Berkovich 2-motives and normed ring stacks

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1. Introduction

In this paper, an n-category means an (∞, n)-category.

This paper is devoted to the study of 2-motives. Most importantly, we prove universal properties. We explain the ideas motivating this concept in Section 1.1, state the main results in Section 1.2, and discuss applications in Section 1.3.

1.1. **Ideas.** In arithmetic geometry, varieties are studied using various cohomology theories. For a variety X over a field k, one might study its Betti cohomology, de Rham cohomology, étale cohomology, crystalline cohomology, etc. These theories are interconnected by comparison isomorphisms, yet they are constructed in entirely different ways. To govern this complexity, Grothendieck envisioned the concept of *motives*, positing the existence of a universal, "absolute" cohomology theory. This would take the form of an abelian category of motives $\mathrm{Mot}(k)$,[1] such that any variety X has a corresponding motive $[X]$. Each known cohomology theory H^* would then arise as a *realization functor* from $\mathrm{Mot}(k)$ to the category of vector spaces. This concept sought to capture the cohomological essence of a variety, an object that would explain all its different cohomological "manifestations." While Grothendieck's original vision of $\mathrm{Mot}(k)$ has remained conjectural, it was later substantially realized by Voevodsky [54], whose construction of the triangulated category of motives $\mathrm{DM}(k)$ led to his celebrated proof of the Milnor conjecture. More relevant to the present work is the stable motivic homotopy theory $\mathrm{SH}(k)$ of Morel–Voevodsky [42].

In parallel, Grothendieck, with key contributions from Verdier, considered six operations. This is the essential toolkit for working with these cohomology theories. It describes the fundamental structural properties such as functoriality, Poincaré duality, and the Künneth formula, that a well-behaved "coefficient theory" (like the category of étale sheaves or D-modules) must have. Nowadays, these six operations can be packaged as a functor from spans (aka correspondences). This perspective was famously explained in a volume by Gaitsgory–Rozenblyum [28], where this idea of using spans was attributed to Lurie.

This brings us to the modern context. If Grothendieck's motives classify cohomology theories, it is natural to consider classifying coefficient theories instead. Drew–Gallauer [21] showed that SH is a universal six-functor formalism. Scholze [45] instead considered this via categorification: There should be a 2-category of 2-motives that is a universal recipient of six-functor formalisms. In this framework, the 2-motive $[X]$ captures the sheaf-theoretic essence of X. The category of 2-motives is thus a 2-category in which varieties determine objects and operations on sheaves determine morphisms. This unified picture, however, raises a technical problem regarding the precise mathematical nature of this 2-category. It should look like the category of linear categories. Stefanich [52] was the first to provide a foundation for such a theory. Our previous work [4]

[1] In this paper, the notation Mot (or 2Mot) does not have a global definition and just stands for some version of (2-)motives.

showed how to achieve this without reference to a universe. The present work builds directly on that foundation. We see a precise formulation of this idea in Theorem A.

Even though we have a good packaging of coefficient theories, the challenge of systematically constructing these coefficient theories still remained. A breakthrough came from the study of de Rham cohomology. The natural coefficient theory for de Rham cohomology is the theory of D-modules due to Sato. In a crucial insight, Simpson [50] realized that D-modules on a variety X over $\mathbf{Q}$ could be reinterpreted as certain quasicoherent sheaves on a new geometric object called the *de Rham stack* X^{dR}. This was the first indication that various coefficient categories are simply quasicoherent sheaves on some modified geometry associated to X. This idea was further refined by Drinfeld [22]. What he emphasized was the ring stack perspective, which is now called *transmutation*, terminology due to Bhatt (cf. [13, Remark 2.3.8]). The idea is that an entire coefficient theory can be generated from a single piece of data; e.g., for D-modules, $(\mathbb{A}^1)^{\mathrm{dR}}$, viewed as a ring stack. By specifying what $\mathbb{A}^1$ maps to, one can "transmute" the base geometry of varieties into a new geometry, and the quasicoherent sheaves on that new geometry give you the coefficient theory.

Given this, Scholze [45] further proposed that $2\mathrm{Mot}(\mathbf{Z})$ should classify ring stacks satisfying certain conditions. In this paper, we prove a precise version of this as Theorem B. In terms of the gestalt theory of Scholze–Stefanich [47], we can say that it provides a moduli description of the gestalt associated to stable motivic homotopy theory.

Furthermore, we prove certain variants of this theorem. Notably, Scholze [46] introduced the category of *Berkovich motives* with applications to local Langlands in mind. It is a certain analytic version of motivic homotopy theory. He constructed six operations for them and proved several desirable properties. One drawback of this theory is that it is difficult to construct realization functors, since the definition already involves arc descent. In Theorem D, we characterize this using ring stacks with an absolute value. This helps to construct realization functors by first establishing 2-categorical realizations.

1.2. **Results.** Let QProj denote the category of quasiprojective static schemes over $\mathbf{Z}$ (see Remark 1.3 for this choice). We take SH to be the Morel–Voevodsky stable motivic homotopy theory (see Section 3.1). This admits six operations due to Ayoub [7, 8]. Applying the machinery of Liu–Zheng [35], we obtain a lax symmetric monoidal functor $\mathrm{SH}\colon \mathrm{Span}(\mathsf{QProj}) \to \mathsf{Pr}_{\mathrm{st}}$.

Definition 1.1. Let S be a divisorial noetherian static scheme.[2] We write $2\mathrm{SH}(S)$ for the presentably symmetric monoidal 2-category of kernels (see Section 2.2) of $\mathrm{SH}\colon \mathrm{Span}(\mathsf{QProj}_S) \to \mathsf{Pr}_{\mathrm{st}}$. Informally, it is freely generated by the image of a symmetric monoidal functor $[-]\colon \mathsf{QProj}_S^{\mathrm{op}} \to 2\mathrm{SH}(S)$, where the image is described as follows:

- The mapping category from $[X]$ to $[Y]$ is $\mathrm{SH}(X \times Y)$.
- The identity $\mathrm{id}_{[X]}$ is $d_!\mathbf{1} \in \mathrm{SH}(X \times X)$ (which is equivalent to $d_*\mathbf{1}$ in this case), where $d\colon X \to X \times X$ is the diagonal.
- For $M\colon [X] \to [Y]$ and $N\colon [Y] \to [Z]$, its composite is $(\mathrm{pr}_{X,Z})_!(\mathrm{pr}_{X,Y}^* M \otimes \mathrm{pr}_{Y,Z}^* N)$, where pr denotes the corresponding projection from $X \times Y \times Z$.

First, we characterize this via its universal property:

Theorem A. *The functor $[-]\colon (\mathsf{QProj}_S)^{\mathrm{op}} \to 2\mathrm{SH}(S)$ is universal among symmetric monoidal functors to stable presentably symmetric monoidal 2-categories satisfying the following axioms:*

(i) Consider a smooth morphism $f\colon Y \to X$. Then $f^*\colon [X] \to [Y]$ admits a left adjoint $f_\natural$. Moreover, the Beck–Chevalley transformation $f'_\natural q^* \to p^* f_\natural$ is an equivalence for any pullback square

$$\begin{array}{ccc} Y' & \xrightarrow{q} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{p} & X. \end{array} \tag{1.2}$$

[2] This assumption is included solely to ensure a unique notion of quasiprojectivity.

(ii) *For a closed subvariety $i\colon Z \hookrightarrow X$ with complement U, the diagram $[U] \leftarrow [X] \rightarrow [Z]$ is a recollement (see Definition 2.9). In particular, $i^*\colon [X] \rightarrow [Z]$ admits a right adjoint i_*.*
(iii) *For the projection $f\colon \mathbb{A}^1_X \rightarrow X$, the counit $f_\natural f^* \rightarrow \mathrm{id}$ is an equivalence.*
(iv) *For the projection $f\colon \mathbb{A}^1_X \rightarrow X$ and its zero section $s\colon X \rightarrow \mathbb{A}^1_X$, the morphism $f_\natural s_*$ is an autoequivalence of $[X]$.*

Remark 1.3. Our choice of QProj_S in Theorem A is not important; we can consider e.g., the category of static schemes separated of finite type over S to obtain the same result. One advantage with this 2-categorical package is that we do not need to worry about which category of geometries to work with.

We now specialize to the case $S = \operatorname{Spec} \mathbf{Z}$. The following is the characterization via the language of ring stacks:

Theorem B. *The stable presentably symmetric monoidal* 2*-category* $2\mathrm{SH}(\mathbf{Z})$ *is freely generated by a homologically trivial smooth sutured ring stack.*

Remark 1.4. By Proposition 4.29, we can equivalently state Theorem B as saying that $2\mathrm{SH}(\mathbf{Z})$ is freely generated by a stable homologically trivial weakly suave ring stack. Now, "stable," "sutured," "homologically trivial," and "weakly suave" correspond to (iv), (ii), (iii), and (i) in Theorem A, respectively. One notable point is that all these conditions are only about $\mathbb{A}^1$. In particular, in Theorem B, we do not mention general schemes at all.

We then consider its étale variant, where $2\mathrm{SH}_{\text{ét}}$ denotes the presentable 2-category of kernels for $\mathrm{SH}_{\text{ét}}$, the étale sheafified version of SH (see Section 8.1):

Theorem C. *The stable presentably symmetric monoidal* 2*-category* $2\mathrm{SH}_{\text{ét}}(\mathbf{Z})$ *is freely generated by a homologically trivial smooth sutured ring stack that is Kummer and Artin–Schreier in the following sense:*
Kummer: *The morphism $R[1/l]^\times \rightarrow R[1/l]^\times$ given by $x \mapsto x^l$ is a cover for any prime l.*
Artin–Schreier: *The morphism $R/p \rightarrow R/p$ given by $x \mapsto x^p - x$ is a cover for any prime p.*

In the statement above, the notion of *cover* is important, and we study it extensively in Section 5 under the name of *descent*.

Remark 1.5. Theorem C shows that there is an idempotent algebra E in $2\mathrm{SH}(\mathbf{Z})$ such that $\mathrm{Mod}_E(2\mathrm{SH}(\mathbf{Z}))$ is equivalent to $2\mathrm{SH}_{\text{ét}}(\mathbf{Z})$.

We then move on to the analytic situation. Scholze [46] introduced Berkovich motives with integral coefficients $\mathrm{D}_{\mathrm{mot}}(-;\mathbf{Z})$ and constructed six operations for them. We here consider Berkovich motives with spherical coefficients $\mathrm{D}_{\mathrm{mot}}(-;\mathbf{S})$; see Section 10.1. Here we consider uniform Banach rings topologically of finite presentation as the class of "varieties" to obtain the presentable 2-category of kernels, for which we write $2\mathrm{D}_{\mathrm{mot}}(\mathbf{Z};\mathbf{S})$. We then characterize this as follows:

Theorem D. *The stable presentably symmetric monoidal* 2*-category* $2\mathrm{D}_{\mathrm{mot}}(\mathbf{Z};\mathbf{S})$ *is freely generated by a Kummer–Artin–Schreier smooth ring stack with an absolute value whose open unit disk is homologically trivial.*

Remark 1.6. Note that $2\mathrm{D}_{\mathrm{mot}}(\mathbf{Z};\mathbf{S})$ admits a class for any seminormed ring. In this situation, these classes are proper, as shown in Proposition 10.19. This means that unlike Scholze's Berkovich motives, proper base change holds without any finiteness assumption.

1.3. **Applications.** An immediate application is to construct motivic realization functors. To do this, we first consider the 2-categorical realization using our universality results. Then, by taking $\mathrm{End}(\mathbf{1})$, we obtain the desired realization 1-functor. For example, we can see that the analytic Habiro stack of Scholze [48] and the Hyodo–Kato stack of Anschütz–Bosco–Le Bras–Rodríguez Camargo–Scholze [1] determine realization functors.

Another application is about the nature of these 2-categories. For example, we see the following:

Theorem 1.7. *Let* $2\mathrm{Mot}(\mathbf{Z})$ *be any of* $2\mathrm{SH}(\mathbf{Z})$*,* $2\mathrm{SH}_{\text{ét}}(\mathbf{Z})$*, or* $2\mathrm{D}_{\mathrm{mot}}(\mathbf{Z};\mathbf{S})$*. Then it is a* 1*-truncated object in* $\mathrm{CAlg}(2\mathsf{Pr})^{\mathrm{op}}$*, i.e., the tautological morphism*

$$2\mathrm{Mot}(\mathbf{Z})^{\otimes S^2} \to 2\mathrm{Mot}(\mathbf{Z})$$

is an equivalence.

Proof. For an object $\mathcal{C} \in \mathrm{CAlg}(2\mathsf{Pr})$, we have

$$\mathrm{Map}_{\mathrm{CAlg}(2\mathsf{Pr})}\big(2\mathrm{Mot}(\mathbf{Z})^{\otimes S^2}, \mathcal{C}\big) \simeq \mathrm{Map}\big(S^2, \mathrm{Map}_{\mathrm{CAlg}(2\mathsf{Pr})}(2\mathrm{Mot}(\mathbf{Z}), \mathcal{C})\big).$$

Therefore, it suffices to show that $\mathrm{Map}_{\mathrm{CAlg}(2\mathsf{Pr})}(2\mathrm{Mot}(\mathbf{Z}), \mathcal{C})$ is 1-truncated. To show this, it suffices to show that every object is 0-truncated, i.e., for a ring stack R satisfying the conditions of Theorem B, the diagonal morphism

$$R \to \mathrm{Map}(S^1, R)$$

is an equivalence. This follows from Lemma 4.25. □

Remark 1.8. Theorem 1.7 for $2\mathrm{Mot}(\mathbf{Z}) = 2\mathrm{SH}(\mathbf{Z})$ can also be deduced simply from Theorem A.

This approach enables us to show some properties without examining the 2-category directly. The proof of the following will appear in [6], where we will introduce the notion of n-rigidity:

Theorem 1.9 ([6])**.** *The stable presentably symmetric monoidal* 2*-categories* $2\mathrm{D}_{\mathrm{mot}}(\mathbf{Z};\mathbf{S}[1/2])$ *and* $2\mathrm{D}_{\mathrm{mot}}(\mathbf{Z}\{1/l\}_1[\zeta_{2l}];\mathbf{S})$ *for any prime* l *are* 2*-rigid.*

In the context of the geometry of gestalts by Scholze–Stefanich (cf. [47]), which treats presentable categorical spectra (see [4, Remark 2.10]) as a building block of geometry, this means that the gestalt of Berkovich motives is almost proper.

Outline. We begin in Section 2 by reviewing some basic tools in higher category theory, which will be essential for the developments to follow. In Section 3, we use these tools to prove Theorem A. Next, we introduce some general machinery. Section 4 covers basic notions of stacks, indispensable for stating our main theorems, while Section 5 develops the theory of descent, a fundamental tool for understanding presentably symmetric monoidal n-categories. We illustrate these concepts with examples in Section 6. With this preparation, we turn to the proofs of the main theorems. In Sections 7 and 8, we prove Theorems B and C, respectively. Finally, we consider the analytic setting. Section 9 studies Berkovich geometry, which provides the framework for the results of Section 10, where we prove Theorem D.

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2. Some 2-category theory

We recall some 2-category theory needed in this paper. In Section 2.1, we review spans. In Section 2.2, we review 2-categories of kernels. In Section 2.3, we commute a hexagon consisting of Beck–Chevalley transformations. In Section 2.4, we do some 2-categorical homological algebra.

We assume familiarity with the basics of presentable n-category theory; see [52] or [4]. What is crucial in this paper is that, as demonstrated in [4, Section 3], presentable 2-category theory behaves well.

2.1. Spans.

For a 1-category $\mathcal{G}$ with finite limits, we obtain the n-category of iterated spans $n\mathrm{Span}(\mathcal{G})$ of Haugseng [31] (see also [31, Section 1.5] for prior works). In this paper, the variants $\mathrm{Span}_E(\mathcal{G})$ and $2\mathrm{Span}_{E;P,J}(\mathcal{G})$ play important roles, where $\mathcal{G}$ is a 1-category and J, P, and E are wide subcategories of $\mathcal{G}$ stable under base change satisfying $J \subset E \supset P$. Concretely, $2\mathrm{Span}_{E;P,J}(\mathcal{G})$ is described as follows; see [19, Construction 4.12] for the precise construction:

- Objects are those of $\mathcal{G}$.
- A 1-morphism from X to X' is a span $X \leftarrow Y \to X'$ in $\mathcal{G}$ such that $Y \to X'$ is in E.

- A 2-morphism from $X \leftarrow Y \rightarrow X'$ to $X \leftarrow Y' \rightarrow X'$ is a diagram

$$
\begin{array}{ccccc}
 & \swarrow & Y & \searrow & \\
 & & \uparrow & & \\
X & \longleftarrow & Z & \longrightarrow & X' \\
 & & \downarrow & & \\
 & \nwarrow & Y' & \nearrow &
\end{array}
\tag{2.1}
$$

in $\mathcal{G}$ such that $Z \to Y$ is in P and $Z \to Y'$ is in J.

By setting J and P to be the class of equivalences, we obtain $\mathrm{Span}_E(\mathcal{G}) = 2\mathrm{Span}_{E;\mathrm{all},\mathrm{all}}(\mathcal{C})$, which is a 1-category. Note the inclusions $\mathcal{G}^{\mathrm{op}} \to \mathrm{Span}_E(\mathcal{G}) \to 2\mathrm{Span}_{E;P,J}(\mathcal{G})$. In [19], the universality of this category was obtained under suitable assumptions. Here we recall from [19, Theorem B] the version that considers the cartesian symmetric monoidal monoidal structure on $\mathcal{G}$:

Theorem 2.2 (Cnossen–Lenz–Linskens)**.** *Let $\mathcal{G}$ be a 1-category with finite products and J, P, and E are subcategories satisfying the following:*

- *Both J and P are closed under base change and satisfy cancellation.*
- *A morphism is in E if and only if it can be written as pj for $j \in J$ and $p \in P$.*
- *Every morphism in $J \cap P$ is n-truncated for some n (depending on the morphism).*

Then for any symmetric monoidal 2-category $\mathcal{C}$, the restriction functor

$$\mathrm{Fun}^{\text{lax-}\otimes}(2\mathrm{Span}_{E;P,J}(\mathcal{G}), \mathcal{C}) \to \mathrm{Fun}^{\text{lax-}\otimes}(\mathcal{G}^{\mathrm{op}}, \mathcal{C})$$

is a subcategory inclusion.

- *Objects are spanned by (J,P)-biadjointable functors D satisfying the projection formula: The morphism $j^* = D(j)$ for any $j \in J$ admits a left adjoint $j_\natural$ satisfying the projection formula and the Beck–Chevalley condition. Similarly, $p^* = D(p)$ for any $p \in P$ admits a right adjoint p_* satisfying the dual conditions. Moreover, these adjoints satisfy the double Beck–Chevalley condition; i.e., for any pullback square*

$$
\begin{array}{ccc}
Y' & \xrightarrow{k} & Y \\
{\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\
X' & \xrightarrow{j} & X
\end{array}
$$

  such that $j \in J$ and $p \in P$, the induced morphism $j_\natural p'_* \to p_* k_\natural$ is an equivalence.
- *Morphisms $\alpha\colon D \to D'$ are natural transformations compatible with the formations of $j_\natural$ and p_* above: For $j\colon Y \to X$ in J, the morphism $D'(j)^{\mathrm{L}}\alpha_X \to \alpha_Y D(j)^{\mathrm{L}}$ is an equivalence and for $p\colon Y \to X$ in P, the morphism $\alpha_X D(p)^{\mathrm{R}} \to D'(p)^{\mathrm{R}}\alpha_Y$ is an equivalence.*

Here, $\mathrm{Fun}^{\text{lax-}\otimes}$ *denotes the category of lax symmetric monoidal functors.*

Remark 2.3. We use Theorem 2.2 throughout this paper, even where weaker results might suffice. We refer the reader to [19, Section 1] for historical accounts about other previous approaches. We can often compare this with a previous construction, such as in Liu–Zheng's work [35], using the uniqueness theorem of Dauser–Kuijper [20].

2.2. **Kernels.** Consider a category with finite limits $\mathcal{G}$ and a lax symmetric monoidal functor $D\colon \mathrm{Span}(\mathcal{G}) \to \mathsf{Cat}$. The symmetric monoidal 2-category we associate to D was used in [26, Section 2.3] based on[3] Lu–Zheng's work [36]. It has the following features:

- Objects are those of $\mathcal{G}$. We write $[X]$ for the object corresponding to $X \in \mathcal{G}$.
- The mapping category from $[X]$ to $[Y]$ is $D(X \times Y)$.
- The identity is $d_!\mathbf{1}$, where $d\colon X \to X \times X$ is the diagonal.
- For $F\colon [X] \to [Y]$ and $G\colon [Y] \to [Z]$, its composite is $(\mathrm{pr}_{X,Z})_!(\mathrm{pr}_{X,Y}^* F \otimes \mathrm{pr}_{Y,Z}^* G)$, where pr denotes the corresponding projection from $X \times Y \times Z$.

[3]See [30, Section 3] and [32, Remark 1.3.11] about the relation.

The following is a precise formulation, which the author learned from Dauser:

Definition 2.4. Let $\mathcal{G}$ be a category with finite limits and $D\colon \mathrm{Span}(\mathcal{G}) \to \mathsf{Cat}$ a lax symmetric monoidal functor. Then since $\mathrm{Span}(\mathcal{G})$ is closed, it is enriched over itself. Base changing along D, we obtain a Cat-enrichment on $\mathrm{Span}(\mathcal{G})$. We call this the *2-category of kernels*.

In the presentable situation, we can make another definition of a similar category. The author learned the following definition from Dauser:

Definition 2.5. Let $\mathcal{G}$ be a category with finite limits and $D\colon \mathrm{Span}(\mathcal{G}) \to \mathsf{Pr}$ a lax symmetric monoidal functor, which we regard as a commutative algebra object of $\mathrm{Fun}(\mathrm{Span}(\mathcal{G}), \mathsf{Pr})$ with the Day convolution symmetric monoidal structure. We consider $\mathrm{Mod}_D(\mathrm{Fun}(\mathrm{Span}(\mathcal{G}), \mathsf{Pr}))$, which we call the *presentable 2-category of kernels.*

An advantage of this definition is that it avoids the use of enriched category theory. Nevertheless, these two constructions can be compared by the following:

Theorem 2.6. *Let $\mathcal{S}$ be a small closed symmetric monoidal category and $\mathcal{V}$ an object of $\mathrm{CAlg}(\mathsf{Pr})$. Let $D\colon \mathcal{S} \to \mathcal{V}$ be a lax symmetric monoidal functor. Then there is a canonical equivalence*

$$\mathrm{PShv}^{\mathcal{V}}(\widetilde{\mathcal{S}}^{\mathrm{op}}) \simeq \mathrm{Mod}_D(\mathrm{Fun}(\mathcal{S}, \mathcal{V})),$$

where $\widetilde{\mathcal{S}}$ is the $\mathcal{V}$-enriched category obtained by base changing the self-enrichment of $\mathcal{S}$ along D.

Corollary 2.7. *In the situation of Definition 2.5, we choose a regular cardinal κ so that D factors through Pr^{κ}. By the procedure in Definition 2.4, we obtain Pr^{κ}-enriched category $\mathcal{K}$. Then $\mathrm{PShv}^{\mathsf{Pr}^{\kappa}}(\mathcal{K})$ is equivalent to $\mathrm{Mod}_D(\mathrm{Fun}(\mathrm{Span}(\mathcal{G}), \mathsf{Pr}^{\kappa}))$.*

Heine helped with the following proof:

Proof of Theorem 2.6. We write $U\colon \mathsf{Ani} \to \mathcal{V}$ for the unit functor. Since $U(\mathrm{Map}_{\mathcal{S}}(\mathbf{1}, -))$ is the unit in $\mathrm{CAlg}(\mathrm{Fun}(\mathcal{S}, \mathcal{V}))$, we obtain a morphism $U(\mathrm{Map}_{\mathcal{S}}(\mathbf{1}, -)) \to D$ there. By base changing the self-enrichment of $\mathcal{S}$ along this morphism, we obtain a $\mathcal{V}$-functor $\mathcal{S} \to \widetilde{\mathcal{S}}$, where the source is enriched trivially. We write f for its opposite. This induces an adjunction $f_! \dashv f^*$ in $\mathrm{Mod}_{\mathcal{V}}(\mathsf{Pr})$. Note that $f^*\colon \mathrm{PShv}^{\mathcal{V}}(\widetilde{\mathcal{S}}^{\mathrm{op}}) \to \mathrm{PShv}^{\mathcal{V}}(\mathcal{S}^{\mathrm{op}}) = \mathrm{Fun}(\mathcal{S}, \mathcal{V})$ is monadic. It suffices to show that this $\mathcal{V}$-linear monad $f^* f_!$ is identified with D. First, by definition, $f^*\mathbf{1}$ is identified with D, from which we obtain a morphism of monads $D \to f^* f_!$. By linearity, to prove that this is an equivalence, it suffices to check this for representables, which is straightforward. □

2.3. **On double Beck–Chevalley transformations.** We prove the following about cubes in a 2-category:

Proposition 2.8. *Let $\mathcal{G} = [1] \times [1]$ be a square category labeled as in (1.2) and $\mathcal{C}$ be a 2-category. We have a functor $D\colon \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$ such that the square*

$$\begin{array}{ccc} D(X) & \xrightarrow{D(p)} & D(X') \\ {\scriptstyle D(f)}\downarrow & & \downarrow{\scriptstyle D(f')} \\ D(Y) & \xrightarrow{D(q)} & D(Y') \end{array}$$

is vertically left adjointable and horizontally right adjointable. We have another diagram D' satisfying the same condition and a natural transformation $\alpha\colon D \to D'$. Then the diagram

$$\begin{array}{ccccc} & \alpha_X D(f)^{\mathrm{L}} D(q)^{\mathrm{R}} & \longrightarrow & \alpha_X D(p)^{\mathrm{R}} D(f')^{\mathrm{L}} & \\ \nearrow & & & & \searrow \\ D'(f)^{\mathrm{L}} \alpha_Y D(q)^{\mathrm{R}} & & & & D'(p)^{\mathrm{R}} \alpha_{X'} D(f')^{\mathrm{L}} \\ \searrow & & & & \nearrow \\ & D'(f)^{\mathrm{L}} D'(q)^{\mathrm{R}} \alpha_{Y'} & \longrightarrow & D'(p)^{\mathrm{R}} D'(f')^{\mathrm{L}} \alpha_{Y'} & \end{array}$$

consisting of four Beck–Chevalley and two double Beck–Chevalley transformations commutes.

We break the symmetry in the following proof; e.g., we do not know if our witness coincides with the one obtained by breaking the symmetry in the other way:

Proof. By pasting Beck–Chevalley transformations, we simplify the diagram to

$$\begin{array}{ccc} D'(f)^{\mathrm{L}}\alpha_Y D(q)^{\mathrm{R}} & \longrightarrow & \alpha_X D(p)^{\mathrm{R}} D(f')^{\mathrm{L}} \\ \downarrow & & \downarrow \\ D'(f)^{\mathrm{L}} D'(q)^{\mathrm{R}} \alpha_{Y'} & \longrightarrow & D'(p)^{\mathrm{R}} \alpha_{X'} D(f')^{\mathrm{L}}, \end{array}$$

which consists of four Beck–Chevalley transformations. We then unpack the rows; we are reduced to commuting the right square of

$$\begin{array}{ccccc} \alpha_Y D(q)^{\mathrm{R}} & \longrightarrow & \alpha_Y D(q)^{\mathrm{R}} D(f') D(f')^{\mathrm{L}} & \xrightarrow{\simeq} & D'(f)\alpha_X D(p)^{\mathrm{R}} D(f')^{\mathrm{L}} \\ \downarrow & & \downarrow & & \downarrow \\ D'(q)^{\mathrm{R}} \alpha_{Y'} & \longrightarrow & D'(q)^{\mathrm{R}} \alpha_{Y'} D(f') D(f')^{\mathrm{L}} & \xrightarrow{\simeq} & D'(f) D'(p)^{\mathrm{R}} \alpha_{X'} D(f')^{\mathrm{L}}. \end{array}$$

By removing $D(f')^{\mathrm{L}}$, now we have to commute

$$\begin{array}{ccccc} \alpha_Y D(q)^{\mathrm{R}} D(f') & \xleftarrow{\simeq} & \alpha_Y D(f) D(p)^{\mathrm{R}} & \xrightarrow{\simeq} & D'(f)\alpha_X D(p)^{\mathrm{R}} \\ \downarrow & & & & \downarrow \\ D'(q)^{\mathrm{R}} \alpha_{Y'} D(f') & \xrightarrow{\simeq} & D'(q)^{\mathrm{R}} D'(f') \alpha_{X'} & \xleftarrow{\simeq} & D'(f) D'(p)^{\mathrm{R}} \alpha_{X'}. \end{array}$$

By pasting Beck–Chevalley transformations again, this simplifies to

$$\begin{array}{ccc} \alpha_Y D(f) D(p)^{\mathrm{R}} & \xrightarrow{\simeq} & D'(f)\alpha_X D(p)^{\mathrm{R}} \\ \downarrow & & \downarrow \\ D'(q)^{\mathrm{R}} \alpha_Y D(f') & \xrightarrow{\simeq} & D'(q)^{\mathrm{R}} D'(f') \alpha_{X'}, \end{array}$$

which commutes. □

2.4. **Recollements and excision squares.** Here we perform some 2-categorical homological algebra.

Definition 2.9. We call a diagram

$$C_U \xleftarrow{j^*} C_X \xrightarrow{i^*} C_Z$$

in a stable presentable 2-category a *recollement* if the following conditions are satisfied:

(i) The morphism j^* admits a left adjoint $j_!$ and the counit $j_! j^* \to \mathrm{id}$ is an equivalence.
(ii) The morphism i^* admits a right adjoint i_* and the unit $\mathrm{id} \to i_* i^*$ is an equivalence.
(iii) The composite $j^* i_*$ is zero[4] (or equivalently, $i^* j_!$ is zero) and the induced square

$$\begin{array}{ccc} j_! j^* & \longrightarrow & \mathrm{id} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & i_* i^* \end{array}$$

in $\mathrm{End}(C_X)$ is a pushout.

We need to do some 2-categorical homological algebra.

[4]Note that this is a condition, not a datum.

Proposition 2.10. *In a stable presentable* 2*-category, we consider a commutative diagram*

$$\begin{array}{ccccc} C_U & \xleftarrow{j^*} & C_X & \xrightarrow{i^*} & C_Z \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle f^*} & & \| \\ C_V & \xleftarrow{l^*} & C_Y & \xrightarrow{k^*} & C_Z \end{array}$$

satisfying the following conditions:

(i) The top row is a recollement.
(ii) The bottom row is a recollement.
(iii) The right square is vertically left adjointable; i.e., f^* *admits a left adjoint* $f_\natural$ *and the Beck–Chevalley transformation* $k^* \to i^* f_\natural$ *is an equivalence.*

Then the left square is cartesian.

Proof. First, $g_\natural = j^* f_\natural l_!$ determines a left adjoint of g^*. With this definition, the left square is also vertically left adjointable.

We apply $\mathrm{Hom}(D, -)$ for an arbitrary object D to reduce this to the case of $\mathsf{Pr}_{\mathrm{st}}$. We write $\mathrm{Hom}(D, C_-)$ as $\mathcal{C}_-$. We write Map_- for $\mathrm{Map}_{\mathcal{C}_-}$.

We first prove that the induced functor from $\mathcal{C}_X$ to the pullback is fully faithful. It is equivalent to the assertion that the square

$$\begin{array}{ccc} \mathrm{Map}_U(j^*C, j^*D) & \longleftarrow & \mathrm{Map}_X(C, D) \\ \downarrow & & \downarrow \\ \mathrm{Map}_V(l^*f^*C, l^*f^*D) & \longleftarrow & \mathrm{Map}_Y(f^*C, f^*D) \end{array}$$

is a pullback for objects C and D of $\mathcal{C}_X$. This is equivalent to the assertion that the square

$$\begin{array}{ccc} j_!j^*C & \longrightarrow & C \\ \uparrow & & \uparrow \\ f_\natural l_! l^* f^* C & \longrightarrow & f_\natural f^* C \end{array}$$

is a pushout. This can be checked by looking at the cofibers of rows.

Then we prove that the induced functor is essentially surjective. We consider $C_U \in \mathcal{C}_U$ and $C_Y \in \mathcal{C}_Y$ with an equivalence $g^*(C_U) \simeq l^*(C_Y)$. We obtain a map

$$i_*k^*(C_Y)[-1] \overset{\text{(iii)}}{\simeq} i_*i^*f_\natural(C_Y)[-1] \xrightarrow{\text{(i)}} j_!j^*f_\natural(C_Y) \simeq j_!g_\natural l^*(C_Y) \simeq j_!g_\natural g^*(C_U) \to j_!(C_U)$$

and its cofiber is the desired object. □

We note the following degenerate case:

Corollary 2.11. *For a recollement in Definition 2.9,*

$$\begin{array}{ccc} 0 & \longleftarrow & C_Z \\ \downarrow & & \downarrow{\scriptstyle i_*} \\ C_U & \xleftarrow{j^*} & C_X \end{array}$$

is a pullback square.

Remark 2.12. Corollary 2.11 means that a recollement is determined by $j^* \colon C_X \to C_U$ (or $i^* \colon C_X \to C_Z$, since we can obtain C_U is the kernel of i^* by passing to the left adjoint).

Definition 2.13. In light of Remark 2.12, we call the left square in Proposition 2.10 an *excision square.*

Remark 2.14. In Definitions 2.9 and 2.13, we have considered the noncommutative situation. We can define the commutative version when the diagrams are in $\mathrm{CAlg}(\mathcal{C})$, simply by considering the same notion in $\mathrm{Mod}_{C_X}(\mathcal{C})$.

3. Algebraic 2-motives

In Section 3.1, we review SH and necessary facts. In Section 3.2, we prove Theorem A. In Section 3.3, we prove the (relative) Künneth formula.

3.1. **A review of algebraic motivic spectra.** The following was introduced in [42]:

Definition 3.1 (Morel–Voevodsky)**.** Let X be a quasicompact quasiseparated static scheme. We write Sm_X for the category of finitely presented smooth X-schemes. The *category of motivic spectra* over X is defined as

$$\mathrm{SH}(X) = \mathrm{Shv}_{\mathrm{Nis}}(\mathsf{Sm}_X;\mathsf{Sp})_{\mathbb{A}^1}[(\Sigma^\infty\mathbb{P}^1)^{\otimes -1}], \tag{3.2}$$

where $\mathrm{Shv}_{\mathrm{Nis}}(\mathsf{Sm}_X;\mathsf{Sp})_{\mathbb{A}^1}$ denotes the full subcategory spanned by $\mathbb{A}^1$-invariant Nisnevich sheaves. See [43, Section 2.1] for the inversion procedure.

Remark 3.3. More precisely, what was considered in [42] was hypercomplete sheaves unlike Definition 3.1. Note that $\mathrm{Shv}_{\mathrm{Nis}}(\mathsf{Sm}_X)$ is hypercomplete when X is of finite type over $\mathbf{Z}$ (cf. [17, Theorem 3.18]). We prefer to work with this version, since $\mathrm{SH}\colon \mathsf{Ring} \to \mathrm{CAlg}(\mathsf{Pr})$ preserves filtered colimits with this definition.

Note the obvious functoriality; SH itself is a contravariant functor to $\mathrm{CAlg}(\mathsf{Pr}_{\mathrm{st}})$. Hence we have four operations. Smooth base change is straightforward from the definition. Proper base change for SH was established in Ayoub's thesis [7, 8].[5] More precisely, he demonstrated in [7, Scholie 1.4.2] that a specific set of axioms implies proper base change. He also considered the symmetric monoidal version in [7, Section 2.3]. We state both versions here:

Theorem 3.4 (Ayoub)**.** *Let S be a divisorial noetherian static scheme. We write QProj_S for the category of static schemes quasiprojective over S. Let $D\colon (\mathsf{QProj}_S)^{\mathrm{op}} \to \mathsf{Pr}_{\mathrm{st}}$[6] be a functor. We assume that it satisfies the following:*

- *(i) Smooth morphisms satisfy base change; i.e., when $f\colon Y \to X$ is smooth, then $f^*\colon D(X) \to D(Y)$ admits a left adjoint $f_\natural$ such that the Beck–Chevalley transformation $f'_\natural q^* \to p^* f_\natural$ is an equivalence for any pullback square (1.2).*
- *(ii) For any closed immersion $i\colon Z \hookrightarrow X$ with its complement $j\colon U \hookrightarrow X$, the diagram $D(U) \leftarrow D(X) \to D(Z)$ is a recollement[7]; see Remark 3.5 below.*
- *(iii) For the tautological map $f\colon \mathbb{A}^1_X \to X$, the functor $f^*\colon D(X) \to D(\mathbb{A}^1_X)$ is fully faithful.*
- *(iv) In the situation of (iii), the functor $f_\natural s_*\colon D(X) \to D(X)$ is an equivalence, where $s\colon X \to \mathbb{A}^1_X$ is the zero section.*

Then projective morphisms satisfy base change; i.e., when $p\colon X' \to X$ is projective, $p^*\colon D(X) \to D(X')$ admits a right adjoint p_* in $\mathsf{Pr}_{\mathrm{st}}$ such that the Beck–Chevalley transformation $f'_\natural q^* \to p^* f_\natural$ is an equivalence for any pullback square (1.2).

Moreover, we consider a lax symmetric monoidal functor $D\colon (\mathsf{QProj}_S)^{\mathrm{op}} \to \mathsf{Pr}_{\mathrm{st}}$ satisfying the above conditions, where (i) is replaced with the following:

- *(v) Smooth morphisms satisfy base change and the projection formula; i.e., (i) holds and moreover $f_\natural$ is $D(X)$-linear.*

In this case, the projection formula holds for projective morphisms; i.e., p_* is also $D(X)$-linear for any projective morphism $p\colon Y \to X$.

Remark 3.5. For Theorem 3.4, Ayoub originally formulated (ii) as $D(\emptyset) = 0$, the full faithfulness of i_*, and the conservativity of $(j^*, i^*)\colon D(X) \to D(U) \oplus D(Z)$. The equivalence follows from applying the smooth base change to $U \to X$ along i.

Remark 3.6. To prove Theorem 3.4, Ayoub first constructed shriek functoriality on the level of triangulated categories. Then he used it to prove the proper base change for $\mathbb{P}^n$. The point is that we cannot use the usual recognition principle based on compactifications to obtain the shriek functoriality, since properness is what we wish to show.

[5] Voevodsky first stated this without proof. Röndigs also independently worked on it, which remains unpublished.

[6] More generally, he worked in the setting of triangulated categories.

[7] Morel–Voevodsky [42, Theorem 2.21] proved that SH satisfies this property.

Remark 3.7. At least the first part of Theorem 3.4 is valid when $\mathsf{Pr}_{\mathrm{st}}$ is replaced with a general stable presentable 2-category. We prove this in Corollary 3.11 below.

Drew–Gallauer [21] identified SH as the universal six-functor formalism:

Theorem 3.8 (Drew–Gallauer)**.** *We use S and QProj_S as in Theorem 3.4 (but see Remark 3.9 below). We consider the following* nonfull *subcategory of* $\mathrm{CAlg}(\mathrm{Fun}((\mathsf{QProj}_S)^{\mathrm{op}}, \mathsf{Pr}_{\mathrm{st}}))$*:*

- *Objects are lax symmetric monoidal functors $D\colon (\mathsf{QProj}_S)^{\mathrm{op}} \to \mathsf{Pr}_{\mathrm{st}}$ satisfying (ii) to (v) of Theorem 3.4.*
- *Morphisms are those natural transformations $\alpha\colon D \to D'$ such that for any smooth morphism $f\colon Y \to X$, the canonical morphism $f_\natural \alpha_Y \to \alpha_X f_\natural$ is an equivalence.*

Then SH *is initial in this category.*

Remark 3.9. More precisely, Drew–Gallauer [21] considered the category of schemes of finite type over a finite-dimensional noetherian base unlike Theorem 3.8. In our situation, their proof a priori produces for X the category in Definition 3.2 with Sm_X being replaced by its full subcategory spanned by those quasiprojective over X, but it coincides with $\mathrm{SH}(X)$ by the standard basis argument. Note that the same remark applies (and hence the same result holds) when we replace QProj_S with the full subcategory spanned by affines.

3.2. **Algebraic 2-motives.** In the proof, we need the following enhancement of Theorem 3.4:

Proposition 3.10. *In the situation of Theorem 3.4, suppose that we have a natural transformation $\alpha\colon D \to D'$ between functors satisfying the conditions. When α is compatible with smooth base change (cf. Theorem 3.8), then α is also compatible with projective base change.*

Proof. We split this into the case of closed immersions and $\mathbb{P}^n_X \to X$. The former case follows from (ii) and smooth base change for open immersions. Therefore, it suffices to consider the latter case.

We instead consider a general smooth projective morphism $f\colon Y \to X$. We form the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{d} & Y \times_X Y & \xrightarrow{p} & Y \\ & & {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ & & Y & \xrightarrow{f} & X. \end{array}$$

Then for D (and similarly for D'), we consider the natural transformation

$$f_\natural \simeq f_\natural p_* d_* \to f_* q_\natural d_*,$$

where the second morphism is induced by the double Beck–Chevalley transformation. Ayoub proved that this is an equivalence and $q_\natural d_*$ is invertible. By the commutative diagram

$$\begin{array}{ccccccc} & & & \overset{\simeq}{\nearrow} & f_\natural p_* d_* \alpha_Y & \xrightarrow{\simeq} & f_* q_\natural d_* \alpha_Y \\ & & & & {\scriptstyle\simeq}\uparrow & & \uparrow{\scriptstyle\simeq} \\ f_\natural \alpha_Y & \xrightarrow{\simeq} & f_\natural \alpha_Y p_* d_* & \longrightarrow & f_\natural p_* \alpha_{Y\times_X Y} d_* & \longrightarrow & f_* q_\natural \alpha_{Y \times_X Y} d_* \\ {\scriptstyle\simeq}\downarrow & & \downarrow{\scriptstyle\simeq} & & & & \downarrow{\scriptstyle\simeq} \\ \alpha_X f_\natural & \xrightarrow{\simeq} & \alpha_X f_\natural p_* d_* & \xrightarrow{\simeq} & \alpha_X f_* q_\natural d_* & \longrightarrow & f_* \alpha_Y q_\natural d_*, \end{array}$$

where the hexagon commutes by Proposition 2.8, the bottom right horizontal morphism $\alpha_X f_* q_\natural d_* \to f_* \alpha_Y q_\natural d_*$ is also an equivalence. Since $q_\natural d_*$ is invertible, the Beck–Chevalley transformation $f_\natural \alpha_Y \to \alpha_X f_\natural$ under consideration is an equivalence. □

Corollary 3.11. *The first part of Theorem 3.4 remains valid when* $\mathsf{Pr}_{\mathrm{st}}$ *is replaced with any stable presentable* 2*-category* $\mathcal{C}$*.*

Proof. We use Theorem 3.4 and Yoneda. We can verify adjointability using Proposition 3.10. □

We now prove Theorem A:

Proof of Theorem A. In this proof, we write $2\mathrm{Mot}(S)$ for the universal target. We wish to prove that the morphism $F\colon 2\mathrm{Mot}(S) \to 2\mathrm{SH}(S)$ given by the universal property is an equivalence.

We first construct a morphism $G\colon 2\mathrm{SH}(S) \to 2\mathrm{Mot}(S)$ under $(\mathsf{QProj}_S)^{\mathrm{op}}$. Recall from Definition 2.5 that $2\mathrm{SH}(S)$ is defined to be $\mathrm{Mod}_{\mathrm{SH}}(\mathrm{Fun}(\mathrm{Span}(\mathsf{QProj}_S, \mathsf{Pr}_{\mathrm{st}})))$. Hence, it is equivalent to constructing a symmetric monoidal functor $\mathrm{Span}(\mathsf{QProj}_S) \to 2\mathrm{Mot}(S)$ extending $[-]\colon \mathsf{QProj}_S^{\mathrm{op}} \to 2\mathrm{Mot}(S)$ and a morphism $\alpha\colon \mathrm{SH} \to \mathrm{Hom}_{2\mathrm{Mot}(S)}(\mathbf{1}, [-])$ in $\mathrm{CAlg}(\mathrm{Fun}(\mathrm{Span}(\mathsf{QProj}_S), \mathsf{Pr}_{\mathrm{st}}))$.

We first construct the desired extension of $[-]$ as a restriction of a functor from $2\mathrm{Span}_{P,J}(\mathsf{QProj}_S)$ using Theorem 2.2, where J and P consist of open immersions and projective morphisms, respectively. We verify the (J, P)-adjointability condition. Base change for J follows from (i). For P, base change follows from Corollary 3.11. The double Beck–Chevalley condition can be seen by considering the complementary closed immersion and using (ii). Since $(\mathsf{QProj}_S)^{\mathrm{op}} \to 2\mathrm{Mot}(S)$ is symmetric monoidal, the projection formula automatically follows from base change.

We then construct α. We write $D = \mathrm{Hom}_{2\mathrm{Mot}}(\mathbf{1}, [-])$. By applying Theorem 3.8, we obtain a natural transformation after restricting to $(\mathsf{QProj}_S)^{\mathrm{op}}$. We wish to invoke Theorem 2.2 again and restrict to $\mathrm{Span}(\mathsf{QProj}_S)$ to obtain α. This requires verifying the condition in Theorem 2.2; namely, we need to show that

$$\begin{array}{ccc} \mathrm{SH}(X) & \xrightarrow{j^*} & \mathrm{SH}(U) \\ {\scriptstyle\alpha_X}\downarrow & & \downarrow{\scriptstyle\alpha_U} \\ D(X) & \xrightarrow{j^*} & D(U) \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{SH}(X) & \xrightarrow{p^*} & \mathrm{SH}(Y) \\ {\scriptstyle\alpha_X}\downarrow & & \downarrow{\scriptstyle\alpha_Y} \\ D(X) & \xrightarrow{p^*} & D(Y) \end{array}$$

for any open immersion $j\colon U \to X$ and projective morphism $p\colon Y \to X$ are left and right adjointable, respectively. The former follows, since it is compatible with smooth base change by construction. The latter follows from Proposition 3.10.

It remains to show that they are mutually inverse. By construction, GF is homotopic to id after composing $[-]\colon \mathsf{QProj}^{\mathrm{op}} \to 2\mathrm{Mot}(S)$, and hence GF is homotopic to id by universality. Hence it suffices to prove that FG is homotopic to id. By construction, it is homotopic to id under $(\mathsf{QProj}_S)^{\mathrm{op}}$, and by Theorem 2.2, under $\mathrm{Span}(\mathsf{QProj}_S)$ as well. We hence obtain the corresponding autoequivalence of D in $\mathrm{CAlg}(\mathrm{Fun}(\mathrm{Span}(\mathsf{QProj}_S), \mathsf{Pr}_{\mathrm{st}}))$, which is homotopic to id by Theorem 3.8. ☐

3.3. **2-motives of pullbacks.** Here we present an axiomatic method to obtain the relative Künneth formula:

Proposition 3.12. *Let $\mathcal{G}$ be a category with finite limits, $\mathcal{C}$ a presentably symmetric monoidal 2-category, and $[-]\colon \mathcal{G}^{\mathrm{op}} \to \mathcal{C}$ a symmetric monoidal functor. Suppose that I is a wide subcategory of $\mathcal{G}$ stable under base change and for $i \in I$, the morphism $i^* = [i]$ admits a fully faithful right adjoint i_* satisfying proper base change. Then for a pullback $Y' = X' \times_X Y$ in $\mathcal{G}$ such that the diagonal of X is in I, the canonical morphism $[X'] \otimes_{[X]} [Y] \to [Y']$ in $\mathcal{C}$ is an equivalence.*

Lemma 3.13. *Let $\mathcal{C}^\bullet$ be a cosimplicial object in* Cat *such that all cofaces are fully faithful. Suppose that*

$$\begin{array}{ccc} \mathcal{C}^m & \xrightarrow{d^0} & \mathcal{C}^{m+1} \\ \downarrow & & \downarrow \\ \mathcal{C}^n & \xrightarrow{d^0} & \mathcal{C}^{n+1} \end{array}$$

is right adjointable for any $[m] \to [n]$ in Δ. Then its totalization $\mathcal{C}^{-1}$ is identified with the full subcategory of $\mathcal{C}^0$ spanned by those objects C such that $d^0(C)$ is equivalent to $d^1(C)$.

Proof. We write $F\colon \mathcal{C}^{-1} \to \mathcal{C}^0$ for the tautological functor. By computing $\mathcal{C}^{-1}$ cosemisimplicially, we see that F is fully faithful. Moreover, it is clear that $C \in \mathcal{C}^0$ is in the image of F, then $d^0(C) \simeq d^1(C)$ holds.

We prove the converse. Suppose that $C \in \mathcal{C}^0$ satisfies $d^0(C) \simeq d^1(C)$. We apply [38, Theorem 4.7.5.2] to see that

$$\begin{array}{ccc} \mathcal{C}^{-1} & \xrightarrow{F} & \mathcal{C}^0 \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle d^1} \\ \mathcal{C}^0 & \xrightarrow{d^0} & \mathcal{C}^1 \end{array}$$

is right adjointable. By $C \simeq d^{0,\mathrm{R}}d^0(C) \simeq d^{0,\mathrm{R}}d^1(C) \simeq FF^{\mathrm{R}}(C)$, we see that C is in the image of F. □

Proof of Proposition 3.12. We realize $[X'] \otimes_{[X]} [Y]$ as the geometric realization of $[X' \times X^{\times \bullet} \times Y]$ in $\mathcal{C}$. Then by considering $\mathrm{Hom}_{\mathcal{C}}(-, C)$ for an arbitrary object C, this is reduced to Lemma 3.13. □

Remark 3.14. Another useful technique we can use is $X' \times_X Y = X \times_{X \times X} (X' \times Y)$; cf. the proof of Proposition 10.14.

Corollary 3.15. *In the situation of Theorem A, for a pullback $Y' = X' \times_X Y$ in QProj_S, the canonical morphism $[X'] \otimes_{[X]} [Y] \to [Y']$ is an equivalence in $2\mathrm{SH}(S)$.*

Corollary 3.16. *In the situation of Theorem A, for $X \in \mathsf{QProj}_S$, the morphism $2\mathrm{SH}(S) \to 2\mathrm{SH}(X)$ induces an equivalence $\mathrm{Mod}_{[X]}(2\mathrm{SH}(S)) \simeq 2\mathrm{SH}(X)$.*

Proof. This follows from Proposition 3.12 and [53, Proposition 4.8]. □

Corollary 3.17. *Let X be a divisorial noetherian static scheme and $X' \to X \leftarrow Y$ be quasiprojective morphisms. Then the canonical morphism*

$$2\mathrm{SH}(X') \otimes_{2\mathrm{SH}(X)} 2\mathrm{SH}(Y) \to 2\mathrm{SH}(X' \times_X Y)$$

is an equivalence.

Remark 3.18. Following [53, Section 3], we can also obtain the presentably symmetric monoidal n-category $n\mathrm{SH}(\mathbf{Z})$ of n-motives for $n \geq 3$. By the result above, it is the iterated module n-category over $2\mathrm{SH}(\mathbf{Z})$. In other words, stable motivic homotopy theory is 2-affine.

4. Stacks over a 2-category

In this section, we fix a presentably symmetric monoidal 2-category $\mathcal{C}$ and study the geometry of the following:

Definition 4.1. We call $\mathrm{CAlg}(\mathcal{C})^{\mathrm{op}}$ the category of *stacks* over $\mathcal{C}$. For a stack X, we write $[X]$ for the corresponding object in $\mathrm{CAlg}(\mathcal{C})$.

Remark 4.2. In the language of Scholze–Stefanich (cf. [47]), up to size restrictions, Definition 4.1 yields the category of 1-affine gestalts over $\mathrm{Gest}(\mathcal{C})$.

The following examples are particularly important:

Example 4.3. Let X be a locale. Then $\mathrm{Shv}(X)$ is 0-truncated stack over Pr; this follows from [5, Corollary 1.10] (it is more straightforward in this particular case). We write X for this where no confusion can arise.

Example 4.4. Let X be an accessible presheaf on $\mathsf{CAlg}^{\mathrm{op}}$, the opposite category of E_∞-rings. Then $\mathrm{D}(X) = \varprojlim_{\mathrm{Spec}(A) \to X} \mathrm{D}(A)$ is a stack over $\mathsf{Pr}_{\mathrm{st}}$.

In Section 4.1, we introduce some classes of morphisms of stacks. In Section 4.2, we explain a certain way to construct six operations. In Section 4.3, we explain the notion of ring stacks used in Theorems B to D. In Section 4.4, we explain the notion of absolute values on ring stacks used in Theorem D.

4.1. **Suave and prim maps.** See [32, Remark 4.4.2] for the origin of these terminologies.

Definition 4.5. We consider a morphism of stacks $Y \to X$. We call it *weakly suave* when $[X] \to [Y]$ admits an $[X]$-linear left adjoint. We call it *suave* if furthermore $[Y]$ is dualizable over $[X]$. We call it *prim* when $[X] \to [Y]$ admits an $[X]$-linear right adjoint.

Remark 4.6. In Definition 4.5, by Lemma 4.7 below, the extra condition for suaveness is equivalent to the assumption that the morphism $\mathrm{Mod}_{[X]}(\mathcal{C}) \to \mathrm{Mod}_{[Y]}(\mathcal{C})$ also admits a $\mathrm{Mod}_{[X]}(\mathcal{C})$-linear left adjoint.

Lemma 4.7. *Consider $A \in \mathrm{CAlg}(\mathcal{C})$. The following are equivalent:*

(i) The underlying object of A is dualizable.
(ii) The functor $\mathcal{C} \to \mathrm{Mod}_A(\mathcal{C})$ admits a $\mathcal{C}$-linear left adjoint.
(iii) The forgetful functor $\mathrm{Mod}_A(\mathcal{C}) \to \mathcal{C}$ admits a $\mathcal{C}$-linear right adjoint.

Proof. The equivalence (i) ⇔ (iii) is well known. The equivalence (i) ⇔ (ii) follows from taking the duals of both sides as $\mathcal{C}$-modules. □

Example 4.8. Let $\mathcal{G}$ be a category with finite limits. Suppose that $\mathcal{C}$ is obtained as the presentable 2-category of kernels from a lax symmetric monoidal functor $D\colon \mathrm{Span}(\mathcal{G}) \to \mathsf{Pr}_{\mathrm{st}}$. Then to check that $Y \to X$ in $\mathcal{G}$ determines a suave (or prim) morphism over $\mathcal{C}$, it suffices to check that $[X] \to [Y]$ has an $[X]$-linear left (or right) adjoint.

Example 4.9. Any étale geometric morphism of toposes $\mathcal{Y} \to \mathcal{X}$, i.e., a morphism of the form $\mathcal{X}_{/X} \to \mathcal{X}$ for some object $X \in \mathcal{X}$, determines a weakly suave morphism of stacks over Pr; see [37, Remark 6.3.5.12].

Example 4.10. Not every proper morphism $\mathcal{Y} \to \mathcal{X}$ of toposes (see [37, Definition 7.3.1.4]) determines a prim morphism of stacks over Pr. However, $\mathrm{Shv}(\mathcal{X}; \mathsf{Sp}) \to \mathrm{Shv}(\mathcal{Y}; \mathsf{Sp})$ is a prim morphism of stacks over $\mathsf{Pr}_{\mathrm{st}}$; see, e.g., [2, Lemma 6.7].

Example 4.11. Let $Y \to X$ be a morphism of quasicompact quasiseparated schemes. Then the corresponding morphism of stacks over $\mathsf{Pr}_{\mathrm{st}}$ is prim.

Definition 4.12. An *open immersion* is a suave monomorphism. A *closed immersion* is a prim monomorphism.

Lemma 4.13. *Let $f\colon Y \to X$ be a morphism of stacks. It is an open (or closed) immersion if and only if $[X] \to [Y]$ admits a fully faithful $[X]$-linear left (or right) adjoint.*

Proof. We only treat the open immersion case; the closed immersion case is easier.

We first prove the "only if" direction. Since f is weakly suave, it admits a linear left adjoint $f_\natural$. The left adjointability of the square

$$\begin{array}{ccc} [X] & \longrightarrow & [Y] \\ \downarrow & & \downarrow \\ [Y] & \longrightarrow & [Y] \end{array}$$

implies that the unit $\mathrm{id} \to f^* f_\natural$ is an equivalence.

We then prove the "if" direction. In this situation, $[Y] \to [Y] \otimes_{[X]} [Y]$ admits a fully faithful left adjoint, and since

$$\begin{array}{ccc} [X] & \longrightarrow & [Y] \\ \downarrow & & \downarrow \\ [Y] & \longrightarrow & [Y] \otimes_{[X]} [Y] \end{array}$$

is left adjointable, it must be an equivalence. Therefore, it is a monomorphism. By definition, it is weakly suave. We see that $[Y]$ is self-dual over $[X]$ in this situation, so it is suave. □

Definition 4.14. We consider a static morphism of stacks. We call a static map of stacks *unramified* if its diagonal is an open immersion. We call it *étale* if it is suave and unramified. We call it *separated* if its diagonal is a closed immersion. We call it *proper* if it is prim and separated.

4.2. **Six operations for open stacks.** We need to construct this in a situation where we do not have access to compactification. We use Theorem 2.2 to construct this.

Definition 4.15. We say a morphism of stacks $Y \to X$ *exceptional* if it is n-truncated for some n and the object $[Y]$ is dualizable over $[Y^{\times_X S^{n-1}}] = [Y]^{\otimes_{[X]} S^{n-1}}$ for any $n \geq 0$.

Proposition 4.16. *Exceptional morphisms are closed under base change and compositions and satisfy cancellation.*

Lemma 4.17. *For morphism $A \to B \to C$ in $\mathrm{CAlg}(\mathcal{C})$, when B is dualizable over A and C is dualizable over B, then C is dualizable over A.*

Proof. This follows from Lemma 4.7. □

Proof of Proposition 4.16. Base change is straightforward. We consider morphisms of stacks $Z \to Y \to X$. Composition follows from Lemma 4.17 and the diagram

$$\begin{array}{ccccc} Z & \longrightarrow & Z^{\times_Y S^{n-1}} & \longrightarrow & Z^{\times_X S^{n-1}} \\ & & \downarrow & & \downarrow \\ & & Y & \longrightarrow & Y^{\times_X S^{n-1}}, \end{array}$$

where the square is a pullback. The cancellation property follows from Lemma 4.17 and the diagram

$$\begin{array}{ccccccc} Z & \longrightarrow & Z^{\times_{Y\times_X Z} S^{n-1}} & = & Z^{\times_{Y\times_X Z} S^{n-1}} & \longrightarrow & Z^{\times_Y S^{n-1}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & Y^{\times_X S^{n}} & & Z & \longrightarrow & Z^{\times_X S^{n-1}}, \end{array}$$

where both squares are pullbacks. □

Consequently, the relevant span category is well defined. We prove the following:

Theorem 4.18. *We write $\mathcal{G}$ for the category of stacks and E for the class of exceptional morphisms. Then we have a canonical symmetric monoidal functor $\mathrm{Span}_E(\mathcal{G}) \to \mathcal{C}$ extending $\mathcal{G}^{\mathrm{op}} \to \mathcal{C}$.*

Proof. We consider $\mathcal{G}^{\mathrm{op}} \to \mathrm{Mod}_{\mathcal{C}}(2\mathsf{Pr})$ given by $X \mapsto \mathrm{Mod}_{[X]}(\mathcal{C})$. We claim that this satisfies (E,E)-biadjointability condition of Theorem 2.2. If this is the case, we obtain a canonical extension $2\mathrm{Span}_{E;E,E}(\mathcal{G}) \to \mathrm{Mod}_{\mathcal{C}}(2\mathsf{Pr})$ by Theorem 2.2, and by looking at $\mathrm{End}(\mathbf{1})$, we obtain the desired functor.

We need to see that the biadjointability condition is satisfied; the projection formula is automatic by symmetric monoidality. The Beck–Chevalley condition for $P = E$ is clear. For $J = E$, when $Y \to X$ is exceptional, we first see that $\mathrm{Mod}_{[X]}(\mathcal{C}) \to \mathrm{Mod}_{[Y]}(\mathcal{C})$ admits a left adjoint given by $[Y]^\vee \otimes_{[Y]} -$, where $[Y]^\vee$ denotes the dual of $[Y]$ over $[X]$. By considering the right adjoints of all morphisms appearing in the Beck–Chevalley transformation, the Beck–Chevalley condition is reduced to the Beck–Chevalley condition for P. To check the double Beck–Chevalley condition, again we can consider the right adjoints. □

4.3. **Ring stacks.** We define these notions using Lawvere theories.

Definition 4.19. We write Lat^+, Lat, and Pol for the full subcategories of abelian monoids, abelian groups, and commutative rings, respectively, spanned by $\mathbf{N}^n$, $\mathbf{Z}^n$, and $\mathbf{Z}[T_1, \dots, T_n]$ for $n \geq 0$.

Definition 4.20. A *monoid stack*, *group stack*, and *ring stack* are functors $\mathsf{Lat}^+ \to \mathrm{CAlg}(\mathcal{C})$, $\mathsf{Lat} \to \mathrm{CAlg}(\mathcal{C})$, and $\mathsf{Pol} \to \mathrm{CAlg}(\mathcal{C})$, respectively, preserving finite coproducts.

Remark 4.21. In Definition 4.20, it is more precise to say that they are strict or animated, since they are not the free ones in the categorical sense. In this paper, we restrict our attention to such situations.

Example 4.22. The left adjoint of the "multiplicative" forgetful functor from commutative rings to abelian monoids is restricted to $\mathsf{Lat}^+ \to \mathsf{Pol}$. By composing this with a ring stack, we obtain the monoid stack of the underlying multiplicative monoid. Similarly, we have the underlying additive group stack by composing with $\mathsf{Lat} \to \mathsf{Pol}$.

Moreover, we also obtain the multiplicative group stack of invertible elements by considering $\mathsf{Lat} \to \mathrm{PShv}_\Sigma(\mathsf{Pol})$ given by $\mathbf{Z}^n \mapsto \mathbf{Z}[\mathbf{Z}^n]$.

Example 4.23. When X is a monoid locale, then $\mathrm{Shv}(X)$ naturally becomes a monoid stack in $2\mathsf{Pr}$.

Two examples are particularly important. First, the Sierpiński space $\{s, \eta\}$, where s is special and η is generic with multiplication given by $s \cdot s = s$, $s \cdot \eta = s$, and $\eta \cdot \eta = \eta = 1$ is a monoid locale. Second, $[0, \infty)$ with the usual multiplication is also a monoid locale. Note that $\{s, \eta\}$ is a quotient of $[0, \infty)$ as a monoid locale by $0 \mapsto s$ and $r \mapsto \eta$ for $r > 0$.

Definition 4.24. We call a ring stack *sutured* if the diagram $R^\times \to R \xleftarrow{0} *$ of stacks induces a recollement in $\mathrm{Mod}_{[R]}(\mathcal{C})$ (cf. Remark 2.14).

Lemma 4.25. *The diagonal of a sutured ring stack is a closed immersion. In particular, a sutured ring stack is static.*

Proof. By considering the shearing map $(x, y) \mapsto (x, x - y)$, the diagonal $R \to R \times R$ is isomorphic to $\mathrm{id} \times 0 \colon R \times * \to R \times R$. Since $0 \colon * \to R$ is a base change of the inclusion of the closed point to the Sierpiński space, the desired result follows. ☐

Definition 4.26. We call a sutured weakly suave ring stack *stable* if $f_\natural s_*$ is invertible, where $s \colon * \to R$ is the zero section.

Lemma 4.27. *A stable sutured weakly suave ring stack is suave and exceptional.*

Proof. We show that $[R]$ is self-dual. We write $f \colon R \to *$ for the tautological map and $d \colon R \to R \times R$ for the diagonal, which is a closed immersion by Lemma 4.25. We claim that

$$\eta \colon [*] \xrightarrow{f^*} [R] \xrightarrow{d_*} [R \times R], \qquad \epsilon \colon [R \times R] \xrightarrow{d^*} [R] \xrightarrow{f_\natural} [*]$$

are the unit and counit of the duality. To prove this, we consider the diagram

$$\begin{array}{ccccc}
R & \xrightarrow{\quad d \quad} & R \times R & \xrightarrow{\quad \mathrm{pr}_2 \quad} & R \\
{\scriptstyle d}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \times d} & & \\
R \times R & \xrightarrow{d \times \mathrm{id}} & R \times R \times R & & \\
{\scriptstyle \mathrm{pr}_1}\downarrow & & & & \\
R, & & & &
\end{array}$$

where the square is a pullback. By proper base change, we obtain one triangle; note that the composite $\mathrm{pr}_1 d$ is homotopic to id by shearing as in the proof of Lemma 4.25. The other triangle is similar.

Since $[R]$ is also self-dual over $[R] \otimes [R]$ by Lemma 4.25, it is exceptional. ☐

Definition 4.28. We call a sutured ring stack *smooth* if it is suave (and then it is automatically exceptional by Lemma 4.25 hence six operations make sense) and $f^! \mathbf{1}$ is invertible in $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, [R])$.

Proposition 4.29. *For a sutured ring stack, it is smooth if and only if it is weakly suave and stable.*

Proof. We write $\mathcal{G}$ for the category of exceptional stacks and consider

$$D = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, [-]) \colon \mathrm{Span}(\mathcal{G}) \to \mathsf{Pr}$$

obtained from Theorem 4.18. Note that in both situations, $R = \mathbb{G}_a$ is exceptional by Lemmas 4.25 and 4.27. By the extension result of Mann [40, Section A.5] (see also [32, Section 3.4]), we extend this to the situation where the tautological morphism $t\colon * \to B\mathbb{G}_a$ is shriekable. We consider the diagram

$$\begin{array}{ccccc} * & \xrightarrow{s} & \mathbb{G}_a & \xrightarrow{f} & * \\ & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle t} \\ & & * & \xrightarrow{t} & B\mathbb{G}_a. \end{array}$$

By $f^!\mathbf{1} \simeq f^!t^*\mathbf{1} \simeq f^*t^!\mathbf{1}$, we see that R is smooth if and only if $L = t^!\mathbf{1}$ is invertible. Since $f_!s_* \simeq f_!s_! \simeq \mathrm{id}$, we have

$$L \simeq s^*f^!\mathbf{1} \simeq f_!s_*s^*f^!\mathbf{1} \simeq f_!(f^!\mathbf{1} \otimes s_*\mathbf{1}) \simeq f_\natural s_*\mathbf{1}.$$

Therefore, stability is equivalent to the invertibility of L, and hence the desired result follows. □

4.4. **Absolute values.**

Definition 4.30. An *absolute semivalue* on a ring stack R over $\mathcal{C}$ is a multiplicative map $N\colon R \to [0,\infty)$, i.e., a 2-cell

$$\begin{array}{ccc} \mathsf{Lat}^+ & & \\ \downarrow & \overset{\mathrm{Shv}([0,\infty))}{\searrow} & \\ \mathsf{Pol} & \xrightarrow[R]{\Downarrow} & \mathrm{CAlg}(\mathcal{C}), \end{array}$$

where we use Examples 4.22 and 4.23, satisfying the following conditions:

- We have $N(0) \leq 0$; i.e., the morphism $0\colon * \to R$ factors through $\{0\} \to [0,\infty)$, which is a monomorphism.
- It satisfies the triangle inequality $N(x+y) \leq N(x) + N(y)$; i.e., we have $N(N^{-1}([0,r]) + N^{-1}([0,s])) \subset [0, r+s]$ for r, $s \in [0,\infty)$.

We call it an *absolute value* if it moreover satisfies the following:

- We have $N^{-1}(\{0\}) = \{0\}$.
- We have $N^{-1}((0,\infty)) = R^\times$.

Remark 4.31. A ring stack with an absolute value is automatically sutured. In fact, a sutured ring stack can be formulated in a similar way to Definition 4.30 using the Sierpiński space S instead of $[0,\infty)$ (see Example 4.23).

Remark 4.32. In Definition 4.30, certain completeness is inherent already when we consider a morphism $R \to [0,\infty)$; this is because $[0,\infty)$ automatically has the overconvergence property and it is hence inherited by R.

We resume the study of this notion in Section 10, where we prove Theorem D. We conclude this section by describing what it means to have a multiplicative map $R \to [0,\infty)$ concretely:

Lemma 4.33. *Let X be a stack. The morphism $X \to [0,\infty)$ is a specification of idempotent algebras $\cdots \to D_r \to \cdots \to D_0$ for $r \in [0,\infty)$ and $\mathbf{1} = E_0 \to \cdots \to E_r \to \cdots$ for $r \in [0,\infty)$ in $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1},[X])$ satisfying the following:*

(i) We have $\varinjlim_{r'>r} D_{r'} = D_r$.
(ii) We have $\varinjlim_{r'<r} E_{r'} = E_r$.
(iii) and $D_r \vee E_r = \mathbf{1}$ for $r \geq 0$.
(iv) We have $D_r \otimes E_s = 0$ for $0 \leq r < s$.

Proof. By [5], we have to check that the universal frame described by these conditions coincides with $[0,\infty)$.

First, (i) describes the topology on $[0,\infty)$ such that (r,∞) is open for $r \geq 0$. Then (ii) describes the topology on $[0,\infty)$ such that $[0,r)$ is open for $r \geq 0$. This means that the universal locale is a sublocale of their product. The conditions (iii) and (iv) restricts this to the diagonal part of this space, which is precisely the usual topology on $[0,\infty)$. □

Lemma 4.34. *Let M be a monoid stack. A multiplicative map $M \to [0,\infty)$ is a datum described in Lemma 4.33 satisfying*

$$\{1\} \subset D_1, \qquad D_r \cdot D_s \subset D_{rs}, \qquad \{1\} \subset E_1, \qquad E_r \cdot E_s \subset E_{rs}$$

for r and $s \in [0,\infty)$, where we identify idempotent algebras with the corresponding closed substacks; e.g., $D_r \cdot D_s \subset D_{rs}$ means that the action of $[M]$ on $\operatorname{Mod}_{D_r}[M] \otimes \operatorname{Mod}_{D_s}[M]$ coming from the binary multiplication on M factors through $\operatorname{Mod}_{D_{rs}}[M]$.

Proof. By the 0-truncatedness of $[0,\infty)$, multiplicativity is a condition, which is precisely the commutativity (which is a condition) of the diagrams

$$\begin{array}{ccc} * & \longrightarrow & * \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ M & \longrightarrow & [0,\infty), \end{array} \qquad\qquad \begin{array}{ccc} M^2 & \longrightarrow & [0,\infty)^2 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\ M & \longrightarrow & [0,\infty), \end{array}$$

where m is the multiplication map. Since $1\colon * \to [0,\infty)$ corresponds to $D_1 \otimes E_1$ in the universal case, we see that the commutativity of the first square is equivalent to $\{1\} \subset D_1$ and $\{1\} \subset E_1$. We consider the second square. Similarly, we can match the rest since

$$m^*(D_t) = \bigvee_{rs\le t} D_r \boxtimes D_s, \qquad\qquad m^*(E_t) = \bigvee_{rs\le t} E_r \boxtimes E_s$$

holds in the universal case. □

Lemma 4.35. *Let R be a ring stack. Then an absolute semivalue on R is a datum as in Lemma 4.34 satisfying $\{0\} \subset D_0$ and $D_r \cdot D_s \subset D_{r+s}$. It is an absolute value if furthermore $\{0\} = R \setminus R^\times = D_0$ is satisfied.*

Proof. This is straightforward. □

5. Descent

We develop an axiomatic theory of descent. In this paper it serves not only as a prerequisite for the statements of Theorem C, but also as a powerful ingredient in the proof of Theorem B.

After introducing the notion in Section 5.1, we establish its basic properties in Section 5.2. In Section 5.3, we show that, in the prim case, the condition admits a simpler formulation, which connects to the notion of descendability, which we discuss in Section 5.4. In Section 5.5 we return to the theory of stacks to provide some applications. We see many examples of descent in Section 6.

5.1. **Descent.**

We first introduce the following condition:

Definition 5.1. In a presentably symmetric monoidal n-category, we say that an E_0-coalgebra, i.e., a morphism $C \to \mathbf{1}$, is *faithful* if the augmented semisimplicial object $C^{\otimes \bullet+1}$ is a colimit diagram.

Proposition 5.2. *Let A be a commutative algebra object in a presentably symmetric monoidal n-category $\mathcal{C}$. We consider the following conditions:*

- *(i) The dual $A^\vee$ is faithful in the sense of Definition 5.1.*
- *(ii) The morphism $|\operatorname{Mod}_{A^{\otimes \bullet+1}}(\mathcal{C})| \to \mathcal{C}$ (where transitions are forgetful functors) admits a fully faithful right adjoint in $\operatorname{Mod}_{\mathcal{C}}(n\mathsf{Pr})$.*
- *(iii) The morphism in (ii) is an equivalence, i.e., $\operatorname{Mod}_A(\mathcal{C}) \to \mathcal{C}$ is faithful in $\operatorname{Mod}_{\mathcal{C}}(n\mathsf{Pr})$ in the sense of Definition 5.1.*

Then we have (iii) ⇒ (ii) ⇒ (i). If A underlies a dualizable object in $\mathcal{C}$, we have (i) ⇒ (ii). If A is moreover dualizable as an $A \otimes A$-module, we also have (ii) ⇒ (iii).

Proof. We assume that A underlies a dualizable object.

The condition (iii) asks the morphism $|\mathrm{Mod}_{A^{\otimes \bullet+1}}(\mathcal{C})^\vee| \to \mathcal{C}$ is an equivalence. By the self-duality of $\mathrm{Mod}_B(\mathcal{C})$ for any commutative algebra object B, this is equivalent to asking $L\colon |\mathrm{Mod}_{A^{\otimes \bullet+1}}(\mathcal{C})| \to \mathbf{1}$ is an equivalence, where the transitions for the geometric realization are forgetful functors. We forget degeneracies and consider the semisimplicial colimit. By the dualizability of A, every transition admits a right adjoint (here we use the fact that we forget degeneracies) in $\mathrm{Mod}_{\mathcal{C}}(n\mathsf{Pr})$. This means that the source of L can be computed as a limit and L admits a right adjoint R in $\mathrm{Mod}_{\mathcal{C}}(n\mathsf{Pr})$, which can be described as $C \mapsto ((A^\vee)^{\otimes \bullet+1} \otimes C)$. With this limit description, L can be expressed as $(M_\bullet) \mapsto |M_\bullet|$.

We first consider the condition that $\epsilon\colon LR \to \mathrm{id}$ is an equivalence, which is exactly (ii). This means that $|(A^\vee)^{\otimes \bullet+1} \otimes C| \to C$ is an equivalence for any $C \in \mathcal{C}$. This is equivalent to (i).

Now we furthermore assume that A is dualizable as an $A \otimes A$-module. We prove that $\eta\colon \mathrm{id} \to RL$ is an equivalence under (i). In this situation, degeneracies also admit right adjoints, so that we can consider the entire simplicial diagram to describe $|\mathrm{Mod}_{A^{\otimes \bullet+1}}(\mathcal{C})|$. The condition means that for any compatible family $M_\bullet$, the morphism $M_n \to (A^\vee)^{\otimes n+1} \otimes |M_\bullet|$ is an equivalence for $n \geq 0$. This reduces to the case $n = 0$, where the right-hand side is equivalent to $|(A^\vee) \otimes M_\bullet|$, which splits. □

Definition 5.3. Let $\mathcal{C}$ be a presentably symmetric monoidal n-category and A a commutative algebra object. We say that A satisfies *descent* if (iii) of Proposition 5.2 holds. We say that a morphism $A \to B$ satisfies *descent* if B satisfies descent as a commutative algebra object of $\mathrm{Mod}_A(\mathcal{C})$.

We record the following from Proposition 5.2 and its proof:

Corollary 5.4. *In the situation of Definition 5.3, the following are equivalent:*

(i) The commutative algebra object A satisfies descent.
(ii) The commutative algebra object $\mathrm{Mod}_A(\mathcal{C})$ in $\mathrm{Mod}_{\mathcal{C}}(n\mathsf{Pr})$ satisfies descent.
(iii) The functor $\mathrm{Mod}_{\mathcal{C}}(n\mathsf{Pr}) \to \mathrm{Tot}(\mathrm{Mod}_{\mathrm{Mod}_{A^{\otimes \bullet+1}}(\mathcal{C})}(n\mathsf{Pr}))$ is fully faithful.
(iv) The functor $\mathrm{Mod}_{\mathcal{C}}(n\mathsf{Pr}) \to \mathrm{Tot}(\mathrm{Mod}_{\mathrm{Mod}_{A^{\otimes \bullet+1}}(\mathcal{C})}(n\mathsf{Pr}))$ is an equivalence.
(v) For any $\mathcal{C}$-module $\mathcal{M}$, the morphism $\mathcal{M} \to \mathrm{Tot}(\mathrm{Mod}_{A^{\otimes \bullet+1}}(\mathcal{M}))$ is an equivalence.

Note that all totalizations above exist for any A.

We conclude this section with the following criterion of descent for suave maps; cf. [49, Proposition 6.19]. We treat prim descent in Section 5.3 below.

Proposition 5.5. *Let $\mathcal{C}$ be a presentably symmetric monoidal $(n+1)$-category such that $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$ is presentable. Let A be a commutative algebra object in $\mathcal{C}$ such that $f^*\colon \mathbf{1} \to A$ admits a left adjoint $f_\natural$ and A is dualizable. Then $A^\vee$ is faithful if and only if $(f^*)^\vee (f_\natural)^\vee$ is faithful in $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$.*

Proof. By arguing as in the proof of (ii) ⇒ (i) in Proposition 5.2, we obtain the "only if" direction.

We prove the "if" direction. By the dualizability of A, we obtain a cocommutative coalgebra $A^\vee$. Therefore, the faithfulness condition is about the augmented simplicial object $(A^\vee)^{\otimes \bullet+1}$ being a colimit diagram. By arguing as in the proof of (i) ⇒ (iii) in Proposition 5.2, we obtain the desired result. □

Since the condition of descent involves higher categories, it is in general not true that we can impose it universally. Still, note the following, which is the case we need in Theorem C:

Lemma 5.6. *Let $\mathcal{C}$ be a presentably symmetric monoidal n-category and $A \to B$ a morphism of commutative algebra objects. If B is dualizable both over A and $B \otimes_A B$, there exists a universal morphism $\mathcal{C} \to \mathcal{D}$ that makes this morphism satisfy descent.*

Proof. This directly follows from Proposition 5.2: Imposing $B_A^\vee$ to be faithful in $\mathrm{Mod}_A(\mathcal{C})$ is the same as imposing $(B_A^\vee)^{\otimes \bullet+1}$ to be a colimit diagram in $\mathcal{C}$. Hence we can localize $\mathcal{C}$ along this condition. □

Remark 5.7. In general, if we wish to impose descent, one must sacrifice affineness: When we work in eventually affine presentable categorical spectra (see [4, Remark 2.9]), we can always impose descent by losing affineness by 1.

5.2. **Basic properties.** We then record some basic permanence properties:

Lemma 5.8. *Let $\mathcal{C} \to \mathcal{D}$ be a morphism of presentably symmetric monoidal n-categories. Then it maps a commutative algebra object satisfying descent to a commutative algebra object satisfying descent.*

Proof. This directly follows from the definition. □

Lemma 5.9. *Let $\mathcal{C}$ be a presentably symmetric monoidal n-category and A and B commutative algebra objects.*

(1) *When A satisfies descent, so does $B \to A \otimes B$.*
(2) *When B and $B \to A \otimes B$ satisfy descent, so does A.*

Proof. First, (1) follows from Lemma 5.8.

We consider (2). We wish to show that $\mathrm{Mod}_A(\mathcal{C})$ is faithful in $\mathrm{Mod}_{\mathcal{C}}(n\mathsf{Pr})$. Since B satisfies descent, it suffices to show that $\mathrm{Mod}_{A\otimes B}(\mathcal{C})$ is faithful in $\mathrm{Mod}_{\mathrm{Mod}_B(\mathcal{C})}(n\mathsf{Pr})$, which follows from the assumption that $B \to A \otimes B$ satisfies descent. □

Lemma 5.10. *Let $\mathcal{C}$ be a presentably symmetric monoidal n-category and $A \to B$ a morphism of commutative algebra objects.*

(1) *If A and $A \to B$ satisfy descent, then so is B.*
(2) *When B satisfies descent, so is A.*

Proof. We first prove (2). By (2) of Lemma 5.9, it suffices to prove that $B \to B \otimes A$ satisfies descent. This morphism admits a retraction and hence satisfies descent by Lemma 5.11 below.

We then prove (1). By (2) Lemma 5.9, we need to show that $A \to A \otimes B$ satisfies descent. Since the composite $A \to A \otimes B \to B$ satisfies descent, the desired result follows from (2). □

For the following, we recall the notion of *n-retract* in an ∞-category, which was introduced as *n-condensation* by Gaiotto–Johnson-Freyd [27]: Consider a pair of morphisms $s\colon C \to D$ and $r\colon D \to C$. We say that s is a 0-*section* and r is a 0-*retraction* if they are equivalences. For $n \geq 0$, we say that s is an $(n+1)$-*section* and r is an $(n+1)$-*retraction* if there is an n-section $\mathrm{id} \to rs$ and an n-retraction $rs \to \mathrm{id}$.

Lemma 5.11. *Let $\mathcal{C}$ be a presentably symmetric monoidal n-category and A a commutative algebra object. When the underlying morphism of $\mathbf{1} \to A$ in $\mathcal{C}$ admits an n-retraction, it satisfies descent.*

Proof. Note that $\mathcal{C} \to \mathrm{Mod}_A(\mathcal{C})$ in $\mathrm{Mod}_{\mathcal{C}}(n\mathsf{Pr})$ admits an $(n+1)$-retraction. Hence by replacing A with $\mathrm{Mod}_A \mathcal{C}$, we can assume that A is rigid. Now, we need to check that the augmented simplicial diagram $A^{\otimes \bullet+1}$ is a colimit diagram. This is an n-retract of $A \otimes A^{\otimes \bullet+1}$, which is a colimit diagram by splitting. Therefore, the desired claim follows. □

5.3. **Prim descent.** By definition, to check descent, we need to go higher. Here we consider how to go lower.

Lemma 5.12. *Let $\mathcal{C}$ be a presentably symmetric monoidal $(n+1)$-category such that $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$ is presentable. Consider a commutative algebra object A in $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$. Then the following are equivalent:*

(i) *The commutative algebra A satisfies descent in $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$.*
(ii) *The commutative algebra $\mathrm{Mod}_A(\mathbf{1})$ satisfies descent in $\mathcal{C}$.*

Proof. Consider the full faithful embedding $\mathrm{Mod}_{\mathrm{End}_{\mathcal{C}}(\mathbf{1})}(n\mathsf{Pr}) \to \mathcal{C}$. Then (i) is equivalent to the condition that the augmented simplicial diagram $\mathrm{Mod}_{A^{\bullet+1}}(\mathcal{C})$ is a colimit diagram in the source. Then (ii) is equivalent to the same condition in the target. □

Theorem 5.13. *Let $\mathcal{C}$ be a presentably symmetric monoidal $(n+1)$-category such that $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$ is presentable. Consider a commutative algebra object A such that the unit $u\colon \mathbf{1} \to A$ admits a right adjoint u^{R}. Then the following are equivalent:*

(i) *The commutative algebra A in $\mathcal{C}$ satisfies descent.*

(ii) The commutative algebra $u^{\mathrm{R}}u$ in $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$ *satisfies descent.*

Proof. We decompose u as $\mathbf{1} \to \mathrm{Mod}_{u^{\mathrm{R}}u}(\mathbf{1}) \to A$. By Lemma 5.12, the condition (ii) is equivalent to that the first map satisfies descent. Hence by Lemma 5.10, it suffices to observe that $\mathrm{Mod}_{u^{\mathrm{R}}u}(\mathbf{1}) \to A$ satisfies descent, which follows from Lemma 5.11. □

Example 5.14. Let $f\colon Y \to X$ be a morphism of quasicompact quasiseparated (spectral) schemes. Then $\mathrm{D}(X) \to \mathrm{D}(Y)$ satisfies descent in $\mathsf{Pr}_{\mathrm{st}}$ if and only if $f_*(\mathbf{1}_Y)$ satisfies descent in $\mathrm{D}(X)$.

5.4. **Descendability.** As we see in Theorem 5.13, in a nice situation, we can reduce the descent condition to something lower categorical. The following notion of Mathew [41] (see also [12]) is useful in such a situation:

Definition 5.15. Let $\mathcal{C}$ be a stable presentably symmetric monoidal category and A a commutative algebra object. We say that A is *descendable* if the Pro-object associated to the Adams tower is equivalent to the constant Pro-object $\mathbf{1}$. By [14, Lemma 11.20], it is equivalent to the condition that the tautological map $I^{\otimes d} \to \mathbf{1}$ is zero for some $d \geq 0$, where $I = \mathrm{fib}(\mathbf{1} \to A)$. In this case we say that A is descendable of index $\leq d$.

First, we observe the following (cf. [41, Corollary 3.42]):

Proposition 5.16. *In the situation of Definition 5.15, if A is descendable, it satisfies descent.*

Proof. We write I for $\mathrm{fib}(\mathbf{1} \to A)$. By Corollary 5.4, it suffices to show that for any $\mathcal{C}$-module $\mathcal{M}$, the morphism

$$\mathcal{M} \to \mathrm{Tot}\big(\mathrm{Mod}_{A^{\otimes \bullet+1}}(\mathcal{M})\big),$$

where the transitions are base change, is an equivalence. By [38, Corollary 4.7.5.3], it suffices to see that $F\colon \mathcal{M} \to \mathrm{Mod}_A(\mathcal{M})$ is conservative and preserve F-split totalizations.

First we prove conservativity. Suppose that $A \otimes M \simeq 0$ for $M \in \mathcal{M}$. Then $I \otimes M \to M$ is an equivalence, but then $I^{\otimes n} \otimes M \to M$ is zero and an equivalence for $n \gg 0$ and therefore $M \simeq 0$.

We then consider an F-split cosimplicial object $M\colon \Delta \to \mathcal{M}$. We claim that the Pro-object associated to M is constant. Similarly in this case, the cofiber of $I^n \otimes M \to M$ satisfies that condition for $n \geq 0$. For $n \gg 0$, this contains M as a direct summand. □

The converse is false:

Example 5.17. Let $R_n \to S_n$ be a morphism of (static) rings that is descendable but $\mathrm{fib}(R_n \to S_n)^{\otimes n} \to R_n$ is nonzero; see [55] or [3] for a construction. We consider nonquasicompact schemes $X = \coprod_n \operatorname{Spec} R_n$ and $Y = \coprod_n \operatorname{Spec} S_n$ and the induced morphism $f\colon Y \to X$. Then Proposition 5.16 implies that $f_*(\mathbf{1}_Y) \in \mathrm{D}(X)$ satisfies descent, but it is not descendable by assumption.

Example 5.18. Fix a prime p and consider $\mathrm{Fun}(BC_p, \mathrm{D}(\mathbf{F}_p))$. Then the regular representation $\mathbf{F}_p[C_p]$ is not descendable. This is because of a morphism $\mathbf{F}_p[-1] \to \mathrm{fib}(\mathbf{F}_p \to \mathbf{F}_p[C_p])$ corresponding to the generator of of the cohomology ring $H^*(BC_p; \mathbf{F}_p)$, whose power does not vanish. However, still by Proposition 6.19, it satisfies descent.

The units in both examples above are not compact. In fact, this is the only obstruction:

Proposition 5.19. *In the situation of Definition 5.15, suppose that $\mathbf{1} \in \mathcal{C}$ is compact.*[8] *Then if A satisfies descent, it is descendable.*

Proof. We write T^n for the Adams tower of A, whose constancy we wish to prove. We write $[-,-]$ for the internal mapping object in $\mathcal{C}$. For $C \in \mathcal{C}$, we have

$$\varinjlim_n \mathrm{Map}_{\mathcal{C}}(T^n, C) \simeq \varinjlim_n \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, [T^n, C]) \simeq \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, \varinjlim_n [T^n, C]) \simeq \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, |[A^{\otimes \bullet+1}, C]|)$$

[8]Or more generally, we can only assume that it is sequentially compact, i.e., $\mathrm{Map}(\mathbf{1}, -)$ commutes with sequential colimits.

by the (sequential) compactness of C. Hence it suffices to show that the morphism

$$|[A^{\otimes \bullet+1}, C]| \to C \tag{5.20}$$

is an equivalence for any $C \in \mathcal{C}$.

We take a regular cardinal κ satisfying the following:

- The presentably symmetric monoidal category $\mathcal{C}$ belongs to $\mathrm{CAlg}(\mathsf{Pr}^{\kappa})$.
- The functor $C \mapsto [A, C]$ preserves κ-filtered colimits.

Under this, it suffices to verify that (5.20) is an equivalence for $C \in \mathcal{C}_{\kappa}$.

We now take a regular cardinal $\lambda \geq \kappa$ satisfying the following:

- The full subcategory $\mathcal{C}_{\lambda}$ is closed under κ-small limits.
- When C is κ-compact and D is λ-compact, $[C, D]$ is λ-compact.

We then consider $\mathcal{D} = \mathrm{Ind}_{\kappa}((\mathcal{C}_{\lambda})^{\mathrm{op}})$ as a $\mathcal{C}$-module such that the action of $\mathcal{C}_{\kappa}$ on $\mathcal{D}$ restricts to $(\mathcal{C}_{\lambda})^{\mathrm{op}}$ and is given by $(C, D) \mapsto [C, D]$. Since $\mathcal{D} \to \mathrm{Tot}(\mathrm{Mod}_{A^{\otimes \bullet+1}}(\mathcal{D}))$ is an equivalence by assumption, we see that $|A^{\otimes \bullet+1} \otimes D| \to D$ is an equivalence for $D \in \mathcal{D}$. Unwinding this for $D \in \mathcal{D}_{\kappa} = (\mathcal{C}_{\lambda})^{\mathrm{op}}$, we obtain the desired result. ☐

We conclude this section by collecting some facts about descendability:

Lemma 5.21. *Let $\mathcal{C}$ be a stable presentably symmetric monoidal category and $A \to B$ a morphism of commutative algebra objects.*

1. *When B is descendable of index $\leq e$, then so is A.*
2. *When A is descendable of index $\leq d$ and B is descendable of index $\leq e$ in $\mathrm{Mod}_A(\mathcal{C})$, then B is descendable of index $\leq de$.*

Proof. We write I, J, and K for the fibers of $\mathbf{1} \to A$, $\mathbf{1} \to B$, and $A \to B$, respectively. We have a cofiber sequence $I \to J \to K$. Since $I \to \mathbf{1}$ factors through $J \to \mathbf{1}$, we obtain (1). Under the assumption of (2), we see that $J^{\otimes e} \to \mathbf{1} \to A$ is zero, which implies $J^{\otimes e} \to \mathbf{1}$ factors through I. Therefore, we see the desired claim. ☐

The following is from [14, Lemma 11.22]:

Lemma 5.22. *Let $\mathcal{C}$ be a stable presentably symmetric monoidal category and $A_0 \to \cdots$ a sequence of commutative algebra objects. If A_n is descendable of index $\leq d$ for any n, then $\varinjlim_n A_n$ is descendable of index $\leq 2d$.*

Proof. We write I_n for $\mathrm{fib}(\mathbf{1} \to A_n)$. By the Milnor sequence

$$0 \to \varprojlim_n{}^1 \pi_1 \mathrm{Map}(I_n^{\otimes d}, A) \to \pi_0 \mathrm{Map}(I^{\otimes d}, A) \to \varprojlim_n \pi_0 \mathrm{Map}(I_n^{\otimes d}, A) \to 0,$$

the obstruction lies in $\varprojlim_n^1 \pi_1 \mathrm{Map}(I_n^{\otimes d}, A)$, which squares to 0. ☐

The following is from [1, Lemma 5.4.6]:

Lemma 5.23. *Let $\mathcal{C}$ be a stable presentably symmetric monoidal category and A a commutative algebra object. Consider an idempotent coalgebra $\mathbf{1}_U$ with the complementary idempotent algebra $\mathbf{1}_Z$, defining $\mathcal{C}_U$ and $\mathcal{C}_Z$. Suppose that $A_U = A \otimes \mathbf{1}_U$ and $A_Z = A \otimes \mathbf{1}_Z$ are descendable of index $\leq d_U$ and $\leq d_Z$ in $\mathcal{C}_U$ and $\mathcal{C}_Z$, respectively. Then A is descendable of index $\leq d_U + d_Z$.*

Proof. Since the composition $F^{\otimes d_Z} \to \mathbf{1} \to \mathbf{1}_Z$ is zero, $F^{\otimes d_Z} \to \mathbf{1}$ factors through $\mathbf{1}_U$. Therefore, $F^{\otimes d_U + d_Z} \to \mathbf{1}$ is zero, since $F_U^{\otimes d_U} \to \mathbf{1}_U$ is zero. ☐

Remark 5.24. We often do not know whether the category of interest has compact unit, precluding the use of Lemma 5.23 when we have descent on each component. Still, we can use Proposition 5.5 in some situations, and then stratify; cf. the proof of Lemma 8.19.

5.5. **Descending suave and prim maps.** We go back to the context of Section 4; we fix a presentably symmetric monoidal 2-category $\mathcal{C}$.

Definition 5.25. We call a morphism of stacks $Y \to X$ over $\mathcal{C}$ a *cover* if $[X] \to [Y]$ satisfies descent.

The following is an immediate consequence of descent for (iterated) module categories:

Proposition 5.26. *Let $Y \to X$ be a morphism of stacks. Let $X' \to X$ be a cover. If the base change $Y' \to X'$ is weakly suave, then so is $Y \to X$. The same holds when we replace all "weakly suave" with "suave" or "prim."*

Proposition 5.27. *Let $Y \to X$ be a morphism of stacks. Let $Y' \to Y$ be a weakly suave cover. If the composite $Y' \to X$ is weakly suave, then $Y \to X$ is weakly suave.*

Proof. In this situation, $[Y]$ can be computed as the totalization of $[Y'^{\times_Y \bullet+1}]$, which is also the *semisimplicial* colimit along the left adjoint. This description gives us the desired left adjoint. □

6. Examples of descent

In this section, we explain some examples of descent. In Sections 6.1 and 6.2, we treat covers by open and closed subsets. In Section 6.3, we consider surjections between compact Hausdorff spaces. In Section 6.4, we explain Clausen's theorem, which we use in the proof of Theorem D.

6.1. **Open covers.** We note that this descent works even in the unstable case:

Proposition 6.1. *Let X be a locale and $(U_i)_{i\in I}$ be an open cover. Then the morphism $\operatorname{Shv}(X) \to \prod_{i\in I} \operatorname{Shv}(U_i)$ satisfies descent in Pr.*

Combining this with Lemma 5.9, we obtain a stronger claim:

Corollary 6.2. *Let $Y \to X$ be a morphism of locales that admits a local section. Then $\operatorname{Shv}(X) \to \operatorname{Shv}(Y)$ satisfies descent in Pr.*

Proof of Proposition 6.1. We write Y for $\coprod_{i\in I} U_i$, which admits a tautological map to X. Since $\operatorname{Shv}(Y)$ is self-dual over $\operatorname{Shv}(X)$ and $\operatorname{Shv}(Y)^{\otimes_{\operatorname{Shv}(X)} 2} \simeq \operatorname{Shv}(Y^{\times_X 2})$ (see [5, Corollary 1.10]), it suffices to show that $\operatorname{Shv}(Y)^{\otimes_n \bullet+1} \simeq \operatorname{Shv}(Y^{\times_X \bullet+1})$ is a colimit diagram in Pr. Since $\operatorname{Shv}(Y^{\times_X n}) \simeq \operatorname{Shv}(X)_{/Y^{\times_X n}}$, this follows from the fact that $Y \to X$ is an effective epimorphism in the topos $\operatorname{Shv}(X)$ and [37, Proposition 6.3.5.14]. □

6.2. **Finite closed covers.** We recall the following:

Definition 6.3. Let $(Z_i)_{i\in I}$ is a family of subsets of a set X. We say that it is of *order* $\le d$, if for any subset $I_0 \subset I$ with cardinality $> d$, we have $\bigcap_{i\in I_0} Z_i = \emptyset$.

Proposition 6.4. *Let X be a locale and $(Z_i)_{i\in I}$ be a closed cover of order $\le d$ then the commutative algebra $\prod_{i\in I} \mathbf{S}_{Z_i}$ in $\operatorname{Shv}(X; \mathsf{Sp})$ is descendable of index $\le d$.*

Proof. We consider a filtration of X by the closed subsets $\emptyset = X_0 \subset X_1 \subset \cdots \subset X_d = X$ such that X_n is the subspace of points that are covered by at most n of Z_i. Then over $Z_n \setminus Z_{n-1}$, for $1 \le n \le d$, the commutative algebra admits a splitting. Therefore, the desired result follows from Lemma 5.23. □

6.3. **Surjections of compact Hausdorff spaces.** We recall here that any surjection between finite-dimensional compact Hausdorff spaces of countable weight satisfies descent.

Proposition 6.5. *The pushforward along the dyadic expansion*

$$\{0,1\}^{\mathbf{N}} \to [0,1]; \qquad (x_n)_n \mapsto \sum_n \frac{x_n}{2^{n+1}}$$

determines a descendable commutative algebra of index ≤ 4 in $\operatorname{Shv}([0,1]; \mathsf{Sp})$.

Proof. It is the limit of

$$\coprod_{k=0}^{2^n-1}\left[\frac{k}{2^n},\frac{k+1}{2^n}\right]\to[0,1].$$

This is a closed cover of order 2 by Proposition 6.4, the desired result follows from Lemma 5.22. □

Corollary 6.6. *Any nonempty compact Hausdorff space of countable weight of dimension $\leq d$ admits a cover by a Cantor set that is of index 2^{4d+2}.*

Proof. By embedding the space into $[0,1]^{2d+1}$, the result follows from Proposition 6.5. □

Corollary 6.7. *Consider a surjection $Y\to X$ from a light compact Hausdorff space to a light compact Hausdorff space of dimension $\leq d$. Then it is descendable of index $\leq 2^{4d+3}$.*

Proof. By Corollary 6.6, we obtain a map $C\to X$ from the Cantor set. Then we consider a surjection $D\to C\times_X Y$ from a Cantor set. By (1) of Lemma 5.21, it suffices to bound the descendability of $D\to X$. The map $D\to C$ is a sequential limit of surjections between finite sets. By Lemma 5.22, this is descendable of index ≤ 2. □

Remark 6.8. The bound in Corollary 6.7 is suboptimal; e.g., one can construct a surjection from the Cantor set to $[0,1]^2$ with descendability index ≤ 6.

Remark 6.9. In Corollary 6.7, the finite-dimensionality assumption on X cannot be dropped. When $X=[0,1]^{\mathbf{N}}$, we cannot find a cover from a profinite set, since even the usual descent fails.

Remark 6.10. In Corollary 6.7, even in the zero-dimensional case, the size restriction cannot be dropped; in general, a surjection between profinite sets is not a cover over $\mathsf{Pr}_{\mathrm{st}}$, as shown in [3].

6.4. **Condensed animas.** We review Clausen's pointwise descendability criterion in [16, Section 6]. First we need to recall the following definition of Clausen–Scholze [18]:

Definition 6.11. We call a profinite set *light* if it is of countable weight. We write $\mathsf{PFin}_{\mathrm{lgt}}$ for the category of light profinite sets. We consider the Grothendieck topology on it generated by finite jointly surjective families. A *(light) condensed anima* is a hypersheaf on $\mathsf{PFin}_{\mathrm{lgt}}$. We write $\mathsf{ConAni}_{\mathrm{lgt}}$ for this category.

Definition 6.12. Let X be a condensed anima. We write $\mathrm{Shv}^{\wedge}(X;\mathsf{Sp})$ for what is obtained by applying descent to $X\mapsto\mathrm{Shv}(X;\mathsf{Sp})$ for light profinite sets X. We define $\mathrm{Shv}^{\wedge}(X;\mathrm{D}(\mathbf{Z}))$ similarly.

Example 6.13. By [29, Corollary 2.8], when X is a light compact Hausdorff space, this is the category of Postnikov-complete sheaves, hence the notation.

We recall the following:

Definition 6.14. Let X be a condensed anima. Its *cohomological dimension* $\dim_{\mathbf{Z}}X$ is the smallest integer d (if it exists) such that for any $M\in\mathrm{Shv}^{\wedge}(X;\mathrm{D}(\mathbf{Z}))$ concentrated in degree 0, the cohomology $\Gamma(X;M)$ belongs to $\mathrm{D}(\mathbf{Z})_{\geq -d}$. For a profinite group G, we simply call $\dim_{\mathbf{Z}}(BG)$ the cohomological dimension of G.

Remark 6.15. What is defined in Definition 6.14 is often called the *strict cohomological dimension* in the literature on profinite groups, where cohomological dimension can mean the dimension with respect to torsion sheaves. By Lemma 6.16 below, we can relate these notions.

Lemma 6.16. *When $\Gamma(X;-)\colon\mathrm{Shv}^{\wedge}(X;\mathrm{D}(\mathbf{Z}))\to\mathrm{D}(\mathbf{Z})$ preserves filtered colimits,* $\dim_{\mathbf{Z}}X\leq\max(\dim_{\mathbf{Q}}X,\sup_p(\dim_{\mathbf{F}_p}X)+1)$ *holds.*

Proof. We assume that the right-hand side is finite and write it as d. We suppose the contrary so that there is $i>d$ and M such that $H^i(X;M)\neq 0$. We take a prime p and we write pM for the image of $p\colon M\to M$. We then consider short exact sequences $0\to M[p]\to M\to pM\to 0$ and $0\to pM\to M\to M/p\to 0$. Now $M[p]$ and M/p are $\mathbf{F}_p$-vector spaces, and hence for $i>d$, the morphism $H^i(X;M)\to H^i(X;pM)$ and $H^i(X;pM)\to H^i(X;M)$ are injective. Therefore, p acts injectively on $H^i(X;M)$. Since this holds for any prime p, we see $H^i(X;M)\otimes\mathbf{Q}=H^i(X;M\otimes\mathbf{Q})\neq 0$, which is a contradiction. □

Example 6.17. Let G be a free profinite group. By freeness, the torsion cohomology of G vanishes in degree ≥ 2. Hence, by Remark 6.15, we obtain $\dim_{\mathbf{Z}}(BG) \leq 2$. This inequality is strict except for the trivial case, since $H^2(\hat{\mathbf{Z}}; \mathbf{Z}) = \mathbf{Q}/\mathbf{Z} \neq 0$.

Theorem 6.18 (Clausen)**.** *Let X be a 1-truncated light profinite anima of finite cohomological dimension; i.e., the cohomological dimension of G_x is uniformly bounded. A commutative algebra object A in $\mathrm{Shv}^{\wedge}(X; \mathsf{Sp})$ is descendable if and only if there is d such that $x^*(A)$ is descendable of index $\leq d$ in Sp for any $x\colon * \to X$.*

Proof. This is a special case of [16, Theorem 6.21]. Indeed, its assumptions are satisfied by [16, Examples 6.14.2 and 6.19.3]. □

Theorem 6.18 is only useful in the situation where descendability works. Here we note the following easy example (cf. Example 5.18):

Proposition 6.19. *Let G be a finite group. Then $\mathrm{Fun}(BG; \mathsf{Sp}) = \mathrm{Shv}^{\wedge}(BG; \mathsf{Sp}) \to \mathsf{Sp}$ satisfies descent in $2\mathsf{Pr}_{\mathrm{st}}$.*

Proof. We can apply Proposition 5.5, since both $\mathrm{Fun}(BG, \mathsf{Sp}) \to \mathsf{Sp}$ and $\mathrm{Fun}(G, \mathsf{Sp}) \to \mathsf{Sp}$ admit left adjoints. By unwinding the definition, it is reduced to showing that the regular representation is faithful in $\mathrm{Fun}(BG, \mathsf{Sp})$. We can compute the colimit in Sp instead, and there we have a splitting. □

7. Algebraic 2-motives and ring stacks

In this section, we prove Theorem B, which characterizes $2\mathrm{SH}(\mathbf{Z})$ in terms of ring stacks. In Section 7.1, we first translate Theorem A in a form suitable for our use. In Section 7.2, we first construct six operations in the universal recipient. In Section 7.3, we perform an axiomatic argument showing that we get a class of smooth morphisms in the universal target. In Section 7.4, we prove Theorem A. In Section 7.5, we prove that cdh descent holds for algebraic 2-motives.

7.1. 2-motives of rings.

To prove Theorem B, we need to have a version of Theorem A that uses animated rings in place of quasiprojective static varieties:

Theorem 7.1. *We write $\mathsf{Ring}^{\heartsuit}$ for the category of static rings. The morphism $[-]\colon (\mathsf{Ring}^{\heartsuit})_{\aleph_0} \to 2\mathrm{SH}(\mathbf{Z})$ defined as $[\mathrm{Spec}(-)]$ is the universal symmetric monoidal functor satisfying the following:*

- *(i) When $A \to B$ is smooth, $f^*\colon [A] \to [B]$ has an $[A]$-linear left adjoint $f_\natural$ satisfying base change.*
- *(ii) For A and an element $a \in A$, the $[A[a^{-1}]] \leftarrow [A] \to [A/a]$ is a recollement.*
- *(iii) The morphism $\mathrm{id} \to f^* f_\natural$ is an equivalence for $f\colon A \to A[T]$ for any A.*
- *(iv) For $f\colon A \to A[T]$ and $s\colon A[T] \to A$ given by $T \mapsto 0$, the morphism $f_\natural s_*$ is an equivalence.*

We write $2\mathrm{SH}'(\mathbf{Z})$ for the universal target of Theorem 7.1. By Theorem A, we obtain a morphism $F\colon 2\mathrm{SH}'(\mathbf{Z}) \to 2\mathrm{SH}(\mathbf{Z})$.

Lemma 7.2. *For $A' \leftarrow A \to B$ in $(\mathsf{Ring}^{\heartsuit})_{\aleph_0}$, the morphism $[A'] \otimes_{[A]} [B] \to [A' \otimes_A B]$ is an equivalence in $2\mathrm{SH}'(\mathbf{Z})$.*

Proof. This follows from Proposition 3.12; any closed immersion is a finite composition of principal closed immersions. □

Lemma 7.3. *A Zariski cover $A \to B$ in $(\mathsf{Ring}^{\heartsuit})_{\aleph_0}$ determines a cover of stacks over $2\mathrm{SH}'(\mathbf{Z})$.*

Proof. We need to consider $A \to \prod_{i=1}^n A[1/a_i]$ with a_i generating the unit ideal. Moreover, it is reduced to the case $n = 0$ and 2. The case $n = 0$ is clear, so we consider the case $n = 2$.

By using Proposition 2.10 and (ii), we see that the morphism $[A] \to [A[1/a_1]] \times_{[A[1/a_1 a_2]]} [A[1/a_2]]$ is an equivalence.

We then use Proposition 5.5. By the limit formula above, it suffices to see that it is a cover on $A[1/a_1]$ and on $A[1/a_2]$, which is clear. □

Proof of Theorem 7.1. We construct $G\colon 2\mathrm{SH}(\mathbf{Z}) \to 2\mathrm{SH}'(\mathbf{Z})$. We right Kan extend $(\mathsf{Ring}^{\heartsuit})_{\aleph_0} \to 2\mathrm{SH}'(\mathbf{Z})$, which is possible by [4, Theorem D],[9] to obtain $\mathrm{PShv}(((\mathsf{Ring}^{\heartsuit})_{\aleph_0})^{\mathrm{op}}) \to 2\mathrm{SH}'(\mathbf{Z})$. By Lemma 7.3, this factors through the category of Zariski sheaves. We verify that its restriction to QProj satisfies the conditions in Theorem A when $S = \operatorname{Spec} \mathbf{Z}$.

We prove that in the situation $U \to X \leftarrow Z$ as in (ii) of Theorem A, we have an $[X]$-linear recollement and smooth base change for $U \hookrightarrow X$ (hence proper base change for $Z \hookrightarrow X$). By assumption, we only know this when X is affine and U is principal open.

We first see this when X is affine and U is quasiaffine inside X. This can be seen by induction on the number of affines needed to cover U.

We then see this in general. This is done by writing X as a colimit of $\operatorname{Spec} A$ (as a Zariski sheaf) and use smooth base change for open immersions and proper base change for closed immersions.

Therefore, we obtain (ii). By running the same argument as in the proof of Lemma 7.3, we see Zariski descent. In particular, for X, there is $\operatorname{Spec} A \to X$ such that $[X] \to [A]$ satisfies descent. This implies (iii) and (iv). For (i), by relative Künneth formula, which follows from Proposition 3.12, we need to show that any smooth morphism $Y \to \operatorname{Spec} A$ induces a weakly suave map. By Proposition 5.27, we can reduce to the case when Y is affine too.

Now we have constructed G. By construction, GF is homotopic to id. The other equivalence follows from the fact that $\mathsf{QProj}^{\mathrm{op}} \to 2\mathrm{SH}(\mathbf{Z})$ is a Zariski sheaf, which follows from the argument of Lemma 7.3. □

7.2. Six operations.

We now come back to proving Theorem B.

Definition 7.4. We write $2\mathrm{Mot}(\mathbf{Z})$ for the universal target. By definition, it comes with a colimit-preserving functor $\mathsf{Ring} \to \mathrm{CAlg}(2\mathrm{Mot}(\mathbf{Z}))$. We write $[-]$ for this.

Lemma 7.5. *The functor in Definition 7.4 factors through* $\mathsf{Ring}^{\heartsuit}$.

Proof. Let $\mathbf{Z}[T_d]$ be the free animated ring generated by an element in degree $d \geq 0$. We first prove that the tautological map $\mathbf{1} \simeq [\mathbf{Z}] \to [\mathbf{Z}[T_d]]$ is an equivalence for $d \geq 1$. We use $\mathbf{Z} \otimes_{\mathbf{Z}[T_{d-1}]} \mathbf{Z} = \mathbf{Z}[T_d]$ and induction to prove this. The base case $d = 1$ follows from the suturedness.

We go back to the statement. We prove that for A, the morphism $A \to \pi_0(A)$ induces an equivalence. We write A as a colimit of a sequence

$$\mathbf{Z} = A_{-1} \to A_0 \to A_1 \to \cdots,$$

where A_n is obtained from A_{n-1} by attaching n-cells. By the previous paragraph, we see that $[A_1] \to \cdots$ are equivalences. From A_1, considering another sequence $A_1 \to A_2' \to \cdots$ killing higher cells, we can also obtain $\pi_0(A)$ as a colimit. This shows the desired result. □

The key in our proof of Theorem B is deducing that smooth maps of rings determine suave morphisms of stacks over $2\mathrm{Mot}(\mathbf{Z})$. It is convenient to use six operations to show that. We use the material developed in Section 4.2 to construct such:

Proposition 7.6. *Every map $A \to B$ in $(\mathsf{Ring}^{\heartsuit})_{\aleph_0}$ determines an exceptional morphism. Therefore, we have a functor $\mathrm{Span}(((\mathsf{Ring}^{\heartsuit})_{\aleph_0})^{\mathrm{op}}) \to 2\mathrm{Mot}(\mathbf{Z})$ by Theorem 4.18.*

Proof. We write R for the universal ring stack over $2\mathrm{Mot}(\mathbf{Z})$. By Proposition 4.16, it suffices to show this for the zero section $s\colon * \to R$ and the tautological map $f\colon R \to *$. For s, it is clear. For f, this follows from Lemma 4.27. □

7.3. Smooth implies suave.

Theorem 7.7. *Let $A \to B$ be a smooth morphism of static rings. Then the corresponding morphism of stacks over $2\mathrm{Mot}(\mathbf{Z})$ is suave.*

We first explain the reduction to the following specific case:

[9] We can avoid using this, since it suffices to extend to QProj, and we can see the existence of limits concretely.

Proposition 7.8. *Let A be a static ring of finite type and P be a monic polynomial. We consider $A \to A[T][(P')^{-1}]/P$. Then the corresponding morphism of stacks over $2\mathrm{Mot}(\mathbf{Z})$ is étale (see Definition 4.14; we know that it is static already).*

Lemma 7.9. *A Zariski cover $A \to B$ in $(\mathsf{Ring}^{\heartsuit})_{\aleph_0}$ determines a cover of stacks over $2\mathrm{Mot}(\mathbf{Z})$.*

Proof. This follows from the same argument as in Lemma 7.3. □

Proof of Theorem 7.7. By Lemma 7.9, suaveness can be checked Zariski locally on the source and target by Proposition 5.27. Therefore, we can assume that $Y \to X$ factors as $Y \to \mathbb{A}^n_X \to X$ where the first morphism is étale. The second morphism is suave, so it suffices to prove that étale morphisms are suave.

We can again argue Zariski locally on the source and target, and by [51, Tag 00UE], we need to consider $A \to B = A[T][Q^{-1}]/P$ such that P is monic and P' is invertible in B. This factors as $A \to A[T][(P')^{-1}]/P \to B$, where the second morphism is suave. Hence the desired claim follows from Proposition 7.8. □

We move on to the proof of Proposition 7.8. We write $f\colon Y \to X$ for the corresponding morphism of stacks. Consider the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{d} & Y \times_X Y & \xrightarrow{p} & Y \\ & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle f} \\ & & Y & \xrightarrow{f} & X. \end{array}$$

We know that f is static and unramified, so we have a canonical identification of $d_!$ as $d_\natural$. To prove that f is étale, we need to show that the morphism

$$\mathrm{id}_{[Y]} \simeq q_! d_! d^* p^* \simeq q_! d_\natural d^* p^* \to q_! p^* \simeq f^* f_!$$

is a unit of an adjunction. However, we have to check less for this by [49, Proposition 6.13]; see also [32, Lemma 4.6.4]:

Lemma 7.10. *In the situation above, to show that f is étale, it suffices to show that*

$$f^! \mathbf{1}_X \simeq d^* p^* f^! \mathbf{1}_X \to d^* q^! f^* \mathbf{1}_X \simeq d^! q^! f^* \mathbf{1}_X \simeq f^* \mathbf{1}_X \simeq \mathbf{1}_Y$$

in $\mathrm{Hom}_{2\mathrm{Mot}(\mathbf{Z})}(\mathbf{1}, [Y])$ is an equivalence.

So from now on, we only consider the lax symmetric monoidal functor

$$D = \mathrm{Hom}_{\mathrm{Mod}_{[A]}(2\mathrm{Mot}(\mathbf{Z}))}([A], -)\colon\ \mathrm{Span}(\mathsf{Aff}_A) \to \mathsf{Pr}_{\mathrm{st}},$$

where Aff_A is the category of affine schemes of finite type over A. We now then conclude by repeating the argument in [49, Section 10]. We use $\operatorname{Spec} A$ as the implicit base.

Lemma 7.11. *In the situation above, let U and V be quasicompact open affine subschemes of $\mathbb{A}^m$ and $\mathbb{A}^n$, respectively. For any morphism $g\colon V \to U$, the object $g^!\mathbf{1}$ is equivalent to $\mathbf{1}(n-m)[2n-2m]$. This formation is compatible with the base change from the point, i.e., for any quasicompact open affine subscheme W of $\mathbb{A}^l$, the canonical morphism $p^* g^! \mathbf{1} \to (\mathrm{id}_W \times g)^! \mathbf{1}$ is an equivalence, where $p\colon W \times V \to V$ is the projection.*

Proof. We write $f\colon U \to *$ and $h\colon V \to *$ for the tautological morphisms. By Proposition 4.29, we see $f^!\mathbf{1} = \mathbf{1}(m)[2m]$ and $h^!\mathbf{1} = \mathbf{1}(n)[2n]$. By $g^! f^! \simeq h^!$,

$$g^!\mathbf{1} \simeq g^! f^! \mathbf{1}(-m)[-2m] \simeq h^! \mathbf{1}(-m)[-2m] \simeq \mathbf{1}(n-m)[2n-2m]$$

holds. The desired compatibility of the base change also follows from this proof. □

Proof of Proposition 7.8. We consider the diagram

$$\begin{array}{ccccc} X & \longrightarrow & U & \hookrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow{\scriptstyle g|_U} & \swarrow{\scriptstyle g} & \\ * & \xrightarrow{0} & \mathbb{A}^1, & & \end{array}$$

where g is determined by the polynomial P. We need to prove that X is étale. Instead, we wish to prove that $g|_U$ is étale. For that, we need to prove that the morphism $(g|_U)^!\mathbf{1} \to \mathbf{1}$ constructed in Lemma 7.10 is an equivalence. By Lemma 7.11, this is an endomorphism of $\mathbf{1}$.

By base changing the whole diagram along $U \to *$, we are now in the situation where we have a section $s\colon * \to U$, depicted as

$$\begin{array}{ccccc} * & \xrightarrow{s} & U & \hookrightarrow & \mathbb{A}^1 \\ & & {\scriptstyle g|_U}\downarrow & \swarrow{\scriptstyle g} & \\ & & \mathbb{A}^1. & & \end{array}$$

By the second part of Lemma 7.11, it suffices to show that $(g|_U)^!\mathbf{1} \to \mathbf{1}$ is an equivalence after applying s^*.

By shifting, we can assume that the composite of the top arrows is zero. This means that $P = T^n + \cdots + a_1T + a_0$ with $a_1 \in A^\times$. We here consider $Q = a_0 + a_1T + S(a_2T^2 + \cdots + T^n) \in A[S,T]$ and the diagram

$$\begin{array}{ccccc} \mathbb{A}^1_S & \xrightarrow{s} & V & \hookrightarrow & \mathbb{A}^1_S \times \mathbb{A}^1 \\ & & {\scriptstyle h|_V}\downarrow & \swarrow{\scriptstyle h} & \\ & & \mathbb{A}^1_S \times \mathbb{A}^1, & & \end{array}$$

where h is given by the projection to S and the polynomial Q, and V is the étale locus. If we prove that $(h|_V)^!\mathbf{1} \to \mathbf{1}$ is an equivalence after s^*, we obtain the desired result by pulling back along $S = 1$, which is possible by the second part of Lemma 7.11. However, by the first part of Lemma 7.11, it is a map between object coming from the point, and by $\mathbb{A}^1$-invariance, it suffices to prove this after pulling back along $S = 0$. There, by the second part of Lemma 7.11 again, we are in the original situation with a section, where P has degree one. In this case, g is an isomorphism, and hence the desired result follows. □

7.4. **Algebraic 2-motives and ring stacks.** We have all the ingredients to complete the proof:

Proof of Theorem B. By Theorem 7.1, both are defined via the universal property mapping from Ring, and therefore it suffices to construct morphisms in both directions compatible with the universal maps from Ring.

The morphism $2\mathrm{Mot}(\mathbf{Z}) \to 2\mathrm{SH}(\mathbf{Z})$ can be readily constructed. We construct a morphism in the other direction. We have a functor $\mathsf{Ring} \to 2\mathrm{Mot}(\mathbf{Z})$. We prove that its restriction to $(\mathsf{Ring}^{\heartsuit})_{\aleph_0}$ satisfies the conditions in Theorem 7.1. We have already seen (i) as Theorem 7.7. The conditions (ii) to (iv) follows from base changing the universal situation. □

7.5. **Cdh descent for algebraic 2-motives.** We here observe that the classical descent result for SH upgrades to 2SH:

Theorem 7.12. *A cdh cover of static affine schemes of finite type determines a cover over* $2\mathrm{SH}(\mathbf{Z})$*. Therefore,* $\mathrm{Mod}_{[-]}(2\mathrm{SH}(\mathbf{Z}))\colon \mathsf{Ring} \to 2\mathsf{Pr}_{\mathrm{st}}$ *is a cdh sheaf.*

Proof. We first treat Nisnevich descent. We consider the Nisnevich square

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array}$$

of affine schemes of finite type. By Proposition 5.5, we need to prove that $[U] \oplus [Y] \to \mathbf{1}$ in $\mathrm{SH}(X)$ is faithful. We can see this after pulling back to $\mathrm{SH}(U)$ and $\mathrm{SH}(Y)$.

We then prove cdp descent. For that, it is convenient to use the 2-motives of quasiprojective schemes. With this, it suffices to show that for a (concrete) blowup square

$$\begin{array}{ccc} E & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X, \end{array}$$

where X is an affine scheme of finite type, the morphism $[X] \to [Y] \times [Z]$ satisfies descent. Now by using Theorem 5.13, it asks whether the pushforward of $\mathbf{1}_{Y \amalg Z}$ in $\mathrm{SH}(X)$ satisfies descent. We can see its descendability using Lemma 5.23. □

8. Étale 2-motives

In this section, we prove Theorem C. We write $2\mathrm{SH}_{\mathrm{KAS}}(\mathbf{Z})$ for the universal target of Theorem C, which exists by Lemma 5.6. We wish to show that this is equivalent to $2\mathrm{SH}_{\text{ét}}(\mathbf{Z})$.

In Section 8.1, we review étale motivic spectra. In Section 8.2, we review the theory of motivic Euler characteristics. In Section 8.3, we prove Galois descent for the axiomatic target of Theorem C. In Section 8.4, we prove Theorem C. In Section 8.5, we prove that h descent holds for étale 2-motives.

8.1. A review of étale motivic spectra.

First, we recall the definition of étale motivic spectra:

Definition 8.1. For a ring A, we define $\mathrm{SH}_{\text{ét}}(A)$ to be the étale sheafified version of $\mathrm{SH}(A)$; i.e., for any étale cover $Y \to X$ of smooth schemes of finite presentation over A, we impose $[Y^{\times_X \bullet+1}]$ to be a colimit diagram. Since Nisnevich descent is already imposed, it suffices to do this for finite Galois covers between smooth affines.

The hypercompleted version is usually considered in literature. The advantage of Definition 8.1 is the following:

Lemma 8.2. *The functor*

$$\mathrm{SH}_{\text{ét}} \colon \mathsf{Ring} \to \mathrm{CAlg}(\mathsf{Pr}_{\mathrm{st}})$$

commutes with filtered colimits.

Proof. This follows from the same observation for SH and the fact any finite Galois cover $Y \to X$ between smooth affines over A comes from the same situation over a ring of finite presentation. □

Nevertheless, it was proven in [11, Theorem 6.29] that these categories coincide under mild assumptions:

Theorem 8.3 (Bachmann–Burklund–Xu)**.** *Let A be a finite-dimensional noetherian ring such that the virtual cohomological dimension of residue fields is uniformly bounded (e.g., A is of finite type over $\mathbf{Z}$). Then $\mathrm{SH}_{\text{ét}}(A)$ coincides with the hypercomplete version.*

We later use the following variant:

Corollary 8.4. *In Theorem 8.3, the same holds if any local rings of A satisfies the hypothesis of Theorem 8.3.*

Proof. It suffices to show that the functors $\mathrm{SH}_{\text{ét}}(A) \to \mathrm{SH}_{\text{ét}}(A_{\mathfrak{m}})$ are jointly conservative, where $\mathfrak{m}$ runs over maximal ideals of A. If an object M maps to 0 over $A_{\mathfrak{m}}$, by Lemma 8.2, we can take $f \notin \mathfrak{m}$ such that it is 0 over $A[f^{-1}]$. This covers A and hence $M \simeq 0$. □

Remark 8.5. In [9, Corollary 5.7], it is shown at least under the assumption of Theorem 8.3 that the category $\mathrm{SH}_{\text{ét}}(A)$ is compactly generated by $\Sigma^\infty_+ X$ when X is a smooth scheme of finite presentation over A with finite cohomological dimension. From this, we see that our category $\mathrm{SH}_{\text{ét}}(A)$ is always compactly generated by using Lemma 8.2.

One subtlety is that $\mathbf{1} \in \mathrm{SH}_{\text{ét}}(A)$ is not necessarily compact. However, we can always cover A by $A[1/2][\zeta_4] \times A[1/3][\zeta_6]$ to make it compact: To see this, again by Lemma 8.2, we can consider the case when A is of finite type. Then, the result follows from the description of a set of compact generators above.

Ayoub constructed six operations for $\mathrm{SH}_{\text{ét}}$ for quasiprojective schemes over $\mathbf{Z}$; namely, we can apply Theorem 3.4 in this situation (note that (ii) was proven in [8, Corollaire 4.5.47]). We here consider the étale variant of $2\mathrm{SH}(\mathbf{Z})$:

Definition 8.6. We write $2\mathrm{SH}_{\text{ét}}(\mathbf{Z})$ for the presentable 2-category of kernels (see Definition 2.5) for $\mathrm{SH}_{\text{ét}}\colon \mathrm{Span}(\mathsf{QProj}) \to \mathsf{Pr}_{\mathrm{st}}$.

Lemma 8.7. *There is a canonical*[10] *morphism* $\mathrm{SH} \to \mathrm{SH}_{\text{ét}}$ *in* $\mathrm{CAlg}(\mathrm{Fun}(\mathrm{Span}(\mathsf{QProj}), \mathsf{Pr}_{\mathrm{st}}))$ *extending the tautological morphism in* $\mathrm{CAlg}(\mathrm{Fun}(\mathsf{QProj}^{\mathrm{op}}, \mathsf{Pr}_{\mathrm{st}}))$.

We can just use Proposition 3.10 for this, but we can also use Theorem A as follows:

Proof. The morphism $[-]\colon \mathsf{QProj}^{\mathrm{op}} \to 2\mathrm{SH}_{\text{ét}}(\mathbf{Z})$ satisfies the hypothesis of Theorem A so that we obtain a morphism $2\mathrm{SH}(\mathbf{Z}) \to 2\mathrm{SH}_{\text{ét}}(\mathbf{Z})$. The composition of this with $2\mathrm{Span}_{\mathrm{all};P,J}(\mathsf{QProj}) \to 2\mathrm{SH}(\mathbf{Z})$, where J consists of open immersions and P of projective morphisms coincides with the tautological morphism. By restricting this to $\mathrm{Span}(\mathsf{QProj})$, we obtain the desired morphism. □

8.2. **A review of Euler characteristic.** We need the following:

Proposition 8.8. *Let A be a static ring containing either ζ_4 or ζ_6 and P a monic polynomial of degree d. We consider $f\colon A \to A[T]/P$. Then there is a morphism $\epsilon\colon f_*f^*\mathbf{1} \to \mathbf{1}$ and an invertible morphism $v\colon \mathbf{1} \to \mathbf{1}$ in $\mathrm{SH}(A)$ such that the composite*

$$\mathbf{1} \to f_*f^*\mathbf{1} \xrightarrow{\epsilon} \mathbf{1} \xrightarrow{v} \mathbf{1} \xrightarrow{d} \mathbf{1}$$

is homotopic to d^2.

For the proof, we recall some motivic homotopy theory. Consider the motivic J-homomorphism $\Omega^\infty K(A) \to \mathrm{SH}(A)$, where $K(A)$ denotes the algebraic K-theory spectrum of A. Since it maps 0 to $\mathbf{1}$, we obtain a morphism $\Omega^{\infty+1}K(A) \to \mathrm{End}_{\mathrm{SH}(A)}(\mathbf{1})$. By composing this with $A^\times \to K_1(A)$, we obtain a morphism $A^\times \to \pi_0 \mathrm{End}_{\mathrm{SH}(A)}(\mathbf{1})$, for which we write $\langle - \rangle$. For $d \geq 0$, we write d_ϵ for $\langle -1\rangle^0 + \cdots + \langle -1\rangle^{d-1}$.

Lemma 8.9. *Suppose that $\zeta_{2l} \in A$ for a prime l. Then we have $2^n(\langle -1\rangle - 1) = 0$ in $\pi_0 \mathrm{End}_{\mathrm{SH}(A)}(\mathbf{1})$ for $n \geq 0$ satisfying $2^n \geq l-1$.*

Proof. We can assume $A = \mathbf{Z}[\zeta_{2l}]$. By [23], we have $\langle u^2\rangle = 1$ for $u \in A^\times$ and $\langle u\rangle + \langle v\rangle = \langle u+v\rangle + \langle (u+v)uv\rangle$ for u, v, $u+v \in A^\times$. We use the first one to see $\langle \zeta_l^k\rangle = 1$ for $k = 1, \ldots, l-1$ and then iteratively apply the second one to $\zeta_l + \cdots + \zeta_l^{l-1} = -1$ to obtain the desired result. □

Lemma 8.10. *Let A be a ring containing either ζ_4 or ζ_6. For $d \geq 0$, the element $dd_\epsilon \in \pi_0 \mathrm{End}_{\mathrm{SH}(A)}(\mathbf{1})$ is d^2 up to unit.*

Proof. If A contains ζ_4, by Lemma 8.9, we have $d_\epsilon = d$.

If A contains ζ_6, by Lemma 8.9, we have $2(\langle -1\rangle - 1) = 0$. We do case-by-case analysis modulo 4. When $d \equiv 0$ or 1, we have $d_\epsilon = d$. When $d \equiv 3$, we have $d_\epsilon = d\langle -1\rangle$. So we are left to treat the case $d \equiv 2$, but since d is even in this case, $2d_\epsilon = 2d$ is sufficient for the conclusion. □

Proof of Proposition 8.8. We apply [24, Proposition B.1.4] (see [25, Proposition 2.2.5] for a more relevant formulation) to obtain a morphism $f_*f^*\mathbf{1} \to \mathbf{1}$ such that its composite with $\mathbf{1} \to f_*f^*\mathbf{1}$ is d_ϵ. By Lemma 8.10, we obtain the desired result. □

8.3. **Étale descent for Kummer–Artin–Schreier 2-motives.** The following is the key in the proof of Theorem C:

Theorem 8.11. *Let G be a finite group and $A \to B$ a G-Galois morphism of rings. Then it induces a cover of stacks over $2\mathrm{SH}_{\mathrm{KAS}}(\mathbf{Z})$.*

We first simplify the Artin–Schreier condition. We use the following theorem from [10]:

Theorem 8.12 (Bachmann)**.** *The tautological morphism $\mathsf{Sp} \to \mathrm{SH}(\mathbf{F}_p)_{(p)}$ in $\mathrm{CAlg}(\mathsf{Pr}_{\mathrm{st}})$ canonically factors through $\mathrm{D}(\mathbf{Z})$.*

[10]This morphism is necessarily unique by [20].

Proposition 8.13. *We write* $E(\mathbf{F}_p) = \operatorname{Hom}_{2\mathrm{SH}_{\mathrm{KAS}}(\mathbf{Z})}(\mathbf{1}, [\mathbf{F}_p])$*. Then* $p \in \operatorname{End}_{E(\mathbf{F}_p)_{(p)}}(\mathbf{1})^{\times}$*.*

Remark 8.14. In Proposition 8.13, a posteriori (after Theorem C is proven), we see $p \in \operatorname{End}_{E(\mathbf{F}_p)}(\mathbf{1})^{\times}$.

Proof. We consider the composite $\mathrm{Shv}_{\mathrm{Nis}}(\mathsf{Sm}_{\mathbf{F}_p}; \mathsf{Sp}) \to \mathrm{SH}(\mathbf{F}_p) \to E(\mathbf{F}_p)$. By the Artin–Schreier condition, $\mathbf{F}_p \to \mathbb{G}_a \xrightarrow{T \mapsto T^p - T} \mathbb{G}_a$ (with the tautological nullhomotopy) becomes a cofiber sequence in $E(\mathbf{F}_p)$. Since the second morphism is equivalent to $\mathrm{id}_{\mathbf{1}}$ in $\mathrm{SH}(\mathbf{F}_p)$, this implies that $\mathbf{F}_p$ becomes 0 in $E(\mathbf{F}_p)$. This amounts to say that $\mathsf{Sp} \to E(\mathbf{F}_p)$ carries $\mathbf{F}_p$ to 0.

Now we use Theorem 8.12 to obtain $\mathrm{D}(\mathbf{Z}) \to E(\mathbf{F}_p)_{(p)}$. The above paragraph shows that this morphisms kills $\mathbf{Z} \otimes_{\mathbf{S}} \mathbf{F}_p$, where we consider the $\mathbf{Z}$-linear structure coming from the left factor. Since $\mathbf{F}_p$ is a retract of $\mathbf{F}_p \otimes_{\mathbf{S}} \mathbf{F}_p$ in $\mathrm{D}(\mathbf{Z})$, it means that $\mathbf{F}_p = \mathbf{Z} \otimes_{\mathbf{S}} \mathbf{S}/p$ is also killed. Therefore, the desired result follows. □

We then prove several cases:

Lemma 8.15. *The morphism of stacks over* $2\mathrm{SH}_{\mathrm{KAS}}(\mathbf{Z})$ *induced by the ring map* $\mathbf{Z}[1/n] \to \mathbf{Z}[1/n][\zeta_{2n}]$ *is a cover for* $n \geq 1$*.*

Proof. This is the base change of the Kummer map along the morphism $\mathbf{Z}[1/n][T^{\pm}] \to \mathbf{Z}[1/n]$ mapping T to -1. □

Lemma 8.16. *The morphism of stacks over* $2\mathrm{SH}_{\mathrm{KAS}}(\mathbf{Z})$ *induced by the ring map* $\mathbf{Z} \to \mathbf{Z}[1/2][\zeta_4] \times \mathbf{Z}[1/3][\zeta_6]$ *is a cover.*

Proof. First, $\mathbf{Z} \to \mathbf{Z}[1/2] \times \mathbf{Z}[1/3]$ is a Zariski cover of rings and hence induces a cover by Theorem 7.12. We obtain the desired result by combining this with Lemma 8.15. □

Lemma 8.17. *Let* A *be a ring and* P *be a monic polynomial of degree* d*. If* $A \to A[T]/P = B$ *is étale, then the morphism* $[A][1/d] \to [B][1/d]$[11] *satisfies descent.*

Proof. By base changing along the map in Lemma 8.16, the desired result follows from Proposition 8.8. □

Lemma 8.18. *For* $n \geq 1$ *invertible in a ring* A*, any* C_n*-Galois cover* $A \to B$ *determines a cover.*

Proof. By Lemma 8.15, we can assume that A contains ζ_n. By Kummer theory, as long as $\mathrm{Pic}(A)$ vanishes, such a cover is a base change of the Kummer map. Since Pic vanishes Zariski locally, the desired result follows from Theorem 7.12. □

Lemma 8.19. *For* $e \geq 0$*, the morphism of stacks over* $2\mathrm{SH}_{\mathrm{KAS}}(\mathbf{Z})$ *induced by any* C_{p^e}*-Galois cover* $A \to B$ *is a cover.*

Proof. We consider $A \to B$. Since it is proper, by Theorem 5.13, it suffices to show that $f_*\mathbf{1}$ in $\operatorname{Hom}_{2\mathrm{SH}_{\mathrm{KAS}}(\mathbf{Z})}(\mathbf{1}, [A])$ satisfies descent. Moreover, since it is finite étale, it is dualizable with a dualizable diagonal. Hence by Proposition 5.2, we need to see that the (E_0-)coalgebra $(f_*\mathbf{1})^{\vee}$ is faithful. This can be checked after mapping to $\operatorname{Hom}_{2\mathrm{SH}_{\mathrm{KAS}}(\mathbf{Z})}(\mathbf{1}, [A[1/p]])$ and $\operatorname{Hom}_{2\mathrm{SH}_{\mathrm{KAS}}(\mathbf{Z})}(\mathbf{1}, [A/p])$. The former follows from Lemma 8.18, and the latter follows from Propositions 8.8 and 8.13. □

Proof of Theorem 8.11. Let $p_1, \ldots, p_n$ be the primes smaller than the cardinality of G. We cover the coefficients $\mathbf{S}$ by $\mathbf{S}[p_i/p_1 \cdots p_n]$ for $i = 1, \ldots, n$. By Proposition 6.4,[12] the resulting cover satisfies descent. We then need to show that $[A][p_i/p_1 \cdots p_n] \to [B][p_i/p_1 \cdots p_n]$ satisfies descent for each i. We write $p = p_i$ and choose a p-Sylow group P of G. Since P is solvable, we can write $A \to B$ as a sequence of extensions $A \to B^P = B_0 \to \cdots \to B_m = B$ where $B_{j-1} \to B_j$ for $j = 1, \ldots, m$ is a $C_{p^{e_j}}$-Galois extension for some $e_j \geq 1$, which, by Lemma 8.19, induces a cover. Hence it suffices to see that $A \to B_0$ induces a cover. Since this is Nisnevich locally monogenic by the primitive element theorem, the desired result follows from Theorem 7.12 and Lemma 8.17. □

[11] Do not confuse this with $[A[1/d]] \to [B[1/d]]$.

[12] Note that quasicompact opens in the Zariski spectrum are closed categorically.

8.4. **Étale 2-motives and ring stacks.** We here prove Theorem C.

Proposition 8.20. *Let G be a finite group and $A \to B$ a G-Galois morphism of rings. Then it determines a cover of stacks over $2\mathrm{SH}_{\text{ét}}(\mathbf{Z})$.*

Proof. By considering $G/H \mapsto A^H$, we obtain a square

$$\begin{array}{ccc} \mathrm{Fun}(BG;\mathsf{Sp}) & \longrightarrow & \mathsf{Sp} \\ \downarrow & & \downarrow \\ \mathrm{SH}_{\text{ét}}(A) & \longrightarrow & \mathrm{SH}_{\text{ét}}(B) \end{array}$$

in $\mathrm{CAlg}(\mathsf{Pr}_{\mathrm{st}})$ and correspondingly, we obtain a pushout square

$$\begin{array}{ccc} \mathrm{Fun}(BG;\mathsf{Sp}) \otimes \mathbf{1} & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow \\ [A] & \longrightarrow & [B] \end{array}$$

in $\mathrm{CAlg}(2\mathrm{SH}_{\text{ét}}(\mathbf{Z}))$. The desired result follows from Proposition 6.19. ☐

Remark 8.21. For any finite étale map $A \to B$, what is used in the proof of Proposition 8.20 generalizes to the pushout square

$$\begin{array}{ccc} \mathrm{Shv}_{\text{fét}}(A) \otimes \mathbf{1} & \longrightarrow & \mathrm{Shv}_{\text{fét}}(B) \otimes \mathbf{1} \\ \downarrow & & \downarrow \\ [A] & \longrightarrow & [B], \end{array}$$

in $2\mathrm{SH}_{\text{ét}}(\mathbf{Z})$. Moreover, when A and B satisfy the assumption of Theorem 8.3, we can replace $\mathrm{Shv}_{\text{fét}}$ with its hypercompleted version, as hypercompletion is smashing by [17, Corollary 4.40].

Proof of Theorem C. By Proposition 8.20, we obtain a morphism $F\colon 2\mathrm{SH}_{\mathrm{KAS}}(\mathbf{Z}) \to 2\mathrm{SH}_{\text{ét}}(\mathbf{Z})$. We construct a morphism $G\colon 2\mathrm{SH}_{\text{ét}}(\mathbf{Z}) \to 2\mathrm{SH}_{\mathrm{KAS}}(\mathbf{Z})$ as in the proof of Theorem A. We already have a symmetric monoidal functor $\mathrm{Span}(\mathsf{QProj}) \to 2\mathrm{SH}_{\mathrm{KAS}}(\mathbf{Z})$ and therefore, we wish to construct a morphism $\mathrm{SH}_{\text{ét}} \to \mathrm{Hom}_{2\mathrm{SH}_{\mathrm{KAS}}(\mathbf{Z})}(\mathbf{1},[-])$ in $\mathrm{CAlg}(\mathrm{Fun}(\mathrm{Span}(\mathsf{QProj}),\mathsf{Pr}_{\mathrm{st}}))$. By Theorem 8.11, we obtain such a morphism when restricted to $\mathsf{QProj}^{\mathrm{op}}$. By Theorem 2.2, to promote this to the desired morphism, we need to check the compatibility condition described in the statement of Theorem 2.2; namely, that the squares

$$\begin{array}{ccc} \mathrm{SH}_{\text{ét}}(X) & \longrightarrow & \mathrm{SH}_{\text{ét}}(U) \\ \downarrow & & \downarrow \\ D(X) & \longrightarrow & D(U), \end{array} \qquad \begin{array}{ccc} \mathrm{SH}_{\text{ét}}(X) & \longrightarrow & \mathrm{SH}_{\text{ét}}(Y) \\ \downarrow & & \downarrow \\ D(X) & \longrightarrow & D(Y) \end{array}$$

for any open immersion $U \to X$ and projective morphism $Y \to X$ are left and right adjointable, respectively, where $D = \mathrm{Hom}_{2\mathrm{SH}_{\mathrm{KAS}}(\mathbf{Z})}(\mathbf{1},[-])$. By Lemma 8.7 and the fact that $\mathrm{SH}_{\text{ét}}(-)$ is a localization of $\mathrm{SH}(-)$, it suffices to verify this when $\mathrm{SH}_{\text{ét}}$ is replaced with SH, which follows from the argument in Lemma 8.7.

We need to see that this construction induces an equivalence. By construction, GF is homotopic to id. For FG, we argue as in the proof of Theorem A. ☐

8.5. **H descent for étale 2-motives.** We observe that the h descent result for $\mathrm{SH}_{\text{ét}}$ upgrades to $2\mathrm{SH}_{\text{ét}}$:

Theorem 8.22. *An h cover of static affine schemes of finite type determines a cover over $2\mathrm{SH}_{\text{ét}}(\mathbf{Z})$. Therefore, $\mathrm{Mod}_{[-]}(2\mathrm{SH}_{\text{ét}}(\mathbf{Z}))\colon \mathsf{Ring} \to 2\mathsf{Pr}_{\mathrm{st}}$ is an h sheaf.*

We prove this by reducing this to the following:

Proposition 8.23. *Finite universal homeomorphisms between static affine schemes of finite type determines a cover over $2\mathrm{SH}_{\text{ét}}(\mathbf{Z})$.*

The reduction is based on the following standard observations:

Lemma 8.24. *Let S be a quasicompact quasiseparated static scheme. Then the h topology on the category of static schemes of finite presentation over S is generated by the cdp and Zariski topologies and by finite locally free surjections.*

Proof. By [14, Theorem 2.9], the h topology is generated by the cdp and fppf topologies. By [51, Tag 05WN], the fppf topology is generated by the Zariski topology and finite locally free surjections. □

Lemma 8.25. *Let $Y \to X$ be a finite locally free morphism between quasicompact quasiseparated static schemes. Then it admits a sequence of closed subschemes of finite presentation $\emptyset = Z_0 \subset \cdots \subset Z_n = X$ such that over $Z_{i+1} \setminus Z_i$, it is a decomposition $Y \to X' \to X$ such that $Y \to X'$ is a universal homeomorphism and $X' \to X$ is a finite étale map.*

Proof. By approximation, we can assume that X is of finite type over $\mathbf{Z}$. By noetherian induction, it suffices to find a nonempty open subscheme over which $Y \to X$ admits the desired decomposition. The desired result follows from Lemma 8.26 below. □

Lemma 8.26. *Let k be a field and B an artinian k-algebra. Then it admits a factorization $k \to A \to B$ such that $k \to A$ is étale and $A \to B$ is a universal homeomorphism.*

Proof. By treating each connected component, we can assume that B is local; we write l for its residue field. We let A to be the subring of B consisting of separable elements over k. It suffices to show that $A \to B$, equivalently, $A \to B \to l$ is a universal homeomorphism. To prove this, it suffices to show that any separable element of l over k lifts to B. This follows from the henselian property of B. □

Proof of Theorem 8.22. Note that cdh descent follows from Theorem 7.12. By Lemma 8.24, it suffices to show descent for finite flat maps.

Then we use Lemma 8.25. By using Remark 8.5, we are in the descendability situation. Therefore, we can use Lemma 5.23 to reduce to the case when the map is a universal homeomorphism, which is Proposition 8.23. □

We now move on to the proof of Proposition 8.23. We recall the following notion from [44, Appendix B]. The formulation here is taken from [51, Tag 0EUR]:

Definition 8.27. Let A be a static ring. The *absolute weak normalization* of A is the final object in the category of A-algebras whose structure morphisms are universal homeomorphisms. We write $A \to A^{\mathrm{awn}}$ for this. For a general ring, its absolute weak normalization is that of π_0.

Lemma 8.28. *In the situation of Definition 8.27, the absolute weak normalization A^{awn} can be written as a filtered colimit of A-algebras whose transitions are composites of finitely many base changes of the following morphisms:*

(1) The morphism $\mathbf{Z}[S,T]/(S^3 - T^2) \to \mathbf{Z}[U]$ mapping S and T to U^2 and U^3, respectively.
(2) The morphism $\mathbf{Z}[S,T]/(p^pS - T^p) \to \mathbf{Z}[U]$ mapping S and T to U^p and pU, respectively.
(3) The morphism $\mathbf{Z}[S,T]/(S^p - T^p, pS - pT) \to \mathbf{Z}[U]$ mapping both S and T to U.

Proof. By [51, Tag 0EUL], a static ring is absolutely weakly normal if and only if the following conditions hold:

- When we have x and y satisfying $x^3 = y^2$, there is z such that $x = z^2$ and $y = z^3$.[13]
- For each prime p, when we have x and y satisfying $p^px = y^p$, there is a unique element z such that $x = z^p$ and $y = pz$.

The first condition corresponds to (1) and the second to (2) and (3). □

Proof of Proposition 8.23. We base change along $\mathbf{Z} \to \mathbf{Z}[1/2][\zeta_4] \times \mathbf{Z}[1/3][\zeta_6]$ to be in the situation where we can use descendability, and in particular Lemma 5.23.

[13]This solution is automatically unique for weakly seminormal rings.

Let $A \to B$ be a finite universal homeomorphism. This induces an isomorphism on the absolute weak normalizations; in particular, B lies between A and its absolute weak normalization. Therefore, by Lemma 8.28, it suffices to show that each map described there is descendable.

We consider (1). We stratify the base by S. Over the open locus, it is an isomorphism. Over the closed locus, both sides reduce to $\mathbf{Z}$.

We consider (2). We stratify the base by p. Over the open locus, it is an isomorphism. Over the closed locus, it becomes $\mathbf{F}_p[S,T]/T^p \to \mathbf{F}_p[U]$. By considering the reduction, it suffices to show that $\mathbf{F}_p[S] \to \mathbf{F}_p[U]$ given by $S \mapsto U^p$ is descendable. This follows from Proposition 8.8, since p is invertible in $\mathrm{SH}_{\acute{e}t}(\mathbf{F}_p)$.

We then treat (3). We stratify the base by p. Over the open locus, it is an isomorphism. Over the closed locus, it becomes $\mathbf{F}_p[S,T]/((S-T)^p) \to \mathbf{F}_p[U]$, whose reduction is an isomorphism. □

Corollary 8.29. *The functor* $\mathsf{Ring} \to 2\mathrm{SH}(\mathbf{Z})$ *factors through the category of absolute weakly normal rings via* $(-)^{\mathrm{awn}}$.

Proof. By Lemma 8.28, it suffices to show that any finite universal homeomorphism induces an isomorphism in $2\mathrm{SH}_{\acute{e}t}(\mathbf{Z})$. Let $Y \to X$ be such a map. By Proposition 8.23, we see $[X] \simeq \mathrm{Tot}[Y^{\times_X \bullet+1}]$. But since the reduction of $Y^{\times_X \bullet+1}$ is the reduction of Y, the cosimplicial object is constant. □

9. Some Berkovich geometry

This section collects some facts on Berkovich geometry. In Section 9.1, we introduce seminormed rings and uniform Banach rings. In Section 9.2, we consider Berkovich spectra. In Section 9.3, we consider the notion of strictly totally disconnecteds, which are important building blocks in Berkovich geometry. In Section 9.4, we introduce arc topology. In Section 9.5, we study uniform Banach rings of topologically finite presentation.

9.1. Seminormed rings and uniform Banach rings.

Here we consider the derived variant for completeness, but it is not important, as the derived variant of normed rings are automatically static.

Definition 9.1. A *seminorm* on a static ring A is a function $|-|\colon A \to [0,\infty)$ such that $|0| \leq 0$, $|a+b| \leq |a| + |b|$, $|1| \leq 1$, and $|ab| \leq |a||b|$ hold for any a and b. A morphism $A \to B$ of seminormed ring is a ring map $f\colon A \to B$ satisfying $|f(a)| \leq |a|$ for any $a \in A$. We write $\mathsf{SNRing}^{\heartsuit}$ for such a category. We then define the category of (animated) *seminormed rings* via the pullback square

$$\begin{array}{ccc} \mathsf{SNRing} & \longrightarrow & \mathsf{Ring} \\ \downarrow & & \downarrow{\scriptstyle \pi_0} \\ \mathsf{SNRing}^{\heartsuit} & \longrightarrow & \mathsf{Ring}^{\heartsuit}. \end{array}$$

in Pr. Concretely, a *seminorm* on an (animated) ring A is a seminorm on $\pi_0 A$.

Example 9.2. The initial object in SNRing is $\mathbf{Z}$ with the usual absolute value $|n| = \max(n,-n)$ for $n \in \mathbf{Z}$.

Example 9.3. The coproduct of A and B in SNRing, which we write as $A \otimes B$ has the underlying ring $A \otimes B$. The seminorm is given by

$$|c| = \inf\left\{ \sum_{i=1}^{n} |a_i||b_i| \;\middle|\; c = \sum_{i=1}^{n} a_i \otimes b_i \right\}.$$

This is commonly called the *projective tensor product.*

Example 9.4. For $r \in [0,\infty)$, we have the seminormed ring $\mathbf{Z}[T]_r$ that corepresents the functor $\mathsf{SNRing} \to \mathsf{Ani}$ that maps A to the anima of elements $a \in A$ satisfying $|a| \leq r$. Concretely, the underlying ring is $\mathbf{Z}[T]$ and the seminorm is given by $|\sum_{i=0}^n a_i T^i| = \sum_{i=0}^n |a_i| r^i$. We have $\varinjlim_{r'>r} \mathbf{Z}[T]_{r'} = \mathbf{Z}[T]_r$.

By taking coproducts, we also obtain $\mathbf{Z}[T_1,\ldots,T_n]_{r_1,\ldots,r_n}$ for $r_i \in [0,\infty)$.

The presentable category SNRing is not compactly generated, but it is close to that:

Proposition 9.5. *Consider the full subcategory* SNPol *of* SNRing *spanned by* $\mathbf{Z}[T_1, \dots, T_n]_{r_1,\dots,r_n}$ *for* $r_i \in [0, \infty)$*. Then the Yoneda functor*

$$\mathsf{SNRing} \to \mathrm{Fun}(\mathsf{SNPol}^{\mathrm{op}}, \mathsf{Ani})$$

is fully faithful and the essential image consists of functors F *that carry finite coproducts to products and the colimits*

$$\mathbf{Z}[T]_s \otimes_{\mathbf{Z}[T]_r} \mathbf{Z}[T]_s \simeq \mathbf{Z}[T]_s \qquad\qquad \varinjlim_{r'>r} \mathbf{Z}[T]_{r'} \simeq \mathbf{Z}[T]_r,$$

where $r \leq s$*, to the corresponding limits. In particular, it is* $\aleph_1$*-compactly generated.*

Proof. Note that the Yoneda functor $\mathsf{Ring} \to \mathrm{Fun}(\mathsf{Pol}^{\mathrm{op}}, \mathsf{Ani})$ is fully faithful and its essential image consists of functors that carry finite coproducts to products.

We first prove that it is fully faithful. For a seminormed ring A, it suffices to show the colimit of the tautological diagram indexed by $\mathsf{SNPol}_{/A}$ is equivalent to A. This index category is sifted. We first consider $\mathsf{SNPol}_{/A} \to \mathsf{Pol}_{/A}$ and observe that it is cofinal. Therefore, the colimit computes the correct underlying ring. The identification of the seminorm is straightforward by using $\mathbf{Z}[T]_r$.

We then check the description of the essential image. Suppose F satisfies the conditions. We define F' to be its left Kan extension along $\mathsf{SNPol} \to \mathsf{Pol}$. Concretely, we have $F'(\mathbf{Z}[T_1, \dots, T_n]) = \varinjlim_r F(\mathbf{Z}[T_1, \dots, T_n]_{r,\dots,r})$. Hence it carries finite coproducts to finite products, and therefore it is representable by a ring A. For $a \in \pi_0(A)$, consider a corresponding element in $F'(\mathbf{Z}[T])$. Since $F(\mathbf{Z}[T]_r) \to F'(\mathbf{Z}[T])$ is a monomorphism by the condition, $F(\mathbf{Z}[T]_r) \times_{F'(\mathbf{Z}[T])} *$ is either final or empty. We set $|a|$ to be the infimum of r such that $F(\mathbf{Z}[T]_r) \times_{F'(\mathbf{Z}[T])} *$ is nonempty, which exists by the condition. We can check that this defines a seminorm on A, and that F is represented by A. ☐

Corollary 9.6. *The category* SNRing *is a retract of a compactly generated category in* Pr*.*

Proof. We consider the full subcategory $\mathcal{C}$ of $\mathrm{Fun}(\mathsf{SNPol}^{\mathrm{op}}, \mathsf{Ani})$ spanned by functors F that carry finite coproducts and $F(\mathbf{Z}[T]_s) \to F(\mathbf{Z}[T]_s) \times_{F(\mathbf{Z}[T]_r)} F(\mathbf{Z}[T]_s)$ is an equivalence for any $r \leq s$. This is compactly generated. We conclude the proof by showing that the left adjoint L the embedding $\mathsf{SNRing} \hookrightarrow \mathcal{C}$ of Proposition 9.5 admits a further left adjoint. It suffices to show that L preserves limits. By Proposition 9.5, we see that

$$\mathrm{Map}_{\mathsf{SNRing}}(\mathbf{Z}[T]_r, L(F)) \simeq \varprojlim_{r'>r} F(\mathbf{Z}[T]_{r'}),$$

which shows the desired result. ☐

Remark 9.7. Corollary 9.6 says that SNRing is *continuous* in the sense of Johnstone–Joyal [33] or *compactly assembled* in the sense of Lurie [39, Section 21.1.2]. One can also prove a symmetric monoidal refinement of Corollary 9.6.

We then introduce its important full subcategories:

Definition 9.8. Let A be a seminormed ring. It is called a *normed ring* if the anima of elements $a \in A$ with $|a| = 0$ is trivial. This forces the underlying animated ring to be static. We furthermore say that it is a *Banach ring* any Cauchy sequence converges. We say that it is *uniform*[14] if $|a^2| = |a|^2$ holds for any $a \in A$. We write URing for the full subcategory of SNRing spanned by uniform Banach rings.

We can characterize these using lifting properties:

Proposition 9.9. *Let* A *be a seminormed ring.*

(1) *It is normed if and only if it is local with respect to* $\mathbf{Z}[T]_0 \to \mathbf{Z}$*.*

[14]This is also called *spectral*, since a Banach ring satisfies this condition if and only if $|a| = \lim_{n\to\infty}|a^n|^{1/n}$ holds for any $a \in A$.

(2) *If it is normed, then it is Banach if and only if it is local with respect to $A \to B$ for $r > 0$, where A is $\mathbf{Z}[T_1, \dots]_{r,\dots}$ with $|T_m - T_{m+k}| \le 1/m$ for $m \ge 1$ and $k \ge 1$ and B is $A[U]_r$ with $|U - T_m| \le 1/m$ for $m \ge 1$. Formally, to construct A, we first add variables $S_{m,k}$ for $m \ge 1$ and $k \ge 1$ with radius $1/m$ and then take the quotient by $S_{m,k} - T_m - T_{m+k}$. Similarly for B.*

(3) *If it is Banach, it is uniform if and only if it is local with respect to*

$$\mathbf{Z}[T, U]_{r,s}/(T^2 - U) \to \mathbf{Z}[T]_{\sqrt{s}}$$

with $r^2 \ge s > 0$.

Proof. This is straightforward. □

Corollary 9.10. *All the full subcategories in Definition 9.8 are $\aleph_1$-compactly generated and localizations live in $\mathsf{Pr}^{\aleph_1}$.*

Proof. This is an immediate consequence of Propositions 9.5 and 9.9. □

Definition 9.11. We call the left adjoint to the inclusion $\mathsf{SNRing} \to \mathsf{UBan}$ *uniform completion* and write it as $A \mapsto A^{\mathrm{ucpl}}$.

Example 9.12. For a uniform Banach ring A, we write $A\{T_1, \dots, T_n\}_{r_1,\dots,r_n}$ for the uniform completion of the tensor product of A with $\mathbf{Z}[T_1, \dots, T_n]_{r_1,\dots,r_n}$.

We conclude this section by performing some calculations, which we use later:

Lemma 9.13. *Let A be a normed ring and $P = T^n + \cdots + a_0$ a monic polynomial. Then the seminormed ring $A[T]_r/P$ is Banach as long as*

$$r^n \ge 2(|a_{n-1}|r^{n-1} + \cdots + |a_0|)$$

is satisfied. Moreover, in this situation, when A is Banach, so is $A[T]_r/P$.

Proof. In this proof, for a polynomial S, we write $S_{<n}$ and $S_{\ge n}$ for the parts of degree $< n$ and $\ge n$, respectively.

To show the claim, it suffices to show that for $R \in A[T]$ of degree $< n$, we have

$$|PQ + R| \ge |R|$$

for any $Q \in A[T]$, where $|-|$ denotes the norm on $A[T]_r$. By

$$|PQ + R| = |(PQ + R)_{<n}| + |(PQ + R)_{\ge n}| \ge (|R| - |(PQ)_{<n}|) + |(PQ)_{\ge n}|,$$

it suffices to show $|(PQ)_{\ge n}| \ge |(PQ)_{<n}|$. We have

$$|(PQ)_{<n}| = \left|\sum_{i=0}^{n-1} a_i(T^iQ)_{<n}\right| \le \sum_{i=0}^{n-1}|a_i||T^iQ| = \sum_{i=0}^{n-1}|a_i|r^i|Q|.$$

On the other hand, we have

$$|(PQ)_{\ge n}| \ge |T^nQ| - \sum_{i=0}^{n-1}|a_i||(T^iQ)_{\ge n}| \ge |T^nQ| - \sum_{i=0}^{n-1}|a_i||T^iQ| = \left(r^n - \sum_{i=0}^{n-1}|a_i|r^i\right)|Q|.$$

Therefore, the result follows from the assumption.

In the Banach case, completeness can be checked directly by using the description of the norm obtained above. □

Remark 9.14. Unlike Lemma 9.13, when we want to extend a uniform Banach ring structure, different considerations are needed; e.g., the uniform completion of $\mathbf{R}[T]_r/T^2$ is $\mathbf{R}$ for any r. We treat a very special case in Lemma 9.32.

9.2. **Berkovich spectrum.** We introduce the notion of fields and hence the spectrum:

Definition 9.15. A seminorm on a ring A is *multiplicative* if $|1| = 1$ and $|ab| = |a||b|$ for any a and $b \in A$. A *normed field* is a normed ring whose underlying ring is a field and the seminorm is multiplicative. We define *Banach field* similarly; note that a Banach field is automatically uniform.

Example 9.16. Fix a prime p and consider $\mathbf{Q}_p$ with the norm $\max(|-|_p, |-|_p^2)$. This is a uniform Banach ring, but not a Banach field.

Definition 9.17 (Berkovich)**.** Let A be a seminormed ring. Its *Berkovich spectrum* is the closed subspace

$$\mathrm{M}(A) \subset \prod_{a \in A} [0, |a|]$$

spanned by multiplicative seminorms on A bounded by the original norm on A.

Remark 9.18. In Definition 9.17, the underlying set of $\mathrm{M}(A)$ can be identified with the set of equivalence classes of morphisms $A \to K$ to normed fields K. The equivalence relation is generated by $(A \to K) \sim (A \to K')$ if there is an isometric embedding $K \to K'$ making the diagram commute.

The following was proven in [46, Proposition 3.2]:

Proposition 9.19. *The following holds for the Berkovich spectrum of a seminormed ring A:*

(1) When A is a filtered colimit $\varinjlim_i A_i$, the map $\mathrm{M}(A) \to \varprojlim_i \mathrm{M}(A_i)$ is a homeomorphism.
(2) When $B' = A' \otimes_A B$ is a pushout of seminormed rings, the map $\mathrm{M}(B') \to \mathrm{M}(A') \times_{\mathrm{M}(A)} \mathrm{M}(B)$ is surjective.
(3) The morphism $A \to A^{\mathrm{ucpl}}$ induces a homeomorphism $\mathrm{M}(A^{\mathrm{ucpl}}) \to \mathrm{M}(A)$.

We do not consider general rational domains in this paper; Weierstraß and Laurent domains suffice for our purpose:

Lemma 9.20. *Let A be a seminormed ring. The Berkovich spectra of $A[T]_r/(T-a)$ and $A[T]_{1/r}/(aT-1)$ are identified with the base changes of $\mathrm{M}(A)$ along the inclusions $[0, r] \to [0, |a|]$ and $[r, |a|] \to [0, |a|]$, respectively. In particular, M is compatible with base change in this case.*

Proof. Since Berkovich spectra are compact Hausdorff, it suffices to show this on the level of underlying sets. This follows from Remark 9.18. □

The following was proven in [46, Theorem 2.18]:

Theorem 9.21 (Scholze)**.** *Let A be a Banach ring. Then clopen subsets of $\mathrm{M}(A)$ correspond bijectively to idempotent elements of A.*

Corollary 9.22. *Let A be a Banach ring. The canonical map $\mathrm{M}(A) \to \mathrm{Spec}(A)$ that maps $|-|$ to its kernel is continuous and induces bijection on connected components.*

Proof. The continuity is straightforward. The claim about connected components requires Theorem 9.21. □

Corollary 9.23. *Let A be a Banach ring such that $\mathrm{M}(A)$ is totally disconnected. Then the map $\mathrm{M}(A) \to \mathrm{Spec}(A)$ is injective and exactly hits the closed points. Moreover, each connected component of $\mathrm{Spec}(A)$ has a unique closed point.*

Proof. This is a direct consequence of Corollary 9.22. □

We bound the size of the Berkovich spectrum:

Lemma 9.24. *Let $\kappa > \aleph_0$ be a regular cardinal and A a κ-compact object in URing. Then $\mathrm{M}(A)$ has weight $< \kappa$.*

Proof. It is a quotient of the uniform completion of the seminormed ring $A = \mathbf{Z}[T_i \mid i \in I]_{r_i}$, where I is a set of cardinality $< \kappa$ and $r_i \geq 0$. Hence it is a closed subset of $\mathrm{M}(A^{\mathrm{ucpl}})$. By (3) of Proposition 9.19, it suffices to observe that $\mathrm{M}(A)$ has weight $< \kappa$, but this follows from the definition and that A is of cardinality $< \kappa$, since $\prod_{i \in I}[0, r_i]$ has weight $< \kappa$. □

We only use the following notation for Banach rings:

Definition 9.25. Let A be a Banach ring and $x \in \mathrm{M}(A)$. The (complete) *residue field* $K(x)$ is the completion of the residue field at $\ker|-|_x$, where $|-|_x$ is the multiplicative seminorm corresponding to x.

9.3. **(Strictly) totally disconnecteds.** We first recall the following:

Lemma 9.26. *Let K be a nondiscrete Banach field. The completion of its separable closure is algebraically closed.*

Proof. The archimedean case is vacuous. The nonarchimedean case follows from [15, Propositions 3.4.1.3 and 3.4.1.6]. □

Definition 9.27 (Scholze)**.** We call a uniform Banach ring *totally disconnected* when the underlying space is totally disconnected and all residue fields are nondiscrete. We call it *strictly totally disconnected* when moreover residue fields are separably closed (hence algebraically closed by Lemma 9.26). We write TDis and STDis for the full subcategories of URing spanned by them, respectively.

We recall the following from usual algebraic geometry:

Definition 9.28. Let A be a totally disconnected. For $x \in \mathrm{M}(A) \subset \mathrm{Spec}(A)$ (see Corollary 9.23), we consider the uncompleted residue field $\kappa(x)$, which is just the Zariski residue field at $\ker|-|_x$.[15] Then we define its local ring A_x to be its Zariski local ring.

Proposition 9.29. *In the situation of Definition 9.28, the local ring A_x is henselian.*

Proof. It suffices to show that $A_x[T]/P$ for any monic polynomial P is a finite product of local rings. By spreading, we can assume that P is defined over A. We take $r \gg 0$ so that $A[T]_r/P$ is Banach by Lemma 9.13. Since $\mathrm{M}(A[T]_r/P) \to \mathrm{M}(A)$ has finite fibers, $A[T]_r/P$ has totally disconnected spectrum. Then by Corollary 9.23 again, $A_x[T]/P$ becomes the product of local rings at points of $\mathrm{M}(A[T]_r/P)$ lying over x. □

Lemma 9.30. *Let A be a totally disconnected. Then there is a morphism $\mathbf{Z}\{T, T^{-1}\}_{s,1/r} \to A$ for some $0 < r \leq s < 1$.*

Proof. We argue locally. So we have $A_x \to \kappa(x) \to K(x)$. When the completion is nondiscrete, then $\kappa(x)$ should be also nondiscrete. Therefore, we have a uniformizer. Then we can spread this to π and $\pi^{-1} \in A$.

By this, we obtain an element $\pi \in A$ such that the map whose norm factors through $(0, 1) \subset [0, \infty]$. We write $[r, s]$ for the subset containing its image and the desired result follows. □

We record some facts about finite étale maps over totally disconnecteds:

Proposition 9.31. *Let A be a totally disconnected and $x \in A$ a point. Consider a finite separable extension $L(x)$ over $K(x)$. Then there is a finite étale map $A \to B$ (cf. Lemma 9.32) that induces a homeomorphism on spectra such that $L(x)$ is the base change of B along $A \to K(x)$.*

We first confirm that finite étale algebras over totally disconnecteds admit a unique structure of a totally disconnected:

Lemma 9.32. *Let A be a totally disconnected. The forgetful functor $\mathsf{TDis}_{A/} \to \mathsf{Ring}_{A/}$ is an equivalence of categories when we restrict to the full subcategory of finite étale algebras on the target.*

[15] For general A and $x \in \mathrm{M}(A)$, this does *not* recover the definition of the uncompleted residue field in Berkovich geometry. Our totally disconnectedness assumption ensures that it does here.

In the following proof, we see an explicit formula for the norm:

Proof. What we show here is that any finite étale algebra B over A admits a unique structure of a totally disconnected so that $A \to B$ is a morphism of totally disconnecteds; the desired equivalence follows from the proof.

We first consider a finite separable extension of nondiscrete Banach fields $A \to B$. The archimedean case is straightforward. The nonarchimedean case follows from [15, Theorem 3.2.4.2]. By uniqueness, the norm on B must be given by $|\mathrm{Nm}_{B/A}(-)|^{1/d}$, where $\mathrm{Nm}_{B/A}$ is the norm map $B \to A$ and d is the degree.

We go back to the general statement. For $x \in \mathrm{M}(A)$, we write $B(x)$, for the (algebraic) tensor product $K(x) \otimes_A B$. This is a finite product of finite separable extensions of $K(x)$, and hence by the argument above, admits a unique norm that makes $K(x) \to B(x)$ a morphism of uniform Banach rings. The norm on B must be spectral, so the only possible candidate is mapping $b \in B$ to $\sup_{x \in \mathrm{M}(A)} |b|_x$, where we write $|-|_x$ for the norm on $B(x)$. The problem is the existence of this supremum, but since it coincides with $|\mathrm{Nm}_{B/A}(-)|^{1/d}$, where d is the degree (as a locally constant function), it is well defined. The fact that this function gives B a structure of a totally disconnected is clear from the description via supremum. ☐

Lemma 9.33. *Let $\kappa \to K$ be the completion morphism of a normed field. Then any separable finite extension of K lifts to a separable finite extension of κ.*

Proof. In the archimedean case, we need to consider $K = \mathbf{R} \to \mathbf{C}$ only and the desired result is clear. The nonarchimedean case follows from [15, Proposition 3.4.2.5]. ☐

Proof of Proposition 9.31. We consider the maps $A_x \to \kappa(x) \to K(x)$. By Proposition 9.29 and Lemma 9.33, the finite étale algebra $B(x)$ lifts to a finite étale algebra over A_x, and hence to a finite étale algebra B' over $A[e^{-1}]$ that induces a homeomorphism on spectra for some idempotent e that is invertible in $K(x)$. Then $B' \times A/e$ is the desired A-algebra. ☐

We introduce a class of totally disconnecteds:

Definition 9.34. We call a (strictly) totally disconnected *light* if it is $\aleph_1$-compact as an object of URing. We write $\mathsf{TDis}_{\mathrm{lgt}}$ and $\mathsf{STDis}_{\mathrm{lgt}}$ for the corresponding full subcategories.

The following bound is important later:

Proposition 9.35. *There are countably many isomorphism classes of finite étale algebras over a light totally disconnected.*

Proof. The question is reduced to the algebraic case by Lemma 9.32. Each finite étale algebra B determines a morphism $\mathrm{M}(B) \to \mathrm{M}(A)$ and there is a partition of A into finitely many components such that B is a free module of finite rank on each component. Moreover, it is a finite product of monogenic ones. Therefore, by Lemma 9.24, it suffices to show that there are countably many isomorphism classes of étale A-algebras of the form $A[T]/(T^n + \cdots + a_0)$.

We view A as a topological space by considering the metric induced by the norm. Since it is $\aleph_1$-compact as a uniform Banach ring, A contains a countable dense subset. Thus, it is Lindelöf.

We consider a separable polynomial $P = T^n + \cdots + a_0$. We wish to find a neighborhood of P in the space of monic polynomials of degree n (which is isomorphic to A^n, which is Lindelöf) such that $A[T]/P \simeq A[T]/Q$ when Q is contained in this neighborhood. To construct such an isomorphism, we need to find an element $\alpha \in A[T]/P$ close enough to T satisfying $Q(\alpha) = 0$; the proximity guarantees the map is an isomorphism. This is possible using the Newton method. ☐

9.4. **Arc topology.**

Definition 9.36 (Scholze)**.** The arc topology on $\mathsf{URing}^{\mathrm{op}}$ is the Grothendieck topology generated by surjections on Berkovich spectra and by finite disjoint unions. This is well defined by (2) of Proposition 9.19. An *arc stack*[16] is an accessible hypersheaf on $\mathsf{URing}^{\mathrm{op}}$.

[16] This is called "small arc stack" in [46]. Our arc stacks are always small.

Example 9.37. The morphism $\mathbf{Z} \to \mathbf{Z}\{T, T^{-1}\}_{r,1/r}$ is an arc cover for $r > 0$. This translates to the fact that for any Banach field K, there is an extension $K \to K'$ of Banach fields such that K' contains an element a with $|a| = r$.

Example 9.38. For an element a of a uniform Banach ring A and $r \geq 0$, the uniform completions of the rings given in Lemma 9.20 determine an arc cover.

Note that strictly totally disconnecteds constitute a nice basis:

Theorem 9.39 (Scholze)**.** *For any uniform Banach ring A, there is a strictly totally disconnected B with an arc cover $A \to B$.*

Theorem 9.40 (Scholze)**.** *Strictly totally disconnecteds are (hyper)subcanonical with respect to the arc topology.*

Definition 9.41. For a uniform Banach ring A, we write $\mathrm{Sparc}(A)$ for the corresponding arc stack.

See [46, Theorem 3.13] for the proof of Theorem 9.40. Since the proof of Theorem 9.39 is important, we recall it here. We first axiomatize the situation, since we use it in a slightly different context later:

Lemma 9.42. *Let A be a ring with a continuous map $S \to \operatorname{Spec} A$ from a profinite set S satisfying the following:*

(i) The map $S \to \operatorname{Spec} A$ exactly hits the closed points.
(ii) The local ring of A at each closed point is henselian.
(iii) Each connected component of $\operatorname{Spec} A$ admits a unique closed point.

Then there is a filtered colimit of finite étale A-algebras $\varinjlim_i A_i = B$ with a compatible map $S \to \operatorname{Spec} B$ such that for each i, the induced map $S \to \operatorname{Spec} A_i$ satisfies the condition above and the local ring of B at each closed point identifies with the strict henselization of A.

Proof. We first construct such B and then realize it as a filtered colimit of finite étale A-algebras. We first fix a well-ordering on S. We construct a transfinite sequence of A-algebras indexed by S with a compatible map $S \to \operatorname{Spec} A_s$ satisfying the condition above by recursion. We first consider $C = \varinjlim_{t<s} A_t$ (so when s is initial, we set $C = A$). Then we have a corresponding closed point $z \in \operatorname{Spec} C$. We take $k = \kappa(z)$. Then there is an ordinal α and a transfinite sequence $(k_\beta)_{\beta<\alpha}$ of fields over k such that $\varinjlim_{\gamma<\beta} k_\gamma \to k_\beta$ is a finite separable extension and $\varinjlim_{\beta<\alpha} k_\beta$ is a separable closure of k. We construct a similar transfinite sequence of C-algebras $(C_\beta)_{\beta<\alpha}$: For each $\beta < \alpha$, we consider $D = \varinjlim_{\gamma<\beta} C_\beta$. We first write w for the image of s in $\operatorname{Spec} D$. By the henselian property, the extension $\varinjlim_{\gamma<\beta} \to k_\beta$ spreads to a finite étale algebra around w. By the conditions, we see that it spreads to a clopen neighborhood of w, and therefore, we can take a finite étale algebra over D with a unique lift of $S \to \operatorname{Spec} D$. We take C_β to be this algebra. Then we take $A_s = \varinjlim_{\beta<\alpha} C_\beta$. We see that $B = \varinjlim_{s\in S} C_s$ satisfies the condition.

Now, we consider the category of finite étale A-algebras with a map $A' \to B$ such that the induced map $S \to \operatorname{Spec} A'$ satisfies the conditions. This category is filtered and its colimit computes B. □

Proof of Theorem 9.39. First, by base changing along $\mathbf{Z} \to \mathbf{Z}\{T, T^{-1}\}_{1/2,2}$ (see Example 9.37), we can assume that A has nondiscrete residue fields.

Then by considering the Cantor set cover described in Proposition 6.5 of $[0, |a|]$ for each a, we can replace A with a totally disconnected. It is an arc cover by Example 9.38.

We then apply Lemma 9.42 for $\mathrm{M}(A) \to \operatorname{Spec}(A)$. Note that (i) and (iii) follow from Corollary 9.23 and (ii) from Proposition 9.29. It is an arc cover, since the topology is unchanged. □

We have the following light variant:

Proposition 9.43. *When A is a $\aleph_1$-compact uniform Banach ring. Then there is a light strictly totally disconnected B with an arc cover $A \to B$.*

Proof. We verify that each step in the proof of Theorem 9.39 can be made to preserve lightness.

The first step is valid, since $\mathbf{Z}\{T, T^{-1}\}_{1/2,2}$ is light.

For the second step of obtaining a totally disconnected, by Lemma 9.24, we can choose a countable sequence $a_0, \ldots$ such that $\mathrm{M}(A) \to \prod_{n=0}^{\infty}[0, |a_n|]$ is injective. We can then do the same process to obtain something light.

For the last step, what the proof produces is light by Proposition 9.35. □

We observe the following immediate consequence:

Corollary 9.44. *A presheaf* $\mathsf{URing}_{\aleph_1} \to \mathsf{Ani}$ *is an arc (hyper)sheaf if and only if it is right Kan extended from* $\mathsf{STDis}_{\aleph_1} \to \mathsf{Ani}$*, which is an arc (hyper)sheaf there. In particular, we have equivalences*

$$\mathrm{Shv}((\mathsf{STDis}_{\mathrm{lgt}})^{\mathrm{op}}) \simeq \mathrm{Shv}((\mathsf{URing}_{\aleph_1})^{\mathrm{op}}), \qquad \mathrm{Shv}^{\mathrm{hyp}}((\mathsf{STDis}_{\mathrm{lgt}})^{\mathrm{op}}) \simeq \mathrm{Shv}^{\mathrm{hyp}}((\mathsf{URing}_{\aleph_1})^{\mathrm{op}}),$$

both given by right Kan extension.

9.5. **Topological finite presentation.** We here consider the following:

Definition 9.45. Let A be a uniform Banach ring. We call a uniform Banach ring B over A *topologically of finite presentation* if it is a quotient of $A\{T_1, \ldots, T_n\}_{r_1,\ldots,r_n}$ by a finitely generated ideal. When $A = \mathbf{Z}$, we simply say that B is topologically of finite presentation. We write $\mathsf{URing}_{\mathrm{tfp}}$ for the full subcategory of URing. Equivalently, this is the smallest full subcategory containing $\mathbf{Z}\{T\}_r$ for $r > 0$ and closed under finite colimits.

We prove a certain finiteness statement about such rings. First, we note the following about residue fields:

Proposition 9.46. *Let O be a totally imaginary number ring. For any complete residue field K of $\mathbf{Z}\{T_1, \ldots, T_n\}_{r_1,\ldots,r_n}$, we have* $\dim_{\mathbf{Z}}(\mathrm{Sparc}(K)) \leq n + 3$.

Proof. We only have to consider finite extensions of of $\mathbf{Q}(T_1, \ldots, T_n)$ or $\mathbf{F}_p(T_1, \ldots, T_n)$; namely, if K is the completion of κ, then the absolute Galois group G_K is a quotient of G_κ. Now we can use the result from étale cohomology (but see Remark 6.15). By [51, Tag 0F0S], we are reduced to the case when $n = 0$ and this is classical. □

We also need the following later; see, e.g., [34, Théorème 7.3.7] for a proof:

Proposition 9.47. *Let O be a number ring. The covering dimension of $O\{T_1, \ldots, T_n\}_{r_1,\ldots,r_n}$ for any $r_1, \ldots, r_n > 0$ is $2n + 1$.*

Remark 9.48. When we bound the arc cohomological dimension (see [46, Remark 4.11]) of a uniform Banach ring of topological finite presentation by bounding its topological dimension and the cohomological dimension of residue fields separately, we obtain a suboptimal bound. With a careful argument, one can show, e.g., that $\mathrm{Sparc}(O[T_1, \ldots, T_n]_{r_1,\ldots,r_n})$ for a totally imaginary number ring O, has cohomological dimension $2n + 4$ for any $r_1, \ldots, r_n > 0$.

10. Berkovich 2-motives

In this section, we prove Theorem D, which characterizes $2\mathrm{D}_{\mathrm{mot}}(\mathbf{Z})$ in terms of ring stacks with an absolute value. Throughout this section, we write $2\mathrm{Mot}(\mathbf{Z})$ for the universal target of Theorem D.

In Section 10.1, we review $\mathrm{D}_{\mathrm{mot}}$. In Section 10.3, we consider the class of a seminormed ring in $2\mathrm{Mot}(\mathbf{Z})$. In Sections 10.2 and 10.4, we prove important descent results. In Section 10.5, we complete the proof of Theorem D.

10.1. **A review of Berkovich motivic spectra.** In [46], Scholze considered integral coefficients. Here, we consider the same category with spherical coefficients.

Definition 10.1. Let X be an accessible presheaf on $\mathsf{STDis}^{\mathrm{op}}$ preserving finite products. A *finitary arc sheaf* over X is a morphism $Y \to X$ from another such presheaf such that for a cofiltered limit $T = \varprojlim_i T_i$, the square

$$\begin{array}{ccc} \varinjlim_i Y(T_i) & \longrightarrow & Y(T) \\ \downarrow & & \downarrow \\ \varinjlim_i X(T_i) & \longrightarrow & X(T) \end{array}$$

is a pullback. It is equivalently just a finitary arc sheaf on $(\mathsf{STDis}^{\mathrm{op}})_{/X}$.

Definition 10.2. Let X be an arc stack. We define

$$\mathrm{D}_{\mathrm{mot}}(X;\mathbf{S}) = \mathrm{Shv}_{\mathrm{fin}}(X;\mathsf{Sp})_{\mathbb{D}^1}[(\Sigma^\infty\mathbb{P}^1)^{\otimes -1}].$$

First, $\mathrm{Shv}_{\mathrm{fin}}(X;\mathsf{Sp})$ is the category of Sp-valued finitary arc sheaves on $(\mathsf{STDis}^{\mathrm{op}})_{/X}$. Then we restrict to $\mathbb{D}^1$-invariant sheaves. This means that for any uniform Banach ring A, the morphism $A \to A\{T\}_1$ induces an equivalence. By [46, Corollary 5.4], for finitary sheaves, it suffices to check this for $C \to C\{T\}_1$ where C is a nondiscrete algebraically closed Banach field (but still note that $C\{T\}_1$ is not totally disconnected). We omit $\mathbf{S}$ and simply write $\mathrm{D}_{\mathrm{mot}}(X)$ when there is no confusion.

The following for $\mathrm{D}_{\mathrm{mot}}(-;\mathbf{Z})$ was proven in [46, Theorem 9.2] (see also [46, Corollary 10.2]). The same proof works in the spherical case:

Theorem 10.3 (Scholze)**.** *Let $A \to B$ be a map of uniform Banach rings such that $Y = \mathrm{Sparc}(B)$ embeds to a finite-dimensional ball over $X = \mathrm{Sparc}(A)$ and $\sup_{x\in \mathrm{M}(A)} \dim_{\mathbf{Z}}(Y_x)$ is finite. Then the functor $f^*\colon \mathrm{D}_{\mathrm{mot}}(A) \to \mathrm{D}_{\mathrm{mot}}(B)$ admits a $\mathrm{D}_{\mathrm{mot}}(A)$-linear right adjoint f_* which is compatible with base change.*

Remark 10.4. One subtle part in the proof of Theorem 10.3 is the projection formula for $\mathbb{P}^1_X \to X$, when $X = \mathrm{Sparc}(C)$ for a nondiscrete algebraically closed Banach field C. This can also be deduced using Theorem 3.4.

Hence we can apply Theorem 2.2 to obtain the following:

Corollary 10.5. *The lax symmetric monoidal functor $\mathrm{D}_{\mathrm{mot}}\colon \mathsf{URing}_{\mathrm{tfp}} \to \mathsf{Pr}_{\mathrm{st}}$ extends (uniquely) to $2\mathrm{Span}_{\mathrm{all;all,iso}}((\mathsf{URing}_{\mathrm{tfp}})^{\mathrm{op}})$.*

Definition 10.6. We write $2\mathrm{D}_{\mathrm{mot}}(\mathbf{Z})$ for the presentable 2-category of kernels for the functor $\mathrm{Span}((\mathsf{URing}_{\mathrm{tfp}})^{\mathrm{op}}) \to \mathsf{Pr}_{\mathrm{st}}$, which is a restriction of the functor in Corollary 10.5.

10.2. **Disk invariance.** We consider the condition of having homologically trivial disks. It suffices to assume that the open unit disk $\mathbb{B}^1$ has this property:

Proposition 10.7. *Consider a stable presentably symmetric monoidal 2-category $\mathcal{C}$ and a stable weakly suave ring stack R with an absolute value $N\colon R \to [0,\infty)$ over $\mathcal{C}$. Assume $\mathbb{B}^1 = N^{-1}([0,1))$ (which is suave by Lemma 4.27) is homologically trivial.*

(1) Then $\mathbb{B}^1_r = N^{-1}([0,r))$ is suave and homologically trivial for any $r > 0$.
(2) Then $\mathbb{A}^1 = R$ is suave and homologically trivial.
(3) Then $\mathbb{D}^1_r = N^{-1}([0,r])$ is proper and cohomologically trivial for any $r \geq 0$.

Proof. We first prove (1). The multiplication map $\mathbb{B}^1 \times \mathbb{B}^1_r \to \mathbb{B}^1_r$ gives a $\mathbb{B}^1$-homotopy between id on $\mathbb{B}^1_r$ and the constant map 0.

Then (2) follows from considering the colimit $\mathbb{A}^1 = \varinjlim_r \mathbb{B}^1_r$. This shows that the specification of the ring stack factors through $2\mathrm{SH}(\mathbf{Z})$ by Theorem B.

We then proceed to (3). Note that $\mathbb{P}^1 = R\amalg_{R^\times R}$ is proper since it is proper over $2\mathrm{SH}(\mathbf{Z})$. Hence the extended norm map $\widetilde{N}\colon \mathbb{P}^1 = R \amalg_{R^\times} R \to [0,\infty]$ is proper. Therefore, $\mathbb{D}^1_r = N^{-1}([0,r]) = \widetilde{N}^{-1}([0,r])$ is proper. Its cohomology can be computed by (1) and the recollement induced by $\mathbb{B}^1_{1/r} \to \mathbb{P}^1 \leftarrow \mathbb{D}^1_r$, which is the base change of $[0,1/r) \to [0,\infty] \leftarrow [1/r,\infty]$ along $\widetilde{N}$. ☐

This proves the existence of a pseudouniformizer up to descent:

Corollary 10.8. *In the situation of Proposition 10.7, the morphism $N^{-1}(\{r\}) \to *$ is a cover for any $r > 0$.*

Proof. We know that $\mathbb{P}^1$ is proper and therefore, the extended norm map $\widetilde{N}$ is also proper. This implies that $f\colon N^{-1}(\{r\}) \to *$ is proper and therefore by Theorem 5.13, it suffices to show that $f_*\mathbf{1} \in \mathrm{End}_{\mathcal{C}}(\mathbf{1})$ satisfies descent. We compute this by considering it as the intersection of $N^{-1}([0,r]) \cap \widetilde{N}^{-1}([r,\infty])$ inside $\mathbb{P}^1$. By (3), we can compute it as $\mathbf{1} \oplus \mathbf{1}(-1)[-1]$, and therefore it admits a section and by Lemma 5.11, it satisfies descent. □

Remark 10.9. Something similar to Corollary 10.8 holds without assuming disk invariance: Given a stable homologically trivial weakly suave ring stack with an absolute value N, one can show that $N^{-1}((r,s)) \to *$ is a cover for any $0 < r < s$. Nevertheless, $\mathbb{A}^1$-invariance is strictly weaker than $\mathbb{B}^1$-invariance.

Remark 10.10. At least after proving Theorem D, one can show with extra work that the norm map $N\colon \mathbb{A}^1 \to [0,\infty)$ is a cover over $2\mathrm{Mot}(\mathbf{Z})$.

10.3. **2-motives of seminormed rings.** By Theorem C, we obtain a morphism $2\mathrm{SH}_{\text{ét}}(\mathbf{Z}) \to 2\mathrm{Mot}(\mathbf{Z})$. Therefore, we obtain the class for each ring:

Definition 10.11. For a ring A, we write $[A]_{\mathrm{alg}} \in \mathrm{CAlg}(2\mathrm{Mot}(\mathbf{Z}))$ for the image of $[A] \in \mathrm{CAlg}(2\mathrm{SH}_{\text{ét}}(\mathbf{Z}))$.

In our situation, we can define the motive of each seminormed ring:

Definition 10.12. For a seminormed ring A, we consider the pushout

$$\begin{array}{ccc} \mathrm{Shv}([0,\infty]^{\pi_0(A)}) \otimes \mathbf{1} & \longrightarrow & [A]_{\mathrm{alg}} \\ \downarrow & & \downarrow \\ \mathrm{Shv}(\mathrm{M}(A)) \otimes \mathbf{1} & \longrightarrow & [A] \end{array}$$

in $\mathrm{CAlg}(2\mathrm{Mot}(\mathbf{Z}))$. Since it $[A]_{\mathrm{alg}} \to [A]$ is an epimorphism, by Lemma 10.13 below, this assembles into a functor $[-]\colon \mathsf{SNRing} \to \mathrm{CAlg}(2\mathrm{Mot}(\mathbf{Z}))$.

Lemma 10.13. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. For $C \in \mathcal{C}$, we have a monomorphism $F'_C \to F(C)$. Assume that for any $C \to C'$, the composite $F'_C \to F(C) \to F(C')$ factors through $F'_{C'}$. Then this family uniquely assembles into a functor $F'\colon \mathcal{C} \to \mathcal{D}$ and a natural transformation $F' \to F$.*

Proof. By Yoneda, we can assume $\mathcal{D} = \mathsf{Ani}$: More precisely, we consider $G\colon \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathsf{Ani}$ given by $(C,D) \mapsto \mathrm{Map}(D, F(C))$ and $G'_{C,D} = \mathrm{Map}(D, F'_C)$. In this case, we can construct $F'_0\colon \mathcal{C} \to \mathsf{Set}$ as a subfunctor of $\pi_0 F\colon \mathcal{C} \to \mathsf{Set}$. The functor $F' = F'_0 \times_{\pi_0(F)} F$ satisfies the desired property. The uniqueness also follows from the construction. □

Proposition 10.14. *The functor $[-]\colon \mathsf{SNRing} \to \mathrm{CAlg}(2\mathrm{Mot}(\mathbf{Z}))$ factors through $\mathsf{SNRing}^{\heartsuit}$ and preserves colimits.*

Proof. The first claim follows directly from the definition. We prove the second claim about colimits. For filtered colimits, it follows from the definition and (1) of Proposition 9.19.

We first prove that it preserves finite coproducts. For the nullary case, we need to show that $\mathrm{Shv}([0,\infty]^{\mathbf{Z}}) \to \mathrm{End}_{2\mathrm{Mot}(\mathbf{Z})}(\mathbf{1})$ factors through $\mathrm{Shv}(\mathrm{M}(\mathbf{Z}))$. For this, we recall that $\mathrm{M}(\mathbf{Z}) \subset \prod_n [0,\infty]$ is cut out by the multiplicative seminorm axioms. Therefore, this fact comes from the definition of an absolute semivalue. Similarly, we can see that it preserves binary coproducts.

We then prove that it preserves pushouts $A' \otimes_A B$ to complete the proof. By $A' \otimes_A B = A \otimes_{A \otimes A} (A' \otimes B)$, it suffices to treat the case when $A \to A'$ is a surjection. In this case, $\mathrm{M}(A') \to \mathrm{M}(A)$ is the pullback of the diagonal inclusion $[0,\infty]^{A'} \to [0,\infty]^A$ and hence $\mathrm{M}(A' \otimes_A B) \simeq \mathrm{M}(A') \otimes_{\mathrm{M}(A)} \mathrm{M}(B)$. From this, we can observe the desired result. □

We see some examples:

Example 10.15. We describe the direct relation between this construction and the norm $N\colon R \to [0,\infty)$. We claim that $[\mathbf{Z}[T]_r]$ for $r \geq 0$ corresponds to $N^{-1}([0,r])$. To see this, we need to see that $[\mathbf{Z}[T]_r]$ inside $[\mathbf{Z}[T]]_{\mathrm{alg}} = [R]$ is solely cut out from T. This follows from a similar argument to the proof of Proposition 10.14; all the inequality is generated by $|T| \leq r$. Similarly, $[\mathbf{Z}[T,T^{-1}]_{s,1/r}]$ for $r \leq s$ corresponds to $N^{-1}([r,s])$.

Remark 10.16. One can give another definition of the 2-motive of a seminormed ring via Example 10.15. Namely, we define $\mathsf{SNPol} \to 2\mathrm{Mot}(\mathbf{Z})$ by sending $\mathbf{Z}[T_1,\dots,T_n]_{r_1,\dots,r_n}$ to $N^{-1}([0,r_1]) \times \dots \times N^{-1}([0,r_n])$. Then we can check the relations described in Proposition 9.5 are satisfied so that this extends to a functor $\mathsf{SNRing} \to 2\mathrm{Mot}(\mathbf{Z})$. By Proposition 10.14, this definition recovers our definition here.

Example 10.17. Let l be a prime. Consider $\mathbf{Z}[1/l]_1$ to be the universal one with $|1/l| \leq 1$. The motive of this is just the base change of $\mathbf{1}$ along the morphism $\mathrm{Shv}(\mathrm{M}(\mathbf{Z}))\otimes\mathbf{1} \to \mathrm{Shv}(\mathrm{M}(\mathbf{Z}[1/l]_1))\otimes \mathbf{1}$.

Example 10.18. Let $A \to A/I$ be a quotient of (static) seminormed rings. In this case,

$$\begin{array}{ccc} [A]_{\mathrm{alg}} & \longrightarrow & [A/I]_{\mathrm{alg}} \\ \big\downarrow & & \big\downarrow \\ [A] & \longrightarrow & [A/I] \end{array}$$

is a pushout in $\mathrm{CAlg}(2\mathrm{Mot}(\mathbf{Z}))$. Since both commutes with colimits, it suffices to show this for $\mathbf{Z}[T]_r \to \mathbf{Z}$ for $r \geq 0$, which follows from Example 10.15.

Proposition 10.19. *For any seminormed ring A, the stack corresponding to $[A]$ is static and proper.*

Proof. We can assume A is static by Proposition 10.14. We write it as a filtered colimit of the algebra of the form $\mathbf{Z}[T_1,\dots,T_n]_{r_1,\dots,r_n}/(P_1,\dots,P_m)$ to reduce to the case of such rings. Moreover, by Example 10.18, we can assume $m=0$. Now we are reduced to the case of $\mathbf{Z}[T]_r$. The corresponding stack is $N^{-1}([0,r])$ by Example 10.15. We now consider the extended norm map $\widetilde{N}\colon \mathbb{P}^1 \to [0,\infty]$, which is obtained by gluing N and N^{-1}. Since $\mathbb{P}^1$ is static and proper already over $2\mathrm{SH}(\mathbf{Z})$, the desired result follows. □

Note that uniform completion is automatic in this process:

Theorem 10.20. *The functor $\mathsf{SNRing} \to 2\mathrm{Mot}(\mathbf{Z})$ in Definition 10.12 factors through the uniform completion functor $\mathsf{SNRing} \to \mathsf{URing}$.*

Proof. We prove that it factors through the categories of normed rings, Banach rings, and then uniform Banach rings. To prove that, it suffices to show that each morphism in Proposition 9.9 is sent to an equivalence.

We consider (1); i.e., we wish to show that $\mathbf{Z}[T]_0 \to \mathbf{Z}$ is sent to an equivalence. This follows from Example 10.15.

Then we consider (2). We fix $r \gg 0$. The situation is that A is $\mathbf{Z}[T_1,\dots]_{r,\dots}$ with $|T_m - T_{m+k}| \leq 1/m$ for $m \geq 1$ and $k \geq 1$ and B is $A[U]_r$ with $|U - T_m| \leq 1/m$ for $m \geq 1$. We wish to show that $[A] \to [B]$ is equivalence. By Proposition 10.19 and considering its diagonal (cf. our proof of Corollary 8.29), it suffices to show that the map $[A] \to [B]$ satisfies descent. By Theorem 5.13, it reduces to the question of $f_*\mathbf{1} \in \mathrm{Hom}_{2\mathrm{Mot}(\mathbf{Z})}(\mathbf{1},[A])$ satisfying descent. By Proposition 5.16, it suffices to show that it is descendable. We first write $A \to B$ as a sequential colimit of $f_n\colon A_n \to B_n$ for $n \geq 0$ by only considering $T_1, \dots, T_n$. But this map $A_n \to B_n$ admits a section given by $U \mapsto T_n$, and hence $f_{n,*}\mathbf{1} \in \mathrm{Hom}_{2\mathrm{Mot}(\mathbf{Z})}(\mathbf{1},[A_n])$ is descendable. Therefore, by Lemma 5.22, we obtain the desired result.

For (3), we have nothing to prove. □

Example 10.21. Let $A \to B$ be a finite étale morphism of totally disconnecteds. Then the square

$$\begin{array}{ccc} [A]_{\mathrm{alg}} & \longrightarrow & [B]_{\mathrm{alg}} \\ \big\downarrow & & \big\downarrow \\ [A] & \longrightarrow & [B] \end{array}$$

is a pushout in CAlg(2Mot($\mathbf{Z}$)).

10.4. **Arc descent.** We prove two descent results in 2Mot($\mathbf{Z}$).

Theorem 10.22. *Let $f\colon A \to B$ be an arc cover in $\mathsf{STDis}_{\mathrm{lgt}}$. Then $f^*\colon [A] \to [B]$ satisfies descent in* 2Mot($\mathbf{Z}$)*. In other words, the functor $\mathsf{STDis}_{\mathrm{lgt}} \to \mathsf{2Pr}$ given by $A \mapsto 2\mathrm{Mot}(A)$ is an arc sheaf.*

Remark 10.23. In Theorem 10.22, by considering uniform Banach rings $\mathscr{C}(X;\mathbf{C})$ for compact Hausdorff spaces X, we can see from Remark 6.10 that the size assumption cannot be dropped and from Remark 6.9 that we cannot take $\mathsf{UBan}_{\aleph_1}$ instead. We see in Corollary 10.25 below that descent is still valid under the topological finite presentation assumption.

Proof. By Proposition 10.19, the morphism is prim and hence by Theorem 5.13, it suffices to show that $f_*\mathbf{1} \in \mathrm{Hom}_{2\mathrm{Mot}(\mathbf{Z})}(\mathbf{1}, [A])$ satisfies descent. By Proposition 5.16, it suffices to show that it is descendable.

By [46, Lemma 4.8], we can write B as a filtered colimit $\varinjlim_i B_i$ of uniform Banach rings over A such that each $A \to B_i$ admits a splitting. By the $\aleph_1$-compactness, up to retract, we can assume that the colimit is sequential. Hence the desired result follows from Lemma 5.22. □

Theorem 10.24. *For $B = \mathbf{Z}\{T_1, \ldots, T_n\}_{r_1,\ldots,r_n}$, there is a map to a light strictly totally disconnected B'' such that $f^*\colon [B] \to [B'']$ satisfies descent in* 2Mot($\mathbf{Z}$).

We need to do a slightly intricate construction for this:

Proof. We first replace B with $B\{U, U^{-1}\}_{1/2,2}$; this substitution is possible by Corollary 10.8. We then consider its algebraic counterpart $A = \mathbf{Z}[T_1, \ldots, T_n, U^{\pm}]$. We have a canonical morphism $A \to B$.

We then consider $B \to B'$ to be the map constructed in Proposition 9.43. With this construction, we obtain a pushout square

$$\begin{array}{ccc} \mathrm{Shv}(\mathrm{M}(B)) \otimes \mathbf{1} & \longrightarrow & \mathrm{Shv}(\mathrm{M}(B')) \otimes \mathbf{1} \\ \big\downarrow & & \big\downarrow \\ [B] & \longrightarrow & [B'] \end{array}$$

in CAlg(2Mot($\mathbf{Z}$)), and therefore, by Proposition 9.47 and Corollary 6.7, the morphism $[B] \to [B']$ satisfies descent.

Next, we construct an algebraic counterpart of B'. We consider all étale A-algebras C together with a morphism $C \to B'$ such that for any point in the image of $\mathrm{M}(B') \to \operatorname{Spec} B' \to \operatorname{Spec} C$, the induced map on residue fields is an isomorphism. The category of such C admits finite colimits, and therefore, we can take its filtered colimit A'.

Note that $\mathrm{M}(B') \to \operatorname{Spec} A'$ satisfies the conditions of Lemma 9.42. Applying Lemma 9.42, we therefore obtain a morphism $A' \to A''$. Here, for (ii), we also note that the local ring is the henselization of the corresponding local ring of A. This shows that A' satisfies the assumption of Corollary 8.4. By Remark 8.21, we obtain a pushout square

$$\begin{array}{ccc} \mathrm{Shv}^{\wedge}_{\mathrm{f\acute{e}t}}(A') \otimes \mathbf{1} & \longrightarrow & \mathrm{Shv}^{\wedge}_{\mathrm{f\acute{e}t}}(A'') \otimes \mathbf{1} \\ \big\downarrow & & \big\downarrow \\ [A'] & \longrightarrow & [A''] \end{array}$$

in $2\mathrm{SH}_{\text{ét}}(\mathbf{Z})$, but note that $\mathrm{Shv}^{\wedge}_{\text{fét}}(A'')$ is simply $\mathrm{Shv}(\mathrm{M}(B'))$. In this situation, by Theorem 6.18, we see that $\mathrm{Shv}^{\wedge}_{\text{fét}}(A') \to \mathrm{Shv}^{\wedge}_{\text{fét}}(A'')$ satisfies descent over $\mathsf{Pr}_{\mathrm{st}}$. Therefore, the morphism $[A'] \to [A'']$ satisfies descent over $2\mathrm{SH}_{\text{ét}}(\mathbf{Z})$, and hence the morphism $[A']_{\mathrm{alg}} \to [A'']_{\mathrm{alg}}$ satisfies descent.

We now base change $A' \to A''$ along $A' \to B'$ to obtain a totally disconnected B''. This is possible by Lemma 9.32 and the fact that A'' is a filtered colimit of finite étale algebras over A'. Then we obtain a square

$$\begin{array}{ccc} [A']_{\mathrm{alg}} & \longrightarrow & [A'']_{\mathrm{alg}} \\ \downarrow & & \downarrow \\ [B'] & \longrightarrow & [B''], \end{array}$$

which is a pushout square by Example 10.21. Hence, the morphism $[B] \to [B'']$ satisfies descent.

We are now reduced to showing that B'' is a light strictly totally disconnected. Since A, A', and A'' are countable by construction, B'' is a light totally disconnected. Therefore, it remains to show that all the complete residue fields of B'' are algebraically closed, which can be checked fiberwise. For each $y' \in \mathrm{M}(B')$, we consider the corresponding points $x' \in \operatorname{Spec} A'$ and $x'' \in \operatorname{Spec}(A'')$, respectively. The fiber is the completion of $K(y') \otimes_{\kappa(x')} \kappa(x'')$, where $\kappa(x'')$ is a separable closure of $\kappa(x')$. Since $K(y')$ is the completion of $\kappa(x')$, all the complete residue fields of this totally disconnected are the completed separable closure (which is algebraically closed by Lemma 9.26) of $K(y')$. ☐

Combining Theorems 10.22 and 10.24, we obtain the following:

Corollary 10.25. *Let $A \to B$ be an arc cover in $\mathsf{URing}_{\mathrm{tfp}}$. Then $[A] \to [B]$ satisfies descent in $2\mathrm{Mot}(\mathbf{Z})$.*

Proof. By Theorem 10.22, it suffices to show that any $A \in \mathsf{URing}_{\mathrm{tfp}}$ admits a morphism $A \to A'$ such that A' is a light strictly totally disconnected. We can take the cover in Theorem 10.24 and take its quotient by the defining ideal. ☐

10.5. **Berkovich 2-motives and normed ring stacks.** We prove Theorem D. We first introduce the following categories of geometric objects: We write $\mathsf{Stk}^{\vee}_{\mathrm{lgt}}$ for the category of arc sheaves on $(\mathsf{STDis}_{\mathrm{lgt}})^{\mathrm{op}}$; the notation indicates that we do not impose hyperdescent. We write Var for the essential image of $(\mathsf{URing}_{\mathrm{tfp}})^{\mathrm{op}}$ in $\mathsf{Stk}^{\vee}_{\mathrm{lgt}}$. We note that $2\mathrm{D}_{\mathrm{mot}}(\mathbf{Z})$ is also the presentable category of kernels for $\mathrm{D}_{\mathrm{mot}} \colon \mathrm{Span}(\mathsf{Var}) \to \mathsf{Pr}_{\mathrm{st}}$.

In Section 10.3, we constructed a symmetric monoidal functor $\mathsf{SNRing} \to 2\mathrm{Mot}(\mathbf{Z})$. We first restrict this to $\mathsf{STDis}_{\aleph_1}$. Its right Kan extension, whose existence follows from [4, Theorem D], determines a symmetric monoidal functor $(\mathsf{Stk}^{\vee}_{\mathrm{lgt}})^{\mathrm{op}} \to 2\mathrm{Mot}(\mathbf{Z})$ by Theorem 10.22.

We begin with the following:

Proposition 10.26. *The morphism $(\mathsf{Stk}^{\vee}_{\mathrm{lgt}})^{\mathrm{op}} \to 2\mathrm{Mot}(\mathbf{Z})$ constructed above carries a $\mathbb{D}^1$-invariant finitary morphism to a weakly suave morphism.*

We recall the following, which is useful in various places:

Lemma 10.27. *Let F be a finitary arc stack over A. Then for any filtered colimit $B = \varinjlim_i B_i$ of uniform Banach rings over a strictly totally disconnected A with uniformly bounded arc cohomological dimension, the morphism*

$$\varinjlim_i \mathrm{Map}(\mathrm{Sparc}(B_i), F) \to \mathrm{Map}(\mathrm{Sparc}(B), F)$$

is an equivalence.

Proof. This is the unstable version of [46, Remark 4.11]: When F is truncated, the claim is straightforward; see [46, Lemma 4.7]. In general, we write $F = \varprojlim_n \tau_{\leq n} F$ and observe that for each k,

$$\big(\pi_k \, \mathrm{Map}(\mathrm{Sparc}(B_i), \tau_{\leq n} F)\big)_n$$

stabilizes uniformly in i; this follows from our assumption on the cohomological dimension. ☐

Lemma 10.28. *Let D be a uniform Banach ring topologically of finite presentation over a strictly totally disconnected A. Then for any finitary arc sheaf, $B \mapsto F(B \otimes_A D)$ determines a finitary arc sheaf.*

Proof. This follows from Lemma 10.27. □

Lemma 10.29. *Any static finitary arc stack F over a strictly totally disconnected A is a colimit of open substacks of $\mathbb{B}^n_{r,A}$.*

Proof. By Lemma 10.27, we can cover F by $U = \coprod_i U_i$, where U_i is an open substack of $\mathbb{B}^n_{r,A}$. We then consider the Čech nerve of $U \to F$. For $m \geq 0$, since $U^{\times_F m+1} \to U^{\times m+1}$ is the base change of $F \to F^{\times m+1}$, which is a finitary monomorphism, we see that $U^{\times_F m+1}$ is a disjoint union of open substacks of $\mathbb{B}^n_{r,A}$. □

Lemma 10.30. *Let F be a finitary arc stack over a strictly totally disconnected A that is (-1)-truncated over the nonarchimedean locus $|2| \leq 1$. Then it is a colimit of open substacks of* $\mathrm{Sparc}(A)$.

Proof. We consider the colimit F' of all open substacks of $\mathrm{Sparc}(A)$ mapping to F. Since it is finitary, it suffices to show this after pulling back to each $x \in \mathrm{M}(A)$. Since open substacks spread, we can assume that $A = C$ is an algebraically closed nondiscrete Banach field. The result is tautological in this case. □

Proof of Proposition 10.26. By writing the target as a colimit, we can assume it is $\mathrm{Sparc}(A)$ for a strictly totally disconnected A.

For any morphism of finitary arc stacks $G \to G'$, by Lemma 10.27, it is an equivalence if and only if $\mathrm{Map}(U, G) \to \mathrm{Map}(U, G')$ is an equivalence for any open substack U of $\mathbb{B}^n_{r,A}$. Therefore, it suffices to show that its $\mathbb{D}^1$-homotopification $L_{\mathbb{D}^1}(U)$, which is finitary by [46, Lemma 5.11], is weakly suave for any such U.

We now fix U together with the embedding $U \subset \mathbb{B}^n_{r,A}$. We take a small positive number ϵ. We consider $U_0 = U$ and the open substack U_1 of $U \times U$ consisting of pairs whose distance is $< \epsilon$. We take its 1-coskeleton to obtain $U_\bullet$. Now, U_k is an open substack of $\mathbb{B}^{n(k+1)}_{r,A}$ and hence weakly suave. Therefore, $F = |U_\bullet|$, which is finitary by an argument similar to [46, Lemma 5.11], is weakly suave. Note that in the nonarchimedean locus $|2| \leq 1$, this is equivalent to the action groupoid of the translation action of $\mathbb{B}^n_{\epsilon,A}$ on U. Therefore, it is static there.

We wish to show that $L_{\mathbb{D}^1} F$ is weakly suave. We write $F^{\mathbb{D}^k}$ for the stack obtained by Lemma 10.28 for $D = A\{T_1, \ldots, T_k\}$. Since $L_{\mathbb{D}^1} F$ can be computed as the geometric realization of $F^{\mathbb{D}^\bullet}$, it suffices to show that they are weakly suave. As in the proof of Lemma 10.29, we first take a surjection from $V = \coprod_i V_i$, where V_i is an open substack of $\mathbb{B}^n_{r,A}$. By considering the Čech nerve, we are now reduced to showing that $V_{i_0} \times_{F^{\mathbb{D}^k}} \cdots \times_{F^{\mathbb{D}^k}} V_{i_l}$ is weakly suave. It suffices to show that the tautological map $V_{i_0} \times_{F^{\mathbb{D}^k}} \cdots \times_{F^{\mathbb{D}^k}} V_{i_l} \to V_{i_0} \times \cdots \times V_{i_l}$, since its target is weakly suave. This map is a base change of $F^{\mathbb{D}^k} \to (F^{\mathbb{D}^k})^{l+1}$, which is weakly suave by Lemma 10.30. □

Proof of Theorem D. We first construct a morphism $F\colon 2\mathrm{Mot}(\mathbf{Z}) \to 2\mathrm{D}_{\mathrm{mot}}(\mathbf{Z})$. To do so, we need to construct a ring stack with an absolute value over $2\mathrm{D}_{\mathrm{mot}}(\mathbf{Z})$. First, by composing $(-)^{\mathrm{ucpl}}\colon \mathsf{SNPol} \to \mathsf{URing}_{\mathrm{tfp}}$, we obtain a functor $\mathsf{SNPol} \to 2\mathrm{D}_{\mathrm{mot}}(\mathbf{Z})$. Then, right Kan extending along the forgetful functor $\mathsf{SNPol} \to \mathsf{Pol}$ (which is possible by [4, Theorem D]), we obtain $\mathsf{Pol} \to 2\mathrm{D}_{\mathrm{mot}}(\mathbf{Z})$, which defines a ring stack over $2\mathrm{D}_{\mathrm{mot}}(\mathbf{Z})$. The norm can then be concretely constructed as a symmetric monoidal functor

$$\mathrm{Shv}([0,\infty)) \to \mathrm{Hom}_{2\mathrm{D}_{\mathrm{mot}}(\mathbf{Z})}(\mathbf{1}, [\mathbb{A}^1]) \simeq \mathrm{D}_{\mathrm{mot}}(\mathbb{A}^1) = \varprojlim_r \mathrm{D}_{\mathrm{mot}}(\mathbb{D}^1_r)$$

compatible with the multiplication; this compatibility is a condition. We can use Lemma 4.35 for this by setting

$$D_r = i_{r,*}\mathbf{1}, \qquad E_r = k_{r,*}\mathbf{1}$$

where $i_r\colon \mathbb{D}^1_r \to \mathbb{A}^1$ and $k_r\colon \mathbb{A}^1 \setminus \mathbb{B}^1_r \to \mathbb{A}^1$, where $\mathbb{B}^1_r$ denotes the open disk of radius r.

Next, we construct a morphism $G\colon 2\mathrm{D}_{\mathrm{mot}}(\mathbf{Z}) \to 2\mathrm{Mot}(\mathbf{Z})$ as in the proof of Theorem A. First, we have ${\mathsf{Stk}^{\vee}_{\mathrm{lgt}}}^{\mathrm{op}} \to 2\mathrm{Mot}(\mathbf{Z})$ by Theorem 10.22. Since any object in Var determines a proper stack over $2\mathrm{Mot}(\mathbf{Z})$ by Corollary 10.25 and Proposition 10.19, we can apply Theorem 2.2 to obtain a symmetric monoidal functor $2\mathrm{Span}_{\mathrm{all;all,iso}}(\mathsf{Var}) \to 2\mathrm{Mot}(\mathbf{Z})$. To construct G, it remains to construct

$$\mathrm{D}_{\mathrm{mot}}(-) \to \mathrm{Hom}_{2\mathrm{Mot}(\mathbf{Z})}(\mathbf{1}, [-])$$

in $\mathrm{CAlg}(\mathrm{Fun}(\mathrm{Span}(\mathsf{Var}), \mathsf{Pr}_{\mathrm{st}}))$. To do this, we apply Theorem 2.2 when J consists of $\mathbb{D}^1$-invariant finitary morphisms to obtain a symmetric monoidal functor $2\mathrm{Span}_{J;\mathrm{iso},J}(\mathsf{Stk}^{\vee}_{\mathrm{lgt}}) \to 2\mathrm{Mot}(\mathbf{Z})$. By considering $\mathrm{Hom}(\mathbf{1}, -)$ on both sides, we obtain functors

$$\mathrm{Shv}_{\mathrm{fin}}(X)_{\mathbb{D}^1} \to \mathrm{Hom}_{2\mathrm{Mot}(\mathbf{Z})}(\mathbf{1}, [X])$$

natural in $X \in (\mathsf{Stk}^{\vee}_{\mathrm{lgt}})^{\mathrm{op}}$. This functor preserves colimits by construction. Moreover, this factors through as

$$\mathrm{D}_{\mathrm{mot}}(X) \to \mathrm{Hom}_{2\mathrm{Mot}(\mathbf{Z})}(\mathbf{1}, [X])$$

by the axioms. To construct G, we still need to extend this natural transformation to $\mathrm{Span}(\mathsf{Var})$. We obtain this as the restriction of the symmetric monoidal functor from $2\mathrm{Span}_{\mathrm{all;all,iso}}(\mathsf{Var})$, which we can construct by Theorem 2.2; we check the condition in Lemma 10.31 below.

We then prove that GF is homotopic to id. We see that it preserves the universal ring stack; formally, this means that it is an equivalence after composing with $\mathsf{Pol} \to 2\mathrm{Mot}(\mathbf{Z})$, which specifies the ring stack. Then we can match the norms to see that GF is homotopic to id. The fact that the norm is preserved is just a condition, and can be checked straightforwardly.

For FG, as in the proof of Theorem A, it suffices to show that this is an equivalence after composing with $\mathsf{Var}^{\mathrm{op}} \to 2\mathrm{D}_{\mathrm{mot}}(\mathbf{Z})$. This amounts to showing that the composite

$$\mathsf{URing}_{\mathrm{tfp}} \to \mathsf{Var}^{\mathrm{op}} \hookrightarrow (\mathsf{Stk}^{\vee}_{\mathrm{lgt}})^{\mathrm{op}} \to 2\mathrm{Mot}(\mathbf{Z}) \xrightarrow{F} 2\mathrm{D}_{\mathrm{mot}}(\mathbf{Z})$$

is equivalent to the structure morphism. This is equivalent to the composite

$$\mathsf{URing}_{\mathrm{tfp}} \hookrightarrow \mathsf{URing} \to 2\mathrm{Mot}(\mathbf{Z}) \xrightarrow{F} 2\mathrm{D}_{\mathrm{mot}}(\mathbf{Z}).$$

by Theorems 10.22 and 10.24. By the construction of F, when we compose this with $\mathsf{SNPol} \to \mathsf{URing}_{\mathrm{tfp}}$, we obtain the structure morphism. Since the structure morphism $\mathsf{URing}_{\mathrm{tfp}} \to 2\mathrm{D}_{\mathrm{mot}}(\mathbf{Z})$ preserves finite colimits by Proposition 3.12, we obtain the desired result. ☐

In the proof above, we postponed the following technical step:

Lemma 10.31. *For any $f\colon Y \to X$ in Var, the square*

$$\begin{array}{ccc} \mathrm{D}_{\mathrm{mot}}(X) & \xrightarrow{f^*} & \mathrm{D}_{\mathrm{mot}}(Y) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{2\mathrm{Mot}(\mathbf{Z})}(\mathbf{1}, [X]) & \xrightarrow{f^*} & \mathrm{Hom}_{2\mathrm{Mot}(\mathbf{Z})}(\mathbf{1}, [Y]) \end{array} \tag{10.32}$$

is right adjointable.

Proof. By construction, the square Lemma 10.32 is left adjointable when f is a $\mathbb{D}^1$-invariant finitary morphism in $\mathsf{Stk}^{\vee}_{\mathrm{lgt}}$.

By considering the complement, we see that Lemma 10.32 is right adjointable for any closed immersion f in Var. Therefore, we are reduced to the case $f\colon \mathbb{P}^1_X \to X$ for $X \in \mathsf{Var}$. By Theorem 10.24, we can replace X with with $\mathrm{Sparc}(A)$ where A is a light strictly totally disconnected.

We check this using Proposition 3.10. We first consider the analytification functor $\mathsf{QProj} \to \mathsf{Stk}^{\vee}_{\mathrm{lgt}}$, which maps X to X_A. By composing this with $\mathrm{D}_{\mathrm{mot}}$ and $\mathrm{Hom}_{2\mathrm{Mot}(\mathbf{Z})}(\mathbf{1}, [-])$, we are at least in a situation where Theorem 3.4 applies, as we can observe by applying Theorem B. The only remaining condition for applying Proposition 3.10 is that Lemma 10.32 is left adjointable when $Y \to X$ is the A-analytification of a smooth morphism in QProj.

Using the left adjointability of Lemma 10.32 for $\mathbb{D}^1$-invariant finitary morphisms, we are reduced to showing this when $Y = \mathbb{A}^1_X$. By compactifying X and using descent again, we are reduced to showing that Lemma 10.32 is left adjointable for $Y = \mathbb{A}^1_X \to X$. It suffices to check this left adjointability at $[V] \in \mathrm{D}_{\mathrm{mot}}(Y)$, where V is an open subspace of $\mathbb{A}^n_Y$ for some n, since they generate $\mathrm{D}_{\mathrm{mot}}(Y)$ under colimits and shifts by [46, Proposition 5.13].

Now we consider $f\colon Y \to X = \mathrm{Sparc}(A)$ to be an open subspace of $\mathbb{A}^n_X$. It suffices to show that the square Lemma 10.32 is left adjointable at $\mathbf{1} \in \mathrm{D}_{\mathrm{mot}}(Y)$. By recollement, it suffices to treat the archimedean and nonarchimedean cases for A separately.

For the archimedean case, we claim that the square

$$\begin{array}{ccc} \mathrm{Shv}(\mathrm{M}(X)) \otimes \mathbf{1} & \longrightarrow & \mathrm{Shv}(\mathrm{M}(Y)) \otimes \mathbf{1} \\ \downarrow & & \downarrow \\ [X] & \longrightarrow & [Y] \end{array}$$

is a pushout in $\mathrm{CAlg}(2\mathrm{Mot}(\mathbf{Z}))$, which implies the desired result. To show this, by compactifying, we can replace Y with $\mathbb{P}^n_X$; this case then follows from descent.

We then treat the nonarchimedean case. We take $r > 0$ small enough so that $\mathbb{B}^n_{r,X}$ acts on Y by translation. Then $g\colon Y/\mathbb{B}^n_{r,X} \to X$ is finitary and static. Moreover, the canonical morphism $f_\natural \mathbf{1} \to g_\natural \mathbf{1}$ is an equivalence in $\mathrm{D}_{\mathrm{mot}}(X)$ and also in $\mathrm{Hom}_{2\mathrm{Mot}(\mathbf{Z})}(\mathbf{1}, [X])$. Therefore, the desired result follows from the left adjointability of Lemma 10.32 for g. □

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