)$, and $g_1(x) > g_2(x)$ when $G_2(x) > \frac{x}{1-2x}$.*

Therefore, the support is symmetric to the auctions, as the curves g_1 and g_2 are cut in half by the 45-degree line. Now, consider the following two functions:

$$F_1(\{(y, g_1(y))|x \in [0, x]\}) = G_1(x)$$
$$F_2(\{(y, g_2(y))|x \in [0, x]\}) = G_2(x).$$

Both F_1 and F_2 are not defined over the entire Borel σ-algebra of $Graph(g_1)$ and $Graph(g_2)$, but they can be extended in the obvious way. Assume also that these extensions take value of zero on the square $[0,1]\times[0,1]$ except on their respective graphs. Call these extensions μ_1 and μ_2. Therefore, for example,

$$\mu_1(\{(x,g_1(x))|x\in[a,b]\}) = F_1(\{(x,g_1(x))|x\in[0,b]\}) - F_1(\{(x,g_1(x))|x\in[0,a]\}).$$

The same goes for μ_2. Observe then that μ_1 and μ_2 are measures over $[0,1]\times[0,1]$[5].

Proposition 2.9. *Let* $G_2(x) > \frac{x}{1-2x}$. *Then*

(1) μ_1 *and* μ_2 *induce the probability measure* $\sigma(1)=\mu_1+\mu_2$ *over the union of the graphs of* g_1 *and* g_2*;*

(2) The points $\{(x,y)|x\in[0,1/4], y\in[g_2(x),g_1(x)]\}$ *provide the same payoff against* σ*, where* $\sigma(0)=[1(0,0)]$*;*

(3) σ *is a symmetric mixed strategy equilibrium.*

Proof. (1) Trivially, μ_1 and μ_2 are measures and therefore σ is a measure. Next, observe

$$\sigma(Graph(g_1)\bigcup Graph(g_2)) = \mu_1(Graph(g_1)) + \mu_2(Graph(g_2)) = G_1(1/4) + G_2(1/4) = \frac{1}{2}+\frac{1}{2} = 1.$$

Hence, σ is a probability measure over the graphs of g_1 and g_2.

(2) Next, let $x\in[0,1/4]$ and $y\in[g_2(x),g_1(x)]$ and consider player I with $\theta_I=1$. The probability of winning at least one object is then

$$\underbrace{\frac{1}{2}\cdot 1}_{\theta_{II}=0} + \underbrace{\frac{1}{2}[\underbrace{(1-G_1(1/4))}_{II\text{ plays on }g_2} + \underbrace{G_1(x)+G_1(y)}_{II\text{ plays on }g_1}]}_{\theta_{II}=1}$$

and the expected payment is

$$\underbrace{\left(\frac{1}{2}+\frac{1}{2}(G_1(x)+G_2(x))\right)x}_{\text{Win auction 1}} + \underbrace{\left(\frac{1}{2}+\frac{1}{2}(G_1(y)+G_2(y))\right)y}_{\text{Win auction 2}}$$

that can be rewritten as

$$\frac{1}{2}x(G_1(x)+G_2(x)) + \frac{1}{2}y(G_1(y)+G_2(y)) + \frac{1}{2}(x+y).$$

Now, observe that for $x\in[0,1/4]$

$$G_1(x) = \frac{x}{1-x}(1+G_2(x)) \Rightarrow G_1(x) = x + x(G_1(x)+G_2(x)).$$

5. The extension on $[0,1]\times[0,1]$ works in the following way: for all $A\subseteq[0,1]\times[0,1]$, $\mu_i(A) = \mu_i(A\cap Graph(g_i))$, for $i\in\{1,2\}$.

Therefore, the expected payment is

$$\frac{1}{2}(G_1(x)-x)+\frac{1}{2}(G_1(y)-y)+\frac{1}{2}(x+y) \;=\; \frac{1}{2}(G_1(x)+G_1(y)).$$

Finally, the expected payoff is

$$\frac{1}{2}+\frac{1}{2}(1-G_1(1/4))+\frac{1}{2}(G_1(x)+G_2(y))-\frac{1}{2}(G_1(x)+G_2(y)) \;=\; \frac{1}{2}+\frac{1}{2}(1-G_1(1/4)),$$

and then it is independent of (x,y) as long as $x\in[0,1/4]$ and $y\in[g_2(x),g_1(x)]$.

(3) We are left to show that σ is an equilibrium, that is, there are no profitable deviations outside the set $\{(x,y)|x\in[0,1/4],y\in[g_2(x),g_1(x)]\}$ against σ. Observe we only have to check deviations such that $x,y\le\frac{1}{4}$.

Obviously $\sigma(0)=[1(0,0)]$ is optimal. For $\theta=1$, let $x\in[0,1/4]$ and $y>g_1(x)$. Observe this action provides the same probability of winning as $g_1(x)$ (i.e., 1) but has a strictly higher expected price. Hence, it cannot be optimal. Next, consider $y<g_2(x)$. The probability of winning one object at least is now

$$\frac{1}{2}+\frac{1}{2}(G_1(x)+G_2(x)+G_1(y)+G_2(y))$$

and together with the expected price we have a payoff of

$$\frac{1}{2}+\frac{1}{2}(G_2(x)+G_2(y)).$$

Recall that $G_1(1/4)+G_2(1/4)=1$, and so we can rewrite the previous expression as

$$\frac{1}{2}+\frac{1}{2}(1-G_1(1/4))-\frac{1}{2}(G_2(1/4)-G_2(x)-G_2(y)),$$

and since $G_2(1/4)-G_2(x)-G_2(y)>0$, it is not profitable to deviate to $y<g_2(x)$.

□

Therefore, any G_2 that satisfies the condition in the previous Proposition gives us an equilibrium. In the following picture, we show an example of the support of this equilibrium and the corresponding transformation into the support of G^{SPA} equilibrium.

The case just showed presents a particular kind of incomplete information. A player who is interested in acquiring the object ($\theta=1$) is uncertain about whether the other player desire or not the object. Therefore, the information she is missing is about the participation decision of the other agent. It is interesting to ask what kind of symmetric equilibrium emerges when the low type is bounded away from zero. Or, equivalently, when both types are interested in winning the object. It turns out that this time, the support of each type can be just one decreasing line. Therefore, let $\Theta=\{a,1\}$, where $a>0$. The auctions are first-price as before. Let g_1 and

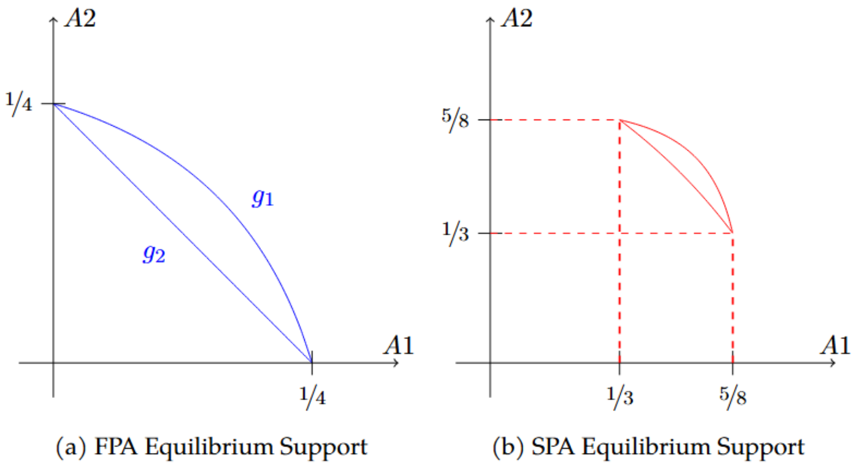

Figure 2.4. Mixed equilibria support

g_a be defined on $[0, a/2]$ and $[0, b]$ ($b < a/2$), respectively. The graph of the first function is the support of the equilibrium strategy of $\theta = 1$ while the second one is the support of the equilibrium strategy of $\theta = a$. These functions are implicitly defined by

$$\begin{aligned} G_1(g_1(x)) &= 1 - G_1(x) \\ G_a(g_a(x)) &= 1 - G_a(x), \end{aligned}$$

where $G_a(0) = 0$, $G_a(b) = 1$, G_a strictly increasing over $[0, b]$, and

$$G_1(x) = \begin{cases} \frac{x}{a-x} G_a(x) & x \in [0, b] \\ \frac{x}{a-x} & x \in (b, a/2]. \end{cases}$$

The distributions on the curves in the equilibrium are

$$\begin{aligned} F_1(\{(y, g_1(y)) | y \in [0, x)\}) &= G_1(x) \\ F_a(\{(y, g_a(y)) | y \in [0, x)\}) &= G_a(x). \end{aligned}$$

We extend these functions to μ_1 and μ_a in the same fashion as before. Observe μ_1 and μ_a are probability measures. This equilibrium looks different to the case $a = 0$. Here every type randomizes over one decreasing line. In fact, $\theta = a$ plays over the graph of g_a while $\theta = 1$ plays over the graph of g_1.

Lemma 2.10. *The curves g_1 and g_a are both symmetric with respect to the objects, strictly decreasing in their domain and $g_1 > g_a$ in the domain of g_a.*

The proof of this Lemma follows the same lines of Lemma 2.8, hence it is skipped.

Proposition 2.11. *The strategy above generates a symmetric mixed strategy equilibrium.*

Proof. We first show that both $\theta = 1$ and $\theta = a$ are indifferent in their support, and then we prove there is no profitable deviation for the players. Then, consider $\theta = 1$. Let $(x, g_1(x))$ be the action she plays. Given that the other player plays (μ_a, μ_1), the payoff is

$$\begin{aligned}
&\frac{1}{2}[1 - G_1(x)x - G_1(g_1(x))g_1(x)] + \frac{1}{2}[1 - G_a(x)x - G_a(g_1(x))g_1(x)] \\
=&\frac{1}{2}[2 - (G_1(x) + G_a(x))x - (G_1(g_1(x)) + G_a(g_1(x)))g_1(x))].
\end{aligned}$$

Observe that for all $x \in [0, a/2]$, we have

$$\begin{aligned}
(G_1(x) + G_a(x))x &= aG_1(x) \\
(G_1(g_1(x)) + G_a(g_1(x)))g_1(x) &= (1 - G_1(x))a.
\end{aligned}$$

Therefore, the payoff reduces to $1 - a/2$. Hence, $\theta = 1$ is indifferent in the graph of g_1.

Next, let $\theta = a$. We show a stronger statement for this type, that is, she is indifferent on the entire set

$$A = \{(x,y) | g_a(x) \leq y \leq g_1(x)\}$$

Therefore, suppose $\theta = a$ plays $(x,y) \in A$. Observe that her payoff is

$$\begin{aligned}
&\frac{1}{2}a[1 + G_1(x) + G_1(y)] - \frac{1}{2}[(G_a(x) + G_1(x))x + (G_a(y) + G_1(y))y] \\
=&\frac{1}{2}a + \frac{1}{2}(aG_1(x) + aG_1(y)) - \frac{1}{2}(aG_1(x) + aG_1(y)) \\
=&\frac{1}{2}a
\end{aligned}$$

Hence, $\theta = a$ is indifferent on the entire set A, which includes the graph of g_a.

Now, we check optimality. Consider again $\theta = 1$. Clearly, (x,y) where $y > g_1(x)$ is not optimal as in the case $a = 0$. Then, let the agent bid (x,y), $g_a(x) < y < g_1(x)$. The expected payoff is then

$$\begin{aligned}
&\frac{1}{2}[G_1(x) + G_1(y) + 1 - (G_1(x) + G_a(x))x - (G_1(y) + G_a(y))y] \\
=&\frac{1}{2}[G_1(x) + G_1(y) + 1 - aG_1(x) - aG_a(y)] \\
=&\frac{1}{2}[(1 - a)(G_1(x) + G_1(y)) + 1] \\
<&1 - \frac{a}{2},
\end{aligned}$$

where the last inequality follows from $G_1(x)+G_1(y)<1$. Hence, (x,y) cannot be a profitable deviation. Assume $y<g_a(x)$. The expected profit is then

$$\frac{1}{2}[(1-a)(G_1(x)+G_1(y))+(G_a(x)+G_a(y))] \;<\; 1-\frac{a}{2}.$$

We conclude that $\theta=1$ is in equilibrium. We are left to show that $\theta=a$ does not want to deviate to any (x,y) with $y<g_a(x)$. If the low type is playing a bid below g_a, her payoff is

$$\begin{aligned}
&\frac{1}{2}a(G_a(x)+G_a(y)+G_1(x)+G_1(y))-\frac{1}{2}(G_a(x)x+G_a(y)y+G_1(x)x+G_1(y)y)\\
=&\frac{1}{2}a(G_a(x)+G_a(y)+G_1(x)+G_1(y))-\frac{1}{2}(aG_1(x)+aG_1(y))\\
=&\frac{1}{2}a(G_a(x)+G_a(y)) \;<\; \frac{1}{2}a,
\end{aligned}$$

where the last inequality follows from $G_a(x)+G_a(y)<1$. Hence, type $\theta=a$ is in equilibrium. Therefore, the suggested strategies form an equilibrium. □

As the proof shows, the high type strictly prefers to play in its support than in any other portion of the action space. The low type, instead, is indifferent among all the points between the curves g_a and g_1. Therefore, the low type is characterized by the same payoff condition as in Szentes (2007), where the agent had complete information.

2.6 Conclusion

In this paper, we have analyzed the bidding behavior of unit-demand buyers who have the opportunity to place bids on multiple sealed-bid auctions. Our analysis reveals several key insights about the strategic behavior of bidders in such environments.

Firstly, we demonstrated that restricting bidders to participate in only one auction is not without loss of generality. In fact, bidders have a strong incentive to place bids on multiple auctions simultaneously. This multi-auction bidding behavior arises due to the strategic trade-offs faced by bidders. While bidding on multiple auctions raises the sum of expected prices, it also increases the likelihood of winning at least one item. However, the correlation among auctions plays a critical role in shaping these incentives. When auctions are highly correlated, the incentive for bidders to participate in all auctions diminishes. Specifically, if one auction is nearly a perfect copy of another, the incentive to bid on both auctions is significantly reduced. Yet, independent auctions, i.e., bidders participate in only one auction, creates incentives to bid on all of them. This leads to the conclusion that in any pure equilibrium, at least one bidder will place bids across multiple auctions.

Secondly, we explored the existence of symmetric equilibria with 'standard' strategies. These strategies include at least one bidding function that is increasing from the lowest type to any arbitrarily higher type. We found that such equilibria are unattainable in our setting. When bidders adopt regular strategies, some auctions become highly correlated, which incentivizes bidders to focus their efforts on a single auction, effectively abandoning the others. To reintroduce symmetry, we must consider mixed strategies. We prove equilibrium existence in the case of two bidders. Moreover, our analysis shows that if we aim to achieve symmetric equilibria through mixed strategies, we should expect all bidders to participate in all auctions with probability one.

Lastly, we provided a detailed characterization of the equilibria in the specific case of two bidders facing two sealed-bid auctions and incomplete information, therefore extending previous literature. As we considered binary type spaces for closed form solutions, future research could extend our model further by investigating equilibria with continuous types, which would offer a better understanding of bidder behavior. Additionally, examining the role of reserve prices in such auctions could provide valuable insights into how sellers can influence bidding strategies and auction outcomes.

Appendix 2.A Asymmetric pure equilibria

In this section we find explicit solutions to the case of $K = 2$ and $N = 3$. We start with the case of binary types. We consider discrete and continuous type spaces and show that we get similar equilibria. Assume there are $K = 2$ sealed-bid second-price auctions and $N = 3$ ex-ante symmetric bidders.

Binary types

Let $\Theta = \{0, 1\}$. Types are equally likely, and then $Pr(\theta = 0) = Pr(\theta = 1) = \frac{1}{2}$. The bid space is $\mathscr{B} = [0, 1]$. Finally, label the players with I, II, and III. Ties are broken evenly and randomly. Then, the following set of strategies constitutes an equilibrium of the game G:

(1) $\beta_I(\theta) = (\theta, 0)$

(2) $\beta_{II}(\theta) = (0, \theta)$

(3) $\beta_{III}(\theta) = \begin{cases} (0,0) & \text{if } \theta = 0 \\ \big(1/2, 1/2\big) & \text{if } \theta = 1 \end{cases}$

Observe the following. The third bidder would be in equilibrium with any strategy that has $\beta_{III}(1) = (\varepsilon, \varepsilon)$ where $\varepsilon \in (0, 1)$. He can only win when the other bidders have a type equal to 0. Hence, whenever he wins one or two objects, he gets them for free. One may think that the previous equilibrium is because with a probability of 1/2 bidders I or II will not bid on the auction, leaving to III the possibility of winning the object for free. This is not entirely true. Consider the following case. Let $\Theta = \{2/5, 1\}$ and $Pr(\theta = 2/5) = 1/2$. Then,

(1) $\beta_I(\theta) = (\theta, 0)$

(2) $\beta_{II}(\theta) = (0, \theta)$

(3) $\beta_{III}(\theta) = \begin{cases} (0,0) & \text{if } \theta = 2/5 \\ \big(1/2, 1/2\big) & \text{if } \theta = 1 \end{cases}$

is an equilibrium of G. Even if the minimum bid from I and II is pretty high, III prefers to bid on both when his type is 1. Moreover, when III has a valuation of 2/5 for the object, he does not participate in any of the auctions, as winning leaves him indifferent.

It turns out that there are conditions on the type space and type distribution under which an equilibrium of this kind always exists. We fix the upper bound of Θ to 1 as this has no qualitative impact on the following result.

Proposition 2.12. *Consider the game G where $K = 2$, $N = 3$ and $\mathscr{B} = [0, 1]$. Let $\Theta = \{a, 1\}$ and $Pr(\theta = a) = P_a$. Then, for any $\varepsilon \in (a, 1)$, whenever $1 - a \geq P_a$, there exists an equilibrium $(\beta_I, \beta_{II}, \beta_{III})$ where*

(1) $\beta_I(\theta) = (\theta, 0)$

(2) $\beta_{II}(\theta) = (0, \theta)$

(3) $\beta_{III}(\theta) = \begin{cases} (0,0) & \text{if } \theta = a \\ (\varepsilon, \varepsilon) & \text{if } \theta = 1 \end{cases}$.

Proof. Consider player III first. Observe that the unique deviation to consider is to play either $(1,0)$ or $(0,1)$, as all the other alternatives are either equivalent/dominated by $(\varepsilon, \varepsilon)$ or dominated by $(1,0)$ and $(0,1)$. Suppose $\theta_{III} = 1$ (optimality for $\theta_{III} = a$ is trivial). Then,

$$u_{III}(1, (1,0), \beta_I, \beta_{II}) \;=\; 1\left(P_a + (1-P_a)\frac{1}{2}\right) - \left(P_a \cdot a + (1-P_a)\frac{1}{2} \cdot 1\right) \;=\; P_a(1-a).$$

Now compute the expected payoff of $(\varepsilon, \varepsilon)$. First, we find the probability of winning at least one object and then the expected prices. We can write Q_{III} as 1 minus the probability of losing all the objects. Observe that the last event happens with a probability of $(1-P_a)^2$. Therefore,

$$Q_{III}((\varepsilon, \varepsilon), \beta_I, \beta_{II}) \;=\; 1 - (1-P_a)^2.$$

By symmetry of β_I and β_{II}, the expected prices of auction 1 and 2 are the same given $\beta_{III}(1) = (\varepsilon, \varepsilon)$, and are equal to $P_a \cdot a + (1-P_a) \cdot 0 = P_a \cdot a$. Therefore, we have

$$u_{III}(1, (\varepsilon, \varepsilon), \beta_I, \beta_{II}) \;=\; 1(1 - (1-P_a)^2) - 2P_a \cdot a.$$

The action $(\varepsilon, \varepsilon)$ is weakly better than $(1,0)$ or $(0,1)$ whenever

$$1 - (1-P_a)^2 - 2P_a \cdot a \;\geq\; P_a(1-a) \iff 1-a \;\geq\; P_a.$$

Therefore, whenever the last weak inequality is satisfied, player III is in equilibrium with β_{III}.

Now consider player I. There are four possible scenarios:

(*i*) $\theta_{II} = a$, $\theta_{III} = a$

(*ii*) $\theta_{II} = a$, $\theta_{III} = 1$

(*iii*) $\theta_{II} = 1$, $\theta_{III} = a$

(*iv*) $\theta_{II} = 1$, $\theta_{III} = 1$

Suppose $\theta_I = a$. In (*i*), the player cannot do better than playing $(a, 0)$. In (*ii*), he cannot obtain more than a payoff of 0, and $(a, 0)$ achieves it. Case (*iii*) is the same as (*i*). Finally, in case (*iv*), he cannot gain more than 0 as in (*ii*). Therefore, $\beta_I(a) = (a, 0)$ is optimal.

Next, suppose $\theta_I = 1$. In case (*i*), $(1,0)$ is trivially optimal. In (*ii*), the minimum price is ε on both auctions. As $1 > \varepsilon$, $(1,0)$ is optimal. In case (*iii*) it is strictly better to play on auction 1 and $(1,0)$ is the optimal bid. In (*iv*), $(1,0)$ is again trivially optimal.

Since I and II are symmetric and play symmetric roles in the equilibrium, II's optimality follows from I's optimality. Therefore, $(\beta_I, \beta_{II}, \beta_{III})$ is an equilibrium under $1-a \geq P_a$. □

The statement tells us that for $a = 0$, any distribution of types sustains such an equilibrium. Observe that if a increases, the expected price of the third player (with $\theta_{III} = 1$) increases as well. To convince the third player to bid on both, the probability of a needs to decrease so that the low type becomes less relevant. When this happens, competition is high, and the event of winning both objects is unlikely. Therefore, $\theta_{III} = 1$ can accept the risk and keep bidding on all the available auctions.

On the other hand, when P_a increases, it becomes easier for θ_{III} to win an object. Then the incentives of bidding on both are lower. A low a reduces the expected prices, and winning all the goods is not too costly. Player III can then bid on both.

Continuum of types

Let $\Theta = [0,1]$. The following is an equilibrium of the game G:

(1) $\beta_I(\theta) = (\theta, 0)$

(2) $\beta_{II}(\theta) = (0, \theta)$

(3) $\beta_{III}(\theta) = \left(\frac{\theta}{1+\theta}, \frac{\theta}{1+\theta}\right)$

Proof. Consider agent III. Her utility is

$$u_{III}(\theta, (b_1, b_2), (\beta_I, \beta_{II})) \;=\; \theta(b_1 + b_2 - b_1 b_2) - \frac{b_1^2}{2} - \frac{b_2^2}{2}.$$

This function is concave and hence FOC will be sufficient. The point that satisfies the FOC is

$$\beta_{III}(\theta) \;=\; \left(\frac{\theta}{1+\theta}, \frac{\theta}{1+\theta}\right).$$

For what regards players I and II, consider the following. Suppose I bids $\beta_I = (x_1, x_2)$. Then, observe that $x_1 \geq x_2$ is not optimal in case I wants to bid on both auctions. In fact, note that I faces $\theta_{III}/(1+\theta_{III})$ on auction 1 and $\theta_{III}/(1+\theta_{III})$ and θ_{II} on auction 2. Hence, when $x_1 \geq x_2$, winning auction 2 immediately implies winning auction 1. Winning both is never desired. Hence, $x_2 = 0$ would be an improvement. Therefore, we are going to assume that $x_2 > x_1$. This condition is not optimal too. To see this, suppose we start from $x_2 = x_1$. We study what happens when we go in the direction $x_2 + \varepsilon$ compared to the direction $x_1 + \varepsilon$. Observe that $\forall \varepsilon > 0$, we increase the probability of winning in auction 1 more than in auction 2 as in auction 1 agent I faces only III and in auction 2 she faces II and III (and III bids the same amount on 1 and 2). Moreover, the increase in the expected price in the second auction is at least as high as in the first one, as in 1 player I faces $\theta_{III}/(1+\theta_{III})$ and in the second one, she faces $\theta_{III}/(1+\theta_{III})$ and θ_{II}. Hence, going in the direction of $x_2+\varepsilon$ is not optimal. Since neither $x_1 \geq x_2$ nor $x_2 > x_1$ is optimal, bidding on both auctions is never optimal. Of course, it is better to bid $(\theta_I, 0)$ than $(0, \theta_I)$, as the first auction features a higher probability of winning and a lower expected price. Note that II would apply the same reasoning. Hence the suggested equilibrium is indeed an equilibrium. □

In this equilibrium, the third bidder bids on both auctions with the same amount. The strategy in each auction is concave with respect to the type. The reason why is the following. Consider for example type $\theta = 1$. This type bids $(1/2, 1/1)$, and therefore the sum of her bids is equal to her type. If we consider type $\theta = 1/2$, then we observe the bid $(1/3, 1/3)$. In this case, the sum of her bids are strictly higher her true valuation. As we approach type 0, we see that the sum of the bids gets closer and closer to 2θ, that is, the bidder is placing a bid of *almost* her type on both auctions. Clearly, this is due to the fact that as the type increases, the probability of winning *both* auctions increases as well if the bidder place her type for both objects. To offset this undesired event, the bidder decreases the sum of the bids with respect to her type.

Four bidders

There are $K = 2$ auctions with $N = 4$ ex-ante symmetric bidders. The type space is $\Theta = \{0, 1\}$ and $P_0 = \frac{1}{2}$. Then,

(1) $\beta_I(\theta) = (\theta, 0)$

(2) $\beta_{II}(\theta) = (0, \theta)$

(3) $\beta_{III}(\theta) = \begin{cases} (0,0) & \text{if } \theta = 0 \\ \big(3/4, 1/2\big) & \text{if } \theta = 1 \end{cases}$

(4) $\beta_{IV}(\theta) = \begin{cases} (0,0) & \text{if } \theta = 0 \\ \big(1/2, 3/4\big) & \text{if } \theta = 1 \end{cases}$

is an equilibrium. This equilibrium is similar to the previous ones, but two bidders are bidding on both auctions this time. Moreover, these bidders do not bid symmetrically in the two auctions. The first one bids more aggressively on the first auction, while the second one does the opposite. Bidding symmetrically for them is not an equilibrium. If they do so, each of them could unilaterally deviate by increasing the bid in the first auction by some small $\varepsilon > 0$ and decrease the amount in the second one by the same $\varepsilon > 0$. This deviation would allow the player, with the same price, to win one object every time he would have tied. Furthermore, they would reduce the probability of winning and paying for both goods. Hence they can be in equilibrium (given I and II behavior) only once they coordinate. Then, not only do bidders coordinate on who should bid where, but also on the amount they bid.

Appendix 2.B Equilibrium existence with $N = 2$

A mixed strategy is a distribution over the product of measurable functions from Θ to $\mathscr{B}$. A game, in what follows, is a vector $G = (X_i, u_i)_{i=1}^N$ where X_i is the strategy set of each player i and u_i is her utility function.

The following definitions and results are fundamental for the proof of the existence of a symmetric mixed equilibrium.

Definition 2.13. Player i can secure a payoff of $\alpha \in \mathbb{R}$ at $x \in X$ if there exists $\overline{x}_i \in X_i$ such that $\exists U \subseteq X_{-i}$ (open) with $x_{-i} \in U$ such that

$$\forall x'_{-i} \in U \quad u_i(\overline{x}_i, x'_{-i}) \geq \alpha.$$

Therefore, when the game is at $x \in X$, i can secure a payoff α if i has a strategy that grants him that payoff even when the other players deviate slightly from x_{-i}. Next, let $u(x) = (u_1(x), \ldots, u_N(x))$ be the vector payoff function which, for each $x \in X$, gives the utility of all players.

Definition 2.14. A game $G = (X_i, u_i)_{i=1}^N$ is better-reply secure if whenever (x^*, u^*) is in the closure of the graph of its vector payoff function and x^* is not an equilibrium, some player i can secure a payoff strictly above u_i^* at x^*.

Hence, a game is better-reply secure when i can secure a payoff strictly above u_i^* whenever x^* is not an equilibrium. Reny (1999) observes that any continuous game is better-reply secure. Any better reply will provide (at least) one agent with a payoff that is strictly above the payoff of a non-equilibrium and its neighborhood.

Now, let G be a quasi-concave game whenever X_i is convex for each player i, and, for each i, for each $x_{-i} \in X_{-i}$, $u_i(\cdot, x_{-i})$ is quasi-concave on X_i. The following is Reny (1999)'s main theorem.

Theorem 2.15. *(Reny 1999, Theorem 3.1) If $G = (X_i, u_i)_{i=1}^N$ is compact, quasi-concave, and better-reply secure, it possesses a pure strategy Nash equilibrium.*

The proof of the theorem makes the role of better-reply security in the existence of the equilibrium clearer. Better reply security creates a link between continuous and discontinuous games. When the game possesses this property, it is possible to characterize the set of Nash equilibrium in terms of a particular function $\underline{u}_i$ instead of u_i, which is lower-semicontinuous. Then u_i is approachable from below by continuous functions, and this reduces the existence problem to establishing whether there are strategies that are robust against a finite set of deviations. In the final part of the proof, we show that such strategies do indeed exist.

In any case, what we are interested in is not the set of pure strategy equilibria. As previously shown, if we require symmetry in the equilibrium, we need to abandon increasing strategies. The following definitions will guide us through the existence result of a mixed strategy equilibrium first and the existence of a symmetric mixed strategy equilibrium next.

Let M the set of probability measure over (X, B), where B is the Borel σ-algebra over X, where X is equipped with its weak*-topology. With a slight abuse of notation, call $u_i(\mu) = \int_X u_i d\mu$ for $\mu \in M$ for each $i \in \{1, \ldots, N\}$. Then, let $\overline{G} = (M_i, u_i)_{i=1}^N$ be the mixed extension of G.

Theorem 2.16. *(Reny (1999), Corollary 5.2) Suppose that $G = (X_i, u_i)_{i=1}^N$ is a compact and Hausdorff game. Then G possesses a mixed strategy Nash Equilibrium if its mixed extension $\overline{G}$ is better-reply secure.*

Before showing that our game $\overline{G}$ is better-reply secure, let us see what are the sufficient conditions for a symmetric equilibrium.

Definition 2.17. A game $G = (X_i, u_i)_{i=1}^N$ is quasi-symmetric if

(1) $\forall i, j \in \{1, \ldots, N\}\ X_i = X_j$;
(2) $\forall x, y \in X\ u_1(x, y, \ldots, y) = u_2(y, x, y, \ldots, y) = \cdots = u_N(y, \ldots, y, x)$

Here, $u_i(y, \ldots, y, x, y, \ldots, y)$ denotes the function u_i evaluated at the strategy in which player i chooses x while any other player $j \neq i$ chooses strategy y.
Now, call $v(x) = u_i(x, \ldots, x)$ the diagonal payoff function. Consider the following two definitions.

Definition 2.18. Player i can secure a payoff of $\alpha \in \mathbb{R}$ along the diagonal at $(x, \ldots, x) \in X^N$, if there exists $\overline{x} \in X$ such that $u_i(x', \ldots, \overline{x}, \ldots, x') \geq \alpha$ for all x' in some open neighborhood of $x \in X$.

Definition 2.19. The game $G = (X_i, u_i)_{i=1}^N$ is diagonally better-reply secure if whenever $(x^*, u^*) \in X \times \mathbb{R}$ is in the closure of the graph of its diagonal payoff function and $(x^*, \ldots, x^*)$ is not an equilibrium, some player i can secure a payoff strictly above u^* along the diagonal at $(x^*, \ldots, x^*)$

As Reny (1999) points out, diagonal better-reply security is strictly weaker than better-reply security when $N \geq 3$. Hence, showing better-reply security implies that the game is diagonally better-reply secure when the game is quasi-symmetric. The next theorem, along the lines of the preceding ones, tells us that diagonal better-reply security is sufficient for the existence of a symmetric mixed strategy equilibrium.

Theorem 2.20. *(Reny (1999), Corollary 5.3) Suppose that $G = (X_i, u_i)_{i=1}^N$ is a quasi-symmetric, compact, Hausdorff game. Then G possesses a symmetric mixed strategy Nash Equilibrium if its mixed extension, $\overline{G}$, is better reply secure along the diagonal.*

Now that all the definitions and results have been introduced, let us summarize what we need to prove to show the existence of a mixed symmetric equilibrium:

A. X_i has to be compact for each i;
B. X_i has to be Hausdorff for each i;
C. G has to be quasi-symmetric;
D. $\overline{G}$ has to be better-reply secure.

While point B. and C. will be trivial, point A. and D. requires a bit of work.

A. X_i is compact.

In a single auction, each player would have to choose a strategy beforehand, which consists of a measurable function from the interval $[0, 1]$ to the interval $[0, 1]$. Here, instead, the bidder has to choose K strategies of the kind $x : [0, 1] \longrightarrow [0, 1]$. For analytical purposes, these functions have to be measurable. Hence, X_i is the product of K spaces, in particular, K copies of the set of measurable functions $x : [0, 1] \longrightarrow [0, 1]$. We will consider one of these spaces at the time.

Consider the space $L^\infty_\lambda([0, 1])$, that is, the space of all (equivalence classes of) measurable functions with domain $[0, 1]$ which are λ-essentially bounded, where λ is the Lebesgue measure. Observe that the set of all measurable functions $x : [0, 1] \longrightarrow [0, 1]$, say $\tilde{X}_i$, is strictly contained in $L^\infty_\lambda([0, 1])$. Moreover, if we consider $L^\infty_\lambda([0, 1])$ as a normed vector space (equipped with the supremum norm $|| \cdot ||_\infty$), we have that

$$\tilde{X}_i \subseteq B_\infty := \{f \in L^\infty_\lambda([0, 1]) : ||f||_\infty \leq 1\},$$

that is, $\tilde{X}_i$ is contained in the unit ball of the space.

Now, let us topologize $L^\infty_\lambda([0, 1])$ with its weak*-topology. By the Banach-Alaoglu's Theorem, B_∞ is weakly*-compact. Therefore, to prove that $\tilde{X}_i$ is weakly*-compact, we need to show that it is weakly*-closed. We can use sequences instead of nets to characterize the closedness of $\tilde{X}_i$.

Proposition 2.21. *$\tilde{X}_i$ is a weakly*-closed subset of B_∞.*

Proof. Observe that $\forall x \in \tilde{X}$, $||x||_\infty \leq 1$ λ-a.e.

Now, take a sequence $x_n \in \tilde{X}_i$ such that $x_n \overset{\star}{\rightharpoonup} x$ (that is, x_n weakly* converges to x). By the Riesz Representation Theorem, we have that

$$x_n \overset{\star}{\rightharpoonup} x \Leftrightarrow \int_{[0,1]} x_n g d\lambda \to \int_{[0,1]} x g d\lambda \quad \forall g \in L^1_\lambda([0, 1]). \tag{2.B.1}$$

Claim 1: $x \leq 1$ λ-a.e.
Suppose not, i.e., $\exists A \in B$ such that $\lambda(A) > 0$ and $x(a) > 1$ $\forall a \in A$. Then, consider the function $g = \mathbb{1}_A$, i.e., the index function of A. Since A is a measurable set, g is a measurable function. Moreover,

$$\int_{[0,1]} g d\lambda = \int_{[0,1]} \mathbb{1}_A d\lambda = \int_A 1 d\lambda = \lambda(A) \leq \lambda([0, 1]) < \infty.$$

Hence, g is λ-integrable and therefore $g \in L^1_\lambda([0,1])$. Moreover, observe

$$\int_{[0,1]} xg d\lambda = \int_{[0,1]} x \mathbb{1}_A d\lambda = \int_A x d\lambda > \int_A 1 d\lambda = \lambda(A),$$

where the inequality sign comes from the monotonicity of the integral. Therefore, by equation (2.*B*.1), we have that

$$\exists N \in \mathbb{N} \text{ such that } \forall n \geq N, \qquad \int_{[0,1]} x_n g d\lambda > \lambda(A).$$

But

$$\int_{[0,1]} x_n g d\lambda = \int_{[0,1]} x_n \mathbb{1}_A d\lambda = \int_A x_n d\lambda \leq \int_A 1 d\lambda = \lambda(A),$$

a contradiction. Therefore, $x \leq 1$ λ-a.e.

Claim 2: $x \geq 0$ λ-a.e.
The proof follows the same lines of the previous claim. Suppose $x < 0$ on a set $A \in B$ such that $\lambda(A) > 0$. Consider again $g = \mathbb{1}_A \in L^1_\lambda([0,1])$. Then,

$$\int_{[0,1]} xg d\lambda = \int_{[0,1]} x \mathbb{1}_A d\lambda = \int_A x d\lambda < 0 = \int_A 0 d\lambda.$$

As before, since $\int x_n g d\lambda \to \int xg d\lambda$ in the euclidean topology over $\mathbb{R}$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$

$$\int_{[0,1]} x_n g d\lambda < 0,$$

but again

$$\int_{[0,1]} x_n g d\lambda = \int_{[0,1]} x_n \mathbb{1}_A d\lambda = \int_A x_n d\lambda \geq 0 = \int_A 0 d\lambda,$$

a contradiction. Hence, $x \geq 0$ λ-a.e.

Therefore, $x \in \tilde{X}_i$. This implies that $\tilde{X}_i$ is weakly*-closed and therefore weakly*-compact. □

Now, equip X_i with the product topology (recall X_i is the product of K copies of $\tilde{X}_i$). By the Tychonoff Theorem, X_i is compact if and only if every component of the product is compact. Since this is indeed the case, X_i is compact.

B. X_i is Hausdorff.

Observe that the weak*-topology over $\tilde{X}_i$ is metrizable, as B_∞ is metrizable.

Hence, $\tilde{X}_i$ is Hausdorff. Since the product of Hausdorff spaces is Hausdorff (Aliprantis and Border (2006)), X_i is a Hausdorff space.

C. G is quasi-symmetric.

Since $X_i = X_j \;\; \forall i \neq j$, and the bidders are endowed with the same utility function, the game is trivially quasi-symmetric.

D. $\overline{G}$ is better-reply secure.

Reny proves in his paper that the pay-your-bid auction is a better-reply secure game. We are going to follow the same lines. The only difference is that he only deals with strictly increasing strategies, which is not our case, unfortunately. In any case, we can fix it by allowing players to play strategies that, given the others' strategies, do not permit ties with strictly positive probability. It causes no loss of generality, as the following Lemma states.

Lemma 2.22. *Let $N = 2$. For $G \in \{G^{FPA}, G^{SPA}\}$, bidders can always use a pure strategy that induces ties with zero probability and lose an arbitrarily small utility.*

Proof. Let $(\beta_i^1(\theta_i), \beta_i^2(\theta_i), \dots, \beta_i^K(\theta_i))$ be i's bid in the game G when his type is θ_i. Without loss of generality, assume that it induces a tie on auction 1 with strictly positive probability. By assumption, this is true for a set of positive measure $A \subseteq [0,1]$. Therefore, $\theta_i \in A$. Define $\Pr(k \neq 1|\text{tie } 1)$ as the probability of winning any auction $k \in \{2, \dots, K\}$ given that i ties on auction 1 using the bid $\beta_i^1(\theta_i)$. Then, in the event of the tie, i's utility is

$$u_i(\theta_i, (\beta_i^1(\theta_i), \dots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) \; =$$

$$(\theta_i - 1/2\beta_i^1(\theta_i))\Pr(k \neq 1|\text{tie } 1) + 1/2(\theta_i - \beta_i^1(\theta_i))(1 - \Pr(k \neq 1|\text{tie } 1)) - \sum_{k=2}^{K} E[P_k|\beta_i^k(\theta_i), \beta_{-i}^k, \text{tie } 1].$$

Consider a small $\varepsilon > 0$ and bid $\beta_i^1(\theta_i) + \varepsilon$ on auction 1 such that it does not induce ties with positive probabilities. Clearly, such ε exists. Then, the strategy $(\beta_i^1(\theta_i) + \varepsilon, \beta_i^2(\theta_i), \dots, \beta_i^K(\theta_i))$ provides a conditional utility of

$$\begin{aligned}
&u_i(\theta_i, (\beta_i^1(\theta_i) + \varepsilon, \dots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) \; = \\
&(\theta_i - \beta_i^1(\theta_i) - c^{FPA}\varepsilon)\Pr(k \neq 1|\text{tie } 1) + \theta_i - \beta_i^1(\theta_i) - c^{FPA}\varepsilon)(1 - \Pr(k \neq 1|\text{tie } 1)) \\
&- \sum_{k=2}^{K} E[P_k|\beta_i^k(\theta_i), \beta_{-i}^k, \text{tie } 1],
\end{aligned}$$

where

$$c^{FPA} = \begin{cases} 1 & \text{if } G = G^{FPA} \\ 0 & \text{if } G = G^{SPA}. \end{cases}$$

Then, we have

$$u_i(\theta_i, (\beta_i^1(\theta_i) + \varepsilon, \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) - u_i(\theta_i, (\beta_i^1(\theta_i), \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) = \\ -(1/2\beta_i^1(\theta_i) + c^{FPA}\varepsilon)\Pr(k \neq 1|\text{tie } 1) + 1/2(\theta_i - \beta_i^1(\theta_i))(1 - \Pr(k \neq 1|\text{tie } 1)) \\ - c^{FPA}\varepsilon(1 - \Pr(k \neq 1|\text{tie } 1)).$$

If this difference is strictly positive, then i strictly prefers $\beta_i^1(\theta_i) + \varepsilon$ over $\beta_i^1(\theta_i)$. In fact, in the event of tie he can get a strictly higher utility, while in the event of no ties he can have an arbitrarily small loss, controlled through ε. As ε diminishes, the utility gain in the first event does not decreases over a certain constant amount, while the maximum loss in the second event converges to zero. Therefore, assume otherwise that $\forall \varepsilon > 0$, the difference is strictly negative. Therefore, we have

$$-(1/2\beta_i^1(\theta_i))\Pr(k \neq 1|\text{tie } 1) + 1/2(\theta_i - \beta_i^1(\theta_i))(1 - \Pr(k \neq 1|\text{tie } 1)) \leq 0.$$

For the final step, consider the bid $\beta_i^1(\theta_i) - \varepsilon$ on auction 1, for some small ε. Again, an ε that produces no ties exists. The conditional payoff is then

$$u_i(\theta_i, (\beta_i^1(\theta_i) - \varepsilon, \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1)) = \theta_i \Pr(k \neq 1|\text{tie } 1) - \sum_{k=2}^{K} E[P_k|\beta_i^k(\theta_i), \beta_{-i}^k, \text{tie } 1].$$

Hence, the difference is

$$u_i(\theta_i, (\beta_i^1(\theta_i) - \varepsilon, \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) - u_i(\theta_i, (\beta_i^1(\theta_i), \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) = \\ 1/2\beta_i^1(\theta_i)\Pr(k \neq 1|\text{tie } 1) - \frac{1}{2}(\theta_i - \beta_i^1(\theta_i))(1 - \Pr(k \neq 1|\text{tie } 1)) \geq 0.$$

Since as before on the event of no ties the loss can be made arbitrarily small, we obtain that i can avoid ties (through $\beta_i^1(\theta_i) + \varepsilon$ or $\beta_i^1(\theta_i) - \varepsilon$) and lose an arbitrarily small utility. As this can be applied to any auction $k \in \{1, \ldots, K\}$, where ties happen with positive probability, we get the desired result. □

Before moving into the proof, we need another technical detail. Recall that each $\tilde{X}_i$ equipped with the relative weak*-topology of the unit ball is metrizable. This implies that X_i with the product topology is metrizable (Theorem 3.36, Aliprantis and Border (2006)). Hence, since X_i is compact and metrizable, the set of probability measures over X_i is compact and metrizable (Theorem 15.11, Aliprantis and Border (2006)). This means that the topological properties of our new set of strategies, M_i, can be expressed in terms of sequences without loss of generality. Now, let us show that our game is better-reply secure. The proof follows the same lines as the example in Reny (1999). We include this proof for completeness.

Let $m^* \in M_i$ and suppose it is not an equilibrium and does not imply ties with strictly positive probability. Moreover, suppose that (m^*, u^*) is an element of the closure of the graph of the mixed extensions vector (ex-ante) payoff function. Now, consider a sequence m^n that converges to m^*. By definition, $\lim u(m^n) = u^*$. Since m^*

is not an equilibrium, $\exists i \in \{1,\dots,N\}$ that has a profitable deviation. Observe that, by definition of the supremum, $\forall m_{-i} \in M_{-i}$ $\forall \varepsilon > 0$ i can use a pure strategy x_i^ε that provides a payoff within ε of her supremum and, by the previous argument, does not imply any tie (Lemma 2.22). If there are no ties, u_i is continuous at $(x_i^\varepsilon, m_{-i}^*)$. Now, since x_i^ε is a profitable deviation, $u_i(x_i^\varepsilon, m_{-i}^*) > u_i(m_i^*, m_{-i}^*) = u_i(m^*)$. By continuity, there exists a neighborhood of m_{-i}^* such that $u_i(x_i^\varepsilon, m'_{-i}) > u_i(m_i^*, m_{-i}^*)$ for each m'_{-i} in the neighborhood. Since there are no ties by assumption, u_i is also continuous at m^*, which implies $u(m^*) = u(\lim m^n) = \lim u(m^n) = u^*$. Hence, $u_i(x_i^\varepsilon, m'_{-i}) > u_i^*$ in the neighborhood, that is, i can secure a payoff strictly above u_i^* at m^*.

Now suppose that ties happen with strictly positive probability at m^*. Then, the function is not continuous at m^* and then $u^* = \lim u(m^n) \neq u(\lim m^n) = u(m^*)$. Moreover, one of the bidders loses with a probability strictly higher than zero for an infinite amount of times along the sequence m^n. This bidder can strictly increase her payoff with x_i^ε for sufficiently small ε as it does not produce ties. Hence, $u_i(x_i^\varepsilon, m_{-i}^n)$ is bounded away from $u_i(m^n)$ for large n. Hence, $u_i(x_i^\varepsilon) > u_i^*$ in the limit and by continuity of u_i at m_{-i}^*, there exists a neighborhood of m_{-i}^* where $u_i(x_i^\varepsilon, m_{-i}^*) > u_i^*$. Then, again, i can secure a payoff strictly higher than u_i^* at m^*. Therefore, $\overline{G}$ is better-reply secure.

G with more than two players We discuss here why we cannot obtain the same existence result when $N > 2$. It pins down to the proof of Lemma 2.22. In order to show why the Lemma does not apply with more than two players, consider an N-player game G^{SPA} and suppose that all of the i's opponent play the same strategy, so that the event "tie" does not convey different information based on the identity of the players who tied. Call the probability of tying with $\ell \in \{1,\dots,N-1\}$ players $Tie(\ell)$. As before, we assume without loss of generality that on auction 1 player i can tie with positive probability using $(\beta_i^1(\theta_i),\dots,\beta_i^K(\theta_i))$. The conditional utility is

$$
\begin{aligned}
&u_i(\theta_i, (\beta_i^1(\theta_i),\dots,\beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) = \\
&\sum_{\ell=2}^{N}[(\theta_i - {}^1\!/\!_\ell\,\beta_i^1(\theta_i))\Pr(k \neq 1|\text{tie } 1,\ell) + {}^1\!/\!_\ell(\theta_i - \beta_i^1(\theta_i))(1-\Pr(k \neq 1|\text{tie } 1,\ell))]Tie(\ell) \\
&-\sum_{k=2}^{K} E[P_k|\beta_i^k(\theta_i), \beta_{-i}^k, \text{tie } 1].
\end{aligned}
$$

Now, consider the bid $\beta_i^1(\theta_i) + \varepsilon$, where ε is chosen to not induce ties, as before. We have

$$u_i(\theta_i, (\beta_i^1(\theta_i) + \varepsilon, \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) = \sum_{\ell=2}^{N} [(\theta_i - \beta_i^1(\theta_i))\Pr(k \neq 1|\text{tie } 1, \ell) + (\theta_i - \beta_i^1(\theta_i))(1 - \Pr(k \neq 1|\text{tie } 1, \ell))]Tie(\ell) - \sum_{k=2}^{K} E[P_k|\beta_i^k(\theta_i), \beta_{-i}^k, \text{tie } 1].$$

Therefore, the difference is

$$u_i(\theta_i, (\beta_i^1(\theta_i) + \varepsilon, \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) - u_i(\theta_i, (\beta_i^1(\theta_i), \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) = \sum_{\ell=2}^{N} [\frac{1-\ell}{\ell}\beta_i^1(\theta_i)\Pr(k \neq 1|\text{tie } 1, \ell) + \frac{\ell-1}{\ell}(\theta_i - \beta_i^1(\theta_i))(1 - \Pr(k \neq 1|\text{tie } 1, \ell))]Tie(\ell) = \sum_{\ell=2}^{N} \frac{1-\ell}{\ell}[\beta_i^1(\theta_i) - \theta_i(1 - \Pr(k \neq 1|\text{tie } 1, \ell))]Tie(\ell).$$

As in the previous Lemma, now we consider the bid $\beta_i^1(\theta_i) - \varepsilon$. After some computations, we get

$$u_i(\theta_i, (\beta_i^1(\theta_i) - \varepsilon, \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) - u_i(\theta_i, (\beta_i^1(\theta_i), \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) = \sum_{\ell=2}^{N} \frac{1}{\ell}[\beta_i^1(\theta_i) - \theta_i(1 - \Pr(k \neq 1|\text{tie } 1, \ell))]Tie(\ell).$$

Let $\beta_i^1(\theta_i) - \theta_i(1 - \Pr(k \neq 1|\text{tie } 1, \ell)) = a_\ell$. We have

$$u_i(\theta_i, (\beta_i^1(\theta_i) + \varepsilon, \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) - u_i(\theta_i, (\beta_i^1(\theta_i), \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) = \sum_{\ell=2} \frac{1-\ell}{\ell} a_\ell Tie(\ell),$$

and

$$u_i(\theta_i, (\beta_i^1(\theta_i) - \varepsilon, \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) - u_i(\theta_i, (\beta_i^1(\theta_i), \ldots, \beta_i^K(\theta_i)), \beta_{-i}|\text{tie } 1) = \sum_{\ell=2} \frac{1}{\ell} a_\ell Tie(\ell).$$

Therefore, we can have, in principle, a set of strategies that allows for

$$\sum_{\ell=2} \frac{1-\ell}{\ell} a_\ell Tie(\ell) < 0 \quad \text{and} \quad \sum_{\ell=2} \frac{1}{\ell} a_\ell Tie(\ell) < 0,$$

that is, a bid that induces ties with positive probability is strictly better than any bid that reduces this probability to zero. This fact blocks Lemma 2.22 and so the game cannot have, in principle, enough continuity so that better-reply security goes through.

Appendix 2.C Symmetric mixed strategy equilibrium

Lemma 2.23. *Suppose one player is playing $\sigma(0) = [1(0,0)]$ and $\sigma(1)$ is atomless. Then, the set of best-replies of the other player is an anti-lattice*[6] *when her type is $\theta = 1$.*

Proof. Suppose the other player is playing $\sigma(0) = [1(0,0)]$ and that $\sigma(1)$ is an atomless distribution represented by the cdf F. Take $(x_1,y_1) > (x_2,y_2)$ best responses against σ and consider type $\theta = 1$. Suppose this type plays the strategy $\hat{\sigma} = [0.5(x_1,y_2), 0.5(x_2,y_1)]$. The expected gain for this type is then

$$0.5\left[\frac{1}{2}(F_x(x_1) + F_y(y_2) - F(x_1,y_2)) + \frac{1}{2}\right] + 0.5\left[\frac{1}{2}(F_x(x_2) + F_y(y_1) - F(x_2,y_1)) + \frac{1}{2}\right].$$

The expected gain of playing $\tilde{\sigma} = [0.5(x_1,y_1), 0.5(x_2,y_2)]$ is instead

$$0.5\left[\frac{1}{2}(F_x(x_1) + F_y(y_1) - F(x_1,y_1)) + \frac{1}{2}\right] + 0.5\left[\frac{1}{2}(F_x(x_2) + F_y(y_2) - F(x_2,y_2)) + \frac{1}{2}\right].$$

Observe the expected prices of $\hat{\sigma}$ and $\tilde{\sigma}$ are exactly the same and therefore they do not count in the difference. Thus, the difference in terms of expected payoffs in terms of the two strategies is

$$\frac{1}{4}[F(x_1,y_1) + F(x_2,y_2) - F(x_1,y_2) - F(x_2,y_1)].$$

Since F is a cdf, this difference is positive and by the assumption of optimality of (x_1,y_1) and (x_2,y_2), we must have that it is exactly zero, i.e., F puts mass of 0 on the square. Then, we have that (x_1,y_2) and (x_2,y_1) are also best replies, and therefore the best-response set is an anti-lattice. □

The proof, as noted in Szentes (2007), suggests that the support cannot be any two-dimensional set, the union of any number of increasing lines or the union of more than two decreasing lines. In his paper, it is also noted that only one decreasing line cannot constitute the entire support. While we can have a support with one decreasing line here, we analyzed the two-line case.

Proof of Lemma 2.8
Symmetry with respect of the object is equivalent to proving that $g_i(g_i(x)) = x$, $i = 1,2$. Then,

$$G_i(g_i(g_i(x))) = \frac{1}{2} - G_i(g_i(x)) = \frac{1}{2} - \left(\frac{1}{2} - G_i(x)\right) = G_i(x).$$

6. A set $X \subseteq \mathbb{R}^2$ is an anti-lattice if whenever $a \wedge b, a \vee b \in X$ then $a, b \in X$.

Since both G_1 and G_2 are strictly increasing, it must be that $g_i(g_i(x)) = x$.
In order to prove that the two curves are strictly decreasing, consider $x > y$. We have

$$G_i(g_i(x)) \;=\; \frac{1}{2} - G_i(x) \;<\; \frac{1}{2} - G_i(y) \;=\; G_i(g_i(y)).$$

Since G_i is strictly increasing, $g_1(x) < g_2(y)$. Finally, we prove $g_1 > g_2$. First, observe that

$$G_2(x) \;>\; G_1(x) \iff G_2(x) \;>\; \frac{x}{1-2x}.$$

Suppose so. Then,

$$G_1(g_1(x)) \;=\; \frac{1}{2} - G_1(x) \;>\; \frac{1}{2} - G_2(x) \;=\; G_2(g_2(x)),$$

and since $G_2 > G_1$, it must be that $g_1(x) > g_2(x)$. □

Appendix 2.D Equilibrium Transformation

Szentes (2005) provides a method to transform FPA equilibria into SPA equilibria in frameworks like the current one. The intuition is simple: we assume that both equilibria (FPA and SPA) have equal expected costs; from this equivalence, we derive the optimal bidding function for the SPA format. We change the strategy description to make the transformation easier. Therefore, a strategy is now a vector (F, p) where F is a probability measure and p is a function $p : Supp(F) \longrightarrow [0, 1]^2$. F will describe the randomization process; p transforms the outcomes of the randomization into bids. Observe that this kind of description is without loss of generality. For example, in the previous equilibrium, $F = \sigma$ and $p = id$ (the identity function). In the SPA case, we fix F and find the function q such that (F, q) is an equilibrium strategy of SPA. Of course, when $\theta = 0$, the action $(0, 0)$ is still optimal, hence we focus on $\theta = 1$. Once we have found q, we need to check whether it is strictly increasing. If so, (F, q) has the same payoffs of (F, id) in the FPA case, so it is an equilibrium. If a profitable deviation exists in (F, q), then we would be able to recover a profitable deviation in (F, id), which would lead to a contradiction.

Since the equation to compute q is slightly different than in Szentes (2005), we write its derivation explicitly. So, let σ_i be the marginal distribution of the distribution of σ over the bids on auction i, $i = 1, 2$. Then, bidding x on auction i has an expected payoff of $\frac{1}{2}x + \frac{1}{2}\sigma_i(x)x$. If we use (σ, q) on the SPA format, we would get $\frac{1}{2}0 + \frac{1}{2}\int_0^x q_i(y)d\sigma_i(y)$. Therefore, the cost equivalence condition requires

$$\int_0^x q_i(y)d\sigma_i(y) \;=\; x(1 + \sigma_i(x)).$$

We assume that σ_i has a density f_i, and so we have

$$\begin{aligned}
\int_0^x q_i(y)d\sigma_i(y) &= x(1+\sigma_i(x)) \\
\Rightarrow \int_0^x q_i(y)f_i(y)dy - x &= x\sigma_i(x) \\
\Rightarrow \int_0^x (q_i(y)f_i(y)-1)dy &= x\sigma_i(x) \\
\Rightarrow \int_0^x \left(q_i(y)-\frac{1}{f_i(y)}\right)f_i(y)dy &= x\sigma_i(x).
\end{aligned}$$

Denote $z_i(y) = q_i(y) - \frac{1}{f_i(y)}$, so that

$$\begin{aligned}
\int_0^x z_i(y)f_i(y)dy &= x\sigma_i(x) \\
\Rightarrow \int_0^x z_i(y)d\sigma_i(y) &= x\sigma_i(x) \\
\Rightarrow z_i(x) &= \frac{d(x\sigma_i(x))/dx}{d\sigma_i(x)/dx}.
\end{aligned}$$

Observe that in our case $\sigma_i(x) = G_1(x) + G_2(x)$ for $i = 1, 2$. Let $G_2(x) = 2x$ (which satisfies all the assumptions imposed on $G_2(x)$). Then, we have

$$z_i(x) = 2x - x^2 \Rightarrow q_i(x) = \frac{4x-2x^2+1}{3},$$

and since q_i is strictly increasing between $[0,1]$, we get the equivalence of payoffs. Therefore $\sigma(0) = [1(0,0)]$ together with $(\sigma(1), q)$, where $q = (q_1, q_2)$, constitutes a symmetric mixed strategy equilibrium for the SPA game.

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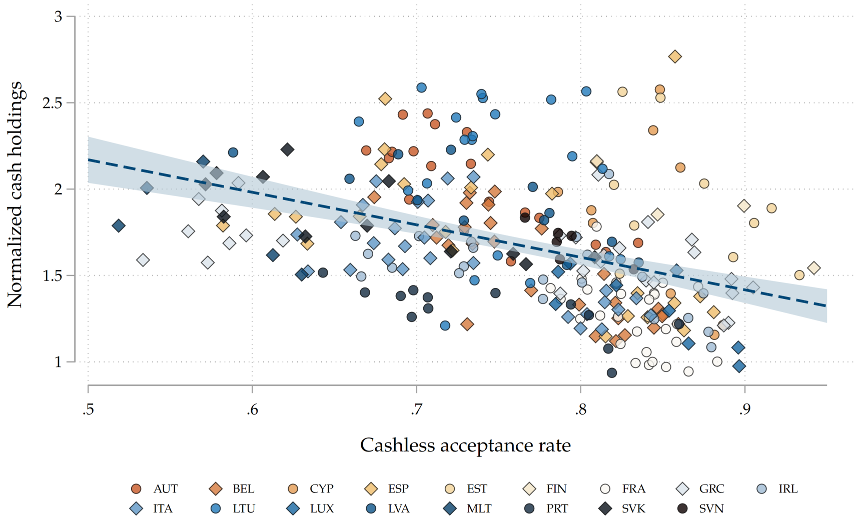
Chapter 3

Payment Choices and Cash Demand in an Equilibrium Model of Card Acceptance

Joint with Elia Moracci

3.1 Introduction

The advent of electronic payment systems has revolutionized the way transactions are conducted, prompting shifts in consumer behavior and merchant practices. While card payments offer advantages appreciated by some consumers (convenience, speed, and enhanced security), other people might be reluctant to use them because of privacy concerns. Moreover, imperfect acceptance of cards by merchants remains a critical obstacle to widespread usage. For these and other reasons, cash is still a prevalent medium of exchange in many economies, albeit with varying degrees across different areas. As 3.2 shows, cash usage is still widespread around Europe, as in all regions in the sample people hold on average largely positive amounts of cash. Moreover, the Figure illustrates that the amount of cash people carry with them is higher when card acceptance rates are low. This is consistent with studies relating lower card acceptance rates to increased cash usage (Arango, Huynh, and Sabetti (2015), Bagnall et al. (2016)): people need to carry enough cash if they frequently encounter merchants who only accept physical money in exchange for goods and services.. Huynh, Schmidt-Dengler, and Stix (2014) have highlighted the reciprocal nature of this relationship: if people carry less cash with them, merchants find it optimal to accept card payments. Therefore, the relationship plotted in 3.2 cannot be interpreted causally, as the scattered points represent different outcomes of the interaction between sellers and buyers. In this paper, we aim to rationalize how these different equilibria emerge by constructing an equilibrium model that accounts for the feedback effects between card acceptance rates and cash holdings.



The figure shows average cash holdings reported by households (as a fraction of daily expenditure) and cashless acceptance rates, grouped by region at the NUTS-2 level and for different years. Each dot is a region×year combination. Average cash holdings are measured as the mean level of cash balances during the day, which is derived from payment diaries; cashless acceptance rates are computed using the share of shops visited by each respondent in which it was possible to carry out the transaction using cards.
Source: Own calculations based on the Survey on the Use of Cash by Households (2016) and on the Study on the Payment Attitudes of Consumers in the Euro Area (2019, 2023).

Figure 3.1. Card acceptance rates and cash holdings across European regions

Our research is particularly relevant in the present environment, characterized by the co-existence of cash and cheap, secure cashless alternatives: understanding the factors that drive merchants' decisions to accept card payments and the impact of these choices on buyers' cash management and payment behavior is essential for appropriately targeting policies aimed at fostering card usage, which has been found to deter tax evasion (Immordino and Russo (2017), Giammatteo, Iezzi, and Zizza (2022)) and criminal activities (Rogoff (2017)).

In this chapter, we study the acceptance and usage of cashless payment methods and interaction with the transactions demand for cash in a model of the payments market that features strategic interaction among merchants and equilibrium linkages between buyers and sellers. We model an economy where buyers and sellers are randomly matched to exchange goods and services of different values. Buyers have the option to use cash or cards to settle purchases, and they decide how much cash to hold. Cash and cards differ in several aspects: cash is costly to store (in terms of time spent handling it and due to the risk of theft), and can be used only if the size of the transaction is small enough; cards, instead, can be used to settle purchases of any value, but they might require the payment of a fixed cost, and they are not universally accepted. Sellers decide whether to accept cash payments only or to allow their customers to use cards: the key trade-off is that each transaction generates less profits for sellers when settled using cards, but the total *number* of transactions increases when accepting. Indeed, we allow buyers to search for shops that accept cards, inducing competition among sellers in their payment method acceptance choices; however, we assume that with some probability agents are *captive* (forced to purchase in a certain shop), introducing a source of imperfect competition/market power in the model. We also allow for heterogeneity on the buyers' side, by assuming that a fraction of them prefer to pay using cards, while the others use cash whenever they have the chance. Since buyers' decisions on how much cash to hold depend on the overall card acceptance rate and, at the same time, sellers' acceptance policies are optimally targeted in response to their customers' money balances, the model features equilibrium interactions between the two sides of the market.

We start by highlighting the main tradeoffs faced by sellers in a simple model with two merchants, showing that the degree of strategic complementarity/substitutability in acceptance choices (which is related to the extent of search frictions) is a key driver of equilibrium outcomes. For equilibria with imperfect acceptance (some sellers accept cards, others don't) to arise, search frictions should be small enough, to foster competition among sellers. In fact, when search frictions are too high, sellers do not face any competition, and since they are symmetric they adopt the same strategy (either accepting or not card payments). We then move to a model with N sellers, where we micro-found the cash holding decision of buyers and we include buyer heterogeneity. The presence of agents who prefer cards (which seems empirically relevant from survey data on payments) generates a mechanism that discourages sellers' acceptance. When a store starts to accept cards, indeed, two things happen: on one hand, it benefits from the fact that its customers can now purchase goods that have a price higher than their cash balances; however, this gain is counterbalanced by the fact that their customers that prefer cards will stop using cash even for *small* purchases. We show equilibrium existence for our model economy, and we derive a uniqueness result for the game with a large enough number of sellers N exploiting its nonatomic limit with $N \to +\infty$. Then, we perform comparative statics exercises to evaluate how equilibria are shifted by parameter changes, showing that (i) higher competition between shops leads to higher equilibrium card acceptance and lower cash balances, (ii) a larger share of agents who prefer to pay using cards reduces equilibrium card acceptance (perhaps surprisingly), and (iii) a rise in the opportunity cost of holding cash shifts the economy towards higher acceptance.

Finally, we present an extended version of the model that we can bring to the data, featuring a dynamic cash management problem for buyers and optimal search behavior, which we can bring to the data. We discipline our model using data drawn from ECB payment diaries and surveys of companies, that provide information on the size and frequency of purchases, average cash holdings, the number of cash withdrawals per year, the card share of expenditure, as well as card acceptance rates at points of sale. We then use our calibrated model to perform a simple exercise, studying the effects of a subsidy to card usage resembling the *cashback policy* rolled out by the Italian government in fall 2020, and comparing the response that one would obtain with a partial equilibrium model that abstract from sellers' choices with that given by our model with strategic interaction among sellers. Our results show that neglecting the impact that policy changes have on sellers' incentives to accept or reject card payments may lead to choosing policies that ultimately have unintended effects: when sellers adjust their acceptance choices in response to the card subsidy, the overall acceptance rate falls, and the increase in card usage in shops that keep accepting cards is more than compensated by a fall in the overall use of electronic payments, as a result of tighter supply-side constraints.

3.2 Related literature

In this paper, we contribute to two strands of the literature. First, we contribute to the literature that jointly studies the adoption, usage, and acceptance of payment methods through equilibrium models of the payments market: a recent example is the work by Huynh, Nicholls, and Shcherbakov (2022), who use detailed data on buyers and merchants to estimate a structural model of the Canadian payments market. While their model also features an equilibrium relationship between merchants and buyers, we also allow for strategic interactions between different sellers, that take into account economy-wide acceptance probabilities before deciding whether to accept or not. Second, we contribute to the literature that theoretically studies the transactions demand for cash (Alvarez and Lippi (2009), Alvarez and Lippi (2013)): in particular, our work is particularly related to studies of cash management and payment choices by households when cashless alternatives are available (Alvarez and Lippi (2017)). Recent empirical papers by Huynh, Schmidt-Dengler, and Stix (2014), Arango, Huynh, and Sabetti (2015), Bagnall et al. (2016) showed that imperfect acceptance of cashless payments creates a precautionary motive to hold cash. Our work builds on this literature and allows for imperfect cashless acceptance in a model of cash demand, by also featuring feedback effects between these two aggregates. Not only imperfect acceptance drives up cash usage, but the opposite channel is also relevant: if individuals always have enough cash to carry out transactions, merchants have little incentive to accept cards.

We are not the first to jointly model the interaction between merchants and buyers in a payments market while simultaneously studying cash holding choices: in a closely related paper, Masters and Rodríguez-Reyes (2005) employ a setup similar to ours to rationalize observed differences in the intensity of card usage across countries. We extend their framework to allow for competition among merchants in their payment method acceptance decisions, by allowing for buyer search à la

Burdett and Judd (1983): as buyers might visit multiple shops, merchants have an incentive to accept their preferred payment methods in order not to lose sales. This helps us to rationalize equilibria with imperfect acceptance without resorting to heterogeneity in the population of sellers as done by Masters and Rodríguez-Reyes (2005).

Finally, our paper also speaks to the IO literature on payment systems that studies the determinants of acceptance (Rochet and Tirole (2002), Rochet and Tirole (2014), Li, McAndrews, and Wang (2019)), by explicitly modeling the choice problem faced by buyers that need to decide how much cash to hold.
The chapter is organized as follows. In 3.3 we introduce a simple acceptance game between two sellers, to illustrate the fundamental strategic trade-offs involved in card acceptance decisions. In **??** we introduce our theoretical model with buyers and sellers: after presenting the decision problems faced by both types of agents, we derive the properties of optimal choices and we define our equilibrium concept; then, we provide results on the existence of equilibrium and on the uniqueness of imperfect acceptance equilibria, and we present some comparative statics. In 3.5 we present a quantitative extensios of our framework which enable us to take the model to the data, calibrating its parameters. In 3.6 we perform a counterfactual exercise, simulating the introduction of a subsidy to card usage, and we compare its partial and general equilibrium effects. 3.7 concludes.

3.3 A simple acceptance game

In this Section, we present a simple card acceptance game between two sellers ($N = 2$), ignoring almost entirely from the buyers' cash-holding problem, to explain the sellers' main trade-offs they face when choosing whether to accept cards or not.

There are two identical sellers and two buyers endowed with a debit card. Each seller decides on whether to accept or reject cards as a payment method in her shop. Cash is always accepted. The buyers observe the sellers' choices and decide the amount of cash they want to hold. Then, buyers need to purchase a good whose price is drawn from a distribution with cdf F and nonnegative support. Let u denote the profit merchants obtain when selling a good[1]. If the purchase is carried out by card, the profit shrinks to $u - t$ as the merchant needs to pay fees/taxes. Buyers observe the acceptance choices of both shops and then decide how much cash m to hold. If both shops accept cards, they hold $m = 0$; if one shop accepts cards and the other doesn't, they hold $m' > 0$; finally, if nobody accepts cards, they hold a level $m'' > m'$ of cash balances[2] such that $F(m'') = 1$, i.e., such that they can finance any

1. For simplicity purposes, we assume that u is independent of the value of the good/service sold.

2. At this stage, we don't explicitly model money demand by buyers as we want to focus on the fundamental determinants of interaction between sellers. We will include optimal cash demand in

purchase. With probability α each buyer only visits a random seller among the two (we sometimes refer to the buyer as *captive* in such situations), while with probability $1-\alpha$ they can choose in which shop to carry out the purchase, after observing the size of the transaction. When the buyers face a transaction with price smaller than the level of cash balances (with probability $F(m)$), they will use cash[3]; therefore, if given a choice among the two shops, they will pick a random one. When the transaction price is larger than the level of cash balances (with probability $1-F(m)$) they need to use their payment card; therefore, they visit a shop that accepts the card when having the option to do so.

Therefore, in this model the buyers' actions are exogenously given (they are not in the next sessions) and we investigate the sellers' game only. In 3.1 we display the payoffs of each seller $i \in \{1,2\}$ given his action $\Phi_i \in \{c, cd\}$ and the action of the other seller $\Phi_{-i} \in \{c, cd\}$, where cd denotes acceptance (it is possible to use *c*ash and *d*ebit cards) and c denotes rejection (it is possible to use *c*ash only). From now on, we denote i's profits when strategies played are (Φ_i, Φ_{-i}) by $\Pi^{\Phi_i}(\Phi_{-i})$. Let's look at all four possible pairs of pure strategies from the perspective of i. If both i and $-i$ accept the card, they will attract one buyer each in expectation and, in case they make a sale, pay the fee t for sure (as buyers don't carry cash with them and $F(0)=0$), getting an expected profit of $\Pi^{cd}(cd)=u-t$. If seller i decides to reject card transactions while his opponent is accepting, she will get an expected payoff $\Pi^{c}(cd)=F(m')u$, as she will only be visited by one buyer in expectation, which will complete the transaction only if it's small. If the seller decides to be the unique shop that accepts, she will get an expected payoff $\Pi^{cd}(c)=F(m')u+\big(1-F(m')\big)(u-t)(2-\alpha)$: if the purchase is small, she will share it with his non-accepting competitor and get one client in expectation; if it is large, she will capture one extra client with probability $1-\alpha$, totaling $2-\alpha$ clients. Finally, when nobody accepts the card, the expected payoff is $\Pi^{c}(c)=u$.

Two things are worth noticing about the above payoffs. First, we have that $\Pi^{c}(c) > \Pi^{c}(cd)$: since money demand is higher when nobody accepts (recall the buyers observe the sellers' choices), each seller will benefit from the non-acceptance of her competitor, as she will be able to secure more transactions. Second, notice that $\Pi^{cd}(c) > \Pi^{c}(cd)$: when only one shop is accepting cards, it is better to be the owner of the shop who accepts rather than the owner of the shop that does not.

Properties of the game. We assume that there are no dominant strategies, which would make the game uninteresting as there would be no relevant strategic interaction. Hence, either (i) $\Pi_{cd}(cd) \geq \Pi_c(cd) \wedge \Pi_c(c) \geq \Pi_{cd}(c)$ or (ii) $\Pi_{cd}(cd) \leq \Pi_c(cd) \wedge$

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the next Sections. The fact that cash demand is decreasing in the level of acceptance is easily derived from the optimizing behavior of buyers.

3. The fact that cash is always used when the agent has enough can be motivated through the existence of a preference for cash. We explicitly consider preferences and we model payment method choices in the next Sections.

Table 3.1. Payoffs of seller i in the acceptance game

Φ_i \ Φ_{-i}	cd	c
cd	$u-t$	$F(m')u+\big(1-F(m')\big)(u-t)(2-\alpha)$
c	$F(m')u$	u

$\Pi_c(c) \le \Pi_{cd}(c)$ holds. If (i) holds, it is optimal to accept the card when the opponent does so and to reject the card in the opposite case. In this case, the game features strategic complementarity in acceptance decisions [4], as

$$\Pi_{cd}(cd) \;\ge\; \Pi_c(cd) \wedge \Pi_c(c) \;\ge\; \Pi_{cd}(c) \quad\Longrightarrow\quad \Pi_{cd}(cd) - \Pi_c(cd) \;\ge\; 0 \;\ge\; \Pi_{cd}(c) - \Pi_c(c).$$

If (ii) holds, it is optimal to accept the card when the opponent does not accept and to reject the card in case he does so. In this case, the game features strategic substitutability in acceptance decisions. We now study the conditions on parameters determining which class the acceptance game belongs to. Acceptance decisions are strategic substitutes if

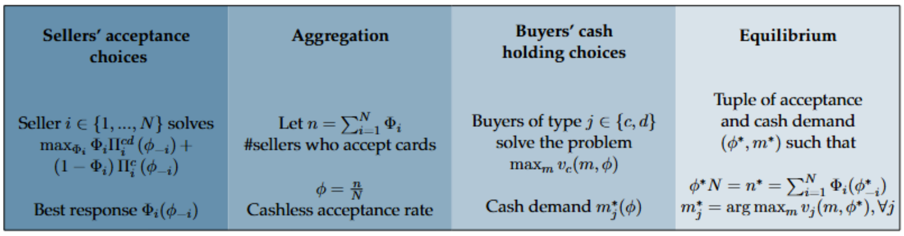
$$\alpha \;\le\; \overline{\alpha}(t) \;=\; 1 - \frac{t}{u-t}. \tag{3.1}$$

$$t \;\ge\; (1-F(m'))u, \tag{3.2}$$

while they are strategic complements if $\alpha \ge \overline{\alpha}(t) = 1 - \frac{t}{u-t}$ and $t \le (1-F(m'))u$. It is worth exploring the above conditions further. First, notice that the condition for strategic substitutes/complements involves α, the extent of search frictions. Strategic substitutes require search frictions to be small enough, to foster competition among buyers in their acceptance decisions. When search frictions are too large, there is no incentive to accept when other shops do not since buyers are captive, and cannot be captured through card acceptance. Notice that $\overline{\alpha}'(t) < 0$, i.e., since the margin on large transactions is decreasing in t, a higher value of t requires even smaller levels of α to induce strategic substitutability.

Equilibria of the game. We now characterize the equilibria of the acceptance game. As a solution concept, we use pure-strategy Nash equilibrium. Let ϕ^* denote the card acceptance rate in the economy.

4. Notice that in our context acceptance decisions are *strategic complements* if $\Pi_{cd}(cd) - \Pi_c(cd) \ge \Pi_{cd}(c) - \Pi_c(c)$, while they are *strategic substitutes* if $\Pi_{cd}(cd) - \Pi_c(cd) \le \Pi_{cd}(c) - \Pi_c(c)$.



As described in the main text, let $\phi_{-i} = \sum_{j \neq i}^N \Phi_j / N$. Sellers optimally respond to their competitors' acceptance choices. Buyers observe the cashless acceptance rate and hold an amount m^* accordingly. In equilibrium, i) each seller has no incentive to deviate given the actions of the other $N - 1$ sellers and given money demand m^* by buyers, and ii) money demand is optimal given the aggregate acceptance rate ϕ^*. Let $j \in \{c, cd\}$ denote buyers' types, defined according to their payment preferences.

Figure 3.2. Timing and structure of the model

Proposition 3.1. *If acceptance decisions are strategic complements, the game features only two Nash equilibria, $\phi^* = 1$ (full acceptance) and $\phi^* = 0$ (no acceptance). If acceptance decisions are strategic substitutes, the game features two imperfect acceptance equilibria with $\phi^* = 1/2$.*

Proof. Trivial. □

This simple example illustrates that imperfect acceptance equilibria can arise only in the presence of strategic substitutability, a result that we will obtain again under a more general setting in the next Section.

3.4 Theoretical model

In this Section, we outline, solve, and describe the properties of a model that builds upon the simple example presented above in two respects: first, we allow for many identical sellers; second, we explicitly analyze the cash-holding problem of heterogeneous buyers.

3.4.1 The model

There are N identical sellers and a continuum of buyers with measure N. Sellers make simultaneous acceptance decisions. Let n be the number of sellers who decide to accept and let $\phi = n/N$ denote the cashless acceptance rate. Buyers need to purchase a good or service from sellers. Both sellers and buyers derive utility u from completing a single transaction. The price of the purchase is drawn from the distribution F, whose properties we already described. We also assume that F is twice differentiable, with density denoted by f. With probability α, buyers can shop in a unique store, randomly selected. With probability $1 - \alpha$, they can freely choose which one of the N stores to visit. A fraction $1 - \omega$ of buyers prefer using cash when paying, whereas the remaining ω exhibit a preference for card payments. Completing a purchase using the least preferred payment method entails a utility cost κ for both types. Buyers observe the acceptance decisions of sellers and decide how much cash to hold. Carrying m units of cash entails an opportunity cost Rm. Both types of agents fail to get utility u when they don't complete the transaction, which happens when cash balances are smaller than the value of the transaction and it is impossible to use cards. We make some simplifying assumptions on the parameters of the model, which simplify our analysis.

Assumption 4. *We make the following four assumptions on parameters.*

$$0 < R < \kappa < u, \tag{A1.a}$$

$$0 < t < u, \tag{A1.b}$$

$$f'(s) < 0, \textit{for all } s \in [0, +\infty), \tag{A2}$$

$$f(0) > \max\left\{\frac{R}{\kappa}, \frac{R}{u-\kappa}\right\}. \tag{A3}$$

$$0 \leq \alpha < 1 \tag{A4}$$

Assumption (A1.a) clarifies that R is small relative to κ (one is a variable cost proportional to m, the other one is fixed), and that κ is smaller than u (even though people dislike to pay with cards, they still prefer it to lose the purchase). Assumption (A1.b) makes sure that fees paid by merchants when receiving a card payment are small enough that the net benefit of receiving a card payment is still positive. Assumption (A2) specifies that small purchase sizes are more likely than large ones, which has been shown empirically (Boeschoten (1992)). Assumption (A3) says that small payments should be sufficiently likely. From a practical point of view, (A2) and (A3) can be seen as regularity conditions on the buyer's cash holding problem which ensure interior solutions. In particular, $f(0) > R/\kappa$ makes sure that agents who prefer cash always bring some with them, even under full acceptance of cards, while $f(0) > R/(u-\kappa)$ makes sure that agents who prefer cards bring some cash with them when no one accepts cards. Finally, (A4) makes sure that there exists some competition among sellers.

3.4.1.1 Buyer's problem

We start by analyzing the cash-holding decision of buyers. Let n be the number of sellers that accept cards. Each buyer observes the share of sellers $\phi = n/N$ accepting cards and solves a cost minimization problem. Buyers who prefer to use cash for transactions face a different problem than those who enjoy more paying by card. Buyers who favor using cards utilize them whenever they can do so, whereas those who prefer cash resort to using cards only when their cash balances are lower than the transaction amount. First, we describe the cash-holding problem for the $1-\omega$

buyers who prefer paying with cash. When a fraction $\phi > 0$ of sellers are accepting cards, these buyers solve the problem

$$\min_m v_c(m,\phi) = \min_m \quad Rm + \alpha\,(1-F(m))\big(\phi\kappa + (1-\phi)u\big) + (1-\alpha)\,(1-F(m))\,\kappa, \tag{3.3}$$

When no sellers are accepting cards, i.e., when $\phi = 0$, they instead solve the problem

$$\min_m v_c(m,0) = \min_m \quad Rm + (1-F(m))\,u. \tag{3.4}$$

Let's now consider the ω buyers that prefer using cashless methods to settle transactions. When a fraction $\phi > 0$ of sellers are accepting cards, these buyers solve the problem

$$\min_m v_d(m,\phi) = \min_m \quad Rm + \alpha(1-\phi)\,(F(m)\kappa + (1-F(m))\,u)\,, \tag{3.5}$$

When no sellers are accepting cards, i.e., when $\phi = 0$, they instead solve the problem

$$\min_m v_d(m,0) = \min_m \quad Rm + F(m)\kappa + (1-F(m))\,u. \tag{3.6}$$

The following Proposition characterizes buyers' optimal choices.

Proposition 3.2. *Let ϕ be the share of merchants accepting cards. Optimal cash holdings of agents who prefer cash are given by*

$$m_c^*(\phi) = \arg\min_m v_c(m,\phi) = \begin{cases} f^{-1}\left(\frac{R}{u-(u-\kappa)\left(\alpha\phi+(1-\alpha)\right)}\right) & \phi > 0 \\ f^{-1}\left(R/u\right) & \phi = 0 \end{cases} \tag{3.7}$$

Optimal cash holdings of agents who prefer cards are given by

$$m_d^*(\phi) = \arg\min_m v_d(m,\phi) = \begin{cases} f^{-1}\left(\frac{R}{\alpha(1-\phi)(u-\kappa)}\right) & \phi > 0, \\ f^{-1}(R/(u-\kappa)) & \phi = 0. \end{cases} \tag{3.8}$$

Proof. See 3.A.1. □

Some observations are in order. First, notice that, as f^{-1} is a decreasing function, for any cashless acceptance rate ϕ, type-c buyers (who prefer to use cash for payments) hold higher cash balances than their type-d counterparts. Second, both types increase the amount of cash held as ϕ decreases, as the precautionary value of holding cash gets higher. Moreover, money demand functions are discontinuous at $\phi = 0$ as under full rejection of cards they will find no store that allows them to pay cashless even when they are free to shop wherever they prefer. Third, when

$\phi = 1$, $m_c^*(\phi) = f^{-1}(R/\kappa) > 0$, therefore agents who prefer to pay in cash always bring some cash with them. This does not apply to agents who prefer to pay using cards, which could decide to hold zero

In what follows, we often assume that the distribution of transaction sizes follows an exponential with parameter λ, i.e., that $f(s) = \lambda e^{-\lambda s}$. Notice that under this specification, the inverse pdf f^{-1} is given by $f^{-1}(s) = -\frac{1}{\lambda}\ln\left(\frac{s}{\lambda}\right)$, and that (A3) becomes $\lambda > \max\{R/\kappa, R/(u-\kappa)\}$. From Proposition 3.2, we get that money demand functions are

$$m_c^*(\phi) = \begin{cases} -\frac{1}{\lambda}\ln\left(\frac{R}{\lambda u - \lambda(u-\kappa)(\alpha\phi + (1-\alpha))}\right) & \phi > 0, \\ -\frac{1}{\lambda}\ln\left(\frac{R}{\lambda u}\right) & \phi = 0, \end{cases} \tag{3.9}$$

and

$$m_d^*(\phi) = \begin{cases} 0 & \phi \geq \hat{\phi}, \\ -\frac{1}{\lambda}\ln\left(\frac{R}{\lambda\alpha(1-\phi)(u-\kappa)}\right) & \phi \in (0, \hat{\phi}), \\ -\frac{1}{\lambda}\ln\left(\frac{R}{\lambda(u-\kappa)}\right) & \phi = 0, \end{cases} \tag{3.10}$$

where $\hat{\phi} = 1 - \frac{R}{\lambda\alpha(u-\kappa)}$ is the threshold level of card acceptance above which agents who prefer cards decide to hold no cash balances at all. Also notice that the exponential distribution has the desirable property that, when optimal cash balances take the form $m^* = -\frac{1}{\lambda}\ln\left(\frac{x}{\lambda}\right)$, where x is one of the objects defined above, the probability that the purchase size is smaller than cash balances is given by

$$F(m^*) = 1 - e^{-\lambda m^*} = 1 - e^{-\lambda\left(-\frac{1}{\lambda}\ln\left(\frac{x}{\lambda}\right)\right)} = 1 - \frac{x}{\lambda}.$$

3.4.1.2 Seller's problem

Consider sellers indexed by $i \in \{1, \ldots, N\}$. Let $\Phi_i \in \{0, 1\}$ denote the acceptance decision of each seller, with $\Phi_i = 1$ meaning that seller i accepts cashless payments in her shop. The utility of seller i depends both on her decision and on the actions chosen by the other sellers through the average cashless acceptance rate. Denote by $n_{-i} = \sum_{j \neq i} \Phi_j$ the number of competitors of i that accept cards, and let $\phi_{-i} = n_{-i}/N$ be the cashless acceptance rate if i were not to accept cards.

The expected profit for i if she rejects card payments when the cashless acceptance rate is $\phi_{-i} > 0$ is given by

$$\begin{aligned} \Pi_i^c(\phi_{-i}) &= (1-\omega)\left[\alpha F(m_c^*(\phi_{-i}))u + (1-\alpha)F(m_c^*(\phi_{-i}))u\right] + \omega\left[\alpha F(m_d^*(\phi_{-i}))u\right] = \\ &= (1-\omega)F(m_c^*(\phi_{-i}))u + \omega\alpha F(m_d^*(\phi_{-i}))u. \end{aligned} \tag{3.11}$$

When $\phi_{-i} = 0$, rejecting card payments guarantees i the expected profit

$$\begin{aligned} \Pi_i^c(0) &= (1-\omega)\left[\alpha F(m_c^*(0))u + (1-\alpha)F(m_c^*(0))u\right] + \\ &+ \omega\left[\alpha F(m_d^*(0))u + (1-\alpha)F(m_d^*(0))u\right] \\ &= (1-\omega)F(m_c^*(0))u + \omega F(m_d^*(0))u. \end{aligned} \tag{3.12}$$

Notice that the last expression is different from the case in which $\phi_{-i} > 0$ as when nobody accepts the seller might still be able to attract agents who prefer cashless payments even if they can search for their preferred shop, as they would visit a random shop if no stores accept cards.

On the other hand, for any given level of acceptance ϕ_{-i}, accepting card payments yields to seller i the following expected profits[5]

$$\begin{aligned} \Pi_i^{cd}(\phi_{-i}) &= (1-\omega)\left\{\alpha\left[F\left(m_c^*\left(\phi_{-i}+\frac{1}{N}\right)\right)u + \left(1-F\left(m_c^*\left(\phi_{-i}+\frac{1}{N}\right)\right)\right)(u-t)\right]\right. \\ &\quad \left. + (1-\alpha)\left[F\left(m_c^*\left(\phi_{-i}+\frac{1}{N}\right)\right)u + \frac{N}{\phi_{-i}N+1}\left(1-F\left(m_c^*\left(\phi_{-i}+\frac{1}{N}\right)\right)\right)(u-t)\right]\right\} \\ &\quad + \omega\left[\alpha(u-t) + (1-\alpha)\frac{N}{\phi_{-i}N+1}(u-t)\right] = \\ &= (1-\omega)F\left(m_c^*\left(\phi_{-i}+\frac{1}{N}\right)\right)u \\ &\quad + (1-\omega)\left(1-F\left(m_c^*\left(\phi_{-i}+\frac{1}{N}\right)\right)\right)(u-t)\left(\alpha + (1-\alpha)\frac{N}{\phi_{-i}N+1}\right) \\ &\quad + \omega(u-t)\left(\alpha + (1-\alpha)\frac{N}{\phi_{-i}N+1}\right) \end{aligned} \tag{3.13}$$

The optimal choice of seller i is given by

$$\Phi_i^*(\phi_{-i}) = \arg\max_{\Phi_i}\left(1-\Phi_i\right)\Pi_i^c(\phi_{-i}) + \Phi_i\Pi_i^{cd}(\phi_{-i}). \tag{3.14}$$

A few observations are in order. First, notice that by deciding to accept cards, each store influences the overall card acceptance rate in the economy, increasing it by a factor of $1/N$. Second, recall that in expectation each seller is matched with a continuum of buyers with measure one. A share α these buyers are captive, i.e., they cannot visit other shops, while the remaining $1-\alpha$ buyers can shop wherever they like. When accepting cards, no customers matched with the store will choose to search for another one, as there is no gain from doing so. Moreover, the store that accepts cards can attract buyers matched with other stores, in two situations. First, it attracts non-captive buyers who prefer to pay with cash but don't have enough.

5. Notice that differently from Π_i^c, we can write Π_i^{cd} with just one expression because for $\phi_{-i} = 0$, there is still one seller who accepts card payments in the market, which is seller i.

Notice that there are $N(1-\alpha)(1-\omega)\left(1-F\left(m_c^*\left(\phi_{-i}+\frac{1}{N}\right)\right)\right)$ purchases of this kind in the economy every period. The share of such purchases accruing to each of the accepting stores is

$$\underbrace{\left(1+\frac{\left(1-\left(\phi_{-i}+1/N\right)\right)N}{\left(\phi_{-i}+1/N\right)N}\right)}_{\frac{N}{\phi_{-i}N+1}}(1-\alpha)(1-\omega)\left(1-F\left(m_c^*\left(\phi_{-i}+\frac{1}{N}\right)\right)\right),$$

where $\left(1-\left(\phi_{-i}+1/N\right)\right)N$ is the number of stores that do not accept cards, and $\left(\phi_{-i}+1/N\right)N$ is the number of accepting stores. The accepting stores get their random share of the total number of purchases, *plus* their share of the number of purchases from buyers matched with non-accepting stores that look for a shop that accepts. Second, it attracts non-captive buyers who prefer to pay with their cards. There are $N(1-\alpha)\omega$ purchases of this kind in the economy every period. As before, the share of such purchases accruing to each of the accepting stores is $\frac{N}{\phi_{-i}N+1}(1-\alpha)\omega$.

3.4.2 Equilibrium

We use pure Nash equilibrium as a solution concept. We define an equilibrium as a combination of card acceptance and cash holding decisions such that i) given the prevailling acceptance rate, buyers' cash holding choices are optimal, and ii) no seller has an incentive to deviate from their current acceptance policy. A more formal definition of equilibrium in this game follows.

Definition 3.3. An equilibrium is a tuple of acceptance rate and average cash balances (ϕ^*, m^*) such that

(1) average cash balances are given by $m^* = (1-\omega)m_c^* + \omega m_d^*$, where $m_t^* = m_t^*(\phi^*)$ for all $t \in \{c, d\}$;
(2) sellers' acceptance policies are a best response, i.e, for all $i \in \{1, \dots, N\}$, $\Phi_i = \Phi_i^*(\phi_{-i})$. Hence, $\phi^* N = n^* = \sum_{i=1}^N \Phi_i\left(\phi_{-i}^*\right)$.

We only focus on pure strategy Nash equilibria[6], and for simplicity we simply refer to equilibria through the acceptance rate (ignoring the equilibrium level of money demand), i.e, we denote an equilibrium by ϕ^*. A key object in our analysis of equilibria of the game is the function

6. Notice that the presence of N sellers allows for $\phi^* \notin \{0, \frac{1}{2}, 1\}$ without relying on mixed strategies.

$$\begin{aligned}\Delta_i\left(\phi_{-i}\right) &= \Pi_i^{cd}(\phi_{-i}) - \Pi_i^c(\phi_{-i}) \\ &= (1-\omega)\left(F\left(m_c^*\left(\phi_{-i}+\frac{1}{N}\right)\right) - F\left(m_c^*\left(\phi_{-i}\right)\right)\right)u \\ &\quad + (1-\omega)\left(1 - F\left(m_c^*\left(\phi_{-i}+\frac{1}{N}\right)\right)\right)(u-t)\left(\alpha + (1-\alpha)\frac{N}{\phi_{-i}N+1}\right) \\ &\quad + \omega(u-t)\left(\alpha+(1-\alpha)\frac{N}{\phi_{-i}N+1}\right) - \omega\alpha F(m_d^*(\phi_{-i}))u\end{aligned} \tag{3.15}$$

which yields the *net benefit of accepting cards* relative to accepting cash only as a function of the share of opponents who accept cards. We have that $\phi^* \in \{1/N, 2/N, \ldots, (N-1)/N\}$ is an equilibrium card acceptance rate if the conditions

$$\Pi_i^{cd}\left(\phi^* - \frac{1}{N}\right) \geq \Pi_i^c\left(\phi^* - \frac{1}{N}\right) \implies \Delta_i\left(\phi^* - \frac{1}{N}\right) \geq 0, \tag{3.16}$$

$$\Pi_i^{cd}\left(\phi^*\right) < \Pi_i^c\left(\phi^*\right), \implies \Delta_i\left(\phi^*\right) < 0, \tag{3.17}$$

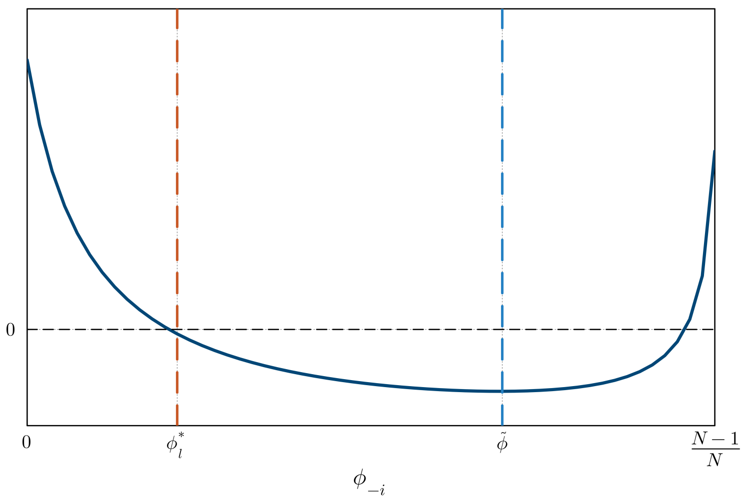
are both satisfied. These conditions make sure that i) it is optimal to accept cards if $\phi_{-i} = \phi^* - 1/N$, which makes sure that at least a fraction ϕ^* of merchants accept, and ii) it is not optimal to accept cards if $\phi_{-i} = \phi^*$, which makes sure that at most a fraction ϕ^* of merchants accept. We label equilibria characterized by $\phi^* \in (0,1)$ as *imperfect acceptance equilibria* (IAE). In 3.3 we display the function $\Delta_i(\phi_{-i})$ graphically for a reasonable parametrization of the model, for illustrative purposes. From the graph, it is possible to spot the unique imperfect acceptance equilibrium of the model for the chosen parameter vector, which we label as ϕ_I^*. The condition for a *pure cash* equilibrium $\phi^* = 0$ is instead

$$\Pi_i^{cd}(0) < \Pi_i^c(0), \implies \Delta_i(0) < 0, \tag{3.18}$$

while a *full acceptance* equilibrium $\phi^* = 1$ exists if

$$\Pi_i^{cd}\left(\frac{N-1}{N}\right) \geq \Pi_i^c\left(\frac{N-1}{N}\right), \implies \Delta_i\left(\frac{N-1}{N}\right) \geq 0. \tag{3.19}$$

Notice that in the example depicted in 3.3, full acceptance is an equilibrium outcome, as $\Delta_i\left(\frac{N-1}{N}\right) > 0$, while a pure cash equilibrium does not exist. Also notice that, in continuity with the result of 3.3, the presence of strategic substitutability is essential in order to generate imperfect acceptance equilibria. At the equilibrium ϕ_I^*, acceptance choices are indeed strategic substitutes, as an increase in the number of accepting shops makes acceptances less attractive than non-acceptance for everybody. Notice from the shape of the function $\Delta_i(\phi_{-i})$ in 3.3 that our games features either complementarity or substitutability depending on the level of ϕ_{-i} itself, keeping other parameters constant. The reason is that the acceptance game has features of both coordination and congestion games. For levels of ϕ smaller than $\widetilde{\phi}$, congestion effects dominate and the benefit of accepting cards is reduced



The above Figure displays the function $\Delta_i(\phi_{-i})$ for any $\phi_{-i} \in \{0, 1/N, 2/N, ..., (N-1)/N\}$. The orange line marks the imperfect acceptance equilibrium ϕ_l^*. Notice that $\phi^* = 1$ is also an equilibrium of the model. Baseline parametrization with $N = 60$ sellers. F is an exponential distribution with parameter $\lambda = 2$. Other parameters are $u = 1$, $\kappa = 0.03$, $R = 0.025$, $t = 0.5$, $\alpha = 0.8$, $\omega = 0.3$.

Figure 3.3. The function $\Delta_i(\phi_{-i})$

by other people entering the market. For higher levels of ϕ, coordination effects start to dominate: as buyers start carrying much less cash when ϕ goes up, the value of accepting relative to non-accepting rises again as buyers are only able to complete very small purchases in shops that don't accept cards.

We now analyze the existence and uniqueness of equilibria in our model, starting from an existence result.

Proposition 3.4. *In the acceptance game with N sellers, at least one equilibrium* (ϕ^*, m^*) *exists.*

Proof. To rule out *full acceptance* and *no acceptance* equilibria, (3.18) and (3.19) must hold simultaneously. The function $\Delta_i(\phi_i)$ must be positive for $\phi_{-i} = 0$ and negative for $\phi_{-i} = (N-1)/N$. But if that is the case, we must have that for at least one $\phi^* \in \{0, 1/N, 2/N, ..., (N-1)/N\}$, it must be that both (3.16) and (3.17) hold, and that an *imperfect acceptance equilibrium* exists. □

The above result makes sure that the model has at least one equilibrium. If $\phi^* = 0$ and $\phi^* = 1$ are not equilibria, there has to be an IAE $\phi^* \in \{1/N, 2/N, ..., (N-1)/N\}$. We now discuss uniqueness. We start by analyzing a nonatomic version of the game in which the number of sellers $N \to +\infty$, to eliminate each seller's impact on buyers' money demand. Let $\Delta(\phi) = \lim_{N \to +\infty} \Delta_i^N(\phi)$, where we write $\Delta_i^N(\phi)$ to underscore the dependence of the net benefit of accepting cards on the number of sellers in the finite-player version of the game. It is given by

$$\begin{aligned}\Delta(\phi) &= \lim_{N\to+\infty} \Delta_i^N(\phi) = \\ &= (1-\omega)[F(m_c^*(\phi)) - F(m_c^*(\phi))]u + (1-\omega)(1-F(m_c^*(\phi)))(u-t)\left(\alpha + \frac{1-\alpha}{\phi}\right) \\ &\quad + \omega((u-t)\left(\alpha + \frac{1-\alpha}{\phi}\right) - \alpha F(m_d^*(\phi))u) \\ &= (1-\omega)(1-F(m_c^*(\phi)))(u-t)\left(\alpha + \frac{1-\alpha)}{\phi}\right) \\ &\quad + \omega((u-t)\left(\alpha + \frac{1-\alpha}{\phi}\right) - \alpha F(m_d^*(\phi))u).\end{aligned} \tag{3.20}$$

In the nonatomic version of the game, imperfect acceptance equilibria are given by $\phi^* \in (0,1)$ for which $\Delta(\phi) = 0$. Moreover, $\phi^* = 0$ is an equilibrium if $\lim_{\phi\to 0} \Delta(\phi) = 0$ and $\phi^* = 1$ is an equilibrium if $\Delta(1) \geq 0$. Notice that the function $\Delta(\phi)$ is not defined for $\phi = 0$ as $\Pi^{cd}(\phi)$ diverges when $\phi \to 0$. In the next Lemma, we study the properties of the function $\Delta(\phi) = \lim_{N\to+\infty} \Delta_i^N(\phi)$ to investigate how many equilibria does the nonatomic acceptance game have. As obtaining results for a generic distribution of transactions F is hard, we focus on the exponential distribution.

Lemma 3.5. *Let transaction sizes be exponentially distributed. For generic parameters, the nonatomic acceptance game has at most three equilibria: the full acceptance equilibrium $\phi^* = 1$, and either zero or two imperfect acceptance equilibria, a low-acceptance one (ϕ_l^*) and a high-acceptance one (ϕ_h^*), with $\phi_l^* < \phi_h^*$. Only the low acceptance equilibrium and the full acceptance one are stable*[7].

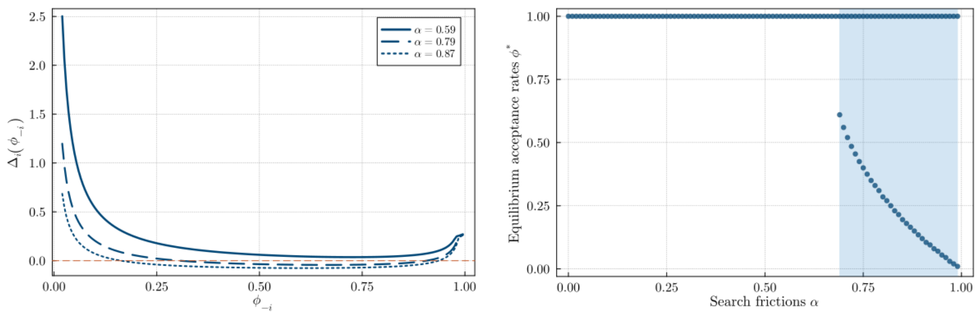
Proof. See 3.A.2 □

Given that $\Delta_i^N(\phi)$ converges pointwise to $\Delta(\phi)$ for $\phi \in (0,1)$, we obtain the following result on the number of equilibria finite-player acceptance games with *enough* sellers.

Proposition 3.6. *Let transaction sizes be exponentially distributed. Generically, there exists N' sufficiently large such that for all $N > N'$ the acceptance game with N sellers has at most two equilibria: the full acceptance equilibrium $\phi^* = 1$ and either zero or one imperfect acceptance equilibria $\phi_l^* \in \{1/N, 2/N, ..., (N-1)/N\}$.*

Proof. Suppose $\Delta(\phi) = 0$ has no solutions. Then $\exists N' \in \mathbb{N}$ such that for all $N > N'$, $\Delta_i^N(\phi_{-i}) > 0$ for all $\phi_{-i} \in \{0, 1/N, 2/N, \ldots, (N-1)/N\}$. In this case, no $\phi^* \in \{0, 1/N, 2/N, \ldots, (N-1)/N\}$ is an equilibrium. Now suppose $\Delta(\phi) = 0$ has two solutions. Then $\exists N' \in \mathbb{N}$ such that for all $N > N'$, $\exists! \phi_l^* \in \{0, 1/N, 2/N, \ldots, 1\}$ that satisfies conditions (3.16)-(3.17). In this case, ϕ_l^* is the unique IAE of the game. □

7. We say an equilibrium is *stable* if, for a small perturbation (a share $\epsilon > 0$ of sellers deviate from equilibrium play), the economy converges back to equilibrium under best-response dynamics.



In the left Panel, we display how the function $\Delta_i(\phi_{-i})$ changes as α rises, for three different values of α. In the right Panel, we display equilibrium acceptance rates ϕ^* for all values of $\alpha \in [0,1)$. Baseline parametrization with $N = 60$ sellers. F is an exponential distribution with parameter $\lambda = 2$. Other parameters are $u = 1$, $\kappa = 0.03$, $R = 0.025$, $t = 0.5$, $\omega = 0.3$.

Figure 3.4. Comparative statics: search frictions α

The intuition for the above result is the following: since $\Delta_i^N(\phi)$ approaches $\Delta(\phi)$ for large enough N, when $\Delta(\phi) = 0$ has two solutions there exist a $\phi_l^* \in \{1/N, 2/N, ..., (N-1)/N\}$ for which $\Delta_i\left(\phi_l^* - 1/N\right) \geq 0$ and $\Delta_i\left(\phi_l^*\right) < 0$, as well as a $\phi_h^* \in \{1/N, 2/N, ..., (N-1)/N\}$ for which $\Delta_i\left(\phi_h^* - 1/N\right) < 0$ and $\Delta_i\left(\phi_h^*\right) \geq 0$. However, only the low acceptance one is an IAE of the N-player game. For the nonatomic game, both ϕ_l^* and ϕ_h^* are equilibria, even though the only the low acceptance one is stable. In other words, imposing a stability requirement to equilibria of the nonatomic game leaves us with the same equilibria of the N-player acceptance game with large enough N.

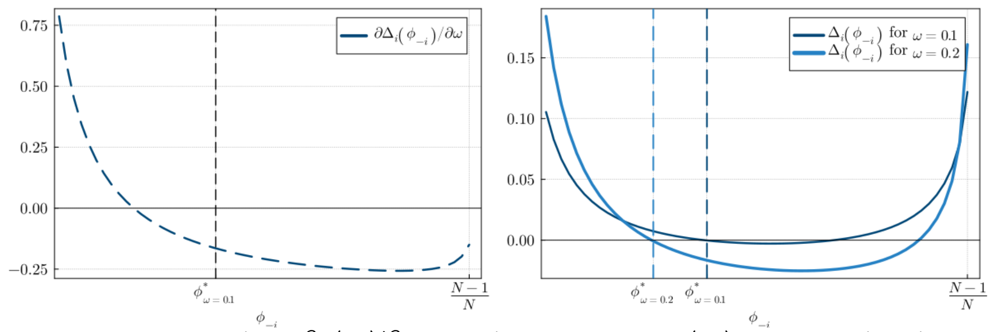
3.4.3 Comparative statics

We now analyze properties of equilibria and in particular we focus on describing how do they change as a function of model parameters.
Search frictions α. We start by analyzing the effect of α on equilibria. Recall that α is the probability that each buyer can visit one seller only. It can be interpreted as the degree of search frictions in this economy, i.e., the likelihood that buyers are constrained in their search and they can only visit one shop. When $\alpha \to 0$, search frictions vanish and the buyers can potentially visit all the shops in the economy.

Proposition 3.7. *There exists $N' \in \mathbb{N}$ sufficiently large such that for all $N > N'$ and for any ϕ_{-i}, $\partial\Delta_i(\phi_{-i})/\partial\alpha < 0$. Let (m^*, ϕ^*) be an imperfect acceptance equilibrium. Then, $\partial\phi^*/\partial\alpha \leq 0$ and $\partial m^*/\partial\alpha \geq 0$.*

Proof. See 3.A.3. □

The above Proposition states that when N is large enough, an increase in α reduces the relative value of accepting cards with respect to the no-acceptance alternative, independently of the share of accepting merchants in the economy. When



In the left Panel, we display $\partial\Delta_i(\phi_{-i})/\partial\omega$. In the right Panel, we how $\Delta_i(\phi_{-i})$ and the associated imperfect acceptance equilibrium change for an increase in ω from 0.2 to 0.3. Baseline parametrization with $N = 60$ sellers. F is an exponential distribution with parameter $\lambda = 2$. Other parameters are $u = 1$, $\kappa = 0.03$, $R = 0.025$, $t = 0.5$, $\alpha = 0.8$.

Figure 3.5. Comparative statics: payment preferences ω

α rises, indeed, consumers are able to search for their preferred shop less often, and merchants' incentives to start accepting to increase their client base become less relevant. In 3.4 we show how the function $\Delta_i(\phi_{-i})$ changes as search frictions α increase, as well as how the set of equilibrium acceptance rates is affected by α. Notice that imperfect acceptance equilibria only arise when the extent of search frictions, which generate strategic substitutability, is large enough. When an imperfect acceptance equilibrium $\phi^* \in (0, 1)$ exists, it is decreasing in α. For a range of values of α, we have multiple equilibria: both full acceptance ($\phi^* = 1$) and imperfect acceptance can arise in equilibrium.

Payment preferences ω. In the model, a fraction ω of agents prefer to use their cards to settle purchases when they have the option to do so. We now investigate how equilibria are affected by the share of agents preferring cards as a payment method.

Proposition 3.8. *Let (m^*, ϕ^*) be an imperfect acceptance equilibrium. Then, there exists $N' \in \mathbb{N}$ sufficiently large such that for all $N > N'$, $\partial\Delta_i(\phi^*)/\partial\omega \le 0$, and therefore $\partial\phi^*/\partial\omega \le 0$, while $\partial m^*/\partial\omega$ is of ambiguous sign.*

Proof. It is possible to show that

$$\begin{aligned}\frac{\partial\Delta_i(\phi_{-i})}{\partial\omega} = & \frac{\Delta_i(\phi_{-i})}{\omega} \\ & -\frac{1}{\omega}\left(1 - F\left(m_c^*\left(\phi_{-i} + \frac{1}{N}\right)\right)\right)(u-t)\left(\alpha + (1-\alpha)\frac{N}{\phi_{-i}N+1}\right) \\ & -\frac{1}{\omega}\underbrace{\left(F\left(m_c^*\left(\phi_{-i} + \frac{1}{N}\right)\right) - F\left(m_c^*(\phi_{-i})\right)\right)}_{\approx 0 \text{ when } N\to+\infty} u,\end{aligned} \tag{3.21}$$

i.e., that $\partial \Delta_i(\phi_{-i})/\partial \omega \geq 0 \implies \Delta_i(\phi_{-i}) > 0$. Since $\Delta_i(\phi^*) < 0$, we get that $\partial \Delta_i(\phi^*)/\partial \omega < 0$, and the conclusion $\partial \phi^*/\partial \omega \leq 0$ follows. Since $m^* = \omega m_d^*(\phi^*(\omega)) + (1-\omega)m_c^*(\phi^*(\omega))$, one can show that the effect on $m^*(\phi^*)$ is ambiguous. See 3.A.4 for the complete proof. □

The interpretation of 3.8 is as follows. As we show in detail in 3.A.4, an increase in ω generates two effects, which have opposite implications for the convenience of accepting card payments. On one hand, they increase the store's potential to attract clients, as agents who prefer to use cashless payments always search for a shop that accepts, independently on the size of the transaction they face, differently from buyers who prefer to pay in cash. On the other hand, shops receive only $u - t$ (instead of u) when selling to agents who prefer to pay using cards, even if they have enough cash with them. In other words, a higher ω is associated with a larger client base for shops who accept (which makes acceptance more attractive), but with smaller utility derived from each transaction (which has the opposite effect). Depending on which of the two effects dominates, Δ might grow or shrink as ω rises. We prove that the latter effect dominates at the equilibrium acceptance rate ϕ^*, and that the former only dominates for $\phi_{-i} < \phi^*$. This is also shown graphically in 3.5. In the left panel, we display the derivative $\partial \Delta_i/\partial \omega$ for all values of ϕ_{-i}. The graph shows that the derivative gets negative for levels of ϕ_{-i} slightly below the equilibrium one. The values of ϕ_{-i} for which an increase in ω makes acceptance relatively more profitable are those at which acceptance is so profitable that more people start accepting. When few sellers accept, an increase in ω enhances the potential client base which can be obtained through acceptance much more than when a large number of sellers accept. In the right panel of the Figure, we show how Δ_i changes in response to a 100% increase in ω, and how the equilibrium acceptance rate falls. We also show that the effect on average cash balances is ambiguous since i) both types of agents hold more cash as ω increases, in response to lower acceptance, but ii) agents who prefer paying with cards, who hold less cash than agents of the other type, are now a larger share of the population of buyers.

Opportunity costs *R*. We now analyze how the opportunity cost of holding cash influences equilibria. Recall that R only affects sellers' profits through optimal cash demand bu buyers.

Proposition 3.9. *Let (m^*, ϕ^*) be an imperfect acceptance equilibrium. Then, for large enough N, $\partial \Delta_i(\phi^*)/\partial R \geq 0$, and therefore $\partial \phi^*/\partial R \geq 0$, while $\partial m^*/\partial R \leq 0$.*

Proof. See 3.A.5. □

An increase in opportunity costs of holding cash reduces the level of cash balances for any card acceptance rate ϕ, and for both types of agents. Therefore, with higher probability agents won't have enough money to settle transactions in cash. The above Proposition states that when N is large enough, the value of accepting cards relative to cash increases since (i) the profits from acceptance Π^{cd} go up as more transactions with agents who prefer cash are completed, and (ii) the profits from non-acceptance Π^{c} fall as fewer transactions with captive buyers that prefer cards are completed.

3.5 A quantitative extension

In this Section, we present an augmented version of our framework which enables us to bring the model to the data. We start by summarizing the main differences between the framework outlined below and the theoretical model we studied in **??**. In the model outlined previously, agents faced a simple static money demand problem, which however had several limitations, namely (i) the absence of a dynamic component in payment method choices, and (ii) the absence of model implications on the frequency and size of withdrawals. In this Section, we model buyers' behavior by adapting the dynamic cash management problem with a means of payment choice by Lippi and Moracci (2024) to our framework with heterogeneity among buyers and search across shops. We slightly modify the search protocol as well: in the model of **??**, we assumed out of simplicity that non-captive agents searched for shops that accepted cards if and only if (i) the purchase was too large to be settled using cash, or (ii) they were of type d, i.e., they preferred card payments. In what follows, instead, we freely allow agents of both types to choose whether to search for shops that accept cards (by paying a cost) or to visit a random store, depending on the level of cash on hand they have. The problem faced by sellers is similar to the one described above, even though they don't face anymore a deterministic level of cash balances, but a stationary distribution of cash holdings in the population of consumers.

3.5.1 The model

We now describe formally the problems faced by buyers and sellers in the extended model.

3.5.1.1 Buyer's problem

We now formally present the buyer's problem, describe optimal decision rules, and characterize moments implied by the solution.

Setup. Time is continuous and the exponential discount factor is ρ. Buyers need to finance an expenditure stream given by a compound Poisson process with arrival rate λ and size distribution F, with $F(0) = 0$. Buyers differ in their preferred payment method: a share $1 - \omega$ of agents are of type c and they face a fixed cost $\kappa_c > 0$ when paying by card, while ω agents of type d face a fixed cost $\kappa_d > 0$ when using cash. Buyers take the share of merchants who accept cards ϕ as given, and they obtain utility u whenever they purchase something. Let m denote the level of cash balances. Holding m units of cash entails an instantaneous opportunity cost Rm, with $R > 0$, and agents need to pay $b > 0$ to withdraw cash from ATMs. When $\phi > 0$, the value of the problem for agents of type c inside the range of inaction is given by

$$(\rho + \lambda)v_c(m) = Rm + \lambda\left[\alpha\tilde{v}_c(m) + (1-\alpha)\min\{\bar{v}_c(m) + \eta(1-\phi), \tilde{v}_c(m)\}\right] \quad (3.22)$$

where $\eta(1-\phi)$ is the fixed cost of searching for a shop that accepts cards. Notice that the cost of searching is zero when all shops accept ($\phi = 1$) and it is maximized and equal to η when no shops accept. When the card acceptance rate is $\phi = 0$, clearly agents will never search. The value of searching is given by

$$\bar{v}_c(m) = (1-F(m))\left(v_c(m) + \kappa_c\right) + \int_0^m \min\{v_c(m-s), v_c(m) + \kappa\}dF(s), \quad (3.23)$$

and the value of not searching is given by

$$\begin{aligned}\tilde{v}_c(m) = \; & (1-F(m))\left(v_c(m) + \phi\kappa + (1-\phi)u\right) \\ & + \phi\int_0^m \min\{v_c(m-s), v_c(m) + \kappa\}dF(s) \\ & + (1-\phi)\int_0^m v_c(m-s)dF(s).\end{aligned} \quad (3.24)$$

The problem for agents of type d is similar, but they bear the fixed cost κ when paying with cash. The full problem for agents of type $j \in \{c, d\}$ is given by

$$v_j(m) = \min\left\{b + v_j^*, \frac{Rm + \lambda\alpha\tilde{v}_c(m) + \lambda(1-\alpha)\min\{\bar{v}_c(m) + \eta(1-\phi), \tilde{v}_c(m)\}}{\rho + \lambda}\right\}, \quad (3.25)$$

where b is the cost of adjusting cash balances and

$$v_j^* = v_j(m^*), \text{ where } m_j^* = \arg\min_{\hat{m}} v_j(\hat{m}). \quad (3.26)$$

Optimal policy. We consider a withdrawal policy of the following kind: agents of type j withdraw as soon as their cash balances fall below a trigger level $\underline{m}_j$, and when that happens they withdraw up to the target level m_j^*. When they have the option to search (which happens with probability α) they do it whenever $\bar{v}_c(m) - \eta(1-\phi)$. Let $\ell_j(m)$ denote the search policy function of type-j agents, with $\ell_j(m) = 1$ meaning that agents of type j search for a shop that accepts when they have m units of cash on hand and $\ell_j(m) = 0$ meaning that they visit a random shop. Finally, let $\ell_j(m, s)$ denote payment choice policies of type-j agents, with $p_j(m, s) = 1$ meaning that cards are used to settle a purchase of size s when having m units of cash on hand and $p_j(m,s) = 0$ meaning that cash is used to pay for such a purchase. Clearly, $p_j(m,s) = 1$ whenever $s > m$.

Implied moments. From the set of policy functions $\{\underline{m}_j, m_j^*, \ell_j, p_j\}_{j\in\{c,d\}}$, given a certain card acceptance rate ϕ, it is possible to derive a set of model-implied moments which summarize features of the behavior of type-j buyers. Of course, given any statistic X_j, the aggregate counterpart for the whole economy adjusting for heterogeneity across buyers is $X = (1-\omega)X_c + \omega X_d$. In what follows, we adjust the results of Lippi and Moracci (2024) (Section 3.3) to account for optimal search behavior. We start by denoting with $\overline{\phi}_j(m) = \phi + (1-\alpha)(1-\phi)\ell_j(m)$ the effective acceptance probability faced by type j consumers when having m units of cash on hand. We now characterize the steady-state distribution of cash balances for agents of type j, which we label $h_j(m)$. It is easy to show that for any $m \in [\underline{m}_j, m_j^*)$ the steady-state distribution satisfies the functional equation

$$h_j(m) = \frac{\displaystyle\int_m^{m_j^*} h_j(m')\chi_j(m', m'-m)dm' + h_j(m_j^*)\chi_j(m_j^*, m_j^*-m)}{\displaystyle\int_0^m \chi_j(m,s)ds}, \tag{3.27}$$

where (with a slight abuse of notation) $h_j(m_j^*)$ denotes a mass point at $m = m^*$, and $\chi_j(m,s) = f(s)\left(1-\overline{\phi}_j(m)p_j(m,s)\right)$ for any $m' > m$. The mass point $h_j(m_j^*)$ is pinned down by $h_j(m_j^*) = 1 - \int_{\underline{m}_j}^{m_j^*} h_j(m)dm$. Notice that the expression

$$\underbrace{\overline{\phi}_j(m)}_{\text{Pr(Card possible)}} \cdot \underbrace{p_j(m,s)}_{\text{Choose cards}}$$

inside (3.27) represents the probability of paying with cards for payments of size s when having a level m of cash balances, taking into account the probability of randomly entering a shop that accepts, plus the probability of being able to search $1-\alpha$ for such a shop and doing so (if $\ell_j(m) = 1$), times the decision to pay with cards when having the chance $p_j(m,s)$. Equipped with $h_j(m)$, it is possible to compute the effective acceptance rate for type j buyers, i.e., the probability that a shop visited by this type of agent accepts cards, given by

$$\overline{\phi}_j = \int_{\underline{m}_j}^{m_j^*} h_j(m)\overline{\phi}_j(m)dm + h_j(m_j^*)\overline{\phi}_j(m_j^*). \tag{3.28}$$

Clearly, $\overline{\phi}_j$ is an upper bound for the true ϕ, as agents will visit disproportionately shops that accept cards when $\alpha < 1$ and search is not too expensive. Average cash balances held by type j buyers are given by $M_j = \int_{\underline{m}_j}^{m_j^*} mh_j(m)dm + h_j(m_j^*)m_j^*$, and median cash holdings can be computed trivially using the cdfs $H_j(m) = \int_{\underline{m}_j}^{m_j^*} h_j(m)dm + h_j(m_j^*)\mathbb{1}\left(m = m_j^*\right)$. The average number of withdrawals per unit of time is given by $n_j = 1/t_j(m_j^*)$, where $t_j(m)$ is the expected time before the next withdrawal when current cash balances are equal to m, which obeys the functional equation

$$t_j(m) = \frac{1 + \lambda \displaystyle\int_0^{m-\underline{m}_j} \chi_j(m,s) t_j(m-s) ds}{\lambda \int_0^m \chi_j(m,s) ds}. \tag{3.29}$$

Average cash balances at withdrawal $\underline{M}_j$ are given by

$$\underline{M}_j = \int_{\underline{m}_j}^{m_j^*} h_j^w(m) \left[\frac{\int_{m-\underline{m}_j}^{m} f(s)(m-s)ds}{\int_{m-\underline{m}_j}^{m} f(s)ds} \right] dm + h_j(m_j^*) \left[\frac{\int_{m_j^*-\underline{m}_j}^{m_j^*} f(s)(m-s)ds}{\int_{m_j^*-\underline{m}_j}^{m_j^*} f(s)ds} \right], \tag{3.30}$$

where $h_j^w(m)$ denotes the stationary distribution of cash holdings conditional on a withdrawal taking place immediately after, which is given by

$$h_j^w(m) = \frac{h_j(m) \left(\int_{m-\underline{m}_j}^{m} \chi_j(m,s) ds \right)}{\displaystyle\int_{\underline{m}_j}^{m_j^*} h_j(m') \left(\int_{m'-\underline{m}_j}^{m'} \chi_j(m',s) ds \right) dm' + h_j(m_j^*) \left(\int_{m_j^*-\underline{m}_j}^{m_j^*} \chi_j(m_j^*,s) ds \right)},$$

with a mass point at $m = m_j^*$ which is pinned down by $h_j^w(m_j^*) = 1 - \int_{\underline{m}_j}^{m_j^*} h_j^w(m) dm$. The average withdrawal size is given by $W_j = m_j^* - \underline{M}_j$. Finally, we derive model-implied payment method shares. Let e_j denote total expenditures per unit of time for agents of type j, given by

$$e_j = \lambda \int_0^{m_j^*} sf(s) \left(\int_{\underline{m}_j}^{s} h_j(m) \overline{\phi}_j(m) dm + \left(1 - H_j(s)\right) \right) ds + \lambda \overline{\phi}_j \int_{m_j^*}^{+\infty} sf(s) ds,$$

where the term inside parentheses denotes the proportion of payments of size s which are completed by the buyer, either because $m \geq s$ (which happens with probability $1 - H_j(s)$), or (when $m < s$) because the consumer visits a store that accepts cards[8]. The number of completed purchases is given by

$$\hat{\lambda} = \lambda \left(\int_{\underline{m}_j}^{m_j^*} h_j(m) \left(\overline{\phi}_j(m) + \left(1 - \overline{\phi}_j(m)\right) F(m) \right) dm + h_j(m_j^*) \left(\overline{\phi}_j(m_j^*) + \left(1 - \overline{\phi}_j(m_j^*)\right) F(m_j^*) \right) \right).$$

The share of expenditure settled using cards is given by

$$\gamma_j = \frac{\lambda \left(\int_{\underline{m}_j}^{m_j^*} h_j(m) \overline{\phi}_j(m) \gamma_j(m) dm + h_j(m_j^*) \overline{\phi}_j(m_j^*) \gamma_j(m^*) \right)}{e_j}, \tag{3.31}$$

8. In the above formula, the last integral on the right-hand side does not include m_j^* in the sum, as the event $s = m_j^*$ has probability zero.

where $\gamma_j(m) = \int_0^m sf(s)p_j(m,s)ds + \int_m^{+\infty} sf(s)ds$ is the share of expenditure paid with cards by type-j agents when having m units of cash available. The share of expenditure settled using cards conditional on having both options available is given by

$$\widetilde{\gamma}_j = \frac{\lambda\left(\int_{\underline{m}_j}^{m_j^*} h_j(m)\overline{\phi}_j(m)\widetilde{\gamma}_j(m)dm + h_j(m_j^*)\overline{\phi}_j(m_j^*)\widetilde{\gamma}_j(m^*)\right)}{\lambda\left(\int_{\underline{m}_j}^{m_j^*} h_j(m)\overline{\phi}_j(m)\left(\int_0^m sf(s)ds\right)dm + h_j(m_j^*)\overline{\phi}_j(m_j^*)\left(\int_0^{m_j^*} sf(s)ds\right)\right)}, \quad (3.32)$$

where $\widetilde{\gamma}_j(m) = \int_0^m sf(s)p_j(m,s)ds$ is the share of expenditure paid with cards by type-j agents when having m units of cash available if s is smaller or equal than m.

3.5.1.2 Seller's problem

We consider the nonatomic version of the game with a continuum of sellers. Sellers maximize total profits by deciding whether to accept or reject card payments, as before. Unit profits when receiving a cash payment are u, while they shrink to $u-t$ in the case of a card transaction. We allow for a fixed cost T in accepting cards, which is meant to represent the fixed amount that sellers need to pay to be able to accept card payments, such as the fees associated with installing a POS terminal, the cost of setting up a contract with the service provider, and so on. The parameter T is meant to capture all the costs related to acceptance that arise independently from the share of purchases which are effectively paid for using cards. The main difference is that now sellers do not face buyers who hold a fixed amount of cash, but they face a stationary distribution of cash balances in the economy, as well as optimal search choices. When a fraction $\phi > 0$ of sellers accept, the value of accepting is given by

$$\begin{aligned}\Pi^{cd}(\phi) = \quad & (1-\omega)\alpha\left(\int_{\underline{m}_c}^{m_c^*} h_c(m)\pi_c^{cd}(m)dm + h_c(m_c^*)\pi_c^{cd}(m_c^*)\right) \\ & +(1-\omega)(1-\alpha)\left(\int_{\underline{m}_c}^{m_c^*} h_c(m)\widetilde{\phi}_c(m)\pi_c^{cd}(m)dm + h_c(m_c^*)\widetilde{\phi}_c(m_c^*)\pi_c^{cd}(m_c^*)\right) \\ & +\omega\alpha\left(\int_{\underline{m}_d}^{m_d^*} h_d(m)\pi_d^{cd}(m)dm + h_d(m_d^*)\pi_d^{cd}(m_d^*)\right) \\ & +\omega(1-\alpha)\left(\int_{\underline{m}_d}^{m_d^*} h_d(m)\widetilde{\phi}_d(m)\pi_d^{cd}(m)dm + h_d(m_d^*)\widetilde{\phi}_d(m_d^*)\pi_d^{cd}(m_d^*)\right) - T,\end{aligned} \quad (3.33)$$

where

$$\widetilde{\phi}_j(m) = \frac{\phi + (1-\phi)\ell_j(m)}{\phi}, \quad (3.34)$$

and

$$\pi_j^{cd}(m) = \int_0^m \left(u - p_j(m,s)t\right) dF(s) + (1-F(m))\,(u-t) \tag{3.35}$$

is the expected profit for a shop that accepts cards conditional on having a client of type j with m units of cash balances entering the shop. Notice that

$$(1-\omega)(1-\alpha)\int_{\underline{m}_c}^{m_c^*} h_c(m)\left(\frac{\phi+(1-\phi)\ell_c(m)}{\phi}\right)dm$$

is the expected number of type c clients that can search who visit the shop. Notice that, when everybody accepts ($\phi = 1$) or nobody searches ($\ell_c(m) = 0$ for all m), the expected number of clients that can search who visit the shop is one. The same holds for the expected number of type d searchers who visit the shop. Notice that $\Pi^{cd}(\phi)$ is not defined for $\phi = 0$, and $\lim_{\phi\to 0^+}\Pi^{cd}(\phi) = +\infty$, i.e., sellers get infinitely high profits from accepting when nobody else does, as they attract a large mass of searching customers. The value of not accepting cards when a fraction $\phi > 0$ of sellers accept is given by

$$\begin{aligned}\Pi^c(\phi) = {} & (1-\omega)\alpha\left(\int_{\underline{m}_c}^{m_c^*} h_c(m)\pi_c^c(m)dm + h_c(m_c^*)\pi_c^c(m_c^*)\right)\\ & +(1-\omega)(1-\alpha)\left(\int_{\underline{m}_c}^{m_c^*} h_c(m)\left(1-\ell_c(m)\right)\pi_c^c(m)dm + h_c(m_c^*)(1-\ell_c(m_c^*)\pi_c^c(m_c^*)\right)\\ & +\omega\alpha\left(\int_{\underline{m}_d}^{m_d^*} h_d(m)\pi_d^c(m)dm + h_d(m_d^*)\pi_d^c(m_d^*)\right)\\ & +\omega(1-\alpha)\left(\int_{\underline{m}_d}^{m_d^*} h_d(m)\left(1-\ell_d(m)\right)\pi_d^c(m)dm + h_d(m_d^*)(1-\ell_d(m_d^*))\pi_d^c(m_d^*)\right),\end{aligned} \tag{3.36}$$

where

$$\pi_j^c(m) = F(m)u, \quad \text{for } j \in \{c,d\}. \tag{3.37}$$

3.5.1.3 Equilibrium

We now define our an equilibrium for our economy.

Definition 3.10. A *payments market equilibrium* is a tuple of value functions $\{v_c, v_d\}$, policy functions $\{\underline{m}_c, m_c^*, \ell_c, p_c, \underline{m}_d, m_d^*, \ell_d, p_d\}$, stationary distributions $\{h_c, h_d\}$, sellers' profit functions $\{\Pi_c, \Pi_{cd}\}$ and a card acceptance rate ϕ^* such that:

(1) The value function v_j the policy functions $\{\underline{m}_j, m_j^*, \ell_j, p_j\}$ solve the cash management and payment choice problem of type j households, for $j \in \{c,d\}$, when $\phi = \phi^*$.
(2) Given the policy functions $\{\underline{m}_j, m_j^*, \ell_j, p_j\}$ and the implied stationary distributions of cash holdings h_j for $j \in \{c,d\}$, $\Delta(\phi^*) = \Pi_{cd}(\phi^*) - \Pi_c(\phi^*) = 0$.

The arguments for equilibrium existence that we used in **??** apply to this version of the model as well. As before, we cannot guarantee uniqueness, but after solving the model numerically, for each vector of parameters, we can check the existence of a unique stable imperfect acceptance equilibrium $\phi_l^* \in (0,1)$, in addition to the full acceptance equilibrium $\phi^* = 1$.

3.5.2 Calibration

We now discuss how to calibrate our model and present our preferred calibration of model parameters.

Calibration strategy. We calibrate the model at the yearly frequency to reproduce aggregate statistics for the Euro Area that we derive from the Study on the Payment Attitudes of Consumers in the Euro Area (SPACE from now on). The model has a total of twelve scalar and functional parameters, given by $\{\rho, \omega, F, b, R, \kappa, \lambda, u, \alpha, \eta, t, T\}$. Some of these parameters can be externally calibrated without solving the model. We start by setting ρ to 36.0 to target a monthly discount rate of .96 which we deem appropriate for cash holding and payment decisions. The parameter ω is set to 60 percent, which is the percentage of SPACE respondents who reportedly prefer using cards to carry out their purchases[9] As in Lippi and Moracci (2024), we assume that F is lognormal and we calibrate its parameters μ_s and σ_s^2 to target a daily expenditure of one and to match the coefficient of variation of purchase sizes. We calibrate the remaining parameters of the model, i.e. $\{R, \kappa, \lambda, u, \alpha, \eta, t, T\}$, through a two-step minimum distance procedure. We start by assuming that the data we observe describes an imperfect acceptance equilibrium outcome with card acceptance rate ϕ_l^*. Then, we set $\phi = \widehat{\phi}$ (which we observe from the Survey on the Use of Cash by Companies in the Euro Area (2022)[10]) and we compute model implied moments for a large grid of possible parameter vectors for the buyer problem $\{r, \kappa, \lambda, u, \alpha, \eta\}$, searching for the parameter vector that minimizes the distance between six model-implied moments and their empirical counterparts. After finding the minimizer, we performed a grid search over the vector of parameters that only enter the seller's problem, i.e. $\{t, T\}$, to find a combination of the two parameters that make $\widehat{\phi}$ a stable imperfect acceptance equilibrium, given the stationary distributions and policy functions implied by the solution of the

9. Since reported payment preferences vary a lot across different regions in SPACE, estimating the model at the Euro Area level might be an issue, since type-c and type-d buyers identified through the survey question might differ in a lot more than their payment preferences. A more appropriate exercise would be to choose a country and estimate the model at that level. Still, we cannot do that as we observe the acceptance rate reported by companies (hence, the true level $\widehat{\phi}$) only for the Euro Area as an aggregate, through the Companies' Survey on the Use of Cash, and the associated microdata is not publicly available. We are currently in touch with the ECB Banknotes Directorate to have access to this supply-side acceptance data and considerably improve estimation.

10. As a measure of the rate of card acceptance, we took the minimum between the acceptance rate for credit cards, debit cards and contactless cards.

buyer's problem for the best parameter vector found above. To calibrate the parameter set $\{r, \kappa, \lambda, u, \alpha, \eta\}$, we target moments that we deem informative about each parameter. In particular, we target the average amount of cash holdings (divided by expenditure) M/e for both type c and type d buyers, which we identify from the answer to the survey question on preferences reported above. The level of these two statistics should be informative about R, the opportunity cost of holding cash. The difference between M_c/e and M_d/e should be informative about κ, since any difference between the two groups must stem from κ being higher than zero: the largest is κ in absolute value, the more the two types of individuals are different. As the other model-implied moments for type d individuals are quite extreme (they never use cash when they can avoid it, while type c agents often use cards even when they have enough cash, to postpone withdrawals[11]), we then use two moments on type c buyers, the average number of withdrawals per year they perform n_c, as well as the share of purchases they pay with cards when having both options available $\widetilde{\gamma}_c$. The moment n_c is informative about the parameter u: the higher the cost of losing a purchase, the more frequently agents will withdraw cash to insure against missed purchase opportunities. For a given level of κ, the moment $\widetilde{\gamma}_c$ is instead informative about α: with a higher α, agents know that they will be able to search less effectively in the future, therefore they might pay using cards more frequently when they have both options, to save cash for future shopping trips in which cards might not be accepted. The fifth moment we target is the effective acceptance rate $\overline{\phi}$ (which we get from the SPACE diary through reported card acceptance in stores visited), which is the demand-side counterpart of ϕ, the average acceptance rate. This moment is informative about η: as search costs fall, agents disproportionately visit stores that accept cards, and $\overline{\phi}$ becomes much higher than the true card acceptance rate ϕ. The last moment we target is $\hat{\lambda}$, the number of completed purchases, which is informative about the arrival rate of purchase opportunities λ.

Results. In 3.2 we display our calibration for the model and how well the calibrated model fits the statistics used in estimation. The model successfully replicates some salient features of the payments market, including patterns of cash management by households. Our parameter estimates for R, κ, and λ are broadly in line with Lippi and Moracci (2024), while we estimate a much smaller value for u. The degree of search frictions is sizeable: to match the data, we need our agents to be unable to search across shops for 72% of their transactions. Estimated search costs, however, are low, i.e., about a fifth of the cost of a cash withdrawal. As for the parameters of the merchant problem, we calibrate a value of t such that the profit obtained from a card transaction is 36% smaller than that yielded from a cash transaction. Finally, the fixed cost of accepting cards accruing to each purchase is around 14% of the profits derived from a cash transaction. We plan to improve our characterization of

11. See Lippi and Moracci (2024) for a more accurate description of payment choices by individuals who prefer using cash.

Table 3.2. Estimated parameters and model fit

Parameter	Calibrated value	
Opportunity cost R	0.05	
Cost using least pref. payment method κ/b	0.73	
Purchase oppurt. per day $\lambda/365$	2.02	
Utility cost lost purchase u/b	47.74	
Search frictions α	0.72	
Search cost $\eta(1-\phi)/b$	0.22	
Acceptance fee t/u	0.36	
Acceptance fixed cost T/u	0.14	
Moment	**Data**	**Model**
Cash balances, M_c/e (prefer card)	1.36	1.24
Cash balances, M_d/e (prefer cash)	1.00	1.09
N. cash withdrawals per year, n_c (prefer cash)	106.51	106.83
Cashless share of expenditure $\widetilde{\gamma}$ (both poss.)	0.25	0.26
Effective acceptance rate $\overline{\phi}$	0.84	0.86
N. purchases per day $\hat{\lambda}/365$	1.93	1.99
Card acceptance rate ϕ	0.80	0.80

the magnitude of each parameter of the seller problem by computing, for instance, the average amount paid in card fees and fixed costs as a proportion of the equilibrium profits of accepting sellers.

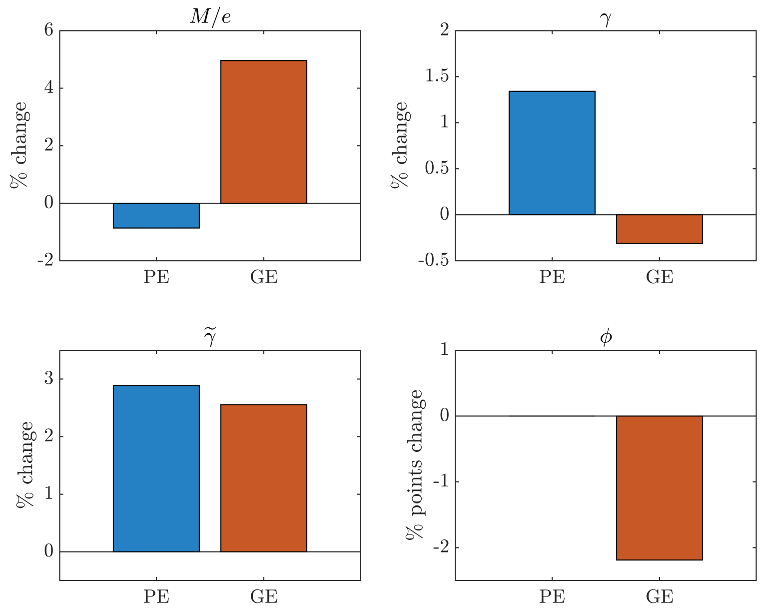
3.6 An application: a subsidy to card usage

We now use our model to analyze the effects of a small subsidy to card usage given to all agents in our economy. Such policies have been implemented several times, with a recent example being the Italian *cashback policy* rolled out in late 2020, to encourage card usage and make the payment system "more efficient, more transparent and traceable", with the additional goal of "recovering the underground economy, discouraging payments 'in the black'"[12]. The policy consisted in a reimbursement up to EUR 150 every six months for consumers that completed at least a minimum amount of card payments (50) in that time window, plus multiple EUR 1,500 prizes for the first 100,000 people that completed the most number of digital payments in the same six-month window. Within the context of our model, this is equivalent to shifting the fixed cost κ down for type c agents, and up for type d agents. We assume that the shift is identical and we parametrize it as follows: the fixed cost of using cards shrinks to $\kappa - \xi$ for type c agents, while the fixed cost of using cash increases to $\kappa + \xi$ for type d agents. Notice that this exercise is similar to an increase in ω, for which we performed a comparative statics exercise in the simpler model of **??**.

We compute the partial and general equilibrium effects[13] of such a subsidy to card usage when $\xi = 0.1\kappa$ (i.e., it is equal to ten percent of the average inconvenience of using cards for type c agents), in terms of changes in average cash held by households, in the share of expenditure settled using cards (both overall and when cards are available), and in the average acceptance rate. As a benchmark, we use our baseline parametrization. By partial equilibrium effects, we mean that we assume that ϕ is fixed and that only buyers' choices can change in response to the increased convenience of electronic payments. Under this assumption, the policy achieves the proposed effects: it lowers the amount of cash held by households by approximately 1%, and it raises the share of expenditure paid for using cards by around 1.5% (3% when focusing on purchases where both options were available). When allowing for an optimal response by sellers to the policy change, though, the average card acceptance rate changes, falling by around 2 percentage points. The

12. Quote by the Italian PM Giuseppe Conte, see the article by *Il Sole 24 ore* here.

13. We abstract from the role of the government in our exercise. In a model with tax evasion where tax revenues are endogenous and depend positively on the share of expenditure settled using cards (following the logic of Immordino and Russo (2017), according to which merchants are more prone to evade taxes when paid in cash), the policy could self-finance itself, but at the same time could still have the same unintended consequences shown here.



The above Figure displays the changes in M/e, γ, $\widetilde{\gamma}$ and in the equilibrium acceptance rate ϕ when introducing a card subsidy of value ξ, both in a partial equilibrium setting (keeping ϕ fixed and simply solving again the buyer's problem), and in a general equilibrium setting in which we allow sellers to respond optimally by adjusting their acceptance policies, possibly affecting ϕ.

Figure 3.6. Partial and general equilibrium effects of a cashback policy

reason is that sellers' internalize that now their customers will use cards more often not only to settle large purchases that they could not pay with cash (as they don't have enough), but also to pay for tiny transactions in order to get the rewards related to the cashback policy. This fall in acceptance more than compensates for the increased benefit of paying with cards, and households increase their average cash balances by around 4% relative to the benchmark, as a result of stronger precautionary motives. The card share of expenditure falls by 0.25%, despite an increase in the intensity of card usage when both options are available by around 2.5% (as a result of the increased convenience). In short, the general equilibrium effect in this example dominates the partial equilibrium one, and the card share of expenditure falls despite agents pay more often with cards when they can, simply because they have this option less often. This exercise provides a backing for our claim that when evaluating policy proposals that affect payment and cash management choices of households (such as cash bans or limits to cash usage), general equilibrium effects that affect the other side of the payments market (sellers) should be taken into account.

3.7 Conclusion

In this paper, we presented an equilibrium model of the payments market in which buyers and sellers interact to settle purchases. Buyers decide how much cash to hold and how to pay for goods and services bought (between cash and a non-cash option such as a payment card), while sellers choose whether to accept payment cards or not. Agents can search for shops that accept their preferred payment methods, even though we allow for frictions that make search imperfect and generate market power for sellers. We showed that our model may feature either strategic complementarity or substitutability in acceptance decisions, depending on the overall level of acceptance, and the existence of strategic substitutability is a requirement to obtain imperfect acceptance equilibria such as the ones we observe in the data. We outlined existence and uniqueness results for the equilibrium of our model economy, and performed comparative statics exercises showing analytically the equilibrium responses to changes in the cost of holding cash, in the extent of search frictions, and in preferences for card versus cash payments. We presented an extended version of our model that we can bring to the data, matching aggregate statistics from 2021-22 payment diaries for the Euro Area. Finally, we used our calibrated model to perform a policy exercise, computing the partial and general equilibrium effects of a subsidy to card usage, and showing that policies intended to boost card usage may have unintended effects if one neglects the equilibrium response of sellers. Our results show that payments acceptance decisions by sellers (and the related strategic interactions) need to be included in models of cash management and payment choices, if one is interested in computing the equilibrium effects of policy changes that directly or indirectly affect merchants' incentives to accept card payments.

Appendix 3.A Proofs

3.A.1 Proof of 3.2

We start from agents who prefer to pay using cash. When $\phi > 0$, we have that

$$\begin{aligned}\frac{\partial V^c(m,\phi)}{\partial m} &= -R + \alpha\phi f(m)u - \alpha\phi f(m)(u-\kappa) + \alpha(1-\phi)f(m)u + (1-\alpha)f(m)u - (1-\alpha)f(m)(u-\kappa)\\ &= -R + pf(m)u - \alpha\phi f(m)(u-\kappa) + (1-\alpha)f(m)u - (1-\alpha)f(m)(u-\kappa) =\\ &= -R + f(m)u - \alpha\phi f(m)(u-\kappa) - (1-\alpha)f(m)(u-\kappa).\end{aligned}$$

The first-order condition yields

$$f(m) = \frac{R}{u-(u-\kappa)\big(\alpha\phi + (1-\alpha)\big)}.$$

As we assume that $f'(s) < 0$ for all $s \in [0, +\infty]$, this equation either has one solution (if $f(0) > \frac{R}{u-(u-\kappa)(\alpha\phi+(1-\alpha))}$) or zero solutions. When the condition does not hold, it is optimal to hold zero cash. Notice, however, that this cannot happen as we assume that $f(0) > R/\kappa$ (see (A2) in Assumption 4), and $\frac{R}{u-(u-\kappa)(\alpha\phi+(1-\alpha))} < \frac{R}{\kappa}$. When everybody accepts cards ($\phi = 1$), the solution is given by $f(m) = R/\kappa$. When $\phi = 0$, we similarly get that the solution is given by $f(m) = R/u$. Notice that the money demand function is discontinuous at $\phi = 0$, as $\lim_{\phi\to 0^+} m_c^*(\phi) = f^{-1}\left(\frac{R}{u-(u-\kappa)(1-\alpha)}\right) \neq m_c^*(0)$.

Moving to agents who prefer to pay with cards, when $\phi > 0$, we have that

$$\frac{\partial V^d(m,\phi)}{\partial m} = -R + \alpha(1-\phi)f(m)(u-\kappa).\cdot$$

The equation has one solution if $f(0) > \frac{R}{\alpha(1-\phi)(u-\kappa)}$, while in the opposite case agents will hold zero cash. This is not ruled out by assumptions as $\frac{R}{\alpha(1-\phi)(u-\kappa)}$ might be larger than $\max\{R/\kappa, R/(u-\kappa)\}$.

3.A.2 Proof of 3.5

In the nonatomic acceptance game, the function that yields the difference in payoffs between accepting and not accepting cards as a function of the average card acceptance rate ϕ is

$$\begin{aligned}\Delta(\phi) &= \lim_{N\to+\infty} \Delta_i^N(\phi) =\\ &= (1-\omega)[F(m_c^*(\phi)) - F(m_c^*(\phi))]u + (1-\omega)(1-F(m_c^*(\phi)))(u-t)(\alpha + (1-\alpha)/\phi)\\ &\quad + \omega((u-t)(\alpha + (1-\alpha)/\phi) - \alpha F(m_d^*(\phi))u)\\ &= (1-\omega)(1-F(m_c^*(\phi)))(u-t)(\alpha + (1-\alpha)/\phi)\\ &\quad + \omega((u-t)(\alpha + (1-\alpha)/\phi) - \alpha F(m_d^*(\phi))u).\end{aligned}$$

Now, let the distribution of payment sizes s be exponential, i.e., let $F(s) = 1 - e^{-\lambda s}$. from (3.9) and (3.10) we get

$$\begin{aligned}\Delta(\phi) \;=\; & (1-\omega)\frac{1}{\lambda}\frac{R}{\alpha\phi\kappa + \alpha(1-\phi)u + (1-\alpha)\kappa}(u-t)\left(\alpha + \frac{1-\alpha}{\phi}\right) \\ & + \omega(u-t)\left(\alpha + \frac{1-\alpha}{\phi}\right) - \omega\alpha u\left(1 - \frac{1}{\lambda}\frac{R}{\alpha(1-\phi)(u-\kappa)}\right)\end{aligned}$$

for $0 < \phi \leq \hat{\phi}$, where $\hat{\phi} = 1 - \frac{R}{\lambda\alpha(u-\kappa)}$ as defined above and

$$\begin{aligned}\Delta(\phi) \;=\; & (1-\omega)\frac{1}{\lambda}\frac{R}{\alpha\phi\kappa + \alpha(1-\phi)u + (1-\alpha)\kappa}(u-t)\left(\alpha + \frac{1-\alpha}{\phi}\right) \\ & + \omega(u-t)\left(\alpha + \frac{1-\alpha}{\phi}\right)\end{aligned}$$

for $\phi > \hat{\phi}$. Now denote the first branch $\widetilde{\Delta}(\phi)$ and the second branch $\overline{\Delta}(\phi)$. We study these two functions separately, and each in the domain $(0,1)$.
First, observe $\lim_{\phi\to 0^+}\widetilde{\Delta}(\phi) = +\infty$ and $\lim_{\phi\to 1^-}\widetilde{\Delta}(\phi) = +\infty$. Next, we check how many solutions the equation $\widetilde{\Delta}(\phi) = 0$ can have in the interval $(0,1)$. Note that

$$\begin{aligned}\widetilde{\Delta}(\phi) \;=\; 0 \;\Rightarrow\; & (1-\omega)\tfrac{1}{\theta}R(u-t)(\alpha\phi + (1-\alpha))(\alpha(1-\phi)(u-\kappa)) \\ & +\omega(u-t)(\alpha\phi + (1-\alpha))(\alpha(1-\phi)(u-\kappa))(\alpha\phi\kappa + \alpha(1-\phi)u + (1-\alpha)\kappa) \\ & -\omega\alpha u\phi(\alpha\phi\kappa + \alpha(1-\phi)u + (1-\alpha)\kappa)(\alpha(1-\phi)(u-\kappa)) \\ & +\omega\alpha\tfrac{1}{\theta}R\phi(\alpha\phi\kappa + \alpha(1-\phi)u + (1-\alpha)\kappa) \;=\; 0,\end{aligned} \tag{3.A.1}$$

i.e., that any solution to $\widetilde{\Delta}(\phi) = 0$ is also a solution to the equation on the right, which is a cubic equation and hence cannot have more than three solutions in the interval $(0,1)$. Therefore, we conclude that $\widetilde{\Delta}(\phi) = 0$ has at most three solutions in the $(0,1)$. Suppose it has exactly three solutions in $(0,1)$. Call them $\phi_1 < \phi_2 < \phi_3$. Since we are considering generic parameters only, we can assume that each solution $\phi^* \in \{\phi_1, \phi_2, \phi_3\}$ is such that

(1) either $\widetilde{\Delta}(\phi^* - \varepsilon) < 0 \wedge \widetilde{\Delta}(\phi^* + \varepsilon) > 0$,

(2) or $\widetilde{\Delta}(\phi^* - \varepsilon) > 0 \wedge \widetilde{\Delta}(\phi^* + \varepsilon) < 0$,

for sufficiently small $\varepsilon > 0$. Since $\lim_{\phi\to 0^+}\widetilde{\Delta}(\phi) = +\infty$, we have that for ϕ_1 point (2) must hold. Then, by continuity, ϕ_2 is such that (1) holds and, finally, ϕ_3 is such that (2) holds again. Since (2) holds at ϕ_3, there exists $\phi' > \phi_3$ such that $\widetilde{\Delta}(\phi') < 0$. Yet, since $\lim_{\phi\to 1^-}\widetilde{\Delta}(\phi) = +\infty$ there exists $\phi'' > \phi'$ such that $\widetilde{\Delta}(\phi'') > 0$, and by continuity $\exists\phi_4 \in (\phi', \phi'')$ with $\widetilde{\Delta}(\phi_4) = 0$, a contradiction since $\{\phi_1, \phi_2, \phi_3\}$ includes all the solutions to $\widetilde{\Delta}(\phi) = 0$. We just proved that $\widetilde{\Delta}(\phi) = 0$ cannot have three solutions.
We can apply the same logic to $\widetilde{\Delta}(\phi) = 0$ having only one solution. Therefore, $\widetilde{\Delta}(\phi)$ can either have two solutions or zero. We analyze both cases individually.

(*i*) Suppose the function has two solutions in $(0,1)$. Observe, moreover, that $\overline{\Delta}(\phi) > 0$ for each $\phi \in (0,1)$. Since Δ is continuous, we must have $\widetilde{\Delta}(\hat{\phi}) > 0$. Hence, $\Delta(\phi) = 0$ has, in this case, either two solutions (the solutions of $\widetilde{\Delta}(\phi) = 0$) or zero solutions (when $\hat{\phi} < \phi_1$). Notice that the case $\phi_1 < \hat{\phi} < \phi_2$ is ruled out by the fact that $\widetilde{\Delta}(\hat{\phi}) > 0$.

(*ii*) Suppose it has zero solutions. Then $\widetilde{\Delta}(\phi) > 0$ for all $\phi \in (0,1)$. Since $\overline{\Delta}(\phi) > 0$ as well, it must be the case that $\Delta(\phi) = 0$ has no solutions in the interval.

We conclude that $\Delta(\phi) = 0$ has either zero or two solutions in the $(0,1)$ interval. If it has two solutions, they are both equilibria of the nonatomic acceptance game, denoted by (ϕ_l^*, ϕ_h^*). Moreover, given that $\overline{\Delta}(1) > 0$, we have that $\phi^* = 1$ is always an equilibrium of the nonatomic acceptance game.

3.A.3 Proof of 3.7

The derivative of the function Δ_i with respect to α is given by

$$\begin{aligned}
\frac{\partial \Delta_i(\phi_{-i})}{\partial \alpha} &= (1-\omega)\left(\frac{\partial F\left(m_c^*\left(\phi_{-i}+\frac{1}{N}\right)\right)}{\partial \alpha} - \frac{\partial F\left(m_c^*\left(\phi_{-i}\right)\right)}{\partial \alpha}\right) \\
&\quad + (1-\omega)\left(1 - F\left(m_c^*\left(\phi_{-i}+\frac{1}{N}\right)\right)\right)(u-t)\left(1-\frac{N}{\phi_{-i}N+1}\right) \\
&\quad - (1-\omega)(u-t)\left(\alpha + (1-\alpha)\frac{N}{\phi_{-i}N+1}\right)\frac{\partial F\left(m_c^*\left(\phi_{-i}+\frac{1}{N}\right)\right)}{\partial \alpha} \\
&\quad + \omega(u-t)\left(1-\frac{N}{\phi_{-i}N+1}\right) - \omega F\left(m_d^*\left(\phi_{-i}\right)\right)u - \omega\alpha\frac{\partial F\left(m_d^*\left(\phi_{-i}\right)\right)}{\partial \alpha}u.
\end{aligned}$$

First, notice that given that $1-\frac{N}{\phi_{-i}N+1} \leq 0$, $\frac{\partial F\left(m_c^*(\phi_{-i})\right)}{\partial \alpha} > 0$ and $\frac{\partial F\left(m_d^*(\phi_{-i})\right)}{\partial \alpha} > 0$, all terms except the first one are unambiguously negative. Hence, given that for large enough N, $\left(\frac{\partial F\left(m_c^*(\phi_{-i}+\frac{1}{N})\right)}{\partial \alpha} - \frac{\partial F\left(m_c^*(\phi_{-i})\right)}{\partial \alpha}\right) \approx 0$, we can say that there exists $N' \in \mathbb{N}$ such that, for all $N \geq N'$, $\partial \Delta_i\left(\phi_{-i}\right)/\partial \alpha < 0$ for all ϕ_{-i}. The desired result on ϕ^* follows naturally.

3.A.4 Proof of 3.8

For $\phi_{-i} > 0$ we have that

$$\begin{aligned}
\frac{\partial \Delta_i(\phi_{-i})}{\partial \omega} &= u\left(F(m_c^*(\phi_{-i})) - pF(m_d^*(\phi_{-i}))\right) \\
&\quad + F\left(m_c^*\left(\phi_{-i}+\frac{1}{N}\right)\right)\left[(u-t)\left(\alpha+(1-\alpha)\frac{N}{\phi_{-i}N+1}\right) - u\right].
\end{aligned}$$

The first term is larger than zero, as individuals who prefer cash hold more of it than agents who prefer cards, for the same acceptance rate. The second term, however, is smaller than zero, since $u-t < u$ and given that $\alpha/N + (1-\alpha)/(\phi_{-i}N+1)$ is smaller than one.

For $\alpha = 0$, the expression becomes

$$\begin{aligned}\frac{\partial \Delta_i(\phi_{-i})}{\partial \omega} =& u\left(F(m_c^*(\phi_{-i})) - F\left(m_c^*\left(\phi_{-i} + \frac{1}{N}\right)\right)\right) \\ &+ F\left(m_c^*\left(\phi_{-i} + \frac{1}{N}\right)\right)\left[(u-t)\left(\frac{N}{\phi_{-i}N+1}\right)\right].\end{aligned}$$

The first term is positive as money demand decreases in ϕ and F is a cdf. The second term is also positive whenever $t < u$. Hence, $\partial \Delta_i(\phi_{-i})/\partial \omega > 0$ when $\alpha = 0$, for any value of $t < u$.

For $\alpha = 1$, the expression is instead given by

$$\frac{\partial \Delta_i(\phi_{-i})}{\partial \omega} = u\left(F(m_c^*(\phi_{-i})) - F\left(m_d^*(\phi_{-i})\right)\right) - tF\left(m_c^*\left(\phi_{-i} + \frac{1}{N}\right)\right).$$

As before, the first term is positive. The second term, however, is now negative. If t is sufficiently close to u, we have that $\partial \Delta_i(\phi_{-i})/\partial \omega < 0$. Hence, $\partial \Delta_i(\phi_{-i})/\partial \omega$ is of ambiguous sign. However, notice that

$$\begin{aligned}\Delta_i\left(\phi_{-i}\right) =& (1-\omega)\left(F\left(m_c^*\left(\phi_{-i} + \frac{1}{N}\right)\right) - F\left(m_c^*\left(\phi_{-i}\right)\right)\right)u \\ &+ (1-\omega)\left(1 - F\left(m_c^*\left(\phi_{-i} + \frac{1}{N}\right)\right)\right)(u-t)\left(\alpha + (1-\alpha)\frac{N}{\phi_{-i}N+1}\right) \\ &+ \omega(u-t)\left(\alpha + (1-\alpha)\frac{N}{\phi_{-i}N+1}\right) - \omega\alpha F(m_d^*(\phi_{-i}))u \\ =& \omega\left(F\left(m_c^*\left(\phi_{-i}\right)\right) - pF(m_d^*(\phi_{-i}))\right)u \\ &+ \omega F\left(m_c^*\left(\phi_{-i} + \frac{1}{N}\right)\right)\left[(u-t)\left(\alpha + (1-\alpha)\frac{N}{\phi_{-i}N+1}\right) - u\right] \\ &+ \left(1 - F\left(m_c^*\left(\phi_{-i} + \frac{1}{N}\right)\right)\right)(u-t)\left(\alpha + (1-\alpha)\frac{N}{\phi_{-i}N+1}\right) \\ &+ \left(F\left(m_c^*\left(\phi_{-i} + \frac{1}{N}\right)\right) - F\left(m_c^*\left(\phi_{-i}\right)\right)\right)u \\ =& \omega\frac{\partial \Delta_i(\phi_{-i})}{\partial \omega} + \left(1 - F\left(m_c^*\left(\phi_{-i} + \frac{1}{N}\right)\right)\right)(u-t)\left(\alpha + (1-\alpha)\frac{N}{\phi_{-i}N+1}\right) \\ &+ \left(F\left(m_c^*\left(\phi_{-i} + \frac{1}{N}\right)\right) - F\left(m_c^*\left(\phi_{-i}\right)\right)\right)u.\end{aligned} \tag{3.A.2}$$

The second term is positive and the third term tends to zero as $N \to +\infty$. Therefore, there exists $N' \in \mathbb{N}$ such that, for all $N \geq N'$, $\Delta_i\left(\phi_{-i}\right)$ is positive whenever $\partial \Delta_i\left(\phi_{-i}\right)/\partial \omega$ is positive. We can also rewrite the above as

$$\begin{aligned}\frac{\partial \Delta_i(\phi_{-i})}{\partial \omega} =& \frac{\Delta_i\left(\phi_{-i}\right)}{\omega} \\ &- \frac{1}{\omega}\left(1 - F\left(m_c^*\left(\phi_{-i} + \frac{1}{N}\right)\right)\right)(u-t)\left(\alpha + (1-\alpha)\frac{N}{\phi_{-i}N+1}\right) \\ &- \frac{1}{\omega}\left(F\left(m_c^*\left(\phi_{-i} + \frac{1}{N}\right)\right) - F\left(m_c^*\left(\phi_{-i}\right)\right)\right)u.\end{aligned}$$

Recall that if ϕ^* is an IAE, (3.17) implies that $\Delta_i(\phi^*) < 0$. Hence, we obtain that there exists $N' \in \mathbb{N}$ such that, for all $N \geq N'$, $\partial \Delta_i\left(\phi^*\right)/\partial \omega < 0$. To understand how the equilibrium changes when ω rises, we need to understand what is the new level ϕ' for which (3.16) and (3.17) are satisfied. Notice that for ϕ^* (3.17) is still satisfied, therefore it cannot be that the new imperfect acceptance equilibrium $\phi' \in \{\phi^* + 1/N, \ldots, 1\}$. If there exists $\phi' \in \{0, 1/N, \ldots \phi^* - 1/N\}$ (with $\Delta_i(\phi') \geq 0$ such that $\Delta_i(\phi') - \frac{\partial \Delta_i(\phi')}{\partial \omega} < 0$, that ϕ' satisfies both (3.16) and (3.17) and it is the new equilibrium. We just showed that if ϕ^* is an IAE, then $\partial \phi^*/\partial \omega \leq 0$. As for $\partial m^*/\partial \omega$, observe that

$$m^*(\phi^*(\omega)) = \omega m_d^*(\phi^*(\omega)) + (1-\omega) m_c^*(\phi^*(\omega)),$$

where we write $\phi^*(\omega)$ to make the dependence of the equilibrium acceptance rate on ω explicit. Therefore,

$$\begin{aligned}\frac{\partial m^*(\phi^*(\omega))}{\partial \omega} &= m_d^*(\phi^*(\omega)) + \omega \frac{\partial m_d^*(\phi^*)}{\partial \phi^*}\frac{\partial \phi^*}{\partial \omega} - m_c^*(\phi^*(\omega)) + (1-\omega)\frac{\partial m_c^*(\phi^*)}{\partial \phi^*}\frac{\partial \phi^*}{\partial \omega} \\ &= \underbrace{\left(m_d^*(\phi^*(\omega)) - m_c^*(\phi^*(\omega))\right)}_{<0} + \underbrace{\omega \frac{\partial m_d^*(\phi^*)}{\partial \phi^*}\frac{\partial \phi^*}{\partial \omega}}_{>0} + \underbrace{(1-\omega)\frac{\partial m_c^*(\phi^*)}{\partial \phi^*}\frac{\partial \phi^*}{\partial \omega}}_{>0}.\end{aligned}$$

3.A.5 Proof of 3.9

For $\phi_{-i} > 0$ we have that

$$\begin{aligned}\frac{\partial \Delta_i(\phi_{-i})}{\partial R} &= (1-\omega)\left(\frac{\partial F\left(m_c^*\left(\phi_{-i} + \frac{1}{N}\right)\right)}{\partial R} - \frac{\partial F\left(m_c^*\left(\phi_{-i}\right)\right)}{\partial R}\right) u \\ &\quad - (1-\omega)(u-t)\left(\alpha + (1-\alpha)\frac{N}{\phi_{-i}N + 1}\right)\frac{\partial F\left(m_c^*\left(\phi_{-i} + \frac{1}{N}\right)\right)}{\partial R} \\ &\quad - \omega\alpha \frac{\partial F\left(m_d^*\left(\phi_{-i}\right)\right)}{\partial R} u,\end{aligned}$$

First, notice that given that $\frac{\partial F(m_c^*(\phi_{-i}))}{\partial R} < 0$ and $\frac{\partial F(m_d^*(\phi_{-i}))}{\partial R} < 0$, all terms except the first one are unambiguously positive. Hence, given that for large enough N, $\left(\frac{\partial F(m_c^*(\phi_{-i} + \frac{1}{N}))}{\partial R} - \frac{\partial F(m_c^*(\phi_{-i}))}{\partial R}\right) \approx 0$, we can say that there exists $N' \in \mathbb{N}$ such that, for all $N \geq N'$, $\partial \Delta_i\left(\phi_{-i}\right)/\partial R > 0$ for all ϕ_{-i}. The desired result on ϕ^* follows naturally.

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