On the Conformal Anomaly

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Abstract

The conformal anomaly is the distinguishing feature of a conformal field theory as a two-dimensional quantum field theory. Recently, it also appears in conformally covariant random geometry. This thesis studies the conformal anomaly mathematically as a real determinant line bundle over infinite-dimensional moduli spaces of Riemann surfaces with analytically parametrized boundary components, summarizing three works of the author on this topic. As an introduction, the conformal anomaly is presented in the context of mathematical physics and probability theory, including a detailed breakdown of the relevant geometry of sewing operations involving said Riemann surfaces.

The main results of the contained works are, respectively, the derivation of the Virasoro algebra from the conformal anomaly, generalizations of loop Loewner energy for Schramm–Loewner evolution, and a universal property for real one-dimensional modular functors. The latter is an abstraction of the real determinant line bundle inspired by Segal's definitions in the complex case, to which assumptions of locality and modular invariance are added. The main tools developed are results on complex deformations of the unit circle, which come with a local composition law integrating the Lie algebra of complex-valued vector fields on the unit circle. Since complex deformations act on the moduli spaces by deformation of boundary components, the real determinant line bundle pulls back to a central extension of the Lie algebra, which facilitates the algebraic study of the conformal anomaly.

I dedicate this thesis to the artists whose music complements my mathematical thinking in mysterious ways.

Acknowledgements

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Contents

1 Introduction **1**

1.1 Anomalies in quantum field theory . . . 2

1.2 The conformal anomaly . . . 6

1.2.1 Weyl transformations . . . 6

1.2.2 Conformal field theory . . . 9

1.2.3 Conformal covariance in random geometry . . . 10

1.3 Computing with Riemann surfaces . . . 11

1.3.1 Moduli spaces . . . 15

1.3.2 Complex deformations of the unit circle . . . 19

1.4 The real determinant line bundle . . . 23

1.4.1 Construction . . . 24

1.4.2 Central extensions of complex deformations . . . 27

1.4.3 The functorial perspective . . . 29

2 Summary of results **37**

2.1 From the conformal anomaly to the Virasoro algebra . . . 37

2.1.1 Interlude: Bypassing the stress-energy tensor construction 40

2.2 Two-loop Loewner potentials . . . 42

2.3 Universality of the conformal anomaly . . . 44

Bibliography **49**

A Publication: From the conformal anomaly to the Virasoro algebra **57**

B Publication: Two-loop Loewner potentials **59**

C Manuscript: Universality of the conformal anomaly **61**

Chapter 1

Introduction

Very specifically, *conformal anomaly* refers to the following functional:

$$S_L^0(\sigma, g) = \frac{1}{12\pi} \iint_\Sigma \left(\frac{1}{2}|\nabla_g \sigma|_g^2 + R_g \sigma\right) \mathrm{d}V_g + \frac{1}{12\pi} \int_{\partial\Sigma} k_g \sigma \, \mathrm{d}\ell_g. \qquad (1.0.1)$$

It is defined for $\sigma \in C^\infty(\Sigma, \mathbb{R})$, where (Σ, g) is a compact surface with a smooth Riemannian metric g and possibly with boundary. In the equation, ∇_g, R_g, and k_g are, respectively, the divergence, the Gaussian curvature, and the boundary curvature with respect to the metric g.

In the larger scheme of conformal field theory (CFT), the conformal anomaly may also be called *Weyl anomaly* or *trace anomaly*, and it is an integral part of the theory. As the main subject of this thesis, I relate several perspectives on the conformal anomaly: The formula above, the Virasoro algebra, and the real determinant line bundle — axiomatized as a real one-dimensional modular functor. I also relate the conformal anomaly to recent developments in conformally invariant random geometry. This thesis does not cover the perspective of chiral CFT, related to the complex determinant line bundle [Seg04, Hua97, Hen24].

This thesis presents an overview of two articles and one manuscript,

[MP25a] Sid Maibach and Eveliina Peltola. "From the conformal anomaly to the Virasoro algebra". In: *Proceedings of the London Mathematical Society* 130.4 (2025), e70040.
DOI: 10.1112/plms.70040,

[LM25] Yan Luo and Sid Maibach. "Two-Loop Loewner Potentials". In: *International Mathematics Research Notices* 2025.11 (2025).
DOI: 10.1093/imrn/rnaf133,

[MP25b] Sid Maibach and Eveliina Peltola. "Universality of the conformal anomaly". 2025.

Chapter 1 is an introduction the general topic from the perspectives of theoretical physics and random geometry. I also discuss the main objects of the included works: Weyl transformations, sewing operations on moduli spaces of Riemann surfaces, and the real determinant line bundle. This provides a common setup for the results in the included works summarized in Chapter 2, highlighting the contributions of the author.

1.1 Anomalies in quantum field theory

I would like to start with a short introduction to my work from the perspective of theoretical physics. For details, see the physics works [Fra99, Pes19, Mus20, Tal22, EI23], and the mathematical introductions [Gaw99, Sch08, GKR24].

The central objects of most classical, that is, non-quantum, physics are the equations of motion. In the Lagrangian formalism, these are derived from a variational problem of an action functional, which is a real-valued function $S : \mathcal{E} \to \mathbb{R}$ on a state space $\mathcal{E}$ which is a set of functions, or more generally, a set of sections of a bundle over space-time — a smooth (pseudo) Riemannian manifold (Σ, g). Many theories of quantum physics may be viewed as the "quantization" of their classical counterparts. In quantum mechanics, this procedure is called canonical quantization, and it yields a Schrödinger equation, whose solutions form the Hilbert space of quantum states of the system. Carrying forward from here, the transition from quantum mechanics to a quantum field theory is a further construction, called second quantization.

Other approaches, such as the path integral formalism, attempt to define a quantum field theory directly from the classical action functional. In this thesis, we mainly take this perspective, since we are motivated by the quantum physics arising from statistical mechanics, and since it has recently been established in mathematics for a particular non-free quantum field theory called Liouville conformal field theory. Indeed, by the end of this section, I specialize to two-dimensional Euclidean conformal field theory, which is the setting for the conformal anomaly.

Here, we assume already that the theory is Euclidean, that is, it does not have a time component, such that (Σ, g) is just a Riemannian manifold. The Euclidean path integral approach to "quantization" departs from the idea of finding an exact minimizer of the action functional S. Instead, a measure on the state space $\mathcal{E}$ is defined such that more weight is given to states $\varphi \in \mathcal{E}$ whose action $S(\varphi)$ is closer to the minimizer. With respect to this measure, one is interested in the moments and correlations of observables — such as the value of the field or its derivatives at a certain point. If $F : \mathcal{E} \to \mathbb{R}$ is a product of such observables, then the path integral takes the form

$$\text{“}\, \langle F \rangle_g = \int_{\mathcal{E}} F(\varphi)\, e^{-\frac{1}{\hbar} S(\varphi)}\, \mathcal{D}_g(\varphi) \,\text{”}, \tag{1.1.1}$$

where $\mathcal{D}_g(\varphi)$ is supposed to be a "uniform" measure on the state space $\mathcal{E}$, and $\hbar$ is the Planck constant. To specify what is meant by the latter, one usually analyzes symmetries of the classical system. For example, if the classical action functional is invariant under translations in Σ, it makes sense to require the same translation symmetry for $\mathcal{D}_g(\varphi)$, which in finite dimensions characterizes the Lebesgue measure. Usually, a measure with all the required properties does not exist. It is the quantization of the classical symmetries and controlled breaking of their constraints on the choice of the measure $\mathcal{D}_g(\varphi)$ that leads to *anomalies*. Before addressing anomalies in more detail, I would like to illustrate the path integral approach with an analogy in probability theory.

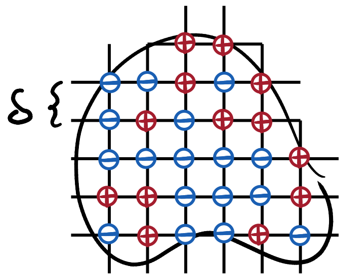


Figure 1.1: Discretization of (Σ, g) with lattice constant δ. The discretized field assigns values $\varphi(x)$ to vertices $x \in \Sigma_\delta$. Displayed are values ± 1 like in the Ising model [CHI15].

Example 1.1.1 (Gibbs measures)**.** In probability theory, or statistical mechanics, one often considers a discretization Σ_δ of compact Σ, that is, a finite graph. The parameter $\delta > 0$ in Σ_δ, for instance a lattice constant as in Figure 1.1, is chosen such that Σ is recovered as $\delta \to 0$. The state space $\mathcal{E}_\delta$ comes with a product measure of a probability measure μ_E on a fixed target space E, which is put, for example, at each of the finitely many vertices $x \in \Sigma_\delta$. In this setup, an analogue of the Euclidean path integral is well-defined, and is called a *Gibbs measure*, with (unnormalized) expectations

$$\langle F \rangle_\delta = \int_{\mathcal{E}_\delta} F(\varphi)\, e^{-\beta S_\delta(\varphi)} \prod_{x \in \Sigma_\delta} \mathrm{d}\mu_E(\varphi(x)) \tag{1.1.2}$$

of functions F on $\mathcal{E}_\delta$. Here, the inverse $\beta = \frac{1}{T}$ of the temperature $T \geq 0$ of the system plays the role of the factor $\frac{1}{\hbar}$. Possibly, the measure μ_E has discrete support, such as in the case of the Ising model, where $\varphi(x)$ takes values ± 1, in which case the integral becomes a sum. The discretized action functional S_δ usually needs a specific dependence on δ in order that a continuum theory can be obtained in the limit $\delta \to 0$. A probability measure is obtained by considering indicator functions $F = \mathbb{1}_A$ for events $A \subset \mathcal{E}$,

$$\mathbb{P}_\delta(A) = \frac{\langle \mathbb{1}_A \rangle_\delta}{Z_\delta}. \tag{1.1.3}$$

The normalization factor $Z_\delta = \langle 1 \rangle_\delta$ is called the *partition function* of the system. Its continuous analogue takes a central role in this work, and keeping a discrete approximation in mind often leads to useful heuristics.

Returning to the question of symmetry, and the "uniform" measure $\mathcal{D}_g(\varphi)$, we observe that in the discrete version (1.1.2), the product measure is maximally symmetric with respect to the graph, that is, symmetric under all permutations over vertices. There may, however, be additional symmetries of the action S with respect to the value of the fields φ called gauge symmetries, also to be reflected by the measure. If the expectation values (1.1.2) are to converge as $\delta \to 0$, further dependence on δ might need to be introduced not just to $S_\delta \to S$

but also to the measure, or to the observables F. The consequence is that the "uniform" measures $\mathcal{D}_g(\varphi)$ in the path integrals (1.1.1) might not be able to preserve all the symmetries of the action functional. It is a general observation in physics that in every way of quantization, there is the possibility that a symmetry of the classical physics is not quite preserved in the quantum physics. Such mild forms of symmetry breaking in quantum physics are called *anomalies*, and they take many forms — see [Pes19, Chapter 19] for an overview.

Example 1.1.2 (Projective representations of Lie algebras)**.** One of the mathematical notions related to anomalies, prevalent in the canonical quantization in quantum mechanics, is that of projective representations of Lie algebras and their correspondence to Lie algebra central extensions. A continuous (Lie group) symmetry in classical mechanics leads to a (usually unitary) representation of the Lie algebra $\mathfrak{g}$ on the Hilbert space $\mathcal{H}$. This representation $\rho : \mathfrak{g} \to \mathfrak{gl}(\mathcal{H})$, however, might be a projective representation, that is, ρ is a Lie algebra homomorphism only up to a cocycle Ω with values in the center of $\mathfrak{gl}(\mathcal{H})$,

$$\rho([X,Y]) = [\rho(X), \rho(Y)] + \Omega(X,Y), \qquad X, Y \in \mathfrak{g}. \tag{1.1.4}$$

In the case of finite-dimensional unitary representations of finite-dimensional Lie algebras, projective representations may always be lifted to actual representations of a Lie algebra central extension of $\mathfrak{g}$ by the cocycle Ω, see [Hal13, Proposition 16.46]. More generally, a central extension of $\mathfrak{g}$ by an abelian Lie algebra $\mathfrak{h}$ is a Lie algebra $\hat{\mathfrak{g}}$ and an exact sequence

$$\{0\} \longrightarrow \mathfrak{h} \longrightarrow \hat{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow \{0\}, \tag{1.1.5}$$

such that the kernel of the map $\hat{\mathfrak{g}} \to \mathfrak{g}$ is contained in the center of $\hat{\mathfrak{g}}$. Central extensions, in turn, are in one-to-one correspondence to the Lie algebra cohomology $H^2(\mathfrak{g}, \mathfrak{h})$. The cocycle Ω above is obtained by identifying the center of $\mathfrak{gl}(\mathcal{H})$ with $\mathfrak{h}$. In this work, we consider the case $\mathfrak{h} = \mathbb{R}$, which corresponds to the cocycle acting by multiples of the identity $\mathbb{1}_{\mathcal{H}}$ in Equation (1.1.4). Note that for finite-dimensional simple Lie algebras, we have $H^2(\mathfrak{g}, \mathbb{R}) = 0$. In CFT, however, we are confronted with an infinite-dimensional Lie algebra, which is the Witt algebra introduced in Section 1.3.2, and its unique central extension by $\mathbb{R}$ which is the Virasoro algebra.

In mathematical physics, another approach to quantum physics abstracts the properties of the path integral to a category-theoretical framework by working with a family of manifolds instead of the fixed space (Σ, g). The family of action functionals $S_{\Sigma,g}(\varphi)$ additionally depends on the space, and the fields $\varphi \in \mathcal{E}(\Sigma)$ are defined on the respective space (e.g. as a sheaf). One usually considers a *local* family of action functionals, that is, with the restriction property for certain $U \subset \Sigma$,

$$S_{\Sigma,g}(\varphi) = S_{U,\, g|_U}(\varphi|_U) + S_{\Sigma\setminus U,\, g|_{\Sigma\setminus U}}(\varphi|_{\Sigma\setminus U}). \tag{1.1.6}$$

Then, the path integral (1.1.1) has interesting properties with respect to decomposition of (Σ, g) into regular submanifolds with boundary — leading to an axiomatization in terms of a functor from a cobordism category to a category

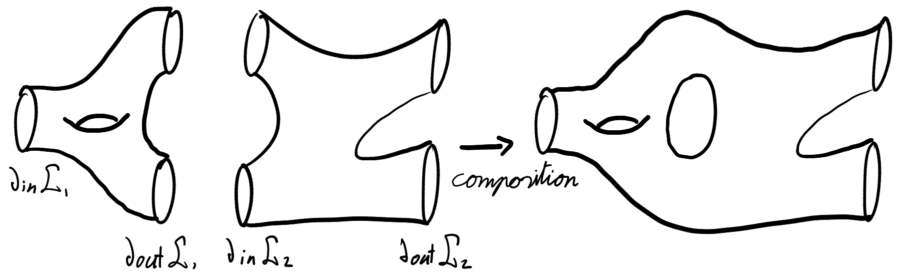


Figure 1.2: Two morphisms in a category of two-dimensional cobordisms are composed by gluing their in and out boundary components.

of vector spaces, ideally Hilbert spaces. Since the introduction by Atiyah and Witten for topological quantum field theory [Ati88, Wit91], by Segal for CFT [Seg04], and also by Kontsevich, there have been many developments in this direction, and we only sketch the most basic idea.

A category of cobordisms of dimension d has manifolds of dimension $d-1$ as objects. If two objects are boundary components of a d-dimensional manifold Σ, separated into two components $\partial_{\mathrm{in}}\Sigma$ and $\partial_{\mathrm{out}}\Sigma$, then the manifold Σ is a morphism from $\partial_{\mathrm{in}}\Sigma$ to $\partial_{\mathrm{out}}\Sigma$ in the cobordism category. Depending on the physical theory under consideration, the precise manifold structure can differ, be it smooth or Riemannian manifolds, eventually with extra structure, or as in the case of two-dimensional CFT, Riemann surfaces. A composition of morphisms is defined by "gluing" two manifolds along their respective incoming and outgoing boundaries; see Figure 1.2. If the cobordism category is concrete enough, this can be achieved by pointwise identification of the boundary components. In Section 1.3, we consider an alternative "sewing operation" in the case of Riemann surfaces up to isomorphisms, where boundary components are identified via parametrizations of the boundary components by certain reference manifolds — additional information that is part of every morphism.

The anticipated functor maps the objects, or boundaries, to vector spaces of boundary conditions $\varphi|_{\partial\Sigma}$ of the fields, and cobordisms to linear maps between the "in" and "out" vector spaces, where the path integral (1.1.1) with given boundary conditions is the matrix element for the operator,

$$\mathcal{A}(\partial\Sigma) = \mathcal{H}_{\partial\Sigma}, \qquad \mathcal{A}(\Sigma) : \mathcal{H}_{\partial_{\mathrm{in}}\Sigma} \to \mathcal{H}_{\partial_{\mathrm{out}}\Sigma}. \tag{1.1.7}$$

I mention this functorial axiomatization, because there is a possibility for anomalies also here. Namely, the composition of the linear maps $\mathcal{A}(\Sigma)$ may only be projective in the sense of Example 1.1.2. Similar to the Lie algebra setting, such projectiveness may be compensated on the cobordism side of the functor by extending the cobordism category. This yields an object, which in the context of two-dimensional CFT is sometimes called a modular functor. The goal of this work is the mathematical study of the conformal anomaly, the anomaly common to all CFTs as exhibited by the formula in Equation (1.0.1), as a real one-dimensional modular functor, a notion that is defined in Section 1.4.3.

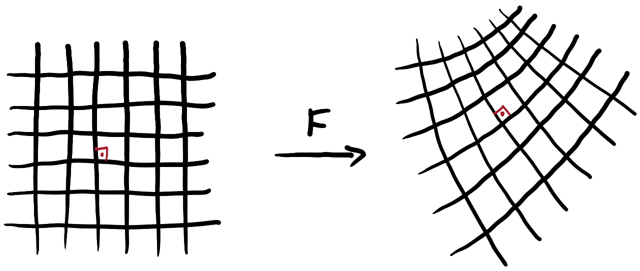


Figure 1.3: A conformal map preserves angles but distorts lengths. In two dimensions, conformal maps are biholomorphisms.

1.2 The conformal anomaly

Being an anomaly associated with conformal symmetry — the change of lengths, yet fixed angles — the conformal anomaly can have a global or local character. On the one hand, there are globally conformal maps such as automorphisms of Riemann surfaces, Möbius transformations of the Riemann sphere $\hat{\mathbb{C}}$, and their higher-dimensional analogues.

Example 1.2.1 (Scaling covariance)**.** One important family of Möbius transformations are global scaling transformations, which are of particular interest in statistical mechanics at critical temperature. There, it exhibits an anomaly with respect to scaling. For example, the observable $\varphi(z)$, which just evaluates the field φ at the points $z \in \mathbb{C}$, is *scaling covariant* if the (unnormalized) two-point correlation function $\langle \varphi(z)\varphi(w)\rangle$ in the Euclidean metric g_0 on $\mathbb{C}$ is such that, for $\lambda > 0$,

$$\frac{\langle \varphi(z/\lambda)\varphi(w/\lambda)\rangle_{g_0}}{Z_{g_0}} = \frac{\langle \varphi(z)\varphi(w)\rangle_{\lambda^2 g_0}}{Z_{\lambda^2 g_0}} = \lambda^{2\Delta} \frac{\langle \varphi(z)\varphi(w)\rangle_{g_0}}{Z_{g_0}}, \tag{1.2.1}$$

where the constant Δ is called the conformal weight or scaling dimension. Note that Equation (1.2.1) implies scale invariance for $\Delta = 0$, but otherwise the scale invariance is broken in a very specific, anomalous way.

On the other hand, there are infinite-dimensional conformal symmetries, in which the scale-change of the metric is position-dependent [BPZ84, Pol81]. Such symmetries are called Weyl transformations, and we introduce them in Section 1.2.1. Then, in Sections 1.2.2 and 1.2.3 we see how Weyl transformations pertain to CFT and random geometry.

1.2.1 Weyl transformations

Let (Σ, g) be an Riemannian manifold, for now of general dimension $d \geq 1$, with smooth metric g. A *Weyl transformation* is a local rescaling of the metric g by a positive smooth function $e^{2\sigma}$ formed by $\sigma \in C^\infty(\Sigma, \mathbb{R})$, resulting in a new metric $e^{2\sigma} g$. Geometrically, the transformed metric $e^{2\sigma} g$ measures lengths

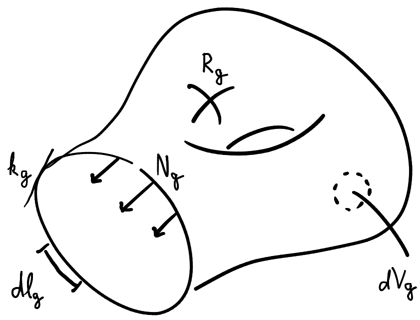


Figure 1.4: The basic quantities of two-dimensional Riemannian geometry have simple transformation laws under Weyl transformations. See Equations (1.2.4) and (1.2.5).

differently, while angles stay the same. Note that the scaling transformation in Example 1.2.1 is a special case of a Weyl transformation where the function σ is constant.

Example 1.2.2. Let $g_0 = \mathrm{d}z\mathrm{d}\bar{z} = \mathrm{d}x^2 + \mathrm{d}y^2$ denote the Euclidean flat metric in the complex plane $\mathbb{C}$ with respect to the standard coordinate $z = x = \mathrm{i}\,y$. Given a conformal transformation, for instance a biholomorphism $F : \Omega_1 \to \Omega_2$ between domains in the complex plane $\Omega_1, \Omega_2 \subseteq \mathbb{C}$, the pullback of g_0 by F is

$$F^* g_0 = e^{2\log|F'|} g_0, \tag{1.2.2}$$

which is a Weyl transformation with $\sigma = \log|F'(z)|$ where $F'(z) = \partial_z F(z)$ denotes the Wirtinger derivative. See Figure 1.3.

Most quantities in Riemannian geometry have relatively simple transformation laws under Weyl transformations. Let $g = g_{jk}\,\mathrm{d}x^j \otimes \mathrm{d}x^k$ denote the metric tensor in coordinates $x^1, \ldots, x^d$ on Σ, and g^{jk} the pointwise inverse matrix. For instance, the positive[1] Laplace–Beltrami operator, or simply *Laplacian*, is defined by

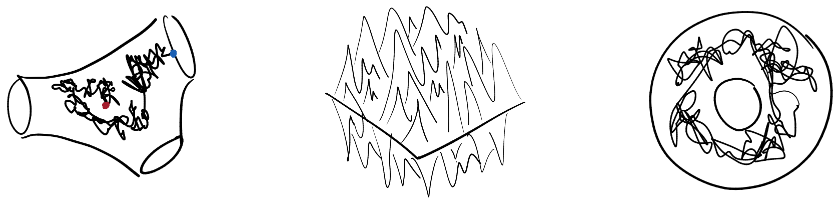
$$\Delta_g = -\frac{1}{\sqrt{\det(g)}} \sum_{j,k=1}^{d} \partial_j \sqrt{\det(g)} g^{jk} \partial_k, \qquad g \in \mathrm{Conf}(\Sigma). \tag{1.2.3}$$

Acting for example on $C^\infty(\Sigma, \mathbb{R})$, the Laplacian transforms as

$$\Delta_{e^{2\sigma}g}\, f = e^{-2\sigma} \Big(\Delta_g f + (d-2)\, g\big(\nabla_g \sigma, \nabla_g f\big)\Big), \qquad f \in C^\infty(\Sigma, \mathbb{R}). \tag{1.2.4}$$

The factor $(d-2)$ already hints at the particular interest in Weyl transformations on surfaces, $d = 2$. In this case, the volume form $\mathrm{d}V_g$ and Gaussian curvature R_g also have simple formulas for the change under Weyl transformations. Moreover, if Σ is orientable and has a non-empty boundary $\partial\Sigma$, the boundary volume

[1] The positivity of Δ_g refers to the fact that with this sign convention, the spectrum of eigenvalues of Δ_g on a compact manifold Σ is discrete and non-negative.



(a) Brownian motion killed at the boundary.

(b) Gaussian free field with Dirichlet boundary condition.

(c) Brownian loop measure winding once around an annulus.

Figure 1.5: The conformally invariant objects of random geometry.

form $\mathrm{d}\ell_g$ and the boundary curvature k_g have analogous properties,

$$(d=2) \qquad \begin{aligned} \mathrm{d}V_{e^{2\sigma}g} &= e^{2\sigma}\mathrm{d}V_g, & R_{e^{2\sigma}g} &= e^{-2\sigma}R_g + \Delta_g\sigma, \\ \mathrm{d}\ell_{e^{2\sigma}g} &= e^{\sigma}\mathrm{d}\ell_g, & k_{e^{2\sigma}g} &= e^{-\sigma}(k_g + N_g\sigma), \end{aligned} \tag{1.2.5}$$

where N_g denotes the (outward pointing) normal derivative at the boundary. Notably, the combination $\Delta_g f\,\mathrm{d}V_g$ of Laplacian and volume form for $f \in C^\infty(\Sigma,\mathbb{R})$ is invariant under Weyl transformations on surfaces. More advanced observations of this kind lead to the essential conformally invariant objects in two-dimensional random geometry.

1. *Brownian motion (BM).* As an operator on more general function spaces, $-\frac{1}{2}\Delta_g$ is the infinitesimal generator of the canonical diffusion process on Σ, also known as Brownian motion. If Dirichlet boundary conditions are used, the BM is killed at the boundary. The conformal invariance is a special property in two dimensions. In the complex plane $\mathbb{C}$, to keep things simple, it states that if B_t is a BM with starting point $z_0 \in \mathbb{C}$, and $F : U \to \mathbb{C}$ a conformal map on a domain $U \subset \mathbb{C}$, then $F(B_t)$ is, up to reparametrization of the time, again a BM started at $F(z_0)$, and defined until the exit time of B_t in U, see [Le 92, Chapter 2] and [Dav79].
2. *Gaussian free field (GFF).* The Gaussian free field is a random distribution on functions on a compact Riemannian manifold (Σ, g) in any dimension. One way to define the GFF is as a random series over an orthonormal basis of eigenfunctions $e_0, e_1, e_2, \dots$ of Δ_g with Dirichlet boundary conditions, and eigenvalues $\lambda_0, \lambda_1, \lambda_2, \dots$, where the coefficients are normal random variables with mean zero and variance $\frac{2\pi}{\lambda_j}$. On orientable surfaces, $d = 2$, the GFF is invariant under Weyl transformations of the metric, see [GRV19, Lemma 3.1] for a proof and a more detailed construction of the GFF. Note that for $\partial\Sigma = \emptyset$, the coefficient of the constant eigenfunction e_0 with eigenvalue $\lambda_0 = 0$ needs to be replaced by a Lebesgue measure.
3. *Brownian loop measure (BLM).* The Brownian loop measure (BLM) is an infinite measure on the set of continuous loops in a Riemannian surface (Σ, g). It may be defined from BM with respect to Δ_g started at $z_0 \in \Sigma$ by first disintegrating it with respect to the endpoint $z_1 \in \Sigma$ after a fixed time $t > 0$. This yields a measure called the (unnormalized) Brownian bridge on paths from z_0 to z_1 in time t. Then, a Brownian loop rooted at

z_0 is obtained from setting $z_1 = z_0$ and integrating over the time $t \in [0, \infty)$. Finally, the (unrooted) BLM is obtained by integrating over z_0 using the volume measure $\mathrm{d}V_g$. See [LW04] for a detailed definition in the plane, and [WX25] for an introduction to the construction on Riemann surfaces, and a proof that BLM is invariant under Weyl transformations.

The conformal invariance of these objects in two dimensions makes it natural to regard them as being defined on Riemann surfaces, which may be viewed as Riemannian surfaces up to Weyl transformations.

1.2.2 Conformal field theory

A conformal field theory (CFT) is a special type of the quantum field theory as discussed in Section 1.1. The local conformal symmetry, that is, symmetry under Weyl transformations, of a CFT exhibits an anomaly through the formula (1.0.1) called the conformal anomaly. Here, we make this precise by covering an axiomatic definition of CFT.

A CFT implements a special type of observables, the primary fields $\phi(z)$, each of which comes with a constant $\Delta \in \mathbb{R}$ called conformal weight or scaling dimension (not to be confused with the Laplacian Δ_g). In terms of the path integral (1.1.1) one takes products $F = \phi_1(z_1) \cdots \phi_n(z_n)$ of primary fields with conformal weights $\Delta_1, \dots, \Delta_n$ as functions on the space of fields. In mathematics, we often interpret the expression

$$\langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle_g \in \mathbb{R}, \tag{1.2.6}$$

just as a notation for function, the *correlation function*, of the parameters $\Delta_1, \dots, \Delta_n$ and distinct points $z_1, \dots, z_n \in \Sigma$. The defining property of the primary fields is Weyl covariance

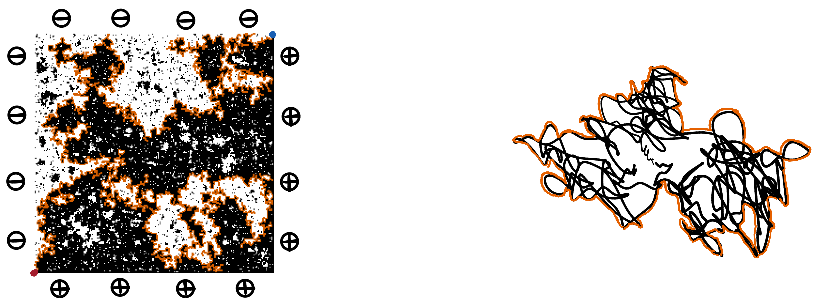
$$\langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle_{e^{2\sigma} g} = e^{\mathbf{c} S_L^0(\sigma, g) - \sum_{j=1}^n \Delta_j \, \sigma(z_j)} \langle \phi_1(z_1) \cdots \phi_n(z_n) \rangle_g, \tag{1.2.7}$$

which is the essential axiom of CFT as in [Gaw99]. The constant $\mathbf{c} \in \mathbb{R}$ is called the *central charge*, and it determines the strength of the conformal anomaly. If the correlation functions are normalized by the partition function $Z_g = \langle 1 \rangle_g$, like in the probability measures (1.1.3), the conformal anomaly $S_L^0(\sigma, g)$ in (1.2.7) of the ratio cancels. Thus, the conformal anomaly is fully captured by the partition function.

Example 1.2.3. Consider a Weyl transformation by the constant function $\sigma(z) = \lambda$. Since $\nabla_g \sigma = 0$ and by the Gauss–Bonnet theorem, the conformal anomaly (1.0.1) takes the following simple form,

$$S_L^0(\sigma, g) = \frac{\lambda}{12\pi} \left(\iint_\Sigma R_g \, \mathrm{d}V_g + \int_{\partial\Sigma} k_g \, \mathrm{d}\ell_g \right) = \frac{\lambda}{6} \, \chi(\Sigma), \tag{1.2.8}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ.



(a) Chordal SLE as an interface of the Ising model; see [CDHKS14].

(b) Loop SLE as the outer boundary of BLM; see [Wer07].

Figure 1.6: Schramm–Loewner evolution (SLE) is a measure on self-avoiding, globally conformally invariant curves with a fractal structure controlled by the central charge in the range $\mathbf{c} \in (-\infty, 1]$.

1.2.3 Conformal covariance in random geometry

Based on the conformally invariant construction of Brownian motion (BM), Gaussian free field (GFF), and Brownian loop measure (BLM) briefly introduced in Section 1.2.1, there are constructions in random geometry which break the conformal symmetry in a mild way, only exhibiting the conformal anomaly.

1. *Schramm–Loewner evolution (SLE).* Using the theory of Loewner chains and BM, stochastic Loewner evolution or Schramm–Loewner evolution (SLE) is originally defined as a family of random simple curves between two points on the boundary of a simply connected domain [Sch00]. In this setup, it has the conformal invariance property that if $F : U \to V$ is a Riemann mapping, the pushforward of the SLE measure of curves between $z_1, z_2 \in \partial U$ is the corresponding SLE measure of curves in V between $F(z_1)$ and $F(z_2)$. Since we do not treat Riemann surfaces with marked boundary points or corners in this work, it makes more sense to consider the loop version of SLE, see [Wer07, KS07]. As a family $\mu^{\mathbf{c}}_{\hat{\mathbb{C}}}$ with $\mathbf{c} \in (-\infty, 1]$ of measures on Jordan loops γ in $\hat{\mathbb{C}}$, loop SLE is uniquely characterized by a conformal restriction property involving BLM; see [BJ24]. The restrictions to simply connected domains U involve the central charge $\mathbf{c}$,

$$\mathrm{d}\mu^{\mathbf{c}}_U(\gamma) = \mathbb{1}_{\gamma \subset U}\, e^{\frac{\mathbf{c}}{2}\Lambda^*(\gamma, \partial U)} \mathrm{d}\mu^{\mathbf{c}}_{\hat{\mathbb{C}}}(\gamma), \tag{1.2.9}$$

   where Λ^* denotes the renormalized total mass of loops touching two sets under BLM; see also Section 2.2 where we relate it to the conformal anomaly. The real-valued function on such loops called *universal Liouville action* or *Loewner energy*, see [TT06, Wan19], is sort of an action functional for loop SLE, or to be precise, an Onsager–Machlup functional [CW23]. Here, we define the unnormalized version called the *Loewner potential*,

$$\mathcal{H}(\gamma) = \log \frac{\det_\zeta \Delta_{g|_{\hat{\mathbb{C}}}}}{\det_\zeta \Delta_{g|_{D_1}} \det_\zeta \Delta_{g|_{D_2}}}. \tag{1.2.10}$$

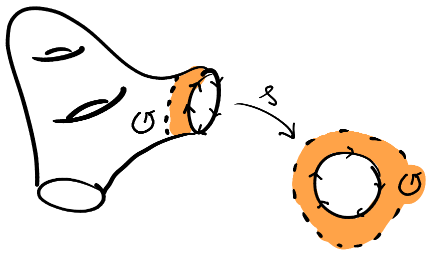


Figure 1.7: A collar chart acting as a real-analytic boundary parametrization with negative orientation.

The distinction between Loewner energy and potential was made in [PW23]. Whereas the latter is unnormalized, the energy is

$$\mathrm{I}^{\mathrm{L}}(\gamma) = 12(\mathcal{H}(\gamma) - \inf_{\eta} \mathcal{H}(\eta)). \tag{1.2.11}$$

Zeta-regularized determinants of Laplacians $\det_\zeta \Delta_{g|_\Sigma}$ appear again in Example 1.4.2, where they are related to the conformal anomaly. They are also related to BLM, see [Dub09, APPS22] and Section 2.2.

2. *Gaussian multiplicative chaos (GMC).* The exponential of the GFF X_g with coefficient $\gamma \in (0,2)$ is defined by a certain renormalization [Kah85]

$$M_{g,\gamma} = \lim_{\varepsilon \to 0} \varepsilon^{\frac{\gamma^2}{2}} e^{\gamma X_{g,\varepsilon}}. \tag{1.2.12}$$

This quantity, called Gaussian multiplicative chaos, may be regarded as a random scaling factor in Weyl transformations. The resulting random volume measure $M_{g,\gamma}\,\mathrm{d}V_g$ satisfies the Weyl covariance

$$M_{e^{2\sigma}g,\gamma}\mathrm{d}V_{e^{2\sigma}g} = e^{(1+\frac{\gamma^2}{4})2\sigma} M_{g,\gamma}\mathrm{d}V_g, \tag{1.2.13}$$

see [GRV19, Section 3.2], which induces the Weyl covariance (1.2.7) of correlation functions in the probabilistic construction of Liouville conformal field theory [GKR24] with central charge given by $\mathbf{c} = 1 + 6Q^2$ and $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$.

1.3 Computing with Riemann surfaces

One way to define a Riemann surface is as a smooth manifold Σ, possibly with boundary, and a choice of conformal class. The latter is an equivalence class of smooth Riemannian metrics on Σ, where two metrics g_1 and g_2 are considered equivalent if they are related by a Weyl transformation $g_2 = e^{2\sigma} g_1$ for some $\sigma \in C^\infty(\Sigma, \mathbb{R})$ as introduced in Section 1.2.1. For a given Riemann surface, we denote the surface and the conformal class by the same symbol, say Σ, and if we are explicitly using the conformal class, we denote it by $\mathrm{Conf}(\Sigma)$. If Σ is a

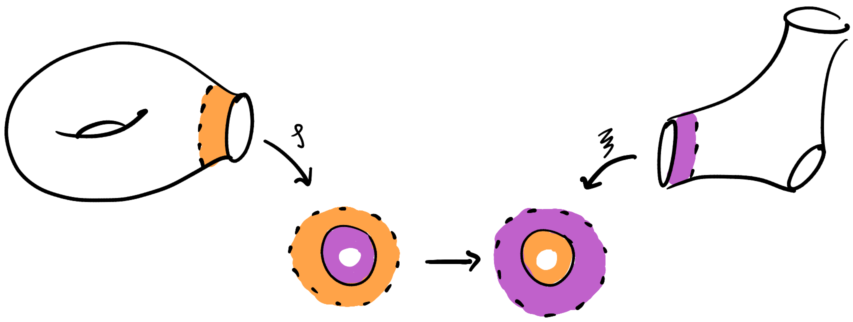


Figure 1.8: Two surfaces are sewn along their boundary components as parameterized by ζ and ξ.

closed Riemann surface, $\partial\Sigma = \emptyset$, by the variant of the classical uniformization theorem concerning metrics, there exists a unique constant curvature metric in the conformal class $\mathrm{Conf}(\Sigma)$ of curvature -1, 0, or 1 depending on the Euler characteristic. On compact Riemann surfaces with boundary, there are two ways of uniformizing the metric; see [OPS88].

- *Type I.* Constant curvature and geodesic boundary components.
- *Type II.* Zero curvature and constant boundary curvature.

In the case of negative Euler characteristic, that is, in the cases

$$\mathsf{g} \geq 2, \qquad \text{or } \mathsf{g} = 1 \text{ and } \mathsf{b} \geq 1, \qquad \text{or } \mathsf{g} = 0 \text{ and } \mathsf{b} \geq 3, \tag{1.3.1}$$

the theory of genus g Riemann surfaces with b boundary components may be phrased in terms of hyperbolic geometry.

In an equivalent definition, a Riemann surface with boundary is a smooth manifold Σ with an atlas such that the charts take values in the closed upper half plane $\bar{\mathbb{H}} = \{z \in \mathbb{C} \mid \operatorname{Im} z \geq 0\}$, or equivalently, the closed unit disk $\bar{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$, and such that the transition functions are biholomorphic. The definitions are equivalent by considering the conformal class of the flat metric in the charts, which, as shown in Example 1.2.2, is preserved by biholomorphisms; see also [Jos06].

We are particularly interested in *collar charts*, that is, complex-analytic charts whose domain is a neighborhood of a boundary component, mapping it real-analytically to the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. The inverse of such a chart is a real-analytic parametrization of the boundary component by S^1 as depicted in Figure 1.7. Let Σ_1 and Σ_2 denote two Riemann surfaces, both with at least one boundary component, and moreover, ζ and ξ such real-analytic parametrizations of one boundary component each, which we assume to be negatively oriented (see explanation below). We define the *sewing operation* on Σ_1 and Σ_2 along the parametrizations ζ and ξ as the result of a construction of

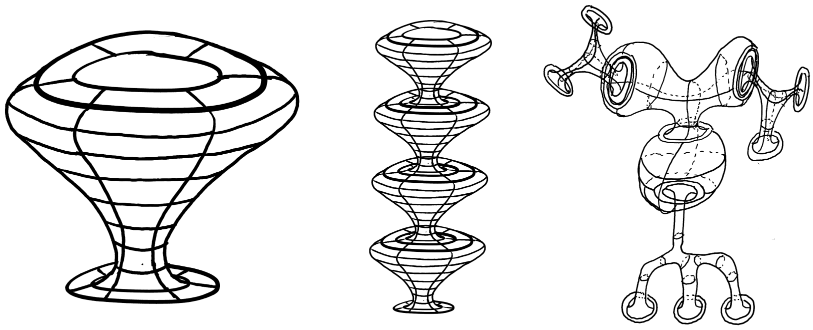


Figure 1.9: The conformal class contains metrics that induce smooth conformal metrics on the sewn surface.

a new Riemann surface[2]

$$\Sigma_1 \infty \Sigma_2 = (\Sigma_1 \sqcup \Sigma_2)/_{\sim}, \tag{1.3.2}$$

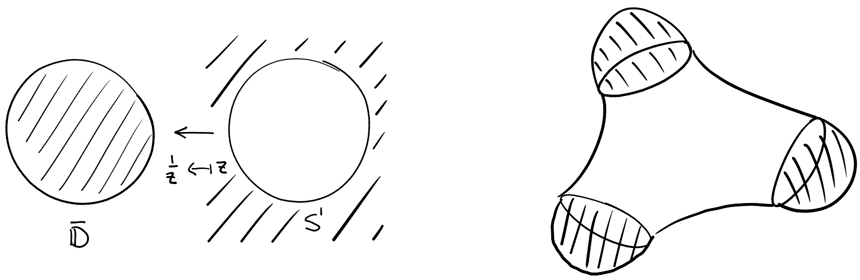
where the relation $\sim$ identifies the respective boundary components pointwise through the equivalence relation generated by $\zeta(z) \sim \xi(\mathrm{J}(z))$ for $z \in S^1$. The insertion of the inversion $\mathrm{J}(z) = \frac{1}{z}$ ensures that the interior of the surfaces are aligned opposite of the unit circle in the new chart $\zeta \sqcup (\mathrm{J} \circ \xi)$ of the sewn surface $\Sigma_1 \infty \Sigma_2$, see Figure 1.8. The complex-analytic atlas on $\Sigma_1 \infty \Sigma_2$ is generated by charts of Σ_1 and Σ_2 away from the seam, and a new chart across the seam formed by joining the parametrizations into $\zeta^{-1} \sqcup (\xi \circ \mathrm{J})^{-1} : \Sigma_1 \infty \Sigma_2 \to \mathbb{C}$.

In terms of the conformal classes of Σ_1, Σ_2, the conformal class on the sewn surface $\Sigma_1 \infty \Sigma_2$ is the conformal class of a metric $g = g_1 \sqcup g_2$ where $g_1 \in \mathrm{Conf}(\Sigma_1)$ and $g_1 \in \mathrm{Conf}(\Sigma_1)$ chosen such that g is smooth. Such a choice can always be made by applying an appropriate Weyl transformation. In the collar charts ζ^{-1} and ξ^{-1}, the metrics g_1 and g_2 are of the form $\zeta^* g_1 = e^{2\sigma_1} \mathrm{d}z\mathrm{d}\bar{z}$, and $\zeta^* g_2 = e^{2\sigma_2} \mathrm{d}z\mathrm{d}\bar{z}$, where $\mathrm{d}z\mathrm{d}\bar{z}$ is as in Example 1.2.2. By applying Weyl transformations, we can find g_1 and g_2 in the conformal class such that σ_1 vanishes and $\sigma_2(z) = \log |\mathrm{J}'(z)| = -2 \log |z|$, and thus $\xi_* \mathrm{J}^* \zeta^* g_1 = g_2$. This means that, we make the metric flat near the boundary, as depicted in Figure 1.9.

In the definition the sewing operation (1.3.2), choices were made as to which boundary components surfaces are sewn, and which charts are used as boundary parametrizations. To establish the sewing operation as an algebraic structure on the entirety of Riemann surfaces, we use a notion that keeps track of these choices by enumerating the boundary components and including the choice of parametrization for each boundary component. The full definition reads as follows.

Definition 1.3.1. A *Riemann surface with analytically parametrized boundary components* $(\Sigma, \zeta_1, \ldots, \zeta_{\mathsf{b}})$ is a connected compact Riemann surface Σ with b boundary components enumerated $\partial_1\Sigma, \ldots, \partial_{\mathsf{b}}\Sigma$, and parametrized by real-analytic maps $\zeta_j : S^1 \to \partial_j \Sigma$ with negative orientation.

[2]I use the notation using the infinity sign for sewing from Vafa [Vaf87] and Huang [Hua97] to distinguish it from the topological gluing operation often denoted #.



(a) The unit disk. (b) A capped pair of pants.

Figure 1.10: The unit disk as a Riemann surface with analytically parametrized boundary component has the role of a cap in the correspondence to closed Riemann surfaces with special coordinate neighborhoods.

Given enumerations of boundary components and their parametrizations, we denote the sewing operation identifying $\partial_j \Sigma_1$ with $\partial_k \Sigma_2$ by $\Sigma_1 {}_j\infty_k \Sigma_2$. Instead of analytic boundary parametrizations, other choices of regularity include smooth [Hen24], quasisymmetric, or Weil–Petersson quasisymmetric [RSS17]. We also consider the case of closed Riemann surfaces, $\mathsf{b} = 0$. Next up, we have the first example of a surface with nontrivial boundary.

Example 1.3.2. The closed unit disk $\bar{\mathbb{D}} = \{z \in \mathbb{C} \mid |z| \leq 1\}$ with the boundary parametrization $\mathrm{J}(z) = 1/z$ defines a Riemann surface with a single analytically parametrized boundary component $(\bar{\mathbb{D}}, \mathrm{J})$. See also Figure 1.10.

The unit disk has a special role, because it is used to transition to the equivalent notion of a capped Riemann surface, obtained by sewing b unit disks to a given surface Σ as in the Definition 1.3.1,

$$\Sigma \,\underline{\infty}\, \underline{\mathbb{D}} = \Sigma \ {}_1\infty_1 \, \mathbb{D} \ \cdots \ {}_{\mathsf{b}}\infty_1 \, \mathbb{D}. \tag{1.3.3}$$

See also Figure 1.10. The choice of negative orientation of the boundary parametrizations is such that these parametrizations extend to conformal maps $\zeta_j : \bar{\mathbb{D}} \to \Sigma \,\underline{\infty}\, \underline{\mathbb{D}}$ where $1 \leq j \leq \mathsf{b}$ on the closed unit disk, and not, as would be the case with positive orientation, anti-conformal maps. Remembering these conformal maps allows one to recover the surface with boundary Σ by "cutting out" their images. See [RSS17] for more details on this correspondence. The notation with the underline as in Equation (1.3.3) stands for a multiple application of sewing operations, which may be used in this work from time to time, and is always explained in the respective context. Note also that we have to relabel the boundary components in some lexicographic order after applying a sewing operation. Sometimes, especially when applying many sewing operations in sequence — since the sewing operation is associative — it is more convenient to postpone this relabeling until after the last sewing operation is applied. For example, in the case of $\mathsf{b} = 2$, we should have written $(\Sigma \ {}_1\infty_1 \, \mathbb{D}) \ {}_1\infty_1 \, \mathbb{D}$ instead of $\Sigma \ {}_1\infty_1 \, \mathbb{D} \ {}_2\infty_1 \, \mathbb{D}$.

If the surface is of genus 0, then the capped surface (1.3.3) is a topological sphere, and thus biholomorphic to the Riemann sphere. Applying such a

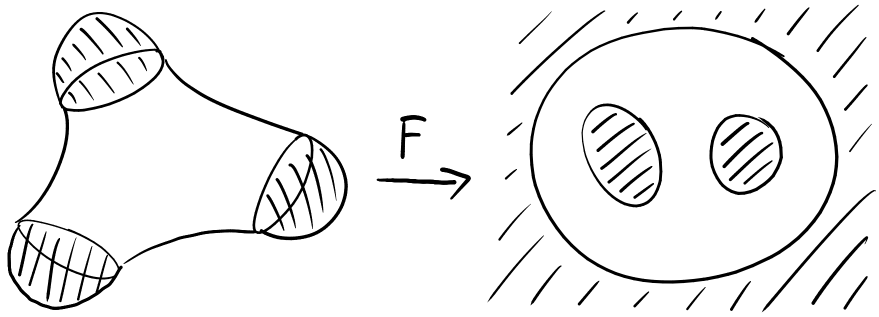


Figure 1.11: An isomorphism of a pair of pants embedding it into the Riemann sphere.

biholomorphism $F : \Sigma \,\underline{\infty}\, \underline{\mathbb{D}} \to \hat{\mathbb{C}}$ to just Σ yields a Riemann surface $F(\Sigma)$ which is a subset of $\hat{\mathbb{C}}$ with boundary parametrizations $F \circ \zeta_1, \ldots, F \circ \zeta_{\mathsf{b}}$, also called a "sphere with tubes" in [Hua97]. Since in this case, the boundary parametrizations determine the surface uniquely as a closed subset in $\hat{\mathbb{C}}$, we use the notation

$$\big(\ \cdot\, , F \circ \zeta_1, \ldots, F \circ \zeta_{\mathsf{b}}\,\big), \tag{1.3.4}$$

where the dot stands for the closed subset of $\hat{\mathbb{C}}$ bounded by the images of S^1 under the boundary parametrizations.

The biholomorphism F above is an example of the more general notion of *isomorphism of Riemann surfaces with analytically parametrized boundary*. Given two such surfaces, $(\Sigma_1, \zeta_1, \ldots, \zeta_{\mathsf{b}_1})$ and $(\Sigma_2, \xi_1, \ldots, \xi_{\mathsf{b}_2})$, an isomorphism F from one to the other is a biholomorphism $F : \Sigma_1 \to \Sigma_2$ such that $\zeta_j \circ F = \xi_j$ for $1 \leq j \leq \mathsf{b}_1 = \mathsf{b}_2$. In particular, it preserves the labels of the boundary components. This implies that there are very few isomorphisms for $\mathsf{b} \neq 0$ compared to automorphisms of compact Riemann surfaces without the fixed boundary parametrizations. In fact, since for an automorphism F of $(\Sigma, \zeta_1, \ldots, \zeta_{\mathsf{b}})$ we have $\zeta_j \circ F = \zeta_j$, the identity theorem implies that $F = \mathbb{1}$.

1.3.1 Moduli spaces

So far, we have considered Riemann surfaces with fixed modulus, that is, a fixed conformal class. It turns out that the moduli spaces, that is, the sets of Riemann surfaces with fixed topology up to isomorphism, come with geometric structure on their own. Leaving the boundary parametrizations from the previous section out of the picture for now, the moduli spaces, which we denote by $\check{\mathcal{M}}_{\mathsf{g},\mathsf{b}}$ fixing the surface topology, are well-known finite-dimensional orbifolds, see [Mir06, Wol09] and references therein. However, since in this work we study properties of the sewing operation (1.3.2), we introduce moduli spaces $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ of Riemann surfaces together with enumerated and analytically parametrized boundary components,

$$\mathcal{M}_{\mathsf{g},\mathsf{b}} = \left\{ \begin{array}{l} \text{genus } \mathsf{g} \text{ Riemann surfaces with} \\ \mathsf{b} \text{ analytically parametrized boundary} \\ \text{components (Definition 1.3.1)} \end{array} \right\}_{/\ \text{isomorphism}} \tag{1.3.5}$$

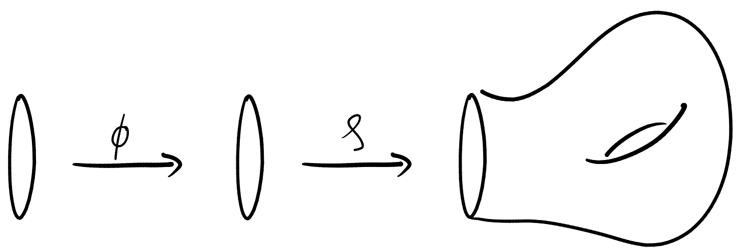


Figure 1.12: Reparametrization of a boundary component by an orientation-preserving real-analytical diffeomorphism $\phi \in \mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$ of the unit circle.

with the notion of isomorphism as defined at the end of the previous section. If $(\Sigma, \zeta_1, \ldots, \zeta_{\mathsf{b}})$ is a Riemann surface with analytically parametrized boundary components as in Definition 1.3.1, we denote the equivalence class in the moduli space by

$$\left[\, \Sigma, \zeta_1, \ldots, \zeta_{\mathsf{b}} \,\right] \in \mathcal{M}_{\mathsf{g},\mathsf{b}}. \tag{1.3.6}$$

I find that this notation is quite amenable to computations, as shown in the examples following in this section.

Example 1.3.3. The closed complement $\hat{\mathbb{C}} \setminus \mathbb{D}$ of the unit disk with analytical boundary parametrization given by the identity $\mathbb{1} : z \mapsto z$ is isomorphic to $\mathbb{D} = (\bar{\mathbb{D}}, \mathrm{J})$ through the isomorphism $\mathrm{J} : z \mapsto 1/z$,

$$\mathbb{D} = \left[\, \bar{\mathbb{D}}, \mathrm{J} \,\right] = \left[\, \hat{\mathbb{C}} \setminus \mathbb{D}, \mathbb{1} \,\right] \in \mathcal{M}_{0,1}. \tag{1.3.7}$$

In contrast to the case without boundary parametrization, the moduli spaces $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ are infinite-dimensional. This may be understood by considering reparametrizations of boundary components by the infinite-dimensional Lie group $\mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$ of orientation-preserving real-analytical diffeomorphisms of the unit circle. Given a surface, a choice of boundary component $1 \leq j \leq \mathsf{b}$, and a diffeomorphism $\phi \in \mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$, we denote such a reparametrization by

$$\left(\, \Sigma, \zeta_1, \ldots, \zeta_{\mathsf{b}} \,\right) \underset{j}{*} \phi = \left(\, \Sigma, \zeta_1, \ldots, \zeta_j \circ \phi, \ldots, \zeta_{\mathsf{b}} \,\right). \tag{1.3.8}$$

Since each real-analytic diffeomorphism extends to a biholomorphism on an annular neighborhood of S^1, reparametrization descends to a right action of $\mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$ on $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ denoted $[\Sigma, \zeta_1, \ldots, \zeta_{\mathsf{b}}] \underset{j}{*} \phi = [\Sigma, \zeta_1, \ldots, \zeta_j \circ \phi, \ldots, \zeta_{\mathsf{b}}]$. In Section 1.3.2, I provide more details on these actions in the context of complex deformations of S^1, which generalize the notion of diffeomorphism.

Example 1.3.4. Let $[D, \zeta] \in \mathcal{M}_{0,1}$ be any disk with analytically parametrized boundary component. By a Riemann mapping $F : D \to \bar{\mathbb{D}}$, we put it into the standard form

$$\left[\, D, \zeta \,\right] = \left[\, \bar{\mathbb{D}}, F \circ \zeta \,\right] = \mathbb{D} \underset{1}{*} \phi \in \mathcal{M}_{0,1}. \tag{1.3.9}$$

where $\phi = \mathrm{J} \circ F \circ \zeta \in \mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$ is a real-analyttical diffeomorphism.

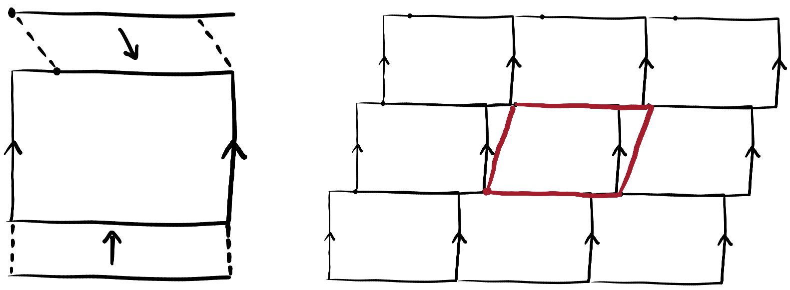


Figure 1.13: Self-sewing of a standard annulus $\mathbb{A}_\tau \underset{1}{*} R_\theta$ with twisted boundary component results in a torus with geodesic seam.

In [MP25b], we define a geometric structure — a Frölicher structure generated by smooth curves — on the moduli spaces $\mathcal{M}_{\mathsf{g},\mathsf{b}}$. See also the summary in Section 2.1. A curve in $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ rooted at Σ_0 may be thought of as a one-parameter deformation $t \mapsto \Sigma_t$ of Σ_0. These geometric structures are precisely such that the sewing operation (1.3.2) descends for $1 \leq j \leq \mathsf{b}_1$ and $1 \leq k \leq \mathsf{b}_2$ to smooth functions

$$\begin{aligned} \mathcal{M}_{\mathsf{g}_1,\mathsf{b}_1} \times \mathcal{M}_{\mathsf{g}_2,\mathsf{b}_2} &\to \mathcal{M}_{\mathsf{g}_1+\mathsf{g}_2,\mathsf{b}_1+\mathsf{b}_2-2} \\ (\Sigma_1, \Sigma_2) &\mapsto \Sigma_1 \, {}_j\infty_k \, \Sigma_2. \end{aligned} \tag{1.3.10}$$

Let me also mention the self-sewing operation, which is defined analogously to Equation (1.3.2), except that it identifies two boundary components of the same surface,

$$\begin{aligned} \mathcal{M}_{\mathsf{g},\mathsf{b}} &\to \mathcal{M}_{\mathsf{g}+1,\mathsf{b}-2} \\ \Sigma &\mapsto \infty_{j,k} \Sigma. \end{aligned} \tag{1.3.11}$$

Notably, the self-sewing operation provides a way to obtain genus 1 surfaces from genus 0 surfaces.

Example 1.3.5 (Self-sewing of annuli)**.** Sewing the two boundary components of an annulus results in a torus

$$T = \infty_{1,2} A \in \mathcal{M}_{1,0}, \qquad A \in \mathcal{M}_{0,2}. \tag{1.3.12}$$

Both the annulus and the torus come with (type I) flat metrics. In Section 1.4.3, we are interested in the case where these metrics agree via the embedding of A into T. Since the embedding is conformal and the flat metric is unique, this depends on the annulus metric extending smoothly over the seam inside T. This is the case if the seam is a geodesic with respect to the flat metric on T, and the parametrization is of constant speed. See Figure 1.13 for an example. This defines the set of *annuli with geodesic property*, $\mathcal{M}_{0,2}^{\mathsf{geod}} \subseteq \mathcal{M}_{0,2}$. Basic examples are the standard annuli for $\tau > 0$.

$$\mathbb{A}_\tau = \big[\, \{z \in \mathbb{C} \mid e^{-2\pi\tau} \leq |z| \leq 1\}, \quad \mathrm{J}, \quad e^{-2\pi\tau}\, \mathbb{1} \,\big] \in \mathcal{M}_{0,2}^{\mathsf{geod}}, \tag{1.3.13}$$

and $\mathbb{A}_\tau \underset{1}{*} R_\theta \in \mathcal{M}_{0,2}^{\mathsf{geod}}$ where one boundary parametrization is twisted by a rotation $R(z) = e^{\mathrm{i}\,\theta} z$ for $\theta \in \mathbb{R}$. Any other annulus with geodesic property is of

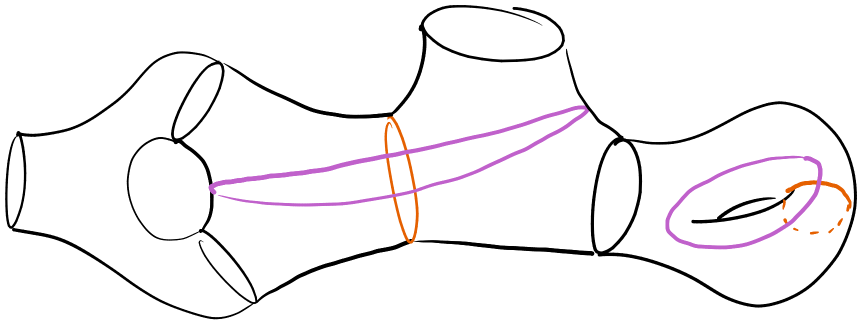


Figure 1.14: Pants decomposition of a hyperbolic surface, highlighting one possible A- and S-moves each.

the form

$$\mathbb{A}_\tau \underset{1}{*} R_\theta \underset{1}{*} \phi \underset{2}{*} (\mathrm{J} \circ \phi^{-1} \circ \mathrm{J}) \tag{1.3.14}$$

for some diffeomorphism $\phi \in \mathrm{Diff}_+^{\mathrm{an}}(\mathrm{S}^1)$ which cancels in the sewing operation (1.3.12).

Surfaces of higher genus may be obtained by combining several genus 1 surfaces. This is important for the inductive procedure used in the proof of the main theorem in [MP25b]; see also Theorem 2.3.1.

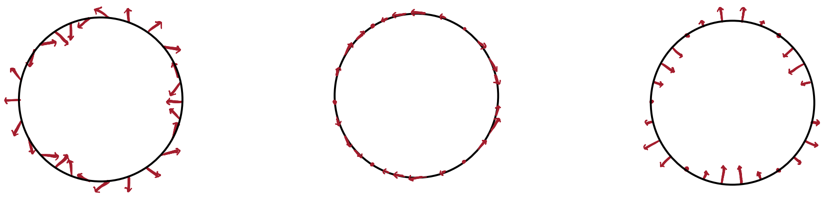
Example 1.3.6 (Hyperbolic surfaces)**.** If g and b are such that the Euler characteristic is negative, see Equation (1.3.1), there is a unique hyperbolic metric (type I) in the conformal class of any surface $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$. If Σ has a representative $(\Sigma, \zeta_1, \ldots, \zeta_{\mathsf{b}})$ such that each parametrization has constant speed $|\partial_\theta \zeta_j(e^{\mathrm{i}\,\theta})|_g$ in the hyperbolic metric g, we call Σ itself *hyperbolic.* Note that this gives a notion of boundary length

$$l_j(\Sigma) = \int_0^{2\pi} |\partial_\theta \zeta_j(e^{\mathrm{i}\,\theta})|_g \mathrm{d}\theta, \qquad 1 \leq j \leq \mathsf{b}. \tag{1.3.15}$$

We denote the subspace of hyperbolic surfaces of $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ by $\mathcal{M}_{\mathsf{g},\mathsf{b}}^{\mathsf{hyp}}$. Note that $\mathcal{M}_{\mathsf{g},0}^{\mathsf{hyp}} = \mathcal{M}_{\mathsf{g},0}$ for $\mathsf{g} \geq 2$. Any hyperbolic surface $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}^{\mathsf{hyp}}$ can be constructed from a finite number of hyperbolic pairs of pants $P_1, \ldots, P_{2\mathsf{g}-2+\mathsf{b}} \in \mathcal{M}_{0,3}^{\mathsf{hyp}}$. Collectively denoting them $\underline{P}$, the sewn surface

$$\Sigma = \underline{\infty}\, \underline{P} \tag{1.3.16}$$

is called a *pants decomposition* if the boundary lengths of each pair of sewn boundary components agree. By $\underline{\infty}$ we denote the $3\mathsf{g}-3+\mathsf{b}$ sewing operations on the pairs of pants, each of which either sews a pair of boundary components from two separate pairs of pants, or self-sews the boundary components of the same pair of pants. Note that even though we only consider pants decompositions with geodesic seams, there is an infinite number of inequivalent decompositions related to the homotopy classes of the seams. One way to move between pants decompositions is by A- and S-moves, see [HT80, Hat99] and Figure 1.14, to which we come back in Section 2.3.



(a) The vector field ℓ_3. (b) The tangent vector field $a_3^{\|}$. (c) The perpendicular vector field $b_3^{\perp}$.

Figure 1.15: The Witt algebra realized by vector fields on the unit circle.

1.3.2 Complex deformations of the unit circle

The diffeomorphism group of the unit circle with real-analytic regularity $\mathrm{Diff}_+^{\mathrm{an}}(\mathrm{S}^1)$ already appeared in Equation (1.3.8) where it acts on the moduli spaces $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ in b ways by reparametrization of one of the b boundary components. It is an infinite-dimensional Lie group whose complexified Lie algebra is the *Witt algebra* typically defined in terms of generators ℓ_n,

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m}, \qquad n, m \in \mathbb{Z}. \tag{1.3.17}$$

The Witt algebra may be realized as a Lie algebra $\mathrm{Vect}_{\mathbb{C}}^{\mathrm{an}}(\mathrm{S}^1)$ of complex-valued vector fields on the unit circle by setting

$$\ell_n = -z^{n+1}\partial_z \tag{1.3.18}$$

in the standard coordinate z in $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. To a vector field $v \in \mathrm{Vect}_{\mathbb{C}}^{\mathrm{an}}(\mathrm{S}^1)$ we associate a function $v(z)$ such that $v = v(z)\partial_z$, for example, $\ell_n(z) = -z^{n+1}$. Then, the usual Lie bracket on $\mathrm{Vect}_{\mathbb{C}}^{\mathrm{an}}(\mathrm{S}^1)$ is

$$[v, w] = \big(v'(z)w(z) - v(z)w'(z)\big)\partial_z, \tag{1.3.19}$$

where the prime denotes the derivative in z. A computation shows that this Lie bracket agrees with that of the Witt algebra (1.3.17) via the generators (1.3.18) which span the Lie subalgebra $\mathbb{C}[z, z^{-1}]\partial_z \subseteq \mathrm{Vect}_{\mathbb{C}}^{\mathrm{an}}(\mathrm{S}^1)$ of Laurent polynomials.

The space of vector fields $\mathrm{Vect}_{\mathbb{C}}^{\mathrm{an}}(\mathrm{S}^1)$ has another set of generators over $\mathbb{R}$ given by the following vector fields, respectively, tangent and normal to S^1,

$$\begin{aligned} a_n^{\|} &= \frac{\ell_n - \ell_{-n}}{2}, \qquad & b_n^{\|} &= \frac{\ell_n + \ell_{-n}}{2\mathrm{i}}, \\ a_n^{\perp} &= \frac{\ell_n - \ell_{-n}}{2\mathrm{i}}, \qquad & b_n^{\perp} &= \frac{\ell_n + \ell_{-n}}{2}. \end{aligned} \tag{1.3.20}$$

See also Figure 1.15. Before complexification, the Lie algebra of $\mathrm{Diff}_+^{\mathrm{an}}(\mathrm{S}^1)$ is isomorphic to the Lie subalgebra of $\mathrm{Vect}_{\mathbb{C}}^{\mathrm{an}}(\mathrm{S}^1)$ of all tangent vector fields, which we denote by $\mathrm{Vect}_{\mathbb{R}}^{\mathrm{an}}(\mathrm{S}^1)$, since in the coordinate θ on S^1 defined by $z = e^{\mathrm{i}\,\theta}$ they become $a_n^{\|}(e^{\mathrm{i}\,\theta}) = \mathrm{i}\,e^{\mathrm{i}\,\theta}\sin(n\theta)$, and $b_n^{\|}(e^{\mathrm{i}\,\theta}) = \mathrm{i}\,e^{\mathrm{i}\,\theta}\cos(n\theta)$, where the rotation by $\mathrm{i}\,e^{\mathrm{i}\,\theta}$ takes the real-valued functions of θ to tangent vector fields. Identification of the Lie algebra of $\mathrm{Diff}_+^{\mathrm{an}}(\mathrm{S}^1)$ with tangent vector fields

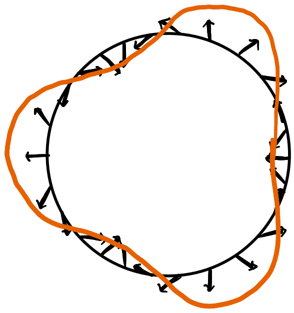


Figure 1.16: The flow of ℓ_3 after a small time, resulting in a complex deformation of the unit circle.

$\mathrm{Vect}^{\mathrm{an}}_{\mathbb{R}}(\mathrm{S}^1)$ has a natural geometric interpretation since the exponential map is given by intagration of the the flow equation of a vector field $v \in \mathrm{Vect}^{\mathrm{an}}_{\mathbb{R}}(\mathrm{S}^1)$

$$\partial_t \Phi_v(t,z) = v(t, \Phi_v(t,z)), \qquad \Phi_v(t,z) = z, \tag{1.3.21}$$

until $t = 1$. Note that this exponential map is where we start seeing the particularities of $\mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$ being an infinite-dimensional Lie group. It is well known that the exponential map is not surjective, that is, not every diffeomorphism is the finite-time flow of a fixed vector field; see [Mil85] for a counterexample. Note that if the vector field in Equation (1.3.21) is allowed to be time-dependent, this changes the situation and now any diffeomorphism can be reached at time $t = 1$. This observation is essential for the infinite-dimensional geometric structure that we put on $\mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$ and related spaces in [MP25b]; see Section 2.3.

Another infinite-dimensional particularity is the absence of Lie's third theorem — not every infinite-dimensional Lie algebra is the Lie algebra of an infinite-dimensional Lie group. This is the case with the Witt algebra $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$. In other words, there is no Lie group that is the complexification of $\mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$ [Lem97]. However, it still makes sense to integrate the flow equation (1.3.21) for general complex-valued vector fields $v \in \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$, at least, up to a small positive time. Again, it is useful to also consider time-dependent vector fields $v = v(t,z)\partial_z$. Then, by the real-analyticity of v in z, the flow $\Phi_v(t,z)$ is a real analytic map of S^1 into the complex plane. The flow starts with the identity map $\Phi(0,z)$, and as time increases, it deforms the circle $\Phi_v(t,S^1)$ within the complex plane. Therefore, we call $\Phi_v(t,z)$ a deformation of the unit circle in the complex plane, or just a *complex deformation*. See Figure 1.16 for an example. Abstracting from the flow equation, we define the set of complex deformations as follows:

$$\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1) = \left\{ \begin{array}{l} \phi : S^1 \to \mathbb{C} \setminus \{0\} \text{ positively oriented around } 0, \\ \text{and extending biholomorphically to an annular} \\ \text{neighborhood } A \text{ of } S^1 \text{ such that } S^1 \subset \phi(A) \end{array} \right\} \tag{1.3.22}$$

Note that a choice was made here to make every complex deformation invertible by requiring that ϕ^{-1} extends conformally back to the unit circle. The homotopy

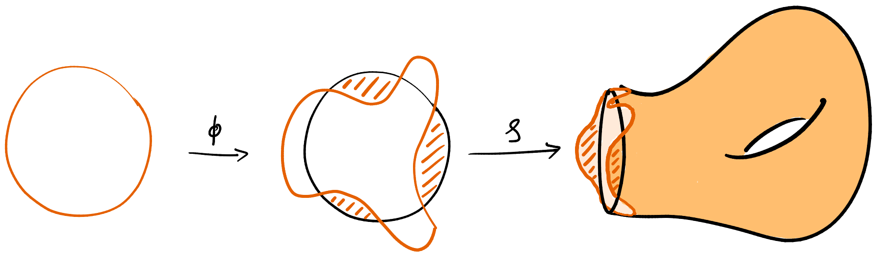


Figure 1.17: A complex deformation acting on a surface, adding and subtracting area near the boundary component.

condition is such that the definition is compatible with the slightly different one in [MP25a], see also Section 2.1, where the unit circle is deformed within the infinite cylinder $S^1 \times \mathbb{R}$.

If the image $\psi(S^1)$ of a second complex deformation $\psi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ is contained in an annular domain[3] $A \subset \mathbb{C} \setminus \{0\}$ for ϕ, then they may be composed into a third complex deformation $\phi \circ \psi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$. This yields a partially defined, but associative and invertible composition law on $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$. Since for two flows $\Phi_v(t, z)$ and $\Phi_w(s, z)$ of time-dependent vector fields v and w may be composed to $\Phi_v(t, \Phi_w(s, z))$ for $t, s \geq 0$ small enough, the set of complex deformations comes with a local composition law, not quite forming a Lie group since the multiplication is only be defined for complex deformations close to the identity. The smooth structure on $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ is the aforementioned Frölicher structure, and the composition law is smooth with respect to this structure. See [MP25b, Section 1] for more details. Note also that the structure of complex deformations is closely related to the concept of pseudogroups going back to Lie [LE93] and Cartan [Car04]. The tangent space at $\mathbb{1} \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ is, analogous to the case of $\mathrm{Diff}_+^{\mathrm{an}}(\mathrm{S}^1)$, the Lie algebra of vector fields $\mathrm{Vect}_{\mathbb{C}}^{\mathrm{an}}(\mathrm{S}^1)$, that is, the Witt algebra. However, since we are in the infinite-dimensional setting, this statement requires more care, and the details can be found in [MP25b, Section 1] as well.

Remark 1.3.7. Even though $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ is not quite a local Lie group (since both complex deformations need to be close to the identity to make them composable), the following remark concerning the notion of isomorphism for local Lie groups still applies. On the one hand, "local" isomorphisms only need to be invertible in a neighborhood of the identity. In this case, the category of finite-dimensional local Lie groups is equivalent to that of finite-dimensional Lie algebras, so there is generally not much reason to consider this category of local Lie groups except in some proof of Lie's third theorem. On the other hand, if the isomorphism of the local Lie group is required to be "global", there

[3]The topological restriction of the domain being an annulus around 0 is sufficient for the uniqueness of the composition. Other domains may result in multivalued compositions subject to the choice of analytic extension of ϕ around singularities.

are many more local Lie groups, even finite-dimensional ones, which do not extend to global Lie groups. In the case of complex deformations, $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ retracts to its subgroup of rotations, and is thus not simply connected. Hence, it makes sense to consider neighborhoods of not just the identity, but including all rotations, for example, like in our computation of the group-level cohomology in [MP25b, Section 2].

As already introduced with Equation (1.3.8), the diffeomorphism group $\mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1) \subseteq \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ acts on the moduli spaces $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ in b ways by reparametrization of the boundary components. This action extends to the complex deformations, which can, in contrast to the diffeomorphisms, change the moduli of the surface as well by adding and subtracting from the surface near the boundary component. Letting " $\cdot$ " stand for the deformed surface, we define

$$\left(\, \Sigma, \zeta_1, \ldots, \zeta_{\mathsf{b}} \,\right) \underset{j}{*} \phi = \left(\ \cdot\, , \zeta_1, \ldots, \zeta_j \circ \phi, \ldots, \zeta_{\mathsf{b}} \,\right). \tag{1.3.23}$$

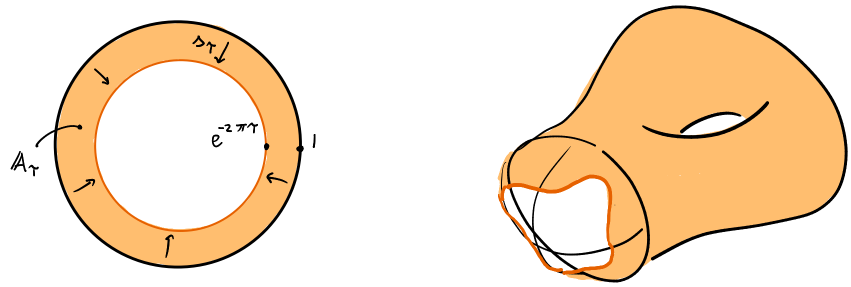
See the illustration in Figure 1.17. However, just like how the composition of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ is only defined locally, this action is only local. The following obstructions can prevent $\Sigma \underset{j}{*} \phi$ from existing.

1. The part of the deformation $\phi(S^1) \cap (\hat{\mathbb{C}} \setminus \mathbb{D})$ cutting into the surface is outside the radius of convergence of the boundary parametrization ζ_j.
2. The deformations would cause boundary components to intersect.

The observation that for $\phi = \Phi_v(t, \cdot\,)$ with $v \in \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ there always exists some enough small time t such that the deformed surface (1.3.23) exists shows that the action still differentiable, yielding an action of the Lie algebra. This action is essential to our construction of the Frölicher structure on moduli spaces.

The following list of special types of complex deformations demonstrates their versatility.

1. *Rotations.* Already mentioned above, the rotations
$$R_\theta(z) = e^{\mathrm{i}\,\theta} z, \qquad \theta \in \mathbb{R}, \tag{1.3.24}$$
form a subgroup of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$.
2. *Scaling transformations.* Another one-dimensional subgroup of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ is given by the scaling transformations
$$\mathrm{Sc} = \left\{ s_\tau : \hat{\mathbb{C}} \to \hat{\mathbb{C}},\ z \mapsto e^{-2\pi\tau} z \,\middle|\, \tau \in \mathbb{R} \right\}. \tag{1.3.25}$$
The scaling transformations with $\tau > 0$ are related to the standard annuli (1.3.13) by
$$\mathbb{A}_\tau = \left[\, \{ z \in \mathbb{C} \mid e^{-2\pi\tau} \le |z| \le 1 \}, \quad \mathrm{J}, \quad s_\tau \,\right] \in \mathcal{M}^{\mathsf{geod}}_{0,2}, \tag{1.3.26}$$
also in the sense that $\Sigma \underset{j}{*} s_\tau = \Sigma \;{}_j\!\infty_1\, \mathbb{A}_\tau$. See also Figure 1.18.
3. *Möbius transformations.* More generally, any Möbius transformation $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $F(S^1)$ winds around 0 in positive orientation is a complex



(a) A scaling transformation and the corresponding annulus.

(b) The action of a contraction extends the surface.

Figure 1.18: Examples of complex deformations and their properties.

deformations. In particular, we do not consider the inversion $\mathrm{J}(z) = 1/$ to be a complex deformation. The conjugation $\mathrm{J} \circ \phi^{-1} \circ \mathrm{J}$ of $\phi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$, however, is a complex deformation, and it is often useful since

$$(\Sigma_1 \underset{k}{*} \phi)\ {}_j\infty_k\ \Sigma_2 = \Sigma_1\ {}_j\infty_k\ (\Sigma_2 \underset{k}{*} (\mathrm{J} \circ \phi^{-1} \circ \mathrm{J})). \tag{1.3.27}$$

4. *Contractions.* We call complex deformations that map S^1 inside the open unit disk *contractive*, denoting their set by $\mathrm{Def}_{\mathbb{D}}(\mathrm{S}^1) \subseteq \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$. Generalizing Equation (1.3.26), they are associated to annuli

$$\iota(\phi) = \left[\ \cdot\ ,\quad \mathrm{J},\quad \phi\ \right] \in \mathcal{M}_{0,2}, \qquad \phi \in \mathrm{Def}_{\mathbb{D}}(\mathrm{S}^1), \tag{1.3.28}$$

with the property that the sewing of annuli corresponds to composition of contractive deformations. Note that the action of contractive deformations ϕ only adds to the surface, and thus $\Sigma \underset{j}{*} \phi$ is always defined; see Figure 1.18.

5. *Univalent functions.* Some complex deformations extend conformally $\bar{\mathbb{D}}$ or $\hat{\mathbb{C}} \setminus \mathbb{D}$. The former are just univalent functions and have the property that their action preserves the uniformized representative of genus 0 surfaces in Equation (1.3.4).

1.4 The real determinant line bundle

Since a Riemann surface comes only with a conformal class, a new mathematical tool is needed to describe objects with conformally covariant dependence on the metric — objects like CFT and SLE. Note that the conformal class had too many degrees of freedom in the first place, since conformal covariance is fully characterized by the quantity $S_L^0(\sigma, g)$. The determinant line bundle is a construction that retains precisely this information. The idea originates from the work of Friedan and Shenker [FS87] in the context of CFT, and Kontsevich and Suhov [KS07] in the context of SLE. Further details are also spelled out in the work of Benoist and Dubédat [Dub15, BD16]. This section summarizes the presentation in [MP25a].

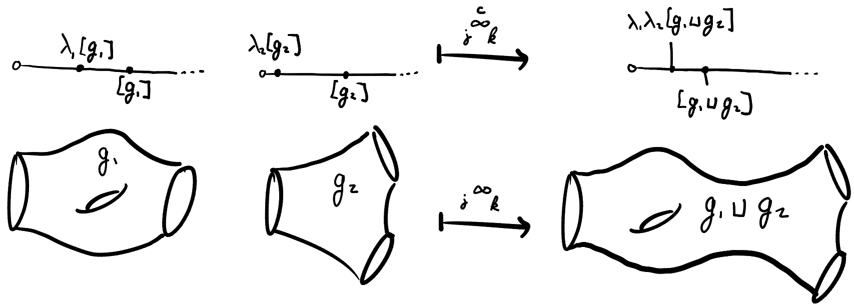


Figure 1.19: The sewing operation of real determinant lines.

1.4.1 Construction

The real determinant line of central charge $\mathbf{c} \in \mathbb{R}$ over Σ is a quotient space of $\mathbb{R}_+ \times \mathrm{Conf}(\Sigma)$,

$$\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma) = \{\lambda[g] \mid \lambda \in \mathbb{R}_+,\ g \text{ conformal metric on } \Sigma\}\,, \tag{1.4.1}$$

where the first component $\lambda \in \mathbb{R}_+$ represents the conformal anomaly picked up by any previous Weyl transformations applied to the current metric $g \in \mathrm{Conf}(\Sigma)$. We only keep track of the metric up to the information needed to consistently carry the factor λ through several Weyl transformations. More precisely, we identify any two pairs by the equivalence relation

$$\lambda[e^{2\sigma} g] = (\lambda e^{\mathbf{c} S^0_L(\sigma,g)})[g], \tag{1.4.2}$$

defining the equivalence classes $\lambda[g]$. The well-definedness of this relation follows from properties of the conformal anomaly $S^0_L(\sigma, g)$, namely, the cocycle property for two Weyl transformations $\sigma_1, \sigma_2 \in C^\infty(\Sigma, \mathbb{R})$, and $g \in \mathrm{Conf}(\Sigma)$,

$$S^0_L(\sigma_1, g) + S^0_L(\sigma_2, e^{2\sigma_1} g) = S^0_L(\sigma_1 + \sigma_2, g). \tag{1.4.3}$$

Detailed computations are carried out in the Sections 2.5, 3.1, and Appendix A of the article [MP25a], reproduced in Appendix A of this thesis. The real determinant line comes with an $\mathbb{R}_+$-linear structure

$$\lambda_1[g_1] + \lambda_2[g_2] = (\lambda_1 + \lambda_2\, e^{\mathbf{c} S^0_L(\sigma,g_1)})[g_1], \quad g_2 = e^{2\sigma} g_1 \in \mathrm{Conf}(\Sigma), \tag{1.4.4}$$

making it a one-dimensional real half-line.

The sewing operation (1.3.2) on Riemann surfaces Σ_1 and Σ_2 extends to the real determinant line bundles bilinearly by joining the metrics of the two parts of to surface

$$\begin{aligned} \cdot \overset{\mathbf{c}}{\infty} \cdot \ : \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma_1) \otimes \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma_1) &\to \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma_1 \infty \Sigma_2) \\ \lambda_1[g_1] \otimes \lambda_2[g_2] &\mapsto \lambda_1 \lambda_2 [g_1 \sqcup g_2]. \end{aligned} \tag{1.4.5}$$

Note that generally the metric $g_1 \sqcup g_2$ might not be smooth across the seam. However, by applying a Weyl transformation to the metrics near the boundary, representatives can be found such that $g_1 \sqcup g_2$ becomes smooth — see Figure 1.9 and Figure 1.19. The independence of choice of such representatives is a direct consequence of the locality of the conformal anomaly, see [MP25a] for details.

Remark 1.4.1. One might as well use $\mathbb{R}$ instead of $\mathbb{R}_+$ in the definition of $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma)$, Equation (1.4.1), and then the linear structure in Equation (1.4.4) turns the determinant line into a one-dimensional $\mathbb{R}$-vector space, which is the arguably more natural object for a line bundle. However, we observe that the scalar relating basic elements of the form $[g], [e^{2\sigma} g] \in \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma)$ is positive, that is, $e^{\mathbf{c} S^0_L(\sigma, g)} > 0$. This has the advantage that an $\mathbb{R}_+$-bundle is always a trivial bundle, which is as expected since the only nontrivial structure of the real determinant line bundles $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}$ is encoded in the sewing isomorphisms. On the contrary, the complex determinant line bundle $\mathrm{Det}^{\mathbf{c}}_{\mathbb{C}}$ that applies to chiral CFT [Seg04, Hua97] is a line bundle over $\mathbb{C}$, and might have nontrivial holonomy. The relationship between this complex determinant line bundle is yet to be studied, but is expected to be $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+} = |\,\mathrm{Det}^{\mathbf{c}}_{\mathbb{C}}\,|$, that is, the real determinant line bundle is the absolute value of the complex determinant line bundle. This fits into the CFT framework, where a chiral and anti-chiral CFT — holomorphic and anti-holomorphic — are combined to obtain a full CFT. Beware that this is an oversimplification since the combination does not happen at the level of partition functions, but rather one has to consider the spectrum of the chiral CFT and combine the modules of the Virasoro algebra at each level, giving a much more complex relationship between partition functions.

A trivialization of $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}$ over $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ is a section

$$Z : \mathcal{M}_{\mathsf{g},\mathsf{b}} \to \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\mathcal{M}_{\mathsf{g},\mathsf{b}}) \tag{1.4.6}$$

of the $\mathbb{R}_+$-bundles $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\mathcal{M}_{\mathsf{g},\mathsf{b}})$. A family of trivializations for each genus g and number of boundary components b is collectively denoted Z. At a given surface $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$, a trivialization encodes a conformally covariant quantity — given a metric $g \in \mathrm{Conf}(\Sigma)$, we obtain a number $Z_g(\Sigma) \in \mathbb{R}_+$ that is the coefficient of

$$Z(\Sigma) = Z_g(\Sigma)\ [g] \in \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma) \tag{1.4.7}$$

with respect to the basis $[g] \in \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma)$.

Example 1.4.2. Relating to previous sections of this work, there are several natural ways to trivialize the real determinant line bundle:

1. *CFT partition functions.* As the notation suggests, the family of partition functions of a given CFT over all surfaces defines a trivialization

$$Z(\Sigma) = Z_g(\Sigma)[g] \tag{1.4.8}$$

   of $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}$, where $\mathbf{c}$ is the central charge of the CFT. This definition is independent of the choice of metric $g \in \mathrm{Conf}(\Sigma)$ by the Weyl covariance property (1.2.7).

2. *Zeta-regularized determinants of the Laplacian.* Already mentioned in Section 1.2.3, the zeta-regularized determinant of the Laplacian is defined as the derivative of the analytic continuation of a spectral zeta function [RS71].

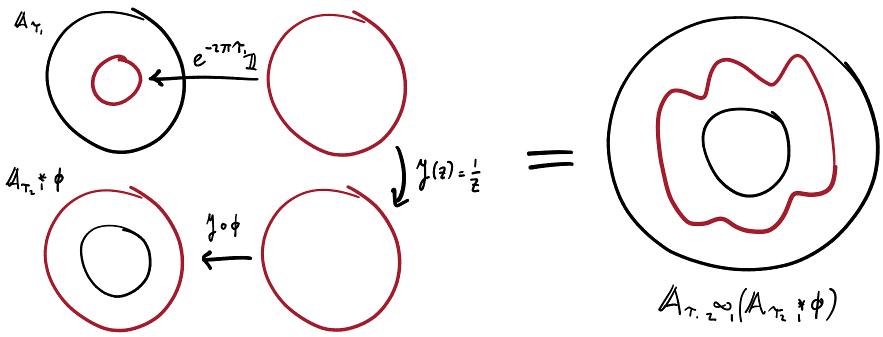


Figure 1.20: Two standard annuli are sewn after one boundary component is reparametrized by a diffeomorphism $\phi \in \mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$. The cocycle (1.4.11) is a measure of deformation due to the reparametrization, especially with respect to the trivialization by uniformized metrics.

The resulting real number $\det_\zeta \Delta_{g|_\Sigma}$ exhibits Weyl covariance under conformal change for $\partial\Sigma \neq \emptyset$,

$$\frac{\det_\zeta \Delta_{e^{2\sigma} g|_\Sigma}}{e^{\frac{1}{4\pi}\int_{\partial\Sigma} k_{e^{2\sigma} g} \mathrm{d}\ell_{e^{2\sigma} g}}} = e^{2\, S^0_L(\sigma, g)}\, \frac{\det_\zeta \Delta_{g|_\Sigma}}{e^{\frac{1}{4\pi}\int_{\partial\Sigma} k_g \mathrm{d}\ell_g}}. \tag{1.4.9}$$

Without the normalization, this anomaly is better known as the Polyakov–Alvarez anomaly formula [Pol81, Alv83, OPS88]. For a closed surface, the anomaly formula still holds, but one needs to exclude the zero eigenvalue in the definition of $\det_\zeta \Delta_{g|_\Sigma}$ and normalize by the volume instead, see [LM25, Appendix B]. One may think of $\det_\zeta \Delta_{g|_\Sigma}$ as the partition function of the free boson CFT, defining the trivialization

$$\left(\frac{\det_\zeta \Delta_{g|_\Sigma}}{e^{\frac{1}{4\pi}\int_{\partial\Sigma} k_g \mathrm{d}\ell_g}}\right)^{-\frac{\mathbf{c}}{2}} [g] \in \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma) \tag{1.4.10}$$

of the real determinant line bundle.

3. *Uniformized metrics.* Another way to produce a trivialization of $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}$ is to pick a metric $g_0(\Sigma)$ in the conformal class of each surface Σ, resulting in $[g_0(\Sigma)] \in \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma)$. Uniformized metrics type I or type II make interesting choices for this type of trivialization. For instance, in [MP25a], see also Section 2.1, we use the cylindrical metric on annuli. With respect to this type of trivialization, the coefficients λ in $\lambda[g_0(\Sigma)]$ may be interpreted as a measure of deformation due to sewing operations. See Figure 1.20. ⌟

Given any trivialization Z of $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}$, the sewing isomorphisms (1.4.5) define the *cocycle* on pairs of surfaces and boundary labels by the equation

$$Z(\Sigma_1)\, {}_j\overset{\mathbf{c}}{\infty}_k\, Z(\Sigma_2) = e^{\Omega^Z_{j,k}(\Sigma_1,\Sigma_2)} Z(\Sigma_1\, {}_j\infty_k\, \Sigma_2). \tag{1.4.11}$$

Since the constants λ in an element $\lambda[g] \in \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma)$ are always positive, we can denote the coefficient on the right-hand side as the exponential of a real number $\Omega^Z_{j,k}(\Sigma_1, \Sigma_2) \in \mathbb{R}$. This formulation has the advantage that, considering

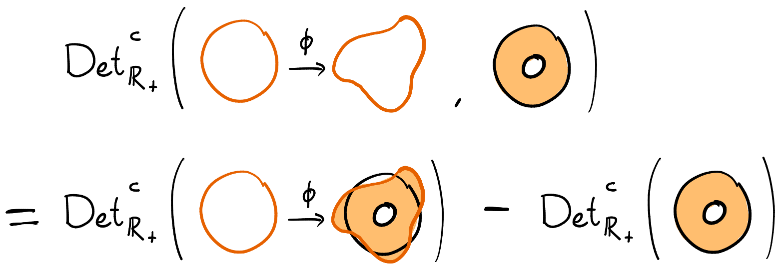


Figure 1.21: Illustration of the definition of the real determinant line of a complex deformation relative to a standard annulus.

the associativity of multiple sewing isomorphisms, we obtain additive *cocycle identities* such as

$$\Omega^{Z}_{j,k}(\Sigma_1,\Sigma_2)+\Omega^{Z}_{l,m}(\Sigma_1 \infty \Sigma_2,\Sigma_3) = \Omega^{Z}_{j,k}(\Sigma_1,\Sigma_2 \infty \Sigma_3)+\Omega^{Z}_{l,m}(\Sigma_2,\Sigma_3), \quad (1.4.12)$$

corresponding to the sewing of three surfaces Σ_1, Σ_2, Σ_2 such that Σ_3 attaches to Σ_2. In fact, by considering the sewing operations along the various boundary components, and also self-sewing, we find many cocycles and cocycle identities, which are spelled out in detail in Section 3.4 of [MP25b]; see also Appendix C.

1.4.2 Central extensions of complex deformations

In this section we explain how through the action of complex deformations on a surface $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$ by deforming a boundary component $1 \leq j \leq \mathsf{b}$, see Equation (1.3.23), the real determinant lines $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma)$ and $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma \underset{j}{*} \phi)$ pull back to the complex deformation $\phi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$. The idea is that the action of the real determinant lines would result in a map

$$\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma) \otimes \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\phi) \to \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma \underset{j}{*} \phi). \quad (1.4.13)$$

Anticipating such an action, we define the determinant line of ϕ relative to Σ and j, such that $\Sigma \underset{j}{*} \phi$ exists, as follows,

$$\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\phi,\Sigma,j) = \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma \underset{1}{*} \phi) \otimes \big(\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma)\big)^{\vee}, \quad (1.4.14)$$

where the $\vee$ stands for the dual. One way to view the idea behind this definition we take the real determinant line of the deformed surface $\Sigma \underset{j}{*} \phi$ and then "divide out" that of the original surface Σ by tensoring with the dual of the real determinant line; see also Figure 1.21.

Remark 1.4.3. The construction in [MP25a] uses annuli, and in particular the standard annulus $\mathbb{A}_1$, for the construction of the central extension, which goes back to Segal [Seg04] for the complex determinant line bundle, and was also used by Huang [Hua97, Appendix D]. In [MP25b], we find that instead of an annulus, as above, any surface $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$ and boundary components $1 \leq j \leq \mathsf{b}$ may be used in this definition. This yields the same central extension, but it

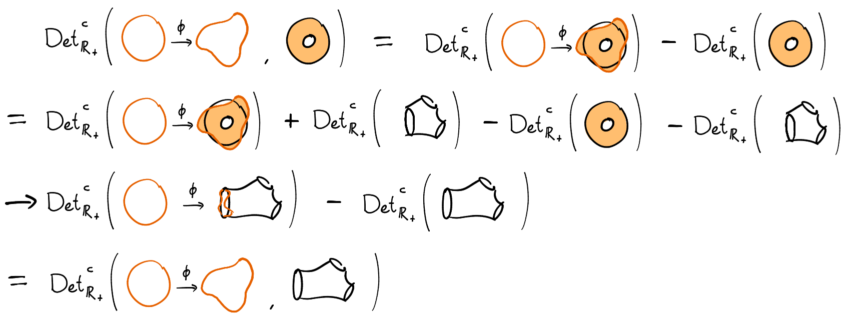


Figure 1.22: Construction of the isomorphisms (1.4.16) relating real determinant lines of complex deformations relative to various surfaces.

simplifies the action of the central extension on $\gamma \in \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma)$. In this setup, the action essentially replaces γ by α where $\alpha \otimes \gamma^{\vee} \in \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\phi, \Sigma, j)$. Especially $\Sigma = \mathbb{D}$ has the advantage that $\mathbb{D} \underset{1}{*} \phi$ is always defined, leading to a global trivialization

$$Z_{\mathbb{D},1}(\phi) = Z(\mathbb{D} \underset{1}{*} \phi) \otimes \big(Z(\mathbb{D})\big)^{\vee} \in \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\gamma, \mathbb{D}, j), \tag{1.4.15}$$

given a trivialization Z of $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\mathcal{M}_{0,1})$. ⌟

By defining the real determinant line of ϕ in the way of Equation (1.4.14), the question presents itself how $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\phi, \Sigma, j)$ depends on Σ and j. The locality of the conformal anomaly (1.0.1) suggests that, somehow, it only sees a neighborhood of $\partial_j \Sigma$. Indeed, by identifying two surfaces Σ_1 and Σ_2 in a neighborhood of boundary components $\partial_j \Sigma_1$ and $\partial_k \Sigma_2$, see Figure 1.22, we find isomorphisms $\mathrm{I}^{\Sigma_1,j}_{\Sigma_2,k,\phi} : \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\phi, \Sigma_1, j) \to \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\phi, \Sigma_2, k)$, which are natural in the sense that

$$\mathrm{I}^{\Sigma_1,j}_{\Sigma_1,j,\phi} = \mathbb{1}_{E_j(\phi,\Sigma_1)} \qquad \text{and} \qquad \mathrm{I}^{\Sigma_2,k}_{\Sigma_3,l,\phi} \circ \mathrm{I}^{\Sigma_1,j}_{\Sigma_2,k,\phi} = \mathrm{I}^{\Sigma_1,j}_{\Sigma_3,l,\phi}\,. \tag{1.4.16}$$

See [MP25a, Section 3.3] and [MP25b, Section 3] for details on these isomorphisms. By identifying elements of real determinant lines of complex deformations with respect to different surfaces through the isomorphisms, we define surface-independent real determinant lines

$$\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\phi) = \bigsqcup_{\Sigma \underset{j}{*} \phi \text{ exists}} \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\phi, \Sigma, j) \Big/ (1.4.16). \tag{1.4.17}$$

These form an $\mathbb{R}_+$-bundle over $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$, fitting into the exact sequence

$$0 \to \mathbb{R}_+ \to \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)) \to \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1) \to 0, \tag{1.4.18}$$

where the map $\mathbb{R}_+ \to \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ maps into $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\mathbb{1})$, which may be represented, for instance, by $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\mathbb{D}) \otimes (\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\mathbb{D}))^{\vee}$, which in turn is isomorphic to $\mathbb{R}_+$ by evaluating the first component in the second (said map is the identity after composition with the evaluation). The sequence (1.4.18) becomes a central extension of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ by $\mathbb{R}_+$ after introducing the composition law $\overset{\mathbf{c}}{*}$

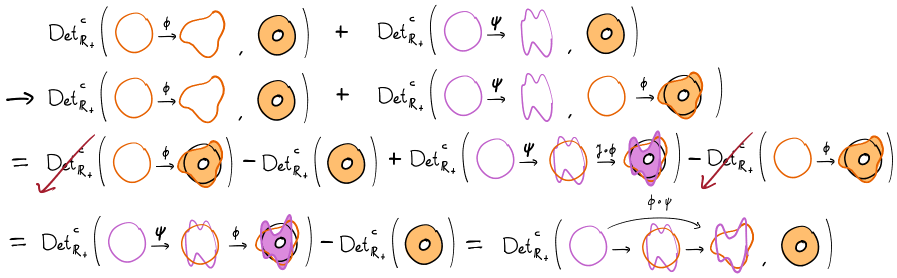


Figure 1.23: Illustration of the composition law $\overset{\mathbf{c}}{*}$ of the central extension.

on real determinant lines of composable complex deformations $\phi, \psi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$. This composition law is most concretely explained in terms of representatives of $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\phi)$ and $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\psi)$, see also Figure 1.23. Let $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\phi)$ be represented by $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma \underset{j}{*} \phi) \otimes (\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma \underset{j}{*} \phi))^\vee$, and let $\alpha \otimes \beta^\vee$ be an element of this tensor product. The other real determinant line $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\psi)$ may then be represented by

$$\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\psi, \Sigma \underset{j}{*} \phi, j) = \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma \underset{j}{*} (\phi \circ \psi)) \otimes \left(\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma \underset{j}{*} \phi)\right)^\vee \tag{1.4.19}$$

involving ϕ as well, and any element of this tensor product can be taken of the form $\gamma \otimes \alpha^\vee$ where $\alpha^\vee(\alpha) = 1$. For such representatives, the composition law is defined by

$$(\alpha \otimes \beta^\vee) \overset{\mathbf{c}}{*} (\gamma \otimes \alpha^\vee) = \gamma \otimes \beta^\vee, \tag{1.4.20}$$

which one can think of as evaluating the first component of the left-hand side of $\overset{\mathbf{c}}{*}$ in the second component of the right-hand side.

Trivializations of $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$, such as those of the form in Equation (1.4.15) above, define cocycles on $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ by

$$Z_{\Sigma,j}(\phi) \overset{\mathbf{c}}{*} Z_{\Sigma,j}(\psi) = e^{\Omega^{Z}_{\Sigma,j}(\phi,\psi)} Z_{\Sigma,j}(\phi \circ \psi). \tag{1.4.21}$$

Then, in [MP25a] we compute the corresponding Lie algebra cocycle, see also Theorem 2.1.1. In [MP25b], we compute the cohomology $H^2(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1), \mathbb{R})$ of all cocycles on $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$. Also in [MP25b], we generalize the construction of this central extension to the setting of a real one-dimensional modular functor, a notion defined in the next section.

1.4.3 The functorial perspective

In this section, I make an effort to put the conformal anomaly as described by the real determinant line bundle into a category-theoretic perspective. However, instead of narrowing down the precise categories and functors, my goal is to list abstract properties of the real determinant line bundle, which can then be used to select an appropriate setting. In the work [MP25b], see also Section 2.3 and Appendix C, it is shown that the real determinant line bundle is characterized just by these properties.

First, we identify the overall structure of the real determinant line bundle. As an $\mathbb{R}_+$-bundle over the moduli spaces $\mathcal{M}_{\mathsf{g},\mathsf{b}}$, it assigns a fiber $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma)$ isomorphic to $\mathbb{R}_+$ to each surface $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$. If we regard this as a functorial assignment with respect to the composition of cobordisms — the sewing operation defined in Equation (1.3.10) — we expect that there is a way to compose the fibers of any two surfaces. Indeed, such isomorphisms are defined by the Equation (1.4.5). Just this structure may be called a real one-dimensional modular functor, building on terminology in [Seg04, Bak01], and also in [Hua97] used in the context of vertex operator algebras. In this context, complex modular functors as considered with any dimension. Let us define the real one-dimensional case as follows.

Definition 1.4.4. A *real one-dimensional modular functor* E consists of $\mathbb{R}_+$-bundles

$$E(\mathcal{M}_{\mathsf{g},\mathsf{b}}) \longrightarrow \mathcal{M}_{\mathsf{g},\mathsf{b}}, \qquad \mathsf{g},\mathsf{b} \geq 0, \tag{1.4.22}$$

and bilinear maps called *sewing isomorphisms*,

$$\begin{aligned} \cdot \ {}_j\overset{E}{\infty}_k \ \cdot &: E(\mathcal{M}_{\mathsf{g}_1,\mathsf{b}_2}) \boxtimes E(\mathcal{M}_{\mathsf{g}_2,\mathsf{b}_2}) \to E(\mathcal{M}_{\mathsf{g}_1+\mathsf{g}_2,\mathsf{b}_1+\mathsf{b}_2-2}), \\ \overset{E}{\infty}_{j,k} \ \cdot &: E(\mathcal{M}_{\mathsf{g},\mathsf{b}}) \to E(\mathcal{M}_{\mathsf{g}+1,\mathsf{b}-2}), \end{aligned} \tag{1.4.23}$$

which are linear isomorphisms on fibers. They extend the respective sewing operations $\cdot \ {}_j\infty_k \ \cdot$, and $\infty_{j,k} \ \cdot$ including associativity

$$\cdot \ {}_j\overset{E}{\infty}_k \left(\cdot \ {}_l\overset{E}{\infty}_m \ \cdot \right) = \left(\cdot \ {}_j\overset{E}{\infty}_k \ \cdot \right) {}_l\overset{E}{\infty}_m \ \cdot, \tag{1.4.24}$$

$$\overset{E}{\infty}_{j,k} \left(\cdot \ {}_l\overset{E}{\infty}_m \ \cdot \right) = \left(\overset{E}{\infty}_{j,k} \ \cdot \right) {}_l\overset{E}{\infty}_m \ \cdot, \tag{1.4.25}$$

$$\overset{E}{\infty}_{j,k} \left(\overset{E}{\infty}_{l,m} \ \cdot \right) = \overset{E}{\infty}_{l,m} \left(\overset{E}{\infty}_{j,k} \ \cdot \right), \tag{1.4.26}$$

and symmetry

$$\begin{aligned} \alpha \ {}_j\overset{E}{\infty}_k \ \beta &= \beta \ {}_k\overset{E}{\infty}_j \ \alpha \\ \overset{E}{\infty}_{j,k} \ \alpha &= \overset{E}{\infty}_{k,j} \ \alpha \end{aligned} \qquad \alpha \in E(\mathcal{M}_{\mathsf{g}_1,\mathsf{b}_2}), \quad \beta \in E(\mathcal{M}_{\mathsf{g}_2,\mathsf{b}_2}). \tag{1.4.27}$$

In [MP25b] we include Frölicher smoothness into the definition, a property to which we come back in Section 2.3. In fact, the notion of bundle is meaningless here since we did not even have a topology on $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ so far — for now, assume that bundle just means fiber bundle with whatever regularity is given to the moduli spaces and the sewing operation. In the specific case of $E = \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}$, the bundle structure may be defined by a global trivialization; see Example 1.4.2.

So far, the real determinant line bundle clearly is a real one-dimensional modular functor. Part of the significance of the article [MP25a] is the verification that $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}$ is nontrivial as a real one-dimensional modular functor. Indeed, for different $\mathbf{c} \in \mathbb{R}$, we show that the real one-dimensional modular functors defined by $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}$ are not isomorphic. See also Theorem 2.1.1 in Section 2.1, and the comments in [MP25a, Section 1.3] for this specific application of the theorem. The notion of isomorphism used is the following.

Definition 1.4.5. An *isomorphism of real one-dimensional modular functors* $\Psi : E \to D$ consists of $\mathbb{R}_+$-bundle isomorphisms

$$\Psi_{\mathsf{g},\mathsf{b}} : E(\mathcal{M}_{\mathsf{g},\mathsf{b}}) \longrightarrow D(\mathcal{M}_{\mathsf{g},\mathsf{b}}), \qquad \mathsf{g},\mathsf{b} \geq 0, \tag{1.4.28}$$

preserving the sewing isomorphisms, that is,

$$\Psi_{\mathsf{g}_1+\mathsf{g}_2,\mathsf{b}_1+\mathsf{b}_2-2}(\,\cdot\,\,{}_j\overset{E}{\infty}_k\,\,\cdot\,) = \Psi_{\mathsf{g}_1,\mathsf{b}_1}(\,\cdot\,)\,{}_j\overset{D}{\infty}_k\,\Psi_{\mathsf{g}_2,\mathsf{b}_2}(\,\cdot\,), \tag{1.4.29}$$

$$\overset{D}{\infty}_{j,k}\,\Psi_{\mathsf{g},\mathsf{b}}(\,\cdot\,) = \Psi_{\mathsf{g}+1,\mathsf{b}-2}(\,\overset{E}{\infty}_{j,k}\,\,\cdot\,). \tag{1.4.30}$$

Again, the notion of bundle isomorphisms should follow the respective regularity of the modular functors.

Contrary to the complex case, where literature [Seg04, Hua97] suggests that the complex determinant line bundle should be unique up to choice of a central charge, we find that in the real case more properties of $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}$ need to be axiomatized for uniqueness to hold. Below, I make a few observations about the real determinant line bundle, attaching a name to each property. Then, definitions of the generalized properties satisfied by $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}$ follow.

1. *Locality.* Note that the definition of $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma)$ only depends on the conformal class of the surface, but not on the boundary parametrizations. Therefore, we can identify the fibers over surfaces related by reparametrization. In other words, $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\mathcal{M}_{\mathsf{g},\mathsf{b}})$ descends to the finite-dimensional moduli space $\check{\mathcal{M}}_{\mathsf{g},\mathsf{b}}$. Also, none of the trivializations in Example 1.4.2 depend on the boundary parametrizations — in this sense, they are pullbacks via the projection

$$\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\mathcal{M}_{\mathsf{g},\mathsf{b}}) \to \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\check{\mathcal{M}}_{\mathsf{g},\mathsf{b}}), \tag{1.4.31}$$

   of a corresponding trivialization of the latter. We call such trivializations *reparametrization invariant.* Note that the sewing operation $\cdot\,{}_j\infty_k\,\cdot$ on the moduli spaces $\mathcal{M}_{\mathsf{g},\mathsf{b}}$, like in Equation (1.3.2), only involves the boundary parametrizations at j and k. The same holds for the sewing isomorphisms of $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}$, since the new metric only needs a local modification on the seam to become a smooth metric, and by locality of the conformal anomaly, this change can be made in a neighborhood of the seam. The latter justifies the name *locality* for the property that the sewing isomorphisms factor through moduli spaces $\check{\mathcal{M}}^{j}_{\mathsf{g}_1,\mathsf{b}_1}$ and $\check{\mathcal{M}}^{k}_{\mathsf{g}_2,\mathsf{b}_2}$, whose elements respectively carry a boundary parametrization only at the jth and kth boundary component. Another way to express locality is as reparametrization invariance of the cocycles,

$$\Omega^{Z}_{j,k}(\Sigma_1 \underset{l}{*} \phi, \Sigma_2) = \Omega^{Z}_{j,k}(\Sigma_1 \underset{l}{*} \phi, \Sigma_2), \qquad \phi \in \mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1), \quad l \neq j. \tag{1.4.32}$$

   See [MP25b, Section 3.4] for more details.

2. *Flat metrics and modular invariance.* By sewing the boundaries of an annulus $A \in \mathcal{M}_{0,2}$, we obtain a torus $T = \infty_{1,2}\,A \in \mathcal{M}_{1,0}$. If $Z(A) = [g_0(A)]$ is the trivialization of $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(A)$ in Example 1.4.2 using cylindrical metrics, and the seam inside the torus T is a geodesic, then the sewing

operation induces a flat metric on the torus. See Figure 1.13. The case of annuli and tori is special in the sense that, because of their vanishing Euler characteristic, and by Example 1.2.3, the conformal anomaly vanishes for constant Weyl transformations. Hence, the scale of the flat metric on $\infty_{1,2} A$ does not change the element $[g_0(A)|_T] \in \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\infty_{j,k} A)$, making all conformal flat metrics on a torus equivalent. Therefore, here we find a property that could be called *modular invariance* stating the following: If a torus has two decompositions $T = \infty_{1,2} A = \infty_{1,2} B$ such that both seams are geodesic, then

$$[g_0(A)|_T] = [g_0(B)|_T] \in \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(T). \tag{1.4.33}$$

The significance here is that the geodesic seams can belong to different homotopy classes. To find an equivalent abstract property, the question remains how to algebraically characterize the trivialization $Z(A) = [g_0(A)]$ on the semigroup of annuli $\mathcal{M}_{0,2}$. Any reparametrization invariant trivialization W of $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\mathcal{M}_{0,2})$ is related to Z by a function $f : \mathbb{R}_+ \to \mathbb{R}$ of the modulus $\tau(A)$,

$$W(A) = e^{f(\tau(A))} Z(A), \qquad A \in \mathcal{M}_{0,2}. \tag{1.4.34}$$

The trivialization Z on standard annuli $\mathbb{A}_\tau$ has the additivity property

$$Z(\mathbb{A}_{\tau_1}) \, {}_1\overset{\mathbf{c}}{\infty}_2 \, Z(\mathbb{A}_{\tau_2}). = Z(\mathbb{A}_{\tau_1+\tau_2}). \tag{1.4.35}$$

This implies the following equivalence, making the function f additive in the modulus,

$$\Omega^W_{1,2}(\mathbb{A}_{\tau_1}, \mathbb{A}_{\tau_2}) = 0 \qquad \Longleftrightarrow \qquad f(\tau_1 + \tau_2) = f(\tau_1) + f(\tau_2). \tag{1.4.36}$$

This condition constrains the possible functions to $f(\tau) = \lambda\tau$ for some $\lambda \in \mathbb{R}$. Finding a further condition that forces $\lambda = 0$ is slightly more involved, and we state it as a cohomological condition. The left-hand side in Equation (1.4.36) implies that the cocycle $\Omega^W_{1,\mathbb{A}_\tau,1}(\phi_1, \phi_2)$ on complex deformations $\phi_1, \phi_2 \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ is relative to the subgroup of scaling transformations. In [MP25b, Section 2] we compute the relevant relative cohomology groups. On the one hand, we find that for the Lie algebra cocycle $\omega^W_{1,\mathbb{A}_\tau,1} = \mathrm{D}\,\Omega^W_{1,\mathbb{A}_\tau,1}$ there is some $\mathbf{h}_W \in \mathbb{R}$ such that

$$\omega^W_{1,\mathbb{A}_\tau,1} = \mathbf{c}\,\mathrm{Im}\,\omega_{\mathrm{GF}} + \mathbf{h}_W\,\mathrm{Im}\,\omega_{\mathrm{rot}} + \mathrm{d}\,\alpha, \tag{1.4.37}$$

where α is a Lie algebra 1-cycle relative to diffeomorphisms and scaling transformations. On the other hand, we know from [MP25a, Theorem 1.1], see also Theorem 2.1.1, that the Lie algebra cocycle $\omega^Z_{1,\mathbb{A}_\tau,1}$ for the cylindrical trivialization equals $\mathbf{c}\,\mathrm{Im}\,\omega_{\mathrm{GF}}$ precisely, not just up to coboundary, and therefore has $\mathbf{h}_Z = 0$. By the definition of the trivializations of $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ induced by Z and W, see Section 1.4.2, the difference between the respective Lie algebra cocycles for Z and W is given by the Lie algebra cohomology differential of the derivative

$$\beta = \mathrm{d}_{\mathbb{1}}\left(f(\mathbb{A}_\tau \underset{1}{*} \cdot\,) - f(\mathbb{A}_\tau)\right). \tag{1.4.38}$$

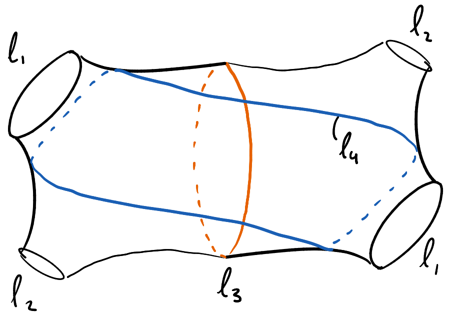


Figure 1.24: Sewing a hyperbolic pair of pants to another copy of itself along the same boundary components, and then cutting along the geodesic such that the boundary components of equal lengths are in the same connected component, results in pairs of pants with two equal boundary lengths each.

Generally speaking, we expect the Lie algebra cohomology differential $\mathrm{d}\,\beta$ to be a coboundary. However, the derivative in the scaling direction ℓ_0 is $\beta(\ell_0) = \lambda$, and thus, in the cohomology relative to the subalgebra $\mathbb{R}\ell_0$ it is nontrivial. In fact, since $\operatorname{Im}\omega_{\text{rot}}$ is the differential of a similar function with ℓ_0 derivative $1/24$, we find that $\mathrm{d}\,\beta$ is cohomologous to $\mathbf{h}_W \operatorname{Im}\omega_{\text{rot}}$ with $\mathbf{h}_W = 24\lambda$. We conclude that $f = 0$ if and only if the condition (1.4.36) holds and $\mathbf{h}_W = 0$, which is the anticipated cohomological condition.

3. *Hyperbolic metrics and crossing invariance.* For hyperbolic surfaces, see Equation (1.3.1), let Z denote the trivialization of $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}$ given by hyperbolic metrics. It has interesting invariance properties similar to Equation (1.4.35). Namely, if two hyperbolic surfaces in are sewn along boundary components of equal length, the hyperbolic metric on the sewn surface is just the union of the two hyperbolic metrics, and the seam is still a geodesic. See Figure 1.14. A special case is the situation where four hyperbolic pairs of pants $P_1, P_2, P_3, P_4 \in \mathcal{M}^{\mathsf{hyp}}_{0,3}$ with geodesic boundary are sewn pairwise with matching boundary lengths such that

$$P_1 \,{}_j\!\infty_k\, P_2 = P_3 \,{}_l\!\infty_m\, P_4. \tag{1.4.39}$$

In the topology of pants decompositions, see also Example 1.3.6, such a relation is called an A-move [HT80]. It is one of two elementary moves between pants decompositions by which any pair of pants decompositions is related through a finite number of moves. The other, called S-move, is related to the self-gluing of two hyperbolic pairs of pants $P_1, P_2 \in \mathcal{M}^{\mathsf{hyp}}_{0,3}$, respectively with matching boundary length, such that

$$\infty_{j,k} P_1 = \infty_{l,m} P_2. \tag{1.4.40}$$

For the trivialization Z, the relations (1.4.39) and (1.4.40) respectively imply

$$Z(P_1) \,{}_j\!\overset{\mathbf{c}}{\infty}_k\, Z(P_2) = Z(P_3) \,{}_l\!\overset{\mathbf{c}}{\infty}_m\, Z(P_4), \tag{1.4.41}$$

$$\overset{\mathsf{c}}{\infty}_{j,k} Z(P_1) = \overset{\mathsf{c}}{\infty}_{l,m} Z(P_2). \tag{1.4.42}$$

Now, how to characterize the trivialization Z without referring to the hyperbolic metric? Any other reparametrization invariant trivialization W over hyperbolic pairs of pants is related to Z by a function f of the boundary lengths $l_1, l_2, l_3 > 0$,

$$W(P) = e^{f(l_1,l_2,l_3)} Z(P), \qquad P \in \mathcal{M}^{\mathsf{hyp}}_{0,3}. \tag{1.4.43}$$

Let us only assume the property (1.4.41), which we call *crossing invariance*, for the trivialization W. By sewing any fixed hyperbolic pair of pants $P \in \mathcal{M}^{\mathsf{hyp}}_{0,3}$ to itself, say along $\partial_3 P$, we find a surface $P\ {}_3\!\infty_3\ P$ with four boundary components, divided into two pairs of equal length. By cutting $P\ {}_3\!\infty_3\ P$ along the geodesic separating these pairs from each other, like in Figure 1.24, we obtain an identity for the function f,

$$2f(l_1, l_2, l_3) = f(l_1, l_1, l_4) + f(l_2, l_2, l_4), \tag{1.4.44}$$

where $l_4 > 0$ is the length of said geodesic. The choice of the geodesic, and hence the length l_4, is not uniquely determined by l_3, which implies that $f(l_1, l_2, l_3)$ is independent of l_3. By applying the same argument to the other boundary components, we find that f is constant. Hence, the second property, Equation (1.4.42), which we call *hyperbolic modular invariance*, follows from the crossing invariance.

Now we turn toward a general real one-dimensional modular functor as in Definition 1.4.4. To reproduce the above properties of $\mathrm{Det}^{\mathsf{c}}_{\mathbb{R}_+}$, we make the following definitions. Again, refer to [MP25b, Section 3] for definitions including the Frölicher smoothness.

Definition 1.4.6. A real one-dimensional modular functor E is *local* if the bundles $E(\mathcal{M}_{\mathsf{g},\mathsf{b}})$ are pullbacks of bundles $E(\check{\mathcal{M}}_{\mathsf{g},\mathsf{b}})$, and the sewing isomorphisms descend to the bundles $E(\check{\mathcal{M}}^{j}_{\mathsf{g},\mathsf{b}})$ in the sense that the maps, also denoted $\cdot\ {}_j\!\infty^E_k\ \cdot$, and defined by the following diagrams, are independent of the choice of lift: For sewing,

$$\begin{array}{ccc} E(\mathcal{M}_{\mathsf{g}_1,\mathsf{b}_2}) \boxtimes E(\mathcal{M}_{\mathsf{g}_2,\mathsf{b}_2}) & \xrightarrow{{}_j\!\overset{E}{\infty}_k} & E(\mathcal{M}_{\mathsf{g}_1+\mathsf{g}_2,\mathsf{b}_1+\mathsf{b}_2-2}) \\ \downarrow & & \downarrow \\ E(\check{\mathcal{M}}^{j}_{\mathsf{g}_1,\mathsf{b}_2}) \boxtimes E(\check{\mathcal{M}}^{k}_{\mathsf{g}_2,\mathsf{b}_2}) & \overset{{}_j\!\overset{E}{\infty}_k}{\dashrightarrow} & E(\check{\mathcal{M}}_{\mathsf{g}_1+\mathsf{g}_2,\mathsf{b}_1+\mathsf{b}_2-2}) \end{array} \tag{1.4.45}$$

and for self-sewing,

$$\begin{array}{ccc} E(\mathcal{M}_{\mathsf{g},\mathsf{b}}) & \xrightarrow{\overset{E}{\infty}_{j,k}} & E(\mathcal{M}_{\mathsf{g}+1,\mathsf{b}-2}) \\ \downarrow & & \downarrow \\ E(\check{\mathcal{M}}^{j,k}_{\mathsf{g},\mathsf{b}}) & \overset{\overset{E}{\infty}_{j,k}}{\dashrightarrow} & E(\check{\mathcal{M}}_{\mathsf{g}+1,\mathsf{b}-2}) \end{array} \tag{1.4.46}$$

are independent of the choice of lift.

Note that [MP25b, Proposition 3.6] lists a number of equivalent properties, for example, the one in Equation (1.4.38).

For the modular invariance property, we are mainly interested in a generalization of Equation (1.4.33), which is part of the following definition. However, since we do not necessarily know whether a suitable trivialization exists, we need to make this assumption as well.

Definition 1.4.7. A local real one-dimensional modular functor is *flatly modular invariant* if there exists a reparametrization invariant trivialization Z such that

1. $\Omega^Z_{1,2}(\mathbb{A}_{\tau_1}, \mathbb{A}_{\tau_2}) = 0$ for any $\tau_1, \tau_2 > 0$.
2. The cocycle $\Omega^Z_{1,\mathbb{A}_\tau,1}$ which in the basis [MP25b, Section 2] of relative cohomology $H^2(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1); \mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1), \mathrm{Sc}; \mathbb{R})$ is a linear combination

$$\Omega^Z_{1,\mathbb{A}_\tau,1} = \mathbf{c}\,\mathrm{Im}\,\omega_{\mathrm{GF}} + \mathbf{h}_W\,\mathrm{Im}\,\omega_{\mathrm{rot}} + \mathrm{d}\,\alpha \tag{1.4.47}$$

   satisfies $\mathbf{h}_Z = 0$,

and every such trivialization has the property that for $A, B \in \mathcal{M}^{\mathsf{geod}}_{0,2}$ the following implication holds,

$$\infty_{1,2}\, A = \infty_{1,2}\, B \qquad \Longrightarrow \qquad \overset{E}{\infty}_{1,2}\, Z(A) = \overset{E}{\infty}_{1,2}\, Z(B). \tag{1.4.48}$$

We call such a trivialization Z *modular invariant* as well.

The definitions in the hyperbolic case follow a similar structure. A certain kind of trivialization, following (1.4.41), is required, which then automatically has the additional property (1.4.42). However, since we can conceptually separate these properties, we split the two parts related to the A- and S-moves in topology into two definitions, yet the second definition assumes the first.

Definition 1.4.8. A local real one-dimensional modular functor is *crossing invariant* if there exists a reparametrization invariant trivialization Z of $E(\mathcal{M}_{0,3})$ such that

$$Z(P_1)\,{}_j\infty_k\, Z(P_2) = Z(P_3)\,{}_l\infty_m\, Z(P_4), \tag{1.4.49}$$

for hyperbolic pairs of pants $P_1, P_2, P_3, P_4 \in \mathcal{M}^{\mathsf{hyp}}_{0,3}$ such that $P_1\,{}_j\infty_k\, P_2 = P_3\,{}_l\infty_m\, P_4 \in \mathcal{M}_{0,4}$ where the left and right hand sides have equal boundary lengths at the seams.

Then, assuming crossing invariance, we can define the following.

Definition 1.4.9. A local crossing invariant real one-dimensional modular functor E is *hyperbolically modular invariant* if any crossing invariant trivialization has the property that

$$\overset{E}{\infty}_{j,k}\, Z(P_1) = \overset{E}{\infty}_{l,m}\, Z(P_2), \tag{1.4.50}$$

for hyperbolic pairs of pants $P_1, P_2 \in \mathcal{M}^{\mathsf{hyp}}_{0,3}$ such that

$$\infty_{j,k}\, P_1 = \infty_{l,m}\, P_2 \in \mathcal{M}_{1,1} \tag{1.4.51}$$

where the left and right hand sides have equal boundary lengths at the seams.

Chapter 2

Summary of results

In this chapter, I summarize my work on the articles [MP25a] and [LM25] included in this thesis as Appendices A and B, as well as the manuscript [MP25b] in Appendix C.

2.1 From the conformal anomaly to the Virasoro algebra

This article concerns the relationship between the conformal anomaly, as defined by the formula in Equation (1.0.1), and the Virasoro algebra. The latter is an infinite-dimensional Lie algebra, that is a central extension of the Witt algebra $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ introduced in Section 1.3.2. The Virasoro algebra also characterizes the conformal anomaly on an infinitesimal level since a CFT Hilbert space carries a representation of it, determining the central charge $\mathbf{c}$ of the CFT. To highlight the difference with our method, I sketch a typical CFT textbook derivation of the relation between the two in Section 2.1.1. Even if these arguments are made rigorous in specific cases, such as in Liouville CFT [GKR24], they become quite involved and still only apply to a specific choice of CFT. Our method, on the other hand, is both concrete and mathematically rigorous, and moreover, agnostic to the choice of CFT. The trade-off is that, as such, it does not reveal new information about any specific CFT.

As a vector space, the Virasoro algebra is a direct sum

$$\mathfrak{Vir}_{\mathbf{c}} = \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1) \oplus \mathbb{C}, \qquad L_n = \ell_n + 0, \tag{2.1.1}$$

and we denote the generators ℓ_n of the Witt algebra by $L_n \in \mathfrak{Vir}_{\mathbf{c}}$ to highlight the new commutation relations[1],

$$[L_n, L_m] = (n-m)L_{n+m} + \mathbf{c}\,\frac{n^3-n}{12}\delta_{n+m,0}, \qquad n,m \in \mathbb{Z}. \tag{2.1.2}$$

Up to a coboundary, the central term is the Gel'fand–Fuks cocycle ω_{GF}

$$\omega_{\mathrm{GF}}(v,w) = \frac{1}{24\pi}\int_0^{2\pi} v'(\theta)w''(\theta)\,\mathrm{d}\theta, \qquad \omega_{\mathrm{GF}}(\ell_n,\ell_m) = \frac{\mathrm{i}}{12}n^3\delta_{n+m}, \qquad n,m\in\mathbb{Z}. \tag{2.1.3}$$

evaluated on the vector fields ℓ_n and ℓ_m.

[1] Whether the central charge $\mathbf{c} \in \mathbb{C}$ is part of the definition of the Virasoro algebra is a matter of convention — for different values of $\mathbf{c}$, they are isomorphic as Lie algebras, yet, as central extensions they are not isomorphic since that requires an isomorphism of exact sequences (1.1.5), see [MP25a, Section 2.2] for details.

The real determinant line bundle $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}$ is already discussed in Friedan and Shenker [FS87] and defined by Kontsevich and Suhov [KS07], using the conformal anomaly $S^0_L(\sigma, g)$ in Equation (1.0.1). In [MP25a, Section 2.5 and Appendix A] give a more detailed construction, proving all the necessary identities of $S^0_L(\sigma, g)$. Moreover, in [MP25a, Section 3.1], the sewing isomorphisms of $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}$ are introduced, proving in detail how the locality of $S^0_L(\sigma, g)$ assures that it is well-defined. See also Equation (1.4.5) above.

In [MP25a, Section 3.3], the central extension $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ mentioned in Section 1.4.2 is constructed with reference to cylinders, with particular emphasis on the standard cylinder $\mathbb{A} = (S^1 \times [0,1], \theta, \theta + \mathrm{i}\,)$. See the comments on the use of cylinders instead of annuli further below. While for the diffeomorphism group $\mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$ and complex determinant line bundle, the general idea is already present in the work of Segal [Seg04] and Huang [Hua97], we spell out all the definitions, proofs of well-definedness and associativity, and constructions of the exact sequences for the first time. The result then yields the group-level cocycle $\Omega^Z_{1,\mathbb{A}_1,a}(\phi_1, \phi_2)$ — denoted $\log \Gamma_{\mathbf{c}}(\phi_1, \phi_2)$ there — on complex deformations, also mentioned in Equation (1.4.21) in this work. The trivialization used is that of uniformized metrics in Example 1.4.2 in the special case of cylinders.

The main theorem of the paper is the computation of the Lie algebra cocycle.

Theorem 2.1.1. With respect to the trivialization on annuli defined by the unique flat (type II) metric of $Z(A) = [g_0(A)] \in \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\mathcal{M}_{0,2})$, the Lie algebra cocycle associated to $\Omega^Z_{1,\mathbb{A}_1,1}$ equals the imaginary part of the Gel'fand-Fuks cocycle at central charge $\mathbf{c}$, that is, for $v, w \in \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$,

$$\begin{aligned}&\frac{\partial^2}{\partial s \partial t}\bigg|_{t=s=0} \Big(\Omega^Z_{1,\mathbb{A}_1,1}\big(\Phi_v(t,\,\cdot\,), \Phi_w(s,\,\cdot\,)\big) - \Omega^Z_{1,\mathbb{A}_1,1}\big(\Phi_w(s,\,\cdot\,), \Phi_v(t,\,\cdot\,)\big)\Big) \\ &= \mathbf{c}\,\mathrm{Im}\,\omega_{\mathrm{GF}}(v,w).\end{aligned} \tag{2.1.4}$$

Note that the theorem does not only show that the central extension is isomorphic to a variant of the Virasoro algebra only using the imaginary part of the cocycle, but it is also a direct calculation of the Lie algebra cocycle, which determines it not just up to a coboundary. This may be relevant in future work on Kähler structures, where in the case of the universal Teichmüller space, many cocycles on the Witt algebra are Kähler forms, but certain coboundaries are preferred, see [BR87].

The main technical effort of this paper is the computation of the derivative in Equation (2.1.4), see [MP25a, Section 4] for the proof. The conceptual part of this proof is to find the right setup, reducing it to computations of elementary integrals. The setup involves choosing particular families of metrics on the cylinders

$$\mathbb{A} \underset{1}{*} \Phi_v(t,\,\cdot\,), \qquad \mathbb{A} \underset{1}{*} \Phi_w(s,\,\cdot\,), \qquad \mathbb{A} \underset{1}{*} \Phi_v(t, \Phi_w(s,\,\cdot\,)), \tag{2.1.5}$$

which enables a direct computation of the derivative of $S^0_L(\sigma, g)$ where g is the flat reference metric on the cylinder and σ becomes a one-parameter family of Weyl transformations resulting from the choices of metric. These choices may be summarized as follows.

1. The use of cylinders $S^1 \times [0, L]$ of length $L \geq 0$ with coordinates (θ, x) offers significant computational simplification over the use of annuli in the complex plane, which conversely provide more geometric intuition towards their complex structure. While the complex structure on cylinders is slightly unconventional with $z = \theta + \mathrm{i}\, x$, it does treat both the periodic and the radial part on equal footing as opposed to $z = Re^{\mathrm{i}\,\theta}$ in the complex plane. Otherwise, cylinders and annuli are equivalent through the exponential map. This perspective carries through for complex deformations, which are defined here as deformations of S^1 inside the infinite cylinder $S^1 \times \mathbb{R}$ instead of the complex plane.

2. On the first two cylinders in Equation (2.1.5), the same coordinates may still be used at $t, s \neq 0$ by viewing them as subsets of the infinite cylinder $S^1 \times \mathbb{R}$. Note that the action of the flows both cuts from and adds to the surface, see Figure 1.3.23. The metrics on these cylinders are defined in two parts. First, we consider the reference flat metric on the infinite cylinder. Then we consider the pushforward of the flat metric via the parametrization, given by the flow, near the deformed boundary of the surface. Since the flow is conformal, the pushforward metric differs from the reference metric by a simple conformal factor. The convenient metric is one that interpolates both these metrics using a smooth bump function, whose derivative has compact support in a small cylindrical subset. That is, it changes values from 0 to 1 in this subset, and is constantly 0 or 1 everywhere else. See also Figures 2 and 3 in [MP25a, Section 4].

3. Perhaps the most important choice is the support for the bump functions. Since via $\Phi_v(t, \,\cdot\,)$ part of the surface $\mathbb{A} \underset{1}{*} \Phi_w(s, \,\cdot\,)$ maps into $\mathbb{A} \underset{1}{*} \Phi_v(t, \Phi_w(s, \,\cdot\,))$, we have two metrics on the latter, one from the pushforward of this mapping, and another from extending that on $\mathbb{A} \underset{1}{*} \Phi_v(t, \,\cdot\,)$. The trick is to choose the supports such that we can turn these two metrics into a two-step interpolating metric. Finally, some care is needed to ensure that these choices can be made uniformly for t and s small enough. See also Figure 4 in [MP25a, Section 4].

With these families of metrics at hand, the next step is to identify how the cocycle $\Omega^{Z}_{1,\mathbb{A},1}(\Phi_v(t, \cdot), \Phi_w(s, \cdot))$ is expressed in terms of the conformal anomaly of their conformal factors with respect to the flat reference metric. This involves careful analysis of how the anomalies carry through the multiplication, see [MP25a, Figure 5]. Finally, one carries out the aforementioned elementary computation.

Aside from the main theorem, several small contributions are made in the paper at hand, which we summarize here.

1. The complex deformations discussed in Section 1.3.2 of this thesis are first introduced in [MP25a], but without geometric structure, and in the slightly different (yet equivalent) cylindrical setup. Conceptually, complex deformations offer a complementary perspective on the semigroup of annuli. The latter has the advantages that composability (sewing in this case) is guaranteed to hold, and there is a well-defined concept of surface, which

allows making further mathematical constructions on top of the semigroup of annuli, such as the GFF; See e.g. [BGKR24]. On the other hand, complex deformations are more flexible in their infinitesimal description. Emerging naturally from the flow equations, they are more amenable to the computation of derivatives, which, as exhibited in the proof of Theorem 2.1.1, can be very useful.

2. In [MP25a, Section 1.3], we briefly discuss the role of $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}$ as a real one-dimensional modular functor, drawing the conclusion that it is nontrivial as such since by the main theorem it has non-zero central charge. This aspect is elaborated more in Section 1.4.3 of this thesis.
3. In [MP25a, Section 3.2] we discuss the zeta-regularized determinant of Laplacian as a trivialization of $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}$ (see also Example 1.4.2). While this alone is not new, see [Dub15], it leads to an interesting observation. Formalizing ideas of Kontsevich and Suhov [KS07], we define the determinant line of an analytic loop in Σ by

$$\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma,\gamma) = \mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma) \otimes \bigotimes_{A\in\pi_0(\Sigma\setminus\gamma)} \big(\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(A)\big)^{\vee}, \tag{2.1.6}$$

where the tensor product is taken over the connected components of $\Sigma \setminus \gamma$, and $\vee$ denotes the dual. There is a natural evaluation function on $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma,\gamma)$, identifying it with $\mathbb{R}_+$ by applying the sewing isomorphisms (1.4.5) to the determinant lines of said connected components, and subsequently evaluating the resulting element of $(\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}(\Sigma))^{\vee}$ at the first component. The observation is that when this evaluation function is applied to the trivialization using zeta-regularized determinants of the Laplacian, this results in the loop Loewner energy, as discussed in Section 1.2.3 as the presumptive action functional of loop SLE.

The last item is the starting point of the second paper [LM25], also included in this thesis in Section 2.2.

2.1.1 Interlude: Bypassing the stress-energy tensor construction

In this section, I briefly sketch the relationship between the conformal anomaly and the Virasoro algebra as often explained in CFT textbooks. There, both the conformal anomaly as in formula in Equation (1.0.1), and the Virasoro algebra, see Example 1.1.2, are two faces of the same anomaly, and they may be related through a special observable called the *stress-energy tensor*. Please note that the lack of mathematical rigor in this section is part of the motivation for our work in [MP25a], where we provide a rigorous relation to the Virasoro algebra by different means.

Following Gawedzki [Gaw99], one way to define the stress energy tensor is via variations of correlation functions (1.2.6) of primary fields with respect to the metric g. Using Einstein notation in real coordinates x^1, x^2 around a point $z \in \Sigma$, the metric is given by real-valued functions g_{jk} such that

$$g = g_{jk}\, \mathrm{d}x^j \otimes \mathrm{d}x^k. \tag{2.1.7}$$

A correlation function involving the jk-component of the stress-energy tensor $T_{jk}(z)$ at $z \in \Sigma$ is defined as a variation with respect to the function g^{jk} obtained by inverting the tensor g_{jk} pointwise,

$$\langle T_{jk}(z) F \rangle_g = 4\pi \left. \frac{\partial}{\partial t} \right|_{t=0} \langle F \rangle_{g_t}, \tag{2.1.8}$$

where F is a product of primary fields as in Equation (1.2.6), and g_t is a suitable one-parameter family of metrics varying only the g^{jk} coordinate localized around z. Note that these variations may deform g outside the conformal class $\mathrm{Conf}(\Sigma)$. Multiple insertions of the stress-energy at distinct points are defined by multiple consecutive variations.

Since the metric is a symmetric tensor, the stress-energy tensor is symmetric as well, and hence defined by three components T_{11}, T_{22} and T_{12}. Assume for the moment that the reference metric g is flat around z, that is $g = \mathrm{d}x^1 \otimes \mathrm{d}x^1 + \mathrm{d}x^2 \otimes \mathrm{d}x^2$. In this case the trace of the stress energy tensor is a variation by Weyl transformations localized at z. By Weyl covariance (1.2.7) and the Dirichlet principle, these variations of $\langle F \rangle_g$ vanish, that is,

$$\langle (T_{11}(z) + T_{22}(z)) F \rangle_g = 0. \tag{2.1.9}$$

Note that this is not true if the reference metric is not locally flat. Instead, the trace becomes $-\frac{\mathbf{c}}{6} R_g$, where R_g is the Gaussian curvature, whence the conformal anomaly is also called the trace anomaly.

Keeping the assumption of a flat metric near z, let $z = x^1 + \mathrm{i}\, x^2$ also denote the complex local coordinate in which we define

$$\begin{aligned} \langle T_{zz} F \rangle_g &= \frac{1}{4} \langle (T_{11}(z) - T_{22}(z) - 2\mathrm{i}\, T_{12}(z))\ F \rangle_g, \\ \langle T_{z\bar{z}} \rangle &= \frac{1}{4} \langle (T_{11}(z) + T_{22}(z)) F \rangle_g = 0. \end{aligned} \tag{2.1.10}$$

We assumed multiple insertions of the stress-energy tensors to be at distinct points, since for example, the function $\langle T_{zz} T_{ww} \rangle_g$ is singular as $w \to z$. However, this specific function is generally assumed to have asymptotics in the limit $w \to z$ of the form

$$\langle T(z)\, T(w) \rangle_g = \frac{\mathbf{c}/2}{(z-w)^4} + \frac{2}{(z-w)^2} \langle T(w) \rangle_g + \frac{1}{z-w} \partial_w \langle T(w) \rangle_g + \mathcal{O}\big(|z-w|\big), \tag{2.1.11}$$

see [Gaw99] for a computation. This type of asymptotics is essential for the algebraic approaches to CFT, and is called *operator product expansion*. Now to find the Virasoro algebra, the asymptotics (2.1.11) are analyzed for a Laurent-type expansion of T_{zz}. In a specific setup of $\Sigma = \hat{\mathbb{C}}$, and with the notations still making sense only inside $\langle \cdot F \rangle_g$ one defines

$$L_n = \frac{1}{2\pi \mathrm{i}} \int_{S^1} z^{n+1} T(z)\, \mathrm{d}z, \qquad n \in \mathbb{Z}. \tag{2.1.12}$$

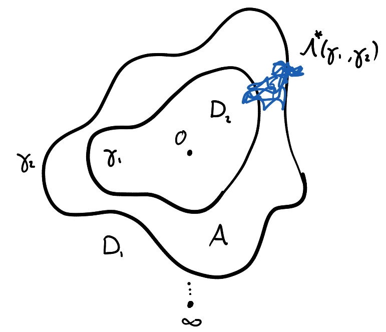


Figure 2.1: A pair of non-intersecting simple analytical loops, and a Brownian loop intersecting both.

Using the operator product expansion as a product, the commutator of the modes is computed to be

$$\begin{aligned}[L_n, L_m] &= \frac{1}{2\pi \mathrm{i}} \int \int z^{n+1} w^{m+1} \Big(T(z)T(w) - T(w)T(z) \Big) \, \mathrm{d}z \, \mathrm{d}w \\ &= (n-m) L_{n+m} + \mathbf{c} \, \frac{n^3 - n}{12} \delta_{n+m,0},\end{aligned} \tag{2.1.13}$$

which are the relations of the Virasoro algebra as defined in Equation (2.1.2). Going from Equations (2.1.11) and (2.1.12) to Equation (2.1.13) is an informal computation of contour integrals, see for example [Fra99, Section 6.2].

2.2 Two-loop Loewner potentials

In some way, this article is a continuation of the observation in the first article concerning the loop Loewner energy of a single loop in the Riemann sphere. At least in the context of this thesis focusing on the conformal anomaly, the two-loop Loewner potentials defined in [LM25] are natural generalizations of the loop Loewner energy (1.2.10) following from the general theory in [LM25, Appendix B] using the real determinant line bundle. The initial definition of the two-loop potential, however, is a probabilistic one.

The general setup of this article involves two non-intersecting simple smooth or analytic loops γ_1 and γ_2 in the Riemann sphere $\hat{\mathbb{C}}$. Generally, we assume that both loops separate 0 and ∞ such that γ_1 is nested inside γ_2 Moreover, we denote by D_1 and D_2 the simply connected subsets of $\hat{\mathbb{C}}$ bounded respectively by γ_1 and γ_2 such that $0 \in D_1$ and $\infty \in D_2$. We denote the annulus between the loops by A, and its modulus by τ; see Figure 2.1. The two-loop Loewner potential is defined as

$$\mathcal{H}(\gamma_1, \gamma_2) = \mathcal{H}(\gamma_1) + \mathcal{H}(\gamma_2) + \Lambda^*(\gamma_1, \gamma_2). \tag{2.2.1}$$

It is a combination of the respective one-loop Loewner potentials and a new probabilistic interaction term: $\Lambda^*(\gamma_1, \gamma_2)$ is a renormalization of the total mass of continuous loops intersecting both γ_1 and γ_2 under Brownian loop measure (BLM) introduced in Section 1.2.1. The renormalization was introduced in [LW04, FL13], and is necessary since said total mass is infinite under BLM due to increasingly long loops.

With [LM25, Theorem 2.1], we prove that this probabilistic two-loop Loewner potential has the unique interaction term such that the two-loop SLE satisfies a conformal restriction property similar to the single loop case in Equation (1.2.9). We also generalize the main theorem of [CW23] to the two-loop case, that is, we show that $\mathcal{H}(\gamma_1, \gamma_2)$ is an Onsager–Machlup functional for the newly defined two-loop SLE. This is a mathematically rigorous way of saying that the Loewner potential is an action functional for SLE.

Perhaps the most interesting technical contribution of this article is the expression of the interaction term $\Lambda^*(\gamma_1, \gamma_2)$ by zeta-regularized determinants of Laplacians (defined in Example 1.4.2). The proof builds on a formula due to Dubédat [Dub09, Proposition 2.1] in the unnormalized case, and an approximation of zeta-regularized determinants of Laplacians by BLM with restricted quadratic variation [APPS22, Theorem 1.3]. By combining the resulting formula,

$$\Lambda^*(\gamma_1, \gamma_2) = \log \frac{\det_\zeta \Delta_{g|_{D_1 \cup A}} \det_\zeta \Delta_{g|_{A \cup D_2}}}{\det_\zeta \Delta_{g|_{\hat{\mathbb{C}}}} \det_\zeta \Delta_{g|_A}} + \log 2 - \log 4\pi, \tag{2.2.2}$$

see [LM25, Corollary 3.2], with the definition of the one-loop Loewner potential (1.2.10), we find another expression of the two-loop Loewner potential[2], see [LM25, Theorem 3.1],

$$\mathcal{H}_{\hat{\mathbb{C}},2}(\gamma_1, \gamma_2) = \log \frac{\det_\zeta \Delta_{g|_{\hat{\mathbb{C}}}}}{\det_\zeta \Delta_{g|_{D_1}} \det_\zeta \Delta_{g|_A} \det_\zeta \Delta_{g|_{D_2}}} + \log 2 - \log 4\pi. \tag{2.2.3}$$

We also find two other formulas for $\mathcal{H}_{\hat{\mathbb{C}},2}(\gamma_1, \gamma_2)$, but since that work in Sections 3.2 and 3.3 of [LM25] originated from the master's thesis of Yan Luo [Luo23], my contribution to these sections is limited to restructuring the presentation of the proofs.

Indeed, the formula (2.2.3) is related to the real determinant line bundle through the trivialization in Example 1.4.2 also involving zeta-regularized determinants of Laplacians, see Section 5 of [LM25] for details. The possibility of having other trivializations of $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}$ such as CFT partition functions results in different two-loop Loewner potentials, as defined in [LM25, Appendix A]. This brings us to the main conceptual insight of this article. While SLE universally appears as the law of interfaces in CFT, it may differ from the established probabilistic definition of SLE by a Radon-Nikodym derivative depending on the moduli of the involved surfaces, such as the annulus A in the two-loop case. Of course, this is only possible if multiple connected domains are involved. In the introduction of this article, we provide extensive motivation for the perspective of using CFT partition functions for the Loewner potential, also using heuristics of discrete statistical mechanics similar to Example 1.1.1 in this thesis, we also point out possible relations to other works, such as on random annuli [ARS22].

[2] The constants are kept separate because they might change depending on conventions used for the zeta-regularized determinants.

To show that the dependence on the modulus makes a difference, we investigate this for the probabilistic two-loop Loewner potential (2.2.1). We find that there is a problem in the interpretation of the probabilistic two-loop Loewner potential as an action functional since it is bounded neither from above nor from below. Therefore, a definition of a two-loop Loewner energy by normalization of the potential — by subtracting its minimum — is not possible. To prove this, we use the following two-step strategy.

1. *[LM25, Section 4.1].* There is a variational formula proven in [TT06, SW24] for the one-loop Loewner potential. We use it to derive a variational formula of $\mathcal{H}(\gamma_1, \gamma_2)$ for analytic deformations of γ_1 and γ_2 keeping the modulus τ fixed. These variations vanish if and only if both loops are circles.
2. *[LM25, Section 4.2].* By explicit formulas found in [Wei87], we compute $\mathcal{H}_{e^{-2\pi\tau}S^1,S^1}$, showing that it diverges to $-\infty$ as the circles move further apart ($\tau \to \infty$) and to $+\infty$ as the circles merge ($\tau \to 0$).

This suggests that in any CFT application of SLE in the presence of multiply connected surfaces needs to take into account the dependence of the partition functions on moduli. To illustrate how this might look, we give a very basic example using the theory of boundary CFT by Cardy [Car08], see [LM25, Example 5.4].

Finally, I would like to mention that the application to the real determinant line bundle suggests making some adjustments to the normalization of zeta-regularized determinants of Laplacians, such that the Polyakov–Alvarez anomaly formula [Pol81, Alv83, OPS88] agrees with the conformal anomaly as in Equation (1.0.1). On the one hand, for a compact surface without boundary, as explained in [LM25, Appendix A], the zeta-regularized determinant of the Laplacian excluding the zero-mode may be normalized by the volume of the surface. On the other hand, in the presence of a boundary, the zeta-regularized determinant of the Laplacian may be normalized by an integral of the boundary curvature, see [LM25, Appendix B].

2.3 Universality of the conformal anomaly

The main result in the manuscript [MP25b] is the classification of real one-dimensional modular functors with additional properties of locality and modular invariance, as introduced in Section 1.4.3 of this thesis. Moreover, the result holds with regularity assumptions relating to the Frölicher structure on the infinite-dimensional moduli spaces $\mathcal{M}_{\mathtt{g},\mathtt{b}}$ also introduced in the manuscript, see [MP25b, Section 1], and see [MP25b, Section 3] for the corresponding definition of real one-dimensional modular functor. Here, we formulate the result by relating to the real determinant line bundle $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}$.

Theorem 2.3.1. Up to the choice of a central charge $\mathbf{c} \in \mathbb{R}$, there exists only one local, flatly modular invariant, crossing invariant, and hyperbolically modular invariant (Frölicher smooth) real one-dimensional modular functor up to isomorphism, and it is isomorphic to $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}$.

This way of interpreting the results portrays it as a universal property of the real determinant line bundle. Since the latter is an elementary construction using conformal anomaly $S_L^0(\sigma, g)$, this perspective explains mathematically the appearance of the formula (1.0.1) in CFT.

There are multiple technical contributions in the manuscript, which are prerequisites to the proof of Theorem 2.3.1, but are also of interest on their own. Perhaps most noteworthy, we find a characterization of reparametrization invariant disk-disk cocycles $\Omega^Z_{1,1}(D_1, D_2)$ in terms of the Loewner energy of loop SLE, or equivalently universal Liouville action on universal Teichmüller space. It is the one-loop case of the Loewner energy or potential, which we discussed in Section 2.2 and around Equation (1.2.10). Our result [MP25b, Theorem 4.2] states that

$$12\ \Omega^Z_{1,1}(\mathbb{D} \underset{1}{*} \phi, \mathbb{D}) = \frac{\mathsf{c}}{2}\, \mathrm{I}^{\mathrm{L}}(\gamma) + (\text{const.}), \qquad \phi \in \mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1), \tag{2.3.1}$$

up to a constant independent of ϕ. Here, the analytical diffeomorphism ϕ and the Jordan loop γ are related via conformal welding.

The proof of (2.3.1) and other results in the manuscript use applications of the theory of Frölicher spaces to the moduli spaces $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ introduced in Section 1.3.1. See [MP25b, Section 1.1] for a brief introduction to Frölicher structures. The action of complex deformations on the moduli spaces, possibly in combination with sewing, fully determines the tangent spaces — a result which in algebraic geometry is often called Virasoro uniformization [Kon87, BS88]. By integrating the flow equations (1.3.21) of smoothly time-dependent vector fields, we first define a Frölicher structure on complex deformations. This complements the introduction of complex deformations in [MP25a], associating rigorously an infinite-dimensional Lie algebra to $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$. The Lie algebra exists by checking the general condition of Laubinger [Lau11], and as expected, we find that it is isomorphic to the Witt algebra $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$. Coming back to the proof of Equation (2.3.1), we apply our result [MP25b, Proposition 1.8] on integral representations of Frölicher smooth functionals on the Lie algebra of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$. The integral representation is achieved using a Cauchy–Hilbert transform, leading back to the Grothendieck–Köthe–Sebastião e Silva duality [Mor93]. We find the connection between the functional analysis and the Frölicher structures through the work of Kriegl and Michor [KM97].

Through the actions of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ on the moduli spaces $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ by deformation of the boundary components, see Equation (1.3.23), smooth curves in $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ generate curves in $\mathcal{M}_{\mathsf{g},\mathsf{b}}$, which in turn generate a Frölicher structure on $\mathcal{M}_{\mathsf{g},\mathsf{b}}$. Naturally, this induced Frölicher structure is sufficient to differentiate the actions of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$, yielding Lie algebra homomorphisms

$$\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1) \to \mathrm{Vect}(\mathcal{M}_{\mathsf{g},\mathsf{b}}). \tag{2.3.2}$$

This aspect of the Frölicher structures is used to make sure that the central extensions $\mathrm{Det}^{\mathsf{c}}_{\mathbb{R}_+}(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ as defined in Section 1.4.2, or $E(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ for general real one-dimensional modular functors, are indeed Frölicher smooth central extensions of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$, which correspond to Frölicher smooth cocycles. In [MP25b, Section 3], we generalize the construction outlined in Section 1.4.2

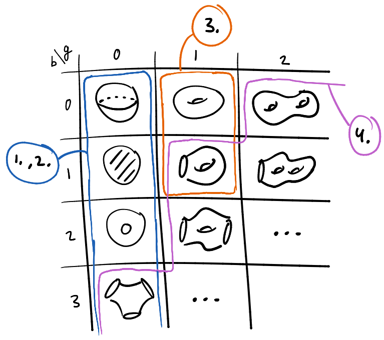


Figure 2.2: Overview of the proof of Theorem 2.3.1.

to real one-dimensional modular functors. We also list a number of cocycle identities of the cocycles (1.4.11), showcasing the algebraic side of the interaction between complex deformations and surfaces, and give several equivalent characterizations of the locality property in Definition (1.4.6).

To be able to generalize the flat modular invariance property of $\mathrm{Det}^{\mathbf{c}}_{\mathbb{R}_+}$ to real one-dimensional modular functors, see 1.4.7, we make a detailed computation of the cohomology of complex deformations in degrees 1 and 2, also relative to the subgroups of diffeomorphisms and scaling transformations, and including the cohomology on the Lie algebra level. To this end, we generalize a result found in [Nee04] to the Frölicher space and relative cohomology setting, aiding our computations by putting the fundamental group, the group-level cohomology, and the Lie algebra cohomology into an exact sequence. Note, however, that our result is at the same time less general since we use the trivial module $\mathbb{R}$ for the coefficients. The computation of the cohomologies reveals an interesting relationship between the classical rotation number of diffeomorphisms, which we generalize to complex deformations, and the also classically defined conformal radius applied to complex deformations, combining them into a complex-valued function. In the relative cohomology, the differential of this function becomes a nontrivial cocycle Ω_{Rot}, the Lie algebra cocycle of which appears in the definition of flat modular invariance.

Finally, there is an alternative way of getting an isomorphism in genus 0 without modular invariance. To explain this further, we first sketch the steps of the proof of Theorem 2.3.1. See Figure 2.2 for a graphical overview of the proof structure.

1. The disk-disk cocycle is determined by Equation (2.3.1). After normalization, this defines the isomorphism at the level of the Riemann sphere and disks.
2. The isomorphism for surfaces with $\mathsf{b} \geq 2$ boundary components follows by an induction step. In fact, this induction step works in any genus provided that the isomorphism is known for any number of boundary components in lower genera.
3. For genus 1, initially tori and handles (tori with one boundary component), flat modular invariance, Definition 1.4.7, is used to define a trivialization

over tori. Since the property of a trivialization being flat modular invariant is a genus 0 property, the induced trivialization of $D(\mathcal{M}_{0,2})$ is flat modular invariant as well — defining the isomorphism on tori. The isomorphism on handles is constructed similarly to the induction step, but with slightly modified arguments.

4. All further isomorphisms are constructed using crossing invariant trivalizations, again a genus 0 condition, see Definition 1.4.8, and pants decompositions. By hyperbolic modular invariance, see Definition 1.4.9, we have invariance under both A- and S-moves, which implies independence of the pants decomposition.

Note that up to the second step, only locality was used. Thus, we have the corollary that in genus 0, locality is enough to characterize real one-dimensional modular functors up to isomorphism by the central charge. Similarly, by applying the induction step after the third step, only flat modular invariance is needed to obtain an isomorphism up to genus 1.

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Appendix A

Publication: From the conformal anomaly to the Virasoro algebra

Please refer to the following publication:

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Appendix B

Publication: Two-loop Loewner potentials

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Appendix C

Manuscript: Universality of the conformal anomaly

Universality of the conformal anomaly

Sid Maibach* and Eveliina Peltola†

December 2, 2025

Abstract

We prove a universal property of real one-dimensional modular functors: Given that two such modular functors are also local and modular invariant, they are isomorphic if they have the same central charge $\mathbf{c} \in \mathbb{R}$. The guiding example is the real determinant line bundle over infinite-dimensional moduli spaces of connected compact Riemann surfaces with analytically parametrized boundary components, defined by the conformal anomaly of conformal field theories with central charge $\mathbf{c}$.

Algebraically, the modular functors are characterized by real-valued cocycles on pairs of surfaces. The proof of the universal property crucially depends on the identification of the disk-disk cocycle as the loop Loewner energy, also known as universal Liouville action.

Moreover, we study a local group-like space of complex deformations of the unit circle as an infinite-dimensional space with a Frölicher structure such that it has a Lie algebra, which is given by the Witt algebra. Since complex deformations act naturally on the aforementioned moduli spaces by deformation of boundary components, our computations of the group-level cohomology of complex deformations are also essential for the proof of the universal property. In the cohomology groups, we find the Bott–Thurston cocycle, related to the Gel'fand–Fuks cocycle of the Virasoro algebra, and further cocycles related to the rotation number and conformal radius of a complex deformation.

This manuscript is produced solely to be included in the PhD thesis of Sid Maibach. An introduction and a summary of the results in this manuscript is provided in the thesis. Future readers are advised to look up an updated version.

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Contents

1 Frölicher structures on moduli spaces and complex deformations **3**
1.1 A brief introduction to Frölicher structures 3
1.2 Vector fields on the unit circle and their flow 5
1.3 Complex deformations of the unit circle 9
1.4 The Lie algebra of complex deformations 13
1.5 Infinite-dimensional moduli spaces of Riemann surfaces 16
1.5.1 Tori with geodesic seam 19
1.5.2 Hyperbolic surfaces 19
1.5.3 Möbius surfaces 19

2 Cohomology of complex deformations **20**
2.1 Relative group-level cohomology 20
2.2 Cocycles on complex deformations 24
2.3 Computation of the cohomology 28

3 Real one-dimensional modular functors **30**
3.1 Definition and locality 30
3.2 Central extensions of complex deformations 31
3.3 Action of the central extensions on the modular functor 34
3.4 Various cocycle identities 36
3.5 Central charge and modular invariance 38

4 Disk-disk cocycles and loop Loewner energy **40**

5 Isomorphisms of real one-dimensional modular functors **44**
5.1 Induction step on the number of boundary components 46
5.2 Spheres and disks, and complex deformations 47
5.3 Tori and flat modular invariance 48
5.4 Higher genus 50

1 Frölicher structures on moduli spaces and complex deformations

In this section, we introduce the theory of Frölicher spaces that is needed to obtain the analytical results in this work, including the Lie algebra of the complex deformations introduced in Section 1.3, and the representation of functionals on said Lie algebra of real-analytic complex-valued vector fields on S^1 in Section 4. Moreover, we put a Frölicher structure on the infinite-dimensional moduli spaces of Riemann surfaces with analytically parametrized boundary components, see Section 1.5, and compute the cohomology of complex deformations in the Frölicher smooth setting (Section 2). For a more detailed introduction to Frölicher structures and relation to other geometric structures, see [KM97, chapter 23] and [Sta11].

1.1 A brief introduction to Frölicher structures

Frölicher structures are generalizations of manifolds where, instead of charts, the structure consists of sets of functions from $\mathbb{R}$ into a set X, and functions from X into $\mathbb{R}$ satisfying a completeness relation based on $C^\infty(\mathbb{R},\mathbb{R})$, the usual set of smooth functions.

Definition 1.1. A *Frölicher space* is a set X with a *Frölicher structure* $(X, \mathcal{C}(X), \mathcal{F}(X))$ consisting of X and sets of *curves* $\gamma \in \mathcal{C}(X)$, $\gamma : \mathbb{R} \to X$ and *functions* $f \in \mathcal{F}(X)$, $f : X \to \mathbb{R}$ such that

$$\mathcal{C}(X) = \{\gamma : \mathbb{R} \to X \mid f \circ \gamma \in C^\infty(\mathbb{R},\mathbb{R})\ \forall f \in \mathcal{F}(X)\} \tag{1.1}$$

$$\mathcal{F}(X) = \{f : X \to \mathbb{R} \mid f \circ \gamma \in C^\infty(\mathbb{R},\mathbb{R})\ \forall \gamma \in \mathcal{C}(X)\} \tag{1.2}$$

A map $\varphi : X \to Y$ is Frölicher smooth (Fr-smooth) with respect to Frölicher structures $(X, \mathcal{C}(X), \mathcal{F}(Y))$ and $(Y, \mathcal{C}(Y), \mathcal{F}(Y))$ if one of the following equivalent conditions holds:

$$\gamma \in \mathcal{C}(X) \implies \varphi \circ \gamma \in \mathcal{C}(Y) \tag{1.3}$$

$$f \in \mathcal{F}(Y) \implies f \circ \varphi \in \mathcal{F}(X) \tag{1.4}$$

$$\gamma \in \mathcal{C}(X), f \in \mathcal{F}(Y) \implies f \circ \varphi \circ \gamma \in \mathbb{C}^\infty(\mathbb{R},\mathbb{R}) \tag{1.5}$$

A finite-dimensional smooth manifold structure on a set X may be reconstructed from the induced Frölicher structure $(X, C^\infty(\mathbb{R}, X), C^\infty(X,\mathbb{R}))$ using Boman's theorem [KM97, Theorem 3.4]. However, Frölicher structures are also suitable for spaces of inhomogeneous or infinite dimension. They form a categorical framework in which many other types of geometric structures can be compared. For example, finite-dimensional smooth manifolds, or Fréchet manifolds [Frö82, Theorem 3.2], form full subcategories of Frölicher spaces. Generally speaking, differential geometric notions that only involve differentiation have generalizations defined entirely in terms of the Frölicher structure; below, we define tangent spaces, Lie algebras, and differential forms in this way. Integration, however, typically needs some form of coordinate charts. Thus, it is helpful if a given Frölicher space $(X, \mathcal{C}(X), \mathcal{F}(X))$ also has a manifold structure such that the smooth curves and functions agree respectively with $\mathcal{C}(X)$ and $\mathcal{F}(X)$. For example, with Equation (1.35), we use the surrounding manifold structure of a space of real-analytic maps to realize a Frölicher structure. This allows the application of the Poincaré lemma in Section 2.

The Frölicher structure generated by a set of curves $\mathcal{C}_0(X)$ is given by the functions

$$\mathcal{F}(X) = \{f : X \to \mathbb{R} \mid f \circ \gamma \in C^\infty(\mathbb{R},\mathbb{R})\ \forall \gamma \in \mathcal{C}_0(X)\} \tag{1.6}$$

and the curves $\mathcal{C}(X)$ defined by (1.1). Analogously given a set of functions $\mathcal{F}_0(X)$,

$$\mathcal{C}(X) = \{\gamma : \mathbb{R} \to X \mid f \circ \gamma \in \mathbb{C}^\infty(\mathbb{R},\mathbb{R})\ \forall f \in \mathcal{F}_0(X)\}$$

and $\mathcal{F}(X)$ defined by (1.2) generate a Frölicher structure. The two ways of generating Frölicher structures are related as follows.

Proposition 1.2. *Let $\mathcal{C}_0(X)$ and $\mathcal{F}_0(X)$ be sets of curves and functions. Assume both*

1. *Given $\gamma \in \mathcal{C}_0(X)$, $f \in \mathcal{F}_0(X)$, it follows that $f \circ \gamma \in C^\infty(\mathbb{R}, \mathbb{R})$.*
2. *Given $\gamma : \mathbb{R} \to X$ such that for every $f \in \mathcal{F}_0(X)$ we have $f \circ \gamma \in C^\infty(\mathbb{R}, \mathbb{R})$, and $g : X \to \mathbb{R}$ such that for every $\eta \in \mathcal{C}_0(X)$ we have $g \circ \eta \in C^\infty(\mathbb{R}, \mathbb{R})$, it follows that $g \circ \gamma \in C^\infty(\mathbb{R}, \mathbb{R})$.*

Then, $\mathcal{C}_0(X)$ and $\mathcal{F}_0(X)$ generate the same Frölicher structure.

Proof. Let $(X, \mathcal{C}_1, \mathcal{F}_1)$ denote the Frölicher structure generated by $\mathcal{C}_0(X)$ and $(X, \mathcal{C}_2, \mathcal{F}_2)$ the Frölicher structure generated by $\mathcal{F}_0(X)$. The first condition ensures that $\mathcal{F}_0(X) \subseteq \mathcal{F}_1$, and thus $\mathcal{C}_1 \subseteq \mathcal{C}_2$. The second condition reads that for $\gamma \in \mathcal{C}_2$ and $g \in \mathcal{F}_1$, we have $f \circ \gamma \in C^\infty(\mathbb{R}, \mathbb{R})$. Since $\mathcal{C}_1$ is determined by $\mathcal{F}_1$ we have $\mathcal{C}_2 \subseteq \mathcal{C}_1$, and therefore $\mathcal{C}_1 = \mathcal{C}_2$. The proposition follows since the Frölicher structures are determined by their sets of curves. □

For Fr-smoothness, it is sufficient to check condition (1.3) on a generating set of curves:

Proposition 1.3. *For a map $Q : X \to Y$, assume that*

$$\gamma \in \mathcal{C}_0(X) \implies Q \circ \gamma \in \mathcal{C}(Y). \tag{1.7}$$

Then, Q is Fr-smooth.

Proof. Since $f \circ Q \circ \gamma_0 \in C^\infty(\mathbb{R}, \mathbb{R})$ for all $f \in \mathcal{F}(Y)$ and $\gamma_0 \in \mathcal{C}_0(X)$, by (1.6), it follows that $f \circ Q \in \mathcal{F}(X)$. Thus Q is Frölicher smooth by (1.4). □

Another way to define Frölicher structures is through a family of maps $(Q_j : X \to Y_j)_{j \in J}$ to Frölicher spaces $(Y, \mathcal{C}(Y), \mathcal{F}(Y))$. The *initial Frölicher strucutre* on X is then generated by the functions $\mathcal{F}_0(X) = \{f \circ Q_j \mid f \in \mathcal{F}(Y_j), j \in J\}$. If X also comes with an $\mathbb{R}$-vector space structure, we denote the Fr-smooth dual by

$$X^\vee = \{F \in \mathcal{F}(X) \mid F \text{ is linear}\}. \tag{1.8}$$

A Frölicher structure $(X, \mathcal{C}(X), \mathcal{F}(X))$ induces two topologies on the set X that might not agree. On the one hand, there is the functional topology defined as the weakest topology such that all functions $\mathcal{F}(X)$ are continuous. On the other hand, and this is the case that we will mostly use, there is the strongest topology such that all curves $\mathcal{C}(X)$ are continuous, which may equivalently be defined as

$$\mathcal{TC}(X) = \left\{U \subseteq X \,\middle|\, \gamma^{-1}(U) \text{ is open in } \mathbb{R}\ \forall \gamma \in \mathcal{C}(X)\right\}, \tag{1.9}$$

and is called the *curvaceous topology*. Concretely, this means that $U \subseteq X$ is open if and only if for every curve $\gamma \in \mathcal{C}(X)$ such that $\gamma(0) \in U$ there exists some $\varepsilon > 0$ such that $\gamma(-\varepsilon, \varepsilon) \subset U$.

Similar to the topology, there exist two concepts of tangent space on Frölicher spaces that do not necessarily agree. The functional definition uses derivations on in $\mathcal{F}(X)$, and the curvaceous definition — the one we are mostly interested in — reads

$$T_x X = \{\gamma \in \mathcal{C}(X) \mid \gamma(0) = x\} / \sim, \qquad x \in X, \tag{1.10}$$

where

$$\gamma_1 \sim \gamma_2 \quad \iff \quad (f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0) \quad \forall f \in \mathcal{F}(X). \tag{1.11}$$

The full tangent bundle is $TX = \sqcup_{x \in X} T_x X$, and for $Q : X \to Y$ a Fr-smooth function, the derivative is defined by

$$\begin{aligned} \mathrm{d}\, Q : TX &\longrightarrow TY, \\ [\gamma]_\sim &\mapsto [Q \circ \gamma]_\sim. \end{aligned} \tag{1.12}$$

The tangent bundle then comes with a Frölicher structure generated by the functions $Tf : TX \to T\mathbb{R} \cong \mathbb{R}^2$ for $f \in \mathcal{F}(X)$. Note that this notion of tangent space does not always yield a vector space. A curvaceous tangent vector $v = [\gamma]_\sim \in T_x X$ still acts on a function $f \in \mathcal{F}(X)$ as a derivation via $vf = (f \circ \gamma)'(0)$. This induces a Lie bracket on $\mathrm{Vect}(X)$, which is the Frölicher space of Fr-smooth sections of the tangent bundle $TX \to X$, given by the usual Lie bracket of vector fields in terms of derivations $[v, w] = vw - wv$ for $v, w \in \mathrm{Vect}(X)$. Note that the vector field $[v, w]$ might take values in the functional tangent space defined using derivations.

Similar to the definition of the curvaceous tangent space in Equation (1.10), there is also a cotangent space

$$T^x X = \mathcal{F}(X)/\sim_x, \qquad x \in X, \tag{1.13}$$

where the equivalence relation $\sim_x$ is defined by

$$f \sim_x g \iff (f \circ \gamma)'(0) = (g \circ \gamma)'(0)\ \forall \gamma \in \mathcal{C}(X) \text{ such that } \gamma(0) = x. \tag{1.14}$$

There is a canonical pairing of tangent and cotangent spaces

$$\mathsf{ev}([f]_{\sim_x}, [\gamma]_\sim) = (f \circ \gamma)'(0) \in \mathbb{R}, \qquad [\gamma] \in T^x X, \tag{1.15}$$

and we let it generate the Frölicher structure on the cotangent bundle $T^\vee X = \sqcup_{x \in X} T^x X$ by requiring the functions $\mathsf{ev}(\,\cdot\,, v) : \mathcal{F}(TX) \to \mathcal{F}(X)$ for $v \in \mathrm{Vect}(X)$ to be Fr-smooth. The definition of cotangent space leads to n-forms $\omega \in \Omega^n(X)$ as sections of $\bigwedge^n T^\vee X$ where $\bigwedge^0 T^\vee X = \mathcal{F}(X)$. On finite-dimensional smooth manifolds, the exterior derivative can be computed using the invariant formula; see [Lee12, Proposition 14.32]. Here, we define the exterior derivative $\mathrm{d}\,\omega$ of $\omega \in \Omega^n(X)$ evaluated on vector fields $v_1, \dots v_{n+1} \in \mathrm{Vect}(X)$ using the a generalization of the invariant formula,

$$\begin{aligned}(\mathrm{d}\,\omega)(v_1, \dots, v_{n+1}) = &\sum_{j=1}^{n+1} (-1)^{j-1} v_j \omega(v_1, \dots, \hat{v}_j, \dots v_{n+1}) \\ &+ \sum_{j=1}^{n+1} \sum_{k=j+1}^{n+1} (-1)^{j+k} \omega([v_j, v_k], v_1, \dots, \hat{v}_j, \dots \hat{v}_k, v_{n+1}),\end{aligned} \tag{1.16}$$

where the hat stands for the absence of the symbol.

1.2 Vector fields on the unit circle and their flow

In this section, we introduce a Frölicher structure on the Lie algebra of real-analytic complex-valued vector fields on the unit circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, which we denote by $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$. In the standard coordinate z on S^1, we denote a vector field $v \in \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ by $v = v(z)\partial_z$. Then, the Lie bracket on $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ is the standard Lie bracket of vector fields given by

$$[v, w] = (w(z)v'(z) - v(z)w'(z))\partial_z, \qquad v = v(z)\partial_z. \tag{1.17}$$

In the $\mathbb{C}$-basis,

$$\ell_n = \ell_n(z)\partial_z = -z^{n+1}\partial_z, \qquad [\ell_n, \ell_m] = (n - m)\ell_{n+m}, \qquad n, m \in \mathbb{Z}, \tag{1.18}$$

it is also known as the Witt algebra. Note that for $n \geq -1$, $\ell_n(z) = -z^{n+1}$ extends holomorphically to $\bar{\mathbb{D}}$. We identify the subalgebra generated by ℓ_{-1}, ℓ_0, and ℓ_{-1} with $\mathfrak{sl}(2, \mathbb{C})$. Since we are only concerned with real-analytic structures in this work, we often use the $\mathbb{R}$-basis given by ℓ_n and $\mathrm{i}\,\ell_n$ for $n \in \mathbb{Z}$. Another convenient choice of $\mathbb{R}$-basis on $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ is expressed in terms of vector fields which are respectively tangential and normal to S^1,

$$\begin{aligned} a_n^{\parallel} &= \frac{\ell_n - \ell_{-n}}{2}, & b_n^{\parallel} &= \frac{\ell_n + \ell_{-n}}{2\mathrm{i}}, \\ a_n^{\perp} &= \frac{\ell_n - \ell_{-n}}{2\mathrm{i}}, & b_n^{\perp} &= \frac{\ell_n + \ell_{-n}}{2}. \end{aligned} \tag{1.19}$$

The tangential vector fields $a_n^{\|}$ and $b_n^{\|}$ for $n \in \mathbb{Z}$ span a Lie algebra which we think of as the real-analytic real-valued vector fields on S^1 and denote by $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{R}}(\mathrm{S}^1)$, since in the coordinate $z = e^{\mathrm{i}\,\theta}$ they become $a_n^{\|}(e^{\mathrm{i}\,\theta}) = \mathrm{i}\,e^{\mathrm{i}\,\theta}\sin(n\theta)$, and $b_n^{\|}(e^{\mathrm{i}\,\theta}) = \mathrm{i}\,e^{\mathrm{i}\,\theta}\cos(n\theta)$, where the rotation by $\mathrm{i}\,e^{\mathrm{i}\,\theta}$ takes the real-valued functions of θ to tangent vector fields.

Define the complex linear projections onto vector fields respectively extending holomorphically to $\bar{\mathbb{D}}$ and $\hat{\mathbb{C}} \setminus \mathbb{D}$ respectively — while removing the modes $\ell_{-1}, \ell_0, \ell_1$ which are holomorphic on all of $\hat{\mathbb{C}}$ completely — in the $\mathbb{C}$-basis in Equation (1.18) by

$$P^+(\ell_n) = \begin{cases} \ell_n & n > 1, \\ 0 & n \leq 1, \end{cases} \qquad P^-(\ell_n) = \begin{cases} 0 & n \geq -1, \\ \ell_n & n < -1. \end{cases} \tag{1.20}$$

Given a biholomorphism $F : A \to B$ between annular neighborhoods $A, B \subset \hat{\mathbb{C}}$ of S^1, the pullback of a vector field $v \in \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ which is holomorphic on B by F is is a holomorphic vector field on A given by

$$F^* v = \frac{v(F(z))}{F'(z)} \partial_z. \tag{1.21}$$

In particular, pullback by the inversion

$$\mathrm{J} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}, \qquad z \mapsto \frac{1}{z} \tag{1.22}$$

acts on the generators ℓ_n by

$$\mathrm{J}^*\, \ell_n = -z^{-(n+1)}(-z^2)\partial_z = z^{-n+1}\partial_z = -\ell_{-n}, \qquad n \in \mathbb{Z} \tag{1.23}$$

Then, the projections P^+ and P^- are conjugate by J^*, that is

$$\mathrm{J}^*\, P^+\, \mathrm{J}^* = P^-. \tag{1.24}$$

In particular, we have for $n > 1$,

$$\begin{aligned} P^+ a_n^{\|} &= \frac{1}{2}\ell_n, & P^+ b_n^{\|} &= \frac{1}{2\mathrm{i}}\ell_n, \\ -\,\mathrm{J}^*\, P^- a_n^{\|} &= -\frac{1}{2}\ell_n, & -\,\mathrm{J}^*\, P^- b_n^{\|} &= \frac{1}{2\mathrm{i}}\ell_n. \end{aligned} \tag{1.25}$$

The Frölicher structure on $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ is defined as follows.

Proposition 1.4. *The set of smoothly time-dependent vector fields*

$$\mathcal{C}(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)) = \left\{ v(t,z)\partial_z \;\middle|\; v(t,z) \in \mathbb{C} \text{ is smooth in } t \in \mathbb{R} \text{ and real-analytic in } z \in S^1 \right\}. \tag{1.26}$$

together with $\mathcal{F}(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))$ *defined by Equation (1.2) is a Frölicher structure on* $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$*. Moreover,* $\mathcal{F}(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))$ *is generated by the functions*

$$\begin{aligned} a_n : \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1) &\longrightarrow \mathbb{C}, \\ v &\longmapsto \frac{1}{2\pi\mathrm{i}} \int_{S^1} \frac{v(z)\, \ell_{-n}(z)}{z} \mathrm{d}z. \end{aligned} \tag{1.27}$$

Proof. Any $v = v(t,z)\partial_z \in \mathcal{C}(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))$ has a Laurent expansion

$$v(t,z) = \sum_{n\in\mathbb{Z}} a_n(t)\ell_n(z), \qquad a_n(t) = \frac{1}{2\pi\mathrm{i}} \int_{S^1} \frac{v(t,z)\, \ell_{-n}(z)}{z} \mathrm{d}z. \tag{1.28}$$

By smoothness of $v(t,z)$ in t and z and the Leibniz integral rule, we can exchange derivatives in t with the integral over z in the definition of $a_n(t)$. It follows that $a_n(t)$ is smooth in t for all $n \in \mathbb{Z}$. Hence, by Equation (1.2) the functions a_n are Fr-smooth for each $n \in \mathbb{Z}$ and we have $v = \sum_{n\in\mathbb{Z}} a_n(v)\ell_n$. Now consider a curve $\gamma : \mathbb{R} \to \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ as in Equation (1.1). In particular, the functions $a_n(\gamma(t))$ are smooth in t. Since the Laurent expansion of the curve $\gamma = \gamma(t,z)\partial_z$ at time t given by $\gamma(t,z) = \sum_{n\in\mathbb{Z}} a_n(\gamma(t))\, \ell_n(z)$, it follows that $\gamma(t,z)$ for fixed z depends smoothly on t. Since for fixed $t \in \mathbb{R}$, the function $\gamma(t,z)$ is real-analytic in z, it follows that $\gamma \in \mathcal{C}(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))$. Since only smoothness of the functions a_n was used to prove this, they indeed generate $\mathcal{F}(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))$. □

We use the following property of $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ in Section 4.

Proposition 1.5. *For Fr-smooth $\mathbb{R}$-linear functional $F \in (\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))^\vee$ vanishing on $\mathfrak{sl}(2,\mathbb{C})$ there exists unique holomorphic quadratic differentials ρ^+ on $\mathbb{D}$, and ρ^- on $\hat{\mathbb{C}} \setminus \mathbb{D}$ such that $\rho^-(\infty) = 0$, and*

$$F(v) = \mathrm{Re}\left(\int_{(1-\varepsilon)S^1} P^- v\, \rho^+ + \int_{(1+\varepsilon)S^1} P^+ v\, \rho^- \right). \tag{1.29}$$

where $\varepsilon > 0$ must be chosen depending on v such that the integral exists (but the value of $F(v)$ does not depend on ε).

Proof. Since we would like to represent Fr-smooth $\mathbb{R}$-linear functions $F : \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1) \to \mathbb{R}$ by integrals using the Cauchy–Hilbert transform, see [Mor93, Definition 2.1.7], we first consider the vector space of real-analytic complex-valued functions on S^1, denoted $\mathcal{O}(S^1)$. It comes with the inductive limit topology with respect to restriction of holomorphic functions $\mathcal{O}(U_n)$ on the annuli $U_n = \{z \in \mathbb{C} \mid 1 - \frac{1}{n} \le |z| \le 1 + \frac{1}{n}\}$, $n \ge 1$, which, in turn, come with the topology of uniform converence on compact sets in U_n. We then identify $\mathcal{O}(S^1)$ with $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ via the map

$$\begin{aligned} Q : \mathcal{O}(S^1) &\to \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1) \\ f(z) &\mapsto f(z)\partial_z. \end{aligned} \tag{1.30}$$

which is clearly $\mathbb{C}$-linear, bijective, and we verify that it is an isomorphism of Frölicher spaces where the Frölicher structure on $\mathcal{O}(S^1)$ is defined by the inductive limit topology.

The topology under consideration makes $\mathcal{O}(S^1)$ a convenient vector space (see [KM97, Theorem 8.4]), and hence the smooth curves $C^\infty(\mathbb{R}, \mathcal{O}(S^1))$ with the notion of smoothness defined by the topology (see [KM97, Section 1.2]) define a Frölicher structure (see [KM97, Theorem 2.14]). In fact, the the set of smooth curves $C^\infty(\mathbb{R}, \mathcal{O}(S^1))$ depends only on the bornology defined induced by the topology, and all the $\mathcal{O}(U_n)$ embed bornologically as a closed substpace into $C^\infty(U, \mathbb{C})$ which is the usual set of smooth functions (see [KM97, Theorem 8.2]). Therefore a curve $\gamma : \mathbb{R} \to \mathcal{O}(S^1)$ is in $C^\infty(\mathbb{R}, \mathcal{O}(S^1))$ if and only if it is smooth as a function $(t,z) \mapsto \gamma(t,z)$ on $\mathbb{R} \times S^1$. We conclude that γ is in $C^\infty(\mathbb{R}, \mathcal{O}(S^1))$ if and only if $Q(\gamma(t, \cdot\,)) = \gamma(t,z)\partial_z$ is a smooth curve with respect to the Frölicher structure on $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ defined in Proposition 1.4. Hence, Q and Q^{-1} are Fr-smooth by Equation (1.3).

Let $G \in \mathcal{O}(S^1)^\vee$ be a Fr-smooth real-valued $\mathbb{R}$-linear functional, and consider the complexification $H(f) = G(f) - \mathrm{i}\, G(\mathrm{i}\, f)$, which is a Fr-smooth complex-valued $\mathbb{C}$-linear functional such that $G(f) = \mathrm{Re}\, H(f)$. Now, we apply [Mor93, Theorem 2.1.9] to a functional H, where we use that smooth functionals are, in particular, continuous, and thus "analytic functionals". By the theorem, there exists unique holomorphic functions $\rho^+ : \mathbb{D} \to \mathbb{C}$, and $\rho^+ : \hat{\mathbb{C}} \setminus \mathbb{D} \to \mathbb{C}$ such that $\rho^-(\infty) = 0$, and H has the integral representation

$$H(f) = \int_{(1-\varepsilon)S^1} f(z)\, \rho^+(z)\, \mathrm{d}z + \int_{(1+\varepsilon)S^1} f(z)\, \rho^-(z)\, \mathrm{d}z, \qquad f \in \mathcal{O}(S^1), \tag{1.31}$$

where the integral does not depend on $\varepsilon > 0$, but it must be chosen small enough depending on f such that it is defined. Now consider the integral representation (1.31) for $G = F \circ Q$ where

$F \in (\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))^\vee$, which is again Fr-smooth by Fr-smoothness of Q. Regarding the functions $\rho^+(z)$ and $\rho^-(z)$ as quadratic differentials $\rho^+ = \rho^+(z)\,\mathrm{d}z^2$ and $\rho^- = \rho^-(z)\,\mathrm{d}z^2$, pairing them with the vector field $v = f(z)\,\partial_z$, the formula (1.31) yields a (coordinate independent) integral representation for F in Equation (1.29). Note that we also inserted the projections P^+ and P^- defined in Equation (1.20), since the functional vanishes on $\mathfrak{sl}(2,\mathbb{C})$ by definition, and both integrals each vanish if we insert the respective other projection. □

We would like to integrate the time-dependent vector fields $v \in \mathcal{C}(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))$ for small time $t \in (-\varepsilon, \varepsilon)$ as the flow $\Phi_v(t,\,\cdot\,)$ solving the flow equations

$$\partial_t \Phi_v(t,z) = v(t, \Phi_v(t,z)), \qquad \Phi_v(0,z) = z. \tag{1.32}$$

Since a real-analytic vector field extends holomorphically to a neighborhood of S^1, the flow $\Phi_v(t,z)$ for fixed $t \in (-\varepsilon,\varepsilon)$ is biholomorphic in z in a neighborhood of S^1. Thus, the solution $\Phi_v(t,\,\cdot\,)$ maps S^1 to analytic loops near S^1 inside $\mathbb{C}$, deforming the circle as time increases. Fixing $z \in S^1$, the trajectory $\Phi_v(\,\cdot\,,z) : (-\varepsilon,\varepsilon) \to \mathbb{C}$ is smooth since v depends smoothly on time. However, the flow might not be defined for all $(t,z) \in \mathbb{R} \times S^1$ since with increasing time, a point might run into a singularity of v. Moreover, the analytic continuation stops being biholomorphic if $\Phi_v(t,S^1)$ runs into itself.

We say that the flow Φ_v exists for all time if its domain as a time-dependent biholomorphism is an open neighborhood of $\mathbb{R} \times S^1$ in $\mathbb{R} \times \mathbb{C}$. For instance, any time-dependent vector fields may be mollified such that the flow exists for all time, e.g. by localizing it around $t = 0$ using a smooth bump function, stopping the flow after a finite time. Since the curves of a Frölicher structure are defined for all $t \in \mathbb{R}$, we identify a set of time-dependent vector fields such that Φ_v exists for all time, and which at the same time generates the Frölicher structure $\mathcal{C}(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))$ of Proposition (1.4). Anticipating our definitions in the next section, would also like to guarantee the following:

1. At any time t, the inverse $\Phi_v^{-1}(t,z)$ in the z-coordinate has a unique biholomorphic analytic continuation to S^1.
2. At any time t, The curve $\Phi_v(t,\,\cdot\,) : S^1 \to \mathbb{C}$ winds around $0 \in \mathbb{C}$ with positive orientation.

For the first condition to hold, $\Phi_v^{-1}(t,\,\cdot\,)$ cannot have any singularity between $\Phi_v(t,S^1)$ and S^1. With the following notation, which assumes the second condition, this is equivalent to $\Phi_v^{-1}(t,z)$ extending to a neighborhood of the newly defined set $U(\Phi(t,\,\cdot\,))$ as a biholomorphism.

Definition 1.6. Given a real-analytic map $\phi : S^1 \to \mathbb{C}\setminus\{0\}$, denote by $U(\phi)$ the annular closed set bounded by the inner and outer boundary of $S^1 \cup \phi(S^1)$.

Moreover, the second condition on $v \in \mathcal{C}(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))$ above implies that if another flow Φ_w satisfies the same two conditions above, and is such that at a fixed time $t \in \mathbb{R}$ the flow $\Phi_v(t,\,\cdot\,)$ extends to a neighborhood of $U(\Phi_w(s,\,\cdot\,))$ as a biholomorphism, then the composition $\Phi_v(t,\,\cdot\,) \circ \Phi_w(s,\,\cdot\,)$ of the two flows is uniquely defined. Finally, the set of time-dependent vector fields we are interested in is

$$\mathcal{C}_0(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)) = \left\{ v \in \mathcal{C}(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)) \;\middle|\; \begin{array}{l} \Phi_v^{-1}(t,\,\cdot\,) \text{ exists and extends to a} \\ \text{neighborhood of } U(\Phi_v(t,\,\cdot\,)) \text{ as a} \\ \text{biholomorphism for all } t \in \mathbb{R} \end{array} \right\}. \tag{1.33}$$

As mentioned above, any time-dependent vector field $v \in \mathcal{C}(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))$ may be localized such that the flow exists for all time. Then, it may be localized further such that the deformed circle avoids 0 and stays within a neighborhood of S^1 where v does not have any singularities. Since smoothness of curves in $\mathcal{C}(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))$ is a local property, we find that the set of vector fields $\mathcal{C}_0(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))$ generates the Frölicher structure on $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$.

Remark 1.7. Note that for $v \in \mathcal{C}_0(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))$, the possible values of $v(t,\,\cdot\,)$ may become more restricted as t increases since $v(t,\,\cdot\,)$ is required to be holomorphic not just on S^1, but on a neighborhood of $U(\Phi_v(t,\,\cdot\,))$.

1.3 Complex deformations of the unit circle

We define the set of *complex deformations* of the unit circle inside the (punctured) complex plane as

$$\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1) = \left\{ \phi \in C^\omega(S^1, \mathbb{C}\setminus\{0\}) \,\middle|\, \begin{array}{l} \phi \text{ is positively oriented around 0 and} \\ \phi^{-1} \text{ extends biholomorphically to a} \\ \text{neighborhood of } U(\phi). \end{array} \right\}, \tag{1.34}$$

using the closed set bounded by S^1 and $\phi(S^1)$ as in Definition 1.6. Here, $C^\omega(S^1, \mathbb{C}\setminus\{0\})$ is the space of real-analytic maps from S^1 to $\mathbb{C}\setminus\{0\}$, which as a subspace of $\mathcal{O}(S^1)$ comes with an infinite-dimensional manifold; see the proof of Proposition 1.5 and also [KM97, Theorem 42.6], or view it as a real-analytic loop group [PS03, Section 3.5]. Thus, the inclusions

$$\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1) \subset C^\omega(S^1, \mathbb{C}\setminus\{0\}) \subset \mathcal{O}(S^1), \tag{1.35}$$

relate $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ to an infinite-dimensional manifold.

Remark 1.8. The inclusion (1.35) is not open since for any $\phi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ such that $\phi(S^1) \neq S^1$, there exist smooth curves through ϕ that exit $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ immediately. Neither is the inclusion closed since there Cauchy sequences ϕ_n in $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ with respect to the topology of $\mathcal{O}(S^1)$ can break the regularity of $\phi = \lim_{n\to\infty} \phi_n \in \mathcal{O}(S^1)$ at the boundary of $U(\phi)$.

We proceed to define a Frölicher structure on $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$, which agrees with the restriction of the Frölicher structure on $\mathcal{O}(S^1)$ induced by the manifold structure.

By construction, the flow Φ_v of a time-dependent vector field $v \in \mathcal{C}_0(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))$ according to Equation (1.33) is at any time a complex deformation, thus defining a curve in $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$. Such curves generate a Frölicher structure on $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ in the sense of Equation (1.6),

$$\mathcal{C}_0(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)) = \left\{ t \mapsto \Phi_v(t,\,\cdot\,) \,\middle|\, v \in \mathcal{C}_0(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)) \right\}. \tag{1.36}$$

Since all of these curves are rooted at the identity $\mathbb{1} \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$, it might seem that this set of smooth curves is relatively sparse. However, the following result shows that Fr-smooth curves in $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ may be locally represented by the flows of vector fields.

Proposition 1.9. *Any curve $\gamma \in \mathcal{C}(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ in the Frölicher structure generated by $\mathcal{C}_0(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ locally agrees with a curve $\Phi_v \in \mathcal{C}_0(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$, that is, for any $t_0 \in \mathbb{R}$, there $\varepsilon > 0$ and $s \in \mathbb{R}$ such that*

$$\gamma(t) = \Phi_v(s+t,\,\cdot\,), \qquad t \in (t_0 - \varepsilon, t_0 + \varepsilon). \tag{1.37}$$

Moreover, the Frölicher structure generated by $\mathcal{C}_0(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ agrees with that generated by the functions

$$\begin{aligned} A_n : \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1) &\to \mathbb{C}, \\ \phi &\mapsto \frac{1}{2\pi\mathrm{i}} \int_{S^1} \frac{\phi(z)}{z^{n+1}} \mathrm{d}z, \end{aligned} \qquad n \in \mathbb{Z}. \tag{1.38}$$

The space of curves for this Frölicher structure is

$$\mathcal{C}(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)) = \left\{ \gamma : \mathbb{R} \to \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1) \,\middle|\, \gamma(t,z) \textit{ is smooth in } t \textit{ and analytic in } z. \right\}. \tag{1.39}$$

Proof. We first prove the second statement, which is similar to Proposition 1.4 for vector fields. The curve $\Phi_v(t,z)$ has a Laurent expansion in z at any time $t \in \mathbb{R}$,

$$\Phi_v(t,z) = \sum_{n\in\mathbb{Z}} A_n(\Phi_v(t,z))\, z^n. \tag{1.40}$$

By smoothness of $\Phi_v(t,z)$ in t and z and the Leibniz integral rule, we can exchange derivatives in t and the integral over z in the definition of $A_n(\Phi_v(t,z))$. It follows that the coefficients $A_n(\Phi_v(t,z))$ are smooth in t, and therefore the functions A_n on $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ are Fr-smooth with respect to the Frölicher structure generated by $\mathcal{C}_0(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$.

Let $\gamma : \mathbb{R} \to \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ be a curve such that $A_n \circ \gamma$ is smooth for every $n \in \mathbb{Z}$. As a function $\gamma(t,z)$ this curve has a Laurent expansion $\gamma(t,z) = \sum_{n\in\mathbb{Z}} A_n(\gamma(t))z^n$ converging for z in an annular neighborhood A_t of S^1. Since all the coefficients depend smoothly on t, the function $\gamma(t,z)$ is smooth in t. For any $t \in \mathbb{R}$, the complex deformation $\gamma(t,\,\cdot\,) : A_t \to B_t$ is a biholomorphism where $B_t = \gamma(t, A_t)$ is also an annular neighborhood of S^1. Because $S^1, \gamma(t, S^1) \subset B_t$, the inverse $\gamma^{-1}(t,\,\cdot\,) : B_t \to A_t$ is a a complex deformation. The time derivative defines a smoothly time-dependent vector field

$$v(t,\,\cdot\,) = \big(\partial_t \gamma(t,\,\cdot\,)\big) \circ \gamma^{-1}(t,\,\cdot\,) \tag{1.41}$$

which at time t is analytic on B_t. The Equation (1.41) is equivalent to $\partial_t\gamma(t,z) = v(t,\gamma(t,z))$, which, in turn, is equivalent to the flow equation (1.32) if γ is rooted at the identity. Since segments of Fr-smooth curves may be concatenated into new Fr-smooth curves by letting the curve smoothly come to a halt for a small time, if γ is not rooted at the identity, we can re-root it. On the one hand, this proves Equation (1.37). On the other hand, since we assumed γ to be smooth only on the functions A_n, we can proceed to prove the equivalence of the Frölicher structures generated by $\mathcal{C}_0(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ and the functions A_n by checking the conditions of Proposition 1.2.

Let $f : \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1) \to \mathbb{R}$ be a function such that $t \mapsto f(\Phi_w(t,\,\cdot\,))$ is smooth for every $w \in \mathcal{C}_0(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))$, that is, $f \in \mathcal{F}(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ in the Frölicher structure generated by $\mathcal{C}_0(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$. Given any γ as above, only assuming that the compositions $A_n \circ \gamma$ are smooth, we may represent it locally by a flow as in Equation (1.37). We find that

$$f(\gamma(t,\,\cdot\,)) = f(\Phi_v(s+t,\,\cdot\,)), \qquad t \in (t_0 - \varepsilon, t_0 + \varepsilon) \tag{1.42}$$

is a smooth function of t. Since t_0 may be chosen arbitrarily, $f \circ \gamma \in C^\infty(\mathbb{R},\mathbb{R})$. Hence, the curve γ, which is Fr-smooth with respect to the Frölicher structure generated by the A_n, is also in $\mathcal{C}(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$. Conversely, a curve $\gamma \in \mathcal{C}(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ is in particular Fr-smooth with respect to the A_n since $A_n \in \mathcal{F}(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$. Thus, by Proposition 1.2, the set of curves $\mathcal{C}_0(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ and the functions A_n generate the same Frölicher structure.

The identification as the Fr-smooth surface in Equation (1.39) is a direct consequence of the facts that we have already proven. On the one hand, any curve $\gamma : \mathbb{R} \to \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ for which $A_n \circ \gamma \in C^\infty(\mathbb{R},\mathbb{R})$ is smooth in t, and by definition of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ analytic in z. On the other hand, any curve as in Equation (1.39) has a Laurent expansion $\gamma(t,z) = \sum_{n\in\mathbb{Z}} A_n(\gamma(t))z^n$ with smooth coefficients. ☐

Complex deformations $\phi, \psi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ can be composed as $\phi \circ \psi$ if ϕ analytically extends to $\psi(S^1)$ as a biholomorphism, and the composition is unique if ϕ has no singularity between S^1 and $\psi(S^1)$. That is, there are annular neighborhoods of S^1 such that the biholomorphic extensions $\phi : A_\phi \to B_\phi$ and $\psi : A_\psi \to B_\psi$ satisfy $\psi(S^1) \subset A_\phi$. Since we would like the composition to be a complex deformation again, we need that $B_\psi \cap A_\phi$ contains an annular neighborhood B of S^1 such that its image under ϕ contains S^1. In particular, we can set $B_\psi = A_\phi = B$ and let $A = \psi^{-1}(B)$, $C = \phi(B)$. Thus, we consider composition only on the following set of *composable* pairs,

$$M = \left\{ (\phi,\psi) \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1) \times \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1) \;\middle|\; \begin{array}{l} \exists \text{ biholmorphic extensions} \\ \phi : B \to C \text{ and } \psi : A \to B \text{ to} \\ \text{annular neighborhoods of } S^1 \end{array} \right\}. \tag{1.43}$$

Indeed, by the considerations above, the composition of a composable pair is a complex deformation again, yielding a map

$$\begin{aligned} M &\to \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1), \\ (\phi, \psi) &\mapsto \phi \circ \psi. \end{aligned} \tag{1.44}$$

The set M comes with the initial Frölicher structure with respect to the inclusion $M \subset \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1) \times \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$.

Remark 1.10. The subset M of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1) \times \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ is not open in the curvaceous topology on $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1) \times \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$. For example, we always have $(\mathbb{1}, \phi) \in M$. However, if the vector field $v \in \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ has a singularity in $U(\phi^{-1}(S^1))$, then for any $t \neq 0$, the complex deformations ϕ and Φ_v are not composable. The same situation occurs for the pair $(\mathbb{1}, \phi) \in M$. If $v \in \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ has a singularity in $U(\phi(S^1))$, then the pair (Φ_v, ϕ) is not composable for any $t \neq 0$. Thus, $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ is not a local Lie group in a way that is adapted, e.g. from [Nee05], to Frölicher structures but still requires M to be open in the curvaceous topology. See also the notion of (global) Frölicher Lie group in [Lau11].

Despite the remark, for a curve $(\Phi_v(t, \cdot), \Phi_w(s, \cdot)) \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1) \times \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ rooted at $(\mathbb{1}, \mathbb{1})$, the pair is composable if both t and s sufficiently small. This is because both v and w have no singularities in a neighborhood of S^1, and by keeping t and s small, we can ensure that $\Phi_v(S^1)$ and $\Phi_w^{-1}(S^1)$ are contained in the intersection of these two neighborhoods.

Proposition 1.11. *The inversion $\phi \mapsto \phi^{-1}$ and composition (1.44) of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ are Fr-smooth, and the latter is associative.*

Proof. Using the characterization of curves as in (1.39), and the fact that inversion preserves smoothness in t and analyticity in z, we find that the inversion is smooth. Similarly, composition preserves smoothness in t and analyticity in z, showing that the composition of curves of the form (1.39) in M (which generate the Frölicher structure on M) is of the same form. □

There are a few interesting subsets of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$.

1. As finite-dimensional subgroups, there are the rotations, which play the special role of capturing the fundamental group of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$. The rotations are contained in $\mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$, the subgroup of Möbius transformations preserving the unit circle. There is also the full Möbius group $\mathrm{PSL}(2,\mathbb{C}) \subset \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$, which also includes the group of scaling transformations
$$\mathrm{Sc} = \left\{ s_\tau : \hat{\mathbb{C}} \to \hat{\mathbb{C}},\ z \mapsto e^{-2\pi\tau} z \,\middle|\, \tau \in \mathbb{R} \right\} \subset \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)\,. \tag{1.45}$$
2. The group $\mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$ of real-analytical orientation-preserving diffeomorphisms of S^1 is a subgroup of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$. As such, the Frölicher structure of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ induces a Frölicher structure on $\mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$ generated by curves $\gamma \in \mathcal{C}(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ contained in $\mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$. In addition, the diffeomorphism group $\mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$ is an infinite-dimensional Lie group, regular and real-analytic in the sense of [KM97, Theorem 43.4]. The smooth curves and functions of the manifold structure agree with those of the Frölicher structure.
3. Univalent functions with real-analytic boundary behaviour, and such that the inverse may be analytically extended to $\bar{\mathbb{D}}$, are complex deformations, forming the subset
$$\mathcal{V} = \left\{ F : \mathbb{D} \to \mathbb{C} \,\middle|\, \begin{array}{l} F^{-1} \text{ extends biholomorphically to a} \\ \text{neighborhood of } \bar{\mathbb{D}} \cup F(\bar{\mathbb{D}}) \text{ and } F(S^1) \\ \text{is positively oriented around } 0 \end{array} \right\} \subset \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)\,. \tag{1.46}$$
Note that the real-analyticity of the boundary and the fact that $S^1 \subset F(\mathbb{D})$ already imply that $F \in \mathcal{V}$. However, $\mathcal{V}$ also includes univalent functions that map parts of the circle

inside $\mathbb{D}$, given that the inverse extends back to S^1. As a subset, $\mathcal{V} \subset \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ comes with the Frölicher structure generated by curves in $\mathcal{C}(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ contained in $\mathcal{V}$. The curves in $\mathcal{V}$ are generated by the flows of time-dependent vector fields which are holomorphic in a neighborhood of the disk bounded by $\Phi_v^{-1}(t, S^1)$ and of $\bar{\mathbb{D}}$. We also define the set of normalized univalent functions

$$\mathcal{V}_0 = \{F \in \mathcal{V} \mid \; F(0) = 0 \text{ and } F'(0) > 0 \; \} \subset \mathcal{V}, \tag{1.47}$$

and the sets of all (normalized) univalent functions with real-analytic boundary behaviour, including those that are not complex deformations,

$$\mathcal{U} = \left\{F : \mathbb{D} \to \mathbb{C} \;\middle|\; F \text{ is univalent, real-analytic on } S^1 \; \right\}, \tag{1.48}$$
$$\mathcal{U}_0 = \{F \in \mathcal{U} \mid \; F(0) = 0 \text{ and } F'(0) > 0 \; \}. \tag{1.49}$$

Any univalent function $F \in \mathcal{U}_0$ may be composed by a scaling transformation $s_\tau \in \mathrm{Sc}$ such that $S^1 \subset s_\tau(\mathbb{F}(\mathbb{D}))$. Since the positive orientation requirement is fulfilled automatically, we have $s_\tau \circ F \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$. By the Koebe 1/4-theorem, we can take

$$\tau = \min\left\{\frac{1}{2\pi} \log \frac{|F'(0)|}{4}, 0\right\}. \tag{1.50}$$

Diffeomorphisms, univalent functions, and complex deformations are also related through the following decomposition.

Proposition 1.12. *Any complex deformation $\phi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ may be uniquely decomposed into a diffeomorphism $D_\phi \in \mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$ and a univalent function $F_\phi \in \mathcal{U}$ such that*

$$\phi = F_\phi \circ D_\phi. \tag{1.51}$$

Proof. Note that by the definition of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$, see Equation (1.34), the analytical loop $\phi(S^1)$ surrounds 0. Let U be the domain with positively oriented boundary $\phi(S^1)$ and $F_\phi : \mathbb{D} \to U$ the Riemann mapping uniquely determined by $F_\phi(0) = 0$ and $F'_\phi(0) > 0$. Since the boundary of U is analytical, F_ϕ is analytical in a neighborhood of $\bar{\mathbb{D}}$. The composition $D_\phi = F_\phi^{-1} \circ \phi$ is an analytical diffeomorphism of S^1. We precompose with F_ϕ to obtain the decomposition. □

Note that this decomposition is not helpful to study the composition of complex deformations, since generally it is rather difficult to decompose $F_1 \circ \phi_1 \circ F_2 \circ \phi_2$ for $F_1, F_2 \in \mathcal{V}$, and $\phi_1, \phi_2 \in \mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$. However, it improves our analytic understanding of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ because of the aforementioned manifold structure on $\mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$, which is compatible with the Frölicher structure. A well-known result on diffeomorphisms and univalent functions is their relation through conformal welding decompositions. For real-analytic boundary behaviour of the univalent functions and $\mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$, we provide the rather elementary proof using the Riemann mapping theorem below; see [TT06] for the more general statement using quasiconformal boundary behaviour. We do add some results on Fr-smoothness.

Proposition 1.13 (Conformal welding for analytical diffeomorphisms)**.** *With the following normalizations, either the diffeomorphism or one of the univalent functions, all related through*

$$\phi = \left((\mathrm{J} \circ \zeta_2 \circ \mathrm{J})^{-1} \circ \zeta_1\right)|_{S^1}, \qquad \phi \in \mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1), \quad \zeta_1, \zeta_2 \in \mathcal{U}, \tag{1.52}$$

determines the other two:

1. *Given $\phi \in \mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$ and $a \in \mathbb{C} \setminus \{0\}$, there exist unique $\zeta_1, \zeta_2 \in \mathcal{U}$ such that*

$$\zeta_1(S^1) = (\mathrm{J} \circ \zeta_2 \circ \mathrm{J})(S^1), \quad \zeta_1(0) = 0, \quad \zeta_1'(0) = a, \quad \zeta_2(0) = 0, \tag{1.53}$$

and Equation (1.52) holds. Moreover, for $|a| \in (4, \infty)$*, the map*

$$\mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1) \to \mathcal{V}_0, \qquad \phi \mapsto \zeta_1. \tag{1.54}$$

is Fr-smooth.

2. *Given* $\phi \in \mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$ *and* $a \in \mathbb{C} \setminus \{0\}$*, there exist unique* $\zeta_1, \zeta_2 \in \mathcal{U}$ *such that*

$$\zeta_1(S^1) = (\mathrm{J} \circ \zeta_2 \circ \mathrm{J})(S^1), \quad \zeta_1(0) = 0, \quad \zeta_2(0) = 0, \quad \zeta_2'(0) = a, \tag{1.55}$$

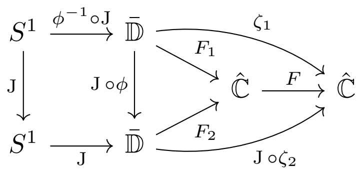
and Equation (1.52) holds. Moreover, for $|a| \in (4, \infty)$*, the map*

$$\mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1) \to \mathcal{V}_0, \qquad \phi \mapsto \zeta_2. \tag{1.56}$$

is Fr-smooth.

3. *Given* $\zeta_1 \in \mathcal{V}_0$*, there exist unique* $\phi \in \mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$ *and* $\zeta_2 \in \mathcal{U}_0$ *such that Equation (1.52) holds. Moreover, the map* $\zeta_1 \to \phi$ *is Fr-smooth.*
4. *Given* $\zeta_2 \in \mathcal{V}_0$*, there exist unique* $\phi \in \mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$ *and* $\zeta_1 \in \mathcal{U}_0$ *such that Equation (1.52) holds. Moreover, the map* $\zeta_2 \to \phi$ *is Fr-smooth.*

Proof. Given $\phi \in \mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$, consider the disks with analytically parameterized boundary[1] $\mathbb{D} \underset{1}{*} (\mathrm{J} \circ \phi^{-1} \mathrm{J}) = (\bar{\mathbb{D}}, \phi^{-1} \circ \mathrm{J})$ and $\mathbb{D} = (\bar{\mathbb{D}}, \mathrm{J})$. The sphere $(\mathbb{D} \underset{1}{*} \phi)\ {}_1\infty_1\ \mathbb{D} \in \mathcal{M}_{0,0}$ obtained by sewing these two disks is isomorphic to the Riemann sphere; by an isomorphism which is unique only up to Möbius transformations. Given a choice of isomorphism, it restricts to the two embeddings of the closed unit disk into $\hat{\mathbb{C}}$ given by the embeddings of the first and second surfaces, which we denote by F_1 and F_2. Then, the normalization may be changed by post-composition with a Möbius transformation $F \in \mathrm{PSL}(2, \mathbb{C})$:



(1.57)

Since the identification of the boundary components of the two disks is given by $\mathrm{J} \circ \phi$, we find that the compositions $F \circ F_1$ and $\mathrm{J} \circ F_2 \circ F$ should respectively become the univalent functions ζ_1 and ζ_2 for the relation (1.52) to hold. Thus, we let F be the unique Möbius transformation such that $\zeta_1(0) = F(F_1(0)) = 0$, $\zeta_1(0) = F'(F_1(0))F_1'(0) = a$, and $\mathrm{J}(\zeta_2(0)) = F(F_2(0)) = \infty$, for the first case, and analogously for the second case.

Given $\zeta_1 \in \mathcal{V}_0$, let F be the unique Riemann mapping from $\bar{\mathbb{D}}$ to the complement of $\zeta_1(\bar{\mathbb{D}})$ such that $F(0) = \infty$ and $F'(0) < 0$, and define $\zeta_2 = \mathrm{J} \circ F$. Note that since $\zeta_2(0) = \mathrm{J}(\infty) = 0$, and $\zeta_2'(0) = -F'(0)/F(0)^2 > 0$, the defintion matches the normalization, and $(\mathrm{J} \circ \zeta_2 \circ \mathrm{J})^{-1} \circ \zeta_1 = \mathrm{J} \circ F^{-1} \circ \zeta_1$ indeed restricts to an orientation-preserving diffeomorphism $\phi \in \mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$.

For the Fr-smoothness on the functions, note that a one-parameter family of Riemann maps with the same normalization associated to a smooth one-parameter family of smooth parametrizations of a boundary curve depends smoothly on the parameter; see e.g. [Bel15, Theorem 28.1]. □

1.4 The Lie algebra of complex deformations

First, we identify the curvaceous tangent space $T_{\mathbb{1}} \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ as defined by Equation (1.10) with $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$.

[1]See section 1.5 for details on this terminology and the following sewing operation.

Proposition 1.14. *The tangent space at the identity of* $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ *is identified with* $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ *by a Fr-smooth* $\mathbb{R}$*-linear isomorphism* $[\Phi_v]_\sim \mapsto v(0,z)\,\partial_z$.

Proof. Any tangent vector at $\mathbb{1}$ may be represented by the flow of a smoothly time-dependent vector field $v \in \mathcal{C}_0(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))$, that is, $[\Phi_v]_\sim \in T_{\mathbb{1}}\,\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$. Since for such flows $\Phi_v(t,\,\cdot\,)$ and $\Phi_w(s,\,\cdot\,)$ for $v, w \in \mathcal{C}_0(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))$, the flows become composable as complex deformations for $s,t \in \mathbb{R}$ close enough to 0, the tangent space $T_{\mathbb{1}}\,\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ is a vector space with the addition and scalar multiplication given by

$$\lambda[\Phi_v]_\sim + [\Phi_w]_\sim = [t \mapsto \Phi_v(\lambda t)\Phi_w(t)]_\sim, \qquad , \lambda \in \mathbb{R}. \tag{1.58}$$

With the Fr-smooth function A_n on $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ for $n \in \mathbb{Z}$ defined in Equation (1.38) we compute

$$\frac{\partial}{\partial t}\bigg|_{t=0} A_n(\Phi_v(t,\,\cdot\,)) = \frac{\partial}{\partial t}\bigg|_{t=0} \frac{1}{2\pi i}\oint_{S^1} \frac{\Phi_v(t,z)}{z^{n+1}}dz = \frac{1}{2\pi i}\oint_{S^1} \frac{v(0,z)}{z^{n+1}}dz = a_{n+1}$$

where $v(0,z) = \sum_{n\in\mathbb{Z}} a_n \ell_n$. By relation (1.11), this implies that each representative of $[\Phi_v]_\sim$ gives the same $v(0,z)$. Thus, the map

$$\begin{aligned} Q : T_{\mathbb{1}}\,\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1) &\to \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1) \\ [\Phi_v]_\sim &\mapsto v(0,z)\partial_z \end{aligned} \tag{1.59}$$

is well-defined and injective. Surjectivity follows by defining for $v \in \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ the time-dependent vector field $w(t,z) = \rho(t)v(z)\partial_z$ where the smooth function $\rho(t)$ is constantly 1 in a neighborhood of 0 and has compact support such that the flow of $w \in \mathcal{C}_0(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))$.

In particular, we have shown that any vector in $[\gamma]_\sim \in T_{\mathbb{1}}\,\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ may be represented by a time-constant vector field as $[\gamma]_\sim = Q^{-1}(Q([\gamma]_\sim)) = [\Phi_v]_\sim$ vector field $v \in \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ and the curve $\Phi_v(t,z)$ is defined in a neighborhood of $t = 0$ and extended to all time by means of a cut-off function like ρ above. For the rest of this proof, we use this result implicitly.

For the linearity of Q, we compute

$$\begin{aligned} \frac{\partial}{\partial t}\bigg|_{t=0} \Phi_v(\lambda t, \Phi_w(t,z)) &= \Big(\lambda\Big(\partial_t\Phi_v\Big)(t,\Phi_w(t,z)) + \Big(\partial_z\Phi_v\Big)(\lambda t, \Phi_w(t,z))\cdot\partial_t\Phi_w(t,z)\Big)\bigg|_{t=0} \\ &= \lambda v(0,z) + w(0,z) \end{aligned} \tag{1.60}$$

With the vector space structure defined by (1.58) and relation (1.11), this shows that Q is linear:

$$Q(\lambda[\Phi_v]_\sim + [\Phi_w]_\sim) = \lambda v(0,\,\cdot\,) + w(0,\,\cdot\,), \qquad \Phi_v, \Phi_w \in \mathcal{C}_0(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)), \lambda \in \mathbb{C}.$$

Let γ be a Fr-smooth curve in $T_{\mathbb{1}}\,\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$. For each fixed $t \in \mathbb{R}$, $\gamma(t) \in T_{\mathbb{1}}\,\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ may be represented by a time-constant vector field $v_t = v_t(z)\,\partial_z$ as $\gamma(t) = [\Phi_{v_t}]_\sim$, that is, the flow $\Phi_{v_t}(s,\,\cdot\,)$ is taken with respect to the vector field v_t for fixed $t \in \mathbb{R}$ and a new independent time variable $s \in \mathbb{R}$. It is not clear whether v_t is smooth in t since we only know that $[\Phi_{v_t}]_\sim$ is smooth in t. The curve γ in $T_{\mathbb{1}}\,\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ is Fr-smooth if and only if for every function of the form $\mathrm{d}\,f$ for $f \in \mathcal{F}(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$, the composition $(\mathrm{d}\,f)\circ\gamma$ is smooth. Considering the functions $\mathrm{d}\,A_n \in \mathcal{F}_0(T_{\mathbb{1}}\,\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$, we have the smooth function of t,

$$\mathrm{d}\,A_n([\Phi_{v_t}]_\sim) = [A_n(\Phi_{v_t})]_\sim = \frac{\partial}{\partial\varepsilon}\bigg|_{\varepsilon=0} A_n(\Phi_{v_t}(\varepsilon,\,\cdot\,)) = a_{n+1}(t), \tag{1.61}$$

where $v_t = \sum_{n\in Z} a_{n+1}(t)\ell_n$. We conclude that since the functions a_{n+1} as defined in Equation (1.27) are Fr-smooth and generate the Frölicher structure, v_t is also smooth in t, and thus, v_t as a function of t is a Fr-smooth curve in $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$. This implies that Q is smooth.

For smoothness of Q^{-1}, take a curve $v \in \mathcal{C}_0(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))$, that is, a smoothly time-dependent vector field such that Φ_v exists for all time. To obtain a curve in $T_{\mathbb{1}} \operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1)$, we must consider a two-parameter family of complex deformations. First fix $s \in \mathbb{R}$ and let $w_s(t,z) = \rho(t)\, v(s,z)$ with ρ as above. This is a time-dependent vector field such that the flow Φ_{w_s} exists for all time — however, the time parameter s in $v(s,z)$ is fixed. This defines an element $[\Phi_{w_s}]_\sim \in T_{\mathbb{1}} \operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1)$ such that $Q([\Phi_{w_s}]_\sim) = w_s(0,\,\cdot\,) = v(s,\,\cdot\,)$ for small t. Taking the parameter s into account again, we now have a curve $[s \mapsto [\Phi_{w_s}]_\sim]_\sim$ in $T_{\mathbb{1}} \operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1)$. Q^{-1} is Fr-smooth if and only if this (two-parameter) curve is Fr-smooth. To show this, we reverse the argument around Equation (1.61). The Frölicher structure on $T_{\mathbb{1}} \operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1)$ is generated differentials $\mathrm{d}_{\mathbb{1}} A_n$ of functions $A_n \in \mathcal{F}_0(\operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1))$. Since the composition of $[s \mapsto [\Phi_{w_s}]_\sim]_\sim$ with $\mathrm{d}_{\mathbb{1}}$ yields the Fr-smooth functions $s \mapsto a_{n+1}(s)$ of $v = \sum_{n \in Z} a_n \ell_n$, we find that indeed the curve $[s \mapsto [\Phi_{w_s}]_\sim]_\sim$ is Fr-smooth and thus Q^{-1} is Fr-smooth.

□

Even though $\operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1)$ does not have a (local) Lie group structure, we show that the Fr-smooth composition found in Proposition 1.11 is sufficient to induce the expected Lie algebra structure on $\mathfrak{g} = T_{\mathbb{1}} \operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1)$. We follow a strategy similar to that in [Lau11] for a Frölicher–Lie group. Namely, if the map

$$\begin{aligned} \Xi : \mathfrak{g} &\hookrightarrow T_0\mathfrak{g}, \\ v &\mapsto [t \mapsto tv]_\sim. \end{aligned} \tag{1.62}$$

is bijective, then the Lie bracket on $\mathfrak{g}$ may be defined as

$$[v,w] = \Xi^{-1}([s \mapsto [t \mapsto \gamma(s)\, \eta(t)\, \gamma^{-1}(s)\, \eta^{-1}(t)]_\sim]_\sim) \tag{1.63}$$

where $\gamma, \eta \in \mathcal{C}(\operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1))$ represent the tangent vectors $v = [\gamma]_\sim$ and $w = [\eta]_\sim$. Since $[t \mapsto 0]_\sim \in T_0\mathfrak{g}$ is the 0 vector, Ξ is clearly injective. Note that the proof of [Lau11, Theorem 3.12] only depends on the local structure of the Frölicher–Lie group at the identity, and thus also applies to the case of $\operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1)$. In particular, the Lie bracket (1.63) is well defined (if Ξ is surjective) since the complex deformation in the formula becomes composable for s and t both small enough.

Proposition 1.15. *The Lie bracket on $T_{\mathbb{1}} \operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1)$ in Equation (1.63) is well-defined and agrees with the usual Lie bracket on $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$.*

Proof. We check the surjectivity of Ξ as in Equation (1.62). Let $[\gamma]_\sim$ be an element of $T_0 T_{\mathbb{1}} \operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1)$, represented by a curve $\gamma \in \mathcal{C}(T_{\mathbb{1}} \operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1))$. We have already seen above that such a γ again is represented by $\gamma(t) = [\Phi_{v_t}]_\sim$ where v_t is a smoothly time-dependent vector field but the flow $\Phi_{v_t}(s,\,\cdot\,)$ integrated for fixed $t \in \mathbb{R}$ and a new time variable $s \in \mathbb{R}$. Define the smoothly time-dependent vector field $w = \rho(t)\, \partial_t v_t(z)\, \partial_z$, where ρ is a smooth bump function as before. It has the property that $\Xi([\Phi_w]_\sim) = [s \mapsto s[\Phi_w]_\sim]_\sim$. To compare it to the original vector $[\gamma]_\sim = [t \mapsto [\Phi_{v_t}]_\sim]_\sim$, we differentiate the function $A_n \in \mathcal{F}_0(\operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1))$, which yields a function $\mathrm{d}_0\, \mathrm{d}_{\mathbb{1}} A_n$ on $T_0 T_{\mathbb{1}} \operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1)$, and compute on the one hand,

$$(\mathrm{d}_0\, \mathrm{d}_{\mathbb{1}} A_n)([s \mapsto s[\Phi_w]_\sim]_\sim) = \partial_s \partial_t s A_n(\Phi_w)\Big|_{t=s=0} = a_{n+1}(w(t,\,\cdot\,)) = \frac{\partial}{\partial s}\Big|_{s=0} a_{n+1}(v_s), \tag{1.64}$$

and on the other hand

$$(\mathrm{d}_0\, \mathrm{d}_{\mathbb{1}} A_n)([s \mapsto [\Phi_{v_s}]_\sim]_\sim) = \partial_s \partial_t A_n(\Phi_{v_s})\Big|_{t=s=0} = \frac{\partial}{\partial s}\Big|_{s=0} a_{n+1}(v_s). \tag{1.65}$$

Finally, the Lie bracket on $T_{\mathbb{1}} \operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1)$ given by Equation (1.63) is precisely the expression for the Lie bracket of vector fields in terms of their flows

$$[v,w] = \frac{\partial^2}{\partial t \partial s}\Big|_{t=s=0} \Phi_v(t,\,\cdot\,) \circ \Phi_w(s,\,\cdot\,) \circ (\Phi_v(t,\,\cdot\,))^{-1} \circ (\Phi_w(s,\,\cdot\,))^{-1} \tag{1.66}$$

□

Any tangent vector in $T_\phi \operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1)$ may be represented by $[t \mapsto \Phi_v(s+t,\,\cdot\,)]_\sim$ for some time-dependent vector field $v \in \mathcal{C}_0(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))$ and $s \in \mathbb{R}$ such that $\phi = \Phi_v(s,\,\cdot\,)$. Since the time-independent vector field $w = v(s,z)\,\partial_z \in \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ is real-analytic on $\phi(S^1)$, the pullback $\phi^* w$ via ϕ is real-analytic on S^1; see also Equation (1.21). Adding to Remark 1.7, the pullback maps $\phi^* : T_\phi \operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1) \to T_0 \operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1)$ are isomorphisms if and only if $\phi \in \mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$, since otherwise there exists a vector field in $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ with a singularity between $\phi^{-1}(S^1)$ and S^1, and this vector field cannot be pushed forward. At the level of the flow, the pullback by a conformal map $F : A \to B$ between annular neighborhoods $A, B \subset \hat{\mathbb{C}}$ of S^1 such that v is holomorphic on B is given by

$$\Phi_{F^* v}(t,\,\cdot\,) = F^{-1} \circ \Phi_v(t,\,\cdot\,) \circ F, \tag{1.67}$$

for t close enough to 0.

We use these pullbacks to define left-invariant differential forms on $\operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1)$. For $F \in \bigwedge^n \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)^\vee$, the n-form $\alpha_F \in \Omega^n(I)$ on $\operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1)$ defined by

$$\alpha_F(v_1, \ldots, v_n) = F(\phi^* v_1, \ldots, \phi^* v_n), \qquad v_j \in T_\phi \operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1), \tag{1.68}$$

is the left-invariant n-form which reduces to F at $\phi = \mathbb{1}$. Since holomorphicity of the vector fields representing the tangent vectors is only needed on $\phi(S^1)$, the differential form (1.68) may be continued to $C^\omega(S^1, \mathbb{C} \setminus \{0\})$ by the same formula (1.68). Then, we can apply the Poincaré lemma [KM97, Lemma 33.20].

1.5 Infinite-dimensional moduli spaces of Riemann surfaces

In this section, we obtain results on the moduli spaces $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ of Riemann surfaces with enumerated and analytically parametrized boundary components,

$$\mathcal{M}_{\mathsf{g},\mathsf{b}} = \left\{ \begin{array}{c} \text{connected compact genus } \mathsf{g} \text{ Riemann surfaces } \Sigma \text{ with } \mathsf{b} \\ \text{enumerated and analytically parametrized boundary} \\ \text{components } \partial_1\Sigma, \ldots, \partial_{\mathsf{b}}\Sigma \text{ in negative orientation} \end{array} \right\}_{/\text{ isom.}} \tag{1.69}$$

The surfaces in $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ come with negatively oriented[2] boundary parametrizations $\zeta_j : S^1 \to \partial_j \Sigma$. We denote them by tuples $\Sigma = (\Sigma, \zeta_1, \ldots, \zeta_{\mathsf{b}})$ and their equivalence classes in $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ by $[\Sigma] = [\Sigma, \zeta_1, \ldots, \zeta_{\mathsf{b}}]$. Two surfaces $(\Sigma_1, \zeta_1, \ldots, \zeta_{\mathsf{b}})$ and $(\Sigma_2, \xi_1, \ldots, \xi_{\mathsf{b}})$ are isomorphic if there exists a biholomorphism $F : \Sigma_1 \to \Sigma_2$ such that $F \circ \zeta_j = \xi_j$ for $1 \le j \le \mathsf{b}$. We define the commonly used surfaces

$$\mathbb{D} = \big[\, \bar{\mathbb{D}}, \mathrm{J} \,\big] \qquad \in \mathcal{M}_{0,1}, \tag{1.70}$$

$$\mathbb{A}_\tau = \big[\, \{z \in \mathbb{C} \mid e^{-2\pi\tau} \le |z| \le 1\}, \quad \mathrm{J}, \quad e^{-2\pi\tau}\,\mathbb{1} \,\big] \qquad \in \mathcal{M}_{0,2}, \qquad \tau > 0. \tag{1.71}$$

The main interest in the infinite-dimensional moduli spaces $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ is their algebraic structure with respect to the sewing (or gluing) operations $\mathcal{M}_{\mathsf{g}_1,\mathsf{b}_1} \times \mathcal{M}_{\mathsf{g}_2,\mathsf{b}_2} \to \mathcal{M}_{\mathsf{g}_1+\mathsf{g}_2,\mathsf{b}_1+\mathsf{b}_2-2}$, defined by

$$\begin{aligned} &\big(\, \Sigma_1, \zeta_1, \ldots, \zeta_{\mathsf{b}_1} \,\big) \;{}_j\!\infty_k\; \big(\, \Sigma_2, \xi_1, \ldots, \xi_{\mathsf{b}_2} \,\big) \\ &= \big(\, (\Sigma_1 \sqcup \Sigma_2)/_\sim, \zeta_1, \ldots, \zeta_{j-1}, \zeta_{j+1}, \ldots \zeta_{\mathsf{b}_1}, \xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots \xi_{\mathsf{b}_2} \,\big) \end{aligned} \tag{1.72}$$

where $\sim$ identifies the boundaries $\partial_j \Sigma_1$ and $\partial_k \Sigma_2$ via $\xi_k \circ \mathrm{J} \circ \zeta_j^{-1}$, with J defined by Equation (1.22). Since all boundary parametrizations are negatively oriented, the inversion J ensures

[2]The negative orientation is such that if $\Sigma \subset \hat{\mathbb{C}}$ is a genus 0 surface, then if any boundary parametrization $\zeta_j : S^1 \to \partial_j \Sigma$ extends to the unit disk $\mathbb{D}$, then we may "fill in" that boundary component, such that the seam as viewed from $\zeta_j(0)$ is positively oriented.

that we are identifying the inside of S^1 in the parametrization of $\partial_j \Sigma_1$ to the outside of S^1 in the parametrization of $\partial_k \Sigma_2$. A self-sewing operation $\infty_{j,k} \; \cdot \; : \mathcal{M}_{\mathsf{g},\mathsf{b}} \to \mathcal{M}_{\mathsf{g}+1,\mathsf{b}-2}$ is defined analogously.

The complex deformations $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$, defined in Section 1.3, act on $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ in b ways by deformation of the boundary component. Below, we provide a Frölicher structure on $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ precisely such that these actions and the sewing operations (1.72) are Fr-smooth. We initially define the actions in a pointwise manner. For a fixed surface $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$ with representative $(\Sigma, \zeta_1, \ldots, \zeta_{\mathsf{b}})$ and $1 \leq j \leq \mathsf{b}$, consider the surface $\Sigma \,{}_j\infty_1\, \mathbb{D}$. The representative comes with an open neighborhood U of $\bar{\mathbb{D}}$ in $\mathbb{C}$ such that ζ_j has a conformal extension $\hat{\zeta}_j : U \to \Sigma \,{}_j\infty_1\, U$ of $\zeta_j : U \setminus \mathbb{D} \to \Sigma$ such that $\hat{\zeta}_j(z) = z \in \bar{\mathbb{D}} \subset \Sigma \,{}_j\infty_1\, \bar{\mathbb{D}}$ for $z \in \bar{\mathbb{D}}$. Let $\phi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ be a complex deformation such that $\phi(S^1) \subset U$. Denote by $\mathring{U}_\phi$ the domain in U bounded by $\phi(S^1)$. Define the surface

$$\Sigma \underset{j}{*} \phi = \big((\Sigma \,{}_j\infty_1\, \bar{\mathbb{D}}) \setminus \hat{\zeta}_j(\mathring{U}_\phi), \zeta_1, \ldots, \zeta_{j-1}, \zeta_j \circ \phi, \zeta_{j+1}, \ldots, \zeta_{\mathsf{b}} \big). \tag{1.73}$$

This is the surface Σ after deformation of the jth boundary component by the complex deformation ϕ. Note that this deformation does not exist if either $\phi(S^1)$ is outside the radius of convergence of $\hat{\zeta}_j$ or if the deformation would cause the boundary components to overlap. On the contrary, it is possible that, besides changing the boundary parametrization, the deformation adds or subtracts parts of the surface.

Proposition 1.16. *If $\Sigma \underset{j}{*} \phi$ exists, it is independent of the representative of $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$. For $j \neq k$ and $\phi_1, \phi_2 \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ such that $\Sigma \underset{j}{*} \phi_1$ and $\Sigma \underset{k}{*} \phi_2$ exist, the surfaces*

$$\Sigma \underset{j}{*} \phi_1 \underset{k}{*} \phi_2 = \Sigma \underset{k}{*} \phi_2 \underset{j}{*} \phi_1 \in \mathcal{M}_{\mathsf{g},\mathsf{b}} \tag{1.74}$$

exist and agree. For $j = k$, we have

$$\Sigma \underset{j}{*} \phi_1 \underset{j}{*} \phi_2 = \Sigma \underset{j}{*} (\phi_2 \circ \phi_1) \in \mathcal{M}_{\mathsf{g},\mathsf{b}}, \tag{1.75}$$

if all compositions exist.

Proof. Let $F : \Sigma_1 \to \Sigma_2$ be an isomorphism of representatives $\Sigma_1 = (\Sigma_1, \zeta_1, \ldots, \zeta_{\mathsf{b}})$ and $\Sigma_1 = (\Sigma_1, \zeta_1, \ldots, \zeta_{\mathsf{b}})$ for the same element of $\mathcal{M}_{\mathsf{g},\mathsf{b}}$. Then, F extends to an isomorphism $\hat{F} : \Sigma_1 \,{}_j\infty_1\, \bar{\mathbb{D}} \to \Sigma_2 \,{}_j\infty_1\, \bar{\mathbb{D}}$ by defining it as the identity on the chart $\bar{\mathbb{D}}$ because the transition maps are respectively ζ_j and ξ_j, which $\hat{F}$ is compatible with since $F \circ \zeta_j = \xi_j$. The same identity shows that if ζ_j extends to U_1, then ξ_j extends to $U_2 = \hat{F}(U_1)$, where U_1 and U_1 are defined like U above. Therefore, $\hat{F}$ restricts to an isomorphism of $\Sigma_1 \underset{1}{*} \phi$ and $\Sigma_2 \underset{1}{*} \phi$.

The simultaneous existence and commutativity of the actions at different boundary components are immediate since the actions are constructed locally in a neighborhood of the boundary components. Multiple deformations at the same boundary component compose since, by Equation (1.73), the deformations act by composing with the boundary parametrization. □

For $1 \leq j \leq \mathsf{b}$, we consider the set of pairs of surfaces and complex deformations such that the deformed surface (1.73) exists:

$$\mathcal{U}_{\mathsf{g},\mathsf{b},j} = \left\{ (\Sigma, \phi) \in \mathcal{M}_{\mathsf{g},\mathsf{b}} \times \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1) \,\middle|\, \Sigma \underset{j}{*} \phi \text{ exists} \right\}. \tag{1.76}$$

The actions of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ on $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ are then defined by the maps

$$\cdot \underset{j}{*} \cdot : \mathcal{U}_{\mathsf{g},\mathsf{b},j} \to \mathcal{M}_{\mathsf{g},\mathsf{b}}, \qquad 1 \leq j \leq \mathsf{b}. \tag{1.77}$$

They have the following interactions with the sewing operation defined in Equation (1.72) for $j \neq k$,

$$(\Sigma_1 \underset{j}{*} \phi) \,{}_k\infty_l\, \Sigma_2 = (\Sigma_1 \,{}_k\infty_l\, \Sigma_2) \underset{j}{*} \phi, \tag{1.78}$$

$$(\Sigma_1 \underset{j}{*} \phi) \; {}_j\infty_l \; \Sigma_2 = \Sigma_1 \; {}_j\infty_l \; (\Sigma_2 \underset{l}{*} (\mathrm{J} \circ \phi^{-1} \circ \mathrm{J})), \tag{1.79}$$
$$\infty_{k,l} \; (\Sigma \underset{j}{*} \phi) = (\; \infty_{k,l} \; \Sigma) \underset{j}{*} \phi \tag{1.80}$$
$$\infty_{j,l} \; (\Sigma \underset{j}{*} \phi) = \; \infty_{j,l} \; (\Sigma \underset{l}{*} (\mathrm{J} \circ \phi^{-1} \circ \mathrm{J})) \tag{1.81}$$

which only hold if all deformations exist.

Remark 1.17. Note that for a given vector field $v \in \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$, a surface $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$, and $1 \leq j \leq \mathsf{b}$, there exists some $\varepsilon > 0$ such that for all $t \in (-\varepsilon, \varepsilon)$ the complex deformations given by the flow $\Phi_v(t, \,\cdot\,)$ can act on $\partial_j\Sigma$, that is, $(\Sigma, \Phi_v(t, \,\cdot\,)) \in \mathcal{U}_{\mathsf{g},\mathsf{b},j}$.

We endow $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ with a Frölicher structure, after introducing the following cutting operation. Consider an analytically parametrized loop $\eta : S^1 \to \Sigma$ in a fixed surface $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$ represented by $(\Sigma, \zeta_1, \ldots, \zeta_{\mathsf{b}})$. Assume that $\eta(S^1) \cap \partial\Sigma = \emptyset$ holds (unless $\eta = \zeta_j$ for some $1 \leq j \leq \mathsf{b}$, in which case the following is already defined in Equation (1.77) as the action on the boundary parametrization). We can consider η and $\eta \circ \mathrm{J}$ as boundary parametrizations of surfaces Σ_1 and Σ_2 which are the left and right hand sides of Σ along the loop $\eta(S^1)$ with the respective boundary parametrizations from $\zeta_1, \ldots, \zeta_n, \eta, \eta \circ \mathrm{J}$. In this setup we have $\Sigma_1 \; {}_j\infty_k \; \Sigma_2$ where η has become the jth boundary parametrization of Σ_1 and $\eta \circ \mathrm{J}$ has become the kth boundary parametrization of Σ_2. For $\phi, \psi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$, such that $\Sigma_1 \underset{j}{*} \phi$ and $\Sigma_2 \underset{k}{*} \psi$ exist, define that action of the pair (ϕ, ψ) at η by

$$\Sigma \underset{\eta}{*} (\phi, \psi) = (\Sigma_1 \underset{j}{*} \phi) \; {}_j\infty_k \; (\Sigma_2 \underset{k}{*} \psi). \tag{1.82}$$

Then, we let simultaneous deformations of the boundary components and at finitely many interior curves generate a Frölicher structure on $\mathcal{M}_{\mathsf{g},\mathsf{b}}$:

$$\mathcal{C}_0(\mathcal{M}_{\mathsf{g},\mathsf{b}}) = \left\{ t \mapsto \Sigma_t \;\middle|\; \begin{array}{c} \eta_1, \ldots \eta_n \text{ are } n \geq 0 \text{ disjoint analytic loops interior to } \Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}, \\ \phi_{1,t}, \ldots, \phi_{n,t}, \psi_{1,t}, \ldots, \psi_{n,t}, \gamma_{1,t}, \ldots, \gamma_{\mathsf{b},t} \in \mathcal{C}(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)) \text{ such that} \\ \Sigma_t = \Sigma \underset{\eta_1}{*} (\phi_{1,t}, \psi_{1,t}) \cdots \underset{\eta_n}{*} (\phi_{n,t}, \psi_{1,t}) \underset{1}{*} \gamma_{1,t} \cdots \underset{\mathsf{b}}{*} \gamma_{\mathsf{b},t} \text{ exists } \forall t \in \mathbb{R}. \end{array} \right\} \tag{1.83}$$

This Frölicher structure is defined exactly such that $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ acts on $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ in a Fr-smooth way, and such that the sewing operations (1.72) are Fr-smooth. More precisely, we prove the following theorem.

Theorem 1.18. *The maps*

$$\begin{aligned} \cdot \underset{j}{*} \cdot : \mathcal{U}_{\mathsf{g},\mathsf{b},j} &\longrightarrow \mathcal{M}_{\mathsf{g},\mathsf{b}} \\ (\Sigma, \phi) &\longmapsto \Sigma \underset{j}{*} \phi. \end{aligned} \tag{1.84}$$

and the sewing operations

$$\cdot \; {}_j\infty_k \; \cdot : \mathcal{M}_{\mathsf{g}_1,\mathsf{b}_1} \times \mathcal{M}_{\mathsf{g}_2,\mathsf{b}_2} \to \mathcal{M}_{\mathsf{g}_1+\mathsf{g}_2,\mathsf{b}_1+\mathsf{b}_2-2}, \qquad 1 \leq j \leq \mathsf{b}_1, 1 \leq k \leq \mathsf{b}_2 \tag{1.85}$$

as defined by Equation (1.72) *are Fr-smooth. Moreover, the actions define Fr-smooth Lie algebra homomorphisms*

$$\begin{aligned} \varphi_j : \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1) &\longrightarrow \mathrm{Vect}(\mathcal{M}_{\mathsf{g},\mathsf{b}}) \\ v &\longmapsto \big(\Sigma \mapsto [\Sigma \underset{j}{*} \Phi_v]_\sim\big). \end{aligned} \tag{1.86}$$

Proof. At this point, we know that the Frölicher smooth structure on $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ is such that for a fixed surface, the action is smooth as a function of the complex deformation. Consider a curve $t \mapsto (\Sigma_t, \phi_t)$ in $\mathcal{U}_{\mathsf{g},\mathsf{b},j}$, where the curve Σ_t is of the form $\mathcal{C}_0(\mathcal{M}_{\mathsf{g},\mathsf{b}})$. Then, $\Sigma_t \underset{j}{*} \phi_t$ is a smooth curve as well since, by Proposition 1.16, ϕ_t just composes with the deformation at the jth boundary component and thus the smoothness follows from the smoothness of the composition in $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$. Smoothness of the sewing is immediate since the deformations at the sewn boundary components compose to a deformation in the interior, which by definition is a smooth curve as in (1.82). Finally, the Fr-smooth action induces a Lie algebra homomorphism by replicating [Lau11, Theorem 3.12]. □

Let us briefly discuss embeddings of Riemann surfaces with analytically parameterized boundary components. Of course, there are the canonical maps

$$\Sigma_1 \hookrightarrow \Sigma_1 \, {}_j\!\infty_k \, \Sigma_2 \qquad \text{and} \qquad \Sigma_2 \hookrightarrow \Sigma_1 \, {}_j\!\infty_k \, \Sigma_2 \tag{1.87}$$

into a surface sewn from two components, and analogously for $\Sigma \to \infty_{j,k} \Sigma$ which, however, is not injective on the seam. Despite that, we consider the following notion of embedding, continuing the two relations above into a partial order as long as there is at least one common boundary component. We also consider the case where the labels of the common boundary components do not match.

Definition 1.19. A surface $\Sigma_1 \in \mathcal{M}_{\mathsf{g}_1,\mathsf{b}_1}$ embeds into a surface $\Sigma_2 \in \mathcal{M}_{\mathsf{g}_2,\mathsf{b}_2}$ at the pairs of boundary components $(j_1, k_1), \dots, (j_n, k_n)$ if Σ_2 can be obtained from Σ_1 by a finite number of sewing operations, sewing a tuple of surfaces $\underline{\Sigma}$ to Σ_1 not at $j_1, \dots, j_n$, and finally relabeling the boundary components $j_1, \dots, j_n$ to $k_1, \dots, k_n$. We denote this relation by

$$\Sigma_1 \, {}_{j_1,\dots j_n}\!\subseteq_{k_1,\dots k_n} \Sigma_2 \qquad \Longleftrightarrow \qquad \exists \, \underline{\Sigma} \text{ such that } \Sigma_1 \, \underline{\infty} \, \underline{\Sigma} = \Sigma_2. \tag{1.88}$$

In the following, we introduce several interesting subspaces of the moduli spaces $\mathcal{M}_{\mathsf{g},\mathsf{b}}$.

1.5.1 Tori with geodesic seam

Sewing the two boundary components of an annulus results in a torus

$$T = \infty_{1,2} A \in \mathcal{M}_{1,0}, \qquad A \in \mathcal{M}_{0,2}. \tag{1.89}$$

Up to scale, both the annulus and the torus come with a unique conformal flat metric. Since the embedding is conformal, the seam inside T defined by the parametrizations of A is a geodesic with respect to the flat metric on T if the parametrization is of constant speed. This defines the set of *annuli with geodesic property*, $\mathcal{M}_{0,2}^{\mathsf{geod}} \subseteq \mathcal{M}_{0,2}$. Basic examples are the standard annuli $\mathbb{A}_\tau$ for $\tau > 0$ defined in Equation (1.71), and $\mathbb{A}_\tau \underset{1}{*} R_\theta \in \mathcal{M}_{0,2}^{\mathsf{geod}}$ where one boundary parametrization is twisted by a rotation $R(z) = e^{\mathrm{i}\,\theta} z$ for $\theta \in \mathbb{R}$. Any other annulus with the geodesic property is of the form

$$\mathbb{A}_\tau \underset{1}{*} R_\theta \underset{1}{*} \phi \underset{2}{*} (\mathrm{J} \circ \phi^{-1} \circ \mathrm{J}) \tag{1.90}$$

for some diffeomorphism $\phi \in \mathrm{Diff}_+^{\mathrm{an}}(\mathrm{S}^1)$ which cancels in the sewing operation (1.89).

1.5.2 Hyperbolic surfaces

If g and b are such that the Euler characteristic is negative, there is a unique conformal hyperbolic metric (curvature -1) in the conformal class of any surface $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$ such that the boundary components are geodesics. If Σ has a representative $(\Sigma, \zeta_1, \dots, \zeta_{\mathsf{b}})$ such that each parametrization has constant speed $|\partial_\theta \zeta_j(e^{\mathrm{i}\,\theta})|_g$ in the hyperbolic metric g, we call Σ itself *hyperbolic*. We denote the subspace of hyperbolic surfaces of $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ by $\mathcal{M}_{\mathsf{g},\mathsf{b}}^{\mathsf{hyp}}$.

1.5.3 Möbius surfaces

Another special type of surface is one where a boundary parameterization is given by a Möbius transformation. A priori, this property is only meaningful in genus zero, where a surface $\Sigma \in \mathcal{M}_{0,\mathsf{b}}$ may be embedded into the Riemann sphere by uniformizing the surface with the b boundary components capped off by sewing on $\bar{\mathbb{D}} \in \mathcal{M}_{0,1}$,

$$\Sigma \longrightarrow \Sigma \, {}_1\!\infty_1 \, \bar{\mathbb{D}} \cdots {}_{\mathsf{b}}\!\infty_1 \, \bar{\mathbb{D}} \xrightarrow{F} \hat{\mathbb{C}}, \tag{1.91}$$

where F is the uniformizing isomorphism. Then, a boundary parametrization $\zeta_j : S^1 \to \Sigma$ maps to $\hat{\mathbb{C}}$ by composing with F, and we can ask for $F \circ \zeta_j$ to be a Möbius transformation. Since F is unique up to Möbius transformations, this property is independent of the choice of embedding. We say that a representative of $\Sigma \in \mathcal{M}_{0,\mathsf{b}}$ is *uniformized* if it is of the form

$$\big(\hat{\mathbb{C}} \setminus \big(\mathrm{J}(\zeta_1(\mathbb{D})) \cup \zeta_2(\mathbb{D}) \cup \cdots \cup \zeta_{\mathsf{b}}(\mathbb{D}) \big), \mathrm{J} \circ \zeta_1, \zeta_2, \ldots, \zeta_{\mathsf{b}} \big), \tag{1.92}$$

for complex deformations $\zeta_1, \ldots, \zeta_{\mathsf{b}} \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ extending conformally to $\bar{\mathbb{D}}$, and $\zeta_1(0) = 0$, that is, $\mathrm{J}(\zeta_1(0)) = \infty$.

Slightly more generally, a surface with this notion of Möbius parametrization satisfies the Definition 1.20 below. For higher genus surfaces, we can now ask for a pants decomposition such that an external boundary component may be represented as a Möbius transformation mapping into the respective pair of pants. Moreover, we consider the case where all boundaries are simultaneously Möbius.

Definition 1.20. A surface $\Sigma \in \mathcal{M}_{0,\mathsf{b}}$ is *j-Möbius* for $1 \leq j \leq \mathsf{b}$ if there exists a representative $(A, \zeta_1, \ldots, \zeta_{\mathsf{b}})$ with $A \subset \hat{\mathbb{C}}$ such that the parametrization $\zeta_j : S^1 \to \hat{\mathbb{C}}$ extends to a Möbius transformation. For $\mathsf{g} > 0$, $A \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$ is j-Möbius if there exists a decomposition of A into genus 0 surfaces such that the surface attached to $\partial_j A$ is Möbius at the boundary component corresponding to $\partial_j A$. A surface $A \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$ is *Möbius* if there exists a representative $(A, \zeta_1, \ldots, \zeta_{\mathsf{b}})$ such that the parametrizations ζ_j for all $1 \leq j \leq \mathsf{b}$ extend to Möbius transformations (in the same decomposition and representatives). Denote by

$$\mathcal{M}_{\mathsf{g},\mathsf{b}}^{j\text{-Möb}} \subset \mathcal{M}_{\mathsf{g},\mathsf{b}}, \qquad \mathcal{M}_{\mathsf{g},\mathsf{b}}^{\text{Möb}} \subset \mathcal{M}_{\mathsf{g},\mathsf{b}}, \tag{1.93}$$

respectively the moduli spaces of j-Möbius surfaces and Möbius surfaces.

2 Cohomology of complex deformations

In this section, we first introduce group-level cohomology on $G = \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ or a subspace

$$G = \left\{ \phi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1) \,\middle|\, \phi(S^1) \subset e^{2\pi\tau} \bar{\mathbb{D}} \right\} \subset \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1), \qquad \tau > 0, \tag{2.1}$$

of complex deformations with bounded deformation, which act on the annuli $\mathbb{A}_\tau$ defined in Equation (1.71). The cohomology may also be relative to finitely many (local) subgroups $H_1, \ldots, H_N$. We are mainly interested in two subgroups: On the one hand, the diffeomorphism group $H_1 = \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$, and, on the other hand, the group $H_2 = \mathrm{Sc}$ of scaling transformations defined in Equation (1.45), possibly restricted to (2.1). For the cohomologies of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ relative to the subgroups, there are four exact sequences of the form (2.7), which conveniently fit into a braided diagram (Figure 2). In Section 2.3, we compute the terms of this diagram up to $n = 2$, with special interest in finding a basis of $H^2(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1); \mathrm{Diff}_+^{\mathrm{an}}(\mathrm{S}^1), \mathrm{Sc}; \mathbb{R})$. In addition to the braided diagram, our method involves an exact sequence relating the group cohomology to the respective Lie algebra cohomology and characters of the fundamental groups. The respective group- and algebra-level cocycles are presented beforehand, in Section 2.2.

2.1 Relative group-level cohomology

Let $\underline{H} = (H_1, \ldots H_N)$ any subgroups of G such that $H_j \cap H_k = \{\mathbb{1}\}$ for $1 \leq j < k \leq N$. Then, the n-cochains on G relative to H with coefficients in $\mathbb{R}$ are defined as

$$C^n(G; \underline{H}; \mathbb{R}) = \left\{ \Omega \in \mathcal{F}(M_n(G)) \,\middle|\, \Omega|_{H_j^n} = 0 \;\forall\; 1 \leq j \leq N \right\}. \tag{2.2}$$

where $M_n(G)$ is the subset of tuples in $(g_1, \ldots, g_n) \in G^n$ such that products $g_j g_{j+1} \cdots g_{j+k}$ of any number of consecutive elements exist, $1 \leq j \leq n$ and $1 \leq k \leq n - j$. In particular, for

$G = \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ we have $M_2(G) = M$, where the latter is the subset of $G \times G$ in Equation (1.43). The non-relative cochains are obtained as $C^n(G;\mathbb{R}) = C^n(G;\{\mathbb{1}\};\mathbb{R})$. By forgetting a single subgroup H_j, leaving $\underline{\hat{H}}$ we have a short exact sequence

$$\{0\} \longrightarrow C^n(G;\underline{H};\mathbb{R}) \longrightarrow C^n(G;\underline{\hat{H}};\mathbb{R}) \longrightarrow C^n(H_j;\mathbb{R}) \longrightarrow \{0\} \tag{2.3}$$

where the maps are defined by restriction of the cocycles. The differential $\delta\,\Omega$ of a cochain $\Omega \in C^n(G,\underline{H},\mathbb{R})$ is defined by

$$\begin{aligned}(\delta\,\Omega)(g_1,\dots,g_{n+1}) = \Omega(g_2,\dots,g_{n+1}) + (-1)^{n+1}\Omega(g_1,\dots,g_n)\\ + \sum_{j=1}^{n}(-1)^j\Omega(g_1,\dots,g_{j-1},g_jg_{j+1},g_{j+1},\dots,g_{n+1}).\end{aligned} \tag{2.4}$$

In the special cases of $n = 1$ and $n = 2$ this reduces to

$$\begin{aligned}(\delta\,f)(g_1,g_2) &= f(g_1) + f(g_2) - f(g_1g_2), & f \in C^1(G,H,\mathbb{R}),\\ (\delta\,\Omega)(g_1,g_2,g_3) &= \Omega(g_2,g_3) + \Omega(g_1,g_2g_3) - \Omega(g_1,g_2) - \Omega(g_1g_2,g_3), & \Omega \in C^2(G,H,\mathbb{R}).\end{aligned} \tag{2.5}$$

The relative group cohomology groups are then defined by

$$\begin{aligned}Z^n(G;\underline{H};\mathbb{R}) &= \ker\delta,\\ B^n(G;\underline{H};\mathbb{R}) &= \delta\,C^{n-1}(G;\underline{H};\mathbb{R}),\\ H^n(G;\underline{H};\mathbb{R}) &= Z^n(G;\underline{H};\mathbb{R})/B^n(G;\underline{H};\mathbb{R}).\end{aligned} \tag{2.6}$$

The short exact sequence (2.3) leads to a long exact sequence of cohomology groups

$$\cdots \to H^{n-1}(H_j,\mathbb{R}) \to H^n(G;\underline{H};\mathbb{R}) \to H^n(G;\underline{\hat{H}};\mathbb{R}) \to H^n(H_j,\mathbb{R}) \to H^{n+1}(G;\underline{H};\mathbb{R}) \to \cdots \tag{2.7}$$

where the transgression maps are defined by the usual zig-zag lemma.

In the special case where all G is a finite-dimensional and simply connected Lie group, the relative group cohomology is isomorphic to the relative Lie algebra cohomology. The isomorphism $H^n(G;H_1,\dots,H_N;\mathbb{R}) \cong H^n(\mathfrak{g};\mathfrak{h}_1,\dots,\mathfrak{h}_N;\mathbb{R})$ is called the van Est isomorphism [Van53a, Van53b]. In the case of G, which is connected but not simply connected, we prove a similar statement for the universal cover. By using that $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ is naturally a subset of $C^\omega(S^1,\mathbb{C}\setminus\{0\})$, the universal cover of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ exists [KM97, Paragraph 27.14], and we denote it by $\mathcal{UC}(G)$. Recall the standard construction of the universal cover by considering all continuous, or equivalently, Fr-smooth paths in G, see e.g. [Ful99]. In our setup, the universal cover comes with a Frölicher structure given by lifts of curves $\mathcal{C}(X)$ and the projection $\pi : \mathcal{UC}(X) \to X$ is a Fr-smooth covering map. The path-construction also gives an embedding $\pi_1(G,\mathbb{1}) \subseteq \mathcal{UC}(G)$ of the (Fr-smooth) fundamental group as the group of equivalence classes of paths returning to the basepoint $\mathbb{1} \in G$.

Conceptually, the result below is obtained from a similar exact sequence proven by Neeb to hold in a quite general setup of infinite-dimensional Lie groups [Nee04]. However, the setup of Neeb does not fit our setting of Frölicher structures and local composition laws, whence we adapt the proof of Neeb. Note that, while choosing a partially more general setting, our result is at the same time less general compared to that of Neeb, since our coefficients are in the additive group $\mathbb{R}$, which is regarded as a trivial $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$-module, as opposed to a possibly infinite-dimensional nontrivial module of the group in Neeb's theorem. Note also that in our infinite-dimensional setting, Lie's third theorem, which allows integration of Lie algebra cocycles to the group-level, might not hold. Hence, we assume this part of the statement, and explicitly formulate a group-level cocycle for each Lie algebra cocycle in our application of the result to complex deformations in the next section.

Proposition 2.1. *If the derivative* $\mathrm{D} : H^2(G;\underline{H};\mathbb{R}) \to H^2(\mathfrak{g};\underline{\mathfrak{h}};\mathbb{R})$ *is surjective, then the following sequence is exact,*

$$\{0\} \longrightarrow \operatorname{Hom}(\pi_1(G;\underline{H}),\mathbb{R}) \xrightarrow{b} H^2(G;\underline{H};\mathbb{R}) \xrightarrow{\mathrm{D}} H^2(\mathfrak{g};\underline{\mathfrak{h}};\mathbb{R}) \longrightarrow \{0\} \tag{2.8}$$

where $b(\gamma)$ *is the cohomology class of the central extension defined by*

$$(\mathcal{UC}(G)\times\mathbb{R})/\left\{(\alpha,\gamma(\alpha)) \mid \alpha\in\pi_1(G;\underline{H})\right\}. \tag{2.9}$$

Proof. Note that the central extension of any subroup H_j, which is the quotient of $\mathcal{UC}(H_j)\times\mathbb{R}$ by $\{(\alpha,\gamma(\alpha)) \mid \alpha\in\pi_1(G;\underline{H})\}$ is the trivial central extension $H_j\times\mathbb{R}$, and thus, $b(\gamma)$ is indeed a relative cohomology class. Since the Lie algebra cocycle of a central extension of G only depends on a simply connected neighbourhood of the identity, it only depends on the restriction of $b(\gamma)$ to such a neighborhood, where it is trivial. Thus, we have $\mathrm{D}(b(\gamma)) = 0$. For injectivity of b, observe that $b(\gamma) = 0$ only if the factor $\mathbb{R}$ is unaffected by the relation in (2.9), that is, $\gamma = 0$.

Now we integrate a Lie algebra coboundary to the universal cover of G with the goal of constructing the corresponding function $\gamma\in\operatorname{Hom}(\pi_1(G;\underline{H}),\mathbb{R})$. Let Ω represent any group-level cocycle in $H^2(G;\underline{H};\mathbb{R})$ such that the Lie algebra cocycle $\mathrm{D}\,\Omega$ is a coboundary, that is, $\mathrm{D}\,\Omega = \delta F$ for some $F\in\mathfrak{g}^\vee$ vanishing on $\mathfrak{h}_1^\vee,\dots,\mathfrak{h}_N^\vee$. Concretely, given $v,w\in\mathfrak{g}$ we have $(\mathrm{D}\,\Omega)(v,w) = F([v,w])$. Consider the Lie algebra central extension $\mathfrak{g}\times_{\mathrm{D}\,\Omega}\mathbb{R}$ as an exact sequence split by F,

$$\{0\} \longrightarrow \mathbb{R} \underset{p}{\leftrightarrows} \mathfrak{g}\times_{\mathrm{D}\,\Omega}\mathbb{R} \underset{v\mapsto(v,F(v))}{\leftrightarrows} \mathfrak{g} \longrightarrow \{0\} \tag{2.10}$$

Where p is the $\mathbb{R}$-linear map

$$\begin{aligned} p : \mathfrak{g}\times_{\mathrm{D}\,\Omega}\mathbb{R} &\to \mathbb{R} \\ (v,a) &\mapsto a - F(v) \end{aligned} \tag{2.11}$$

with kernel $\ker p = \{(v,a) \mid F(v) = a\}$ being the graph of F.

Since Lie algebra 1-cochains are just linear functionals on the Lie algebra, we can consider $p\in(\mathfrak{g}\times_{\mathrm{D}\,\Omega}\mathbb{R})^\vee$ as such. Taking into account that F and thus also p vanish on the subalgebras $\mathfrak{h}_j\times\{0\}$, it actually is a relative 1-chain $p\in C^1(\mathfrak{g}\times_{\mathrm{D}\,\Omega}\mathbb{R};\underline{\mathfrak{h}\times\{0\}};\mathbb{R})$ on the Lie algebra central extension. Hence, we can apply the Lie algebra cohomology differential to p, and find

$$(\delta\,p)((v,a),(w,b)) = p([(v,a),(w,b)]) = p(([v,w],(\mathrm{D}\,\Omega)(v,w))) = 0. \tag{2.12}$$

Thus, the map is actually a 1-cocycle $p\in Z^1(\mathfrak{g}\times_{\mathrm{D}\,\Omega}\mathbb{R};\underline{\mathfrak{h}\times\{0\}};\mathbb{R})$.

The universal cover of the central extension may be identified as

$$\mathcal{UC}(G\times_\Omega\mathbb{R}) = \mathcal{UC}(G)\times_{\hat\Omega}\mathbb{R}, \tag{2.13}$$

where the cocycle lifed cocycle $\hat\Omega$ is defined by pullback, that is, $\hat\Omega(\hat\phi,\hat\psi) = \Omega(\phi,\psi)$ if $\hat\phi$ and $\hat\psi$ are lifts of $\phi,\psi\in G$ respectively. We proceed to define the invariant differential 1-form α_p of p on this universal cover of the central extension as in Equation (1.68). Since the Lie algebra cohomology differential $\mathrm{d}\,p$ and the exterior derivative $\mathrm{d}\,\alpha_p$ are defined by the same formula (1.16), we have

$$\mathrm{d}\,\alpha_p = \alpha_{\delta\,p}. \tag{2.14}$$

Since $\delta\,p = 0$, we conclude that the form is closed, that is, $\mathrm{d}\,\alpha_p = \alpha_{\delta\,p} = 0$. Since $\mathcal{UC}(G)\times_{\hat\Omega}\mathbb{R}$ is simply connected, and the invariant differential 1-form α_p extends to the universal cover of the manifold $C^\omega(S^1,\mathbb{C}\setminus\{0\})$ (as explained at the end of Section 1.4), we can apply the Poincaré lemma [KM97, Lemma 33.20] to find a unique function $\varphi\in\mathcal{F}(\mathcal{UC}(G)\times_{\hat\Omega}\mathbb{R})$ such that

$$\mathrm{d}\,\varphi = \alpha_p, \qquad \varphi(\mathbb{1}) = 0. \tag{2.15}$$

Since the invariant differential 1-form α_p restricts to the trivial 1-form on the subgroups $\mathcal{UC}(H_j) \times \{0\}$, the function φ is trivial on these subgroups as well. We claim that the function φ is a 1-cocycle on $\mathcal{UC}(G) \times_{\hat{\Omega}} \mathbb{R}$ relative to the subgroups, that is, a homomorphism

$$\varphi\big((\hat{\phi}, \lambda) \star (\hat{\psi}, \mu)\big) = \varphi\big((\hat{\phi}, \lambda)\big) + \varphi\big((\hat{\psi}, \mu)\big) = \varphi\big((\hat{\phi} \circ \hat{\psi}, \lambda + \mu + \Omega(\phi, \psi)\big), \tag{2.16}$$

where $(\hat{\phi}, \lambda), (\hat{\psi}, \mu) \in \mathcal{UC}(G) \times_{\hat{\Omega}} \mathbb{R}$ and $\cdot \star \cdot$ denotes the multiplication in the central extension of the universal cover. The method to prove this claim is to define a function

$$\begin{aligned} Q : \mathcal{UC}(G) \times_{\hat{\Omega}} \mathbb{R} &\longrightarrow \mathbb{R} \\ (\hat{\psi}, \mu) &\longmapsto \varphi\big((\hat{\phi}, \lambda) \star (\hat{\psi}, \mu)\big) - \varphi\big((\hat{\psi}, \mu)\big) - \varphi\big((\hat{\phi}, \lambda)\big). \end{aligned} \tag{2.17}$$

for fixed $(\hat{\phi}, \lambda) \in \mathcal{UC}(G) \times_{\hat{\Omega}} \mathbb{R}$. By invariance of α_p we find the the differential of Q vanishes,

$$\mathrm{d}\, Q = \mathrm{d}\left(\varphi\big((\hat{\phi}, \lambda) \star \,\cdot\,\big) - \varphi(\,\cdot\,) - \varphi\big((\hat{\phi}, \lambda)\big)\right) = \big((\hat{\phi}, \lambda) \star \,\cdot\,\big)^* \,\mathrm{d}\,\varphi - \mathrm{d}\,\varphi = 0. \tag{2.18}$$

Hence, Q is constant, and moreover, the normalization $\varphi(\mathbb{1}) = 0$ makes the constant zero.

Note that $\lambda \mapsto (\mathbb{1}, \lambda) \in \mathcal{UC}(G) \times_{\hat{\Omega}} \mathbb{R}$ for $\lambda \in \mathbb{R}$ is a homomorphism, that is, a linear map, and consider the derivative

$$\partial_\lambda|_{\lambda=0} \varphi\big((\mathbb{1}, \lambda)\big) = \mathrm{d}_{(\mathbb{1},0)}\, \varphi(0, 1) = (\alpha_p)_{(\mathbb{1},0)}(0, 1) = p(0, 1) = 1 - F(0) = 1. \tag{2.19}$$

Thus, we have $\varphi\big((\mathbb{1}, \lambda)\big) = \lambda$, which splits the central extension of the universal cover,

$$\{0\} \longrightarrow \mathbb{R} \underset{\varphi}{\rightleftarrows} \mathcal{UC}(G) \times_{\hat{\Omega}} \mathbb{R} \underset{\sigma}{\rightleftarrows} \mathcal{UC}(G) \longrightarrow \{0\} \tag{2.20}$$

where σ is defined by

$$\begin{aligned} \sigma : \mathcal{UC}(G) &\to \mathcal{UC}(G) \times_{\hat{\Omega}} \mathbb{R} \\ \hat{\phi} &\mapsto \Big(\mathbb{1}, \varphi\big((\hat{\phi}, 0)^{-1}\big)\Big) \star (\hat{\phi}, 0), \end{aligned} \tag{2.21}$$

using the fact that every element in G has an inverse. The map σ is evidently a section of the projection onto $\mathcal{UC}(G)$, and we claim that it is a homomorphism. Since φ is a homomorphism, we find

$$\varphi((\mathbb{1}, \lambda) \star (\hat{\phi}, \mu)) = \lambda + \varphi((\hat{\phi}, \mu)). \tag{2.22}$$

In the special case of $\mu = 0$ and $\lambda = \varphi\big((\hat{\phi}, 0)^{-1}\big)$, we find that $\varphi(\sigma(\phi)) = \varphi((\sigma(\phi))^{-1}) + \varphi(\sigma(\phi)) = 0$, where we also used that φ is a homomorphism, see (2.16). Now, we can confirm that for composable $\hat{\phi}, \hat{\psi} \in \mathcal{UC}(G)$,

$$\begin{aligned} \sigma(\hat{\phi}\hat{\psi}) &= \big(\mathbb{1}, \varphi((\hat{\phi}\hat{\psi}, 0)^{-1})\big) \star \big(\hat{\phi}\hat{\psi}, 0\big) \\ &= \Big(\mathbb{1}, \varphi\big((\mathbb{1}, \Omega(\phi, \psi)) \star (\hat{\psi}, 0)^{-1}\big) \star (\hat{\phi}, 0)^{-1}\Big) \star \Big((\hat{\phi}, 0) \star (\hat{\psi}, 0) \star (\mathbb{1}, -\Omega(\phi, \psi))\Big) \\ &= \big(\mathbb{1}, \varphi((\hat{\psi}, 0)^{-1})\big) \star \big(\mathbb{1}, \varphi((\hat{\phi}, 0)^{-1})\big) \star (\hat{\phi}, 0) \star (\hat{\psi}, 0) \\ &= \big(\mathbb{1}, \varphi((\hat{\phi}, 0)^{-1})\big) \star (\hat{\phi}, 0) \star \big(\mathbb{1}, \varphi((\hat{\psi}, 0)^{-1})\big) \star (\hat{\psi}, 0) \\ &= \sigma(\hat{\phi}) \star \sigma(\hat{\psi}), \end{aligned} \tag{2.23}$$

where Equation (2.22) is applied in the third equality to take the cocycle out of φ and cancel it with the cocycle in the last term. We also use that the elements in the central extension of the universal cover of the form $(\mathbb{1}, \lambda)$ are central.

The split central extension (2.20) is isomorphic to the trival central extension $\mathcal{UC}(G) \times \mathbb{R}$ with the isomorphism given by $\Psi((\hat{\phi}, \lambda)) = \big(\hat{\phi}, \lambda - \varphi((\hat{\phi}, 0))\big)$, yielding the following morphisms

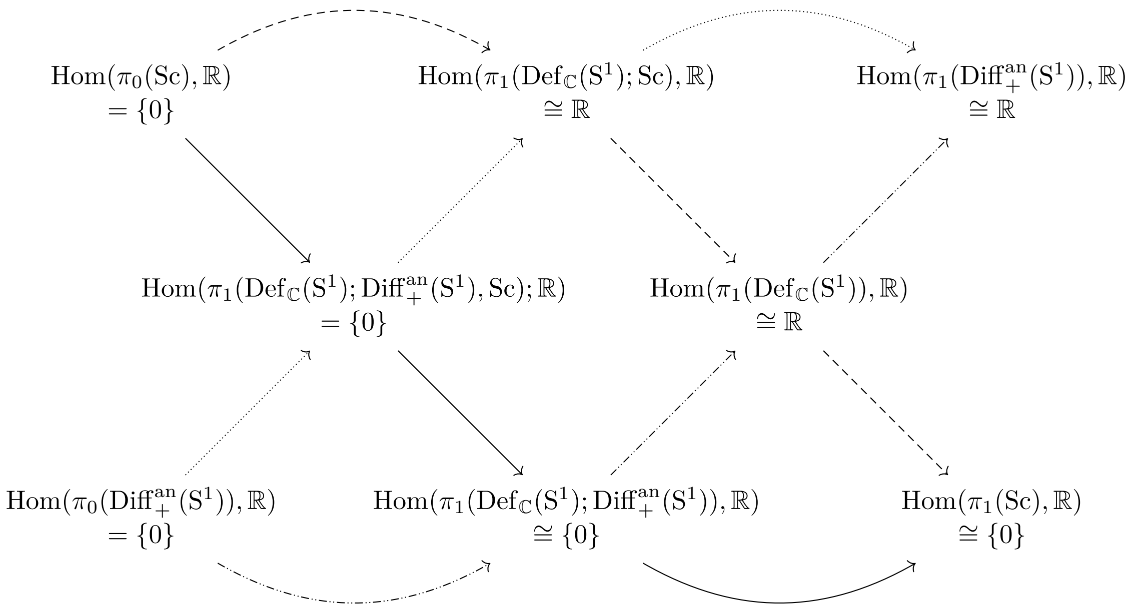


Figure 1: Characters of the relative homotopy groups of complex deformations and the subgroups of diffeomorphisms and scaling transformations.

of exact sequences

$$\begin{array}{ccccccccc}
\{0\} & \longrightarrow & \mathbb{R} & \longrightarrow & G \times_\Omega \mathbb{R} & \longrightarrow & G & \longrightarrow & \{0\} \\
 & & \| & & {\scriptstyle P}\uparrow & & {\scriptstyle \pi}\uparrow & & \\
\{0\} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathcal{UC}(G) \times_{\hat{\Omega}} \mathbb{R} & \longrightarrow & \mathcal{UC}(G) & \longrightarrow & \{0\} \\
 & & \| & & {\scriptstyle \Psi}\uparrow & & \| & & \\
\{0\} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathcal{UC}(G) \times \mathbb{R} & \longrightarrow & \mathcal{UC}(G) & \longrightarrow & \{0\}
\end{array} \tag{2.24}$$

The kernel of the composition of Ψ with the projection $P : \mathcal{UC}(G) \times_{\hat{\Omega}} \mathbb{R} \to G \times_\Omega \mathbb{R}$ is the set of pairs

$$\ker(P \circ \Psi) = \left\{ (\hat{\phi}, \lambda) \,\middle|\, \pi(\hat{\phi}) = \mathbb{1}, \lambda = \varphi((\hat{\phi}, 0)) \right\} \tag{2.25}$$

Note that this is the graph of the homomorphism φ restricted to $\ker \pi \times \{0\}$, which may be identified withe the fundamental group $\pi_1(G) \subset \mathcal{UC}(G)$ as a discrete subgroup of $\mathcal{UC}(G) \times \mathbb{R}$, showing that the central extension $G \times_\Omega \mathbb{R}$ is indeed a quotient of the form (2.9). □

2.2 Cocycles on complex deformations

In this section, we introduce the various cocycles on $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ and subgroups that appear in the cohomologies in Figures 2 and 3, which will be computed in the next section. All the cocycles are complex-valued at first, and then we take their real and imaginary parts.

On the group level, we generalize the Bott–Thurston cocycle, usually defined on $\mathrm{Diff}_+^{\mathrm{an}}(\mathrm{S}^1)$, to complex deformations. This results in the following complex-valued cocycle,

$$\Omega_{\mathrm{BT}}(\phi_1, \phi_2) = \frac{1}{24\pi} \int_{S^1} \log((\phi_1 \circ \phi_2)') \, \mathrm{d} \log(\phi_2'(z)). \tag{2.26}$$

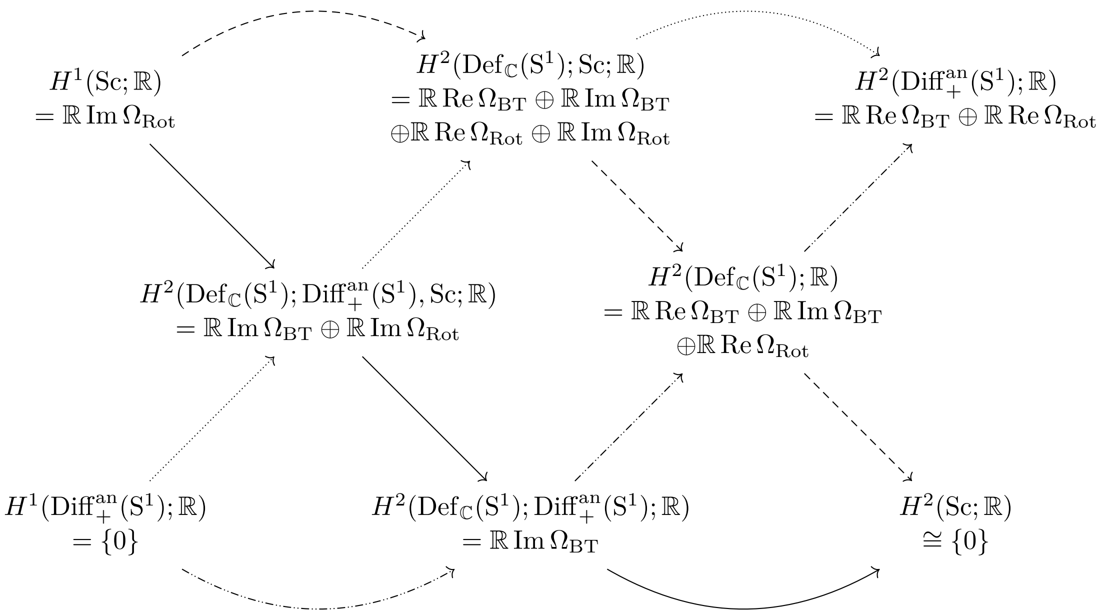


Figure 2: Relative Lie group cohomology of complex deformations and the subgroups of diffeomorphisms and scaling transformations.

The associated Lie algebra cocycle on $v, w \in \mathrm{Vect}_{\mathbb{C}}^{\mathrm{an}}(\mathrm{S}^1)$ is defined as

$$\mathrm{D}\,\Omega_{\mathrm{BT}}(v,w) = \frac{1}{2}\,\frac{\partial}{\partial s}\bigg|_{s=0}\frac{\partial}{\partial t}\bigg|_{t=0}\Bigg(\Omega_{\mathrm{BT}}(\Phi_v(t,\,\cdot\,),\Phi_w(s,\,\cdot\,)) - \Omega_{\mathrm{BT}}(\Phi_w(s,\,\cdot\,),\Phi_v(t,\,\cdot\,))\Bigg). \tag{2.27}$$

Note the prefactor of $\frac{1}{2}$ as in [MP25], and as opposed to [Khe09]. Computing one of the terms, we find

$$\begin{aligned}
&\frac{\partial}{\partial s}\bigg|_{s=0}\frac{\partial}{\partial t}\bigg|_{t=0}\Omega_{\mathrm{BT}}(\Phi_v(t,\,\cdot\,),\Phi_w(s,\,\cdot\,))\\
&= \frac{\partial}{\partial s}\bigg|_{s=0}\frac{\partial}{\partial t}\bigg|_{t=0}\frac{1}{24\pi}\int_{S^1}\Big(\log\big(\Phi_v'(t,\Phi_w(s,w))\big)+\log\big(\Phi_w'(s,z)\big)\Big)\frac{\Phi_w''(s,z)}{\Phi_w'(s,z)}\,\mathrm{d}z.\\
&= \frac{\partial}{\partial s}\bigg|_{s=0}\frac{1}{24\pi}\int_{S^1}v'(\Phi_w(s,z))\frac{\Phi_w''(s,z)}{\Phi_w'(s,z)}\,\mathrm{d}z.\\
&= \frac{1}{24\pi}\int_{S^1}v'(z)w''(z)\,\mathrm{d}z.
\end{aligned} \tag{2.28}$$

Note that this expression is already antisymmetric. Therefore, also expressed in the basis (1.18), the Lie algebra cocycle is

$$\mathrm{D}\,\Omega_{\mathrm{BT}}(v,w) = \frac{1}{24\pi}\int_{S^1}v'(z)w''(z)\,\mathrm{d}z, \qquad \mathrm{D}\,\Omega_{\mathrm{BT}}(\ell_n,\ell_m) = \frac{\mathrm{i}}{12}(n^3-n)\delta_{n+m}, \qquad n,m\in\mathbb{Z} \tag{2.29}$$

Up to a coboundary given by the term in Equation (2.29) which is linear in n, this agrees with the Gel'fand–Fuks cocycle. We express the later in the coordinate $\theta \mapsto e^{i\theta} \in S^1$, $\theta \in [0,2\pi)$, where $v = v(\theta)\partial_\theta$ and $w = w(\theta)\partial_\theta$. Since $\partial_\theta = \mathrm{i}\,e^{\mathrm{i}\,\theta}\partial_z$, we find for v and ℓ_n in the basis (1.18)

$$\begin{aligned}
v(\theta) &= \mathrm{i}\,e^{\mathrm{i}\,n\theta}, & v'(\theta) &= -ne^{\mathrm{i}\,n\theta}, & v''(\theta) &= -\mathrm{i}\,n^2e^{\mathrm{i}\,n\theta}, & v &\in \mathrm{Vect}_{\mathbb{C}}^{\mathrm{an}}(\mathrm{S}^1),\\
\ell_n(\theta) &= \mathrm{i}\,e^{\mathrm{i}\,n\theta}, & \ell_n'(\theta) &= -ne^{\mathrm{i}\,n\theta}, & \ell_n''(\theta) &= -\mathrm{i}\,n^2e^{\mathrm{i}\,n\theta}, & n &\in \mathbb{Z}.
\end{aligned} \tag{2.30}$$

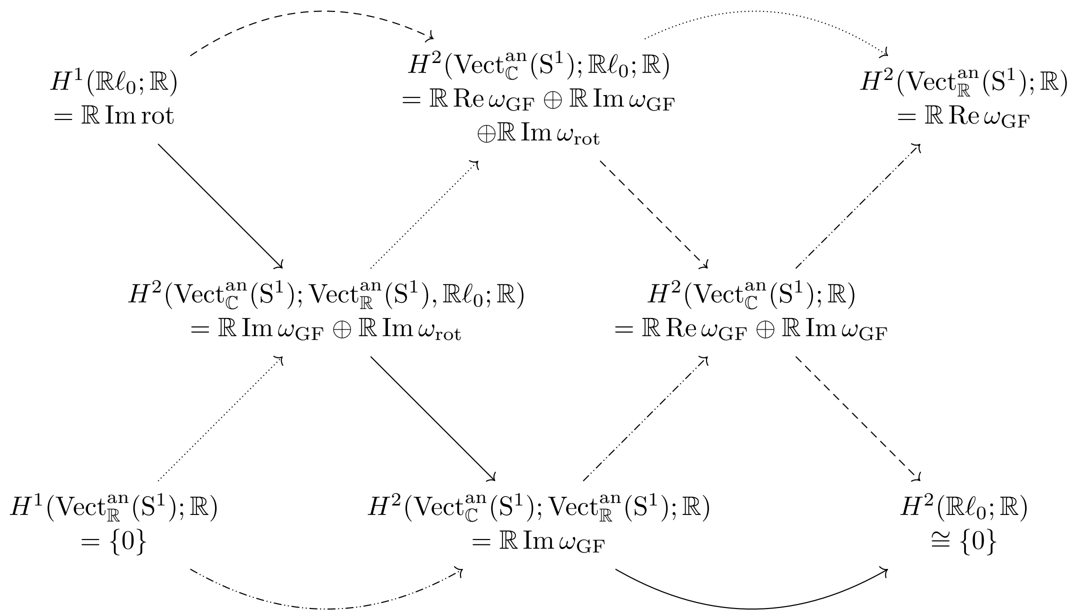


Figure 3: Relative Lie algebra cohomology of complex deformations and the subgroups of diffeomorphisms and scaling transformations.

In these coordinates, the Gel'fand–Fuks cocycle is defined by

$$\omega_{\mathrm{GF}}(v,w) = \frac{1}{24\pi}\int_0^{2\pi} v'(\theta)w''(\theta)\,\mathrm{d}\theta, \qquad \omega_{\mathrm{GF}}(\ell_n,\ell_m) = \frac{\mathrm{i}}{12}n^3\delta_{n+m}, \qquad n,m\in\mathbb{Z}. \tag{2.31}$$

These cohomologous cocycles $\mathrm{D}\,\Omega_{\mathrm{BT}}$ and ω_{GF} correspond to the unique central extension of $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ called the Virasoro algebra.

The second complex-valued cocycle we consider is precisely the Lie algebra coboundary mentioned above. On the group-level, we define it as the differential of a complex-valued function RCR on $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ utilizing the decomposition of a complex deformation in Proposition 1.12 into a diffeomorphism and an univalent function. The argument of RCR is the rotation number of the diffeomorphism, whereas the absolute value of RCR is the conformal radius of the univalent function. We give the usual definitions and explain how they apply to complex deformations.

The rotation number of a diffeomorphism is the average position of a point $z\in S^1$ under repeated application of the diffeomorphism,

$$e^{\mathrm{i}\,\mathrm{Rot}(\phi)} = \lim_{n\to\infty}\left(\phi^{\circ n}(z)\right)^{\frac{1}{n}}, \qquad \phi\in\mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1), \tag{2.32}$$

where the power $\circ n$ stands for n-fold composition[3]. It is well-known to be independent of the starting point z. In particular, a rotation $R_\alpha(z) = e^{\mathrm{i}\,\alpha}z$ has rotation number $\mathrm{Rot}(R_\alpha)=\alpha$, and if $\phi\in\mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$ has a fixed point, then $\mathrm{Rot}(\phi)=0$. To extend the rotation number to complex deformations, we define

$$\mathrm{Rot}(\phi) = \mathrm{Rot}(D_\phi), \qquad \phi\in\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1) \tag{2.33}$$

in terms of the decomposition in Proposition 1.12.

[3]By considering the exponential, we avoid taking a lift of the diffeomorphism of S^1 to a diffeomorphism of the universal cover $\mathbb{R} = \mathcal{UC}(S^1)$. Moreover, we use the coordinate z in the complex plane instead of θ as defined in Section 1.2 in this definition.

We define the conformal radius of a complex deformation $\phi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ as that of the univalent function F_ϕ in the decomposition in Proposition 1.12,

$$\mathrm{CR}(\phi) = \frac{1}{F'_\phi(0)}. \tag{2.34}$$

Note that the normalization of F_ϕ makes the conformal radius a positive real number. From the decomposition we see that if $\psi \in \mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$, then $\mathrm{CR}(\phi \circ \psi) = \mathrm{CR}(\phi)$. A scaling transformation $s_\tau \in \mathrm{Sc}$, see Equation (1.45), has conformal radius $\mathrm{CR}(s_\tau) = e^{2\pi\tau}$.

Finally, the function $\mathrm{RCR} : \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1) \to \mathbb{C}$ is defined by

$$\mathrm{RCR}(\phi) = \frac{e^{\mathrm{i}\,\mathrm{Rot}(D_\phi)}}{\mathrm{CR}(F_\phi)}, \qquad \phi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)\,. \tag{2.35}$$

Note that the definition of RCR was chosen such that for a scaling transformation and rotation we have $\mathrm{RCR}(s_\tau \circ R_\alpha) = e^{-2\pi\tau + \mathrm{i}\,\alpha}$, which is the image of 1 under this complex deformation. A 2-cocycle on $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ is defined by taking the group-level differential of the function RCR

$$\Omega_{\mathrm{Rot}}(\phi_1, \phi_2) = \frac{\mathrm{i}}{24}\,\delta\,\mathrm{RCR}(\phi_1, \phi_2) = \frac{\mathrm{i}}{24}\Big(\mathrm{RCR}(\phi_1) + \mathrm{RCR}(\phi_2) - \mathrm{RCR}(\phi_1 \circ \phi_2)\Big). \tag{2.36}$$

To compute the corresponding Lie algebra cocycle, we need the derivative of RCR.

Proposition 2.2. *The derivative of* RCR *at the identity is* $\mathbb{C}$*-linear and*

$$\mathrm{d}\,\mathrm{RCR}(\ell_n) = -\delta_{n,0}, \qquad n \in \mathbb{Z}. \tag{2.37}$$

Proof. The flow of ℓ_n for $n \geq 1$ permits a formal solution

$$\Phi_{\ell_n}(t, z) = (z^{-n} + nt)^{-\frac{1}{n}}, \tag{2.38}$$

for which branch choices can be made such that it becomes conformal in a neighborhood of $\bar{\mathbb{D}}$ with derivative $\partial_z\big|_{z=0}\Phi_{\ell_n}(t,z) = 1$, and since $\ell_n(0) = 0$ also $\Phi_{\ell_n}(t, 0) = 0$. Thus, we we have $F_{\Phi_{\ell_n}(t,\,\cdot\,)} = \Phi_{\ell_n}(t,\,\cdot\,)$, and hence, $D_{\Phi_{\ell_n}(t,\,\cdot\,)} = \mathbb{1}$ making the rotation number vanish. Moreover, the conformal radius is $\mathrm{CR}(\Phi_{\ell_n}(t,\,\cdot\,)) = 1$. We conclude that $\mathrm{d}_{\mathbb{1}}\,\mathrm{RCR}(\ell_n) = 0$ for $n \geq 1$. Because the derivative $\mathrm{d}\,\mathrm{RCR}$ in our setup is only $\mathbb{R}$-linear, we have to consider $\mathrm{i}\,\ell_n$ separately. In this case we have $\Phi_{\mathrm{i}\,\ell_n}(t,z) = (z^{-n} + \mathrm{i}\,nt)^{-\frac{1}{n}}$, also with fixed point 0, and derivative $\partial_z\big|_{z=0}\Phi_{\mathrm{i}\,\ell_n}(t,z) = 1$.

The tangent vector fields $a^{\|}_n$ and $b^{\|}_n$ for $n \geq 1$ mix the positive and negative modes for ℓ_n and $\mathrm{i}\,\ell_n$ respectively. Both have zeroes on S^1, and therefore their flows — which are diffeomorphisms — have fixed points, making the rotation number vanish. Hence, we find $\mathrm{d}_{\mathbb{1}}\,\mathrm{RCR}(a^{\|}_n) = 0$ and $\mathrm{d}_{\mathbb{1}}\,\mathrm{RCR}(b^{\|}_n) = 0$. We conclude that $\mathrm{d}_{\mathbb{1}}\,\mathrm{RCR}(\ell_n) = 0$ and $\mathrm{d}_{\mathbb{1}}\,\mathrm{RCR}(\mathrm{i}\,\ell_n) = 0$ for all $n \neq 0$.

The vector field $\ell_0 = -z\partial_z$ generates a scaling transformation,

$$\frac{\partial}{\partial t}\bigg|_{t=0} s_{\frac{t}{2\pi}} = \frac{\partial}{\partial t}\bigg|_{t=0} e^{-t}z = \ell_0(z), \tag{2.39}$$

and thus

$$\mathrm{d}\,\mathrm{RCR}(\ell_0) = \frac{\partial}{\partial t}\bigg|_{t=0} e^{-t} = -1 \tag{2.40}$$

The rotated vector field $\mathrm{i}\,\ell_0 = -\mathrm{i}\,z\partial_z$ generates a rotation

$$\frac{\partial}{\partial t}\bigg|_{t=0} R_{-t} = \frac{\partial}{\partial t}\bigg|_{t=0} e^{-\mathrm{i}\,t}z = \mathrm{i}\,\ell_0(z), \tag{2.41}$$

leading to the variation

$$\mathrm{d}\,\mathrm{RCR}(\mathrm{i}\,\ell_0) = \frac{\partial}{\partial t}\bigg|_{t=0} e^{-\mathrm{i}\,t} = -\mathrm{i}\,. \tag{2.42}$$

□

Indeed, we find the Lie algebra cocycle with the linear dependence in n,

$$\mathrm{D}\,\Omega_{\mathrm{Rot}}(\ell_n,\ell_m) = \frac{\mathrm{i}}{24}(n-m)\delta_{n+m,0} = \frac{\mathrm{i}}{12}n\delta_{n+m}, \qquad n,m\in\mathbb{Z} \tag{2.43}$$

It agrees with the Lie algebra cohomology differential of the linear function of $v \in \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$,

$$\mathrm{rot}(v) = \frac{1}{48\pi}\int_0^{2\pi} v(\theta)\,\mathrm{d}\theta, \qquad \mathrm{rot}(\ell_n) = \frac{\mathrm{i}}{24}\delta_{n,0}. \tag{2.44}$$

defining the Lie algebra cocycle $\omega_{\mathrm{rot}} = \mathrm{D}\,\Omega_{\mathrm{Rot}} = \delta\,\mathrm{rot}$,

$$\omega_{\mathrm{rot}}(v,w) = \frac{1}{24\pi}\int_0^{2\pi} v(\theta)w'(\theta)\,\mathrm{d}\theta, \qquad \omega_{\mathrm{rot}}(\ell_n,\ell_m) = \frac{\mathrm{i}}{12}n\delta_{n+m,0}. \tag{2.45}$$

2.3 Computation of the cohomology

Theorem 2.3. *The group- and algebra-level cohomologies of* $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ *relative to* $\mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$ *and* Sc *are respectively as in Figure 2 and Figure 3.*

Proof. The Lie algebra cohomologies, Figure 3, follow from a variation on a classical computation of the second cohomology of the Witt algebra. The computation is well-known for complex-valued vector fields $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ with complex coefficients or for real-valued vector fields $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{R}}(\mathrm{S}^1)$ with real coefficients. The latter gives $H^2(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{R}}(\mathrm{S}^1);\mathbb{R})$ in the top right corner of the diagram. However, here we also need the second cohomology of $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ with coefficients in $\mathbb{R}$, which we compute as follows. Let $\alpha \in Z^2(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1),\mathbb{R})$ be any cocycle. Consider the coboundary $\mathrm{d}f$ for the function $f \in (\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1))^\vee$ defined by

$$\begin{aligned} f(\ell_n) &= \frac{\alpha(\ell_n,\ell_0)}{n}, \\ f(\mathrm{i}\,\ell_n) &= \frac{\alpha(\mathrm{i}\,\ell_n,\ell_0)}{n}, \end{aligned} \qquad n \neq 0, \qquad f(\ell_0) = f(\mathrm{i}\,\ell_0) = 0, \tag{2.46}$$

in the $\mathbb{R}$-basis $\{\ell_n, \mathrm{i}\,\ell_n \mid n \in \mathbb{Z}\}$ of $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$. We expand the cocycle property of α in this basis for $n,m,k \in \mathbb{Z}$,

$$\begin{aligned} (n-m)\,\alpha(\ell_{n+m},\ell_k) + (m-k)\,\alpha(\ell_{m+k},\ell_n) + (k-n)\,\alpha(\ell_{k+n},\ell_m) &= 0 \\ (n-m)\,\alpha(\mathrm{i}\,\ell_{n+m},\ell_k) + (m-k)\,\alpha(\ell_{m+k},\mathrm{i}\,\ell_n) + (k-n)\,\alpha(\mathrm{i}\,\ell_{k+n},\ell_m) &= 0 \\ -(n-m)\,\alpha(\ell_{n+m},\ell_k) + (m-k)\,\alpha(\mathrm{i}\,\ell_{m+k},\mathrm{i}\,\ell_n) + (k-n)\,\alpha(\mathrm{i}\,\ell_{k+n},\mathrm{i}\,\ell_m) &= 0 \\ -(n-m)\,\alpha(\ell_{n+m},\mathrm{i}\,\ell_k) - (m-k)\,\alpha(\ell_{m+k},\mathrm{i}\,\ell_n) - (k-n)\,\alpha(\ell_{k+n},\mathrm{i}\,\ell_m) &= 0 \end{aligned} \tag{2.47}$$

By considering $k=0$, we find that the cohomologous cocycle $\beta = \alpha - \mathrm{d}f$ satisfies for $n+m \neq 0$,

$$\begin{aligned} \beta(\ell_n,\ell_m) &= \alpha(\ell_n,\ell_m) - \frac{n-m}{n+m}\,\alpha(\ell_{n+m},\ell_0) = 0, \\ \beta(\mathrm{i}\,\ell_n,\mathrm{i}\,\ell_m) &= \alpha(\mathrm{i}\,\ell_n,\mathrm{i}\,\ell_m) + \frac{n-m}{n+m}\,\alpha(\ell_{n+m},\ell_0) = 0, \\ \beta(\mathrm{i}\,\ell_n,l_m) &= \alpha(\mathrm{i}\,\ell_n,\ell_m) - \frac{n-m}{n+m}\,\alpha(\mathrm{i}\,\ell_{n+m},\ell_0) = 0. \end{aligned} \tag{2.48}$$

Thus β is supported on the anti-diagonal $n+m=0$ for $n,m \in \mathbb{Z}$, where the values are given by constants $C_n, D_n, E_n \in \mathbb{R}$, $n \in \mathbb{Z}$,

$$\begin{aligned} \beta(\ell_n,\ell_m) &= C_n\delta_{n+m}, & C_{-n} &= -C_n, \\ \beta(\mathrm{i}\,\ell_n,\mathrm{i}\,\ell_m) &= D_n\delta_{n+m}, & D_{-n} &= -D_n, \\ \beta(\mathrm{i}\,\ell_n,\ell_m) &= E_n\delta_{n+m}, & E_{-n} &= -E_n, \end{aligned} \tag{2.49}$$

already determining $C_0 = D_0 = E_0 = 0$. By considering the cocycle property (2.47) for β instead of α, we find for $n + m + k = 0$,

$$\begin{aligned} (n-m)C_{n+m} + (m-k)C_{m+k} + (k-n)C_{k+n} &= 0,\\ (n-m)E_{n+m} + (m-k)E_{m+k} + (k-n)E_{k+n} &= 0,\\ -(n-m)C_{n+m} + (m-k)D_{m+k} + (k-n)D_{k+n} &= 0. \end{aligned} \tag{2.50}$$

Note how the third relation in Equation (2.50) mixes the constants C_n and D_n. By setting $k = n$ and $m = -2n$ we find that $C_n = -D_n$ for all $n \in \mathbb{Z}$. Moreover, by setting $k = 1$, $m = -(n+1)$ for $n \geq 2$, we that the C_n and E_n are solutions of equal and independent recursion relations

$$\begin{aligned} C_{n+1} &= \frac{1}{n-1}\Big((n+2)\, C_n - (2n+1)\, C_1\Big),\\ E_{n+1} &= \frac{1}{n-1}\Big((n+2)\, E_n - (2n+1)\, E_1\Big). \end{aligned} \tag{2.51}$$

The solutions are determined by $(C_1, C_2) \in \mathbb{R}^2$ and $(E_1, E_2) \in \mathbb{R}^2$ respectively. We may check the initial conditions $(1, 2)$ and $(1, 8)$, correspond to the linearly independent solutions n and n^3 respectively. Since the recursion equations are linear, all solutions are linear combinations of these two. The solutions $C_n = n$, $E_n = 0$ and $C_n = 0$, $E_n = n$ are respectively propotional to $\operatorname{Im}\omega_{\mathrm{rot}}$ and $\operatorname{Re}\omega_{\mathrm{rot}}$ in equation (2.45). The other two solutions $C_n = n^3$, $E_n = 0$ and $C_n = 0$, $E_n = n^3$ are proportional to $\operatorname{Im}\omega_{\mathrm{GF}}$ and $\operatorname{Re}\omega_{\mathrm{GF}}$, the real and imaginary parts of the Gel'fand–Fuks cocycle as in Equation (2.31). Note that the couboundary δf with f as in Equation (2.46) above vanishes on the subalgebra $\mathbb{R}\ell_0$ and thus it is also a coboundary on the cohomology relative to $\mathbb{R}\ell_0$.

Since $\operatorname{Im}\omega_{\mathrm{rot}} = \operatorname{Im}\delta\,\mathrm{rot}$ and $\operatorname{Re}\omega_{\mathrm{rot}} = \operatorname{Re}\delta\,\mathrm{rot}$ are coboundaries in $Z^2(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1), \mathbb{R})$, the Lie algebra cohomology of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ is spanned by $\operatorname{Im}\omega_{\mathrm{GF}}$ and $\operatorname{Re}\omega_{\mathrm{GF}}$. Relative to $\mathbb{R}\ell_0$ however, $\operatorname{Im}\omega_{\mathrm{rot}}$ is not a coboundary anymore since $\operatorname{Im}\mathrm{rot}(\ell_0) \neq 0$. Thus the cohomology $H^2(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1); \mathbb{R}\ell_0; \mathbb{R})$ has it as an additional generator. The cocycle $\mathbb{R}\omega_{\mathrm{GF}}$ does not vanish on $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{R}}(\mathrm{S}^1)$, thus $H^2(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1); \mathrm{Vect}^{\mathrm{an}}_{\mathbb{R}}(\mathrm{S}^1); \mathbb{R})$ only containes the cocycle $\operatorname{Im}\omega_{\mathrm{GF}}$. For the cohomology relative to both $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{R}}(\mathrm{S}^1)$ and $\mathbb{R}\ell_0$, both of the arguments above hold and thus it is spanned by $\operatorname{Im}\omega_{\mathrm{GF}}$ and $\operatorname{Im}\omega_{\mathrm{rot}}$.

The first Lie algebra cohomology is given by derivations modulo inner derivations. For the simple Lie algebra $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{R}}(\mathrm{S}^1)$ — that is, $[\mathrm{Vect}^{\mathrm{an}}_{\mathbb{R}}(\mathrm{S}^1), \mathrm{Vect}^{\mathrm{an}}_{\mathbb{R}}(\mathrm{S}^1)] = \mathrm{Vect}^{\mathrm{an}}_{\mathbb{R}}(\mathrm{S}^1)$ — every derivation is trivial. On the other hand, for the abelian Lie algebra $\mathbb{R}\ell_0$, any linear function is a derivation, and no derivation is inner. We may view this one-dimensional space as the span of the function $\operatorname{Im}\mathrm{rot}$. This concludes the characterization of the Lie algebra cohomologies in Figure 3.

Turning to the group-level cohomology, Figure 2, observe that for every Lie algebra cocycle above, we have found a corresponding group-level cocycle in Section 2.2 that integrates the respective Lie algebra cocycle. Moreover, Proposition 2.1 yields a short exact sequence between the diagrams

$$\{0\} \to \text{Figure 1} \to \text{Figure 2} \to \text{Figure 3} \to \{0\}, \tag{2.52}$$

and the indicival the four braided chain complexes in each of the figures each are exact by the respective long exact sequences of relative homotopy groups, group-level cohomology, and Lie algebra cohomology.

For homotopy groups, Figure 1, the fundamental groups each are either trivial or isomorphic to $\mathbb{Z}$. In case of the latter, $\mathrm{Hom}(\mathbb{Z}, \mathbb{R}) \cong \mathbb{R}$. This is the case for $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ and $\mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$. However, the fundamental group of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ relative to $\mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$ is trivial again since $\mathrm{Diff}^{\mathrm{an}}_{+}(\mathrm{S}^1)$ contains the subgroup of rotations, curves in which represent homotopy classes of noncontractible loops in $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$.

Returning to the group-level cohomology, the additional cocycle coming from the homotopy group is $\operatorname{Re}\Omega_{\mathrm{Rot}}$, which indeed differentiates to a trival Lie algebra cocycle. Finally, exactness of the all the sequences shows that we have found all group-level cohomology classes. □

3 Real one-dimensional modular functors

In the following, we define the main notion in this work, real one-dimensional modular functors (Section 3.1), and several constructions building on this definition. Namely, a Fr-smooth real one-dimensional modular functor induces a central extension of the complex deformations introduced in Section 1.3 by the multiplicative group $\mathbb{R}_+$ (Section 3.2), and this central extension, in turn, acts on the modular functor (Section 3.3). Moreover, we list a number of identities involving cocycles of the form

$$\Omega^{Z}_{j,k} : \mathcal{M}_{\mathsf{g}_1,\mathsf{b}_1} \times \mathcal{M}_{\mathsf{g}_2,\mathsf{b}_2} \to \mathbb{R} \tag{3.1}$$

obtained from spelling out the sewing isomorphisms of the modular functor with respect to a trivialization Z (Section 3.4). Finally, we define several additional notions specializing the definition of real one-dimensional modular functor: Locality, flat modular invariance, crossing invariance, and hyperbolic modular invariance (Section 3.5). All of these need to be assumed for our main theorem — the classification of such modular functors in Theorem 5.1, Section 5.

3.1 Definition and locality

Definition 3.1. A *Fr-smooth real one-dimensional modular functor* E consists of Fr-smooth $\mathbb{R}_+$-bundles

$$E(\mathcal{M}_{\mathsf{g},\mathsf{b}}) \xrightarrow{\pi_{E,\mathsf{g},\mathsf{b}}} \mathcal{M}_{\mathsf{g},\mathsf{b}}, \qquad \mathsf{g},\mathsf{b} \geq 0, \tag{3.2}$$

and bilinear Fr-smooth maps called *sewing isomorphisms*,

$$\begin{aligned} \cdot\ {}_j\overset{E}{\infty}_k\ \cdot &: E(\mathcal{M}_{\mathsf{g}_1,\mathsf{b}_2}) \boxtimes E(\mathcal{M}_{\mathsf{g}_2,\mathsf{b}_2}) \to E(\mathcal{M}_{\mathsf{g}_1+\mathsf{g}_2,\mathsf{b}_1+\mathsf{b}_2-2}), \\ \overset{E}{\infty}_{j,k}\ \cdot &: E(\mathcal{M}_{\mathsf{g},\mathsf{b}}) \to E(\mathcal{M}_{\mathsf{g}+1,\mathsf{b}-2}), \end{aligned} \tag{3.3}$$

which are isomorphisms on fibers. They extend the respective sewing operations $\cdot\ {}_j\infty_k\ \cdot$, and $\infty_{j,k}\ \cdot$ including associativity

$$\cdot\ {}_j\overset{E}{\infty}_k\ (\ \cdot\ {}_l\overset{E}{\infty}_m\ \cdot\ ) = (\ \cdot\ {}_j\overset{E}{\infty}_k\ \cdot\ )\ {}_l\overset{E}{\infty}_m\ \cdot\ , \tag{3.4}$$

$$\overset{E}{\infty}_{j,k}\ (\ \cdot\ {}_l\overset{E}{\infty}_m\ \cdot\ ) = (\ \overset{E}{\infty}_{j,k}\ \cdot\ )\ {}_l\overset{E}{\infty}_m\ \cdot\ , \tag{3.5}$$

$$\overset{E}{\infty}_{j,k}\ (\ \overset{E}{\infty}_{l,m}\ \cdot\ ) = \overset{E}{\infty}_{l,m}\ (\ \overset{E}{\infty}_{j,k}\ \cdot\ ), \tag{3.6}$$

and symmetry

$$\begin{aligned} \alpha\ {}_j\overset{E}{\infty}_k\ \beta &= \beta\ {}_k\overset{E}{\infty}_j\ \alpha \\ \overset{E}{\infty}_{j,k}\ \alpha &= \overset{E}{\infty}_{k,j}\ \alpha \end{aligned} \qquad \alpha \in E(\mathcal{M}_{\mathsf{g}_1,\mathsf{b}_2}), \beta \in E(\mathcal{M}_{\mathsf{g}_2,\mathsf{b}_2}). \tag{3.7}$$

Generally, the sewing isomorphisms can depend on the boundary parametrizations of the respective surfaces in any way — as long as associativity holds. Following the example of the real determinant line, however, it makes sense that they should only depend on the boundary parametrizations that are involved in the sewing. In other words, if a the boundary parametrization that is external to the sewing operation is reparametrized, by a diffeomorphism in $\mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$ say, then the sewing isomorphism should be somehow invariant under this reparametrization. The way we realize this is to think of the $\mathbb{R}_+$-bundles as defined over the moduli spaces $\check{\mathcal{M}}_{\mathsf{g},\mathsf{b}}$

without boundary parametrization instead of $\mathcal{M}_{\mathsf{g},\mathsf{b}}$. Then, the sewing isomorphism is a bilinear map on these bundles which also depends on the choice of boundary parametrizations — but only at the boundaries involved in the sewing operation. We formalize this property as a new notion which we call "local" since in the case of the real determinant line bundle it follows from the locality of the conformal anomaly formula. The moduli spaces $\check{\mathcal{M}}_{\mathsf{g},\mathsf{b}}^{j_1,\ldots,j_n}$ denote moduli spaces of surfaces which have analytical boundary parametrizations only at the boundary components labeled by $j_1, \ldots, j_n$.

Definition 3.2. A Fr-smooth real one-dimensional modular functor E is *local* if the bundles $E(\mathcal{M}_{\mathsf{g},\mathsf{b}})$ are pullbacks of bundles $E(\check{\mathcal{M}}_{\mathsf{g},\mathsf{b}})$, and the sewing isomorphisms descend to the bundles $E(\check{\mathcal{M}}_{\mathsf{g},\mathsf{b}}^{j})$ in the sense that the maps also denoted $\cdot\, {}_j\infty_k^E \,\cdot$, and defined by the following diagrams, are independent of the choice of lift,

$$\begin{array}{ccc} E(\mathcal{M}_{\mathsf{g}_1,\mathsf{b}_2}) \boxtimes E(\mathcal{M}_{\mathsf{g}_2,\mathsf{b}_2}) & \xrightarrow{{}_j\overset{E}{\infty}_k} & E(\mathcal{M}_{\mathsf{g}_1+\mathsf{g}_2,\mathsf{b}_1+\mathsf{b}_2-2}) \\ \downarrow & & \downarrow \\ E(\check{\mathcal{M}}^{j}_{\mathsf{g}_1,\mathsf{b}_2}) \boxtimes E(\check{\mathcal{M}}^{k}_{\mathsf{g}_2,\mathsf{b}_2}) & \overset{{}_j\overset{E}{\infty}_k}{\dashrightarrow} & E(\check{\mathcal{M}}_{\mathsf{g}_1+\mathsf{g}_2,\mathsf{b}_1+\mathsf{b}_2-2}) \end{array} \qquad \begin{array}{ccc} E(\mathcal{M}_{\mathsf{g},\mathsf{b}}) & \xrightarrow{\overset{E}{\infty}_{j,k}} & E(\mathcal{M}_{\mathsf{g}+1,\mathsf{b}-2}) \\ \downarrow & & \downarrow \\ E(\check{\mathcal{M}}^{j,k}_{\mathsf{g},\mathsf{b}}) & \overset{\overset{E}{\infty}_{j,k}}{\dashrightarrow} & E(\check{\mathcal{M}}_{\mathsf{g}+1,\mathsf{b}-2}) \end{array} \tag{3.8}$$

The notion of isomorphisms for Fr-smooth real one-dimensional modular functors is defined in the usual way. The main result of this work, Theorem 5.1, is the classification of Fr-smooth real one-dimensional modular functors which are local and modular invariant (a notion introduced below in Section 3.5) up to these isomorphisms.

Definition 3.3. A *Fr-smooth isomorphism of Fr-smooth real one-dimensional modular functors* $\Psi : E \to D$ consists of Fr-smooth $\mathbb{R}_+$-bundle isomorphisms

$$\Psi_{\mathsf{g},\mathsf{b}} : E(\mathcal{M}_{\mathsf{g},\mathsf{b}}) \longrightarrow D(\mathcal{M}_{\mathsf{g},\mathsf{b}}), \qquad \mathsf{g}, \mathsf{b} \geq 0, \tag{3.9}$$

preserving the sewing isomorphisms, that is,

$$\Psi_{\mathsf{g}_1+\mathsf{g}_2,\mathsf{b}_1+\mathsf{b}_2-2}(\,\cdot\;{}_j\overset{E}{\infty}_k\;\cdot\,) = \Psi_{\mathsf{g}_1,\mathsf{b}_1}(\,\cdot\,)\;{}_j\overset{D}{\infty}_k\;\Psi_{\mathsf{g}_2,\mathsf{b}_2}(\,\cdot\,), \tag{3.10}$$

$$\overset{D}{\infty}_{j,k}\;\Psi_{\mathsf{g},\mathsf{b}}(\,\cdot\,) = \Psi_{\mathsf{g}+1,\mathsf{b}-2}(\,\overset{E}{\infty}_{j,k}\;\cdot\,). \tag{3.11}$$

In a construction of the maps $\Psi_{\mathsf{g},\mathsf{b}}$ case by case, such as in Section 5, it is helpful to call Ψ an isomorphism *up to* genus g and b boundary components if only the maps $\Psi_{\mathsf{g}_1,\mathsf{b}_1}$ are defined only for $\mathsf{g}_1 < \mathsf{g}$ and any b_1, or $\mathsf{g}_1 = \mathsf{g}$ and $\mathsf{b}_1 \leq \mathsf{b}$. Then, we require only that the sewing isomorphisms are preserved for surfaces such that the genus and number of boundary components after sewing do not exceed g and b.

3.2 Central extensions of complex deformations

In this section, we introduce the central extension by the multiplicative group $\mathbb{R}_+$ that a Fr-smooth real one-dimensional modular functor E induces on complex deformations $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$. The idea of this construction goes back to Segal [Seg04], and was also used by Huang in [Hua97, Appendix D]. There, central extensions of the diffeomorphism group $\mathrm{Diff}_+^{\mathrm{an}}(\mathrm{S}^1)$ are constructed for the complex determinant line bundles or complex one-dimensional modular functors, relative to a standard annulus such as $\mathbb{A}_1$, In [MP25], we carry out this construction concretely for the real determinant line bundle and complex deformations, also relative to $\mathbb{A}_1$. In the following, we show that the central extension can be defined with respect to arbitrary surfaces $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$, for any real one-dimensional modular functor E.

Conceptually, a complex deformation $\phi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ encodes the difference between a surface $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$ and a deformed surface $\Sigma \underset{j}{*} \phi$. Applying this to E, given that $\Sigma \underset{j}{*} \phi$ exists, we define

$$E_j(\phi, \Sigma) = E(\Sigma \underset{j}{*} \phi) \otimes \big(E(\Sigma)\big)^{\vee}. \tag{3.12}$$

Tensoring with the dual of $E(\Sigma)$ is a way of "dividing out" the original surface Σ after the deformation by ϕ, retaining only the structure of E on the difference between the surfaces $\Sigma \underset{j}{*} \phi$ and Σ. In fact, $E(\phi, \Sigma)$ does not actually depend on Σ in the sense that we can define

$$E(\phi) = \bigsqcup_{\Sigma \underset{j}{*} \phi \text{ exists}} E_j(\phi, \Sigma) \Big/ \sim \tag{3.13}$$

where disjoint union is taken over all surfaces $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$, $\mathsf{g} \geq 0$, $\mathsf{b} \geq 1$ with at least one boundary component and $1 \leq j \leq \mathsf{b}$ such that $\Sigma \underset{j}{*} \phi$ exists, that is $(\Sigma, \phi) \in \mathcal{U}_{\mathsf{g},\mathsf{b},j}$ according to Equation (1.76). The equivalence relation $\sim$ is generated by a family of isomorphisms

$$\mathrm{I}^{\Sigma_1,j}_{\Sigma_2,k,\phi} : E_j(\phi, \Sigma_1) \to E_k(\phi, \Sigma_2) \tag{3.14}$$

which are natural in the sense that for three surfaces $\Sigma_1, \Sigma_2, \Sigma_2$, we have

$$\mathrm{I}^{\Sigma_1,j}_{\Sigma_1,j,\phi} = \mathbb{1}_{E_j(\phi,\Sigma_1)} \qquad \text{and} \qquad \mathrm{I}^{\Sigma_2,k}_{\Sigma_3,l,\phi} \circ \mathrm{I}^{\Sigma_1,j}_{\Sigma_2,k,\phi} = \mathrm{I}^{\Sigma_1,j}_{\Sigma_3,l,\phi}\,. \tag{3.15}$$

We define these isomorphisms first for surfaces such that $\Sigma_1\,{}_j\!\subseteq_k \Sigma_2$ in the sense of the embedding relations in Definition 1.19. In these cases there exist a finite number of surfaces collectively denoted $\underline{\Sigma}$ such that $\Sigma_1\,{}_{\underline{j}}\!\infty_{\underline{l}}\,\underline{\Sigma} = \Sigma_2$, where the tuples $\underline{j}$ (which does not include j) and $\underline{l}$ denote the pairwise sewing operations collectively denoted ${}_{\underline{j}}\!\infty_{\underline{l}}$. The isomorphism is then defined by the composition

$$\mathrm{I}^{\Sigma_1,k}_{\Sigma_2,l,\phi} = \Big(\big(\,\cdot\, {}_{\underline{j}}\!\overset{E}{\infty}_{\underline{l}} \,\cdot\, \big) \otimes \big(\,\cdot\, {}_{\underline{j}}\!\overset{E}{\infty}_{\underline{l}} \,\cdot\, \big)^{\vee} \Big) \circ \Big(\mathsf{ev}^{E(\underline{\Sigma})}_{2,4} \Big)^{-1}, \qquad \mathrm{I}^{\Sigma_2,k}_{\Sigma_1,l,\phi} = (\mathrm{I}^{\Sigma_1,j}_{\Sigma_2,k,\phi})^{-1}. \tag{3.16}$$

In words, this isomorphism takes a vector $\alpha \otimes \beta^{\vee} \in E_j(\phi, \Sigma_1)$ and then tensors it with vectors $\underline{\delta} \in E(\underline{\Sigma})$ and their duals $\underline{\delta}^{\vee} \in E(\underline{\Sigma})^{\vee}$ such that we get $\alpha \otimes \underline{\delta} \otimes \beta^{\vee} \otimes \underline{\delta}^{\vee}$. Then, the sewing isomorphisms of E are applied to sew α to $\underline{\delta}$ and $\beta^{\vee}$ to $\underline{\delta}^{\vee}$. The latter is the uniquely defined sewing isomorphism induced on the dual of E. This results in an element $\alpha\,{}_{\underline{j}}\!\infty^{E}_{\underline{k}}\,\underline{\delta} \otimes (\beta\,{}_{\underline{j}}\!\infty^{E}_{\underline{k}}\,\underline{\delta})^{\vee}$ of $E_k(\phi, \Sigma_2)$ which is independent of the choice of $\underline{\delta}$. For $\Sigma_1\,{}_j\!\subseteq_k \Sigma_2\,{}_k\!\subseteq_l \Sigma_3$ it follows from associativity of the sewing isomorphisms of E that Equation (3.15) holds.

If Σ_1 does not embed into Σ_2 or vice versa, we can still find a surface Σ_3 and l such that both $\Sigma_3\,{}_l\!\subseteq_j \Sigma_1$ and $\Sigma_3\,{}_l\!\subseteq_k \Sigma_2$ — for example a standard annulus $\mathbb{A}_\tau$ for small $\tau > 0$. The isomorphism is then defined as

$$\mathrm{I}^{\Sigma_1,j}_{\Sigma_2,k,\phi} = \mathrm{I}^{\Sigma_3,l}_{\Sigma_2,k,\phi} \circ \mathrm{I}^{\Sigma_1,j}_{\Sigma_3,l,\phi}\,. \tag{3.17}$$

In the case that $\Sigma_1\,{}_j\!\subseteq_k \Sigma_2$ also holds, this reduces to the isomorphism already defined in Equation (3.16) by Equation (3.15). Moreover, the isomorphism in Equation (3.17) does not depend on the choice of Σ_3 by repetition of the argument that for Σ_4 and m being another choice for Σ_3 and l, there exist a fifth surface Σ_5 and n such that $\Sigma_5\,{}_n\!\subseteq_l \Sigma_3$ and $\Sigma_5\,{}_n\!\subseteq_m \Sigma_4$. Indeed, we have

$$\begin{aligned}\mathrm{I}^{\Sigma_3,l}_{\Sigma_2,k,\phi} \circ \mathrm{I}^{\Sigma_1,j}_{\Sigma_3,l,\phi} &= \mathrm{I}^{\Sigma_3,l}_{\Sigma_2,k,\phi} \circ \mathrm{I}^{\Sigma_5,n}_{\Sigma_3,l,\phi} \circ \mathrm{I}^{\Sigma_3,l}_{\Sigma_5,n,\phi} \circ \mathrm{I}^{\Sigma_1,j}_{\Sigma_3,l,\phi} = \mathrm{I}^{\Sigma_5,n}_{\Sigma_2,k,\phi} \circ \mathrm{I}^{\Sigma_1,j}_{\Sigma_5,n,\phi} \\ &= \mathrm{I}^{\Sigma_5,n}_{\Sigma_2,k,\phi} \circ \mathrm{I}^{\Sigma_4,m}_{\Sigma_5,n,\phi} \circ \mathrm{I}^{\Sigma_5,n}_{\Sigma_4,m,\phi} \circ \mathrm{I}^{\Sigma_1,j}_{\Sigma_5,n,\phi} = \mathrm{I}^{\Sigma_4,m}_{\Sigma_2,k,\phi} \circ \mathrm{I}^{\Sigma_1,j}_{\Sigma_4,m,\phi},\end{aligned} \tag{3.18}$$

confirming the relation (3.16).

Now, we combine the fibers $E(\phi)$ into a $\mathbb{R}_+$-bundle over $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$,

$$E(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)) = \bigsqcup_{\phi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)} E(\phi). \tag{3.19}$$

It comes with a Frölicher structure generated by local trivializations of the form

$$Z_{\Sigma,j}(\phi) = Z(\Sigma \underset{j}{*} \phi) \otimes Z(\Sigma), \tag{3.20}$$

where Z is a Fr-smooth trivialization of E and Σ is kept fixed. These trivializations are local because they only make sense where $\Sigma \underset{j}{*} \phi$ is defined.

For the bundle $E(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)) \to \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ to become a central extension of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$, the composition law on it must extend the composition law $\phi \circ \psi$ of $\phi, \psi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ composable. We obtain it from the sewing isomorphisms of E, first relative to a surface Σ such that $\Sigma \underset{j}{*} \phi$ and $\Sigma \underset{j}{*} (\phi \circ \psi)$ exist, by defining multiplication isomorphisms $\mathrm{m}_{\phi,\psi,\Sigma} : E_j(\phi,\Sigma) \otimes E_j(\psi,\Sigma) \to E_j(\phi \circ \psi, \Sigma)$ through the composition

$$\mathrm{m}_{\phi,\psi,\Sigma,j} = \mathsf{flip} \circ \mathsf{ev}_{1,4}^{E(\Sigma \underset{j}{*} \phi)} \circ (\mathbb{1} \otimes \mathrm{I}^{\Sigma,j}_{\Sigma \underset{j}{*} \phi, j, \psi}). \tag{3.21}$$

Let us concretely spell this out for vectors $\lambda \otimes \alpha^\vee \in E_j(\phi,\Sigma)$ and $\mu \otimes \alpha^\vee \in E_j(\psi,\Sigma)$, where the respective second factors may be chosen to agree by adjusting the first by a respective scalar multiplication. The first step is to change the surface in the latter by finding

$$\mathrm{I}^{\Sigma,j}_{\Sigma \underset{j}{*} \phi, j, \psi}(\mu \otimes \alpha^\vee) = \nu \otimes \lambda^\vee, \tag{3.22}$$

with ν chosen such that the second component matches $\lambda^\vee$. Then, the evaluation and flip result in

$$\mathrm{m}_{\phi,\psi,\Sigma,j}(\lambda \otimes \alpha^\vee \otimes \mu \otimes \alpha^\vee) = \nu \otimes \alpha^\vee. \tag{3.23}$$

Note that this depends on the sewing isomorphisms of E through $\mathrm{I}^{\Sigma,j}_{\Sigma \underset{j}{*} \phi, j, \psi}$.

Theorem 3.4. *The composition law defined for $\alpha \otimes \beta^\vee \in E_j(\phi,\Sigma)$ and $\gamma \otimes \alpha^\vee \in E_j(\psi, \Sigma \underset{j}{*} \phi)$ by*

$$(\alpha \otimes \beta^\vee) \overset{E}{*} (\gamma \otimes \alpha^\vee) = \mathrm{m}_{\phi,\psi,A}(\alpha \otimes \beta^\vee \otimes \gamma \otimes \alpha^\vee) = \gamma \otimes \beta^\vee \tag{3.24}$$

is well-defined and associative on $E(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$. Moreover,

$$\{0\} \to \mathbb{R}_+ \to E(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)) \to \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1) \to \{0\}, \tag{3.25}$$

is a central extension. The second map is the inverse $\lambda \mapsto \lambda\ \alpha \otimes \alpha^\vee$ of the evaluation map $\mathsf{ev} : E(\mathbb{1}) \to \mathbb{R}_+$.

Proof. We first show that the isomorphisms $\mathrm{m}_{\phi,\psi,\Sigma,j}$ are canonical and associative in the sense that

$$\mathrm{I}^{\Sigma_1,j}_{\Sigma_2,k,\phi\circ\psi} \circ \mathrm{m}_{\phi,\psi,\Sigma_1,j} = \mathrm{m}_{\phi,\psi,\Sigma_2,k} \circ (\mathrm{I}^{\Sigma_1,j}_{\Sigma_2,k,\phi} \otimes \mathrm{I}^{\Sigma_1,j}_{\Sigma_2,k,\psi}) \tag{3.26}$$

$$\mathrm{m}_{\phi_1\phi_2,\phi_3,\Sigma,j} \circ (\mathrm{m}_{\phi_1,\phi_2,\Sigma,j} \otimes \mathbb{1}) = \mathrm{m}_{\phi_1,\phi_2\phi_3,\Sigma,j} \circ (\mathbb{1} \otimes \mathrm{m}_{\phi_2,\phi_3,\Sigma,j}). \tag{3.27}$$

For the former, start with the case where $\Sigma_1 \,{}_j{\subseteq}_k\, \Sigma_2$, that is $\Sigma_1 \,{}_{\underline{j}}{\infty}_{\underline{l}}\, \underline{\Sigma} = \Sigma_2$ as above. Let $\underline{\beta}$ denote elements of the fibers of E over the surfaces $\underline{\Sigma}$ and define $\underline{\beta}^\vee$ accordingly. Overall, by Equation (3.23), the left hand side of Equation (3.27) becomes

$$(\mathrm{I}^{\Sigma_1,j}_{\Sigma_2,k,\phi\psi} \circ \mathrm{m}_{\phi,\psi,\Sigma_1,j})(\lambda \otimes \alpha^\vee \otimes \mu \otimes \alpha^\vee) = (\nu \,{}_{\underline{j}}\overset{E}{\infty}_{\underline{l}}\, \underline{\beta}) \otimes (\alpha \,{}_{\underline{j}}\overset{E}{\infty}_{\underline{l}}\, \underline{\beta})^\vee. \tag{3.28}$$

Now, for the right-hand side, we first insert and sew the vectors $\underline{\beta}$ to obtain

$$(\lambda \,{}_{\underline{j}}\overset{E}{\infty}_{\underline{l}}\, \underline{\beta}) \otimes (\alpha \,{}_{\underline{j}}\overset{E}{\infty}_{\underline{l}}\, \underline{\beta})^\vee \otimes (\mu \,{}_{\underline{j}}\overset{E}{\infty}_{\underline{l}}\, \underline{\beta}) \otimes (\alpha \,{}_{\underline{j}}\overset{E}{\infty}_{\underline{l}}\, \underline{\beta})^\vee. \tag{3.29}$$

To this, we apply $\mathbb{1}$ tensored with the right hand side of Equation (3.16), giving

$$(\lambda \,{}_{\underline{j}}\overset{E}{\infty}_{\underline{l}}\, \underline{\beta}) \otimes (\alpha \,{}_{\underline{j}}\overset{E}{\infty}_{\underline{l}}\, \underline{\beta})^\vee \otimes (\nu \,{}_{\underline{j}}\overset{E}{\infty}_{\underline{l}}\, \underline{\beta}) \otimes (\lambda \,{}_{\underline{j}}\overset{E}{\infty}_{\underline{l}}\, \underline{\beta})^\vee, \tag{3.30}$$

where the defining relation (3.22) for ν was used again. After applying the final evaluation and flip here, we find that the right-hand side of Equation (3.27) becomes equal to (3.28). The general case follows from stacking the diagrams for a common surfaces $\Sigma_3 \, {}_l{\subseteq}_j \, \Sigma_1$ and $\Sigma_3 \, {}_l{\subseteq}_k \, \Sigma_2$. Associativity follows from the arguments as in [MP25, Theorem 3.13], which generalize to the setting of the general modular functor E. Fr-smoothness of the composition law follows from the Fr-smoothness of the sewing isomorphisms of E, since other than those, the composition law is given by linear operations which are Fr-smooth as well.

Note that the identity $\mathbb{1} \in E(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ may be represented by $\alpha \otimes \alpha^\vee$ for any $\alpha \in E_j(\mathbb{1}, \Sigma)$ for any surface Σ and boundary component j. The inverse with respect to $\cdot \overset{E}{*} \cdot$ of a representative $\alpha \otimes \beta^\vee \in E_j(\phi, \Sigma) = E(\Sigma \underset{j}{*} \phi) \otimes E(\Sigma)$ is best represented by

$$(\alpha \otimes \beta^\vee)^{-1} = \beta \otimes \alpha^\vee \in E_j(\phi^{-1}, \Sigma \underset{j}{*} \phi)^\vee = E(\Sigma) \otimes E(\Sigma \underset{j}{*} \phi)^\vee, \tag{3.31}$$

since then we have already found the right hand side of Equation (3.22) such that

$$\mathrm{m}_{\phi,\phi^{-1},\Sigma,j}(\alpha \otimes \beta^\vee \otimes \beta \otimes \alpha^\vee) = \mathsf{flip} \circ \mathsf{ev}_{1,4}^{E(\Sigma \underset{j}{*} \phi)}(\alpha \otimes \beta^\vee \otimes \beta \otimes \alpha^\vee) = \beta \otimes \beta^\vee \tag{3.32}$$

is indeed the identity.

For the central extension, consider the map

$$\begin{aligned} \mathsf{ev} : E_j(\mathbb{1}, \Sigma) = E(\Sigma) \otimes E(\Sigma)^\vee &\to \mathbb{R} \\ \alpha \otimes \beta^\vee &\mapsto \mathsf{ev}(\beta, \alpha), \end{aligned} \tag{3.33}$$

which is well-defined on $E(\mathbb{1})$ since we can write any $\alpha \otimes \beta^\vee \in E_j(\mathbb{1}, \Sigma_1)$ as $\lambda \ \alpha \otimes \alpha^\vee$ for some $\lambda \in \mathbb{R}_+$, and then $\mathrm{I}^{\Sigma_1,j}_{\Sigma_2,k,\mathbb{1}}(\lambda \ \alpha \otimes \alpha^\vee) = \lambda \ \gamma \otimes \gamma^\vee$ for any $\gamma \in E(\Sigma_2)$, hence $\mathsf{ev}(\alpha \otimes \beta^\vee) = \lambda = \mathsf{ev}(\lambda \ \gamma \otimes \gamma^\vee)$. Then, the map $\lambda \mapsto \lambda \ \alpha \otimes \alpha^\vee$ for any $\alpha \in E(\Sigma)$ for any Σ is independent of Σ and α, and indeed inverts the evaluation map. The kernel of the projection $E(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)) \to \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ is the fiber $E(\mathbb{1})$ over $\mathbb{1} \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$, which is clearly the image of this map, making the Sequence (3.25) exact. Finally, the composition law on $E(\mathbb{1})$ given $\lambda\alpha \otimes \alpha^\vee$ and $\mu\alpha \otimes \alpha^\vee$ for $\lambda, \mu \in \mathbb{R}_+$ and $\alpha \in E(\Sigma)$, for any surface Σ and j, is given by $\mathrm{m}_{\mathbb{1},\mathbb{1},\Sigma,j}(\lambda \ \alpha \otimes \alpha^\vee \otimes \mu \ \alpha \otimes \alpha^\vee) = \lambda \ \mu \ \alpha \otimes \alpha^\vee$, since in Equation (3.22) we have $\nu = \alpha$, and therefore it agrees with the multiplication in $\mathbb{R}_+$. Finally, $E(\mathbb{1})$ is central in $E(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ since given any $\alpha \otimes \beta^\vee \in E(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$, any element of $E(\mathbb{1})$ may be represented by $\lambda \ \alpha \otimes \alpha^\vee = \lambda \ \beta \otimes \beta^\vee$ for some $\lambda \in \mathbb{R}_+$, and then, we have

$$(\lambda \ \beta \otimes \beta^\vee) \overset{E}{*} (\alpha \otimes \beta^\vee) = \lambda \ \alpha \otimes \beta^\vee = (\alpha \otimes \beta^\vee) \overset{E}{*} (\lambda \ \alpha \otimes \alpha^\vee). \tag{3.34}$$

□

3.3 Action of the central extensions on the modular functor

Consider the fiber $E(\Sigma)$ over any surface $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$, and $\phi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$, and $1 \leq j \leq \mathsf{b}$ such that $(\Sigma, \phi) \in \mathcal{U}_{\mathsf{g},\mathsf{b},j}$, that is, the deformed surface $\Sigma \underset{j}{*} \phi$ exists. Take $\gamma \in E(\Sigma)$, and any element of $E(\phi)$ which in our setup may be represented in $E_j(\phi, \Sigma)$ as $\alpha \otimes \gamma^\vee$ by adjusting α. Conceptually, we can read the formula $\alpha \otimes \gamma^\vee$ as "replace γ by α", as this is what ϕ does geometrically when deforming the surface Σ. Indeed, this defines actions of $E(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ on bundles $E(\mathcal{M}_{\mathsf{g},\mathsf{b}})$ at each boundary component.

Proposition 3.5. *For $1 \leq j \leq \mathsf{b}$, the maps*

$$\begin{aligned} \cdot \overset{E}{\underset{j}{*}} \cdot : E(\mathcal{U}_{\mathsf{g},\mathsf{b},j}) &\to E(\mathcal{M}_{\mathsf{g},\mathsf{b}}), \\ (\gamma, \alpha \otimes \gamma^\vee) &\mapsto \alpha. \end{aligned} \tag{3.35}$$

define Fr-smooth actions of $E(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ on the Frölicher spaces $E(\mathcal{M}_{\mathsf{g},\mathsf{b}})$. They are bundle morphisms over $\cdot \underset{j}{*} \cdot$ (see Theorem 1.18), and isomorphisms on fibers. Moreover, the following compatibilities with the sewing isomorphisms of E for $1 \le j, k, l \le \mathsf{b}$ hold:

$$\left(\cdot \overset{E}{\underset{j}{*}} \cdot\right) \overset{E}{\underset{k}{*}} \cdot = \left(\cdot \overset{E}{\underset{k}{*}} \cdot\right) \overset{E}{\underset{j}{*}} \cdot \qquad \textit{and} \qquad \left(\overset{E}{\infty}_{j,k} \cdot\right) \underset{l}{*} \cdot = \overset{E}{\infty}_{j,k} \left(\cdot \underset{l}{*} \cdot\right) \tag{3.36}$$

$$\left(\cdot {}_j\overset{E}{\infty}_k \cdot\right) \overset{E}{\underset{l}{*}} \cdot = \cdot {}_j\overset{E}{\infty}_k \left(\cdot \overset{E}{\underset{l}{*}} \cdot\right) \qquad \textit{or} \qquad \left(\cdot {}_j\overset{E}{\infty}_k \cdot\right) \overset{E}{\underset{l}{*}} \cdot = \left(\cdot \overset{E}{\underset{l}{*}} \cdot\right) {}_j\overset{E}{\infty}_k \cdot \tag{3.37}$$

Here, the case in the second line depends on which of the surfaces the complex deformations acts on.

Proof. For two consecutive actions at the same boundary component we have on the one hand

$$\left(\gamma \overset{E}{\underset{j}{*}} (\alpha \otimes \gamma^\vee)\right) \overset{E}{\underset{j}{*}} (\beta \otimes \alpha^\vee) = \alpha \overset{E}{\underset{j}{*}} (\beta \otimes \alpha^\vee) = \beta, \tag{3.38}$$

and on the other hand

$$\gamma \underset{j}{*} \left((\alpha \otimes \gamma^\vee) \overset{E}{*} (\beta \otimes \alpha^\vee)\right) = \gamma \underset{j}{*} (\beta \otimes \gamma^\vee) = \beta, \tag{3.39}$$

where $\beta \otimes \alpha^\vee$ is already in the representation on the right hand side of Equation (3.22). Fr-smoothness of the actions follows directly from the choice of Frölicher structure on $E(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$.

Let $\Sigma_1 {}_j{\subseteq}_k \Sigma_2$, such that both $\Sigma_1 \underset{j}{*} \phi$ and $\Sigma_2 \underset{k}{*} \phi$ exist. For any $\gamma \in E(\Sigma_1)$ and $\delta \in E(\Sigma_2)$ there exists $\underline{\mu} \in E(\underline{\Sigma}))$ where $\Sigma_1 {}_{\underline{j}}\infty_{\underline{l}} \underline{\Sigma} = \Sigma_2$ such that $\gamma {}_{\underline{j}}\infty^E_{\underline{l}} \underline{\mu} = \delta$. If $\underline{\Sigma}$ is just one surface, then $\underline{\mu}$ is unique; otherwise the $\underline{\mu}$ may exchange scalars. Then, an element of $E(\phi)$ may be represented either as $\alpha \otimes \gamma^\vee \in E_j(\phi, \Sigma_1)$ or as $\beta \otimes \delta^\vee \in E_k(\phi, \Sigma_2)$ related by $\mathrm{I}^{\Sigma_1,j}_{\Sigma_2,k,\phi}(\alpha \otimes \gamma^\vee) = \beta \otimes \delta^\vee$, or equivalently, $\alpha {}_{\underline{j}}\infty^E_{\underline{l}} \underline{\mu} = \beta$. The relations in Equation (3.37) follow immediately from this. The second relation in Equation (3.36) follows analogously, for instance with Σ_2 the surface under consideration, Σ_1 an annulus at $\partial_l \Sigma_2$ and a single surface in $\underline{\Sigma}$ which is self sewn in the way of Σ_2.

For the first relation in Equation (3.36), we need a single surface $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$ and $1 \le j < k \le \mathsf{b}$, and two deformations $\phi, \psi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ such that each of the surfaces $\Sigma \underset{j}{*} \phi$, $\Sigma \underset{k}{*} \psi$ and $\Sigma \underset{j}{*} \phi \underset{k}{*} \psi$ exists. Then, we find a decomposition $\Sigma = \Sigma_1 {}_l\infty_m \Sigma_3 {}_n\infty_o \Sigma_2$ such that $\Sigma_1 {}_p{\subseteq}_j \Sigma$ and $\Sigma_2 {}_q{\subseteq}_k \Sigma$, for instance, small annuli. Take any $\gamma \in E(\Sigma)$ and decompose it as $\gamma = \mu_1 {}_l\infty^E_m \delta {}_n\infty^E_o \mu_2$ like above. Any two vectors in $E(\phi)$ and $E(\psi)$ may respectively be represented by $\alpha_1 \otimes \mu_1^\vee$ and $\alpha_2 \otimes \mu_2^\vee$. With this setup the first relation in Equation (3.36) becomes

$$\begin{aligned}
\left(\gamma \overset{E}{\underset{j}{*}} (\alpha_1 \otimes \mu_1^\vee)\right) \underset{k}{*} (\alpha_2 \otimes \mu_2^\vee) &= \left((\mu_1 {}_l\overset{E}{\infty}_m \delta {}_n\overset{E}{\infty}_o \mu_2) \overset{E}{\underset{j}{*}} (\alpha_1 \otimes \mu_1^\vee)\right) \underset{k}{*} (\alpha_2 \otimes \mu_2^\vee) \\
&= \left(\alpha_1 {}_l\overset{E}{\infty}_m \delta {}_n\overset{E}{\infty}_o \mu_2\right) \underset{k}{*} (\alpha_2 \otimes \mu_2^\vee) \\
&= \alpha_1 {}_l\overset{E}{\infty}_m \delta {}_n\overset{E}{\infty}_o \alpha_2
\end{aligned} \tag{3.40}$$

on the one hand, and equally on the other because of the associativity relation (3.4) of the sewing isomorphisms of E. □

If the action takes place between the sewn boundary components, for $\gamma \in E(\Sigma_1)$, $\alpha \otimes \gamma^\vee \in E_j(\phi, \Sigma_1)$, and $\delta \in E(\Sigma_2)$, we solve the following linear equations for a unique vector $\beta \in E(\Sigma_2 \underset{k}{*} (\mathrm{J} \circ \phi^{-1} \circ \mathrm{J}))$,

$$\left(\gamma \overset{E}{\underset{j}{*}} (\alpha \otimes \gamma^\vee)\right) {}_j\overset{E}{\infty}_k \delta = \alpha {}_j\overset{E}{\infty}_k \delta = \gamma {}_j\overset{E}{\infty}_k \beta = \gamma {}_j\overset{E}{\infty}_k \left(\delta \underset{j}{*} (\beta \otimes \delta^\vee)\right). \tag{3.41}$$

to find a relation extending Equation (1.79). If the action takes place between self-sewn boundary components instead, for $\gamma \in E(\Sigma)$, and $\alpha \otimes \gamma^\vee \in E_j(\phi, \Sigma)$, we find a relation extending Equation (1.81) by solving the following linear equation for $\beta \in E(\Sigma \underset{k}{*} (\mathrm{J} \circ \phi^{-1} \circ \mathrm{J}))$

$$\overset{E}{\infty}_{j,k} \left(\gamma \overset{E}{\underset{j}{*}} (\alpha \otimes \gamma^\vee\right) = \overset{E}{\infty}_{j,k} \alpha = \overset{E}{\infty}_{j,k} \beta = \overset{E}{\infty}_{j,k} \left(\gamma \overset{E}{\underset{k}{*}} (\beta \otimes \gamma^\vee)\right). \tag{3.42}$$

3.4 Various cocycle identities

Let E be a Fr-smooth real one-dimensional modular functor. Let Z collectively denote a trivialization of E, which consists of Fr-smooth trivializations $Z_{\mathsf{g},\mathsf{b}} : \mathcal{M}_{\mathsf{g},\mathsf{b}} \to E(\mathcal{M}_{\mathsf{g},\mathsf{b}})$ for $\mathsf{g}, \mathsf{b} \geq 0$. For each choice of boundary labels $1 \leq j \leq \mathsf{b}_1$, $1 \leq k \leq \mathsf{b}_2$, the trivialization leads to Fr-smooth functions $\Omega^Z_{j,k} \in \mathcal{F}(\mathcal{M}_{\mathsf{g}_1,\mathsf{b}_1} \times \mathcal{M}_{\mathsf{g}_2,\mathsf{b}_2})$ such that

$$Z(\Sigma_1) \, {}_j\overset{E}{\infty}_k \, Z(\Sigma_2) = e^{\Omega^Z_{j,k}(\Sigma_1,\Sigma_2)} Z(\Sigma_1 \, {}_j\infty_k \, \Sigma_2). \tag{3.43}$$

Associated to self-sewing at distinct boundary labels $1 \leq j \leq \mathsf{b}$ and $1 \leq k \leq \mathsf{b}$, we define the functions $\Omega^Z_{j,k} \in \mathcal{F}(\mathcal{M}_{\mathsf{g},\mathsf{b}})$ by

$$\overset{E}{\infty}_{j,k} \, Z(\Sigma) = e^{\Omega^Z_{j,k}(\Sigma)} Z(\, \infty_{j,k} \, \Sigma). \tag{3.44}$$

We call these functions the *cocycles* of E with respect to the trivialization Z, and they satisfy *cocycle identities* arising from the associativity of the sewing isomorphisms of E,

$$\Omega^Z_{j,k}(\Sigma_1, \Sigma_2) + \Omega^Z_{l,m}(\Sigma_1 \, {}_j\infty_k \, \Sigma_2, \Sigma_3) = \Omega^Z_{j,k}(\Sigma_1, \Sigma_2 \, {}_l\infty_m \, \Sigma_3) + \Omega^Z_{l,m}(\Sigma_2, \Sigma_3), \tag{3.45}$$

$$\Omega^Z_{j,k}(\Sigma_1) + \Omega^Z_{l,m}(\, \infty_{j,k} \, \Sigma_1, \Sigma_2) = \Omega^Z_{j,k}(\Sigma_1 \, {}_l\infty_m \, \Sigma_2) + \Omega^Z_{l,m}(\Sigma_1, \Sigma_2), \tag{3.46}$$

$$\Omega^Z_{j,k}(\Sigma_1 \, {}_l\infty_m \, \Sigma_2) + \Omega^Z_{l,m}(\Sigma_1, \Sigma_2) = \Omega^Z_{l,m}(\Sigma_1 \, {}_j\infty_k \, \Sigma_2) + \Omega^Z_{j,k}(\Sigma_1, \Sigma_2), \tag{3.47}$$

$$\Omega^Z_{j,k}(\Sigma) + \Omega^Z_{l,m}(\, \infty_{j,k} \, \Sigma) = \Omega^Z_{j,k}(\, \infty_{l,m} \, \Sigma) + \Omega^Z_{l,m}(\Sigma). \tag{3.48}$$

As already mentioned with Equation (3.20), the trivialization Z also induces local trivializations on the central extension of complex deformations $E(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ by

$$Z_{\Sigma,j}(\phi) = Z(\Sigma \underset{j}{*} \phi) \otimes Z(\Sigma)^\vee, \tag{3.49}$$

for each surface $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$ and $\phi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ such that $\Sigma \underset{j}{*} \phi$ exists. The composition law $\overset{E}{*}$ of $E(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ defines a deformation-deformation cocycle for $\phi, \psi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ by

$$Z_{\Sigma,j}(\phi) \overset{E}{*} Z_{\Sigma,j}(\psi) = e^{\Omega^Z_{\Sigma,j}(\phi,\psi)} Z_{\Sigma,j}(\phi \circ \psi), \tag{3.50}$$

defined where the section is defined at all three deformations. If $\Sigma_1 \, {}_j\subseteq_k \, \Sigma_2$, that is, $\Sigma_1 \, {}_{\underline{j}}\infty_{\underline{l}} \, \underline{\Sigma} = \Sigma_2$, the trivializations are related by

$$Z_{\Sigma_1,j}(\phi) = e^{\Omega^Z_{\underline{j},\underline{l}}(\Sigma_1 \underset{j}{*} \phi, \underline{\Sigma}) - \Omega^Z_{\underline{j},\underline{l}}(\Sigma_1, \underline{\Sigma})} Z_{\Sigma_2,j}(\phi), \tag{3.51}$$

and by Equation (3.26), the cocycles are related by the identity

$$\begin{aligned} &\Omega^Z_{\Sigma_1,j}(\phi_1, \phi_2) - \Omega^Z_{\Sigma_2,k}(\phi_1, \phi_2) \\ &= \Omega^Z_{\underline{j},\underline{l}}(\Sigma_1 \underset{j}{*} \phi_1, \underline{\Sigma}) + \Omega^Z_{\underline{j},\underline{l}}(\Sigma_1 \underset{j}{*} \phi_2, \underline{\Sigma}) - \Omega^Z_{\underline{j},\underline{l}}(\Sigma_1 \underset{j}{*} (\phi_1 \circ \phi_2), \underline{\Sigma}) - \Omega^Z_{\underline{j},\underline{l}}(\Sigma_1, \underline{\Sigma}), \end{aligned} \tag{3.52}$$

where the surface-surface cocycles with $\underline{\Sigma}$ in the argument are just sums of consecutive cocycles sewing on the surfaces in $\underline{\Sigma}$ one by one, which does not depend on the order of sewing by the cocycle identities above. Note that this identity has the form of a "coboundary" for the function $\Omega^Z_{\underline{j},\underline{l}}(\Sigma_1 \underset{j}{*} \cdot, \underline{\Sigma}) - \Omega^Z_{\underline{j},\underline{l}}(\Sigma_1, \underline{\Sigma})$ on $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$. By associativity of the multiplication in $E(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$, see Equation (3.27), these cocycle satisfy the following deformation-deformation-deformation cocycle identity,

$$\Omega^Z_{\Sigma,j}(\phi_1 \circ \phi_2, \phi_3) + \Omega^Z_{\Sigma,j}(\phi_1, \phi_2) = \Omega^Z_{\Sigma,j}(\phi_1, \phi_2 \circ \phi_3) + \Omega^Z_{\Sigma,j}(\phi_2, \phi_3). \tag{3.53}$$

From the action of $E(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ on $E(\mathcal{M}_{\mathsf{g},\mathsf{b}})$ we also obtain the mixed cocycles

$$Z(\Sigma) \overset{E}{\underset{k}{*}} Z_{\Sigma_1,j}(\phi) = e^{\Omega^Z_{\Sigma_1,j,k}(\Sigma,\phi)} Z(\Sigma \underset{k}{*} \phi), \tag{3.54}$$

where Σ_1 is the surface relative to which the trivialization of $E(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ is defined, and Σ is an additional surface on whose kth boundary component we act on by $\phi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$. For $\Sigma_1 \,{}_j{\subseteq}_k\, \Sigma_2$ as above the cocycles are related by

$$\Omega^Z_{\Sigma_1,j,k}(\Sigma,\phi) - \Omega^Z_{\Sigma_2,j,k}(\Sigma,\phi) = \Omega^Z_{\underline{j},\underline{l}}(\Sigma_1 \underset{j}{*} \phi, \underline{\Sigma}) - \Omega^Z_{\underline{j},\underline{l}}(\Sigma_1,\underline{\Sigma}), \tag{3.55}$$

which is independent of Σ and equal to the function that the right hand side of (3.52) is a differential of. If we take $\Sigma = \Sigma_1$ and $j = k$ in Equation (3.54) we find by Equation (1.77) in Propostion 3.5 that

$$\Omega^Z_{\Sigma_1,j,j}(\Sigma_1,\phi) = 0, \qquad \phi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)\,. \tag{3.56}$$

Also by Proposition 3.5, the mixed cocycles satisfy mixed cocycle identities

$$\begin{aligned}
&\Omega^Z_{j,k}(\Sigma_1,\Sigma_2 \underset{l}{*} \phi) + \Omega^Z_{\Sigma,m,l}(\Sigma_2,\phi) = \Omega^Z_{\Sigma,m,l}(\Sigma_1 \,{}_j{\infty}_k\, \Sigma_2,\phi) + \Omega^Z_{j,k}(\Sigma_1,\Sigma_2),\\
&\Omega^Z_{j,k}(\Sigma_1 \underset{l}{*} \phi,\Sigma_2) + \Omega^Z_{\Sigma,m,l}(\Sigma_1,\phi) = \Omega^Z_{\Sigma,m,l}(\Sigma_1 \,{}_j{\infty}_k\, \Sigma_2,\phi) + \Omega^Z_{j,k}(\Sigma_1,\Sigma_2),\\
&\Omega^Z_{j,k}(\Sigma_1 \underset{j}{*} \phi,\Sigma_2) + \Omega^Z_{\Sigma,m,j}(\Sigma_1,\phi) = \Omega^Z_{j,k}(\Sigma_1,\Sigma_2 \underset{k}{*} (\mathrm{J}\,\phi^{-1}\,\mathrm{J})) + \Omega^Z_{\Sigma,m,k}(\Sigma_2,\mathrm{J}\,\phi^{-1}\,\mathrm{J}).
\end{aligned} \tag{3.57}$$

depending on which boundary ϕ acts relative with the trivialization on to complex deformations relative to Σ, m. Moreover, by compatibility of the action,

$$\begin{aligned}
&\Omega^Z_{\Sigma,l,k}(\Sigma_1 \underset{j}{*} \phi_1,\phi_2) + \Omega^Z_{\Sigma,l,j}(\Sigma_1,\phi_1) = \Omega^Z_{\Sigma,l,j}(\Sigma_1 \underset{k}{*} \phi_2,\phi_1) + \Omega^Z_{\Sigma,l,k}(\Sigma_1,\phi_2),\\
&\Omega^Z_{\Sigma,l,j}(\Sigma_1 \underset{j}{*} \phi_1,\phi_2) + \Omega^Z_{\Sigma,l,j}(\Sigma_1,\phi_1) = \Omega^Z_{\Sigma,l,j}(\Sigma_1,\phi_1\circ\phi_2) + \Omega^Z_{\Sigma_1,l}(\phi_1,\phi_2).
\end{aligned} \tag{3.58}$$

and more cocycles from self-sewing.

In the case that E is local, we can take a trivialization that is a pullback of a trivialization of $E(\check{\mathcal{M}}_{\mathsf{g},\mathsf{b}})$. For such trivializations the cocyles have additional symmetries under reparametrization by diffeomorphisms. We identify these and show that they in turn characterize the locality condition.

Proposition 3.6. *The following are equivalent:*

1. *E is local.*
2. *There exists a trivialization of E such that for any $\Sigma_1 \in \mathcal{M}_{\mathsf{g}_1,\mathsf{b}_1}, \Sigma_2 \in \mathcal{M}_{\mathsf{g}_2,\mathsf{b}_2}, \phi \in \mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$, $l \neq j$,*
$$\Omega^Z_{j,k}(\Sigma_1 \underset{l}{*} \phi,\Sigma_2) = \Omega^Z_{j,k}(\Sigma_1,\Sigma_2) \tag{3.59}$$
3. *There exists a trivialization of E such that for any $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$, $1 \leq j \leq \mathsf{b}$, $\phi \in \mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$, and $\Sigma_1 \in \mathcal{M}_{\mathsf{g}_1,\mathsf{b}_1}$ and $1 \leq j \leq \mathsf{b}_1$,*
$$\Omega^Z_{\Sigma,j,k}(\Sigma_1,\phi) = 0. \tag{3.60}$$
4. *There exists a trivialization of E such that*
$$\Omega^Z_{\Sigma,j}(\phi,\psi) = 0, \qquad \phi,\psi \in \mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1), \tag{3.61}$$
for all surfaces Σ with boundary component j.
5. *There exists a trivialization of E such that*
$$\Omega^Z_{\Sigma_1,j}(\phi,\psi) = \Omega^Z_{\Sigma_2,k}(\phi,\psi), \qquad \phi,\psi \in \mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1), \tag{3.62}$$
for any surfaces Σ_1, Σ_2 with choice boundary components j and k respectively.
6. *There exists a trivialization of E such that*
$$\Omega^Z_{j,k}(\Sigma_1 \underset{j}{*} \phi,\Sigma_2) = \Omega^Z_{j,k}(\Sigma_1,\Sigma_2 \underset{k}{*} (\mathrm{J}\circ\phi^{-1}\circ\mathrm{J})) \tag{3.63}$$

Morover, if E is local a trivialization Z satisisfying any of the above also satisfies the other conditions.

Proof. We prove the following implications: 1. $\iff$ 2. $\implies$ 3. $\implies$ 4. $\implies$ 5. $\implies$ 2. and 3. $\iff$ 6..

1. $\implies$ 2. Let Z denote trivializations of $E(\mathcal{M}_{\mathsf{g},\mathsf{b}})$ lifted from $E(\check{\mathcal{M}}_{\mathsf{g},\mathsf{b}})$. Let Σ_1 and Σ_2 be any surfaces and $\phi \in \mathrm{Diff}_+^{\mathrm{an}}(\mathrm{S}^1)$. $Z(\Sigma_1)$ and $Z(\Sigma_1 \underset{j}{*} \phi$ project to the same element in $E(\check{\mathcal{M}}_{\mathsf{g},\mathsf{b}})$. Then, by the factorization in the diagram (3.8), the sewn vectors

$$\begin{aligned} Z(\Sigma_1 \underset{l}{*} \phi)\ {}_j\overset{E}{\infty}_k\ Z(\Sigma_2) &= e^{\Omega^Z_{j,k}(\Sigma_1 \underset{l}{*} \phi, \Sigma_2)} Z((\Sigma_1\ {}_j\infty_k\ \Sigma_2) \underset{l}{*} \phi) \\ Z^E(\Sigma_1)\ {}_j\overset{E}{\infty}_k\ Z^E(\Sigma_2) &= e^{\Omega^E_{j,k}(\Sigma_1, \Sigma_2)} Z^E(\Sigma_1\ {}_j\infty_k\ \Sigma_2) \end{aligned} \tag{3.64}$$

project to the same vectors in $E(\check{\mathcal{M}}_{\mathsf{g},\mathsf{b}})$ and thus the cocyles in Equation (3.59) agree.

2. $\implies$ 1. Let Σ_1 and Σ_2 be any two surfaces and j, k respectively a choice of boundary component. Generally consider $\Sigma_1 \underset{\underline{j}}{*} \underline{\phi}$ and $\Sigma_1 \underset{\underline{k}}{*} \underline{\psi}$ where $\underline{\phi}$, $\underline{\psi}$ each denote multiple elements of $\mathrm{Diff}_+^{\mathrm{an}}(\mathrm{S}^1)$ respectively acting at boundary components $\underline{j}$ and $\underline{k}$ which do not include j and k. By choice of the diffemorphisms these parametrize all other surfaces mapping to the same equivalence classes respectively in $\check{\mathcal{M}}^j_{\mathsf{g}_1,\mathsf{b}_1}$ and $\check{\mathcal{M}}^k_{\mathsf{g}_2,\mathsf{b}_2}$ retaining only the jth and kth boundary parametrizations. Now let Z^E be a trivialization as in 2. Vectors in $E(\check{\mathcal{M}}^j_{\mathsf{g}_1,\mathsf{b}_1})$ and $E(\check{\mathcal{M}}^k_{\mathsf{g}_2,\mathsf{b}_2})$ lift to elements of the form $\lambda_1 Z^E(\Sigma_1)$ and $\lambda_1 Z^E(\Sigma_1 \underset{\underline{j}}{*} \underline{\phi})$, and respectively $\lambda_2 Z^E(\Sigma_2)$ and $\lambda_2 Z^E(\Sigma_2 \underset{\underline{k}}{*} \underline{\psi})$, which then sew into

$$\begin{aligned} \lambda_1 Z^E(\Sigma_1)\ {}_j\overset{E}{\infty}_k\ \lambda_2 Z^E(\Sigma_2) &= \lambda_1\lambda_2 e^{\Omega^E_{j,k}(\Sigma_1,\Sigma_2)} Z^E(\Sigma_1\ {}_j\infty_k\ \Sigma_2), \\ \lambda_1 Z^E(\Sigma_1 \underset{\underline{j}}{*} \underline{\phi})\ {}_j\overset{E}{\infty}_k\ \lambda_2 Z^E(\Sigma_2 \underset{\underline{k}}{*} \underline{\psi}) &= \lambda_1\lambda_2 e^{\Omega^E_{j,k}(\Sigma_1 \underset{\underline{j}}{*} \underline{\phi}), \lambda_2 Z^E(\Sigma_2 \underset{\underline{k}}{*} \underline{\psi})} Z^E(\Sigma_1 \underset{\underline{j}}{*} \underline{\phi}\ {}_j\infty_k\ \Sigma_2 \underset{\underline{k}}{*} \underline{\psi}). \end{aligned} \tag{3.65}$$

The trivializations project to the same vectors and by repeated application of Equation (3.59) the cocycles agree. Therefore, the sewing operation in diagram (3.8) is indeed independent of the lifts and a projection from $E(\mathcal{M}_{\mathsf{g},\mathsf{b}})$ to $E(\check{\mathcal{M}}_{\mathsf{g},\mathsf{b}})$ may be defined using the trivialization Z^E such that $E(\mathcal{M}_{\mathsf{g},\mathsf{b}})$ be comes he pullback bundle along this projection.

2. $\implies$ 3. By Equations (3.59) and (3.55), the cocycle in 3. does not depend on on Σ and j. Thus, by changing to $\Sigma = \Sigma_1$ and $j = k$, Equation (3.56) implies 3.

3. $\implies$ 4. This implication follows directly from the second equation in (3.58).

4. $\implies$ 5. If all the cocycles on $\mathrm{Diff}_+^{\mathrm{an}}(\mathrm{S}^1)$ vanish, they also agree.

5. $\implies$ 2. Assuming 5., the coboundaries in Equation (3.52) vanish, and thus define homomorphisms $\mathrm{Diff}_+^{\mathrm{an}}(\mathrm{S}^1) \to \mathbb{R}$, where $\mathbb{R}$ denotes the abelian addditive group. Since $\mathrm{Diff}_+^{\mathrm{an}}(\mathrm{S}^1)$ is a perfect group, see e.g. [GR07, Theorem 4.4.2], any such homomorphism vanishes, and thus we have

$$\Omega^E_{\underline{j},\underline{l}}(\Sigma_1 \underset{j}{*} \phi_1, \underline{\Sigma}) - \Omega^E_{\underline{j},\underline{l}}(\Sigma_1, \underline{\Sigma}) = 0, \tag{3.66}$$

which by choosing $\underline{\Sigma}$ to be a single surface is precisely Equation (3.59) in 2.

3. $\implies$ 6. follows directly from the third equation in (3.57).

6. $\implies$ 3. also follows from the third equation in (3.57) by choosing $\Sigma_2 = \Sigma$ and $k = m$.

Finally, observe that none of the implications above require changing the trivialization. □

3.5 Central charge and modular invariance

Given a trivialization Z of a Fr-smooth real one-dimensional modular functor E, by Equation (3.53), the cocycles $\Omega^Z_{\Sigma,j}(\phi, \psi)$ relative to the jth boundary component of a surface $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$ are cocycles on those complex deformations such that $\Sigma \underset{j}{*} \phi$, $\Sigma \underset{j}{*} \psi$ and $\Sigma \underset{j}{*} (\phi \circ \psi)$ exist.

The unit disk with standard parametrization $\mathbb{D} = (\mathbb{D}, \mathrm{J})$ is s special surface such that these deformations always exist (if ϕ and ψ are composable). Thus, the cocycle $\Omega^Z_{\mathbb{D},1}$ defines a cohomology class in the cohomology $H^2(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1); \mathbb{R})$, which was computed in Theorem 2.3; see also Figure 2. There are unique $\mathbf{a}_E, \mathbf{b}_E, \mathbf{c}_E \in \mathbb{R}$ such that in the given basis of cocycles, we have

$$\Omega^Z_{\mathbb{D},1} = \mathbf{a}_E \operatorname{Re}\Omega_{\mathrm{BT}} + \mathbf{b}_E \operatorname{Re}\Omega_{\mathrm{Rot}} + \mathbf{c}_E \operatorname{Im}\Omega_{\mathrm{BT}} + \delta A_Z, \tag{3.67}$$

where A_Z is a 1-cycle on $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ depending on the choice of trivialization Z of E. Note that since changing the trivialization Z only changes the cocycle $\Omega^Z_{\mathbb{D},1}$ by a coboundary, the constants $\mathbf{a}_E, \mathbf{b}_E, \mathbf{c}_E$ do not depend on the choice of Z. We are interested in the constant $\mathbf{c}_E$, which we call the *central charge* of E.

The central charge is also accessible on the Lie algebra level. Let $\omega^Z_{\mathbb{D},1}$ be the Lie algebra 2-cocycle on $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ obtained from differentiation of $\Omega^Z_{\mathbb{D},1}$. By the characterization of the Lie algebra cohomology $H^2(\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1), \mathbb{R})$, see Figure 3, there is a decomposition

$$\omega^Z_{\mathbb{D},1} = \mathbf{a}_E \operatorname{Re}\omega_{\mathrm{GF}} + \mathbf{c}_E \operatorname{Im}\omega_{\mathrm{GF}} + \delta \alpha_Z, \tag{3.68}$$

where α_Z is a Lie algebra 1-cycle, which is the derivative of A_Z above.

If E is local, and Z is reparametrization invariant, then by Equation (3.61) in Proposition 3.6, the cocycle $\Omega^Z_{\mathbb{D},1}$ vanishes on diffeomorphisms. Thus, it is a cocycle relative to $\mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$. Using the characterization of the relative cohomolog $H^2(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1); \mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1); \mathbb{R})$ we find the decompositions

$$\Omega^Z_{\mathbb{D},1} = \mathbf{c}_E \operatorname{Im}\Omega_{\mathrm{BT}} + \delta B_Z, \qquad \omega^Z_{\mathbb{D},1} = \mathbf{c}_E \operatorname{Im}\omega_{\mathrm{GF}} + \delta \beta_Z, \tag{3.69}$$

where B_Z is a 1-chain on $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ relative $\mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$ and $\beta = \mathrm{D}\, B$. Note that the assumption of locality singles out the central charge as the only coefficient, that is, $\mathbf{a}_E = \mathbf{b}_E = 0$.

Even though the cohomology class of the cocycle $\Omega^Z_{\mathbb{D},1}$ of a local Fr-smooth real one-dimensional modular functor is fully determined by the central charge, more details on which coboundaries δB can appear are needed for a full characterization of E up to isomorphism. In particular, we are interested in the cocycle $\operatorname{Im}\Omega_{\mathrm{Rot}}$ introduced in Section 2.2, Equation (2.36). This is a coboundary in the non-relative cohomology, but becomes nontrivial relative to the subgroup of scaling transformations Sc defined in Equation (1.45). Consider the family of standard annuli $\mathbb{A}_\tau$, $\tau > 0$ as defined in (1.71). The cocycle $\Omega^Z_{\mathbb{A}_\tau,1}$ is only defined on complex deformations which deform S^1 at most by $e^{-2\pi\tau}$, see Equation (2.1). Since this subset has the same fundamental group, the exact sequences in Proposition 2.1 yield isomorphisms of the relative second cohomology groups to those of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$, which are computed in Theorem 2.3. In other words, we can use the characterization of the second cohomology groups in Figure 2 also for the cocycle $\Omega^Z_{\mathbb{A}_\tau,1}$. Assume momentarily that for the trivialization Z we have

$$\Omega^Z_{1,2}(\mathbb{A}_{\tau_1}, \mathbb{A}_{\tau_2}) = 0, \qquad \tau_1, \tau_2 > 0. \tag{3.70}$$

Since in terms of scaling transformations $s_{\tau_1}, s_{\tau_2} \in \mathrm{Sc}$ with $\tau_1, \tau_2 > 0$,

$$\Omega^Z_{\mathbb{A}_\tau,1}(s_{\tau_1}, s_{\tau_2}) = \Omega^Z_{1,2}(\mathbb{A}_{\tau_1}, \mathbb{A}_{\tau_2}) + \Omega^Z_{1,2}(\mathbb{A}_\tau, \mathbb{A}_{\tau_1+\tau_2}) - \Omega^Z_{1,2}(\mathbb{A}_\tau, \mathbb{A}_{\tau_1}) - \Omega^Z_{1,2}(\mathbb{A}_\tau, \mathbb{A}_{\tau_1}), \tag{3.71}$$

the assumption (3.70) implies that the cocycles $\Omega^Z_{\mathbb{A}_\tau,1}$ are relative to the subgroup Sc of $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$. Thus, we can resolve the difference to $\mathbf{c}_E \operatorname{Im}\Omega_{\mathrm{BT}}$ more precisely:

$$\Omega^Z_{\mathbb{A}_\tau,1} = \mathbf{c}_E \operatorname{Im}\Omega_{\mathrm{BT}} + \mathbf{h}_Z \operatorname{Im}\Omega_{\mathrm{Rot}} + \delta C_{Z,\tau}. \tag{3.72}$$

Here, $C_{Z,\tau}$ is a 1-cycle on $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ relative to both $\mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$ and Sc. When changing τ, the cocycle is changed by the coboundary in Equation (3.52). By the assumption (3.70), this coboundary is relative to Sc, and thus the constant $\mathbf{h}_Z \in \mathbb{R}$ does not depend on τ. The notion that we define requires that $\mathbf{h}_Z = 0$, and we call it "flat modular invariance" since it relates to the flat metrics in the example of the real determinant line bundle.

Definition 3.7. A local Fr-smooth real one-dimensional modular functor is *flatly modular invariant* if there exists a reparametrization invariant trivialization Z such that

1. $\Omega^Z_{1,2}(\mathbb{A}_{\tau_1}, \mathbb{A}_{\tau_2}) = 0$ for any $\tau_1, \tau_2. > 0$.
2. $\mathbf{h}_Z = 0$ in Equation (3.72),

and every such trivialization has the property that for $A, B \in \mathcal{M}^{\mathsf{geod}}_{0,2}$ the following implication holds,

$$\infty_{1,2}\, A = \infty_{1,2}\, B \qquad \Longrightarrow \qquad \overset{E}{\infty}_{1,2}\, Z(A) = \overset{E}{\infty}_{1,2}\, Z(B). \tag{3.73}$$

We call such a trivialization Z *modular invariant* as well.

Remark 3.8. Infinitesimally in the ℓ_n, $\mathrm{i}\,\ell_n$ basis, the special coboundary $\operatorname{Im}\Omega_{\mathrm{Rot}}$ corresponds to the term which is linear in n for $m = -n$ (as opposed to the n^3 in the Gel'fand–Fuks cocycle. Conceptually, it makes sense to require $\approx_E$ since the same cocycles appear when considering Kähler structures on $\mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$ as in [BR87]. There, it is explained that while one gets a Kähler structure also for $\mathbf{c}_E = 0$ and $\mathbf{h}_E = 1$, the opposite choice makes more sense geometrically.

The cocycles $\Omega^Z_{\mathbb{D},1}$ and $\Omega^Z_{\mathbb{A}_\tau,1}$ are related by the following special case of Equation (3.52) using $\mathbb{A}_\tau\, {}_2\infty_1\, \mathbb{D} = \mathbb{D}$,

$$\begin{aligned} &\Omega^Z_{\mathbb{D},1}(\phi_1, \phi_2) - \Omega^Z_{\mathbb{A}_\tau,1}(\phi_1, \phi_2) \\ &= \Omega^Z_{1,2}(\mathbb{D}, \mathbb{A}_\tau \underset{1}{*} \phi_1) + \Omega^Z_{1,2}(\mathbb{D}, \mathbb{A}_\tau \underset{1}{*} \phi_2) - \Omega^Z_{1,2}(\mathbb{D}, \mathbb{A}_\tau \underset{1}{*} (\phi_1 \circ \phi_2)) - \Omega^Z_{1,2}(\mathbb{D}, \mathbb{A}_\tau). \end{aligned} \tag{3.74}$$

Definition 3.9. A local Fr-smooth real one-dimensional modular functor is *crossing invariant* if there exists a reparametrization invariant trivialization Z of $E(\mathcal{M}_{0,3})$ such that

$$Z(P_1)\, {}_j\infty_k\, Z(P_2) = Z(P_3)\, {}_l\infty_m\, Z(P_4), \tag{3.75}$$

for hyperbolic pairs of pants $P_1, P_2, P_3, P_4 \in \mathcal{M}^{\mathsf{hyp}}_{0,3}$ such that $P_1\, {}_j\infty_k\, P_2 = P_3\, {}_l\infty_m\, P_4 \in \mathcal{M}_{0,4}$ where the left and right hand sides have equal boundary lengths at the seams.

Definition 3.10. A local crossing invariant Fr-smooth real one-dimensional modular functor E is *hyperbolically modular invariant* if any crossing invariant trivialization has the property that

$$\overset{E}{\infty}_{j,k}\, Z(P_1) = \overset{E}{\infty}_{l,m}\, Z(P_2), \tag{3.76}$$

for hyperbolic pairs of pants $P_1, P_2 \in \mathcal{M}^{\mathsf{hyp}}_{0,3}$ such that $\infty_{j,k}\, P_1 = \infty_{l,m}\, P_2 \in \mathcal{M}_{1,1}$ where the left and right hand sides have equal boundary lengths at the seams.

4 Disk-disk cocycles and loop Loewner energy

Consider the special case of the disk-disk cocycle $\Omega^Z_{1,1} : \mathcal{M}_{0,1} \times \mathcal{M}_{0,1} \to \mathbb{R}$ of a local Fr-smooth real one-dimensional modular functor E with respect to a reparametrization invariant trivialization Z over $\mathcal{M}_{0,0}$ and $\mathcal{M}_{0,1}$. If $\mathbf{c}_E \neq 0$, this is the cocycle for the sewing of two disks, and it defines a function

$$\begin{aligned} \mathcal{H} : \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1) &\to \mathbb{R} \\ \phi &\mapsto \frac{2}{\mathbf{c}_E}\, \Omega^Z_{1,1}(\mathbb{D} \underset{1}{*} \phi, \mathbb{D}). \end{aligned} \tag{4.1}$$

Since any disk in $\mathcal{M}_{0,1}$ may be represented in the form $(\mathbb{D}, \mathrm{J} \circ \phi) = \mathbb{D} \underset{1}{*} \phi$ for a diffeomorphism $\phi \in \mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$, the disk-disk cocycle is fully characterized by the restriction of the function $\mathcal{H}$ to $\mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$. By locality, see Proposition 3.6, the restriction of $\mathcal{H}$ to diffeomorphisms has the symmetry

$$\mathcal{H}(\phi) = \mathcal{H}(\mathrm{J} \circ \phi^{-1} \circ \mathrm{J}), \qquad \phi \in \mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)\,. \tag{4.2}$$

This symmetry leads to the critical point of $\mathcal{H}$.

Lemma 4.1. *We have* $(\mathrm{d}_{\mathbb{1}}\,\mathcal{H})(v) = 0$ *for any vector field* $v \in \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$.

Proof. First, for a vector field generating a Möbius transformation, the lemma holds since $\mathbb{D} \underset{1}{*} \phi = \mathbb{D} \in \mathcal{M}_{0,1}$ for any Möbius transformation $\phi \in \mathrm{PSL}(2,\mathbb{C})$. Now, for $n > 1$, consider the vector field ℓ_{-n}. Since it extends holomorphically to $\hat{\mathbb{C}} \setminus \{0\}$, we again have

$$\mathbb{D} \underset{1}{*} \Phi_{\ell_{-n}} = \left[\, \infty \in \,\cdot\,, \Phi_{\ell_n} \,\right] = \mathbb{D} \in \mathcal{M}_{0,1}, \tag{4.3}$$

and thus the lemma holds for $v = \ell_{-n}$. For ℓ_n, start with the tangential vector field $a_n^{\|}$ defined in Equation (1.19). By $\mathbb{R}$-linearity and the case above,

$$(\mathrm{d}_{\mathbb{1}}\,\mathcal{H})(a_n^{\|}) = \frac{1}{2}(\mathrm{d}_{\mathbb{1}}\,\mathcal{H})(\ell_n), \qquad (\mathrm{d}_{\mathbb{1}}\,\mathcal{H})(\mathrm{J}^*\, a_n^{\|}) = \frac{1}{2}(\mathrm{d}_{\mathbb{1}}\,\mathcal{H})(\ell_n). \tag{4.4}$$

By the symmetry (4.2) we also have $(\mathrm{d}_{\mathbb{1}}\,\mathcal{H})(a_n^{\|}) = -(\mathrm{d}_{\mathbb{1}}\,\mathcal{H})(\mathrm{J}^*\, a_n^{\|})$. Both equalities can only hold if $(\mathrm{d}_{\mathbb{1}}\,\mathcal{H})(\ell_{-n}) = 0$.

Since $\mathrm{d}_{\mathbb{1}}\,\mathcal{H}$ is only $\mathbb{R}$-linear, we also have to treat the case of the vector fields $\mathrm{i}\,\ell_n$ and $\mathrm{i}\,\ell_{-n}$ separately. However, the argument for ℓ_n does not work for $\mathrm{i}\,\ell_n$ since the tangential vector $b_n^{\|}$ does not have a sign difference between $\mathrm{i}\,\ell_n$ and $\mathrm{i}\,\ell_{-n}$. It is useful to introduce the rotation $R_\alpha(z) = e^{\mathrm{i}\,\alpha} z$ by $\alpha \in [0, 2\pi)$, which restricted to $z \in S^1$ is an element of $\mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$. Conjugating ℓ_n by a rotation yields

$$R_\alpha^* \ell_n = -(e^{\mathrm{i}\,\alpha} z)^{n+1} e^{-\mathrm{i}\,\alpha} \partial_z = e^{\mathrm{i}\,n\alpha} \ell_n. \tag{4.5}$$

Thus, for $n \neq 0$ and $\alpha_n = \frac{\pi}{2n}$, we have $R_{\alpha_n}^* \ell_n = \mathrm{i}\,\ell_n$. Considering the identities

$$\mathrm{J} \circ R_\alpha \circ \mathrm{J} = R_{-\alpha}, \qquad R_\alpha^{-1} = R_{-\alpha}, \qquad \mathbb{D} \underset{1}{*} R_\alpha = \mathbb{D} \in \mathcal{M}_{0,1}, \tag{4.6}$$

and reparametrization invariance (3.63) applied to $\Omega^Z_{1,1}$, the function $\mathcal{H}$ is invariant under both pre- and postcomposition by rotations at any $\phi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$. That is,

$$\mathcal{H}(R_\alpha \circ \phi \circ R_\beta) = \frac{2}{\mathbf{c}_E}\, \Omega^Z_{1,1}(\mathbb{D} \underset{1}{*} (R_\alpha \circ \phi), \mathbb{D} \underset{1}{*} (\mathrm{J} \circ R_\beta^{-1} \circ \mathrm{J})) = \mathcal{H}(\phi), \qquad \alpha, \beta \in [0, 2\pi). \tag{4.7}$$

Returning to the variation, we find

$$(\mathrm{d}_{\mathbb{1}}\,\mathcal{H})(\mathrm{i}\,\ell_n) = (\mathrm{d}_{\mathbb{1}}\,\mathcal{H})(R_{\alpha_n}^* \ell_n) = (\mathrm{d}_{\mathbb{1}}\,\mathcal{H})([R_{\alpha_n} \circ \Phi_{\ell_n} \circ R_{-\alpha_n}]_\sim) = (\mathrm{d}_{\mathbb{1}}\,\mathcal{H})(\ell_n) = 0 \tag{4.8}$$

and analogously for $\mathrm{i}\,\ell_{-n}$. □

At other $\phi \in \mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$, the differential $\mathrm{d}_\phi\,\mathcal{H}$ may be identified with a certain Fr-smooth functional on $\mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ through conformal welding (Proposition 1.13). This functional has an integral representation by a quadratic differential through the Cauchy–Hilbert transform (Proposition 1.5). The main result of this section is that the quadratic differential turns out to be the Schwarzian derivative of a factor in the conformal welding decomposition. This agrees with the variational formula for the universal Liouville action, or loop Loewner energy I^{L} [TT06, Wan19, PWW25],

$$\mathrm{I}^{\mathrm{L}}(\gamma) = -\frac{2}{\pi}\,\mathrm{Im} \int_\gamma v\, \mathcal{S}[(\mathrm{J} \circ \zeta_2 \circ \mathrm{J})^{-1}], \tag{4.9}$$

where ζ_2 is as in the conformal welding for the analytical loop γ; see Proposition 1.13. Therefore, we state our result as follows.

Theorem 4.2. *Up to a constant depending on E and Z, but not on ϕ, we have*

$$12\mathcal{H}(\phi) = \mathrm{I}^{\mathrm{L}}(\phi) + (\text{const.}), \qquad \phi \in \mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)\,. \tag{4.10}$$

Proof. Let $[\gamma]_\sim \in T_\phi \operatorname{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$ be a tangent vector such that $\gamma(0,z) = \phi(z)$. In the conformal welding decompositions (see Proposition 1.13),

$$\phi = (\mathrm{J} \circ \zeta_2 \circ \mathrm{J})^{-1} \circ \zeta_1, \qquad \gamma(t,\,\cdot\,) = \Big(\mathrm{J} \circ \eta_2(t,\,\cdot\,) \circ \mathrm{J}\Big)^{-1} \circ \eta_1(t,\,\cdot\,) \tag{4.11}$$

the complex deformations ζ_1 and ζ_2 and also the time-dependent complex deformations η_1 and η_2 are conformal on $\bar{\mathbb{D}}$. By Proposition 1.9, η_1 defines a time-dependent vector field whose flow yields back η_1. Denote by $v \in \operatorname{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ this vector field at the time where the flow equals η_1. It has the property $P^- v = 0$. The variation of $\mathcal{H}$ by the tangent vector $[\gamma]_\sim$ then is a function $\frac{2}{\mathbf{c}_E}\,\Theta_\phi(v) = (\mathrm{d}_\phi\,\mathcal{H})([\gamma]_\sim)$ of the vector field v, which we extend to $\operatorname{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ by

$$\Theta_\phi(v) = \frac{\partial}{\partial t}\Big|_{t=0} \Omega^Z_{1,1}(\mathbb{D} \mathbin{\underset{1}{*}} (\Phi_v \circ \zeta_1), \mathbb{D}), \qquad v \in \operatorname{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1), \tag{4.12}$$

such that it vanishes if $v = P^- v$. By diffeomorphism invariance, the same as used for the symmetry (4.2), we may also express it in terms of ζ_2 as

$$\begin{aligned}
\Theta_\phi(v) &= \frac{\partial}{\partial t}\Big|_{t=0} \Omega^Z_{1,1}(\mathbb{D} \mathbin{\underset{1}{*}} \Phi_v \circ (\mathrm{J} \circ \zeta_2 \circ \mathrm{J}) \circ \phi), \mathbb{D}) \\
&= \frac{\partial}{\partial t}\Big|_{t=0} \Omega^Z_{1,1}(\mathbb{D} \mathbin{\underset{1}{*}} \Phi_v \circ (\mathrm{J} \circ \zeta_2 \circ \mathrm{J}), \mathbb{D} \mathbin{\underset{1}{*}} \mathrm{J} \circ \phi^{-1} \mathrm{J})) \\
&= \frac{\partial}{\partial t}\Big|_{t=0} \Omega^Z_{1,1}(\mathbb{D} \mathbin{\underset{1}{*}} \Phi_v \circ (\mathrm{J} \circ \zeta_2 \circ \mathrm{J}), \mathbb{D} \mathbin{\underset{1}{*}} \zeta_2))
\end{aligned} \tag{4.13}$$

If v is in $\mathfrak{sl}(2,\mathbb{C})$, the flow Φ_v is a Möbius transformation and thus $\mathbb{D} \mathbin{\underset{1}{*}} \Phi_v = \mathbb{D}$ and $\mathbb{D} \mathbin{\underset{1}{*}} \Phi_{\mathrm{J}^* v} = \mathbb{D}$ in $\mathcal{M}_{0,1}$. Then, the variation Θ_ϕ vanishes on these vector fields. Similarly, for Möbius transformations, we have by Lemma 4.1 that

$$\Theta_\phi(v) = \Theta_{\mathbb{1}}(\phi^* v) = 0, \qquad \phi \in \mathrm{PSL}(2,\mathbb{C}), v \in \operatorname{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)\,. \tag{4.14}$$

Thus, variation of $\mathcal{H}$ is fully determined by the subset of vector fields $v \in \operatorname{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ such that $v = P^+ v$.

By Fr-smoothness of $\Omega^Z_{1,1}$ and the action of $\operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1)$ on $\mathcal{M}_{0,1}$, the $\mathbb{R}$-linear functional $\Theta_\phi : \operatorname{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1) \to \mathbb{R}$ is Fr-smooth. By Proposition 1.5 has a unique integral representation

$$\Theta_\phi(v) = \operatorname{Re}\left(\int_{(1-\varepsilon)S^1} P^- v\, \rho^+_\phi + \int_{(1+\varepsilon)S^1} P^+ v\, \rho^-_\phi\right), \tag{4.15}$$

where ρ^+_ϕ and ρ^-_ϕ are holomorphic quadratic differentials respectively on $\mathbb{D}$ and $\hat{\mathbb{C}} \setminus \mathbb{D}$, such that $\rho^-_\phi(\infty) = 0$. However, we already know that $\rho^+_\phi = 0$. By Equation (4.14), we have $\rho^-_\phi = 0$ for $\phi \in \mathrm{PSL}(2,\mathbb{C})$.

Now, we compute ρ^-_ϕ. Let $\phi_1, \phi_2 \in \operatorname{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$ be diffeomorphisms with conformal welding decompositions

$$\phi_1 = (\mathrm{J} \circ \zeta_2 \circ \mathrm{J})^{-1} \circ \zeta_1|_{S^1}, \qquad \phi_2 = (\mathrm{J} \circ \xi_2 \circ \mathrm{J})^{-1} \circ \xi_1|_{S^1}. \tag{4.16}$$

By changing the normalization of the complex deformations $\zeta_1, \zeta_2, \xi_1, \xi_2 \in \operatorname{Def}_{\mathbb{C}}(\mathrm{S}^1)$, we can let ξ_2 map $\bar{\mathbb{D}}$ into $\bar{\mathbb{D}}$ such that it becomes composable with ζ_2. Let the complex deformation $\zeta_2 \circ \xi_2 : \bar{\mathbb{D}} \to \hat{\mathbb{C}}$ define a diffeomorphism $\phi_3 \in \operatorname{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$ via conformal welding. The functional from (4.13) in terms $\zeta_2 \circ \xi_2$ of becomes

$$\Theta_{\phi_3}(v) = \frac{\partial}{\partial t}\Big|_{t=0} \Omega^Z_{1,1}\Big(\mathbb{D} \mathbin{\underset{1}{*}} (\Phi_v \circ \mathrm{J} \circ \zeta_2 \circ \xi_2 \circ \mathrm{J}), \mathbb{D} \mathbin{\underset{1}{*}} (\zeta_2 \circ \xi_2)\Big). \tag{4.17}$$

We apply the disk-deformation-disk cocycle identity, the line in Equation (3.57), to the deformation ξ_2 and the second surface $\mathbb{D} \mathbin{\underset{1}{*}} \zeta_2$ in $\Theta_{\phi_3}(v)$ at the second argument of the cocycle,

$$\begin{aligned}\Theta_{\phi_3}(v) = \frac{\partial}{\partial t}\bigg|_{t=0} \Bigg(& - \Omega^Z_{\mathbb{D},1,1}\Big(\mathbb{D} \mathbin{\underset{1}{*}} \zeta_2, \xi_2\Big) \\ & + \Omega^Z_{1,1}\Big(\big(\mathbb{D} \mathbin{\underset{1}{*}} (\Phi_v \circ \mathrm{J} \circ \zeta_2 \circ \xi_2 \circ \mathrm{J})\big) \mathbin{\underset{1}{*}} (\mathrm{J} \circ \xi_2 \circ \mathrm{J})^{-1}, \mathbb{D} \mathbin{\underset{1}{*}} \zeta_2\Big) \\ & + \Omega^Z_{\mathbb{D},1,1}\Big(\big(\mathbb{D} \mathbin{\underset{1}{*}} (\Phi_v \circ \mathrm{J} \circ \zeta_2 \circ \xi_2 \circ \mathrm{J})\big), (\mathrm{J} \circ \xi_2 \circ \mathrm{J})^{-1}\Big)\Bigg).\end{aligned} \tag{4.18}$$

Note that the first term is time-independent, and the second term is just $\Theta_{\phi_1}(v)$. For the third term, apply the same cocycle identity to $\mathbb{D}$ and ξ_1 in $\Theta_{\phi_2}((\mathrm{J} \circ \zeta_2 \circ \mathrm{J})^* v)$,

$$\begin{aligned}\Theta_{\phi_2}(\zeta_1^* \, \mathrm{J}^* \, v) = \frac{\partial}{\partial t}\bigg|_{t=0} \Bigg(& - \Omega^Z_{\mathbb{D},1,1}\Big(\mathbb{D}, \xi_2\Big) \\ & + \Omega^Z_{1,1}\Big(\big(\mathbb{D} \mathbin{\underset{1}{*}} (\Phi_{(\mathrm{J} \circ \zeta_2 \circ \mathrm{J})^* v} \circ \mathrm{J} \circ \xi_2 \circ \mathrm{J})\big) \mathbin{\underset{1}{*}} (\mathrm{J} \circ \xi_2 \circ \mathrm{J})^{-1}, \mathbb{D}\Big) \\ & + \Omega^Z_{\mathbb{D},1,1}\Big(\mathbb{D} \mathbin{\underset{1}{*}} (\Phi_{(\mathrm{J} \circ \zeta_2 \circ \mathrm{J})^* v} \circ \mathrm{J} \circ \xi_2 \circ \mathrm{J}), (\mathrm{J} \circ \xi_2 \circ \mathrm{J})^{-1}\Big)\Bigg).\end{aligned} \tag{4.19}$$

Again, the first term is time-independent. The second term is the variation at the identity $\Omega^Z_{1,1}(\mathbb{D} \mathbin{\underset{1}{*}} \Phi_{\zeta_2^* \, \mathrm{J}^* \, v}, \mathbb{D})$, which vanishes by Lemma 4.1, and the third term agrees with the third term in (4.18) since the deformation to the left of Φ_v in $\Phi_{(\mathrm{J} \circ \zeta_2 \circ \mathrm{J})^* v} = \mathrm{J} \circ \zeta_2^{-1} \, \mathrm{J} \circ \Phi_v \circ \mathrm{J} \circ \zeta_2 \circ \mathrm{J}$, see Equation (1.67), can be absorbed by the unit disk. In summary, we have shown that

$$\Theta_{\phi_3}(v) = \Theta_{\phi_2}((\mathrm{J} \circ \zeta_2 \circ \mathrm{J})^* v) + \Theta_{\phi_1}(v). \tag{4.20}$$

For the quadratic differential, this implies that

$$\rho^-_{\phi_3} = (\mathrm{J} \circ \zeta_2 \circ \mathrm{J})_* \rho^-_{\phi_2} + \rho^-_{\phi_1}. \tag{4.21}$$

This equation together with the $\mathrm{PSL}(2, \mathbb{C})$ invariance characterizes ρ^-_ϕ as a scalar multiple of the Schwarzian derivative $\mathcal{S}[(\mathrm{J} \circ \zeta_2 \circ \mathrm{J})^{-1}]$. Note that the quadratic differential $\mathcal{S}[(\mathrm{J} \circ \zeta_2 \circ \mathrm{J})^{-1}]$ is holomorphic in a neighborhood of $\hat{\mathbb{C}} \setminus \mathbb{D}$. Hence, the integrals in (4.15) may just be taken over S^1, that is, we may set $\varepsilon = 0$. What remains is to compute this constant $C \in \mathbb{C}$ such that

$$\mathrm{d}_\phi \, \mathcal{H}([\gamma_\sim]) = \frac{2}{\mathbf{c}_E} \, \Theta_\phi(v) = \operatorname{Re} C \int_{S^1} v \, \mathcal{S}[(\mathrm{J} \circ \zeta_2 \circ \mathrm{J})^{-1}], \qquad v \in \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)\,. \tag{4.22}$$

We compute the variation of $\Theta_\phi(v)$ for a tangent vector at $\phi = \mathbb{1} \in \mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$. It is described by a curve $\gamma \in \mathcal{C}(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ where in the conformal welding we have $[(\mathrm{J} \circ \zeta_2 \circ \mathrm{J})^{-1}]_\sim = [\Phi_w]_\sim$ for a vector field $w \in \mathrm{Vect}^{\mathrm{an}}_{\mathbb{C}}(\mathrm{S}^1)$ such that $w = P^- w$.

$$\begin{aligned}\frac{\partial}{\partial t}\bigg|_{s=0} \frac{2}{\mathbf{c}_E} \, \Theta_{\gamma(s,\,\cdot\,)}(v) &= \frac{\partial}{\partial s}\bigg|_{s=0} \operatorname{Re} C \int_{S^1} v \, \mathcal{S}[\Phi_w(s, \,\cdot\,)] \\ &= \operatorname{Re} C \int_{S^1} v(z) \, w'''(z) \, \mathrm{d}z \\ &= -\operatorname{Re} C \int_{S^1} v'(z) \, w''(z) \, \mathrm{d}z\end{aligned} \tag{4.23}$$

Note that for $C \in \mathrm{i}\,\mathbb{R}$ this is proportional to the cocycle $\operatorname{Im} \omega_{\mathrm{GF}}(v, w)$ defined in Equation (2.31) up to a coboundary, that is,

$$\frac{\partial}{\partial t}\bigg|_{s=0} \Theta_{\gamma(s,\,\cdot\,)}(v) = -12\pi \, \mathbf{c}_E \operatorname{Re} C \, \omega_{\mathrm{GF}}(v, w) + (\text{coboundaries}) \tag{4.24}$$

On the other hand, by Equation (4.13), we have

$$
\begin{aligned}
\frac{\partial}{\partial s}\Big|_{s=0} \Theta_{\gamma(s,\,\cdot\,)}(v) &= \frac{\partial}{\partial s}\Big|_{s=0} \frac{\partial}{\partial t}\Big|_{t=0} \Omega^E_{1,1}\Big(\mathbb{D} \underset{1}{*} (\Phi_v(t,\,\cdot\,) \circ \Phi_{-w}(s,\,\cdot\,)), \mathbb{D} \underset{1}{*} \Phi_{-\,\mathrm{J}^*\,w}(s,\,\cdot\,)\Big) \\
&= \frac{\partial}{\partial s}\Big|_{s=0} \frac{\partial}{\partial t}\Big|_{t=0} \Bigg(\Omega^E_{1,1}\Big(\mathbb{D} \underset{1}{*} (\Phi_v(t,\,\cdot\,) \circ \Phi_{-w}(s,\,\cdot\,)), \mathbb{D}\Big) \\
&\qquad + \Omega^E_{1,1}\Big(\mathbb{D} \underset{1}{*} \Phi_v(t,\,\cdot\,), \mathbb{D} \underset{1}{*} \Phi_{-\,\mathrm{J}^*\,w}(s,\,\cdot\,)\Big)\Bigg)
\end{aligned} \tag{4.25}
$$

To the term in the last line we can apply the disk-deformation-deformation-disk cocycle identity obtained from combining equations in (3.57) and (3.58),

$$
\begin{aligned}
\Omega^E_{1,1}\Big(\mathbb{D} \underset{1}{*} \Phi_v(t,\,\cdot\,), \mathbb{D} \underset{1}{*} \Phi_{-\,\mathrm{J}^*\,w}(s,\,\cdot\,)\Big) &= -\,\Omega^E_{\mathbb{D},1,1}\Big(\mathbb{D}, \Phi_v\Big) - \Omega^E_{\mathbb{D},1,1}\Big(\mathbb{D}, \Phi_{-\,\mathrm{J}^*\,w}\Big) \\
&\quad + \Omega^E_{\mathbb{D},1}\Big(\Phi_v, \Phi_w\Big) + \Omega^E_{\mathbb{D},1,1}\Big(\mathbb{D}, \Phi_v \circ \Phi_w\Big) \\
&\quad + \Omega^E_{1,1}\Big(\mathbb{D} \underset{1}{*} (\Phi_v \circ \Phi_w), \mathbb{D}\Big),
\end{aligned} \tag{4.26}
$$

where we take the section on $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ with respect to the surface $\mathbb{D}$, such that the function $\Omega_{\mathbb{D},1,1}(\mathbb{D},\,\cdot\,)$ on $\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$ is zero (see Equation (3.56)). The latter causes the first, second and fourth terms on the right-hand side to vanish. Since $[\Phi_{-w}]_\sim = -[\Phi_w]_\sim$, the last term in (4.26) cancels with the term in the second line of (4.25). This leaves us with

$$
\frac{\partial}{\partial s}\Big|_{s=0} \Theta_{\gamma(s,\,\cdot\,)}(v) = \frac{\partial}{\partial s}\Big|_{s=0} \frac{\partial}{\partial t}\Big|_{t=0} \Omega^E_{\mathbb{D},1}\Big(\Phi_v(t,\,\cdot\,), \Phi_w(s,\,\cdot\,)\Big). \tag{4.27}
$$

By repeating all of the computations so far with the first vector field associated to η_2 in Equation (4.11) instead, we also find that

$$
\frac{\partial}{\partial s}\Big|_{s=0} \Theta_{\gamma(s,\,\cdot\,)}(v) = -\frac{\partial}{\partial s}\Big|_{s=0} \frac{\partial}{\partial t}\Big|_{t=0} \Omega^E_{\mathbb{D},1}\Big(\Phi_w(s,\,\cdot\,), \Phi_v(t,\,\cdot\,)\Big). \tag{4.28}
$$

for v and w the same vector fields as in Equation (4.27). Thus, by the defintion of the central charge of E in Equation (3.69), we find

$$
\begin{aligned}
&\frac{\partial}{\partial s}\Big|_{s=0} \Theta_{\gamma(s,\,\cdot\,)}(v) \\
&= \frac{\partial}{\partial s}\Big|_{s=0} \frac{\partial}{\partial t}\Big|_{t=0} \Bigg(\Omega^E_{\mathbb{D},1}\Big(\Phi_v(t,\,\cdot\,), \Phi_w(s,\,\cdot\,)\Big) - \Omega^E_{\mathbb{D},1}\Big(\Phi_w(s,\,\cdot\,), \Phi_v(t,\,\cdot\,)\Big)\Bigg) \\
&= 2\Big(\mathbf{c}_E \operatorname{Im} \omega_{\mathrm{GF}}(\Phi_v, \Phi_w) + \delta\,\beta(\Phi_v, \Phi_w)\Big),
\end{aligned} \tag{4.29}
$$

where the factor of 2 arises from the differentiation of group-level cocycles as in Equation (2.27). By linear independence of the Gel'fand–Fuks cocycle ω_{GF} from the coboundaries, we find that $C = \frac{\mathrm{i}}{6\pi}$ by comparing to Equation (4.24). With this constant, Equation (4.22) yields

$$
12\,\mathrm{d}_\phi\,\mathcal{H}([\gamma_\sim]) = -\frac{2}{\pi} \operatorname{Im} \int_{S^1} v\,\mathcal{S}[(\mathrm{J} \circ \zeta_2 \circ \mathrm{J})^{-1}] = \frac{1}{2}\,\mathrm{d}_\phi\,\mathrm{I}^{\mathrm{L}}(\phi)([\gamma_\sim]). \tag{4.30}
$$

□

5 Isomorphisms of real one-dimensional modular functors

Let E and D be Fr-smooth real one-dimensional modular functors with equal central charge $\mathbf{c}_E = \mathbf{c}_D \in \mathbb{R}$. In this section, we construct — under further assumptions — an isomorphism between E and D according to Definition 3.3, proving the main result of this work, Theorem 5.1. The strategy is to find trivializations Z and W, respectively of E and D, such that all cocycles defined in Section 3.4 agree. Then, the isomorphism is given by

$$\begin{aligned} \Psi_{\mathsf{g},\mathsf{b}} : E(\mathcal{M}_{\mathsf{g},\mathsf{b}}) &\to D(\mathcal{M}_{\mathsf{g},\mathsf{b}}), \\ Z(\Sigma) &\mapsto W(\Sigma). \end{aligned} \tag{5.1}$$

We construct the components $\Psi_{\mathsf{g},\mathsf{b}}$ one by one. With each new case, there are new cocycles to be considered, for each of which we show that they agree for Z and W before moving to the next case.

Preliminarily, we consider an induction step. Given that the isomorphism $\Psi_{\mathsf{g},\mathsf{b}}$ is defined up to a certain number of boundary components at a certain genus, and for all boundary components for lower genera, we construct the isomorphism for one more boundary component in that genus. However, this induction step requires that the new $\Psi_{\mathsf{g},\mathsf{b}}$ has $\mathsf{b} \geq 2$, that is, the cases $\mathsf{b} = 0$ and $\mathsf{b} = 1$ for each next higher genus need to be considered separately. In fact, we apply this inductive procedure only for $\mathsf{g} = 0$ and $\mathsf{g} = 1$. The induction step itself assumes that E and D are local (Definition 3.2).

First, we consider the cases of the sphere and disks. Using that $\mathbf{c}_E = \mathbf{c}_D$ and locality, Theorem 4.2 defines $\Psi_{0,0}$ and $\Psi_{0,1}$. This suffices to get an isomorphism for all genus 0 surfaces using the induction step.

For genus 1, we first define a new trivialization X on annuli, which is flatly modular invariant, assuming the respective notion in Definition 3.7 for both E and D. We let X agree with Z on $\mathcal{M}_{0,0}$ and $\mathcal{M}_{0,1}$. Since $\Psi_{0,2}$ is already an isomorphism, $Y(A) = \Psi_{0,2}(X(A))$ is also flatly modular invariant. Then, both have the necessary properties to define trivializations on tori and handles (tori with one boundary component) such that $\Psi_{1,0}$ and $\Psi_{1,1}$ are isomorphisms. The trivializations on handles are defined as in the induction step (note that Proposition 5.4 only requires $\mathsf{b} \geq 1$), and equality of the cocycles follows the case of tori.

Finally, our construction of the isomorphism for genus $\mathsf{g} \geq 2$ assumes crossing invariance (Definition 3.9) and hyperbolic modular invariance (Definition 3.10) for both E and D. We extend the new trivializations X and Y to pairs of pants using crossing invariance. Since crossing invariance and hyperbolic modular invariance respectively correspond to the invariance under A- and S-moves between pants decompositions of hyperbolic surfaces, the new trivializations extend canonically to any hyperbolic surface in terms of pants decompositions. The equality of cocycles then follows by reduction to the genus 0 and 1 cases, extending the isomorphisms Ψ to any genus and number of boundary components.

This proves the main theorem of this work.

Theorem 5.1 (Universal property of real one-dimensional modular functors)**.** *Let E and D be local, flatly modular invariant, crossing invariant, and hyperbolically modular invariant Fr-smooth real one-dimensional modular functors with equal central charges $\mathbf{c}_E = \mathbf{c}_D$. Then, there exists a Fr-smooth isomorphism of real one-dimensional modular functors $\Psi : E \to D$.*

Only considering the construction for genus 0 where modular invariance was not used, we obtain a weaker result.

Corollary 5.2. *Let E and D be local Fr-smooth real one-dimensional modular functors with equal central charges $\mathbf{c}_E = \mathbf{c}_D$. Then, there exists a Fr-smooth isomorphism of real one-dimensional modular functors $\Psi : E \to D$ in genus 0.*

Taking only flat modular invariance into account, we can only go up to genus 1:

Corollary 5.3. *Let E and D be local and flatly modular invariant Fr-smooth real one-dimensional modular functors with equal central charges $\mathbf{c}_E = \mathbf{c}_D$. Then, there exists a Fr-smooth isomorphism of real one-dimensional modular functors $\Psi : E \to D$ up to genus 1.*

5.1 Induction step on the number of boundary components

To extend an isomorphism Ψ to moduli spaces $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ with a higher number of boundary components $\mathsf{b} \geq 2$ and any genus $\mathsf{g} \geq 0$, we proceed by induction — keeping the genus fixed, and increasing the number of boundary components by one.

The construction of the new isomorphism $\Psi_{\mathsf{g},\mathsf{b}}$ involves choosing particular trivializations of E and D. Since we are looking to define reparametrization invariant trivializations, we let them be determined by the pullback of trivializations over $\check{\mathcal{M}}_{\mathsf{g},\mathsf{b}}$, the finite-dimensional moduli spaces without boundary parametrizations. Any such surface in $\check{\mathcal{M}}_{\mathsf{g},\mathsf{b}}$ may be lifted to $\mathcal{M}^{\mathsf{Möb}}_{\mathsf{g},\mathsf{b}}$, the moduli spaces of surfaces with Möbius boundary paramerizations. We define trivializations over $\mathcal{M}^{\mathsf{Möb}}_{\mathsf{g},\mathsf{b}}$ and let them project to $\check{\mathcal{M}}_{\mathsf{g},\mathsf{b}}$, and thereafter, pull them back to $\mathcal{M}_{\mathsf{g},\mathsf{b}}$. However, in this process, we will have to show invariance under Möbius reparametrizations. While it does not seem essential how exactly these trivializations are chosen, it is important that they are chosen in the same manner for E and D. We make this choice as follows.

Proposition 5.4. *Let Z be a reparametrization invariant trivialization of a local real one-dimensional modular functor E over $\mathcal{M}_{0,1}$, and $\mathcal{M}_{\mathsf{g}-1,\mathsf{b}}$ for $\mathsf{b} \geq 1$. Then, the trivialization over $\Sigma \in \mathcal{M}^{\mathsf{Möb}}_{\mathsf{g},\mathsf{b}}$ defined by solving the linear equations*

$$Z(\Sigma) \,{}_1\overset{E}{\infty}_1\, Z(\mathbb{D}) = Z(\Sigma \,{}_1\infty_1\, \mathbb{D}), \tag{5.2}$$

yields a well-defined parametrization invariant trivialization over $\mathcal{M}_{\mathsf{g},\mathsf{b}}$.

Proof. The trivializations over $\Sigma \in \mathcal{M}^{\mathsf{Möb}}_{\mathsf{g},\mathsf{b}}$ are defined by solving the linear equations (5.2), singling out $\partial_1\Sigma$ by attaching a disk to that boundary component, which reduces the number of boundary components by 1. Note that here we require $\mathsf{b} \geq 1$. These linear equations (5.2) are equivalent to $\Omega^Z_{1,1}(\Sigma, \mathbb{D}) = 0$. Invariance under Möbius transformations $\phi \in \mathrm{PSL}(2, \mathbb{R})$ acting as reparametrizations is equivalent to $\Omega^Z_{\mathbb{D},1,j}(\Sigma, \phi)$ vanishing for any $1 \leq j \leq \mathsf{b}$. For $j \neq 1$, consider the surface-deformation-disk cocycle identity

$$\Omega^Z_{1,1}(\Sigma \underset{j}{*} \phi, \mathbb{D}) + \Omega^Z_{\mathbb{D},1,j}(\Sigma, \phi) = \Omega^Z_{1,1}(\Sigma, \mathbb{D}) + \Omega^Z_{\mathbb{D},1,j}(\Sigma \,{}_1\infty_1\, \mathbb{D}, \phi). \tag{5.3}$$

The first term on both sides vanishes by the linear equation (5.2) for the Möbius surfaces $\Sigma \underset{j}{*} \phi$ and Σ respectively. The second term on the right-hand side vanishes since by the induction hypothesis we have reparametrization invariance for any surface with fewer boundary components, such as $\Sigma \,{}_1\infty_1\, \mathbb{D} \in \mathcal{M}_{\mathsf{g},\mathsf{b}-1}$, see also Equation (3.60). Therefore, also $\Omega^Z_{\mathbb{D},1,j}(\Sigma, \phi) = 0$. For $j = 1$, we likewise have a cocycle identity

$$\Omega^Z_{1,1}(\Sigma \underset{1}{*} \phi, \mathbb{D}) + \Omega^Z_{\mathbb{D},1,1}(\Sigma, \phi) = \Omega^Z_{\mathbb{D},1,1}(\mathbb{D}, (\mathrm{J} \circ \phi^{-1} \circ \mathrm{J})) + \Omega^Z_{1,1}(\Sigma, \mathbb{D} \underset{1}{*} (\mathrm{J} \circ \phi^{-1} \circ \mathrm{J})). \tag{5.4}$$

Here, the first term vanishes again by (5.2), and the first term on the right-hand side vanishes by reparametrization invariance (3.60). For the second term on the right-hand side, observe that $\mathbb{D} \underset{1}{*} (\mathrm{J} \circ \phi^{-1} \circ \mathrm{J}) = \mathbb{D}$ since ϕ is a Möbius transformation, and then it vanishes by (5.2) as well. □

For the induction step, the concrete assumption is that the isomorphism Ψ is defined in terms of trivializations Z and W for all surfaces with any number of boundary components for genus $< \mathsf{g}$. For genus g surfaces, we assume that Ψ is only defined for surfaces with $< \mathsf{b}$ boundary components. Then, we also assume that $\Omega^Z_{j,k}(\Sigma_1, \Sigma_2) = \Omega^W_{j,k}(\Sigma_1, \Sigma_2)$ for for all surfaces $\Sigma_1 \in \mathcal{M}_{\mathsf{g}_1,\mathsf{b}_1}$ and $\Sigma_2 \in \mathcal{M}_{\mathsf{g}_2,\mathsf{b}_2}$ such that either $\mathsf{g}_1 + \mathsf{g}_2 < \mathsf{g}$ and b_1, b_2 arbitrary, or $\mathsf{g}_1 + \mathsf{g}_2 = \mathsf{g}$ and $\mathsf{b}_1 + \mathsf{b}_2 - 2 < \mathsf{b}$. Moreover, we assume for the self-sewing cocycles that $\Omega^Z_{j,k}(\Sigma_1) = \Omega^W_{j,k}(\Sigma_1)$ for $\Sigma_1 \in \mathcal{M}_{\mathsf{b}_1,\mathsf{g}_1}$ such that either $\mathsf{g}_1 + 1 < \mathsf{g}$ and b_1 arbitrary, or $\mathsf{g}_1 + 1 = \mathsf{g}$ and $\mathsf{b}_1 - 2 < \mathsf{b}$.

Proposition 5.5. *Let $\Psi : E \to D$ be an isomorphism of local real one-dimensional modular functors up to genus $\mathsf{g} \geq 0$ and (not including) $\mathsf{b} \geq 2$ boundary components, and let Z and W respectively be trivializations up to genus g and (not including) $\mathsf{b} \geq 2$ of E and D such that $\Psi \circ Z = W$. Then, the reparametrization invariant extensions of both trivializations Z and W to $\mathcal{M}_{\mathsf{g},\mathsf{b}}$ in Proposition 5.4 induce an extension of the isomorphism Ψ to b boundary components by defining $\Psi_{\mathsf{g},\mathsf{b}}(Z(\Sigma)) = W(\Sigma)$ for $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$.*

Proof. We check that all new cocycles involving surfaces with one additional boundary component agree for Z and W. The first case to consider is $\Omega^Z_{1,1}(\Sigma, D)$ for $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$ and $D \in \mathcal{M}_{0,1}$. The surface Σ is related to a Möbius surface $\check{\Sigma} \in \mathcal{M}^{\mathsf{Möb}}_{\mathsf{g},\mathsf{b}}$ by $\Sigma = \hat{\Sigma} \underset{1}{*} \phi_1 \cdots \underset{\mathsf{b}}{*} \phi_{\mathsf{b}}$ for diffeomorphisms $\phi_1, \ldots, \phi_{\mathsf{b}} \in \mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$. By diffeomorphism invariance, we only have to consider the case $\Sigma = \check{\Sigma} \underset{1}{*} \phi$ for $\phi \in \mathrm{Diff}^{\mathrm{an}}_+(\mathrm{S}^1)$ and $D = \mathbb{D}$. Then, we apply the surface-disk-disk cocycle identity attaching a unit disk at the second boundary component (this is why we assume $\mathsf{b} \geq 2$),

$$\Omega^Z_{1,1}(\check{\Sigma} \underset{1}{*} \phi, \mathbb{D}) + \Omega^Z_{2,1}(\check{\Sigma} \underset{1}{*} \phi \,{}_1\!\infty_1\, \mathbb{D}, \mathbb{D}) = \Omega^Z_{2,1}(\check{\Sigma} \underset{1}{*} \phi, \mathbb{D}) + \Omega^Z_{1,1}(\check{\Sigma} \underset{1}{*} \phi \,{}_2\!\infty_1\, \mathbb{D}, \mathbb{D}). \tag{5.5}$$

The second term on both sides agrees with the respective cocycle for W by the induction hypothesis, since the surface has one less boundary component. Thus, $\Omega^Z_{1,1}(\check{\Sigma} \underset{1}{*} \phi, \mathbb{D})$ equals the same cocycle of W if and only if this holds for $\Omega^Z_{2,1}(\check{\Sigma} \underset{1}{*} \phi, \mathbb{D}) = \Omega^Z_{2,1}(\check{\Sigma}, \mathbb{D})$. To the latter, apply the same cocycle identity again now with $\phi = \mathbb{1}$. Then, the result follows since $\Omega^Z_{1,1}(\check{\Sigma}, \mathbb{D}) = 0$. Note that by changing the index 2 to any other boundary component and putting back in the other diffeomorphisms, we also have shown that

$$\Omega^Z_{j,1}(\Sigma, D) = \Omega^W_{j,1}(\Sigma, D), \qquad \Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}, D \in \mathcal{M}_{0,1}, 1 \leq j \leq \mathsf{b}. \tag{5.6}$$

Any other new cocycles without self-sewing are of the form $\Omega^Z_{j,k}(\Sigma_1, \Sigma_2)$ and $\Omega^W_{j,k}(\Sigma_1, \Sigma_2)$ for surfaces $\Sigma_1 \in \mathcal{M}_{\mathsf{g}_1,\mathsf{b}_1}$ and $\Sigma_2 \in \mathcal{M}_{\mathsf{g}_2,\mathsf{b}_2}$ with $\mathsf{g}_1 + \mathsf{g}_2 = \mathsf{g}$ and $\mathsf{b}_1 + \mathsf{b}_2 - 2 = \mathsf{b}$. If either of the surfaces is a disk, this case is covered by Equation (5.6) above. Since $\mathsf{b} \geq 2$, either Σ_1 or Σ_2 has at least one more boundary component. Without loss of generality, assume that Σ_1 has another boundary component $l \neq j$. We consider the cocycle identity of attaching a unit disk to this boundary component,

$$\Omega^Z_{j,k}(\Sigma_1, \Sigma_2) + \Omega^Z_{l,1}(\Sigma_1 \,{}_j\!\infty_k\, \Sigma_2, \mathbb{D}) = \Omega^Z_{l,1}(\Sigma_1, \mathbb{D}) + \Omega^Z_{j,k}(\Sigma_1 \,{}_l\!\infty_1\, \mathbb{D}, \Sigma_2). \tag{5.7}$$

Note that all the terms except the first involve surfaces with at least one fewer boundary component in total. Therefore, the agreement of the cocycles follows from the induction hypothesis. Finally, the self-sewing cocycles $\Omega^Z_{j,k}(\Sigma)$ to consider involve $\Sigma \in \mathcal{M}_{\mathsf{g}-1,\mathsf{b}+2}$ since then $\infty_{j,k} \Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$. Let $1 \leq l \leq \mathsf{b}+2$ be a boundary component of Σ other than j or k. Attaching the unit disk at l, we find the following cocycle identity,

$$\Omega^Z_{j,k}(\Sigma) + \Omega^Z_{l,1}(\,\infty_{j,k} \Sigma, \mathbb{D}) = \Omega^Z_{j,k}(\Sigma \,{}_l\!\infty_1\, \mathbb{D}) + \Omega^Z_{l,1}(\Sigma, \mathbb{D}). \tag{5.8}$$

The second term on the right-hand side equals the same cocycle for D by Equation (5.6), and the same holds for the terms on the left-hand side by the induction hypothesis. Hence, $\Omega^Z_{j,k}(\Sigma) = \Omega^W_{j,k}(\Sigma)$, and we have covered all new cocycles. □

5.2 Spheres and disks, and complex deformations

For the case of spheres $\mathcal{M}_{0,0}$, and by reparametrization invariance also for disks $\mathcal{M}_{0,1}$, the trivializations Z and W are unique up to functions on the moduli spaces $\mathcal{M}_{0,0} = \{\hat{\mathbb{C}}\}$ and $\check{\mathcal{M}}_{0,1} = \{\mathbb{D}\}$, that is, up to constants. Thus, at this level, for both E and D, there are two degrees of freedom. We fix one degree of freedom each by normalizing them in the same way, such that

$$\Omega^Z_{1,1}(\mathbb{D}, \mathbb{D}) = \Omega^W_{1,1}(\mathbb{D}, \mathbb{D}) = 0. \tag{5.9}$$

By the assumption that E and D have equal central charge and Theorem 4.2, it follows that the disk-disk cocycles agree up to a constant, which is fixed by the normalization (5.9).

The trivializations on disks define trivializations $Z_{\mathbb{D},1}$ and $W_{\mathbb{D},1}$ of the central extensions $E(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ and $D(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ of complex deformations. By combining the disk-deformation-deformation cocycle

$$\Omega^Z_{\mathbb{D},1}(\mathbb{D}, \phi \circ \psi) + \Omega^Z_{\mathbb{D},1}(\phi, \psi) = \Omega^Z_{\mathbb{D},1}(\mathbb{D} \underset{1}{*} \phi, \psi) + \Omega^Z_{\mathbb{D},1}(\mathbb{D}, \phi), \tag{5.10}$$

where by Equation (3.56) the first and the last terms vanish, and the disk-deformation-disk cocycle of $\mathbb{D} \underset{1}{*} \phi$, ψ, and $\mathbb{D}$, given by

$$\Omega^Z_{\mathbb{D},1}(\mathbb{D} \underset{1}{*} \phi, \psi) + \Omega^Z_{1,1}(\mathbb{D} \underset{1}{*} (\phi \circ \psi), \mathbb{D}) = \Omega^Z_{\mathbb{D},1}(\mathbb{D}, \mathrm{J} \circ \psi^{-1} \circ \mathrm{J}) + \Omega^Z_{1,1}(\mathbb{D} \underset{1}{*} \phi, \mathbb{D} \underset{1}{*} \mathrm{J} \circ \psi^{-1} \circ \mathrm{J}) \tag{5.11}$$

where the third term vanishes, we find that the cocycle $\Omega^Z_{\mathbb{D},1}(\phi, \psi)$ with respect to the composition law of $E(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1))$ is fully expressed in terms of the disk-disk cocycle:

$$\Omega^Z_{\mathbb{D},1}(\phi, \psi) = \Omega^Z_{1,1}(\mathbb{D} \underset{1}{*} \phi, \mathbb{D} \underset{1}{*} \mathrm{J} \circ \psi^{-1} \circ \mathrm{J}) - \Omega^Z(\mathbb{D} \underset{1}{*} (\phi \circ \psi), \mathbb{D}) \tag{5.12}$$

Since the disk-disk cocycle of E agrees with that of D, we find that $\Omega^Z_{\mathbb{D},1}(\phi, \psi) = \Omega^D_{\mathbb{D},1}(\phi, \psi)$ for all $\phi, \psi \in \mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)$, and we have an isomorphism

$$\begin{aligned} \hat{\Psi} : E(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)) &\longrightarrow D(\mathrm{Def}_{\mathbb{C}}(\mathrm{S}^1)) \\ Z^E_{\mathbb{D},1}(\phi) &\longmapsto Z^D_{\mathbb{D},1}(\phi). \end{aligned} \tag{5.13}$$

5.3 Tori and flat modular invariance

To define trivializations on tori, we assume that both E and D are flatly modular invariant. Then, we replace the trivialization Z on annuli with a flatly modular invariant trivialization denoted X. Since Ψ is already an isomorphism in genus 1, the trivialiation

$$Y(A) = \Psi_{0,2}(X(A)), \qquad A \in \mathcal{M}_{0,2}, \tag{5.14}$$

of $D(\mathcal{M}_{0,2})$ is flatly modular invariant as well. For tori $T \in \mathcal{M}_{1,0}$, let $X(T) = \infty^E_{1,2} X(A)$ and $Y(T) = \infty^E_{1,2} Y(A)$ defined by any $A \in \mathcal{M}^{\mathsf{geod}}_{0,2}$ such that $\infty_{1,2} A = T$. The independence of the choice of A is precisely the assumption of flat modular invariance. The resulting cocycles $\Omega^X_{1,2}(A)$ and $\Omega^Y_{1,2}(A)$ for $A \in \mathcal{M}_{0,2}$ have the property that

$$\Omega^X_{1,2}(A) = \Omega^Y_{1,2}(A) = 0, \qquad A \in \mathcal{M}^{\mathsf{geod}}_{0,2}. \tag{5.15}$$

By the following proposition, we have $\Omega^X_{1,2}(A) = \Omega^Y_{1,2}(A)$ for any annuli.

Proposition 5.6. *Extending the isomorphism $\Psi_{0,2}$ on $\mathcal{M}_{0,2}$ to $\Psi_{1,0}$ on $\mathcal{M}_{1,0}$ by sending X to Y makes that the following diagram commutes:*

$$\begin{array}{ccc} E(\mathcal{M}_{0,2}) & \xrightarrow{\Psi_{0,2}} & D(\mathcal{M}_{0,2}) \\ \Big\downarrow \infty^E_{1,2} & & \Big\downarrow \infty^D_{1,2} \\ E(\mathcal{M}_{1,0}) & \xrightarrow{\Psi_{1,0}} & D(\mathcal{M}_{1,0}). \end{array} \tag{5.16}$$

Proof. Let $A, B \in \mathcal{M}_{0,2}$ be annuli such that $\infty_{1,2} A = \infty_{1,2} B$ are equivalent tori. First consider A and B such that the seams in $\infty_{1,2} A = \infty_{1,2} B$ are homotopic and disjoint. In this case, we can decompose A and B such that $A = C \,{}_1\infty_2\, D$ and $B = D \,{}_1\infty_2\, C$ for some $C, D \in \mathcal{M}_{0,2}$ defined by cutting the torus at both seams. By the cocycle identity (3.47) in the case of two annuli and a torus,

$$\Omega^X_{1,2}(C \,{}_1\infty_2\, D) + \Omega^X_{1,2}(C, D) = \Omega^X_{1,2}(D \,{}_1\infty_2\, C) + \Omega^X_{1,2}(D, C), \tag{5.17}$$

and analogously Y. Because the cocycles on annuli for E and D agree, taking the difference between the cocycle identities results in

$$\Omega^X_{1,2}(C\ {}_1\infty_2\ D) - \Omega^Y_{1,2}(C\ {}_1\infty_2\ D) = \Omega^X_{1,2}(D\ {}_1\infty_2\ C) - \Omega^Y_{1,2}(D\ {}_1\infty_2\ C), \tag{5.18}$$

which is equivalent to

$$\Omega^X_{1,2}(A) - \Omega^Y_{1,2}(A) = \Omega^X_{1,2}(B) - \Omega^Y_{1,2}(B). \tag{5.19}$$

Note that the sides of the equation are precisely the factors picked up by mapping from from the fiber over the torus $E({}_1\infty_2 A) = E(\infty_{1,2} B)$ to $D(\infty_{1,2} A) = D({}_1\infty_2 B)$ in Diagram (5.16). This makes the diagram invariant under changing a lift from $E(\mathcal{M}_{1,0})$ to $E(\mathcal{M}_{0,2})$ by homotopic and disjoint seams. We now prove that Equation (5.19) holds for any annuli such that $\infty_{1,2} A = \infty_{1,2} B$.

Using the already known case, we can for any $A \in \mathcal{M}_{0,2}$, find a finite sequence of annuli $A = A_1, A_2, \dots, A_n$ in which we apply Equation (5.19) to the pairs A_j, A_{j+1} ending with A_n having geodesic seam in ${}_1\infty_2\ A$, that is, $A_n \in \mathcal{M}^{\mathsf{geod}}_{0,2}$ as defined in Section 1.5.1. With this method, we can reduce the proof of Equation (5.19) for arbitrary A and B to the case of $A, B \in \mathcal{M}^{\mathsf{geod}}_{0,2}$. By the assumption of flat modular invariance and the definition of the trivializations on tori above, the self-sewing cocycles of the annuli with geodesic seam vanish for both X and Y. Therefore, the Equation (5.19) holds trivially for these annuli and the Diagram (5.16) commutes. Moreover, we have $\Omega^X_{1,2}(A) = \Omega^Y_{1,2}(A)$ for any annuli by using (5.19) to reduce the difference to an annulus with geodesic seam. □

To apply the inductive procedure in genus 1, we are missing the $\mathsf{b} = 1$ case to start the induction. We now treat this case separately. Let the respective trivializations X and Y agree with Z and W on $\mathcal{M}_{0,1}$ and $\mathcal{M}_{0,0}$. The trivializations X and Y over $\mathcal{M}^{\mathsf{Möb}}_{1,1}$ are defined in the same way as in the induction step by Proposition 5.4, which works for the trivializations X and Y in the case $\mathsf{g} = 1$ and $\mathsf{b} = 1$. Hence, we have reparametrization invariant trivializations X and Y over $\mathcal{M}_{1,1}$. We temporarily, just for this section, define the trivializations X and Y over pairs of pants $\mathcal{M}_{0,3}$ using Proposition 5.4 as well. Note that in Section 5.4 we replace them by hyperbolic modular invariant trivializations.

We prove equality of cocycles for X and Y in a different order, starting with the self-sewing cocycles $\Omega^X_{j,k}(P)$ and $\Omega^Y_{j,k}(P)$ for a pair of pants $P \in \mathcal{M}_{0,3}$. First consider the case where $P \in \mathcal{M}^{\mathsf{Möb}}_{0,3}$ and $P\ {}_l\infty_1\ \mathbb{D} \in \mathcal{M}^{\mathsf{geod}}_{0,2}$, that is, the case where by putting a cap $\mathbb{D}$ at the lth boundary component (which is not j or k), we get an annulus which has geodesic seam in the torus $\infty_{j,k}\ (P\ {}_l\infty_1\ \mathbb{D}) = (\ \infty_{j,k}\ P)\ {}_1\infty_1\ \mathbb{D} \in \mathcal{M}_{1,0}$. Associated with this sewing operation, we have the cocycle identity

$$\Omega^X_{1,1}(\ \infty_{j,k}\ P, \mathbb{D}) + \Omega^X_{j,k}(P) = \Omega^X_{l,1}(P, \mathbb{D}) + \Omega^X_{j,k}(P\ {}_l\infty_1\ \mathbb{D}), \tag{5.20}$$

and analogously for Y. Since the torus with one boundary component $\infty_{j,k} P \in \mathcal{M}^{\mathsf{Möb}}_{1,1}$ is again Möbius, the first term vanishes by the definition of the trivializations on $\mathcal{M}^{\mathsf{Möb}}_{1,1}$ above. The terms on the right-hand side vanish as well — respectively by the definition in Equation (5.2) in genus 0, and by Equation (5.15) as we have $P\ {}_l\infty_1\ \mathbb{D} \in \mathcal{M}^{\mathsf{geod}}_{0,2}$. Thus, we find $\Omega^X_{j,k}(P) = 0 = \Omega^Y_{j,k}(P)$.

Now consider any pair of pants $P \in \mathcal{M}_{0,3}$ and an annulus $A \in \mathcal{M}_{0,2}$. In the cocycle identity sewing the annulus in between two legs of the pants,

$$\Omega^X_{j,k}(P\ {}_j\infty_1\ A) + \Omega^X_{j,1}(P, A) = \Omega^X_{j,k}(P\ {}_k\infty_2\ A) + \Omega^X_{k,2}(P, A), \tag{5.21}$$

the pants-annulus cocycles agree with those for Y. Taking the difference we find that the cocycles for $P\ {}_j\infty_1\ A$ agree if and only if they agree for $P\ {}_k\infty_2\ A$:

$$\Omega^X_{j,k}(P\ {}_j\infty_1\ A) - \Omega^Y_{j,k}(P\ {}_j\infty_1\ A) = \Omega^X_{j,k}(P\ {}_k\infty_2\ A) - \Omega^Y_{j,k}(P\ {}_k\infty_2\ A). \tag{5.22}$$

Similar to the proof of Proposition 5.6, we may apply this step a finite number of times to reduce to the case of an annulus with geodesic seam, and by reparametrization invariance, we may apply any diffeomorphism to the outer boundary to obtain a Möbius pair of pants. Finally, other cocycles involve sewing a disk $D \in \mathcal{M}_{0,1}$ or an annulus $A \in \mathcal{M}_{0,2}$ to a handle $H \in \mathcal{M}_{1,1}$. Consider the cocycle identities that combine these sewing operations with self-sewing of the form $H = \infty_{2,3} P$,

$$\Omega^X_{2,3}(P) + \Omega^X_{1,1}(H, D) = \Omega^X_{2,3}(P \,{}_1\!\infty_1\, D) + \Omega^X_{1,1}(P, D), \tag{5.23}$$

$$\Omega^X_{2,3}(P) + \Omega^X_{1,1}(H, A) = \Omega^X_{2,3}(P \,{}_1\!\infty_1\, A) + \Omega^X_{1,1}(P, A). \tag{5.24}$$

Note that these express the new cocycles in terms of those for which we already know that they agree for X and Y, making them agree as well.

5.4 Higher genus

Remark 5.7. The trivializations Z and W on $\mathcal{M}_{0,3}$ defined so far are most likely not crossing-invariant since they are defined to be invariant under sewing of a unit disk at the first boundary component. Aside from this being an asymmetric definition with respect to permutations of the boundary labels, we may also consider what this condition means for the real determinant line bundle. There, we equip the unit disk with the flat or round metric, the annulus with the flat metric, and the pair of pants with the hyperbolic metric. Then, sewing the unit disk to a pair of pants, we do not obtain the flat metric on the annulus, which contradicts the triviality of the disk-pants cocycle.

So far, the isomorphism Ψ is defined in genus 0 and 1 for any number of boundary components. For higher genus, we redefine X and on pairs of pants $\mathcal{M}_{0,3}$ by a crossing invariant trivialization (Definition 3.9). The isomorphism $\Psi_{0,3} : E(\mathcal{M}_{0,3}) \to D(\mathcal{M}_{0,3})$ defines $Y(P) = \Psi_{0,3}(X(P))$. Since $\Psi_{0,3}$ is compatible with the genus 0 operation of sewing two pairs of pants, the trivialization Y of $D(\mathcal{M}_{0,3})$ is also crossing invariant.

Let Σ be a hyperbolic surface and $\Sigma = \underline{\infty}\, \underline{P}$ a pants decomposition, and consider

$$X(\Sigma) = \overset{E}{\underline{\infty}}\, X(\underline{P}), \qquad Y(\Sigma) = \overset{E}{\underline{\infty}}\, Y(\underline{P}). \tag{5.25}$$

This is independent of the pants decomposition, since it is possible to transition between pants decompositions using a finite number of elementary moves, namely, the S- and A-moves as defined by Hatcher and Thurston [Hat99, HT80]. Respectively, the hyperbolic modular invariance and crossing invariance precisely say that (5.25) is invariant under these moves. For surfaces with general boundary parametrizations, X and Y are defined by reparametrization invariance.

Since $Y(P) = \Psi_{0,3}(X(P))$ holds by definition, for surfaces $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$ of genus $\mathsf{g} = 0$ and $\mathsf{b} \geq 2$, or $\mathsf{g} = 1$ and $\mathsf{b} \geq 1$, we find

$$\Psi(W^E(\Sigma)) = \overset{D}{\underline{\infty}}\, \Psi(W^E(\underline{P})) = \overset{E}{\underline{\infty}}\, W^D(\underline{P}) = W^D(\Sigma), \tag{5.26}$$

that is, the new trivialization X is mapped to the trivialization Y where the isomorphism Ψ was already defined.

Since X and Y are now defined for any genus, in particular $\mathsf{g} \geq 2$, we use them to extend the isomorphism Ψ to all moduli spaces. To show that Ψ extends as an isomorphism, we have to show that a number of cocycles agree. For hyperbolic surfaces Σ_1 and Σ_2 we clearly have $\Omega^X_{j,k}(\Sigma_1, \Sigma_2) = \Omega^Y_{j,k}(\Sigma_1, \Sigma_2) = 0$ and also for self-sewing. For general boundary parametrizations away from the seam, we use reparametrization invariance. If there is a different parametrization at the seam, however, some more arguments are needed. We proceed by induction on the genus $\mathsf{g} \geq 2$, relying on the base cases $\mathsf{g} = 0$ and $\mathsf{g} = 1$.

Given any two surfaces $\Sigma_1 \in \mathcal{M}_{\mathsf{g}_1,\mathsf{b}_1}$ and $\Sigma_2 \in \mathcal{M}_{\mathsf{g}_2,\mathsf{b}_2}$ in the cocycle $\Omega^X_{j,k}(\Sigma_1, \Sigma_2)$, we open up a seam in one of the surfaces, say $\Sigma_1 = {}_l\infty_m \hat{\Sigma}_1$, such that $\hat{\Sigma}$ is of genus one lower than Σ_1. Note that since we start the induction knowing genus 0 and 1, we can assume without loss of generality that $\mathsf{g}_1 \geq 1$. Since $\mathsf{g}_1 + \mathsf{g}_2 \geq 2$, the sewn surface $\Sigma_1 \, {}_j\infty_k \, \Sigma_2$ has a hyperbolic metric, and thus we can assume that the new seam is a hyperbolic geodesic in the total surface $\Sigma_1 \, {}_j\infty_k \, \Sigma_2$. Considering the cocycle identity

$$\Omega^X_{j,k}(\Sigma_1, \Sigma_2) + \Omega^X_{l,m}(\hat{\Sigma}_1) = \Omega^X_{j,k}(\hat{\Sigma}_1, \Sigma_2) + \Omega^X_{l,m}(\hat{\Sigma}_1 \, {}_j\infty_k \, \Sigma_2), \tag{5.27}$$

we find that the second and third cocycles involve surfaces of a total genus at most $\mathsf{g}_1 + \mathsf{g}_2 - 1$. The last term vanishes since we assumed the seam to be a hyperbolic geodesic. Thus, by the induction hypothesis we have $\Omega^X_{j,k}(\Sigma_1, \Sigma_2) = \Omega^Y_{j,k}(\Sigma_1, \Sigma_2)$.

Finally, let $\Sigma \in \mathcal{M}_{\mathsf{g},\mathsf{b}}$ be any surface of genus $\mathsf{g} \geq 1$ and number of boundary components $\mathsf{b} \geq 2$. We consider the cocycle $\Omega^X_{j,k}(\Sigma)$. Let $\Sigma = \infty_{l,m} \hat{\Sigma}$ where the seam is a hyperbolic geodesic in the sewn surface $\infty_{j,k} \Sigma$, in the unique hyperbolic metric on Σ, which exists since $\mathsf{g} + 1 \geq 2$. Such a seam must be non-separating in Σ such that $\hat{\Sigma} \in \mathcal{M}_{\mathsf{g}-1,\mathsf{b}}$ is of genus one lower. In the cocycle identity

$$\Omega^X_{j,k}(\Sigma) + \Omega^X_{l,m}(\hat{\Sigma}) = \Omega^X_{j,k}(\hat{\Sigma}) + \Omega^X_{l,m}(\infty_{j,k} \hat{\Sigma}), \tag{5.28}$$

the last term vanishes since the seam is a hyperbolic geodesic. Then, the second and third terms are cocycles for one genus lower, so by the induction hypothesis, we know that these cocycles agree with those for Y. It follows that $\Omega^X_{j,k}(\Sigma) = \Omega^Y_{j,k}(\Sigma)$.

This completes the proof of Theorem 5.1.

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