The differentiable stack cohomology associated to a regular and proper Lie groupoid

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Annika Kraasch-Tarnowsky

aus

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Gutachterin/Betreuerin: Prof. Dr. Catharina Stroppel

Gutachter: Prof. Rui Loja Fernandes

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Abstract

This thesis establishes a model for the differentiable stack cohomology associated to a regular and proper Lie groupoid.

The 2-category of differentiable stacks is equivalent to a 2-category of Lie groupoids, which means that definitions such as cohomology on the former can be transferred to a corresponding notion on the latter. This establishes the concept of the differentiable stack cohomology associated to a Lie groupoid, which has a well-known interpretation as a generalisation of equivariant cohomology. The search for models computing this differentiable stack cohomology can therefore be motivated by pre-existing results about similar objects for equivariant cohomology. Generalising their constructions from the setting of equivariant cohomology to describe a model for the differentiable stack cohomology associated to a general Lie groupoid has been an ongoing effort. A key object in previous research has been the Bott–Shulman–Stasheff bicomplex, which is an established non-infinitesimal model computing the differentiable stack cohomology associated to a Lie groupoid.
In this thesis, it will be shown that in the case of a proper and regular groupoid, the cohomology of the columns of the Bott–Shulman–Stasheff bicomplex can be computed explicitly. For this, a notion of invariance is introduced through a construction using the first jet groupoid and the concept of a multiplicative Ehresmann connection with respect to the isotropy Lie algebroid. The first page of the spectral sequence associated to the induced filtration is computed by analysing the de Rham differential. Furthermore, it is proven that the associated spectral sequence collapses at the second page. It then can be concluded that we obtain a model for the differentiable stack cohomology associated to a proper and regular groupoid, which can be explicitly constructed and computed.

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Introduction

The concept of a “differentiable stack” is a geometric notion generalising the concept of a manifold, but examples also include orbifolds, quotients of Lie group actions and many moduli spaces. While understanding differentiable stacks by unravelling their definition as categories is possible, there is also another way to approach their study: It can be seen that the 2-category of differentiable stacks is equivalent to the 2-category of Lie groupoids with certain bibundles as morphisms. Lie groupoids are quite well-known objects in higher differential geometry and we can now hope that this knowledge can be translated into results on differentiable stacks.

Of course, sometimes we only have statements on a subclass of Lie groupoids. For example, the geometric realisation of the nerve is a certain simplicial construction starting with a Lie groupoid and yielding a topological space. If the Lie groupoid is an action Lie groupoid, i.e. a Lie groupoid constructed out of a Lie group action on a manifold, the cohomology of the thus associated space is precisely the equivariant cohomology of the action. When this statement is translated to the category of differentiable stacks, we find that the cohomology of the topological space agrees with the differentiable stack cohomology of the associated stack (with values in $\mathbb{R}$). This means that all of the results we know for equivariant cohomology now also hold for the differentiable stack cohomology of a stack associated to an action Lie groupoid.

One of the results on equivariant cohomology (and hence, in certain cases, differentiable stack cohomology) that is essential in practical calculations but also relevant for many theoretical considerations is the fact that there are models for equivariant cohomology. To be precise, there are two differently constructed, but isomorphic and very strongly related differential graded algebras - one called the Weil model and one called the Cartan model - whose cohomology agrees with the equivariant cohomology of a given Lie group action by a compact group. One great strength of the Weil and Cartan model is that both are relatively easy to compute. They are sometimes referred to as infinitesimal models, because under mild assumptions they even only depend on the induced Lie algebra action (the infinitesimal counterpart of the Lie group action).

So considering all of this together, we have very well-behaved models for computing the cohomology of a stack associated to an action groupoid of a compact Lie group action. However, showing that a stack is associated to an action groupoid is both tedious and not always possible. It has been a long-standing belief in the community of higher differential geometers, that a generalisation of the models corresponding to more general groupoids should be possible.

Thesis result

This thesis aims to contribute to this quest of finding a (possibly infinitesimal) model for the differentiable stack cohomology associated to a Lie groupoid. The Lie groupoids that

I consider are not a superset of action Lie groupoids arising from compact groups, but rather an entirely different class: Proper Lie groupoids for which the foliation on the object space induced by the orbits is a regular foliation. This can be equivalently expressed by the condition that the anchor ρ of the associated Lie algebroid has constant rank. This is an interesting case as in general, any proper groupoid admits some sort of decomposition by the rank of ρ (or equivalently the dimension of the orbits). It also turns out that it is a strong enough assumption to provide models for the differentiable stack cohomology. The models developed in this thesis differ from the precise notions originally conjectured in some cases, but there are strong relations.

Methodologically, the first step in our construction for the models will be given by replacing the notion of differentiable stack cohomology by the cohomology of the associated Bott–Shulman–Stasheff complex. This is an object defined by a somewhat less categorical set-up, as it is given by a bicomplex of differential forms on the manifolds G_q of q composable arrows. This is a standard approach, as the Bott–Shulman–Stasheff complex is very well-behaved, although unfortunately still too large for efficient computations. We will then show the following theorem:

Thesis result: Theorem 5.2.31

Let G be a proper and regular Lie groupoid and E be a multiplicative Ehresmann connection on G. Then we have a graded isomorphism of $\mathbb{R}$-algebras

$$\Phi : H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))) \to H^\bullet(\mathrm{Tot}(\mathcal{R}_E)) \cong H^\bullet(\mathrm{Tot}(\mathfrak{R}_E)).$$

In particular, both $\mathrm{Tot}(\mathcal{R}_E)$ and $\mathrm{Tot}(\mathfrak{R}_E)$ are models that can be used to compute the cohomology of the differentiable stack associated to G as

$$H^\bullet_{\mathrm{DiffStack}}([G_0//G_1]) \cong H^\bullet(\mathrm{Tot}(\mathcal{R}_E)) \cong H^\bullet(\mathrm{Tot}(\mathfrak{R}_E)).$$

Here, $\mathcal{R}_E$ and $\mathfrak{R}_E$ are isomorphic bicomplexes, where $\mathfrak{R}_E$ takes a form similar to the Cartan model for equivariant cohomology and $\mathcal{R}_E$ is its isomorphic counterpart consisting of actual differential forms similar to the Bott–Shulman–Stasheff complex. From the notation it already becomes apparent that the notion of a multiplicative Ehresmann connection is a key concept in defining the models, even if we will find out later in the thesis that the latter algebra $\mathfrak{R}_E$ is actually independent of the choice of E. A multiplicative Ehresmann connection is one of many possible choices to generalise the notion of a connection to a Lie groupoid. Connections are commonly defined on vector bundles, fibre bundle, principal bundles or submersions, where there is a unique notion of vertical tangent vectors, and essentially describes an additional notion of horizontal vectors. On Lie groupoids, it is not clear what a vertical vector should be defined as - in this thesis, it will usually be a vector in the kernel of both Ts and Tt for the source and target maps of the groupoid.

A multiplicative Ehresmann connection E now precisely decomposes the tangent space of the arrow manifold G_1 into a vertical and horizontal summand, in a way that is nicely compatible with the multiplication of the Lie groupoid. Using it, we will be able to define a notion of invariance on elements of $\Omega^p(G_q)$ and therefore on the vector spaces giving rise to the Bott–Shulman–Stasheff-complex. This notion will heuristically replace the notion of $\mathcal{G}$-invariance when comparing to an action groupoid, although it is in some sense more restrictive. The so-called $J^E G$-invariant forms will be used to define $\mathcal{R}_E$, $\mathfrak{R}_E$ and the isomorphism between both algebras through various constructions.

Summary and structure of the thesis

The proof of the main thesis result spans the entirety of this thesis and requires multiple steps and different techniques. The only exception is Chapter 0, which first covers equivariant cohomology, the correspondence between Lie groupoids and differentiable stack and other topics mentioned in the beginning of this introduction in far greater detail, along with examples and explanations regarding existing generalisations. Chapter 1 serves as the actual beginning of the proof of the previous theorem - at which we will start by laying the groundwork both regarding notational conventions, motivational observations, as well as important concepts such as the jet groupoid which will appear frequently throughout the thesis. It will also contain a proof on the existence of a special type of cover of a regular Lie groupoid, which we will often use in explicit constructions throughout all chapters.

In Chapter 2 we then turn to the aforementioned notion of $J^E G$-invariance, which combines jet groupoids with a multiplicative Ehresmann connection. We prove in this chapter, that the inclusion of invariant elements into a column of the Bott–Shulman–Stasheff-complex is a quasi-isomorphism. Chapter 3 slightly improves this result, by considering the inclusion of $J^E G$-isotropy Cartan forms instead. These are $J^E G$-invariant forms satisfying additional properties regarding their values at the unit section. This is strongly inspired by analogous well-known arguments using a generalisation of the Eilenberg–Zilber theorem.

In Chapter 4, we make a first attempt at computing the total cohomology of the Bott–Shulman–Stasheff complex instead of the cohomology of "just" a column. For this, a precise analysis of the relation between the de Rham differential and the splitting of E is needed, yielding an explicit computation for some specific component of d_{dR}. This component induces a differential on the $J^E G$-isotropy Cartan forms, which we can then use to define the isomorphic bicomplexes $\mathcal{R}_E$ and $\mathfrak{R}_E$. However, the relation between the cohomology of $\mathrm{Tot}(\mathcal{R}_E)$ (or $\mathrm{Tot}(\mathfrak{R}_E)$) and the cohomology of the Bott–Shulman–Stasheff-bicomplex is not trivial. There is a spectral sequence relating one with the other - however, the collapsing of this spectral sequence (or more directly, the isomorphism between the cohomologies) depends on a technical condition. In the case of a flat multiplicative Ehresmann connection, this condition is easier to prove than in the general case, leading to some interesting examples already.

In the general case, the condition referred to is also satisfied. This is more involved to prove, however, which is the content of Chapter 5. It is closely related to the existence of multiplicative Ehresmann connections which are flat when restricted to a certain type of subspace and the fact that the specific construction we can give for vertical preimages in the Bott–Shulman–Stasheff complex has many useful properties, e.g. that it commutes with one of the components of d_{dR} and also retains forms which are supported on a controlled set.

Conventions

On geometric notions

Geometry, roughly speaking, is the study of manifolds and their properties. However, depending on the author and field of study, different objects are admitted (or not admitted) as such - which then again affects other notions such as Lie groups. In this thesis, we will stick to the following conventions:

Definition 0.0.1 ([30], pp. 2-15)**.** A **manifold** M is an underlying Hausdorff second-countable topological space $|M|$ with a smooth atlas, i.e. a collection of local homeomorphisms $\{U_n \subset M \to \mathbb{R}^d\}_{n\in\mathbb{N}}$ (or $\{U_n \subset M \to \mathbb{R}^d\}_{0\leq n\leq f}$ if finite) called charts. These coordinate charts must be smoothly compatible.

Remark. Note that in some sources, the property of second-countability is not assumed for a manifold.

Definition 0.0.2 ([30], p.151)**.** A **Lie group** $\mathcal{G}$ is a group equipped with a manifold structure such that the structure maps (multiplication and inversion of elements) are smooth.

Remark. Note that sometimes, authors admit infinite dimensional spaces in the definition of Lie groups. In these definitions there is always some choice convenient setting such as diffeological spaces, that allows the concept of a smooth manifold to be generalised to infinite dimensional spaces adhering somewhat to the methods of differential geometry. While there are many interesting examples of infinite dimensional Lie groups, we will not include these objects in our studies.

On Gaussian sums

Similar to physics, the Gaussian sum convention is quite popular in the field of geometry when computing expressions in coordinates. We will use the Gaussian sum convention in this thesis when turning to compute certain coordinate expressions, in which case we will make this transparent.

Consider two families of objects $\{X_i\}_{i\in I}$ and $\{u^i\}_{i\in I}$, which are indexed by the same family I, one index is denoted at the base and one index is denoted in the exponent. Then in the Gaussian sum convention, any expression containing both objects u^j and X_j with the same index variable j represents the sum over all such terms.

Example 0.0.3. In Gaussian sum convention, the term

$$\langle u^i, Y\rangle X_i = Y$$

should be read as

$$\sum_{i\in I}\langle u^i, Y\rangle X_i = Y.$$

Example 0.0.4. In Gaussian sum convention, the term

$$[a_i, a_j] = c^k_{i,j} a_k$$

should be read as

$$[a_i, a_j] = \sum_{k\in I} c^k_{i,j} a_k.$$

Example 0.0.5. In Gaussian sum convention, the term

$$\mathrm{d}x^\alpha \Gamma^i_{\alpha,j} X_i$$

should be read as

$$\sum_{i\in I,\alpha\in J} \mathrm{d}x^\alpha \Gamma^i_{\alpha,j} X_i.$$

On the names of variables

In this paper, I aim to follow the following conventions in regard to naming objects. Some of them will be defined more precisely in the first section.

$\mathbb{N}_0$, $\mathbb{N}_{\geq 1}$	natural numbers starting from 0 or 1 respectively
G	Lie groupoid with object space G_0 and morphism space G_1, target morphism t, source morphism s and multiplication μ
m	a point in G_0, i.e. an object
x	a point in G_1, i.e. a morphism
z	a point in G_q, i.e. a higher morphism
1_m	the unit at $m \in G_0$
G_q	level of the simplicial nerve $\mathcal{N}(\mathcal{G})$, i.e. $G_1 \times_{G_0} G_1 \times_{G_0} \dots G_1$
$1_{G_0} \subset G_q$	the image of the unit section
k_q	dimension of G_q
pr_i	the projection of G_q to the i-th copy of G_1 in the fibre product
p_q^i	the projection of G_q to G_0 induced by the inclusion of the 0-th vertex into Δ^q (explicitly given by t applied to the left most component of $G_1 \times_{G_0} G_1 \times_{G_0} \dots G_1$ in case of p_q^0, analogous for p_q^q, and by the value at the i-th pull-back for $0 < i < q$)
O_m	orbit of $m \in G_0$
$\mathcal{F}_n$	orbit foliation on G_n
l_n	dimension of $\mathcal{F}_n$ in case of regularity
A	Lie algebroid associated to G, where $X \in A$ corresponds to a right invariant vector field with $T_x s(X) = 0$ for any $x \in G_1$
A_{iso}	isotropy Lie subalgebroid in the case that A is regular
$\mathrm{Iso}^0(G)$	Lie groupoid given by the connected component(s) of the isotropy containing the units
$\mathrm{Sat}(S)$	saturation of a set $S \subset G_0$
$\delta = \mathrm{d}_{\mathcal{N}}$	vertical differential of the Bott–Shulman–Stasheff bicomplex, induced by the nerve
$d = \mathrm{d}_{\mathrm{dR}}$	horizontal differential of the Bott–Shulman–Stasheff bicomplex, given by the de Rham differential
$\mathbf{d}$	total differential of the Bott–Shulman–Stasheff bicomplex
$\mathfrak{C}^p$	chain complex given by $(\Omega^p(G_\bullet), \mathrm{d}_{\mathcal{N}})$
E	a multiplicative Ehresmann connection
∇	a TG_0-connection on a Lie algebroid

∇^E	the connection on A_{iso} induced by E
$\mathrm{d}_{\mathrm{dR}}^{-1}, \mathrm{d}_{\mathrm{dR}}^{\pm 0}, \mathrm{d}_{\mathrm{dR}}^{+1}$	decomposition of the de Rham differential with respect to a multiplicative Ehresmann connection

On left and right (source and target)

Let me also make the following note on conventions regarding the reading direction:

Given a Lie groupoid $G_1 \rightrightarrows G_0$ with multiplication μ, I will follow the convention that

$$\mu : G_1 \times_{G_0}^{s,t} G_1 \to G_1.$$

Graphically, this means that the source of an arrow corresponds to its right side, while the target corresponds to its left, i.e. that the elements of G_1 follow the same convention of source and target as functions do and the compositions of two elements is denoted from right to left.

However, when passing to the simplicial nerve of the groupoid, note that d_0 corresponds to “the omittance of the left-most component”, which on G_1 only leaves the information of the base-point associated to the right side of an arrow, i.e. its source. So even though d_0 is the map I will denote on the left of d_1, we have $d_0 = s$ and $d_1 = t$.

I strongly doubt that I can hold myself to the standard of never mislabelling an index, but I will try.

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This thesis is the result of my work during the time as a PhD student, which has been an interesting, challenging and overall enjoyable environment. In the last years, I have met many wonderful people who have help shape me on my way to becoming a mathematician, but also have been my friends, mentors, cheerleaders, teachers, role-models and occasionally therapists. The center of these formative interactions, without doubt, has always been the Max-Planck Institute for Mathematics in Bonn, where I have been a guest since 2021 and an employee since 2024. I have benefited greatly from the friendly climate, the active community building and the exposure to many other guests over the years. Furthermore, I have held a Hausdorff PhD position funded by the Rheinische Friedrich-Wilhelms Universität Bonn between 2021 and 2024 and profited from my affiliation to the university through skill trainings, special funding to organise conferences and an exchange travel program. Especially the latter has had a great impact on the content and framework of this thesis.

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About the author

My current affiliations, publications and contact information can be found associated to my OrcID at `https://orcid.org/0009-0007-3761-8129`

Chapter 0

Exposition

This chapter concerns itself with motivating the search for infinitesimal models for differentiable stack cohomology and will mostly contain references to theorems, examples and counterexamples. While some of the latter are original work, this chapter is intended as a survey.

We will briefly cover equivariant cohomology, differentiable stacks and the generalised Weil algebra (as defined by Mehta in [32] and Abas in [1]). The content discussed here will not be a necessary prerequisite for the original results of this thesis in the following chapters, but it has been a relevant part of my research process and I believe it to be helpful to understanding the larger picture.

0.1 Equivariant cohomology

Equivariant cohomology is a concept applicable to general topological spaces, but in particular interesting when considering manifolds. It defines a cohomology on $\mathcal{G}$-manifolds, i.e. the category of manifolds with an action of a fixed Lie group $\mathcal{G}$, which is motivated by considering the corresponding quotient. There is also a nice higher categorical point of view on the construction. In fact, I personally find equivariant cohomology to be one of the most tangible examples of where a geometer may come across model category structures naturally.

The concept of equivariant cohomology has been considered for at least the last 60 years (in [24], which supplies very detailed bibliographical notes, it is attributed to Borel with reference to [7]) and is topic of countless papers. In the following, I will introduce the main definitions and results, along with some examples. I will mainly follow Guillemin and Sternberg (cf. [24]) and a series of introductory lectures by Tu (cf. [41]) along this section. The latter contains a very thorough but accessible survey on equivariant cohomology particularly suited to an in depth introduction to the topic.

Definition 0.1.1. A **smooth left $\mathcal{G}$-action** of a group $\mathcal{G}$ on a manifold M is a map

$$L : \mathcal{G} \times M \to M$$

which is smooth and satisfies the axioms of a left $\mathcal{G}$-action, i.e.

$$L(1, m) = m$$

and

$$L(g, L(h, m)) = L(\mu(g, h), m)$$

for any $m \in M$ and $g, h \in \mathcal{G}$.

We will also denote the action as $L : G \circlearrowright M$. There is an analogous notion of a smooth right $\mathcal{G}$-action, denoted $R : M \times \mathcal{G} \to M$ or $R : M \circlearrowleft \mathcal{G}$.

Definition 0.1.2. (cf. [41], p. 31) Let M be a manifold, $\mathcal{G}$ a Lie group, $L : \mathcal{G} \times M \to M$ a Lie group action and $\mathcal{R}$ be a coefficient ring. Then the **equivariant cohomology** $H_{\mathcal{G}}(M;\mathcal{R})$ is defined to be

$$H^\bullet_{\mathcal{G}}(M;\mathcal{R}) := H^\bullet(M_{\mathcal{G}};\mathcal{R}),$$

where $M_{\mathcal{G}}$ is the topological space constructed by

$$M_{\mathcal{G}} = (E\mathcal{G} \times M) / \left((x, m) \sim (R(x, g^{-1}), L(g, m))\right).$$

Here, $E\mathcal{G}$ denotes an arbitrary contractible topological space with a free right $\mathcal{G}$-action $R : E\mathcal{G} \times \mathcal{G} \to E\mathcal{G}$.

Remark. Note that in the definition, we can also replace M by a topological space X with a continuous $\mathcal{G}$-action (which is also the way it is defined in [41]).

Remark. The fact that $E\mathcal{G}$ of the Definition 0.1.2 is only characterised as topological space which is contractible and carries a free $\mathcal{G}$-action poses two immediate questions: Why does such a space exist? And why is the definition independent of the choice?

The first question can be answered by giving an explicit construction (e.g. by Milnor, cf. [41], Section 5.3) and is tied to the context of universal bundles and classifying spaces. The second one can be inferred from certain facts about fibre bundles (cf. [41], Section 4.4) that can be applied to

$$E\mathcal{G} \times E'\mathcal{G} \times M,$$

where $E'\mathcal{G}$ is a (potentially different) possible choice of a contractible topological space with a free right $\mathcal{G}$-action. (This, again, can be interpreted in a higher categorical framework as $E\mathcal{G}$ and $E'\mathcal{G}$ constructing different choices of cofibrant replacements.)

Definition 0.1.3. We will denote

$$\mathrm{H}^\bullet_{\mathcal{G}}(M) := H^\bullet_{\mathcal{G}}(M;\mathbb{R}) = H^\bullet(M_{\mathcal{G}};\mathbb{R}).$$

One of the results mentioned in the previous remark is the following one, that we can also apply to another context.

Proposition 0.1.4 ([41], Lemma 4.9)**.** *If E is a weakly contractible space on which a topological group $\mathcal{G}$ acts and $P \to P/\mathcal{G}$ is a principal $\mathcal{G}$-bundle, then $(E \times P)/\mathcal{G}$ and $P/\mathcal{G}$ are weakly homotopy equivalent.*

This will now imply an important relation.

Corollary 0.1.5. *Let $\mathcal{G}$ be a Lie group acting freely and properly on a manifold M. Then*

$$H^\bullet_{\mathcal{G}}(M;\mathcal{R}) = H^\bullet(M/\mathcal{G};\mathcal{R})$$

and in particular

$$H^\bullet_{\mathcal{G}}(M) = H^\bullet(M/\mathcal{G}).$$

Proof. $M/\mathcal{G}$ is well-defined as a quotient of a free and proper action and $M \to M/\mathcal{G}$ automatically has the structure of a principal $\mathcal{G}$-bundle ([30], Theorem 21.10.), so that Proposition 0.1.4 applies. $\square$

This highlights that equivariant cohomology is still very closely tied to computing the cohomology of the quotient of an action. $E\mathcal{G} \times M/\mathcal{G}$ can be thought of as some sort of idealised quotient of the action, it is called the homotopy quotient in [41]. Unfortunately, while the description of $E\mathcal{G}$ in theory is elegant, in practice it is quite hard to work with. One of the problems that arises is, that while $E\mathcal{G}$ can be constructed as a CW-complex, it will often need to have cells of arbitrarily high dimensions. This can be inferred from the fact that $B\mathcal{G}$, over which $E\mathcal{G}$ is a principal $\mathcal{G}$-bundle, can have cohomology classes supported in arbitrarily high degrees.

In particular, $E\mathcal{G}$ can often not be chosen with an appropriate manifold structure. This means that the (singular) cohomology of the space can not be computed using differential forms, even when the ring of coefficients is $\mathbb{R}$. Therefore we often find that $H_{\mathcal{G}}(M)$ is quite difficult to compute via the original definition, where "difficult" is of course somewhat subjective. In this case it is used as one often needs to apply technical results or methods, e.g. spectral sequences.

Example 0.1.6. Consider $\mathcal{G} = \mathbb{S}^1$ and $M = \{*\}$ with the trivial action. Then we obtain

$$H_{\mathcal{G}}(M) = E\mathbb{S}^1/\mathbb{S}^1.$$

It is well known, that $E\mathbb{S}^1 = \mathbb{S}^\infty$ with the action such that

$$B\mathbb{S}^1 = E\mathbb{S}^1/\mathbb{S}^1 = \mathbb{C}P^\infty.$$

This implies that $H^\bullet_{\mathcal{G}}(M) = H^\bullet(\mathbb{C}P^\infty)$ (cf. [41], p.37).

Luckily, in the case where the group $\mathcal{G}$ is compact, there is an easier way to compute equivariant cohomology. It was first published by Cartan around 1950. (cf. [24], Section 4.9 which names [10], pp. 62-63 as the original source. It is referred to as follows: "These two sections [...] don't make for easy reading." The fact that the content of these two pages of Cartan spans an entire chapter in [24] does underline their point.) It replaces the problem of computing the cohomology of a topological space by the problem of computing the cohomology of a differential graded $\mathbb{R}$-algebra.

Theorem 0.1.7 (Cartan, cf. [10] p. 64)**.** *Let $\mathcal{G}$ be a compact Lie group and $\mathfrak{g}$ its Lie algebra. Let $L : \mathcal{G} \circlearrowright M$ be a smooth action. We have that*

$$H^\bullet_{\mathcal{G}}(M) = H\left(\bigoplus_{\bullet=p+2q} \left(\Omega^p(M) \otimes S^q(\mathfrak{g}^*)\right)^{\mathcal{G}}, \mathrm{d}_{\mathrm{Cart}}\right),$$

where

$$\left(\Omega^p(M) \otimes S^q(\mathfrak{g}^*)\right)^{\mathcal{G}} \subset \Omega^p(M) \otimes S^q(\mathfrak{g}^*)$$

denotes the subalgebra of $\mathcal{G}$-invariant elements. Note that the bigrading is induced by the grading on $\Omega^\bullet(M)$ and $S^\bullet(\mathfrak{g}^*)$ with a weight of 2 (i.e. the degree of an element $u \in S^1(\mathfrak{g}^*)$ is 2).

An element $g \in \mathcal{G}$ acts on the algebra $\Omega(M)$ by the action induced by L. This is constructed for any $g \in \mathcal{G}$ by considering the expression

$$L_g(m) := L(g, m),$$

which defines a diffeomorphism on M and the pull-backs $(L_g)^*$ act on $\Omega(M)$ accordingly. On $S^\bullet(\mathfrak{g}^*)$ we instead have the coadjoint action (induced by the coadjoint action on its generator $\mathfrak{g}^*$).

The Cartan differential $\mathrm{d}_{\mathrm{Cart}}$ *is defined by*

$$\begin{aligned} \omega \in \Omega(M) &\mapsto \mathrm{d}_{\mathrm{dR}}(\omega) - \iota_{\rho(X_i)}\omega \otimes u^i, \\ u^i \in S^1(\mathfrak{g}^*) &\mapsto 0, \end{aligned}$$

where $\{u^i\}_{i\in I}$ *is a basis of* $\mathfrak{g}^*$ *corresponding to a dual basis* $\{X_i\}_{i\in I}$ *of* $\mathfrak{g}$.

Remark. Similar to previous notational conventions, we understand $\Omega(M)$ to mean $\Omega(M;\mathbb{R})$.

Remark. Note that the definition of the differential in Theorem 0.1.7 is actually independent of the choice of the basis. Because we are using the Gaussian summing convention (for details consider the introductory remarks following page 12) summing over all terms, any base change of the u^i and X^i will affect the sum simultaneously in inverse ways.

Definition 0.1.8. The differential graded $\mathbb{R}$-algebra

$$\left((\Omega(M) \otimes S^\bullet(\mathfrak{g}^*))^{\mathcal{G}}, \mathrm{d}_{\mathrm{Cart}} \right)$$

of Theorem 0.1.7 is also referred to as the **Cartan model for equivariant cohomology** (or just as the Cartan model).

Remark. Also note that in particular, if $\mathcal{G}$ is not only compact but also connected we have that the invariant subcomplex satisfies

$$(\Omega(M) \otimes S^\bullet(\mathfrak{g}^*))^{\mathcal{G}} = (\Omega(M) \otimes S^\bullet(\mathfrak{g}^*))^{\mathfrak{g}}.$$

By this, we mean that the invariance of a form $\omega \in \Omega(M) \otimes S^\bullet(\mathfrak{g}^*)$ under the action of $\mathcal{G}$ is equivalent to the "infinitesimal" invariance under "infinitesimal" elements of $\mathcal{G}$. Mathematically, we differentiate the action of $\mathcal{G}$ on $\Omega(M)$ to a Lie algebra action of $\mathfrak{g}$, which yields the Lie derivative with respect to fundamental vector fields. Similarly, we differentiate the coadjoint action of $\mathcal{G}$ on $S^\bullet(\mathfrak{g}^*)$ to the coadjoint action of $\mathfrak{g}$, which can also be expressed via the Lie bracket. Because of this, the Cartan model is referred to as an infinitesimal model, as in this case it only depends on the infinitesimal data of the group action around the unit.

Of course, there may be instances in which the original definition of equivariant cohomology leads to a more convenient computation. However, as $\Omega^\bullet(M)$ vanishes in sufficiently high degrees and $\mathfrak{g}^*$ is a finite dimensional vector space, the Cartan model can reduce the complexity of computing equivariant cohomology significantly.

Example 0.1.9. Let $\mathcal{G}$ be any compact Lie group. Then

$$H(B\mathcal{G}) = H_{\mathcal{G}}(\{*\})$$

can be computed by the Cartan model as

$$H\left((S^\bullet(\mathfrak{g}^*))^{\mathcal{G}}, \mathrm{d}_{\mathrm{Cart}} \right).$$

However, we know that the elements of $S^1(\mathfrak{g}^*)$ are supported in degree 2, meaning that all elements of $S^\bullet(\mathfrak{g}^*)$ are supported in even degree. This in particular implies $\mathrm{d}_{\mathrm{Cart}} = 0$.

So we have that

$$H^{2k}(B\mathcal{G}) = \left(S^k(\mathfrak{g}^*) \right)^{\mathcal{G}}$$

and

$$H^{2k+1}(B\mathcal{G}) = 0$$

for all $k \in \mathbb{N}_0$.

Example 0.1.10. We will now reconsider the set-up of Example 0.1.6: Let $\mathcal{G} = \mathbb{S}^1$ acting trivially on $M = \{*\}$. This means that $\mathfrak{g} = \mathbb{R}$ and as $\mathbb{S}^1$ is abelian, the induced adjoint representation vanishes.

We can apply the considerations of the previous Example 0.1.9 to compute

$$H^{2k}_{\mathcal{G}}(\{*\}) = \left(S^k(\mathfrak{g}^*)\right)^{\mathcal{G}} = S^k(\mathbb{R} \cdot u) = \mathbb{R} \cdot u^k$$

and

$$H^{2k+1}_{\mathcal{G}}(\{*\}) = 0$$

for all $k \in \mathbb{N}_0$, meaning $H^{\bullet}_{\mathcal{G}}(\{*\})$ has the structure of a polynomial ring in one variable of degree 2.

The Cartan model as described is actually part of a larger structure, the Weil algebra, which will play a role in the generalisation conjecture we will concern ourselves with in this thesis.

The general reason to consider the Weil algebra is that, just as the Cartan model relates to $E\mathcal{G} \times M/\mathcal{G}$, the Weil algebra can be used to construct an algebraic version of $E\mathcal{G} \times M$ together with an inclusion of algebras that behaves similarly to the pull-back along

$$E\mathcal{G} \times M \to E\mathcal{G} \times M/\mathcal{G}.$$

The reason why it is particularly interesting in the context of this thesis is that there is a version to generalise it to Lie algebroids which we will consider later.

Definition 0.1.11 (cf. [24], pp. 34-35 and [41], pp. 151-156)**.** Given a Lie algebroid $\mathfrak{g}$, the **Weil algebra** $\mathcal{W}(\mathfrak{g})$ is defined as the differential graded $\mathbb{R}$-algebra given by

$$\mathcal{W}^{\bullet}(\mathfrak{g}) := \bigoplus_{\bullet = p+2q} \wedge^p \mathfrak{g}^* \otimes S^q(\mathfrak{g}^*).$$

where the algebra structure is induced by the one of the exterior and symmetric algebras.

The coadjoint representation of $\mathfrak{g}$ on the Weil algebra is the representation induced by the coadjoint representation on each factor simultaneously. If $\mathcal{G}$ is a group with Lie algebra $\mathfrak{g}$, the coadjoint representation of $\mathcal{G}$ on the Weil algebra is defined analogously.

This concept now only depends on a Lie group or Lie algebra, not on any action on a manifold. This is why in our context, the Weil algebra will frequently appear together with $\Omega(M)$, where M is the manifold that the Lie group $\mathcal{G}$ acts upon. For notational reasons, we will now also introduce a way to abbreviate this.

Definition 0.1.12. Consider a Lie algebra $\mathfrak{g}$ and a manifold M. The **extended Weil algebra** $\mathcal{W}(\mathfrak{g}, M)$ associated to $\mathfrak{g}$ and M is given by

$$\mathcal{W}^{\bullet}(\mathfrak{g}, M) := \bigoplus_{\bullet = p+r+2q} \Omega^p(M) \otimes \wedge^r \mathfrak{g}^* \otimes S^q(\mathfrak{g}^*).$$

Consider a Lie group $\mathcal{G}$ acting on M such that $\mathfrak{g}$ is the Lie algebra associated to $\mathcal{G}$. Then the $\mathcal{G}$-action on M induces an action on $\Omega(M)$, which together with the coadjoint action defines a $\mathcal{G}$-action on $\mathcal{W}(\mathfrak{g}, M)$.

Definition 0.1.13. (cf. [24], Def. 2.3.1) Let $\mathcal{G}$ be a Lie group and $(\mathcal{A}, d_{\mathcal{A}})$ a differential graded $\mathbb{R}$-algebra. Then a **$\mathcal{G}^\star$-algebra structure** on $\mathcal{A}$ is a representation $P : \mathcal{G} \to \mathrm{Aut}(\mathcal{A})$ of $\mathcal{G}$ on $\mathcal{A}$ together with a Lie algebra morphism

$$\begin{aligned} \mathfrak{g} &\to \mathrm{Der}(\mathcal{A}) \\ X &\mapsto \iota_X \end{aligned}$$

into the space $\mathrm{Der}(\mathcal{A})$ of graded derivations of $\mathcal{A}$ of degree -1. This data needs to satisfy that for all $g \in \mathcal{G}$ and $X \in \mathfrak{g}$ holds

- that P retains the differential, i.e.
$$P(g) \circ d_{\mathcal{A}} \circ P(g^{-1}) = d_{\mathcal{A}}.$$
- that
$$\frac{\partial}{\partial t} P(\exp(t \cdot X))|_{t=0} = [d_{\mathcal{A}}, \iota_X] := \mathcal{L}_X.$$
- that
$$P(g) \circ \iota_X \circ P(g^{-1}) = \iota_{\mathrm{Ad}_g(X)}.$$
- that consequently
$$P(g) \circ \mathcal{L}_X \circ P(g^{-1}) = \mathcal{L}_{\mathrm{Ad}_g(X)}.$$

There is also a similar notion related to Lie algebras, which we will not use in this section, but refer to again at a later point.

Definition 0.1.14 ([41], Definition 18.3.)**.** Let $\mathfrak{g}$ be a Lie algebra. A **$\mathfrak{g}$-differential graded $\mathbb{R}$-algebra** $\mathcal{A}$ is a differential graded $\mathbb{R}$-algebra $(\mathcal{A}, d_{\mathcal{A}})$ that has two actions of $\mathfrak{g}$:

$$\iota : \mathfrak{g} \times \mathcal{A} \to \mathcal{A}$$

and

$$\mathcal{L} : \mathfrak{g} \times \mathcal{A} \to \mathcal{A},$$

where for $X \in \mathfrak{g}$ both ι_X and $\mathcal{L}_X$ are $\mathbb{R}$-linear in X, ι_X acts on $\mathcal{A}$ as an anti-derivation of degree -1 (in particular such that $(\iota_X)^2 = 0$) and $\mathcal{L}_X$ acts on $\mathcal{A}$ as a derivation of degree 0; furthermore, d, ι_X, and $\mathcal{L}_X$ satisfy Cartan's homotopy formula:

$$\mathcal{L}_X = [d_{\mathcal{A}}, \iota_X].$$

Remark. Note that this notion is not the infinitesimal counterpart to Definition 0.1.13. It is notably weaker, in that we don't require all equations of the Cartan calculus to hold, which immediately follows when $\mathcal{L}_X$ comes from a group action.

Proposition 0.1.15 (cf. [24], p. 19, pp. 34-35 and pp. 41-45)**.** *Let $\mathcal{G}$ be a Lie group acting on a manifold M. Consider the extended Weil algebra $\mathcal{W}(\mathfrak{g}, M)$ together with the differential $\mathrm{d}_{\mathrm{Cart}}$ given by*

$$\begin{aligned} \omega \in \Omega(M) &\mapsto \mathrm{d}_{\mathrm{dR}}(\omega) + (-1)^{|\omega|} \mathcal{L}_{\rho(X_i)} \otimes \theta^i - \iota_{\rho(X_i)} \omega \otimes u^i \\ \theta^i &\mapsto u^i - \frac{1}{2} c^i_{j,k} \theta^j \wedge \theta^k \\ u^i &\mapsto -c^i_{j,k} \theta^j \otimes u^k \end{aligned}$$

Then $(\mathcal{W}(\mathfrak{g}, M), \mathrm{d}_{\mathrm{Cart}})$ carries a $\mathcal{G}^\star$-algebra structure with respect to the coadjoint representation of $\mathcal{G}$ on $\mathcal{W}(\mathfrak{g}, M)$. The action ι of this structure is given by

$$\begin{aligned}\iota : \mathfrak{g} \times \mathcal{W}(\mathfrak{g}, M) &\to \mathcal{W}(\mathfrak{g}, M)\\ (X, \omega) \in \mathfrak{g} \times \Omega(M) &\mapsto 0\\ (X, \theta) \in \mathfrak{g} \times \wedge^1(\mathfrak{g}^*) &\mapsto \langle \theta, X\rangle\\ (X, u) \in \mathfrak{g} \times S^1(\mathfrak{g}^*) &\mapsto 0.\end{aligned}$$

If we consider the inclusion

$$(\Omega(M) \otimes S^\bullet(\mathfrak{g}^*))^{\mathcal{G}} \subset \mathcal{W}(\mathfrak{g}, M)$$

we furthermore find that $\mathrm{d}_{\mathrm{Cart}}$ indeed restricts to the notion of $\mathrm{d}_{\mathrm{Cart}}$ used in the Cartan model of Theorem 0.1.7. Its image are precisely the elements which are $\mathcal{G}$-invariant and horizontal under the given $\mathcal{G}^\star$-algebra structure, i.e. those $\nu \in \mathcal{W}(\mathfrak{g}, M)$ for which

$$(L_g)^*\nu = \nu,$$

where $(L_g)^*$ denotes the action of an arbitrary $g \in \mathcal{G}$ and

$$\iota_X \nu = 0.$$

for all $X \in \mathfrak{g}$.

Remark. Note that we have defined the extended Weil algebra a priori for any Lie algebra $\mathfrak{g}$ and manifold M. If we additionally have a Lie algebra action of $\mathfrak{g}$ on M, we can actually still define the $\mathfrak{g}$-differential algebra structure, even though it is not induced by a $\mathcal{G}^\star$-algebra structure.

For this work, we could now conclude the survey of the Weil and Cartan algebra. We will later cover how to generalise them in the sense of the definitions previously presented. However, I find it both interesting and important to note, that there is an equivalent construction of the Weil algebra that can actually motivate a lot of the equations we will encounter while generalising, even though it does not appear directly.

Proposition 0.1.16 ([24], p. 19, pp. 34-35)**.** *Let $\mathcal{G}$ be a Lie group acting on a manifold M. Then there is a differential $\mathrm{d}_{\mathrm{Weil}}$ on the extended Weil algebra $\mathcal{W}(\mathfrak{g}, M)$ given by*

$$\begin{aligned}\omega \in \Omega(M) &\mapsto \mathrm{d}_{\mathrm{dR}}(\omega)\\ \theta^i &\mapsto u^i - \frac{1}{2} c^i_{j,k} \theta^j \wedge \theta^k\\ u^i &\mapsto -c^i_{j,k} \theta^j \otimes u^k\end{aligned}$$

The extended Weil algebra $(\mathcal{W}(\mathfrak{g}, M), \mathrm{d}_{\mathrm{Weil}})$ carries a $\mathcal{G}^\star$-algebra structure with respect to the coadjoint representation of $\mathcal{G}$ on $\mathcal{W}(\mathfrak{g}, M)$. The action ι of this structure is given by

$$\begin{aligned}\iota : \mathfrak{g} \times \mathcal{W}(\mathfrak{g}, M) &\to \mathcal{W}(\mathfrak{g}, M)\\ (X, \omega) \in \mathfrak{g} \times \Omega(M) &\mapsto \iota_{\rho(X)}\omega\\ (X, \theta) \in \mathfrak{g} \times \wedge^1(\mathfrak{g}^*) &\mapsto \langle \theta, X\rangle\\ (X, u) \in \mathfrak{g} \times S^1(\mathfrak{g}^*) &\mapsto 0.\end{aligned}$$

Here, ρ denotes the map

$$\rho : \mathfrak{g} \to \mathfrak{X}(M)$$

that associates fundamental vector fields to the Lie algebra elements.

Remark. When considering

$$z^i := u^i - \frac{1}{2}c^i_{j,k}\theta^j \wedge \theta^k,$$

we can rewrite the equations for the differential above as

$$\begin{aligned} \omega \in \Omega(M) &\mapsto \mathrm{d}_{\mathrm{dR}}(\omega) \\ \theta^i &\mapsto z^i \\ z^i &\mapsto 0 \end{aligned}$$

In some sources (e.g. [24], cf. p. 34), the Weil algebra is actually introduced in terms of these coordinates. The advantage of using these coordinates is that the Weil algebra becomes the differential graded symmetric algebra generated by

$$0 \to \mathfrak{g}^*[-1] \to \mathfrak{g}^*[-2] \to 0 \to 0 \to \dots$$

where the only non-trivial morphism is the identity, which is useful for the intuition of it being acyclic as well as for understanding the expressions in the generalisations that are to follow.

The advantage of the coordinate expressions with respect to the u^i is the simplicity of writing down the horizontal subcomplex, but also a deep connection to group cohomology and the Bott–Shulman–Stasheff complex, which will become apparent later in the thesis but motivates the convention chosen here.

Theorem 0.1.17 ([24], pp. 41-44)**.** *The Mathai-Quillen-isomorphism ϕ is an isomorphism between the $\mathcal{G}^\star$-structure on the Weil algebra of Proposition 0.1.15 and the $\mathcal{G}^\star$-structure on the Weil algebra of Proposition 0.1.16.*

Corollary 0.1.18. *The Mathai-Quillen-isomorphism ϕ is an isomorphism between the $\mathfrak{g}$-differential graded algebra structure on the Weil algebra induced by the $\mathcal{G}^\star$-structure of Proposition 0.1.15 and the $\mathfrak{g}$-differential graded algebra structure on the Weil algebra induced by the $\mathcal{G}^\star$-structure of Proposition 0.1.16.*

Corollary 0.1.19. *The Mathai-Quillen-isomorphism ϕ maps the Cartan algebra, when understood as the basic subalgebra of the $\mathcal{G}^\star$-algebra structure on $\mathcal{W}(\mathfrak{g}, M)$ of Proposition 0.1.15, to the basic subalgebra of the $\mathcal{G}^\star$-algebra structure on $\mathcal{W}(\mathfrak{g}, M)$ of Proposition 0.1.16. In particular,*

$$\mathcal{W}_{\mathrm{bas}}(\mathfrak{g}, M) := \{\nu \in \mathcal{W}(\mathfrak{g}, M) \ \mathcal{G}\textit{-invariant and horizontal}\} \subset \mathcal{W}(\mathfrak{g}, M)$$

also computes $H_{\mathcal{G}}(M)$. By $\mathcal{G}$-invariant and horizontal we mean that

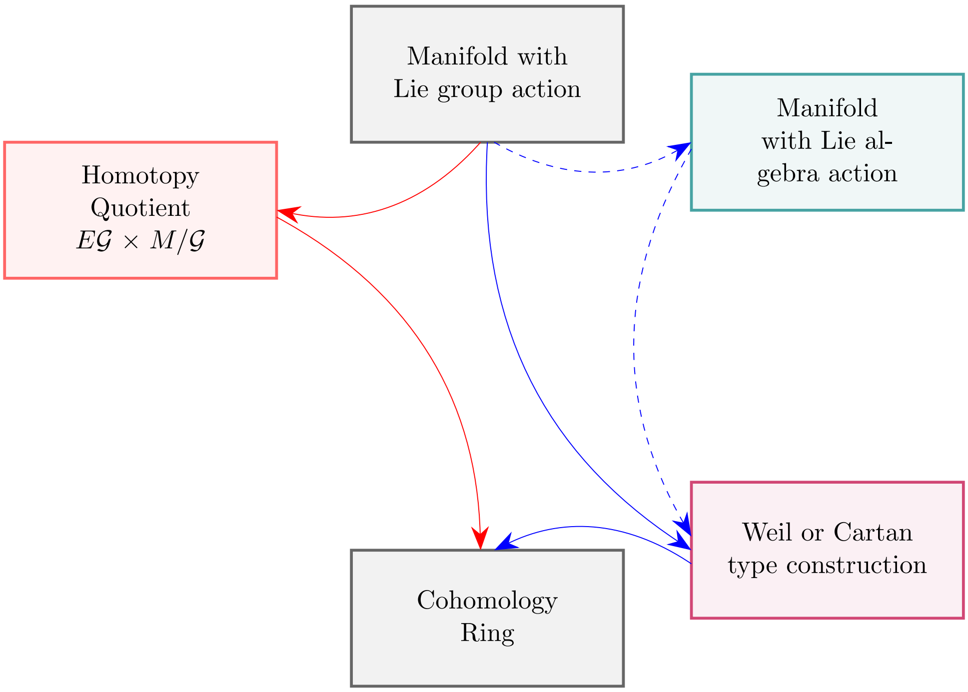
$$(L_g)^*\nu = \nu,$$

where $(L_g)^*$ denotes the action of an arbitrary $g \in \mathcal{G}$ and

$$\iota_X \nu = 0.$$

for all $X \in \mathfrak{g}$.

Conceptually, we may summarise the previous content on models for equivariant cohomology in the following diagram



The red arrows indicate that the procedures on the left are generally considered difficult (at least by geometers) and the blue arrows on the right indicate that these procedures are, in comparison, considered easier.

0.2 Differentiable Stacks

The analogue to algebraic stacks or topological stacks in differential geometry is called a differentiable stack. The intent behind this notion is to create a category that fully faithfully contains the category Mfld of smooth manifolds, but also allows for geometric notions of, for example, the objects $B\mathcal{G}$ and $E\mathcal{G}$ we considered previously.

Differentiable stacks have arisen more recently as an object of interest. (Recently is used as a somewhat relative term here. They have been studied since around the 1990, for example I was made aware by David Miyamoto that the concept was already discussed by Pronk in 1993 in [35].) We will mostly follow [3] and [20] to introduce them, where the former is a more original source, while the latter is a set of lecture notes covering the basics of the topic.

Definition 0.2.1 (cf. [20], pp. 2-3)**.** A **pre-stack over** Mfld is a category $\mathcal{D}$ together with a functor

$$\pi : \mathcal{D} \to \mathsf{Mfld}$$

such that:

- For every morphism $f : V \to U$ and every $\overline{U}$ with $\pi(\overline{U}) = U$, there exists some $\overline{V}$ with $\pi(\overline{V}) = V$ and a lift
$$\overline{f} : \overline{V} \to \overline{U}$$
which is a lift in the sense that $\pi(\overline{f}) = f$.

- For every commutative triangle

$$\begin{tikzcd} U \arrow[d] \arrow[dr] & \\ & W \\ V \arrow[ur] & \end{tikzcd}$$

of manifolds and smooth maps and any diagram in $\mathcal{D}$ of the form

$$\begin{tikzcd} \overline{U} \arrow[dr] & \\ & \overline{W} \\ \overline{V} \arrow[ur] & \end{tikzcd}$$

which maps to the corresponding part of the triangle under π, there is a unique morphism $\overline{U} \to \overline{V}$ extending the diagram accordingly.

Definition 0.2.2 (cf. [20], pp. 3-4)**.** For any inclusion $V \subset U$ and any object $\overline{U} \in \mathcal{D}$ we will use the notation of $\overline{U}|_V$ for any element for which there is a morphism

$$\bar{\imath} : \overline{U}|_V \to \overline{U}$$

lifting the inclusion.

Remark. The name for this notion varies in the literature. Ginot refers to it as a pre-stack (cf. [20], pp. 2-3) while Behrend and Xu refer to it as a category fibred in groupoids (cf. [3], pp. 290-291). There is also an equivalent definition of a (pre-)stack (cf. [26], Definition 1.1.) as follows: A pre-stack over Mfld is a 2-functor

$$\widehat{\mathcal{D}} : \mathsf{Mfld}^{\mathsf{op}} \to \mathsf{Grpd}.$$

The relation between the two definitions comes from the fact that when considering a pre-stack $\mathcal{D}$ according to Definition 0.2.1, $\pi^{-1}(U)$ together with morphisms covering id_U for each $U \in \mathsf{Mfld}$ is a groupoid defining $\widehat{\mathcal{D}}(U)$. The construction of $\mathcal{D}$ when given $\widehat{\mathcal{D}}$ corresponds to choosing an explicit $f^*(\overline{U})$ for $\overline{U} \in \widehat{\mathcal{D}}(U) \subset \mathcal{D}_0$ such that

$$f^*(\overline{U}) \to \overline{U}$$

maps to f under π. Note that these two constructions describe a natural equivalence of categories, but not an isomorphism.

A way to think of stacks is as a category of structures parametrised by a manifold. When considering Definition 0.2.1, the first property can be thought of as the existence of pull-backs of the structure, their uniqueness up to isomorphism corresponding to the second. However, such a parametrisation also has a locality property, that is not yet included in the framework and constitutes one of the differences between a pre-stack over Mfld and a differentiable stack.

Definition 0.2.3 (cf. [20], pp. 4-5)**.** A **stack $\mathcal{D}$ over** Mfld is a pre-stack on which the following holds:

- Objects can be glued, i.e. consider $\{U_i \subset M\}_{i\in I}$ an open cover of M and $\{A_i \in \mathcal{D}\}_{i\in I}$ a family of objects with $\pi(A_i) = U_i$. Assume that that there are isomorphisms

$$\psi_{ij} : A_i|_{U_i\cap U_j} \to A_j|_{U_i\cap U_j}$$

with

$$\psi_{jk} \circ \psi_{ij} \equiv \psi_{ik}$$

on $U_i \cap U_j \cap U_k$, then we have an object $A \in \mathcal{D}$ with $\pi(A) = M$ where each A_i is a pull-back of A along $U_i \subset M$.

- Morphisms can be glued, i.e. consider objects $A, B \in \mathcal{D}$ such that

$$\pi(A) = \pi(B) =: M$$

together with $\{U_i \subset M\}$ an open cover of M. Assume that there is a family

$$\{\phi_i : A|_{U_i} \to B|_{U_i}\}$$

for some choice of $A|_{U_i}$ and B_{U_i} and a choice of $A_{U_i \cap U_j}$ and $B_{U_i \cap U_j}$ for any i and j such that the induced morphisms $\phi_i|_{U_i \cap U_j}$ and $\phi_j|_{U_i \cap U_j}$ satisfy

$$\phi_i|_{U_i \cap U_j} = \phi_j|_{U_i \cap U_j}.$$

Then there exists a unique morphism $\phi : A \to B$ such that $\phi|_{U_i} = \phi_i$ for all i.

Example 0.2.4 (cf. [20], p. 4)**.** Any manifold M defines the category

$$\mathcal{D} = \mathsf{Mfld}/M.$$

This category is defined by

$$\mathcal{D}_0 = \{f : N \to M \text{ morphism of manifolds}\}$$

and morphisms given by commutative diagrams of the form

$$\begin{tikzcd} N_1 \arrow[d] \arrow[dr] & \\ N_2 \arrow[r] & M. \end{tikzcd}$$

The functor which makes this into a differentiable stack is given on objects by

$$(f : N \to M) \mapsto N$$

and on morphisms analogously.

This construction induces a fully faithful functor

$$\mathsf{Mfld} \to \mathsf{Stack}.$$

This can be worked out explicitly by considering $\mathrm{id} : M \to M$ as an object of $\mathcal{D}$. The stack induced by M will also be denoted as $[M]$.

Definition 0.2.5 (cf. [20], p. 5)**.** A **morphism of stacks** is a functor of the respective categories

$$F : \mathcal{D}_1 \to \mathcal{D}_2$$

such that

$$\pi_1 \circ F = \pi_2 \circ F.$$

Remark. Note that the notions of isomorphisms between categories, natural transformations of functors and natural equivalences between categories can be immediately transferred to differentiable stacks due to this definition. This defines the structure of a 2-category, where all 2-morphisms are invertible (cf. [26], p. 4).

With this, we can finally write down what it means to be a differentiable stack.

Definition 0.2.6 (cf. [20], p. 7)**.** A stack $\mathcal{D}$ over Mfld is a **differentiable stack** if it admits a representable epimorphism

$$p : [G_0] \to \mathcal{D},$$

where G_0 is a manifold. Here, $[G_0]$ denotes the stack constructed as in Example 0.2.4. That p is a representable epimorphism is equivalent to the following condition: For any manifold U and its associated stack $[U]$ together with any morphism $[U] \to \mathcal{D}$, the fibred product $[G_0] \times_{\mathcal{D}} [U]$ is isomorphic to a manifold denoted $G_0 \times_{\mathcal{D}} U$ (when identified with its associated stack $[G_0 \times_{\mathcal{D}} U]$) and the induced map $G_0 \times_{\mathcal{D}} U \to U$ is a surjective submersion.

Remark. The notion of fibre product referred to here is not the ordinary notion, but rather a 2-categorical one, which satisfies a 2-categorical universal property.

Remark. Similar to ordinary fibre products, we can see that

$$[M] \times_{[M]} [U] \simeq [U],$$

meaning that the identity $[M] \to [M]$ is a representable epimorphism and $[M]$ itself is a differentiable stack (cf. [20], Example 6.).

Definition 0.2.7 (cf. [20], p. 7)**.** Consider a differentiable stack $\mathcal{D}$ with a representable epimorphism $p : [G_0] \to \mathcal{D}$. Then p is called an **atlas of $\mathcal{D}$**.

Proposition 0.2.8 (cf. [20], Proposition 2.4.)**.** *Let $p : [G_0] \to \mathcal{D}$ be an atlas of a differentiable stack. Consider G_1 the manifold such that*

$$[G_1] \cong [G_0 \times_{\mathcal{D}}^{p,p} G_0]$$

Then there is a naturally associated Lie groupoid structure on $G_1 \rightrightarrows G_0$.

In practice, we will not have to think about this more technical condition often, because we will use a different viewpoint on the category of differentiable stacks. For now, let us focus on examples. As we have remarked above, stacks of the form $[M]$ for a manifold M are always differentiable. So we can still think of the category of differentiable stacks as a generalisation of the category of manifolds. At the same time, it also encompasses spaces such as differential versions of $E\mathcal{G}$ and $B\mathcal{G}$.

Example 0.2.9 (cf. [20], p. 6)**.** Let $\mathcal{G}$ be a Lie group. Consider the category $\mathcal{D}$ where the objects are given by

$$\mathcal{D}_0 = \{P \to M \text{ where } P \text{ has a fixed } \mathcal{G}\text{-principal bundle structure}\}$$

and morphisms given by commutative diagrams of the form

$$\begin{tikzcd} Q \arrow[r] \arrow[d] & P \arrow[d] \\ N \arrow[r] & M \end{tikzcd}$$

where the top arrow is $\mathcal{G}$-equivariant.

The functor defining the pre-stack structure is given on objects by

$$(P \to M \text{ where } P \text{ has a fixed } \mathcal{G}\text{-principal bundle structure}) \mapsto M$$

and on morphisms analogously.

Now consider a morphism of differentiable stacks (i.e. functor) of the form

$$F : [M] \to \mathcal{D},$$

i.e. from the differentiable stack associated to a manifold M as described in Example 0.2.4 to our current example $\mathcal{D}$. We can evaluate the morphism as a functor of categories on the final object $\mathrm{id} : M \to M$ of $[M]$. By construction, its image is given by a $\mathcal{G}$-principal bundle. Furthermore, any natural equivalence between two functors defining such a morphism of differentiable stacks induces an isomorphism between the associated principal bundles. Hence any homotopy class of maps from a manifold to $\mathcal{D}$ defines a principal bundle up to isomorphism over M.

The converse is also true, which can be seen as $\mathrm{id} : M \to M$ is final and all pull-backs exist uniquely up to unique isomorphism. This is the reason why the stack $\mathcal{D}$ of this construction is also referred to as $B\mathcal{G}$, as it satisfies the universal property (only in the category of stacks instead of the category of topological spaces). If we include the category of differentiable stacks in the category of topological stacks, $\mathcal{D}$ and the topological $[B\mathcal{G}]$ become equivalent as stacks (cf. [26], p. 3, 5).

As it turns out, this construction can be generalised to produce even more interesting differentiable stacks. For this, we will introduce the notion of principal bundles with respect to a Lie groupoid instead of a Lie group. Lie groupoids are central objects in the field of higher differential geometry and therefore the concept but also many examples are well-known. For completeness, we will review it here.

Definition 0.2.10. A **Lie groupoid** G is a groupoid internal to the category of manifolds (with morphisms being smooth maps) where both source map s and target map t are surjective submersions. We denote its object manifold by G_0, its arrow or morphism manifold by G_1 and the corresponding multiplication map by

$$\mu : G_1 \times^{s,t} G_1 \to G_1.$$

The inverse of $x \in G_1$ will be denoted by x^{-1} and the unit corresponding to $m \in G_0$ will be denoted 1_m. We will also refer to $1_{G_0} \subset G_1$ as the submanifold of G_1 that corresponds to the embedding of G_0 by the unit section u_1.

Example 0.2.11. Consider any manifold M. The object and morphism spaces

$$G_0 := M$$

and

$$G_1 := M$$

together with source and target map $s = t = \mathrm{id}_M$ can be equipped with a unique Lie groupoid structure where all arrows are automatically units.

Example 0.2.12. Let M be any manifold. Then the object and morphism spaces

$$G_0 := M$$

and

$$G_1 := M \times M$$

together with the source map $s = \mathrm{pr}_2$ and target map $t = \mathrm{pr}_1$ can be equipped with a unique Lie groupoid structure where for every $m, n \in M$ there exists a unique arrow (m, n) with source n and target m.

Example 0.2.13. Let $\mathcal{G}$ be a Lie group with multiplication map $\mu_{\mathcal{G}}$, M a manifold and $L : \mathcal{G} \circlearrowright M$ a smooth action. Then we can consider

$$G_0 = M$$

and

$$G_1 = \mathcal{G} \times M$$

which we can equip with the structure maps of a groupoid. For this, we have $s = \mathrm{pr}_M$, $t \equiv L : \mathcal{G} \times M \to M$ and μ the map that corresponds to $\mu_{\mathcal{G}} \times \mathrm{id}$ on the right-hand side when we identify

$$\begin{aligned} G_1 \times^{s,t} G_1 &\cong \mathcal{G} \times \mathcal{G} \times M \\ ((h, n), (g, m)) &\mapsto (h, g, m) \\ ((h, L(g, m)), (g, m)) &\mapsfrom (h, g, m). \end{aligned}$$

This example is known as the action Lie groupoid of L. More explicitly, we have

$$\mu((h, n), (g, m)) = (\mu_{\mathcal{G}}(h, g), m)$$

for composable elements of G_1.

There is a very deep connection between Lie groupoids and differentiable stacks, as there are constructions that relate both concepts. One of those constructions is the following:

Example 0.2.14 (cf. [20], p. 9)**.** Let G be a Lie groupoid. Consider the category $\mathcal{D}$ where the objects are given by

$$\mathcal{D}_0 \subset \{(M, P \text{ manifolds}, q : P \to M, \psi : P \to G_0, \rho : G_1 \times^{s,\psi} P \to P)\},$$

which can be visualised as

$$\begin{tikzcd} G_1 \arrow[r, phantom, "\overset{\rho}{\circlearrowright}"] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & P \arrow[dl, "\psi"] \arrow[d, "q"] \\ G_0 & M \end{tikzcd}$$

The precise space of objects $\mathcal{D}_0$ consists of the tuples where q is a surjective submersion and where the maps of the data are compatible in the sense that

$$q \circ \rho \equiv q \circ \mathrm{pr}_2$$

and

$$\psi \circ \rho \equiv t \circ \mathrm{pr}_1$$

on all of $G_1 \times^{s,\psi} P$, that

$$\rho((1_{\psi(p)}, p)) = p$$

for all $p \in P$ and that
$$\rho \circ (\mu, \mathrm{id}) \equiv \rho \circ (\mathrm{id}, \rho)$$
on all of $G_1 \times^{s,t} G_1 \times^{s,\psi} P$ and furthermore satisfy that
$$\rho|_{G_1 \times_{G_0} \{p\}} : G_1 \times_{G_0} \{p\} \to P$$
maps bijectively onto $q^{-1}(\{q(p)\})$ for any $p \in P$. The morphisms are given by commutative diagrams
$$\begin{tikzcd}
 & P \arrow[r, "F"] \arrow[dl, "\psi"'] \arrow[d, "q"'] & P' \arrow[d, "q'"] \arrow[dll, "\psi'"] \\
G_0 & M \arrow[r, "f"] & M'
\end{tikzcd}$$
where the top arrow must be equivariant with respect to the action of G, i.e. it must satisfy that
$$F \circ \rho \equiv \rho' \circ (\mathrm{id}, F)$$
as maps
$$G_1 \times^{s,\psi} P \to P'.$$

Mapping an element of $\mathcal{D}_0$ to the manifold which is in the above definition referred to as M defines a functor $\mathcal{D} \to \mathsf{Mfld}$. With this structure, the category $\mathcal{D}$ describes a stack. Furthermore, we can look at the map
$$p : [G_0] \to \mathcal{D}$$
given as a functor by mapping any
$$f : N \to G_0 \in [G_0]$$
to
$$\begin{tikzcd}
G_1 \arrow[r, bend left] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & G_1 \times^{s,f} N \arrow[dl, "t \circ \mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] \\
G_0 & N
\end{tikzcd}$$
where the action is given by
$$\rho := (\mu, \mathrm{id}) : G_1 \times^{s,t} G_1 \times^{s,f} N \to G_1 \times^{s,f} N.$$
We can now explicitly check that for any manifold U and the associated stack $[U]$ with any morphism $[U] \to \mathcal{D}$, the fibred product $[G_0] \times_{\mathcal{D}} [U]$ is isomorphic to a manifold. For this, we can consider the image of $\mathrm{id} : U \to U$ under the functor describing $[U] \to \mathcal{D}$, which is given by a G-principal bundle described by the data as above, in particular by a manifold P. Calculations can show that the stack $[G_0] \times_{\mathcal{D}} [U]$ is isomorphic to precisely $[P]$. Therefore, $\mathcal{D}$ is indeed a differentiable stack. This also shows that $[G_0]$ is an atlas for $\mathcal{D}$ and that $[G_0] \times_{\mathcal{D}} [G_0]$ precisely recovers $[G_1]$.

Remark. In this definition, we have precisely spelled out what it means for ρ to define a left G-action on P. We will, for now, not introduce the notion of left (and right) Lie groupoid actions further, apart from giving some examples. A review of the notion can be found in Chapter 1, where it will be much more central.

Definition 0.2.15. The **differentiable stack associated to a Lie groupoid** G as described in Example 0.2.14 will be denoted $[G_0//G_1]$.

Remark. Again, the language and notation is not unified throughout the literature. We adhere here to the conventions of [20] (cf. p. 9), while in [3] (cf. pp. 298-299) the arising stack is called BG instead of $[G_0//G_1]$. Also, the objects of the stack are sometimes referred to as torsors (cf. [3], Definition 2.6.), but they are also known as G-principal bundles (cf. [33] pp. 144-145) and are referred to as such in [20].

This means we have now seen both how to associate a Lie groupoid to a differentiable stack using an atlas and how to construct a differentiable stack from a Lie groupoid. From the viewpoint of a higher differential geometer, it makes sense to try to study differentiable stacks using these "auxiliary" geometric objects. Note, that an atlas is not necessarily unique (as we will also see in Example 0.2.21). Furthermore, a morphism of differentiable stacks does not necessarily induce a groupoid morphism on the charts. However, it turns out that the relation between differentiable stacks and Lie groupoids still assemble into a very concise structure using higher category theory.

Definition 0.2.16 ([5], Definition 2.10.)**.** Let G and H be Lie groupoids. Consider

$$\begin{tikzcd} G_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & P \arrow[dl] \arrow[dr] & H_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ G_0 & & H_0 \end{tikzcd}$$

where P is a G-H-bibundle (i.e. it comes with commuting left and right actions). P is called **right principal** if the following three conditions hold:

- The left moment map of P is a surjective submersion.
- The right H-action is free.
- The right H-action is transitive on the fibres of the left multiplication.

Example 0.2.17. Consider G to be the identity groupoid of Example 0.2.11 and H to be the pair groupoid of Example 0.2.12. We have

$$\begin{tikzcd} M \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & M \times M \arrow[dl] \arrow[dr] & M \times M \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ M & & M, \end{tikzcd}$$

where the two bundle maps are given by the projections on the respective factor of $M \times M$. The left action is trivial, in the sense that any s-fibre only contains the identity element that acts trivially. The right action is defined by acting on the second factor of the product with the action analogous to the action on the object manifold $H_0 = M$. This is indeed a right principal G-H-bibundle.

Definition 0.2.18 ([5], Proposition 2.12.)**.** Define the **2-category $\mathcal{C}$ of Lie groupoids** with Morita morphisms as the 2-category with

$$\mathcal{C}_0 = \{G \text{ which is a Lie groupoid}\}$$

and

$$\mathcal{C}_1 = \left\{ \begin{tikzcd} G_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & P \arrow[dl] \arrow[dr] & H_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ G_0 & & H_0 \end{tikzcd} \text{ right principal bibundle} \right\}$$

and

$$\mathcal{C}_2 = \{P \to Q \text{ which is a biequivariant map of bibundles}\}.$$

The weakly invertible 1-morphisms are referred to as **Morita equivalences of Lie groupoids**.

Remark. The notion of 2-category used here is the notion of bicategory (or weak 2-category).

Example 0.2.19. Consider the pair groupoid of Example 0.2.12. We have

$$\begin{tikzcd} M \times M \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & M \arrow[dl, "\mathrm{id}"] \arrow[dr] & \{*\} \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ M & & \{*\} \end{tikzcd}$$

This is indeed a biprincipal G-H-bibundle for G the pair groupoid and H the identity groupoid. We can associate an biprincipal H-G-bibundle given by

$$\begin{tikzcd} \{*\} \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & M \arrow[dl] \arrow[dr, "\mathrm{id}"] & M \times M \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ \{*\} & & M \end{tikzcd}$$

and explicitly computing the compositions of the bibundles yields

$$\begin{tikzcd} M \times M \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & M \times M \arrow[dl, "\mathrm{id}"] \arrow[dr, "\mathrm{id}"] & M \times M \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ M & & M \end{tikzcd}$$

and

$$\begin{tikzcd} \{*\} \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & \{*\} \arrow[dl] \arrow[dr] & \{*\} \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ \{*\} & & \{*\} \end{tikzcd}$$

proving that these biprincipal bibundles are (weak) inverses in $\mathcal{C}_1$.

Theorem 0.2.20 ([5], Theorem 2.18.)**.** *There is an equivalence of 2-categories between the 2-category of Lie groupoids and the 2-category of differentiable stacks.*

Remark. The equivalence of categories is indeed constructed by using the method of Example 0.2.14 to associate to every Lie groupoid G the stack $\mathcal{D} = [G_0//G_1]$ and for its inverse the construction of Proposition 0.2.8.

Example 0.2.21. The biprincipal bibundle of Example 0.2.19 can be used to construct an equivalence of categories between the stacks $[M//M \times M]$ and $[\{*\}//\{*\}] \cong [\{*\}]$. The earlier example of Example 0.2.17 describes a map of stacks from $[M//M] \cong [M]$ to $[M//M \times M]$. The referred to isomorphisms can be explicitly spelled out, they are given by forgetful maps from $[\{*\}//\{*\}]$ to $[\{*\}]$ (or $[M//M \times M]$ to $[M]$). Post-composing with the Morita equivalence of Example 0.2.19 yields a map from $[M]$ to $[\{*\}]$ that can be interpreted as the smooth map

$$M \to \{*\}.$$

This can be checked by evaluating the functors on the object $(\mathrm{id}_M : M \to M) \in [M]$.

Both the isomorphism from $[\{*\}]$ to $[\{*\}//\{*\}]$ inverse to the map appearing in this construction and the map from $[M]$ to $[M//M \times M]$ similarly constructed from the map of Example 0.2.19 can be shown to be representable epimorphisms (cf. [26], Lemma 3.1.). So both of them define atlases of this stack, in particular, atlases are not unique.

A consequence of this is that a lot of concepts related to differentiable stacks can be related to a construction on Lie groupoids. In this thesis, the concept we will focus on is that of differentiable stack cohomology.

Definition 0.2.22 (cf. [3], pp. 306-307)**.** Let $\mathcal{D}$ be a differentiable stack and $\mathcal{F}_{\mathcal{D}}$ be a sheaf of differential graded $\mathbb{R}$-algebras over $\mathcal{D}$. Then we define the **$\mathcal{F}_{\mathcal{D}}$-valued cohomology of** $\mathcal{D}$ as

$$H^\bullet_{\mathrm{DiffStack}}(\mathcal{D}, \mathcal{F}_{\mathcal{D}}) = H^\bullet(R\Gamma(\mathcal{D}, \mathcal{F}_{\mathcal{D}})),$$

where Γ denotes the global sections functor and $R\Gamma$ denotes its right derived functor.

Furthermore, we will denote

$$H^\bullet_{\mathrm{DiffStack}}(\mathcal{D}) = H^\bullet_{\mathrm{DiffStack}}(\mathcal{D}, \mathbb{R}_{\mathcal{D}}),$$

where $\mathbb{R}_{\mathcal{D}}$ denotes the sheaf of locally constant functions on $\mathcal{D}$ (i.e. any $\overline{U} \in \mathcal{D}$ will be assigned the set of locally constant functions on its corresponding $U \in \mathsf{Mfld}$).

Remark. Recall (cf. [3], p. 306) that a presheaf of differential graded $\mathbb{R}$-algebras on a category $\mathcal{D}$ is defined as a functor

$$\mathcal{D}^{\mathsf{op}} \to \mathsf{dg}\text{-}\mathbb{R}\text{-}\mathsf{Alg}.$$

Such a functor is called a sheaf if it is compatible with the Grothendiek topology on $\mathcal{D}$. However, we can define this using the Grothendiek topology on Mfld and the functor π contained in the data of $\mathcal{D}$.

Furthermore, recall that the global section functor of a sheaf over $\mathcal{D}$ can be defined (in analogy to the general theory) as

$$\Gamma(\mathcal{D}, \mathcal{F}) := \mathrm{Hom}(\mathbb{R}_{\mathcal{D}}, \mathcal{F}).$$

Definition 0.2.23. Consider a Lie groupoid G. The **nerve of** G is defined as the simplicial manifold $\mathcal{N}(G)$ defined by

$$[q] \mapsto G_q := G_1 \times^{s,t}_{G_0} G_1 \times^{s,t}_{G_0} \cdots \times^{s,t}_{G_0} G_1,$$

where G_q is referred to as **the manifold of q-composable arrows**, where there are q copies of G_1 in the pull-back.

The corresponding face maps are given by

$$\begin{aligned} d_i : G_q &\to G_{q-1} \\ (z_1, \dots, z_q) &\mapsto (\dots, \mu(z_i, z_{i+1}), \dots) \end{aligned}$$

and the corresponding degeneracy maps by

$$\begin{aligned} s_i : G_q &\to G_{q+1} \\ (z_1, \dots, z_q) &\mapsto (\dots, z_i, 1_m, z_{i+1}, \dots) \end{aligned}$$

where $m = s(z_i) = t(z_{i+1})$.

Definition 0.2.24. The maps $u_q : G_0 \to G_q$ defined by

$$m \mapsto (1_m, \dots, 1_m)$$

are referred to as the **unit maps**.

Remark. We have that

$$s_0 = 1_{(\bullet)} = u_1 : G_0 \to G_1.$$

Similarly, consider any sequence

$$s_{i_1} \circ \dots s_{i_q} : G_0 \to G_q.$$

In component expressions, it is easy to check that one recovers the map u_q. Using this, the definition of u_q can be generalised to arbitrary simplicial manifolds.

Definition 0.2.25. In a similar vein to Definition 0.2.10, we will use the notation $1_{G_0} \subset G_q$ to refer to the embedded manifold defined by the image of u_q.

Definition 0.2.26 (cf. [9], p. 44)**.** Let X be a simplicial manifold. We define the **Bott–Shulman–Stasheff bicomplex** $\Omega_{\mathrm{BSS}}(X; \mathbb{R})$ (with the convention of anti-commuting differentials) as the differential graded bicomplex given by

$$\Omega^{(p,q)}_{\mathrm{BSS}}(X; \mathbb{R}) = \Omega^p(X_q; \mathbb{R})$$

together with the horizontal differential given by the de Rham differential $d = \mathrm{d}_{\mathrm{dR}}$ and the vertical differential given by the simplicially induced differential

$$\delta = \mathrm{d}_{\mathcal{N}} := \sum (-1)^{p+i} (d_i)^*.$$

We consider $\Omega^p(X_q; \mathbb{R}) = 0$ where $p < 0$ or $q < 0$.

Remark. The fact that the signs chosen here define anti-commuting differentials is clear immediately, as d_{dR} commutes with pull-backs.

Remark. Technically, we can also define the Bott–Shulman–Stasheff bicomplex for values in a representation, which is why the trivial coefficients are included in the definition. Similar to the abbreviation $\Omega(M) := \Omega(M; \mathbb{R})$ we will also denote

$$\Omega_{\mathrm{BSS}}(X) := \Omega_{\mathrm{BSS}}(X; \mathbb{R}).$$

Remark. In [9], the definition of the Bott–Shulman–Stasheff bicomplex is not explicit on the sign conventions or on which direction should be the horizontal or vertical one. For any considerations on the cohomology, all of these conventions yield the same results. There are different conventions throughout the literature, for example, in [31] the grading

$$\Omega^{(p,q)}_{\mathrm{BSS}}(X;\mathbb{R}) = \Omega^q(X_p;\mathbb{R})$$

is used instead, together with the vertical differential

$$\delta = (-1)^p \, \mathrm{d}_{\mathrm{dR}}$$

and horizontal differential

$$d = \sum (-1)^i (d_i)^*.$$

For our constructions, we choose to follow the conventions as in [32] (cf. pp. 96-97). It will be helpful that the de Rham differential appears as the horizontal differential without any signs in Chapter 4 of this thesis. For all conventions, we can easily construct bigraded vector space isomorphisms that transform the given complex into a complex of any other sign convention. This is why we can nonetheless apply statements and methods that were proven using different conventions.

Definition 0.2.27 (cf. [31], p. 97)**.** The **cup product** on two forms $\omega \in \Omega^p(G_q)$ and $\nu \in \Omega^r(G_s)$ of the Bott–Shulman–Stasheff bicomplex is given by

$$\omega \cup \nu := \mathrm{pr}^*_{1,\dots,q}\, \omega \wedge \mathrm{pr}^*_{q+1,\dots,q+s}\, \nu.$$

Here, $\mathrm{pr}_{1,\dots,q}$ denotes the projection

$$\begin{aligned} G_{q+s} &\to G_q \\ z = (z_1, \dots, z_{q+s}) &\mapsto (z_1, \dots, z_q) \end{aligned}$$

and $\mathrm{pr}_{q+1,\dots,q+s}$ analogously denotes the projection

$$\begin{aligned} G_{q+s} &\to G_q \\ z = (z_1, \dots, z_{q+s}) &\mapsto (z_{q+1}, \dots, z_{q+s}). \end{aligned}$$

The **cup product** on two arbitrary elements of the Bott–Shulman–Stasheff complex is defined by extension.

Lemma 0.2.28 (cf. [31], p. 97)**.** *The differentials of the Bott–Shulman–Stasheff bicomplex are both derivations with respect to the cup product, i.e. we have that for $\omega \in \Omega^p(G_q)$ and $\nu \in \Omega^r(G_s)$ holds that*

$$\delta(\omega \cup \nu) = \delta(\omega) \cup \nu + (-1)^{p+q}\omega + \delta(\nu)$$

and similarly

$$d(\omega \cup \nu) = d(\omega) \cup \nu + (-1)^{p+q}\omega + d(\nu).$$

Example 0.2.29. The nerve $\mathcal{N}(G)$ of a Lie groupoid is a simplicial manifold that gives rise to the following Bott–Shulman–Stasheff bicomplex:

$$\begin{array}{ccccccccc}
\cdots & & \cdots & & \cdots & & \cdots & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
C^\infty(G_2) & \longrightarrow & \Omega^1(G_2) & \longrightarrow & \Omega^2(G_2) & \longrightarrow & \Omega^3(G_2) & \longrightarrow & \cdots \\
{\scriptstyle \mathrm{d}_{\mathcal{N}}}\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
C^\infty(G_1) & \longrightarrow & \Omega^1(G_1) & \longrightarrow & \Omega^2(G_1) & \longrightarrow & \Omega^3(G_1) & \longrightarrow & \cdots \\
{\scriptstyle \mathrm{d}_{\mathcal{N}}}\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
C^\infty(G_0) & \xrightarrow{\mathrm{d}_{\mathrm{dR}}} & \Omega^1(G_0) & \longrightarrow & \Omega^2(G_0) & \longrightarrow & \Omega^3(G_0) & \longrightarrow & \cdots
\end{array}$$

Definition 0.2.30. The **totalisation of a bicomplex** $C^{\bullet,\bullet}$ is given by the cochain complex

$$\mathrm{Tot}(C)^r = \bigoplus_{r=p+q} C^{p,q}$$

together with the total differential

$$\mathbf{d} = d + \delta.$$

Example 0.2.31. Consider the Bott–Shulman–Stasheff bicomplex of Example 0.2.29. Its totalisation is given by

$$\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))^r = \bigoplus_{r=p+q} \Omega^p(G_q).$$

The total differential $\mathbf{d}$ is given by

$$\mathbf{d} = \mathrm{d}_{\mathrm{dR}} + \mathrm{d}_{\mathcal{N}}.$$

The totalisation has an algebra structure with respect to the cup product.

Theorem 0.2.32 (cf. [3], pp. 308, 310-11)**.** *Let $\mathcal{D}$ be a differentiable stack and G a Lie groupoid which was constructed from an atlas $[G_0] \to \mathcal{D}$. Then the following are isomorphic as graded rings:*

$$H^\bullet_{\mathrm{DiffStack}}(\mathcal{D}) \cong H^\bullet_{\mathrm{DiffStack}}(\mathcal{D}, \Omega_{\mathcal{D}}) \cong H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(\mathcal{N}(G)))).$$

Remark. The ring structure of

$$H^\bullet_{\mathrm{DiffStack}}(\mathcal{D})$$

and

$$H^\bullet_{\mathrm{DiffStack}}(\mathcal{D}, \Omega_{\mathcal{D}})$$

is immediately induces as the sheafs are valued in $\mathbb{R}$-algebras. For

$$H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(\mathcal{N}(G))))$$

the ring structure is induced by the cup product and the compatibility of the ring structures is an important aspect of the theorem.

This result can now be used to translate known facts about simplicial manifolds and bicomplexes into statements about differentiable stack cohomology. This is not only useful on a conceptual level, but also on a computational one. At least in low degrees, the Bott–Shulman–Stasheff complex can be quite explicitly written down.

Lemma 0.2.33. *Consider a Lie groupoid G such that G_0 is simply connected and G_1 is connected. Then*

$$H^1_{\mathrm{DiffStack}}([G_0//G_1]) = 0.$$

Proof. Consider a closed 1-form $\omega \in \Omega_{\mathrm{BSS}}(G)$. It has two components $\omega_{(0,1)} \in \Omega^0(G_1)$ and $\omega_{(1,0)} \in \Omega^1(G_0)$ in the respective bidegrees. One of the equations that needs to be satisfied is

$$\mathrm{d}_{\mathrm{dR}}(\omega_{(1,0)}) = 0.$$

However, we know that G_1 is simply connected, meaning that $H^1(G_0) = 0$. This means that there is some $\nu \in C^\infty(G_0)$ with $\mathrm{d}_{\mathrm{dR}}(\nu) = \omega_{(1,0)}$.

Now consider

$$\omega - \mathbf{d}(\nu),$$

which is still closed in Ω_{BSS}, as we have only subtracted an exact term. By definition, it only has a non-zero component in bidegree $(0,1)$ given by

$$\omega_{(0,1)} - \mathrm{d}_{\mathcal{N}}(\nu).$$

This must now be closed under d_{dR}, hence locally constant. As G_1 is connected, this means it must be a globally constant function.

At the same time, the component must vanish under $\mathrm{d}_{\mathcal{N}}$. But if we choose any $m \in G_0$ and consider $1_m \in G_2$ at the identity section, we have that

$$\mathrm{d}_{\mathcal{N}}(f)(1_m) = \sum_{i=0}^{2} d_i^* f(1_m) = f(1_m)$$

for any function $f \in C^\infty(G_1)$, in particular for

$$f = \omega_{(0,1)} - \mathrm{d}_{\mathcal{N}}(\nu).$$

So we find that as $\mathrm{d}_{\mathcal{N}}(f)$ vanishes at 1_m at the identity section in G_2, we already find that f vanishes at the identity section in G_1. As f was constant, this means

$$f \equiv 0$$

and

$$\omega = \mathbf{d}(\nu).$$

In particular,

$$[\omega] = [0] \in H^1(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))) \cong H^1_{\mathrm{DiffStack}}([G_0//G_1]).$$

$\square$

Now look at an example that is slightly more involved - and will become important later.

Lemma 0.2.34. *Consider a Lie groupoid G such that $G_0 = \mathbb{S}^1$ and G_1 is connected and satisfies that*

$$u_1^* : H^1(G_1) \to H^1(G_0) = H^1(\mathbb{S}^1)$$

is an isomorphism. Now consider a closed element ω of total degree 2 in the Bott–Shulman–Stasheff bicomplex. Then ω must be exact.

Remark. We will later see a construction of a groupoid with precisely these properties. The definition of the groupoid requires some prerequisites that we will only encounter at a later point. Nonetheless, the explicit computations here do highlight possible advantages of computing with the Bott–Shulman–Stasheff complex nicely.

Proof. The element ω a priori has up to three non-vanishing components: of degree $(2,0)$, $(1,1)$ and $(0,2)$. However there can not be one of the first type as $\Omega^2(\mathbb{S}^1)=0$. This implies that the component $\omega_{(1,1)}$ of bidegree $(1,1)$ is closed under the de Rham differential and thus represents a cohomology class $[\omega_{(1,1)}]\in H^1(G_1)$.

By our assumption this cohomology class is uniquely defined by its image under u_1^*. Also, we find that on 1-forms we have

$$u_2^*\circ \mathrm{d}_{\mathcal{N}} \equiv -u_1^*,$$

which we can compute as each of the three components d_i^* of $\mathrm{d}_{\mathcal{N}}$ satisfies an analogous relation (and the signs add up correctly). As we assumed ω to be closed under the total differential, we have that

$$\mathrm{d}_{\mathcal{N}}(\omega_{(1,1)}) = -\,\mathrm{d}_{\mathrm{dR}}(\omega_{(0,2)})$$

and

$$u_1^*(\omega_{(1,1)}) = -u_2^*(\mathrm{d}_{\mathcal{N}}(\omega_{(1,1)})) = \mathrm{d}_{\mathrm{dR}}(u_2^*(\omega_{(0,2)})).$$

So by the assumption that u_1^* is an injective map on $H^1(G_1)$ we find that $\omega_{(1,1)}$ must represent a trivial cohomology class. Furthermore, any preimage ν' of $\omega_{(1,1)}$ under d_{dR} satisfies that

$$\omega - \mathbf{d}(\nu')$$

is a closed element of total degree 2 with vanishing component in bidegree $(1,1)$.

Let us assume without loss of generality that already $\omega_{(1,1)}=0$. This implies that

$$\mathrm{d}_{\mathrm{dR}}(\omega_{(0,2)}) = 0$$

and hence $\omega_{(0,2)}$ is a locally constant function. As we assumed G_1 to be connected the function is globally constant.

If we let $\nu_{(0,1)}\in\Omega^0(G_1)=C^\infty(G_1)$ be the corresponding constant function on G_1, we obtain that

$$\mathrm{d}_{\mathrm{dR}}(\nu_{(0,1)}) = 0$$

and

$$\mathrm{d}_{\mathcal{N}}(\nu_{(0,1)}) = -\omega_{(0,2)},$$

as all of the components of $\mathrm{d}_{\mathcal{N}}$ map constant functions to the corresponding constant functions of G_2 and the signs add up accordingly.

So we find

$$\omega + \mathbf{d}(\nu_{(0,1)}) = 0$$

implying that ω is exact with respect to the total differential of the Bott–Shulman–Stasheff complex. □

Interestingly, there is another interpretation of the totalisation of the bicomplex $\Omega_{\mathrm{BSS}}(G)$ aside from the cohomology of $[G_0//G_1]$.

Definition 0.2.35. Let X be a simplicial manifold. Then the **geometric realisation** $||X||$ of X is given by the topological space

$$||X|| = \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}}\,\Delta_n\times X_n.$$

Theorem 0.2.36 ([9], p. 51)**.** *Let X be a simplicial manifold. Then we have that*

$$H^\bullet(||X||;\mathbb{R}) = H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(X;\mathbb{R}))).$$

Remark. The precise statement of [9] requires all X_q to be paracompact. As we require manifolds to be second countable, we may omit this condition.

Corollary 0.2.37. *Let G be a Lie groupoid and $[G_0//G_1]$ its associated differentiable stack. We have that*

$$H^\bullet_{\mathrm{DiffStack}}([G_0//G_1];\mathbb{R}) = H^\bullet(||\mathcal{N}(G)||;\mathbb{R}).$$

This statement has multiple consequences. We can use it for example to relate differentiable stack cohomology back to equivariant cohomology, which we will see via the following examples.

Example 0.2.38. Consider the action of a Lie group $\mathcal{G}$ on itself. The associated action groupoid G is Morita equivalent to the trivial groupoid $\{*\} \rightrightarrows \{*\}$ by

$$\begin{tikzcd} \mathcal{G}\times\mathcal{G} \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & \mathcal{G} \arrow[dl] \arrow[dr] & \{*\} \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ \mathcal{G} & & \{*\} \end{tikzcd}$$

While the stack corresponding to $[*//*]$ is quite easy to spell out (see Example 0.2.4), considering the isomorphic stack $[G_1//G_0]$ has some geometric advantages. This stack is sometimes referred to as $E\mathcal{G}$ because its connection to the corresponding topological space.

We know that G has its simplicial nerve given by

$$G_n = \mathcal{G}\times\mathcal{G}\times\cdots\times\mathcal{G},$$

where there are $n+1$ copies of $\mathcal{G}$. The simplicial face maps are given by either omitting the first coordinate in case of d_0 or multiplying adjacent elements in case of d_i for $1\le i\le n$. This means that if we define the right action of $\mathcal{G}$ on G_n by multiplying the right-most coordinate from the right, i.e.

$$G_n\times\mathcal{G}\to G_n$$
$$(z,g)=(z_1,\dots,z_n,z',g)\in G_n\mapsto (z_1,\dots,z_n,\mu(z',g))$$

all face maps are $\mathcal{G}$-equivariant. The same holds for the degeneracy maps that introduce a unit in one of the first n coordinates (from left to right, i.e. any but the right-most component). So the $\mathcal{G}$-action will induce a well-defined action on $||\mathcal{N}(G)||$. However, we also have that $||\mathcal{N}(G)||$ is contractible. This can either be seen explicitly or by applying the earlier fact that $[G_0//G_1]$ is equivalent to $[*]$ (cf. [40], Proposition 2.1.).

This means, that we can consider $||\mathcal{N}(G)||$ to be a model of the topological space $E\mathcal{G}$ and its quotient by the action of $\mathcal{G}$ to be the corresponding model for $B\mathcal{G}$. However, by our explicit construction

$$B\mathcal{G} := E\mathcal{G}/\mathcal{G} = ||\mathcal{N}(G)||/\mathcal{G}$$

has a very explicit form. It is given by the simplicial realisation $||X||$ of the simplicial manifold for which

$$X_n = \mathcal{N}(G)_n/\mathcal{G}$$

with the induced simplicial maps. This description, however, is precisely given by

$$X = \mathcal{N}(H)$$

for the action Lie groupoid H of the trivial action of $\mathcal{G}$ on $\{*\}$. So in particular, we have now seen that for coefficients in $\mathbb{R}$ holds that

$$H^\bullet_{\mathcal{G}}(\{*\}) = H^\bullet(B\mathcal{G}) = H^\bullet(||\mathcal{N}(H)||).$$

Example 0.2.39. Let M be a manifold. We can consider the left action of a Lie group $\mathcal{G}$ on the manifold $\mathcal{G} \times M$ by only acting on the first component of the product. Consider the associated action Lie groupoid $G_{\mathcal{G}\times M}$. Similarly to Example 0.2.38, we can then construct $||\mathcal{N}(G_{\mathcal{G}\times M})||$ and relate it to $E\mathcal{G}$. As taking the product with M and taking the colimit over $[n] \in \Delta^{\mathsf{op}}$ commutes, we find that

$$||\mathcal{N}(G_{\mathcal{G}\times M})|| = ||\mathcal{N}(G_{\mathcal{G}})|| \times M = E\mathcal{G} \times M,$$

where $G_{\mathcal{G}}$ denotes the action Lie groupoid of $\mathcal{G}$ acting on itself.

If we further have a left $\mathcal{G}$-action on M, this is given by a map

$$\mu_{(\mathcal{G}\circlearrowleft M)} : \mathcal{G} \times M \to M.$$

If we interpret this as an equivariant map (due to associativity) with respect to the corresponding $\mathcal{G}$-actions on each space, we obtain a morphism $\phi_\mu : G_{\mathcal{G}\times M} \to G_M$ of the associated action Lie groupoids. It can be extended to a morphism of the respective simplicial nerves and therefore to a continuous map

$$E\mathcal{G} \times M = ||\mathcal{N}(G_{\mathcal{G}\times M})|| \to ||\mathcal{N}(G_M)||.$$

Additionally, we can consider the induced left $\mathcal{G}$-action on $(G_{\mathcal{G}\times M})_n$ given by

$$\begin{gathered} ||\mathcal{N}(\phi_\mu)|| : \mathcal{G} \times (G_{\mathcal{G}\times M})_n \to (G_{\mathcal{G}\times M})_n \\ (z, g) = (z_1, \dots, z_n, z', m, g) \in G_n \mapsto (z_1, \dots, z_n, \mu_{(\mathcal{G}\circlearrowleft\mathcal{G})}(z', g^{-1}), \mu_{(\mathcal{G}\circlearrowleft M)}(g, m)). \end{gathered}$$

By construction, this is precisely the diagonal action on $E\mathcal{G} \times M$. Furthermore, we have that the image of an element under ϕ_μ is invariant under the action. Because the action of $\mathcal{G}$ on the fibres of ϕ_μ is free and transitive, we have that $||\mathcal{N}(\phi_\mu)||$ exhibits $||\mathcal{N}(G_M)||$ as the quotient of $E\mathcal{G} \times M$ by the diagonal action.

In particular, this implies that for coefficients in $\mathbb{R}$ holds that

$$H^\bullet_{\mathcal{G}}(\{*\}) = H^\bullet(E\mathcal{G}/\mathcal{G}) = H^\bullet(||\mathcal{N}(G_M)||).$$

This example shows that equivariant cohomology agrees with the differentiable stack cohomology of the stack defined by an action groupoid.

Corollary 0.2.40 (cf. [2], p. 10)**.** *Let G be the action groupoid of an action $L : \mathcal{G} \circlearrowleft M$. Then*

$$H_{\mathrm{DiffStack}}([G_0//G_1]) \cong H_{\mathcal{G}}(M).$$

Proof. By Corollary 0.2.37 we have that

$$H^\bullet_{\mathrm{DiffStack}}([G_0//G_1]; \mathbb{R}) = H^\bullet(||\mathcal{N}(G)||; \mathbb{R}).$$

However, in the previous Example 0.2.39 we have seen that

$$||\mathcal{N}(G)|| = E\mathcal{G} \times M/\mathcal{G},$$

such that we recover equivariant cohomology. □

The close relations between the different notions explored here can help to compute the differentiable stack cohomology of differentiable stacks associated to Lie groupoids. The following is an example that we will encounter again later.

Example 0.2.41. Consider the action groupoid S associated to the action of $\mathbb{T}^2$ with its product group structure acting trivially on $\mathbb{S}^1$. Then we can apply Corollary 0.2.40 to obtain

$$H_{\text{DiffStack}}([S_0//S_1]) = H_{\mathbb{T}^2}(\mathbb{S}^1)$$

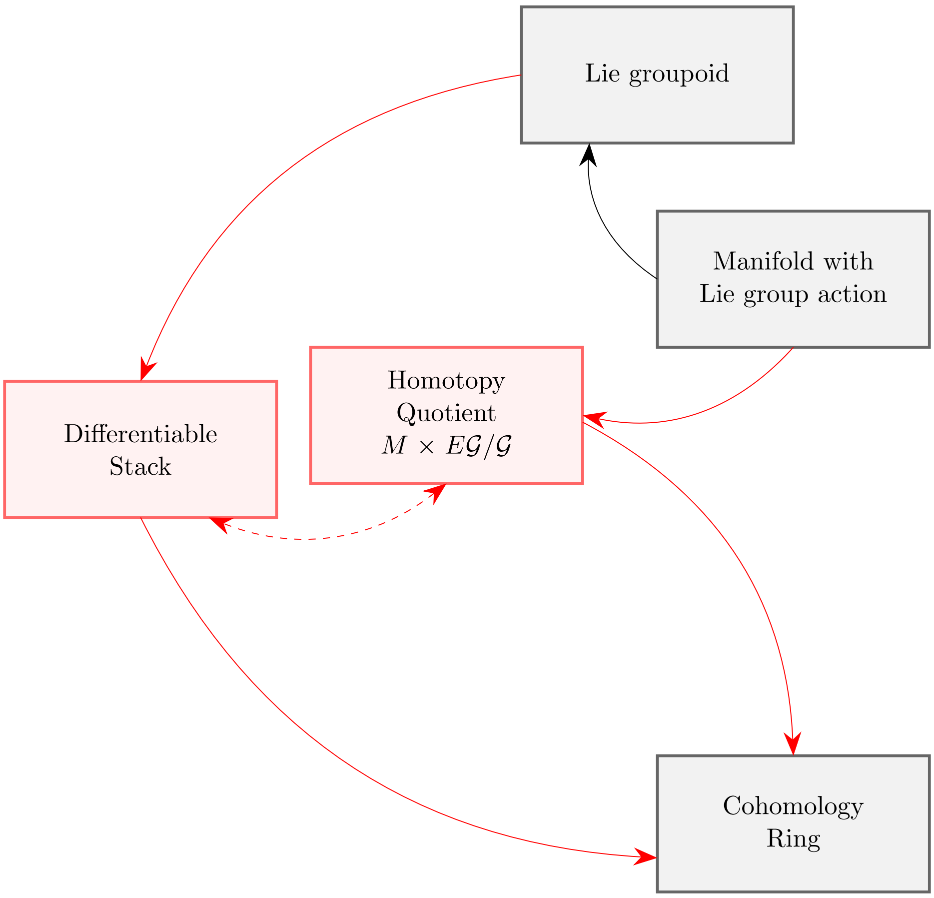
and then Theorem 0.1.7 (simplified in analogy to Example 0.1.10) to obtain

$$H_{\mathbb{T}^2}(\mathbb{S}^1) \cong H\left(\Omega(\mathbb{S}^1) \otimes S^\bullet(\mathbb{R}^2), \mathrm{d}_{\text{Cart}}\right).$$

As $\mathrm{d}_{\text{Cart}} \equiv 0$ on $S^\bullet(\mathbb{R}^2)$ and $\mathrm{d}_{\text{Cart}} \equiv \mathrm{d}_{\text{dR}}$ on $\Omega(\mathbb{S}^1)$, we have

$$H\left(\Omega(\mathbb{S}^1) \otimes S^\bullet(\mathbb{R}^2), \mathrm{d}_{\text{Cart}}\right) = (\mathbb{R} \oplus (\mathrm{d}x) \cdot \mathbb{R}) \otimes \mathbb{R}[u, v].$$

So we may consider differentiable stack cohomology as a generalisation of equivariant cohomology in the sense of the following schematic picture:



Note that the "inner" one of the half-circles describes a diagram that is already familiar: It describes the construction of equivariant cohomology covered earlier in the schematic on page 25. The "outer" diagram describes its generalisation to the differentiable stack cohomology associated to a Lie groupoid.

In the diagram we constructed for equivariant cohomology, we have now replaced manifolds with Lie group actions by Lie groupoids. The construction of an action Lie groupoid (cf. Example 0.2.13) is a natural way of understanding this as a generalisation of the objects we concern ourselves with. The findings of Example 0.2.39 also indicates that we can understand differentiable stacks as a generalisation of the homotopy quotients of the previous diagram, however, the result was only obtained for the stack associated to an action groupoid. This is indicated by the dotted line.

If we return to the original diagram on page 25, we find however, that the construction described and generalised here was the "difficult" one from the definition. There was also an "easy" computation using the Cartan model. The question that we concern ourselves with in this thesis is now natural: Is there a generalisation of the "easy" half of the original diagram? In other words: Is there a construction to generalise the models for equivariant cohomology, such that we can start from a groupoid and obtain a model for its differentiable stack cohomology?

0.3 Generalising the Weil and Cartan models

The geometric equivalent of differentiating a Lie group action to a Lie algebra action is differentiating a Lie groupoid to a Lie algebroid. However, when trying to generalise or even describe the Weil and Cartan model constructions in this setting, there is a hindrance: The Cartan and Weil models of a Lie algebra action do at first glance not only depend on the action Lie algebroid, but on the Lie algebra action, which can not be recovered from an action Lie algebroid. This is why, at least intermittently, the product structure of the action Lie algebroid is remembered through the extra datum of a connection.

The same arguments can a priori be made with regards to the ambient Weil algebra. However, for this, there is indeed an independent description. We will mainly follow [32] by Mehta and [1] by Abad, who described different viewpoints on this algebra, one using supergeometry, the other representations up to homotopy.

Definition 0.3.1. (cf. [1], p.41) Let $A \to G_0$ be a Lie algebroid and ∇ a TG_0-connection on A, i.e. a connection on A as an ordinary vector bundle. Then the **associated Weil algebra** $\mathcal{W}(A, \nabla)$ is given by

$$\mathcal{W}(A, \nabla) = \bigoplus_{p+r+2q} \Gamma(G_0, \wedge^p T^* G_0 \otimes \wedge^r \mathfrak{g}^* \otimes S^q(\mathfrak{g}^*)).$$

This definition is actually not the preferred one in either source, even though it will for our purposes in this section be the most convenient one. The conceptual picture has its own advantages, however, especially with defining the differential.

Definition 0.3.2 (cf. [32], p. 71)**.** Let $A \to G_0$ be a Lie algebroid. Then the **associated Weil algebra** $\mathcal{W}(A)$ is given by

$$\mathcal{W}(A) := C^\infty(T[1](A[1])).$$

Here, $T[1](A[1])$ denotes the $\mathbb{Z}$-graded supermanifold obtained by applying the shifted tangent functor $T[1]$ to the supermanifold $A[1]$. This, in turn, is obtained through a construction that associates a supermanifold to a vector bundle, where the bundle coordinates are functions of degree 1.

Remark. There are different conventions on how to denote the functor $T[1]$. In the original, the functor is actually denoted $[-1]T$, while we adhere to the notation used for example in [6] (cf. p. 69).

Lemma 0.3.3 (cf. [32], p. 74)**.** *The choice of a connection ∇ induces an algebra isomorphism*

$$\mathcal{W}(A) \cong \mathcal{W}(A, \nabla).$$

Again, there will be two equivalent differentials on this generalised Weil algebra. We will first denote them in coordinates and then refer to the more conceptual description.

Proposition 0.3.4 (cf. [32], pp. 23, 36)**.** *Let A be a Lie algebroid and ∇ a connection on A. There is a canonical differential on $\mathcal{W}(A)$ induced by the structure of $\mathcal{W}(A)$ as differential forms on a supermanifold. This induces a corresponding differential on $\mathcal{W}(A,\nabla)$. We will denote it $\mathrm{d}_{\mathrm{Cart}}$.*

Consider a local basis $X_i \in \Gamma(U,A)$ of A as a vector bundle and the corresponding dual basis $\theta^i \in \Gamma(U,A^*)$. The equations

$$\begin{aligned}
x^a \mapsto & dx^a + \rho^a_i \theta^i \\
\theta^i \mapsto & dx^a \cdot \Gamma^i_{a,j}\theta^j + u^i - \frac{1}{2}\left(c^i_{k,l} - \rho^{a,k}\Gamma^i_{a,l} + \rho^{a,l}\Gamma^i_{a,l}\right)\theta^k \wedge \theta^l \\
u^i \mapsto & \Gamma^i_{a,j}\left(dx^a \otimes u^j\right) \\
& + \frac{1}{2}\left(\frac{\partial c^i_{k,l} - \rho^a_l\Gamma^i_{a,k} + \rho^a_k\Gamma^i_{a,l}}{\partial x^a} - \Gamma^i_{a,j}\cdot(c^j_{k,l} - \rho^a_l\Gamma^j_{a,k} + \rho^a_k\Gamma^j_{a,l})\right)\left(dx^a \otimes \theta^k \wedge \theta^l\right) \\
& + \left(c^i_{m,l} - \rho^a_l\Gamma^i_{a,m} + \rho^a_m\Gamma^i_{a,l}\right)\theta^l \otimes u^m \\
& - \frac{1}{2}c^i_{m,l}\cdot(c^m_{j,k} - \rho^a_k\Gamma^m_{a,j} + \rho^a_l\Gamma^m_{a,k})\left(\theta^k \wedge \theta^l \wedge \theta^m\right)
\end{aligned}$$

describe this differential on $\mathcal{W}(A,\nabla)$.

Remark. In Mehta, these equations are not spelled out as explicitly for the given case. They are spelled out in local coordinates on a supermanifold, which correspond to a local connection on A which is flat.

In this case we have that the Christoffel symbols vanish. For the more general case, we have to precisely analyse how the structure coefficients $c^i_{j,k}$ behave under coordinate changes. The terms

$$\overline{c}^i_{j,k} := c^i_{k,l} - \rho^a_k\Gamma^i_{a,j} + \rho^a_j\Gamma^i_{a,k}$$

that appear precisely satisfy the equation

$$\overline{c}^i_{j,k}X_i \equiv [\overline{X_j}, \overline{X_k}]$$

at some point $m \in G_0$ whenever $\overline{X_j} \equiv X_j$ at m and $\overline{X_j}$ is ∇-horizontal and the analogous condition holds for $\overline{X_k}$.

These computations and equations may look complicated at first, but note that if all X_i are ∇-horizontal at $m \in U$, we have that the differential reduces to

$$\begin{aligned}
\theta^i &\mapsto z^i \text{ at } m \\
z^i &\mapsto 0 \text{ at } m
\end{aligned}$$

where

$$z^i = u^i - \frac{1}{2}\left(c^i_{k,l} - \rho^a_l\Gamma^i_{a,k} + \rho^a_k\Gamma^i_{a,l}\right)\theta^k \wedge \theta^l.$$

This term generally corresponds to the z^i that are mentioned in a previous remark on page 24 as an alternate choice of coordinates for the Weil algebra. It will become more important for the Weil differential.

Also, note that we can compute that under the Cartan differential

$$dx^a \mapsto -\frac{\partial \rho^a_i}{\partial x^b}dx^b \otimes \theta^i - \rho^a_i u^i - \rho^a_i\Gamma^i_{b,j}(dx^b \otimes \theta^j) - \frac{1}{2}\rho^a_i(\rho^a_k\Gamma^i_{a,l} - \rho^a_l\Gamma^i_{a,k})(\theta^k \wedge \theta^l).$$

In analogy to the two differentials on the extended Weil algebra as in Proposition 0.1.15 and Proposition 0.1.16, there is a second canonical differential on $\mathcal{W}(A)$. However, it is less important for our context.

Proposition 0.3.5 (cf. [32], p. 72)**.** *The canonically defined differential on $\mathcal{W}(A)$ of Proposition 0.3.4 is related to a different differential, denoted* $\mathrm{d}_{\mathrm{Weil}}$*, by the Mathai-Quillen isomorphism*

$$\gamma = \exp(\iota_{d_A}).$$

Remark. If you consider the Mathai-Quillen isomorphism mentioned as Theorem 0.1.17 (i.e. the one of [24], cf. pp. 41-42), it corresponds to the identity on elements of the second tensor factor, i.e. the Weil algebra of the Lie algebroid. This Mathai-Quillen isomorphism of Mehta does not do the same - its morphism on the part corresponding to the algebroid is given precisely by the change of coordinates from the μ^i to the z^i mentioned in a previous remark on page 44.

Definition 0.3.6 (cf. [32], p. 74)**.** Consider $\mathcal{W}(A, \nabla)$ and let $X \in \Gamma(G_0, A)$ be a section of the Lie algebroid. Define the $C^\infty(G_0)$-linear degree -1 derivation

$$\iota_X : \mathcal{W}(A, \nabla) \to \mathcal{W}(A, \nabla)$$

by

$$\begin{aligned} dx^a \mapsto & 0 \\ \theta^i \mapsto & \langle \theta^i, X \rangle \\ u^i \mapsto & 0. \end{aligned}$$

Furthermore define the degree 0 derivation

$$\mathcal{L}_X := [\mathrm{d}_{\mathrm{Cart}}, \iota_X].$$

A form $\omega \in \mathcal{W}(A, \nabla)$ is **∇-horizontal** if

$$\iota_X \omega = 0$$

for all $X \in \Gamma(G_0, A)$. A form ω is **∇-basic**, if it is horizontal and satisfies

$$\mathcal{L}_X \omega = 0$$

for all $X \in \Gamma(G_0, A)$.

Remark. Note that the Cartan calculus does not hold any more, at least in part. However, by definition, we still have the required identities such that the Cartan differential maps basic forms to basic forms. This can be seen as a generalisation of the structure of a $\mathfrak{g}$-differential graded $\mathbb{R}$-algebra of Definition 0.1.14.

Definition 0.3.7. Define the **Mehta generalisation of the Cartan model** as the differential graded algebra given by

$$\mathcal{A}^{\nabla}_{\text{Mehta-Cartan}}(A) := \{\omega \in \mathcal{W}(A, \nabla) \text{ where } \omega \text{ is basic}\} \subseteq \mathcal{W}(A, \nabla).$$

Define the **Mehta generalisation of the Weil model** $\mathcal{A}^{\nabla}_{\text{Mehta-Weil}}(A)$ as the differential graded algebra that corresponds to $\mathcal{A}^{\nabla}_{\text{Mehta-Cartan}}(A)$ under the Mathai-Quillen-isomorphism of Proposition 0.3.5.

These algebras are now the counterparts to the classical Weil and Cartan models established in Proposition 0.1.15 and Proposition 0.1.16. However, the definition of ι_X depends a priori on the choice of the connection ∇.

It is now a natural question to ask if there again is a generalised interpretation of ι_X. One could hope that even if $\nabla \neq \nabla'$ the corresponding dg-algebras $\mathcal{A}^{\nabla}_{\text{Mehta-Cartan}}(A)$ and $\mathcal{A}^{\nabla'}_{\text{Mehta-Cartan}}(A)$ are isomorphic. The answer to this is negative, as we are about to see.

Example 0.3.8. Consider $\mathbb{S}^1$ acting on the first component of $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. The associated algebroid is given by

$$\mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1.$$

For the canonical connection $\nabla^\times$ on A induced by the product structure of this action algebroid we have (cf. [32], p. 75) that

$$\mathcal{W}(A, \nabla^\times) \cong \mathcal{W}(\mathfrak{g}, \mathbb{T}^2)$$

and that

$$\mathcal{A}^{\nabla^\times}_{\text{Mehta-Cartan}}(A) \cong \left(\Omega(\mathbb{T}^2) \otimes S^\bullet(\mathfrak{g}^*)\right)^{\mathbb{S}^1}.$$

This means that we can compute

$$H^\bullet(\mathcal{A}^{\nabla^\times}_{\text{Mehta-Cartan}}(A)) \simeq H^\bullet_{\mathbb{S}^1}(\mathbb{T}^2) \simeq H^\bullet(\mathbb{S}^1)$$

by Theorem 0.1.7 and Corollary 0.1.5 (about equivariant cohomology of free actions). In particular, we can note that

$$H^2(\mathcal{W}(A, \nabla^\times)) = 0.$$

Example 0.3.9. Consider the algebroid

$$\mathbb{R} \times \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1.$$

of the previous example (Example 0.3.8). Any non-constant, non-vanishing function

$$f_1 : \mathbb{S}^1 \to \mathbb{R}$$

induces a function $f = \mathrm{pr}_1^*(f_1)$ and a connection ∇^f on A given by

$$\begin{aligned} \mathfrak{X}(G_0) \times \Gamma(G_0, A) &\to \Gamma(G_0, A) \\ (X, a) &\mapsto f \cdot \nabla^\times_X \left(\frac{1}{f} \cdot a\right), \end{aligned}$$

where $\nabla^\times$ denotes the connection with respect to the product structure.

The Mehta-Cartan model for (A, ∇^f) is given by

$$\mathcal{A}^{\nabla^f}_{\text{Mehta-Cartan}}(A) = \left(\Gamma(\mathbb{S}^1 \times \mathbb{S}^1, \wedge^\bullet T^*(\mathbb{S}^1 \times \mathbb{S}^1) \otimes S^\bullet(A^*))\right)^{(A, \nabla^f)}.$$

As the connection is flat and has trivial holonomy, with constant global sections of the form $f \cdot c$ where c is constant with respect to the product structure, we can express the invariance of a form

$$\omega \in \Gamma(\mathbb{S}^1 \times \mathbb{S}^1, \wedge^\bullet T^*(\mathbb{S}^1 \times \mathbb{S}^1) \otimes S^\bullet(A^*))$$

as

$$\mathcal{L}_{f \cdot c}(\omega) = 0$$

for any $c \in \mathbb{R}$ when interpreted as a constant global section of the product.

We can compute that

$$\mathcal{L}_{f \cdot c} = f \cdot \mathcal{L}_c + \mathrm{d}_{\mathrm{dR}}(f) \cdot \iota_c.$$

In particular, on $\Gamma(G_0, S^\bullet(A^*))$ the action is given by

$$\mathcal{L}_{f \cdot c} \equiv f \cdot \mathcal{L}_c.$$

This means that $\omega \in \Gamma(G_0, S^\bullet(A^*))$ is invariant (and thus basic) if on each orbit $\mathbb{S}^1 \times \{t\}$ it corresponds to a constant section with respect to the product structure.

If we now consider the global section

$$\gamma = f \cdot \mathrm{d}u \in \Gamma(\mathbb{S}^1 \times \mathbb{S}^1, A^*) \subset \Gamma(\mathbb{S}^1 \times \mathbb{S}^1, \wedge^0 T^*(\mathbb{S}^1 \times \mathbb{S}^1) \otimes S^1(A^*)),$$

it satisfies this condition. Here, we denote by u the frame of A^* induced by the product structure corresponding to the unit covector of $\mathbb{S}^1$. Furthermore, as it is horizontal with respect to ∇^f, we have that

$$\mathrm{d}_{\mathrm{Cart}}(\gamma) = 0.$$

Let us now consider a ∇-basic form

$$\omega \in \left(\Gamma(\mathbb{S}^1 \times \mathbb{S}^1, \wedge^\bullet T^*(\mathbb{S}^1 \times \mathbb{S}^1) \otimes S^\bullet(A^*))\right)^{(A,\nabla^f)}$$

of total degree 1. Automatically, $\omega \in \left(\Omega^1(\mathbb{S}^1 \times \mathbb{S}^1)\right)^{(A,\nabla^f)}$. Furthermore

$$\mathcal{L}_{f \cdot c}\omega = 0,$$

which implies that

$$L^*_{\exp(f\cdot c)}\omega = 0$$

for the left action L of $\mathbb{S}^1$ on $\mathbb{T}^2$ that we derived the algebroid from.

However, we have that when $c = \frac{2\pi}{f_1(n_1)}$ for some $n_1 \in \mathbb{S}^1$ then at any point $m = (n_1, n_2) \in \mathbb{T}^2$ we have $\exp(f \cdot c)(m) = 1_m$. Furthermore, whenever $\mathrm{d}_{\mathrm{dR}}(f)(m) \neq 0$, we have that $\exp(f \cdot c)$ is not tangent to the unit section at m. If we analyse

$$(L_{\exp(f\cdot c)})_* : T_m\mathbb{T}^2 \to T_m\mathbb{T}^2,$$

we find that

$$(L_{\exp(f\cdot c)})_* - \mathrm{id} : T_m\mathbb{T}^2 \to T_m\mathbb{T}^2$$

is contained in the bundle $T_{n_1}\mathbb{S}^1 \times \{0\} \subset T_m\mathbb{T}^2$ as the action on $\mathbb{T}^2$ leaves its second component fixed. Note that this bundle coincides with the foliation bundle $\mathcal{F}_0$ of the orbits. The latter map is non-vanishing at m whenever $\mathrm{d}_{\mathrm{dR}}(f)$ is non-vanishing at m: We can consider the associated maps to $T_{(2\pi,m)}A$, where $f \cdot c$ and the constant section 2π induce maps $T_m\mathbb{T}^2 \to T_{(2\pi,m)}A$ that only differ by vertical vectors precisely corresponding to the directional derivatives of f. Furthermore, $T\exp$ is an immersion on each fibre of the algebroid as the group is abelian and $Tt \circ T\exp$ is injective as the action is furthermore free. In particular, as $\mathcal{F}_0$ is one-dimensional, the image of $(L_{\exp(f\cdot c)})_* - \mathrm{id}$ in that case is $(\mathcal{F}_0)_m$ entirely.

Because

$$\left(L^*_{\exp(f\cdot c)} - \mathrm{id}\right)(\omega) = \omega - \omega = 0$$

it follows that for any $Y \in T_m\mathbb{T}^2$ holds

$$\iota_{((L_{\exp(f\cdot c)})_* - \mathrm{id})(Y)}\omega = \iota_Y\left(L^*_{\exp(f\cdot c)} - \mathrm{id}\right)(\omega) = 0$$

and therefore $\omega(m)$ must vanish under ι_X for any $X \in (\mathcal{F}_0)_m$. So when we restrict ω to the saturated open subset

$$\begin{aligned} U :=& \{m = (n_1, n_2) \in \mathbb{T}^2 \text{ s.t. } \mathrm{d}_{\mathrm{dR}}(f_1)(n_1) \neq 0\} \subset \mathbb{T}^2 \\ =& \{n_1 \in \mathbb{S}^1 \text{ s.t. } \mathrm{d}_{\mathrm{dR}}(f_1)(n_1) \neq 0\} \times \mathbb{S}^1 \subset \mathbb{T}^2 \end{aligned}$$

we actually obtain a form

$$\omega|_U \in \Omega^1_{\text{bas}}(U),$$

meaning that also

$$\mathrm{d}_{\text{Cart}}(\omega|_U) = \mathrm{d}_{\text{dR}}(\omega|_U) \in \Omega^1_{\text{bas}}(U).$$

In particular, we have that $\gamma|_U$ is not in the image of the Cartan differential on U. As the Cartan differential is local, we find that also γ is not in the image of the Cartan differential on $\mathbb{T}^2$. This means that

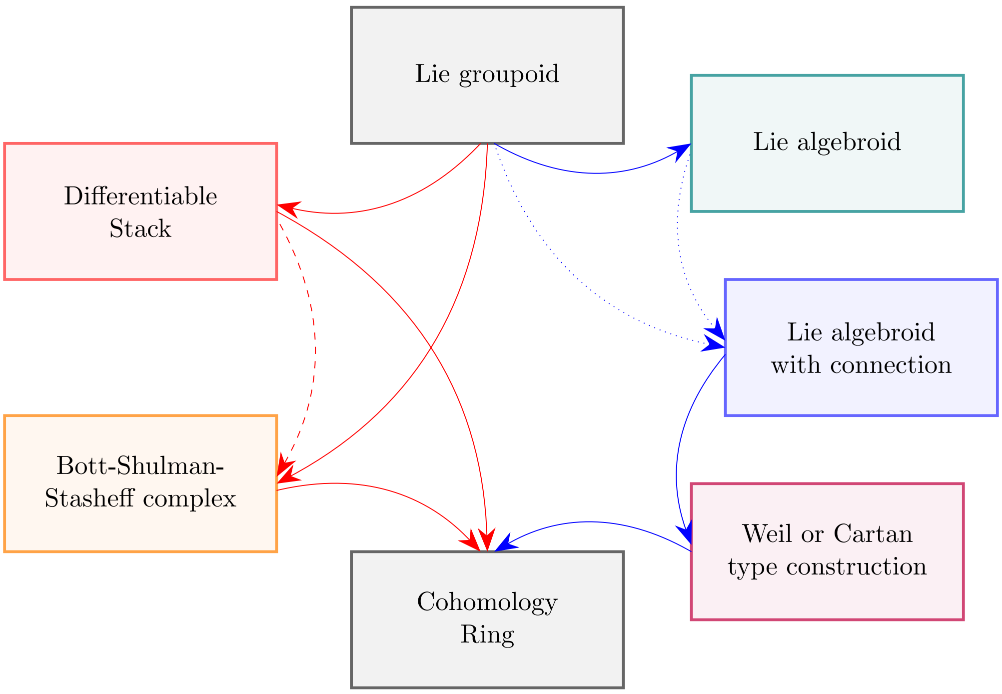
$$[\gamma] \in H^2\left(\mathcal{A}^{\nabla^f}_{\text{Mehta-Cartan}}(A)\right)$$

is a non-trivial cohomology class.

These examples show, that the choice of ∇ is vital not only for the isomorphism class of $\mathcal{A}^{\nabla}_{\text{Mehta-Cartan}}(A)$ as a chain complex or differential graded $\mathbb{R}$-algebra (and subsequently the isomorphism class of $\mathcal{A}^{\nabla}_{\text{Mehta-Weil}}(A)$), but also for their cohomology.

0.4 A counterexample to infinitesimality

We can summarise the previous results on generalising our construction in the schematic picture



where on the right side, we only know how to construct the solid blue arrows. To complete the construction via the generalised Weil algebra, we still need to construct one of the dotted blue ones. This means, we want to consider constructions that assign a Lie algebroid with a connection to a Lie groupoid (or directly to a Lie algebroid). The latest considerations have shown that this can not be solved trivially by an arbitrary assignment.

There is a choice to be made here on how to proceed: Which one of the dotted arrows should one aim to construct? Or mathematically: Should we aim for a construction that associates a suitable connection to a Lie algebroid or instead aim only for a construction of a suitable Lie algebroid with a connection from a Lie groupoid? Note that the latter option is more general because it means that the associated connection may also depend on global data.

In the following, we will give an example that implies that this dependence on global data is indeed necessary. Any construction of a connection on infinitesimal data alone (meaning only on the Lie algebroid) can not succeed on a sufficiently large class of Lie groupoids.

Definition 0.4.1. A **Lie group bundle** is a Lie groupoid satisfying that $s = t$, i.e. source and target map agree.

Lemma 0.4.2. *Consider G_1 and G_0 manifolds and $\pi : G_1 \to G_0$ a surjective submersion. Then the smooth maps μ, 1, $(-)^{-1}$ equip $G_1 \rightrightarrows G_0$ with $s = t = \pi$ with the structure of a Lie group bundle if and only if they equip every fibre of π with a group structure.*

Proof. This follows immediately, as the conditions of a Lie groupoid are either given by source-target-compatibilities, which become trivial for $s = t$, or unitality, associativity and inverses, which are precisely the properties of a group. □

Example 0.4.3. Consider
$$p : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{S}^1.$$
Then the fibrewise group structure of $\mathbb{R}^2$ with addition defines a Lie group bundle R. We can furthermore consider the standard Riemannian metric on $\mathbb{R}^2$ as a smooth family of Riemannian metrics on each fibre of the bundle.

Example 0.4.4. Consider
$$p : \mathbb{T}^2 \times \mathbb{S}^1 \to \mathbb{S}^1.$$
Then the fibrewise group structure of $\mathbb{T}^2 \cong (\mathbb{R}/2\pi\mathbb{Z})^2$ with addition defines a Lie group bundle S. By construction, the map
$$\mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{T}^2 \times \mathbb{S}^1$$
is a quotient map which is a group homomorphism in every fibre. In particular, it induces a Lie group homomorphism $R \to S$, where R denotes the Lie group bundle of Example 0.4.3. The preimage of the unit section is given by the groupoid
$$(2\pi\mathbb{Z})^2 \times \mathbb{S}^1 \to \mathbb{S}^1,$$
which in every fibre is given by a discrete subset of $\mathbb{R}^2$. The Riemannian metric of $\mathbb{R}^2$ descends to $\mathbb{T}^2$, as it is translation invariant.

Lemma 0.4.5. *Consider the groupoid S of Example 0.4.4. We have that its associated Lie algebroid A_S is trivial.*

Proof. Let R be the Lie group bundle of Example 0.4.3. We have that $A_R \cong A_S$, as the open sets
$$\{x \in R_1 \text{ such that } \mu(x, 1_{p(x)}) < \pi\}$$
and
$$\{x \in S_1 \text{ such that } \mu(x, 1_{p(x)}) < \pi\}$$
are diffeomorphic by the quotient map (because the minimal distance between two points that are identified in the quotient is 2π, the restriction has trivial fibres). The restriction of the groupoid structure on both opens is compatible with the quotient map, hence the induced map on algebroids must be an isomorphism.

For R, we have that as it is a trivial group bundle the Lie algebroid will be trivial as a vector bundle and $\rho \equiv 0$. Furthermore, as R has abelian fibres, the Lie bracket vanishes. So $A_R \cong A_S$ is also trivial in that regard. $\square$

Remark. We could have applied a similar argument to A_S itself, but the method above will generalise to non-trivial torus bundles.

Example 0.4.6. Consider the Lie group bundle R given by

$$p : \mathbb{R}^2 \times \mathbb{S}^1 \to \mathbb{S}^1.$$

as in Example 0.4.3 and consider the following subset:

$$L_1 = \{(X, \phi) \text{ for } X \in \Lambda\left(\begin{pmatrix} 2\pi \cdot \cos(\phi/4) \\ 2\pi \cdot \sin(\phi/4) \end{pmatrix}, \begin{pmatrix} -2\pi \cdot \sin(\phi/4) \\ 2\pi \cdot \cos(\phi/4) \end{pmatrix}\right) \text{ and } \phi \in \mathbb{S}^1\}.$$

Note that this is indeed well-defined: Even though we have that

$$\begin{pmatrix} 2\pi \cdot \cos((\phi + 2\pi)/4) \\ 2\pi \cdot \sin((\phi + 2\pi)/4) \end{pmatrix} \neq \begin{pmatrix} 2\pi \cdot \cos(\phi/4) \\ 2\pi \cdot \sin(\phi/4) \end{pmatrix},$$

as well as

$$\begin{pmatrix} -2\pi \cdot \sin((\phi + 2\pi)/4) \\ 2\pi \cdot \cos((\phi + 2\pi)/4) \end{pmatrix} \neq \begin{pmatrix} -2\pi \cdot \sin(\phi/4) \\ 2\pi \cdot \cos(\phi/4) \end{pmatrix},$$

meaning that the expressions in the definition are not well-defined, we have instead that

$$\begin{pmatrix} 2\pi \cdot \cos((\phi + 2\pi)/4) \\ 2\pi \cdot \sin((\phi + 2\pi)/4) \end{pmatrix} = \begin{pmatrix} -2\pi \cdot \sin(\phi/4) \\ 2\pi \cdot \cos(\phi/4) \end{pmatrix},$$

as well as

$$\begin{pmatrix} -2\pi \cdot \sin((\phi + 2\pi)/4) \\ 2\pi \cdot \cos((\phi + 2\pi)/4) \end{pmatrix} = \begin{pmatrix} -2\pi \cdot \cos(\phi/4) \\ -2\pi \cdot \sin(\phi/4) \end{pmatrix},$$

meaning that the lattice is spanned by the same set of vectors up to scaling by -1. This makes it well-defined.

We have that $L_1 \rightrightarrows \mathbb{S}^1$ defines a subgroupoid of R: Locally, a smooth choice of the two generators (for example the one from the definition outside of $\varphi = 0$) define a trivialisation of $p|_L : L \to \mathbb{S}^1$ as a fibre bundle, in which all the fibres are isomorphic to $\mathbb{Z}^2$. Hence L_1 carries the structure of an embedded submanifold. Note that the generalisation also induce a (non-standard) compatible trivialisation of R. Furthermore, the lattice carries the structure of a fibrewise subgroup by definition. The fibrewise structure maps are compatible with the trivialisation of L_1 previously described, such that they are indeed not only fibrewise but globally smooth. Hence restricting the groupoid structure of R to L gives indeed rise to an inclusion of groupoids $L \subset R$.

Example 0.4.7. Consider the Lie group bundle R with fibre $\mathbb{R}^2$ of Example 0.4.3 and the Lie group subbundle $L \subset R$ with lattice fibres of Example 0.4.6. As over every fibre, L defines a normal subgroup of R, we can consider $T := R/L$, which at $\varphi \in \mathbb{S}^1$ has the fibre

$$\mathbb{R}^2 \Big/ \left(\Lambda\left(\begin{pmatrix} 2\pi \cdot \cos(\varphi/4) \\ 2\pi \cdot \sin(\varphi/4) \end{pmatrix}, \begin{pmatrix} -2\pi \cdot \sin(\varphi/4) \\ 2\pi \cdot \cos(\varphi/4) \end{pmatrix}\right)\right) \cong \mathbb{T}^2.$$

As we have discussed in Example 0.4.6 that L is a locally trivial bundle and its trivialisation (the parametrisation of the basis vectors spanning the lattice) is compatible with a trivialisation of R, we have that the same trivialisation establishes T a locally trivial smooth torus bundle. Furthermore, these trivialisations are compatible with the fibrewise group structures, such that there is a well-defined Lie groupoid structure on T such that the canonical map $R \to T$ is a Lie groupoid homomorphism.

Furthermore, we have that the smooth family of metrics on the fibres of R descend to a smooth family of metrics on the fibres of T as the standard Riemannian metric on $\mathbb{R}^2$ is translation and rotation invariant (the rotation invariance is relevant for the local trivialisation of the lattice, as it induces a non-standard local trivialisation of R by fibrewise rotation).

Lemma 0.4.8. *Consider the Lie groupoid T of Example 0.4.7. Then the corresponding Lie algebroid A_T is trivial.*

Proof. Let R denote the Lie groupoid of Example 0.4.3. We have that $A_R \cong A_T$, as the open sets

$$\{x \in R_1 \text{ such that } \mu(x, 1_{p(x)}) < \pi\}$$

and

$$\{x \in T_1 \text{ such that } \mu(x, 1_{p(x)}) < \pi\}$$

are diffeomorphic under the quotient map (because the minimal distance between two points that are identified in the quotient is 2π the quotient map is an isomorphism on these). The restriction of the groupoid structure on both opens is compatible with the quotient map, and hence this describes a diffeomorphism inducing an isomorphism on the algebroids. As we have that $A_R \simeq \mathbb{R}^2 \times \mathbb{S}^1$ and $\rho \equiv 0$ for A_R (the Lie algebroid of a trivial Lie group bundle is a trivial Lie algebra bundle), and furthermore the brackets on A_R vanish as R has abelian fibres, the same must be true for A_T. $\square$

Lemma 0.4.9. *For T_1 the morphisms of the groupoid in Example 0.4.7, we have that $H^1(T_1) \simeq \mathbb{R}$ and*

$$(u_1)^* : H^1(T_1) \to H^1(\mathbb{S}^1)$$

is an isomorphism.

Proof. We have that T_1 is compact as $\mathbb{S}^1$ is compact and T_1 is a fibre bundle with compact fibres. As additionally, the unit section is a global section of the projection, we have that the homology $H_1(T_1)$ is generated by three elements c, t_1, t_2, where c is the image of the generator of $H_1(\mathbb{S}^1)$ under the unit section and t_1 and t_2 are the homology classes corresponding to generators of $H_1(\mathbb{T}^2)$ of the torus fibre at an arbitrary but fixed point $m \in \mathbb{S}^1$. However, we have that the holonomy of L imposes relations on the generators. We have that $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ acts via torus automorphisms on the fibre over m, where the explicit form of L implies that the automorphism corresponding to 1 is given by

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

This tells us that in $H_1(T_1)$ all homology classes generated by t_1 and t_2 are equivalent that correspond to each other via this action. However, this in particular identifies t_1 with t_2 and then $-t_1$ and t_2 similarly with $-t_2$. As the coefficient ring $\mathbb{R}$ we are considering is torsion-free, this implies $[t_1] = [t_2] = [0]$.

Hence we have that $H_1(T_1)$ has at most one generator, c, but as this projects to the generator of $H_1(\mathbb{S}^1)$ under the projection, it can not be 0. So indeed

$$H_1(T_1) \cong c \cdot \mathbb{R}.$$

As $H^1(T_1;\mathbb{R})$ is dual to $H_1(T_1;\mathbb{R})$ by the universal coefficient theorem (cf. [25], Theorem 3.2.), we hence have

$$H^1(T_1) \cong c^* \cdot \mathbb{R}.$$

By construction, c was precisely the image of the generator of $H_1(\mathbb{S}^1)$, such that $(u_1)^*(c^*)$ is the dual generator of $H^1(\mathbb{S}^1)$. In particular, the map $(u_1)^*$ induces an isomorphism between the two cohomology groups. □

Proposition 0.4.10. *The two Lie groupoids S and T of Example 0.4.4 and Example 0.4.7 differentiate to Lie algebroids which satisfy that*

$$A_S \cong A_T,$$

while we also have

$$H_{\text{DiffStack}}([S_0//S_1]) \neq H_{\text{DiffStack}}([T_0//T_1]).$$

Proof. We have by Lemma 0.4.5 and Lemma 0.4.8 that

$$A_S \cong A_R \cong A_T$$

are trivial Lie algebroids and isomorphic.

Furthermore, we know by Lemma 0.4.9 that the necessary conditions are satisfied to apply Lemma 0.2.34. This implies that

$$\mathrm{H}^2_{\text{DiffStack}}([T_0//T_1]) = 0.$$

At the same time we know by Example 0.2.41 that

$$\mathrm{H}^2_{\text{DiffStack}}([S_0//S_1]) = u \cdot \mathbb{R} \oplus v \cdot \mathbb{R},$$

such that the cohomology rings can not be isomorphic. □

We can note here, that while we were able to compute all of $\mathrm{H}^\bullet_{\text{DiffStack}}([S_0//S_1])$ in this proof, we have not yet been able to determine more of $\mathrm{H}^\bullet_{\text{DiffStack}}([T_0//T_1])$ than its degree 2 component. Later in this thesis, we will be able to be more specific about this cohomology ring.

Thesis result: Proposition 4.3.21

The cohomology ring of the differentiable stack associated to the torus bundle of Example 0.4.7 is given by

$$H(\Omega(\mathbb{S}^1)) \otimes \mathbb{R}[x, y, z]/(4z^2 - y(x^2 - y)),$$

where x is a generator of degree 4 and y and z are generators of degree 8.

Corollary 0.4.11. *Let $\mathcal{C}$ be any full subcategory of the category* $\mathsf{LieGrpd}$ *containing both groupoids S and T of Example 0.4.4 and Example 0.4.7 as objects. The functor from $\mathcal{C}$ to the category* $\mathsf{dg}\text{-}\mathbb{R}\text{-}\mathsf{Alg}$ *of differential graded $\mathbb{R}$-algebras given by*

$$G \mapsto H_{\text{DiffStack}}([G_0//G_1])$$

does not factor through the functor

$$\begin{aligned} \mathsf{LieGrpd} &\to \mathsf{LieAlgd} \\ G &\mapsto A_G, \end{aligned}$$

where A_G denotes the Lie algebroid of G.

Even more, the underlying maps of sets of the objects

$$\mathcal{C}_0 \to (\mathsf{dg}\text{-}\mathbb{R}\text{-}\mathsf{Alg})_0$$

does not factor through the underlying map of sets

$$\mathcal{C}_0 \to (\mathsf{LieAlgd})_0 \,.$$

In particular, we can not have an assignment

$$A \mapsto \nabla^A$$

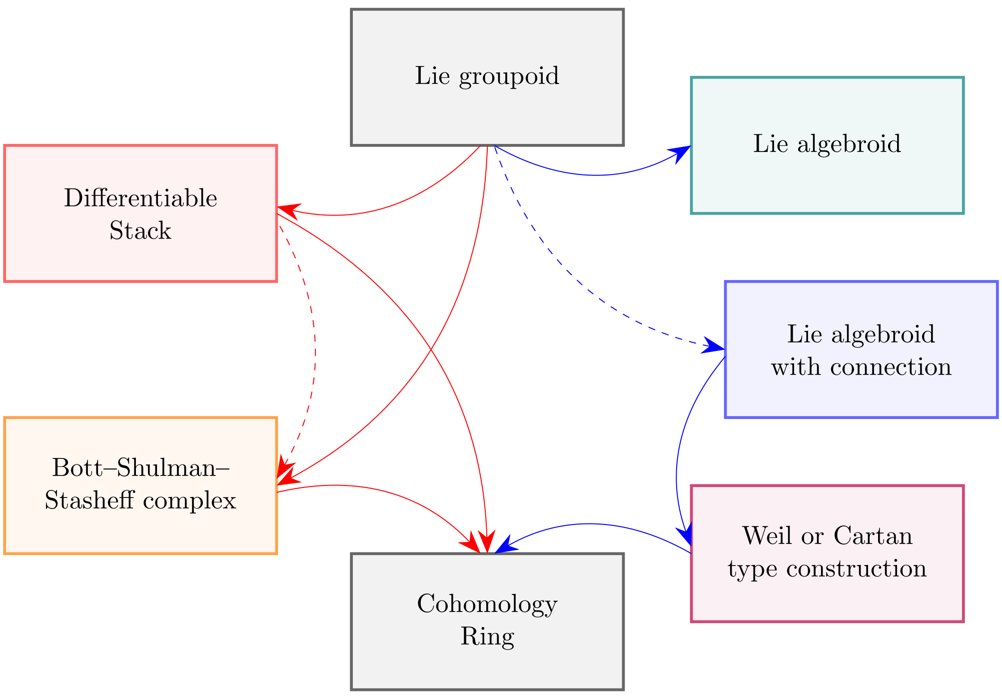
where ∇^A is a connection on A such that for $G \in \mathcal{C}$ a Lie groupoid and A_G its associated Lie algebroid we have

$$\mathrm{H}^\bullet_{\text{DiffStack}}(G) = H^\bullet\left(\mathcal{A}^{\nabla^{A_G}}_{\text{Mehta-Cartan}}(A_G)\right).$$

Proof. The second claim (about the maps of sets) implies both the first and the third, as they would induce such maps. However, these maps can not exist, as S and T have different images under the cohomology map and the same images under the differentiation map. □

Unfortunately, the Lie groupoids S and T are extremely well-behaved. They are proper, have compact object and morphism spaces, have orbits only containing a single point each and admit flat Cartan connections. So the previous fact applies to a lot of categories $\mathcal{C}$ that might be willing to restrict ourselves to.

In consequence, we will aim to generalise the differentiation functor from Lie groups to Lie algebras in a different way than usual. This will turn out to be an endeavour that, albeit complicated, will yield new results on models for differentiable stack cohomology.



To be precise, we will in this thesis construct the missing dashed arrow of the right side of the diagram - at least for any regular groupoid - and show that there is a model for differentiable stack cohomology in this case, which is in some sense related to the Mehta-Cartan model. The construction will not be canonical, but depend on a choice of connection on the Lie groupoid. We will later see that result of the Cartan type model we will describe is, however, canonical.

With this outlook, we will conclude this chapter. This exposition, of course, can not be understood as a complete overview of the ideas and methods that we will build upon. For example, Abad constructed a van-Est-map from the normalised subcomplex of the Bott–Shulman–Stasheff complex to the Weil algebra with certain cohomological properties (cf. [1], Theorem 3.5.1.) and a spectral sequence computing differentiable stack cohomology (cf. [1], Theorem 5.3.2.) which is a technique that we will also use in Chapter 4. However, I do hope that this chapter has been a helpful survey of both the setting and aim of the problem of finding infinitesimal models for differentiable stack cohomology, as well as of the most crucial notions, advances and examples.

Chapter 1

Structures on a regular groupoid

In this chapter we will turn from the theoretical background of computing differentiable stack cohomology to the world of Lie groupoids - in particular proper and regular ones.

This section will cover everything from notational conventions about the simplicial nerve, to concepts we will aim to use, such as the jet groupoids, from motivational examples and auxiliary definitions to very general and strong linearisation results about Lie groupoids. The relation of many of these concepts to the Bott–Shulman–Stasheff bicomplex or differentiable stack cohomology will not always be apparent in this chapter yet.

1.1 Groupoid actions and other preliminaries

Aim of this section: Definitions and Concepts

What is a regular groupoid? What are interesting structures on groupoids? What are groupoid actions?

The beginning of this thesis is mostly an overview of concepts that are well-known in higher differential geometry and a set-up of notational conventions. It will also include a lot of examples, in particular regarding Lie groupoid actions. These are a concept central to this thesis and many of the examples will be reoccurring throughout all chapters.

We will start this chapter by establishing some general facts about Lie groupoids, in particular proper and regular ones. We will then turn to study Lie groupoid actions. The main concepts in this section are often well-known and either commonly taught or easily checked. I claim no originality of the results presented here.

Remark. Recall the definition of the nerve of a Lie groupoid from Definition 0.2.23:

Consider a Lie groupoid G. The **nerve of** G is defined as the simplicial manifold $\mathcal{N}(G)$ defined by

$$[q] \mapsto G_q := G_1 \times_{G_0}^{s,t} G_1 \times_{G_0}^{s,t} \cdots \times_{G_0}^{s,t} G_1,$$

where G_q is referred to as **the manifold of q-composable arrows**, where there are q copies of G_1 in the pull-back.

The corresponding face maps are given by

$$\begin{aligned} d_i : G_q &\to G_{q-1} \\ (z_1, \dots, z_q) &\mapsto (\dots, \mu(z_i, z_{i+1}), \dots) \end{aligned}$$

and the corresponding degeneracy maps by

$$\begin{aligned} s_i : G_q &\to G_{q+1} \\ (z_1, \dots, z_q) &\mapsto (\dots, z_i, 1_m, z_{i+1}, \dots) \end{aligned}$$

where $m = s(z_i) = t(z_{i+1})$.

Definition 1.1.1. For each G_q with $q \geq 1$ and each $1 \leq i \leq q$ we denote by

$$\mathrm{pr}_i : G_q \to G_1$$

the projection to i-th component, i.e. the map respectively given by

$$\begin{tikzcd} G_1 \arrow[d, dashed, "\mathrm{pr}_1"] & \times^{s,t} & G_1 \arrow[d, dashed, "\mathrm{pr}_2"] & \times^{s,t} & \dots & \times^{s,t} & G_1 \arrow[d, dashed, "\mathrm{pr}_q"] \\ G_1 & & G_1 & & \dots & & G_1. \end{tikzcd}$$

Definition 1.1.2. For each G_q with $q \geq 1$ and each $0 \leq i \leq q$ we denote by

$$p_q^i : G_q \to G_0$$

the projection to i-th vertex, i.e. the map given by

$$\begin{tikzcd} & G_1 \arrow[d] \arrow[ddl, bend right, "p_q^0"'] & \times^{s,t} \arrow[dd, dash, "p_q^1"] & G_1 \arrow[d] & \times^{s,t} \arrow[dd, dash, "p_q^2"] & \dots & \times^{s,t} \arrow[dd, dash, "p_q^{(q-1)}"] & G_1 \arrow[d] & \\ & G_1 \arrow[dl, "t"'] \arrow[dr, "s"] & & G_1 \arrow[dl, "t"'] \arrow[dr, "s"] & & \dots & & G_1 \arrow[dl, "t"'] \arrow[dr, "s"] & \\ G_0 & & G_0 & & G_0 & \dots & G_0 & & G_0. \end{tikzcd}$$

For $q = 0$, we define $p_0^0 = \mathrm{id} : G_0 \to G_0$.

Example 1.1.3. Consider a Lie group bundle G, i.e. a Lie groupoid with $s = t$. We then have

$$p_q^i = t \circ \mathrm{pr}_i$$

and

$$p_q^{i+1} = s \circ \mathrm{pr}_i \,.$$

But the codomain of pr_i is G_1, on which we have $s = t$. This then implies $p_q^i = p_q^{i+1}$ for all $0 \leq i \leq q-1$ and therefore $p_q^i = p_q^j$ for all $0 \leq i \leq q$ and $0 \leq j \leq q$.

Remark. Recall the definition of the unit maps u_q to general levels of the nerve from Definition 0.2.24:

The maps $u_q : G_0 \to G_q$ defined by

$$m \mapsto (1_m, \dots, 1_m)$$

are referred to as the **unit maps**.

Lemma 1.1.4. *Consider any $q \in \mathbb{N}_{\geq 1}$. Then there is an isomorphism*

$$(u_q)^* TG_q \cong TG_0 \oplus A^{\oplus q}$$

as vector bundles over G_0.

Proof. For $q = 1$, we have that Ts is a surjective vector bundle map between $(u_1)^*TG_1$ and TG_0 for which Tu_1 defines a section. Therefore

$$(u_1)^*TG_1 \cong TG_0 \oplus (\ker(Ts) \cap (u_1)^*TG_1)\,,$$

where the latter is precisely A. Similarly for G_q, we have that Tp_q^q defines a projection to TG_0 and Tu_q a section of it, such that

$$(u_q)^*TG_q \cong TG_0 \oplus (\ker(Tp_q^q) \cap (u_q)^*TG_q).$$

Now under the projection $T\,\mathrm{pr}_q$ to the right-most copy of TG_1, we have that $\ker(Tp_q^q)$ projects to A. We can define a section of this projection by mapping

$$X \in A_m \mapsto (Tu_1(\rho(X)), \dots, Tu_1(\rho(X)), X)$$

such that

$$(u_q)^*TG_q \cong TG_0 \oplus (\ker(T\,\mathrm{pr}_q) \cap (u_q)^*TG_q) \oplus A.$$

However, we have that being in the kernel of $T\,\mathrm{pr}_q$ automatically implies that the tangent vector vanishes under Tp_q^{q-1} as well. We can inductively apply the same argument with the analogous construction of a section of the form

$$X \in A_m \mapsto (Tu_1(\rho(X)), \dots, Tu_1(\rho(X)), X, 0, \dots).$$

Note that this defines a section to $\ker(Tp_q^k) \cap \bigcap_{k+1 \le i \le q} \ker(T\,\mathrm{pr}_i)$, which is necessary for the induction. The procedure ends for $k = 1$, where we obtain

$$\ker(Tp_q^1) \cap \bigcap_{2 \le i \le q} \ker(T\,\mathrm{pr}_i) \cong \left(\bigcap_{1 \le i \le q} \ker(T\,\mathrm{pr}_i) \right) \oplus A = A,$$

such that the decomposition yields q copies of A in total. □

Lemma 1.1.5. *All maps* pr_i *and* $p_q^j : G_q \to G_0$ *are surjective submersions. In particular, for every* $m \in G_0$*, the space* $(p_q^j)^{-1}(m)$ *is a natural embedded submanifold of* G_q*.*

Proof. Pull-backs of surjective submersions along arbitrary functions are surjective submersions as well (this follows from the local form as a projection). We can apply this fact to find that all projections pr_i from G_q to the G_1, which are pull-backs of the identity, are surjective submersions as well. Hence also the concatenations of pr_i with either s or t are surjective submersions, implying the statement for p_q^j with $q > 0$. For $q = 0$, the identity is a surjective submersion as well.

The fact on the embedding of the fibres follows from the local form of a surjective submersion (cf. [30], Theorem 4.12). □

Lemma 1.1.6. *Let* G *be a Lie groupoid,* $q \in \mathbb{N}_{\ge 1}$ *and*

$$d_{i_1} \circ \dots \circ d_{i_q}$$

the concatenation of any choice of face maps. Then there is some $0 \le l \le q$ *such that*

$$d_{i_1} \circ \dots \circ d_{i_q} = p_q^l.$$

Proof. For $q = 1$, this follows by definition. For $q > 1$, we can see this inductively, as almost all components are retained under d_j, meaning so are their sources and targets, and multiplying two adjacent arrows z_j and z_{j+1} will map to an element of which its source agrees with $s(z_{j+1})$ and its target agrees with $t(z_j)$, which are both maps described by p^l_q for either $l = j - 1$ or $l = j + 1$. □

We will now turn from basic notions related to the simplicial structure of G to basic notions about its geometry.

Definition 1.1.7. A Lie groupoid G is **proper** if the map

$$(t, s) : G_1 \to G_0 \times G_0$$

is proper.

In this thesis, we are mainly concerning ourselves with proper and regular groupoids. Before we define regularity in the context of groupoids, let us recall the definition of a foliation and the Frobenius theorem.

Definition 1.1.8 (cf. [30], p. 501)**.** We define **a (regular) foliation of dimension** l on a manifold G_0 to be a collection F of disjoint, connected, non-empty, immersed k-dimensional submanifolds of G_0 (called the leaves of the foliation), whose union is G_0. They need to satisfy that in a neighbourhood of each point $m \in G_0$ there exists a flat chart for F, i.e. a chart $U \to G_0$ where $U \subset \mathbb{R}^n$ is a cube, and each submanifold in F intersects U in either the empty set or a countable union of l-dimensional slices of the form

$$x^{l+1} = c^{k+1}, \quad \ldots, \quad x^n = c^n.$$

Remark. For any point $m \in G_0$, the definition $(\mathcal{F})_m = T_m L \subset T_m G_0$, where $L \in F$ is the leaf containing m, defines a regular subbundle $\mathcal{F}$ of TG_0 which is involutive. In local coordinates, it takes the form

$$T\mathbb{R}^l \times \{0_x\}_{x \in \mathbb{R}^{n-l}} \subset T\mathbb{R}^n.$$

Theorem 1.1.9 (Frobenius, cf. [30], Theorem 19.12., Theorem 19.21.)**.** *Every involutive distribution is completely integrable. Let $\mathcal{F}_0$ be an involutive distribution on a smooth manifold G_0. The collection of all maximal connected integral manifolds of $\mathcal{F}_0$ forms a foliation of G_0.*

These considerations now allow us to think of (regular) foliations as (regular) involutive distributions.

Definition 1.1.10. Let G be a groupoid (not necessarily a Lie groupoid). For any $m \in G_0$, we denote the **orbit** O_m as the set given by

$$O_m = \{n \in G_0 \text{ s.t. } \exists x \in G_1 \text{ with } t(x) = n, s(x) = m\}.$$

Lemma 1.1.11. *We have that*

$$m \sim n :\Leftrightarrow n \in \mathcal{O}_m$$

is an equivalence relation.

Proof. Consider $x \in G_1$ such that $t(x) = n$ and $s(x) = m$. Then x^{-1} satisfies that $t(x) = m$ and $s(x) = n$ and hence witnesses $m \in \mathcal{O}_n$. Therefore, the relation is symmetric. Unitality follows from the existence of units. Transitivity can be inferred by multiplying x with $t(x) = n$ and $s(x) = m$ and y with $t(x) = l$ and $s(x) = n$ to $\mu(y, x)$. □

Definition 1.1.12. A Lie groupoid is **regular** if the foliation F_0 on G_0 induced by the orbits is a regular foliation, i.e. if the bundle $\mathcal{F}_0$ defined at any $m \in G_0$ by the space

$$(T_{1_m}t)(T_{1_m}s^{-1}(m)) = \rho(A_m) \subset T_mG_0$$

is a vector bundle of constant rank.

Remark. We could equivalently (using the Frobenius theorem) describe the conditions necessary for the orbits to be the leaves of a regular foliation. Note that using the description as above we do not need to state that $\mathcal{F}_0$ is involutive as a requirement, as the anchor ρ is a morphism of Lie algebroids. Without applying the Frobenius theorem, it is not a priori clear that orbits define submanifolds of G_0. In case of regularity of $\mathcal{F}_0$ it can be seen as an automatic consequence, but it can also be shown in the non-regular case, where they are not necessarily embedded but initial (cf. [14], pp. 177-178).

Lemma 1.1.13. *Let G be a regular groupoid. The subset of the associated Lie algebroid A given by*

$$\{X \in A \text{ s.t. } \rho(X) = 0\}$$

defines a Lie subalgebroid.

Proof. The image of the fibre over $m \in G_0$ under ρ is given precisely by

$$(\mathcal{F}_0)_m := (T_{1_m}t)(T_{1_m}s^{-1}(m)) = \rho(A_m)$$

and by assumption of regularity these spaces construct a vector bundle (in particular the fibre-wise rank of the map is constant). So we obtain that the rank of $\ker(\rho)$ is constant and has a vector bundle structure as well. That $\ker(\rho)$ is closed under the Lie bracket follows immediately as ρ is a Lie algebra morphism and the zero section is the trivial Lie subalgebra of $TG_0 \to G_0$. □

Definition 1.1.14. Consider a regular Lie groupoid G. We call the subalgebroid

$$A_{\text{iso}} = \{X \in A \text{ s.t. } \rho(X) = 0\} \subset A$$

the **isotropy Lie algebroid** of G.

Lemma 1.1.15. *For every regular groupoid G, the path-connected components of the isotropy in G_1, i.e. the space given by*

$$\left(\mathrm{Iso}^0(G)\right)_1 := \left\{ x \in G_1 \text{ s.t. } \begin{array}{r} \exists \varphi : [0,1] \to G_1 \text{ with } \varphi(0) = 1_m, \varphi(1) = x \\ \text{and } s(\varphi(r)) = t(\varphi(r)) = m \text{ for } 0 \leq r \leq 1 \end{array} \right\},$$

describes the manifold of morphisms of a Lie groupoid over

$$\left(\mathrm{Iso}^0(G)\right)_0 := G_0.$$

If G is proper, then $\mathrm{Iso}^0(G)$ *is proper as well.*

Proof. Many properties are already induced by the groupoid structure of G. We need to check if $\left(\mathrm{Iso}^0(G)\right)_1$ is a manifold, if t and s are submersions and if the statement on properness holds. We can see this by studying the Lie groupoids in the way found in [11], i.e. by thinking of elements in the Lie groupoid as the endpoints of smooth paths in A.

In a small neighbourhood of the unit section in G_1, the subset of $\left(\mathrm{Iso}^0(G)\right)_1$ is precisely the image of the exponential map restricted to A_{iso}, regardless of the choice of connection in the construction. This is because by construction, A_{iso} maps to $\mathrm{Iso}^0(G)$, but also because locally around the 0-section, elements of $A \backslash A_{\mathrm{iso}}$ get mapped along paths projecting under t to extensions of non-zero tangent vectors and therefore to elements satisfying $s \not\equiv t$. The exponential map is a diffeomorphism locally around the zero section $0_{G_0} \subset A$. Hence restricted to this neighbourhood, the manifold structure of A_{iso} induces a smooth manifold structure on $\mathrm{Iso}^0(G)$.

Furthermore, any element of $\mathrm{Iso}^0(G)$ can be expressed as the endpoint of a path in some fibre of A_{iso} and any such path can always be integrated to a path in $\left(\mathrm{Iso}^0(G)\right)_1$. Hence any local frame on A_{iso} can be used to extend this element to a smooth local section of G contained in $\mathrm{Iso}^0(G)$. We can explicitly check that $\mathrm{Iso}^0(G)$ is closed under multiplication, which implies that a local neighbourhood of any section is isomorphic to a local neighbourhood of the identity section. Thus $\mathrm{Iso}^0(G)$ has the local structure of a manifold at every point. By the same argument, we can also see that s is a submersion (as every element extends to a smooth section) and if s is a submersion, so is t.

When considering how the smooth structure on $\left(\mathrm{Iso}^0(G)\right)_1$ is induced by the exponential map, we find that as A_{iso} is closed in A, hence so is $\left(\mathrm{Iso}^0(G)\right)_1$ in G_1 (and even embedded as a submanifold). Hence if G is proper, so is $\mathrm{Iso}^0(G)$. □

Unsurprisingly, if a groupoid is regular, there are also implications for the associated nerve.

Proposition 1.1.16. *Let G be a regular groupoid and F_0 the orbit foliation on G with associated distribution $\mathcal{F}_0$. Consider the quotient*

$$Q = G_0/F_0 = G_0/\sim,$$

where $\sim$ denotes the equivalence relation of Lemma 1.1.11, (i.e. $m \sim n \Leftrightarrow n \in \mathcal{O}_m$) and the unique canonical projections $\pi_q : G_q \to Q$. Then the fibres $(\pi_q)^{-1}$ induce a regular foliation on G_q, with distribution denoted $\mathcal{F}_q$ satisfying

$$\mathcal{F}_q = (Tp_q^i)^{-1}(\mathcal{F}_0)$$

for any $0 \leq i \leq q$. The foliation has rank

$$l_q = q \cdot (k_1 - k_0) + l_0,$$

where k_0 is the dimension of G_0 and k_1 is the dimension of G_1 (and l_0 is the rank of $\mathcal{F}_0$).

Proof. Fist of all, let us start with showing that the π_q are indeed canonically defined: Consider the maps p_q^i for $0 \leq i \leq q$ as defined in Definition 1.1.2. Note that for any $z \in G_q$, all $p_q^i(z)$ lie in the same orbit, hence in the same leaf of F_0. So their image in Q is well defined. This also automatically implies that π_q factors through any $p_q^i : G_q \to G_0$, and the preimage of $q \in Q$ under π_q agrees with the preimage of the associated leaf $(\pi_0)^{-1}(q)$ under any p_q^i.

But these leaves of the regular foliation on G_0 can be described by their associated tangent subbundle $\mathcal{F}_0$ and the map p_q^i is a surjective submersion, hence the preimage of

$\mathcal{F}_0$ in TG_q defines a regular foliation as well. Thus the fibres $(\pi_q)^{-1}$ can indeed be written as the leaves of a regular foliation. The dimension of the foliation is obtained through this argument as well: The submersion has source G_q, which has dimension $q \cdot k_1 - (q-1) \cdot k_0$, target G_0 (which has dimension k_0) and hence the pull-back of a foliation of dimension l_0 increases its rank by $q \cdot k_1 - q \cdot k_0$. □

Definition 1.1.17. Let G be a Lie groupoid. Then TG denotes its **associated tangent groupoid**, which is defined as the groupoid with

- object space TG_0
- morphism space TG_1
- source map Ts and target map Tt
- the structure maps inherited from G, i.e. multiplication map $T\mu$, unit map Tu_1 and inverse map $T(\bullet)^{-1}$.

Remark. This groupoid is sometimes also referred to as the tangent prolongation groupoid (e.g. in [32], p. 3). Its well-definedness can be easily checked. We can also note that the tangent functor commutes with pull-backs, such that

$$(TG)_q = TG_1 \times_{TG_0}^{Ts,Tt} \cdots \times_{TG_0}^{Ts,Tt} TG_1 = TG_q.$$

Lemma 1.1.18. *Let G be a Lie groupoid and consider the bundle of vector spaces over G_0 defined by*

$$(\mathcal{F}_0)_m := (T1_m t)(T1_m s^{-1}(m)) = \rho(A_m).$$

Then we have

$$(Ts)^{-1}(\mathcal{F}_0) = (Tt)^{-1}(\mathcal{F}_0) := \mathcal{F}_1$$

defining a bundle of vector spaces over G_1. These spaces assemble into a groupoid

$$\begin{array}{c} \mathcal{F}_1 \\ {\scriptstyle Tt}\downdownarrows{\scriptstyle Ts} \\ \mathcal{F}_0 \end{array}$$

with the structure of a subgroupoid of TG. If G is regular, it is an embedded Lie subgroupoid of TG.

Remark. In case G is regular, we recover precisely the definition of $\mathcal{F}_1$ as in Proposition 1.1.16, as that notion was shown to satisfy $\mathcal{F}_1 = (Ts)^{-1}(\mathcal{F}_0) = (Tt)^{-1}(\mathcal{F}_0)$.

Proof. As the unit map Tu_1 is a section of both Ts and Tt and the inverse map $T(\bullet)^{-1}$ exchanges the image under Ts and Tt we can restrict the entire groupoid structure to

$$(Ts)^{-1}(\mathcal{F}_0) \cap (Tt)^{-1}(\mathcal{F}_0) \rightrightarrows TG_0$$

as a subgroupoid of the tangent groupoid. However, if $X \in (Ts)^{-1}(\mathcal{F}_0) \subset TG_1$ is a vector at some base-point $x \in G_1$, we have that by assumption $Ts(X) = Tt(Y)$ for $Y \in A \subset T_{1_{s(x)}}G_0$. In particular, $T\mu(X,Y)$ is well-defined and satisfies

$$Tt(X) = Tt(T\mu(X,Y)).$$

However, we have that $Ts(T\mu(X,Y)) = Ts(Y) = 0_{s(x)}$ by the choice of Y, such that the composition $T\mu(X,Y,0_{x^{-1}})$ is again well-defined. We can immediately check that

$$T\mu(X,Y,0_{x^{-1}}) \in T_{1_{t(x)}}G_1 \cap \ker(Ts) = A_{t(x)},$$

which implies

$$Tt(X) = \rho(T\mu(X,Y,0_{x^{-1}})) \in \mathcal{F}_0$$

such that indeed $s^{-1}(\mathcal{F}_0) = t^{-1}(\mathcal{F}_0)$.

If G is furthermore regular, we have by Proposition 1.1.16 that both $\mathcal{F}_0$ and $\mathcal{F}_1$ are involutive vector bundles of constant rank, meaning that in particular they are embedded submanifolds of TG_0 and TG_1. As Ts and Tt are furthermore vector bundle maps projecting to the surjective submersions s and t, they are surjective submersions themselves if and only if they are fibrewise surjective. However, this is also implied by the condition that s and t are surjective submersions respectively such that we obtain a Lie groupoid structure. $\square$

Lemma 1.1.19. *Let G be a regular Lie groupoid. The orbit foliation bundle $\mathcal{F}_q$ defined in Proposition 1.1.16 is precisely the q-th level nerve of the Lie groupoid $\mathcal{F}_1 \rightrightarrows \mathcal{F}_0$ of Lemma 1.1.18.*

Proof. We have that the nerve $\mathcal{F}_q$ of the groupoid of Lemma 1.1.18 is by definition given by

$$\mathcal{F}_1 \times_{\mathcal{F}_0} \cdots \times_{\mathcal{F}_0} \mathcal{F}_1 \subset TG_1 \times_{TG_0} \cdots \times_{TG_0} TG_1 = TG_q,$$

i.e. by the subset of TG_q given by those elements for which all $T\,\mathrm{pr}_i$ map to $\mathcal{F}_1$. The foliation of Proposition 1.1.16 is in the proof described by those elements $Z \in TG_q$ for which any Tp_q^i maps to $\mathcal{F}_0$. However, by definition $\mathcal{F}_1$ consists of precisely those tangent vectors mapping to $\mathcal{F}_0$ under Ts (or equivalently under Tt) such that the image of $X \in TG_q$ under $T\,\mathrm{pr}_i$ is contained in $\mathcal{F}_1$ if and only if the image under Tp_q^i (or equivalently Tp_q^{i-1}) is contained in $\mathcal{F}_0$. $\square$

We find that some of these properties also hold in the non-regular case.

Lemma 1.1.20. *Let G be a Lie groupoid and $\mathcal{F}_1 \rightrightarrows \mathcal{F}_0$ the groupoid defined in Lemma 1.1.18. Then the nerve of this groupoid satisfies that its q-th level, i.e. the bundle of vector spaces over G_q given by*

$$\mathcal{F}_q := \mathcal{F}_1 \times_{\mathcal{F}_0}^{Ts,Tt} \cdots \times_{\mathcal{F}_0}^{Ts,Tt} \mathcal{F}_1 \subset TG_q$$

agrees with $(Tp_q^i)^{-1}(\mathcal{F}_0)$ for all $0 \leq i \leq q$.

Proof. We can note that $(Ts)^{-1}(\mathcal{F}_0) = (Tt)^{-1}(\mathcal{F}_0) = \mathcal{F}_1$ already implies that $(Tp_q^i)^{-1}(\mathcal{F}_0)$ agrees with both $(T\,\mathrm{pr}_i)^{-1}(\mathcal{F}_1)$ and $(Tp_q^{i-1})^{-1}(\mathcal{F}_0)$. Similarly it implies that $(Tp_q^i)^{-1}(\mathcal{F}_0)$ agrees with both $(T\,\mathrm{pr}_{i+1})^{-1}(\mathcal{F}_1)$ and $(Tp_q^{i+1})^{-1}(\mathcal{F}_0)$. By finite descent, this means that

$$(Tp_q^i)^{-1}(\mathcal{F}_0) = (Tp_q^j)^{-1}(\mathcal{F}_0)$$

for any $0 \leq i \leq q$ and $0 \leq j \leq q$ and

$$(Tp_q^i)^{-1}(\mathcal{F}_0) = (T\,\mathrm{pr}_k)^{-1}(\mathcal{F}_1)$$

for any $1 \leq k \leq q$. In particular

$$(Tp_q^i)^{-1}(\mathcal{F}_0) = \bigcap_{1 \leq k \leq q} (T\,\mathrm{pr}_k)^{-1}(\mathcal{F}_1),$$

which is precisely the description of $\mathcal{F}_q$ as a subset of $(TG)_q$. $\square$

At this point of the section, we will turn our aim to understanding constructions associated with Lie groupoid actions. The notion can be motivated by left multiplication, i.e. the action of $x \in G_1$ on an appropriate $z \in G_q$ by left-multiplying the left-most component of z with x (we will in particular formalise this notion). There is one family of actions that we aim to understand in particular: The action of $x \in G_1$ on G_q by diagonal action at the i-th pull-back of

$$G_q = G_1 \times_{G_0} G_1 \times_{G_0} \dots G_1,$$

which we are going to generalise later.

Definition 1.1.21. For a groupoid G define a **left groupoid action of G on a manifold N** with $p : N \to G_0$ as a smooth action map

$$L : G_1 \times^{s,p} N \to N$$

such that it satisfies unitality for any $n \in N$:

$$L(1_{p(n)}, n) = n$$

as well as associativity (whenever defined):

$$L(g_1, L(g_2, n)) = L(\mu(g_1, g_2), n).$$

Similarly, define a **right groupoid action of G on a manifold N** with $p : N \to G_0$ as a smooth action map

$$R : N \times^{p,t} G_1 \to N$$

such that it satisfies unitality for any $n \in N$:

$$L(n, 1_{p(n)}) = n$$

as well as associativity (whenever defined):

$$R(R(n, g_1), g_2) = R(n, \mu(g_1, g_2)).$$

Remark. Left and right Lie groupoid actions can, similar to Lie group actions, be transformed into another by associating Lie groupoid elements to their inverses. We will in the following mostly define notions for left actions, but their analogues can be inferred in precisely the same way.

Definition 1.1.22. Let G be a Lie groupoid left acting by L_1 on $p_1 : N_1 \to G_0$ and by L_2 on $p_2 : N_2 \to G_0$. Let $f : N_1 \to N_2$ be a smooth map. Then f is **G-equivariant** if $p_2 \circ f = p_1$ and

$$f(L_1(g, n)) = L_2(g, f(n))$$

for any $n \in N$ and $g \in G_1$ with $s(g) = p_1(n)$.

Furthermore, let G be acting by L on $p : N \to G_0$ and consider $f \in C^\infty(N, M)$. We call f a **G-invariant function** if the induced map $(p, f) : N \to G_0 \times M$ is equivariant with respect to the trivial action of G on $\mathrm{pr}_1 : G_0 \times M \to G_0$. Here, the trivial action is given by the unique action

$$\begin{aligned} G_1 \times^{s,\mathrm{pr}_1} G_0 \times M &\to G_0 \times M \\ (x, m, a) &\mapsto (t(x), a) \end{aligned}$$

such that a G-equivariant function is a function for which

$$f(L(g, n)) = f(n).$$

Example 1.1.23. We can consider $\mathrm{id} : N = G_0 \to G_0$. Then we can act with an element $x \in G_1$ on $m \in N = G_0$ if $s(x) = n$. This action can be defined as $L(x, n) = t(x)$. This is the unique action we can define with respect to the identity map.

Example 1.1.24. Consider $p_q^i : G_q \to G_0$ for $q \geq 1$. Then we can define a left action of G. At an element $z \in G_q$ given in components as $z = (z_1, \ldots, z_q)$ it is given by

$$L(x, z) = (z_1, \ldots, \mu(z_i, x^{-1}), \mu(x, z_{i+1}), z_{i+2}, \ldots).$$

For $i = 0$, we understand this term to be of the form

$$L(x, z) = (\mu(x, z_1), z_2, \ldots).$$

and for $i = q$, we understand this term to be of the form

$$L(x, z) = (\ldots, z_{q-1}, \mu(z_q, x^{-1}),).$$

In particular, if $q = 1$ and $i = 0$ we recover what is often called left multiplication with x as

$$L(x, z) = \mu(x, z)$$

and if $q = 1$ and $i = 1$ we recover the induced left action by right multiplication as

$$L(x, z) = \mu(z, x^{-1}).$$

Remark. Even though the notation of

$$L_x(z) := \mu(x, z)$$

and

$$R_{x^{-1}}(z) := \mu(z, x^{-1}).$$

is not uncommon, we will abstain from using it for notational clarity considering the many definitions of this thesis. We will rather use the names instead which we will define in the following (to be precise, $D^0_{x,1}$ and $D^1_{x,1}$).

Example 1.1.25. Consider the action L of G on G_q described in Example 1.1.24 for some $0 \leq i \leq q$. Consider the projection $\mathrm{pr}_j : G_q \to G_1$ to the j-th component. Then if $j \neq i$ and $j \neq i + 1$ we have that pr_j is G-invariant, as the projection of

$$(z_1, \ldots, \mu(z_i, x^{-1}), \mu(x, z_{i+1}), z_{i+2}, \ldots)$$

to its j-th component precisely recovers z_j and therefore

$$\mathrm{pr}_j(L(x, z)) = \mathrm{pr}_j(z).$$

For $j = i$, we have that pr_j is G-equivariant with respect to the action L_1 of G on G_1 constructed in Example 1.1.24 for $i = 1$, as in this case

$$\mathrm{pr}_i(L(x, z)) = \mu(z_i, x^{-1}) = L_1(x, \mathrm{pr}_i(z)).$$

Similarly, for $j = i + 1$, we have that pr_j is G-equivariant with respect to the action L_2 of G on G_1 constructed in Example 1.1.24 for $i = 0$, as in this case

$$\mathrm{pr}_{i+1}(L(x, z)) = \mu(x, z_{i+1}) = L_2(x, \mathrm{pr}_{i+1}(z)).$$

This example is crucial to us, as this action and the concepts we derive from it will later yield notions of invariance on differential forms on G_q. Therefore, we want to give these actions a name depending on the choices of q and i.

Definition 1.1.26. Consider the left action on $p^i_q : G_q \to G_0$ of either Example 1.1.23 (for $q = 0$, as $p^0_0 = id$) or Example 1.1.24 (for $q \geq 1$). We will call an action of this form the **diagonal action at the i-th pull-back** and denote it as $D^i_{x,q}(\bullet)$, i.e. we denote

$$D^i_{x,0}(m) = t(x)$$

for $m \in G_0$ with $s(x) = m$ and

$$D^i_{x,q}(z) = (z_1, \dots, \mu(z_i, x^{-1}), \mu(x, z_{i+1}), z_{i+2}, \dots)$$

for $z \in G_q$ with $s(x) = p^i_q(z)$.

Remark. For notational reasons, I have here decided to denote the element of G_1 that "acts" in the index. This is in part to shorten the formulas in this thesis and in part because we also aim to use the notation of $\widetilde{D}^i_{\gamma,q}(\bullet)$ and $\dot{D}^i_{V,q}(\bullet)$ for the associated induced actions we will later encounter. However, we still have that $D^i_{x,q}(\bullet)$ depends smoothly on x.

Lemma 1.1.27. *Let L be the action of the groupoid G on $p : N \to G_0$, where p is a surjective submersion and consider $x \in G_1$. Then the submanifolds of G_q given by $p^{-1}(s(x))$ and $p^{-1}(s(x))$ are diffeomorphic.*

Proof. As p is a surjective submersion, we have that its fibres define smooth manifolds. The map

$$\begin{aligned} L|_{\{x\}\times p^{-1}(s(x))} : \{x\} \times p^{-1}(s(x)) &\to \{x\} \times p^{-1}(t(x)) \\ z &\mapsto L(x, z) \end{aligned}$$

defines a smooth map between $p^{-1}(s(x))$ and $p^{-1}(s(x))$. Its analogue defined with respect to x^{-1} is precisely the inverse and hence both maps are diffeomorphisms. ☐

Example 1.1.28. By Lemma 1.1.5 we have that all of the maps p^i_q are surjective submersions and we can apply Lemma 1.1.27 to the diagonal action at the i-th pull-back of Example 1.1.24. On the submanifolds of G_q of the form $(p^i_q)^{-1}(m)$ we therefore have that for any $x \in G_1$ we obtain a smooth map

$$\begin{aligned} D^i_{x,q} : (p^i_q)^{-1}(s(x)) &\to (p^i_q)^{-1}(t(x)) \\ z &\mapsto D^i_{x,q}(z). \end{aligned}$$

Lemma 1.1.29. *We have that*

$$D^0_{x,0} \circ d_0 \equiv d_0 \circ D^1_{x,1}$$

as maps on $(p^1_1)^{-1}(s(x))$ and

$$D^0_{x,0} \circ d_1 \equiv d_1 \circ D^0_{x,1}$$

as maps on $(p^0_1)^{-1}(s(x))$.

Proof. First of all, note that $p_1^0 \equiv d_1 \equiv t$ and $p_1^1 \equiv d_0 \equiv s$, such that the left hand side of both equations evaluates to

$$D^0_{x,0}(s(x)) = t(x)$$

by definition.

The fact that

$$s(D^1_{x,1}(z)) = s(\mu(z, x^{-1})) = t(x)$$

and

$$t(D^0_{x,1}(z)) = t(\mu(x, z)) = t(x)$$

follow by inserting the definitions of the $D^i_{x,1}(z)$. □

The examples we have studied up to this point were all very focussed on defining the diagonal actions at the different pull-backs. However, there is a second, very important class of examples in this thesis, given by different "adjoint actions".

When trading Lie groups and Lie algebras in for Lie groupoids and Lie algebroids, it is a well-known and lamented problem that the notions of adjoint action and representation can not be easily transferred. Approaches to fix this often lead to interesting higher structures such as representations up to homotopy (cf. [1], p. 7).

The approach used here is a radically simple one: We restrict the space we are acting on to a subspace, where the definitions generalise more easily. Nonetheless it will prove a useful concept later.

Example 1.1.30. Let G be any Lie groupoid, $H \subset G$ any Lie subgroupoid for which $s \equiv t$ on H_1 and consider H_q for $q \geq 1$. By Example 1.1.3, all of the maps

$$p^i_q : H_q \to G_0$$

agree. We can attempt to define a left action of G on this bundle. At an element $z \in H_q$ with components $z = (z_1, \ldots, z_q)$ we would like it to be given by

$$L(x, z) = (\mu(x, z_1, x^{-1}), \mu(x, z_2, x^{-1}), \ldots, \mu(x, z_q, x^{-1})).$$

Well-definedness of this expression follows from the fact that all $p^i_q : H_q \to G_0$, i.e. sources and targets of the components z_j agree. However, it is not generally the case that $L(x, z) \in H$. We need to add this to our assumptions on H.

If G is regular, this in particular can be applied to $H = (\mathrm{Iso}^0(G))_q$. Here, we obtain that $L(x, z) \in H_q$ (or equivalently $\mu(x, z_i, x^{-1}) \in H_1$ for all $1 \leq i \leq q$) as the action can be applied to the entire path that establishes some z_i as an element of $\mathrm{Iso}^0(G)$. For $q = 0$, we consider L to be the action of Example 1.1.23, as $(\mathrm{Iso}^0(G))_0 = G_0$.

If $\mathcal{G}$ is a Lie group and $G = H = \mathcal{G} \rightrightarrows *$, we furthermore find that the action of G on $H_1 = G_1$ recovers the usual notion of the adjoint action of $\mathcal{G}$ on itself.

Example 1.1.31. Consider the adjoint action L_q of G on H_q as described in Example 1.1.30 and in particular consider the adjoint action L_1 of G on H_1. Consider the projection $\mathrm{pr}_j : G_q \to G_1$ to the j-th component for any $1 \leq j \leq q$. Then we have that pr_j is G-equivariant, as we can explicitly compute

$$\mathrm{pr}_j(L_q(x, z)) = \mu(x, z_j, x^{-1}) = L_1(x, \mathrm{pr}_j(z)).$$

We similarly can consider a simplicial face map d_i and explicitly compute

$$d_i(L_q(x, z)) = \big(\mu(x, z_1, x^{-1}), \ldots, \mu(x, z_i, z_{i+1}, x^{-1}), \mu(x, z_{i+2}, x^{-1}), \ldots\big) = L_{q-1}(x, d_i(z)),$$

meaning that d_i is G-equivariant. Also we can consider a simplicial degeneracy map s_j and explicitly compute

$$s_j(L_q(x,z)) = \big(\mu(x,z_1,x^{-1}),\dots,\mu(x,z_j,x^{-1}),1_{t(x)},\mu(x,z_{j+1},x^{-1}),\dots\big) = L_{q-1}(x,s_j(z)),$$

meaning that s_j is G-equivariant.

Lemma 1.1.32. *Let L be the action of a Lie groupoid G on a bundle $p : N \to G_0$, where p has constant rank. Then there is a canonical induced action L' of G on*

$$(\mathrm{pr}_{G_0} \circ Tp)|_{\ker(Tp)} : \ker(Tp) \to G_0.$$

It extends L in the sense that on the image of the zero section $N \subset \ker(Tp)$ the action restricts to L.

Remark. The proof will reveal that p being of constant rank only matters for the statement that $\ker(Tp)$ is a manifold, however for general p we still obtain a well-defined action of G on $\ker(Tp)$ as a topological space.

Proof. Firstly, let us note that by the regularity of p the space $\ker(Tp)$ is actually a regular subvector bundle of TN and therefore a manifold. We can define the action L' as

$$\begin{aligned} L' : G \times^{s,\mathrm{pr}_{G_0}\circ Tp} \ker(Tp) &\to \ker(Tp)\\ x, X &\mapsto TL(0_x, X). \end{aligned}$$

Note that this is well-defined as $TL(0_x,X)$ satisfies that

$$Tp\,(TL(0_x,X)) = Tt(0_x) = 0_{t(x)}$$

and furthermore, for $X = 0_n$ in the image of the zero section, we obtain

$$L'(x,X) = TL(0_x,0_n) = 0_{L(x,n)}.$$

The fact that this satisfies all the axioms of an action follows immediately using

$$T\mu(0_x,0_y) = 0_{\mu(x,y)}$$

and

$$0_{u_1(p(n))} = Tu_1(0_p(n)).$$

□

Example 1.1.33. Consider Example 1.1.30, i.e. a Lie groupoid G acting on H_q, where $H \subset G$ is in particular a subgroupoid on which we have $s \equiv t$. Denote the algebroid associated to H by A_H. We have already noted that all p_q^i in this example agree so they define a unique map to G_0, and by Lemma 1.1.5 this is a surjective submersion which is in particular regular. By Lemma 1.1.32, we have an induced action of G on $\ker(Tp_q^q) \subset TH_q$.

We know by Lemma 1.1.4 that we have

$$(u_q)^*TH = TG_0 \oplus A_H^{\oplus q},$$

where Tp_q^q projects to the first summand. This means that

$$(u_q)^* \ker(Tp_q^q) = A_H^{\oplus q},$$

such that the domain of L' contains

$$G_1 \times^{s,\mathrm{pr}_{G_0}} A_H^{\oplus q} \subset G_1 \times^{s,\mathrm{pr}_{G_0} \circ Tp_q^q} \ker(Tp_q^q)$$

and therefore restricts to a map

$$L'|_{G_1 \times^{s,\mathrm{pr}_{G_0}} A_H^{\oplus q}} : G_1 \times^{s,\mathrm{pr}_{G_0}} A_H^{\oplus q} \to \ker(Tp_q^q).$$

Furthermore for any $z \in u_q(G_0) \subset H_q$, i.e. $z = 1_m$ for some $m \in G_0$, we have that any $x \in s^{-1}(\{m\})$ satisfies

$$L(x, z) = \big(\mu(x, 1_{s(x)}, x^{-1}), \dots, \mu(x, 1_{s(x)}, x^{-1})\big) = u_q(t(x)).$$

So it immediately follows that the image of the restriction is contained in

$$\ker(Tp_q^q) \cap (u_q)^* TG_q,$$

which as above before is $A_H^{\oplus q}$ itself. Hence we obtain a well-defined restriction

$$L'_{A_H^{\oplus q}} : G_1 \times^{s,\mathrm{pr}_{G_0}} A_H^{\oplus q} \to A_H^{\oplus q}.$$

Note that in the case mentioned in Example 1.1.30, where $G = H = \mathcal{G} \rightrightarrows *$ we have that $A_H = \mathfrak{g}$. In particular, the action of G on A_H precisely recovers the adjoint representation of $\mathcal{G}$ on $\mathfrak{g}$.

We know, that at least in the case of the adjoint action of a Lie group $\mathcal{G}$, there is also a corresponding notion of a coadjoint action. We can replicate it in this setting, however, in this case, a more general framework will become useful later.

Lemma 1.1.34. *Let G be a Lie groupoid and consider a bundle $p : N \to G_0$ and a vector bundle $\mathrm{pr}_N : V \to N$. Assume there is a left G-action L on $p \circ \mathrm{pr}_N$ which is linear, in the sense that for any $x \in G$ and $n \in N$ with $p(n) = s(x)$ the map*

$$L(x, \bullet)|_{V_n} : V_n \to V$$

has its image contained in a unique fibre V_m and is linear as a map of these vector spaces. Then L extends a unique left action of G on N and there is a canonical right action L^* of G on V^* extending its corresponding right action.

Proof. First of all, note that the fact that L is linear immediately implies that for any $n \in N$ with $p(n) = s(x)$ we have that $L(x, 0_n) = 0_m$, such that we may restrict L onto the zero section (which is isomorphic to N) and obtain

$$\underline{L} : G_1 \times^{s,p} N \to N$$

which still satisfies the axioms of an action as the restriction of an action. Its corresponding right action is given by the map

$$(n, x) \in N \times^{p,t} G_1 \mapsto \underline{L}(x^{-1}, n).$$

Now we aim to define

$$L^*(\theta, x) := \big(X \in V_{\underline{L}(x^{-1}, \mathrm{pr}_N(\theta))} \mapsto \theta(L(x, X)) \in \mathbb{R}\big) \in (V)^*_{\underline{L}(x^{-1}, \mathrm{pr}_N(\theta))}.$$

For this, we need to check that $\theta(L(x, X))$ is a well-defined expression for $X \in V_{\underline{L}(x^{-1}, \mathrm{pr}_N(\theta))}$. Let $n \in N$ be our notation for the base-point of θ. Note that we only need to define $L^*(\theta, x)$ under the condition that

$$(p \circ \mathrm{pr}_N)(\theta) = p(n) = t(x).$$

This implies that

$$s(x^{-1}) = t(x) = n$$

and the expression $\underline{L}(x^{-1}, n)$ is actually well-defined. Furthermore, we have that

$$\mathrm{pr}_N(X) = \underline{L}(x^{-1}, n)$$

and therefore

$$(p \circ \mathrm{pr}_N)(X) = t(x^{-1}) = s(x).$$

So the expression $L(x, X)$ is indeed well-defined. As it is linear in X and θ is linear by assumption, we have that the expression indeed defines an element of V^* with base-point $\underline{L}(x^{-1}, n)$. Analogously, the action is linear in θ and in particular if $\theta = 0_n$ then

$$L^*(0_n, x) = 0_{\underline{L}(x^{-1}, n)}.$$

This immediately implies that it extends the right action induced by $\underline{L}$.

The group action properties now essentially follow from those of L, where we note that for example

$$L^*(L^*(\theta, x), y)(X) = \theta(L(x, L(y, X))) = \theta(L(\mu(x, y), X))$$

and all other properties analogously. □

Example 1.1.35. Let L' denote the action of G on $A_H^{\oplus q}$ as defined in Example 1.1.33, where $A_H^{\oplus q}$ denotes the Lie algebroid of H. We now claim that we can apply Lemma 1.1.34 to obtain an action of G on $(A_H^{\oplus q})^* = (A_H^*)^{\oplus q}$ defined by

$$L^*(\theta, x) := \big(X \mapsto \theta(L'(x, X))\big) \in (A_H^*)^{\oplus q}_{t(x)}$$

for $\theta \in (A_H^*)^{\oplus q}_{s(x)}$. For this, we need to see that the expression $L'(x, X)$ is linear in X which directly follows from the linearity of $L'(x, X)$ in X, which holds by its definition of

$$L'(x, X) = T_{1_m}(L(x^{-1}, \bullet))(X)$$

where m is the base-point of X.

Note that in the case where $G = \mathcal{G} \rightrightarrows *$ and $q = 1$ as mentioned in Example 1.1.30 and Example 1.1.33, we have that the action is defined precisely as the right action corresponding to the coadjoint action of $\mathcal{G}$ on $\mathfrak{g}^*$.

Of course, vector bundles admit other constructions than the dual as well. These will, in part, become relevant later.

Lemma 1.1.36. *Let G be a groupoid and consider a bundle $p : N \to G_0$ and two vector bundles $\mathrm{pr}_1 : V \to N$ and $\mathrm{pr}_2 : W \to N$. Assume there is a G-action L_1 on $p \circ \mathrm{pr}_1$ which is linear, in the sense that for any $n \in N$ and $x \in G$ the map*

$$L(x, \bullet)|_{V_n} : V_n \to V$$

has its image contained in a unique fibre V_m and is linear as a map of these vector spaces. Assume that there is a G-action L_2 on $p \circ \mathrm{pr}_2$ which is linear in the analogous sense. Further assume that both actions extend the same underlying action $\underline{L}$ of G on N at the respective zero section of V and W. Then there is a left G-action L on $V \oplus W$ extending both L_1 and L_2 respectively.

Proof. Consider any $n \in N$, then the fibre over n of $V \oplus W$ is given by $V_n \oplus W_n$ and for any $x \in G_1$ with $s(x) = p(n)$ we can define

$$\begin{aligned} V_n \oplus W_n &\to V_{\underline{L}(x,n)} \oplus W_{\underline{L}(x,n)} \\ X_1 \oplus X_2 &\mapsto L_1(x, X_1) \oplus L_2(x, X_2). \end{aligned}$$

The action axioms are immediately inherited from L_1 and L_2 respectively. □

Lemma 1.1.37. *Let G be a Lie groupoid and $H \subset G$ be a subgroupoid as in Example 1.1.30, in particular on which $s \equiv t$. Let A be the Lie algebroid associated to G and A_H the Lie algebroid associated to H. Consider the adjoint actions L and $L^{(q)}$ constructed as actions of G on H_1 and H_q respectively as in Example 1.1.30. Further consider the actions L' and $(L')^{(q)}$ as the actions as described in Example 1.1.33, i.e. the adjoint actions of G on A_H and $A_H^{\oplus q}$ respectively. Then $(L')^{(q)}$ agrees with the action given by the direct sum of q copies of the action L' of G on A_H. Here, we refer to the direct sum of linear actions as defined in Lemma 1.1.36.*

Proof. Explicitly spelling out the definition yields that we need to prove the following: If we consider $Y \in T_{1_m} H_q$ such that $Tp_q^i(Y) = 0$ and Y is given in components by

$$Y = (Y_1, \dots, Y_q)$$

we have that

$$(L')^{(q)}(x, Y) = (L'(x, Y_1), \dots, L'(x, Y_q)).$$

This statement follows directly from the fact that the actions L and $L^{(q)}$ on H_1 and H_q respectively by definition satisfy that

$$L^{(q)}(x, z) = (L(x, z_1), \dots, L(x, z_q))$$

and from the fact that for A_H the inclusion of some copy of A_H in $A_H^{\oplus q}$ actually corresponds in components to the map

$$X \mapsto (0, \dots, 0, X, 0, \dots)$$

This comes from the fact that in the construction of Lemma 1.1.4, the map is defined as

$$X \mapsto (Tu_q(\rho(X)), \dots, Tu_q(\rho(X)), X, 0, \dots)$$

and $\rho \equiv 0$ on $A_H \subset A_{\text{iso}}$. □

Lemma 1.1.38. *Let G be a groupoid and consider a bundle $p : N \to G_0$ and a vector bundle $\mathrm{pr}_N : V \to N$. Assume there is a G-action L on $p \circ \mathrm{pr}_N$ which is linear, in the sense that for any $n \in N$ and $x \in G$ the map*

$$L(x, \bullet)|_{V_n} : V_n \to V$$

has its image contained in a unique fibre V_m and is linear as a map of these vector spaces. Then there is a canonical extension of the action to an action of G on $\mathcal{T}^\bullet V$ as well as an action of G on $S^\bullet V$ and an action of G on $\wedge^\bullet V$.

Proof. Let us start by noting that L by Lemma 1.1.34 extends an underlying action $\underline{L}$ of G on N. We can define an action on the tensor power $\mathcal{T}^k(V)$ on every fibre $n \in N$ by

$$\begin{aligned} V_n \otimes \cdots \otimes V_n &\to V_{\underline{L}(x,n)} \otimes \cdots \otimes V_{\underline{L}(x,n)} \\ X_1 \otimes \cdots \otimes X_k &\mapsto L(x, X_1) \otimes \dots L(x, X_k). \end{aligned}$$

As any $L(x, \bullet)$ is linear when restricted to V_n by assumption, we have that the map is well-defined. The action axioms are immediately inherited by L. It is also clear from the definition that the map is equivariant under any action of the permutation group Σ_k by permuting the factors of the tensor power in domain and codomain. Therefore, the description of the action immediately descends to the quotients $\wedge^k V$ as well as $S^k V$. As we defined the action this way for an arbitrary degree k, the direct sum over all degrees defines an action on the entire space in each of these examples. □

Definition 1.1.39. Consider the actions of Example 1.1.30 and Example 1.1.33 of G on H_q or A_H respectively. We will refer to the actions as the **adjoint actions** on the respective spaces and denote them as

$$L(x, \bullet) = \mathrm{Ad}^H_{x,q}(\bullet)$$

and

$$L'(x, \bullet) = \mathrm{Ad}_x(\bullet)$$

respectively. In the case where G is regular and $H = \mathrm{Iso}^0(G)$, we will use the notation

$$L(x, \bullet) = \mathrm{Ad}^0_{x,q}(\bullet).$$

If q is clear from the context, we will also use the notation $\mathrm{Ad}^H_x(\bullet)$ and $\mathrm{Ad}^0_x(\bullet)$.

We will refer to the action of Example 1.1.35 as the **coadjoint action** of G on A^*_H and denote it as

$$L^*(x, \bullet) = \mathrm{Ad}^*_x(\bullet).$$

We will also use the notation of $\mathrm{Ad}^*_x(\bullet)$ for the extension of the action to $\wedge^\bullet A^*_H$ or $S^\bullet(A^*_H)$ when applicable.

Remark. Recall that in the proof of Lemma 1.1.15 we have seen that for a regular Lie groupoid G the algebroid associated to $\mathrm{Iso}^0(G)$ is A_{iso}. At the same time $s \equiv t$ on $H \subset G$ implies that $A_H \subset A_{\mathrm{iso}}$. This corresponds to the fact that $\mathrm{Iso}^0(H) \subset \mathrm{Iso}^0(G)$ is an embedding of the connected component of H containing the unit section into $\mathrm{Iso}^0(G)$. In particular, we have that the action of G on A_H can be entirely recovered from the action of G on A_{iso}. As we are mostly interested in the case where G is indeed regular, this motivates why we do not distinguish H notationally when considering the adjoint action.

Now that we have defined the actions that are most important for our context, we will turn to induced actions. These will exist for general Lie groupoid actions, but again, we will be particularly interested in the examples we have already established. There is one induced notion which can easily be defined using only standard differential geometric tools, which is the induced action of an s-section $\gamma \in \Gamma(G_0, G_1)$ on the space N. It can even be defined for local sections as well.

Definition 1.1.40. Consider a Lie groupoid G. Let us denote its space of **local sections of** s as

$$\Gamma_{s,\mathrm{loc}}(G_0, G_1) = \{\gamma : U \subset G_0 \to G_1 : s \circ \gamma = \mathrm{id}_U\}$$

Definition 1.1.41. Consider local sections $\gamma_1, \gamma_2 \in \Gamma_{s,\mathrm{loc}}(G_0, G_1)$, defined on U_1 and U_2 respectively. Then, if $(t \circ \gamma_2)^{-1}(U_1) \neq \emptyset$ we define the **concatenation**

$$\mu \circ (\gamma_1, \gamma_2) \in \Gamma_{s,\mathrm{loc}}(G_0, G_1)$$

as

$$\begin{aligned} \mu \circ (\gamma_1, \gamma_2) : (t \circ \gamma_2)^{-1}(U_1) &\to G_1 \\ m &\mapsto \mu(\gamma_2(t(\gamma_1(m))), \gamma_1(m)). \end{aligned}$$

Definition 1.1.42. Let G be a Lie groupoid. Consider $\gamma \in \Gamma_{s,\mathrm{loc}}(G_0, G_1)$ defined on $U \subset G_0$. Then we call γ a **local bisection** if

$$\psi := t \circ \gamma : U \to G_0$$

is a diffeomorphism onto its image V. In that case, we will denote by

$$\gamma^{-1} : V \to G_1 \in \Gamma_{\mathrm{loc}}(G_0, G_1)$$

the local section given by

$$\gamma^{-1} := (\bullet)^{-1} \circ \gamma \circ \psi^{-1}.$$

Lemma 1.1.43. *With the concatenation of local sections as in Definition 1.1.41 we have that*

$$\mu \circ (\gamma, \gamma^{-1}) \equiv u_1$$

and

$$\mu \circ (\gamma^{-1}, \gamma) \equiv u_1.$$

Proof. We only have to check the formulas, which yield

$$\begin{aligned}
&\mu \circ (\gamma, \gamma^{-1})(m) \\
=&\mu(\gamma(t(\gamma^{-1}(m))), \gamma^{-1}(m)) \\
=&\mu(\gamma(t((\gamma(\psi^{-1}(m)))^{-1})), (\gamma(\psi^{-1}(m)))^{-1}) \\
=&\mu(\gamma(s(\gamma(\psi^{-1}(m)))), (\gamma(\psi^{-1}(m)))^{-1}) \\
=&\mu(\gamma(\psi^{-1}(m)), (\gamma(\psi^{-1}(m)))^{-1}) \\
=&1_{t(\gamma(\psi^{-1}(m)))} = 1_{\psi(\psi^{-1}(m))} = 1_m
\end{aligned}$$

and the calculation for the other concatenation follows the same process. □

Lemma 1.1.44. *Let G be a Lie groupoid. Consider a local bisection $\gamma_1 \in \Gamma_{s,\mathrm{loc}}(G_0, G_1)$ defined on $U_1 \subset G_0$ and*

$$\psi_1 := t \circ \gamma_1 : U_1 \to G_0$$

the associated diffeomorphism onto its image V_1. Similarly, consider a local bisection $\gamma_2 \in \Gamma_{s,\mathrm{loc}}(G_0, G_1)$ defined on $U_2 \subset G_0$ and

$$\psi_2 := t \circ \gamma_2 : U_2 \to G_0$$

the associated diffeomorphism onto its image V_2. Assume that $U_1 \cap V_2 \neq 0$.

Then the concatenation

$$\mu \circ (\gamma_1, \gamma_2) : \psi_2^{-1}(U_1 \cap V_2) \to G_1$$

is a local bisection and its inverse is given by

$$(\mu \circ (\gamma_1, \gamma_2))^{-1} = \mu \circ (\gamma_2^{-1}, \gamma_1^{-1}).$$

Proof. By definition, we have that the map associated to $\mu \circ (\gamma_1, \gamma_2)$ is given by

$$\begin{aligned}
m \mapsto & t\,(\mu \circ (\gamma_1, \gamma_2)(m)) \\
&= t(\mu(\gamma_1(\psi_2(m)), \gamma_2(m))) = t(\gamma_1(\psi_2(m))) = (\psi_1|_{U_1 \cap V_2} \circ \psi_2|_{\psi_2^{-1}(U_1)})(m)
\end{aligned}$$

where all expressions well-defined on $\psi_2^{-1}(U_1)$. As the compositions of diffeomorphisms is a diffeomorphism, we have that $\mu \circ (\gamma_1, \gamma_2)$ is indeed a bisection.

The associated $(\mu \circ (\gamma_1, \gamma_2))^{-1}$ is defined on the image $\psi_1(U_1 \cap V_2)$ which can be checked explicitly to be the domain of $\mu \circ (\gamma_2^{-1}, \gamma_1^{-1})$. It is given by

$$m \mapsto \Big(\mu \circ (\gamma_1, \gamma_2)((\psi_1|_{U_1 \cap V_2} \circ \psi_2|_{\psi_2^{-1}(U_1)})^{-1}(m))\Big)^{-1}$$

where the expression can be computed as

$$\begin{aligned}
&\big(\mu(\gamma_1(\psi_2((\psi_2)^{-1}((\psi_1)^{-1}(m)))), \gamma_2((\psi_2)^{-1}((\psi_1)^{-1}(m))))\big)^{-1} \\
=&\mu\left((\gamma_2((\psi_2)^{-1}((\psi_1)^{-1}(m))))^{-1}, (\gamma_1((\psi_1)^{-1}(m)))^{-1}\right) \\
=&\mu\left((\gamma_2((\psi_2)^{-1}(t((\gamma_1((\psi_1)^{-1}(m)))^{-1}))))^{-1}, (\gamma_1)^{-1}(m)\right) \\
=&\mu\left((\gamma_2((\psi_2)^{-1}(t((\gamma_1)^{-1}(m)))))^{-1}, (\gamma_1)^{-1}(m)\right) \\
=&\mu\left((\gamma_2)^{-1}(t((\gamma_1)^{-1}(m)))^{-1}, (\gamma_1)^{-1}(m)\right) \\
=&\mu \circ \left((\gamma_2)^{-1}, (\gamma_1)^{-1}\right)(m).
\end{aligned}$$

□

Definition 1.1.45. Given the left action L of a Lie groupoid G on $p : N \to G_0$, we can define the **associated local left action of** $\Gamma_{s,\mathrm{loc}}(G_0, G_1)$. For a local section

$$\gamma : U \subset G_0 \to G_1$$

of s it is given by

$$\begin{aligned}
\widetilde{L}_\gamma : p^{-1}(U) &\to N \\
n &\mapsto L(\gamma(p(n)), n).
\end{aligned}$$

Remark. As $\Gamma_{s,\mathrm{loc}}(G_0, G_1)$ is not a group this is in particular not a group action. However, it behaves similarly to a group action with respect to the concatenation partially defined on $\Gamma_{s,\mathrm{loc}}(G_0, G_1)$ and behaves as a pseudogroup action when restricted to the pseudogroup of local bisections. This has a structural reason:

Note that

$$T_z\widetilde{L}_\gamma : T_z G_1 \to T_{\widetilde{L}_\gamma(z)} G_1$$

only depends on γ infinitesimally in the sense that it is determined by

$$T_{p(z)}\gamma : T_{p(z)} G_0 \to T_{\gamma(p(z))} G_1.$$

This in particular means that restricted to the group of globally defined bisections, we indeed obtain a map

$$\Gamma_{s,t}(G_0, G_1) \times N \to N$$

which is local in the sense that it factors through the first jet prolongation. This implies in particular that it is also local in the topological sense, where the germ of a bisection γ at $p(n)$ determines its action on $n \in N$. It is straightforward to check that this map describes a group action on N.

Example 1.1.46. Consider the Lie groupoid left action of a Lie groupoid G on G_0 with respect to id of Example 1.1.23. For $\gamma \in \Gamma_{s,\mathrm{loc}}(G_0, G_1)$ defined on $U \subset G_0$ we obtain the local action given by

$$\begin{aligned}
\widetilde{L}_\gamma : U &\to G_0 \\
z &\mapsto t(\gamma(z)).
\end{aligned}$$

We will refer to it as

$$\widetilde{D}^0_{\gamma,0} := \widetilde{L}_\gamma.$$

Example 1.1.47. Let G be a Lie groupoid. Consider the action of Example 1.1.24 for $q = 1$ and $i = 0$, i.e. the left action by left multiplication of G on G_1. For $\gamma \in \Gamma_{s,\mathrm{loc}}(G_0, G_1)$ defined on $U \subset G_0$ we obtain the local action given by

$$\begin{aligned} \widetilde{L}_\gamma : t^{-1}(U) &\to G_1 \\ z &\mapsto \mu(\gamma(t(z)), z). \end{aligned}$$

For consistency with the general notation, we will denote

$$\widetilde{D}^0_{\gamma,1} := \widetilde{L}_\gamma.$$

Similarly we can look at the example of right multiplication (i.e. $q = 1$ and $i = 1$) which induces the following local left action

$$\begin{aligned} \widetilde{D}^1_{\gamma,1} : s^{-1}(U) &\to G_1 \\ z &\mapsto \mu(z, (\gamma(s(z)))^{-1}). \end{aligned}$$

We will now extend this notation to all possible indices in a similar way. The previous examples can often be used to check intuitions about the new notions.

Definition 1.1.48. Let G be a Lie groupoid. On G_q for $q \neq 0$, we define $\widetilde{D}^i_{\gamma,q}$ for $0 \leq i \leq q$ as

$$\widetilde{D}^i_{\gamma,q} : (p^i_q)^{-1}(U) \to G_q$$

the local left action of a section $\gamma : U \subset G_0 \to G_1$ on G_q associated to the diagonal action at the i-th pull-back of Example 1.1.24. In case $q = 0$ we refer to the notion associated to the action of Example 1.1.23, which is the case covered in Example 1.1.46.

As this definition is closely related to the definition of $D^i_{x,q}$, we have that some of our earlier results immediately transfer.

Corollary 1.1.49. *Let G be a Lie groupoid. We have that on the corresponding domains of definition holds*

$$\widetilde{D}^0_{\gamma,0} \circ d_0 \equiv d_0 \circ \widetilde{D}^1_{\gamma,1}$$

and

$$\widetilde{D}^0_{\gamma,0} \circ d_1 \equiv d_1 \circ \widetilde{D}^0_{\gamma,1}.$$

Proof. We have noted in Lemma 1.1.29 that these equations hold for the corresponding Lie groupoid actions. As we have that for any point $z \in G_1$ in the domain of definition that $\widetilde{D}^l_{\gamma,q}(z) = \widetilde{D}^l_{x,q}$ for the appropriate $x = \gamma(p^l_q(z))$, they are immediately transferable. □

Definition 1.1.50. Let G be a Lie groupoid and $H \subset G$ be a Lie subgroupoid as in Example 1.1.30, in particular satisfying that $s \equiv t$ on H_1. On H_q for $q \geq 1$, we define $\widetilde{\mathrm{Ad}}^H_{\gamma,q}$ as the local left action of a section $\gamma : U \subset G_0 \to G_1$ on H_q associated to the adjoint action of Example 1.1.30. In case $q = 0$ we refer to the notion associated to the action of Example 1.1.23. If q is apparent from context, we also use the notation $\widetilde{\mathrm{Ad}}^H_\gamma$. In case $H = \mathrm{Iso}^0(G)$, we also use the notation $\widetilde{\mathrm{Ad}}^0_{\gamma,q}$ (or $\widetilde{\mathrm{Ad}}^0_\gamma$).

Analogously, we define $\widetilde{\mathrm{Ad}}_\gamma$ as the local left action of a section $\gamma : U \subset G_0 \to G_1$ on $A_H \subset A_{\mathrm{iso}}$ associated to the adjoint action of Example 1.1.33.

Remark. As we have already noted on page 71, the action on A_H is actually the restriction of the action on A_{iso} obtained for $H = \text{Iso}^0(G)$.

We have already noted previously that this action (at least for $q = 1$) actually depends on γ in a way that is point-wise only dependent on infinitesimal data. In the following, we will formalise this observation. To this goal, we will construct a new groupoid, the jet groupoid J^1G, from an arbitrary Lie groupoid. The morphisms of the jet groupoid are given by infinitesimal bisections, expressed as subspaces of TG_1.

However, we can already note here, that our definitions did not require γ to be a bisection. This makes the statements and definitions applicable to an even larger space of objects $\Gamma_{s,\text{loc}}(G_0, G_1)$ when compared to local bisections. This is an important observation, which we will also mirror when transferring the structures from sections to jets.

1.2 Jet groupoids and induced actions

Aim of this section: Definitions and Concepts

What is a jet groupoid? What do we mean by induced jet groupoid actions? How does this relate to differential forms?

A jet manifold is a mathematical concept, popular for example in mathematical physics, that describes the space of infinitesimal sections of a vector bundle or fibre bundle. They can also be defined with respect to a surjective submersion. But what precisely should a jet groupoid be, where we have two maps, source and target? There is a satisfying answer to this question, which will lead us to define J^1G, the (first) jet groupoid.

In this section, we will study these jet groupoids (which are an infinitesimal version of the bisection groupoid) and their properties. The most important in our context will be, that jet groupoids come with a natural construction to extend a Lie groupoid action. Unfortunately, the notion of a jet groupoid is not always used for the exact same concept, but there is a relation between the different versions and overlap in some proofs and techniques. We will here refer to the notion used for example by Crainic, Salazar and Struchinger in [12] (cf. Section 3.1).

One of the goals of this section will be to add to the notions of $(D^i_{x,q}(\bullet))_*$ and $(\widetilde{D}^i_{\gamma,q}(\bullet))_*$ a third notion $(\dot{D}^i_{V,q}(\bullet))_*$. Recall that $(\widetilde{D}^i_{\gamma,q}(\bullet))_*$ was induced by $(D^i_{x,q}(\bullet))_*$, which also explains their similar notation. Their difference lies in the fact that a point $x \in G_1$ can only act by $D^i_{x,q}(\bullet)$ on an embedded submanifold of G_q (almost always of lower dimension than G_q). A local section γ of s can instead act by $\widetilde{D}^i_{\gamma,q}(\bullet)$ on an open subset and - if x is in the image of γ - extend the previous notion. This works well with the restriction to jets, too, which act on an infinitesimal neighbourhood of the embedded manifold in the same way as a corresponding local section would. It is recommended to be aware that we will not work with only one of these three types of actions, but with all three throughout the thesis, using the relations between them.

Lemma 1.2.1 (cf. [34], p. 32-33)**.** *Let G be a Lie groupoid. The space consisting of bisection jets*

$$J^1(s,t) := \left\{V_x \subset T_xG_1 \text{ s.t. } T_xs : V_x \to T_{s(x)}G_0 \text{ and } T_xt \text{ are isos}\right\}$$

is an open submanifold of

$$J^1(s : G_1 \to G_0) = \left\{V_x \subset T_xG_1 \text{ s.t. } T_xs : V_x \to T_{s(x)}G_0 \text{ is an iso}\right\}.$$

Remark. For completeness, I have included the proof in Lemma A.3.1.

Also note that this is not the definition of a jet which is commonly used. One usually rather defines jets as equivalence classes of local sections. However, as s is a surjective submersion, we find that every subspace $V_x \subset T_x G$ for which Ts restricts to an isomorphism can be extended to a section and by the inverse function theorem (cf. [30], Theorem 4.5.) if Tt restricts to an isomorphism on V, this section is locally a bisection.

The space $J^1(s : G_1 \to G_0)$ can be thought of as the infinitesimal versions of the space of s-sections $\Gamma_s(G_0, G_1)$. This space of "global" objects intrinsically carry a notion of composition, which we defined as $\mu \circ (\gamma_1, \gamma_2)$ in Definition 1.1.41. When considering the space of global bisections of s and t, this composition satisfies the group axioms, and we can define analogue notions on local or infinitesimal versions of sections and bisections (cf. [33], pp. 114-115).

For us, it will actually be very relevant that we can define the notion of composition not only on the space of bisections or bisection jets, but actually on the space of jets itself. This is why I have chosen to arrange the statements leading up to Definition 1.2.5 in an order that might seem somewhat unintuitive to a reader familiar with the notion of a jet groupoid. Usually, it is not necessary to use structure on $J^1(s : G_1 \to G_0)$ to define the first jet groupoid (as is true for the cited sources [12] and [22]).

Definition 1.2.2. Consider a Lie groupoid G and

$$V, W \in J^1(s : G_1 \to G_0).$$

If $V \in T_x G_1$ and $W \in T_y G_1$ with $s(x) = t(y)$, we can define their **composition** $\widehat{\mu}(V, W)$ as

$$\widehat{\mu}(V, W) := \left\{ T\mu(X, Y) \in T_{\mu(x,y)} G_1 \text{ where } X \in V \text{ and } Y \in W \text{ with } Ts(X) = Tt(Y) \right\}.$$

Lemma 1.2.3. *Consider a Lie groupoid G and*

$$V, W \in J^1(s : G_1 \to G_0).$$

Then $\widehat{\mu}(V, W)$ indeed defines an element of $J^1(s : G_1 \to G_0)$. Let γ_1 and γ_2 be local sections of s through $x \in G_1$ and $y \in G_1$ where $s(x) = t(y)$. Assume that

$$V = T_{s(x)} \gamma_1 (T_{s(x)} G_0)$$

and

$$W = T_{s(y)} \gamma_2 (T_{s(y)} G_0)$$

be the corresponding jets. Then $\widehat{\mu}(V, W)$ is precisely the jet corresponding to $\mu \circ (\gamma_1, \gamma_2)$ at $\mu(x, y)$.

Proof. The fact that $\widehat{\mu}(V, W)$ is indeed a jet (i.e. that Ts restricts to an isomorphism) is implied by the fact that

$$Ts(T\mu(X, Y)) = Ts(Y)$$

uniquely recovers Y by assumption and $Tt(Y) = Ts(X)$ uniquely recovers X analogously. Furthermore, by the same argument and the fact that V and W are jets we can construct X and Y associated to any $Z \in T_{s(y)} G_0$ this way where y denotes the point satisfying that $W \subset T_y G_1$. The second claim directly follows from the definition of $\mu \circ (\gamma_1, \gamma_2)$ as

$$m \mapsto \mu(\gamma_1(t(\gamma_2(m))), \gamma_2(m)),$$

as by the standard rules for derivatives this implies

$$X \mapsto T\mu(T\gamma_1(T(t \circ \gamma_2)(X)), T\gamma_2(X)),$$

where by definition

$$T\gamma_1(T(t \circ \gamma_2)(X)) \in V$$

and

$$T\gamma_2(X) \in W.$$

□

This immediately implies a number of properties of $\widehat{\mu}$.

Lemma 1.2.4. *Consider a Lie groupoid G and the multiplication map $\widehat{\mu}$ as defined in Definition 1.2.2. Then*

- *$\widehat{\mu}$ is associative.*
- *The jets of the unit section $Tu_1(T_mG_0)$ act as neutral elements with respect to $\widehat{\mu}$.*
- *If $T(\bullet)^{-1} : TG_1 \to TG_1$ induces a bijection between V and W, then $\widehat{\mu}(V, W)$ is tangent to the unit section.*

Furthermore, if $V, W \in J^1(s,t)$, we have $\widehat{\mu}(V, W) \in J^1(s,t)$.

Proof. Firstly, recall that we have already noted in the remark on page 76 that any jet can be equivalently be understood as an equivalence class of local sections (because s is a surjective submersion), so for any $V \subset T_xG_1$ we can always find some γ for which

$$V = T_{s(x)}\gamma(T_{s(x)}G_0).$$

This immediately implies that $\widehat{\mu}$ is associative, that jets of the unit section act as neutral elements and that if two jets correspond to γ and γ^{-1} (as defined in Lemma 1.1.43, which precisely corresponds to applying $(\bullet)^{-1}$ on the graph and therefore the jet), we have that they multiply to a jet tangent to the unit section. Furthermore, we can note that $J^1(s,t)$ precisely consists of the invertible elements of $J^1(s : G_1 \to G_0)$ in the sense that their images under $T(\bullet)^{-1}$ still define jets. However, as taking inverses and multiplying commutes on sections by Lemma 1.1.44, the same is true for jets, in particular meaning that

$$T(\bullet)^{-1}(\widehat{\mu}(V, W)) = \widehat{\mu}(T(\bullet)^{-1}(V), T(\bullet)^{-1}(W)),$$

where the right-hand side is well-defined by the assumption of V and W being bijets and a jet by Lemma 1.2.3. □

Definition 1.2.5 (cf. [12], p. 950, [22] pp. 776-777)**.** Given a groupoid G, the **(first) jet groupoid** J^1G is defined as the groupoid described by the following data:

- the object space $(J^1G)_0 = G_0$,
- the morphism space given by $(J^1G)_1 := J^1(s,t)$ of Lemma 1.2.1, i.e. by

$$(J^1G)_1 := \left\{V_x \subset T_xG_1 \text{ s.t. } T_xs : V_x \to T_{s(x)}G_0 \text{ and } T_xt \text{ are isos}\right\}$$

- source and target maps are induced by s and t via precomposition with the bundle projection $TG_1 \to G_1$

- multiplication of V_x and W_y with $s(x) = t(y)$ given by $\widehat{\mu}(V_x, W_y)$ as in Definition 1.2.2, i.e by

$$\{T\mu(X,Y) \text{ where } X \in V_x \text{ and } Y \in W_y \text{ with } Ts(X) = Tt(Y)\}$$

- identity elements for any $m \in G_0$ are given by

$$\big\{(Tu_1)(X) \subset T_{(1_m)}G_1 \text{ where } X \in T_mG_0\big\}$$

- inversion of elements is given as

$$(V_x)^{-1} = \big\{(T(\bullet)^{-1})(X) \subset T_{x^{-1}}G_1 \text{ where } X \in V_x\big\}$$

Remark. As the name suggests, there are also higher jet groupoids, which are not relevant in our context.

Definition 1.2.6. Let G be a Lie groupoid. The map on $J^1(s : G_1 \to G_0)$ described by

$$V \subset T_xG_1 \mapsto x \in G_1$$

maps any $V \in J^1(s : G_1 \to G_0)$ to a corresponding point which we will call the **base-point** of V.

Lemma 1.2.7. *The map of Definition 1.2.6 (i.e. mapping jets to their base-points) induces a Lie groupoid morphism*

$$\begin{tikzcd}
(J^1G)_1 \arrow[r] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & G_1 \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
(J^1G)_0 = G_0 \arrow[r] & G_0
\end{tikzcd}$$

Proof. This follows immediately from the definitions, as all of the structure maps of J^1G are defined using the derivatives of the structure maps of G and hence retain the groupoid structure on the base-points. □

Lemma 1.2.8. *Let G be a Lie groupoid. For any $x \in G_1$ there is some $V \in (J^1G)_1$, such that x is the base-point of V.*

Proof. For any z we can immediately find some $V' \in J^1(s : G_1 \to G_0)$ with base-point z, as s is a submersion. We can compute

$$Tt : V' \to T_{t(z)}G_0.$$

Note that as t is a submersion, we have that $Tt(V')$ must be surjective up to values in $Tt(T_zs^{-1}(\{s(z)\})) = \rho(A_{t(z)}) = (\mathcal{F}_0)_{t(z)}$. In particular, if the image $Tt(V)$ contains $(\mathcal{F}_0)_{t(x)}$, then Tt is surjective and as the dimensions agree, it is bijective.

We can now choose an arbitrary isomorphism

$$\psi : (\mathcal{F}_0)_{s(z)} \to (\mathcal{F}_0)_{t(z)}$$

and construct V as follows: First, we choose any basis $\{X_i\}_{i\in I}$ of $(\mathcal{F}_0)_{s(z)}$. Then, we lift the basis vectors according to the chosen isomorphism, meaning that we find a Y_i for each X_i such that $Ts(Y_i) = X_i$ and $Tt(Y_i) = \psi(X_i)$. Lifting some $X \in (\mathcal{F}_0)_{s(z)}$ to $Y \in T_zG_1$ such that the image is a fixed arbitrary vector of $(\mathcal{F}_0)_{t(z)}$ is possible, as any lift is defined up to terms in $T_z s^{-1}(\{s(z)\})$. The space of all lifts of X therefore describes an affine vector space $Y + T_z s^{-1}(\{s(z)\})$ with affine image under Tt, where we have already seen that it takes the for of $Tt(Y) + (\mathcal{F}_0)_{t(x)}$ but as we have that $Y + T_z s^{-1}(\{s(z)\}) \subset \mathcal{F}_1$, its image under Tt is $\mathcal{F}_0$ itself.

We now choose an extension of the basis $\{X_i\}_{i\in I}$ of $(\mathcal{F}_0)_{s(z)}$ to a basis of $T_{s(z)}G_0$. For the remaining basis vectors, there is a unique lift along Ts contained in V. We can now extend the span of the Y_i accordingly to a subspace V of $T_z G_0$ and as it is spanned by lifts of a basis of $T_{s(z)}G_0$, we have that it indeed describes $V \in J^1(s : G_1 \to G_0)$. By construction $Tt(V)$ now has the property that it contains all of $(\mathcal{F}_0)_{t(z)}$. By our previous considerations, this implies that Tt restricts to an isomorphism on the resulting V. □

Corollary 1.2.9. *Let G be a Lie groupoid. For any $x \in G_1$ there is some local bisection $\gamma \in \Gamma_{s,\mathrm{loc}}(G_0, G_1)$ such that $\gamma(s(x)) = x$.*

Proof. By Lemma 1.2.8, we have that there is some $V \subset T_x G_1$ such that $V \in (J^1G)_1$. Consider the associated isomorphism $(Ts|_V)^{-1} : T_{s(x)}G_0 \to V$. By the local form of a submersion, we find that we can extend any such section of Ts to a local section of s. This section satisfies that the map $Tt \circ T\gamma$ at the fibre of $s(x)$ is precisely induced by $Tt \circ (Ts|_V)^{-1}$, which is the concatenation of two isomorphisms, hence an isomorphism itself. So restricted to a small open around $s(x)$, $t \circ \gamma$ must be a diffeomorphism e.g. by the inverse function theorem (cf. [30], Theorem 4.5.). □

Jets and jet manifolds, especially higher jet manifolds, can be used for example when considering PDEs, as it can reduce complexity to consider formal solutions of PDEs, where the function values and formal derivative values are considered independently. However, in this context, we need a different property of jets that already appears when going to jets of the first order. Even though I was not able to find the following statement in the literature, it is known in the research community. I learned of it from Rui Fernandes during a discussion in the beginning of 2024.

Proposition 1.2.10. *Consider a Lie groupoid G acting on $p : N \to G_0$ by*

$$L : G \times^{s,p} N \to N.$$

Then there exists a canonical action $\widehat{L}$ of J^1G on $p \circ \mathrm{pr}_N : TN \to G_0$ induced by TL.

Proof. The action is defined as

$$\widehat{L}(V, X) = (TL)(Y, X),$$

where $Y \in V$ is the unique vector satisfying $(Y, X) \in T(G \times^{s,p} N)$, i.e. the unique vector such that $Ts(Y) = Tp(X)$.

The properties we need to verify for a groupoid action of J^1G on TN follow directly from the same properties for the action of G on N and the definition of the composition in J^1G: The compatibility of the source and target maps with the bundle map can be seen by unravelling

$$(p \circ \mathrm{pr}_N)\,(TL(Y, X)) = p(L(y, x)),$$

where y and x are the corresponding base-points of Y and X. The units of $(J^1G)_1$ are given by jets tangent to the unit section, meaning

$$(TL)(Y, X) = (TL)(Tu_1(Tp(X)), X) = (T\,\mathrm{id}_N)(X) = X$$

in this case.

The associativity law follows as

$$\widehat{L}(W, \widehat{L}(V, X)) = (TL)(Z, (TL)(Y, X)),$$

where $Z \in W$ is the unique vector satisfying

$$Ts(Z) = Tp((TL)(Y, X)) = Tt(Y),$$

meaning that $\widehat{\mu}(W, V)$ contains the vector $(T\mu)(Z, Y)$ as the unique vector with

$$Ts((T\mu)(Z, Y)) = Ts(Y) = Tp(X).$$

□

Lemma 1.2.11. *Let L be the action of a Lie groupoid G on $p : N \to G_0$, where p is of constant rank. Then the action $\widehat{L}$ defined in Proposition 1.2.10 extends the action L' as defined in Lemma 1.1.32 in the sense that on an element $X \in \ker(Tp)$ the action of a jet $V_x \in (J^1G)_1$ is given by*

$$\widehat{L}(V_x, X) = L'(x, X).$$

Proof. By definition, we have that

$$\widehat{L}(V_x, X) = (TL)(Y, X)$$

where $Y \in V_x$ is the unique vector satisfying $(Y, X) \in T(G \times^{s,p} N)$, i.e. the unique vector such that $Ts(Y) = Tp(X)$. We have that

$$L'(x, X) = TL(0_x, X).$$

However, under the assumption that $X \in \ker(Tp)$, we have that $Ts(Y) = 0$ and 0_x is the unique vector contained in V_x with that property. Therefore, we have by definition that

$$\widehat{L}(V_x, X) = L'(x, X).$$

□

Lemma 1.2.12. *The action of Proposition 1.2.10 is linear with respect to the vector bundle structure of $TN \to N$. In particular, there is an induced dual right action $\widehat{L}^*$ of J^1G on T^*N and a canonical extension to an action of J^1G on $\wedge^\bullet T^*N$. This extension satisfies that for any $n \in N$ and $\omega_n \in \wedge^r T^*_n N$ holds that*

$$\widehat{L}^*(V, \omega_n)(X_1, \dots, X_r) = \omega_n(\widehat{L}(V, X_1), \dots, \widehat{L}(V, X_r))$$

for any $V \in (J^1G)_1$ with base-point x such that $t(x) = n$ and any choice of $X_i \in T_{L(x^{-1},n)}N$ for $1 \leq i \leq r$.

Proof. As derivatives are linear, we have that

$$\widehat{L}(V, X) = (TL)(Y, X),$$

is a linear expression in X. By Lemma 1.1.34, we have that for every linear action there is a dual action, which is defined by precisely the given formula

$$\widehat{L}^*(V, \theta_n) = \Big(X \mapsto \theta_n(\widehat{L}(V, X))\Big)$$

for any $n \in N$ and $\theta_n \in V_n^*$, or equivalently

$$\widehat{L}^*(V,\theta_n)(X) = \theta_n(\widehat{L}(V,X)).$$

By definition, the induced action on $\wedge^\bullet T^*N$ of Lemma 1.1.38 acts on

$$\theta_n^{(1)} \wedge \cdots \wedge \theta_n^{(r)}$$

component-wise and we obtain

$$\widehat{L}^*(V,\theta_n^{(1)} \wedge \cdots \wedge \theta_n^{(r)})(X_1,\ldots,X_r) = \widehat{L}^*(V,\theta_n^{(1)}) \wedge \cdots \wedge \widehat{L}^*(V,\theta_n^{(r)})(X_1,\ldots,X_r).$$

As this depends only on expressions of the form

$$\widehat{L}^*(V,\theta_n^{(i)})(X_j) = \theta_n^{(i)}(\widehat{L}(V,X_j)),$$

we can see that the formula of the statement indeed holds true. □

Example 1.2.13. Consider the action of a Lie groupoid G on $\mathrm{id} : G_0 \to G_0$ of Example 1.1.23. Let us now study the induced action of J^1G on $\mathrm{pr}_{G_0} : TG_0 \to G_0$ constructed in Proposition 1.2.10. Consider any $V_x \in (J^1G)_1$. Then V_x acts on $X \in T_{s(x)}G_0$ by mapping it to $Tt(Y)$ for the unique $Y \in V_x$ with $Ts(Y) = X$. This way the action of V_x produces an isomorphism $T_{s(x)}G_0 \to T_{t(x)}G_0$. The inverse can be constructed from $(V_x)^{-1}$ implying the bijectivity.

The dual action is defined similarly, where we consider the isomorphism $T_{s(x)}G_0 \to T_{t(x)}G_0$ induced by V_x and dualise to a morphism $T^*_{t(x)}G_0 \to T^*_{s(x)}G_0$.

Example 1.2.14. Let G be a regular Lie groupoid with associated Lie algebra A. Consider the action of the previous Example 1.2.13 and the subvector bundle $\mathcal{F}_0 \subset TG_0$ defined by

$$(\mathcal{F}_0)_m := (T_{1_m}t)(T_{1_m}s^{-1}(m)) = \rho(A_m).$$

We can explicitly check, that if $V_x \in (J^1G)_1$ acts on $X \in \mathcal{F}_0$, the auxiliary tangent vector $Y \in V_x$ (as in the previous example) satisfies that $Y \in \mathcal{F}_1$ as $Ts(Y) = X \in \mathcal{F}_0$. This implies that $Tt(Y) \in \mathcal{F}_0$. In particular, the action of Example 1.2.13 restricts to $\mathcal{F}_0$.

When dualising, we have that the inclusion $\mathcal{F}_0 \subset TG_0$ induces a projection $T^*G_0 \to \mathcal{F}_0^*$ and the fact that the action restricts to $\mathcal{F}_0$ can be explicitly checked to be equivalent to the existence of an action of J^1G on $\mathcal{F}_0^*$ for which the projection is J^1G-equivariant.

Example 1.2.15. Let G be a regular Lie groupoid as in the previous Example 1.2.14. We can consider the quotient of vector bundles.

$$TG_0/\mathcal{F}_0 \to G_0.$$

We have seen that the action of Example 1.2.13 restricts to $\mathcal{F}_0$ and by definition the induced action of J^1G on TG_0 is linear with respect to the vector bundle structure of TG_0. Hence the J^1G-action on TG_0 descends to an action on $TG_0/\mathcal{F}_0$.

We can similarly consider

$$(TG_0/\mathcal{F}_0)^* \to G_0$$

which is isomorphic to the kernel of the projection mentioned in Example 1.2.14. By the fact that the projection is J^1G-equivariant and fibre-wise linear, it follows immediately that the J^1G-action restricts to this kernel as a subvector bundle of T^*G_0. This precisely recovers a notion dual to the action of J^1G on $TG_0/\mathcal{F}_0$ as by definition both are induced respectively by dual J^1G-actions on TG_0 and T^*G_0.

However, this bundle has a very specific interpretation: It is the normal bundle of the foliation. On this bundle, there is a well-known action of the Lie groupoid G (specifically, a representation) called the normal representation (cf. [14], pp. 177-178). As it turns out, this action of J^1G is closely related to it.

Lemma 1.2.16. *The action of J^1G on $TG_0/\mathcal{F}_0$ defined in the previous Example 1.2.15 is actually given by a G-action on $TG_0/\mathcal{F}_0$ that has been pulled back along the map of Lemma 1.2.7, i.e. the action of any $V_x \in (J^1G)_1$ is uniquely determined by its base-point $x \in G_1$. The G-action on $TG_0/\mathcal{F}_0$ is precisely given by the normal representation. The analogous statements hold for the actions of J^1G on $(TG_0/\mathcal{F}_0)^*$ and $\wedge^\bullet(TG_0/\mathcal{F}_0)^*$.*

Proof. We can explicitly check that if V_x and W_x are two jets acting on $[X] \in T_{s(x)}G_0/\mathcal{F}_0$ represented by $X \in T_{s(x)}G_0$, the auxiliary vectors Y_V and Y_W as in Example 1.2.13 satisfy that

$$Ts(Y_V - Y_W) = X - X = 0.$$

This implies that $Y_V - Y_W \in \mathcal{F}_1$ and therefore that

$$Tt(Y_V) \in T_{t(x)}G_0/\mathcal{F}_0 = Tt(Y_W) \in T_{t(x)}G_0/\mathcal{F}_0$$

So the descended action of V_x on $[X]$ actually only depends on the base-point x of V_x. This means, that the action can be understood as an action of G that has been pulled back to an action of J^1G. The description of an element x acting by any choice of jet V_x (which can be represented by an s-section) precisely agrees with the geometric description of the normal representation in [14] (cf. p. 178).

The analogous statement for the action of J^1G on $(TG_0/\mathcal{F}_0)^*$ follows immediately from the observation of Example 1.2.15 that the two notions are dual and the fact that the extension of an action is defined by the simultaneous action on all tensor factors. □

Definition 1.2.17. Consider the left action L of a Lie groupoid G on $p^i_q : G_q \to G_0$ induced of Example 1.1.23 (for $q = 0$) or Example 1.1.24 (for $q \geq 1$). Furthermore, let $\widehat{L}$ be the J^1G-action on $Tp^i_q : TG_q \to TG_0$ constructed from L according to Proposition 1.2.10. Then we will denote

$$(\dot{D}^i_{V,q})_*(\bullet) = \widehat{L}(V, \bullet).$$

Analogously, let $\widehat{L}^*$ be the J^1G-action constructed from L according to Lemma 1.2.12. We will denote

$$(\dot{D}^i_{V,q})^*(\bullet) = \widehat{L}^*(V, \bullet).$$

Lemma 1.2.18. *Let G be a Lie groupoid left acting by L_1 on $p_1 : N_1 \to G_0$ and by L_2 on $p_2 : N_2 \to G_0$. Furthermore, let $f \in C^\infty(N_1, N_2)$ be a G-equivariant map, i.e. a map for which*

$$f(L_1(g, n)) = L_2(g, f(n)).$$

Then the induced actions $\widehat{L_1}$ and $\widehat{L_2}$ of Proposition 1.2.10 satisfy that Tf is J^1G-equivariant.

Proof. The property that $p_2 \circ \mathrm{pr}_{N_2} \circ Tf = p_1 \circ \mathrm{pr}_{N_1}$ is immediate as Tf is a map of vector bundles covering f. We have seen in the proof of Proposition 1.2.10 that the action is of the form

$$\widehat{L_1}(V, X) = (TL_1)(Y, X)$$

such that

$$(Tf)(\widehat{L_1}(V, X)) = (T(f \circ L_1))(Y, X) = (T(L_2 \circ (\mathrm{id}, f)))(Y, X) = (TL_2)(Y, (Tf)(X)).$$

As the property that $Y \in V$ is retained and $p_2(\mathrm{pr}_{N_2}(Tf(X))) = p_1(\mathrm{pr}_{N_1}(X))$ the latter expression agrees with
$$\widehat{L_2}(V, Tf(X)),$$
such that Tf is indeed J^1G-equivariant. $\square$

Example 1.2.19. Consider the diagonal action at the i-th pull-back L of a Lie groupoid G on G_q as described in Example 1.1.24 for some $q \geq 1$ and $0 \leq i \leq q$. Consider the projection $\mathrm{pr}_j : G_q \to G_1$ to the j-th component. Then we have already seen in Example 1.1.25 that if $j \neq i$ and $j \neq i+1$ we have that pr_j is G-invariant under L and for $j = i$, we have that pr_j is G-equivariant with respect to L and the action L_1 of G on G_1 constructed in Example 1.1.24 for $i = 1$. Similarly, for $j = i+1$, we have that pr_j is G-equivariant with respect to L and the action L_2 of G on G_1 constructed in Example 1.1.24 for $i = 0$.

Therefore Lemma 1.2.18 implies that if $j \neq i$ and $j \neq i+1$ we have that $T\,\mathrm{pr}_j$ is J^1G-invariant with respect to $\widehat{L}$. Furthermore for $j = i$, we have that $T\,\mathrm{pr}_j$ is J^1G-equivariant with respect to $\widehat{L}$ and the action $\widehat{L}_1$ and for $j = i+1$, we have that $T\,\mathrm{pr}_j$ is J^1G-equivariant with respect to $\widehat{L}$ and the action $\widehat{L}_2$.

Example 1.2.20. Consider any $q \geq 1$ and $0 \leq i \leq q$. Let L be the diagonal action at the i-th pull-back of G on G_q of Example 1.1.24. Let $\widehat{L}$ be the associated action of J^1G on TG_q. We can compute for the action L that
$$p_q^j((z_1, \ldots, \mu(z_i, x^{-1}), \mu(x, z_{i+1}), \ldots, z_q)) = t(z_j) = p_q^j((z_1, \ldots, z_q)),$$
so that for the action L of G on G_q the map p_q^j is L-invariant. By Lemma 1.2.18 this implies that $Tp_q^j : TG_q \to TG_0$ for $j \neq i$ is then J^1G-invariant.

Example 1.2.21. Let G be a Lie groupoid and H be a subgroupoid as in Example 1.1.30. Consider the G-action L defining Ad_x^H as described in that example (and Definition 1.1.39). This action induces a J^1G-action $\widehat{L}$ on TH_q as constructed in Proposition 1.2.10.

We can compute in analogy to Example 1.1.33 that
$$L(x, 1_{s(x)}) = (\mu(x, 1_{s(x)}, x^{-1}), \ldots, \mu(x, 1_{s(x)}, x^{-1})) = 1_{t(x)},$$
such that $\widehat{L}$ can be restricted to
$$\widehat{L}|_{(u_q)^*TH_q} : (J^1G)_1 \times^{s, p_q^i \circ \mathrm{pr}_H} (u_q)^*TH_q \to (u_q)^*TH_q.$$
However, we know that at the unit section of H we have a splitting
$$(u_q)^*TH_q \cong TH_0 \oplus (A_H)^{\oplus q},$$
such that we obtain two maps
$$\widehat{L}|_{TH_0} : (J^1G)_1 \times^{s, \mathrm{pr}_{G_0}} TH_0 \to (u_q)^*TH_q$$
and
$$\widehat{L}|_{(A_H)^{\oplus q}} : (J^1G)_1 \times^{s, \mathrm{pr}_{G_0}} (A_H)^{\oplus q} \to (u_q)^*TH_q.$$

If we consider the first map, the property that L restricts to the unit section immediately implies that
$$TL : TG_1 \times^{Ts, Tp_q^i} TH_q \to TH_q$$

can in domain and target be restricted to $TH_0 \subset TH_q$ where the inclusion is given by Tu_q. Hence the first map $\widehat{L}|_{TH_0}$ also restricts to TH_0 and describes a J^1G-action as the axioms are retained by restriction.

For the second map, we have already seen explicitly in Example 1.1.33 that TL restricts to a map on $(A_H)^{\oplus q}$ induced by a G-action. When compared to the induced jet action $\widehat{L}$, we may note that Tp_q^i vanishes on A_H, such that every jet $V \in (J^1G)_1$ acts on $(A_H)^{\oplus q}$ only by

$$(V, X) \mapsto (TL)(0_x, X),$$

where x is the base-point of V. So the action of V on $(A_H)^{\oplus q}$ agrees with that of $T(L(x, \bullet))$ for its base-point. Therefore, we obtain that $\widehat{L}$ is actually the pull-back of the adjoint G-action on $(A_H)^{\oplus q}$ which by Lemma 1.1.37 is precisely induced by the adjoint G-action of Example 1.1.33 of G on A_H acting on each component of the direct sum simultaneously.

Definition 1.2.22. Let G be a Lie groupoid and consider the adjoint action action L of G on H_q for any subgroupoid $H \subset G$ with $s \equiv t$ as defined in Example 1.1.30. Consider the associated induced action $\widehat{L}$ as defined in Proposition 1.2.10. We will denote the action of J^1G on TH_q as

$$\widehat{\mathrm{Ad}}^H_V(\bullet) = \widehat{L}(V, \bullet).$$

If $H = \mathrm{Iso}^0(G)$, we will also use the notation

$$\widehat{\mathrm{Ad}}^0_V(\bullet) = \widehat{L}(V, \bullet).$$

We will denote the adjoint action of J^1G on A_H by

$$\widehat{\mathrm{Ad}}_V(\bullet) = \widehat{L}(V, \bullet)|_{(u_1)^*TH_1}$$

and the dual coadjoint action of J^1G on A_H^* by

$$\widehat{\mathrm{Ad}}^*_V(\bullet) = \widehat{L}^*(V, \bullet)|_{(u_1)^*T^*H_1}$$

Example 1.2.23. Using this notation, we can write the result for $q = 1$ of Example 1.2.21 as

$$\widehat{\mathrm{Ad}}_V \equiv (\mathrm{Ad}_x)_*,$$

where both expressions are defined as maps $(A_H)_{s(x)} \to (A_H)_{t(x)}$ for x the base-point of V.

One of the ways to think about $J^1(s : G_1 \to G_0)$ is as a bundle over the space G_1 where the bundle map is the projection to the corresponding base-points. The fibre over $x \in G_1$ can then be interpreted as the space of local sections of s through x, where we identify those sections which are tangent to each other. Choosing jets corresponding to sections or sections corresponding to jets is a very natural construction from this point of view.

We could therefore wonder if

$$(\widetilde{D}^i_{\gamma,q})_* \equiv (\dot{D}^i_{V,q})_*$$

when we choose the appropriate V and consider the appropriate domains, as both are induced maps on TG_q. We will find that this actually holds, with the caveat, that we can (currently) only obtain an appropriate V for bisections.

Lemma 1.2.24. *Let G be a Lie groupoid. Consider a local bisection $\gamma : U \subset G_0 \to G_1$ and any $z \in G_q$ with $p^i_q(z) \in U$. Then the space*

$$V = (T_{p^i_q(z)}\gamma)(T_{p^i_q(z)}G_0) \subset T_{\gamma(p^i_q(z))}G_1$$

is an element of $(J^1G)_1$.

Proof. We need to show that on $V = T_{p^i_q(z)}\gamma(T_{p^i_q(z)}G_0)$ the maps Ts and Tt both restrict to isomorphisms. However, we know that as a bisection $T\gamma$ postcomposed with Ts is given by the identity on TG_0 and the composition with Tt corresponds to the derivative of a diffeomorphism. This implies the statement. □

However, while bisections do correspond to elements of the jet groupoid, we have that our original definition of $\widetilde{D}^i_{\gamma,q}$ (or more generally the induced action $L_\gamma : p^{-1}(U) \to N$ of Definition 1.1.45) was also applicable to sections that do not induce diffeomorphisms. This difference will be meaningful in some of the proofs in the later chapters. If we consider

$$V = (T_{p^i_q(z)}\gamma)(T_{p^i_q(z)}G_0) \subset T_{\gamma(p^i_q(z))}G_1$$

for these sections, we only obtain that they are jets of the source map.

The solution for this is quite simple: Just as we have already seen that on the space of jets of local sections of s we have a multiplication-like structure, even if we don't obtain an actual group, we will be able to define an action-like structure induced by a groupoid action, where jets of the source map act on elements of TN for the chosen bundle.

Definition 1.2.25. Let G be a Lie groupoid. Consider the jet manifold

$$J^1(s : G_1 \to G_0) = \left\{V_x \subset T_xG_1 \text{ s.t. } T_xs : V_x \to T_{s(x)}G_0 \text{ is an iso}\right\}.$$

Given a left action L of a Lie groupoid G on $p : N \to G_0$, we extend the induced action of Proposition 1.2.10 to a map

$$\widehat{L} : J^1(s : G_1 \to G_0) \times^{s \circ \mathrm{pr}_{G_1}, p \circ \mathrm{pr}_N} TN \to TN.$$

On a sub-vector space $V \subset T_xG_1$ which satisfies $V \in J^1(s : G_1 \to G_0)$ and $X \in T_nN$ with $p(n) = s(x)$ it is defined as

$$\widehat{L}(V, X) = (TL)(Y, X),$$

where $Y \in V$ is the unique vector satisfying $Ts(Y) = Tp(X)$.

We denote

$$\begin{aligned}(\dot{D}^i_{V,q})_* : T_zG_q &\to T_{D^i_{x,q}(z)}G_q\\ X &\mapsto \widehat{L}(V, X)\end{aligned}$$

extending the definition of Definition 1.2.17.

Of course, the claims contained in this definition of extending some previous definitions needs to be proven.

Lemma 1.2.26. *The map of Definition 1.2.25 is well-defined and indeed extends the map of Proposition 1.2.10. It is compatible with the operation $\widehat{\mu}$ of Definition 1.2.2 in the sense that the unitality and associativity axioms hold.*

Proof. If we consider the proof of Proposition 1.2.10, we can directly see that the definitions agree. The proof of well-definedness follows the same arguments and is completely analogous. □

Lemma 1.2.27. *Consider a Lie groupoid G left acting on some $p : N \to G_0$ by L. Consider the extended induced action $\widehat{L}$ of $J^1(s : G_1 \to G_0)$ on TN. Then the statements of Lemma 1.2.11, Lemma 1.2.12, Lemma 1.2.16 and Lemma 1.2.18 hold analogously.*

Proof. Considering each proof, we have never relied on the fact that the $V_x \in (J^1G)_1$ were invertible and only employed the definition of

$$\widehat{L}(V_x, X) = TL(Y, X),$$

which coincides with the definition of the extension, even if $V_x \in J^1(s : G_1 \to G_0)$ is not invertible. □

Lemma 1.2.28. *Let G be a Lie groupoid. Consider a local section $\gamma : U \subset G_0 \to G_1$ and any $z \in G_q$ with $p^i_q(z) \in U$. Then the space*

$$V = (T_{p^i_q(z)}\gamma)(T_{p^i_q(z)}G_0) \subset T_{\gamma(p^i_q(z))}G_1$$

is actually an element of $J^1(s : G_1 \to G_0)$. For a Lie groupoid action $L : G \times^{s,p} N \to N$ and the associated notions

$$\widetilde{L}_\gamma : p^{-1}(U) \to N$$

of Definition 1.1.45 and

$$\widehat{L} : J^1(s : G_1 \to G_0) \times^{s\circ \mathrm{pr}_{G_1}, p\circ \mathrm{pr}_N} TN$$

of Definition 1.2.25, we obtain that at any given point $(x, n) \in G \times^{s,p} N$ holds

$$(\widetilde{L}_\gamma)_* \equiv \widehat{L}$$

as maps of the form

$$T_nN \to T_{L(x,n)}N.$$

In particular

$$(\widetilde{D}^i_{\gamma,q})_* \equiv (\dot{D}^i_{V,q})_*$$

as maps of the form

$$T_zG_q \to T_{D^i_{x,q}(z)}G_q.$$

Proof. We need to show that on $V = T_{p^i_q(z)}\gamma(T_{p^i_q(z)}G_0)$ the map Ts indeed restricts to an isomorphism. However, we know that as a section, $T\gamma$ postcomposed with Ts is given by the identity on TG_0. Furthermore, when defining

$$\widehat{L}(V, X) = (TL)(Y, X),$$

we characterised Y as the element of V with $Ts(Y) = Tp^i_q(X)$. However, we have that $T\gamma(Tp^i_q(X)) \in V$ by definition and $Ts(T\gamma(Tp^i_q(X))) = Tp^i_q(X)$, so actually

$$Y = T\gamma(Tp^i_q(X))$$

and

$$\widehat{L}(V, X) = (TL)(T\gamma(Tp^i_q(X)), X),$$

meaning that for fixed V it is precisely the derivative at the base-point x of X of

$$z \mapsto L(\gamma(p^i_q(z)), z),$$

which is precisely the definition of $\widetilde{L}_\gamma$. □

Corollary 1.2.29. *Let G be a Lie groupoid. Consider a local section $\gamma : U \subset G_0 \to G_1$. We can compute the entire map $(\widetilde{D}^i_{\gamma,q})_*$ in terms of the action of $J^1(s : G_1 \to G_0)$ on $p^i_q : G_q \to G_0$ by evaluating $(\dot{D}^i_{V,q})_*$ for the family of jets corresponding to γ. If γ is a bisection, the action of J^1G on $p^i_q : G_q \to G_0$ is sufficient.*

Analogously, for any Lie subgroupoid $H \subset G$ with $H_0 = G_0$ for which $s \equiv t$ and any $q \in \mathbb{N}_0$ we can compute the entire map $(\widetilde{\mathrm{Ad}}^H_\gamma)_*$ from the action of $J^1(s : G_1 \to G_0)$ on $p^i_q : H_q \to G_0$ by evaluating $\widehat{\mathrm{Ad}}^H_V$ for the family of jets corresponding to γ. If γ is a bisection, the action of J^1G on $p^i_q : H_q \to G_0$ is sufficient.

Proof. By Lemma 1.2.28, we know that for any $(x, z) \in G_1 \times^{s,p^i_q} G_q$ holds

$$(\widetilde{D}^i_{\gamma,q})_* \equiv (\dot{D}^i_{V,q})_*$$

as maps

$$T_zG_q \to T_{D^i_{x,q}(z)}G_q$$

for $V = (T_{p^i_q(z)}\gamma)(T_{p^i_q(z)}G_0)$. This means that the entire map $(\widetilde{D}^i_{\gamma,q})_*$ can be computed by evaluating $(\dot{D}^i_{V,q})_*$ on $J^1(s : G_1 \to G_0)$. The case of $\widetilde{\mathrm{Ad}}^H_\gamma$ can be argued analogously. By Lemma 1.2.24, we know that if γ is a bisection, the corresponding jets are indeed elements of $(J^1G)_1$. □

Corollary 1.2.30. *Let G be a Lie groupoid. Consider two local sections γ and γ' around m with $\gamma(m) = \gamma'(m)$ and*

$$T_m\gamma = T_m\gamma' : T_mG_0 \to T_{\gamma(m)}G_1 = T_{\gamma'(m)}G_1$$

Then for any $z \in G_q$ with $p^i_q(z) = m$ we have that

$$(\widetilde{D}^i_{\gamma,q})_* \equiv (\widetilde{D}^i_{\gamma',q})_*$$

as maps

$$T_zG_q \to T_{D^i_{\gamma(m),q}(z)}G_q = T_{D^i_{\gamma'(m),q}(z)}G_q.$$

Analogously, if $H \subset G$ is a Lie subgroupoid as in Example 1.1.30, in particular satisfying $s \equiv t$ on H_1, we have that at any $z \in H_q$ with $p^i_q(z) = m$ holds that

$$(\widetilde{\mathrm{Ad}}^H_\gamma)_* \equiv (\widetilde{\mathrm{Ad}}^H_{\gamma'})_*$$

as maps

$$T_zH_q \to T_{D^i_{\gamma(m),q}(z)}H_q = T_{D^i_{\gamma'(m),q}(z)}H_q.$$

Proof. By assumption

$$V = T_m\gamma(T_mG_0) = T_m\gamma'(T_mG_0)$$

and by Lemma 1.2.28 we have

$$(\widetilde{D}^i_{\gamma,q})_* \equiv (\dot{D}^i_{V,q})_* \equiv (\widetilde{D}^i_{\gamma',q})_*.$$

The case of $(\widetilde{\mathrm{Ad}}^H_\gamma)_* \equiv (\widetilde{\mathrm{Ad}}^H_{\gamma'})_*$ can be argued analogously. □

Now that we have established the definition in the most general way possible, we will turn to studying properties of these actions. Let us first note a property that will look unassuming for now, but will be key later.

Lemma 1.2.31. *Let G be a Lie groupoid and $Y \in T_zG_q$ be a tangent vector such that $(Tp_q^i)(Y) = 0$. Given $x \in G_1$ with $s(x) = p_q^i(z)$ and some $V \in J^1(s : G_1 \to G_0)$ with base-point x, then*

$$(\dot{D}^i_{V,q})_*(Y) \equiv (D^i_{x,q})_*Y,$$

where $D^i_{x,q}$ is seen as a diffeomorphism from $(p_q^i)^{-1}(s(x))$ to $(p_q^i)^{-1}(t(x))$. The latter is an embedded submanifold of G_q and therefore $(D^i_{x,q})_*Y$ can be seen as an element of TG_q.

Furthermore if $Y \in \ker(Tp_q^j)$ for some $0 \leq j \leq q$, then so is $(\dot{D}^i_{V,q})(Y)$. In particular, if $Y \in \bigcap_{0 \leq i \leq q} \ker(Tp_q^i)$, so is $(\dot{D}^i_{V,q})(Y)$.

Proof. By definition, we have that

$$(\dot{D}^i_{V,q})_*(Y) = (TL)(Z, Y),$$

where $Z \in V$ is the unique vector with $Ts(Z) = Tp_q^i(Y)$ and L is the action through which $L(x, \bullet) = D^i_{x,q}(\bullet)$ is defined. As this term vanishes by assumption, we have that $Z = 0_x$. Hence

$$(TL)(Z, Y) = (TL)(0_x, Y) = (T(L(x, \bullet)))(Y) = (TD^i_{x,q})(Y).$$

This automatically implies that $Tp_q^i((\dot{D}^i_{V,q})_*(Y)) = 0$, as the image of $D^i_{x,q}$ projects to a point under p_q^i. Furthermore we have that the maps Tp_q^j are J^1G-invariant by Example 1.2.20 if $i \neq j$, meaning that we retain

$$Tp_q^j((D^i_{V,q})(Y)) = Tp_q^j(Y) = 0.$$

□

Other very notable properties are the behaviour of the actions in relation to the face and degeneracy maps d_i and s_j. They will span most of the remainder of this section. Similar to Lemma 1.1.29 implying Corollary 1.1.49, we can start by noting the following immediate consequence.

Corollary 1.2.32. *Let G be a Lie groupoid. We have that*

$$(\dot{D}^0_{V,0})_* \circ (d_0)_* = (d_0)_* \circ (\dot{D}^1_{V,1})_*$$

on T_zG_1 whenever the base-point x of V satisfies $s(x) = s(z)$ and

$$(\dot{D}^0_{V,0})_* \circ (d_1)_* = (d_1)_* \circ (\dot{D}^0_{V,1})_*.$$

on T_zG_1 whenever the base-point x of V satisfies $s(x) = t(z)$.

The proof can be found in A.1.1. This is by far not the only relation between actions and simplicial maps.

Lemma 1.2.33. *Let G be a Lie groupoid and $x \in G_1$. For any $q \in \mathbb{N}_0$ and $0 \leq l \leq q+1$ as well as $0 \leq i \leq q+1$ we have that on $(p^l_{q+1})^{-1}(\{s(x)\})$ holds that*

$$d_i \circ D^l_{x,q+1} = \begin{cases} D^l_{x,q} \circ d_i & \text{if } i > l \\ D^{(l-1)}_{x,q} \circ d_i & \text{if } i < l \\ d_i & \text{if } i = l. \end{cases}$$

Furthermore, for any $q \in \mathbb{N}_0$ with $q \geq 1$ and $0 \leq l \leq q$ as well as $0 \leq i \leq q+1$ we have that on $(p^l_{q+1} \circ d_i)^{-1}(\{s(x)\})$ holds that

$$D^l_{x,q} \circ d_i = \begin{cases} d_i \circ D^l_{x,q+1} & \text{if } i > l \\ d_i \circ D^{(l+1)}_{x,q+1} & \text{if } i \leq l. \end{cases}$$

Also, for any $q \in \mathbb{N}_0$ and $0 \leq l \leq q$ as well as $0 \leq j \leq q$ we have that on $(p^l_q)^{-1}(\{s(x)\})$ holds that

$$s_j \circ D^l_{x,q} = \begin{cases} D^l_{x,q+1} \circ s_j & \text{if } j > l \\ D^{l+1}_{x,q+1} \circ D^l_{\gamma,q+1} \circ s_j & \text{if } j = l \\ D^{(l+1)}_{x,q+1} \circ s_j & \text{if } j < l. \end{cases}$$

Lastly, for any $q \in \mathbb{N}_0$ and $0 \leq l \leq q+1$ as well as $0 \leq j \leq q$ we have that on $(p^l_q \circ s_j)^{-1}(\{s(x)\})$ holds that

$$D^l_{x,q+1} \circ s_j = \begin{cases} s_j \circ D^l_{x,q} & \text{if } j > l \\ s_j \circ D^{(l-1)}_{x,q} & \text{if } j+1 < l. \end{cases}$$

The proof can be found in A.1.2.

Corollary 1.2.34. *Let G be a Lie groupoid and $\gamma \in \Gamma_{s,\mathrm{loc}}(G_0, G_1)$. For any $q \in \mathbb{N}_0$ and $0 \leq l \leq q+1$ as well as $0 \leq i \leq q+1$ we have that on the respective domains of definition it holds that*

$$d_i \circ D^l_{\gamma,q+1} \equiv \begin{cases} D^l_{\gamma,q} \circ d_i & \text{if } i > l \\ D^{(l-1)}_{\gamma,q} \circ d_i & \text{if } i < l \\ d_i & \text{if } i = l. \end{cases}$$

and on the respective domains of definition it holds that

$$D^l_{\gamma,q} \circ d_i \equiv \begin{cases} d_i \circ D^l_{\gamma,q+1} & \text{if } i > l \\ d_i \circ D^{(l+1)}_{\gamma,q+1} & \text{if } i \leq l. \end{cases}$$

The statements on the relations between $\widetilde{D}^l_{\gamma,q}$ and s_j are analogous to those of Lemma 1.2.33.

The proof can be found in A.1.3.

Corollary 1.2.35. *Let G be a Lie groupoid and $V \in (J^1G)_1$ with base-point $x \in G_1$. For any $q \in \mathbb{N}_0$ and $0 \leq l \leq q+1$ as well as $0 \leq i \leq q+1$ we have that on T_zG_q for z satisfying $p^j_q(z) = s(x)$ holds that*

$$(d_i)_* \circ (\dot{D}^l_{V,q+1})_* = \begin{cases} (D^l_{V,q})_* \circ (d_i)_* & \text{if } i > l \\ (D^{(l-1)}_{V,q})_* \circ (d_i)_* & \text{if } i < l \\ (d_i)_* & \text{if } i = l, \end{cases}$$

and on T_zG_q for z satisfying $p^j_q(d_i(z)) = s(x)$ holds that

$$(\dot{D}^l_{V,q})_* \circ (d_i)_* = \begin{cases} (d_i)_* \circ (D^l_{V,q+1})_* & \text{if } i > l \\ (d_i)_* \circ (D^{(l+1)}_{V,q+1})_* & \text{if } i \leq l. \end{cases}$$

The statements on the relations between $(\dot{D}^l_{V,q})_*$ and s_j are analogous to those of Lemma 1.2.33.

The proof can be found in A.1.4.

Lemma 1.2.36. *Let G be a Lie groupoid and $x \in G_1$. For any $q \in \mathbb{N}$ and $0 \leq i \leq q$ we have that on $(p_q^i)^{-1}(\{s(x)\})$ holds that*

$$p_q^i \circ D_{x,q}^i = D_{x,0}^0 \circ p_q^i.$$

The analogous statements

$$p_q^i \circ \widetilde{D}_{\gamma,q}^i \equiv \widetilde{D}_{\gamma,0}^0 \circ p_q^i$$

and

$$(p_q^i) \circ (\dot{D}_{V,q}^i)_* = (\dot{D}_{V,0}^0)_* \circ (p_q^i)_*$$

hold as well on the appropriate spaces.

The proof can be found in A.1.5.

Lemma 1.2.37. *Let G be a Lie groupoid and $x \in G_1$ as well as $y \in G_1$. For any $q \in \mathbb{N}$ and $0 \leq i \leq q$ and for any $0 \leq j \leq q$ with $i \neq j$ we have that on $(p_q^i)^{-1}(\{s(x)\}) \cap (p_q^j)^{-1}(\{s(y)\})$ holds that*

$$D_{y,q}^j \circ D_{x,q}^i = D_{x,q}^i \circ D_{y,q}^j.$$

The analogue statements

$$\widetilde{D}_{\beta,q}^j \circ \widetilde{D}_{\gamma,q}^i \equiv \widetilde{D}_{\gamma,q}^i \circ \widetilde{D}_{\beta,q}^j.$$

and

$$(\dot{D}_{W,q}^j)_* \circ (\dot{D}_{V,q}^i)_* = (\dot{D}_{V,q}^i)_* \circ (\dot{D}_{W,q}^j)_*.$$

hold as well on the appropriate spaces.

The proof can be found in A.1.6.

Lemma 1.2.38. *Let G be a Lie groupoid, $H \subset G$ a Lie subgroupoid as in Example 1.1.30 and $x \in G_1$. For any $q \in \mathbb{N}$ we have that on $H_q \cap (p_q^i)^{-1}(\{s(x)\})$ holds that*

$$\mathrm{Ad}_{x,q}^H \equiv D_{x,q}^q \circ \cdots \circ D_{x,q}^1 \circ D_{x,q}^0.$$

The analogue statements

$$\widetilde{\mathrm{Ad}}_{\gamma,q}^H \equiv \widetilde{D}_{\gamma,q}^q \circ \cdots \circ \widetilde{D}_{\gamma,q}^1 \circ \widetilde{D}_{\gamma,q}^0.$$

and

$$\widehat{\mathrm{Ad}}_{V,q}^H \equiv \dot{D}_{x,q}^q \circ \cdots \circ \dot{D}_{x,q}^1 \circ \dot{D}_{x,q}^0.$$

hold as well on the appropriate spaces.

The proof can be found in A.1.7.

Lemma 1.2.39. *Let G be a Lie groupoid and $x \in G_1$. For any $q \in \mathbb{N}$ we have that on $(\mathrm{Iso}^0(G))_0 \cap (p_0^0)^{-1}(\{s(x)\})$ holds that*

$$u_q \circ D_{x,0}^0 = D_{x,q}^q \circ \cdots \circ D_{x,q}^1 \circ D_{x,q}^0 \circ u_q.$$

The analogue statements

$$u_q \circ \widetilde{D}_{\gamma,0}^0 = \widetilde{D}_{\gamma,q}^q \circ \cdots \circ \widetilde{D}_{\gamma,q}^1 \circ \widetilde{D}_{\gamma,q}^0 \circ u_q$$

and

$$(u_q)_* \circ (\dot{D}_{V,0}^0)_* = (\dot{D}_{V,q}^q)_* \circ \cdots \circ (\dot{D}_{V,q}^1)_* \circ (\dot{D}_{V,q}^0)_* \circ (u_q)_*$$

hold as well on the appropriate spaces

The proof can be found in A.1.8.

This concludes the list of rules we will concern ourselves with, although there are likely many more. We have that these many interchanging rules make a lot of computations immediately easier.

Lemma 1.2.40. *Let G be a Lie groupoid and $\mathcal{F}_q$ be defined as in Lemma 1.1.20 for $\mathcal{F}_0$ and $\mathcal{F}_1$ as defined in Lemma 1.1.18. Consider any $V_x \in (J^1G)_1$ with base-point x and $X \in \mathcal{F}_q$ with base-point z satisfying that $s(x) = p^i_q(z)$. Then*

$$(\dot{D}^i_{V_x,q})_* X \in \mathcal{F}_q.$$

Remark. For $q = 0$, we have already explicitly noted this in Example 1.2.14.

Proof. We have that

$$Tp^i_q((\dot{D}^i_{V_x,q})_*(X)) = (\dot{D}^0_{V_x,q})_*(Tp^i_q(X))$$

by Lemma 1.2.36. By definition (and in analogy to Example 1.2.14), this is defined to be

$$Tt(Y)$$

for the unique $Y \in V_x$ satisfying $Ts(Y) = Tp^i_q(X)$. But as $X \in \mathcal{F}_q$ we have $Tp^i_q(X) \in \mathcal{F}_0$, which in turn implies $Y \in \mathcal{F}_1$ and $Tt(Y) \in \mathcal{F}_0$. Therefore, $(\dot{D}^i_{V_x,q})_*(X)$ maps to $\mathcal{F}_0$ under Tp^i_q and hence $(\dot{D}^i_{V_x,q})_*(X) \in \mathcal{F}_q$. □

1.3 Basic Forms

Aim of this section: Proposition 1.3.14

Let G be a Lie groupoid. The forms $\omega \in \Omega(G_q)$ which are J^1G-invariant are precisely the basic forms on G_q. Hence we can identify

$$(\Omega(G_q))^{J^1G} \cong \Omega_{\text{bas}}(G_0).$$

The definitions and properties previously established will give us a new understanding of the basic forms on the object space G_0 of a Lie groupoid G, especially when interpreted as forms on G_q for varying levels of the nerve. It is not relevant to the rest of the thesis and can be seen as more of a nice result motivating a few helpful observations at a later point.

Definition 1.3.1. Let G be a Lie groupoid. We can define the **basic forms** on G_0 as

$$\Omega_{\text{bas}}(G_0) := \{\omega \in \Omega(G_0) \text{ s.t. } (d_0)^*(\omega) = (d_1)^*(\omega)\}.$$

Lemma 1.3.2. *Let G be a Lie groupoid and $\mathcal{F}_0$ be the (not necessarily smooth) bundle of vector spaces over G_0 defined by*

$$(\mathcal{F}_0)_m = (T_{1_m}t)(T_{1_m}s^{-1}(m)) = \rho(A_m).$$

We have that the basic forms on G_0 can be equivalently described by

$$\Omega_{\text{bas}}(G_0) = \left\{ \omega \in \Omega(G_0) \text{ s.t. } \begin{array}{r} \iota_X\omega = 0 \text{ for } X \in \mathcal{F}_0 \text{ and where defined:} \\ (\widetilde{D}^0_{\gamma,0})^*\omega \equiv \omega \text{ for } \gamma \in \Gamma_{s,\text{loc}}(G_0, G_1) \end{array} \right\}$$

or

$$\Omega_{\text{bas}}(G_0) = \left\{ \omega \in \Omega(G_0) \text{ } s.t. \begin{array}{r} \iota_X\omega = 0 \text{ for } X \in \mathcal{F}_0 \text{ and where defined:} \\ (\widetilde{D}^0_{\gamma,0})^*\omega \equiv \omega \text{ for } \gamma \in \Gamma_{s,t,\text{loc}}(G_0, G_1) \end{array} \right\}.$$

Here, we use $\Gamma_{s,t,\text{loc}}(G_0, G_1)$ *to describe the space of local bisections.*

Remark. The first condition is actually implied by the second, but its useful to note explicitly. Also note that in both conditions, we only concern ourselves with a single action $(\widetilde{D}^0_{\gamma,0})^*$ as for $q = 0$ our family $\{(\widetilde{D}^i_{\gamma,q})^*\}$ only consists of a single element.

Proof. For this proof, we will need to show the following implications: Firstly, we will show that $\omega \in \Omega_{\text{bas}}(G_0)$ according to Definition 1.3.1 implies that

$$\iota_X\omega = 0$$

for $X \in \mathcal{F}_0$ and

$$(\widetilde{D}^0_{\gamma,0})^*\omega \equiv \omega$$

for $\gamma \in \Gamma_{s,\text{loc}}(G_0, G_1)$. That this implies the same for $\gamma \in \Gamma_{s,t,\text{loc}}(G_0, G_1)$ is trivial. So if we then show that $\iota_X\omega = 0$ for $X \in \mathcal{F}_0$ and $(\widetilde{D}^0_{\gamma,0})^*\omega \equiv \omega$ for $\gamma \in \Gamma_{s,t,\text{loc}}(G_0, G_1)$ implies that $\omega \in \Omega_{\text{bas}}(G_0)$, all of the definitions are equivalent.

Let us start with the fact that $(d_0)^*\omega = (d_1)^*\omega$ implies that $\iota_X\omega = 0$ for $X \in \mathcal{F}_0$. For this, consider that

$$\iota_X\omega = (u_1)^*(d_i)^*\iota_X\omega$$

for both $i = 0$ and $i = 1$. If we consider the base-point $m \in G_0$ of X, we have by definition that there is $Y \in A_m$ such that $\rho(A) = Y$, so if we understand $A_m \subset T1_m G_1$ this translates to $Ts(Y) = 0$ and $Tt(Y) = X$.

As d_0 and d_1 are precisely defined as s and t, this implies

$$(u_1)^*(d_1)^*\iota_X\omega = (u_1)^*\iota_Y(d_1)^*\omega = (u_1)^*\iota_Y(d_0)^*\omega = (u_1)^*(d_0)^*\iota_0\omega,$$

however $\iota_0\omega = 0$, and this is retained under the pull-back.

Now consider $\gamma \in \Gamma_{s,\text{loc}}(G_0, G_1)$. If we have ω with $(d_0)^*\omega = (d_1)^*\omega$, we know that also

$$(\gamma)^*(d_0)^*\omega = (\gamma)^*(d_1)^*\omega.$$

But as $d_0 = s$, we have that

$$(\gamma)^*(d_0)^*\omega = \omega,$$

while by definition

$$(\gamma)^*(d_1)^*\omega = (\widetilde{D}^0_{\gamma,0})^*\omega,$$

which indeed proves that ω also satisfies $(\widetilde{D}^0_{\gamma,0})^*\omega = \omega$. As noted in the beginning, this concludes the first major step of the proof.

What remains to be seen is the converse, but when restricting ourselves to local bisections, i.e. that $(d_0)^*\omega = (d_1)^*\omega$ is automatically implied by $\iota_X\omega = 0$ for $X \in \mathcal{F}_0$ and $(\widetilde{D}^0_{\gamma,0})^*\omega \equiv \omega$ for $\gamma \in \Gamma_{s,t,\text{loc}}(G_0, G_1)$. Firstly note that using the notation of Lemma 1.1.18 both pull-backs $(d_i)^*\omega$ for $i = 0$ and $i = 1$ satisfy

$$\iota_Y(d_i)^*\omega = 0 \text{ for } Y \in \mathcal{F}_1$$

as $(d_i)_*(Y) \in \mathcal{F}_0$ for $Y \in \mathcal{F}_1$. Due to the second property we have that both $(d_0)^*\omega$ and $(d_1)^*\omega$ are invariant under $(\widetilde{D}^0_{\gamma,1})^*$ and $(\widetilde{D}^1_{\gamma,1})^*$ for any $\gamma \in \Gamma_{s,t,\text{loc}}(G_0, G1)$ by Corollary 1.1.49. By Corollary 1.2.9, we can for any z find a local bisection γ of s and t that satisfies $\gamma(s(z)) = z$. Then the values $(d_i)^*\omega$ at z can each be expressed by the values of $(D^1_{\gamma,1})^*(d_i)^*\omega$ at z. However, these are fully determined by the value of each $(d_i)^*\omega$ at $1_{t(z)}$ as $(D^1_{z,1})(z) = \mu(z, z^{-1})$. Any tangent vector at $1_{t(z)}$ can be decomposed to a vector tangent to the identity section and a vector Y satisfying $(d_0)_*Y = 0$. As this implies $Y \in \mathcal{F}_1$ and on the identity section we have $d_0 \equiv d_1$, we obtain that the $(d_i)^*\omega$ agree at $1_{t(z)}$. This in turn, shows that indeed $(d_0)^*\omega = (d_1)^*\omega$ on the entire G_1, meaning that indeed $\omega \in \Omega_{\mathrm{bas}}(G_0)$ and concluding the proof. □

Corollary 1.3.3. *Let G be a Lie groupoid and $\mathcal{F}_0$ be the (not necessarily smooth) bundle of vector spaces over G_0 defined by*

$$(\mathcal{F}_0)_m = (T_{1_m}t)(T_{1_m}s^{-1}(m)) = \rho(A_m).$$

We have that the basic forms on G_0 can be equivalently described by

$$\Omega_{\mathrm{bas}}(G_0) = \left\{ \omega \in \Omega(G_0) \ s.t. \ \begin{matrix} \iota_X\omega = 0 \text{ for } X \in \mathcal{F}_0 \text{ and where defined:} \\ (\dot{D}^0_{V,0})^*\omega \equiv \omega \text{ for } V \in J^1(s : G_1 \to G_0) \end{matrix} \right\}$$

or

$$\Omega_{\mathrm{bas}}(G_0) = \left\{ \omega \in \Omega(G_0) \ s.t. \ \begin{matrix} \iota_X\omega = 0 \text{ for } X \in \mathcal{F}_0 \text{ and where defined:} \\ (\dot{D}^0_{V,0})^*\omega \equiv \omega \text{ for } V \in (J^1G)_1 \end{matrix} \right\}.$$

Proof. Let G be a Lie groupoid. We know by Lemma 1.3.2 that the analogous statements hold for local sections of s and local bisections of s and t. However, as s is a submersion, we can extend any $V \in J^1(s : G_1 \to G_0)$ to a local section of s in analogy to Corollary 1.2.9, where we have seen that if $V \in (J^1G)_1$, then we may without loss of generality assume γ to be a bisection. By Lemma 1.2.28, we know that the pull-backs along the action of jets and the pull-backs along the action of local sections agree. □

Lemma 1.3.4. *Any basic form $\omega \in \Omega^p_{\mathrm{bas}}(G_0)$ satisfies that its value at some point of an orbit $n \in \mathcal{O}_m$ can be inferred from its value at m. For $p = 0$, we have that*

$$C^\infty_{\mathrm{bas}}(G_0) := \Omega^0_{\mathrm{bas}}(G_0)$$

can equivalently be defined as

$$C^\infty_{\mathrm{bas}}(G_0) := \{f \in C^\infty(G_0) \text{ such that } f(n) = f(m) \text{ for any } n \in \mathcal{O}_m\}.$$

Proof. We will show the statement using the equivalent definition of C^∞_{bas} of Lemma 1.3.2. Consider $f \in C^\infty_{\mathrm{bas}}(G_0)$ or $\omega \in \Omega^p_{\mathrm{bas}}(G_0)$ and $m \in G_0$. If $n \in \mathcal{O}_m$, we can by definition choose some $x \in G_1$ with $s(x) = m$ and $t(x) = n$. If we extend x to any local section $\gamma : U \to G_1$ of s (which is possible as s is a surjective submersion), we have that

$$(\widetilde{D}^0_{\gamma,0})^*f \equiv f$$

as functions on U or

$$(\widetilde{D}^0_{\gamma,0})^*\omega \equiv \omega$$

as forms on U. As $\gamma(m) = x$ by definition, this immediately implies that

$$\omega_n = (\widetilde{D}^0_{\gamma,0})^*\omega_{D^0_{x,0}(n)} = (\widetilde{D}^0_{\gamma,0})^*\omega_m$$

and similarly

$$(\widetilde{D}^0_{\gamma,0})^*f(m) = f(D^0_{x,0}(m)) = f(n),$$

implying $f(n) = f(m)$.

The converse regarding functions, i.e. the fact that a function constant along orbits is basic, can be proved the same way: As functions are of degree 0, we have that $\iota_X f = 0$ for degree reasons for any $X \in TG_0$. So when we have some function f with the property that $f(n) = f(m)$ if $n \in \mathcal{O}_m$, we can explicitly check that

$$(\widetilde{D}^0_{\gamma,0})^* f(a) = f(D^0_{\gamma(a),0}(a)) = f(t(\gamma(a))) = f(a),$$

as $t(\gamma(a)) \in \mathcal{O}_a$. This means that

$$(\widetilde{D}^0_{\gamma,0})^* f \equiv f$$

on its domain of definition. □

The definition itself does not explain why these forms are called "basic" - implying that they come from a base. This is the case in the motivating example of a free and proper action of a group on a manifold, in which case the manifold becomes a principal bundle over the quotient of the action (cf. [30], Theorem 21.10.), in fact, the definition we use here can be seen as a generalisation of the usual notion of basic form.

Proposition 1.3.5 (cf. [41], Theorem 12.5)**.** *Assume that G is the action groupoid of a free and proper action of a group $\mathcal{G}$ on a manifold M. Then the basic forms are precisely the forms which arise as pull-backs along*

$$\mathrm{pr}_Q : M \to Q$$

where $Q = M/\mathcal{G}$ is the quotient by the group action.

Proof. In Theorem 12.5 of [41], it is shown to be equivalent for a form $\omega \in \Omega(M)$ to arise as the pull-back of a form on Q and to be horizontal and invariant, i.e. to vanish under insertion of vectors contained in $\mathcal{F}_0$ and to be $\mathcal{G}$-invariant. This characterisation of basic forms as horizontal and invariant agrees with the alternate definition of Lemma 1.3.2, as for any

$$\gamma \in \Gamma_{s,\mathrm{loc}}(M, \mathcal{G} \times M) = C^\infty_{\mathrm{loc}}(M, \mathcal{G})$$

and any horizontal $\omega \in \Omega(M)$ we have

$$((\widetilde{D}^0_{\gamma,0})^*\omega)_m = ((\widetilde{D}^0_{\gamma(m),0})^*\omega)_m,$$

where $\gamma(m)$ denotes the constant function or s-section. This is precisely because the derivatives of any two s-sections can only differ in their values by elements vanishing under Ts, meaning that the induced maps $(\widetilde{D}^0_{\gamma,0})_*$ and $(\widetilde{D}^0_{\gamma(m),0})_*$ at m only differ by values in $\mathcal{F}_0$, under which ω vanishes. □

Proposition 1.3.6. *Let G be a Lie groupoid and consider $\omega \in \Omega_{\mathrm{bas}}(G_0)$. Then for any $q \in \mathbb{N}$ the pull-back of ω along any choice of compositions of face maps from G_q to G_0 agree and define a unique form in $\Omega(G_q)$. The form ω can be uniquely recovered.*

Proof. For $q = 1$ this statement reduces to $d_0 = d_1$, which is true by definition. For $q > 0$ we can now use induction together with the simplicial identities.

Compare

$$(d_{i_1} \circ \cdots \circ d_{i_q})^* = d^*_{i_q} \circ (d_{i_1} \circ \cdots \circ d_{i_{q-1}})^*$$

with

$$(d_{j_1} \circ \cdots \circ d_{j_q})^* = d^*_{j_q} \circ (d_{j_1} \circ \cdots \circ d_{j_{q-1}})^*$$

then we may by induction over q assume that

$$d_{j_{q-1}} = d_{i_{q-1}} = d_0$$

as replacing the pull-backs along the first $q-1$ face maps will not change the result.

However, we have

$$d_{i_{q-1}} \circ d_{i_q} = \begin{cases} d_{i_q-1} \circ d_{i_{q-1}} & \text{if } i_{q-1} \leq i_q \\ d_{i_q} \circ d_{i_{q-1}+1} & \text{if } i_{q-1} > i_q \end{cases}$$

and similar relations for $d_{j_{q-1}}$ and d_{j_q}.

If we apply these, we obtain

$$(d_{i_1} \circ \cdots \circ d_{i_q})^* = (\cdots \circ d_0)^*$$

and

$$(d_{j_1} \circ \cdots \circ d_{j_q})^* = (\cdots \circ d_0)^*$$

By induction, these expressions agree.

We can recover $\omega \in \Omega^p_{\text{bas}}(G_0)$ by considering the unit maps $u_q : G_0 \to G_q$ of Definition 0.2.24. As we can explicitly compute

$$\left(d_{i_1} \circ \cdots \circ d_{i_q} \circ u_q\right)(m) = m,$$

for any concatenation of face maps we obtain that

$$(u_q)^* \circ \left(d_{i_1} \circ \cdots \circ d_{i_q}\right)^* \equiv \text{id}$$

on $\Omega(G_0)$. □

Definition 1.3.7. Let G be a Lie groupoid and consider $\omega \in \Omega_{\text{bas}}(G_0)$. We will denote the unique associated form of Proposition 1.3.6 by $(\pi_q)^*(\omega)$. The subalgebra

$$\{(\pi_q)^*(\omega) \text{ for } \omega \in \Omega_{\text{bas}}(G_0)\} \subset \Omega(G_q)$$

will be called the subalgebra of **basic forms** on G_q.

Remark. By Proposition 1.3.6, we already know that

$$\{(\pi_q)^*(\omega) \text{ for } \omega \in \Omega_{\text{bas}}(G_0)\} \cong \Omega_{\text{bas}}(G_0).$$

Lemma 1.3.8. *Let G be a Lie groupoid. The vertical differential $\mathrm{d}_{\mathcal{N}}$ of the Bott–Shulman–Stasheff-bicomplex is $C^\infty_{\text{bas}}(G_0)$-linear in the following sense: If $f \in C^\infty_{\text{bas}}(G_0)$ then we have that*

$$\mathrm{d}_{\mathcal{N}}((\pi_q)^* f \cdot \omega) = (\pi_{q+1})^* f \cdot \mathrm{d}_{\mathcal{N}}(\omega),$$

where we follow the notion of $(\pi_k)^*$ as established in Definition 1.3.7. In particular, for any $p, q \in \mathbb{N}_0$, we have that

$$\Omega^p(G_q) \to \Omega^p(G_{q+1})$$

is a morphism of $C^\infty_{\text{bas}}(G_0)$-modules.

Proof. This simply follows from

$$(d_i)^*((\pi_q)^* f) = (d_0 \circ \cdots \circ d_0 \circ d_i)^* f = (\pi_{q+1})^* f$$

by the definition of $(\pi_{q+1})^* f$ as the pull-back along an arbitrary choice of face maps. Therefore

$$(d_i)^*((\pi_q)^* f \cdot \omega) = (\pi_{q+1})^* f \cdot (d_i)^*(\omega).$$

As this relation holds for any d_i and the fact that $\mathrm{d}_{\mathcal{N}}$ is a linear combination of different face maps, we obtain that the analogue holds for $\mathrm{d}_{\mathcal{N}}$. □

Definition 1.3.9. Consider a left action L of a Lie groupoid G on $p : N \to G_0$. If an element $\omega \in \Omega(N)$ satisfies that for any $n \in N$ and $x \in G_1$ with $s(x) = p(n)$ and $V_x \in (J^1G)_1$ we have that

$$\omega_n(X_1, \ldots, X_p) = \omega_{L(x,n)}(\widehat{L}(V_x, X_1), \ldots, \widehat{L}(V_x, X_p)),$$

then we call ω a **J^1G-invariant form of $p : N \to G_0$ with respect to L.**

If $\omega \in \Omega(G_0)$, we call it **J^1G-invariant** if it is a J^1G-invariant form of $\mathrm{id} : G_0 \to G_0$ with respect to the action of Example 1.1.23.

If $\omega \in \Omega(G_q)$, we call it **J^1G-invariant** if it is a J^1G-invariant form of any choice of $p_q^i : G_q \to G_0$ with respect to the corresponding action of the form as in Example 1.1.24.

Remark. In case $\omega = f \in C^\infty(N)$, we recover the notion of a G-invariant function with respect to L as given in Definition 1.1.22. However, the action of J^1G on $\wedge^0 T^*N = N$ agrees with the pull-back of this action along the base-point projection by definition, meaning that we also precisely recover the notion of a J^1G-invariant function as given in Definition 1.1.22.

Lemma 1.3.10. *A J^1G-invariant form is also invariant under $(\widetilde{D}^i_{\gamma,q})^*$ for any local bisection γ.*

Proof. We have already seen in Corollary 1.2.29, that $(\widetilde{D}^i_{\gamma,q})^*$ can be fibre-wise recovered by the expression $(\dot{D}^i_{V,q})^*$ for the appropriate jet $V \in (J^1G)_1$. So at every point $z \in p_q^i(U)$ for U the domain of definition of γ the restriction of $(\widetilde{D}^i_{\gamma,q})^*$ to a map

$$\wedge^\bullet T^*_{D^i_{\gamma(p_q^i(z)),q}(z)} G_q \to T^*_z G_q$$

satisfies that

$$\omega_{D^i_{\gamma(p_q^i(z)),q}(z)} \mapsto \omega_z$$

we automatically obtain that

$$(\widetilde{D}^i_{\gamma,q})^* \omega = \omega.$$

□

Lemma 1.3.11. *Let G be a Lie groupoid. For any $y, z \in G_q$ for which $p_q^0(y) \in \mathcal{O}_{p_q^0(z)}$ we can find $x^1, \ldots, x^q \in G_1$ such that*

$$D^q_{x^q,q}\left(D^{q-1}_{x^{q-1},q}\left(\ldots D^1_{x^1,q}\left(D^0_{x^0,q}(y)\right)\right)\right) = z.$$

Remark. If G is regular, the lemma precisely requires y and z to be in the same leaf of the orbit foliation F_q (i.e. in the same fibre of the map $\pi_q : G_q \to Q$ of Proposition 1.1.16).

Proof. We have that y, z are of the form

$$(y_1, \dots, y_q),\ (z_1, \dots, z_q) \in G_1 \times_{G_0} \cdots \times_{G_0} G_1,$$

where there is in particular an element x^0 with $s(x^0) = t(y_1)$ and $t(x^0) = t(z_1)$. So we have that

$$D^0_{x^0,q}(y) = y'$$

where $y' = (\mu(x^0, y_1), y_2, \dots, y_q)$. In particular, $p^0_0(y') = t(x^0) = t(z_1)$.

Now we can inductively consider the following procedure: As y' and z satisfy that $p^0_q(y') = p^0_q(z)$, we can multiply their first components and obtain

$$x^1 := \mu((z_1)^{-1}, y'_1).$$

We can then compute

$$y'' := D^1_{x^1,q}(y') = (z_1, \mu((z_1)^{-1}, y'_1, y_2), y_3, \dots).$$

As we have that $(y'')_1 = z_1$, we also find that $p^1_q(y'') = p^1_q(z)$. Therefore we can define

$$x^2 := \mu((z_2)^{-1}, y''_2)$$

and describe the analogous procedure for $D^2_{x^2,q}$, for which we obtain

$$D^2_{x^2,q}\left(D^1_{x^1,q}\left(D^0_{x^0,q}(y)\right)\right) = (z_1, z_2, \mu((z_2)^{-1}, y''_2, y_3), y_4, \dots).$$

We can analogously iterate until we obtain x^q, for which

$$D^q_{x^q,q}\left(D^{q-1}_{x^{q-1},q}\left(\dots D^1_{x^1,q}\left(D^0_{x^0,q}(y)\right)\right)\right) = z,$$

proving the statement. □

Corollary 1.3.12. *For any $z \in G_q$ we can find $m \in G_0$ and $x^1, \dots, x^q \in G_1$ such that*

$$D^q_{x^q,q}\left(D^{q-1}_{x^{q-1},q}\left(\dots D^1_{x^1,q}(z)\right)\right) = u_q(m).$$

Proof. We can apply Lemma 1.3.11 to z and $1_{p^0_0(z)} = u_q(p^0_0(z))$, which already map to the same point under p^0_0. Reviewing the proof will show that in this case, we also may assume that $x^0 = 1_{p^0_0(z)} = u_1(p^0_0(z))$, such that by unitality of the action we have $D^0_{x^0,q} \equiv \mathrm{id}$ wherever defined. □

Lemma 1.3.13. *Let $\omega \in (\Omega(G_q))^{J^1G}$ and $\nu \in (\Omega(G_q))^{J^1G}$. If we have that*

$$\omega \equiv \nu$$

along the unit section $G_0 \subset G_q$, then we find that $\omega = \nu$.

Proof. Consider any point $z \in G_q$. By Corollary 1.3.12, we have that

$$D^q_{x^q,q}\left(D^{q-1}_{x^{q-1},q}\left(\dots D^1_{x^1,q}(z)\right)\right) = u_q(m)$$

for some $m \in G_0$. We can now choose $V_i \in (J^1G)_1$ with base-points x^i by Lemma 1.2.8. The concatenation

$$\left(D^1_{V_1,q}\right)^* \dots \left(D^{q-1}_{V_{q-1},q}\right)^* \left(D^q_{V_q,q}\right)^*$$

induces a map between

$$\wedge^\bullet T^*_{u_q(m)} G_q \to \wedge^\bullet T^*_z G_q,$$

which by assumption of J^1G-invariance maps $\omega_{u_q(m)}$ to ω_z and analogously for ν. But as we have assumed that $\omega_{u_q(m)} = \nu_{u_q(m)}$, this implies that

$$\omega_z = \nu_z$$

and as $z \in G_q$ was chosen arbitrarily, we obtain $\omega = \nu$. □

While not strictly necessary for our work, the following observation will be a good point of intuition for us later, when we restrict ourselves only to a more specific invariance.

Proposition 1.3.14. *Let G be a Lie groupoid. The forms $\omega \in \Omega(G_q)$ which are J^1G-invariant are precisely the basic forms on G_q. Hence we can identify*

$$(\Omega(G_q))^{J^1G} \cong \Omega_{\text{bas}}(G_0).$$

Proof. For this proof, we have to show the two inclusions

$$\{(\pi_q)^*(\omega) \text{ for } \omega \in \Omega_{\text{bas}}(G_0)\} \subseteq (\Omega(G_q))^{J^1G}$$

and

$$(\Omega(G_q))^{J^1G} \subseteq \{(\pi_q)^*(\omega) \text{ for } \omega \in \Omega_{\text{bas}}(G_0)\}.$$

The first direction, i.e. that pull-backs of basic forms define J^1G-invariant forms, will be straightforward. The second direction will rely on Lemma 1.3.13 and the fact that we will show that

$$\iota_X \omega \equiv 0$$

at the unit section for any J^1G-invariant form and any vector field $X \in \mathfrak{X}(G_q)$ contained in the orbit foliation bundle $\mathcal{F}_q$.

To start off, let us show that the basic forms on G_q are indeed J^1G-invariant. For $q = 0$ this is already implied by Corollary 1.3.3. Otherwise, we can inductively assume that the statement holds for $q-1$. Then we can write $(\pi_q)^*\omega = (d_0)^*(\pi_{q-1})^*\omega$ and apply the rules of Lemma 1.2.33, to find that

$$(\dot{D}^i_{V,q})^*(\pi_q)^*\omega \equiv (d_i)^*(\pi_{q-1})^*\omega \equiv (\pi_q)^*\omega.$$

So $(\pi_q)^*\omega$ is indeed J^1G-invariant.

Now let us prove the converse: Assume that we have some form $\nu \in (\Omega(G_0))^{J^1G}$. Define

$$\omega = (u_q)^*\nu.$$

We claim that $\omega \in \Omega_{\text{bas}}(G_0)$ and further that $\nu = (\pi_q)^*\omega$. We immediately know that $\omega \in (\Omega(G_0))^{J^1G}$ from Lemma 1.2.39. To show this, we will first assume the following claim: For any J^1G-invariant form and any vector field $X \in \mathfrak{X}(G_q)$ contained in the orbit foliation bundle $\mathcal{F}_q$, we have

$$\iota_X \nu \equiv 0$$

along the unit section $1_{G_0} \subset G_q$. This will immediately imply that $\omega \in \Omega_{\text{bas}}(G_0)$ by Corollary 1.3.3 and it will also imply that

$$(\pi_q)((u_q)^*\nu) \equiv \nu$$

at the unit section $1_{G_0} \subset G_q$, as the tangent space at any 1_m is spanned by T_mG_0 and $\mathcal{F}_q$ (this can be seen for example by studying their dimensions and noting that their intersection corresponds precisely to $\mathcal{F}_0$). Together with Lemma 1.3.13, this implies that indeed

$$\nu = (\pi_q)^*\omega$$

in this case and concludes the proof.

Now let us turn to showing the claim: To see this, let us first show $\iota_Y\omega_{1_m} = 0$ at the unit section for any $Y = (u_q)_*(\underline{Y}) \in \mathcal{F}_q \cap (Tu_q)(T_mG_0) \subset T_{1_m}G_q$. Let us commence this argument by choosing an arbitrary splitting of $T_mG_0 = H \oplus \mathcal{F}_0$ and then choosing a lift of $\mathcal{F}_0$ to A along ρ. This defines a map

$$T_mG_0 \to T_{1_m}G_1,$$

which by definition is a section of Tt and can be inverted to a section of Ts defining a subspace $V \subset T_{1_m}G_1$. By construction, V satisfies that $V \cap \mathcal{F}_1$ gets mapped to 0 under Tt. It is therefore not an element of $(J^1G)_1$, but of $J^1(s : G_0 \to G_1)$. However, we can use linear interpolation between the jet we constructed and the jet induced by the map

$$Tu_1 : T_mG_0 \to T_{1_m}G_1$$

to obtain a family of elements of $(J^1G)_1$ converging to V. We can explicitly compute that all affine combinations between the jets map elements of $\mathcal{F}_1$ under Tt to a linear multiple of their image under Ts and the scaling constant is only 0 for V itself. As $\ker(Tt) \subset \mathcal{F}_1$, this means Tt is injective on these jets. By continuity of the respective actions, we must have that ν is invariant under any action of the form $(D^i_{V,q})^*$ as well. However, we have by construction that by Lemma 1.2.39 and construction

$$(D^q_{V,q})_* \dots (D^1_{V,q})_*(D^0_{V,q})_*Y = (u_q)_*(D^0_{V,q})_*\underline{Y} = 0$$

for any $Y = (u_q)_*\underline{Y}$ with $Y \in \mathcal{F}_q$ or equivalently $\underline{Y} \in \mathcal{F}_0$. Therefore

$$\iota_Y\omega = \iota_Y(D^q_{V,q})^* \dots (D^1_{V,q})^*(D^0_{V,q})^*\omega = (D^q_{V,q})^* \dots (D^1_{V,q})^*(D^0_{V,q})^*(\iota_0\omega) = 0.$$

The general statement can now be inferred as follows: Consider $Z \in \Gamma(G_0, A)$ arbitrary and any $0 \le i \le q$. We have that around any $m \in G_0$ there is some open neighbourhood $U_m \subset G_0$ and open interval I_m containing 0 such that the expression $\exp(t \cdot Z)$ defines a local bisection on U_m for $t \in I_m$. Therefore, $\widetilde{D}^i_{\exp(t\cdot Z),q}$ defines a family of local diffeomorphisms on $(p^i_q)^{-1}(U_m)$, which can be derived to a vector field $\widehat{Z} \in \mathfrak{X}(G_q)$. If ω denotes a J^1G-invariant form, it is retained under this family of diffeomorphisms and therefore also satisfies that

$$\mathcal{L}_{\widehat{Z}}\omega = 0.$$

Now let us analyse the form of $\widehat{Z}$: We have that it is given at $z \in G_q$ by

$$\frac{\partial}{\partial t}(z_1, \dots, \mu(z_i, (\exp(t \cdot Z)(p^i_q(z)))^{-1}), \mu(\exp(t \cdot Z)(p^i_q(z)), z_{i+1}), \dots)\bigg|_{t=0}.$$

From the formulas, we can directly see that this corresponds to

$$D^q_{z_q,q}\big(\dots(D^i_{z_i,q}(D^{i-2}_{z_{i-1},q}\big(\dots D^0_{z_1,q}\big(\dots, 0, -Z_{p^i_q(z)} + (Tu_q)(\rho(Z_{p^i_q(z)})), Z_{p^i_q(z)}, 0, \dots\big)\dots\big)$$

where we have used that all expressions vanish under Tp^l_q for any $0 \le l \le q$ with $l \ne i$.

In particular, we have that

$$\widehat{f \cdot Z} = (p_q^i)^* f \cdot \widehat{Z}$$

and

$$\mathcal{L}_{\widehat{f \cdot Z}} \omega = \mathrm{d}((p_q^i)^* f) \wedge \iota_{\widehat{Z}} \omega + (p_q^i)^* f \cdot \mathcal{L}_{\widehat{Z}} \omega,$$

such that in particular

$$\mathrm{d}((p_q^i)^* f) \wedge \iota_{\widehat{Z}} \omega = 0.$$

However, we may choose any f for this to hold, such that we may choose it satisfying

$$\iota_Y \, \mathrm{d}\left((p_q^i)^* f\right) = 1$$

at m, where we recall the notion of $Y \in \mathcal{F}_q \cap (Tu_1)_*(T_m G_0)$ from earlier. (This is always possible, as we can choose any preimage of Y under Tu_1 and enforce the analogue condition). This means that

$$0 = \iota_Y \left(\mathrm{d}((p_q^i)^* f) \wedge \iota_{\widehat{Z}} \omega\right)_{1_m} = \left(\iota_{\widehat{Z}} \omega\right)_{1_m} - \mathrm{d}((p_q^i)^* f) \wedge \iota_Y \left(\iota_{\widehat{Z}} \omega\right)_{1_m} = \left(\iota_{\widehat{Z}} \omega\right)_{1_m},$$

where we have used the earlier fact that insertion of Y in ω (and therefore also $\iota_{\widehat{Z}} \omega$) vanishes. So indeed, we find that

$$\iota_{\widehat{Z}} \omega \equiv 0$$

at the unit section. As we can explicitly find from the fact that $\mathcal{F}_1$ is spanned by $\mathcal{F}_0$ and A, we obtain that the $\widehat{Z}$ and Y span $\mathcal{F}_q$ at the unit section of G_q. Therefore

$$\iota_X \omega \equiv 0$$

for any $X \in \Gamma(G_q, \mathcal{F}_q)$ and we can conclude the proof.

□

1.4 Submanifolds complementary to orbits

Aim of this section: Definitions and Concepts

What are embedded submanifolds complementary to the orbit foliation of a regular manifold? What are nice properties of these? We can prove a local linearisation result. What are additional properties needed for additional results?

In this section, we will study regular Lie groupoids by constructing certain subgroupoids as auxiliary structures. They will help us to obtain results on the original groupoids.

For the motivation behind this simplification let us look at transitive Lie groupoids, i.e. Lie groupoids, where the orbit of any point is the entire object space G_0. Up to Morita equivalence, any transitive Lie groupoid corresponds to a group acting on a point (which can be obtained through a similar argument to Example 0.2.19 and Example 0.2.21). So in this case any complexity of the object space (e.g. a complicated fundamental group, non-orientability, etc.) is completely irrelevant when computing the associated stack cohomology.

Returning to general regular groupoids, we could, in a similar vein, try to pick a single point representing each orbit and hope to construct a Morita equivalent Lie groupoid in this case as well. Unfortunately, there is a major flaw in this strategy: Picking a single point for each orbit in a smooth way is equivalent to computing the quotient space of G_0

by G_1 and further embedding it as a submanifold of G_0. The quotient space, however, is generally not even a manifold. Luckily, the idea also works out when the procedure is done locally: The orbits define leafs of a foliation and it is possible to locally choose a manifold complementary to it, intersecting each local leaf exactly once.

We will study precisely these types of embedded manifolds and their associated subgroupoids. It will become apparent, that they are particularly useful if we may assume and additional property, s-properness, on those subgroupoids. This will lead to the notion of "admissible" embedded manifolds, which we will use in multiple proofs at crucial points in this thesis.

Definition 1.4.1. Let G be a groupoid (not necessarily a Lie groupoid). We denote the **saturation** of a subset $S \subset G_0$ as

$$\mathrm{Sat}(S) = \{m \in G_0 \text{ s.t. } \exists x \text{ with } t(x) = m, s(x) \in S\}.$$

A set S is **saturated** if $\mathrm{Sat}(S) = S$.

This concept of saturation and many of its properties can be studied in a more general context (cf. [30], p.605). For completeness, we will present some well-known results for the setting of Lie groupoids here without any claim of originality.

Lemma 1.4.2. *Let G be a groupoid (not necessarily a Lie groupoid). The following are equivalent:*

- *The set S is saturated.*
- *The set S satisfies $t^{-1}(S) = s^{-1}(S)$.*

The proof of this can be found in A.2.1.

Definition 1.4.3. Let G be a groupoid (not necessarily a Lie groupoid) and H be a subgroupoid. Then H is called a **full subgroupoid** of G, if for every $m, n \in H_0$ we have that

$$(t|_H)^{-1}(\{m\}) \cap (s|_H)^{-1}(\{n\}) \subset H_1 \subset G_1$$

agrees with

$$t^{-1}(\{m\}) \cap s^{-1}(\{n\}).$$

Definition 1.4.4. Let G be a Lie groupoid and S a saturated subset. $G|_S$ denotes the **full subgroupoid of G over S** given

- object space $(G|_S)_0 := S$
- morphism space $(G|_S)_1 := s^{-1}(S) = t^{-1}(S)$.

Lemma 1.4.5. *Consider a Lie groupoid G and a saturated embedded submanifold $U \subset G_0$. Then the groupoid $G|_U$ defined in Definition 1.4.4, i.e. given by*

- *object space $(G|_U)_0 = U$*
- *morphism space $(G|_U)_0 = s^{-1}(U) = t^{-1}(U)$*

is a Lie groupoid.

The proof of this can be found at A.2.2.

Lemma 1.4.6. *Let G be a Lie groupoid. The saturation of an open subset of G_0 is open.*

The proof of this statement can be found in A.2.3

Lemma 1.4.7. *Let G be a proper Lie groupoid. The saturation of any compact set $C \subset G_0$ is closed.*

Remark. The assumption of C being compact can not be replaced by the assumption of C being only closed. For example, consider the (regular and proper) foliation groupoid defined by the foliation

$$\{\mathbb{R} \times \{y\}\}_{y \in \mathbb{R}}$$

on $\mathbb{R}^2$. The graph of the exponential function is closed in $\mathbb{R}^2$, but its saturation is given by

$$\mathbb{R} \times \mathbb{R}_{>0},$$

which is not closed.

The proof of this statement can be found in A.2.4.

Lemma 1.4.8. *Let G be a groupoid (not necessarily a Lie groupoid). For any $m \in G_0$, we have that the orbit of Definition 1.1.10 satisfies*

$$O_m = \mathrm{Sat}(\{m\}).$$

Proof. This follows directly as the definitions agree. □

Corollary 1.4.9. *If G is a proper Lie groupoid, then O_m is closed for any $m \in G_0$.*

Proof. This follows from applying Lemma 1.4.7 to $\{m\}$ in the alternative definition of $\mathcal{O}_m$ of Lemma 1.4.8. □

The following is a result for which I am especially certain that it can be found in the pre-existing literature or at least is "folklore". The proof, although lengthy, resembles precisely the classical result for manifolds and their topology.

Proposition 1.4.10 (A.2.5)**.** *Consider a proper Lie groupoid G. Any two closed saturated subsets $C \subset G_0$ and $D \subset G_0$ which are disjoint can be separated by open saturated sets. This means that there exist open sets $N \subset G_0$ and $O \subset G_0$ with $C \subset N$ and $D \subset O$ which also satisfy*

$$N \cap O = \emptyset.$$

Corollary 1.4.11. *Consider a proper Lie groupoid G. Any two closed saturated subsets $C \subset G_0$ and $D \subset G_0$ which are disjoint can be separated by open saturated sets N and O which satisfy*

$$\overline{N} \cap \overline{O} = \emptyset.$$

The proof of this statement can be found in A.2.6

Corollary 1.4.12. *Let G be a proper Lie groupoid. Consider the space*

$$Q = G_0/F_0 = G_0/\sim,$$

where $\sim$ denotes the equivalence relation of Lemma 1.1.11. Then the quotient topology on Q is T_4 (i.e. normal and T_1).

The proof of this statement can be found in A.2.7.

We have already described at the beginning of the section, that we aim to reduce the complexity of studying a Lie groupoid G by constructing auxiliary Lie subgroupoids, where we aim to only include very few (preferably one) points per orbit in the new restricted object space. On regular groupoids, we have that the orbit foliation of G_0 has a constant rank - so we can hope to use the (non-canonical) foliation charts. They will produce a (non-canonical) local embedding of a vector space spanned by the coordinate vectors which are fixed in the leaf of 0 in the given chart - and hence intersecting all the leafs of the chart precisely once.

Definition 1.4.13. Let G be a Lie groupoid. A manifold R is an **embedded submanifold complementary to the orbits** of G_0 if it is an embedded submanifold of G_0 and the embedding $\iota : R \subset G_0$ satisfies that $T\iota : TR \to TG_0$ is complementary to the orbit foliation in the sense that for the foliation bundle $\mathcal{F}_0$ we have that

$$(T\iota)(TR) \cap \mathcal{F}_0 = \emptyset.$$

Remark. It might seem a natural choice to use the word "transversal" instead of "complementary" here. However, transversality of two embeddings Q and R in some G_0 is usually (cf. [30], p. 143) defined as satisfying

$$T_mQ \oplus T_mR = T_mG_0,$$

which is not necessarily the case here. We will sometimes in the following consider embedded submanifolds R for which both conditions are satisfied. It is immediately clear that this requires the codimension of R to be precisely the dimension of the foliation. Under this assumption, the notions of "complementary" and "transversal" become equivalent and the embedded manifolds satisfying them are also known as "maximally transversal". However, some of the following statements can be proven independently of the assumption of a certain codimension of R.

Lemma 1.4.14. *Consider a manifold G_0 with a regular foliation F_0 of rank l_0, i.e. where we can locally around any $m \in G_0$ choose coordinates such that the chosen coordinate chart is given by a cube*

$$U \subset \mathbb{R}^{k_0} = \mathbb{R}^{l_0} \times \mathbb{R}^{k_0 - l_0}$$

and the leaves of the foliation are given by

$$x^{l_0+1} = c^{l_0+1}, \quad \dots, \quad x^{k_0} = c^{k_0}$$

and its associated involutive distribution by

$$\Big(T(\mathbb{R}^{l_0}) \times \{0_y\}_{y \in \mathbb{R}^{k_0 - l_0}} \Big) |_U.$$

Then for any embedded manifold R of codimension l_0 (i.e. dimension $k_0 - l_0$) complementary to the foliation we can choose a coordinate chart

$$\phi : V \subset G_0 \to U \subset \mathbb{R}^{k_0}$$

of the form described above with the additional property that

$$\phi(R \cap V) = (\{0\}^{l_0} \times \mathbb{R}^{k_0 - l_0}) \cap U.$$

The proof of this statement can be found in A.2.8.

Lemma 1.4.15. *Let G be a regular Lie groupoid where the rank of the orbit foliation is l_0. The saturation of a embedded manifold R of codimension l_0 complementary to the orbits is an open subset of G_0 and in particular an embedded manifold.*

The proof of this statement can be found in A.2.9.

We now want to use these embedded submanifolds to study "global" phenomena on the entire groupoid by only looking at the restriction to a submanifold only.

The philosophy will relate the properties (e.g. values of forms) along orbits, like we already have done for example in Lemma 1.3.4 for basic functions, where the value at a point determines the value at each point in an orbit. This means that when studying some complementary embedded manifold R, we will only obtain information on its saturation, which we can continue to a groupoid structure.

Lemma 1.4.16. *For a Lie groupoid G, consider $R \subset G_0$ an embedded manifold complementary to the orbit foliation $\mathcal{F}_0$, such that $\operatorname{Sat}(R)$ is an embedded manifold. Then for any $r \in R$ we have $T_r \operatorname{Sat}(R) = T_r R \oplus \mathcal{F}_0$ and*

$$s|_{t^{-1}(R)} : t^{-1}(R) \to \operatorname{Sat}(R)$$

is a submersion.

The proof of this statement can be found in A.2.10.

Corollary 1.4.17. *For a Lie groupoid G, consider $R \subset G_0$ an embedded manifold complementary to the orbit foliation $\mathcal{F}_0$, such that $\operatorname{Sat}(R)$ is an embedded manifold.*

Then we have that $t^{-1}(R) \cap s^{-1}(R)$ is a manifold embedded in G_1. If G was proper, then the map

$$(t, s) : t^{-1}(R) \cap s^{-1}(R) \to R \times R$$

is proper as well.

The proof of this statement can be found in A.2.11.

Definition 1.4.18. Let G be a Lie groupoid and $R \subset G_0$ an embedded manifold complementary to the orbits satisfying that $\operatorname{Sat}(R) \subset G_0$ is also embedded. Then $G|_R$ denotes the **full subgroupoid of G over R** given by

- object space $(G|_R)_0 := R$
- morphism space $(G|_R)_1 := t^{-1}(R) \cap s^{-1}(R)$.

Remark. This is in some sense an extension of Definition 1.4.4. Also, with this definition the statement of Corollary 1.4.17 can be formulated as: If G is proper, then $G|_R$ is proper.

Corollary 1.4.19. *If G is a regular Lie groupoid with an orbit foliation $\mathcal{F}_0$ of dimension l_0 and further $R \subset G_0$ is an embedded manifold of codimension l_0 complementary to the orbits, then*

$$s|_{t^{-1}(R)} : t^{-1}(R) \to \operatorname{Sat}(R)$$

is a submersion and $t^{-1}(R) \cap s^{-1}(R)$ is a manifold embedded in G_1 defining a subgroupoid of G which we will denote $G|_R$. If G was proper, then so is $G|_R$.

The proof of this statement can be found in A.2.12

Corollary 1.4.20. *If G is a regular Lie groupoid with an orbit foliation $\mathcal{F}_0$ of dimension l_0 and further $R_1 \subset G_0$ and $R_2 \subset G_0$ are embedded manifolds of codimension l_0 complementary to the orbits, then for any $m \in R_1 \cap \mathrm{Sat}(R_2)$ there is a local s-section defined on $U \subset R_1$ around m inducing a diffeomorphism between U and an open submanifold of R_2.*

The proof of this statement can be found in A.2.13.

Intuitively, if we compare $\mathrm{Sat}(R)$ and R, we have that for the Lie groupoid $G|_{Sat(R)}$ the embedding $R \subset \mathrm{Sat}(R)$ behaves similarly to the embedding of a point in a transitive groupoid. In fact, we can check that that $G|_R$ is Morita equivalent to $G|_{\mathrm{Sat}(R)}$ (using the bibundles $s^{-1}(R)$ and $t^{-1}(R)$). However, this simplification of the structure from $G|_{\mathrm{Sat}(R)}$ to $G|_R$ is not quite sufficient for our purposes yet. What we will also need is a result that tells us more about the structure on $G|_R$, specifically, we will be looking for a result that will tell us, that locally, $G|_R$ has the structure of an action Lie groupoid. For this aim, we will look at results on linearising Lie groupoids. The question on how to linearise a general proper Lie groupoid has been already been considered by Weinstein in [43] and Zung in [44] and there is a family of results constantly improving the existing linearisation theorems.

Definition 1.4.21. Consider a Lie groupoid G, a saturated subset $H_0 \subset G_0$ and the associated full subgroupoid $H = G|_{H_0}$. Then the normal groupoid of H in G is denoted $\nu(H)$ and defined by the vector bundle quotients

$$(\nu(H))_0 = TG_0|_{H_0}/TH_0$$

and

$$(\nu(H))_1 = TG_1|_{H_1}/TH_1.$$

Remark. We can see immediately from the construction that $TG|_H$ is a subgroupoid of TG and $TH \subset TG|_H$ is a morphism of VB-groupoids over H. Therefore, the groupoid structure of TG actually descends to $\nu(H)$ and furthermore, $\nu(H)$ has the structure of a VB-groupoid.

Definition 1.4.22. A Lie groupoid G is called **s-proper** if its source map s is a proper map.

Lemma 1.4.23. *Let G be an s-proper Lie groupoid. Then for any $m \in G_0$, the orbit $\mathcal{O}_m$ only has finitely many connected components.*

Proof. Consider the connected components of $s^{-1}(\{m\})$. By assumption of s-properness this is a compact manifold. Therefore it can only have finitely many connected components. As t is continuous, it maps connected components onto connected components and the image under t is given by precisely $\mathcal{O}_m$. □

Lemma 1.4.24. *If a Lie groupoid G is s-proper and an open subset $U \subset G_0$ is saturated, then $G|_U$ is s-proper as well.*

Proof. Consider a sequence $\{x_n \in (G|_U)_1\}$ such that the values of $s(x_n)$ converge. Then as G was s-proper, we have that there is a subsequence converging to $x \in G_1$. However, we have that $s(x) \in U$ is the limit of the $s(x_n)$ and as U is saturated, we have that also $t(x) \in U$. Hence $x \in (G|_U)_1$, implying that $G|_U$ is s-proper. □

Theorem 1.4.25 ([15], Corollary 5.14.)**.** *Consider an s-proper Lie groupoid G and a saturated embedded manifold $S_0 \subset G_0$. Then G is invariantly linearisable around S_0.*

This means when considering the groupoid S with

$$S_1 = t^{-1}(S_0) = s^{-1}(S_0)$$

as a subgroupoid of G, then there is an open saturated subgroupoid U of G (i.e. U_0 is open and saturated and $U_1 = t^{-1}(U_0) = s^{-1}(U_0)$) containing S and an isomorphism $U \simeq V \subset \nu(S)$ where V is an open saturated subgroupoid of $\nu(S)$.

Remark. The main theorem of [15] is much stronger - this result is merely a consequence of it. Furthermore, in the original statement, the additional assumption on G being Hausdorff was made, which we have dropped as it is included in our definition of Lie groupoids.

Corollary 1.4.26. *Given an s-proper Lie groupoid G with discrete orbits and $m \in G_0$ with orbit $O_m = \{m = m_1, m_2, \ldots, m_k\}$. Then there are disjoint $U_1, \ldots, U_k$ which are open neighbourhoods of the respective m_i such that $U = \cup U_i$ is saturated and $G|_U$ is isomorphic to a saturated open subset of $\nu(G|_{O_m})$.*

In particular, both source and target map induce a fibre bundle structure on $G|_U$ which each admit a trivialisation over each U_i. Furthermore, every connected component of $G|_U$ induces a unique diffeomorphism between some U_i and U_j. Also, the trivialisations on the connected component containing the units of some U_i with respect to either $s \equiv t$ agree and the fibre bundle structure is compatible with the groupoid structure, such that we obtain a trivial Lie group bundle structure on $\mathrm{Iso}^0(G|_{U_i})$.

Proof. As all conditions are met, we can apply Theorem 1.4.25 for $S_0 = O_m$. In this case, we have that TO_m is the trivial bundle over O_m, meaning that $(\nu(G|_{O_m}))_0 = TG_0|_{O_m}$, which decomposes into disjoint connected components which are tangent bundle fibres over every m_i in the orbit of m.

Furthermore, we have that any $X \in T_xG_1/T_xt^{-1}(O_m)$ for $x \in t^{-1}(O_m) = s^{-1}(O_m)$ is uniquely determined by x and its image under Ts, which is a tangent vector at $s(x)$. This means that for any m_i the source map of $\nu(G|_{O_m})$ when restricted to $T_mG_0 \simeq U_i$ is the projection of a trivial fibre bundle with fibre $s^{-1}(\{m_i\})$. An analogous result holds for the target.

When expressed according to the trivialisation with respect to s, any connected component of the bundle can be expressed as a product of some connected component of some $s^{-1}(\{m_i\})$ and the associated U_i. In particular, the connected components of the source fibres agree with the source fibres of the connected components (and the same for target fibres).

Also note that under t, any connected component given by the product of U_i and some connected component of $s^{-1}(\{m_i\})$ needs to have connected image, hence it must be contained in some U_j. Furthermore the image of each source fibre in this connected component needs to be connected, but as the orbits are discrete the image is actually constant. Hence if we choose any section of the source map on U_i with image in the chosen connected component, the induced map from U_i to U_j is unique.

The inversion of elements induces an involution on the set of connected components of $(\nu(G|_{O_m}))_1$. Because of inversion exchanging source and target the morphisms induced by two accordingly related connected components on the respective components of U are inverse to each other. In particular, all induced maps are diffeomorphisms.

If we consider a connected component including all units of some U_i it is immediately clear that the induced diffeomorphism is the identity and that $s \equiv t$ on the entire component. If we consider the trivialisation of this connected component, its fibre is given by a group, namely the connected component of the identity in $s^{-1}(m) \cap t^{-1}(m)$ and the trivialisation is compatible with the multiplication, such that we obtain a trivial Lie group bundle, which is equivalent to the action groupoid of the Lie group acting trivially on U_i. $\square$

Corollary 1.4.27. *Let G be an s-proper groupoid with discrete orbits, i.e. where the orbit foliation bundle $\mathcal{F}_0$ vanishes. Let $f \in C^\infty(G_0)$ be any smooth function. Then the function g defined by*

$$m \mapsto g(m) := \frac{1}{|O_m|} \sum_{n \in \mathcal{O}_m} f(n)$$

is a well-defined and smooth function in $C^\infty_{\text{bas}}(G_0)$.

Proof. We know that as G is s-proper, any $m \in R$ only has a finite orbit in R given by $\mathcal{O}_m = \{m = m_1, m_2, \dots, m_k\}$. Hence the values of $g(m)$ are well-defined. It is also clear that for m and m' with $\mathcal{O}_m = \mathcal{O}_{m'}$ holds $g(m) = g(m')$, meaning that by Lemma 1.3.4 the function is basic (assuming that it is smooth). We will now show, that g indeed satisfies smoothness, which will be an application of Corollary 1.4.26.

Firstly, note that $t^*f \in C^\infty(G_1)$ is a smooth function that is locally constant on s-fibres of G, because the orbits are discrete. Further note that instead of averaging over the values of f on $\mathcal{O}_m$, we could have been averaging over the constant values of f on the connected components of $s^{-1}(m)$ instead. In that sum, we would replaced the term of $f(m_i)$ for any $1 \leq i \leq k$ by the term $f(m_i)$ for each connected component of $t^{-1}(m_i) \cap s^{-1}(m)$. However, independently of i, this space is non-canonically isomorphic to $s^{-1}(m) \cap t^{-1}(m)$, meaning the number of connected components is independent of i. Therefore, each term $f(m_i)$ appears with the same frequency and as we are averaging over all terms, this does not change the result.

If we now apply Corollary 1.4.26, we have that $s : G_1 \to G_0$ is a trivial fibre bundle when restricted to some saturated open around $\mathcal{O}_m$, where on each connected component the function f is fibre-wise constant. For any m' we have just shown that averaging over the values of f on the s-fibres yields g. As the bundle is trivial, we have that each connected component induces a unique smooth function on the open component around m, which corresponds to the value of f in the fibre of the component. Averaging over these finitely many smooth functions will result in g, which shows that it is a smooth function as well on an open around m. As smoothness is a local property, we have that f is indeed smooth everywhere on G_0. $\square$

Remark. This corollary will be our go-to procedure to construct basic functions throughout the rest of the chapter. It will become apparent later how to apply it (at least locally) to general proper and regular groupoids. Note that we can also reframe the construction as averaging over a Haar system that is normalised. This is a construction that could be generalised to general s-proper groupoids, but not to proper groupoids.

Unfortunately, properness and s-properness of a groupoid are not equivalent, even when each orbit is compact or even finite. A good example to highlight that is the covering groupoid of a manifold: For a finite but not locally finite cover this has discrete finite orbits and is proper, but not s-proper.

Definition 1.4.28. Consider a proper regular groupoid G with orbit foliation of rank l_0. A complementary embedded manifold $R \subset G_0$ is **admissible** if it is of codimension l_0 and when restricted to $G|_R$ the source map $s|_R$ is proper.

Remark. Due to the maximal codimension, we have that this manifold is automatically also transversal to the foliation in the usual sense, specifically minimally transversal (see the previous remark on page 103).

Note, that this is a technical definition. The term admissible in this context was chosen to express that the manifolds have the necessary properties so that the constructions of Chapter 2 (and later Chapter 5) will work. Generally, there are already similar pre-existing notions for maximally transversal embedded submanifolds. For example, there is the notion of slice as in [13] (cf. p. 738), which denotes a maximally transversal embedded manifold R, but it furthermore must satisfy that for some previously fixed point $m \in G_0$ holds that $R \cap \mathcal{O}_m = \{m\}$. This concept is very useful when considering the linearisation of Lie groupoids which are proper (or even only proper at x). It will turn out that when later describing a general procedure to construct admissible embedded manifolds, we will often end up with slices coincidentally. Nonetheless, in the confines of this thesis, the s-properness required for admissibility is key to a number of proofs, explaining the need for this alternate notion.

Lemma 1.4.29. *Let G be a proper regular Lie groupoid. Consider an embedded manifold $Q \subset G_0$ complementary to the orbit foliation of codimension l_0 and $R \subset Q$ an open subset. Then the following are equivalent:*

- *$R \subset Q$ is admissible as an embedded manifold complementary to the orbit foliation with respect to $G|_Q$.*
- *$R \subset G_0$ is admissible as an embedded manifold complementary to the orbit foliation with respect to G.*

Proof. First of all, the composition of the embedding from R to Q and the embedding from Q to G_0 is still an embedding and as the image is contained in the image of the latter it is also still complementary to the orbit foliation. The codimension of R with respect to G_0 is l_0, as the dimension of R agrees with the dimension of Q. Furthermore, as $G|_R = (G|_Q)|_R$ the s-properness of the groupoid can be checked on either. □

Corollary 1.4.30. *Let G be a proper and regular Lie groupoid and $Q \subset G_0$ an admissible embedded manifold. Then any union $R = \cup_{i \in I} Q_i \subset Q$ of connected components $Q_i \subset Q$ is also an admissible embedded manifold $R \subset G_0$.*

Proof. By Lemma 1.4.29 it is sufficient to show that $R \subset Q$ is admissible with respect to $G|_Q$. The dimension of R agrees with the dimension of Q as an open subset, so it only remains to check s-properness. However, as both s and t map connected components to connected components as they are smooth, we have that $G|_R \subset G|_Q$ is an inclusion of connected components as well. In particular, it is a closed subset meaning that for any compact $C \subset Q$ we have that $s^{-1}(C) \cap G|_R$ is a closed subset of $s^{-1}(C) \cap G|_Q$. The latter is compact, as Q was admissible, and therefore, so is the former. Hence, $G|_R$ is s-proper and R is admissible. □

Lemma 1.4.31. *Let G be a Lie groupoid. Consider an admissible embedded manifold $Q \subset G_0$ and $R \subset Q$ a saturated open subset. Then $R \subset Q$ is admissible as an embedded manifold (complementary to the orbit foliation) with respect to $G|_Q$ and $R \subset G_0$ is admissible as an embedded manifold (complementary to the orbit foliation) with respect to G.*

Proof. As $R \subset Q$ open, the dimensions of Q and R agree, which means that R has codimension 0 with respect to Q and codimension l_0 with respect to G_0. Hence it only remains to check s-properness of $G|_R$.

Assume that we have a sequence of points $\{x_n\}_{n\in\mathbb{N}_0}$ for which the sequence of the values $s(x_n)$ converges in R. Then they also converge in Q, meaning there is a limit $x \in (G|_Q)_1$. However, we have that $s(x) \in R$ by assumption and as R is saturated, this implies $t(x) \in R$ and $x \in (G|_R)_1$. Hence $G|_R$ is s-proper and R is admissible in both of the respective cases. □

Lemma 1.4.32. *Consider a proper and regular groupoid with a coordinate chart U as described in Lemma 1.4.14 and assume there is an open $V \subset \mathbb{R}^{k_0-l_0}$ such that*

$$\{0\} \times V$$

is admissible. Then if

$$[0,p] \cdot b \times V \subset U$$

for some $p \in \mathbb{R}$ and $b \in \mathbb{R}^{l_0}$ we have that

$$\{p \cdot b\} \times V$$

is admissible as well. Furthermore, $r \in [0,p]$ parametrises a family of isomorphisms

$$G|_{\{r\cdot b\}\times V} \cong G|_{\{0\}\times V}$$

that smoothly depend on r.

Proof. We have that $\{r\} \times V$ is a complementary embedded manifold of codimension l_0 for any $r \in [0,p]$. Furthermore the fact that

$$[0,p] \cdot b \times V \subset U$$

implies that locally around every point $m \in V$ we can find a section

$$\gamma : \{0\} \times V \to G_1$$

of s which, when composed with t, maps to $\{r\} \times V$ covering the identity on V. We can construct this section as follows: Choose any vector bundle map of $\mathcal{F}_0$ to A which is a left-inverse to ρ (as ρ is linear, we can do this locally and then glue). We have that this maps the vector field

$$\{\frac{\partial}{\partial b} \in T\mathbb{R}^{l_0}\} \times \{0 \in T\mathbb{R}^{k_0-l_0}\}$$

to a local section $\Gamma_{\mathrm{loc}}(G_0, A)$ defined on U. Continuing this to a local right invariant vector field (defined on $t^{-1}(U)$) and integrating to a flow ϕ_a (parametrised by a) yields γ_r defined by

$$\gamma_r(n) := \phi_r(1_n) \in G_1.$$

It is not a priori clear that the result is well-defined on an open around m. However, as G is proper, we have that $\phi_a(x)$ for $s(x) \in U$ is well-defined precisely as long as the flow of

$$\{\frac{\partial}{\partial b} \in T\mathbb{R}^{l_0}\} \times \{0 \in T\mathbb{R}^{k_0-l_0}\}$$

starting at $s(x)$ is defined at time a. For $x = 1_n$ with $n \in V$ this is precisely the case as long as

$$[0,a] \cdot b \times \{n\} \in U,$$

which in particular holds for all $0 \leq a \leq p$ and implies that γ_r is indeed well defined.

As the vector field we are integrating is contained in $\ker(Ts)$, we also have that γ is a section of s, while by construction, $t \circ \gamma_r$ maps the first coordinate from 0 to $r \cdot b$. As this defines a diffeomorphism between

$$\{0\} \times V \cong \{r \cdot b\} \times V$$

we have that there is also a well-defined section

$$\gamma_r^{-1} : \{r\} \times V \to G_1$$

which is a left and right inverse to γ_r under the multiplication of Definition 1.1.41 by an argument analogous to Lemma 1.1.43.

By construction, this implies that

$$\begin{aligned} (G|_{\{0\}\times V})_1 &\to (G|_{\{r\cdot b\}\times V})_1 \\ x &\mapsto \mu(\gamma_r(t(x)), x, (\gamma_r(s(x)))^{-1}) \end{aligned}$$

and

$$\begin{aligned} (G|_{\{p\cdot b\}\times V})_1 &\to (G|_{\{0\}\times V})_1 \\ x &\mapsto \mu(\gamma_r^{-1}(t(x)), x, (\gamma_r^{-1}(s(x)))^{-1}) \end{aligned}$$

induce inverse groupoid morphisms. By construction, the family of sections γ_r and therefore also the family of induced isomorphisms smoothly depend on r. Furthermore, this shows that $G|_{\{r\cdot b\}\times V}$ is precisely s-proper if and only if $G|_{\{0\}\times V}$ is. As the other properties required for an embedded manifold to be admissible are clear from the explicit form of the embedding, it follows that if $\{0\} \times V$ is admissible, so is $\{r \cdot b\} \times V$.

□

Definition 1.4.33. Consider an embedded manifold $R \subset G_0$. Then the **G_0-boundary** $\partial R \subset G_0$ is given by

$$\overline{R}/R \subset G_0.$$

Lemma 1.4.34. *Consider an admissible embedded manifold $R \subset G_0$. Then*

$$\partial R \subset G_0/\operatorname{Sat}(R).$$

Proof. Assume that

$$\partial R \cap \operatorname{Sat}(R) \neq \emptyset.$$

This would mean that there is some $x \in G_1$ with $s(x) \in R$ and $t(x) \in \partial R$. We can now choose a sequence of points $m_n \in R$ such that they converge to $t(x)$. As t is a surjective submersion we can lift them to a converging sequence $x_n \in G_1$. We now aim to show that we may choose the x_n with $s(x_n) \in R$ for sufficiently large n.

For this note that according to Lemma 1.4.14, we can choose a coordinate chart around $s(x)$ in which R corresponds to

$$\{0\} \times \mathbb{R}^{k_0 - l_0}$$

and all leaves of the form

$$\mathbb{R}^{l_0} \times \{x\}$$

for a fixed $x \in \mathbb{R}^{k_0-l_0}$ belong to the same orbit, meaning in particular that orbits are retained by the standard orthogonal projection to R.

In particular, for every x_n with $s(x_n)$ contained in this chart, we can replace x_n by

$$x'_n = \mu(x_n, y_n)$$

for $y_n \in G_1$ some morphism with $t(y_n) = s(x_n)$ and $s(y_n) \in R$ the unique value to which $s(x_n)$ projects. This constructs a sequence which satisfies that $t(x'_n) = t(x_n) \in R$ converges to a value in ∂R, while $s(x'_n) \in R$ converges to $s(x)$, as in coordinates limit and projection to R can be interchanged.

However, we assumed that $G|_R$ was s-proper. Hence the x'_n have a converging subsequence in $(G|_R)_1$. As limits are unique, it follows that $x \in (G|_R)_1$ and $t(x) \in R$. This is a contradiction by definition of the boundary. □

Lemma 1.4.35. *Consider a proper Lie groupoid G where the leaves of the orbit foliation F_0 are of dimension 0, i.e. all orbits are discrete. Then for any $m \in G_0$ we can choose a precompact coordinate chart $U' \subset G_0$ with $m \in U'$ and local coordinates $U' \cong \mathbb{R}^{k_0}$, such that $\mathcal{O}_n \cap U'$ only contains finitely many points for any $n \in G_0$.*

Proof. Choose any coordinate chart around n. By assumption we have that G is proper, meaning that

$$t^{-1}(\overline{B_1(0)}) \cap s^{-1}(\{n\})$$

is a compact set. In particular, it can only have finitely many connected components. Furthermore, as connected components of this set map to connected components of $\overline{\mathcal{O}_n \cap B_1(0)}$ surjectively, we know that the latter also only has finitely many connected components.

However, as the orbits on G are discrete, this implies that the coordinate chart we can define on $B_1(0)$ satisfies the property of the statement, as it only intersects $\mathcal{O}_n$ in finitely many points. □

Lemma 1.4.36. *Given a proper Lie groupoid G where the orbit foliation $\mathcal{F}_0$ vanishes and for some $m \in G_0$ a coordinate chart $U' \subset G_0$ with $m \in U'$ and local coordinates $U' \cong \mathbb{R}^{k_0}$, such that $\mathcal{O}_m \cap U'$ only contains finitely many points. Furthermore, let $p \in \mathbb{R}$ be such that*

$$p \neq |m_i|,$$

where we consider the standard metric on $\mathbb{R}^{k_0}$.

Then there exists a $0 < q' < p$ such that for any $0 < q \leq q'$ we have

$$\mathrm{Sat}(B_q(0)) \cap \partial B_p(0) = \emptyset.$$

Proof. As we have

$$\mathrm{Sat}(B_q(0)) \cap \partial B_p(0) \subset \mathrm{Sat}(B_{q'}(0)) \cap \partial B_p(0)$$

whenever $q < q'$, it is sufficient to show that $\mathrm{Sat}(B_{q'}(0)) \cap \partial B_p(0) = \emptyset$ for some q'.

Assume that there was no such q'. Then for every $n \in \mathbb{N}_{>\frac{1}{p}}$ we would have

$$\mathrm{Sat}(B_{\frac{1}{n}}(0)) \cap \partial B_p(0) \neq \emptyset$$

implying that there would be some $x_n \in s^{-1}(B_{\frac{1}{n}}(0))$ with $t(x_n) \in \partial B_p(0)$. We have that $\{s(x_n)\}$ is a sequence that converges to $0 \in \mathbb{R}^{k_0}$, which corresponds to m. Furthermore, as $\partial B_p(0)$ is a compact embedded submanifold of G_0, we have that $\{t(x_n)\}$ contains a subsequence that converges to an element of $\partial B_p(0)$. However, as G was assumed proper this means that there is $x \in G_1$ with $s(x) = m$ and $t(x) \in \partial B_p(0)$, i.e. $|t(x)| = p$. This is a contradiction that $p \neq |m_i|$, as $t(x)$ must be in the orbit of m. □

Lemma 1.4.37. *Given a proper Lie groupoid G with discrete orbits and $m \in G_0$. Then there is an open U with $m \in U$ which is admissible. We may choose U connected and contractible and such that $U \subset O$ for any open neighbourhood of m.*

We may additionally choose U such that $G|_U$ is isomorphic to a full open subgroupoid $V \subset \nu(G|_{\mathcal{O}_m})$. This may be done such that both source and target on $G|_U$ induce fibre bundle structures which each admit a trivialisation. In particular, every connected component of $G|_U$ induces a unique diffeomorphism on U. Furthermore, the trivialisations with respect to $s \equiv t$ agree on the connected component containing the units of U, where they induce the structure of a trivial Lie group bundle.

Proof. First of all, let us note that the only reason the source map of a proper Lie groupoid with finite orbits is not proper is the existence of sequences $\{x_n \in G_1\}_{n\in\mathbb{N}_0}$ for which the images under the source map converge, but images under the target map do not. So if we apply Lemma 1.4.35, we can choose local coordinates $U' \simeq \mathbb{R}^{k_0}$ on some open neighbourhood $U' \subset G_0$ of m, where the orbits are finite. We will denote the points in the intersection of U' and O_m by $\{m = m_1, \dots, m_{k'}\}$. Now we may choose $0 < p \in \mathbb{R}$ such that

$$p \neq |m_i|$$

for any $1 \leq i \leq k'$ and consider $B_p(0) \subset \mathbb{R}^{k_0} \simeq U'$.

We can now apply Lemma 1.4.36 to obtain that there is a $q' \in \mathbb{R}$ with $0 < q' < p$, such that $B_q(0) \subset \mathbb{R}^{k_0} \simeq U'$ satisfies that

$$\mathrm{Sat}(B_q(0)) \cap \partial B_p(0) = \emptyset$$

for any $0 < q \leq q'$. We will now show that any choice of q is sufficient for $G|_{\mathrm{Sat}(B_q(0))\cap B_p(0)}$, i.e. the full groupoid over $\mathrm{Sat}(B_q(0)) \cap B_p(0)$, to be s-proper. For this, consider a sequence

$$\{x_n \in \left(G|_{\mathrm{Sat}(B_q(0))\cap B_p(0)}\right)_1\}_{n\in\mathbb{N}}$$

such that $s(x_n)$ converges to

$$a \in \mathrm{Sat}(B_q(0)) \cap B_p(0).$$

By boundedness, a subsequence of $t(x_n)$ converges in

$$\overline{\mathrm{Sat}(B_q(0)) \cap B_p(0)} \subset \overline{B_p(0)}$$

and by properness of G, this means that a sub-subsequence of the x_n converges to x with $s(x) = a$ and $t(x) \in \overline{B_p(0)}$. However, by assumption we have that

$$O_a \cap \overline{B_p(0)} \subset \mathrm{Sat}(B_q(0)) \cap \overline{B_p(0)} = \mathrm{Sat}(B_q(0)) \cap B_p(0)$$

as the intersection with the boundary $\partial B_p(0)$ is empty. Thus

$$x \in (G|_{\mathrm{Sat}(B_q(0))\cap B_p(0)})_1.$$

This concludes the proof that $G|_{\mathrm{Sat}(B_q(0))\cap B_p(0)}$ is indeed s-proper and therefore

$$\mathrm{Sat}(B_q(0)) \cap B_p(0)$$

is admissible.

It remains to show that we can find $U \subset \mathrm{Sat}(B_q(0)) \cap B_p(0)$ which is not only admissible, but also connected and contractible and can be chosen arbitrarily small. However, we are now in the situation where we may apply Corollary 1.4.26 to

$$G|_{O_m \cap B_p(0)} \subset G|_{\mathrm{Sat}(B_q(0)) \cap B_p(0)}.$$

This gives us disjoint open sets $U_i \subset \mathrm{Sat}(B_q(0)) \cap B_p(0)$ for $1 \leq i \leq k'$ with $m_i \in U_i$ such that $G|_{\cup U_i}$ is isomorphic to a saturated open subgroupoid of $\nu(G|_{O_m \cap B_p(0)})$.

As the U_i are disjoint, we have that the inclusion

$$G|_{U_1} \subset G|_{\cup U_i}$$

is an inclusion of connected components both on the level of objects and morphisms. As connected components are closed as well as open, convergence of a sequence in $(G|_{U_1})_1$ is equivalent to convergence of the sequence in $(G|_{\cup U_i})_1$ meaning that in particular $G|_{U_1}$ is s-proper.

Furthermore, we have that $G|_{U_1}$ corresponds under the isomorphism to an open subgroupoid $V' \subset \nu(G|_{\{m\}})$. However, $G|_{\{m\}}$ is given by a compact group acting trivially on a point. So if we consider the trivialisation of $(V')_1$ according to s as in Corollary 1.4.26, we obtain a group action of $(G|_{\{m\}})_1$ on $(V')_0$, which by definition corresponds to the group action of $(G|_{\{m\}})_1$ on $\nu((G|_m)_0)$ induced by the canonical trivialisation of $\nu((G|_m)_1) \cong (G|_m)_1 \times \nu((G|_m)_0)$ along the source map.

However, as we were linearising around a discrete orbit, we have

$$\nu((G|_m)_0) = T_m G_0$$

such that the action of $(G|_{\{m\}})_1$ on $\nu((G|_m)_0)$ is linear with respect to its vector space structure. As $(G|_{\{m\}})_1$ is compact, we can define an inner product on $T_m G_0$ that is invariant under the given action, meaning that in particular all open balls

$$B_r(0) \subset T_m G_0$$

for $r > 0$ are saturated.

If we now choose $r \in \mathbb{R}$ such that additionally

$$B_r(0) \subset (V')_0,$$

we have that $B_r(0)$ is connected and contractible and $\nu(G|_{\{m\}})|_{B_r(0)} \subset V'$ is a well-defined subgroupoid as $B_r(0)$ was saturated. Furthermore, $\nu(G|_{\{m\}})|_{B_r(0)}$ is s-proper as $B_r(0) \subset (V')_0$ is admissible by Lemma 1.4.31. Under the isomorphism with $G|_{U_1}$, we have that $B_r(0)$ corresponds to an open connected contractible subset $U \subset U_1$ and $\nu(G|_{\{m\}})|_{B_r(0)}$ to $G|_U$, which is then an open and s-proper subgroupoid of $G|_{U_1}$, meaning that U is admissible.

The form of the trivialisation of $G|_U$ follows as in Corollary 1.4.26. □

Corollary 1.4.38. *For any proper and regular groupoid G and any $m \in G_0$ there exists an admissible embedded manifold $R \subset G_0$ with $m \in R$.*

Proof. By definition, we have that there are coordinates on G_0 given by a chart $U \to G_0$ where $U \subset \mathbb{R}^n$ is a cube, such that each orbit intersects U in either the empty set or a countable union of l-dimensional slices of the form

$$x^{l+1} = c^{k+1}, \quad \ldots, \quad x^n = c^n.$$

In particular
$$U \cap \{x^i = 0 \text{ for } 1 \leq i \leq k_0 - l_0\}$$
is an embedded manifold of codimension l_0 complementary to the orbits. Therefore, by Corollary 1.4.19, we have that $G|_R$ is well-defined and by Lemma 1.4.37 there exists an admissible manifold $Q \subset R$. By Lemma 1.4.29 this means $Q \subset G_0$ is an admissible embedded manifold. $\square$

1.5 The existence of a specific cover

Aim of this section: Proposition 1.5.9 (abbreviated)

For any proper and regular groupoid G there exists an admissible cover, i.e. a cover that satisfies specific properties. For example, it is comprised of sets of the form $U_n = \operatorname{Sat}(R_n)$ where R_n are admissible embedded manifolds as defined in the previous section.

This section does not aim to produce results that are of use in the general study of a proper and regular Lie groupoid, but rather a specific construction that has explicit properties which will become extremely relevant throughout this thesis. The goal itself is to construct quite ordinary objects, such as covers, subcovers, partitions of unity or cut-off functions, but in a specific framework tied to our previous notions of admissible manifolds and basic functions.

The statements in this section are therefore often quite easy to apply without a deeper understanding of the underlying construction, while the proofs are mostly technical.

Definition 1.5.1. Consider a regular and proper Lie groupoid G. A **cover** $\{U_n\}_{n\in N}$ for
$$N = \mathbb{N}_0 \quad \text{or} \quad N = \{n \in \mathbb{N}_0 \text{ s.t. } 0 \leq n \leq f\}$$
is **admissible with admissible embedded manifolds** $\{R_n\}_{n\in N}$ if

- $\{R_n\}_{n\in N}$ is a family of admissible embedded manifolds.
- $\operatorname{Sat}(R_n) = U_n$ for each $n \in N$.
- $\{U_n\}_{n\in N}$ is a locally finite cover of G_0.

If the closed sets $\{\overline{R_n}\}_{n\in N}$ are additionally pairwise disjoint, we call it **admissible with properly disjoint admissible embedded manifolds**.

An **admissible subordinate refinement** of such an admissible cover is an admissible cover $\{V_n\}_{n\in N}$ with admissible embedded manifolds $\{Q_n\}_{n\in N}$ where we have that
$$V_n \subset U_n$$
for all $n \in N$.

Remark. We will adhere to the notion
$$N = \mathbb{N}_0 \quad \text{or} \quad N = \{n \in \mathbb{N}_0 \text{ s.t. } 0 \leq n \leq f\}$$
throughout this section to simplify the statements.

As it will turn out, finding an admissible cover by properly disjoint admissible manifold is almost always possible. Only when the dimension of $\mathcal{F}_0$ is given by $l_0 = 0$ we will not be able to give a general construction. However, we often have that in the case where the orbits are discrete, the trivial cover $\{R\}$ for $R = G_0$ is admissible, in particular when G is a Lie group bundle.

Definition 1.5.2. Consider a regular and proper Lie groupoid G and an admissible cover $\{U_n\}_{n\in N}$ with admissible embedded manifolds $\{R_n\}_{n\in N}$. If all R_n are connected and contractible, we call the cover **admissible with connected contractible admissible embedded manifolds**.

Definition 1.5.3. Consider a Lie groupoid G and a cover $\{U_n\}_{n\in N}$ by saturated open sets, e.g. an admissible cover. Then a **partition of unity** $\{\chi_n\}_{n\in N}$ **is admissible** if

$$\chi_n \in C^\infty_{\text{bas}}(G_0)$$

for all $n \in N$.

Definition 1.5.4. Consider a Lie groupoid G. Further consider a cover $\{U_n\}_{n\in N}$ by saturated sets and a subordinate refinement $\{V_n\}_{n\in N}$ by saturated sets satisfying that

$$\overline{V_n} \subset U_n$$

as subsets of G_0. Then a **family of cut-off-functions** $\{\xi_n\}_{n\in N}$ associated respectively to the inclusion $V_n \subset U_n \subset G_0$ is **admissible** if

$$\xi_n \in C^\infty_{\text{bas}}(G_0)$$

for all $n \in N$.

Remark. By a cut-off-function for two open sets $V \subset U \subset G_0$ with $\overline{V} \subset U$, we mean a non-negative function $\xi \in C^\infty(G_0)$ such that

$$\text{supp}(\xi) \subset U$$

and

$$\xi \equiv 1$$

on V and

$$\xi \equiv 0$$

on G_0/U. Technically, this is a cut-off-function between $\overline{V}$ and G_0/U.

Lemma 1.5.5. *Consider G an proper and regular groupoid with $R \subset G_0$ admissible and $Q \subset R$ open, such that $\overline{\operatorname{Sat}(Q)} \subset G_0$ is contained in $\operatorname{Sat}(R)$. Then there exists*

$$\xi \in C^\infty_{\text{bas}}(G_0)$$

which is a cut-off-function between $\overline{\operatorname{Sat}(Q)} \subset G_0$ and $G_0 \setminus \operatorname{Sat}(R)$.

Proof. Note that we may without loss of generality assume that $Q \subset R$ is saturated already (as we may otherwise replace Q by $\operatorname{Sat}(Q) \cap R$, as this is still open with the same saturation). We will start the construction of ξ by applying Corollary 1.4.11 to $\overline{Q}$ and $G_0/\operatorname{Sat}(R)$ to find open saturated sets N and O with disjoint closure separating the saturated closed ones. Then, we choose an arbitrary cut-off-function $\xi' \in C^\infty(G_0)$ between

$\overline{N}$ and $\overline{O}$ (cf. [30], Proposition 2.25.). We have that $\mathrm{supp}(\xi') \subset \mathrm{Sat}(R)\backslash\overline{O}$, implying that also

$$\overline{\mathrm{Sat}(\mathrm{supp}(\xi'))} \cap O = \emptyset.$$

As $G|_R$ is s-proper, we can apply Corollary 1.4.27 to obtain

$$\overline{\xi}(m) = \frac{1}{|O_m|} \sum_{n \in \mathcal{O}_m} \xi'(n),$$

where $\overline{\xi} \in C^\infty(R)$. Furthermore, we have $\overline{\xi} \equiv 1$ on Q as $\xi' \equiv 1$ on $\mathrm{Sat}(Q)$, such that we only average over 1 on these orbits.

Also, as subsets of R we have that

$$\mathrm{supp}(\overline{\xi}) \subset \overline{\mathrm{Sat}(\mathrm{supp}(\xi'|_R))}$$

as the latter is closed and contains all points where $\overline{\xi}(m) > 0$, as this implies $\xi'(n) > 0$ for some $n \in \mathcal{O}_m$. We have, by construction, that

$$\overline{\mathrm{Sat}(\mathrm{supp}(\xi'|_R))} \cap O = \emptyset,$$

as $O \cap R$ is saturated and open.

Lastly, we have that $\overline{\xi}$ defines a function $\xi \in C^\infty_{\mathrm{bas}}(\mathrm{Sat}(R))$ by

$$\xi(m) = \overline{\xi}(n)$$

for n any point in $\mathcal{O}_m \cap R$. To see that this function is well-defined and smooth, we can consider the surjective submersions $s : s^{-1}(R) \to R$ and $t : s^{-1}(R) \to \mathrm{Sat}(R)$.

If we pull back $\overline{\xi}$ along s, we obtain $\widehat{\xi} \in C^\infty(s^{-1}(R))$. Because $\overline{\xi}$ was basic, we obtain by Lemma 1.3.4 that $\widehat{\xi}$ is fibre-wise constant along t. As t is surjective, this uniquely defines a function $\xi \in C^\infty(\mathrm{Sat}(R))$, where smoothness is implied as t is a submersion. Furthermore, if we consider two points in the same orbit, the corresponding values of ξ are both defined to agree with the value of $\overline{\xi}(n)$, where n is any point in the intersection of the common orbit with R. Again using Lemma 1.3.4, this means that the resulting function is basic.

Lastly, let us note that as subsets of $\mathrm{Sat}(R)$, we have

$$\mathrm{supp}(\xi) \subset \overline{\mathrm{Sat}(\mathrm{supp}(\overline{\xi}))},$$

as any m with $\xi(m) > 0$ lies in the orbit of some $r \in R$ with $\overline{\xi}(r) > 0$ by construction. However, it is also disjoint from O, which implies

$$\mathrm{supp}(\xi) \subset G_0\backslash O \subset \mathrm{Sat}(R).$$

This means that we can continue ξ smoothly by 0 outside of $\mathrm{Sat}(R)$ and that we have indeed constructed an admissible cut-off-function for $\mathrm{Sat}(Q) \subset \mathrm{Sat}(R) \subset G_0$. $\square$

Corollary 1.5.6. *Let G be a proper and regular Lie groupoid. Consider an admissible embedded manifold $R \subset G_0$ and a saturated open $U \subset G_0$ such that $\mathrm{Sat}(R) \cup U = G_0$. Then there exist non-negative basic functions $f_1, f_2 \in C^\infty_{\mathrm{bas}}(G_0)$ such that*

$$f_1 \equiv 0$$

on an open containing $G_0 \setminus U$ and

$$f_2 \equiv 0$$

on an open containing $G_0 \setminus \mathrm{Sat}(R)$ as well as

$$f_1 + f_2 = 1.$$

Proof. We can apply Corollary 1.4.11 to the closed sets $G_0 \setminus U$ and $G_0 \setminus \mathrm{Sat}(R)$. Let O be the open around $G_0 \setminus U$, then we immediately have that $O \subset \mathrm{Sat}(R)$ and $\overline{O} \subset \mathrm{Sat}(R)$. Therefore we can apply Lemma 1.5.5 to $O \cap R$, which by definition satisfies

$$\mathrm{Sat}(O \cap R) = O.$$

Therefore we obtain a cut-off-function ξ between $\overline{O}$ and $G_0 \setminus \mathrm{Sat}(R)$. By postcomposing with a suitable function if necessary we can ensure that ξ only takes values in $[0, 1]$. In particular, we have that

$$f_1 = 1 - \xi$$

and

$$f_2 = \xi$$

satisfy the properties as claimed in the statement. □

Lemma 1.5.7. *Every admissible cover $\{U_n\}_{n \in N}$ with admissible embedded manifolds $\{R_n\}_{n \in N}$ admits an admissible subordinate refinement $\{V_n\}_{n \in N}$ with admissible embedded manifolds $\{Q_n\}_{n \in N}$ such that*

$$\overline{Q_n} \subset R_n$$

as subsets of G_0 and

$$\overline{V_n} \subset U_n$$

for all $n \in N$.

Proof. We will inductively define $Q_n \subset R_n$ for $n \in \mathbb{N}_0$. Both the case $n = 0$ and the induction step follow the same procedure, which is why they will be simultaneously studied.

Consider the partial cover of R_n by $V_k \cap R_n$ for $k < n$ and $U_l \cap R_n$ for $l > n$. The points that are not covered are those of

$$C = R_n / \left(\bigcup V_k \cup \bigcup U_l \right) = \overline{R_n} / \left(\bigcup V_k \cup \bigcup U_l \right),$$

which is a closed set in G_0 with closed saturation

$$G_0 / \left(\bigcup V_k \cup \bigcup U_l \right).$$

By definition, it is disjoint from $G_0 / \mathrm{Sat}(R_n)$ which is closed and contains ∂R_n as a subset by Lemma 1.4.34. We can now apply that closed disjoint sets can be separated by disjoint open neighbourhoods, yielding $O \supset C$ open with $\overline{O} \cap \partial R_n = \emptyset$. We may assume O to be saturated by Proposition 1.4.10.

We define

$$Q_n := O \cap R_n \supset C,$$

which is open in R_n, and also satisfies

$$\overline{Q_n} \cap \partial R_n = \emptyset$$

in G_0, implying $\overline{Q_n} \subset R_n$. Also, as O is saturated with respect to G we have that Q_n is saturated with respect to $G|_{R_n}$ and thus admissible by Lemma 1.4.31.

Furthermore, if we consider

$$\mathrm{Sat}(Q_n) \cup \{V_k\}_{k<n} \cup \{U_l\}_{l>n},$$

we obtain a cover of G_0. This is because outside of U_n, we have that

$$\{V_k\}_{k<n} \cup \{U_l\}_{l>n}$$

is a cover by assumption and as all sets are saturated, we may check the cover property on U_n on R_n instead, where it holds by construction.

It only remains to argue why the sets

$$\{V_k := \mathrm{Sat}(Q_k)\}_{k\in\mathbb{N}_0}$$

themselves form a cover of G_0. However, the cover by the $\{U_k\}$ was assumed to be locally finite. So for any point $m \in G_0$ there is a $n \in \mathbb{N}_0$ such that $m \notin U_l$ for $l > n$. This means that as

$$\{V_k\}_{k\leq n} \cup \{U_l\}_{l>n}$$

is a cover of G_0, we have that $m \in V_k$ for some $k \leq n$. So all points are contained in the union $\cup_{k\in\mathbb{N}_0} V_k = G_0$. □

Lemma 1.5.8. *Consider G a proper and regular groupoid. Then every admissible cover of G admits an admissible partition of unity.*

Proof. By Lemma 1.5.7, we may choose for every admissible cover $\{U_n\}_{n\in N}$ with embedded manifolds $\{R_n\}_{n\in N}$ an admissible subordinate refinement $\{V_n\}_{n\in N}$ with embedded manifolds $\{Q_n\}_{n\in N}$ such that

$$\overline{V_n} \subset U_n$$

for all $n \in N$.

We can now apply Lemma 1.5.5 to obtain cut-off-functions $\xi_n \in C^\infty_{\mathrm{bas}}(G_0)$ for each pair $\overline{V_n} \subset U_n$. As the cover by the U_n is locally finite and $\mathrm{supp}(\xi_n) \subset U_n$, we find that the sum

$$f = \sum_{n\in N} \xi_n$$

is well-defined. As the V_n form a subcover of G_0, we have that

$$f(m) \geq 1 > 0$$

for all $m \in G_0$. Hence the term

$$\chi_n(m) := \frac{1}{f(m)}\xi_n(m)$$

is well-defined and by construction the collection $\{\chi_n\}_{n\in N}$ constitute an admissible partition of unity. Furthermore, we can apply Lemma 1.3.4 (on the ξ_n as well as the newly constructed functions) to find that f and also each χ_n are basic, such that the partition of unity is admissible. □

Proposition 1.5.9. *Let G be a proper and regular Lie groupoid. Then G admits an admissible cover $\{U_n\}_{n\in N}$ with admissible embedded connected and contractible manifolds $\{R_n\}_{n\in N}$. If we have that the dimension of the orbit foliation F_0 is given by $l_0 \neq 0$, the admissible manifolds can be chosen properly disjoint.*

Remark. This proof is perhaps more complicated then necessary. However, the construction given has some additional properties which we will prove at a later point (to be precise, in Lemma 5.2.27).

Proof. This proof will adhere to the following structure:

Step 1: Describe how to cover each point by an open generated by an admissible embedded manifold adhering to different restrictions.

Step 2: Describe how to construct a locally finite cover where each open set is generated by an admissible embedded manifold. The cover will admit a specific refinement also generated by admissible manifolds.

Step 3: Describe how to construct a refinement of the previous cover satisfying that the admissible embedded manifolds have pairwise disjoint closure.

Step 1:

As G was assumed regular, we know that locally around any $m \in G_0$ we can choose coordinates on some open neighbourhood O_m of m satisfying that the chart takes values in a cube of $\mathbb{R}^{k_0}$ and the foliation bundle is given by

$$T\mathbb{R}^{l_0} \times \{0_x\}_{x \in \mathbb{R}^{k_0 - l_0}}.$$

For any closed set C with $m \notin C$, we may assume that $O_m \cap C = \emptyset$ without loss of generality. For any open set O with $m \in O$ we may without loss of generality assume that $O_m \subset O$ (as cubes form an open basis of the topology of $\mathbb{R}^{k_0}$). Then, for any $m \in G_0$ we can choose a radius r_m and parameter ε_m such that in coordinates

$$[-\varepsilon_m, \varepsilon_m]^{l_0} \times \overline{B_{r_m}(0)} \subset O_m$$

implying that

$$\{y\} \times B_{r_m}(0) \subset O_m$$

is a complementary embedded manifold for $y \in [-\varepsilon_m, \varepsilon_m]^{l_0}$. By Lemma 1.4.37 (also considering the proof, where we pick an invariant metric making balls saturated), we may without loss of generality assume that the coordinates are chosen in a way that any $\{0\} \times B_{r'}(0)$ for $r' \leq r_m$ is admissible (and actually even the full groupoid over it is trivialisable). Let us denote

$$\widehat{R}_m = \{0\} \times B_{r_m}(0).$$

and

$$U'_m = \mathrm{Sat}(\{0\} \times B_{r_m}(0)) = \mathrm{Sat}(\widehat{R}_m)$$

and

$$V'_m = \mathrm{Sat}(\{0\} \times B_{\frac{1}{2}r_m}(0)).$$

By Lemma 1.4.15, both U'_m and V'_m are open. By Lemma 1.4.7, we also have that

$$\mathrm{Sat}(\{0\} \times \overline{B_{r_m}(0)})$$

is closed and hence

$$\overline{V'_m} \subset \mathrm{Sat}(\{0\} \times \overline{B_{r_m}(0)}) \subset U'_m.$$

Step 2:

We start off by exhausting G_0 by precompact open subsets $\{P_k\}_{k \in \mathbb{N}_0}$, where the compact closures satisfy

$$C_k = \overline{P_k} \subset P_{k+1}.$$

We will now iteratively pick finite covers for the C_k by extending a given cover for C_{k-1}. Each open of the cover will be of the form of some V'_m (for some r_m and ϵ_m that have arisen during the construction) as defined in the previous step. Let I be the finite indexing set of the cover and denote

$$U' = \cup_{n\in I} U'_n.$$

and

$$V' = \cup_{n\in I} V'_n.$$

Now we aim to apply the construction of Step 1. For any compact set C_{k-1} we have that $\mathrm{Sat}(C_{k-1})$ is closed by Lemma 1.4.7. As we have discussed, this means that for any $m \notin \mathrm{Sat}(C_{k-1})$, we can choose O_m such that

$$O_m \cap \mathrm{Sat}(C_{k-1}) = \emptyset.$$

implying directly that

$$\widehat{R}_m \cap \mathrm{Sat}(C_{k-1}) = \emptyset$$

and as $\mathrm{Sat}(C_{k-1})$ is saturated

$$U'_m \cap \mathrm{Sat}(C_{k-1}) = \emptyset.$$

So the open set V' (which covers $\mathrm{Sat}(C_{k-1})$) together with the open sets V'_m for all $m \notin \mathrm{Sat}(C_{k-1})$ form a cover of C_k. By compactness of C_k, there exists a finite subcover. So in particular, we have now found an open cover of $C_{k-1}\backslash V'$. If we extend this cover by the cover of C_{k-1} given by V_n for $n \in I$, we obtain a cover of C_k.

The union of all of these iteratively constructed covers results by construction in a countable cover of G_0. The union of all the associated U'_m by construction also form a countable cover of G_0, for which the cover by the V'_m automatically defines a refinement. We claim that both of these covers are locally finite: For any $m \in G_0$ we have that $m \in C_k$ for some $k \in \mathbb{N}_0$. Hence $P_k \subset C_{k+1}$ is an open around m. We have that by the $(k+1)$-st iteration of the procedure, we have constructed a finite cover of C_{k+1}. After the $(k+1)$-st iteration, we only consider sets such that $U'_n \cap \mathrm{Sat}(C_{k+1}) = \emptyset$ to add to the cover. So indeed $m \notin U'_n$ for all but finitely many U'_n for the chosen cover.

Step 3:

Let us consider the locally finite cover constructed in Step 2. It is comprised of a countable number of open sets, which we will denote U'_k for $k \in \mathbb{N}_0$ with a subordinate refinement given by V'_k. For each of these $V'_k \subset U'_k$ there is a point $m_k \in G_0$ and an open coordinate chart O_k such that

$$U'_k = \mathrm{Sat}\left(\{0\} \times B_{r_k}(0)\right) = \mathrm{Sat}(\widehat{R}_k)$$

and

$$V'_k = \mathrm{Sat}\left(\{0\} \times B_{\frac{1}{2}r_k}(0)\right).$$

Now we aim to iteratively replace each U'_k and V'_k by a finite cover of V'_k by sets of the form U_m and V_m, where $\overline{V_m} \subset U_m$ and $V_m = \mathrm{Sat}(R_m)$. We will construct these sets such that each $\overline{R_n}$ intersects each orbit only finitely many times. Assume that we have already replaced all U'_l and V'_l for $0 \leq l < k$. Let us denote the collection of all the obtained open sets by $\{V_n\}_{n\in I}$ and $\{U_n\}_{n\in I}$ and the admissible embedded manifolds by $\{R_n\}_{n\in I}$.

For every $m \in \overline{V_k} \cap \widehat{R}_k = \{0\} \times \overline{B_{\frac{1}{2}r_k}(0)}$ we will now describe a construction that yields admissible connected and contractible embedded manifolds R'_m and Q'_m and open saturated sets U_m and V_m such that the following holds:

- We have
$$\mathrm{Sat}(R'_m) = V_m.$$

- We have
$$\mathrm{Sat}(Q'_m) = U_m.$$
- We have
$$R'_m \subset Q'_m \subset \widehat{R}_k$$
and
$$\overline{Q'_m} \subset \widehat{R}_k.$$
- If $l_0 \neq 0$, we have an embedding
$$(0,1) \times \widehat{R}_k \subset O_k$$
constructed from the natural identification
$$\{p \cdot B\} \times B_{r_k}(0) \cong \{0\} \times B_{r_k}(0).$$
for some $B \in \mathbb{R}^{l_0}$ such that for infinitely many values of $p \in (0,1)$ we have that
$$\{p\} \times \overline{R'_m} \cap \overline{R_n} = \emptyset$$
for any $n \in I$ and R_n the associated embedded manifold constructed in a previous iteration for $0 \leq l < k$.
- Consider any $n \in I$ and the associated pair $V_n \subset U_n$ constructed in a previous iteration for $0 \leq l < k$. We have that if $m \in U_n$ then $Q'_m \subset U_n$ and if $m \notin U_n$ then $Q'_m \cap V_n = \emptyset$.

Let us now describe a construction with these properties: First of all, from the $V'_l \subset U'_l$ for $0 \leq l < k$ we have obtained a collection of finitely many finite open covers (of different sets). Their union is therefore still finite. This means for every $m \in \overline{V_k} \cap \widehat{R}_k$ we can start by choosing an open subset $O_m \subset \widehat{R}_k$ satisfying that $O_m \subset G_0 \backslash \overline{V_n}$ if $m \notin U_n$(as these are finitely many constraints) and similarly we may choose $O_m \subset U_n$ if $m \in U_n$ for any U_n contained in some cover of the V'_l. In this step, we could even consider a stricter set of conditions, for example, we could ask that there exists a local bisection defined on all of O_m with image contained in Q'_n by Corollary 1.4.20 (this will be relevant in the proof of Lemma 5.2.27).

Then we consider $G|_{\widehat{R}_k}$. By Lemma 1.4.37, we can choose some $Q'_m \subset O_m$ which is a connected and contractible admissible embedded open submanifold of $\widehat{R}_k$ with $\overline{Q'_m} \subset \widehat{R}_k$. By Lemma 1.4.29, we know that this means that $Q'_m \subset G_0$ is admissible as well. If we are in the case of $l_0 = 0$, we can skip the following steps and define $R_m := R'_m := Q'_m$. They, by construction, satisfy the properties that we have claimed. Note in particular, that $\overline{R_m}$ and $\overline{R'_m}$ are subsets of $\widehat{R}_k$, which is compact, meaning it intersects each orbit only finitely many times.

In the following, we will now focus on the case where $l_0 \neq 0$. For this consider the local leaf of m in the coordinate chart O_k. It contains precisely those points of $\mathbb{R}^{l_0} \times \{m\}$ intersecting the cube containing the coordinates. We have that the $\overline{R_n}$ are closed, meaning that their (finite) union is closed as well. So if there is some $B_m \in \mathbb{R}^{l_0}$ with
$$\{B_m\} \times \{m\} \in O_k \backslash \overline{R_n}$$

for all of the $n \in I$, then the same is true on an open neighbourhood of $\{B_m\} \times \{m\}$. However, if $l_0 \neq 0$ each $\overline{R_n}$ only intersects the (infinite) local leaf in finitely many points, so it is indeed possible to find B_m as described. The induced embedding

$$(0,1) \times \widehat{R}_k \in O_k$$

is in particular disjoint from any $\overline{R_n}$ on an open around around the endpoint 1.

So for any $R'_m \subset Q'_m$ small enough, we find that

$$\{p \cdot B_m\} \times \overline{R'_m} \cap \left(\cup_{n \in I} \overline{R_n}\right) = \emptyset.$$

for all p in some open subset of $(0,1)$. In particular, we may use Lemma 1.4.37 to choose R'_m admissible with this property. By construction, we have that R'_m and Q'_m satisfy the properties that we claimed earlier. If we now define $V_m = \mathrm{Sat}(R'_m)$ and $U_m = \mathrm{Sat}(Q'_m)$, we have constructed the required data.

Now consider the cover of $\{0\} \times \overline{B_{\frac{1}{2}r_k}(0)} \subset \widehat{R}_k$ by the V_m. As the set is compact, we can find a finite subcover. It only remains to choose p_m for the finitely many V_m contained in this cover, such that under the embedding associated to m (depending on B_m) of

$$(0,1) \times \widehat{R}_k \subset O_k$$

we have

$$\{p_m\} \times \overline{R'_m} \cap \overline{R_n} = \emptyset.$$

As we have infinitely many possibilities for p_m, we may choose them such that $p_m \cdot B_m \in \mathbb{R}^{l_0}$ is a unique vector for all m indexing the cover. This immediately implies that if we define

$$R_m := \{p_m\} \times R'_m \subset O_k$$

the closures

$$\overline{R_m} := \{p_m\} \times \overline{R'_m} \subset O_k$$

are pairwise disjoint. If we furthermore note that R_m is isomorphic to R'_m by translation along a local leaf, it immediately follows, that $\mathrm{Sat}(R_m) = V_m$ and that R_m intersects each orbit the same number of times as R'_m. In analogy to the argument in case $l_0 = 0$, we have that $R'_m \subset \widehat{R}_k$ implies, that this number is finite for any orbit.

Therefore, the described method is indeed well-defined and we can use it to construct a cover as claimed.

$\square$

Chapter 2

A notion of invariance

In the case of a Lie group $\mathcal{G}$ acting freely and properly on a manifold M, we know that there exists a quotient $Q = M/\mathcal{G}$ such that $M \to Q$ is a principal $\mathcal{G}$-bundle. We have already seen that in this case, the equivariant cohomology of the associated action agrees with the cohomology of Q (Corollary 0.1.5). We can actually even check that Q, when interpreted as a trivial groupoid, is Morita equivalent to the action groupoid (by considering M as the right principal bundle describing both inverses). Hence we know we can compute the differentiable stack cohomology of the action groupoid G associated to the action $\mathcal{G} \circlearrowright M$ by computing the cohomology of $\Omega(Q) = \Omega_{\text{bas}}(M)$. As we have previously seen, a different way to interpret $\Omega_{\text{bas}}(M)$ is as the J^1-invariant forms on G_0 or any other G_q.

That a notion of invariance appears here does not seem surprising, as it often does when working with Lie group actions both in general and in the context of equivariant cohomology. Therefore we might intuit that differentiable stack cohomology can be computed using some concept of invariance and perhaps even J^1G-invariant forms. However, we have already found that J^1G-invariant forms do not depend on the degree q. Especially in the example of a group acting on a point, this would only leave us with constants, not with $(S^\bullet(\mathfrak{g}^*))^G$, which we know to be the correct result. So invariance under all jets can not be the appropriate notion for our purposes, but it turns out that asking for invariance under a certain subset of jets leads to interesting results. In this chapter, we will show that the cohomology of the columns of the Bott–Shulman–Stasheff complex only depends on these so called J^EG-invariant forms.

2.1 Multiplicative Ehresmann connections

Aim of this section: Definitions and Concepts

What is a multiplicative Ehresmann connection? How does it define a subgroupoid of J^1G? What do we know about forms which are invariant under elements of J^EG?

Multiplicative Ehresmann connections are a recent tool in higher differential geometry. They go back to [29] by Laurent-Gengoux, Stiénon and Xu but have been in broader generality defined by Fernandes and Marcut in [19].

As it will turn out later in this chapter, the multiplicative Ehresmann connections with respect to the isotropy algebroid of a regular groupoid are a tool that can define a useful subvector space of the Bott–Shulman–Stasheff bicomplex. It is both large enough to capture its vertical cohomology and small enough so we can find relatively explicit expressions.

Definition 2.1.1 (cf. [19], Definition 2.3.)**.** Consider a Lie groupoid $G_1 \rightrightarrows G_0$ with a Lie algebroid $A \to G_0$ where $\rho : A \to TG_0$ denotes the anchor. We call a vector subbundle $B \subset A$ a **bundle of ideals** of G_1 if $B \subset \ker(\rho)$ and B is invariant under conjugation by any $x \in G_1$ defined on $\ker(\rho) \subset \ker(Ts) \cap \ker(Tt) \subset TG_1$ by

$$(D^1_{x,1})_* \circ (D^0_{x,1})_* : (\ker(\rho))_{s(x)} \to (\ker(\rho))_{t(x)}.$$

Remark. The notion of "invariant" used in the definition describes the requirement that under the given map, the image of $B_{s(x)} \subset (\ker(\rho))_{s(x)}$ lies in $B_{t(x)}$. Regarding the notation, recall that in accordance with Definition 1.1.26 we denote left multiplication with an element $x \in G_1$ as

$$D^0_{x,1}(z) = \mu(x, z)$$

(a priori defined on $t^{-1}(s(x))$, while right multiplication with x^{-1} is given by the left action

$$D^1_{x,1}(z) = \mu(z, x^{-1}),$$

defined on $s^{-1}(s(x))$). Note that the conjugation by x of the definition is well-defined by Example 1.1.28 and Example 1.2.20 as well as the fact that $\mu(x, 1_{s(x)}, x^{-1}) = 1_{t(x)}$.

We can see that in case of G regular, the adjoint action of G on A_{iso} is precisely the action by conjugation. For this, recall the adjoint action of G on $\text{Iso}^0(G)$ was defined in Example 1.1.30 and Definition 1.1.39 explicitly as

$$\text{Ad}^0_x(z) = \mu(x, z, x^{-1})$$

on $\text{Iso}^0(G)$. This was derived in Example 1.1.33 to an adjoint action on $A_{\text{iso}} = \ker(\rho)$ and explicitly writing out the corresponding expressions yields

$$X \in (A_{\text{iso}})_{s(x)} \mapsto T\mu(0_x, X, 0_{x^{-1}}),$$

which agrees with the result under $(D^1_{x,1})_* \circ (D^0_{x,1})_*$.

Lemma 2.1.2. *Let G be a Lie groupoid. For every $x \in G_1$ and every bundle of ideals B we have that*

$$(T1_{s(x)} D^0_{x,1})(B_{s(x)}) = (T1_{t(x)} D^1_{x^{-1},1})(B_{t(x)}) =: V_x \subset T_x G_1$$

agree as vector spaces. The resulting subbundle V of $\ker(Ts) \cap \ker(Tt)$ is invariant under both left and right multiplication.

Remark. If we are given a vector bundle $V \subset TG_1$ such that

$$V \subset \ker(Ts) \cap \ker(Tt)$$

and V is invariant under left and right multiplication, we can also restrict to 1_{G_0} to obtain a bundle of ideals B. In other words, this construction is a 1-to-1 correspondence.

Proof. We have that the inverse to $T1_{s(x)} D^0_{x,1}$ is given by $T_x D^0_{x^{-1},1}$ (analogously for right multiplication). So the claim is equivalent to

$$(T_x D^1_{x,1})(T1_{s(x)} D^0_{x,1})(B_{s(x)}) = (D^1_{x,1})_*(D^0_{x,1})_*(B_{s(x)}) = B_{t(x)}.$$

However, by definition $B \subset \ker(\rho)$ is a bundle that is precisely invariant under this map.

Furthermore, the claim that V is invariant under left multiplication follows directly from the equation

$$(T1_{s(x)} D^0_{x,1})(B_{s(x)}) = V_x$$

and the associativity axiom, for right multiplication the analogue argument holds. $\square$

Definition 2.1.3 ([19], Definition 2.4)**.** Let G be a Lie groupoid. Consider a bundle of ideals B and its induced vector bundle $V \subset TG_1$ as in Lemma 2.1.2. A **multiplicative Ehresmann connection** is a splitting

$$TG_1 \cong E \oplus V$$

such that the $E \rightrightarrows TG_0$ is naturally a subgroupoid of $TG_1 \rightrightarrows TG_0$.

Remark. It is also part of the definition of [19] that in the case a multiplicative Ehresmann connection with respect to B exists, the bundle B is called partially split.

Definition 2.1.4. Let G be a Lie groupoid and E a multiplicative Ehresmann connection with respect to a bundle of ideals B. An element $V_x \in J^1(s : G_1 \to G_0)$, where $x \in G_1$ denotes the base-point of V_x, is called **E-horizontal** if $V_x \subset E_x$. We will denote the subset of all E-horizontal s-jets by

$$J^E(s : G_1 \to G_0) \subset J^1(s : G_1 \to G_0).$$

Lemma 2.1.5. *Consider a Lie groupoid G together with a multiplicative Ehresmann connection E with respect to a bundle of ideals B. Then the subspace*

$$J^E(s : G_1 \to G_0) \subset J^1(s : G_1 \to G_0)$$

has the structure of a closed embedded submanifold.

For the proof, we refer to A.3.2, as the proof adds little to the understanding of multiplicative Ehresmann connections (in fact, it does not depend on the multiplicativity at all).

Corollary 2.1.6. *Consider a Lie groupoid G together with a multiplicative Ehresmann connection E with respect to a bundle of ideals B. The subspace of the morphisms of J^1G given by*

$$\Big\{W_x \subset E_x \text{ s.t. } T_x s : W_x \xrightarrow{\sim} T_{s(x)}G_0 \text{ and } T_x t : W_x \xrightarrow{\sim} T_{t(x)}G_0 \text{ isomorphically}\Big\} \subset (J^1G)_1$$

is a closed submanifold.

Proof. We know by Lemma 2.1.5 that $J^E(s : G_1 \to G_0)$ is a closed embedded submanifold. We also know that $(J^1G)_1 \subset J^1(s : G_1 \to G_0)$ is an open subset. This directly proves the claim. □

Lemma 2.1.7. *Let G be a Lie groupoid and E be a multiplicative Ehresmann connection with respect to a bundle of ideals B. Then the submanifold $J^E(s : G_1 \to G_0)$ is closed under the multiplication on $J^1(s : G_1 \to G_0)$ as defined in Definition 1.2.2. The Lie groupoid structure of J^1G restricts to the submanifold defined in Corollary 2.1.6.*

Proof. For the multiplication, we know that

$$\widehat{\mu}(V_x, W_y) \in J^1(s : G_1 \to G_0),$$

is a jet which only contains vectors of the form

$$T\mu(X, Y)$$

for $X \in V_x \subset E_x$ and $Y \in W_y \subset E_y$. Hence $T\mu(X,Y) \in E_{\mu(x,y)}$ and

$$\widehat{\mu}(V_x, W_y) \in J^E(s : G_1 \to G_0).$$

It remains to show that for J^1G the restrictions of the other Lie groupoid structure maps are well-defined. However, the argument for inverses is analogous. The unit map are contained in the tangent space to the unit section, which by definition is a subspace of E. □

We will give this restricted structure a specific name: The E-jet groupoid.

Definition 2.1.8. Given a Lie groupoid G together with a multiplicative Ehresmann connection E with respect to a bundle of ideals B. The **E-jet groupoid** J^EG is defined as the groupoid with

- object space $(J^EG)_0 := G_0$,
- morphism space
$$(J^EG)_1$$
$$:= \left\{ V_x \subset E_x \text{ s.t. } T_xs : V \xrightarrow{\sim} T_{s(x)}G_0 \text{ and } T_xt : V \xrightarrow{\sim} T_{t(x)}G_0 \text{ isomorphically} \right\}$$
- source, target, composition, inversion and unit maps are inherited by J^1G.

Lemma 2.1.9. *Given a Lie groupoid G together with a multiplicative Ehresmann connection E with respect to a bundle of ideals B. There is a canonical inclusion $J^EG \to J^1G$ and a canonical projection $J^1G \to J^EG$. The latter establishes J^1G as a VB-groupoid over J^EG.*

For the proof, we refer to A.3.3, as this is a nice observation, but not of central importance to this chapter.

Definition 2.1.10. Let G be a Lie groupoid and E be a multiplicative Ehresmann connection with respect to some bundle of ideals B. Consider a left action of G on $p : N \to G_0$ and the induced left action of J^1G on $p \circ \mathrm{pr}_N : TN \to G_0$ of Proposition 1.2.10. If an element $\omega \in \Omega^p(N)$ satisfies that for any $n \in N$ and $x \in G_1$ with $s(x) = p(n)$ and $V_x \in (J^EG)_1$ we have that

$$\omega_n(X_1, \dots, X_p) = \omega_{L(x,n)}(\widehat{L}(V_x, X_1), \dots, \widehat{L}(V_x, X_p)),$$

then we call ω a **J^EG-invariant form of** $p : N \to G_0$ **with respect to** L.

Lemma 2.1.11. *For a regular Lie groupoid G, the bundle V as constructed in Lemma 2.1.2 associated to the bundle of ideals $A_{\mathrm{iso}} = \ker(\rho)$ is given by*

$$\ker(Ts) \cap \ker(Tt).$$

Proof. Note that, by definition, the action by $x \in G_1$ has domain $\ker(\rho)_{s(x)} = (A_{\mathrm{iso}})_{s(x)}$ and target $\ker(\rho)_{t(x)} = (A_{\mathrm{iso}})_{t(x)}$, so (A_{iso}) is an invariant bundle. We have that $\ker(Ts)$ is a subbundle of TG_1 (because s is a submersion), which at each point x is isomorphic to $A_{t(x)}$ by right translation. Under this isomorphism, $\ker(Tt) \cap \ker(Ts)$ is identified with $(A_{\mathrm{iso}})_{t(x)}$ as $t \circ D^1_{x,1} = t$ and Tt at a unit element is precisely the map defining ρ. □

Definition 2.1.12. Let G be a regular Lie groupoid. We will call the bundle

$$IG_1 := \ker(Tt) \cap \ker(Ts)$$

the **isotropy tangent subbundle**.

Corollary 2.1.13. *The isotropy tangent subbundle of a regular Lie groupoid G is indeed a vector bundle on G_1.*

Proof. As A_{iso} is locally trivial over G_0, and $V = IG_1$ at each point $g \in G_1$ is isomorphic to $(A_{\mathrm{iso}})_{t(g)}$ by right translation, we have that IG_1 is also locally trivial. □

Theorem 2.1.14 ([19], Theorem 4.2)**.** *Any bundle of ideals in a proper Lie groupoid G is partially split. In particular, if G is proper and regular the bundle given by A_{iso} is partially split, meaning that there exists a multiplicative Ehresmann connection with respect to the isotropy algebroid.*

Remark. Let G be a proper regular Lie groupoid. Spelling out the definition, a multiplicative Ehresmann connection with respect to the isotropy algebroid A_{iso} consists of a splitting

$$TG_1 \cong E_1 \oplus IG_1$$

such that $E_1 \rightrightarrows TG_0$ inherits a groupoid structure.

Lemma 2.1.15. *Let G be a regular Lie groupoid. We have that*

$$\bigcap \ker(Tp_q^i) = IG_1 \times_{TG_0} \cdots \times_{TG_0} IG_1,$$

which is a vector bundle over G_q and in particular a smooth submanifold of TG_q. Furthermore, it splits into a direct sum of components

$$\bigcap \ker(Tp_q^i) \cong (\mathrm{pr}_1)^* IG_1 \oplus (\mathrm{pr}_2)^* IG_1 \oplus \cdots \oplus (\mathrm{pr}_q)^* IG_1.$$

Proof. Note that each projection

$$\mathrm{pr}_i : G_q \to G_1$$

given by mapping onto the i-th component (as defined in Definition 1.1.1) induces a derivative $T\,\mathrm{pr}_i$ with the property that it map $\bigcap \ker(Tp_q^i)$ to a subset of $\ker(Ts) \cap \ker(Tt)$, such that we obtain that

$$\bigcap \ker(Tp_q^i) \subset IG_1 \times_{TG_0} \cdots \times_{TG_0} IG_1,$$

where both are a priori only defined as topological subspaces of TG_q. However, by the analogous argument the maps Tp_q^i vanish on the pull-back, such that we obtain that both sets agree. Note that for this reason, we also have that

$$IG_1 \times_{TG_0} \cdots \times_{TG_0} IG_1 = IG_1 \times_{G_0} \cdots \times_{G_0} IG_1,$$

which immediately implies that this set is indeed a vector bundle over G_q and in particular, a manifold. Furthermore, it directly implies that for any base-point $z \in G_q$ given in components as $(z_1, \ldots, z_q)$ the induced map

$$I_z G_q \to I_{z_1} G_1 \oplus \cdots \oplus I_{z_q} G_1$$

is a bijection. □

Lemma 2.1.16. *Consider a regular Lie groupoid G. Any multiplicative Ehresmann connection with respect to the isotropy bundle induces a splitting of TG_q that can be expressed by*

$$TG_q \cong E_q \oplus IG_q,$$

where IG_q is the vector bundle over G_q that is given by $\bigcap \ker(Tp^i_q)$. The bundle E_q is given by

$$E \times_{TG_0} \cdots \times_{TG_0} E,$$

i.e. by the nerve of

$$\begin{tikzcd} E \arrow[d, shift right, "Tt"'] \arrow[d, shift left, "Ts"] \\ TG_0. \end{tikzcd}$$

Proof. Here, we can apply Lemma 2.1.15, such that it remains to show that

$$TG_q = E_q \oplus IG_q.$$

As $E \rightrightarrows G_0$ is given by a vector subbundle $E \subset TG_1$ with linear structure maps to TG_0 by definition, we automatically find that $E_q \subset TG_q$ is a vector subbundle over G_q. We furthermore find that the horizontal projection induces a vector bundle map

$$\mathrm{pr}_E : TG_q = TG_1 \times_{TG_0} \cdots \times_{TG_0} TG_1 \to E \times_{TG_0} \cdots \times_{TG_0} E = E_q$$

which is given by the horizontal projection in each component of TG_1. It is clear that this maps E_q to itself. The kernel of the projection is also given by precisely those elements which in each component are given by an E-vertical element, i.e. an element in IG_1. So

$$TG_q = E_q \oplus \ker(\mathrm{pr}_E) = E_q \oplus IG_1 \times_{TG_0} \cdots \times_{TG_0} IG_1 = E_q \oplus IG_q.$$

□

Definition 2.1.17. Let G be a regular Lie groupoid and E a multiplicative Ehresmann connection with respect to the isotropy algebroid. A vector $X \in TG_q$ will be called **E-horizontal** if $X \in E_q$ and **E-vertical** if $X \in IG_q$.

Corollary 2.1.18. *Let G be a regular Lie groupoid and E be a multiplicative Ehresmann connection with respect to the isotropy algebroid. Any E-horizontal vector in G_q is uniquely characterised by its base-point z and its image under the Tp^i_q.*

Proof. Consider two vectors $X_1, X_2 \in T_z G_q$ which agree under all projections

$$Tp^i_q(X_1) = Tp^i_q(X_2).$$

Then their difference satisfies

$$X_1 - X_2 \in \bigcap \ker(Tp^i_q) = IG_q.$$

But if both vectors are horizontal, then their difference is horizontal and hence

$$X_1 - X_2 = 0.$$

□

Corollary 2.1.19. *Let G be a regular groupoid and E be a multiplicative Ehresmann connection with respect to the isotropy algebroid. Consider an element $x \in G_1$ and an isomorphism $\psi : T_{s(x)}G_0 \to T_{t(x)}G_0$ induced by some $V_x \in (J^E G)_1$ as*

$$T_{s(x)}G_0 \xrightarrow{(Ts)^{-1}} V_x \xrightarrow{Tt} T_{t(x)}G_0,$$

where the arrows denote the isomorphisms induced by Ts and Tt when restricted to V_x.

Then V_x is uniquely determined by ψ in the sense that if V_x and $V'_x \in (J^E G)_1$ induce the same isomorphism ψ, then $V_x = V'_x$.

Proof. By definition, the vectors both in V_x and V'_x are in bijection with the vectors in $T_{s(x)}$ precisely in a way where they are E-horizontal and their images under Ts (given by the identity) and Tt (given by φ) are predetermined.

By Corollary 2.1.18 this in particular implies that the image of any $X \in T_{s(x)}G_0$ in V_x under $(Ts)^{-1}$ must agree with the image of X in V'_x and hence that $V_x = V'_x$. □

We will now introduce a definition that is central to the thesis. It ties together the induced jet groupoid actions of Chapter 1 with multiplicative Ehresmann connections. Understanding the condition we obtain this way in terms of jet groupoids is thanks to Rui Fernandes. (In earlier versions of these notes, there was an alternate definition using local sections instead. Compared with the following, it was much longer and less elegant.)

Definition 2.1.20. Let G be a regular Lie groupoid and E be a multiplicative Ehresmann connection with respect to the isotropy algebroid. Consider a left action L of G on some bundle $p : N \to G_0$. If an element $\omega \in \Omega(N)$ satisfies that for any $n \in N$ and $x \in G_1$ with $s(x) = p(n)$ and $V_x \in (J^E G)_1$ we have that

$$\omega_n(X_1, \dots, X_p) = \omega_{L(x,n)}(\widehat{L}(V_x, X_1), \dots, \widehat{L}(V_x, X_p)),$$

then we call ω a **$J^E G$-invariant form of $p : N \to G_0$ with respect to L.**

If $\omega \in \Omega(G_0)$, we call it **$J^E G$-invariant** if it is a form $J^E G$-invariant with respect to the G-action on $\mathrm{id} : G_0 \to G_0$ of Example 1.1.23 (inducing the $J^1 G$-action defined in Definition 1.2.17).

If $\omega \in \Omega(G_q)$, we call it **$J^E G$-invariant** if it is a form $J^E G$-invariant with respect to any of the G-actions on any choice of $p^i_q : G_q \to G_0$ of Example 1.1.24 (inducing the $J^1 G$-actions defined in Definition 1.2.17).

Remark. Note that as a subgroupoid of $J^1 G$, the induced action of Proposition 1.2.10 always restricts to a groupoid left action of $J^E G$.

We will also adhere to the notation previously established for the action of $J^1 G$ on G_q and use the notation of a jet V_x acting on some $X \in T_z G_1$ with $s(x) = p^i_q(z)$ as

$$(\dot{D}^i_{V_x,q})_*(X) = \widehat{L}(V_x, X)$$

when describing this action (in analogy to Definition 1.2.17 and Definition 1.2.25).

In particular, we can spell out the definition of $J^E G$-invariance of a form $\omega \in \Omega(G_q)$ as

$$\omega_z(X_1, \dots, X_p) = \omega_{D^i_x(z)}((\dot{D}^i_V)_*(X_1), \dots, (\dot{D}^i_V)_*(X_p))$$

for all $0 \le i \le q$ and all compatible $V \in (J^E G)_1$, $x \in G_1$ and $z \in G_q$.

Lemma 2.1.21. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy. If $\omega \in \Omega^p(G_q)$ is $J^E G$-invariant, so is $(d_i)^*\omega$ for $0 \le i \le (q+1)$ and $(s_j)^*\omega$ for $0 \le j \le (q-1)$.*

Proof. We aim to show that

$$\left(\dot{D}^l_{V,q}\right)^* \left(((d_i)^*\omega)_{D^l_{x,q}(z)}\right) = ((d_i)^*\omega)_z$$

for any $0 \leq l \leq q$ and $z \in G_q$ and $V \in (J^E G)_1$ with base-point $x \in G_1$ satisfying $p^l_q(z) = s(x)$. This would precisely show that $(d_i)^*\omega$ is $J^E G$-invariant.

Firstly, by definition, we have that on the right hand

$$((d_i)^*\omega)_z = (d_i)^* \left(\omega_{d_i(z)}\right)$$

and on the left hand

$$\left(\dot{D}^l_{V,q}\right)^* \left(((d_i)^*\omega)_{D^l_{x,q}(z)}\right) = \left(d_i \circ \dot{D}^l_{V,q}\right)^* \left(\omega_{d_i(D^l_{x,q}(z))}\right)$$

We further have that

$$(d_i)_* \circ (\dot{D}^l_{V,q})_* = \begin{cases} (\dot{D}^l_{V,q})_* \circ (d_i)_* & \text{if } i > l \\ (\dot{D}^{(l-1)}_{V,q})_* \circ (d_i)_* & \text{if } i < l \\ (d_i)_* & \text{if } i = l. \end{cases}$$

according to Corollary 1.2.35. So

$$\left(\dot{D}^l_{V,q}\right)^* (d_i)^* (\omega_z) = \begin{cases} (d_i)^* \left(\dot{D}^l_{V,q}\right)^* \left(\omega_{D^l_{x,q}(d_i(z))}\right) & \text{if } i > l \\ (d_i)^* \left(\dot{D}^{(l-1)}_{V,q}\right)^* \left(\omega_{D^{(l-1)}_{x,q}(d_i(z))}\right) & \text{if } i < l \\ (d_i)^* \left(\omega_{d_i(z)}\right) & \text{if } i = l. \end{cases}$$

We can see that the result equals $(d_i)^* \left(\omega_{d_i(z)}\right)$ in all three of these cases by $J^E G$-invariance of ω. For well-definedness, it is important to note in the case $i > l$ that

$$p^l_q(z) = p^l_q(d_i(z))$$

and in the case $i < l$ that

$$p^l_q(z) = p^{(l-1)}_q(d_i(z)).$$

The argument for $(s_j)^*\omega$ is analogous under employment of the interchange rules given in Corollary 1.2.35. □

Corollary 2.1.22. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy. If ω is $J^E G$-invariant, so is $\mathrm{d}_{\mathcal{N}}(\omega)$.*

Proof. This follows immediately from Lemma 2.1.21. □

Lemma 2.1.23. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy. If $\omega \in \Omega^p(G_1)$ is $J^E G$-invariant, then for any of the component projections $\mathrm{pr}_i : G_q \to G_1$ for $1 \leq i \leq q$, we have that $(\mathrm{pr}_i)^*(\omega)$ is also $J^E G$-invariant.*

Proof. We have by Example 1.2.19 that

$$(\dot{D}^j_{V,q})^*(\mathrm{pr}_i)^*(\omega) = (\mathrm{pr}_i)^*(\omega)$$

immediately for any $0 \leq j \leq q$ unless $j = i - 1$ or $j = i$. We also have that

$$(\dot{D}^{i-1}_{V,q})^*(\mathrm{pr}_i)^*(\omega) = (\mathrm{pr}_i)^*(\dot{D}^0_{V,1})^*(\omega) = (\mathrm{pr}_i)^*(\omega)$$

and

$$(\dot{D}^i_{V,q})^*(\mathrm{pr}_i)^*(\omega) = (\mathrm{pr}_i)^*(\dot{D}^1_{V,1})^*(\omega) = (\mathrm{pr}_i)^*(\omega),$$

such that $(\mathrm{pr}_i)^*(\omega)$ is invariant under each separate action of $J^E G$. Therefore $(\mathrm{pr}_i)^*(\omega)$ is indeed $J^E G$-invariant. □

Lemma 2.1.24. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy. Consider $X \in T_z G_q$ a E-horizontal tangent vector at some $z \in G_q$ and $V_x \in J^E(s : G_1 \to G_0)$ a jet with its base-point x satisfying $p^i_q(z) = s(x)$. Then $(\dot{D}^i_{V_x,q})_*(X)$ is E-horizontal as well.*

Remark. This in particular applies for $V_x \in (J^E G)_1$.

Proof. By definition, $X = (X^{(1)}, \ldots, X^{(q)})$ is E-horizontal if all its components $X^{(j)}$ are E-horizontal. The vector $(\dot{D}^i_{V_x,q})_*(X)$ has likewise components given by either $X^{(j)}$ (if $j < i$ or $j > i+1$) or are given by

$$(T\mu)(X^{(i)}, Y)$$

or

$$(T\mu)(Z, X^{(i+1)})$$

for $Y, Z \in V_x \subset E_x$. But as E is a subgroupoid of TG_1 it is closed under $T\mu$ and hence all components of $(\dot{D}^i_{V_x,q})_*(X)$ are E-horizontal. □

As we have noted already in Chapter 1 (for the precise result, see Proposition 1.3.14), the invariance under jets corresponds to the invariance under bisections. However, it will be advantageous to relate $J^E G$-invariance to invariance under sections, too. Simultaneously, we will find that we can replace $(J^E G)_1$ by $J^E(s : G_1 \to G_0)$ in the invariance condition. We will do this in the following.

Lemma 2.1.25. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy. Consider any $x \in G_1$ and a linear isomorphism $\psi : T_{s(x)} G_0 \to T_{t(x)} G_0$ induced by some $V_x \in (J^E G)_1$.*

Then for any $\lambda \in \mathbb{R}$ there is a $V^\lambda_x \in (J^E G)_1$ inducing an isomorphism ψ^λ with

$$\psi^\lambda(X) = \lambda \cdot (\psi(X))$$

for any $X \in \mathcal{F}_0$. They can be constructed in a way that the limit $\lim_{\lambda \to 0} \psi^\lambda$ is well-defined and the limit $\lim_{\lambda \to 0} V^\lambda_x \in J^E(s : G_0 \to G_1)$ is well-defined.

We refer the proof to A.3.4, as it is mainly a technical explicit construction.

Proposition 2.1.26. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy. Consider a form $\omega \in \Omega^p(G_q)$ and a multiplicative Ehresmann connection E with respect to the isotropy bundle. Then if ω is $J^E G$-invariant (according to Definition 2.1.20) we have that*

$$\iota_X \omega = 0$$

for any $X \in \mathcal{F}_q$ (the orbit foliation on G_q as defined in Proposition 1.1.16 or Lemma 1.1.19) which is E-horizontal.

Proof. We can start by decomposing X into its components

$$X = (X^{(1)}, \ldots, X^{(q)}).$$

and its base-point $z \in G_q$ into its components

$$z = (z_1, \ldots, z_q).$$

By assumption, we have $X \in \mathcal{F}_q$, which implies that $X^{(i)} \in \mathcal{F}_1$ for all $1 \leq i \leq q$.

Now consider $p_q^0(z) = t(z_1)$. We can consider the corresponding unit

$$V_0 = Tu_q(T_{p_q^0(z)}G_0) = j^1(u_q)_{p_q^0(z)} \in (J^E G)_1$$

and by Lemma 2.1.25, it induces a family V_0^λ of E-jets. We have that

$$\iota_X \omega_z(Y_1, \ldots, Y_{p-1}) = \iota_{(\dot{D}^0_{V_0^\lambda})_*(X)} \omega_z((\dot{D}^0_{V_0^\lambda})_*(Y_1), \ldots, (\dot{D}^0_{V_0^\lambda})_*(Y_{p-1}))$$

by the definition of $J^E G$-invariance. Let V_0' be the limit of V_0^λ. As the action of $J^E(s : G_1 \to G_0)$ on the term is continuous, we find that the right-hand side converges, yielding

$$\iota_X \omega_z(Y_1, \ldots, Y_{p-1}) = \iota_{(\dot{D}^0_{V_0'})_*(X)} \omega_z((\dot{D}^0_{V_0'})_*(Y_1), \ldots, (\dot{D}^0_{V_0'})_*(Y_{p-1})).$$

We have that $(\dot{D}^0_{V_0'})(X)$ when written in components is given by

$$(\dot{D}^0_{V_0'})_*(X) = ((X')^{(1)}, X^{(2)}, \ldots, X^{(q)})$$

where $(X')^{(1)}$ is by definition given by $T\mu(Z, X^{(1)})$ for the unique compatible $Z \in V_0'$. Note that $(X')^{(1)}$ is still E-horizontal. We have that $Tt((X')^{(1)}) = \psi_0'(Ts(X^{(1)}))$ for ψ_0' the limit of the isomorphisms ψ_0^λ associated to the V_0^λ. However, as $X^{(1)} \in \mathcal{F}_1$, we have that $Tt(X^{(1)}) \in \mathcal{F}_0$, such that $\psi^\lambda(Ts(X^{(1)}))$ depends linearly on λ and in the limit we obtain $\psi_0'(Ts(X^{(1)})) = Tt((X')^{(1)}) = 0$.

We can iteratively define V_i, V_i^λ and V_i' for $1 \leq i \leq q$ and iteratively apply the invariance equation for $(\dot{D}^i_{V_i^\lambda})$. Because of continuity on all of $J^E(s : G_1 \to G_0)$ for all of the multiplication actions, we have that taking limits over the different V_i^λ in the expression

$$(\dot{D}^q_{V_q^\lambda})_* \left(\ldots (\dot{D}^0_{V_0^\lambda})_*(Z) \right)$$

commutes and the limit is given by

$$(\dot{D}^q_{V_q'})_* \left(\ldots (\dot{D}^0_{V_0'})_*(Z) \right).$$

This also means that

$$\begin{aligned} &\iota_X \omega_z(Y_1, \ldots, Y_{p-1}) \\ =&\iota_{(D^q_{V_q'})_* \left(\ldots (\dot{D}^0_{V_0'})_*(X) \right)} \omega_z \left((\dot{D}^q_{V_q'})_* \left(\ldots (\dot{D}^0_{V_0'})_*(Y_1) \right), \ldots, (\dot{D}^q_{V_q'})_* \left(\ldots (\dot{D}^0_{V_0'})_*(Y_{p-1}) \right) \right). \end{aligned}$$

Note that analogous to the previous considerations

$$X' = (\dot{D}^q_{V_q'})_* \left(\ldots (\dot{D}^0_{V_0'})_*(X) \right)$$

is an E-horizontal vector satisfying that $Tp^i_q(X') = 0$ for all $0 \leq i \leq q$. By Corollary 2.1.18, this implies $X' = 0$. So we find that

$$\iota_X \omega_z(Y_1, \dots, Y_{p-1}) = \iota_0 \omega_z(Y'_1, \dots, Y'_{p-1}) = 0$$

where we denote

$$Y'_i = (\dot{D}^q_{V'_q})_* \left(\dots (\dot{D}^0_{V'_0})_*(Y_i) \right),$$

such that (since the Y_i were chosen arbitrarily) indeed $\iota_X \omega_z = 0$.

□

Lemma 2.1.27. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy. When considered as subsets of*

$$J^1(s : G_1 \to G_0),$$

we have that

$$J^E(s : G_1 \to G_0)$$

is the closure of

$$(J^E G)_1 \subset (J^1 G)_1 \subset J^1(s : G_1 \to G_0).$$

We refer the proof to A.3.5, as the proof is similar to that of Lemma 2.1.25 and mostly technical without additional insight.

Corollary 2.1.28. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy. Consider $\omega \in \Omega^p(G_q)$ which is $J^E G$-invariant. Then ω satisfies*

$$\omega_z(X_1, \dots, X_p) = \omega_{D^i_x(z)}((\dot{D}^i_{V_x})(X_1), \dots, (\dot{D}^i_{V_x})(X_p))$$

as defined in Definition 1.2.25 for any $V_x \in J^E(s : G_1 \to G_0)$. In particular, $J^E G$-invariance is equivalent to invariance under all $V_x \in J^E(s : G_1 \to G_0)$.

Proof. By Lemma 2.1.27 we have that $J^E(s : G_1 \to G_0)$ is precisely the closure of $(J^E G)_1 \subset J^1(s : G_1 \to G_0)$. As the action is continuous, we have that for any fixed ω each invariance condition (w.r.t. each D^i-action for $1 \leq i \leq q$) defines a closed subset of $J^1 G$ where it is satisfied.

□

Corollary 2.1.29. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy. Consider a form $\omega \in \Omega^p(G_q)$ and a multiplicative Ehresmann connection E with respect to the isotropy bundle. Then ω is $J^E G$-invariant if and only if it satisfies that for all $0 \leq i \leq q$ and all $z \in G_q$ we have that under*

$$(\widetilde{D}^i_{\gamma,q})^* : \wedge^\bullet T^*_{D^i_{x,q}(z)} G_q \to \wedge^\bullet T^*_z G_q$$

holds

$$(\widetilde{D}^i_{\gamma,q})^*(\omega_{D^i_{x,q}(z)}) = \omega_z$$

for any local bisection γ which is horizontal at $p^i_q(z)$ with respect to the multiplicative Ehresmann connection, where we denote $\gamma(p^i_q(z)) = x \in G_1$.

By a bisection horizontal at $p^i_q(z)$, we mean that

$$\left(T_{p^i_q(z)}\gamma\right)(T_{p^i_q(z)} G_0) \subset (E_1)_x \subset T_x G_1,$$

i.e. the infinitesimal image of γ at $p^i_q(z)$ is contained in the horizontal subbundle of the multiplicative Ehresmann connection.

Proof. Let γ be a bisection as in the statement. We know by Lemma 1.2.24 and Lemma 1.2.28 that $(\widetilde{D}^i_{\gamma,q})^* \equiv (\dot{D}^i_{V,q})^*$ as maps

$$\wedge^\bullet T^*_{D^i_{\gamma,q}(z)} G_q \to \wedge^\bullet T^*_z G_q$$

if we choose

$$(T_{p^i_q(z)}\gamma)(T_{p^i_q(z)}G_0) = V \in (J^1G)_1.$$

We also have by definition that $T_{p^i_q(z)}\gamma(T_{p^i_q(z)}G_0) \subset E$ and hence $V \in (J^EG)_1$. So if a form ω is J^EG-invariant, it must be invariant under the infinitesimal action of the bisection γ as well.

On the other hand, if we are given a $V \in (J^EG)_1$, we have that $V \subset E_x$ for some $x \in G_1$. As we further have that Ts is an isomorphism onto $T_{s(x)}G_0$ we can find a local section γ of s that satisfies $\gamma(T_{p^i_q(z)}G_0) = V$.

We have already seen in Corollary 1.2.9 that such a γ can be assumed to be a local bisection. So if a form is invariant under the infinitesimal action of any horizontal bisection this includes γ. Therefore it will also be invariant under the action of V, which can be chosen as any element of $(J^EG)_1$, and hence ω is J^EG-invariant. □

Corollary 2.1.30. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy. Consider a form $\omega \in \Omega^p(G_q)$ and a multiplicative Ehresmann connection E with respect to the isotropy bundle. Then ω is J^EG-invariant if and only if it satisfies that for all $0 \leq i \leq q$ we have that under*

$$(\widetilde{D}^i_{\gamma,q})^* : \wedge^\bullet T^*_{\widetilde{D}^i_{\gamma,q}(z)} G_q \to \wedge^\bullet T^*_z G_q$$

holds

$$(\widetilde{D}^i_{\gamma,q})^*(\omega_{\widetilde{D}^i_{\gamma,q}(z)}) = \omega_z$$

for any local section γ of s which is horizontal at $p^i_q(z)$ with respect to the multiplicative Ehresmann connection.

By a section of s horizontal at $p^i_q(z)$, we mean that

$$\left(T_{p^i_q(z)}\gamma\right)(T_{p^i_q(z)}G_0) \subset (E_1)_{\gamma(p^i_q(z))} \subset T_{\gamma(p^i_q(z))}G_1,$$

i.e. the infinitesimal image of γ at $p^i_q(z)$ is contained in the horizontal subbundle of the multiplicative Ehresmann connection.

Proof. The proof of this is analogous to the proof of Corollary 2.1.29, only that the corresponding infinitesimal action is that of an element of $J^E(s : G_1 \to G_0)$. However, J^EG-invariance is equivalent to invariance under infinitesimal sections of s by Corollary 2.1.28. □

Definition 2.1.31. Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Consider $\omega_z \in \wedge^p T_z G_q$. The identification

$$T_z G_q = E_z \oplus I_z G_q$$

induces a decomposition

$$\wedge^p T_z G_q = \bigoplus_{p_0+p_1=p} \wedge^{p_0} E_z \otimes \wedge^{p_1} I_z G_q.$$

If ω_z is 0 or contained in a single component of the direct sum, we will call ω_z to be **homogeneous with respect to** E (or E-homogeneous) of E-horizontal degree p_0 and E-vertical degree p_1.

Consider $\omega \in \Omega^p(G_q)$. If for any $z \in G_q$ we have that ω_z is homogeneous with respect to E (or E-homogeneous) of E-horizontal degree p_0 and E-vertical degree p_1, we will call ω to be **homogeneous with respect to** E (or E-homogeneous) of E-horizontal degree p_0 and E-vertical degree p_1.

Remark. Note that this means that we now have two different notions of homogeneous components of a bidegree - the notion from the definition of the Bott–Shulman–Stasheff bicomplex, where any $\omega \in \Omega^p(G_q)$ is homogeneous of bidegree (p, q) and the E-horizontal and E-vertical bidegree, which can be used to decompose this space again. We will use the expressions "horizontal/vertical degree" and "bidegree" to reference p and q and the expressions "E-horizontal/E-vertical degree" and "E-homogeneous bidegree" to reference p_0 and p_1. We could also denote the multidegree corresponding to an E-homogeneous component by (p_0, p_1, q) (where we can recover $p = p_0 + p_1$) and at a later point, we will actually use even more intricate multidegrees following this principle.

This definition will be vital for many results of this thesis, as we will often use the decomposition of a form into its E-homogeneous components. We will in the following lay down a lot of the necessary groundwork for this method to be viable.

Lemma 2.1.32. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. The splitting of $TG_q = E_q \oplus IG_q$ of Proposition 2.1.16 is retained through $(d_i)_*$ for all $0 \leq i \leq q$ and $(s_j)_*$ for all $0 \leq j \leq q$.*

Proof. As the degeneracy maps are given by inserting the unit section at the corresponding position of the iterated pull-back, the horizontal bundle is mapped to the pull-back of the copies of E_1, where at the corresponding position there is an additional factor given by the tangent space to the unit section. As this is horizontal by definition, the image of a horizontal tangent vector is horizontal.

Furthermore, we have that $p^k_{q+1} \circ s_j = p^{k'}_q$, so any element of $\bigcap \ker(Tp^k_q)$ is mapped to $\bigcap \ker(Tp^k_{q+1})$, so verticality is also retained.

Regarding the face maps, we have that by definition

$$\mu(E_1 \times_{TG_0} E_1) \subset E_1,$$

as E_1 was supposed to inherit a groupoid structure, which also implies that all face maps that are given by multiplication of adjacent factors retain horizontality. Clearly the omittance of either the left or the right factor of the iterated pull-back will also map the horizontal bundle to the horizontal bundle.

Lastly, we have that $p^k_{q+1} \circ d_i = p^{k'}_q$, so any element of $\bigcap \ker(Tp^k_q)$ is mapped to $\bigcap \ker(Tp^k_{q-1})$. Hence the image of a vertical vector is again vertical. □

Corollary 2.1.33. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Let $\omega \in \Omega^p(G_q)$ be an E-homogeneous form of E-horizontal degree p_0 and E-vertical degree p_1. Then $\mathrm{d}_{\mathcal{N}}(\omega)$ is also an E-homogeneous form of the respective E-horizontal and E-vertical degrees.*

Proof. By Lemma 2.1.32, we have that all $(d_i)^*$ for $0 \leq i \leq q$ retain the splitting of Proposition 2.1.16. Therefore all terms $(d_i)^*(\omega)$ occurring in the definition of $\mathrm{d}_{\mathcal{N}}$ are E-homogeneous and therefore, so is their sum. □

Lemma 2.1.34. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Consider a form $\omega \in \Omega^p(G_q)$ of E-horizontal degree p_0 and subsequently E-vertical degree $p_1 := p - p_0$. Further consider any subspace $V \in J^E(s : G_1 \to G_0)$.*

We then have that

$$(D^i_{V,q})^*\omega$$

is homogeneous with respect to E of E-horizontal degree p_0 and E-vertical degree p_1.

Proof. We know by Lemma 2.1.24 that for any $X \in TG_q$ an E-horizontal tangent vector we have

$$(\dot{D}^i_{V,q})_*(X)$$

is E-horizontal.

We also know by Lemma 1.2.31 that for any $X \in TG_q$ an E-vertical tangent vector we have

$$(D^i_{V,q})_*(Y) = (D^i_{x,q})_*(Y)$$

is E-vertical.

As $(D^i_{V,q})^*$ is precisely dual to the notion on tangent vectors, we obtain that it maps E-horizontal cotangent vectors to E-horizontal cotangent vectors and E-horizontal forms to E-horizontal forms. □

Corollary 2.1.35. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. The space of $J^E G$-invariant forms decomposes into the direct sum along the E-homogeneous bidegree, i.e.*

$$(\Omega^p(G_q))^{J^E G} = \bigoplus_{p_0+p_1=p} \left(\Gamma(G_0, \wedge^{p_0} E_q \oplus \wedge^{p_1} IG_q)\right)^{J^E G}.$$

Proof. By Lemma 2.1.34 the action of $J^E G$ restricts to the E-homogeneous subspaces, which immediately implies that each E-homogeneous component of a $J^E G$-invariant form is $J^E G$-invariant as well. □

Lemma 2.1.36. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Consider a form $\omega \in \Omega^p(G_q)$ of E-horizontal degree p_0 and subsequently E-vertical degree $p_1 := p - p_0$. Further consider any subspace $V \in J^1(s : G_1 \to G_0)$.*

Then the decomposition of

$$(\dot{D}^i_{V,q})^*\omega$$

into components of homogeneous E-horizontal and E-vertical degree satisfies that the E-vertical degree of each component is at most p_1. Furthermore, the component of E-vertical degree p_1 is given by

$$(\dot{D}^i_{W,q})^*\omega,$$

where W is the E-horizontal projection of V, i.e. the space given by

$$W = \{\mathrm{pr}_E(X) \in T_x G_1 \text{ for } X \in V\},$$

where $\mathrm{pr}_E : TG_q \to E_q$ denotes the E-horizontal projection.

Proof. First of all, we may note that if we are given E-horizontal vector fields $X_1, \dots, X_k$ and E-vertical vector fields $Y_1, \dots, Y_l$ for $k + l = p$ and $l > p_1$ it holds that for degree reasons

$$\begin{aligned}
&(\dot{D}^k_{V,q})^*(\omega)(X_1, \dots, X_k, Y_1, \dots, Y_l)\\
=&\omega((\dot{D}^k_{V,q})_*(X_1), \dots, (\dot{D}^k_{V,q})_*(X_k), (\dot{D}^k_{V,q})_*(Y_1), \dots, (\dot{D}^k_{V,q})_*(Y_l))\\
=&0
\end{aligned}$$

as by Lemma 1.2.31 we have

$$(\dot{D}^k_{V,q})_*(Y_j) = (\dot{D}^k_{x,q})_*(Y_j).$$

which is E-vertical and hence we evaluate ω on more vertical tangent vectors than it has vertical degree.

Furthermore, if $l = p_1$, we find

$$\begin{aligned}
&(\dot{D}^k_{V,q})^*(\omega)(X_1, \dots, X_k, Y_1, \dots, Y_l)\\
=&\omega((\dot{D}^k_{V,q})_*(X_1), \dots, (\dot{D}^k_{V,q})_*(X_k), (\dot{D}^k_{V,q})_*(Y_1), \dots, (\dot{D}^k_{V,q})_*(Y_l))\\
=&\omega(\mathrm{pr}_E((\dot{D}^k_{V,q})_*(X_1)), \dots, \mathrm{pr}_E((\dot{D}^k_{V,q})_*(X_k)), (\dot{D}^k_{V,q})_*(Y_1), \dots, (\dot{D}^k_{V,q})_*(Y_l))
\end{aligned}$$

for the same reason. Now for any $1 \leq i \leq k$ let us compare

$$\mathrm{pr}_E((\dot{D}^k_{V,q})_*(X_i))$$

to the expression

$$(\dot{D}^k_{W,q})_*(X_i)$$

By definition,

$$(\dot{D}^k_{V,q})_*(X_i) = (TD^k_{\bullet,q})(Z, X_i)$$

for some $Z \in V \subset TG_1$ agrees with $(\dot{D}^k_{W,q})_*(X_i)$ under all Tp^j_q, as horizontal projection from Z to $\mathrm{pr}_E(Z)$ changes neither the image under Ts nor under Tt. However, also by definition X_i is E-horizontal and $((\dot{D}^k_{W,q})_*$ retains E-horizontality as $W \in J^E G$. Hence we have that

$$((\dot{D}^k_{W,q})_*(X_i) = \mathrm{pr}_E(((\dot{D}^k_{V,q})_*(X_i))$$

which proves the claim. □

Corollary 2.1.37. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Consider a $J^E G$-invariant form $\omega \in \Omega^p(G_q)$ of E-horizontal degree p_0 and E-vertical degree p_1. Consider $\gamma : U \subset G_0 \to G_1$ any local section of s. Then if we consider the component of $(\widetilde{D}^i_{\gamma,q})^*\omega$ of E-horizontal degree p_0 (and E-vertical degree p_1), it agrees with ω on its domain $(p^i_q)^{-1}(U)$.*

Proof. By Lemma 1.2.28, we have that $\left((\widetilde{D}^i_{\gamma,q})^*\omega\right)_z = (\dot{D}^i_{V,q})^*\omega$ for $V = T\gamma(T_{p^i_q(z)}G_0)$ for any $z \in (p^i_q)^{-1}(U)$. By Lemma 2.1.36, we find that this means

$$\left((\widetilde{D}^i_{\gamma,q})^*\omega\right)_z \equiv (\dot{D}^i_{W,q})^*\omega$$

in E-horizontal degree p_0 for some $W \in J^E(s : G_1 \to G_0)$. However, as ω is $J^E G$-invariant, we can apply Corollary 2.1.28 to obtain the claim. □

Lemma 2.1.38. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Let $\omega \in \Omega^p(G_q)$ be a $J^E G$-invariant element. Then the values of ω along any orbit (i.e. on any leaf of the foliation F_q defined by the full preimage of $n \in Q$ as in Proposition 1.1.16) are determined by the value of ω at a single point in that orbit.*

Proof. Consider any two elements $y, z \in G_q$ in the same orbit. This means that by Lemma 1.3.11 we can find $x^0, \dots, x^q \in G_1$ which satisfy that

$$D^q_{x^q,q}\left(D^{q-1}_{x^{q-1},q}\left(\dots D^1_{x^1,q}\left(D^0_{x^0,q}(z)\right)\right)\right) = y.$$

Now we have that for each x^i we may choose by Lemma 1.2.8 some $V^i \in (J^1 G)_1$ with base-point x^i. By Lemma 2.1.9 we may without loss of generality assume $V^i \in (J^E G)_1$. By $J^E G$-invariance, we now have that

$$(\dot{D}^0_{V^0,q})^* (\dot{D}^1_{V^1,q})^* \dots (\dot{D}^q_{V^q,q})^* \omega_y = \omega_z.$$

So indeed, the value of ω_y determines ω_z and as z was chosen arbitrarily, ω_y determines the values of ω along the entire orbit of y.

□

Lemma 2.1.39. *Let G be a regular Lie groupoid. Let $m, n \in G_0$ and $x^0, \dots, x^q \in G_1$ such that*

$$(D^q_{x^q,q}) \dots (D^1_{x^1,q})(D^0_{x^0,q})(1_m) = 1_n.$$

Then $x^0 = \dots = x^q$ and

$$(D^q_{x^q,q}) \dots (D^1_{x^1,q})(D^0_{x^0,q})(1_m) = \mathrm{Ad}^0_{x^0}(1_m).$$

Proof. If we explicitly express 1_m in components and evaluate the expression, we obtain

$$(D^q_{x^q,q}) \dots (D^1_{x^1,q})(D^0_{x^0,q})(1_m) = (\mu(x^0, 1_m (x^1)^{-1}), \dots, \mu(x^{q-1}, 1_m (x^q)^{-1})),$$

where we know for each term that

$$\mu(x^i, 1_m, (x^{i+1})^{-1}) = 1_n.$$

However, by associativity and unitality, we find that this implies

$$x^i = x^{i+1}.$$

The first statement follows immediately from this, the second is implied by Lemma 1.1.37.

□

Lemma 2.1.40. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. The $J^E G$-invariant elements of bidegree (p, q) are determined by their restriction to the image of the identity section, where they are contained in the subspace*

$$\bigoplus_{p_0+p_1=p} \left(\Gamma\left(G_0, \wedge^{p_0} T^*(G_0/\mathcal{F}_0) \otimes \wedge^{p_1}\left((A^*_{\mathrm{iso}})^{\oplus q}\right)\right)\right)^G \subset \Gamma(G_0, \wedge^p (u_q)^* T G_q).$$

Here, the inclusion $(A^*_{\mathrm{iso}})^{\oplus q} \subset T^* G_q$ is induced by the E-vertical projection map and the G-invariance denotes invariance (when evaluated at the corresponding points) with respect to the extension of the action of Lemma 1.2.16 and the extension of the direct sum of q copies of the coadjoint action of Definition 1.1.39.

Proof. As any orbit intersects the unit section (e.g. x and $1_{p_0^0(x)}$ are always in the same orbit as seen in Corollary 1.3.12), the values at the unit section determine all of ω by Lemma 2.1.38. A priori, we obtain

$$\omega 1_{G_0} := \omega \circ u_q \in \Gamma(G_0, \wedge^p u_q^* T^* G_q).$$

First, we will now apply the splitting of the tangent space in E-horizontal and E-vertical tangent vectors. The E-vertical vectors at the unit section correspond to the q-fold sum of the isotropy in G_1 by Lemma 2.1.16, such that we obtain an element in

$$\Gamma(G_0, \wedge^{p_0} E^* \otimes \wedge^{p_1} \left((A_{\mathrm{iso}}^*)^{\oplus q}\right))$$

with $p_0 + p_1 = p$. Now Proposition 2.1.26 implies that the values are actually contained in

$$\Gamma(G_0, \wedge^{p_0} T^* G_0 \otimes \wedge^{p_1} \left((A_{\mathrm{iso}}^*)^{\oplus q}\right)),$$

as at the unit section E splits into TG_0 and $(A \cap E)^{\oplus q}$. The lemma further implies that the values are contained in

$$\Gamma(G_0, \wedge^{p_0} (TG_0/\mathcal{F}_0)^* \otimes \wedge^{p_1} \left((A_{\mathrm{iso}}^*)^{\oplus q}\right)).$$

Next, let us show, that the values we have obtained this way are G-invariant in the sense of the statement.

If ω is a $J^E G$-invariant form, it is invariant under all sequences of actions generated by the $D_{V,q}^i$ for $V \in (J^E G)_1$ that map the tangent space at a unit into the tangent space at another unit. So they in particular satisfy the invariance

$$(\dot{D}_{V,q}^q)^* (\dot{D}_{V,q}^{q-1})^* \dots (\dot{D}_{V,q}^1)^* (\dot{D}_{V,q}^0)^* \omega 1_m = \omega 1_n,$$

where the base-point x of V must satisfy $s(x) = m$ and $t(x) = n$ for well-definedness. By Lemma 1.2.38 we have on $T(\mathrm{Iso}^0(G))_q \subset TG_q$ holds

$$(\dot{D}_{V,q}^q)_* (\dot{D}_{V,q}^{q-1})_* \dots (\dot{D}_{V,q}^1)_* (\dot{D}_{V,q}^0)_* = (\widehat{\mathrm{Ad}}_V^0)_*,$$

which identifies the $J^E G$-action on the components $TG_0 \oplus (A_{\mathrm{iso}})^{\oplus q}$ of the splitting. This means that the $J^E G$-invariance implies an invariance of $\omega|_{(\mathrm{Iso}^0(G))_q}$ under $(\widehat{\mathrm{Ad}}_V^0)^*$ for any $V \in J^E G$. In terms of the splitting

$$(u_q)^* T^* G_q = T^* G_0 \oplus ((A \cap E)^*)^{\oplus q} \oplus (A_{\mathrm{iso}}^*)^{\oplus q}$$

the inclusion precisely corresponds to the projection to the components of $T^* G_0$ and $(A_{\mathrm{iso}}^*)^{\oplus q}$. In particular, as $\omega 1_{G_0}$ already vanishes under any $X \in (A \cap E)^{\oplus q}$, we have that the projection of $\omega 1_{G_0}$ to

$$\begin{aligned} \mathrm{pr}_{(u_q)^* T^* (\mathrm{Iso}^0(G)_q))} \circ \omega 1_{G_0} = \omega|_{(\mathrm{Iso}^0(G))_q} \circ u_q \in & \Gamma(G_0, \wedge^p (u_q)^* T^* (\mathrm{Iso}^0(G)_q)) \\ \cong & \bigoplus_{p_0' + p_1' = p} \Gamma(G_0, \wedge^{p_0'} T^* G_0 \oplus \wedge^{p_1'} (A_{\mathrm{iso}}^*)^{\oplus q}) \end{aligned}$$

corresponds to the expression of the same form as $\omega 1_{G_0}$ itself. Furthermore, we have seen in Lemma 1.2.39 that on $Tu_q : TG_0 \subset TG_q$ the action is given by $(D_{V,0}^0)_*$ and descends to a G-action on $TG_0/\mathcal{F}_0$ by Example 1.2.15. We can furthermore see from Lemma 1.2.11 and Example 1.1.33 that $((\widehat{\mathrm{Ad}^0})_V)_*$ restricts to $(A_{\mathrm{iso}})^{\oplus q}$, where it is also given by a G-action, namely $\mathrm{Ad}^{\oplus q}$ by Lemma 1.1.37. □

Proposition 2.1.41. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. The map of Lemma 2.1.40 from $J^E G$-invariant forms of bidegree (p,q) to their values on the image of the identity section is an isomorphism. In particular*

$$(\Omega^p(G_q))^{J^E G} \cong \bigoplus_{p_0+p_1=p} \left(\Gamma\left(G_0, \wedge^{p_0} T^*(G_0/\mathcal{F}_0) \otimes \wedge^{p_1}\left((A^*_{\mathrm{iso}})^{\oplus q}\right)\right)\right)^G.$$

Proof. We need to show that each element of

$$\left(\Gamma\left(G_0, \wedge^{p_0} T^*(G_0/\mathcal{F}_0) \otimes \wedge^{p_1}\left((A^*_{\mathrm{iso}})^{\oplus q}\right)\right)\right)^G$$

actually defines a $J^E G$-invariant form.

Now suppose that we have an element

$$\omega_{1_{G_0}} \in \Gamma(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \wedge^{p_1}\left((A^*_{\mathrm{iso}})^{\oplus q}\right)) \subset \Gamma(G_0, (u_q)^* TG_q),$$

where the inclusion is as those forms which vanish on $\mathcal{F}_q \cap E$. We could aim to extend it to a form on G_q by reversing the construction described in the proof of Lemma 2.1.38. What we need to show is that this reconstruction process is indeed well-defined (assuming that $\omega_{1_{G_0}}$ is G-invariant) and that the arising form is $J^E G$-invariant.

For this, recall that by Lemma 1.3.12 any $z \in G_q$ is in the same orbit as $1_{p^0_q(z)} \in G_q$. By the proof of Lemma 2.1.38, we found that $J^E G$-invariance implies that whenever z and 1_m are contained in the same orbit, we have

$$(\dot{D}^0_{V^0,q})^*(\dot{D}^1_{V^1,q})^* \dots (\dot{D}^{q-1}_{V^{q-1},q})^*(\dot{D}^q_{V^q,q})^* \omega_{1_m} = \omega_z$$

for some choice of $V^i \in (J^E G)_1$ for $0 \le i \le q$. So as ω_{1_m} is determined by $\omega_{1_{G_0}}$, this equation can simply be taken as a definition for ω_z.

Now we need to check that this describes a well defined form. For this, let $n \in G_0$ be a different point such that 1_n is also in the same orbit as z (meaning in particular that n and m are in the same orbit in G_0) and W^i for $0 \le i \le q$ be a different set of E-horizontal jets chosen such that

$$(\dot{D}^0_{W^0,q})^*(\dot{D}^1_{W^1,q})^* \dots (\dot{D}^{q-1}_{W^{q-1},q})^*(\dot{D}^q_{W^q,q})^* \omega_{1_n} \in \wedge^p T^*_z G_q.$$

For well-definedness of ω we actually only need to show that

$$(\dot{D}^0_{V^0,q})^* \dots (\dot{D}^q_{V^q,q})^* \omega_{1_m} = (\dot{D}^0_{W^0,q})^* \dots (\dot{D}^q_{W^q,q})^* \omega_{1_n} \in \wedge^\bullet T_z G_q$$

in the case of $n = m$, but we can show it in this generality as it automatically proves a number of other useful properties.

We have that for the base-points $x^i \in G_1$ of V^i and y^i of W^i holds by construction that

$$(D^q_{x^q,q}) \circ \dots \circ (D^0_{x^0,q})(1_m) = (D^q_{y^q,q}) \circ \dots \circ (D^0_{y^0,q})(1_n) = z$$

implying (by Example 1.2.20 and Lemma 1.2.36 or simply evaluating the expressions) that

$$t(x^i) = p^i_q(z) = t(y^i)$$

This means, first of all, that the jets V^i and $(W^i)^{-1}$ can each be multiplied, and furthermore by the commutativity of Lemma 1.2.37 and the associativity of the actions, the statement we aim to show becomes equivalent to

$$(\dot{D}^q_{\widehat{\mu}((W^q)^{-1},V^q),q})^* \dots (\dot{D}^1_{\widehat{\mu}((W^1)^{-1},V^1),q})^*(\dot{D}^0_{\widehat{\mu}((W^0)^{-1},V^0),q})^* \omega_{1_m} = \omega_{1_n}$$

Note, that by assumption we have that both ω_{1_m} and ω_{1_n} satisfy that they vanish on all of $\mathcal{F}_q \cap E$. As by Lemma 1.2.40, $\mathcal{F}_q$ is retained under any of the $(\dot{D}^i_{\widehat{\mu}((W^i)^{-1},V^i),q})_*$ and by Lemma 2.1.24, so is E, we obtain that it is actually sufficient to compare the values of ω_{1_n} and

$$(\dot{D}^q_{\widehat{\mu}((W^q)^{-1},V^q),q})^* \dots (\dot{D}^1_{\widehat{\mu}((W^1)^{-1},V^1),q})^* (\dot{D}^0_{\widehat{\mu}((W^0)^{-1},V^0),q})^* \omega_{1_m}$$

after projection to $T^* \operatorname{Iso}^0(G)_q$.

However, if we consider the base-points of the $\widehat{\mu}((W^i)^{-1}, V^i)$ they need to satisfy

$$(D^q_{\mu((y^q)^{-1},x^q),q}) \circ \dots \circ (\dot{D}^0_{\mu((y^0)^{-1},x^0),q})(1_m) = (1_n)$$

such that we can apply Lemma 2.1.39 to obtain that all the base-points of the compositions agree. We have already seen that if we additionally assume that all $\widehat{\mu}((W^i)^{-1}, V^i)$ agree, that for the projection of $\omega_{1_{G_0}}$ to the respective values of $\Gamma(G_0, (u_q)^* T(\operatorname{Iso}^0(G))_q)$ the expression of this form satisfies

$$(\dot{D}^0_{\widehat{\mu}((W^0)^{-1},V^0),q})^* \dots (\dot{D}^q_{\widehat{\mu}((W^0)^{-1},V^0),q})^* (\omega_{1_m}|_{\operatorname{Iso}^0(G)}) = ((\widehat{\operatorname{Ad}}^0)_{\widehat{\mu}((W^0)^{-1},V^0)})^* (\omega_{1_m}) \Big|_{\operatorname{Iso}^0(G)},$$

and we have furthermore seen in the proof of Lemma 2.1.40 that this latter action takes the form of the G-action described in the statement, such that by assumption

$$((\widehat{\operatorname{Ad}}^0)_{\widehat{\mu}((W^0)^{-1},V^0)})^* (\omega_{1_m}|_{\operatorname{Iso}^0(G)}) = (\omega_{1_n}|_{\operatorname{Iso}^0(G)}).$$

This in particular implies that

$$(\dot{D}^q_{\widehat{\mu}((W^0)^{-1},V^0),q})^* \dots (\dot{D}^1_{\widehat{\mu}((W^0)^{-1},V^0),q})^* (\dot{D}^0_{\widehat{\mu}((W^0)^{-1},V^0),q})^* \omega_{1_m} = \omega_{1_n}.$$

However, even if not all $\widehat{\mu}((W^i)^{-1}, V^i)$ agree, we can apply this fact to reformulate

$$(\dot{D}^q_{\widehat{\mu}((W^q)^{-1},V^q),q})^* \dots (\dot{D}^1_{\widehat{\mu}((W^1)^{-1},V^1),q})^* (\dot{D}^0_{\widehat{\mu}((W^0)^{-1},V^0),q})^* \omega_{1_m} = \omega_{1_n}$$

as

$$\begin{aligned}&(\dot{D}^q_{\widehat{\mu}((W^q)^{-1},V^q),q})^* \dots (\dot{D}^1_{\widehat{\mu}((W^1)^{-1},V^1),q})^* (\dot{D}^0_{\widehat{\mu}((W^0)^{-1},V^0),q})^* \omega_{1_m}\\ =&(\dot{D}^q_{\widehat{\mu}((W^0)^{-1},V^0),q})^* \dots (\dot{D}^1_{\widehat{\mu}((W^0)^{-1},V^0),q})^* (\dot{D}^0_{\widehat{\mu}((W^0)^{-1},V^0),q})^* \omega_{1_m}\end{aligned}$$

which means that the claim that ω is invariant can now be written as

$$(\dot{D}^q_{U^q,q})^* \dots (\dot{D}^1_{U^1,q})^* (\dot{D}^0_{U^0,q})^* \omega_{1_m} = \omega_{1_m}$$

for the appropriate $U^i \in (J^E G)_1$ with $U^0 = \widehat{u_1}(1_m)$ being the identity jet. In particular, we can apply the same reasoning as above to find that the base-points of all U^i are given by 1_m. For the same reasons as above (any $(\dot{D}^i_{U^i,q})_*$ retains the splitting of E by Lemma 2.1.24 and the foliation $\mathcal{F}_q$ by Lemma 1.2.40) it is sufficient to check that the endomorphism of $T_{1_m} G_q$ described by any such $(\dot{D}^i_{U^i,q})_*$ induces the identity on $T_{1_m} G_q/(\mathcal{F}_q \cap E)$.

We can note that on A_{iso}, the map

$$(\dot{D}^i_{U^i,q})_* : (A_{\text{iso}})_m \to T_{1_m} G_q$$

is already given by the identity due to Lemma 1.2.31. We also have that on $TG_0 \subset TG_q$ the image of any $(Tu_q)(X)$ under $(\dot{D}^i_{U^i,q})_*$ satisfies that

$$Tp^i_q((\dot{D}^i_{U^i,q})_*((Tu_q)(X))) = (\dot{D}^i_{U^i,q})_*((Tp^i_q)(Tu_q)(X)) = (\dot{D}^0_{U^i,0})_*(X) = Tt(Y)$$

by Lemma 1.2.36, where $Y \in U^i$ is the unique tangent vector with $Ts(Y) = X$. However, as U^i has its base-point at a unit, we can analyse Y under the splitting of

$$T_{1_m}G_1 = T_{1_m}G_0 \oplus A$$

we know that the component of Y in $T_{1_m}G_0$ must be given by X, such that $Tt(Y)-X \in \mathcal{F}_0$. Therefore, also

$$(\dot{D}^i_{U^i,q})_*((Tu_q)(X)) - (Tu_q)(X) \in \mathcal{F}_q$$

and furthermore both terms are contained in E. So even though

$$(\dot{D}^i_{U^i,q})_* : T_mG_0 \to T_{1_m}G_q$$

is not given by the identity map per se, the quotient map

$$(\dot{D}^i_{U^i,q})_* : T_mG_0/\mathcal{F}_0 \to T_{1_m}G_q/\mathcal{F}_q$$

is. So indeed, we have that

$$(\dot{D}^i_{U^i,q})^*\omega_{1_m} = \omega_{1_m}$$

for any of the $0 \leq i \leq q$ and therefore

$$(\dot{D}^q_{U^q,q})^* \dots (\dot{D}^1_{U^1,q})^*(\dot{D}^0_{U^0,q})^*\omega_{1_m} = \omega_{1_m}.$$

This concludes the proof of well-definedness.

The fact that any ω defined this way is J^EG-invariant now holds by construction, as by the commutativity of Lemma 1.2.37, we have that the action of any $(\dot{D}^i_{W,q})^*$ on an expression of the form

$$(\dot{D}^0_{V^0,q})^*(\dot{D}^1_{V^1,q})^* \dots (\dot{D}^{q-1}_{V^{q-1},q})^*(\dot{D}^q_{V^q,q})^*\omega_{1_m}$$

can be commuted with all the pull-backs up to $(\dot{D}^i_{V^i,q})^*$, where we obtain

$$(\dot{D}^i_{W,q})^* \circ (\dot{D}^i_{V^i,q})^* = (\dot{D}^i_{\widehat{\mu}(W,V^i),q})^*$$

such that the resulting expression takes an analogous form and therefore agrees with the definition of ω.

$\square$

So algebraically, the J^EG-invariant forms can be constructed from an expression which only depends on G_0, the orbit foliation $\mathcal{F}_0$ and the isotropy algebroid, where the invariance is technically an invariance under elements of G_1 and not jets. This makes it relatively easy to express.

Lemma 2.1.42. *Consider a proper and regular Lie groupoid G and let A be the associated Lie algebroid. Let E be a multiplicative Ehresmann connection with respect to the isotropy algebroid. Then the J^EG-invariant forms of bidegree (p,q) are a subalgebra of*

$$\bigoplus_{p_0+p_1=p} \left(\Gamma\left(G_0, \wedge^{p_0}T^*(G_0/\mathcal{F}_0) \otimes \wedge^{p_1}\left((A^*_{\mathrm{iso}})^{\oplus q}\right)\right)\right)^{\Gamma(G_0,A)}$$

with equality in the case where G is s-connected. By the invariance under $\Gamma(G_0, A)$ we denote the subspace of

$$\Gamma\left(G_0, \wedge^{p_0}T^*(G_0/\mathcal{F}_0) \otimes \wedge^{p_1}\left((A^*_{\mathrm{iso}})^{\oplus q}\right)\right)$$

on which

$$\mathcal{L}_X \equiv 0,$$

where $\mathcal{L}_X$ is the local derivative which acts on each section $\Gamma(G_0, (TG_0/\mathcal{F}_0)^*)$ as $\mathcal{L}_{\rho(X)}$ and on each section $\Gamma(G_0, (A^*_{\mathrm{iso}}))$ corresponding to one copy of A^*_{iso} as the dual to $[X, \bullet]$.

Proof. Note that in the proof of Lemma 2.1.40, we have shown that the G-invariance of the expression actually stemmed from a J^EG-invariance which was induced by the adjoint action $\widehat{\mathrm{Ad}}^0$ on $(\mathrm{Iso}^0(G))_q$. However, we have then seen that for the J^1G-actions on the respective cotangent spaces that they were already determined by the base-points of the acting jets. Hence we can write the space of J^EG-invariant forms as

$$\bigoplus_{p_0+p_1=p} \left(\Gamma\left(G_0, \wedge^{p_0}T^*(G_0/\mathcal{F}_0) \otimes \wedge^{p_1}\left((A^*_{\mathrm{iso}})^{\oplus q}\right)\right)\right)^{J^1G}$$

for that action instead. Note that using J^1G instead of J^EG here is deliberate!

Now consider an invariant element $\omega_{1_{G_0}}$. By Lemma 1.2.28, we know that we can recover the action associated to Ad^0 of any local bisection γ entirely by the action of the local jets. In particular, for any $X \in \Gamma(G_0, A)$, we can consider a neighbourhood U of some $m \in G_0$ such that $\exp(t\cdot X)$ defines a family of local bisections on U for all $t \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$. We then have that

$$(\widetilde{\mathrm{Ad}}^0_{\exp(t\cdot\gamma)})^*\omega_{1_{G_0}} = \omega_{1_{G_0}},$$

where we need to interpret $\omega_{1_{G_0}} \in \Gamma(G_0, (u_q)^*(T(\mathrm{Iso}^0(G))_q))$ for well-definedness.

This implies that then also

$$\frac{\partial}{\partial t}\left((\widetilde{\mathrm{Ad}}^0_{\exp(t\cdot X)})^*\omega_{1_{G_0}}\right)_m\bigg|_{t=0} = 0,$$

which we now claim to be $\mathcal{L}_X(\omega_{1_{G_0}})_m$. This can be seen as follows: We have by Lemma 1.2.39 as well as Lemma 1.2.38, that

$$\widetilde{\mathrm{Ad}}^0_{\exp(t\cdot X)} \circ (u_q) \equiv \widetilde{D}^0_{\exp(t\cdot X),q}$$

where $\widetilde{D}^0_{\exp(t\cdot X),q}$ defines a family of local diffeomorphisms of G_0 that can be explicitly derived to $\rho(X)$ (actually not only at $t = 0$).

Furthermore, we have that for any $Y \in \Gamma(G_0, A)$, the Lie bracket

$$[X, Y]$$

is the unique vector field obtained as

$$m \mapsto -\frac{\partial}{\partial a}\frac{\partial}{\partial b}\phi^Y_b \circ \phi^X_a \circ (\phi^Y_b)^{-1} \circ (\phi^X_a)^{-1}(m)\bigg|_{t=0},$$

which can also be written as

$$m \mapsto \frac{\partial}{\partial a} - (\phi^X_a)_* \circ (D^1_{((\phi^X_a)^{-1}(m))^{-1},1})_*(-Y)_{t((\phi^X_a)^{-1}(m))} - Y_m\bigg|_{a=0},$$

where $D^1_{((\phi^X_a)^{-1}(m))^{-1},1}$ is understood as a map of the manifolds

$$s^{-1}(\{t((\phi^X_a)^{-1}(m))\}) \to s^{-1}(\{s((\phi^X_a)^{-1}(m))\}).$$

We have that

$$\phi^X_a = \widetilde{D}^0_{\exp(a\cdot X),1},$$

as both derive to the right invariant vector field induced by X, meaning in particular that

$$(\phi_a^X)^{-1} = \widetilde{D}^0_{(\exp(a\cdot X))^{-1},1}.$$

and

$$(\phi_a^X)^{-1}(m) = (\exp(a\cdot X)(\psi_a(m)))^{-1},$$

where ψ_a is the local diffeomorphism induced by $\exp(a\cdot X)$ around m. This implies that

$$\exp(a\cdot X)(\psi_a(m)) = ((\phi_a^X)^{-1}(m))^{-1}$$

and therefore $\widetilde{D}^1_{\exp(a\cdot X),1}$ restricts on $s^{-1}(\{\psi_a(m)\})$ to $D^1_{((\phi_a^X)^{-1}(m))^{-1},1}$.

So rewriting our previous statements, we have that

$$[X,Y]$$

is the unique vector field obtained as

$$m \mapsto \frac{\partial}{\partial a}(\widetilde{D}^0_{\exp(a\cdot X),1})_* \circ (\widetilde{D}^1_{\exp(a\cdot X),1})_* Y_{\psi_a(m)} - Y_m \Big|_{t=0},$$

and if $Y \in \Gamma(G_0, A_{\mathrm{iso}})$ we have that this agrees by Lemma 1.2.38 with

$$m \mapsto \frac{\partial}{\partial a}(\widetilde{\mathrm{Ad}}^0_{\exp(a\cdot X),1})_* Y_{\psi_a(m)} - Y_m \Big|_{a=0}$$

and as the constant term vanishes, this agrees with

$$m \mapsto \frac{\partial}{\partial a}\left((\widetilde{\mathrm{Ad}}^0_{\exp(a\cdot X),1})_* Y\right)_m \Big|_{a=0}$$

which is by definition dual to $(\widetilde{\mathrm{Ad}}^0_{\exp(a\cdot X),1})^*$.

It only remains to note that by Example 1.1.31 we have

$$(\mathrm{pr}_i)_* \circ (\widetilde{\mathrm{Ad}}^0_{\exp(a\cdot X),q})_* = (\widetilde{\mathrm{Ad}}^0_{\exp(a\cdot X),1})_* \circ (\mathrm{pr}_i)_*,$$

such that on each $Y \in \Gamma(G_0, (A_{\mathrm{iso}})^{\oplus q})$ contained in a single copy the Lie derivative for the action on G_q (instead of G_1) acts by $[X, \bullet]$ as well.

The statement on equality in case of s-connectedness can be inferred as following: By our considerations on the Lie derivative, we have actually seen that invariance under $\Gamma(G_0, A)$ is equivalent to invariance under the local bisections we constructed locally as $\exp(t\cdot X)$. We have also already used previously that invariance under the J^1G-action and G-action are equivalent, meaning that from invariance with respect to the adjoint J^1G-action under all the local bisections $\exp(t\cdot X)$ we obtain invariance with respect to the adjoint G-action under all the element $\exp(t\cdot X)(m)$. As we can construct all elements locally around the unit section of G_1 as elements of this form and invariance under two composable elements means invariance under their composition, we find that s-connectedness implies that the invariance holds under the adjoint G-action of any element. □

In conclusion, this means that the algebraic expression that we have found is actually independent of the chosen multiplicative Ehresmann connection. This is pleasing both from a conceptual perspective of not having to concern ourselves with “better” or “worse” choices in hindsight and from a practical perspective of the formula being computable without even having to choose. Of course, when considering the map

$$\bigoplus_{p_0+p_1=p} \left(\Gamma\left(G_0, \wedge^{p_0} T^*(G_0/\mathcal{F}_0) \otimes \wedge^{p_1}\left((A^*_{\mathrm{iso}})^{\oplus q}\right)\right)\right)^G \cong \left(\Omega^p(G_q)\right)^{J^E G} \subset \Omega^p(G_q).$$

this very strongly depends on the choice of the multiplicative Ehresmann connection.

We will now aim to show that these $J^E G$-invariant forms can be used to compute the cohomology of the columns of the Bott–Shulman–Stasheff complex.

2.2 Some kind of averaging

Aim of this section: Definitions and Concepts

How (and on what space) can we define a measure or a family of measures that will allow us to average over functions and forms? What will be the properties of this averaging process?

If we have a chain complex of $\mathbb{R}$-vector spaces with a compact Lie group $\mathcal{G}$ acting on it, we often find that the inclusion of the subcomplex of $\mathcal{G}$-invariant elements is a quasi-isomorphism with quasi-inverse given by averaging over $\mathcal{G}$. One of the key facts for this procedure to be possible is that any compact Lie group $\mathcal{G}$ admits a Haar measure, which is the unique normalised measure that is left and right invariant as well-as normalised.

On groupoids, there are similar concepts, like Haar systems, which are a collection of measures on the source fibres (cf. [37], pp. 16-17). However, source fibres are not generally compact, so even with our new notion of invariance, there is still no well-defined averaging procedure. Due to this, we will alter the concept slightly and define measures not on $s^{-1}(\{m\})$ for all $m \in G_0$, but on $t^{-1}(R) \cap s^{-1}(\{m\})$, where R is an admissible embedded manifold. As this space is compact, an averaging procedure will be possible.

Proposition 2.2.1. *Consider a proper and regular Lie groupoid G with an admissible embedded manifold R.*

Then for any $m \in \mathrm{Sat}(R)$ there is a unique measure μ_{Haar} on $t^{-1}(R) \cap s^{-1}(m)$ for any $0 \leq l \leq q$ which satisfies that

- *μ_{Haar} is invariant under right multiplication with any element of $t^{-1}(m) \cap s^{-1}(m)$.*
- *μ_{Haar} is invariant under left multiplication with any element of $t^{-1}(r_2) \cap s^{-1}(r_1)$ when μ_{Haar} is restricted to $t^{-1}(r_1) \cap s^{-1}(m)$ and $t^{-1}(r_2) \cap s^{-1}(m)$ respectively.*
- *We have that*
$$\mu_{\mathrm{Haar}}(t^{-1}(R) \cap s^{-1}(m)) = 1.$$

The measure is smooth in m, meaning that for the surjective submersion (according to Lemma 1.4.16)

$$s|_{t^{-1}(R)} : t^{-1}(R) \to G_0$$

the measure varies smoothly in the fibres.

It also satisfies that for any $x \in (G|_{\mathrm{Sat}(R)})_1$, the map

$$\begin{aligned} t^{-1}(R) \cap s^{-1}(t(x)) &\to t^{-1}(R) \cap s^{-1}(s(x)) \\ a &\mapsto \mu(a, x) \end{aligned}$$

is an isomorphism retaining the Haar measures.

Proof. Consider any $m \in G_0$, then

$$t^{-1}(R) \cap s^{-1}(\{m\}) = \bigcup_{r \in R \cap O_m} t^{-1}(\{r\}) \cap s^{-1}(\{m\}),$$

where all components of the union are mutually disjoint and each is isomorphic to

$$t^{-1}(\{m\}) \cap s^{-1}(\{m\})$$

by left multiplication with some choice of

$$a_r \in t^{-1}(\{r\}) \cap s^{-1}(\{m\})$$

and its inverse respectively.

As G was assumed proper, $t^{-1}(\{m\}) \cap s^{-1}(\{m\})$ is a compact group and hence comes with a Haar measure μ_m, which is both left and right invariant. We can push the measure forward through the isomorphism and obtain μ_r on each component $t^{-1}(\{r\}) \cap s^{-1}(\{m\})$. As left and right multiplication commute, μ_r is invariant under right multiplication with elements of

$$t^{-1}(\{m\}) \cap s^{-1}(\{m\}).$$

We now can see that μ_r is independent of a_r, as any other choice a'_r can be written as

$$a'_r = \mu(a_r, g)$$

for

$$g = \mu((a_r)^{-1}, a'_r) \in t^{-1}(\{m\}) \cap s^{-1}(\{m\}),$$

and hence the push-forward by left multiplication with a'_r can be interpreted as the push-forward under left multiplication with g and then a_r, but the Haar measure is invariant under the first of these operations.

For a similar reason, the measure also satisfies that it is invariant under right multiplication with an element

$$g \in t^{-1}(\{r\}) \cap s^{-1}(\{r\}),$$

as under the isomorphism, this corresponds to left multiplication with $\mu((a_r)^{-1}, g, a_r)$. Given two different components indexed by r_1 and r_2, we further have that any

$$x \in t^{-1}(\{r_2\}) \cap s^{-1}(\{r_1\})$$

can be written as

$$x = \mu(a_{r_1}, g, (a_{r_2})^{-1}),$$

such that left push-forward induced by left multiplication will map the measure μ_{r_1} to the Haar measure on $t^{-1}(\{m\}) \cap s^{-1}(\{m\})$, which is retained by left multiplication with g and then pushed forward to the measure μ_{r_2} by definition.

Note that for this reason the first two conditions of the statement actually imply that on each component $t^{-1}(\{r\}) \cap s^{-1}(\{m\})$ the measure we aim to construct has to be induced by a linear multiple of the Haar measure on $t^{-1}(\{m\}) \cap s^{-1}(\{m\})$ and that the scaling factor has to be uniform for all $r \in R$. Now we can define

$$\mu_{\mathrm{Haar}}(S) = \frac{\sum_{r \in R \cap \mathcal{O}_m} \mu_r \left(S \cap \left(t^{-1}(\{r\}) \cap s^{-1}(\{m\})\right)\right)}{\sum_{r \in R \cap \mathcal{O}_m} \mu_r(t^{-1}(\{r\}) \cap s^{-1}(\{m\}))}$$

as the unique measure on $t^{-1}(R) \cap s^{-1}(\{m\})$ which satisfies all required properties, because the property on $\mu_{\mathrm{Haar}}(t^{-1}(R) \cap s^{-1}(m))$ uniquely defines the rescaling constant. We can also notice, that the measure on $t^{-1}(\{r\}) \cap s^{-1}(\{m\})$ can not only be inferred from the Haar measure on $t^{-1}(\{m\}) \cap s^{-1}(\{m\})$, but also from the Haar measure on $t^{-1}(\{r\}) \cap s^{-1}(\{r\})$ for any $r \in O_m \cap R$, which immediately implies the last statement of the proposition.

Furthermore, around any $m \in G_0$ we may choose a local section γ of s with target in R. By our latest remark on our construction this describes a family of isomorphisms of each $t^{-1}(R) \cap s^{-1}(m)$ with $t^{-1}(R) \cap s^{-1}(t(\gamma(m)))$, where this family retains the Haar measures and smoothly depends on m. This implies that when checking the smoothness of the measure depending on $m \in G_0$, it is sufficient if we can show smoothness of the measure for points $m_R \in R$. Now we can also realise that the constructed Haar measure restricted to $t^{-1}(\{m_R\}) \cap s^{-1}(\{m_R\}) \subset t^{-1}(R) \cap s^{-1}(m_R)$ is in particular invariant under left an right multiplication, hence a linear multiple of the group Haar measure. This implies that on any connected component the measure is induced by a linear multiple of the Haar measure on the connected component of the unit, which we will denote μ_m^0. We know that $\mathrm{Iso}^0(G)$ is a Lie groupoid according to Lemma 1.1.15, and as on this subgroupoid $s \equiv t$, it is actually a Lie group bundle of compact Lie groups, hence the Haar measures μ_m^0 of $\mathrm{Iso}^0(G)|_R$ vary smoothly for any $m \in R$.

By the assumption that R is admissible, we have that $G|_R$ is an s-proper Lie groupoid. By Corollary 1.4.26, we know that locally around m_R, the source fibres of $t^{-1}(R) \cap s^{-1}(R)$ are isomorphic to $t^{-1}(R) \cap s^{-1}(\{m_R\})$. Under this identification $\mathrm{Iso}^0(G)$ is identified with $t^{-1}(\{m_R\}) \cap s^{-1}(\{m_R\})$ and as the trivialisation is compatible with t trivialised over each connected component of R.

Hence the measure $\mu_{\mathrm{Haar},m}$ on each connected component of every fibre is a linear multiple of μ_m^0 by a factor which is uniform over every fibre. Hence smoothness of μ_{Haar} in m is determined by smoothness of the scaling factor. However, as the measure on every fibre is normalised, the scaling factor is the inverse of the number of components, which is constant as the bundle is locally trivial. □

Definition 2.2.2. Consider a proper and regular Lie groupoid G and an admissible embedded manifold $R \subset G_0$. Then the measure on $t^{-1}(R) \cap s^{-1}(m)$ described in Proposition 2.2.1 will be called the **R-Haar measure**.

Lemma 2.2.3. *Consider a proper and regular Lie groupoid G equipped with a multiplicative Ehresmann connection E with respect to the isotropy algebroid, together with an admissible embedded submanifold $R \subset G_0$. For any $x \in G_1$ with $t(x) \in R$, there exists a unique $V_x \in J^E(s : G_1 \to G_0)$ which under Tt maps to $T_{t(x)}R \subset T_{t(x)}G_0$.*

Proof. By Corollary 1.4.19, we have that $s : t^{-1}(R) \to G_0$ is a submersion which is surjective onto $\mathrm{Sat}(R)$. This means that at $x \in t^{-1}(R)$, we can choose an arbitrary lift W_x which projects under Ts to $T_{s(x)}G_0$ isomorphically. We further know that it maps to $T_{t(x)}R$ under t. Consider now the E-horizontal projection $V_x \subset E_x$ of W_x. As it only changes the elements contained in W_x by adding E-vertical vectors (which both vanish under Ts and Tt) both properties are retained.

For the claim of uniqueness, assume that there is V_x' with the same properties. Now consider $X \in T_{s(x)}G_0$ and compare the unique lifts $Z \in V_x$ and $Z' \in V_x'$. We have that $Z - Z'$ vanishes under Ts. This implies in particular that $Z - Z'$ maps to an element of $\mathcal{F}_0$ under Tt. But as we have that the vector is also contained in $T_{t(x)}R$, meaning that it must be 0. So $Z - Z' \in \ker(Tt) \cap \ker(Ts) = IG_1$. At the same time, we have $Z - Z' \in E_x$, such that indeed $Z = Z'$. As this procedure works for any $X \in T_{s(x)}G_0$ and Ts induces an isomorphism from $T_{s(x)}G_0$ to V_x (and V_x' respectively), we find that indeed V_x is unique. □

Definition 2.2.4. Consider a proper and regular Lie groupoid G equipped with a multiplicative Ehresmann connection E with respect to the isotropy algebroid, together with an admissible embedded manifold $R \subset G_0$. Then for every $\omega \in \Omega(G_q)$ the **R-average of ω over the diagonal action at the l-th pull-back** is the form $(\overline{D}^l_{R,q})^*(\omega)$ on the open submanifold $(G|_{\mathrm{Sat}(R)})_q \subset G_q$ point-wise defined by

$$\left(\overline{D}^l_{R,q}\right)^*(\omega) : (G|_{\mathrm{Sat}(R)})_q \to \wedge^p(T^*G_q)$$
$$z \mapsto \int_{a \in s^{-1}(p^l_q(z)) \cap t^{-1}(R)} \left((\dot{D}^l_{V_a})^*(\omega)\right)_z \mathrm{d}\mu_{\mathrm{Haar}},$$

where we choose V_a depending on a as the unique subspace of E_a as defined in Lemma 2.2.3. The measure μ_{Haar} refers to the R-Haar measure (as in Proposition 2.2.1).

Remark. Under insertion of vector fields, this definition means that

$$\left(\left(\overline{D}^l_{R,q}\right)^*(\omega)\right)_z (X_1, \ldots, X_p)$$
$$= \int_{a \in s^{-1}(p^l_q(z)) \cap t^{-1}(R)} \omega_{(D^l_a(z))} \left((\dot{D}^l_{V_a})_* X_1, \ldots, (\dot{D}^l_{V_a})_* X_p\right) \mathrm{d}\mu_{\mathrm{Haar}}.$$

Lemma 2.2.5. *Consider a proper and regular Lie groupoid G equipped with a multiplicative Ehresmann connection E with respect to the isotropy algebroid, together with an admissible embedded manifold $R \subset G_0$. If $\omega \in \Omega^p(G_q)$ is $J^E G$-invariant, then $(\overline{D}^l_{R,q})^*(\omega) \equiv \omega$ as forms on $(G|_{\mathrm{Sat}(R)})_q \subset G_q$.*

Proof. As ω is $J^E G$-invariant, we have that in Definition 2.2.4 the expression defining $(\overline{D}^l_{R,q})^*(\omega)$ can be simplified using $\left(\left(\dot{D}^l_{V_a}\right)^* \omega\right)_z = \omega_z$ because $V_a \subset E_a$. Hence we are integrating a constant value with respect to a normalised measure. □

Lemma 2.2.6. *Consider a proper and regular Lie groupoid G equipped with a multiplicative Ehresmann connection E with respect to the isotropy algebroid, together with an admissible embedded manifold $R \subset G_0$. Assume that $\omega \in \Omega^p(G_q)$ is E-homogeneous of E-horizontal degree p_0 and E-vertical degree p_1. Then $(\overline{D}^l_{R,q})^*(\omega)$ is also E-homogeneous of the same degrees.*

Proof. In Definition 2.2.4, we have that for each $z \in G_q$ the terms which we integrate to obtain $(\overline{D}^l_{R,q})^*(\omega)$ at z is E-homogeneous. This holds by Lemma 2.1.34 as any $(\dot{D}^l_{V_a})^*$ respects the splitting with respect to E. As integration is linear (and well-defined due to the compact domain), we obtain the same for the result. □

Lemma 2.2.7. *Consider a proper and regular Lie groupoid G equipped with a multiplicative Ehresmann connection E with respect to the isotropy algebroid, together with an admissible embedded manifold $R \subset G_0$. Furthermore consider $\omega \in \Omega^p(G_q)$. Let us denote the restriction of $(\overline{D}^k_{R,q})^*\omega$ to $G|_R$ by*

$$\left((\overline{D}^k_{R,q})^*\omega\right)\Big|_R$$

and the restriction of ω to $G|_R$ by $\omega|_R$.

We have that $\left((\overline{D}^k_{R,q})^*\omega\right)|_R$ is completely determined by $\omega|_R$ and the dependence is linear. In particular, if $\omega|_R = 0$, we find also $\left((\overline{D}^k_{R,q})^*\omega\right)|_R = 0$.

Proof. By definition, we have that for $z \in G|_R$ the value of $\left((\overline{D}^k_R)^*\omega\right)_z$ is determined by its evaluations

$$(X_1, \dots, X_p) \mapsto \left((\overline{D}^k_R)^*\omega\right)_z (X_1, \dots, X_p)$$

which is computed from the terms of the form

$$\omega_{(D^k_a(z))}\left((D^k_{V_a})_* X_1, \dots, (D^k_{V_a})_* X_p\right)$$

for $a \in t^{-1}(R) \cap s^{-1}(p^l_q(z))$. As we are only interested in values of the restriction, we may furthermore assume that all $X_i \in T(G|_R)_q$, which is equivalent to stating that $Tp^j_q(X_i) \in TR$ for all $0 \leq j \leq q$.

However, we know by Example 1.2.20 that for any $a \in t^{-1}(R) \cap s^{-1}(p^k_q(z))$ holds

$$Tp^j_q(X_i) = Tp^j_q((D^k_{V_a})_*(X_i))$$

if $j \neq k$. If $j = k$ then we know by Lemma 1.2.36 that

$$Tp^j_q((D^k_{V_a})_*(X_i)) = (\dot{D}^k_{V,0})_*(Tp^j_q(X_i)),$$

i.e. it is given by the image under Tt of the unique element $Y \in V_a$ satisfying that

$$Ts(Y) = Tp^j_q(X_i).$$

By the choice of V_a satisfying that $Tt(Y) \in TR$, we find that $(D^k_{V_a})_*(X_i)$ is also contained in $T(G|_R)_q$. This automatically implies that $\omega|_R$ completely determines all values of $\left((D^k_R)^*\omega\right)|_R$. As the integral is linear, we find that so is the correspondence and in particular, if we are integrating 0, we obtain 0. □

Lemma 2.2.8. *Consider a proper and regular groupoid G with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Furthermore assume that there is an admissible embedded manifold R with $\mathrm{Sat}(R) = G_0$. Let $\omega \in \Omega^p(G_q)$ be a form for $p > \dim(R) + q \cdot (k_1 - k_0)$, where k_0 is the dimension of G_0 and k_1 is the dimension of G_1. Then*

$$(\overline{D}^l_{R,q})^*\omega = 0.$$

Proof. By definition, we have that for any $z \in G_q$ the value of $\left((\overline{D}^l_R)^*\omega\right)_z$ is determined by its evaluations

$$(X_1, \dots, X_p) \mapsto \left((\overline{D}^l_R)^*\omega\right)_z (X_1, \dots, X_p)$$

which is computed from the terms of the form

$$\omega_{(D^l_a(z))}\left((D^l_{V_a})_* X_1, \dots, (D^l_{V_a})_* X_p\right)$$

for $a \in t^{-1}(R) \cap s^{-1}(p^l_q(z))$. As we have already noted in the proof of Lemma 2.2.7, all the vectors $(D^l_{V_a})_* X_i$ satisfy that

$$Tp^l_q\left((D^l_{V_a})_* X_i\right) \in T_{t(a)}R,$$

such that $\left((\overline{D}^k_R)^*\omega\right)_z$ only depends on $\omega|_{(p^l_q)^{-1}(R)}$.

We have that $p^l_q : G_q \to G_0$ is a surjective submersion by Lemma 1.1.5, such that $(p^l_q)^{-1}(R)$ is a submanifold of G_q of codimension l_0 (the same as the codimension of R as a submanifold of G_0). This implies that

$$\dim((p^l_q)^{-1}(R)) = q \cdot k_1 - (q-1) \cdot k_0 - l_0 = (k_0 - l_0) + q \cdot (k_1 - k_0).$$

So as $p > \dim((p^l_q)^{-1}(R))$, we find that trivially $\omega|_{(p^l_q)^{-1}(R)} = 0$, meaning that all terms we have discussed in this proof vanish. $\square$

Lemma 2.2.9. *Consider a proper and regular Lie groupoid G equipped with a multiplicative Ehresmann connection E with respect to the isotropy algebroid, together with an admissible embedded manifold $R \subset G_0$. Whenever defined, we have*

$$(\overline{D}^k_{R,q+1})^* \circ (d_i)^* = \begin{cases} (d_i)^* \circ (\overline{D}^k_{R,q})^* & \text{if } i > k \\ (d_i)^* \circ (\overline{D}^{(k-1)}_{R,q})^* & \text{if } i < k \\ (d_i)^* & \text{if } i = k, \end{cases}$$

and

$$(d_i) \circ (\overline{D}^l_{R,q})^* = \begin{cases} (\overline{D}^l_{R,q+1})^* \circ (d_i)^* & \text{if } i > l \\ (\overline{D}^{(l+1)}_{R,q+1})^* \circ (d_i)^* & \text{if } i \leq l \end{cases}$$

as well as

$$(\overline{D}^l_{R,q})^* \circ (s_j)^* = \begin{cases} (s_j)^* \circ (\overline{D}^l_{R,q+1})^* & \text{if } j > l \\ (s_j)^* \circ (\overline{D}^{l+1}_{R,q+1})^* & \text{if } j < l, \end{cases}$$

and

$$(s_j)^* \circ (\overline{D}^l_{R,q+1})^* = \begin{cases} (\overline{D}^l_{R,q})^* \circ (s_j)^* & \text{if } j > l \\ (\overline{D}^{(l-1)}_{R,q})^* \circ (s_j)^* & \text{if } j + 1 < l. \end{cases}$$

Furthermore, we for any γ and any $l \neq k$, we have that

$$(\widetilde{D}^l_{\gamma,q})^* \circ (\overline{D}^k_{R,q})^* \omega = (\overline{D}^k_{R,q})^* \circ (\widetilde{D}^l_{\gamma,q})^* \omega.$$

and analogously for any $V_x \in J^1(s : G_1 \to G_0)$ and any $l \neq k$, we have that

$$(\dot{D}^l_{V_x,q})^* \circ (\overline{D}^k_{R,q})^* \omega = (\overline{D}^k_{R,q})^* \circ (\dot{D}^l_{V_x,q})^* \omega,$$

where both expressions are defined on $(p^l_q)^{-1}(s(x))$.

Proof. As pull-backs are defined fibre-wise, they interchange with the integral in the expression defining $(\overline{D}^k_{R,q})^*$, such that we can apply the interchanging rules of Corollary 1.2.35 and Lemma 1.2.37 (where for section, we additionally employ Lemma 1.2.28). Also note that, in almost all of these cases above the domains of integration agree on the left-hand and right-hand side, meaning that we recover precisely the equations above. The exception is $(\overline{D}^k_{R,q})^* \circ (d_k)^* = (d_k)^*$, where after the application of the interchanging rules on the left-hand side, we obtain an integral over a constant with respect to a normalised measure. $\square$

Lemma 2.2.10. *Consider a proper and regular Lie groupoid G equipped with a multiplicative Ehresmann connection E with respect to the isotropy algebroid, together with an admissible embedded manifold $R \subset G_0$. If V_x is E-horizontal and $s(x) = p^l_q(z)$, we have that*

$$\left((\dot{D}^l_{V_x,q})^* \circ (\overline{D}^l_{R,q})^* \omega\right)_z = \left((\overline{D}^l_{R,q})^* \omega\right)_z,$$

i.e. $(\overline{D}^l_{R,q})^* \omega$ is invariant under $(\dot{D}^l_{V_x,q})^*$ whenever defined.

Proof. We have

$$\left((\overline{D}^l_{R,q})^*\omega\right)_{\dot{D}^l_{V_x,q}(z)} = \int_{a\in s^{-1}(p^l_q(D^l_{x,q}(z)))\cap t^{-1}(R)} \left((\dot{D}^l_{V_a})^*(\rho)\right)_{D^l_{x,q}(z)} \mathrm{d}\mu_{\mathrm{Haar}}$$

and hence

$$\left((\dot{D}^l_{V_x,q})^* \circ (\overline{D}^l_{R,q})^*\omega\right)_z = (\dot{D}^l_{V_x,q})^* \left(\int_{a\in s^{-1}(p^l_q(D^l_{x,q}(z)))\cap t^{-1}(R)} \left((\dot{D}^l_{V_a})^*(\rho)\right)_{D^l_{x,q}(z)} \mathrm{d}\mu_{\mathrm{Haar}}\right).$$

However, as pull-backs are fibre-wise linear, this can be computed as

$$\begin{aligned}
&(\dot{D}^l_{V_x,q})^* \left(\int_{a\in s^{-1}(p^l_q(D^l_{x,q}(z)))\cap t^{-1}(R)} \left((\dot{D}^l_{V_a})^*(\rho)\right)_{D^l_{x,q}(z)} \mathrm{d}\mu_{\mathrm{Haar}}\right)\\
&= \int_{a\in s^{-1}(p^l_q(D^l_{x,q}(z)))\cap t^{-1}(R)} (\dot{D}^l_{V_x,q})^* \left(\left((\dot{D}^l_{V_a})^*(\rho)\right)_{D^l_{x,q}(z)}\right) \mathrm{d}\mu_{\mathrm{Haar}}\\
&= \int_{a\in s^{-1}(p^l_q(D^l_{x,q}(z)))\cap t^{-1}(R)} \left((\dot{D}^l_{V_x,q})^*(\dot{D}^l_{V_a})^*(\rho)\right)_z \mathrm{d}\mu_{\mathrm{Haar}}\\
&= \int_{a\in s^{-1}(p^l_q(D^l_{x,q}(z)))\cap t^{-1}(R)} \left((\dot{D}^l_{\mu(V_a,V_x)})^*(\rho)\right)_z \mathrm{d}\mu_{\mathrm{Haar}}.
\end{aligned}$$

Note, that we have already seen in Lemma 1.2.33 that

$$p^l_q(D^l_{x,q}(z)) = D^0_{x,0}(p^l_q(z)) = t(x).$$

Now, by multiplicativity of E, we have that $\mu(V_a, V_x)$ is an E-horizontal subspace of $T_{\mu(a,x)}G_1$ and satisfies that the image under Tt is contained in $T_{t(a)}R$, as this is true for V_a. Hence we can choose $\mu(V_a, V_x) = V_b$ for

$$b = \mu(a, x).$$

This defines a bijection between $s^{-1}(t(x)) \cap t^{-1}(R)$ and $s^{-1}(s(x)) \cap t^{-1}(R)$, as we can recover a by inverse multiplication with x^{-1} and by the properties of μ_{Haar} according to Proposition 2.2.1, we have that the measure is invariant under it. Thus

$$\begin{aligned}
&\int_{a\in s^{-1}(p^l_q(D^l_{x,q}(z)))\cap t^{-1}(R)} \left((\dot{D}^l_{\mu(V_a,V_x)})^*(\rho)\right)_z \mathrm{d}\mu_{\mathrm{Haar}}\\
&= \int_{b\in s^{-1}(p^l_q(z))\cap t^{-1}(R)} \left((\dot{D}^l_{\mu(V_a,V_x)})^*(\rho)\right)_z \mathrm{d}\mu_{\mathrm{Haar}}\\
&= \int_{b\in s^{-1}(p^l_q(z))\cap t^{-1}(R)} \left((\dot{D}^l_{V_b})^*(\rho)\right)_z \mathrm{d}\mu_{\mathrm{Haar}}\\
&= \left((\overline{D}^l_{R,q})^*\omega\right)_z.
\end{aligned}$$

□

2.3 A model for the columns

Aim of this section: Theorem 2.3.13 (abbreviated):

Let G be a proper and regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. The inclusion

$$\left((\Omega^p(G_\bullet))^{J^E G}, \mathrm{d}_{\mathcal{N}}\right) \subset (\Omega^p(G_\bullet), \mathrm{d}_{\mathcal{N}}) = \mathfrak{C}_p$$

is a quasi-isomorphism, meaning the cohomology of each column of the Bott–Shulman–Stasheff-complex $H^\bullet(\mathfrak{C}_p)$ can be computed on the $J^E G$-invariant elements.

We can now commence with the actual proof of the result mentioned as the aim of this section and chapter. The entirety of this section will be dedicated to prove that the $J^E G$-invariant forms compute the cohomology of the columns of the Bott–Shulman–Stasheff complex, however, we also include some additional details. Most of this section will be technical, as the proof heavily relies on using simplicial identities to find explicit preimages for exact terms.

We will define the notation needed for this first and then start on the propositions and lemma containing the core proof steps.

Definition 2.3.1. Let G be a Lie groupoid. We define $\mathfrak{C}_p$ as the **cochain complex of the p-th column of** $\Omega_{\mathrm{BSS}}(G)$ given by

$$(\mathfrak{C}_p)^\bullet = (\Omega^p(G_\bullet), \mathrm{d}_{\mathcal{N}})$$

Remark. In the following, we will have to use a number of simplicial identities, but also the relation of face and degeneracy maps with the $\dot{D}^i_{V,q}$ and $\overline{D}^i_{R,q}$. We would like to refer the reader to the collection of computation rules we established between pages 88 and 91.

Proposition 2.3.2. *Consider a proper and regular Lie groupoid G together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid, and assume there is an admissible embedded submanifold $R \subset G_0$ with* $\mathrm{Sat}(R) = G_0$.

Then for any $0 \leq j \leq q$ we can express $\omega \in \Omega^p(G_q)$ as

$$\omega = (-1)^{p+j}\left(A_j + B_j + C_j - (s_j)^*\left((\overline{D}^{j+1}_{R,q+1})^*(d_{\mathcal{N}}(\omega))\right)\right),$$

where

$$A_j = \sum_{i=0}^{j-1}(-1)^{p+i}\left((d_i)^*(s_{j-1})^*(\overline{D}^j_{R,q})^*(\omega)\right)$$

and

$$B_j = (-1)^{p+j}\left((\overline{D}^j_{R,q})^*(\omega)\right)$$

and

$$C_j = \sum_{i=j+2}^{q+1}(-1)^{p+i}\left((d_{i-1})^*(s_j)^*(\overline{D}^{j+1}_{R,q})^*(\omega)\right).$$

Proof. As all maps (including $(\overline{D}^l_{R,q})^*$, which is clear from the definition) are defined such that their value at a point is linearly dependent on their argument, we have that

$$\begin{aligned}
&(s_j)^*\left((\overline{D}^{j+1}_{R,q+1})^*\left(\mathrm{d}_{\mathcal{N}}(\omega)\right)\right)\\
=&(s_j)^*\left((\overline{D}^{j+1}_{R,q+1})^*\left(\sum_{i=0}^{q+1}(-1)^{p+i}(d_i)^*(\omega)\right)\right)\\
=&\sum_{i=0}^{q+1}(-1)^{p+i}\left((s_j)^*(\overline{D}^{j+1}_{R,q+1})^*(d_i)^*(\omega)\right)\\
=&\sum_{i=0}^{j}(-1)^{p+i}\left((s_j)^*(d_i)^*(\overline{D}^{j}_{R,q})^*(\omega)\right)\\
&+(-1)^{p+j+1}\left((s_j)^*(d_{j+1})^*(\omega)\right)\\
&+\sum_{i=j+2}^{q+1}(-1)^{p+i}\left((s_j)^*(d_i)^*(\overline{D}^{j+1}_{R,q})^*(\omega)\right)\\
=&\sum_{i=0}^{j-1}(-1)^{p+i}\left((d_i)^*(s_{j-1})^*(\overline{D}^{j}_{R,q})^*(\omega)\right)\\
&+(-1)^{p+j}\left((\overline{D}^{j}_{R,q})^*(\omega)\right)\\
&+(-1)^{p+j+1}\omega\\
&+\sum_{i=j+2}^{q+1}(-1)^{p+i}\left((d_{i-1})^*(s_j)^*(\overline{D}^{j+1}_{R,q})^*(\omega)\right)
\end{aligned}$$

Thus we can express

$$\omega=(-1)^{p+j}\left(A_j+B_j+C_j-(s_j)^*\left((\overline{D}^{j+1}_{R,q+1})^*(d_{\mathcal{N}}(\omega))\right)\right)$$

where

$$A_j=\sum_{i=0}^{j-1}(-1)^{p+i}\left((d_i)^*(s_{j-1})^*(\overline{D}^{j}_{R,q})^*(\omega)\right)$$

and

$$B_j=(-1)^{p+j}\left((\overline{D}^{j}_{R,q})^*(\omega)\right)$$

and

$$C_j=\sum_{i=j+2}^{q+1}(-1)^{p+i}\left((d_{i-1})^*(s_j)^*(\overline{D}^{j+1}_{R,q})^*(\omega)\right).$$

□

Example 2.3.3. Consider a proper and regular Lie groupoid G together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid, and assume there is an admissible embedded submanifold $R \subset G_0$ with $\mathrm{Sat}(R) = G_0$. Consider $f \in C^\infty(G_0)$. Then we can compute the terms from the decomposition of Proposition 2.3.2: We have

$$A_0 = C_0 = 0$$

trivially, as the sums are empty. We have

$$B_0 = (-1)^{0+1}\left((\overline{D}^0_{R,q})^*(f)\right)$$

which at $m \in G_0$ is given by

$$-\int_{a\in t^{-1}(R)\cap s^{-1}(m)} ((\dot{D}^0_{V_a,0})^*(f))(m)\mathrm{d}\mu_{\mathrm{Haar}},$$

simplifying to

$$-\int_{a\in t^{-1}(R)\cap s^{-1}(m)} f(t(a))\mathrm{d}\mu_{\mathrm{Haar}} = -\sum_{r\in R\cap\mathcal{O}_m} \mu_{\mathrm{Haar}}(t^{-1}(r)\cap s^{-1}(m))\cdot f(r).$$

However, as the Haar measure is invariant under right multiplication and normalised, we have

$$\mu_{\mathrm{Haar}}(t^{-1}(r)\cap s^{-1}(m)) = \frac{1}{|R\cap\mathcal{O}_m|},$$

such that the formula describes an averaging procedure. We have already seen in Corollary 1.4.27 that this procedure produces a well-defined and basic function. This, together with the result of Proposition 2.3.2, implies in particular that any $\mathrm{d}_{\mathcal{N}}$-closed function is automatically basic.

Furthermore, we can consider the decomposition in the case in which $\mathrm{d}_{\mathcal{N}}(f)$ does not necessarily vanish. We then have

$$(s_0)^*\left((\overline{D}^1_{R,1})^*(\mathrm{d}_{\mathcal{N}}(f))\right) = (u_1)^*\left((\overline{D}^1_{R,1})^*((s^*-t^*)(f))\right)$$

which at m is given by

$$\begin{aligned}&\int_{a\in t^{-1}(R)\cap s^{-1}(m)} ((s^*(f))(a^{-1}))\mathrm{d}\mu_{\mathrm{Haar}} - \int_{a\in t^{-1}(R)\cap s^{-1}(m)} ((t^*(f))(a^{-1}))\mathrm{d}\mu_{\mathrm{Haar}}\\ =&\int_{a\in t^{-1}(R)\cap s^{-1}(m)} ((s^*(f))(a^{-1}))\mathrm{d}\mu_{\mathrm{Haar}} - f(m)\end{aligned}$$

and as $s(a^{-1}) = t(a)$ the first term agrees (up to sign) with B_0.

Example 2.3.4. Consider a proper and regular Lie groupoid G together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid, and assume there is an admissible embedded submanifold $R \subset G_0$ with $\mathrm{Sat}(R) = G_0$. Consider $\omega \in \Omega^p(G_q)$ where $p > \dim(R) + q\cdot(k_1 - k_0)$, where k_0 is the dimension of G_0 and k_1 is the dimension of G_1. Then we can compute the terms of the decomposition of Proposition 2.3.2: We have

$$A_j = B_j = C_j = 0,$$

which follows from Lemma 2.2.8. This implies that whenever an ω of this form is closed, it automatically must be 0 and more generally

$$(-1)^{p+j+1}\left((s_j)^*\left((\overline{D}^{j+1}_{R,q+1})^*(d_{\mathcal{N}}(\omega))\right)\right) = \omega.$$

Example 2.3.5. Consider a proper and regular Lie groupoid G together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid, and assume there is an admissible embedded submanifold $R \subset G_0$ with $\mathrm{Sat}(R) = G_0$. Consider $\omega \in (\Omega^p(G_q))^{J^E G}$ a $J^E G$-invariant form, and assume that

$$(s_j)^*(\omega) = (s_{j-1})^*(\omega) = 0.$$

Then we can compute the terms of Proposition 2.3.2 as follows: We have

$$A_j = C_j = 0,$$

as both of these terms are linear in either $(s_j)^*(\omega)$ or $(s_{j-1})^*(\omega)$. We further have

$$B_j = (-1)^{p+j}\omega.$$

This in particular implies that

$$(s_j)^*\left((\overline{D}^{j+1}_{R,q+1})^*(d_{\mathcal{N}}(\omega))\right) = 0$$

and that the equation of Proposition 2.3.2 takes the form

$$\omega = (-1)^{p+j}\left(0 + (-1)^{p+j}\omega + 0 - 0\right).$$

Lemma 2.3.6. *Consider a proper and regular Lie groupoid G together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid, and assume there is an admissible embedded submanifold $R \subset G_0$ with $\mathrm{Sat}(R) = G_0$. Further assume some ω is invariant (when comparing values at the appropriate $z \in G_q$) with respect to the action by $(\dot{D}^l_{V_x,q})^*$ for any $0 \leq l < k$ of any $V_x \in J^E(s : G_1 \to G_0)$.*

Then the term A_k defined as in Proposition 2.3.2 is invariant with respect to the action by $(\dot{D}^l_{V_y,q})^*$ for any $0 \leq l \leq k$ of any $V_y \in J^E(s : G_1 \to G_0)$.

Proof. We have that when $i < k$ (which holds for all terms appearing in A_k) and $l < k$ it holds

$$\begin{aligned}
&(\dot{D}^l_{V_y,q})^*(d_i)^*(s_{k-1})^*(\overline{D}^k_{R,q})^*(\omega') \\
&= \begin{cases} (d_i)^*(\dot{D}^{l-1}_{V_y,q})^*(s_{k-1})^*(\overline{D}^k_{R,q})^*(\omega') & \text{if } i < l \\ (d_i)^*(s_{k-1})^*(\overline{D}^k_{R,q})^*(\omega') & \text{if } i = l \\ (d_i)^*(\dot{D}^l_{V_y,q})^*(s_{k-1})^*(\overline{D}^k_{R,q})^*(\omega') & \text{if } i > l \end{cases} \\
&= \begin{cases} (d_i)^*(s_{k-1})^*(\dot{D}^{l-1}_{V_y,q})^*(\overline{D}^k_{R,q})^*(\omega') & \text{if } i < l \\ (d_i)^*(s_{k-1})^*(\overline{D}^k_{R,q})^*(\omega') & \text{if } i = l \\ (d_i)^*(s_{k-1})^*(\dot{D}^l_{V_y,q})^*(\overline{D}^k_{R,q})^*(\omega') & \text{if } i > l \end{cases} \\
&= \begin{cases} (d_i)^*(s_{k-1})^*(\overline{D}^k_{R,q})^*(\dot{D}^{l-1}_{V_y,q})^*(\omega') & \text{if } i < l \\ (d_i)^*(s_{k-1})^*(\overline{D}^k_{R,q})^*(\omega') & \text{if } i = l \\ (d_i)^*(s_{k-1})^*(\overline{D}^k_{R,q})^*(\dot{D}^l_{V_y,q})^*(\omega') & \text{if } i > l \end{cases} \\
&= (d_i)^*(s_{k-1})^*(\overline{D}^k_{R,q})^*(\omega'),
\end{aligned}$$

where we have used the assumption that ω is in particular invariant under $(\dot{D}^l_{V_y,q})^*$ and the interchanging rules of Corollary 1.2.35 as well as Lemma 2.2.9. This implies

$$(\dot{D}^l_{V_y,q})^* A_k = A_k$$

for $l < k$.

For $l = k$ we have

$$\begin{aligned}
&(\dot{D}^k_{V_y,q})^*(d_i)^*(s_{k-1})^*(\overline{D}^k_{R,q})^*(\omega') \\
&= (d_i)^*(\dot{D}^{k-1}_{V_y,q-1})^*(s_{k-1})^*(\overline{D}^k_{R,q})^*(\omega') \\
&= (d_i)^*(s_{k-1})^*(\dot{D}^k_{V_y,q})^*(\dot{D}^{k-1}_{V_y,q})^*(\overline{D}^k_{R,q})^*(\omega') \\
&= (d_i)^*(s_{k-1})^*(\dot{D}^k_{V_y,q})^*(\overline{D}^k_{R,q})^*(\dot{D}^{k-1}_{V_y,q})^*(\omega') \\
&= (d_i)^*(s_{k-1})^*(\dot{D}^k_{V_y,q})^*(\overline{D}^k_{R,q})^*(\omega') \\
&= (d_i)^*(s_{k-1})^*(\overline{D}^k_{R,q})^*(\omega'),
\end{aligned}$$

where we have additionally used Lemma 1.2.37 and Lemma 2.2.10 such that also

$$(\dot{D}^l_{V_y,q})^* A_k = A_k$$

in this case. □

Lemma 2.3.7. *Consider a proper and regular Lie groupoid G together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid, and assume there is an admissible embedded submanifold $R \subset G_0$ with $\mathrm{Sat}(R) = G_0$. Assume some ω is invariant (when comparing values at the appropriate $z \in G_q$) with respect to the action by $(\dot{D}^l_{V_x,q})^*$ for any $0 \leq l < k$ of any $V_x \in J^E(s : G_1 \to G_0)$.*

Then the term B_k defined as in Proposition 2.3.2 is invariant with respect to the action by $(\dot{D}^l_{V_y,q})^*$ for any $0 \leq l \leq k$ of any $V_y \in J^E(s : G_1 \to G_0)$.

Proof. According to Lemma 2.2.9, we have that

$$(\dot{D}^l_{V_y,q})^*(\overline{D}^k_{R,q})^*(\omega') = (\overline{D}^k_{R,q})^*(\dot{D}^l_{V_y,q})^*(\omega') = (\overline{D}^k_{R,q})^*(\omega')$$

for $l < k$, where we use the assumption of ω being invariant under $(\dot{D}^l_{V_y,q})^*$, and Lemma 2.2.10 implying

$$(\dot{D}^k_{V_y,q})^*(\overline{D}^k_{R,q})^*(\omega') = (\overline{D}^k_{R,q})^*(\omega'),$$

such that B_k is also invariant under $(\dot{D}^k_{V_y,q})^*$. □

Lemma 2.3.8. *Consider a proper and regular Lie groupoid G together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid, and assume there is an admissible embedded submanifold $R \subset G_0$ with $\mathrm{Sat}(R) = G_0$. Assume some ω is invariant (when comparing values at the appropriate $z \in G_q$) with respect to the action by $(\dot{D}^l_{V_x,q})^*$ for any $0 \leq l < k$ of any $V_x \in J^E(s : G_1 \to G_0)$.*

Then the term $\widetilde{C}_k$ given by

$$\widetilde{C}_k = \begin{cases} C_k + \mathrm{d}_{\mathcal{N}}\left((s_k)^*(\overline{D}^{k+1}_{R,q})^*(\omega)\right) & \text{if } k < q \\ C_k = 0 & \text{if } k = q \end{cases}$$

where C_k is defined as in Proposition 2.3.2, is invariant with respect to the action by $(\dot{D}^l_{V_y,q})^*$ for any $0 \leq l \leq k$ of any $V_y \in J^E(s : G_1 \to G_0)$.

Proof. First of all, note that the sum defining C_k is trivially empty if $k = q$ and 0 is trivially invariant under all actions. So the case that is difficult to consider is the case where $k < q$.

If we spell out the latter term, we obtain

$$\mathrm{d}_{\mathcal{N}}\left((s_k)^*(\overline{D}^{k+1}_{R,q})^*(\omega)\right) = \sum_{i=0}^{q}(-1)^{p+i}\left((d_i)^*(s_k)^*(\overline{D}^{k+1}_{R,q})^*(\omega)\right).$$

Considering that C_k was defined as

$$\begin{aligned} C_k &= \sum_{i=k+2}^{q+1}(-1)^{p+i}\left((d_{i-1})^*(s_k)^*(\overline{D}^{k+1}_{R,q})^*(\omega)\right) \\ &= \sum_{i=k+1}^{q}(-1)^{p+i+1}\left((d_i)^*(s_k)^*(\overline{D}^{k+1}_{R,q})^*(\omega)\right), \end{aligned}$$

we obtain

$$\widetilde{C}_k = \sum_{i=0}^{k}(-1)^{p+i}\left((d_i)^*(s_k)^*(\overline{D}^{k+1}_{R,q})^*(\omega)\right).$$

We can check that when $i \le k$ and $l \le k$ we have

$$\begin{aligned} &(\dot{D}^l_{V_y,q})^*(d_i)^*(s_k)^*(\overline{D}^{k+1}_{R,q})^*(\omega) \\ &= \begin{cases} (d_i)^*(\dot{D}^{l-1}_{V_y,q})^*(s_k)^*(\overline{D}^{k+1}_{R,q})^*(\omega) & \text{if } i < l \\ (d_i)^*(s_k)^*(\overline{D}^{k+1}_{R,q})^*(\omega) & \text{if } i = l \\ (d_i)^*(\dot{D}^{l}_{V_y,q})^*(s_k)^*(\overline{D}^{k+1}_{R,q})^*(\omega) & \text{if } i > l \end{cases} \\ &= \begin{cases} (d_i)^*(s_k)^*(\dot{D}^{l-1}_{V_y,q})^*(\overline{D}^{k+1}_{R,q})^*(\omega) & \text{if } i < l \\ (d_i)^*(s_k)^*(\overline{D}^{k+1}_{R,q})^*(\omega) & \text{if } i = l \\ (d_i)^*(s_k)^*(\dot{D}^{l}_{V_y,q})^*(\overline{D}^{k+1}_{R,q})^*(\omega) & \text{if } i > l \end{cases} \\ &= \begin{cases} (d_i)^*(s_k)^*(\overline{D}^{k+1}_{R,q})^*(\dot{D}^{l-1}_{V_y,q})^*(\omega) & \text{if } i < l \\ (d_i)^*(s_k)^*(\overline{D}^{k+1}_{R,q})^*(\omega) & \text{if } i = l \\ (d_i)^*(s_k)^*(\overline{D}^{k+1}_{R,q})^*(\dot{D}^{l}_{V_y,q})^*(\omega) & \text{if } i > l \end{cases} \\ &= (d_i)^*(s_k)^*(\overline{D}^{k+1}_{R,q})^*(\omega), \end{aligned}$$

where we have used the assumption that ω is invariant at any $z \in G_q$ with respect to $(\dot{D}^l_{V_y,q})^*$ for any $0 \le l < k$, the interchanging rules as in Corollary 1.2.35 and Lemma 2.2.9 and the fact that $i \le k$, such that the last case ($i > l$) is only possible if $l < i \le k$ and in particular $l < k$.

Hence $\widetilde{C}_k$ is invariant at any $z \in G_q$ under $(\dot{D}^l_{V_y,q})^*$ for any $0 \le l \le k$. □

Example 2.3.9. In the setting of Example 2.3.3 (i.e. $\omega = f \in C^\infty(G_0)$), we have that $\widetilde{C}_0 = 0$, simply because $k = q = 0$.

In the setting of Example 2.3.4 (i.e. $\omega \in \Omega^p(G_0)$ with $p > \dim(R)$), we have that $\widetilde{C}_0 = 0$, as similar to the example, Lemma 2.2.8 implies that $(\overline{D}^{k+1}_{R,q})^*(\omega) = 0$.

In the setting of Example 2.3.5 (i.e. $\omega \in (\Omega^p(G_q))^{J^E G}$ with $(s_j)^*(\omega) = (s_{j-1})^*(\omega) = 0$), we find that $\widetilde{C}_j = 0$ as $C_j = 0$ and $(\overline{D}^{k+1}_{R,q})^*(\omega) = \omega$ by Lemma 2.2.5.

Proposition 2.3.10. *Consider a proper and regular Lie groupoid G with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Assume we have an element $\omega \in \Omega^p(G_q) = (\mathfrak{C}_p)^q$ such that $\mathrm{d}_{\mathcal{N}}(\omega)$ is $J^E G$-invariant.*

Then there is an element $\rho \in (\mathfrak{C}_p)^{q-1}$ such that $\omega + \mathrm{d}_{\mathcal{N}}(\rho)$ is $J^E G$-invariant.

If ω is homogeneous with respect to the splitting into E-vertical and E-horizontal subspaces (i.e. at each $z \in G_q$ we have

$$\omega_z \in \left(\wedge^{p_0} \left(E_q\right)_z \otimes \wedge^{p_1} I_z G_q\right)$$

such that $p_0 + p_1 = p$) then ρ can also be chosen homogeneously with respect to the splitting induced by E. Its E-horizontal degree in that case is p_0 and its E-vertical degree is p_1 (which implies the same E-homogeneous degrees for $\mathrm{d}_{\mathcal{N}}(\rho)$ and $\omega + \mathrm{d}_{\mathcal{N}}(\rho)$).

Proof. In the first part of the proof, let us assume that there is an admissible embedded submanifold $R \subset G_0$ with $\mathrm{Sat}(R) = G_0$. We will eliminate the assumption in the end.

The proof of the statement will proceed inductively (iterating over $0 \leq k \leq q+1$) prove the following statement: There is an exact term ν_k that we can add to ω such that the result $\omega + \nu_k$ when evaluated at the corresponding points is invariant under

$$(\dot{D}^l_{V_y,q})^* : \wedge^\bullet T^*_{D^l_{y,q}(z)} G_q \to \wedge^\bullet T^*_z G_q$$

for any $V_y \in J^E(s : G_1 \to G_0)$ and any $0 \leq l < k$. For $k = 0$, the condition is trivially met for $\nu_0 = 0$.

Let us now prove the induction step from k to $k+1$. By induction hypothesis there is an exact ν such that $\omega' = \omega + \nu_k$ satisfies the given condition.

As ν_k is exact we have that

$$d_{\mathcal{N}}(\omega) = d_{\mathcal{N}}(\omega')$$

and hence $d_{\mathcal{N}}(\omega')$ is $J^E G$-invariant.

By Proposition 2.3.2 we can express

$$\omega' = (-1)^{p+k} \left(A_k + B_k + C_k - (s_k)^* \left((\overline{D}^{k+1}_{R,q+1})^*(\mathrm{d}_{\mathcal{N}}(\omega'))\right)\right),$$

where

$$A_k = \sum_{i=0}^{k-1} (-1)^{p+i} \left((d_i)^*(s_{k-1})^*(\overline{D}^k_{R,q})^*(\omega')\right)$$

and

$$B_k = (-1)^{p+k} \left((\overline{D}^k_{R,q})^*(\omega')\right)$$

and

$$C_j = \sum_{i=k+2}^{q} (-1)^{p+i} \left((d_{i-1})^*(s_k)^*(\overline{D}^{k+1}_{R,q})^*(\omega')\right).$$

By assumption, ω' is invariant under $(\dot{D}^l_{V_y,q})^*$ for any $V_y \in J^E(s : G_1 \to G_0)$ and any $0 \leq l < k$. According to Lemma 2.3.6 and Lemma 2.3.7, this implies that A_k and B_k are invariant under $(\dot{D}^l_{V_y,q})^*$ for any $0 \leq l < k+1$. By Lemma 2.3.8, we have that

$$C_k + \mathrm{d}_{\mathcal{N}} \left((s_k)^*(\overline{D}^{k+1}_{R,q})^*(\omega)\right)$$

is also invariant under $(\tilde{D}^l_{R,q})^*$ for $0 \leq l < k+1$.

As $\mathrm{d}_{\mathcal{N}}(\omega')$ is $J^E G$-invariant by assumption, we have by Corollary 2.2.5, that

$$(\overline{D}^{k+1}_{R,q+1})^*(\mathrm{d}_{\mathcal{N}}(\omega')) = \mathrm{d}_{\mathcal{N}}(\omega')$$

and by Lemma 2.1.21, we obtain that

$$(s_k)^* \left((\overline{D}^{k+1}_{R,q+1})^*(\mathrm{d}_{\mathcal{N}}(\omega')) \right) = (s_k)^*((\mathrm{d}_{\mathcal{N}}(\omega')))$$

is $J^E G$-invariant as well, meaning that it is in particular invariant under $(\dot{D}^l_{V_y,q})^*$ for any $0 \leq l < k+1$. This means that for

$$\nu' = \mathrm{d}_{\mathcal{N}} \left((-1)^{p+k} (s_k)^* (\overline{D}^{k+1}_{R,q})^*(\omega') \right)$$

we have that

$$\omega' + \nu' = (-1)^{p+k} \left(A_k + B_k + \widetilde{C}_k - (s_k)^* \left((\overline{D}^{k+1}_{R,q+1})^*(\mathrm{d}_{\mathcal{N}}(\omega')) \right) \right),$$

where all terms are invariant under $(\dot{D}^l_{V_y,q})^*$ for any $0 \leq l < k+1$.

As $\omega' + \nu' = \omega + \nu_k + \nu'$ and $\nu_k + \nu'$ is exact by construction, this proves the induction step for $\nu_{k+1} = \nu_k + \nu'$. Note that for the terms to be well-defined, we need to assume $k \leq q$, hence in the end we obtain ν_{q+1} such that $\omega + \nu_{q+1}$ is invariant under $(\dot{D}^l_{V_y,q})^*$ for $0 \leq l < q+1$, i.e. for $0 \leq l \leq q$, recovering the notion of $J^E G$-invariance.

When writing down the formulas for $\nu = \nu_{q+1}$ and ρ such that $\nu = \mathrm{d}_{\mathcal{N}}(\rho)$, we find that

$$\rho = \sum_{i=0}^{q} \rho^{(k)}$$

where

$$\rho^{(k)} = (-1)^{p+k} (s_k)^* (\overline{D}^{k+1}_{R,q})^*(\omega^{(k)})$$

for the iterations $\omega^{(k)} = \omega + \nu_k$ of the induction. This also implies that if $\omega = \omega^{(0)}$ is E-homogeneous of E-horizontal degree p_0 and E-vertical degree p_1 then so is $\rho^{(0)}$ by Lemma 2.2.6 and Lemma 2.1.32. Subsequently, $\omega^{(1)}$ is E-homogeneous of the same degrees, which implies that $\rho^{(1)}$ is, too. Analogously, we can see that during the entire induction, we have that $\omega^{(k)}$ and $\rho^{(k)}$ retain the splitting. So the same will be also true for $\omega^{(q+1)}$ and for ρ.

We can now complete the proof by showing that we can use this to recover the result if R only exists locally as in Proposition 1.5.9. For this, note that the vertical differential $\mathrm{d}_{\mathcal{N}}$ is $C^\infty_{\mathrm{bas}}(G_0)$-linear, as shown in Lemma 1.3.8.

Now if we do not have an admissible embedded manifold $R \subset G_0$ with $\mathrm{Sat}(R) = G_0$, we can instead apply Proposition 1.5.9 for a collection $\{R_n\}_{n \in \mathbb{N}_0}$ (or $\{R_n\}_{0 \leq n \leq f}$) of admissible embedded manifolds. We can consider an admissible partition of unity $\{\chi_n\}$ underlying the cover $\{U_n = \mathrm{Sat}(R_n)\}$. If we have an element $\omega \in \Omega^p(G_q)$ such that $\mathrm{d}_{\mathcal{N}}(\omega)$ is $J^E G$-invariant, we find that $(\pi_q)^* \chi_n \cdot \omega$ is also $J^E G$-invariant under the vertical differential due to the $C^\infty_{\mathrm{bas}}(G_0)$-linearity and $J^E G$-invariance of basic functions (by Proposition 1.3.14, even $J^1 G$-invariance). We can now consider the groupoid $G|_{U_n}$, for which by construction R_n is an admissible embedded manifold with $\mathrm{Sat}(R_n) = (G|_{U_n})_0$.

If we denote the restriction of ω to $G|_{U_n}$ by $\omega|_{U_n}$, we have that $(\pi_q)^* \chi_n \cdot \omega|_{U_n}$ is an element to which we can apply the construction described previously: As we have shown, we can find an exact term $\mathrm{d}_{\mathcal{N}}(\rho_n)$ such that adding it to $(\pi_q)^* \chi_n \cdot \omega|_{U_n}$ yields a $J^E G$-invariant element. We have also explicitly constructed a formula for ρ_n, which shows that the value of $\rho_n(x)$ for any $x \in G_{q-1}$ linearly depends on the collection of values ω_z for $z \in (\pi_q)^{-1}(\pi_q(x))$, i.e. z a point in the leaf of the orbit foliation on G_q which corresponds (under the pull-back along any face map) to the leaf of the orbit foliation on G_{q-1} containing x. Therefore, we have that $\rho_n(x) = 0$ whenever $(\pi_q)^*\chi_n \cdot \omega|_{U_n} \equiv 0$ on $(\pi_q)^{-1}(\pi_q(x))$, which is in particular the case for x outside of the support of $(\pi_{q-1})^*\chi_n$. So $\rho_n \equiv 0$ on an open collar of U_n and hence can be extended by 0 to an element ρ_n' on all of G_{q-1}.

The image $\mathrm{d}_{\mathcal{N}}(\rho_n')$ is (by $C^\infty_{\mathrm{bas}}(G_0)$-linearity of $\mathrm{d}_{\mathcal{N}}$) is precisely the extension by 0 of the constructed $\mathrm{d}_{\mathcal{N}}(\rho_n)$ from $(G|_{U_n})_q$ to G_q. As χ_n is supported on U_n, we find that this implies that $(\pi_q)^*\chi_n \cdot \omega + \mathrm{d}_{\mathcal{N}}(\rho_n')$ is the extension of $(\pi_q)^*\chi_n \cdot \omega|_{U_n} + \mathrm{d}_{\mathcal{N}}(\rho_n)$ by 0, which is in particular still $J^E G$-invariant. So as

$$\rho = \sum_{n \in \mathbb{N}_0} \rho_n'$$

is a locally finite sum, it is well-defined and satisfies

$$\omega + \mathrm{d}_{\mathcal{N}}(\rho) = \sum_{n \in \mathbb{N}_0} (\pi_q)^*\chi_n \cdot \omega + \mathrm{d}_{\mathcal{N}}(\rho_n')$$

where the sum is locally finite and consists only of $J^E G$-invariant terms - hence the result must be $J^E G$-invariant as well.

□

Lemma 2.3.11. *Consider a proper and regular Lie groupoid G together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Fix an admissible cover of G_0 with an underlying admissible partition of unity. Then the assignment*

$$\omega \in \Omega^p(G_q) \text{ s.t. } \mathrm{d}_{\mathcal{N}}(\omega) = 0 \mapsto \rho \in \Omega^p(G_{q-1})$$

constructed in Proposition 2.3.10 is $C^\infty_{\mathrm{bas}}(G_0)$-linear and can be extended to all of $\Omega^p(G_q)$. If ω is $J^E G$-invariant, so is ρ.

If the admissible cover of G_0 used in the construction is trivial, i.e. only consists of the open $U = G_0 = \mathrm{Sat}(R)$ for R an admissible embedded manifold, we have that $\rho|_R$ only depends on the values of $\omega|_R$, i.e. the restriction of ω to $G|_R$.

Proof. We have in the previous proof established formulas for the construction of $\omega^{(k)}$ and $\rho^{(k)}$ following inductive procedures with respect to some admissible embedded manifold R. These formulas can be checked to be $C^\infty_{\mathrm{bas}}(G_0)$-linear in ω and furthermore, they can be evaluated as well for $\omega \in \Omega^p(G_q)$ whenever $\mathrm{d}_{\mathcal{N}}(\omega)$ is not $J^E G$-invariant. Furthermore, we have that for a $J^E G$-invariant form ω all the pull-backs, the vertical differential and the operation $(\overline{D}^i_{R,q})^*$ retains $J^E G$-invariance by Lemma 2.1.32, Corollary 2.1.33 and Lemma 2.2.5. For a cover given by a family of admissible manifolds, it remains to glue these constructions, which is also $C^\infty_{\mathrm{bas}}(G_0)$-linear in the data and as the diagonal action at the i-th pull-back retains the orbits, we can explicitly check that the conditions for $J^E G$-invariance are $C^\infty_{\mathrm{bas}}(G_0)$-linear using Lemma 1.3.4.

Under the assumption of a trivial cover, let us now show that $\rho|_R$ only depends on $\omega|_R$: For $\rho^{(0)} = 0$, this is trivially the case. By evaluating the formulas for $\omega^{(k)}$ and $\rho^{(k)}$, we find that each $\rho^{(k)}|_R$ depends on $(\overline{D}^{j+1}_{R,q})^*(\omega^{(k)})|_R$, which by Lemma 2.2.7 only depends on $\omega^{(k)}|_R$. As

$$\omega^{(k)}|_R = \omega|_R + \sum_{l=0}^{k-1} \mathrm{d}_{\mathcal{N}}(\rho^{(l)}|_R)$$

we have that this also only depends on $\omega|_R$ by induction. So the same must be true for all $\rho^{(k)}|_R$ as well as $\rho|_R = \sum \rho^{(k)}|_R$. $\square$

Corollary 2.3.12. *Consider a proper and regular Lie groupoid G together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Consider $\omega \in \Omega^p(G_q)$ such that $\mathrm{d}_{\mathcal{N}}(\omega)$ is J^EG-invariant. Assume that the admissible cover of G_0 used in the construction of Proposition 2.3.10 extended as in Lemma 2.3.11 is trivial, i.e. only consists of the open $G_0 = \mathrm{Sat}(R)$ for R an admissible embedded manifold. Then the expression*

$$\omega + \mathrm{d}_{\mathcal{N}}(\rho)$$

can be uniquely recovered from $\omega|_R$.

In general, we have that if $\mathrm{d}_{\mathcal{N}}(\omega)$ is J^EG-invariant, the expression $\omega + \mathrm{d}_{\mathcal{N}}(\rho)$ only depends on the collection of values of $\omega|_{R_n}$, i.e. the restriction of ω to $G|_{R_n}$. The dependence is linear and in particular $\omega|_{R_n} = 0$ for all of the R_n implies that

$$\omega + \mathrm{d}_{\mathcal{N}}(\rho) = 0.$$

Proof. We know by Lemma 2.1.38 that the values of a J^EG-invariant element on an orbit are determined by the values at a point in the orbit. So as $\omega + \mathrm{d}_{\mathcal{N}}(\rho)$ is J^EG-invariant by Proposition 2.3.10, it is possible to recover all values from those at the units 1_r for $r \in R$. Furthermore, from the explicit form of Proposition 2.1.41, we can see that the restriction to $(\omega + \mathrm{d}_{\mathcal{N}}(\rho))|_R$ can uniquely recover these values. But $(\omega + \mathrm{d}_{\mathcal{N}}(\rho))|_R$ only depends on $\omega|_R$ and $\rho|_R$, so by Lemma 2.3.11, we can recover it from $\omega|_R$ completely.

As $\omega + \mathrm{d}_{\mathcal{N}}(\rho)$ only depends on $\omega|_R$ in the case of a trivial cover, the claim about $\omega + \mathrm{d}_{\mathcal{N}}(\rho)$ in general only depending on the family $\omega|_{R_n}$ follows from the construction immediately (as it is given by a sum of elements only depending on $\omega|_{R_n}$). $\square$

Theorem 2.3.13. *Let G be a proper and regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. The inclusion*

$$\left((\Omega^p(G_\bullet))^{J^EG}, \mathrm{d}_{\mathcal{N}}\right) \subset (\Omega^p(G_\bullet), \mathrm{d}_{\mathcal{N}}) = \mathfrak{C}_p$$

is a quasi-isomorphism, meaning the cohomology of each column of the Bott–Shulman–Stasheff-complex $H^\bullet(\mathfrak{C}_p)$ can be computed on the J^EG-invariant elements as

$$H^\bullet(\mathfrak{C}_p) \cong H^\bullet\left((\Omega^p(G_\bullet))^{J^EG}, \mathrm{d}_{\mathcal{N}}\right)$$

or under the isomorphism of Proposition 2.1.41 as

$$H^\bullet(\mathfrak{C}_p) \cong H^\bullet\left(\bigoplus_{p_0+p_1=p} \left(\Gamma(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \wedge^{p_1}((A^*_{\mathrm{iso}})^{\oplus\bullet}))\right)^{J^EG}, \mathrm{d}_{\mathcal{N}}|_{1_{G_0}}\right).$$

Proof. By Corollary 2.1.22, we know that J^EG-invariance is retained under the horizontal differential $\mathrm{d}_{\mathcal{N}}$. Hence we have an inclusion of chain complexes of the J^EG-invariant elements of a column into all the elements of that column in the Bott–Shulman–Stasheff complex.

Surjectivity of the inclusion when considering cohomology is the statement of Proposition 2.3.10 under the stricter assumption that $\mathrm{d}_{\mathcal{N}}(\omega) = 0$, as 0 is J^EG-invariant and injectivity follows from the same proposition as any exact J^EG-invariant form $\nu = \mathrm{d}_{\mathcal{N}}(\omega)$ with ω not necessarily J^EG-invariant has a J^EG-invariant preimage $\omega + \mathrm{d}_{\mathcal{N}}(\rho)$.

That the J^EG-invariant forms can equivalently be expressed by their values at the unit section is precisely the statement of Proposition 2.1.41. $\square$

Lemma 2.3.14. *Consider a proper and regular Lie groupoid G together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. We can construct a quasi-inverse to the inclusion of Theorem 2.3.13 by*

$$\omega \in \Omega^p(G_q) \mapsto \omega + r_\omega + \mathrm{d}_{\mathcal{N}}(\rho'_\omega)$$

for $r_\omega \in \Omega^p(G_q)$ the element associated to $\mathrm{d}_{\mathcal{N}}(\omega)$ using the assignment of Lemma 2.3.11 and $\rho'_\omega \in \Omega^p(G_{q-1})$ the element associated to $\omega + r$ using the same assignment.

Proof. First of all, we need to check if this is a well-defined map. We have that $\omega + r_\omega$ by Proposition 2.3.10 satisfies that $\mathrm{d}_{\mathcal{N}}(\omega) + \mathrm{d}_{\mathcal{N}}(r_\omega)$ is $J^E G$-invariant. So by the same result, we have that

$$\omega + r_\omega + \mathrm{d}_{\mathcal{N}}(\rho'_\omega)$$

is $J^E G$-invariant as well. Secondly, we need to check that it is compatible with the differential $\mathrm{d}_{\mathcal{N}}$. As $\mathrm{d}_{\mathcal{N}}(\omega)$ is closed and the assignment of Lemma 2.3.11 is $\mathbb{R}$-linear, we find that

$$r_{(\mathrm{d}_{\mathcal{N}}(\omega))} = 0$$

and $\rho'_{(\mathrm{d}_{\mathcal{N}}(\omega))}$ agrees with the image of $\mathrm{d}_{\mathcal{N}}(\omega)$ under the assignment, which incidentally is the definition of r_ω. This implies that

$$\mathrm{d}_{\mathcal{N}}(\omega + r_\omega + \mathrm{d}_{\mathcal{N}}(\rho'_\omega)) = \mathrm{d}_{\mathcal{N}}(\omega) + \mathrm{d}_{\mathcal{N}}(r_\omega) = \mathrm{d}_{\mathcal{N}}(\omega) + r_{(\mathrm{d}_{\mathcal{N}}(\omega))} + \mathrm{d}_{\mathcal{N}}(\rho'_{(\mathrm{d}_{\mathcal{N}}(\omega))}).$$

Therefore, the map is well-defined.

We also find that more generally on all closed elements, the map agrees with

$$\omega \text{ s.t. } \mathrm{d}_{\mathcal{N}}(\omega) = 0 \mapsto \omega + \mathrm{d}_{\mathcal{N}}(\rho'_\omega).$$

This shows that both pre- and postcomposition with the inclusion of $J^E G$-invariant elements into $\mathfrak{C}$ is on closed $J^E G$-invariant elements given by the map

$$\omega \mapsto \omega + \mathrm{d}_{\mathcal{N}}(\rho'_\omega)$$

As any cohomology class can be represented by some ω of this form, the constructed map is a quasi-inverse to the inclusion if and only if the resulting form is again cohomologically equivalent to ω. We have that $\mathrm{d}_{\mathcal{N}}(\rho'_\omega)$ is trivially exact as an element of $\mathfrak{C}$, however, as ρ'_ω is not necessarily $J^E G$-invariant itself, the same is not immediately true in $(\Omega^p(G_\bullet))^{J^E G}$. However, the existence of a $J^E G$-invariant preimage of $\mathrm{d}_{\mathcal{N}}(\rho'_\omega)$ is one of the consequences of Proposition 2.3.10. □

Example 2.3.15 (The Pair Groupoid)**.** Consider a pair groupoid, i.e. a groupoid G with object space $G_0 = M$ and morphism space $G_1 = M \times M$. As the isotropy is trivial at every point, we have that there is a unique multiplicative Ehresmann connection given by

$$T_x G_1 = T_x G_1 \oplus 0.$$

As the Ehresmann connection encompasses the entire tangent space at each point, the $J^E G$-invariant forms of bidegree (p, q) are given by

$$(\Omega^p(G_q))^{J^E G} = (\Omega^p(G_q))^{J^1 G},$$

which by Proposition 1.3.14 implies

$$(\Omega^p(G_q))^{J^E G} = \Omega^p_{\mathrm{bas}}(G_0).$$

We know by Lemma 1.3.2, that basic forms vanish under insertion of any vector field contained in the orbit foliation. But as the orbit foliation at each point is given by the entire tangent space, we obtain that

$$(\Omega^p(G_q))^{J^E G} = \Omega^p_{\mathrm{bas}}(G_0) = 0$$

unless $p = 0$, in which case by Lemma 1.3.4

$$\left(\Omega^0(G_q)\right)^{J^E G} = C^\infty_{\mathrm{bas}}(G_0) = \mathbb{R}$$

is given by constant functions. In particular, we obtain $H(\mathfrak{C}_p) = 0$ for $p \neq 0$.

We can further compute that in $\mathfrak{C}_0$, the vertical differential is given alternatingly by either 0 or id (as all pull-backs retain constant functions and we sum alternatingly over their results), in a way that the cohomology $H(\mathfrak{C}_0)$ is given by $\mathbb{R}$ in degree 0. In particular, almost every form closed under $\mathrm{d}_\mathcal{N}$ is also in the image of $\mathrm{d}_\mathcal{N}$.

This actually implies that any element in $\Omega_{\mathrm{BSS}}(G)$ of total degree $r > 0$ is closed under the total differential $\mathbf{d}$ if and only if it is $\mathbf{d}$-exact. We will omit the argument here to avoid redundancy and come back to it later in Example 4.2.8. (By that point these considerations will follow without much further proof being necessary. However, the statement regarding this example can be worked out easily with less technical tools.) For total degree $r = 0$, we obtain that all $J^E G$-invariant elements of bidegree $(0, 0)$ define cohomology classes. This is because constant functions are closed under the de Rham differential, hence defining $\mathbf{d}$-closed elements, which for degree reasons define unique cohomology classes. (Analogously, this already describes all the cohomology classes of $H^0(\Omega_{\mathrm{BSS}}(G))$, as those are given by the intersection of $\ker(\mathrm{d}_\mathcal{N})$ and $\ker(\mathrm{d}_{\mathrm{dR}})$, which is in particular a subset of $H^0(\mathfrak{C}^0) = \ker(\mathrm{d}_\mathcal{N})$).

So in conclusion, we have that $H(\Omega_{\mathrm{BSS}}(G)) = \mathbb{R}$, which is concentrated in degree 0. This is an unsurprising result due to the fact that any pair groupoid is Morita equivalent to a point (cf. Lemma 0.2.19), so its differentiable stack cohomology should also be trivial.

Example 2.3.16 (The Free Action)**.** At the beginning of this chapter we have considered the free and proper action of a group $\mathcal{G}$ on a manifold M and the induced action groupoid G, which is equivalent to the trivial groupoid of the quotient Q.

In this case, similar to Example 2.3.15 we also have that IG_1 is the trivial bundle and hence there is a unique multiplicative Ehresmann connection $E = TG_1$ on G with respect to the isotropy. We also have that the $J^E G$-invariant forms in bidegree (p, q) are given by

$$(\Omega^p(G_q))^{J^E G} = (\Omega^p(G_q))^{J^1 G},$$

which we know by Proposition 1.3.14 and Proposition 1.3.5 to be equal to

$$(\Omega^p(G_q))^{J^1 G} = \Omega_{\mathrm{bas}}(G_0) = \Omega(Q).$$

With similar considerations as for $\mathfrak{C}_0$ in the previous example, we can work out that the cohomology of the columns $\mathfrak{C}_p$ is supported only in degree 0 and the cohomology of the total complex is supported in bidegree $(p, 0)$ and that in this example the cohomology is given by

$$H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))) = H^\bullet(\Omega_{\mathrm{bas}}(G_0), \mathrm{d}_{\mathrm{dR}}) = H^\bullet(Q, \mathrm{d}_{\mathrm{dR}}).$$

We will prove this properly in Example 4.2.10. (Similar to the comment on the previous example, we can again alternatively work out the details with less technical tools then we use later.)

Definition 2.3.17. A Lie groupoid G is called **étale** if the source and target maps of G are local diffeomorphisms.

Example 2.3.18. Let G be a proper and étale groupoid. Étale implies that the source and target fibres are discrete, such that the orbit foliation bundle vanishes, hence G is regular, and that $A_{\text{iso}} = 0$. In particular, the multiplicative Ehresmann connection on G is uniquely determined to be the entire tangent space. In analogy to Example 2.3.16 and by Proposition 1.3.14 the $J^E G$-invariant forms in this example are given by

$$(\Omega^p(G_q))^{J^1 G} = \Omega^p_{\text{bas}}(G_0).$$

The computation in the two previous examples actually can be unified in a single setting:

Example 2.3.19. Let G be any proper and regular groupoid satisfying $A_{\text{iso}} = 0$. Then the multiplicative Ehresmann connection on G is uniquely determined as the entire tangent space. Similar as to the argument in Example 2.3.16, we have that this identifies $J^E G = J^1 G$ and by Proposition 1.3.14 we have that

$$(\Omega^p(G_q))^{J^1 G} = \Omega^p_{\text{bas}}(G_0).$$

Example 2.3.20 (The Action on a Point)**.** For the trivial action of a compact group $\mathcal{G}$ on a single point $\{*\}$ we obtain that again, there is a unique multiplicative Ehresmann connection with respect to the isotropy. It is induced by the zero section $\mathcal{G} \subset T\mathcal{G}$ as the unique dimension 0 subbundle, corresponding to the fact that in this case the entire tangent space must be E-vertical as $IG_1 = TG_1$. We have that $A^*_{\text{iso}} = \mathfrak{g}^*$ and hence that the $J^E G$-invariant forms in bidegree (p, q) are according to Proposition 2.1.41 are given by

$$\wedge^p \left((\mathfrak{g}^*)^{\oplus q} \right).$$

A very nice property of this example is that as we are working with a honest group action, we have that d_{dR} maps $\mathcal{G}$-invariant forms to $\mathcal{G}$-invariant forms (which are automatically the $J^E G$-invariant forms, as all jets only contain a single zero vector). This is still a relatively large bicomplex, and computing its cohomology is not trivial. However, we already know that the differentiable stack cohomology of the associated Lie groupoid is given by $S^k(\mathfrak{g}^*)$ in degree $2k$, so this example illustrates very well why we still have to improve the current result.

Example 2.3.21. Consider a compact Lie group $\mathcal{G}$ and a Lie groupoid of the form

$$G_1 = \mathcal{G} \times G_0 \times G_0,$$

where the target map is given by projection to the second component, the source map is given by projection to the third component and the multiplication is in the first component given by $\mu_{\mathcal{G}}$, i.e. the multiplication of $\mathcal{G}$. Then the splitting

$$TG_1 = (T\mathcal{G}) \times T(G_0 \times G_0)$$

induces a multiplicative Ehresmann connection by

$$E_{(g,m,n)} := \{0_g\} \times T_m G_0 \times T_n G_0.$$

We now know that by the Lie groupoid axioms, the diagonal $G_0 \subset G_0 \times G_0$ induces a left and right $\mathcal{G}$-action on G_1 each covering the identity on G_0. This way, every element of $\mathcal{G}$

corresponds to a E-horizontal bisection of G covering the identity. Hence $J^E G$-invariance implies invariance under these bisections and

$$(\Omega^p(G_q))^{J^E G} = \left((\Omega^p(G_q))^{\mathcal{G}}\right)^{J^E G}.$$

Furthermore, if we consider the isomorphism of Proposition 2.1.41, we find that for each term of the form

$$\left(\Gamma\left(G_0, \wedge^p\left((A^*_{\text{iso}})^{\oplus q}\right)\right)\right)^G,$$

we can similarly consider the action of these global sections. By definition of the multiplication on G, we find that the action of any element (g, m, n) on $(A^*_{\text{iso}})^{\oplus q} = (\mathfrak{g}^*)^{\oplus q}$ agrees with the extension of the classical coadjoint action. As all sections cover the identity, we find that we can rewrite the previous term as

$$\left(\Gamma\left(G_0, \left(\wedge^p\left((A^*_{\text{iso}})^{\oplus q}\right)\right)^{\mathcal{G}}\right)\right)^G.$$

If we now rewrite any $(g, m, n) \in G_1$ as

$$(g, m, n) = \mu((1, m, n), (g, n, n)),$$

it becomes apparent that G-invariance of a $\mathcal{G}$-invariant section of the previous expression precisely means that under the trivialisation of $(A^*_{\text{iso}})^{\oplus q} = (\mathfrak{g}^*)^{\oplus q}$ the values at m and n agree. As the groupoid is transitive, this implies

$$\left(\Gamma\left(G_0, \left(\wedge^p\left((A^*_{\text{iso}})^{\oplus q}\right)\right)^{\mathcal{G}}\right)\right)^G \cong \left(\wedge^p\left((\mathfrak{g}^*)^{\oplus q}\right)\right)^{\mathcal{G}},$$

such that this expression characterises the $J^E G$-invariant forms of G. This generalises the findings of Example 2.3.20.

Example 2.3.22. Consider the Lie groupoid $G = T$ of Example 0.4.7. We now aim to equip it with a multiplicative Ehresmann connection.

We have seen in 0.4.6, that $L \subset R$ defines a discrete subgroupoid. We now claim that there is a unique linear multiplicative Ehresmann connection on R such that the embedding of L into R is at each point horizontal and such that it induces a multiplicative Ehresmann connection on T. For this, note that we have already shown that there is a local trivialisation of R as a bundle which is compatible with the fibre-wise addition and scaling and also exhibits L as a trivial Lie group subbundle. This trivialisation induces a vector bundle connection locally and as $s = t$ and the Lie group bundle structure is given by the fibre-wise addition it indeed satisfies the requirements of a multiplicative Ehresmann connection (i.e. that addition of horizontal vectors remains horizontal). Furthermore, as T is exhibited as the quotient by a discrete subgroup (or rather a trivial group subbundle), we have that the connection induces a well-defined connection on T.

Let us now argue, that this connection on R is unique: As the lattice can be generated by two linearly independent vectors, X_1 and X_2, we can write every other vector in R as a linear combination. We obtain that the connection on R at X_1 is precisely given by $T_{X_1}L$ and analogously for X_2. So indeed, the property that the connection is linear and the lattice embeddings are horizontal uniquely define the connection. This is now also the reason, why it is sufficient to prove the existence locally with respect to the bundle structure: On any intersection, two connections defined this way will always agree. As the quotient to T is also given by a fibre-wise construction, it follows that the same holds for the induced connection on T. In particular, it is well-defined.

Therefore, we may now compute the $J^E G$-invariant forms in this example. By Proposition 2.1.41, we have that they are given in bidegree (p,q) by

$$\bigoplus_{p_0+p_1=p} \left(\Gamma(\mathbb{S}^1, \wedge^{p_0}(T^*\mathbb{S}^1) \otimes \wedge^{p_1}((A^*)^{\oplus q}))\right)^G,$$

as the orbits are trivial (meaning $\mathcal{F}_0 = 0$) and $\rho \equiv 0$ on the entire algebroid (meaning $A_{\text{iso}} = A$).

Furthermore, we have that as G is a Lie group bundle, the action of any element on $\mathbb{S}^1$ and its cotangent space is trivial and the adjoint action on A^* is as well, because the fibre of the bundle is $\mathbb{T}^2$, which is abelian. This means the $J^E G$-invariant forms of bidegree (p,q) actually correspond to

$$\bigoplus_{p_0+p_1=p} \left(\Gamma(\mathbb{S}^1, \wedge^{p_0}(T^*\mathbb{S}^1) \otimes \wedge^{p_1}((A^*)^{\oplus q}))\right).$$

Chapter 3

$J^E G$-isotropy Cartan forms

In the previous section, we have shown that the quasi-isomorphism of including $J^E G$-invariant forms into the columns $\mathfrak{C}_p$ of the Bott–Shulman–Stasheff bicomplex reduces the complexity of computing the cohomology of those columns significantly. However, we have also noted that there is still room for improvement. We will now establish an even smaller subcomplex that already computes the cohomology of $\mathfrak{C}_p$.

The differential on that subcomplex will vanish, so it can even be understood as the smallest subcomplex to compute the cohomology of the columns. Furthermore, we will find that computing the cohomology of the columns in this way can, in many examples, already give some relevant insight into the properties of the Bott–Shulman–Stasheff bicomplexes of which we have taken the columns.

3.1 An Eilenberg–Zilber type argument

Aim of this section: Theorem 3.1.20 (abbreviated):

Let G be a regular and proper Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. The inclusions

$$\mathcal{R}_E^{(p,\bullet)} \subset (\Omega^p(G_\bullet))^{J^E G}$$

and

$$\mathcal{R}_E^{(p,\bullet)} \subset (\mathfrak{C}_p)^\bullet$$

are quasi-isomorphisms with respect to the respective restrictions of $\mathrm{d}_{\mathcal{N}}$*. Furthermore, any cohomology class of a column $[\omega] \in H^q(\mathfrak{C}_p)$ can be uniquely represented by a $J^E G$-isotropy Cartan form $\omega \in \mathcal{R}_E^{(p,q)}$.*

This section is dedicated to exploring the bicomplex $\mathcal{R}_E$ mentioned in the theorem, as well as proving the given inclusion to be a quasi-isomorphism. We will first take a look at the space of $J^E G$-invariant forms and decompose it into smaller components, characterised by a multidegree. This will help us understand the inclusion.

Afterwards we will establish all the necessary technical details and properties of this decomposition, to prove the theorem. This will take up the majority of this section. The proof is mostly based on a generalised version of the Eilenberg–Zilber theorem and is constructive, but technical.

Lemma 3.1.1. *Consider a regular Lie groupoid G together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Then the decomposition of IG_q of Lemma 2.1.16 yields the decompositions*

$$\begin{aligned}&\wedge^{p_0} E_q^* \otimes \wedge^{p_1} (IG_q)^* \\ =&\bigoplus_{p'_1+\dots+p'_q=p_1} \wedge^{p_0} E_q^* \otimes \wedge^{p'_1} ((\mathrm{pr}_1)^* IG_1)^* \otimes \dots \otimes \wedge^{p'_q} \left((\mathrm{pr}_q)^* IG_1\right)^*\end{aligned}$$

and

$$\begin{aligned}&\wedge^{p_0} (E_q/\mathcal{F}_q)^* \otimes \wedge^{p_1} (IG_q)^* \\ =&\bigoplus_{p'_1+\dots+p'_q=p_1} \wedge^{p_0} (E_q/\mathcal{F}_q)^* \otimes \wedge^{p'_1} ((\mathrm{pr}_1)^* IG_1)^* \otimes \dots \otimes \wedge^{p'_q} \left((\mathrm{pr}_q)^* IG_1\right)^*\end{aligned}$$

as vector bundles over G_q and at the unit section identifies

$$\begin{aligned}&\wedge^{p_0} (TG_0/\mathcal{F}_0)^* \otimes \wedge^{p_1} \left((A_{\mathrm{iso}}^*)^{\oplus q}\right) \\ =&\bigoplus_{p'_1+\dots+p'_q=p_1} \wedge^{p_0} (TG_0/\mathcal{F}_0)^* \otimes \wedge^{p'_1} (A_{\mathrm{iso}}^*) \otimes \dots \otimes \wedge^{p'_q} (A_{\mathrm{iso}}^*)\end{aligned}$$

as vector bundles over G_0.

Proof. This follows immediately from the fact that the exterior product over a direct sum of vector bundles agrees with the tensor product of the respective exterior powers. $\square$

Definition 3.1.2. Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E. Considering the decomposition of Lemma 3.1.1, we will call any form

$$\omega \in \Gamma\left(G_q, \wedge^{p_0} E_q^* \otimes \wedge^{p'_1} ((\mathrm{pr}_1)^* IG_1)^* \otimes \dots \otimes \wedge^{p'_q} \left((\mathrm{pr}_q)^* IG_1\right)^*\right) \subset \Omega^p(G_q)$$

homogeneous of multidegree $(p_0, p'_1, \dots, p'_q, q)$. Similarly, we will call elements or sections of

$$\wedge^{p_0} E_q^* \otimes \wedge^{p'_1} ((\mathrm{pr}_1)^* IG_1)^* \otimes \dots \otimes \wedge^{p'_q} \left((\mathrm{pr}_q)^* IG_1\right)^*$$

or

$$\wedge^{p_0} (E_q/\mathcal{F}_q)^* \otimes \wedge^{p'_1} ((\mathrm{pr}_1)^* IG_1)^* \otimes \dots \otimes \wedge^{p'_q} \left((\mathrm{pr}_q)^* IG_1\right)^*$$

or

$$\wedge^{p_0} (TG_0/\mathcal{F}_0)^* \otimes \wedge^{p'_1} (A_{\mathrm{iso}}^*) \otimes \dots \otimes \wedge^{p'_q} (A_{\mathrm{iso}}^*)$$

homogeneous of multidegree $(p_0, p'_1, \dots, p'_q, q)$.

Remark. Note that the multidegrees are not indexed over tuples of a fixed length but instead always have $q+2$ components. Also note that by definition, we can recover the E-vertical degree p_1 as well as the entire horizontal degree p from the expression $(p_0, p'_1, \dots, p'_q, q)$.

Lemma 3.1.3. *Let G be a regular Lie groupoid. For the homogeneous elements of multidegree $(p_0, 1, 1, \dots, 1, q)$ of*

$$\wedge^\bullet (TG_0/\mathcal{F}_0)^* \otimes \wedge^\bullet ((A_{\mathrm{iso}}^*)^{\oplus q})$$

i.e. elements of the bundle

$$\wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes (A^*_{\mathrm{iso}}) \otimes \cdots \otimes (A^*_{\mathrm{iso}}) \to G_0$$

there is a canonical action of the permutation group Σ_q on the tensor factors A^*_{iso}. The subset of

$$\Gamma\left(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes (A^*_{\mathrm{iso}}) \otimes \cdots \otimes (A^*_{\mathrm{iso}})\right)$$

given by elements invariant under the induced action can be canonically identified with

$$\Gamma\left(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes S^q(A^*_{\mathrm{iso}})\right).$$

Proof. Consider the action of the permutation group Σ_q on $\wedge^1(A^*_{\mathrm{iso}}) = A^*_{\mathrm{iso}}$ by permuting the tensor factors. As it does not change the projection to G_0, the invariant sections are the sections to the invariant subbundle. The map from the symmetric powers of a vector space or bundle to the tensor powers as the permutation invariant elements is a common construction. □

This subcomplex now already looks quite similar to the subcomplex of the $J^E G$-invariant forms mentioned in the theorem. However, the actual invariance is still missing and we have to check if this notion is well-defined and behaves as intended.

Lemma 3.1.4. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy Lie algebroid. Then for any $0 \leq i \leq q$ the diagonal action at the i-th pull-back of any $V \in J^E(s : G_1 \to G_0)$ on*

$$\Gamma\left(G_q, \wedge^{p_0} E^*_q \otimes \wedge^{p_1}(IG_q)^*\right)$$

by $\dot{D}^i_{V,q}$ retains the multidegree of a form, i.e. the homogeneous components of $\Omega^p(G_q)$ are invariant under the family of diagonal actions of $J^E(s : G_1 \to G_0)$ and in particular under the family of diagonal actions of $J^E G$.

Proof. The fact that any $D^i_{V_x,q}$ retains the multidegree of any form comes in part from the fact that the analogous statement holds for the E-homogeneous degrees p_0 and p_1 and that the additional splitting of p_1 is induced by the projections of G_q to its components.

We have that $D^i_{V_x}$ only depends on the base value $x \in G_1$ when evaluated on IG_q. Under the projections of G_q to G_1 which induce the splitting of IG_q, the action of $D^i_{x,q}$ corresponds to left multiplication, right multiplication or the trivial action. As these are well-defined maps independent of the other components of z, we find that they each induce the action on the respective copy of A^*_{iso} separately. □

Corollary 3.1.5. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy Lie algebroid. We have that the space $(\Omega^p(G_q))^{J^E G}$ decomposes into components*

$$\bigoplus_{p_0+p_1=p} \bigoplus_{p'_1+\cdots+p'_q=p_1} \left(\Gamma\left(G_q, \wedge^{p_0} E_q \otimes \wedge^{p'_1}((\mathrm{pr}_1)^* IG_1) \otimes \cdots \otimes \wedge^{p'_q}\left((\mathrm{pr}_q)^* IG_1\right)\right)\right)^{J^E G}.$$

Proof. This follows directly from Lemma 3.1.4, as it in particular states that invariance of some $\omega \in (\Omega^p(G_q))^{J^E G}$ under $\dot{D}^i_{V,q}$ automatically implies invariance of each component of homogeneous multidegree of ω. □

Corollary 3.1.6. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy Lie algebroid. We have that under the isomorphism of Proposition 2.1.41 given by*

$$\left(\Gamma\left(G_q, \wedge^{p_0} E_q \otimes \wedge^{p_1}\left(IG_q\right)\right)\right)^{J^E G} \cong \Gamma\left(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \wedge^{p_1}\left((A^*_{\text{iso}})^{\oplus q}\right)\right)^G$$

the decomposition into the direct sum of homogeneous components

$$\bigoplus_{p'_1+\cdots+p'_q=p_1} \left(\Gamma\left(G_q, \wedge^{p_0} E_q \otimes \wedge^{p'_1}\left((\mathrm{pr}_1)^* IG_1\right) \otimes \cdots \otimes \wedge^{p'_q}\left((\mathrm{pr}_q)^* IG_1\right)\right)\right)^{J^E G}$$

of Corollary 3.1.5 precisely corresponds to the decomposition into the direct sum of homogeneous components

$$\bigoplus_{p'_1+\cdots+p'_q=p_1} \left(\Gamma\left(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \wedge^{p'_1}\left(A^*_{\text{iso}}\right) \otimes \cdots \otimes \wedge^{p'_q}\left(A^*_{\text{iso}}\right)\right)\right)^G,$$

each representing a form on G_q homogeneous of the same multidegree.

Proof. By definition, the isomorphism of Proposition 2.1.41 is given by restriction to the unit section, where by Lemma 3.1.1 the decomposition precisely takes the described form. □

Lemma 3.1.7. *Let G be a regular Lie algebroid. The action of G on*

$$\Gamma\left(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \wedge^{p'_1}\left(A^*_{\text{iso}}\right) \otimes \cdots \otimes \wedge^{p'_q}\left(A^*_{\text{iso}}\right)\right)$$

given by the extension of the action of Lemma 1.2.16 and the extension of the coadjoint action of Example 1.1.35 commutes with the permutation action of Σ_q.

Proof. As the permutation action is defined as an extension of the trivial action on $(TG_0/\mathcal{F}_0)$, we only need to check that it commutes with the extension of the coadjoint action on $\wedge^{p'_1}\left(A^*_{\text{iso}}\right) \otimes \cdots \otimes \wedge^{p'_q}\left(A^*_{\text{iso}}\right) = \wedge^{p_1}(A^*_{\text{iso}})^{\oplus q}$. Commutativity of the extension of the coadjoint action with Σ_q can be explicitly seen directly, as the adjoint action of any $x \in G_1$ on $IG_q|_{1_{G_0}}$ at $1_{s(x)}$ restricts component-wise to the adjoint action of x on every copy of $IG_1|_{1_{G_0}}$ simultaneously by Lemma 1.1.37. So as the permutation action permutes the components (i.e. it can be characterised by $\mathrm{pr}_{\sigma(i)} \circ L_\sigma = \mathrm{pr}_i$) we find that the two actions commute on each component, and therefore, on all of $(A^*_{\text{iso}})^{\oplus q}$. □

Lemma 3.1.8. *Let G be a regular Lie groupoid. The action of G on*

$$\wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \wedge^1\left(A^*_{\text{iso}}\right) \otimes \cdots \otimes \wedge^1\left(A^*_{\text{iso}}\right) \subset \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \wedge^q\left(A^*_{\text{iso}}\right)^{\oplus q}$$

given by the extension of the action of Lemma 1.2.16 and the coadjoint action on $(A^*_{\text{iso}})^{\oplus q}$ of Example 1.1.35 agrees under the isomorphism

$$\wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \wedge^1\left(A^*_{\text{iso}}\right) \otimes \cdots \otimes \wedge^1\left(A^*_{\text{iso}}\right) \cong \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \mathcal{T}^q(A^*_{\text{iso}}),$$

with the extension of the action of Lemma 1.2.16 and the extension of the coadjoint action on A^*_{iso} of Example 1.1.35.

Proof. This follows immediately from the definition of the respective extensions of Lemma 1.1.38 as well as Lemma 1.1.37. On $\mathcal{T}^q(A^*_{\mathrm{iso}})$, we have that the extension of the G-action is defined by applying Ad component-wise to any decomposition of an element into a product with respect to the tensor algebra structure. At the same time, the inclusion is induced by the inclusion ι_i of each copy of A_{iso} as some respective copy of $(A_{\mathrm{iso}})^{\oplus q}$ and as

$$\iota_1(\mathrm{Ad}_x(\theta^1)) \otimes \cdots \otimes \iota_q(\mathrm{Ad}_x(\theta^q)) = \mathrm{Ad}_x\left(\iota_1(\theta^1)\right) \otimes \cdots \otimes \mathrm{Ad}_x\left(\iota_q(\theta^q)\right)$$

we obtain the result. □

Corollary 3.1.9. *Let G be a regular Lie groupoid. The action of G on*

$$\wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes S^q(A^*_{\mathrm{iso}})$$

as both the subset of Σ_q-invariant elements and the quotient by the Σ_q-action of

$$\wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \wedge^1(A^*_{\mathrm{iso}}) \otimes \cdots \otimes \wedge^1(A^*_{\mathrm{iso}}) \cong \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \mathcal{T}^q(A^*_{\mathrm{iso}})$$

with the action of G given by the extension of the action of Lemma 1.2.16 and the extension of the coadjoint action of Example 1.1.35 on the left-hand side is well-defined and can analogously be described as the extension of the action of Lemma 1.2.16 and the extension of the coadjoint action of Example 1.1.35 to the symmetric algebra.

Proof. The fact that the action of G on

$$\wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \mathcal{T}^q(A^*_{\mathrm{iso}})$$

descends to the quotient by the action of Σ_q follows directly from the commutativity of Lemma 3.1.7. This also shows that invariance under permutation is retained by the G-action. As there is a canonical isomorphism between the invariant and coinvariant elements, we have that the two actions must agree.

The precise form of the action is shown in Lemma 3.1.8, where the Σ_q-action precisely corresponds to commuting the tensor algebra factors, such that both the quotient and the invariant elements precisely describe the construction of $S^q(A^*_{\mathrm{iso}})$ from $\mathcal{T}^q(A^*_{\mathrm{iso}})$. □

Definition 3.1.10. Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy Lie algebroid. A J^EG-invariant element $\omega \in \Omega^p(G_q)$ for which its values at the unit section given by $\omega_{1_{G_0}}$ satisfy that

$$\omega_{1_{G_0}} \in \left(\Gamma\left(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes S^q(A^*_{\mathrm{iso}})\right)\right)^G$$

will be called **homogeneous J^EG-isotropy Cartan form** of bidegree (p,q) or multidegree $(p_0, 1, \ldots, 1, q)$. The space of J^EG-isotropy Cartan forms of bidegree (p,q) will be denoted by $\mathcal{R}_E^{(p,q)}$. The bigraded vector space of all J^EG-isotropy Cartan forms will be denoted by

$$\mathcal{R}_E := \bigoplus_{p,q\in\mathbb{N}_0} \mathcal{R}_E^{(p,q)},$$

where the bigrading is naturally induced.

Let us return to the aim of this section we previously described. We can now understand the claim of the theorem.

Aim of this section: Theorem 3.1.20:

Let G be a regular and proper Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. The inclusions

$$\mathcal{R}_E^{(p,\bullet)} \subset (\Omega^p(G_\bullet))^{J^EG}$$

and

$$\mathcal{R}_E^{(p,\bullet)} \subset (\mathfrak{C}_p)^\bullet$$

are quasi-isomorphisms with respect to the respective restrictions of $\mathrm{d}_\mathcal{N}$. Furthermore, any cohomology class of a column $[\omega] \in H^q(\mathfrak{C}_p)$ can be uniquely represented by a J^EG-isotropy Cartan form $\omega \in \mathcal{R}_E^{(p,q)}$, i.e. an J^EG-invariant form $\omega \in (\Omega^p(G_q))^{J^EG}$ satisfying

$$\omega_{1_{G_0}} \in \left(\Gamma\left(G_0, \wedge^{p-q}(TG_0/\mathcal{F}_0)^* \otimes S^q(A_{\mathrm{iso}}^*)\right)\right)^G.$$

The proof that the J^EG-isotropy Cartan forms compute the same cohomology as the J^EG-invariant elements (on columns) is purely combinatorial and does not require any geometric tools. It will take up the rest of this section.

The first step for this is to understand how the multidegrees behave under the vertical differential $\mathrm{d}_\mathcal{N}$. On the homogeneous components this can be seen quite explicitly, although it involves a lot of combinatorial formulas for some cases. In others, however, the maps are quite easy to describe, especially because many components of the differential vanish.

Lemma 3.1.11. *Let G be a regular groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Consider $\nu \in (\Omega^p(G_q))^{J^EG}$ homogeneous of multidegree $(p_0, p_1', \dots, p_q', q)$ and denote by $\omega_{(r_0,r_1,\dots,r_{q+1},q+1)}$ the homogeneous components of multidegree $(r_0, r_1, \dots, r_{q+1}, q+1)$ of $\omega = \mathrm{d}_\mathcal{N}(\nu)$, where $\mathrm{d}_\mathcal{N}$ denotes the vertical differential. Then we have the following:*

a) $\omega_{(r_0,r_1,\dots,r_{q+1},q+1)} \neq 0$ only when $r_0 = p_0$.

b) $\omega_{(r_0,r_1,\dots,r_{q+1},q+1)} \neq 0$ only when there is some $1 \leq i \leq q$ such that

$$(r_0, \dots, r_{i-1}, (r_i + r_{i+1}), r_{i+2}, \dots, r_{q+1}, q) = (p_0, p_1', \dots, p_q', q)$$

or $r_1 = 0$ and

$$(r_0, r_2, \dots, r_{q+1}, q) = (p_0, p_1', \dots, p_q', q)$$

or $r_{q+1} = 0$ and

$$(r_0, r_1, \dots, r_q, q) = (p_0, p_1', \dots, p_q', q).$$

In particular $\sum_{i=1}^{q+1} r_i = \sum_{i=1}^{q} p_i'$.

c) If $p_j' \neq 0$ and $p_{j+1}' \neq 0$ for $1 \leq j \leq (q-1)$, then $\omega_{(r_0,r_1,\dots,r_{q+1},q+1)} = 0$ for

$$(r_0, r_1, \dots, r_{q+1}, q+1) = (p_0, p_1', \dots, p_j', 0, p_{j+1}', \dots, p_q', q+1).$$

If $p_1' \neq 0$, then $\omega_{(r_0,r_1,\dots,r_{q+1},q+1)} = 0$ for

$$(r_0, r_1, \dots, r_{q+1}, q+1) = (p_0, 0, p_1', \dots, p_q', q+1).$$

If $p'_q \neq 0$, then $\omega_{(r_0,r_1,\dots,r_{q+1},q+1)} = 0$ for

$$(r_0, r_1, \dots, r_{q+1}, q+1) = (p_0, p'_1, \dots, p'_q, 0, q+1).$$

In particular if $p'_j \neq 0$ for all $1 \leq j \leq q$, then $\omega_{(r_0,r_1,\dots,r_{q+1})} \neq 0$ only when $r_i \neq 0$ for all $1 \leq i \leq (q+1)$.

d) *if $i \leq j$ is an interval in $1, \dots, q$ such that the associated $p'_i, \dots, p'_j$ describe a maximal sequence of 0s (i.e. $p'_k = 0$ for $i \leq k \leq j$ as well as either $i = 1$ or $p'_k \neq 0$ for $k = i-1$ and either $j = q$ or $p'_k \neq 0$ for $k = j+1$), then if $j - i$ is odd we have that*

$$\omega_{(p_0,p'_1,\dots,p'_{i-1},0,\dots,0,p'_{j+1},\dots,p'_q,q+1)} = 0,$$

where $r_k = 0$ for $i \leq k \leq (j+1)$ (i.e. we have increased the length of the sequence of 0s by one).

e) *In the setting of d), if $j - i$ is even, we have that $\omega_{(r_0,r_1,\dots,r_{q+1})}$ for*

$$(r_0, r_1, \dots, r_{q+1}, q+1) = (p_0, p'_1, \dots, p'_{i-1}, 0, \dots, 0, p'_{j+1}, \dots, p'_q, q+1)$$

uniquely determines ν. It can be recovered through the isomorphism between homogeneous elements of degree $(r_0, r_1, \dots, r_{q+1}, q+1)$ and degree $(p_0, p'_1, \dots, p'_q, q)$ by contracting the tensor product by a factor of $\Gamma(G_0, \wedge^0(A^*_{\mathrm{iso}})) = C^\infty(G_0)$ and multiplying with $-(-1)^{p+i}$.

Proof. Let us start by noting the following: As each d_i for $0 \leq i \leq q+1$ maps the identity section of G_{q+1} to the identity section of G_q, we can obtain the differential $d_{\mathcal{N}}(\nu)$ by only considering $T_x d_i$ for some point $x = (1_m, \dots, 1_m)$ in the image of the identity section.

Proof of property a):

We have that the Td_i retain the splitting between the isotropy and the horizontal subspace by Lemma 2.1.32, where the latter is spanned (with possibly non-empty intersection) by the vectors tangent to the unit section and the horizontal vectors contained in $\mathcal{F}_q$. As the p_0 and $p_1 = p'_1 + p'_2 + \dots$ indicate the E-horizontal and E-vertical degree of ν, they are retained by $(d_i)^*$.

Hence each $d_{\mathcal{N}}(\nu)$ will only have components in multidegree $(r_0, r_1, \dots, r_{q+1})$, where $r_0 = p_0$.

Proof of property b):

First of all, let us note that by Lemma 2.1.21, ν being invariant implies that all $(d_i)^*(\nu)$ are $J^E G$-invariant as well, such that it is sufficient to identify $(d_i)^*(\nu)$ at the unit section. As the d_i map 1_m in G_{q+1} to 1_m in G_q, it is possible to do this by computing Td_i at the unit section. As the d_i retain the splitting of E by Lemma 2.1.32, we may compute the restriction to the E-horizontal and E-vertical subspace separately.

Note, that E-horizontal vectors at the unit section decompose into vectors tangent to the unit section (on which Td_i corresponds to the identity) and E-horizontal vectors in $\mathcal{F}_1$ (which vanish under insertion into ν or any $(d_i)^*(\nu)$ by Proposition 2.1.26).

We can for $x = 1_m$ explicitly compute $T_x d_i : (A_{\mathrm{iso}})_m^{\oplus(q+1)} \to (A_{\mathrm{iso}})_m^{\oplus q}$. The tangent map of multiplication at the unit is given by addition, thus we have for $1 \leq i \leq q$ that

$$T_x d_i(Z^{(1)}, \dots, Z^{(q+1)}) = (Z^{(1)}, \dots, Z^{(i-1)}, Z^{(i)} + Z^{(i+1)}, \dots, Z^{(q+1)}),$$

for $i = 0$ that

$$T_x d_i(Z^{(1)}, \dots, Z^{(q+1)}) = (Z^{(2)}, \dots, Z^{(q+1)})$$

and for $i = q+1$ that

$$T_x d_i(Z^{(1)}, \dots, Z^{(q+1)}) = (Z^{(1)}, \dots, Z^{(q)}).$$

So assume that the homogeneous component of $\omega = d_{\mathcal{N}}(\nu)$ of some multidegree $(r_0, r_1, \dots, r_{q+1})$ does not vanish (i.e. $\omega_{(r_0,r_1,\dots,r_{q+1})} \neq 0$). Then this fact is witnessed by a family of E-horizontal vectors $X_1, \dots, X_{r_0}$ and a family of E-vertical vectors

$$Y_1^{(1)}, \dots, Y_{r_1}^{(1)}, Y_1^{(2)}, \dots, Y_{r_2}^{(2)}, \dots, Y_1^{(q+1)}, \dots, Y_{r_{q+1}}^{(q+1)}$$

such that each $Y_j^{(i)}$ is contained in the i-th component of the splitting. By witnessing, that $d_{\mathcal{N}}(\nu)$, does not vanish, we mean that

$$d_{\mathcal{N}}(\nu)\left(X_1, \dots, X_{r_0}, Y_1^{(1)}, \dots, Y_{r_1}^{(1)}, \dots, Y_1^{(q+1)}, \dots, Y_{r_{q+1}}^{(q+1)}\right) \neq 0.$$

For some $0 \leq i \leq q$ we must therefore have have

$$\nu((d_i)_*(X_1), \dots, (d_i)_*(Y_{r_{q+1}}^{(q+1)})) \neq 0.$$

We have listed above which form $(d_i)_*$ takes in the three different cases of $2 \leq i \leq q$, $i = 0$ and $i = q+1$. For the first of these cases, we can explicitly check that the family

$$(d_i)_*(X_1), \dots, (d_i)_*(Y_{r_{q+1}}^{(q+1)})$$

has now multidegree $(r_0, \dots, r_{i-1}, (r_i + r_{i+1}), r_{i+2}, \dots)$, in the sense that the family contains r_0 many vectors which are E-horizontal, r_1 many vectors contained in the first copy of $(A_{\text{iso}})^{\oplus q}$ etc. So as we assumed $(d_i)^*\nu(X_1, \dots, Y_{r_{q+1}}^{(q+1)}) \neq 0$, we have that ν as a homogeneous form must be of the according multidegree

$$(p_0, p_1', \dots, p_q', q) = (r_0, \dots, r_{i-1}, (r_i + r_{i+1}), r_{i+2}, \dots)$$

For the other two cases $i = 0$ or $i = q+1$, we need to note that if $(d_i)_*$ vanishes on one vector, then automatically $(d_i)^*\nu(X_1, \dots, Y_{r_{q+1}}^{(q+1)}) = 0$. So as we assumed the contrary, we find that this implies $r_1 = 0$ or $r_{q+1} = 0$ respectively. In this case, too, we can explicitly compute that the induced family is of the multidegrees listed in the statement.

Proof of property c):

Consider a multidegree $(r_0, r_1, \dots, r_{q+1})$ of the form in the statement. We now need to show that $\omega_{(r_0,r_1,\dots,r_{q+1})} = 0$. To consider all three cases of the statement simultaneously, set $j = 0$ if we are in the second case and $j = q$ if we are in the third case.

Analogously to the proof of property b) we can for go through all cases of $0 \leq i \leq q$ in which potentially $(d_i)^*(\nu)$ has a non-vanishing component of multidegree

$$(r_0, r_1, \dots, r_{q+1}, q+1) = (p_0, p_1', \dots, p_j', 0, p_{(j+1)}', \dots p_q', q+1).$$

This analysis shows that this can only be the case for components that satisfy $i = j$ and $i = j+1$, as whenever $i < j$ (which in particular implies $j \geq 1$) we find that the equations of b) would otherwise state

$$\begin{aligned}(p_0, p_1', \dots p_q', q) &= (r_0, r_1, \dots, (r_i + r_{i+1}), \dots, r_j, r_{j+1}, r_{j+2}, \dots, r_{q+1}, q) \\ &= (p_0, p_1', \dots, (p_i' + p_{i+1}'), \dots, p_j', 0, p_{j+1}', \dots, p_q', q)\end{aligned}$$

and in particular imply $p'_j = 0$ (in case $i = 0$ the expression is different but would also imply $p'_j = 0$). Whenever $i > j + 1$ (which in particular implies $j \leq q - 1$) we would analogously find $p'_{j+1} = 0$, which would both contradict the assumptions of the statement.

But if we return to the computation of $T_x d_j : (A_{\mathrm{iso}})_m^{\oplus(q+1)} \to (A_{\mathrm{iso}})_m^{\oplus q}$ on the vertical tangent space, we have that $T_x d_j$ and $T_x d_{j+1}$ agree on E-vertical tangent vectors $Z = (Z^1, \ldots, Z^{q+1})$ where written in components $Z^{j+1} = 0$. Due to the considerations at the beginning of the proof of statement b), this implies that the entire forms obtained by projecting $(d_j)^*(\nu)$ and $(d_{j+1})^*(\nu)$ to their component in multidegree $(r_0, r_1, \ldots, r_{q+1}, q+1)$ agree.

So for the component of $\mathrm{d}_{\mathcal{N}}(\nu)$ in the chosen multidegree $(r_0, r_1, \ldots, r_{q+1}, q+1)$ we obtain that the terms contained in its definition given by

$$((d_i)^*(\nu))_{(r_0,r_1,\ldots,r_{q+1},q+1)} = 0$$

for all cases except $i = j$ and $i = j + 1$ and in these

$$((d_j)^*(\nu))_{(r_0,r_1,\ldots,r_{q+1},q+1)} = ((d_{j+1})^*(\nu))_{(r_0,r_1,\ldots,r_{q+1},q+1)},$$

and as they appear with opposite signs in $(d_{\mathcal{N}})(\nu)$, the component $\omega_{(r_0,r_1,\ldots,r_{q+1},q+1)}$ vanishes too.

Proof of property d):

We follow a similar scheme as in the previous proof. By analysing the multidegrees of $(d_j)^*(\nu)$ and using that $p'_{i-1} \neq 0$ and $p'_{j+1} \neq 0$, we obtain that $\omega_{(r_0,r_1,\ldots,r_{q+1},q+1)}$ consists of precisely the terms obtained by projecting $(d_k)^*(\nu)$ for $i-1 \leq k \leq j$ to the components of the given multidegree.

As now we only need to evaluate $\omega_{(r_0,r_1,\ldots,r_{q+1})}$ on a family of E-horizontal tangent vectors tangent to the identity section and E-vertical tangent vectors all vanishing in the components labelled $i, i+1, \ldots, j$, we obtain that all $((d_k)^*(\nu))_{(r_0,r_1,\ldots,r_{q+1})}$ for $i-1 \leq k \leq j$ actually agree.

As the terms are summed alternatingly and there are $j - i + 1$ of them (an even number), we have $\omega_{(r_0,r_1,\ldots,r_{q+1})} = 0$.

Proof of property e):

To prove property e), we follow the same proof as for property d), only now we have an odd number of terms, such that

$$\omega_{(r_0,r_1,\ldots,r_{q+1})} = (-1)^{p+i-1}\,((d_{i-1})^*(\nu))_{(r_0,r_1,\ldots,r_{q+1})}.$$

If we now consider $T_x d_{i-1}$ on the vertical subspaces, we have

$$T_x d_{i-1}(X^1, \ldots, X^{q+1}) = (X^1, \ldots, X^{i-2}, X^{i-1} + X^i, \ldots, X^{q+1}),$$

which for $X^i = X^{i+1} = \cdots = X^j = 0$ agrees with

$$(X^1, \ldots, X^{i-1}, 0, \ldots, 0, X^{j+1}, \ldots, X^{q+1}),$$

where the sequence of 0s has decreased in length by one. Hence to recover the evaluation of ν on a vector Y with horizontal component Y^0 and vertical components $Y^1, \ldots, Y^q$, we can assume for degree reasons that $Y^i = \cdots = Y^j = 0$, increase the length of the sequence of 0s by one and evaluate. For a family of vectors, we proceed analogously.

Spelling this procedure out in terms of $\omega_{(r_0,r_1,\ldots,r_{q+1})}$, inserting a zero at some point of the sequence corresponds to contracting at the corresponding component of the form $\Gamma(G_0, \wedge^0(A^*_{\mathrm{iso}})) = C^\infty(G_0)$ in the tensor product. $\square$

Corollary 3.1.12. *Let G be a regular groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Then all J^EG-invariant forms of multidegree $(p_0, 1, 1, \ldots, 1, q)$ are closed under the vertical differential $\mathrm{d}_{\mathcal{N}}$. In particular, all J^EG-isotropy Cartan forms are closed under $\mathrm{d}_{\mathcal{N}}$.*

Proof. Consider ν an element of multidegree $(p_0, 1, 1, \ldots, 1)$ and $\mathrm{d}_{\mathcal{N}}(\nu) = \omega$. By Lemma 3.1.11 a) and b) we have $\omega_{(r_0, r_1, \ldots, r_{q+1})} = 0$ whenever $r_0 \neq p_0$ or $r_1 + \cdots + r_{q+1} \neq q$. But if $r_1 + \cdots + r_{q+1} = q$ we have that some $r_i = 0$ as we are summing over non-negative integers, and by c) this implies $\omega_{(r_0, r_1, \ldots, r_{q+1})} = 0$ as well. □

Lemma 3.1.13. *Let G be a regular groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Consider $\omega \in (\Omega^p(G_q))^{J^EG}$ homogeneous of multidegree $(p_0, p_1', \ldots, p_q', q)$, let $\omega_{1_{G_0}}$ denote the values of ω along the unit section and let $1 \leq i \leq (q-1)$. Let*

$$\nu_{1_{G_0}} \in \Gamma\left(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \wedge^{p_1'}(A_{\mathrm{iso}}^*) \otimes \cdots \otimes \wedge^{p_i'+p_{i+1}'}(A_{\mathrm{iso}}^*) \otimes \cdots \otimes \wedge^{p_q'}(A_{\mathrm{iso}}^*)\right)$$

be the expression obtained by identification of the i-th and $i+1$-st respective copy of A_{iso}^* in

$$\omega_{1_{G_0}} \in \Gamma\left(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \wedge^{p_1'}(A_{\mathrm{iso}}^*) \otimes \cdots \otimes \wedge^{p_q'}(A_{\mathrm{iso}}^*)\right),$$

i.e. the form induced by mapping

$$\wedge^{p_i'}(A_{\mathrm{iso}}^*) \otimes \wedge^{p_{i+1}'}(A_{\mathrm{iso}}^*)$$

to

$$\wedge^{p_i'+p_{i+1}'}(A_{\mathrm{iso}}^*)$$

as vector bundles over G_0.

Then $\nu_{1_{G_0}}$ is G-invariant under the action described in Lemma 2.1.40 and in particular it corresponds to a J^EG-invariant form ν.

Proof. The map as described is dual to the map induced by

$$\begin{aligned} S_i : (A_{\mathrm{iso}})^{\oplus q} &\to (A_{\mathrm{iso}})^{\oplus(q+1)} \\ (X^{(1)}, \ldots, X^{(q)}) &\mapsto (X^{(1)}, \ldots, X^{(i)}, X^{(i)}, X^{(i+1)}, \ldots, X^{(q)}), \end{aligned}$$

and the G-action on the respective spaces is dual to the G-action on $(A_{\mathrm{iso}})^{\oplus q}$ and $(A_{\mathrm{iso}})^{\oplus q+1}$ induced by the action Ad^0 on $(\mathrm{Iso}^0(G))_q$ and $(\mathrm{Iso}^0(G))_{q+1}$ as described in Example 1.1.33. This is given by $\mathrm{Ad}^{\oplus q}$ and $\mathrm{Ad}^{\oplus(q+1)}$ respectively by Lemma 1.1.37, i.e. the G-action of Ad applied on each copy of A_{iso} simultaneously. The fact that we can therefore directly compute

$$S_i \circ \mathrm{Ad}_x^{\oplus q} = \mathrm{Ad}_x^{\oplus(q+1)} \circ S_i$$

automatically implies that the dual notion of contracting two forms as defined in the statement retains G-invariance.

The statement on $\nu_{1_{G_0}}$ describing a J^EG-invariant form is now an application of Proposition 2.1.41. □

Lemma 3.1.14. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Consider $\omega \in (\Omega^p(G_q))^{J^EG}$ such that ω*

is E-homogeneous of E-vertical degree p_1 (but not necessarily E-homogeneous of a certain multidegree) and its associated values along the unit section

$$\omega_{1_{G_0}} \in \bigoplus_{p'_1+\cdots+p'_q=p_1} \left(\Gamma\left(G_0, \wedge^{p_0}T^*G_0 \otimes \wedge^{p'_1}\left(A^*_{\mathrm{iso}}\right) \otimes \cdots \otimes \wedge^{p'_q}\left(A^*_{\mathrm{iso}}\right)\right)\right)^G.$$

Furthermore, let $\omega_{(p_0,p'_1,\ldots,p'_q,q)}$ respectively denote its homogeneous components of multidegree $(p_0,p'_1,\ldots,p'_q,q)$. Assume $\mathrm{d}_{\mathcal{N}}(\omega)=0$. Then ω is cohomologically equivalent to the form ω' where its homogeneous components of multidegree $(p_0,p'_1,\ldots,p'_q,q)$ satisfy $\omega'_{(p_0,p'_1,\ldots,p'_q,q)}=0$ whenever $p'_i=0$ for some $1\le i\le q$ and

$$\omega'_{(p_0,p'_1,\ldots,p'_q,q)} = \omega_{(p_0,p'_1,\ldots,p'_q,q)}$$

otherwise. The preimage ν of the difference can be chosen such that $\nu_{(p_0,p'_1,\ldots,p'_{q-1},(q-1))}=0$ unless $p'_i=0$ for some $1\le i\le q-1$.

In particular, if $p_1=q$ we have that $\omega-\omega_{(p_0,1,\ldots,1,q)}$ is exact.

Proof. We will use an argument of finite descent: For each ω, consider all pairs of i and j for which $i\le j$, together with the associated multidegrees $(p_0,p'_1,\ldots,p'_q,q)$ for which holds $p'_i=p'_{i+1}=\cdots=p'_j=0$. Then, if possible, choose a multidegree such that

- $\omega_{(p_0,p'_1,\ldots,p'_q)}\neq 0$
- i is minimal under this constraint and
- $j-i$ is maximal under both of the previous constraints.

If it is not possible to choose $(p_0,p'_1,\ldots,p'_q)$, we can conclude with the descent as $\omega_{(p_0,p'_1,\ldots,p'_q)}\neq 0$ only when $p'_i\neq 0$ for all $1\le i\le q$.

In each step of the descent we will construct a cohomologically equivalent ω' that satisfies $\omega'_{(p_0,p'_1,\ldots,p'_q,q)}=0$ for the chosen multidegree $(p_0,p'_1,\ldots,p'_q,q)$. Furthermore, ω' only differs from ω in specific multidegrees $(u_0,u'_1,\ldots,u'_q,q)$. Either

- $(u_0,u'_1,\ldots,u'_q,q)=(p_0,p'_1,\ldots,p'_q,q)$
- $(u_0,u'_1,\ldots,u'_q,q)$ satisfies that $u'_k\neq 0$ for $1\le k\le i$
- $(u_0,u'_1,\ldots,u'_q,q)$ satisfies that $u'_k\neq 0$ for $1\le k<i$ and the sequence of zero-valued index components $u'_i=u'_{i+1}=\cdots=u'_l=0$ has smaller length than $j-i+1$ (i.e. $l-i<j-i$)

Furthermore, all of these multidegrees in which the form differs will satisfy that there is some index l with vanishing degree $u'_l=0$.

Hence after one step we can again consider all pairs of i' and j' for which $i'\le j'$ together with the multidegrees $(p_0,p'_1,\ldots,p'_q,q)$ for which holds $p'_{i'}=p'_{i'+1}=\cdots=p'_{j'}=0$. Then, if possible, choose i',j' and a multidegree such that

- $\omega'_{(p_0,p'_1,\ldots,p'_q)}\neq 0$
- i' is minimal under this constraint and
- $j'-i'$ is maximal under both of the previous constraints.

We now have that $i' \geq i$ because of the restrictions on the multidegrees $(u_0, u_1', \ldots, u_q', q)$ in which ω' can differ from the previous form ω and for the same reason, we have that if $i' = i$, we have $j' - i' \leq j - i$.

If this value also has stayed constant, we can then infer that the indices satisfying that $u_i' = u_{i+1}' = \cdots = u_j' = 0$ and $\omega'_{(p_0, u_1', \ldots, u_q')} \neq 0$ are a strict subset of those indices where $u_i' = u_{i+1}' = \cdots = u_j' = 0$ and $\omega'_{(p_0, u_1', \ldots, u_q')} \neq 0$. In particular, the number of multi-indices of this form corresponding to non-vanishing components has decreased. So in conclusion, iterating this procedure will in each step either increase i to $i' > i$, or leave $i' = i$ constant and decrease $j - i$ to $j' - i' < j - i$, or leave $i = i'$ constant and $j' - i' = j - i$ constant and decrease the number of non-vanishing components associated to a certain family of multidegrees.

As this number is finite, we must either increase i to $i' > i$ or decrease $j - i$ to $j' - i' < j - i$ after a finite number of iterations. As both of these values are also bounded (by q and 0 respectively), we obtain that the descent must end eventually.

We will now prove that the descent step is possible as previously described. For this, choose i, j and $(p_0, p_1', \ldots, p_q', q)$ with the desired minimality condition.

As we know that $\mathrm{d}_{\mathcal{N}}(\omega) = 0$, we have that in particular

$$\mathrm{d}_{\mathcal{N}}(\omega)_{(r_0, r_1, \ldots, r_{q+1}, q+1)} = 0$$

for

$$(r_0, r_1, \ldots, r_{q+1}, q+1) = (p_0, p_1', \ldots, p_{i-1}', 0, p_i', \ldots p_q', q),$$

where we increased the length of the zero sequence by one.

By Lemma 3.1.11 b), this component of $\mathrm{d}_{\mathcal{N}}(\omega)$ only depends on the components $\omega_{(u_0, u_1', \ldots, u_q')}$, where

$$(u_0, u_1', \ldots, u_q', q) = (r_0, \ldots, r_k + r_{k+1}, \ldots, q).$$

If we analyse this expression to find out properties of $(u_0, u_1', \ldots, u_q', q)$, we obtain the following:

- if $k < i - 1$, we have that $(u_1', \ldots, u_q')$ contains a sequence of 0s of starting at index $i - 1$
- if $i - 1 \leq k \leq j + 1$, we have that $(u_0, u_1', \ldots, u_q', q) = (p_0, p_1', \ldots, p_q', q)$
- if $j + 1 < k$, we have that $(u_1', \ldots, u_q')$ contains a sequence of 0s of length $j - i + 1$ starting at index i

By assumption, however, the components of ω corresponding to indices satisfying either the first or third property vanish.

Therefore the component of multidegree $(r_0, r_1, \ldots, r_{q+1}, q+1)$ of $\mathrm{d}_{\mathcal{N}}(\omega)$ actually only depends on $\omega_{(p_0, p_1', \ldots, p_q', q)}$. As we have assumed $\mathrm{d}_{\mathcal{N}}(\omega) = 0$, it follows that also

$$\mathrm{d}_{\mathcal{N}}(\omega_{(p_0, p_1', \ldots, p_q', q)})_{(r_0, r_1, \ldots, r_{q+1}, q+1)} = 0.$$

We can now apply Lemma 3.1.11 e) to obtain, that $j - i$ must be odd, or we would already $\omega_{(p_0, p_1', \ldots, p_q')} = 0$ contrary to our choice.

In particular, the sequence $i, i+1, \ldots, j$ has at least two elements. Hence we can consider $\omega_{(p_0, p_1', \ldots, p_q')} \neq 0$ and contract at one of the tensor factors $\Gamma(G_0, \wedge^0(A^*_{\mathrm{iso}})) = C^\infty(G_0)$. This will give us a homogeneous element ν of multidegree

$$(p_0, p_1', \ldots, p_{i-1}', 0, \ldots, 0, p_{j+1}', \ldots, p_q', q - 1),$$

where the sequence of 0s has decreased length, but is still of at least length one.

If we compute $d_{\mathcal{N}}(\nu)$, we obtain in analogy to the proof of Lemma 3.1.11 e) that it has a non-vanishing component in multidegree $(p_0, p'_1, \ldots, p'_q)$, where it agrees with ω up to sign. Other non-vanishing components of $d_{\mathcal{N}}(\nu)$ according to Lemma 3.1.11 b) and c) only occur in multidegrees where the first zero of the sequence $r'_1, \ldots, r'_{q+1}$ is at either index i or $i+1$ and the length of the corresponding sequence of 0s agrees with the length of the zero sequence in $(p_0, p'_1, \ldots, p'_{i-1}, 0, \ldots, 0, p'_{j+1}, \ldots, p'_q, q-1)$, meaning its length is strictly shorter than $j - i + 1$.

Hence $\omega' = \omega \pm \mathrm{d}_{\mathcal{N}}(\nu)$ satisfies the conditions formulated in the finite descent and thus proves the original statement. $\square$

Lemma 3.1.15. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Consider the contraction of a form ω at index i and $i+1$ and the result ν as in Lemma 3.1.13 when $p'_i \neq 0$ and $p'_{i+1} \neq 0$. Then the component of $d_{\mathcal{N}}(\nu)_{1_{G_0}}$ in degree $(p_0, p'_1, \ldots, p'_q, q)$ is given by the graded symmetrisation of $\omega_{1_{G_0}}$ up to a scaling constant.*

By this we mean the following: Let

$$X_1, \ldots, X_{p_0} \in \Gamma(G_0, (TG_0/\mathcal{F}_0))$$

and let

$$Y_1^{(1)}, \ldots, Y_{p_1}^{(1)}, \ldots, Y_1^{q}, \ldots, Y_{p_q}^{(q)} \in \Gamma(G_0, (A_{\mathrm{iso}})^{\oplus q})$$

where the image of each $Y_k^{(j)}$ is contained in the j-th copy of A_{iso}. For each $\sigma \in \Sigma_{(p'_i + p'_{i+1})}$ construct

$$\overline{Y}_1^{(\sigma,i)}, \ldots, \overline{Y}_{p_i}^{(\sigma,i)}, \overline{Y}_1^{(\sigma,i+1)}, \ldots, \overline{Y}_{p_{i+1}}^{(\sigma,i+1)} \in \Gamma(G_0, (A_{\mathrm{iso}})^{\oplus q}),$$

where for $1 \leq k \leq p'_i$ with $\sigma^{-1}(k) \leq p'_i$ we define $\overline{Y}_k^{(\sigma,i)}$ as the unique section with image in the i-th copy of $(A_{\mathrm{iso}})^{\oplus q}$ satisfying

$$\mathrm{pr}_i \circ \overline{Y}_k^{(\sigma,i)} = \mathrm{pr}_i \circ Y_{\sigma^{-1}(k)}^{(i)}$$

under the projection to the i-th component $\mathrm{pr}_i : (A_{\mathrm{iso}})^{\oplus q} \to A_{\mathrm{iso}}$ and for $1 \leq k \leq p'_i$ with $\sigma^{-1}(k) \geq p'_i + 1$ we define $\overline{Y}_k^{(\sigma,i)}$ as the unique section with image in the i-th copy of $(A_{\mathrm{iso}})^{\oplus q}$ satisfying

$$\mathrm{pr}_i \circ \overline{Y}_k^{(\sigma,i)} = \mathrm{pr}_{i+1} \circ Y_{\sigma^{-1}(k) - p'_i}^{(i+1)}.$$

Similarly, for $1 \leq k \leq p'_{i+1}$ with $\sigma^{-1}(p'_i + k) \leq p'_i$ we define $\overline{Y}_k^{(\sigma,i+1)}$ as the unique section with image in the $i+1$-st copy of $(A_{\mathrm{iso}})^{\oplus q}$ satisfying

$$\mathrm{pr}_{i+1} \circ \overline{Y}_k^{(\sigma,i)} = \mathrm{pr}_i \circ Y_{\sigma^{-1}(k)}^{(i)}$$

and for $1 \leq k \leq p'_{i+1}$ with $\sigma^{-1}(p'_i + k) \geq p'_i + 1$ we define $\overline{Y}_k^{(\sigma,i+1)}$ as the unique section with image in the $i+1$-st copy of $(A_{\mathrm{iso}})^{\oplus q}$ satisfying

$$\mathrm{pr}_{i+1} \circ \overline{Y}_k^{(\sigma,i)} = \mathrm{pr}_{i+1} \circ Y_{\sigma^{-1}(k) - p'_i}^{(i+1)}.$$

Now, if we evaluate

$$d_{\mathcal{N}}(\nu)_{1_{G_0}}(X_1, \ldots, X_{p_0}, Y_1^{(1)}, \ldots, Y_{p_1}^{(1)}, \ldots, Y_1^{q}, \ldots, Y_{p_q}^{(q)}),$$

we obtain the same result as for

$$c \cdot \sum_{\sigma \in \Sigma_{(p'_i+p'_{i+1})}} \operatorname{sgn}(\sigma)\omega\left(\ldots, \overline{Y}_1^{(\sigma,i)}, \ldots, \overline{Y}_{p_i}^{(\sigma,i)}, \overline{Y}_1^{(\sigma,i+1)}, \ldots, \overline{Y}_{p_{i+1}}^{(\sigma,i+1)}, \ldots\right),$$

with $c = \frac{(-1)^{p+i}}{|\Sigma_{p'_i}| \cdot |\Sigma_{p'_{i+1}}|}$.

In particular, for $p'_i = p'_{i+1} = 1$, *we obtain that*

$$\begin{aligned}
&\omega(\ldots, Y_1^{(1)}, \ldots, Y_{p'_1}^{(1)}, \ldots, Y_1^{(q)}, \ldots, Y_{p'_q}^{(q)}) \\
&\quad - (-1)^{p+i}\, \mathrm{d}_{\mathcal{N}}(\nu)(\ldots, Y_1^{(1)}, \ldots, Y_{p'_1}^{(1)}, \ldots, Y_1^{(q)}, \ldots, Y_{p'_q}^{(q)}) \\
=&\omega(\ldots, Y_{p'_{i-1}}^{(i-1)}, \overline{Y}_1^{(-1,i)}, \overline{Y}^{(-1,i+1)}, Y_1^{(i+2)}, \ldots)
\end{aligned}$$

which corresponds to the result of ω *under the action of the element*

$$\begin{pmatrix} 1 & 2 & \ldots & i-1 & i & i+1 & i+2 & \ldots \\ 1 & 2 & \ldots & i-1 & i+1 & i & i+2 & \ldots \end{pmatrix} \in \Sigma_q$$

by permutation of the respective components as defined in Lemma 3.1.3.

Proof. First of all, let us note that the component of $\mathrm{d}_{\mathcal{N}}(\nu)_{1_{G_0}}$ in degree $(p_0, p'_1, \ldots, p'_q, q)$ agrees with the component of $(d_i)^*(\nu)_{1_{G_0}}$ in that degree. This is because, by definition, $\nu_{1_{G_0}}$ is homogeneous of degree $(p_0, p'_1, \ldots, p'_{i-1}, (p'_i+p'_{i+1}), \ldots, p'_q, q)$, and with this we can see from the proof of statement b) of Lemma 3.1.11 that the component of $(d_j)^*(\nu)_{1_{G_0}}$ in the given degree vanishes unless $i = j$. Note that we need to use the assumption that $p_i > 0$ and $p_{i+1} > 0$ to check that

$$(p_0, p'_1, \ldots, p'_{i-1}, (p'_i + p'_{i+1}), \ldots, p'_q, q) \neq (p_0, p'_1, \ldots, p'_{j-1}, (p'_j + p'_{j+1}), \ldots, p'_q, q)$$

unless $i = j$.

Furthermore, we know from the proof of statement b) of Lemma 3.1.11, that

$$\begin{aligned}
&(d_i)^*(\nu)_{1_{G_0}}(X_1, \ldots, X_{p_0}, Y_1^{(1)}, \ldots, Y_{p_1}^{(1)}, \ldots, Y_1^{(q)}, \ldots, Y_{p_q}^{(q)}) \\
=&\nu_{1_{G_0}}(X_1, \ldots, X_{p_0}, \widetilde{Y}_1^{(1)}, \ldots, \widetilde{Y}_{p_1}^{(1)}, \ldots, \widetilde{Y}_1^{(q)}, \ldots, \widetilde{Y}_{p_q}^{(q)})
\end{aligned}$$

where the sections

$$\widetilde{Y}_1^{(1)}, \ldots, \widetilde{Y}_{p_1}^{(1)}, \ldots, \widetilde{Y}_1^{(q)}, \ldots, \widetilde{Y}_{p_q}^{(q)} \in \Gamma(G_0, (A_{\mathrm{iso}})^{\oplus(q-1)})$$

satisfy that each $\widetilde{Y}_k^{(j)}$ for $j \leq i$ has its image contained in the j-th copy of A_{iso} and for $j > i$ has its image contained in the $j-1$-th copy of A_{iso}, while simultaneously such that

$$\widetilde{Y}_k^{(j)} \equiv Y_k^{(j)}$$

under the corresponding projections to $\Gamma(G_0, A_{\mathrm{iso}})$.

Now we can easily see that the construction of

$$\omega \mapsto \mathrm{d}_{\mathcal{N}}(\nu)_{(p_0, p'_1, \ldots, p'_q, q)}$$

commutes with evaluation on the vector fields

$$Y_1^{(1)}, \ldots, Y_{p_1}^{(1)}, \ldots, Y_1^{(i-1)}, \ldots, Y_{p_{i-1}}^{(i-1)}$$

and

$$Y_1^{(i+2)}, \ldots, Y_{p_{i+2}}^{(i+2)}, \ldots, Y_1^{(q)}, \ldots, Y_{p_q}^{(q)}$$

as for these vectors, insertion into ω and then contracting corresponds (because of the reindexing of the remaining components) precisely to inserting the corresponding $\widetilde{Y}_k^{(j)}$ into ν as well. Therefore, we may without loss of generality assume that $p'_j = 0$ for all indices $j \neq i$ and $j \neq i+1$.

Now we have that the evaluation on vector fields on both

$$\alpha \in \Gamma\left(G_0, \wedge^{p'_i}\left(A^*_{\text{iso}}\right) \otimes \wedge^{p'_{i+1}}\left(A^*_{\text{iso}}\right)\right)$$

and

$$\beta \in \Gamma\left(G_0, \wedge^{p'_i + p'_{i+1}}\left(A^*_{\text{iso}}\right)\right).$$

is induced by the evaluation on vector fields as elements $\overline{\alpha}$ and $\overline{\beta}$ of the corresponding spaces

$$\Gamma\left(G_0, \otimes^{p'_i}\left(A^*_{\text{iso}}\right) \otimes \otimes^{p'_{i+1}}\left(A^*_{\text{iso}}\right)\right) \cong \Gamma\left(G_0, \otimes^{p'_i + p'_{i+1}}\left(A^*_{\text{iso}}\right)\right),$$

where evaluating on the family

$$Y_1^{(i)}, \ldots, Y_{p_i}^{(i)}, Y_1^{(i+1)}, \ldots, Y_{p_{i+1}}^{(i+1)}$$

on the left hand side corresponds to evaluating on the family

$$\widetilde{Y}_1^{(i)}, \ldots, \widetilde{Y}_{p_i}^{(i)}, \widetilde{Y}_1^{(i+1)}, \ldots, \widetilde{Y}_{p_{i+1}}^{(i+1)}$$

on the right hand side.

However, the evaluation of the contracted form β on a family of vector fields

$$\widetilde{Y}_1^{(i)}, \ldots, \widetilde{Y}_{p_i}^{(i)}, \widetilde{Y}_1^{(i+1)}, \ldots, \widetilde{Y}_{p_{i+1}}^{(i+1)}$$

is induced by

$$\sum_{\sigma \in \Sigma_{p'_i + p'_{i+1}}} \operatorname{sgn}(\sigma) \cdot \overline{\beta}(\sigma(\widetilde{Y}_1^{(i)}, \ldots, \widetilde{Y}_{p_{i+1}}^{(i+1)}))$$

while the evaluation of α is induced by

$$\sum_{\sigma^{(i)} \in \Sigma_{p'_i}} \sum_{\sigma^{(i+1)} \in \Sigma_{p'_{i+1}}} \operatorname{sgn}(\sigma^{(i)}) \cdot \operatorname{sgn}(\sigma^{(i+1)}) \cdot \overline{\alpha}(\sigma^{(i)}(Y_1^{(i)}, \ldots, Y_{p'_i}^{(i)}), \sigma^{(i+1)}(Y_1^{(i+1)}, \ldots, Y_{p'_{i+1}}^{(i+1)})).$$

We can explicitly check that permutation of the

$$\widetilde{Y}_1^{(i)}, \ldots, \widetilde{Y}_{p_{i+1}}^{(i+1)}$$

is translated under the isomorphism of

$$\Gamma\left(G_0, \otimes^{p'_i}\left(A^*_{\text{iso}}\right) \otimes \otimes^{p'_{i+1}}\left(A^*_{\text{iso}}\right)\right) \cong \Gamma\left(G_0, \otimes^{p'_i + p'_{i+1}}\left(A^*_{\text{iso}}\right)\right).$$

to the notion

$$Y_1^{(i)}, \ldots, Y_{p'_{i+1}}^{(i+1)} \mapsto \overline{Y}_1^{(\sigma,i)}, \ldots, \overline{Y}_{p_{i+1}}^{(\sigma,i+1)}.$$

So if α and β are related by contraction, we have that

$$\overline{\beta}(\sigma(\widetilde{Y}_1^{(i)}, \ldots, \widetilde{Y}_{p_{i+1}}^{(i+1)})) = \overline{\alpha}(\overline{Y}_1^{(\sigma,i)}, \ldots, \overline{Y}_{p_{i+1}}^{(\sigma,i+1)}).$$

This in particular implies that

$$\beta(\widetilde{Y}_1^{(i)}, \dots, \widetilde{Y}_{p_{i+1}}^{(i+1)}) = \sum_{\sigma \in \Sigma_{p'_i + p'_{i+1}}} \operatorname{sgn}(\sigma) \cdot \overline{\alpha}(\overline{Y}_1^{(\sigma,i)}, \dots, \overline{Y}_{p_{i+1}}^{(\sigma,i+1)}).$$

Let us compare this expression to

$$\sum_{\sigma \in \Sigma_{p'_i + p'_{i+1}}} \operatorname{sgn}(\sigma) \cdot \alpha(\overline{Y}_1^{(\sigma,i)}, \dots, \overline{Y}_{p_{i+1}}^{(\sigma,i+1)}).$$

As postcomposition with the action of $\Sigma_{p'_i}$ or $\Sigma_{p'_{i+1}}$ corresponds to two free commuting actions on $\Sigma_{p'_i + p'_{i+1}}$, we find that all terms of the definition corresponding to the second expression appear each $|\Sigma_{p'_i}| \cdot |\Sigma_{p'_{i+1}}|$ times and each such group corresponds to one term in the sum defining $\beta(\widetilde{Y}_1^{(i)}, \dots, \widetilde{Y}_{p_{i+1}}^{(i+1)})$. The additional sign given by $(-1)^{p+i}$ appears in the constant as this is precisely the sign of $(d_i)^*$ (and our calculations only computed the image without a sign). Applying this to ν and ω gives the statements of the Lemma. If $p'_i = 1$ and $p'_{i+1} = 1$, the normalisation equals 1 such that we directly obtain the second statement. □

Lemma 3.1.16. *Let G be a regular Lie groupoid. Consider an element*

$$w \in (A^*_{\text{iso}} \oplus A_{\text{iso}} \oplus \cdots \oplus A^*_{\text{iso}})^{\otimes p_1} = ((A^*_{\text{iso}})^{\oplus q})^{\otimes p_1}$$

for which

$$p_1 \neq q$$

and

$$\sum_{i=0}^{q+1} (-1)^i (T1_{G_0} d_i)^* w = 0 \in ((A^*_{\text{iso}})^{\oplus q+1})^{\otimes p_1},$$

where we understand $(A^*_{\text{iso}})^{\oplus q}$ as a subbundle of the tangent bundle of G_q restricted to 1_{G_0}.

Then there is a

$$v \in ((A^*_{\text{iso}})^{\oplus q-1})^{\otimes p_1}$$

with

$$\sum_{i=0}^{q} (-1)^i (T1_{G_0} d_i)^* v = w.$$

Furthermore, the assignment is smooth in G_0 and linear, such that for any section

$$\widetilde{w} \in \Gamma\left(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes ((A^*_{\text{iso}})^{\oplus q})^{\otimes p_1}\right)$$

for which

$$\sum_{i=0}^{q+1} (-1)^i (T1_{G_0} d_i)^* \widetilde{w} = 0 \in \Gamma\left(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes ((A^*_{\text{iso}})^{\oplus q+1})^{\otimes p_1}\right),$$

there is a

$$\widetilde{v} \in \Gamma\left(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes ((A^*_{\text{iso}})^{\oplus q-1})^{\otimes p_1}\right)$$

with

$$\sum_{i=0}^{q} (-1)^i (T1_{G_0} d_i)^* \widetilde{v} = \widetilde{w}.$$

If we consider the action of Σ_{p_1} on the given space by permutation of the tensor factors and we have $\widetilde{w}$ satisfying that

$$\sigma.\widetilde{w} = \operatorname{sgn}(\sigma) \cdot \widetilde{w}$$

we may chose $\widetilde{v}$ satisfying the same condition.

If $\widetilde{w}$ is furthermore invariant under the G-action induced by the coadjoint action on $(A^*_{\mathrm{iso}})^{\oplus q}$, then so is $\widetilde{v}$.

Remark. This Lemma in its proof follows very closely the arguments of Bott in [8], in which he computes the cohomology of the columns of the Bott–Shulman–Stasheff bicomplex in the case of a trivial action groupoid or equivalently the equivariant cohomology of a trivial action (cf. [8], Theorem 1.).

Proof. The differential for the case $p_1 = 1$ can be computed in a straightforward way, regardless of the condition between p_1 and q: We know that the face maps of the simplicial structure of $T_{1_{G_0}} G_\bullet = \Gamma(G_0, (A^*_{\mathrm{iso}})^{\oplus \bullet})$ are given by multiplication, which agrees with addition on A_{iso}. The only exceptions are d_0 and d_{q+1}

On $(A^*_{\mathrm{iso}})^{\oplus q}$ we thus obtain that any $(T_{1_{G_0}} d_i)^*$ will fibre-wise map cotangent vectors $(v_1, \ldots, v_{q-1})$ to

$$(v_1, \ldots, v_i, v_i, v_{i+1}, \ldots, v_{q-1}).$$

This means that in case that $q > 0$ is even, we obtain that the image of the alternating sum is given by

$$(-v_1, 0, v_2 - v_3, 0 \ldots, v_{q-2} - v_{q-1}, 0, v_q)$$

and if q is odd it is given by

$$(0, v_2, v_2, v_4, v_4, \ldots, v_{q-1}, v_{q-1}, 0).$$

This means that in particular, if $q \geq 2$ is even, we obtain that a preimage of any closed element can be chosen as

$$(0, \frac{1}{2}(v_2 + v_3), 0, \frac{1}{2}(v_4 + v_5), \ldots, 0, \frac{1}{2}(v_{q-2} + v_{q-1}), 0) \in (A^*_{\mathrm{iso}})^{\oplus q-1}$$

and if $q \geq 3$ is odd a preimage of any closed element can be chosen as

$$(-v_1, \frac{1}{2}v_3, -\frac{1}{2}v_3, \frac{1}{2}v_5, -\frac{1}{2}v_5, \ldots, \frac{1}{2}v_{q-2}, -\frac{1}{2}v_{q-2}, v_q).$$

As $(A^*_{\mathrm{iso}})^{\oplus 0} = \{0\}$, we find that the cohomology of

$$\Gamma\left(G_0, (A^*_{\mathrm{iso}})^{\oplus \bullet}\right)$$

is given by

$$\Gamma\left(G_0, A^*_{\mathrm{iso}}\right)$$

in degree 1.

The statement of the theorem is now essentially an application of a generalised version of the Eilenberg–Zilber theorem. In the paper [8], Bott calls it the Cartier extension of the Eilenberg–Zilber theorem (cf. p. 298), which are also the authors referenced for this result in Dold and Puppe (cf. [16] p. 213). However, this name for the theorem seems not to be universally used. Details on the theorem can be found in A.4.

The theorem as stated in Theorem A.4.5 implies that there is a natural isomorphism

$$H^\bullet(C^\bullet(\mathrm{Diag}(B))) \simeq H^\bullet(\mathrm{Tot}(C^\bullet(B)))$$

where B is a bi-cosimplicial abelian group (or more generally a cosimplicial object in an abelian category) and C is the Moore cochain complex functor.

In particular we can isolate the first factor of the tensor product and interpret

$$\Gamma\left(G_0, ((A^*_{\text{iso}})^{\oplus q})^{\otimes p_1}\right) = \Gamma\left(G_0, ((A^*_{\text{iso}})^{\oplus q}) \otimes ((A^*_{\text{iso}})^{\oplus q})^{\otimes p_1 - 1}\right)$$

as $C^\bullet(\mathrm{Diag}(B))$ for

$$B(\Delta^{n_1}, \Delta^{n_2}) = \Gamma\left(G_0, ((A^*_{\text{iso}})^{\oplus n_1}) \otimes ((A^*_{\text{iso}})^{\oplus n_2})^{\otimes p_1 - 1}\right),$$

where the simplicial structure on G and the respective $T_{1_{G_0}} G_q = (A^*_{\text{iso}})^{\oplus q}$ induces the cosimplicial structure on B.

Now according to the Eilenberg–Zilber–Cartier theorem, this cohomology in degree q can be computed as

$$H^q(\mathrm{Tot}(C^\bullet(B))).$$

As we constructed choices of preimages under the differential in the first step of this proof, we obtain maps

$$B(\Delta^{n_1}, \Delta^{n_2}) \to B(\Delta^{n_1 - 1}, \Delta^{n_2})$$

for $n_1 \geq 2$ and these will still map to preimages under the differential in the first component.

As the differential on $\mathrm{Tot}(C^\bullet(B))$ is precisely the sum of the differential acting on the first and the differential acting on the second component, this implies that any closed degree q element is equivalent to one with only a non-zero value in degree $(1, q-1)$ (and theoretically $(0, q)$, but that space only contains 0). Similarly, any preimage of total degree $q - 1$ in case of exactness would be equivalent to one with only a non-zero value in degree $(1, q-2)$.

But as the differential acting on the first part of the tensor product vanishes on elements of these form, we have induced a shifted quasi-isomorphism

$$H^{\bullet - 1}\left(\Gamma\left(G_0, (A^*_{\text{iso}}) \otimes ((A^*_{\text{iso}})^{\oplus \bullet})^{\otimes p_1 - 1}\right)\right) \simeq H^\bullet(\mathrm{Tot}(C^\bullet(B))),$$

where the differential of the left hand side chain complex is the one that only acts on the second part of the tensor product.

However, this is an argument that can be iterated. In the case where $p_1 < q$, we obtain

$$H^q(\mathrm{Tot}(C^\bullet(B))) = H^{q - p_1 + 1}\left(\Gamma\left(G_0, ((A^*_{\text{iso}})^{\oplus q - p_1 - 1}) \otimes ((A^*_{\text{iso}})^{\oplus \bullet})^{\otimes 1}\right)\right),$$

where we can use the preimages we previously constructed to show that this chain complex only has cohomology in degree 1 while $q - p_1 + 1 \geq 2$.

In the case where $p_1 > q$, we can instead obtain

$$H^q(\mathrm{Tot}(C^\bullet(B))) = H^0\left(\Gamma\left(G_0, ((A^*_{\text{iso}})^{\oplus q}) \otimes ((A^*_{\text{iso}})^{\oplus \bullet})^{\otimes p_1 - q}\right)\right),$$

where the degree 0 elements are given by sections to

$$((A^*_{\text{iso}})^{\oplus q}) \otimes ((A^*_{\text{iso}})^{\oplus 0})^{\otimes p_1 - q} = ((A^*_{\text{iso}})^{\oplus q}) \otimes (\{0\})^{\otimes p_1 - q} = \{0\}$$

and hence

$$H^q(\mathrm{Tot}(C^\bullet(B))) = 0.$$

Lastly, we can note that the differential is equivariant under the symmetric action, so if we consider the anti-symmetrisation of $\widetilde{v}$ it will be the preimage of the anti-symmetrisation of $\widetilde{w}$. Here we mean by anti-symmetrisation the respective term

$$\frac{1}{|\Sigma_{p_1}|}\sum_{\Sigma_{p_1}}\operatorname{sgn}(\sigma)\cdot\sigma.\widetilde{v}$$

or

$$\frac{1}{|\Sigma_{p_1}|}\sum_{\Sigma_{p_1}}\operatorname{sgn}(\sigma)\cdot\sigma.\widetilde{w}.$$

So if $\widetilde{w}$ itself is anti-symmetric, we have hence shown the existence of an anti-symmetric preimage.

For the statement on G-invariance, we could aim to reconstruct the results we used to obtain Proposition 2.3.10. But it is actually sufficient to note that in the construction of $\widetilde{v}$ the terms that occurred were already invariant. First of all, we have that the explicitly constructed preimages at the beginning of the proof. These can be explicitly checked to retain G-invariance, and the anti-symmetrisation process commutes with the adjoint action of any element of G. So it only remains to consider the preimages which come from the generalised version of the Eilenberg–Zilber theorem. However, these are constructed using only simplicial maps (see the discussion in A.4). Due to both $\mathrm{Ad}^{\oplus q}$ and $T1_{G_0}d_i$ on $(A_{\mathrm{iso}})^{\oplus q}$ being obtained by the respective notions on $(\mathrm{Iso}^0(G))_q$, we find that Example 1.1.31 implies that the simplicial maps $T1_{G_0}d_i$ (and the analogous terms for the s_j) commute with the adjoint G-action as well. Therefore, they retain the G-invariance. So all in all, if $\widetilde{w}$ was G-invariant, then so is $\widetilde{v}$ as well as the anti-symmetrisation of $\widetilde{v}$, which concludes the proof. □

Corollary 3.1.17. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Let ω be a J^EG-invariant form of E-horizontal degree p_0 and E-vertical degree p_1, for which $p_1 \neq q$, and corresponding under the isomorphism of Proposition 2.1.41 to*

$$\omega_{1_{G_0}} \in \left(\Gamma\left(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \wedge^{p_1}\left((A_{\mathrm{iso}}^*)^{\oplus q}\right)\right)\right)^G.$$

Then we have that $\omega \in (\Omega^p(G_\bullet))^{J^EG}$ is exact if and only if it is closed.

Proof. Assume that ω is exact, then it is automatically closed as $(\mathrm{d}_\mathcal{N})^2 = 0$. The relevant claim to prove here is that ω is exact if it is closed.

In this case, we can canonically include

$$\wedge^{p_1}\left((A_{\mathrm{iso}}^*)^{\oplus q}\right) \subset \left((A_{\mathrm{iso}}^*)^{\oplus q}\right)^{\otimes p_1}$$

as those elements which are invariant under the anti-symmetric action of Σ_{p_1}, yielding

$$\widetilde{\omega} \in \left(\Gamma\left(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \left((A_{\mathrm{iso}}^*)^{\oplus q}\right)^{\otimes p_1}\right)\right)^G.$$

We can now apply Lemma 3.1.16, to obtain a preimage. Note that the differential as described in the lemma only agrees with the differential on $(\Omega^p(G_\bullet))^{J^EG}$ up to the factor $(-1)^p$. Scaling the preimage by the same factor returns an expression $\widetilde{\nu}$.

By the lemma, we may assume we chose $\widetilde{\nu}$ invariant under the anti-symmetric action and invariant under the G-action induced by the coadjoint action, meaning that there is a corresponding

$$\nu \in \left(\Gamma\left(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \wedge^{p_1}\left((A_{\mathrm{iso}}^*)^{\oplus(q-1)}\right)\right)\right)^G.$$

As the differentials of the lemma precisely corresponds to $\mathrm{d}_{\mathcal{N}}$ up to a sign, which is chosen precisely in a way that it cancels, it follows that

$$\mathrm{d}_{\mathcal{N}}(\nu) = \omega.$$

and ω is exact. □

Lemma 3.1.18. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Consider a form $\omega \in (\Omega^p(G_q))^{J^E G}$ which is homogeneous of multidegree $(p_0, 1, \ldots, 1, q)$ and satisfies $\mathrm{d}_{\mathcal{N}}(\omega) = 0$. Then ω is exact if and only if ω vanishes under symmetrisation.*

Proof. Here, we have to prove two things: That all exact terms vanish under symmetrisation and that all terms that vanish under symmetrisation are exact. First, let us prove that all exact terms vanish under symmetrisation: For this, consider any preimage ν. We have that ν decomposes into homogeneous components $\nu_{(p_0, p'_1, \ldots, p'_{q-1}, q-1)}$ with respect to the splitting given by E. By statement b) of Lemma 3.1.11 the only components satisfying

$$\mathrm{d}_{\mathcal{N}}(\nu_{(p_0, p'_1, \ldots, p'_{q-1}, q-1)})_{(p_0, 1, \ldots, 1, q)} \neq 0$$

are those when

$$(p_0, p'_1, \ldots, p'_{q-1}, q-1) = (p_0, 1, \ldots, 1, 2, 1, \ldots, 1, q-1),$$

where the cases $p'_1 = 2$ and $p_{(q-1)} = 2$ (i.e. one of the lists $1, \ldots, 1$ is empty) are included.

If we consider such a term with $p'_i = 2$, we also know from statement b) and c) of Lemma 3.1.11, that its image under $\mathrm{d}_{\mathcal{N}}$ will only be of multidegree $(p_0, 1, \ldots, 1, q)$, meaning that the sum of only these components will yield a preimage of ω as well and we may without loss of generality assume that all other components of ν vanish. Furthermore, it follows from the proof of statement b) that the only non-vanishing term arises from $(d_i)^*$ for the multi-index with $p'_i = 2$.

If we consider the explicit formula, we further find that its image under $\mathrm{d}_{\mathcal{N}} \equiv (d_i)^*$ is antisymmetric in the i-th and $i+1$-st component. This is because on $(A_{\mathrm{iso}})^{\oplus q}$ the image under $(d_i)_*$ is invariant under the corresponding flip of the i-th and $i+1$-st component, however, the order of the components changes by a permutation of negative sign. Hence ω, as a sum of such terms vanishes under symmetrisation.

Now consider that we have any term ω that vanishes under symmetrisation. By Lemma 3.1.15, we obtain that ω is cohomologically equivalent to ω', its image under the flip of the i-th with the $i+1$-st component. As the symmetric group is generated by flips of adjacent elements, this means ω is cohomologically equivalent to all of its images under permutation and therefore also to its average, which is the symmetrisation of ω. But this was supposed to be 0, so ω must be exact. □

Lemma 3.1.19. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Consider the inclusion*

$$\iota : \left(\Gamma\left(G_0, \wedge^{p-q}(TG_0/\mathcal{F}_0)^* \otimes S^q(A^*_{\mathrm{iso}})\right)\right)^G \hookrightarrow \bigoplus_{p_0+p_1=p} \left(\Gamma(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \wedge^{p_1}\left((A^*_{\mathrm{iso}})^{\oplus q}\right))\right)^G$$

and the projection to the multidegree $(p_0, 1, 1, \ldots, 1, q)$ component followed by symmetrisation

$$\pi : \bigoplus_{p_0+p_1=p} \left(\Gamma(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \wedge^{p_1}\left((A^*_{\mathrm{iso}})^{\oplus q}\right))\right)^G \hookrightarrow \left(\Gamma\left(G_0, \wedge^{p-q}(TG_0/\mathcal{F}_0)^* \otimes S^q(A^*_{\mathrm{iso}})\right)\right)^G$$

where p is fixed and the differential is induced by the vertical differential. Then ι and π are maps of chain complexes corresponding to $(\mathcal{R}_E^{(p,\bullet)}, \mathrm{d}_{\mathcal{N}})$ and $((\Omega^p(G_\bullet))^{J^E G}, \mathrm{d}_{\mathcal{N}})$ that induce inverse maps on cohomology.

Proof. First let us note that the domain of ι is indeed a chain complex as the horizontal differential sends values corresponding to $J^E G$-isotropy Cartan forms to 0, which is in particular corresponding to 0 as an $J^E G$-isotropy Cartan form. With this it is also automatically clear that the inclusion is a map of chain complexes.

To see the same for π means to show that for any $\mathrm{d}_{\mathcal{N}}(\nu_{1_{G_0}}) = \omega_{1_{G_0}}$ of some G-invariant $\nu_{1_{G_0}}$ the symmetrisation of its component $\omega_{(r_0,1,1,\dots,1,q)}$ vanishes. This follows from Lemma 3.1.18, under the assumption that $\omega_{(r_0,1,1,\dots,1,q)}$ is exact. But by statement b) of Lemma 3.1.11 we have that restricting to the components of ν of multi-indices $(r_0, 1, \dots, 2, \dots, 1, q-1)$ constructs a preimage of $\omega_{(r_0,1,1,\dots,1,q)}$ under $\mathrm{d}_{\mathcal{N}}$.

As $\pi \circ \iota = \mathrm{id}$, we only need to show that $\iota \circ \pi$ induces an isomorphism in cohomology for the two maps to be inverse quasi-isomorphisms. As ι is just an inclusion, we need to show that any $\omega_{1_{G_0}}$ with $\mathrm{d}_{\mathcal{N}}(\omega_{1_{G_0}}) = 0$ is cohomologically equivalent to its image under π, i.e. its symmetrised component in multidegree $(p_0, 1, 1, \dots, 1, q)$.

For this, let us first note that any closed $\omega'_{1_{G_0}}$ for which the component in multidegree $(p_0, 1, 1, \dots, 1, q)$ vanishes is automatically exact by Lemma 3.1.14 and Lemma 3.1.17. As the differential of $\omega_{(p_0,1,\dots,1)}$ vanishes, by Corollary 3.1.12, we find that if $\omega_{1_{G_0}}$ is closed, so is

$$\omega'_{1_{G_0}} := \omega_{1_{G_0}} - \omega_{(p_0,1,\dots,1,q)}.$$

But now its projection to its degree $(p_0, 1, \dots, 1, q)$ component is 0, hence it is exact by the previous arguments. In particular, $\omega_{1_{G_0}}$ is cohomologically equivalent to

$$\omega_{1_{G_0}} - \omega'_{1_{G_0}} = \omega_{(p_0,1,\dots,1,q)}.$$

Let us now conclude from this that $\iota \circ \pi$ is a quasi-isomorphism. This means that we need to show that any closed $\omega_{1_{G_0}}$ is cohomologically equivalent to the symmetrisation of its component $\omega_{(p_0,1,\dots,1,q)}$. But using our previous findings it is now sufficient to show that

$$\omega_{(p_0,1,\dots,1,q)} - \mathrm{Sym}(\omega_{(p_0,1,\dots,1,q)})$$

is exact. But we know by Corollary 3.1.18 that any element in multidegree $(p_0, 1, \dots, 1, q)$ is exact if it vanishes under symmetrisation, and this element vanishes under symmetrisation by definition. So in conclusion, $\iota \circ \pi$ is indeed a quasi-isomorphism. □

Theorem 3.1.20. *Let G be a regular and proper Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. The inclusions*

$$\mathcal{R}_E^{(p,\bullet)} \subset (\Omega^p(G_\bullet))^{J^E G}$$

and

$$\mathcal{R}_E^{(p,\bullet)} \subset (\mathfrak{C}_p)^\bullet$$

are quasi-isomorphisms with respect to the respective restrictions of $\mathrm{d}_{\mathcal{N}}$*. Furthermore, any cohomology class of a column* $[\omega] \in H^q(\mathfrak{C}_p)$ *can be uniquely represented by a* J^EG*-isotropy Cartan form* $\omega \in \mathcal{R}_E^{(p,q)}$*, i.e. an* J^EG*-invariant form* $\omega \in (\Omega^p(G_q))^{J^EG}$ *satisfying*

$$\omega_{1_{G_0}} \in \left(\Gamma\left(G_0, \wedge^{p-q}(TG_0/\mathcal{F}_0)^* \otimes S^q(A_{\mathrm{iso}}^*)\right)\right)^G.$$

Proof. The inclusion ι of Lemma 3.1.19 is a quasi-isomorphism, which under the identification of Proposition 2.1.41 corresponds to the inclusion

$$\mathcal{R}_E^{(p,\bullet)} \subset (\Omega^p(G_\bullet))^{J^EG},$$

which is therefore also a quasi-isomorphism. So is the inclusion of the J^EG-invariant forms into $\mathfrak{C}_p$ by Theorem 2.3.13. But we have that $\mathrm{d}_{\mathcal{N}} \equiv 0$ on the J^EG-isotropy Cartan forms by Corollary 3.1.12. Therefore, each J^EG-isotropy Cartan form corresponds to a unique cohomology class in the cochain complex of J^EG-isotropy Cartan forms with fixed p, as well as in the cochain complex of J^EG-invariant forms with fixed p as well as $\mathfrak{C}_p$ itself. □

Corollary 3.1.21. *Let* G *be a regular and proper Lie groupoid together with a multiplicative Ehresmann connection* E *with respect to the isotropy algebroid. The map* π *of Lemma 3.1.19 under the isomorphism of Proposition 2.1.41 yields the construction of a quasi-inverse to the inclusion*

$$\mathcal{R}_E^{(p,\bullet)} \subset (\Omega^p(G_q))^{J^EG}.$$

Composing with the morphism of Lemma 2.3.14, we obtain a quasi-inverse of the inclusion

$$\mathcal{R}_E^{(p,\bullet)} \subset (\mathfrak{C}_p)^\bullet.$$

Proof. By Theorem 3.1.20 we have that the inclusion

$$\mathcal{R}_E^{(p,\bullet)} \subset (\Omega^p(G_q))^{J^EG}$$

is a quasi-isomorphism corresponding to the map ι of Lemma 3.1.19, which already states that π describes a quasi-inverse under the isomorphism of Proposition 2.1.41. Similarly, the map of Lemma 2.3.14 is a quasi-inverse to the inclusion

$$(\Omega^p(G_q))^{J^EG} \subset (\mathfrak{C}_p)^\bullet,$$

such that the composition of quasi-inverses is a quasi-inverse of the composition. □

We have, in this chapter, used the identification of Proposition 2.1.41 (i.e. of J^EG-invariant forms with their values at the units) quite frequently. However, while J^EG-invariance depends heavily on the choice of E, the expressions

$$\left(\Gamma(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \wedge^{p_1}\left((A_{\mathrm{iso}}^*)^{\oplus q}\right))\right)^G$$

does not. It is a priori not clear if and how the differential forms we obtain by extending the same value to a J^EG-invariant or J^FG-invariant form respectively are related. On J^EG-isotropy Cartan forms and J^FG-isotropy Cartan forms, the relation is quite strong as we are about to see.

Corollary 3.1.22. *Let G be a regular and proper Lie groupoid. Let E and F be two choices of multiplicative Ehresmann connections with respect to the isotropy algebroid. Then there is a canonical isomorphism of bigraded $C^\infty_{\text{bas}}(G_0)$-modules between the $J^E G$-isotropy Cartan forms*

$$\psi^{E,F} : \mathcal{R}_E \to \mathcal{R}_F$$

induced by the identity on $H^q(\mathfrak{C}_p)$ for any $p, q \in \mathbb{N}_0$.

Proof. We can include the $J^E G$-isotropy Cartan forms canonically into $\mathfrak{C}_p$ and we know that this induces an isomorphism of $C^\infty_{\text{bas}}(G_0)$-modules between $\mathcal{R}_E$ and $H(\mathfrak{C}_p)$. The same holds for the target. Note that as an isomorphism, the map is invertible.

So we can define $\psi^{E,F}$ by the composition of the two isomorphisms at $H(\mathfrak{C}_p)$. This means that we map any $J^E G$-isotropy Cartan form to the unique $J^F G$-isotropy Cartan form such that both are cohomologically equivalent. □

Lemma 3.1.23. *Let G be a regular and proper Lie groupoid and E and F two choices of a multiplicative Ehresmann connection with respect to the isotropy algebroid. The morphism of Corollary 3.1.22 under the identifications*

$$\mathcal{R}_E \cong \left(\Gamma\left(G_0, \wedge^{p-q}(TG_0/\mathcal{F}_0)^* \otimes S^q(A^*_{\text{iso}})\right)\right)^G$$

and

$$\mathcal{R}_F \cong \left(\Gamma\left(G_0, \wedge^{p-q}(TG_0/\mathcal{F}_0)^* \otimes S^q(A^*_{\text{iso}})\right)\right)^G$$

of Proposition 2.1.41 is given by the identity morphism.

Proof. Consider a $J^E G$-isotropy Cartan form ω. By definition, it is of E-vertical degree q. We can now decompose it into F-homogeneous components. They can be of F-vertical degree at most q, as the insertion of any choice of $q+1$ tangent vectors which are F-vertical and therefore E-vertical into ω must vanish by assumption. We claim that the component of F-vertical degree q, given by ω^q is a $J^F G$-invariant form. Let us first assume this was true and compute ω^q.

Its values at the unit section are given by the projection to the F-vertical degree q of $\omega|_{1_{G_0}}$ by definition. By construction, elements of $\wedge^{p-q}(TG_0/\mathcal{F}_0)$ define cotangent vectors by their values on the unit section (which is both E- and F-horizontal) that also vanish on A. So they correspond to F-horizontal forms of the same expression. The elements of A^*_{iso} correspond to E-vertical cotangent vectors which can be written as a sum of an F-vertical and an F-horizontal cotangent vector. However, the expression of the F-vertical cotangent vector as an element of A^*_{iso} only depends on the evaluation of the cotangent vector on A_{iso}. As the F-horizontal component vanishes on A_{iso} by definition, we have that the F-vertical component therefore corresponds to the same element of A^*_{iso} as the E-vertical cotangent vector does. We can now check that ω^q has to (for degree reasons) consist precisely of the product of the product of the F-horizontal cotangent vector corresponding to $\wedge^{p-q}(TG_0/\mathcal{F}_0)^*$ with the F-vertical components of the E-vertical cotangent vectors, each corresponding to an element in a copy of A^*_{iso}, but as we have seen, the expressions for these are all identical. So the resulting F-homogeneous form ω^q is not only $J^F G$-invariant, but even $J^F G$-isotropy Cartan, with the values at the unit section induced by the identity.

Now let us turn to the claim that the projection of ω to ω^q, its F-homogeneous component of F-vertical degree q, is $J^F G$-invariant. For this, we aim to compute the value of ω^q under

$$(\dot{D}^i_{V_x,q})^* : \wedge^p T^*_{D^i_{x,q}(z)} G_q \to \wedge^p T^*_z G_q,$$

where $V_x \in J^F G$. First of all, let us note that by Lemma 2.1.36, we have that

$$(\dot{D}^i_{V_x,q})^*(\omega) = \omega + \theta,$$

where $\theta \wedge^p T^*_z G_q$ is a term with components of E-vertical degree at most $q-1$, so as by our previous considerations also with components of F-vertical degree at most $q-1$. Furthermore, we have that

$$(\dot{D}^i_{V_x,q})^*(\omega^q)$$

is F-homogeneous of F-vertical degree q by Lemma 2.1.34, and for the F-homogeneous terms $\omega^{q-1}, \ldots, \omega^0$ the analogous is true. This means that the component of F-vertical degree q of $(\dot{D}^i_{V_x,q})^*(\omega)$ is simultaneously given by $(\dot{D}^i_{V_x,q})^*(\omega^q)$ and ω^q, meaning these agree and ω^q is indeed $J^F G$-invariant.

Lastly, it remains to note that by Corollary 3.1.17, any F-homogeneous component of vertical degree different from q is exact, meaning that the $J^E G$-isotropy Cartan form ω and the resulting projection ω^q are cohomologically equivalent. So indeed the map induced by identifying $J^E G$- and $J^F G$-isotropy Cartan forms with their unique cohomology classes maps ω to ω^q. □

Lastly, in this section, we will study the quasi-isomorphisms given by the inclusion of $J^E G$-isotropy Cartan forms either into the $J^E G$-invariant forms or the entire space of differential forms and their respective quasi-inverses more closely.

Lemma 3.1.24. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. There is a $C^\infty_{\mathrm{bas}}(G_0)$-linear assignment*

$$\omega \in (\Omega^p(G_q))^{J^E G} \text{ s.t. } \mathrm{d}_{\mathcal{N}}(\omega) = 0 \mapsto \sigma \in (\Omega^p(G_{q-1}))^{J^E G}$$

such that

$$\omega + \mathrm{d}_{\mathcal{N}}(\sigma)$$

is $J^E G$-isotropy Cartan. It can be chosen to retain the E-horizontal degree on homogeneous components.

Proof. We will follow the structure noted in the proof of Lemma 3.1.19. We will for this consider $\omega' \in (\Omega^p(G_q))^{J^E G}$ defined as

$$\omega' := \omega - \omega^{(q)}$$

where $\omega^{(q)}$ denotes the components of ω of E-vertical degree q. As in the case of Corollary 3.1.17 we can use the method outlined in Lemma 3.1.16 to construct σ' with $\omega' + \mathrm{d}_{\mathcal{N}}(\sigma') = 0$. The remaining term is given by $\omega - \omega' = \omega_q$, which is precisely the component of ω of E-vertical degree q. Here, we will now consider

$$\omega^{(q)} - \omega_{(p_0,1,\ldots,1,q)}$$

and use Lemma 3.1.14 to construct σ_q as a preimage of $\omega^{(q)} - \omega_{(p_0,1,\ldots,1,q)}$. Lastly, we will construct a preimage for

$$\omega_{(p_0,1,\ldots,1,q)} - \mathrm{Sym}(\omega_{(p_0,1,\ldots,1,q)})$$

according to Corollary 3.1.18, which we will denote by $\sigma_{(p_0,1,\ldots,1,q)}$.

All the statements referred to above show that respectively

$$\omega' + \mathrm{d}_{\mathcal{N}}(\sigma') = 0$$

and
$$\omega_q + \mathrm{d}_{\mathcal{N}}(\sigma_q) = 0$$
which are in particular $J^E G$-isotropy Cartan. They also show that
$$\omega_{(p_0,1,\dots,1,q)} + \mathrm{d}_{\mathcal{N}}(\sigma_{(p_0,1,\dots,1,q)}) = \mathrm{Sym}(\omega_{(p_0,1,\dots,1,q)}),$$
which is $J^E G$-isotropy Cartan by definition. Furthermore, the constructions in the statements all retain the E-horizontal degree of ω. Lastly, we can note that all constructions depend on ω $C^\infty_{\mathrm{bas}}(G_0)$-linearly, as they come from fibre-wise constructions on vector bundles over G_0. ☐

Corollary 3.1.25. *Let G be a regular and proper Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. There is a $C^\infty_{\mathrm{bas}}(G_0)$-linear assignment*
$$\omega \in \Omega^p(G_q) \ \textit{s.t.}\ \ \mathrm{d}_{\mathcal{N}}(\omega) = 0 \mapsto \nu \in \Omega^p(G_{q-1})$$
such that
$$\omega + \mathrm{d}_{\mathcal{N}}(\nu)$$
is $J^E G$-isotropy Cartan. It satisfies that
$$\omega + \mathrm{d}_{\mathcal{N}}(\nu) = 0$$
whenever ω is exact and it can be chosen to retain the E-horizontal degree on homogeneous components and such that $J^E G$-invariance of ω implies $J^E G$-invariance of ν.

Furthermore, if $G_0 = \mathrm{Sat}(R)$ for some admissible embedded manifold, we have that $\nu|_R$ only depends on $\omega|_R$.

Proof. In Lemma 2.3.11 we have already constructed an assignment
$$\omega \in \Omega^p(G_q) \mapsto \rho$$
such that if $\mathrm{d}_{\mathcal{N}}(\omega)$ is $J^E G$-invariant (in particular including the case where $\mathrm{d}_{\mathcal{N}}(\omega) = 0$), then also
$$\omega + \mathrm{d}_{\mathcal{N}}(\rho)$$
is $J^E G$-invariant. As this is still $\mathrm{d}_{\mathcal{N}}$-closed, we can now apply the assignment
$$\alpha \in (\Omega^p(G_q))^{J^E G} \ \text{s.t.}\ \ \mathrm{d}_{\mathcal{N}}(\alpha) = 0 \mapsto \sigma \in \Omega^p(G_{q-1})$$
of Lemma 3.1.24 to $\alpha = \omega + \mathrm{d}_{\mathcal{N}}(\rho)$. We have by construction, that
$$\alpha + \mathrm{d}_{\mathcal{N}}(\sigma) = \omega + \mathrm{d}_{\mathcal{N}}(\rho) + \mathrm{d}_{\mathcal{N}}(\sigma)$$
is $J^E G$-isotropy Cartan. As
$$[\omega] = [\omega + \mathrm{d}_{\mathcal{N}}(\rho) + \mathrm{d}_{\mathcal{N}}(\sigma)] \in H^q(\mathfrak{C}_p),$$
we have by Theorem 3.1.20 that if $[\omega] = 0$, then
$$\omega + \mathrm{d}_{\mathcal{N}}(\rho) + \mathrm{d}_{\mathcal{N}}(\sigma) = 0.$$

So, all in all, we have that
$$\nu = \rho + \sigma$$

has the desired properties. As the assignments described for

$$\omega \mapsto \rho$$

and

$$\omega \mapsto \alpha \mapsto \sigma$$

are both $C^\infty_{\mathrm{bas}}(G_0)$-linear and retain the E-horizontal degree of an element, the same is true for the assignment as a whole. Also, we know that if ω is $J^E G$-invariant, so is ρ and therefore also ν. The fact that $\rho|_R$ only depends on $\omega|_R$ is also stated in Lemma 2.3.11, the fact that $\sigma|_R$ only depends on $\alpha|_R$ follows immediately from the fact that, as already noted in the proof of Lemma 3.1.24, the construction of σ is obtained by fibre-wise constructions on vector bundles over G_0. □

Corollary 3.1.26. *Let G be a regular and proper Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Consider any $\mathrm{d}_{\mathcal{N}}$-closed form $\omega \in \Omega^p(G_q)$. Then ω is cohomologically equivalent to $\omega^{(q)}$, which denotes its component in E-vertical degree q.*

Proof. By Corollary 2.1.33, we have that $\mathrm{d}_{\mathcal{N}}$ retains the splitting of E, meaning that if ω is $\mathrm{d}_{\mathcal{N}}$-closed, so is $\omega^{(q)}$. We can therefore define

$$\omega' := \omega - \omega^{(q)}$$

and apply Corollary 3.1.25 to obtain ν such that

$$\omega' + \mathrm{d}_{\mathcal{N}}(\nu)$$

is $J^E G$-isotropy Cartan and vanishes in E-vertical degree q. However, $J^E G$-isotropy Cartan forms are only supported in E-vertical degree q, meaning that

$$\omega' + \mathrm{d}_{\mathcal{N}}(\nu) = 0$$

or equivalently

$$\omega + \mathrm{d}_{\mathcal{N}}(\nu) = \omega^{(q)},$$

meaning that the two elements are cohomologically equivalent. □

To conclude this section, we aim to compute the $J^E G$-isotropy Cartan forms in the examples we previously considered and with them the associated results for $H^\bullet(\mathfrak{C}_p)$.

Example 3.1.27. As we have seen in the pair groupoid of Example 2.3.15, the $J^E G$-invariant forms of the pair groupoid are already only given by the constants in bidegree $(0, 0)$, which are all $J^E G$-isotropy Cartan.

Example 3.1.28. When computing the cohomology associated to the free and proper action of a group $\mathcal{G}$ on a manifold M, we have seen in Example 2.3.16, that all $J^E G$-invariant forms of bidegree (p, q) are given by

$$(\Omega^p(G_q))^{J^E G} \cong \Omega^p(Q).$$

However, it is immediately clear that these forms are all of E-vertical degree 0 (as the dimension of the isotropy is 0 in this example). So these $J^E G$-invariant forms are only $J^E G$-isotropy Cartan for $q = 0$.

In particular, we have that

$$H^q(\mathfrak{C}_p) = \begin{cases} \Omega^p(Q) & \text{if } q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The analogue argument holds in the case of a proper and étale groupoid as in Example 2.3.18 or in the more general case of $A_{\mathrm{iso}} = 0$ of Example 2.3.19.

Example 3.1.29. As we have seen for the trivial action of a group $\mathcal{G}$ on a point of Example 2.3.20, the $J^E G$-invariant forms in bidegree (p, q) of the associated Bott–Shulman–Stasheff-complex are given at the unit by values of the form

$$\left(\wedge^p(\mathfrak{g}^*)^{\oplus q}\right)^{\mathcal{G}}.$$

However, the $J^E G$-isotropy Cartan forms are only given by

$$\left(S^q(\mathfrak{g}^*)\right)^{\mathcal{G}} \subset \left(\wedge^q(\mathfrak{g}^*)^{\oplus q}\right)^{\mathcal{G}}$$

in bidegrees for which $p = q$.

Example 3.1.30. Consider the Lie groupoid of Example 2.3.21. We have already seen that its $J^E G$-invariant forms of bidegree (p, q) correspond to the expression

$$\left(\Gamma\left(G_0, \left(\wedge^p\left((A^*_{\mathrm{iso}})^{\oplus q}\right)\right)^{\mathcal{G}}\right)\right)^G \cong \left(\wedge^p\left((\mathfrak{g}^*)^{\oplus q}\right)\right)^{\mathcal{G}},$$

meaning that the associated $J^E G$-isotropy Cartan forms are given by

$$\left(\Gamma\left(G_0, \left(S^p\left(A^*_{\mathrm{iso}}\right)\right)^{\mathcal{G}}\right)\right)^G \cong \left(S^p\left(\mathfrak{g}^*\right)\right)^{\mathcal{G}}.$$

We can see that this is independent from G_0 and also that in the case $G_0 = *$, this generalises Example 3.1.29.

Example 3.1.31. Consider the groupoid of Example 0.4.7. We have already seen in Example 2.3.22 that the $J^E G$-invariant forms in bidegree (p, q) in this example are given by

$$\bigoplus_{p_0+p_1=p} \left(\Gamma(\mathbb{S}^1, \wedge^{p_0}(T^*\mathbb{S}^1) \otimes \wedge^{p_1}((A^*)^{\oplus q}))\right),$$

i.e. the coadjoint action in this example is trivial. Therefore, we find that the $J^E G$-isotropy Cartan forms of bidegree (p, q) are given by

$$\left(\Gamma(\mathbb{S}^1, \wedge^{p-q}(T^*\mathbb{S}^1) \otimes S^q(A^*))\right).$$

Chapter 4

A conditional model

We now understand the cohomology of the columns of the Bott–Shulman–Stasheff bicomplex, but it remains to understand the cohomology of totalisation of the bicomplex. In this chapter, we will analyse the implications of our previous findings and results.

For this, note that we can interpret Bott–Shulman–Stasheff complex as a chain complex of chain complexes. This can happen in two ways: a chain complex of rows $(\Omega(G_q), \mathrm{d}_{\mathrm{dR}})$ with differential $\mathrm{d}_{\mathcal{N}}$ or a chain complex of columns $\mathfrak{C}_p$ with differential d_{dR}. (As the differentials graded commute instead of commuting, both $\mathrm{d}_{\mathcal{N}}$ and d_{dR} are not strictly speaking maps of chain complexes - but for motivational purposes we will not be strict about this here).

In a naive first attempt to compute the cohomology of this chain complex of chain complexes, we could try computing "the cohomology of the cohomologies": First replace each chain complex given by a column by its cohomology, i.e. the graded vector space of $J^E G$-isotropy Cartan forms of the respective column (technically those come with a differential, too, but as that vanishes, we will forget this datum). At the same time, we replace the de Rham differential by a cohomological version of it, too. By this we obtain a chain complex of graded vector spaces. We could then compute the cohomology of it and hope to recover the cohomology of the total bicomplex.

It will turn out that this approach actually works, but we will need not only this chapter, but also the next, to conclude the proof of that claim. In this chapter, the goal is to properly understand the objects appearing in this construction (in particular the cohomological version of the de Rham differential) and the relations between the total cohomology of the bicomplex and this "naive" approximation as explicitly as possible.

In the first section, we will study the de Rham differential in detail, as we have previously not concerned ourselves with its properties and its relation of $J^E G$-invariance and $J^E G$-isotropy Cartan forms. In the second section, we will explore conditions under which the "naive" approach is applicable to reconstruct the total cohomology as described above. It is relatively straightforward to see that this condition is satisfied when the multiplicative Ehresmann connection is flat, which is a case we will explore in more detail in the third section.

4.1 The de Rham differential

Aim of this section: Proposition 4.1.20 (abbreviated):

Let G be a regular and proper Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. The de Rham differential $\mathrm{d}_{\mathrm{dR}}(\omega)$ of a $J^E G$-isotropy Cartan form is cohomologically equivalent to its projection of E-vertical degree q, which is a $J^E G$-isotropy Cartan form that can be computed by a differential $\mathrm{d}^E_{\mathrm{Cart}}(\omega)$ on $\mathcal{R}_E$.

Let $\omega \in \Omega^p(G_q)$ represent a cohomology class of the p-th column, i.e. let it be closed under $\mathrm{d}_{\mathcal{N}}$. Then $\mathrm{d}_{\mathrm{dR}}(\omega)$ is also closed under $\mathrm{d}_{\mathcal{N}}$ and hence also represents a cohomology class, only this object is now a cohomology class of the $(p+1)$-st column. But how are these classes $[\omega] \in H^\bullet(\mathfrak{C}_p)$ and $[\mathrm{d}_{\mathrm{dR}}(\omega)] \in H^\bullet(\mathfrak{C}_{p+1})$ related? Computing the map $[\mathrm{d}_{\mathrm{dR}}]$, meaning the map induced by the de Rham differential on

$$H^\bullet(\mathfrak{C}_p) \to H^\bullet(\mathfrak{C}_{p+1})$$

will turn out to be a vital component for the methods used later in this thesis.

For this, we will study d_{dR} especially when evaluated on $J^E G$-isotropy Cartan forms, as we have established them previously as possible representatives to study the vertical cohomology classes. Unfortunately, we will find that the de Rham differential does not even retain the $J^E G$-invariance of a form. However, it is possible to decompose d_{dR} into three components. Of these, only one, denoted as $\mathrm{d}^{-1}_{\mathrm{dR}}$, does not retain $J^E G$-invariance, and a different one, denoted $\mathrm{d}^{\pm 0}_{\mathrm{dR}}$, determines the cohomology class of $\mathrm{d}_{\mathrm{dR}}(\omega)$. We are able to compute this component quite explicitly.

During this process of studying the result of a $J^E G$-isotropy Cartan form under d_{dR}, we will also obtain additional results on the de Rham differential, which we can subsequently use to our advantage. Nonetheless, this section contains many technical results that are only needed to arrive at the result referenced at the beginning.

Lemma 4.1.1. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. For any $\omega \in \Omega^p(G_q)$ of E-vertical degree p_1 its de Rham differential $\mathrm{d}_{\mathrm{dR}}(\omega)$ only has components in E-vertical degree $p_1 - 1$, p_1 and $p_1 + 1$.*

To be precise, when evaluated on E-horizontal and E-vertical vector fields, denoted by $X_1, \dots, X_k$ and $Y_1, \dots, Y_l$ respectively, the component $\mathrm{d}^{-1}_{\mathrm{dR}}(\omega)$ of $\mathrm{d}_{\mathrm{dR}}(\omega)$ in E-vertical degree $p_1 - 1$ is given by

$$\begin{aligned}&\left(\mathrm{d}^{-1}_{\mathrm{dR}}(\omega)\right)(X_1, \dots, X_{p_0+2}, Y_1, \dots Y_{p_1-1})\\ := &\sum_{i=1}^{p_0+2} \sum_{j=i+1}^{p_0+2} (-1)^{i+j} \omega([X_i, X_j]_{E-\mathrm{vert}}, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots).\end{aligned}$$

the component $\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega)$ *of* $\mathrm{d}_{\mathrm{dR}}(\omega)$ *in* E*-vertical degree* p_1 *is given by*

$$\begin{aligned}
&\left(\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega)\right)(X_1,\dots,X_{p_0+1},Y_1,\dots Y_{p_1})\\
:=&\sum_{i=1}^{p_0+1}(-1)^{i+1}X_i(\omega(\dots,\widehat{X_i},\dots))\\
&+\sum_{i=1}^{p_0+1}\sum_{j=i+1}^{p_0+1}(-1)^{i+j}\omega([X_i,X_j]_{E-\mathrm{hor}},\dots,\widehat{X_i},\dots,\widehat{X_j},\dots)\\
&+\sum_{i=1}^{p_0+1}\sum_{j=1}^{p_1}(-1)^{i+j+p_0+1}\omega([X_i,Y_j]_{E-\mathrm{vert}},\dots,\widehat{X_i},\dots,\widehat{Y_j},\dots).
\end{aligned}$$

and the component $\mathrm{d}_{\mathrm{dR}}^{+1}(\omega)$ *of* $\mathrm{d}_{\mathrm{dR}}(\omega)$ *in* E*-vertical degree* p_1+1 *is given by*

$$\begin{aligned}
&\left(\mathrm{d}_{\mathrm{dR}}^{+1}(\omega)\right)(X_1,\dots,X_{p_0},Y_1,\dots Y_{p_1+1})\\
:=&\sum_{i=1}^{p_1+1}(-1)^{p_0+i+1}Y_i(\omega(\dots,\widehat{Y_i},\dots))\\
&+\sum_{i=1}^{p_0}\sum_{j=1}^{p_1+1}(-1)^{i+j+k}\omega([X_i,Y_j]_{E-\mathrm{hor}},\dots,\widehat{X_i},\dots,\widehat{Y_j},\dots)\\
&+\sum_{i=1}^{p_1+1}\sum_{j=i+1}^{p_1+1}(-1)^{i+j}\omega([Y_i,Y_j],\dots,\widehat{Y_i},\dots,\widehat{Y_j},\dots).
\end{aligned}$$

Proof. We can choose k vector fields $X_1,\dots,X_k$ defined around $z\in G_q$ which are E-horizontal and l vectors fields $Y_1,\dots,Y_l$ defined around the same point which are E-vertical such that $k+l=p+1$. Now we can evaluate

$$\left(\mathrm{d}_{\mathrm{dR}}(\omega)\right)_z(X_1,\dots,X_k,Y_1,\dots Y_l),$$

where our aim is to show that the result vanishes unless l takes one of the distinguished values.

We have by definition

$$\begin{aligned}
&\left(\mathrm{d}_{\mathrm{dR}}(\omega)\right)(X_1,\dots,X_k,Y_1,\dots Y_l)\\
=&\sum_{i=1}^{k}(-1)^{i+1}X_i(\omega(\dots,\widehat{X_i},\dots))\\
&+\sum_{i=1}^{l}(-1)^{k+i+1}Y_i(\omega(\dots,\widehat{Y_i},\dots))\\
&+\sum_{i=1}^{k}\sum_{j=i+1}^{k}(-1)^{i+j}\omega([X_i,X_j],\dots,\widehat{X_i},\dots,\widehat{X_j},\dots)\\
&+\sum_{i=1}^{k}\sum_{j=1}^{l}(-1)^{i+j+k}\omega([X_i,Y_j],\dots,\widehat{X_i},\dots,\widehat{Y_j},\dots)\\
&+\sum_{i=1}^{l}\sum_{j=i+1}^{l}(-1)^{i+j}\omega([Y_i,Y_j],\dots,\widehat{Y_i},\dots,\widehat{Y_j},\dots).
\end{aligned}$$

We can analyse these terms one by one.

For degree reasons, we have

$$X_i(\omega(\dots,\widehat{X_i},\dots)) = X_i(0) = 0$$

unless $l = p_1$. Similarly,

$$Y_i(\omega(\dots,\widehat{Y_i},\dots))) = Y_i(0) = 0$$

unless $l = p_1+1$. As both terms of the form $[X_i, X_j]$ and $[X_i, Y_j]$ can be mixed (i.e. neither purely horizontal nor purely vertical), we have that they decompose into an E-vertical and E-horizontal component each, which we will denote by $[-,-]_{E-\mathrm{vert}}$ and $[-,-]_{E-\mathrm{hor}}$ respectively.

We have

$$\sum_{i=1}^{k}(-1)^{i+j}\omega([X_i,X_j],\dots,\widehat{X_i},\dots,\widehat{X_j},\dots) = 0$$

unless either $l = p_1 - 1$ or $l = p_1$, while

$$\sum_{i=1}^{k}(-1)^{i+j+k}\omega([X_i,Y_j],\dots,\widehat{X_i},\dots,\widehat{Y_j},\dots) = 0$$

unless either $l = p_1$ or $l = p_1 + 1$. The remaining term satisfies that

$$\sum_{i=1}^{k}(-1)^{i+j}\omega([Y_i,Y_j],\dots,\widehat{Y_i},\dots,\widehat{Y_j},\dots) = 0$$

unless $l = p_1 + 1$, as we have that the vertical bundle is involutive as the intersection of kernels of tangent maps (as those are always involutive). □

Lemma 4.1.2. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. For any $\omega \in \Omega^p(G_q)$ we have that*

$$\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\mathrm{d}_{\mathrm{dR}}^{-1}(\omega)) = -\,\mathrm{d}_{\mathrm{dR}}^{-1}(\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega))$$

as well as

$$\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\mathrm{d}_{\mathrm{dR}}^{+1}(\omega)) = -\,\mathrm{d}_{\mathrm{dR}}^{+1}(\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega)).$$

Proof. As the de Rham differential is linear, we may assume that ω is E-homogeneous of E-vertical degree p_1. Then we have

$$0 = \mathrm{d}_{\mathrm{dR}}(\mathrm{d}_{\mathrm{dR}}(\omega)) = (\mathrm{d}_{\mathrm{dR}}^{-1} + \mathrm{d}_{\mathrm{dR}}^{\pm 0} + \mathrm{d}_{\mathrm{dR}}^{+1})((\mathrm{d}_{\mathrm{dR}}^{-1} + \mathrm{d}_{\mathrm{dR}}^{\pm 0} + \mathrm{d}_{\mathrm{dR}}^{+1})(\omega))$$

where the right-hand side expression decomposes into its component in E-vertical degree $p-1$ given by

$$\mathrm{d}_{\mathrm{dR}}^{-1}(\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega)) + \mathrm{d}_{\mathrm{dR}}^{\pm 0}(\mathrm{d}_{\mathrm{dR}}^{-1}(\omega)),$$

its component in E-vertical degree p given by

$$\mathrm{d}_{\mathrm{dR}}^{-1}(\mathrm{d}_{\mathrm{dR}}^{+1}(\omega)) + \mathrm{d}_{\mathrm{dR}}^{\pm 0}(\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega)) + \mathrm{d}_{\mathrm{dR}}^{+1}(\mathrm{d}_{\mathrm{dR}}^{-1}(\omega)),$$

and its component in E-vertical degree $p+1$ given by

$$\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\mathrm{d}_{\mathrm{dR}}^{+1}(\omega)) + \mathrm{d}_{\mathrm{dR}}^{+1}(\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega)).$$

The vanishing of the first and last expressions imply the statement of the lemma. □

We will now analyse these components. For two of them, we will find that they behave quite well with respect to the notion of $J^E G$-invariance.

Lemma 4.1.3. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Let $\omega \in \Omega^p(G_q)$ be of E-horizontal degree p_1 and consider $z \in G_q$ and γ a local s-section E-horizontal at $p^k_q(z)$.*

If ρ is the component of $(\widetilde{D}^k_{\gamma,q})^*\omega$ in E-vertical degree p_1, then

$$\left((\widetilde{D}^k_{\gamma,q})^*(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega))\right)_z - \left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\rho)\right)_z = 0$$

and

$$\left((\widetilde{D}^k_{\gamma,q})^*(\mathrm{d}^{+1}_{\mathrm{dR}}(\omega))\right)_z - \left(\mathrm{d}^{+1}_{\mathrm{dR}}(\rho)\right)_z = 0.$$

Proof. We have that on the compatible domain there holds that

$$(\widetilde{D}^k_{\gamma,q})^*(\mathrm{d}_{\mathrm{dR}}(\omega)) = \mathrm{d}_{\mathrm{dR}}((\widetilde{D}^k_{\gamma,q})^*(\omega)).$$

If we evaluate this at z and decompose into its components of homogeneous E-vertical and E-horizontal degree, we know by Lemma 4.1.1 that the left hand side decomposes into

$$(\widetilde{D}^k_{\gamma,q})^*(\mathrm{d}^{-1}_{\mathrm{dR}}(\omega))$$

in degree $p-1$ and

$$(\widetilde{D}^k_{\gamma,q})^*(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega))$$

in degree p and

$$(\widetilde{D}^k_{\gamma,q})^*(\mathrm{d}^{+1}_{\mathrm{dR}}(\omega))$$

in degree $p+1$. It remains to be seen what happens on the left hand side.

First of all, by Lemma 2.1.36, if we decompose

$$(\widetilde{D}^k_{\gamma,q})^*(\omega) = \rho^0 + \rho^1 + \dots$$

according to E-vertical degree, we find that the sum terminates with the terms ρ^{p_1-1} and ρ^{p_1}. We will also denote the latter as $\rho = \rho^{p_1}$ as defined in the statement.

This means that by Lemma 4.1.1, we have

$$\mathrm{d}_{\mathrm{dR}}((\widetilde{D}^k_{\gamma,q})^*(\omega))$$

only has terms of E-vertical degree up to p_1+1, in which it is given by

$$\mathrm{d}^{+1}_{\mathrm{dR}}(\rho)$$

and in degree p_1 it is given by

$$\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\rho) + \mathrm{d}^{+1}_{\mathrm{dR}}(\rho^{p_1-1}).$$

So for degree p_1+1 we have automatically obtained the statement of the lemma by comparing degree-wise expressions. For degree p_1, we can explicitly compute the following

$$\begin{aligned}
&\left(\mathrm{d}^{+1}_{\mathrm{dR}}(\rho^{(p_1-1)})\right)(X_1,\dots,X_{p_0+1},Y_1,\dots Y_{p_1}) \\
=&\sum_{i=1}^{p_1}(-1)^{p_0+i}Y_i(\rho^{(p_1-1)}(\dots,\widehat{Y_i},\dots)) \\
&+\sum_{i=1}^{p_0+1}\sum_{j=1}^{p_1}(-1)^{i+j+p_0+1}\rho^{p_1-1}([X_i,Y_j]_{E-\mathrm{hor}},\dots,\widehat{X_i},\dots,\widehat{Y_j},\dots) \\
&+\sum_{i=1}^{p_1}\sum_{j=i+1}^{p_1}(-1)^{i+j}\rho^{p_1-1}([Y_i,Y_j],\dots,\widehat{Y_i},\dots,\widehat{Y_j},\dots).
\end{aligned}$$

We now aim to show that this expression vanishes at z. To obtain this, note that ρ^{p_1-1} vanishes at all z' where γ is E-horizontal at $p_q^k(z')$ as under that condition the degree of a form is preserved by $(\widetilde{D}_{\gamma,q}^k)^*$. This is in particular the case for z, meaning the two latter expressions arising in the sum vanish at z. However, it also holds for the entire fibre $(p_q^k)^{-1}(\{m\})$ for $m = p_q^k(z)$. Any Y_i is by definition tangent to that fibre, meaning that in the remaining first term taking the derivative of at z in the direction of Y_i is a derivative of the 0 function.

Hence all components of the previous sum vanish and we have found that

$$\mathrm{d}_{\mathrm{dR}}^{+1}(\rho^{p_1-1})_z = 0$$

implying that when we compare degree-wise, we have

$$(\widetilde{D}_{\gamma,q}^k)^*(\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega)) \equiv \mathrm{d}_{\mathrm{dR}}^{\pm 0}(\rho) + \mathrm{d}_{\mathrm{dR}}^{+1}(\rho^{p_1-1}) \equiv \mathrm{d}_{\mathrm{dR}}^{\pm 0}(\rho)$$

at z.

□

Proposition 4.1.4. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. For any ω which is $J^E G$-invariant of E-vertical degree p_1, the components $\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega)$ of E-vertical degree p_1 and $\mathrm{d}_{\mathrm{dR}}^{+1}(\omega)$ of E-vertical degree $p_1 + 1$ are $J^E G$-invariant.*

Proof. By Corollary 2.1.37, we have that for any $\gamma : U \subset G_0 \to G_1$ a local section of s that

$$(\widetilde{D}_{\gamma,q}^i)^*(\omega)$$

agrees with ω in E-horizontal degree p_0 (i.e. E-vertical degree p_1) on its domain $(p_q^i)^{-1}(U)$, meaning that when we apply Lemma 4.1.3 (i.e. when we have $z \in G_q$ and γ horizontal at $p_q^i(z)$), we have $\rho = \omega$.

In particular, we have that at any $z \in G_q$ and for any local bisection γ which is E-horizontal at $p_q^i(z)$ the relations

$$\Big((\widetilde{D}_{\gamma,q}^i)^*(\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega))\Big)_z = \big(\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega)\big)_z$$

and

$$\Big((\widetilde{D}_{\gamma,q}^i)^*(\mathrm{d}_{\mathrm{dR}}^{+1}(\omega))\Big)_z = \big(\mathrm{d}_{\mathrm{dR}}^{+1}(\omega)\big)_z$$

hold, which by Corollary 2.1.29 is equivalent to $\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega)$ and $\mathrm{d}_{\mathrm{dR}}^{+1}(\omega)$ being $J^E G$-invariant.

□

Our aim in the following will be to compute $\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega)$ explicitly for any $J^E G$-invariant ω. In the computation, a certain expression will appear that we can relate to a connection ∇^E on the isotropy Lie algebroid. We will first define what this connection is and then establish how it relates to our problem at hand.

Lemma 4.1.5. *Let G be a regular Lie groupoid. For every multiplicative Ehresmann connection E, there is a unique vector bundle connection ∇^E on A_{iso}, to which we will refer to as the isotropy reduction of E, that infinitesimally corresponds to the restriction of E to the connected component of the unit in the isotropy subgroupoid $\mathrm{Iso}^0(G)$.*

Furthermore, we can consider the exponential map

$$\exp : A_{\mathrm{iso}} \to (\mathrm{Iso}^0(G))_1.$$

On open subdomains containing the zero section of A_{iso} such that the restriction of any fibre of $A_{\mathrm{iso}} \to G_0$ is convex and exp *is a diffeomorphism onto its image, the pull-back of E as a splitting of $T(\mathrm{Iso}^0(G))_1$ agrees with the linear Ehresmann connection of the vector bundle A_{iso} corresponding to ∇^E.*

Proof. According to Lemma 1.1.15, the connected component of the the isotropy subgroupoid containing the unit, denoted by $\mathrm{Iso}^0(G)$, is a Lie groupoid. The splitting of TG_1 restricts to a splitting of $T\left(\mathrm{Iso}^0(G)\right)_1$, which is well-defined as $IG_1 \subset T\left(\mathrm{Iso}^0(G)\right)_1$. As the multiplication is inherited, the restriction is a multiplicative Ehresmann connection on $\mathrm{Iso}^0(G)$.

However, as $s = t$ on $\mathrm{Iso}^0(G)$, the multiplicative Ehresmann connection is now connection with respect to s, i.e. a Cartan connection. It can be differentiated to a unique connection on the algebroid of $\mathrm{Iso}^0(G)$, which is precisely the isotropy Lie algebroid A_{iso}. This works as follows (cf. [4], p. 3130): Consider the base-point map $(J^E\,\mathrm{Iso}^0(G))_1 \to (\mathrm{Iso}^0(G))_1$. As for any two jets at base-point $x \in (\mathrm{Iso}^0(G))_1$, the preimages of some $X \in T_{s(x)}G_0$ can only differ by elements in $IG_q \cap E$ (as this agrees with $\ker(s) \cap E$), the map is actually injective and even an immersion. From Lemma 1.2.8 and the projection of Lemma 2.1.9 (also explicitly described in Lemma 2.1.36), we can obtain surjectivity of the map. This means that $(J^E\,\mathrm{Iso}^0(G))_1 \to (\mathrm{Iso}^0(G))_1$ is a diffeomorphism and admits a smooth inverse. As the map was obtained as part of a Lie groupoid morphism covering id_{G_0} it furthermore defines a Lie groupoid isomorphism. The inverse $(\mathrm{Iso}^0(G))_1 \to (J^E\,\mathrm{Iso}^0(G))_1$ also defines a Lie groupoid isomorphism covering id_{G_0}. It can be composed with the inclusion of $(J^E\,\mathrm{Iso}^0(G))_1 \subset (J^1\,\mathrm{Iso}^0(G))_1$ and can then be derived to the corresponding Lie algebroids of $\mathrm{Iso}^0(G)$ and $J^1\,\mathrm{Iso}^0(G)$, which can be seen to be $e : A_{\mathrm{iso}} \to J^1A_{\mathrm{iso}}$, where J^1A_{iso} denotes the jets of $A_{\mathrm{iso}} \to G_0$. Any linear map of this form defines a connection by

$$\nabla^E_\sigma := e - j^1(\sigma),$$

where j^1 denotes the first jet prolongation.

As $\rho = 0$ on A_{iso}, we have that the exponential map is canonically defined on all of A_{iso}. Also, on some open set around the 0-section of A_{iso} it maps diffeomorphically to an open around the unit section in $(\mathrm{Iso}^0(G))_1$.

We have that by definition, pulling E back along the diffeomorphism will return a connection as a bundle (not a vector bundle) on the corresponding domain in A_{iso} such that ∇^E is its linearisation at the 0-section. That the linearisation actually recovers ∇^E comes from the fact that the inclusion $A \subset TG_1$ under the inverse of the exponential map corresponds to $A \subset TA$, where each element of A precisely corresponds to the vertical (with respect to the bundle structure) tangent vector at the zero section. Therefore, the map $G_1 \to (J^1G)_1$ mentioned previously induces a map $A_{\mathrm{iso}} \to J^1A_{\mathrm{iso}}$ for which the derivative at the 0 section restricts to ∇^E on $A \subset TA$ as the vertical tangent bundle at 0. This is precisely the construction referred to as linearisation.

However, as the connection on $\mathrm{Iso}^0(G)$ is multiplicative, we have that the pull-back of E is already linear: As the anchor vanishes, we have that

$$(\exp(X))^n$$

is well-defined and agrees with $\exp(n \cdot X)$ for any $X \in A_{\mathrm{iso}}$ and any fixed $n \in \mathbb{N}_{\geq 1}$ and the same is true for all values in an open neighbourhood of X.

Assume that both X and $n \cdot X$ are contained in the convex domain chosen in the statement to pull back E. By multiplicativity of E, we have that any horizontal lift at $n \cdot X$ is given by the n-fold product of the horizontal lift at X. But as this is the image under taking the n-th power (again, this only holds as $\rho = 0$) which corresponds to linear scaling, we find that the pull-back of E is linear with respect to scaling by $n \in \mathbb{N}_{\geq 1}$, at least whenever X and $n \cdot X$ are contained in the domain on which the pull-back of E is defined.

Linearity with respect to $\lambda \in \mathbb{R}$ follows by the reverse argument for fractions (for $\lambda \in \mathbb{Q}$) and continuity. Here, we need to use that the fibres are connected. Note that the convexity of the fibres of the domain is a convenient choice as it is connected and closed under rescaling with $\lambda \leq 1$. Linearity with respect to addition follows, as multiplication is infinitesimally approximated by addition on A_{iso}. □

Lemma 4.1.6. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. For an E-horizontal vector field X on G_1 tangent to the identity section at units and a E-vertical vector field Y on G_1, the expression $[X,Y]_{E-\mathrm{vert}}$ at 1_m is given by the covariant derivative*

$$\left(\nabla^E\right)_{X|_{G_0}} Y|_{G_0}(m),$$

where ∇^E is defined as in Lemma 4.1.5.

For an E-horizontal vector field X on G_q tangent to the identity section at units and a E-vertical vector field Y on G_q, the expression $[X,Y]_{E-\mathrm{vert}}$ at 1_m is given by the analogue expression for the induced connection on $(A_{\mathrm{iso}})^{\oplus q}$ given by the direct sum of ∇^E on each component.

Proof. Let us first consider the situation on G_1. As we have

$$[f \cdot X, Y] = f \cdot [X,Y] - Y(f) \cdot X$$

and X is horizontal, we have that $[X,Y]_{E-\mathrm{vert}}$ is $C^\infty(G_1)$-linear in the first component and hence its value at 1_m only depends on the value of X at 1_m. Hence we can compute the expression while choosing X in a way that is convenient for us. For example, we may assume that on $(\mathrm{Iso}^0(G))_1$, which in particular contains the unit section, X is tangent to $(\mathrm{Iso}^0(G))_1 \subset G_1$. As any vertical tangent vector has this property automatically, this in particular proves that $[X,Y]$ is tangent to $(\mathrm{Iso}^0(G))_1$, and it further allows us to assume $G = \mathrm{Iso}^0(G)$ for the computation.

For Y, we have that the expression at 1_m only depends on the values of Y at the unit section, as

$$[X, f \cdot Y] = f \cdot [X,Y] + X(f) \cdot Y,$$

which vanishes if $f = 0$ at the unit section. Hence we can also assume any form of Y (and even that Y is only locally defined around the unit section) for our computation, as long as we retain the values along the unit section.

Choose a convex domain where the exponential map (which is universally defined on the isotropy subalgebroid) is a diffeomorphism. Then the form we are going to assume for Y is the following: Consider the pull-back $\exp^* Y$ under the exponential map, which defines a vector field on A_{iso}, such that the values at the zero section correspond to the values at the groupoid units. We will assume Y to be of the form where it corresponds to a fibre-wise constant vector field under the exponential pull-back on this domain.

As by Lemma 4.1.5 we have found ∇^E to satisfy that horizontal vectors get mapped to horizontal vectors under the exponential map, we can choose the precise form of X (locally on the same domain around the unit section) such that the corresponding vector field $\exp^* X$ is at each point of the domain given by the lift of the vector field $X|_{G_0}$ on G_0 according to the connection ∇^E.

But now, we can compute $[X,Y]_{E-\mathrm{vert}}$ by instead computing its pull-back along exp. As it is a vector at the unit section tangent to $(\mathrm{Iso}^0(G))_1$ the splitting of E (which is actually the canonical splitting into TG_0 and A_{iso}) precisely corresponds to the canonical splitting of TA_{iso} at the zero section. We will actually see that by our choice of X and Y, we already have that $\exp^*[X,Y] = [\exp^* X, \exp^* Y]$ is vertical. To simplify notation in the following, we will denote $\exp^* X = X'$ and analogously for Y'.

We know that by construction of X that X' satisfies precisely that the flow line $\gamma_x = \phi_t^{X'}(x)$ integrating X' starting at any point x in the fibre over m is given by horizontal path-lifting of the flow-line $\gamma_m = \phi_t^{X'}(m)$ integrating X' on the zero section of the algebroid with $\gamma_m(0) = m$ (the correct notation would be γ_{0_m}, but we will abbreviate for notational purposes). We furthermore know that by construction, Y corresponds to a Y' that integrates to the flow that shifts the fibres at any $n \in G_0$ linearly by $t \cdot Y'_n$ (where t is the flow parameter).

So if we want compute the preimage of $[X,Y]_{E-\mathrm{vert}}$ under the exponential map diffeomorphism, we can consider the concatenation of the two flows. We obtain that

$$\phi_t^{X'}(\phi_s^{Y'}(m)) = \gamma_{s\cdot Y'_m}(t)$$

which is the horizontal lift of γ_m at $s \cdot Y'_m$, while

$$\phi_s^{Y'}(\phi_t^{X'}(m)) = s \cdot Y'_{(\gamma_m(t))}.$$

If we let t go to 0, we obtain two tangent vectors at $s \cdot Y(m)$, of which the first one is horizontal. Hence

$$\frac{\partial}{\partial t}\phi_s^{Y}(\phi_t^{X}(m)) - \phi_t^{X}(\phi_s^{Y}(m))|_{t=0} = v_E(\frac{\partial}{\partial t}s \cdot Y'_{(\gamma_m(t))}|_{t=0}),$$

where v_E will be the map to the vertical component of $\frac{\partial}{\partial t}s \cdot Y'_{(\gamma_m(t))}|_{t=0}$ with respect to ∇^E. By the linearity of the expression in s, we find

$$v_E(\frac{\partial}{\partial t}s \cdot Y'_{(\gamma_m(t))}|_{t=0}) = v_E(s \cdot \frac{\partial}{\partial t}Y'_{(\gamma_m(t))}|_{t=0}).$$

But ∇^E also has linearity properties, such that

$$v_E(s \cdot \frac{\partial}{\partial t}Y'_{(\gamma_m(t))}|_{t=0}) = s \cdot v_E(\frac{\partial}{\partial t}Y'_{(\gamma_m(t))}|_{t=0}),$$

where the vectors are identified by translation. So

$$\frac{\partial}{\partial s}s \cdot v_E(\frac{\partial}{\partial t}Y'_{(\gamma_m(t))}|_{t=0})|_{s=0} \cong v_E(\frac{\partial}{\partial t}Y'_{(\gamma_m(t))}|_{t=0}) \cong \nabla^E_{X'|_0}Y'|_0 = \nabla^E_{X|_{1_{G_0}}}Y|_{1_{G_0}}.$$

In this chain of identifications we use that the vertical tangent bundle at the zero section of a vector bundle is isomorphic to the vector bundle itself. The second identification is precisely the relation between a connection as a horizontal subspace and a covariant derivative. We actually see in this computation that $[X',Y']$ itself is already a vertical tangent vector. As the exponential map on vertical tangent vectors at the zero section corresponds to the identity on A, we find that indeed

$$[X,Y]_{E-\mathrm{vert}} = \nabla^E_{X|_{1_{G_0}}}Y|_{1_{G_0}}.$$

Now considering a E-vertical vector field on G_q, we can use $\mathbb{R}$-linearity in the second component of $[X,Y]$ to split Y into q vector fields Y^i for $1 \leq i \leq q$ according to the splitting of the isotropy vector bundle of G_q into the q copies of the isotropy vector bundle of G_1.

However, we can consider the embeddings ϵ_q^i of G_1 into G_q for $1 \leq i \leq 1$ given by

$$z \mapsto (1_{t(z)}, \ldots, 1_{t(z)}, z, 1_{s(z)}, \ldots, 1_{s(z)})$$

where the image of the vector bundle IG_1 under the tangent map at any point agrees exactly with the vector bundle induced by A_{iso} supported in the respective i-th component of $(A_{\text{iso}})^{\oplus q}$.

Hence any flow starting at an element of ϵ_q^i along vectors corresponding to elements of the i-th component of $(A_{\text{iso}})^{\oplus q}$ will be contained in the image of ϵ_q^i. But by definition, the vector field Y^i only consists of such vectors, hence the flow induced by Y^i must retain ϵ_q^i.

By the same linearity argument as before we have that $[X, Y]_{E-\text{vert}}$ at $1_m \in G_q$ only depends on X_{1_m}. Hence we may assume any form of X outside of the point 1_m. As X_{1_m} is tangent to the identity section, which is in the image of $\epsilon^i(\text{Iso}^0(G_1))$ (precisely the image of the units), we can assume that similarly, X is tangent to $\epsilon^i(\text{Iso}^0(G_1))$ at any point of the image. This would imply that the flow of X retains $\epsilon^i(\text{Iso}^0(G_1))$ as well.

Hence we can conclude that we can compute $[X, Y^i]_{E-\text{vert}}$ at 1_m on the image of ϵ_q^i, i.e. on G_1. This also implies that the result is contained in the i-th copy of the isotropy vector bundle (as it is the space tangent to the image of ϵ_q^i).

But by our previous findings, $[X, Y^i]_{E-\text{vert}}$ at 1_m on G_1 corresponds to

$$\nabla^E_{X|_{1_{G_0}}}(Y^i|_{1_{G_0}}).$$

So $[X, Y^i]_{E-\text{vert}}$ is given by a tangent vector with its components either given by 0 or $\nabla^E_{X|_{1_{G_0}}}(Y^i|_{1_{G_0}})$. This means $[X, Y]_{E-\text{vert}}$ is given by a tangent vector with its j-th component corresponding to $\nabla^E_{X|_{1_{G_0}}}(Y^j|_{1_{G_0}})$ for each $1 \leq j \leq q$. So indeed

$$[X, Y]_{E-\text{vert}} = \nabla'_{X|_{1_{G_0}}}(Y|_{1_{G_0}})$$

for ∇' the direct sum of the connections. □

At this point, we can recall the earlier result on how to compute $\mathrm{d}^{\pm 0}_{\mathrm{dR}}$, which had the following expression.

$$\begin{aligned}
&\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)\right)(X_1, \ldots, X_{p_0+1}, Y_1, \ldots Y_{p_1}) \\
:= &\sum_{i=1}^{p_0+1} (-1)^{i+1} X_i(\omega(\ldots, \widehat{X_i}, \ldots)) \\
&+ \sum_{i=1}^{p_0+1} \sum_{j=i+1}^{p_0+1} (-1)^{i+j} \omega([X_i, X_j]_{E-\text{hor}}, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots) \\
&+ \sum_{i=1}^{p_0+1} \sum_{j=1}^{p_1} (-1)^{i+j+p_0+1} \omega([X_i, Y_j]_{E-\text{vert}}, \ldots, \widehat{X_i}, \ldots, \widehat{Y_j}, \ldots).
\end{aligned}$$

We can now replace the bracket in the last expression by the covariant derivative, as long as we aim to compute it at a unit and the X_i satisfies the requirements needed.

Example 4.1.7. Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Let ω be a form of E-horizontal degree $p_0 = 0$ and E-vertical degree $p_1 = 1$. Let X be some E-horizontal vector field tangent to the unit section when restricted 1_{G_0} and Y some vertical vector field.

Then the expression of $\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)(X,Y)$ at the unit section simplifies to

$$\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)\right)(X,Y)|_{1_{G_0}} \equiv X|_{1_{G_0}}(\omega|_{1_{G_0}}(Y|_{1_{G_0}})) - \omega|_{1_{G_0}}(\nabla_{X|_{1_{G_0}}}(Y|_{1_{G_0}}))$$

where ∇ is the direct q-fold sum of the connection ∇^E.

This previous result is especially interesting, when we assume ω to be $J^E G$-invariant. In that case, we know that $\mathrm{d}^{\pm 0}_{\mathrm{dR}}$ is $J^E G$-invariant as well - and hence the values at the unit section are sufficient to describe both ω and $\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)$ in their entirety. We may also note that when considering values along the unit section as elements in $\Gamma(G_0, A^*_{\mathrm{iso}})$, we obtain the term defining ∇^*, a covariant derivative on $\Gamma(G_0, A^*_{\mathrm{iso}})$. It is well-known that this notion can be extended to a derivation on $\Omega(G_0, A^*_{\mathrm{iso}})$ which squares to 0 if the connection is flat (cf. [1], p.13). It will turn out, that this indeed corresponds to $\mathrm{d}^{\pm 0}_{\mathrm{dR}}$ on elements of $\Omega(G_0, A^*_{\mathrm{iso}})$ which represent $J^E G$-invariant forms.

Before we show this, we first need to clarify a few facts on the derivation properties and algebra structures of the objects we are working with, as they are important for the proof.

Lemma 4.1.8. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. The maps $\mathrm{d}^{-1}_{\mathrm{dR}}, \mathrm{d}^{\pm 0}_{\mathrm{dR}}$ and $\mathrm{d}^{+1}_{\mathrm{dR}}$ are derivations on the algebra $\Omega(G_q)$ of differential forms (with the standard wedge product).*

Proof. Assume we have ω and ν which are without loss of generality (as $\mathrm{d}^{\pm 0}_{\mathrm{dR}}$ is defined component-wise) E-homogeneous of E-vertical degrees p_1 and r_1 respectively. Then, as d_{dR} is a derivation, we have that

$$\mathrm{d}_{\mathrm{dR}}(\omega \wedge \nu) = (\mathrm{d}^{-1}_{\mathrm{dR}} + \mathrm{d}^{\pm 0}_{\mathrm{dR}} + \mathrm{d}^{+1}_{\mathrm{dR}})(\omega) \wedge \nu + (-1)^{|\omega|}\omega \wedge (\mathrm{d}^{-1}_{\mathrm{dR}} + \mathrm{d}^{\pm 0}_{\mathrm{dR}} + \mathrm{d}^{+1}_{\mathrm{dR}})(\nu).$$

If we now compare the components in E-vertical degrees, e.g. $p_1 + r_1$, we obtain precisely

$$\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega \wedge \nu) = \mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega) \wedge \nu + (-1)^{|\omega|}\omega \wedge \mathrm{d}^{\pm 0}_{\mathrm{dR}}(\nu),$$

showing $\mathrm{d}^{\pm 0}_{\mathrm{dR}}$ is a derivation. The degrees $p_1 + r_1 \pm 1$ yield the analogous results for $\mathrm{d}^{+1}_{\mathrm{dR}}$ and $\mathrm{d}^{-1}_{\mathrm{dR}}$. □

Lemma 4.1.9. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. The inclusion of $J^E G$-invariant forms*

$$(\Omega(G_q))^{J^E G} \subset \Omega(G_q)$$

equips $(\Omega(G_q))^{J^E G}$ with the natural structure of a subalgebra of $\Omega(G_q)$.

Proof. We need to show that if ω and ν are $J^E G$-invariant forms, so is $\omega \wedge \nu$. However, the action of $J^1 G$ (and thus $J^E G$) on $\wedge^\bullet TG_q$ that we refer to with $\dot{D}^i_{V,q}$ is defined as the naturally induced extension of the action on TG_q. This implies that by definition

$$(\dot{D}^i_{V,q})^*(\omega \wedge \nu) = (\dot{D}^i_{V,q})^*\omega \wedge (\dot{D}^i_{V,q})^*\nu = \omega \wedge \nu$$

for any $0 \leq i \leq q$ and $\omega \wedge \nu$ is indeed $J^E G$-invariant. □

Corollary 4.1.10. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. The map $\mathrm{d}_{\mathrm{dR}}^{\pm 0}$ is a derivation on the algebra $(\Omega(G_q))^{J^E G}$ of $J^E G$-invariant differential forms (with the standard wedge product as in Lemma 4.1.9).*

Proof. As $\mathrm{d}_{\mathrm{dR}}^{\pm 0}$ retains $J^E G$-invariance, it restricts to $(\Omega(G_q))^{J^E G}$. The derivation property is inherited from $\Omega(G_q)$. □

Lemma 4.1.11. *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Under the isomorphism*

$$(\Omega(G_q))^{J^E G} \cong \left(\Gamma(G_0, \wedge^\bullet (TG_0/\mathcal{F}_0)^* \otimes \wedge^\bullet ((A_{\mathrm{iso}}^*)^{\oplus q}))\right)^G$$

of Proposition 2.1.41, the algebra structure of $(\Omega(G_q))^{J^E G}$ (with the standard wedge product as in Lemma 4.1.9) corresponds to the $C^\infty(G_0)$-algebra structure inherited by the inclusion

$$\left(\Gamma(G_0, \wedge^\bullet (TG_0/\mathcal{F}_0)^* \otimes \wedge^\bullet ((A_{\mathrm{iso}}^*)^{\oplus q}))\right)^G \subset \Gamma(G_0, \wedge^\bullet (TG_0/\mathcal{F}_0)^* \otimes \wedge^\bullet ((A_{\mathrm{iso}}^*)^{\oplus q}))$$

i.e. given by point-wise wedge product on the fibres of both tensor factors.

Proof. The isomorphism $(\Omega(G_q))^{J^E G} \cong (\Gamma(G_0, \wedge^\bullet (TG_0/\mathcal{F}_0)^* \otimes \wedge^\bullet ((A_{\mathrm{iso}}^*)^{\oplus q})))^{J^E G}$ is given by restriction to the unit section and the algebra structure on $(\Omega(G_q))^{J^E G}$ is given by the wedge in every fibre of $\wedge^\bullet T^* G_q \to G_q$.

Hence, at the unit section, we immediately have that the multiplication is given by the wedge product on the fibres $\wedge^\bullet T_{1_m}^* G_q$. The restricted target of the space of sections comes from the fact that under the decomposition

$$T_{1_m}^* G_q = T_m^* G_0 \oplus (A_m^*)^{\oplus q},$$

where $(TG_0/\mathcal{F}_0)^* \subset T^* G_0$ and $A_{\mathrm{iso}}^* \subset A^*$ (using E and its associated vertical projection) we have that the map of Proposition 2.1.41 factors through these subspaces. □

Unfortunately, this algebra structure does not restrict to the space $\mathcal{R}_E$ of $J^E G$-isotropy Cartan forms. The simplest reason for this is the fact, that q must be fixed for this multiplication to be defined, but the E-vertical degree is additive, meaning that multiplying two $J^E G$-isotropy Cartan forms (of respective E-vertical degree q) will yield a form of E-vertical degree $2q$. Hence this form then can not be $J^E G$-isotropy Cartan.

However, there is of course a canonical fibre-wise multiplication on $S^\bullet(A_{\mathrm{iso}}^*)$ due to its structure as a symmetric tensor power.

Lemma 4.1.12. *Let G be a regular Lie groupoid. The $C^\infty(G_0)$-algebra structure of*

$$\Gamma\left(G_0, \wedge^{p_0} T^* G_0 \otimes S^q\left(A_{\mathrm{iso}}^*\right)\right)$$

induced by tensoring the fibre-wise ring structures of $\wedge^{p-q} T^* G_0$ and $S^q\left(A_{\mathrm{iso}}^*\right)$ restricts to

$$\left(\Gamma\left(G_0, \wedge^{p-q} (TG_0/\mathcal{F}_0)^* \otimes S^q\left(A_{\mathrm{iso}}^*\right)\right)\right)^G$$

where the G-action is defined as the extension of the G-action of Lemma 1.2.16 and the extension of the coadjoint G-action of Example 1.1.35.

Proof. First of all, let us note that as $(TG_0/\mathcal{F}_0)^* \subset T^*G_0$, we have that by definition

$$\Gamma\left(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes S^q\left(A^*_{\mathrm{iso}}\right)\right) \subset \Gamma\left(G_0, \wedge^{p_0}T^*G_0 \otimes S^q\left(A^*_{\mathrm{iso}}\right)\right)$$

is an inclusion of algebras. Note that we have seen in the previous chapter (in particular in Corollary 3.1.6 and Corollary 3.1.9) that the G-action on the subcomplex is well-defined. In particular, we have defined the G-action as an extension of two G-actions on $(TG_0/\mathcal{F}_0)^*$ and A^*_{iso} with respect to the algebra structure of the exterior and symmetric algebra. This immediately implies that the action commutes with multiplication of elements by fibre-wise multiplication in $\wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes S^q\left(A^*_{\mathrm{iso}}\right)$ and that therefore, the subspace is indeed a subalgebra. □

Definition 4.1.13. Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. We will denote the product on the space of J^EG-isotropy Cartan forms $\mathcal{R}^{(p,q)}_E$ induced by the isomorphism to

$$\left(\Gamma\left(G_0, \wedge^{p-q}(TG_0/\mathcal{F}_0)^* \otimes S^q\left(A^*_{\mathrm{iso}}\right)\right)\right)^G$$

and the algebra structure of Lemma 4.1.12 on the target by

$$(a, b) \mapsto a \,\check{\cdot}\, b.$$

Lemma 4.1.14. *Let G be a regular Lie groupoid. Consider a vector bundle connection (i.e. TG_0-connection) ∇ on $(A_{\mathrm{iso}})^{\oplus q}$. The map*

$$C : \Gamma\left(G_0, \wedge^{\bullet}T^*G_0 \otimes \wedge^{p_1}\left((A^*_{\mathrm{iso}})^{\oplus q}\right)\right) \to \Gamma\left(G_0, \wedge^{\bullet+1}T^*G_0 \otimes \wedge^{p_1}\left((A^*_{\mathrm{iso}})^{\oplus q}\right)\right)$$

defined on

$$w \in \Gamma\left(G_0, \wedge^{p_0}T^*G_0 \otimes \wedge^{p_1}\left((A^*_{\mathrm{iso}})^{\oplus q}\right)\right)$$

and $X_1, \dots, X_{p_0+1} \in \mathfrak{X}(G_0)$ and $Y_1, \dots, Y_{p_1} \in \Gamma(G_0, ((A_{\mathrm{iso}})^{\oplus q}))$ by

$$\begin{aligned}
&C(w)(X_1, \dots, X_{p_0+1}, Y_1, \dots, Y_{p_1}) \\
&= \sum_{i=1}^{p_0+1} (-1)^{i+1} X_i(w(\dots, \widehat{X_i}, \dots)) \\
&\quad + \sum_{1 \le i < j \le (p_0+1)} (-1)^{i+j} w([X_i, X_j], \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots) \\
&\quad + \sum_{i=1, j=1}^{(p_0+1), p_1} (-1)^{i+j+p_0+1} w(\nabla_{X_i} Y_j, \dots, \widehat{X_i}, \dots, \widehat{Y_j}, \dots),
\end{aligned}$$

is local and a derivation with respect to the $\Omega(G_0)$-module structure induced by the exterior fibre products.

When considering $p_0 = 0$, i.e. the map where C is evaluated on $\Gamma(G_0, \wedge^{p_1}(A^*_{\mathrm{iso}})^{\oplus q})$ it is induced by the covariant derivative of a connection $(\nabla')^*$ on $\wedge^{p_1}(A^*_{\mathrm{iso}})^{\oplus q})$. It is precisely the connection on $\wedge^{p_1}((A_{\mathrm{iso}})^{\oplus q})^* = \wedge^{p_1}(A^*_{\mathrm{iso}})^{\oplus q}$ dual to ∇' on $\wedge^{p_1}(A_{\mathrm{iso}})^{\oplus q}$, which is defined by

$$\nabla'_X\left(Y_1 \wedge \dots \wedge Y_{p_1}\right) := \sum (-1)^{i+1} (\nabla_X Y_i) \wedge Y_1 \wedge \dots \wedge \widehat{Y_i} \wedge \dots.$$

In case $p_1 = 0$, i.e. where we are considering $\Omega^{p_0}(G_0)$ it is given by the de Rham differential $\mathrm{d_{dR}}$.

Proof. First of all, let us note that ∇' indeed defines a connection, as it is clearly $C^\infty(G_0)$-linear in X and we have that

$$\nabla_X f \cdot (Y_1 \wedge \cdots \wedge Y_{p_1}) = \nabla_X \left(f \cdot Y_1 \wedge \cdots \wedge Y_{p_1}\right)$$

such that the Leibniz rule can be inferred from the Leibniz rule of ∇.

Now consider the exterior covariant derivative $d_{(\nabla')^*}$ induced by $(\nabla')^*$. It is uniquely defined by its properties of being local and a derivation with the map for degree $p_0 = 0$ being precisely induced by the covariant derivative (cf. [28], p. 169). It is also given by a Koszul formula (cf. [1], p. 13), which is of the following form:

$$\begin{aligned}
&d_{(\nabla')^*}(w)(X_1, \ldots, X_{p_0+1}) \\
&= \sum_{1 \leq i < j \leq (p_0+1)} (-1)^{i+j} w([X_i, X_j], \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots) \\
&\quad + \sum_{i=1}^{(p_0+1)} (-1)^{i+1} (\nabla')^*_{X_i} w(\ldots, \widehat{X_i}, \ldots).
\end{aligned}$$

Here, if we unravel the definition of a dual connection, we obtain

$$\begin{aligned}
&d_{(\nabla')^*}(w)(X_1, \ldots, X_{p_0+1})(Y_1 \wedge \cdots \wedge Y_{p_1}) \\
&= \sum_{1 \leq i < j \leq (p_0+1)} (-1)^{i+j} w([X_i, X_j], \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots)(Y_1 \wedge \cdots \wedge Y_{p_1}) \\
&\quad + \sum_{i=1}^{(p_0+1)} (-1)^{i+1} X_i \left(w(\ldots, \widehat{X_i}, \ldots)(Y_1 \wedge \cdots \wedge Y_{p_1}) \right) \\
&\quad - \sum_{i=1}^{(p_0+1)} (-1)^{i+1} w(\ldots, \widehat{X_i}, \ldots) \left(\nabla'_{X_i}(Y_1 \wedge \cdots \wedge Y_{p_1}) \right) \\
&= \sum_{1 \leq i < j \leq (p_0+1)} (-1)^{i+j} w([X_i, X_j], \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots)(Y_1 \wedge \cdots \wedge Y_{p_1}) \\
&\quad + \sum_{i=1}^{(p_0+1)} (-1)^{i+1} X_i \left(w(\ldots, \widehat{X_i}, \ldots)(Y_1 \wedge \cdots \wedge Y_{p_1}) \right) \\
&\quad - \sum_{i=1, j=1}^{(p_0+1), p_1} (-1)^{i+j} w(\ldots, \widehat{X_i}, \ldots) \left((\nabla'_{X_i} Y_j) \wedge (Y_1 \wedge \cdots \wedge \widehat{Y_j}, \ldots) \right).
\end{aligned}$$

However, we can see that this expression precisely coincides with the formula of C as evaluation on terms of the form $Y_1 \wedge \cdots \wedge Y_{p_1}$ agrees with evaluation on $(Y_1, \ldots, Y_{p_1})$ and the signs correspond to each other using that w is by definition anti-symmetric on $((TG_0/\mathcal{F}_0) \oplus A_{\mathrm{iso}})^{\otimes(p_0+p_1)}$. Therefore, we have obtained the statement of the proposition.

The fact that for $p_1 = 0$ we obtain the de Rham differential follows directly from the formula, as in this case C only consists of the terms of the first and second form, which recovers precisely the Koszul formula for the de Rham differential. $\square$

Lemma 4.1.15. *Let G be a regular Lie groupoid. The induced map*

$$C : \Gamma\left(G_0, \wedge^\bullet T^* G_0 \otimes \wedge^\bullet \left((A^*_{\mathrm{iso}})^{\oplus q}\right)\right) \to \Gamma\left(G_0, \wedge^{\bullet+1} T^* G_0 \otimes \wedge^\bullet \left((A^*_{\mathrm{iso}})^{\oplus q}\right)\right)$$

which on components is given by the map of Lemma 4.1.14 is a local derivation with respect to the algebra structure on $\Gamma\left(G_0, \wedge^\bullet T^* G_0 \otimes \wedge^\bullet \left((A^*_{\mathrm{iso}})^{\oplus q}\right)\right)$ of Lemma 4.1.11.

Proof. As we have already seen in Lemma 4.1.14 that the given map is local and a derivation with respect to the $C^\infty(G_0)$-module structure, we only need to show that C is a derivation when multiplying elements of $\Gamma\left(G_0, \wedge^\bullet\left((A^*_{\mathrm{iso}})^{\oplus q}\right)\right)$. However, we have on this subspace that

$$\begin{aligned} &C(w)(X, Y_1, \dots, Y_a)\\ =&X(w(Y_1, \dots))\\ &+\sum_{j=1}^{a}(-1)^j w(\nabla_X Y_j, Y_1, \dots, \widehat{Y_j}, \dots), \end{aligned}$$

and for $v \in \Gamma\left(G_0, (A^*_{\mathrm{iso}})^{\oplus q}\right)$

$$\begin{aligned} &C(v)(X, Y)\\ =&X(v(Y))\\ &+(-1)^1 v(\nabla_X Y), \end{aligned}$$

such that

$$\begin{aligned} &(C(w)\wedge v)(X, Y_1, \dots, Y_{a+1})\\ =&\sum_{i=1}^{a+1}(-1)^{i+a+1}C(w)(X, Y_1, \dots, \widehat{Y_i}, \dots)\cdot v(Y_i). \end{aligned}$$

and similarly for $w \wedge C(v)$.

At the same time, we can compute

$$\begin{aligned} &C(w\wedge v)(X, Y_1, \dots, Y_{a+1})\\ =&\sum_{i=1}^{a+1}(-1)^{i+a+1}X(w(Y_1, \dots, \widehat{Y_i}, \dots)\wedge v(Y_i))\\ &+\sum_{j=1}^{p_1}(-1)^{j+a}w(Y_1, \dots, \widehat{Y_j}, \dots)\cdot v(\nabla_X Y_j)\\ &+\sum_{j=1}^{a}\sum_{i=2}^{j}(-1)^{j+a+i+1}w(\nabla_X Y_j, Y_1, \dots, \widehat{Y_{i-1}}, \dots)\cdot v(Y_{i-1})\\ &+\sum_{j=1}^{a}\sum_{i=j+1}^{a+1}(-1)^{j+a+i+1}w(\nabla_X Y_j, Y_1, \dots, \widehat{Y_i}, \dots)\cdot v(Y_i), \end{aligned}$$

where we can now use the fact that X is a derivation on functions and some combinatorial rearranging of the terms to obtain that this indeed agrees with

$$C(w)\wedge v + (-1)^a w \wedge C(v).$$

By associativity we can now obtain that the same statement holds even if v is of degree p_1 that is not necessarily assumed to be 1. This yields the result of the statement. □

Lemma 4.1.16. *Let G be a regular Lie groupoid. Consider the inclusion*

$$\Gamma\left(G_0, \wedge^\bullet T^*G_0 \otimes S^q\left(A^*_{\mathrm{iso}}\right)\right) \subset \Gamma\left(G_0, \wedge^\bullet T^*G_0 \otimes \wedge^q\left((A^*_{\mathrm{iso}})^{\oplus q}\right)\right)$$

of elements of multidegree $(p_0, 1, \dots, 1, q)$ *which are invariant under the permutation action of* Σ_q*. Let* C *be the map defined in Lemma 4.1.15 for a connection* $\nabla^{\oplus q}$ *on* $(A_{\mathrm{iso}})^{\oplus q}$ *given by the direct sum of identical connections* ∇ *on each copy of* A_{iso}*. Then* C *restricts to this subspace. Furthermore, this restriction of* C *is also local and a derivation when considering the algebra structure described in Lemma 4.1.12 on the subspace.*

Proof. We know by Lemma 4.1.14 and Lemma 4.1.15 that C is local and a derivation with respect to the algebra structure on

$$\Gamma\left(G_0, \wedge^{\bullet} T^* G_0 \otimes \wedge^{\bullet}\left((A_{\mathrm{iso}}^*)^{\oplus q}\right)\right).$$

If we now have an element of

$$\omega \in \Gamma\left(G_0, \wedge^{p_0} T^* G_0 \otimes S^q(A_{\mathrm{iso}}^*)\right),$$

we need to show three things. Firstly, we will show that the multidegree of $C(\omega)$ is of the correct form, secondly, we will show that C commutes with the permutation action (implying that the invariance is retained under C) and thirdly, we will consider the new algebra structure.

We will start by showing that $C(\omega)$ has the appropriate multidegree and commutes with the permutation action of Σ_q. For both of this we will use the properties that C is local and that C is a derivation to check the necessary statements only on $\Omega(G_0)$ and $\Gamma\left(G_0, (A_{\mathrm{iso}}^*)^{\oplus q}\right)$. (To locally decompose any element into a sum of products of those, we may simply choose an arbitrary local frame.)

There, we find, that on $\Omega(G_0)$ the action is trivial and an element of multidegree $(p_0, 0, \dots, 0, q)$ is mapped under $C \equiv \mathrm{d}_{\mathrm{dR}}$ to an element of multidegree $(p_0 + 1, 0, \dots, 0, q)$. On $\Gamma\left(G_0, (A_{\mathrm{iso}}^*)^{\oplus q}\right)$, each homogeneous subspace of multidegree $(0, \dots, 0, 1, 0, \dots, q)$ corresponds to $\Gamma\left(G_0, A_{\mathrm{iso}}^*\right)$ contained in the corresponding copy of $(A_{\mathrm{iso}}^*)^{\oplus q}$. As the covariant derivative of the direct sum connection $\nabla^{\oplus q}$ is given precisely by the covariant derivative of ∇ applied to each component of the direct sum, we obtain that the multidegree of the image is $(1, \dots, 0, 1, 0, \dots, q)$ and corresponds precisely to the image of the corresponding element in $\Gamma\left(G_0, A_{\mathrm{iso}}^*\right)$. In particular, C commutes with the permutation action. This means that, also in general, C maps homogeneous elements (with respect to the multidegree) to homogeneous elements and increases the first component of the multidegree by 1. So the multidegree $(p_0, 1, \dots, 1, q)$ will be mapped to $(p_0 + 1, 1, \dots, 1, q)$. Furthermore, C commutes with the action of Σ_q.

The last assertion, that C is compatible with the algebra structure of Lemma 4.1.11 can now be inferred from the fact that the multiplication with respect to this algebra structure can be constructed equivalently by including $(A_{\mathrm{iso}})^{\oplus q}$ and $(A_{\mathrm{iso}})^{\oplus q'}$ into $(A_{\mathrm{iso}})^{\oplus q+q'}$ applying the original multiplication of elements lastly averaging over $\Sigma_{q+q'}$. These are all procedures which commute with C (the statement on the inclusions follows from the form of $\nabla^{\oplus q}$ in analogue to the argument of C being invariant under permutation). □

Lemma 4.1.17. *Let* G *be a regular Lie groupoid with a multiplicative Ehresmann connection* E *with respect to the isotropy algebroid. Consider* $\omega \in \Omega^p(G_q)$ *which is* E*-homogeneous of* E*-vertical degree* p_1 *and therefore* E*-horizontal degree* $p_0 := p - p_1$*. Let* $X_1, \dots, X_{p_0+1}$ *be vector fields defined on the unit section* $1_{G_0} \subset G_q$ *which are tangent to the unit section and* $Y_1, \dots, Y_{p_1}$ *be* E*-vertical vector fields (i.e. sections of* $(A_{\mathrm{iso}})^{\oplus q}$*) defined at the unit section. The expression*

$$(\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega))_{1_m}(X_1, \dots, X_{p_0+1}, Y_1, \dots, Y_{p_1})$$

is independent of the values of ω outside of the identity section. Its corresponding element of $\Gamma\left(G_0, \wedge^{p_0+1}T^*G_0 \otimes \wedge^{p_1}\left((A^*_{\mathrm{iso}})^{\oplus q}\right)\right)$ is given by $C(\omega_{1_{G_0}})$, where C is computed as in Lemma 4.1.15 for $\nabla = (\nabla^E)^{\oplus q}$ and furthermore

$$\omega_{1_{G_0}} \in \Gamma\left(G_0, \wedge^{p_0}T^*G_0 \otimes \wedge^{p_1}\left((A^*_{\mathrm{iso}})^{\oplus q}\right)\right)$$

is defined as the image of $(u_q)^*(\omega) \in \Gamma(G_0, (u_q)^*(TG_q))$ under the projection induced by $TG_0 \oplus A_{\mathrm{iso}} \subset TG_q$.

Proof. First of all, we know that the value of the expression considered in the claim only depends on the values of X_i and Y_j at 1_m, so we may continue the X_i by E-horizontal vector fields and the Y_j by E-vertical vector fields without loss of generality.

We have by Lemma 4.1.1, that on p_0+1 horizontal and p_1 vertical vector fields, the de Rham differential is defined as follows:

$$\begin{aligned}
&(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega))_{1_m}(X_1, \ldots, X_{p_0+1}, Y_1, \ldots, Y_{p_1})\\
:=&\sum_{i=1}^{p_0+1}(-1)^{i+1}(X_i)_{1_m}(\omega(\ldots, \widehat{X_i}, \ldots))\\
&+\sum_{1\leq i<j\leq (p_0+1)}(-1)^{i+j}\omega_{1_m}([X_i, X_j]_{E-\mathrm{hor}}, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots)\\
&+\sum_{i=1,j=1}^{(p_0+1),p_1}(-1)^{i+j+p-q+1}\omega_{1_m}([X_i, Y_j]_{E-\mathrm{vert}}, \ldots, \widehat{X_i}, \ldots, \widehat{Y_j}, \ldots)
\end{aligned}$$

The first summand only depends on the values of ω along X_i, hence along the unit section. The other two summands even only depend on the value of ω at 1_m. This proves the claim that the result only depends on $\omega_{1_{G_0}}$.

We know that

$$[X_i, X_j]_{\mathrm{hor}} \equiv [X_i, X_j]$$

at 1_m, as both X_i and X_j restrict to vector fields on the unit section meaning $[X_i, X_j]$ does so, too, and any vector tangent to the unit section is horizontal by definition. We also know by Lemma 4.1.6 that $[X_i, Y_j]_{E-\mathrm{vert}} \equiv \nabla_{X_i}Y_j$ at 1_m for $\nabla = (\nabla^E)^{\oplus q}$.

Hence indeed the expression coincides with $C(\omega|_{1_{G_0}})$. $\square$

Proposition 4.1.18. *Let G be a regular Lie groupoid with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Consider $\omega \in \Omega^p(G_q)$ which is J^EG-invariant and E-homogeneous of E-vertical degree p_1. Let $p_0 := p - p_1$ be the corresponding E-horizontal degree. Then $\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)$ is given by the J^EG-invariant form corresponding to*

$$C(\omega|_{1_{G_0}}) \in \left(\Gamma\left(G_0, \wedge^{(p_0+1)}(TG_0/\mathcal{F}_0)^* \otimes \wedge^{p_1}\left((A^*_{iso})^{\oplus q}\right)\right)\right)^G,$$

where C is computed as in Lemma 4.1.15 for $\nabla = (\nabla^E)^{\oplus q}$.

When evaluated on a J^EG-isotropy Cartan form ω the result will again be J^EG-isotropy Cartan. The endomorphism on J^EG-isotropy Cartan forms

$$\omega \mapsto \mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)$$

is local and a derivation with respect to the algebra structure of Lemma 4.1.12. It can be computed as the restriction of the derivation on

$$\Gamma\left(G_0, \wedge^{(p-q)}T^*G_0 \otimes S^q\left(A^*_{\mathrm{iso}}\right)\right)$$

which on $\omega \in \Omega^{p-q}(G_0)$ acts by the de Rham differential and on $\omega \in \Gamma\left(G_0, A^*_{\mathrm{iso}}\right)$ is given by the form locally given by

$$dx^\alpha \otimes \frac{\partial \omega}{\partial x^\alpha},$$

where the derivative is taken with respect to the connection dual to ∇^E.

Proof. We know by Lemma 4.1.1, that if ω is $J^E G$-invariant, so $\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)$. So in particular it is uniquely identified by its restriction $\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)\right)|_{1_{G_0}}$ which we know to satisfy by Proposition 2.1.41 that

$$\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)\right)|_{1_{G_0}} \in \Gamma\left(G_0, \wedge^{p_0+1}(TG_0/\mathcal{F}_0)^* \otimes \wedge^{p_1}\left((A^*_{\mathrm{iso}})^{\oplus q}\right)\right).$$

However, this expression can be reconstructed by evaluating it on a family of vector fields $X_0, \dots, X_{p_0+1}$ defined on the unit section which are tangent to the unit section and a family of E-vertical vector fields $Y_1, \dots, Y_{p_1}$ defined at the unit section. As we know by Lemma 4.1.17, the result agrees with the result when evaluating $C(\omega|_{1_{G_0}})$ on the same vector fields, hence

$$\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)\right)|_{1_{G_0}} \equiv C(\omega|_{1_{G_0}}).$$

In particular, $C(\omega|_{1_{G_0}})$ is contained in

$$\Gamma\left(G_0, \wedge^{(p_0+1)}(TG_0/\mathcal{F}_0)^* \otimes \wedge^{p_1}\left((A^*_{\mathrm{iso}})^{\oplus q}\right)\right) \subset \Gamma\left(G_0, \wedge^{(p_0+1)}T^*G_0 \otimes \wedge^{p_1}\left((A^*_{\mathrm{iso}})^{\oplus q}\right)\right)$$

and furthermore $J^E G$-invariant.

Also, we know by Lemma 4.1.16 that C retains the subspace of elements of multidegree $(p_0, 1, \dots, 1, q)$ which are invariant under the permutation action of Σ_q. So if ω is $J^E G$-isotropy Cartan and hence $\omega|_{1_{G_0}}$ is contained in that subspace, we find that so is $C(\omega|_{1_{G_0}})$. Combining both observations, this implies

$$C(\omega|_{1_{G_0}}) \in \left(\Gamma\left(G_0, \wedge^{(p+1-q)}(TG_0/\mathcal{F}_0)^* \otimes S^q\left(A^*_{\mathrm{iso}}\right)\right)\right)^G.$$

So, $C(\omega|_{1_{G_0}})$ is representing a $J^E G$-isotropy Cartan form as well. Furthermore, we know that C is a derivation with respect to the algebra structure of

$$\Gamma\left(G_0, \wedge^\bullet T^*G_0 \otimes S^\bullet\left(A^*_{\mathrm{iso}}\right)\right),$$

by Lemma 4.1.16, where the computations on $\Omega(G_0)$ and $\Gamma(G_0, A^*_{\mathrm{iso}})$ can be inferred from Lemma 4.1.14. $\square$

Definition 4.1.19. Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. We will denote the endomorphism on $J^E G$-isotropy Cartan forms described in Proposition 4.1.18 by

$$\widehat{\mathrm{d}^E_{\mathrm{Cart}}} : \mathcal{R}^{(p,q)}_E \to \mathcal{R}^{(p+1,q)}_E$$

and the map corresponding to it by

$$\mathrm{d}^E_{\mathrm{Cart}} : \left(\Gamma\left(G_0, \wedge^{p-q}(TG_0/\mathcal{F}_0)^* \otimes S^q(A^*_{\mathrm{iso}})\right)\right)^G \hookrightarrow \left(\Gamma\left(G_0, \wedge^{p-q+1}(TG_0/\mathcal{F}_0)^* \otimes S^q(A^*_{\mathrm{iso}})\right)\right)^G.$$

In cases, where the choice of multiplicative Ehresmann connection is clear, we may also denote it by $\mathrm{d}_{\mathrm{Cart}}$.

Remark. Note that the expression of C (which can be used to compute $\mathrm{d}^E_{\mathrm{Cart}}$) on $\Omega(G_0)$ and $\Gamma(G_0,(A_{\mathrm{iso}})^*)$ shows that the differential itself only depends on ∇^E, not E. So in a sense, the model (as it will turn out to be one) given by this chain complex is only depending on the infinitesimal objects A_{iso} and ∇^E and can be called an infinitesimal model.

We will later even be able to show that $\mathrm{d}^E_{\mathrm{Cart}}$ is independent of the choice of E, further justifying why the index may be suppressed. The map C on the ambient space however is not independent of the choice of E.

Proposition 4.1.20. *Let G be a regular and proper Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. The de Rham differential $\mathrm{d}_{\mathrm{dR}}(\omega)$ of a J^EG-isotropy Cartan form*

$$\omega \in \mathcal{R}_E^{(p,q)}$$

is cohomologically equivalent to its projection of E-vertical degree q. This projection is given by the J^EG-isotropy Cartan form

$$\widehat{\mathrm{d}^E_{\mathrm{Cart}}}(\omega) = \mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega),$$

and can be computed according to Proposition 4.1.18.

Moreover, the preimage of the exact term by which the two elements $\mathrm{d}_{\mathrm{dR}}(\omega)$ and its projection to vertical degree q differ can be chosen such that its E-vertical degree q with respect to the multiplicative Ehresmann connection vanishes.

Proof. For any $\nu \in \Omega^{p+1}(G_q)$ closed under $\mathrm{d}_{\mathcal{N}}$, we have that its vertical cohomology class is determined by the projection to its component of vertical degree q by Corollary 3.1.26.

For a J^EG-isotropy Cartan form ω we have that $\mathrm{d}_{\mathrm{dR}}(\omega)$ is $\mathrm{d}_{\mathcal{N}}$-closed as the two differentials commute. Therefore, it is cohomologically equivalent to $\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)$, as this is the component in E-vertical degree q. The remaining statements directly follow from Proposition 4.1.18. $\square$

Corollary 4.1.21. *Let G be a regular and proper Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. On the algebra $\mathcal{R}_E$ of J^EG-isotropy Cartan forms, we have*

$$(\widehat{\mathrm{d}^E_{\mathrm{Cart}}})^2 = 0.$$

On the associated isomorphic algebra $\left(\Gamma\left(G_0, \wedge^\bullet (TG_0/\mathcal{F}_0)^* \otimes S^\bullet(A^*_{iso})\right)\right)^G$, we have

$$(\mathrm{d}^E_{\mathrm{Cart}})^2 = 0.$$

Remark. On $\omega \in \Gamma\left(G_0, A^*_{\mathrm{iso}}\right)$, the Cartan derivation C does not necessarily square to 0 depending on the curvature. In fact, the exteriour covariant derivative with respect to a vector bundle and its connection precisely only squares to 0 if the connection is flat, something that we may not assume in our case (cf. [1], p. 13).

We can motivate our given result nonetheless by the fact that the Cartan derivation does square to 0 on $\omega \in \Gamma\left(G_0, (S^\bullet\left(A^*_{\mathrm{iso}}\right))^G\right)$, as on this bundle we do obtain a flat connection. This is not an obvious fact but it can be derived by a careful study of how multiplicative Ehresmann connections can be chosen on Lie group bundles (like the isotropy bundle). We will remark on this claim again at a later point of this thesis (page 301).

Proof. Any $J^E G$-isotropy Cartan form ω decomposes into $J^E G$-isotropy Cartan forms $\omega_{(p,q)}$, which are only supported in bidegree (p, q). For each of these, we know by Proposition 4.1.20, that the expressions $\mathrm{d_{dR}}(\omega_{(p,q)})$ and $\widehat{\mathrm{d}^E_{\mathrm{Cart}}}(\omega_{(p,q)})$ yield equivalent cohomology classes in $H^q(\mathfrak{C}^{p+1})$.

This implies that also the cohomology classes of $(\mathrm{d_{dR}})^2(\omega_{(p,q)})$ and $\mathrm{d_{dR}}(\widehat{\mathrm{d}_{\mathrm{Cart}}}(\omega_{(p,q)}))$ in $H^q(\mathfrak{C}^{p+2})$ agree, where we know $(\mathrm{d_{dR}})^2 = 0$.

However, as $\widehat{\mathrm{d}_{\mathrm{Cart}}(\omega_{(p,q)})}$ is again $J^E G$-isotropy Cartan, we can again use Proposition 4.1.20 to identify the cohomology class $\mathrm{d_{dR}}(\widehat{\mathrm{d}_{\mathrm{Cart}}}(\omega_{(p,q)}))$ as precisely the class containing the $J^E G$-isotropy Cartan element $(\widehat{\mathrm{d}_{\mathrm{Cart}}})^2(\omega_{(p,q)})$. As the cohomology class contains 0, we find that this must be the only $J^E G$-isotropy Cartan form by Theorem 3.1.20. Hence $(\mathrm{d}_{\mathrm{Cart}})^2 \equiv 0$ when evaluated on every component $\omega_{(p,q)}$ and on ω as a whole. □

Corollary 4.1.22. *Let G be a regular and proper Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. The bigraded vector space $\mathcal{R}_E$ of $J^E G$-isotropy Cartan forms can be used to construct a bicomplex by defining the vertical differential as $\delta = 0$ and the horizontal differential as $d = \widehat{\mathrm{d}^E_{\mathrm{Cart}}}$.*

Proof. By Corollary 4.1.21, we have that d is indeed a differential. The facts that $\delta^2 = 0$ and $d\delta = -\delta d$ are immediate as $\delta = 0$. □

Definition 4.1.23. Let G be a regular and proper Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. We can define the families of cochain complexes

$$\mathcal{R}_q^\bullet = \left(\mathcal{R}_E^{(\bullet,q)}, \widehat{\mathrm{d}^E_{\mathrm{Cart}}}\right)$$

and

$$\mathfrak{R}_q^\bullet = \left(\left(\Gamma\left(G_0, \wedge^{\bullet-q}(TG_0/\mathcal{F}_0)^* \otimes S^q(A^*_{\mathrm{iso}})\right)\right)^G, \mathrm{d}^E_{\mathrm{Cart}}\right)$$

and the bicomplex

$$\mathfrak{R}_E^{(\bullet,\bullet)} = \left(\bigoplus_{p=\bullet}\bigoplus_{q=\bullet}\left(\Gamma\left(G_0, \wedge^{p-q}(TG_0/\mathcal{F}_0)^* \otimes S^q(A^*_{\mathrm{iso}})\right)\right)^G, \mathrm{d}^E_{\mathrm{Cart}}, 0\right).$$

Remark. Note that all of these complexes depend definitorially on the multiplicative Ehresmann connection E. We have suppressed this in the notation in the first two cases, as they are auxiliary constructions.

Lemma 4.1.24. *Let G be a regular and proper Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. The cochain complexes $\mathcal{R}_q$ and $\mathfrak{R}_q$ are canonically isomorphic for any $q \in \mathbb{N}_0$. The bicomplex $\mathfrak{R}_E$ is canonically isomorphic to the bicomplex described in Corollary 4.1.22.*

Proof. By definition, we have that $\mathcal{R}_E^{(p,q)}$ is isomorphic to

$$\left(\Gamma\left(G_0, \wedge^{p-q}(TG_0/\mathcal{F}_0)^* \otimes S^q(A^*_{\mathrm{iso}})\right)\right)^G$$

under the morphism of Proposition 2.1.41. By definition, this means that $\mathcal{R}_q^p$ is isomorphic to $\mathfrak{R}_q^p$, meaning that $\mathcal{R}_q^\bullet$ and $\mathfrak{R}_q^\bullet$ are isomorphic. Under the same morphism $\mathfrak{R}_E^{p,q}$ corresponds to $\mathcal{R}_E^{(p,q)}$ directly, where the 0 maps trivially correspond to each other under the isomorphism and by definition so do $\mathrm{d}^E_{\mathrm{Cart}}$ and $\widehat{\mathrm{d}^E_{\mathrm{Cart}}}$. This already implies the claim. □

Corollary 4.1.25. *Let G be a proper and regular Lie groupoid and E and F two choices of a multiplicative Ehresmann connections with respect to the isotropy algebroid. The map $\psi^{E,F}$ of Corollary 3.1.22 induces a map of chain complexes between $\mathcal{R}_{q,E}$ associated to E and $\mathcal{R}_{q,F}$ associated to F. On the canonically isomorphic chain complexes $\mathfrak{R}_{q,E}$ and $\mathfrak{R}_{q,F}$ the isomorphism is given by the identity. In particular the differentials on $\mathfrak{R}_{q,E}$ and $\mathfrak{R}_{q,F}$ agree. The analogous statement holds for $\mathfrak{R}_E$ and $\mathfrak{R}_F$.*

Proof. We already obtain from the definition of $\psi^{E,F}$ in Corollary 3.1.22, that the map retains the bidegree of elements of $\mathcal{R}_E$. So it restricts to $\mathcal{R}_{q,E} \to \mathcal{R}_{q,F}$. By definition, the isomorphism is given by identifying

$$\mathcal{R}^p_{q,E} \cong H^q(\mathfrak{C}_p) \cong \mathcal{R}^p_{q,F}.$$

Under these identifications, we have by Proposition 4.1.20 that $\widehat{\mathrm{d}^E_{\mathrm{Cart}}}$ is mapped to $[\mathrm{d}_{\mathrm{dR}}]$ (i.e. the map between $H^q(\mathfrak{C}_p)$ and $H^q(\mathfrak{C}_{p+1})$ induced by d_{dR} on $\mathfrak{C}^q_p$). However, this map is completely independent of any multiplicative Ehresmann connection and therefore corresponds to $\widehat{\mathrm{d}^F_{\mathrm{Cart}}}$ under the second isomorphism as well.

We also have that under the isomorphism of Proposition 2.1.41 the map $\psi^{E,F}$ as a map of vector spaces corresponds in each bidegree to the identity map

$$\mathrm{id} : \mathfrak{R}^p_{q,E} \to \mathfrak{R}^p_{q,F}$$

by Lemma 3.1.23. However, we have just seen, that $\psi^{E,F}$ is compatible with the chain complex structure, and by Lemma 4.1.24, so is the induced isomorphism from $\mathcal{R}_{q,E}$ to $\mathfrak{R}_{q,E}$ and the respective analogue morphism for F. So $\mathrm{d}^E_{\mathrm{Cart}}$ and $\mathrm{d}^F_{\mathrm{Cart}}$ correspond to each other under the identity map and hence agree.

If we consider the same arguments on $\mathcal{R}_E$ and $\mathcal{R}_F$ or $\mathfrak{R}_E$ and $\mathfrak{R}_F$, the same considerations hold. In these cases, it only remains to observe that the vertical differential is given by 0, which is trivially retained by the linearity of $\psi^{E,F}$ and its induced maps. □

Lemma 4.1.26. *Let G be a regular and proper Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. The algebra structure on the bicomplex $\mathfrak{R}_E$ as defined in Lemma 4.1.12 is compatible with its bigrading and both the horizontal and vertical differential. Subsequently, the induced algebra structure on $\mathcal{R}_E$ is also compatible with its bigrading and horizontal and vertical differential.*

Proof. The multiplication on $\mathfrak{R}_E$ is induced by the multiplication on the exterior product and the symmetric product of $\Gamma\left(G_0, \wedge^\bullet T^*G_0 \otimes S^\bullet(A^*_{\mathrm{iso}})\right)$, which are both compatible with the respective degrees in the respective power algebras. The degree of a J^EG-isotropy Cartan form being (p,q) is equivalent to its degree in the exterior power being given by $p_0 = p - q$ and its degree being $p_1 = q$ in the symmetric power, which can equivalently be expressed by the equations $p = p_0 + p_1$ and $q = p_1$. As all those equations are linear, we obtain that indeed, the fact that multiplication adds the degrees of homogeneous elements is transferred to $\mathfrak{R}_E$. The same statement for $\mathcal{R}_E$ follows directly as the bigrading is precisely defined in the way such that it is retained under the isomorphism of Proposition 2.1.41.

For compatibility with the differentials, we may first note that the zero differential, i.e. the vertical differential, is compatible with any algebra structure. The fact that the horizontal differential is also compatible with this structure is already stated in Proposition 4.1.18. □

Lemma 4.1.27. *Let G be a regular and proper Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. A $J^E G$-isotropy Cartan form $\omega \in \mathrm{Tot}(\mathcal{R}_E)$ of total degree t is closed under $\mathrm{d}_{\mathrm{Cart}}$ if and only if all its components $\omega_{(p,q)}$ with $p+q=t$ are closed under $\mathrm{d}_{\mathrm{Cart}}$. In particular, ω is closed under the total differential if and only if all $\omega_{(p,q)}$ are closed as elements of $\mathcal{R}_q$.*

A $J^E G$-isotropy Cartan form ω of total degree t is exact in $\mathrm{Tot}(\mathcal{R}_E)$ if and only if all its components $\omega_{(p,q)}$ with $p+q=t$ are exact in $\mathcal{R}_q$.

Proof. We know that the differential $\mathrm{d}_{\mathrm{Cart}}$ maps each form of bidegree (p,q) to a homogeneous term of bidegree $(p+1,q)$. so the differential of each $\omega_{(p,q)}$ can be recovered from the decomposition of $\mathrm{d}_{\mathrm{Cart}}(\omega)$ into its homogeneous components. Hence if the differential of ω vanishes, so does the differential of $\omega_{(p,q)}$ for all (p,q). By definition and the fact that the vertical differential is 0 on $\mathcal{R}_E$, we find that this precisely describes the differentials of the respective complexes.

The claim about exactness follows analogously from the fact that $\mathrm{d}_{\mathrm{Cart}}$ maps each form of degree $(p-1,q)$ to a homogeneous term of degree (p,q), so the preimage of an exact term can be decomposed into its components which then are preimages of the $\omega_{(p,q)}$. □

Corollary 4.1.28. *Let G be a regular and proper Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. We have that*

$$H^k(\mathrm{Tot}(\mathcal{R}_E)) = \bigoplus_{p+q=k} H^p(\mathcal{R}_q)$$

and

$$H^k(\mathrm{Tot}(\mathfrak{R}_E)) = \bigoplus_{p+q=k} H^p(\mathfrak{R}_q)$$

Proof. This follows from Lemma 4.1.27, as the direct sum can be explicitly described by the decomposition of ω into its components $\omega_{(p,q)}$. □

4.2 The total cohomology of $\Omega_{\mathrm{BSS}}(G)$

Aim of this section: Theorem 4.2.16 (abbreviated):

Let G be a proper and regular groupoid and E a multiplicative Ehresmann connection with respect to the isotropy algebroid. Assume that there was a linear assignment from $J^E G$-isotropy Cartan forms closed under the Cartan differential to closed elements of $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$ satisfying certain compatibility conditions.

Then there exists a graded vector space isomorphism

$$\Phi : H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))) \to H^\bullet(\mathrm{Tot}(\mathcal{R}_E)) \cong H^\bullet(\mathrm{Tot}(\mathfrak{R}_E)).$$

Now that we have computed the differential on the $J^E G$-isotropy Cartan forms quite explicitly, we will return to the problem of computing the cohomology of $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$. At the beginning of this section, we will describe how that cohomology can be computed by a spectral sequence - and we will claim that the spectral sequence collapses at its second page, even though the proof will not be contained in this section. Under that assumption, the cohomology of $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$ is indeed comprised of the cohomology groups of the $\mathcal{R}_q$, meaning they would agree with the cohomology of $\mathcal{R}_E$ (at least as cohomology groups, although not necessarily with respect to the algebra structure).

We will, however, consider the result on spectral sequences noted here as a somewhat motivational approach and take the time to spell out the relation between cohomology classes in $\Omega_{\mathrm{BSS}}(G)$ and $\mathcal{R}_E$ explicitly. This will enable some explicit computations, for example regarding the algebra structure. Therefore, understanding spectral sequences is not necessary to follow the main arguments of this subsection or section. (Note that I, myself, have only very little experience with spectral sequences.)

The fact that we aim to understand precisely how the cohomology classes are related when using the framework of spectral sequences and unravelling the associated definitions also means that there is little originality to the content of this section. It can be seen more as the application of a general framework to the specific context, with some added observations on how expressions may be simplified. Note that also, the sources we use sometimes contain more general statements or constructions, which we then apply to our context-specific setting.

Definition 4.2.1 (cf. [42] Definition 5.6.1)**.** Consider a double complex $C_{(\bullet,\bullet)}$ (with graded commuting differentials). Then the **filtration by columns** on $\mathrm{Tot}(C)$ is given by

$$A_n := \bigoplus_{p\leq n} C_{(p,q)} \subset \mathrm{Tot}(C).$$

For a double cocomplex it is given by the dual notion (i.e. by the same subspaces interpreted as a quotient instead of a subset).

Proposition 4.2.2 (cf. [42], pp.135, 142)**.** *Consider a double complex $C_{(\bullet,\bullet)}$ (with graded commuting differentials) supported in the first quadrant (i.e. where $p \geq 0$ and $q \geq 0$). Then the spectral sequence associated to the horizontal filtration converges to the cohomology of $\mathrm{Tot}(C)$.*

Its 0th page is given by

$$(E_0(C))_{(p,q)} = C_{(p,q)}.$$

Its 1st page is given by

$$(E_1(C))_{(p,q)} = H_q(C_{(p,\bullet)}, \delta),$$

where δ denotes the vertical differential. Its 2nd page is given by

$$(E_2(C))_{(p,q)} = H_p\left(H_q(C_{(\bullet,\bullet)}, \delta), [d]\right),$$

where $[d]$ denotes the differential induced by the horizontal differential.

Dually, on a double cocomplex we obtain that the cohomology of $\mathrm{Tot}(C)$ can be computed with respect to a spectral sequence for which its 0th page is given by

$$\left(E^0(C)\right)^{(p,q)} = C^{(p,q)}.$$

Its 1st page is given by

$$\left(E^1(C)\right)^{(p,q)} = H^q(C^{(p,\bullet)}, \delta),$$

where δ denotes the vertical differential. Its 2nd page is given by

$$\left(E^2(C)\right)^{(p,q)} = H^p\left(H^q(C^{(\bullet,\bullet)}, \delta), [d]\right),$$

where $[d]$ denotes the differential induced by the horizontal differential.

If we apply this fact to our situation at hand, we have already managed to compute the complexes on the first page. They are given precisely by $H^q(\mathfrak{C}^p)$, which we have identified to be the $J^E G$-isotropy Cartan forms.

What we have essentially done in the previous section is computing $d^{(1)} = [d] = [\mathrm{d}_{\mathrm{dR}}]$ more precisely and what we will essentially do in this section and the next chapter is showing that the differential on the second page vanishes - meaning that the spectral sequence collapses. The latter realisation is thanks to a suggestion by Luca Acconero (who also helped me understand the spectral sequence of a double complex).

While we will often refer to this spectral sequence viewpoint of the problem in this chapter, we will nonetheless focus on the explicit structures that go into constructing the spectral sequence for various reasons. One of them is, that the proof that the sequence collapses is constructive and quite explicit. Therefore, we will often consider closed forms in $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$ (or their preimages) and try to relate their cohomology classes to $J^E G$-isotropy Cartan forms explicitly.

Definition 4.2.3. Let G be a Lie groupoid. Consider a element $\omega_{(p,q)} \in \Omega^p(G_q)$ satisfying $\omega_{(p,q)} \neq 0$. A **descending homogeneous extension** of $\omega_{(p,q)}$ is an element $\omega \in \Omega_{\mathrm{BSS}}(G)$ homogeneous with respect to the total grading (in which it has degree $p+q$) such that its component of bidegree (p,q) is given by $\omega_{(p,q)}$ and its components of bidegree (p',q') vanish in case $p' < p$ (or equivalently $q' > q$).

The element $0 \in \Omega_{\mathrm{BSS}}(G)$ is considered to be the descending homogeneous extension of $0 \in \Omega^k(G_0)$ in bidegree $(k,0)$ for any $k \in \mathbb{N}_0$

Remark. In particular, any homogeneous element of the totalisation of the Bott–Shulman–Stasheff complex of degree r can be interpreted as a descending homogeneous extension of its component in bidegree $(k, r-k)$ where $k \in \mathbb{N}_0$ is minimal such that the component does not vanish.

The nomenclature of this definition is not a standard definition. The objects described here can be obtained by unravelling the description of permanent cycles of the spectral sequence associated to a double cocomplex with the filtration by columns, as we will see in the following. However, as we are considering a very explicit case, I found it helpful to give a name to the elements of this form.

Lemma 4.2.4. *Let G be a regular Lie groupoid and E a multiplicative Ehresmann connection with respect to the isotropy algebroid. Any closed descending homogeneous extension of a non-vanishing element $\omega_{(p,q)} \in \Omega^p(G_q)$ in the Bott–Shulman–Stasheff complex is cohomologically equivalent to a closed descending homogeneous extension of a $J^E G$-isotropy Cartan form in bidegree $(p+k, q-k)$ for $k \in \mathbb{N}_0$.*

Remark. Note that the $J^E G$-isotropy Cartan form is not unique.

Proof. Let ω denote the extension of $\omega_{(p,q)}$. Note, that in particular $\mathbf{d}(\omega) = 0$ implies $\mathrm{d}_{\mathcal{N}}(\omega_{(p,q)}) = 0$. By Corollary 3.1.25, we can choose ν such that $\omega_{(p,q)} + d_{\mathcal{N}}(\nu)$ is $J^E G$-isotropy Cartan. But as $d_{\mathcal{N}} = \delta$, we have that either $\omega_{(p,q)} + d_{\mathcal{N}}(\nu) = 0$ or $\omega + \mathbf{d}(\nu)$ is a descending homogeneous extension of

$$(\omega + \mathbf{d}(\rho + \sigma))_{(p,q)} = \omega_{(p,q)} + d_{\mathcal{N}}(\rho + \sigma),$$

which is a $J^E G$-isotropy Cartan form.

If we are in the case where $\omega_{(p,q)} + d_{\mathcal{N}}(\nu) = 0$, we still have that

$$(\omega + \mathbf{d}(\nu))_{(p',q')} = 0$$

for $q' > q$, as ν is homogeneous of bidegree $(q-1, p)$. Hence we obtain a closed descending homogeneous extension of some $(\omega + \mathbf{d}(\rho + \sigma))_{(p+l,q-l)}$ for some $l > 0$. By iterating the previous argument, we obtain a closed homogeneous extension of a $J^E G$-isotropy Cartan element (non-zero or zero) after a finite descent. □

Corollary 4.2.5. *Let G be a regular and proper Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Any closed descending homogeneous extension of a non-vanishing element $\omega_{(p,q)} \in \Omega^p(G_q)$ in the Bott–Shulman–Stasheff complex is cohomologically equivalent to a closed descending homogeneous extension of an element $\omega'_{(p',q')}$ of bidegree (p', q') with $p' \geq q'$.*

Proof. By Lemma 4.2.4, we have that ω is cohomologically equivalent to some ω' which is a closed descending homogeneous extension of a $J^E G$-isotropy Cartan element $\omega'_{(p',q')}$. However, all non-zero $J^E G$-isotropy Cartan forms have bidegrees that satisfy $p' \geq q'$, which proves the statement. □

Lemma 4.2.6. *Let G be a regular and proper Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Assume a $J^E G$-isotropy Cartan form $\omega_{(p,q)}$ has a closed homogeneous descending extension. Then $\omega_{(p,q)}$ is closed under the Cartan differential.*

Proof. Assume that $\omega_{(p,q)}$ has a closed homogeneous extension ω. Then we know that

$$\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)}) = -\,\mathrm{d}_{\mathcal{N}}(\omega_{(p+1,q-1)}).$$

Hence $\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})$ is closed and exact.

We know that the cohomology class of $\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})$ is determined by the $J^E G$-isotropy Cartan form $\mathrm{d}_{\mathrm{Cart}}(\omega_{(p,q)})$ by Proposition 4.1.20. So $\mathrm{d}_{\mathrm{Cart}}(\omega_{(p,q)})$ must be exact, too. But as a $J^E G$-isotropy Cartan form is exact if and only if it vanishes, this means that

$$\mathrm{d}_{\mathrm{Cart}}(\omega_{(p,q)}) = 0.$$

□

Our aim will be to show that the previous relation is actually an equivalence, and that $\mathrm{d}_{\mathrm{Cart}}(\omega_{(p,q)}) = 0$ is sufficient for $\omega_{(p,q)}$ to admit a closed descending homogeneous extension as well. In this section, however, we will mostly focus on analysing, how this assumption will give us an isomorphism between $H(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G)))$ and $H(\mathrm{Tot}(\mathcal{R}_E))$.

In the spectral sequence picture, $J^E G$-isotropy Cartan forms are the elements of the first page, and $\mathrm{d}_{\mathrm{Cart}}$ corresponds to the first page differential $d^{(1)}$. So $H(\mathrm{Tot}(\mathcal{R}_E))$ describes cohomology classes of the first page of the spectral sequence, while $H(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G)))$ describes cohomology classes of the of the limit of that spectral sequence. As such, it is quite unsurprising that for every element of the limit there are corresponding elements on the first page which are not always unique. In some cases which we will consider we will however be able to control the choice of their representatives.

That these maps agree in their construction with the methods used to obtain the previous results is only clear when going into the definition of the spectral sequence associated to a double complex and the definition of convergence ([42], pp. 125-126, 133, 135, 141-142). The same is true about the next proposition - but it can instead also be used as a black box on why studying closed descending homogeneous extensions in $\Omega_{\mathrm{BSS}}(G)$ is important.

Proposition 4.2.7. *Let G be a regular and proper Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Consider a $J^E G$-isotropy Cartan form $\omega_{(p,q)} \in \Omega^p(G_q)$ closed under $\mathrm{d}_{\mathrm{Cart}}$. The following are equivalent:*

- *$\omega_{(p,q)}$ admits a closed descending homogeneous extension ω.*
- *$\omega_{(p,q)}$ is a permanent cycle, i.e. on the n-th page of the spectral sequence for any $n \in \mathbb{N}_0$ we have that $d^{(n)}([\omega_{(p,q)}]) = 0$.*

Proof. For the definition of the spectral sequence associated to a filtration, consider [42] (cf. pp. 133, 141-142). The description is for a chain complex or bicomplex, not for the dual version, but we will describe the dual notions explicitly in this proof. Firstly, let us analyse how $d^{(n)}([\omega_{(p,q)}])$ and ω are related for $n \geq 2$. By spelling out $d^{(2)}$ as the canonical map of the double complex, we find that

$$d^{(2)}([\omega_{(p,q)}]) = [[\mathrm{d}_{\mathrm{dR}}(\nu_{(p+1,q-1)})]_{\mathrm{vert}}]_{\mathrm{hor}} \in H^{p+2}(\mathcal{R}_{q-1}),$$

where $\nu_{(p+1,q-1)}$ is an arbitrary element satisfying

$$-\,\mathrm{d}_{\mathcal{N}}(\nu_{(p+1,q-1)}) = \mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)}),$$

which exists as $\mathrm{d}_{\mathrm{Cart}}(\omega_{(p,q)}) = 0$. The brackets in the equation above denote

$$[\mathrm{d}_{\mathrm{dR}}(\nu_{(p+1,q-1)})]_{\mathrm{vert}} \in H^{q-1}(\mathfrak{C}_{p+2}) \cong \mathcal{R}^{p+2}_{q-1},$$

such that the bracket $[-]_{\mathrm{hor}}$ can then denote the according cohomology class.

If we now assume that $\omega_{(p,q)}$ admits a closed descending homogeneous extension, we find that $\omega_{(p+1,q-1)}$ is a possible choice for $\nu_{(p+1,q-1)}$. However, we know that

$$\mathrm{d}_{\mathrm{dR}}(\omega_{(p+1,q-1)}) = -\,\mathrm{d}_{\mathcal{N}}(\omega_{(p+2,q-2)}),$$

meaning that as

$$[\mathrm{d}_{\mathcal{N}}(\omega_{(p+2,q-2)})] = [0] \in H^{q-1}(\mathfrak{C}_{p+2}) \cong \mathcal{R}^{p+2}_{q-1}$$

the same holds in $H^{p+2}(\mathfrak{R}_{q-1})$.

For the other direction, assume that $d^{(2)}([\omega_{(p,q)}]) = 0$. This precisely means that for any choice of $\nu_{(p+1,q-1)}$, we have that

$$[[\mathrm{d}_{\mathrm{dR}}(\nu_{(p+1,q-1)})]_{\mathrm{vert}}]_{\mathrm{hor}} = [0] \in H^{p+2}(\mathcal{R}_{q-1}),$$

i.e that there is a preimage under $\mathrm{d}_{\mathrm{Cart}}$ of $[\mathrm{d}_{\mathrm{dR}}(\nu_{(p+1,q-1)})]_{\mathrm{vert}}$ as an element of $\mathcal{R}^{p+2}_{q-1}$ given by a $J^E G$-isotropy Cartan form $\rho_{(p+1,q-1)} \in \mathcal{R}^{p+1}_{q-1}$. By Proposition 4.1.20 this implies that $\mathrm{d}_{\mathrm{dR}}(\nu_{(p+1,q-1)})$ and $\mathrm{d}_{\mathrm{dR}}(\rho_{(p+1,q-1)})$ also only differ by a $\mathrm{d}_{\mathcal{N}}$-exact term. If we now choose

$$\omega_{(p+1,q-1)} = \nu_{(p+1,q-1)} - \rho_{(p+1,q-1)},$$

we find that

$$[\mathrm{d}_{\mathrm{dR}}(\omega_{(p+1,q-1)})]_{\mathrm{vert}} = [0] \in H(\mathfrak{C}^{p+2}_{q-1}) \cong \mathcal{R}^{p+2}_{q-1}.$$

This means that in particular the set of $\nu_{(p+2,q-2)}$ for which

$$\mathrm{d}_{\mathrm{dR}}(\omega_{p+1,q-1}) = -\,\mathrm{d}_{\mathcal{N}}(\nu_{p+2,q-2})$$

is non-empty.

Similarly, if $d^{(n-1)}([\omega_{(p,q)}]) = 0$, we have that

$$d^{(n)}([\omega_{(p,q)}]) \equiv [[\mathrm{d_{dR}}(\nu_{(p+n-1,q+n-1)})]_{\mathrm{vert}}]_{\mathrm{hor}} \in H^{p+n}(\mathcal{R}_{q-n+1})/\,\mathrm{Im}(d^{(2)}), \mathrm{Im}(d^{(3)}), \ldots$$

for any $\nu_{(p+n-1,q-n+1)}$ satisfying

$$\mathrm{d_{dR}}(\omega_{(p+n-1,q-n+1)}) = -\,\mathrm{d}_{\mathcal{N}}(\nu_{(p+n-1,q-n+1)}).$$

Again, if ω is a closed descending homogeneous extension, we may choose

$$\nu_{(p+n-1,q-n+1)} = \omega_{(p+n-1,q-n+1)}$$

which then proves the claim.

Conversely, if $d^{(n)}([\omega_{(p,q)}]) = 0$ we can first write

$$0 = [[\mathrm{d_{dR}}(\nu_{(p+n-1,q+n-1)})]_{\mathrm{vert}}]_{\mathrm{hor}} + d^{(n-1)}([\alpha^{(n-1)}]) + \cdots + d^{(2)}([\alpha^{(2)}]) \in H^{p+n}(\mathcal{R}_{q+n-1}).$$

Each representative $\alpha^{(l)} \in \mathcal{R}_E^{(p+n-l,q-n+l)}$ of $[\alpha^{(l)}]$ admits, precisely by the previous construction an extension by terms

$$\alpha^{(l)}_{(p+n-l+a,q-n+l-a)}$$

in bidegree $(p+n-l+a, q-n+l-a)$ for all $1 \leq a \leq l-2$. By definition of the respective $d^{(l)}$, we find that subtracting

$$\omega_{(p+k,q-k)} - \sum_{k=n-l}^{n-1} \alpha^{(l)}_{(p+k,q-k)}$$

yields terms $\omega'_{(p+k,q-k)}$ for $1 \leq k \leq n-2$ with the same properties (relative to each other) as the $\omega_{(p+k,q-k)}$. However, for the $\omega'_{(p+n-2,q+n-2)}$ we find that the associated expression

$$[[\mathrm{d_{dR}}(\nu'_{(p+n-1,q-n+1)})]_{\mathrm{vert}}]_{\mathrm{hor}} \in H^{p+n}(\mathcal{R}_{q+n-1})$$

is actually 0. So in this case, we can apply again the previous method to define

$$\omega_{(p+n-1,q-n+1)} = \nu'_{(p+n-1,q+n-1)} - \rho_{(p+n-1,q+n-1)}$$

for which then again the set of $\nu_{(p+n,q-n)}$ satisfying

$$\mathrm{d_{dR}}(\omega_{(p+n-1,q-n+1)}) = -\,\mathrm{d}_{\mathcal{N}}(\nu_{(p+n,q-n)})$$

is non-empty.

This procedure terminates in the case where $n = q+1$, in which case we find $\omega_{(p+q,0)}$ such that $\mathrm{d_{dR}}(\omega_{(p+q,0)})$ is $\mathrm{d}_{\mathcal{N}}$-exact, which for degree reasons implies

$$\mathrm{d_{dR}}(\omega_{(p+q,0)}) = 0.$$

$\square$

One of the implications of the previous statements is, that if any $J^E G$-isotropy Cartan form admits a closed descending homogeneous extension, then the associated graded to the filtration of the spectral sequence is given precisely by

$$H^\bullet(\mathrm{Tot}(\mathcal{R}_E)) \cong H^\bullet(\mathrm{Tot}(\mathfrak{R}_E)) = \bigoplus_{p+q=\bullet} H^p(\mathfrak{R}_q).$$

This then also computes the cohomology group of $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$.

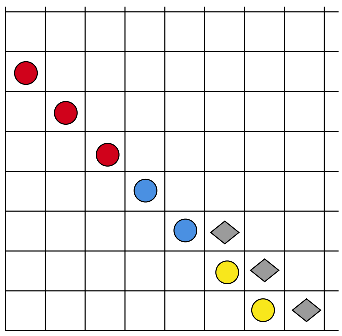
However, even without knowing that the condition holds in general, we can infer a lot from studying how the map from $H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G)))$ to $H^\bullet(\mathrm{Tot}(\mathfrak{R}_E))$ is constructed. Studying these maps explicitly for $\Omega_{\mathrm{BSS}}(G)$ will not only already suffice to compute multiple of our previous examples, but it will also help understanding how the algebra structure of $H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G)))$ relates to $H^\bullet(\mathrm{Tot}(\mathfrak{R}_E))$. We will still make notes every now and then to the relation with the spectral sequence.

Example 4.2.8. Consider the pair groupoids of Example 2.3.15. We now know that any closed element in $\Omega_{\mathrm{BSS}}(G)$ is cohomologically equivalent to the extension of a $J^E G$-isotropy Cartan form. However, we have already seen in Example 3.1.27, that there are no $J^E G$-isotropy Cartan forms in bidegree (p, q) with $p \neq 0$, so the only $J^E G$-isotropy Cartan forms will be in bidegree $(0, 0)$.

So any closed descending homogeneous extension of a $J^E G$-isotropy Cartan form of total degree $r > 0$ can only be an extension of 0, and hence must be 0 itself. Thus, any closed element in Ω_{BSS} of total degree $r > 0$ is exact.

Lemma 4.2.9. *Let G be a regular and proper Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. If there are two closed descending homogeneous extensions of (possibly different) $J^E G$-isotropy Cartan forms $\omega_{(p,q)}$ and $\omega'_{(p',q')}$ of total degree r which only differ by a $\mathbf{d}$-exact term, then the preimage of this term may be chosen as a descending homogeneous extension of an element $\nu_{(k,r-1-k)}$ in bidegree $(k, r-1-k)$ for $k \in \mathbb{N}_0$, where $2k \geq r-1$ (or equivalently $k \geq r-1-k$). The choice can additionally be made such that $\nu_{(k,r-1-k)}$ is $J^E G$-isotropy Cartan if $k \leq \max\{p, p'\}-1$.*

Proof. Let ν' denote a preimage of the exact term. It is a homogeneous descending extension of $\nu'_{(k,r-1-k)}$ for some $k \in \mathbb{N}_0$. Without loss of generality we may assume that ν' was chosen such that it maximises k (as $k \leq r$ by definition the maximum is well-defined). If k already satisfies $2k \geq r-1$ and $k \geq p$ and $k \geq p'$, the claim is trivially satisfied. In the following picture, this schematically corresponds to $(k, r-1-k)$ being one of the bidegrees marked with a yellow circle, where the grey diamonds mark the degrees in which ω and ω' are supported. The red circles denote where $2k < r-1$ (or equivalently $k < r-1-k$.)



In the following, we will simultaneously prove, that $(k, r-1-k)$ cannot correspond to one of the bidegrees schematically marked in red and also prove that if $(k, r-1-k)$

is in one of the bidegrees schematically marked in blue, then the component of ν' may be chosen J^EG-isotropy Cartan.

Consider the difference between the two closed descending homogeneous extensions. It is itself a closed descending homogeneous extension of an element $\omega_{(a,b)}$. We now claim that if one of the conditions $2k \geq r-1$, $k \geq p$ or $k \geq p'$ is violated, we find $k < a$: First note that by construction $a \geq b$, $a \geq p$, $a \geq p'$ and $a+b=r$. If $k < p$ or $k < p'$, it immediately follows that $k < a$. If $2k < r-1$, we can consider that $2a \geq a+b = r$, implying that $2k < 2a-1$ and thus also $k < a$, as k and a are natural numbers. So if $\nu'_{(k,r-1-k)}$ does contradict any one of the conditions $2k \geq r-1$ or $k \geq p$ or $k \geq p'$, we must have $k < a$.

We know that

$$\big(\mathbf{d}(\nu')\big)_{(a',b')} = \omega_{(a',b')} = 0$$

for all $a' < a$ and $b' > b$. This in particular implies that if $k < a$ we have

$$\mathrm{d}_{\mathcal{N}}(\nu'_{(k,r-1-k)}) = \big(\mathbf{d}(\nu')\big)_{(k,r-k)} = \omega_{(k,r-k)} = 0$$

and by Corollary 3.1.25 we can choose $\nu = \nu' + \mathbf{d}(\alpha_{(k,r-2-k)})$ to satisfy that $\nu_{(k,r-1-k)}$ is J^EG-isotropy Cartan, without compromising on the fact that $\nu_{(l,r-l-1)} = 0$ for $l < k$.

If furthermore $\nu_{(k,r-1-k)} = 0$ we have that ν is actually a descending homogeneous extension of an element $\nu_{(k',r-1-k')}$ for $k' > k$, contradicting our assumption of k being maximal. So ν must be actually be a descending homogeneous extension of a (non-zero) J^EG-isotropy Cartan form in bidegree $(k, r-1-k)$. In particular, this implies $k \geq r-1-k$ or equivalently $2k \geq r-1$ as all J^EG-isotropy Cartan forms are supported at and below the counter-diagonal. □

Example 4.2.10. Consider the Lie groupoid of a free and proper group action on a manifold as in Example 2.3.16. We now know that every cohomology class will contain an element given by a closed descending extension of a J^EG-isotropy Cartan form, which by Example 3.1.28 are all contained in bidegrees $(p, 0)$. In particular, the extension is just given by the element itself.

Now it only remains to analyse which of these elements are actually exact. By Lemma 4.2.9, two J^EG-isotropy Cartan forms of degree $(p, 0)$ are cohomologically equivalent if and only if their difference is given by $\mathrm{d}_{\mathcal{N}}(\nu)$ with ν a descending homogeneous extension of an element in bidegree $(k, r-1-k)$, where necessarily $k \leq p-1$ (as $p-1$ is its total degree), such that ν is also an extenstion of a J^EG-isotropy Cartan element and thus can only have leading component in degree $(p-1, 0)$, also making this term its only component.

Hence the cohomology of the entire bicomplex can be indeed computed by

$$H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))) \cong H^\bullet(\Omega_{\mathrm{bas}}(G_0)),$$

on which the total differential of the bicomplex restricts to d_{dR} and by Proposition 1.3.5, we have that

$$\Omega^\bullet_{\mathrm{bas}}(G_0) \cong \Omega^\bullet(Q)$$

where both complexes are equipped with the respective de Rham differentials. Note that this recovers the result of Corollary 0.1.5 using that

$$H_{\mathcal{G}}(M) \cong H^\bullet_{\mathrm{DiffStack}}([G_0//G_1]) \cong H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))).$$

Definition 4.2.11. An **orbifold** is a differentiable stack which can be presented by a proper étale Lie groupoid.

Example 4.2.12. Let G be a Lie groupoid presenting an orbifold, i.e. let G be proper and étale. We have covered this example already in Example 2.3.18 and Example 3.1.28. The multiplicative Ehresmann connection on G is given by the entire tangent space and therefore automatically flat. Its $J^E G$-isotropy Cartan forms are, as we have previously computed given in degree $(p, 0)$ by

$$\Omega^p_{\text{bas}}(G_0)$$

and in degrees where $q > 0$ by 0.

This means that in analogy to Example 4.2.10, we obtain

$$H^\bullet_{\text{DiffStack}}([G_0//G_1]) \cong H^\bullet(\text{Tot}(\Omega_{\text{BSS}}(G))) = H^\bullet(\Omega_{\text{bas}}(G_0)),$$

where the $J^E G$-Cartan differential acts as the de Rham differential d_{dR} on $\Omega_{\text{bas}}(G_0)$. The latter term is even invariant under Morita equivalence, such that the cohomology of the orbifold can be computed as $H^\bullet(\Omega_{\text{bas}}(G'_0))$ for any Lie groupoid representation G'.

This line of argument does not depend on the orbits being discrete, such that the analogue argument holds in the case of Example 2.3.19 where $A_{\text{iso}} = 0$. We already noted this fact when computing the $J^E G$-isotropy Cartan forms in Example 3.1.28.

Lemma 4.2.13. *Let G be a proper and regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. When restricted to homogeneous descending extensions where the leading term is in bidegree (p, q) with $p \geq k$, the projections of $\Omega_{\text{BSS}}(G)$ to $\mathfrak{C}_k$ induces a map*

$$\Phi^k : H^{2k}(\text{Tot}(\Omega_{\text{BSS}})) \to H^k(\mathfrak{C}_k) \cong \mathcal{R}_E^{(k,k)} \cong \left(\Gamma\left(G_0, S^k(A^*_{\text{iso}})\right)\right)^G.$$

Proof. Assume we are given an element $\omega \in \Omega_{\text{BSS}}$ which is closed under the total differential $\mathbf{d}$ to represent a cohomology class of $H^{2k}(\text{Tot}(\Omega_{\text{BSS}}))$. Then we know by Lemma 4.2.4 that we may assume ω to be a closed homogeneous descending extension of a $J^E G$-isotropy Cartan element $\omega_{(p,q)}$ with $p+q = 2k$ and implicitly $p \geq q$. Hence $p \geq k$ and $\omega_{k,k}$ is in all cases $J^E G$-isotropy Cartan (non-zero only if $p = q = k$). So it indeed represents a cohomology class in $H^k(\mathfrak{C}_k)$.

We still need to show that this is a well-defined assignment in the sense that indeed each representative ω' which is a homogeneous descending extension with leading component in bidegree (p', q') with $p' \geq k$ projects to the same cohomology class in $H^k(\mathfrak{C}_k)$.

Firstly note that ω' actually projects to a closed element, as either $\omega'_{(k,k)} = 0$ or $\omega'_{(k,k)} = \omega'_{(p',q')}$ if $p' = q' = k$ and in that case $\mathbf{d}(\omega') = 0$ implies $\mathrm{d}_{\mathcal{N}}(\omega'_{(p,q)}) = 0$. Secondly note that the cohomology class of this element in $H^k(\mathfrak{C}_k)$ is not impacted by adding $\mathbf{d}(\rho)$ for ρ only supported in bidegree $(p', q'-1)$ either, such that we may assume $\omega'_{(p',q')}$ to be $J^E G$-isotropy Cartan as well.

By Lemma 4.2.9 we have that $\omega - \omega'$ can be expressed by $\mathbf{d}(\nu)$, where ν is a descending homogeneous extension of an element in bidegree $(l, p+q-1-l)$ where $2l \geq p+q-1 = 2k-1$, implying $l \geq k$ because both l and k are natural numbers.

This, however, implies that the component of $\mathbf{d}(\nu)$ in bidegree (k, k) is given by $\mathrm{d}_{\mathcal{N}}(\nu_{(k,k-1)})$. Hence $\omega_{(k,k)}$ and $\omega'_{(k,k)}$ only differ by an exact term in $\mathfrak{C}_k$ and represent the same cohomology class in $H^k(\mathfrak{C}_k)$. $\square$

Corollary 4.2.14. *Let G be a proper and regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. The image of the map Φ^k defined in Lemma 4.2.13 is contained in*

$$\left\{\omega \in \mathcal{R}_E^{(k,k)} \text{ s.t. } \widehat{\mathrm{d}^E_{\text{Cart}}}(\omega) = 0\right\} = H^k(\mathcal{R}_k) \cong H^k(\mathfrak{R}_k)$$

Proof. Φ^k is defined by projecting closed descending homogeneous extensions ω to $J^E G$-isotropy Cartan forms $\omega_{(k,k)}$ such that either $\omega_{(k,k)} = 0$ or ω is a closed descending homogeneous extension of $\omega_{(k,k)}$. By Lemma 4.2.6, all of these $\omega_{(k,k)}$ satisfy that $\widehat{\mathrm{d}^E_{\mathrm{Cart}}}(\omega_{(k,k)}) = 0$, thus we can restrict the image. □

Our next aim will be to define an analogous map for the projection of $H^{2k+1}(\Omega_{\mathrm{BSS}}(G))$. From the spectral sequence picture, we already have the intuition that the "correct" target for such a map is $H^{k+1}(\mathcal{R}_k)$ or $H^{k+1}(\mathfrak{R}_k)$ instead of $H^k(\mathfrak{C}_{k+1})$ (despite the indices looking differently, we have that this is the natural choice as both arise from $\Omega^{k+1}(G_k)$). However, we can also see why very explicitly: There may be elements ω which are exact in $H^{2k+1}(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G)))$ and hence agree with 0 cohomologically. But they do not necessarily project to 0 in $H^k(\mathfrak{C}_{k+1})$, making the assignment not well-defined.

This problem for example arises for exact elements of the form $\omega = \mathbf{d}(\nu_{(k,k)})$ where the preimage $\nu_{(k,k)} \in \Omega^k(G_k)$ is some $J^E G$-isotropy Cartan form. Hence ω only has one non-vanishing component $\omega_{(k+1,k)}$ in bidegree $(k+1, k)$ given by

$$\omega_{(k+1,k)} = \mathrm{d}_{\mathrm{dR}}(\nu_{(k,k)})$$

which is not necessarily exact in $\mathfrak{C}$, but would rather project to the $J^E G$-isotropy Cartan form $\mathrm{d}^E_{\mathrm{Cart}}(\nu_{(k,k)})$, as we have seen in the previous chapter. However, if we consider this as an element of

$$H^{k+1}\left(H^k(\mathfrak{C}_\bullet), [\mathrm{d}_{\mathrm{dR}}]\right) = \frac{\ker([\mathrm{d}_{\mathrm{dR}}]) \subset H^k(\mathfrak{C}_{k+1})}{Im([\mathrm{d}_{\mathrm{dR}}]) \subset H^k(\mathfrak{C}_{k+1})}$$

we are quotienting out precisely all terms of this form, such that ω and 0 are equivalent again.

This also corresponds to our claim that $H^\bullet(\mathrm{Tot}(\mathfrak{R}_E))$ is a model for $H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G)))$. In the following, we will examine the maps that extend the previously described Φ^k to each $H^\bullet(\mathfrak{R}_q)$ explicitly and also how they assemble into one unified construction.

Proposition 4.2.15. *Let G be a proper and regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Assume that any $J^E G$-isotropy Cartan form which is closed under* $\mathrm{d}_{\mathrm{Cart}}$ *admits a closed descending homogeneous extension. Then there are well defined maps $\Phi^{(p,q)}$ such that for $p = q = k$ we have*

$$\Phi^{(k,k)} = \Phi^k : H^{2k}(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))) \to H^k(\mathcal{R}_k) \cong H^k(\mathfrak{R}_k) \subset H^{2k}(\mathrm{Tot}(\mathcal{R}_E))$$

for $p = q + 1$ we have

$$\Phi^{(p,q)} : H^{p+q}(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))) \to H^p(\mathcal{R}_q) \cong H^p(\mathfrak{R}_q) \subset H^{p+q}(\mathrm{Tot}(\mathcal{R}_E))$$

and for $p \geq q + 2$

$$\Phi^{(p,q)} : \ker(\Phi^{(p-1,q+1)}) \to H^p(\mathcal{R}_q) \cong H^p(\mathfrak{R}_q) \subset H^{p+q}(\mathrm{Tot}(\mathcal{R}_E)).$$

All of these maps are surjective onto $H^{p+q}(\mathcal{R}_q)$. Furthermore, we have that if any $[\omega] \in H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G)))$ is mapped to 0 under all $\Phi^{(p,q)}$, then $[\omega] = 0$.

Proof. The map $\Phi^{(p,q)}$ for $p = q = k$ is already well-defined due to Lemma 4.2.13 and Corollary 4.2.14. Also, the assumption that any $J^E G$-isotropy Cartan form which is closed under the Cartan differential (which includes all cycles in $\mathfrak{R}_k$) admit a closed descending homogeneous extension by definition implies that the map is surjective after restriction of the target space.

We will define the map $\Phi^{(p,q)}$ for $p = q + 1$ similar to Lemma 4.2.13 on descending homogeneous extensions. First, let us show that any $[\omega] \in \Omega_{\mathrm{BSS}}(G)$ of degree $p+q$ can be represented by a homogeneous descending extension with leading component in bidegree (p', q') for $p' \geq p$:

We know by Lemma 4.2.4, that any ω representing a cohomology class in $\Omega_{\mathrm{BSS}}(G)$ may be chosen as a closed descending homogeneous extension of a $J^E G$-isotropy Cartan element. This in particular means that the leading component is supported in bidegree (p', q') with $p' \geq q'$. This implies $2p' \geq p + q = 2p - 1$, but because both numbers are even integers we obtain $p' \geq p$. So the value of $\Phi^{(p,q)}([\omega])$ can indeed be defined as the projection to bidegree (p, q) which yields either 0 or a non-zero $J^E G$-isotropy Cartan form.

Also, consider any other ω' that is a closed descending homogeneous extension of a $J^E G$-isotropy Cartan form that represents the same cohomology class. By Lemma 4.2.9, we may choose the preimage ν of their difference $\omega - \omega'$ as a descending homogeneous extension of an element in bidegree $(k, p + q - 1 - k)$ for $k \in \mathbb{N}_0$ with $2k \geq p + q - 1$. As $p + q - 1 = 2p - 2$, we obtain $k \geq p - 1$.

If $k \geq p$, this implies that $\omega_{(p,q)}$ and $\omega'_{(p,q)}$ only differ by $\mathrm{d}_{\mathcal{N}}(\nu_{(p,q-1)})$. However, as $J^E G$-isotropy Cartan forms are uniquely determined by their cohomology classes in $\mathfrak{C}_p$ by Theorem 3.1.20 this would imply $\omega_{(p,q)} = \omega'_{(p,q)}$, meaning that the image of $[\omega] = [\omega']$ under $\Phi^{(p,q)}$ is well-defined. So it remains to consider the case $k = p - 1$.

We know that in that case by Lemma 4.2.9 that we may additionally assume ν to be a descending homogeneous extension of a $J^E G$-isotropy Cartan element. In that case, we have that $\omega_{(p,q)}$ and $\omega'_{(p,q)}$ only differ by definition by

$$\mathrm{d}_{\mathrm{dR}}(\nu_{(p-1,q)}) + (-1)^p \,\mathrm{d}_{\mathcal{N}}(\nu_{(p,q-1)})$$

which decomposes into a $\mathrm{d}_{\mathcal{N}}$-exact term and the expression $\mathrm{d}_{\mathrm{Cart}}(\nu_{(p-1,q)})$ by Proposition 4.1.20. But $\mathrm{d}_{\mathrm{Cart}}(\nu_{(p-1,q)})$ is one of the boundaries in $\mathcal{R}^q$ by definition, hence ω and ω' get mapped to the same cohomology class in $H^p(\mathcal{R}^q)$.

Again, as in the case with $p = q$, we have that the map is surjective under our additional assumption, as every closed descending homogeneous extension of a $J^E G$-isotropy Cartan form is mapped to the cohomology class of precisely that $J^E G$-isotropy Cartan form.

The proof for the remaining cases is quite similar. But there are some things to note. Firstly, we can note that a closed descending homogeneous extension ω of a $J^E G$-isotropy Cartan form $\omega_{(p-1,q+1)}$ in degree $(p - 1, q + 1)$ is only in $\ker(\Phi^{(p-1,q+1)})$ if there is a cohomologically equivalent form in $\Omega_{\mathrm{BSS}}(G)$ which satisfies that the leading component is in degree (p', q') with $p' > p - 1$, i.e. $p' \geq p$.

We achieve this by adding $\mathbf{d}(-\nu_{(p-2,q+1)})$, where $\nu_{(p-2,q+1)}$ is some $J^E G$-isotropy Cartan form in bidegree $(p - 2, q + 1)$ of $\Omega_{\mathrm{BSS}}(G)$ corresponding to a preimage of the boundary $\omega_{(p-1,q+1)}$ in $\mathcal{R}^{q+1}$ and then some term $\nu_{(p-1,q)}$ in bidegree $(p - 1, q)$ to cancel the remaining components of $\mathrm{d}_{\mathrm{dR}}(-\nu_{(p-2,q+1)})$ which are by Proposition 4.1.20 exact in $\mathfrak{C}^{p-1}$.

So indeed, we can define the map $\Phi^{(p,q)}$ on the entire kernel as the map induced by projecting a closed descending homogeneous extension ω of an element in bidegree (p', q') which we may assume to be $J^E G$-isotropy Cartan and satisfying $p' \geq p$ to its component in degree (p, q).

Also, if we have a $\mathbf{d}$-exact element, which is a closed descending homogeneous extension of an element in bidegree (p, q) and its preimage ν is a descending homogeneous extension of an element in bidegree (p', q'), as long as $p' < p$, we may assume $\nu_{(p',q')}$ is $J^E G$-isotropy Cartan.

If further, $p' < p-1$, we have that $\mathrm{d}_{\mathrm{dR}}(\nu_{(p',q')})$ is exact as an element of $\mathfrak{C}^{p'+1}$ and hence $\nu_{(p',q')}$ is closed under $\mathrm{d}_{\mathrm{Cart}}$. By assumption, this means that $\nu_{(p',q')}$ admits a closed descending homogeneous extension, which we may subtract from ν without altering its image, hence showing that we can gradually increase the value of p' until $p' \geq p-1$. Then we can proceed analogous to the proof for $p = q+1$.

For the statement regarding the case of $[\omega] \in \Omega_{\mathrm{BSS}}(G)$ which gets mapped to 0 we can note the following: We have already seen in this proof that $[\omega]$ is precisely in $\ker(\Phi^{(p,q)})$ if the representative ω of the cohomology class can be chosen to be a closed descending homogeneous extension of a $J^E G$-isotropy Cartan element in bidegree $(p+1, q-1)$.

So if $[\omega]$ is iteratively mapped to 0 by all maps, this must mean that ω can be chosen as an extension of an element of bidegree degree (p', q') with p' iteratively being increased to $p+q$, at which point $\Phi^{(p,q)}(\omega) = \omega$, such that this representative must be identically 0, and therefore the entire cohomology class vanishes. □

Theorem 4.2.16. *Let G be a proper and regular Lie groupoid and E a multiplicative Ehresmann connection with respect to the isotropy algebroid. Assume that there was a linear assignment from $J^E G$-isotropy Cartan forms closed under the Cartan differential to their closed descending homogeneous extensions in* $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$*, such that cohomologically equivalent forms in* $\mathrm{Tot}(\mathfrak{R}_E)$ *would be mapped to cohomologically equivalent forms in* $\Omega_{\mathrm{BSS}}(G)$*.*

Then there exists a graded vector space isomorphism

$$\Phi : H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))) \to H^\bullet(\mathrm{Tot}(\mathcal{R}_E)) \cong H^\bullet(\mathrm{Tot}(\mathfrak{R}_E))$$

extending the maps $\Phi^{(p,q)}$ of Proposition 4.2.15.

The inverse Φ^{-1} is described precisely by associating $[\omega] \in H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G)))$ to $[\omega_{(p,q)}] \in H^\bullet(\mathrm{Tot}(\mathcal{R}_E))$ where ω is the closed descending homogeneous extension associated to $\omega_{(p,q)}$.

Remark. From the from of Φ^{-1} it is immediately clear that Φ depends strongly on the choice of closed descending homogeneous extensions. As the choice of an extension ω of $\omega_{(p,q)}$ is not unique, and in particular can be replaced by $\omega + \nu$ for ν any closed descending homogeneous extension of $\nu_{(p+k,q-k)}$, we find that there are many different choices for the map Φ that can arise from this construction.

Proof. To define this map, we consider projections from $\ker(\Phi^{(p-1,q+1)})$ (or $H^{p+q}(\Omega_{\mathrm{BSS}}(G))$ for $p = q$ and $p = q+1$) to $\ker(\Phi^{(p,q)})$, which is a subspace by definition. If we have a family of these, we can extend maps from $\ker(\Phi^{(p,q)})$ to $\ker(\Phi^{(p-1,q+1)})$ iteratively until we can extend to $H^{p+q}(\Omega_{\mathrm{BSS}}(G))$ for $p = q+1$ or $p = q$.

We can define them through linear involutions on $\ker(\Phi^{(p-1,q+1)})$ that vanishes on $\ker(\Phi^{(p,q)})$ instead (because subtracting such a map from the identity yields a projection). For this, first apply $\Phi^{(p,q)}$ to obtain a cohomology class in $H^p(\mathcal{R}_q)$ which can be represented by a $J^E G$-isotropy Cartan form and then apply the assignment from $J^E G$-isotropy Cartan forms to their extensions of the assumption. This is not well-defined as a map to $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$, but by assumption on the assignment it is well-defined as a map to $H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G)))$. Furthermore, as the resulting cohomology class is given by the difference of a class in $\ker(\Phi^{(p-1,q+1)})$ and a class given by the extension of an element in bidegree (p,q) its representative can be chosen as a descending homogeneous extension as an element in bidegree (p', q') with $p' \geq p$ as well. Lastly, note that the assignment was assumed to be linear, such that on $\ker(\Phi^{(p,q)})$, we obtain the 0 cohomology class through this construction.

Once we have the projections (which we can concatenate as well), we can extend the maps $\Phi^{(p,q)}$ to all of $H^{p+q}(\Omega_{\mathrm{BSS}}(G))$. By summing up over all the results under these maps $\Phi^{(p,q)}$ we obtain a single map from $H^k(\Omega_{\mathrm{BSS}}(G))$ to

$$\bigoplus_{p+q=k} H^p(\mathcal{R}_q)$$

which agrees with $H^{p+q}(\mathrm{Tot}(\mathcal{R}_E))$ by Corollary 4.1.28. The injectivity and surjectivity follow from Proposition 4.2.15.

Also note that, by definition, all but one $\Phi^{(p,q)}$ vanish on any non-zero closed descending homogeneous extension in the image of the assignment from the assumption: If ω is the closed descending homogeneous extension assigned to some $J^E G$-isotropy Cartan form $\omega_{(p',q')}$, we have that $\omega_{(p',q')} \in \ker(\Phi^{(p,q)})$ as long as $p < p'$ (or equivalently $q > q'$). Furthermore $\Phi^{(p',q')}(\omega) = \omega_{(p',q')}$ by definition, and also by definition ω vanishes under the projection to $\ker(\Phi^{(p',q')})$. This now also proves the fact about the inverse, as summing up over all the images of ω under the different maps will precisely yield $\omega_{(p',q')}$ and therefore $[\omega]$ is the unique preimage of $[\omega_{(p,q)}]$. □

Remark. Under the assumption of the axiom of choice, the assumptions of Proposition 4.2.15 and Theorem 4.2.16 are equivalent.

So we now have a potential model for the cohomology of $\mathrm{Tot}(\Omega_{\mathrm{BSS}})$ and hence a potential model for

$$H_{\mathrm{DiffStack}}([G_0//G_1]).$$

The only caveat is that for it to hold, we need to prove the assumptions of Proposition 4.2.15 and Theorem 4.2.16. This will be of varying difficulty depending on the case. For example, in the next section we are going to consider a flat multiplicative Ehresmann connection, for which the assignment can be constructed more easily.

However, note that we have proven in Corollary 4.1.25 that the model $\mathfrak{R}_E$ is actually independent of the chosen Ehresmann connection. So if we obtain that $\mathcal{R}_E$ is a model for $\Omega_{\mathrm{BSS}}(G)$, we can actually compute $\mathfrak{R}_E$ by using an arbitrary multiplicative Ehresmann connection in the process. As it turns out, we can actually already replace $\mathcal{R}_E$ by $\mathcal{R}_F$ much earlier in the construction of the model.

Lemma 4.2.17. *Let G be a proper and regular Lie groupoid. Assume there is a multiplicative Ehresmann connection E with respect to the isotropy algebroid such that any $J^E G$-isotropy Cartan form closed under $\mathrm{d}^E_{\mathrm{Cart}}$ has a closed descending homogeneous extension in Ω_{BSS}. Then any multiplicative Ehresmann connection F with respect to the isotropy algebroid satisfies that any $J^E G$-isotropy Cartan form closed under $\mathrm{d}^F_{\mathrm{Cart}}$ has a closed descending homogeneous extension.*

Assume that there is a well-defined linear assignment

$$\omega_{(p,q)} \mapsto \omega$$

between $J^E G$-isotropy Cartan forms closed under $\mathrm{d}_{\mathrm{Cart}}$ and their extensions, such that cohomologically equivalent elements of $\mathcal{R}^p_q$ are mapped to cohomologically equivalent elements in $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$. Then there is a well-defined induced assignment from $J^F G$-isotropy Cartan forms to their extensions satisfying the same properties.

Proof. By Corollary 3.1.22, we know that there is a linear isomorphism $\psi^{E,F}$ between $J^E G$-isotropy Cartan forms and $J^F G$-isotropy Cartan forms, where E and F are any two multiplicative Ehresmann connections. It is constructed such that for the cohomology classes of corresponding elements holds

$$[\psi^{E,F}(\omega_{(p,q)})] = [\omega_{(p,q)}] \in H^q(\mathfrak{C}_p).$$

In particular, this implies the existence of a preimage $\nu \in \mathfrak{C}_p^{q-1} = \Omega^p(G_{q-1})$ of their difference $\psi^{E,F}(\omega_{(p,q)}) - \omega_{(p,q)} \in \mathfrak{C}_p^q$ under $\mathrm{d}_{\mathcal{N}}$. While it is not canonical, we can use the construction of Corollary 3.1.25 to choose the preimage, which in particular yields a linear assignment between $J^E G$-isotropy Cartan forms and preimages of this form and an induced linear assignment between $J^F G$-isotropy Cartan forms and preimages of this form.

So if there is a closed descending homogeneous extension ω for the $J^E G$-isotropy Cartan form $\omega_{(p,q)}$, we can add $\mathbf{d}(\nu)$ to obtain a closed descending homogeneous extension of $\psi^{E,F}(\omega_{(p,q)})$. Similarly, we can construct a closed descending homogeneous extension of $\omega_{(p,q)}$ if one is given for $\psi^{E,F}(\omega_{(p,q)})$. Note that by construction, the elements assigned to $\omega_{(p,q)}$ and $\psi^{E,F}(\omega_{(p,q)})$ are cohomologically equivalent in $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$.

This now shows that $\psi^{E,F}(\omega)$ has a closed descending homogeneous extension if and only if ω has a closed descending homogeneous extension. This is, by assumption, exactly the case when $\mathrm{d}^E_{\mathrm{Cart}}(\omega) = 0$. We now need to show that this condition is met whenever $\mathrm{d}^F_{\mathrm{Cart}}(\psi^{E,F}(\omega)) = 0$. This statement suffices, as $\psi^{E,F}$ is an isomorphism and we can reconstruct a corresponding ω for any $J^F G$-isotropy Cartan form.

However, we have by Proposition 4.1.20 that $\mathrm{d}^E_{\mathrm{Cart}}(\omega)$ agrees with $\mathrm{d}_{\mathrm{dR}}(\omega)$ in $H(\mathfrak{C}^{p+1})$ and because $d = \mathrm{d}_{\mathrm{dR}}$ and $\mathrm{d}_{\mathcal{N}}$ anti-commute, we also have that

$$[\mathrm{d}_{\mathrm{dR}}(\psi^{E,F}(\omega))] = [\mathrm{d}_{\mathrm{dR}}(\omega)] \in H^q(\mathfrak{C}^{p+1}).$$

So we obtain that $\psi^{E,F}(\omega)$ has a closed descending homogeneous extension if and only if $\mathrm{d}_{\mathrm{dR}}(\psi^{E,F}(\omega))$ is exact, which is the case exactly if it vanishes under $\mathrm{d}^F_{\mathrm{Cart}}$. □

This now tells us that the condition of the existence of an assignment of the form described in Proposition 4.2.15 and Theorem 4.2.16 does not actually depend on the choice of multiplicative Ehresmann connection. As previously noted, there are particularly well-suited Ehresmann connections to construct the assignments for. Before we will turn to that aspect, we will however make some additional observations on the algebraic structures involved. The aim in this is to understand, under which conditions the map Φ of Theorem 4.2.16 is not only an isomorphism of graded vector spaces, but of graded $\mathbb{R}$-algebras.

Lemma 4.2.18. *Consider a Lie groupoid G and two descending homogeneous extensions ω and ν in $\Omega_{\mathrm{BSS}}(G)$.*

Then the product $\omega \cup \nu$ for the product on the Bott–Shulman–Stasheff bicomplex as defined in Definition 0.2.27 is a descending homogeneous extension. If ω and ν are closed, so is $\omega \cup \nu$.

Furthermore, if $\omega_{(p,q)}$ and $\nu_{(p',q')}$ are the elements that are the leading components of ω and ν, we have that $\omega \cup \nu$ extends

$$\mathrm{pr}^*_{1,\ldots,q}\,\omega_{(p,q)} \wedge \mathrm{pr}^*_{q+1,\ldots,q+q'}\,\nu_{(p',q')}$$

Proof. The first assertion is true, as multiplication on $\Omega_{\mathrm{BSS}}(G)$ works component-wise and the bidegree and algebra structure are all compatible by definition. The second assertion follows from the definition of $\omega_{(p,q)} \cup \nu_{(p',q')}$ and the fact that all other terms have bidegree (a, b) with $a > p + p'$ or equivalently $b < q + q'$. □

Lemma 4.2.19. *Consider a regular Lie groupoid G and assume that it is equipped with a multiplicative Ehresmann connection E (with respect to the isotropy algebroid). Consider $\omega \in \Omega^p(G_q)$ and $\nu \in \Omega^{p'}(G_{q'})$.*

Assume that ω is homogeneous of E-vertical degree p_1, and ν is homogeneous of E-vertical degree r_1. Then $\omega \cup \nu$ is homogeneous of E-vertical degree $p_1 + r_1$, where $\omega \cup \nu$ is the multiplication defined in Definition 0.2.27.

Proof. By definition, we have that

$$\omega \cup \nu = \mathrm{pr}_{1,\dots,q}^* \omega \wedge \mathrm{pr}_{q+1,\dots,q+q'}^* \nu.$$

As the projections (by definition of $E_q \subset TG_q$) retain the splitting of E, we find that indeed,

$$\mathrm{pr}_{1,\dots,q}^* \omega$$

has E-vertical degree p_1 (and E-horizontal degree p_0)

$$\mathrm{pr}_{q+1,\dots,q+q'}^* \nu$$

has E-vertical degree r_1 and their wedge product has E-vertical degree $p_1 + r_1$. □

Lemma 4.2.20. *Consider a proper and regular Lie groupoid G and assume that it is equipped with a multiplicative Ehresmann connection E (with respect to the isotropy algebroid). Let ω and ν be $J^E G$-isotropy Cartan forms. Then*

$$\mathrm{d}_{\mathcal{N}}(\omega \cup \nu) = 0$$

and the unique $J^E G$-isotropy Cartan form representing $[\omega \cup \nu] \in H^{q+s}(\mathfrak{C}_{p+r})$ (i.e. the vertical cohomology class of $\omega \cup \nu$) corresponds to $\omega \breve{\cdot} \nu$ (i.e. the product of $J^E G$-isotropy Cartan forms as in Definition 4.1.13).

Proof. We know by Lemma 0.2.28 that the differentials on $\Omega_{\mathrm{BSS}}(G)$ are compatible with the cup product multiplication. So, indeed, as all $J^E G$-isotropy Cartan forms are closed under $\mathrm{d}_{\mathcal{N}}$, we have

$$\mathrm{d}_{\mathcal{N}}(\omega \cup \nu) = \mathrm{d}_{\mathcal{N}}(\omega) \cup \nu + (-1)^q \omega \cup \mathrm{d}_{\mathcal{N}}(\nu) = 0,$$

where (p, q) is the bidegree of ω.

We also have by Lemma 2.1.23, that projections retain $J^E G$-invariance, such that

$$\mathrm{pr}_{1,\dots,q}^* \omega$$

and

$$\mathrm{pr}_{q+1,\dots,q+s}^* \nu$$

are $J^E G$-invariant elements. Their multidegree is given by

$$(p_0, 1, \dots, 1, 0, \dots, 0, q+s)$$

and

$$(r_0, 0, \dots, 0, 1, \dots, 1, q+s),$$

as the projections that appear retain the unit section, where they respectively correspond on $(A_{\mathrm{iso}})^{\oplus q}$ to projection to the first q or last s copies of A_{iso}.

Now the wedge product of these two forms is a $J^E G$-invariant form of multidegree

$$(p_0 + r_0, 1, \dots, 1, q + s),$$

and by definition its values at the unit section are given by the fibre-wise wedge product on

$$\Gamma\left(G_0, \wedge^\bullet\left((TG_0/\mathcal{F}_0)^* \oplus ((A^*_{\mathrm{iso}})^{\oplus q+s})\right)\right) = \Gamma\left(G_0, \wedge^\bullet (TG_0/\mathcal{F}_0)^* \otimes \wedge^\bullet ((A^*_{\mathrm{iso}})^{\oplus q+s})\right).$$

When identifying the wedge product of disjoint summands with the tensor product of these summands, we have that the $J^E G$-invariant forms of multidegree

$$(p_0, 1, \dots, 1, 0, \dots, 0, q + s)$$

can be identified with elements of

$$\Gamma(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes A^*_{\mathrm{iso}} \otimes A^*_{\mathrm{iso}} \otimes \dots A^*_{\mathrm{iso}}) = \Gamma(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes (A^*_{\mathrm{iso}})^{\otimes q}).$$

The analogous statement holds for elements of the other relevant multidegrees, i.e.

$$(r_0, 0, \dots, 0, 1, \dots, 1, q + s) \quad \text{as well as} \quad (r_0, 1, \dots, 1, q + s).$$

Under this identification, the wedge product of $(\mathrm{pr}^*_{1,\dots,q}\,\omega)|_{1_{G_0}}$ and $(\mathrm{pr}^*_{q+1,\dots,q+s}\,\nu)|_{1_{G_0}}$ which by definition agrees with $\omega \cup \nu$ can be computed by combining the wedge product on $\wedge^\bullet(TG_0/\mathcal{F}_0)^*$ with the map induced by the tensor product on

$$(A^*_{\mathrm{iso}})^{\otimes q} \times (A^*_{\mathrm{iso}})^{\otimes s} \to (A^*_{\mathrm{iso}})^{\otimes q+s}.$$

We can now use this explicit form of $\omega \cup \nu$ to prove the statement: By Lemma 3.1.19, the unique $J^E G$-isotropy Cartan form contained in $[\omega \cup \nu]$ is the symmetrisation of $\omega \cup \nu$ as an element of

$$\Gamma(G_0, (TG_0/\mathcal{F}_0)^* \otimes A_{\mathrm{iso}} \otimes A_{\mathrm{iso}} \otimes \dots A_{\mathrm{iso}}).$$

But the multiplication $\omega \,\check{\cdot}\, \nu$ is induced precisely by the multiplication on symmetric tensor powers

$$S^q(A^*_{\mathrm{iso}}) \times S^s(A^*_{\mathrm{iso}}) \to S^{(q+s)}(A^*_{\mathrm{iso}}),$$

which factors through the multiplication on tensor powers as above when including the terms of the domain e.g. $S^q(A^*_{\mathrm{iso}}) \subset (A^*_{\mathrm{iso}})^{\otimes q}$ as permutation invariant elements and considering $S^{(q+s)}(A^*_{\mathrm{iso}})$ as a quotient of $\mathcal{T}^{(q+s)}(A^*_{\mathrm{iso}})$. So we have that

$$\pi(\omega \cup \nu) = \omega \,\check{\cdot}\, \nu,$$

where π denotes precisely that quotient projection.

Incidentally, π agrees with the map of Lemma 3.1.19, because $\omega \cup \nu$ already is contained in the correct multidegrees. As π is shown to be the quasi-inverse to the inclusion, it follows that

$$[\omega \cup \nu] = [\omega \,\check{\cdot}\, \nu]$$

as cohomology classes in $H^{(q+s)}(\mathfrak{C}_{(p+r)})$. □

Lemma 4.2.21. *Let G be a proper and regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Assume that $[\alpha] \in H^{p+q}(\mathrm{Tot}(\Omega_{\mathrm{BSS}}))$ and $[\beta] \in H^{r+s}(\mathrm{Tot}(\Omega_{\mathrm{BSS}}))$ are two cohomology classes which are in the domains of $\Phi^{(p,q)}$ and $\Phi^{(r,s)}$ respectively. Consider corresponding representatives α*

and β given by closed homogeneous descending extensions of J^EG-isotropy Cartan forms in degrees $(p+k, q-k)$ and $(r+l, q-l)$ for $k, l \in \mathbb{N}_0$ respectively.

Then the cohomology class of their product

$$[\alpha \cup \beta] \in H^{p+q+r+s}(\mathrm{Tot}(\Omega_{\mathrm{BSS}}))$$

with respect to the non-commutative multiplication of the Bott–Shulman–Stasheff complex described in Definition 0.2.27 maps under $\Phi^{(p+r,q+s)}$ to the J^EG-isotropy Cartan form given by

$$\Phi^{(p+r,q+s)}([\alpha \cup \beta]) = \Phi^{(p,q)}([\alpha]) \mathbin{\check{\cdot}} \Phi^{(r,s)}([\beta]).$$

Proof. Let us first note that by construction, we have

$$\Phi^{(p,q)}(\alpha) \equiv [\alpha_{(p,q)}] \in H^\bullet(\mathrm{Tot}(\mathcal{R}_E)),$$

where we denote by $\alpha_{(p,q)}$ the component of α in bidegree (p,q). This was assumed to be either 0 or a non-zero J^EG-isotropy Cartan form, and was therefore interpreted as an element of $\mathrm{Tot}(\mathcal{R}_E)$ in the construction of $\Phi^{(p,q)}$. The analogous holds for $\Phi^{(r,s)}(\beta) \equiv \beta_{(r,s)}$.

Further note that $\alpha \cup \beta$ by Lemma 4.2.18 is a closed descending homogeneous extension of

$$\mathrm{pr}^*_{1,\dots,q}\, \alpha_{(p,q)} \wedge \mathrm{pr}^*_{q+1,\dots,q+s}\, \beta_{(r,s)}$$

As $\alpha_{(p,q)}$ and $\beta_{(r,s)}$ are J^EG-isotropy Cartan, we can apply Lemma 4.2.20 to find that

$$[\mathrm{pr}^*_{1,\dots,q}\, \alpha_{(p,q)} \wedge \mathrm{pr}^*_{q+1,\dots,q+s}\, \beta_{(r,s)}] = [\alpha_{(p,q)} \mathbin{\check{\cdot}} \beta_{(r,s)}] \in H^{(p+r)}(\mathfrak{C}_{(q+s)}).$$

In particular, when following the construction of Lemma 4.2.4 to find a closed descending homogeneous extension cohomologically equivalent to $\alpha \cup \beta$ with leading component being a J^EG-isotropy Cartan form, we will obtain that the component of bidegree $(p+r, q+s)$ of the resulting component in bidegree (p,q) is given by $\alpha_{(p,q)} \mathbin{\check{\cdot}} \beta_{(r,s)}$. As the map $\Phi^{(p+r,q+s)}$ is given precisely by projection to this component, the statement follows.

□

So we have found that the maps $\Phi^{(p,q)}$ are actually also compatible with the multiplication $\check{\cdot}$ of J^EG-isotropy Cartan forms. This leads to the question: When is the same true for Φ?

Lemma 4.2.22. *Let G be a proper and regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Consider a linear assignment from J^EG-isotropy Cartan forms closed under $\mathrm{d}^E_{\mathrm{Cart}}$ to closed descending homogeneous extensions thereof*

$$\alpha_{(a,b)} \in \ker(\widehat{\mathrm{d}^E_{\mathrm{Cart}}}) \subset \mathrm{Tot}(\mathcal{R}_E) \mapsto \alpha \in \mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$$

satisfying that cohomologically equivalent forms of $\mathrm{Tot}(\mathcal{R}_E)$ are mapped to cohomologically equivalent elements of $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$. Then the map Φ of Theorem 4.2.16 constructed using this assignment is an isomorphism of $\mathbb{R}$-algebras if and only if it the assignment satisfies that for every $\omega_{(p,q)} \in \mathcal{R}_E^{(p,q)}$ with image ω and $\nu_{(r,s)} \in \mathcal{R}_E^{(r,s)}$ with image ν the image of

$$\omega_{(p,q)} \mathbin{\check{\cdot}} \nu_{(r,s)} \in \mathcal{R}_E^{(p+r,q+s)}$$

is cohomologically equivalent in $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$ to

$$\omega \cup \nu.$$

Proof. As Φ is an isomorphism, we can check that it retains the ring structure on Φ^{-1}, which is explicitly described as associating $[\alpha]$ to $[\alpha_{(a,b)}]$ for α a closed descending homogeneous extension of the $J^E G$-isotropy Cartan form $\alpha_{(a,b)}$ in the image of the assignment. For the unit axiom, we can note that extensions of elements of bidegree $(0,0)$ are always trivial. The linearity holds by assumption. It only remains to check under which conditions Φ^{-1} retains the multiplication. But the formulation in the statement is equivalent to that condition. $\square$

Lemma 4.2.23. *Let G be a proper and regular Lie groupoid and let E and F be two choices of a multiplicative Ehresmann connection with respect to the isotropy algebroid. Let Ψ be a linear assignment from $J^E G$-isotropy Cartan forms closed under $\mathrm{d}^E_{\mathrm{Cart}}$ to closed descending homogeneous extensions thereof*

$$\alpha_{(a,b)} \in \ker(\widehat{\mathrm{d}^E_{\mathrm{Cart}}}) \subset \mathrm{Tot}(\mathcal{R}_E) \mapsto \Psi(\alpha_{(a,b)}) \in \mathrm{Tot}(\Omega_{\mathrm{BSS}}(G)).$$

Assume that Ψ maps cohomologically equivalent elements of $\mathrm{Tot}(\mathcal{R}_E)$ to cohomologically equivalent elements of $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$. Further assume that Ψ satisfies that the image

$$\Psi(\omega_{(p,q)} \mathbin{\breve{\cdot}} \nu_{(r,s)})$$

is cohomologically equivalent in $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$ to

$$\Psi(\omega_{(p,q)}) \cup \Psi(\nu_{(r,s)}).$$

Let Ψ' be an assignment such that

$$\Psi'(\alpha'_{(a,b)}) - \Psi(\alpha_{(a,b)})$$

is exact for any $\alpha_{(a,b)}$, where $\alpha'_{(a,b)}$ denotes the unique $J^F G$-isotropy Cartan form corresponding to $\alpha_{(a,b)}$ under the isomorphism of Corollary 3.1.22. Then Ψ' also satisfies that

$$\Psi'(\omega'_{(p,q)} \mathbin{\breve{\cdot}} \nu'_{(r,s)})$$

is cohomologically equivalent in $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$ to

$$\Psi'(\omega'_{(p,q)}) \cup \Psi'(\nu'_{(r,s)}).$$

Proof. This is immediate, as we have that both Ψ and Ψ' map to closed elements of $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$, meaning that any ρ and σ with

$$\mathbf{d}(\rho) = \Psi(\omega_{(p,q)}) - \Psi'(\omega'_{(p,q)})$$

and

$$\mathbf{d}(\sigma) = \Psi(\nu_{(r,s)}) - \Psi'(\nu'_{(r,s)})$$

satisfy that

$$\rho \cup \Psi'(\nu'_{(r,s)}) + (-1)^{p+q} \Psi(\omega_{(p,q)}) \cup \sigma$$

is a preimage of

$$\Psi'(\omega'_{(p,q)}) \cup \Psi'(\nu'_{(r,s)}) - \Psi(\omega_{(p,q)}) \cup \Psi(\nu_{(r,s)}),$$

in particular showing that this term is exact. We know that the cohomology classes of $\Psi(\omega_{(p,q)}) \cup \Psi(\nu_{(r,s)})$ and $\Psi(\omega_{(p,q)} \mathbin{\breve{\cdot}} \nu_{(r,s)})$ as well as $\Psi'(\omega'_{(p,q)} \mathbin{\breve{\cdot}} \nu'_{(r,s)})$ agree by the various assumptions. $\square$

Corollary 4.2.24. *Consider a proper and regular Lie groupoid G together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Let Ψ be a linear assignment between $J^E G$-isotropy Cartan forms closed under $\mathrm{d}_{\mathrm{Cart}}$ and their closed descending homogeneous extensions. Assume that Ψ maps cohomologically equivalent elements of $\mathrm{Tot}(\mathcal{R}_E)$ to cohomologically equivalent elements of $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$. Further assume that Ψ satisfies that the image*

$$\Psi(\omega_{(p,q)} \breve{\cdot} \nu_{(r,s)})$$

is cohomologically equivalent in $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$ to

$$\Psi(\omega_{(p,q)}) \cup \Psi(\nu_{(r,s)})$$

for any two $J^E G$-isotropy Cartan forms $\omega_{(p,q)}$ and $\nu_{(r,s)}$.

Then for every multiplicative Ehresmann connection F with respect to the isotropy algebroid there exists an assignment

$$\Psi' : \ker(\widehat{\mathrm{d}^F_{\mathrm{Cart}}}) \subset \mathrm{Tot}(\mathcal{R}_F) \mapsto \mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$$

such that the image

$$\Psi'(\omega_{(p,q)} \breve{\cdot} \nu_{(r,s)})$$

is cohomologically equivalent in $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$ to

$$\Psi'(\omega_{(p,q)}) \cup \Psi'(\nu_{(r,s)}).$$

for any two $J^F G$-isotropy Cartan forms $\omega_{(p,q)}$ and $\nu_{(r,s)}$. In particular, the associated map

$$\Phi : H(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))) \to H(\mathrm{Tot}(\mathcal{R}_F))$$

constructed in Theorem 4.2.16 is an isomorphism of graded $\mathbb{R}$-algebras.

Proof. By Lemma 4.2.22, we know that Ψ yields an isomorphism of graded $\mathbb{R}$-algebras precisely if and only if it satisfies that

$$\Psi(\omega_{(p,q)} \breve{\cdot} \nu_{(r,s)})$$

is cohomologically equivalent in $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$ to

$$\Psi(\omega_{(p,q)}) \cup \Psi(\nu_{(r,s)}).$$

Following the construction of Lemma 4.2.17, we can simply define Ψ' satisfying that

$$\Psi'(\alpha'_{(a,b)}) - \Psi(\alpha_{(a,b)})$$

is exact for any $\widehat{\mathrm{d}^F_{\mathrm{Cart}}}$-closed $J^F G$-isotropy Cartan form and its corresponding $J^E G$-isotropy Cartan form under the isomorphism of Corollary 3.1.22. However, this immediately implies that we may apply Lemma 4.2.23, such that Ψ' also satisfies the prerequisites to apply Lemma 4.2.22 which states that Φ constructed from Ψ' is an isomorphism of graded $\mathbb{R}$-algebras. $\square$

4.3 Flat multiplicative Ehresmann connections

Aim of this section: Applications and Examples

How can we apply the previous results to the setting where the multiplicative Ehresmann connection E is flat? How does this relate to pre-existing concepts such as the van-Est-map? Are there interesting examples?

No matter if we are studying connections on a vector bundle, a fibre bundle or a Lie groupoid, flatness is known as a condition for particularly "nice" results. Their existence often allows for local trivialisations which are compatible in a particularly nice way, making computations that are local, like that of the de Rham differential, behave almost as in the globally trivial examples.

When applied to our study of $\Omega_{\mathrm{BSS}}(G)$, we find that indeed, the flatness of the multiplicative Ehresmann connection has many convenient implications. For example the de Rham differential retains $J^E G$-invariance in this case, meaning in particular that the $J^E G$-invariant forms form a subbicomplex of $\Omega_{\mathrm{BSS}}(G)$. Additionally, the fact that the de Rham differential retains $J^E G$-invariance yields a straightforward assignment that can be used in the construction of $\Phi : H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))) \to H^\bullet(\mathfrak{R}_E)$ of the previous chapter (Theorem 4.2.16), such that Φ is an isomorphism of $\mathbb{R}$-algebras.

Definition 4.3.1. A multiplicative Ehresmann connection E with respect to some bundle of ideals B on a Lie groupoid G is **flat** if E is an integrable distribution, i.e. if

$$[X_1, X_2] \in \Gamma(G_1, E)$$

or equivalently

$$[X_1, X_2]_{E-\mathrm{vert}} = 0$$

for all E-horizontal vector fields $X_1, X_2 \in \Gamma(G_1, E) \subset \mathfrak{X}(G_1)$.

Lemma 4.3.2. *Consider a regular Lie groupoid G equipped with a flat multiplicative Ehresmann connection E with respect to the isotropy algebroid. Then all E-horizontal vector fields $Z_1, Z_2 \in \mathfrak{X}(G_1)$ satisfy that*

$$[Z_1, Z_2]_{E-\mathrm{vert}} = 0.$$

Proof. For any $z = (z_1, \ldots, z_q) \in G_q$, we have that writing the Lie bracket in components yields

$$[Z_1, Z_2]_z = ([Z_1^{(1)}, Z_2^{(1)}]_{z_1}, \ldots, [Z_1^{(q)}, Z_2^{(q)}]_{z_q}),$$

where $Z_i^{(j)}$ are the component of Z_i under projection to the j-th factor G_1. Now consider the case where Z_1 and Z_2 are E-horizontal. This precisely means that all $Z_i^{(j)}$ are E-horizontal as vector fields on G_1. As we have assumed E to be flat, this means all $[Z_1^{(j)}, Z_2^{(j)}]$ are E-horizontal and therefore $[Z_1, Z_2]$ is E-horizontal as well. $\square$

Lemma 4.3.3. *Let G be a regular Lie groupoid and E a flat multiplicative Ehresmann connection with respect to the isotropy algebroid. Then $\mathrm{d}_{\mathrm{dR}}^{-1} \equiv 0$ on all of $\Omega^p(G_q)$.*

In particular, the subspaces of $J^E G$-invariant forms

$$(\Omega^p(G_q))^{J^E G} \subset \Omega_{\mathrm{BSS}}^{(p,q)}$$

define a subbicomplex of Ω_{BSS}.

Proof. Recall that by Lemma 4.1.1 the formula to compute $\mathrm{d}_{\mathrm{dR}}^{-1}$ is given by

$$\begin{aligned}&\left(\mathrm{d}_{\mathrm{dR}}^{-1}(\omega)\right)(X_1,\dots,X_{p_0+2},Y_1,\dots Y_{p_1-1})\\ &:=\sum_{i=1}^{p_0+2}\sum_{j=i+1}^{p_0+2}(-1)^{i+j}\omega([X_i,X_j]_{E-\mathrm{vert}},\dots,\widehat{X_i},\dots,\widehat{X_j},\dots).\end{aligned}$$

where $X_1,\dots,X_{p_0+2}$ denotes a family of E-horizontal vector fields and $Y_1,\dots,Y_{p_1-1}$ denotes a family of E-vertical vector fields. As E is flat, we can now apply Lemma 4.3.2. This means that in the computation of $\mathrm{d}_{\mathrm{dR}}^{-1}(\omega)$, the expression and hence the whole term actually vanishes and $\mathrm{d}_{\mathrm{dR}}^{-1}\equiv 0$.

This in particular (trivially) maps J^EG-invariant forms to J^EG-invariant forms. We already know by Lemma 4.1.1 and Lemma 4.1.4 that there are only three components to d_{dR} and that $\mathrm{d}_{\mathrm{dR}}^{\pm 0}$ and $\mathrm{d}_{\mathrm{dR}}^{+1}$ retain J^EG-invariance. At the same time, the vertical differential $\mathrm{d}_{\mathcal{N}}$ generally retains J^EG-invariance, such that the bicomplex structure maps indeed restrict to the subspaces defined by

$$(\Omega^p(G_q))^{J^EG}\subset\Omega^{(p,q)}_{\mathrm{BSS}}(G).$$

□

Lemma 4.3.4. *Consider a proper and regular Lie groupoid G and assume that it is equipped with a flat multiplicative Ehresmann connection E (with respect to the isotropy algebroid). Then every J^EG-isotropy Cartan form which is closed under* $\mathrm{d}_{\mathrm{Cart}}$ *admits a closed descending homogeneous extension. We can furthermore construct a linear assignment*

$$\omega_{(p,q)}\in\ker(\widehat{\mathrm{d}^E_{\mathrm{Cart}}})\subset\mathrm{Tot}(\mathcal{R}_E)\mapsto\omega\in\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$$

of J^EG-isotropy Cartan forms to their closed descending homogeneous extensions. The assignment may be chosen such that ω is J^EG-invariant.

Proof. First, note that any J^EG-isotropy Cartan form of bidegree $(p,0)$ is given by an ordinary differential form $\omega_{(p,0)}\in\Omega^p(G_0)$, but the Cartan differential agrees with the de Rham differential on $\Omega(G_0)$. So any J^EG-isotropy Cartan form of bidegree $(p,0)$ which is closed under the Cartan differential also defines a closed descending homogeneous extension of itself.

To cover bidegrees (p,q) with $q\geq 1$, let us start by noting

$$\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})=\mathrm{d}_{\mathrm{dR}}^{-1}(\omega_{(p,q)})+\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega_{(p,q)})+\mathrm{d}_{\mathrm{dR}}^{+1}(\omega_{(p,q)}),$$

where the first component vanishes by Lemma 4.3.3 and the second component vanishes by assumption, as it agrees with $\widehat{\mathrm{d}_{\mathrm{Cart}}}(\omega_{(p,q)})$ by Proposition 4.1.20.

So we find that $\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})\in\Omega^{p+1}(G_q)$ is homogeneous of E-vertical degree $q+1$ and so is $-\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})$. Also, we have

$$\mathrm{d}_{\mathcal{N}}(-\mathrm{d}_{\mathrm{dR}}(\omega_{p,q}))=\mathrm{d}_{\mathrm{dR}}(\mathrm{d}_{\mathcal{N}}(\omega_{(p,q)}))=0$$

According to Corollary 3.1.17 and Corollary 3.1.25, we can now choose a preimage of $-\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})$ under $\mathrm{d}_{\mathcal{N}}$ that we denote $\omega_{(p+1,q-1)}\in\Omega^{p+1}(G_{q-1})$, which is also of E-vertical degree $q{+}1$. Note that by Proposition 4.1.4, we have that that $-\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})=-\mathrm{d}_{\mathrm{dR}}^{+1}(\omega_{(p,q)})$ is J^EG-invariant. This means that the preimage $\omega_{(p+1,q-1)}$ chosen according to Corollary 3.1.17 is also J^EG-invariant analogous to the invariance of the preimage in Lemma 3.1.16.

We will now iteratively choose $\omega_{(p+k,q-k)} \in \Omega^{p+k}(G_{q-k})$ until $k = q$, such that for each $k \geq 1$ the components of E-vertical degree q or lower vanish. For $k = 1$, we have already done so.

To iteratively choose $\omega_{(p+k+1,q-k-1)}$ depending on $\omega_{(p+k,q-k)}$, we consider

$$-\,\mathrm{d}_{\mathrm{dR}}(\omega_{(p+k,q-k)}) \in \Omega^{p+k+1}(G_{q-k}).$$

By the decomposition of d_{dR} according to Lemma 4.1.1 (where $\mathrm{d}_{\mathrm{dR}}^{-1}$ vanishes due to flatness), and the assumption on the E-vertical degrees of $\omega_{(p+k,q-k)}$ we have that this expression will also only consist of terms of E-vertical degree $q+1$ or higher. Also

$$\mathrm{d}_{\mathcal{N}}(-\,\mathrm{d}_{\mathrm{dR}}(\omega_{(p+k,q-k)})) = \mathrm{d}_{\mathrm{dR}}(\mathrm{d}_{\mathcal{N}}(\omega_{(p+k,q-k)})) = \mathrm{d}_{\mathrm{dR}}(-\,\mathrm{d}_{\mathrm{dR}}(\omega_{(p+k-1,q-k+1)})) = 0.$$

Hence we can indeed choose $\omega_{(p+k+1,q-k-1)}$ as a preimage of $-\,\mathrm{d}_{\mathrm{dR}}(\omega_{(p+k,q-k)})$ and we can do so in a way that we only obtain components of E-vertical degree $q+1$ and higher, or equivalently, all components of degree q or lower vanish.

Note that if $q \geq 1$, this proof implies that $\omega_{(p+q,0)} = 0$, as any form on G_0 has E-vertical degree 0.

By construction, if we consider

$$\omega = \sum_{k=0}^{q} \omega_{(p+k,q-k)},$$

we have that $\mathbf{d}(\omega) = 0$ and hence ω is a closed descending homogeneous extension of $\omega_{(p,q)}$.

The claim that we can choose these extensions in a way that linearly depends on $\omega_{(p,q)}$ follows from the fact that all constructions described here (or referred to and described in earlier chapters) are linear constructions. Studying the precise constructions referred to above, we have actually chosen $\omega_{(p+1,q-1)}$ to be $J^E G$-invariant and inductively all following forms as well. ☐

Lemma 4.3.5. *Consider a proper and regular Lie groupoid G and assume that it is equipped with a flat multiplicative Ehresmann connection E (with respect to the isotropy algebroid). Then there is a map Ψ of chain complexes from* $\mathrm{Tot}(\mathcal{R}_E)$ *to* $\mathrm{Tot}(\Omega_{\mathrm{BSS}})$ *that assigns to every non-vanishing $J^E G$-isotropy Cartan form $\omega_{(p,q)} \in \Omega^p(G_q)$ a descending homogeneous extensions $\omega = \Psi(\omega_{(p,q)})$. It may be chosen such that it extends the assignment constructed in Lemma 4.3.4. It may also be chosen such that ω is $J^E G$-invariant.*

Proof. As we have already constructed the assignment of Lemma 4.3.4, we now need to construct ω precisely for the elements $\omega_{(p,q)}$ which are not closed under $\mathrm{d}_{\mathrm{Cart}}$.

For such an $\omega_{(p,q)}$, we now now that $\mathrm{d}_{\mathrm{Cart}}(\omega_{(p,q)}) \neq 0$ does satisfies the condition that its image under $\mathrm{d}_{\mathrm{Cart}}$ vanishes and therefore $\mathrm{d}_{\mathrm{Cart}}(\omega_{(p,q)})$ can be extended according to Lemma 4.3.4 to an element ρ, which is a closed descending homogeneous extension of

$$\rho_{(p+1,q)} = \mathrm{d}_{\mathrm{Cart}}(\omega_{(p,q)})$$

in bidegree $(p+1, q)$. Furthermore, we know from the respective proof that apart from this bidegree, $\rho_{(p+1+k,q-k)}$ for $1 \leq k \leq q$ will always be of E-vertical degree at least $q+1$ and $J^E G$-invariant.

So now, we will aim to choose $\omega_{(p+k,q-k)}$ iteratively to obtain

$$\mathbf{d}(\omega) = \rho,$$

where we will choose $\omega_{(p+k,q-k)}$ to consist of components of E-vertical degree of at least $q+1$ for $1 \leq k \leq q$.

For this, we will use the method exactly analogous to the proof of Lemma 4.3.4: We will consider $\omega_{(p+k,q-k)}$ for the largest k that we have previously defined. Then we will compute

$$\rho_{(p+k+1,q-k)} - \mathrm{d}_{\mathrm{dR}}(\omega_{(p+k,q-k)})$$

which is (for $k=0$ by assumption and then inductively by the same arguments as in the reference) only contained in components of E-vertical degree of at least $q+1$.

We have

$$\mathrm{d}_{\mathcal{N}}\left(\rho_{(p+k+1,q-k)} - \mathrm{d}_{\mathrm{dR}}(\omega_{(p+k,q-k)})\right) = -\,\mathrm{d}_{\mathrm{dR}}(\rho_{(p+k,q-k-1)}) + \mathrm{d}_{\mathrm{dR}}(\mathrm{d}_{\mathcal{N}}(\omega_{(p+k,q-k)})).$$

If $k=0$, this expression is 0 as $\omega_{(p,q)}$ is $J^E G$-isotropy Cartan and ρ is a closed descending homogeneous extension of an element in bidegree $(p+1,q)$. If $k>0$ we have constructed $\omega_{(p+k,q-k)}$ to satisfy

$$\mathrm{d}_{\mathcal{N}}(\omega_{(p+k,q-k)}) + \mathrm{d}_{\mathrm{dR}}(\omega_{(p+k-1,q-k+1)}) = \rho_{(p+k,q-k+1)},$$

which equivalently means

$$\rho_{(p+k,q-k+1)} - \mathrm{d}_{\mathcal{N}}(\omega_{(p+k,q-k)}) = \mathrm{d}_{\mathrm{dR}}(\omega_{(p+k-1,q-k+1)}),$$

implying

$$-\,\mathrm{d}_{\mathrm{dR}}(\rho_{(p+k,q-k-1)}) + \mathrm{d}_{\mathrm{dR}}(\mathrm{d}_{\mathcal{N}}(\omega_{(p+k,q-k)})) = 0.$$

So, we find that

$$\rho_{(p+k+1,q-k)} - \mathrm{d}_{\mathrm{dR}}(\omega_{(p+k,q-k)})$$

is closed and of E-vertical degree $q+1$, hence exact, and we can choose a preimage

$$\omega_{(p+k+1,q-k-1)}$$

of E-vertical degree $q+1$.

Hence we will after a finite descent obtain ω with $\mathbf{d}(\omega) = \rho$. Furthermore, note that if $\rho = 0$, the previous construction precisely agrees with the one of Lemma 4.3.4. Furthermore, the constructed map

$$\mathrm{Tot}(\mathcal{R}_E) \to \mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$$

is linear, as all of the used constructions were linear. In analogy to the proof of Lemma 4.3.4, we may again note that by Lemma 4.1.4 as well as Corollary 3.1.17, we have actually chosen $\omega_{(p+1,q-1)}$ to be $J^E G$-invariant and inductively all following forms as well. Here, we additionally use that the $J^E G$-invariance also holds for ρ. $\square$

Corollary 4.3.6. *Let G be a proper and regular Lie groupoid together with a flat multiplicative Ehresmann connection E with respect to the isotropy algebroid. The assignment Ψ of Lemma 4.3.5 satisfies that cohomologically equivalent $J^E G$-isotropy Cartan forms in* $\mathrm{Tot}(\mathcal{R}_E)$ *get mapped to cohomologically equivalent elements of* $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$.

Proof. We may check this claim on homogeneous forms $\omega_{(p,q)}$ and $\omega'_{(p,q)}$ which are cohomologically equivalent in $\mathcal{R}^p_q$ by Corollary 4.1.28. If we choose a preimage $\nu_{(p-1,q)} \in \mathcal{R}^{p-1}_q$ of the term $\omega_{(p,q)} - \omega'_{(p,q)}$, we have that by definition

$$\widehat{\mathrm{d}^E_{\mathrm{Cart}}}(\nu_{(p-1,q)}) = \omega_{(p,q)} - \omega'_{(p,q)}.$$

So the descending homogeneous extension associated to $\nu_{(p-1,q)}$ will map to the closed descending homogeneous extension of $\omega_{(p,q)} - \omega'_{(p,q)}$, which by linearity is $\omega - \omega'$. $\square$

Proposition 4.3.7. *Let G be a proper and regular Lie groupoid and assume that it is equipped with a flat multiplicative Ehresmann connection E (with respect to the isotropy algebroid). Then there exists an isomorphism of graded $\mathbb{R}$-vector spaces*

$$H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))) \cong H^\bullet(\mathrm{Tot}(\mathcal{R}_E)).$$

Proof. By Theorem 4.2.16 there exists a graded isomorphism Φ between $H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G)))$ and $H^\bullet(\mathrm{Tot}(\mathcal{R}_E))$ under the condition that there is an assignment

$$\omega_{(p,q)} \in \ker(\widehat{\mathrm{d}^E_{\mathrm{Cart}}}) \subset \mathrm{Tot}(\mathcal{R}_E) \mapsto \omega \in \mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$$

from $J^E G$-isotropy Cartan forms closed under $\widehat{\mathrm{d}_{\mathrm{Cart}}}$ to their closed descending homogeneous extensions. However, by Lemma 4.3.4 and Corollary 4.3.6, we have constructed an assignment of this form, which is furthermore linear and descends to cohomology classes (i.e. it maps cohomologically equivalent elements of $\mathrm{Tot}(\mathcal{R}_E)$ to cohomologically equivalent elements of $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$). □

Corollary 4.3.8. *Let G be a proper and regular Lie groupoid together with a flat multiplicative Ehresmann connection E with respect to the isotropy algebroid. We have that Ψ constructed in Lemma 4.3.5 induces a map*

$$H(\Psi) : H(\mathrm{Tot}(\mathcal{R}_E)) \to H(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G)))$$

which is inverse to the map Φ constructed in Proposition 4.3.7.

Proof. We have, in Theorem 4.2.16, already described the inverse to Φ by

$$\Phi^{-1}([\omega_{(p,q)}]) = [\omega],$$

where ω is the closed descending homogeneous extension associated to $\omega_{(p,q)}$. This is immediately given by the definition of the map Ψ as an extension of this assignment. □

Lemma 4.3.9. *Let G be a proper and regular Lie groupoid together with a flat multiplicative Ehresmann connection E with respect to the isotropy algebroid. The inclusion*

$$(\Omega_{\mathrm{BSS}}(G))^{J^E G} \subset \Omega_{\mathrm{BSS}}(G)$$

defines a quasi-isomorphism on the totalisations of the respective double complexes.

Proof. By Lemma 4.3.5, we have that the assignment Ψ constructed in that lemma assigns a $J^E G$-invariant form ω to any $J^E G$-isotropy Cartan form $\omega_{(p,q)}$. However, by construction this implies that Ψ factors through $\mathrm{Tot}\left((\Omega_{\mathrm{BSS}}(G))^{J^E G}\right)$ and the inclusion. But as Ψ is a quasi-isomorphism, this means that the inclusion must be surjective on cohomology classes. It also implies that on the image

$$H(\Psi)\left(H(\mathrm{Tot}(\mathcal{R}_E))\right) \subset H\left(\mathrm{Tot}\left((\Omega_{\mathrm{BSS}}(G))^{J^E G}\right)\right),$$

the inclusion induces an isomorphism.

However, we can see that the construction of Lemma 4.2.4 restricts to the subbicomplex of $J^E G$-invariant forms. This implies in particular that $H(\Psi)$ is surjective: Any closed descending homogeneous extension ω of some element $\omega_{(p,q)}$ in bidegree (p,q), is cohomologically equivalent to some closed descending homogeneous extension ω' where either $\omega'_{(p,q)}$ is 0 or $\omega'_{(p,q)}$ is $J^E G$-isotropy Cartan. In either case,

$$\omega' - \Psi(\omega'_{(p,q)})$$

is a closed descending extension of an element in bidegree (p', q') where $p' > q'$. So using an argument of finite descent, we can write ω as a sum of exact terms and terms in the image of Ψ, meaning in particular that the associated cohomology class

$$[\omega] \in H\left(\mathrm{Tot}\left((\Omega_{\mathrm{BSS}}(G))^{J^E G}\right)\right)$$

contains an element in the image of Ψ and therefore is in the image of $H(\Psi)$. $\square$

Lemma 4.3.10. *Consider a proper and regular Lie groupoid G and assume that it is equipped with a flat multiplicative Ehresmann connection E (with respect to the isotropy algebroid).*

Then the restriction to the unit section together with the morphism π of Lemma 3.1.19 induces a map $\widehat{\Phi}$ of bicomplexes between $\left((\Omega_{\mathrm{BSS}}(G))^{J^E G}\right)$ and $\mathfrak{R}_E$. The kernel of $\widehat{\Phi}$ includes all elements of bidegree (p, q) for which the E-vertical component of degree q vanishes.

Proof. The map of Lemma 3.1.19 is defined on

$$\bigoplus_{p_0+p_1=p} \left(\Gamma(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \wedge^{p_1}\left((A^*_{\mathrm{iso}})^{\oplus q}\right))\right)^G$$

which corresponds to the $J^E G$-invariant forms of bidegree (p, q). Its image are precisely the $J^E G$-isotropy Cartan forms of E-horizontal degree $p - q$ given by

$$\left(\Gamma\left(G_0, \wedge^{p-q}(TG_0/\mathcal{F}_0)^* \otimes S^q(A^*_{\mathrm{iso}})\right)\right)^G \subset (\mathfrak{R}_E)^{(p,q)}$$

So, we obtain a map of bigraded vector spaces from $\left((\Omega_{\mathrm{BSS}}(G))^{J^E G}\right)$ to $\mathfrak{R}_E$.

Furthermore, we have that π is given by the concatenation of two maps, the first of which projects a $J^E G$-invariant form to its component of multidegree $(p_0, 1, \ldots, 1, q)$. This implies that the set of elements not contained in the kernel is a subspace of the homogeneous forms of E-vertical degree q. This means all components of a different E-vertical degree will automatically be in the kernel. We will use this fact to show that the bicomplex differentials are retained under this map.

This also shows, that when considering a $J^E G$-invariant form $\omega \in \Omega^p(G_q)$ of E-vertical degree p_1, we will have that both ω and $\mathrm{d}_{\mathrm{dR}}(\omega)$ are in the kernel of $\widehat{\Phi}$ unless $p_1 = q - 1$ or $p_1 = q$ by Lemma 4.1.1 and Lemma 4.3.3 due to the assumption that E is flat.

Let us consider the case where $p_1 = (q - 1)$ first. In this case, we know that the values of ω at the unit section are given by some $\omega_{1_{G_0}}$ that lies in the kernel of π. This implies (as the map to values at the unit is an isomorphism and π is a quasi-isomorphism) that ω is exact with respect to $\mathrm{d}_{\mathcal{N}}$. We now need to show that

$$0 = \mathrm{d}_{\mathrm{Cart}}(\pi(\omega_{1_{G_0}})) = \pi(\mathrm{d}_{\mathrm{dR}}(\omega)_{1_{G_0}}).$$

But by exactness of ω, its preimage ν satisfies that

$$\mathrm{d}_{\mathcal{N}}(\mathrm{d}_{\mathrm{dR}}(-\nu)) = \mathrm{d}_{\mathrm{dR}}(\mathrm{d}_{\mathcal{N}}(\nu)) = \mathrm{d}_{\mathrm{dR}}(\omega),$$

such that $\mathrm{d}_{\mathrm{dR}}(\omega)$ is also exact with respect to $\mathrm{d}_{\mathcal{N}}$. This implies

$$[\pi(\mathrm{d}_{\mathrm{dR}}(\omega)_{1_{G_0}})] = [\mathrm{d}_{\mathrm{dR}}(\omega)_{1_{G_0}}] = [0] \in H(\mathfrak{C}_p),$$

but we have that $\pi(\mathrm{d}_{\mathrm{dR}}(\omega)_{1_{G_0}})$ is an expression corresponding to a $J^E G$-isotropy Cartan form by definition. By Theorem 3.1.20, this form is uniquely determined by the cohomology class of $\mathrm{d}_{\mathrm{dR}}(\omega)$, which implies that it already must be 0 itself.

The argument in the case $p_1 = q$ is similar. Here we know that $\omega - \pi(\omega)$ lies in the kernel of π (as by definition $\pi \equiv \mathrm{id}$ on $J^E G$-isotropy Cartan forms). We can now analogously find that this implies

$$[\pi(\mathrm{d}_{\mathrm{dR}}(\omega - I(\pi(\omega_{1_{G_0}})))_{1_{G_0}})] = [0] \in H(\mathfrak{C}_{p+1})$$

where I denotes the inverse to the map of Proposition 2.1.41. Thus

$$\pi(\mathrm{d}_{\mathrm{dR}}(\omega - I(\pi(\omega_{1_{G_0}})))_{1_{G_0}}) = 0.$$

However, we know that

$$\pi(\mathrm{d}_{\mathrm{dR}}(I(\pi(\omega_{1_{G_0}})))_{1_{G_0}}) = \pi(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(I(\pi(\omega_{1_{G_0}})))_{1_{G_0}})$$

as all other components of d_{dR} lie in the kernel of π by Proposition 4.1.20. Furthermore $\mathrm{d}^{\pm 0}_{\mathrm{dR}} \equiv \widehat{\mathrm{d}_{\mathrm{Cart}}}$ on $J^E G$-isotropy Cartan forms, and $\pi \equiv \mathrm{id}$ on expressions corresponding to $J^E G$-isotropy Cartan forms, implying that

$$\pi(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(I(\pi(\omega_{1_{G_0}})))_{1_{G_0}}) = \mathrm{d}_{\mathrm{Cart}}(\pi(\omega_{1_{G_0}})).$$

Using all our previous findings, this shows that

$$\pi(\mathrm{d}_{\mathrm{dR}}(\omega)_{1_{G_0}}) = \pi(\mathrm{d}_{\mathrm{dR}}(I(\pi(\omega_{1_{G_0}})))_{1_{G_0}}) = \mathrm{d}_{\mathrm{Cart}}(\pi(\omega)).$$

This is precisely what we needed to show for the map to respect the horizontal differential of both bicomplexes.

For the vertical map, it is sufficient to notice that both the restriction to the unit section as well as the map π were defined on chain complexes, such that on the differential on the domain

$$(\Omega^p(G_\bullet))^{J^E G} \subset \mathfrak{C}_p$$

of the concatenation (which agrees with the differential of the Bott–Shulman–Stasheff-complex) is mapped to the 0 differential on the target, which is the vertical differential of $\mathfrak{R}_E$. □

Definition 4.3.11. The map $\widehat{\Phi}$ of Lemma 4.3.10 will be called the $J^E G$**-Van-Est-map**.

When comparing to other notions of the van-Est map (e.g. [1], pp. 60-65), we can see that the target is usually given by

$$\Gamma(G_0, \wedge^{p_0} T^* G_0 \otimes \wedge^r (A^*) \otimes S^q(A^*))$$

In consequence, it might seem more natural to define a map to at least a somewhat more general target, e.g.

$$\Gamma(G_0, \wedge^{p_0} T^* G_0 \otimes \wedge^r (A^*_{\mathrm{iso}}) \otimes S^q(A^*_{\mathrm{iso}})) = \mathcal{W}(A_{\mathrm{iso}}, \nabla^E)$$

in a way where the image of ω depends on $\omega_{1_{G_0}}$ as well as expressions such as $(\mathcal{L}_{\widehat{X}}\omega)_{1_{G_0}}$ for $\widehat{X}$ being obtained by some construction from $X \in \Gamma(G_0, A_{\text{iso}})$.

It is not trivial to compare the map we constructed with notions of a map to $\mathcal{W}(A_{\text{iso}}, \nabla^E)$ or even

$$\mathcal{W}(A, \nabla) = \Gamma\left(G_0, \wedge^{p_0} T^*G_0 \otimes \wedge^r(A^*) \otimes S^q(A^*)\right).$$

This is in part because this complex depends on a choice of a connection ∇. All the choices yield equivalent algebras (see Lemma 0.3.3), however, the isomorphisms are not trivial. The following Lemma can be seen as a tool to relate the notions in the flat case, at least under some compatibility assumptions between ∇ and ∇^E.

Lemma 4.3.12. *Consider a proper and regular Lie groupoid G and assume that it is equipped with a flat multiplicative Ehresmann connection E (with respect to the isotropy algebroid). Let X be the vector field on G_q that is induced by some $\underline{X} \in \Gamma(G_0, A)$ under right translation: At $z \in G_q$ we can consider the action by left multiplication*

$$L : G_1 \times^{s, p_q^0} G_q \to G_q$$

of Example 1.1.24 (for $i = 0$) and define

$$X_z = TL(\underline{X}_{1_{p_q^0(z)}}, 0_z).$$

We have that any $J^E G$-invariant form $\omega \in \Omega^p(G_q)$ satisfies

$$(\mathcal{L}_X \omega)_z = 0$$

if either $\underline{X} \in \Gamma(G_0, A \cap E)$ or $\underline{X} \in \Gamma(G_0, A_{\text{iso}})$ such that it is ∇^E-horizontal at $p_q^0(z)$.

Proof. Assume that $\underline{X} \in \Gamma(G_0, A \cap E)$. As E is closed under multiplication and contains all 0 tangent vectors, we find that X is also an E-horizontal vector field. At the same time, we have that $Tp_q^0(X_z) = Tp_q^0(\underline{X}_{1_{(p_q^0(z))}}) \in \mathcal{F}_0$, implying that X is a vector field contained in $\mathcal{F}_q$.

Furthermore, we have

$$\mathcal{L}_X \omega = [\iota_X, \mathrm{d}_{\mathrm{dR}}]\omega.$$

However, both ω and $\mathrm{d}_{\mathrm{dR}}(\omega)$ are $J^E G$-invariant by Lemma 4.3.3, such that ι_X must vanish on both of them by Proposition 2.1.26.

If we have $\underline{X} \in A_{\text{iso}}$, we know by Lemma 4.1.5, that $\exp(t \cdot \underline{X})$ is well-defined for some $t \in (-\varepsilon, \varepsilon)$ and gives rise to a family of E-horizontal jets V^t. As the right translation of vector fields was induced by the same map that defines the left action of a jet D_V^0, we find that

$$\mathcal{L}_X \omega = \frac{\partial}{\partial t} D^0_{V^t}(\omega)|_{t=0},$$

which vanishes by $J^E G$-invariance. □

Theorem 4.3.13. *Consider a proper and regular Lie groupoid G and assume that it is equipped with a flat multiplicative Ehresmann connection E (with respect to the isotropy algebroid). Furthermore, let $\widehat{\Phi}$ be the $J^E G$-van Est map defined in Lemma 4.3.10 (induced by the restriction to values at the unit and the morphism π of Lemma 3.1.19). Then $\widehat{\Phi}$ defines a quasi-isomorphism of the associated chain complexes*

$$\mathrm{Tot}(\widehat{\Phi}) : \mathrm{Tot}((\Omega_{\mathrm{BSS}}(G))^{J^E G}) \to \mathrm{Tot}(\mathfrak{R}_E).$$

Using the identification of $H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G)))$ and $H^\bullet(\mathrm{Tot}((\Omega_{\mathrm{BSS}}(G))^{J^E G}))$ of Lemma 4.3.9 we have that $\widehat{\Phi}$ lifts the morphism Φ of Proposition 4.3.7.

Proof. Let $\omega_{(p,q)} \in \Omega^p(G_q)$ be a $J^E G$-isotropy Cartan form and ω its associated descending homogeneous extension of Lemma 4.3.5. Then by definition

$$\widehat{\Phi}(\omega_{(p,q)}) = \pi(\omega_{(p,q)}) = \omega_{(p,q)},$$

while for $k > 0$ we have that $\omega_{(p+k,q-k)}$ has only components of E-vertical degree at least $(q+1)$, meaning that $\omega_{(p+k,q-k)} \in \ker(\pi)$ by Lemma 4.3.10 and

$$\widehat{\Phi}(\omega_{(p+k,q-k)}) = \pi(\omega_{(p+k,q-k)}) = 0.$$

This means that on cohomology

$$H(\mathrm{Tot}(\widehat{\Phi}))([\omega]) = [\omega_{(p,q)}]$$

for any $J^E G$-isotropy Cartan form $\omega_{(p,q)}$ closed under $\widehat{\mathrm{d}_{\mathrm{Cart}}}$. But this, by Theorem 4.2.16, is precisely the definition of Φ, only that Φ is defined on cohomology classes of $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$. The difference only lies in the fact that $H(\mathrm{Tot}(\widehat{\Phi}))$ is defined on cohomology classes of $\mathrm{Tot}((\Omega_{\mathrm{BSS}}(G))^{J^E G})$.

Now we only need to apply that by Lemma 4.3.9, the inclusion of double complexes induces a quasi-isomorphism of the associated totalisations and the fact that ω is already an element of $(\Omega_{\mathrm{BSS}}(G))^{J^E G}$ to obtain the statement of the theorem. This also implies that $H(\widehat{\Phi})$ can be written as the concatenation of two isomorphisms, hence it must be an isomorphism itself, meaning $\widehat{\Phi}$ is a quasi-isomorphism. □

Lemma 4.3.14. *Consider a proper and regular Lie groupoid G and assume that it is equipped with a flat multiplicative Ehresmann connection E (with respect to the isotropy algebroid).*

Let ω be a closed descending homogeneous extension of $\omega_{(p,q)}$, where $\omega_{(p,q)}$ is homogeneous of E-vertical degree q and all terms $\omega_{(p+k,q-k)}$ for $k > 0$ only have E-homogeneous components of E-vertical degree at least $q+1$.

Let ν be a closed descending homogeneous extension of $\nu_{(p',q')}$, where $\nu_{(p',q')}$ is homogeneous of E-vertical degree q' and all terms $\nu_{(p'+k,q'-k)}$ for $k > 0$ only have E-homogeneous components of E-vertical degree at least $q'+1$.

Then $\omega \cup \nu$ is a closed descending homogeneous extension of

$$\mathrm{pr}^*_{1,\ldots,q}\,\omega_{(p,q)} \wedge \mathrm{pr}^*_{q+1,\ldots,q+q'}\,\nu_{(p',q')}.$$

where all terms $(\omega \cup \nu)_{(p+p'+l,q+q'-l)}$ for $l > 0$ only have E-homogeneous components of E-vertical degree at least $q+q'+1$.

Proof. We have, by Lemma 4.2.18 that $\omega \cup \nu$ is a closed descending homogeneous extension of

$$\omega_{(p,q)} \cup \nu_{(p',q')} = \mathrm{pr}^*_{1,\ldots,q}\,\omega_{(p,q)} \wedge \mathrm{pr}^*_{q+1,\ldots,q+q'}\,\nu_{(p',q')}.$$

and that all other components consists of sums of terms of the form

$$\omega_{(p+k,q-k)} \cup \nu_{(p'+k',q'-k')} = \mathrm{pr}^*_{1,\ldots,q-k}\,\omega_{(p+k,q-k)} \wedge \mathrm{pr}^*_{q-k+1,\ldots,q+q'-k-k'}\,\nu_{(p'+k',q'-k')},$$

where either $k > 0$ or $l > 0$ or both.

By Lemma 4.2.19 we know, however, that this multiplication is compatible with E-vertical degrees. This means, by our assumption, that all terms have E-vertical degree of at least $q+q'+1$. □

Lemma 4.3.15. *Let G be a proper and regular Lie groupoid together with a flat multiplicative Ehresmann connection E with respect to the isotropy algebroid. The assignment $\Psi : \mathrm{Tot}(\mathcal{R}_E) \to \mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$ satisfies that*

$$\Psi(\omega \check{\cdot} \nu) - \Psi(\omega) \cup \Psi(\nu)$$

is exact for all ω and ν which are closed under $\widehat{\mathrm{d}_{\mathrm{Cart}}}$.

Proof. First of all, by Lemma 4.2.18 we have that

$$\Psi(\omega) \cup \Psi(\nu)$$

is a closed descending homogeneous extension of

$$\omega \cup \nu,$$

where we know that all other components are of E-vertical degree at least $q + s + 1$ by Lemma 4.3.4 and Lemma 4.2.19. We know that $\Psi(\omega \check{\cdot} \nu)$ also has almost only components of E-vertical degrees at least $q + s + 1$ as a closed homogeneous descending extension of $\omega \check{\cdot} \nu$.

So

$$\Psi(\omega \check{\cdot} \nu) - \Psi(\omega) \cup \Psi(\nu)$$

is a closed element of $\Omega_{\mathrm{BSS}}(G)$ which consists of non-zero components in multidegrees of the form $(p + r + k, q + s - k)$ for $k \geq 0$, where for $k = 0$ the component satisfies

$$[\omega \check{\cdot} \nu - (\Psi(\omega) \cup \Psi(\nu))_{(p+r,q+s)}] = [\omega \check{\cdot} \nu - \omega \cup \nu] = 0 \in H(\mathfrak{C}_{p+r})$$

by Lemma 4.2.20. In the corresponding proof we can see that $\omega \cup \nu$ is $J^E G$-invariant itself, that $(\Psi(\omega) \cup \Psi(\nu))_{(p+r,q+s)}$ is of multidegree $(p_0 + r_0, 1, \ldots, 1, q + s)$ and that

$$\pi((\Psi(\omega) \cup \Psi(\nu))_{(p+r,q+s)}) = \omega \check{\cdot} \nu.$$

This means that if we use the method of Lemma 4.2.4 to construct a closed descending homogeneous extension of an $J^E G$-isotropy Cartan form cohomologically equivalent to

$$\Psi(\omega \cdot \nu) - \Psi(\omega) \cup \Psi(\nu),$$

its component in bidegree $(p + r, q + s)$ is given by 0 and the $J^E G$-isotropy Cartan form must be in one of the components $(p + r + k, q + s - k)$ for $k > 0$. We now aim to show that this already implies that the $J^E G$-isotropy Cartan form which is the leading term of this extension is given by 0, such that the whole extension vanishes

In the first step of constructing this closed descending homogeneous extension, we choose a preimage σ of

$$\omega \check{\cdot} \nu - \omega \cup \nu$$

in the cohomology of $\mathfrak{C}_{p+r}$, i.e. with respect to the differential $\mathrm{d}_{\mathcal{N}}$. A possible construction for this preimage is given in Lemma 3.1.18 using the explicit result of Lemma 3.1.15. This construction expresses the preimage σ as a sum of contractions of $\mathrm{d}_{\mathcal{N}}$-closed $J^E G$-invariant forms of multidegree $(p_0 + r_0, 1, \ldots, 1, q + s)$ obtained by elements of Σ_{q+s} acting on $\omega \check{\cdot} \nu - \omega \cup \nu$. We aim to show that this already implies that σ vanishes under $\mathrm{d}^{\pm 0}_{\mathrm{dR}}$.

We have that contraction of adjacent tensor factors as well as the permutation action commutes with $\mathrm{d}^{\pm 0}_{\mathrm{dR}}$ on the corresponding forms. This can be seen as follows: $\mathrm{d}^{\pm 0}_{\mathrm{dR}}$ can be computed as C of Lemma 4.1.15 for the connection $\nabla^{\oplus q}$ on $(A_{\mathrm{iso}})^{\oplus q}$ by Proposition 4.1.18, which is a derivation on

$$\Gamma\left(G_0, \wedge^\bullet T^* G_0 \otimes \wedge^\bullet \left(A^*_{\mathrm{iso}}\right)^{\oplus q}\right)$$

by Lemma 4.1.15. In analogy to the proof of Lemma 4.1.16, the fact that the connection is defined as the sum of identical connections then implies that the computation of C commutes with decomposition of an element into its components of homogeneous multidegree and any kind of change of the index of the copy an element of $\Gamma\left(G_0, \wedge^\bullet A^*_{\mathrm{iso}}\right)$ is contained in. So if $\mathrm{d}^{\pm 0}_{\mathrm{dR}}$ vanishes on a $J^E G$-invariant form, it also vanishes on contractions of that form. Applying this to our situation, if we can show that $\mathrm{d}^{\pm 0}_{\mathrm{dR}}$ vanishes on

$$\omega \mathbin{\check{\cdot}} \nu - \omega \cup \nu$$

it then follows that $\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\sigma) = 0$.

We find that on $J^E G$-isotropy Cartan forms $\omega \mathbin{\check{\cdot}} \nu$ and ω and ν the differential $\mathrm{d}^{\pm 0}_{\mathrm{dR}}$ corresponds to $\widehat{\mathrm{d}_{\mathrm{Cart}}}$ and also that it is a derivation by Lemma 4.1.8. Furthermore, $\mathrm{d}^{\pm 0}_{\mathrm{dR}}$ can be interchanged with pull-backs along face maps by Lemma 2.1.32 and the usual commutativity of d_{dR} and pull-backs. Therefore we have indeed that

$$\mathrm{d}^{\pm 0}_{\mathrm{dR}}\left(\omega \mathbin{\check{\cdot}} \nu - \omega \cup \nu\right) = \widehat{\mathrm{d}_{\mathrm{Cart}}}(\omega) \mathbin{\check{\cdot}} \nu \pm \omega \mathbin{\check{\cdot}} \widehat{\mathrm{d}_{\mathrm{Cart}}}(\nu) - \mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega) \cup \nu \mp \omega \cup \mathrm{d}^{\pm 0}_{\mathrm{dR}}(\nu) = 0.$$

So indeed, we also have that

$$\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\sigma) = 0$$

In particular, by Lemma 4.1.1 and Lemma 4.3.3 we have that

$$\mathbf{d}(\sigma) = \omega \cdot \nu - (\Psi(\omega) \cup \Psi(\nu))_{(p+r, q+s)} + \mathrm{d}^{+1}_{\mathrm{dR}}(\sigma),$$

where $\mathrm{d}^{+1}_{\mathrm{dR}}(\sigma)$ is of E-vertical degree $q + s + 1$.

This means, that in the first step in the construction of Lemma 4.2.4, we show that

$$\Psi(\omega \cdot \nu) - \Psi(\omega) \cup \Psi(\nu),$$

is cohomologically equivalent to a closed element of $\Omega_{\mathrm{BSS}}(G)$ which consists of non-zero components in degree $(p + r + k, q + s - k)$ for $k > 0$ where additionally all components are of E-vertical degree at least $q + s + 1$.

But now we can see, that in all further steps of the construction of Lemma 4.2.4, this property is retained as by Corollary 3.1.25, the preimages of any $\mathrm{d}_{\mathcal{N}}$-exact $J^E G$-invariant form can be chosen to retain the E-vertical degree and furthermore due to flatness of the Ehresmann condition $\mathrm{d}^{-1}_{\mathrm{dR}} \equiv 0$ by Lemma 4.3.3, such that d_{dR} can only increase the E-horizontal degree of a term.

So the leading term of the closed descending homogeneous extenstion we obtain must be a $J^E G$-isotropy Cartan form in bidegree $(p + r + k, q + s - k)$ for $k > 0$, but also have E-vertical degree $q + s + 1$, which is a contradiction unless the form is 0, meaning the entire closed descending homogeneous extension vanishes and

$$\Psi(\omega \cdot \nu) - \Psi(\omega) \cup \Psi(\nu),$$

is cohomologically equivalent to 0. $\square$

Theorem 4.3.16. *Let G be a proper and regular Lie groupoid together with a flat multiplicative Ehresmann connection E with respect to the isotropy algebroid. There is a quasi-isomorphism of chain complexes*

$$\Psi : \mathrm{Tot}(\mathcal{R}_E) \to \mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$$

which establishes $\mathrm{Tot}(\mathcal{R}_E) \cong \mathrm{Tot}(\mathfrak{R}_E)$ *as a model for*

$$H_{\mathrm{DiffStack}}([G_0//G_1]) \cong H(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))).$$

We have that furthermore $H(\Psi)$ is a morphism of graded algebras.

Its inverse on cohomology is given by a morphism of graded algebras Φ which can be lifted to a morphism of bicomplexes $\widehat{\Phi}$ when restricting to

$$(\Omega_{\mathrm{BSS}}(G))^{J^E G} \subset \Omega_{\mathrm{BSS}}(G)$$

where the inclusion is a quasi-isomorphism of double complexes.

Proof. This is simply the collection of all the results of this section. The construction of Ψ is given in Lemma 4.2.4 and Lemma 4.3.5, the fact that it is a quasi-isomorphism is the statement of Corollary 4.3.8 (where we have already stated that on cohomology it is inverse to the map Φ of Proposition 4.3.7) and the fact that it descends to a morphism of graded algebras is the statement of Lemma 4.2.22 and Lemma 4.3.15. The morphism $\widehat{\Phi}$ is the $J^E G$-van-Est-map of Lemma 4.3.10, that it descends to Φ is the statement of Theorem 4.3.13. □

Corollary 4.3.17. *Assume G is a proper and regular Lie groupoid which admits a flat multiplicative Ehresmann connection E with respect to the isotropy algebroid. Then for any multiplicative Ehresmann connection F with respect to the isotropy algebroid, the algebra* $\mathrm{Tot}(\mathcal{R}_F)$ *is a model for the differentiable stack cohomology associated to G and the cohomology of* $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$ *in the sense that*

$$H(\mathrm{Tot}(\mathcal{R}_F)) \cong H(\mathrm{Tot}(\mathfrak{R}_F)) \cong H_{\mathrm{DiffStack}}([G_0//G_1]) \cong H(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G)))$$

are isomorphisms of graded $\mathbb{R}$-algebras.

Proof. By Lemma 4.3.4, we know that there is an assignment

$$\omega_{(p,q)} \mapsto \omega,$$

which maps $J^E G$-isotropy Cartan forms closed under $\mathrm{d}_{\mathrm{Cart}}$ to corresponding closed descending homogeneous extensions. By Lemma 4.2.17 this implies that the same is true for all multiplicative Ehresmann connections, in particular F. Theorem 4.2.16 then implies the statement for isomorphisms of $\mathbb{R}$-vector spaces. But we know by Corollary 4.2.24 and Lemma 4.2.22 that the constructed maps are $\mathbb{R}$-algebra isomorphism as well. □

Of course, constructing a flat multiplicative Ehresmann connection in general is hard - if not impossible. (I suspect that some result on obstructions of the existence of a flat connection on a principal bundle imply that in general it may be impossible to construct a flat multiplicative Ehresmann connection on a Lie groupoid, even one with nice properties such as properness or regularity.)

However, in quite a number of examples we do have a flat multiplicative Ehresmann connection by construction. With this result, we can now look at a few examples of Lie groupoids with flat multiplicative Ehresmann connections.

Example 4.3.18. Recall Example 4.2.8 and Example 4.2.10 we considered earlier in this chapter. We can note that in both of these examples, the chosen multiplicative Ehresmann connections are flat. In both of these cases, we have already seen explicitly that the homogeneous extensions of $J^E G$-isotropy Cartan elements can (and must) be chosen trivially and that d_{dR} and $\mathrm{d}_{\mathrm{dR}}^{\pm 0} = \widehat{\mathrm{d}_{\mathrm{Cart}}}$ actually agree.

Example 4.3.19. Consider the Lie groupoid $G_1 = \mathcal{G} \times G_0 \times G_0$ of Example 2.3.21 and Example 3.1.30. Recall that its $J^E G$-isotropy Cartan forms of bidegree (p, q) are given by

$$\left(\Gamma\left(G_0, \left(S^p\left(A_{\mathrm{iso}}^*\right)\right)^{\mathcal{G}}\right)\right)^G \cong \left(S^p\left(\mathfrak{g}^*\right)\right)^{\mathcal{G}}.$$

Furthermore, we know that by definition of the Ehresmann connection its reduction ∇^E is given precisely by the connection induced by the product structure of $A_{\mathrm{iso}} = \mathfrak{g} \times G_0$ and therefore

$$\mathrm{d}_{\mathrm{Cart}}^E \equiv 0$$

on $(S^\bullet(\mathfrak{g}^*))^{\mathcal{G}}$, as these elements correspond to constant sections. So we obtain

$$H(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))) = H((S^\bullet(\mathfrak{g}^*))^{\mathcal{G}}, 0).$$

For $G_0 = \{*\}$, this recovers the result of Example 0.1.9. Interestingly, one can also see that the Lie groupoid G itself is Morita equivalent to the trivial action groupoid, such that it is very natural that the complex is actually independent of G_0.

Example 4.3.20. Consider the Lie groupoid $G = T$ of Example 0.4.7 and its $J^E G$-isotropy Cartan forms of Example 3.1.31. Its multiplicative Ehresmann connection induced by the lattice is flat, as it locally corresponds to a trivialisation by Example 2.3.22. Hence we can indeed use $\mathrm{Tot}(\mathfrak{R}_E)$ to compute the associated differentiable stack cohomology. We have that by Example 3.1.31 it is given by

$$\Gamma(\mathbb{S}^1, \wedge^\bullet(T^*\mathbb{S}^1) \otimes S^\bullet(A^*)).$$

We would now like to choose a E-horizontal frame of $S^\bullet(A^*)$, because $\mathrm{d}_{\mathrm{Cart}}^E$ vanishes on ∇^E-horizontal sections. However, because of the holonomy this is not possible.

Instead, we will use the following construction: Consider $\pi : \mathbb{R} \to \mathbb{S}^1$ as the universal cover, inducing $B = \pi^* A$ as a new Lie algebroid. We can also pull back the connection ∇^E to B. We now have

$$\Gamma(\mathbb{S}^1, \wedge^\bullet(T^*\mathbb{S}^1) \otimes S^\bullet(A^*)) = \Gamma(\mathbb{R}, \wedge^\bullet(T^*\mathbb{R}) \otimes S^\bullet(B^*))^{2\pi\cdot\mathbb{Z}}.$$

Now let us recall that A and hence B is a bundle of rank 2. As $\mathbb{R}$ is simply connected, we can now choose ∇^E-horizontal frame. We will refer to the elements of this frame as μ_1 and μ_2. Note that for this we may consider an arbitrary point m and choose $\mu_1(m)$ and $\mu_2(m)$ of an arbitrary linearly independent form. For example, we can choose μ_1 and μ_2 according to the generators of the lattice used to define $G = T$, which implies that $2\pi \cdot 1$ acts on μ_1 and μ_2 by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

So we now have that

$$\Gamma(\mathbb{R}, \wedge^\bullet(T^*\mathbb{R}) \otimes S^\bullet(B^*))^{2\pi\cdot\mathbb{Z}} = \Gamma(\mathbb{R}, \wedge^\bullet(T^*\mathbb{R}) \otimes \mathbb{R}[\mu_1, \mu_2])^{2\pi\cdot\mathbb{Z}},$$

where, as we chose μ_1 and μ_2 as ∇^E-horizontal frames, we find

$$\mathrm{d}^E_{\mathrm{Cart}}(\mu_1) = \mathrm{d}^E_{\mathrm{Cart}}(\mu_2) = 0.$$

Furthermore, we know that the $J^E G$-Cartan differential is a derivation and acts on $\Omega^\bullet(\mathbb{R})$ as the de Rham differential. So we can for example see that

$$c \cdot (\mu_1^2 + \mu_2^2)$$

for any constant c or

$$\mathrm{d}f \cdot (\mu_1^2 + \mu_2^2)$$

for any 2π-periodic function $f \in C^\infty(\mathbb{R})$ (equivalently $f \in C^\infty(\mathbb{S})$) describe closed elements (and hence cohomology classes) in

$$\Gamma(\mathbb{R}, \wedge^\bullet(T^*\mathbb{R}) \otimes \mathbb{R}[\mu_1, \mu_2])^{2\pi\cdot\mathbb{Z}}.$$

Proposition 4.3.21. *The cohomology ring of the differentiable stack associated to the torus bundle of Example 0.4.7 is given by*

$$H(\Omega(\mathbb{S}^1)) \otimes \mathbb{R}[x, y, z]/(4z^2 - y(x^2 - y)),$$

where x is a generator of degree 4 and y and z are generators of degree 8.

Remark. In particular, we find that

$$H^2_{\mathrm{DiffStack}}([T_0//T_1]) = 0,$$

which agrees with the result of Proposition 0.4.10.

Proof. We can start by recalling that we have already noted in Example 4.3.20 that the multiplicative Ehresmann connection constructed on T in 2.3.22 is flat. Therefore we can now apply Theorem 4.3.16, such that

$$H^\bullet_{\mathrm{DiffStack}}([T_0//T_1]) = H^\bullet(\mathrm{Tot}(\mathfrak{R}_E))$$

If we again consider the findings of Example 4.3.20, where we have seen that the cohomology of $\mathrm{Tot}(\mathfrak{R}_E)$ is given by

$$\Gamma(\mathbb{R}, \wedge^\bullet(T^*\mathbb{R}) \otimes \mathbb{R}[\mu_1, \mu_2])^{2\pi\cdot\mathbb{Z}},$$

where we have that

$$\mathrm{d}^E_{\mathrm{Cart}}(\mu_1) = \mathrm{d}^E_{\mathrm{Cart}}(\mu_2) = 0.$$

We can use that $T^*\mathbb{R}$ is a bundle of rank 1 to write

$$\Gamma(\mathbb{R}, \mathbb{R} \otimes \mathbb{R}[\mu_1, \mu_2])^{2\pi\cdot\mathbb{Z}} \oplus \Gamma(\mathbb{R}, \mathrm{d}x \cdot \mathbb{R} \otimes \mathbb{R}[\mu_1, \mu_2])^{2\pi\cdot\mathbb{Z}}.$$

Note that written like this, we have seen that the differential only acts on the components in $T^*\mathbb{R}$ and the function coefficients of second component, but vanishes on μ_1 and μ_2. In particular, the entire second summand is closed, while an element in the first summand is only closed when all of its component functions (using $\Gamma(\mathbb{R}, \mathbb{R} \otimes \mathbb{R}[\mu_1, \mu_2]) = C^\infty(\mathbb{R})[\mu_1, \mu_2]$) are constant. Furthermore the image of the differential is contained in the second component entirely. This implies that if an element is exact, its preimage can be chosen to be contained in the first component of the direct sum only. Also, it implies that only elements contained in the second component of the direct sum can be exact. Lastly, it implies that the cohomology groups inherit the structure as a direct sum of cohomology classes represented by elements of the first and second component respectively.

Now let us analyse the first component

$$\Gamma(\mathbb{R}, \mathbb{R} \otimes \mathbb{R}[\mu_1, \mu_2])^{2\pi \cdot \mathbb{Z}} = (C^\infty(\mathbb{R})[\mu_1, \mu_2])^{2\pi \cdot \mathbb{Z}}$$

first. As we have already noted that elements of this subspace cannot be exact, we have that any closed element defines a unique cohomology class. We will now analyse the subspace of closed elements

$$\ker(\mathrm{d}_{\mathrm{Cart}}) \subset (C^\infty(\mathbb{R})[\mu_1, \mu_2])^{2\pi \cdot \mathbb{Z}}.$$

The elements of the first component are of the form

$$\sum_{(n_1,n_2)} f^{(n_1,n_2)} \cdot \mu_1^{n_1} \mu_2^{n_2},$$

where the invariance condition is given by

$$f^{(n_1,n_2)}(x) = (-1)^{n_2} f^{(n_2,n_1)}(x + 2\pi).$$

The closedness condition on each component of the sum is closedness of the $f^{(n_1,n_2)}$ under the de Rham differential, i.e. the functions must be constant. As the invariance condition implies that

$$f^{(n_1,n_2)}(x) = (-1)^{n_1+n_2} f^{(n_1,n_2)}(x + 4\pi),$$

we find that in particular

$$f^{(n_1,n_2)} \equiv 0$$

for $n_1 + n_2$ odd.

So we now in particular know that as a $C^\infty(\mathbb{R})$-algebra, $(C^\infty(\mathbb{R})[\mu_1, \mu_2])^{2\pi \cdot \mathbb{Z}}$ is generated by μ_1^2, μ_2^2 and $\mu_1\mu_2$ which, of course, are subject to equivalence relations. We can also choose the equivalent generating set

$$\{a, b, c\} := \{\mu_1^2 + \mu_2^2, \mu_1^2 - \mu_2^2, \mu_1\mu_2\}.$$

The equivalence condition $4c^2 = (a + b)(a - b)$ is sufficient for the polynomial algebras

$$\mathbb{R}[a, b, c]/(4c^2 - (a + b)(a - b)) \cong \mathbb{R}[\mu_1^2 + \mu_2^2, \mu_1^2 - \mu_2^2, \mu_1\mu_2]$$

to agree, which can be seen as in degree $2k$, there are $2k + 1$ generators of

$$\mathbb{R}[\mu_1^2 + \mu_2^2, \mu_1^2 - \mu_2^2, \mu_1\mu_2]$$

as an $\mathbb{R}$-vector space (given by $\mu_1^{k_1} \cdot \mu_2^{k_2}$), while as $\mathbb{R}$-vector spaces, we have that

$$\mathbb{R}[a, b, c]/(4c^2 - (a + b)(a - b)) \cong \mathbb{R}[a, b] \otimes (\mathbb{R} \oplus c \cdot \mathbb{R}),$$

where there are $k + 1$ generators of degree $2k$ and k generators of degree $2k - 2$ in $\mathbb{R}[a, b]$.

Note that all three of these generators are eigenvectors of the action of $2\pi \cdot \mathbb{Z}$, with eigenvalues $1, -1, -1$ respectively. So if we consider

$$\sum_{(\alpha,\beta,\gamma)} f^{(\alpha,\beta,\gamma)} \cdot a^\alpha b^\beta c^\gamma \in (\mathbb{R}[a, b] \otimes (\mathbb{R} \oplus c \cdot \mathbb{R}))^{2\pi \cdot \mathbb{Z}},$$

we find that they are given by precisely those elements for which

$$f^{(\alpha,\beta,\gamma)} \equiv 0$$

when $\beta+\gamma$ is odd. Equivalently, we can only sum over those values where $\beta+\gamma$ is even, such that we can write the previous element equivalently as

$$\sum_{(\alpha,\beta,\gamma),\beta+\gamma=2\delta} f^{(\alpha,\beta,\gamma)} \cdot a^{\alpha}(b^2)(\delta-\gamma)(bc)^{\gamma} \in (\mathbb{R}[a,b] \otimes (\mathbb{R} \oplus c \cdot \mathbb{R}))^{2\pi\cdot\mathbb{Z}},$$

where the three terms a, b^2 and bc are each invariant under the action.

This means that

$$(\mathbb{R}[a,b] \otimes (\mathbb{R} \oplus c \cdot \mathbb{R}))^{2\pi\cdot\mathbb{Z}} = \mathbb{R}[a,b^2] \otimes (\mathbb{R} \oplus bc \cdot \mathbb{R}),$$

as $\mathbb{R}$-vector spaces and therefore

$$\left(\mathbb{R}[a,b,c]/(4c^2-(a+b)(a-b))\right)^{2\pi\cdot\mathbb{Z}} = \mathbb{R}[a,b^2,bc]/(\sim),$$

as $\mathbb{R}$-algebras, where the equivalence relation $\sim$ is generated by the equivalence relation $(4c^2-(a+b)(a-b))$ on $\mathbb{R}[a,b,c]$ as a superspace of $\mathbb{R}[a,b^2,bc]$.

Spelling out the algebra in terms of these new generators yields

$$\mathbb{R}[a,b^2,bc]/(\sim) \cong \mathbb{R}[u,v,w]/(4w^2-v(u^2-v)),$$

where u is of degree 4 and v and w are of degree 8. That this is indeed an isomorphism of $\mathbb{R}$-algebras follows in an analogous way as in the previous calculations: The map

$$(u,v,w) \mapsto (a,b^2,bc)$$

induces an algebra morphism and by counting the dimensions of the homogeneous subspaces as $\mathbb{R}$-vector spaces, we find that it is indeed bijective. The latter follows directly from the fact that

$$\mathbb{R}[u,v,w]/(4w^2-v(u^2-v)) \cong \mathbb{R}[u,v] \otimes (\mathbb{R} \oplus w \cdot \mathbb{R}) \cong \mathbb{R}[a,b^2] \otimes (\mathbb{R} \oplus bc \cdot \mathbb{R})$$

as $\mathbb{R}$-vector spaces.

This concludes the proof that cohomology classes of

$$\Gamma(\mathbb{R}, \mathbb{R} \otimes \mathbb{R}[\mu_1,\mu_2])^{2\pi\cdot\mathbb{Z}} \oplus \Gamma(\mathbb{R}, \mathrm{d}x \cdot \mathbb{R} \otimes \mathbb{R}[\mu_1,\mu_2])^{2\pi\cdot\mathbb{Z}}$$

represented by elements of the first summand

$$\Gamma(\mathbb{R}, \mathbb{R} \otimes \mathbb{R}[\mu_1,\mu_2])^{2\pi\cdot\mathbb{Z}}$$

correspond to elements of

$$\mathbb{R}[u,v,w]/(4w^2-v(u^2-v)).$$

Recovering u, v and w in terms of μ_1 and μ_2 yields

$$\begin{aligned} u &= \mu_1^2+\mu_2^2 \\ v &= (\mu_1^2-\mu_2^2)^2 = \mu_1^4-2\mu_1^2\mu_2^2+\mu_2^4 \\ w &= (\mu_1^2-\mu_2^2)\mu_1\mu_2 = \mu_1^3\mu_2-\mu_1\mu_2^3. \end{aligned}$$

We will now apply a similar method to obtain an analogous result for the cohomology classes represented by elements of the second summand

$$\Gamma(\mathbb{R}, \mathrm{d}x \cdot \mathbb{R} \otimes \mathbb{R}[\mu_1, \mu_2])^{2\pi\cdot\mathbb{Z}}$$

However, on this space we need to take the quotient by exact terms.

If we ignore the matter of invariance, we find that any element of

$$\Gamma(\mathbb{R}, \mathrm{d}x \cdot \mathbb{R} \otimes \mathbb{R}[\mu_1, \mu_2]) = \Gamma(\mathbb{R}, \mathrm{d}x \cdot \mathbb{R}) \otimes \mathbb{R}[\mu_1, \mu_2] = \Omega^1(\mathbb{R}) \otimes \mathbb{R}[\mu_1, \mu_2]$$

has a preimage under the differential, as it corresponds to the de Rham differential. The preimage is unique up to a closed term (which we already identified to be the terms with constant component functions) and can be described very specifically: The preimage of $\omega \in \Omega^1(\mathbb{R})$ is given by $f \in C^\infty(\mathbb{R})$ defined as

$$f(x) = \int_0^x \omega + c.$$

In the following, we aim to show the following: The inclusion of "constant" forms

$$\mathrm{d}x \cdot \mathbb{R} \otimes \mathbb{R}[\mu_1, \mu_2] \subset \Omega^1(\mathbb{R}) \otimes \mathbb{R}[\mu_1, \mu_2]$$

restricts on the invariant elements to an isomorphism onto the cohomology classes represented by elements of the second summand, where the inverse is induced by

$$\omega \mapsto \mathrm{d}x \cdot \frac{1}{8\pi} \int_0^{8\pi} \omega.$$

To prove this claim, assume that we have a family $\omega^{(n_1,n_2)}$ such that

$$\sum_{n_1,n_2} \omega^{(n_1,n_2)} \cdot \mu_1^{n_1} \mu_2^{n_2} \in \left(\Omega^1(\mathbb{R}) \otimes \mathbb{R}[\mu_1, \mu_2]\right)^{2\pi\cdot\mathbb{Z}},$$

i.e. that

$$\omega^{(n_1,n_2)}(x) = (-1)^{n_2} \omega^{(n_2,n_1)}(x + 2\pi).$$

Let us first analyse, when this element is exact, i.e. when we can define

$$f^{(n_1,n_2)}(x) = \int_0^x \omega^{(n_1,n_2)} + c^{(n_1,n_2)}$$

for an appropriate choice of $c^{(n_1,n_2)}$, and obtain a preimage which is invariant again, i.e.

$$\sum_{n_1,n_2} f^{(n_1,n_2)} \cdot \mu_1^{n_1} \mu_2^{n_2} \in (C^\infty(\mathbb{R})[\mu_1, \mu_2])^{2\pi\cdot\mathbb{Z}}.$$

Because of the form of the transformation matrix (in particular, its fourth power is the identity), the invariance condition can only be satisfied when

$$f^{(n_1,n_2)}(x) = f^{(n_1,n_2)}(x + 8\pi).$$

In particular, the sum $\sum_{n_1,n_2} \omega^{(n_1,n_2)} \cdot \mu_1^{n_1} \mu_2^{n_2}$ can only be exact with an invariant preimage if

$$\int_0^{8\pi} \omega^{(n_1,n_2)} = 0$$

for all indices (n_1, n_2). This shows that each invariant element of

$$\mathrm{d}x \cdot \mathbb{R} \otimes \mathbb{R}[\mu_1, \mu_2] \subset \Omega^1(\mathbb{R}) \otimes \mathbb{R}[\mu_1, \mu_2]$$

is exact if and only if it is given by 0 itself. Therefore, all

$$(\mathrm{d}x \cdot \mathbb{R} \otimes \mathbb{R}[\mu_1, \mu_2])^{2\pi\cdot\mathbb{Z}}$$

define unique cohomology classes.

Now it only remains to show that indeed, each cohomology class corresponding to an element of $\Omega^1(\mathbb{R}) \otimes \mathbb{R}[\mu_1, \mu_2]$ can be represented by an element with constant coefficients. For this, consider the projection

$$\sum_{n_1,n_2} \omega^{(n_1,n_2)} \cdot \mu_1^{n_1}\mu_2^{n_2} \mapsto \sum_{n_1,n_2} \mathrm{d}x \cdot \frac{1}{8\pi}\int_0^{8\pi} \omega^{(n_1,n_2)} \cdot \mu_1^{n_1}\mu_2^{n_2} \in \mathrm{d}x \cdot \mathbb{R} \otimes \mathbb{R}[\mu_1, \mu_2].$$

As invariance implies 8π-periodicity and therefore all integrals of length 8π return the same value, we find that this assignment maps invariant elements to invariant elements. Also it is immediately clear that on the subspace $\mathrm{d}x \cdot \mathbb{R} \otimes \mathbb{R}[\mu_1, \mu_2]$ the map is given by the identity, making it into an actual projection to this subspace.

We aim to show that the map also retains the cohomology class of an element, i.e. that the kernel of

$$\sum_{n_1,n_2} \omega^{(n_1,n_2)} \cdot \mu_1^{n_1}\mu_2^{n_2} \mapsto \sum_{n_1,n_2} \mathrm{d}x \cdot \frac{1}{8\pi}\int_0^{8\pi} \omega^{(n_1,n_2)} \cdot \mu_1^{n_1}\mu_2^{n_2}$$

only contains exact elements. So assume that we have some $\sum_{n_1,n_2} \omega^{(n_1,n_2)} \cdot \mu_1^{n_1}\mu_2^{n_2}$ in the kernel. If we define

$$c^{(n_1,n_2)} = 0,$$

i.e.

$$f^{(n_1,n_2)}(x) = \int_0^x \omega^{(n_1,n_2)},$$

we find that the family of $f^{(n_1,n_2)}(x)$ is indeed 8π-periodic, but not necessarily defines an invariant form. However, we can now look at the functions

$$\begin{aligned}\overline{f}^{(n_1,n_2)}(x) = \frac{1}{4}\Big(&f^{(n_1,n_2)}(x) + (-1)^{n_2} f^{(n_2,n_1)}(2\pi + x)\\ &+ (-1)^{n_1+n_2} f^{(n_1,n_2)}(4\pi + x) + (-1)^{n_1} f^{(n_2,n_1)}(6\pi + x)\Big),\end{aligned}$$

which define an invariant element

$$\sum_{n_1,n_2} \overline{f}^{(n_1,n_2)} \cdot \mu_1^{n_1}\mu_2^{n_2} \in (\mathrm{d}x \cdot \mathbb{R} \otimes \mathbb{R}[\mu_1, \mu_2])^{2\pi\cdot\mathbb{Z}}.$$

This can be checked explicitly: We have that

$$\begin{aligned}
&(-1)^{n_2}\overline{f}^{(n_2,n_1)}(x+2\pi)\\
=&(-1)^{n_2}\Big(\frac{1}{4}\Big(f^{(n_2,n_1)}(x+2\pi)+(-1)^{n_1}f^{(n_1,n_2)}(x+4\pi)\\
&\quad+(-1)^{n_1+n_2}f^{(n_2,n_1)}(x+6\pi)+(-1)^{n_2}f^{(n_1,n_2)}(x+8\pi)\Big)\Big)\\
=&\frac{1}{4}\Big((-1)^{n_2}f^{(n_2,n_1)}(x+2\pi)+(-1)^{n_1+n_2}f^{(n_1,n_2)}(x+4\pi)\\
&\quad+(-1)^{n_1}f^{(n_2,n_1)}(x+6\pi)+f^{(n_1,n_2)}(x+8\pi)\Big)\\
=&\frac{1}{4}\Big((-1)^{n_2}f^{(n_2,n_1)}(x+2\pi)+(-1)^{n_1+n_2}f^{(n_1,n_2)}(x+4\pi)\\
&\quad+(-1)^{n_1}f^{(n_2,n_1)}(x+6\pi)+f^{(n_1,n_2)}(x)\Big)\\
=&\overline{f}(x).
\end{aligned}$$

We can compute that this construction corresponds to choosing

$$c^{(n_1,n_2)}=\frac{1}{4}\left((-1)^{n_2}\int_0^{2\pi}\omega^{(n_2,n_1)}+(-1)^{n_1+n_2}\int_0^{4\pi}\omega^{(n_1,n_2)}+(-1)^{n_1}\int_0^{6\pi}\omega^{(n_2,n_1)}\right),$$

and hence

$$\mathrm{d}^E_{\mathrm{Cart}}\left(\sum_{n_1,n_2}\overline{f}^{(n_1,n_2)}\cdot\mu_1^{n_1}\mu_2^{n_2}\right)=\sum_{n_1,n_2}\omega^{(n_1,n_2)}\cdot\mu_1^{n_1}\mu_2^{n_2}.$$

(If we don't want to check that this constant indeed yields the alternate function as a preimage, we can alternatively use linearity of the Cartan differential to obtain the same conclusion using the invariance of $\sum_{n_1,n_2}\omega^{(n_1,n_2)}\cdot\mu_1^{n_1}\mu_2^{n_2}$.)

So we have now indeed obtained that all of the cohomology classes

$$\sum_{n_1,n_2}\omega^{(n_1,n_2)}\cdot\mu_1^{n_1}\mu_2^{n_2}$$

in the previously described kernel are exact. Hence the cohomology classes represented by elements of

$$\sum_{n_1,n_2}\omega^{(n_1,n_2)}\cdot\mu_1^{n_1}\mu_2^{n_2}\in\left(\Omega^1(\mathbb{R})\otimes\mathbb{R}[\mu_1,\mu_2]\right)^{2\pi\cdot\mathbb{Z}}$$

can be equivalently represented by elements of

$$(\mathrm{d}x\cdot\mathbb{R}\otimes\mathbb{R}[\mu_1,\mu_2])^{2\pi\cdot\mathbb{Z}},$$

which describe unique cohomology classes as we have stated previously. In complete analogy to our considerations for the invariant elements $(\mathbb{R}\otimes\mathbb{R}[\mu_1,\mu_2])^{2\pi\cdot\mathbb{Z}}$, we can identify

$$(\mathrm{d}x\cdot\mathbb{R}\otimes\mathbb{R}[\mu_1,\mu_2])^{2\pi\cdot\mathbb{Z}}\cong\mathrm{d}x\cdot\mathbb{R}[u,v,w]/(4w^2-v(u^2-v)),$$

where the generators correspond to the expressions

$$\begin{aligned}
u&=\mu_1^2+\mu_2^2\\
v&=(\mu_1^2-\mu_2^2)^2=\mu_1^4-2\mu_1^2\mu_2^2+\mu_2^4\\
w&=(\mu_1^2-\mu_2^2)\mu_1\mu_2=\mu_1^3\mu_2-\mu_1\mu_2^3.
\end{aligned}$$

So all in all, we have found that the cohomology classes can indeed be expressed by

$$(\mathbb{R}[\mu_1,\mu_2])^{2\pi\cdot\mathbb{Z}} \oplus \mathrm{d}x \cdot (\mathbb{R}[\mu_1,\mu_2])^{2\pi\cdot\mathbb{Z}} \cong H(\Omega(\mathbb{S}^1)) \otimes \mathbb{R}[u,v,w]/(4w^2 - v(u^2-v)),$$

where by the precise form of the isomorphism (i.e. the fact that the ring structure of $\mathbb{R}[u,v,w]/(4w^2 - v(u^2-v)$ embeds as a subring into $\mathbb{R}[\mu_1,\mu_2]$) proves that this does not only describe the group structure but also the ring structure of $H(\mathrm{Tot}(\mathfrak{R}_E))$. □

Corollary 4.3.22. *We have that for the groupoid of Example 0.4.7 holds*

$$\dim(H^k_{\mathrm{DiffStack}}([T_0//T_1])) = \begin{cases} 2l+1 & \text{if } k = 8l \\ 2l+1 & \text{if } k = 8l+1 \\ 0 & \text{if } k = 8l+2 \\ 0 & \text{if } k = 8l+3 \\ 2l+1 & \text{if } k = 8l+4 \\ 2l+1 & \text{if } k = 8l+5 \\ 0 & \text{if } k = 8l+6 \\ 0 & \text{if } k = 8l+7. \end{cases}$$

Proof. We can apply the result of Proposition 4.3.21 to compute the dimensions of the cohomology ring We can use the $\mathbb{R}$-vector space isomorphism established in the proof, which identifies

$$\mathbb{R}[u,v,w]/(4w^2 - v(u^2-v)) \cong \mathbb{R}[u,v] \otimes (\mathbb{R} \oplus w \cdot \mathbb{R}),$$

where, for $l \geq 0$, we have that there are $l+1$ monomials of $\mathbb{R}[u,v]$ of total degree $8l$ determined by the power of y in the range $\{0,\ldots,l\}$, and the same argument applies in degree $8l+4$. There are no monomials in degrees not divisible by 4. (This implies for $l > 0$, there are l monomials of total degree $8(l-1)$ or $8(l-1)+4$.)

Now, we can use these monomials to choose a basis of the $\mathbb{R}$-vector space (where they are multiplied according to the tensor products by at most one power of $\mathrm{d}x$ or w). This distinguishes four types of basis vectors by whether or not they contain $\mathrm{d}x$ and whether or not they contain w. For each of these four types there are precisely as many basis vectors of degree k as there are monomials of the appropriate degree (either $k, k-1, k-8, k-9$). We have previously described how to compute this number, after which we can sum over the four cases. □

Chapter 5

A construction of the model

In Chapter 4, we have studied how the total cohomology of the Bott–Shulman–Stasheff-complex relates to the cohomology of the bicomplex of $J^E G$-isotropy Cartan forms $\mathcal{R}_E$ and the closely related bicomplex $\mathfrak{R}_E$. In particular, we have found a condition under which they are quasi-isomorphic (after totalisation) in a way that even recovers the algebra structure of $\Omega_{\mathrm{BSS}}(G)$.

The referred to condition was related to $J^E G$-isotropy Cartan forms closed in $\mathcal{R}_E$ (i.e. under the Cartan differential) having corresponding closed elements in the Bott–Shulman–Stasheff complex (i.e. closed descending homogeneous extensions). If we consider the findings of Section 4.3, we can see that flatness of a multiplicative Ehresmann connection is a strong enough assumption to show that the condition is met.

Of course, imposing flatness is often a meaningful obstruction to the existence of connections. However, for admissible embedded manifolds we have a local form induced by the linearisation around a fibre, which automatically implies the existence of a flat connection on some open neighbourhood. If we assume that a flat connection is defined on all of $G|_R$ we can even apply our previous results to this subgroupoid. However, $G|_R$ and $G|_{\mathrm{Sat}(R)}$ are Morita equivalent (with bibundles $s^{-1}(R)$ and $t^{-1}(R)$ describing inverse right principal bundles). This means that both Lie groupoids need to induce the same differentiable stack cohomology, so that that the associated Bott–Shulman–Stasheff complex must have similar properties. We could hope, for instance, that if the technical condition mentioned above is met on $\Omega_{\mathrm{BSS}}(G|_R)$, it is also met on $\Omega_{\mathrm{BSS}}(G|_{\mathrm{Sat}(R)})$.

In this chapter, we aim to use this idea to generalise our findings previously obtained for a proper and regular Lie groupoid with a flat multiplicative Ehresmann connection, to results where we may drop the flatness condition. Specifically, we will show that the condition previously mentioned on the closed descending homogeneous extensions of $\widehat{\mathrm{d}_{\mathrm{Cart}}}$-closed $J^E G$-isotropy Cartan forms is indeed true independently of the groupoid or the chosen multiplicative Ehresmann connection. We will proceed in two steps. Firstly, we will consider groupoids, for which there exists a trivial admissible cover, i.e. an admissible embedded manifold $R \subset G_0$ with $\mathrm{Sat}(R) = G_0$. We will find that even though the multiplicative Ehresmann connection might not be flat, there is a construction similar to that of Section 4.3 to construct an extension of any $J^E G$-isotropy Cartan form. Furthermore, we will study properties of this construction if there is an additional flatness assumption on the restriction to $G|_R$.

In the second step, we will generalise to the setting of a general admissible cover. It will turn out that for this construction, we will need a compatibility condition between the cover and the chosen multiplicative Ehresmann connection. Most of the second chapter will be dedicated to showing that for almost any proper Lie groupoid G it is possible to choose such compatible data (and in the remaining case, there is a trick to enable the construction as well).

5.1 Establishing sufficient conditions

Aim of this section: Theorem 5.1.19:

Consider a regular and proper Lie groupoid G with a multiplicative Ehresmann connection E and an admissible embedded manifold R such that $\mathrm{Sat}(R) = G_0$. Then the associated differentiable stack cohomology groups can be computed as

$$H^\bullet_{\mathrm{DiffStack}}([G_0//G_1]) \cong H^\bullet(\mathrm{Tot}(\mathcal{R}_E)) \cong H^\bullet(\mathrm{Tot}(\mathfrak{R}_E)).$$

If furthermore $E|_R$ is flat, we have that the isomorphisms additionally preserve the ring structure of $H^\bullet_{\mathrm{DiffStack}}([G_0//G_1])$.

As mentioned in the introduction to this chapter, we have that given any proper and regular Lie groupoid G and admissible embedded manifold $R \subset G_0$, the groupoids $G|_R$ and $G|_{\mathrm{Sat}(R)}$ are Morita equivalent. So if the cohomology of $G|_R$ can be modelled using $J^E G$-isotropy Cartan forms, the same result must hold for $G|_{\mathrm{Sat}(R)}$.

Interestingly, studying what this relation corresponds to in the Bott–Shulman–Stasheff-complex reveals that the method described in Section 4.3 can be extended to $G|_{\mathrm{Sat}(R)}$ - no matter if the connection E satisfies that $E|_R$ is flat or not. (In the case where $E|_R$ is flat, there will be some additional results on the multiplicative structures.)

To see this, a major result will be that $(\overline{D}^l_{R,q})^*$ and $\mathrm{d}^{\pm 0}_{\mathrm{dR}}$ commute. This will be a very technical result. As will be the rest of the section, in which we will revisit the construction of how to assign a descending homogeneous extension to the $J^E G$-isotropy Cartan forms, generalising from the case where E is flat.

Lemma 5.1.1. *Let G be an s-proper Lie groupoid where all orbits are discrete and finite. Consider a multiplicative Ehresmann connection E with respect to the isotropy algebroid and $m \in G_0$ and define $N := G|_{\mathcal{O}_m}$. Then there is a neighbourhood of N which is trivialisable to the structure of the infinitesimal neighbourhood $E|_N$ of N.*

By this, we mean that there is an open connected and contractible subset U with $m \in U$, such that its saturation $\mathrm{Sat}(U)$ is diffeomorphic to $\mathcal{O}_m \times B_1(0)$, identifying $\{m\} \times B_1(0)$ with U and $\mathcal{O}_m \times \{0\}$ with $\mathcal{O}_m$, together with a diffeomorphism

$$N \times B_1(0) \to (G|_{\mathrm{Sat}(U)})_1 \subset G_1.$$

In addition, we require that the induced regular Lie groupoid structure on

$$N \times B_1(0) \rightrightarrows \mathcal{O}_m \times B_1(0)$$

has its source given by $(s \circ \mathrm{pr}_1, \mathrm{pr}_2)$ and a target map also compatible with projection to $N \rightrightarrows \mathcal{O}_m$. The diffeomorphism between $N \times B_1(0)$ and $(G|_{\mathrm{Sat}(U)})_1$ furthermore satisfies that the pull-back of E and the splitting induced by the product structure agree when considered on $N \times \{0\}$.

Proof. Firstly note, that as the orbits are discrete, we have that at any $x \in G_1$ the Ehresmann connection E satisfies

$$T_x G_1 = E_x \oplus \ker(Ts),$$

as the image of $\ker(Ts)$ under Tt is contained in the orbit foliation and hence is 0. This means that we actually have a Cartan connection in this case and in particular, we can use the connection to lift vector fields on G_0 along s.

We have already seen in Corollary 1.4.26 that $s : G_1 \to G_0$ locally has the structure of a trivial fibre bundle. The product structure induces a connection which is not necessarily given by the multiplicative Ehresmann connection, but it can help us in our construction. This is because the given trivialisation of G around $N \rightrightarrows \mathcal{O}_m$ being isomorphic to an open subset of the normal bundle $\nu(N) \rightrightarrows \nu(\mathcal{O}_m)$ shows that there are coordinates (precisely given by coordinates compatible with the vector bundle structure of $\nu(\mathcal{O}_m)$) locally around any $m' \in \mathcal{O}_m$ for which any local diffeomorphism induced by a local section of G_1 is a linear map (in coordinates). The construction of U from this will now consist of two steps.

Firstly, analogous to a part of the proof of Lemma 1.4.37 (by choosing an invariant metric and balls sufficiently small to be contained in the corresponding subset of $\nu(\mathcal{O}_m)$), we may assume the domain of the coordinate chart around each point to be isomorphic to $B_{r'}(0) \subset \mathbb{R}^k$ for an appropriate $r' \in \mathbb{R}$ and any local diffeomorphism induced by a local section of G_1 is given by an orthonormal linear map. In particular, for any $r < r'$ we have that $\mathcal{O}_m \subset B_r(0)$ is saturated.

We will now construct simultaneously a morphism from $N \times B_{r'}(0)$ to G_1 and a restricted domain related to the choice of U: Consider this coordinate chart $B_{r'}(0) \subset \mathbb{R}^{k_0} \to G_0$. For every $p \in B_{r'}(0) \subset \mathbb{R}^{k_0}$ we can associate the corresponding vector field $X_p \in \mathfrak{X}(\mathbb{R}^{k_0})$ that is constant when expressed in coordinates and satisfies that if we integrate to a flow $\phi^t_{X_p}$ this flow satisfies

$$\phi^1_{X_p}(m) = p.$$

We can locally lift X_p to $\mathcal{G} \times M$ along s and integrate to a flow $\phi^t_{\widehat{X_p}}$ which is a priori defined only for $t \in [0, \varepsilon)$ at any point in the fibre of $\mathbb{R}^{k_0}$, where $\epsilon > 0$ can be arbitrarily small. As we however have that the groupoid is s-proper and the vector fields X_p can be integrated when starting at m to a path on $(-\epsilon', 1 + \epsilon')$ for some sufficiently small $\epsilon' > 0$, the flow integrating $\widehat{X_p}$ starting in the fibre $s^{-1}(m)$ over m are defined for any $t \in (-\epsilon', 1 + \epsilon')$ as well. Evaluation at 1 defines a morphism

$$s^{-1}(m) \times B_{r'}(0) \to G_1.$$

This morphism is a diffeomorphism locally around $s^{-1}(m)$, this can be seen as the tangent map is bijective by construction. As G is s-proper, we can now find a $B_r(0) \subset B_{r'}(0)$, such that

$$s^{-1}(m) \times B_r(0) \to G_1$$

is a diffeomorphism globally. The corresponding set $U \subset G_0$ will now have the desired properties.

As per our choice, we have that choosing these trivialisations for every $m' \in \mathcal{O}_m$ yields a morphism

$$N \times B_r(0) \to G_1.$$

We now need to check that the groupoid structure that this morphism induces has the desired property with respect to the projection on N, i.e. that for any composable pair (x, k) and (y, l) the first component of the composition is given by $\mu(x, y)$.

However, we have that the (in coordinates) linear path between any $(x, 0)$ and (x, k) is E-horizontal by construction and so is the path between any $(y, 0)$ and (y, l). Furthermore, we have already noted that all induced diffeomorphisms - in particular any diffeomorphism extending this lift locally - induces an orthonormal linear map in our chosen coordinates.

So as we know that (x,k) and (y,l) are compatible, so must be $(x,0)$ and $(y,0)$, as any linear map fixes 0, as well as $(x,\lambda\cdot k)$ and $(y,\lambda\cdot l)$ for any $\lambda\in(-\epsilon,1+\epsilon)$ for some $\epsilon>0$.

This in particular implies that $\mu(x,y)$ is well-defined, as the morphism restricted to $N\times\{0\}$ is given by the identity and $(\mu(x,y),0)$ is the result of composing $(x,0)$ and $(y,0)$. Furthermore, we can consider the entire path defined by the compositions of $(x,\lambda\cdot k)$ and $(y,\lambda\cdot l)$ for $\lambda\in(-\epsilon,1+\epsilon)$. As E is multiplicative, it, too is E-horizontal. It also covers the same path as $(y,\lambda\cdot l)$ under s, which is linear.

So this path is precisely given by the flow of $\widehat{X_l}$ starting at y. This means that under the diffeomorphism

$$N\times B_r(0)\to(G|_{\mathrm{Sat}(U)})_1$$

it corresponds to $(\mu(x,y),\lambda\cdot l)$. In particular, the composition between the images of (x,k) and (y,l) is given by $(\mu(x,y),l)$, for which the first component is indeed given by $\mu(x,y)$.

It remains to analyse the pull-back of E. However, the fact that it agrees along $G|_{\mathcal{O}_m}$ with the product structure is by definition, as the tangent vectors horizontal with respect to the product structure are precisely mapped to the (E-horizontal) value of some $\widehat{X_p}$. $\square$

Lemma 5.1.2. *Let G be a proper and regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Assume that $R\subset G_0$ is an admissible embedded manifold. Let $\omega\in\Omega^p(G_q)$ be of E-vertical degree p_1 and $z\in(G|_{\mathrm{Sat}(R)})_q$. Furthermore let γ be a local section of s which is E-horizontal at $p^i_q(z)$. Then we have*

$$\left((\widetilde{D}^i_{\gamma,q})^*\,\mathrm{d}^{\pm0}_{\mathrm{dR}}\left((\overline{D}^i_{R,q})^*(\omega)\right)-\mathrm{d}^{\pm0}_{\mathrm{dR}}\left((\overline{D}^i_{R,q})^*(\omega)\right)\right)_z=0$$

and

$$\left((\widetilde{D}^i_{\gamma,q})^*\,\mathrm{d}^{+1}_{\mathrm{dR}}\left((\overline{D}^i_{R,q})^*(\omega)\right)-\mathrm{d}^{+1}_{\mathrm{dR}}\left((\overline{D}^i_{R,q})^*(\omega)\right)\right)_z=0.$$

In particular, if

$$\left(\mathrm{d}^{\pm0}_{\mathrm{dR}}\left((\overline{D}^i_{R,q})^*(\omega)\right)-(\overline{D}^i_{R,q})^*\left(\mathrm{d}^{\pm0}_{\mathrm{dR}}(\omega)\right)\right)|_{(p^i_q)^{-1}(R)}=0,$$

the same holds on all of $\mathrm{Sat}(R)$ *and the analogous statement holds for* $\mathrm{d}^{+1}_{\mathrm{dR}}$.

Proof. We aim to apply Lemma 4.1.3 to our situation. For this, note that $(\overline{D}^i_{R,q})^*$ retains the E-homogeneity of ω and its E-vertical degree p_1. It remains to show that the component of

$$(\widetilde{D}^i_{\gamma,q})^*(\overline{D}^i_{R,q})^*(\omega)$$

of E-vertical degree p_1 is given by $(\overline{D}^i_{R,q})^*(\omega)$ itself, not only at z but on an open around it.

However, at any $y\in G_q$ such that γ is defined at $p^i_q(y)$, we have that by Lemma 1.2.28

$$(\widetilde{D}^i_{\gamma,q})^*\equiv(\dot{D}^i_V)^*,$$

where $V\subset TG_1$ is the image of $T_{p^i_q(y)}G_0$ under $T\gamma$. Furthermore, we know by Lemma 2.1.36, that as V is not necessarily E-horizontal, the image of $(D^i_{R,q})^*(\omega)$ under $(\dot{D}^i_{V,q})^*$ decomposes into E-homogeneous components, where the component in E-horizontal degree p_0 agrees with the image under $(\dot{D}^i_W)^*$ for W the E-horizontal projection of V.

However, by Lemma 2.2.10, we know that

$$\left((\dot{D}^i_{W,q})^*(\overline{D}^i_{R,q})^*(\omega)\right)_y=\left((\overline{D}^i_{R,q})^*(\omega)\right)_y.$$

Hence indeed the E-homogeneous component of E-horizontal degree p_0 and E-vertical degree p_1 of $(\widetilde{D}^i_{\gamma,q})^*(\overline{D}^i_{R,q})^*(\omega)$ agrees with $(\overline{D}^i_{R,q})^*(\omega)$. Additionally, we have that

$$\left((\widetilde{D}^i_{\gamma,q})^*(\overline{D}^i_{R,q})^*(\omega)\right)_z = \left((\dot{D}^i_{V,q})^*(\overline{D}^i_{R,q})^*(\omega)\right)_z = \left((\overline{D}^i_{R,q})^*(\omega)\right)_z.$$

The main statement of the lemma now follows from applying Lemma 4.1.3 to $(\overline{D}^i_{R,q})^*(\omega)$, which in this case precisely states that

$$\left((\widetilde{D}^i_{\gamma,q})^*\, \mathrm{d}^{\pm 0}_{\mathrm{dR}}\left((\overline{D}^i_{R,q})^*(\omega)\right) - \mathrm{d}^{\pm 0}_{\mathrm{dR}}\left((\overline{D}^i_{R,q})^*(\omega)\right)\right)_z = 0$$

and

$$\left((\widetilde{D}^i_{\gamma,q})^*\, \mathrm{d}^{+1}_{\mathrm{dR}}\left((\overline{D}^i_{R,q})^*(\omega)\right) - \mathrm{d}^{+1}_{\mathrm{dR}}\left((\overline{D}^i_{R,q})^*(\omega)\right)\right)_z = 0.$$

Now let us turn to the implication mentioned in the lemma. We have that for any $z \in (G|_{\mathrm{Sat}(R)})_q$ we have by definition that $p^i_q(z)$ is in the orbit of some $r \in M$, which is why we can find a γ locally defined around and E-horizontal at $p^i_q(z)$ mapping to $t^{-1}(R)$. As by our previously established formula, $\mathrm{d}_{\mathrm{dR}}((\overline{D}^i_{R,q})^*(\omega))$ and $(\widetilde{D}^i_{\gamma,q})^*(\mathrm{d}_{\mathrm{dR}}((\overline{D}^i_{R,q})^*(\omega)))$ at z agree in E-vertical degrees p_1 and p_1+1.

Using those findings, we have that

$$\begin{aligned}
&\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}\left((\overline{D}^i_{R,q})^*(\omega)\right)\right)_z - \left((\overline{D}^i_{R,q})^*\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)\right)\right)_z \\
&= \left((\widetilde{D}^i_{\gamma,q})^*\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}\left((\overline{D}^i_{R,q})^*(\omega)\right)\right)\right)_z - \left((\widetilde{D}^i_{\gamma,q})^*(\overline{D}^i_{R,q})^*\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)\right)\right)_z \\
&= \left((\widetilde{D}^i_{\gamma,q})^*\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}\left((\overline{D}^i_{R,q})^*(\omega)\right)\right) - (\widetilde{D}^i_{\gamma,q})^*(\overline{D}^i_{R,q})^*\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)\right)\right)_z \\
&= \left((\widetilde{D}^i_{\gamma,q})^*\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}\left((\overline{D}^i_{R,q})^*(\omega)\right) - (\overline{D}^i_{R,q})^*\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)\right)\right)\right)_z,
\end{aligned}$$

using that Lemma 2.2.10 - similar to a previous argument, implies that

$$\left((\widetilde{D}^i_{\gamma,q})^*(\overline{D}^i_{R,q})^*\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)\right)\right)_z = \left((\dot{D}^i_{V,q})^*(\overline{D}^i_{R,q})^*\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)\right)\right)_z = \left((\overline{D}^i_{R,q})^*\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)\right)\right)_z.$$

However, the term

$$(\widetilde{D}^i_{\gamma,q})^*\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}\left((\overline{D}^i_{R,q})^*(\omega)\right) - (\overline{D}^i_{R,q})^*\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)\right)\right)$$

vanishes under the assumption that

$$\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}\left((\overline{D}^i_{R,q})^*(\omega)\right) - (\overline{D}^i_{R,q})^*\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)\right)\right)|_{(p^i_q)^{-1}(R)} = 0$$

as $(p^i_q)^{-1}(R)$ contains the image of $\widetilde{D}^i_{\gamma,q}$ by construction. So we indeed obtain that the statement of the lemma has the implication as claimed. The same line of argument as described here also holds for $\mathrm{d}^{+1}_{\mathrm{dR}}$. □

Lemma 5.1.3. *Let G be a proper and regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Assume that $R \subset G_0$ is an admissible embedded manifold. Let $\omega \in \Omega^p(G_q)$ be of E-vertical degree p_1. Then we have that $(\overline{D}^i_R)^*$ commutes with both $\mathrm{d}^{\pm 0}_{\mathrm{dR}}$ and $\mathrm{d}^{+1}_{\mathrm{dR}}$, i.e.*

$$\mathrm{d}^{\pm 0}_{\mathrm{dR}}((D^i_R)^*(\omega)) = (D^i_R)^*(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega))$$

and

$$\mathrm{d}_{\mathrm{dR}}^{+1}((D_R^i)^*(\omega)) = (D_R^i)^*(\mathrm{d}_{\mathrm{dR}}^{+1}(\omega)).$$

In particular, we have that

$$\mathrm{d}_{\mathrm{dR}}((D_R^i)^*(\omega)) - (D_R^i)^*(\mathrm{d}_{\mathrm{dR}}(\omega))$$

is of E-vertical degree $p_1 - 1$.

Proof. Considering the result of Lemma 5.1.2, let us assume without loss of generality that $z \in G_q$ with $m = p_q^i(z) \in R$. By Lemma 5.1.1, there is locally a trivialisation of $G|_R$ given by a map

$$G|_{\mathcal{O}_m} \times U \to (G|_R)_1$$

inducing a local product structure that on the fibre $s^{-1}(m)$ agrees with E. In particular we obtain that on $U \cong \{m\} \times U \subset R$ there is a product structure induced identification

$$s^{-1}(n) \cap t^{-1}(R) = (s^{-1}(m) \cap G|_{\mathcal{O}_m}) \times \{n\} \cong (s^{-1}(m) \cap G|_{\mathcal{O}_m}) \times \{m\} = s^{-1}(m) \cap t^{-1}(R)$$

for any $n \in U$. This induces sections of s on U corresponding to each element on $s^{-1}(m) \cap t^{-1}(R)$. All those sections are E-horizontal at m. We will refer to them as γ_b' for $b \in s^{-1}(m) \cap t^{-1}(R)$.

Our aim in the following will be to replace the E-jets W_a in the definition of

$$\left((\overline{D}_{R,q}^i)^*(\omega)\right)_y = \int_{a \in s^{-1}(p_q^i(y)) \cap t^{-1}(R)} \left((\dot{D}_{W_a,q}^i)^*(\omega)\right)_y \mathrm{d}\mu_{\mathrm{Haar}}$$

and

$$\left((\overline{D}_{R,q}^i)^*(\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega))\right)_y = \int_{a \in s^{-1}(p_q^i(y)) \cap t^{-1}(R)} \left((\dot{D}_{W_a,q}^i)^*(\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega))\right)_y \mathrm{d}\mu_{\mathrm{Haar}}$$

by jets (not necessarily E-horizontal) corresponding to the s-sections γ_b' we just constructed.

However, the sections γ_b' are defined on $U \subset R$, so the associated jet $V_{(b,n)}'$ for any $n \in U$ only maps isomorphically to $T_n R$ under Ts. We will choose smooth extensions γ_b of γ_b' with image contained in R, using that $t^{-1}(R) \to G_0$ is a surjective submersion. It is possible to choose the extensions uniformly on some $W \subset G_0$ an open neighbourhood of m as the space $s^{-1}(m) \cap G|_R$ is compact. The choices can be explicitly constructed by using the local form of a submersion to locally lift vector fields along s and a procedure locally integrating these vector fields (corresponding to lifts of the complementary directions of R), where we can project all lifted vector fields to their E-horizontal components so we may without loss of generality assume that each γ_b is E-horizontal at m. Let us denote the jets associated to some γ_b as

$$V_{(b,n)} = (T_n \gamma_b)(T_n G_0)$$

for any $n \in (U \cap W)$.

Now for $m = p_q^i(z)$ we have

$$\left((\overline{D}_{R,q}^i)^*(\nu)\right)_z = \int_{a \in s^{-1}(m) \cap t^{-1}(R)} \left((\dot{D}_{W_a,q}^i)^*(\nu)\right)_z \mathrm{d}\mu_{\mathrm{Haar}}.$$

and for $y \in (p_q^i)^{-1}(U \cap W)$ with $n = p_q^i(y)$ similarly

$$\left((\overline{D}_{R,q}^i)^*(\nu)\right)_y = \int_{a \in s^{-1}(n) \cap t^{-1}(R)} \left((\dot{D}_{W_a,q}^i)^*(\nu)\right)_y \mathrm{d}\mu_{\mathrm{Haar}}.$$

We will now compare those terms to the expression

$$\int_{b\in s^{-1}(n)\cap t^{-1}(R)} \left((\dot{D}^i_{V_{(b,n)},q})^*(\nu)\right)_y \mathrm{d}\mu_{\mathrm{Haar}}.$$

(Here, we consider $y = z$ and $n = m$ when comparing to the first term.)

First of all, we can denote the map of Lemma 5.1.1 as

$$\begin{aligned} G|_{\mathcal{O}_m} \times U &\to (G|_R)_1 \\ (b, n) &\mapsto a_{(b,n)}. \end{aligned}$$

This induces the term

$$W_{a_{(b,n)}},$$

which are jets with base-points $a_{(b,n)}$ as in the definition of $(\overline{D}^i_{R,q})^*$ (i.e. the unique jets mapping to TR under Tt). As the map to G_1 is a groupoid isomorphism onto a saturated subset of G_1, we find that the space $s^{-1}(n) \cap t^{-1}(R) \subset G_1$ is isomorphic to $G|_{\mathcal{O}_m} \times \{n\}$ where the R-Haar measure agrees with the $\mathcal{O}_m \times U$-Haar measure. However, we know that as $G|_{\mathcal{O}_m} \times U$ is by definition a trivial bundle with respect to s and the trivialisation is compatible with the multiplication, the Haar measure in each fibre agrees. It is therefore retained by the isomorphism

$$s^{-1}(m) \cap t^{-1}(R) = (s^{-1}(m) \cap G|_{\mathcal{O}_m}) \times \{m\} \cong (s^{-1}(m) \cap G|_{\mathcal{O}_m}) \times \{n\} = s^{-1}(n) \cap t^{-1}(R)$$

mentioned earlier, which can now be written as

$$b \mapsto a_{(b,n)}.$$

In particular, we have that

$$\int_{a\in s^{-1}(n)\cap t^{-1}(R)} \left((\dot{D}^i_{W_a,q})^*(\nu)\right)_y \mathrm{d}\mu_{\mathrm{Haar}} = \int_{b\in s^{-1}(m)\cap t^{-1}(R)} \left((\dot{D}^i_{W_{a_{(b,n)}},q})^*(\nu)\right)_y \mathrm{d}\mu_{\mathrm{Haar}},$$

where the measures refer to the Haar measure of $s^{-1}(n) \cap t^{-1}(R)$ and $s^{-1}(m) \cap t^{-1}(R)$ respectively.

Let us now focus on the implications of this equation. We have that $W_{a_{(b,n)}}$ is the unique E-horizontal subspace at $a_{(b,n)}$ with its image under Tt contained in TR, but the jets $V_{(b,n)}$ have the same property and hence induce the same underlying maps on T_nG_0. Therefore, it turns out that

$$W_{a_{(b,n)}}$$

is precisely the E-horizontal projection of

$$V_{(b,n)}.$$

By Lemma 2.1.36, we find that therefore

$$\left((\dot{D}^i_{W_{a_{(b,n)}},q})^*(\nu)\right)_y$$

is the component of E-vertical degree p_1 of

$$\left((\dot{D}^i_{V_{(b,n)},q})^*(\nu)\right)_y = \left((\widetilde{D}^i_{\gamma_b,q})^*(\nu)\right)_y.$$

We will apply this result twice, once for $\nu = \omega$ and once for $\nu = \mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega)$. In the first case, we obtain that the component of E-vertical degree p_1 of

$$\left((\widetilde{D}^i_{\gamma_b,q})^*(\omega)\right)_y$$

is given by $\left((\dot{D}^i_{W_{a_{(b,n)}},q})^*(\omega)\right)_y$.

In the second case, for $y = z$ and $\nu = \mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega)$, we find that

$$\left((\dot{D}^i_{W_{a_{(b,n)}},q})^*(\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega))\right)_y$$

is the component of E-vertical degree p_1 of

$$\left((\dot{D}^i_{V_{(b,n)},q})^*(\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega))\right)_y = \left((\widetilde{D}^i_{\gamma_b,q})^*(\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega))\right)_y$$

and that (because sums and integrals can be computed component-wise)

$$\left((\overline{D}^i_{R,q})^*(\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega))\right)_y \equiv \int_{b\in s^{-1}(m)\cap t^{-1}(R)} \left(\left((\widetilde{D}^i_{\gamma_b,q})^*(\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega))\right)\right)_y \mathrm{d}\mu_{\mathrm{Haar}}$$

in E-horizontal degree p_1. As each γ_b itself is E-horizontal at $m = p^i_q(z)$, this means

$$\left((\overline{D}^i_{R,q})^*(\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega))\right)_z = \int_{b\in s^{-1}(m)\cap t^{-1}(R)} \left(\left((\widetilde{D}^i_{\gamma_b,q})^*(\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega))\right)\right)_z \mathrm{d}\mu_{\mathrm{Haar}}.$$

Meanwhile, we can apply Lemma 4.1.3 to obtain that for each $b \in s^{-1}(m) \cap t^{-1}(R)$ we have

$$\left((\widetilde{D}^i_{\gamma_b,q})^*(\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega))\right)_z = \left(\mathrm{d}_{\mathrm{dR}}^{\pm 0}\,(\rho_b)\right)_z.$$

where ρ_b is by definition the component of E-vertical degree p_1 of $(\widetilde{D}^l_{\gamma_b,q})^*\omega$. So we can replace the integrand to obtain

$$\left((\overline{D}^i_{R,q})^*(\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega))\right)_z = \int_{b\in s^{-1}(m)\cap t^{-1}(R)} \left(\mathrm{d}_{\mathrm{dR}}^{\pm 0}\,(\rho_b)\right)_z \mathrm{d}\mu_{\mathrm{Haar}}.$$

The right-hand term agrees with

$$\left(\mathrm{d}_{\mathrm{dR}}^{\pm 0}\left(\int_{b\in s^{-1}(m)\cap t^{-1}(R)} \rho_b\right) \mathrm{d}\mu_{\mathrm{Haar}}\right)_z$$

as d_{dR} and hence also its components act linearly on forms and the domain of integration is compact such that integration and taking derivatives commutes.

However, by our earlier considerations (for $\nu = \omega$) we precisely have that

$$(\rho_b)_y = \left((\dot{D}^l_{W_{a_{(b,n)}},q})^*(\omega)\right)_y$$

for $y \in (p^i_q)^{-1}(U \cap W)$ and hence

$$\int_{b\in s^{-1}(m)\cap t^{-1}(R)} \rho_b = \int_{b\in s^{-1}(m)\cap t^{-1}(R)} (\dot{D}^i_{W_{a_{(b,n)}},q})^*(\omega) = (\overline{D}^i_{R,q})^*(\omega).$$

Retracing the equations, this means

$$\left((\overline{D}^i_{R,q})^*(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega))\right)_z = \left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}\left((\overline{D}^i_{R,q})^*(\omega)\right)\right)_z$$

Hence indeed

$$\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}((\overline{D}^i_{R,q})^*(\omega))|_{(p^i_q)^{-1}(R)}\right)_z - \left((\overline{D}^i_{R,q})^*(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega))|_{(p^i_q)^{-1}(R)}\right)_z = 0.$$

As z was chosen arbitrarily, we obtain

$$\left(\mathrm{d}_{\mathrm{dR}}((\overline{D}^i_{R,q})^*(\omega)) - (\overline{D}^i_{R,q})^*(\mathrm{d}_{\mathrm{dR}}(\omega))\right)|_{(p^i_q)^{-1}(R)} \equiv 0$$

on the entire submanifold. By Lemma 5.1.2, the statement of this lemma follows. The same line of argument as described here also holds for $\mathrm{d}^{+1}_{\mathrm{dR}}$.

□

Lemma 5.1.4. *Let G be a proper and regular Lie groupoid and E a multiplicative Ehresmann connection with respect to the isotropy algebroid. Assume there is an admissible embedded submanifold R of G_0 with $Sat(R) = G_0$. Consider a form $\omega \in \Omega^p(G_q)$ of E-vertical degree p_1 and $\rho \in \Omega^p(G_{q-1})$ chosen according to the assignment of Lemma 2.3.11 with respect to R, where ρ is of E-vertical degree p_1.*

Further consider $\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)$ and also chose $\rho_{\pm 0} \in \Omega^{p+1}(G_{q-1})$ according to the assignment of Lemma 2.3.11 with respect to R, where $\rho_{\pm 0}$ is of E-vertical degree p_1. Similarly consider $\mathrm{d}^{+1}_{\mathrm{dR}}(\omega)$ and chose $\rho_{+1} \in \Omega^{p+1}(G_{q-1})$ according to the assignment of Lemma 2.3.11 with respect to R, where ρ_{+1} is of E-vertical degree $p_1 + 1$.

We obtain that

$$\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\rho) = -\rho_{\pm 0}$$

and

$$\mathrm{d}^{+1}_{\mathrm{dR}}(\rho) = -\rho_{+1}.$$

In particular, if $\mathrm{d}_{\mathrm{dR}}(\omega)$ has a vanishing component of E-vertical degree p_1, then $\mathrm{d}_{\mathrm{dR}}(\rho)$ also has a vanishing component of E-vertical degree p_1.

If $\mathrm{d}_{\mathrm{dR}}(\omega)$ has a vanishing component of E-vertical degree $p_1 + 1$, then $\mathrm{d}_{\mathrm{dR}}(\rho)$ also has a vanishing component of E-vertical degree $p_1 + 1$.

Proof. Firstly, let us recall the definition of ρ: Under the assumption of the existence of R, it is given according to Lemma 2.3.8 and the proof of Proposition 2.3.10 as a sum of terms of the form

$$\rho^{(k)} = (-1)^{p+k}(s_j)^*(\overline{D}^{j+1}_{R,q})^*(\omega^{(k)}),$$

where the $\omega^{(k)}$ are iteratively defined as

$$\omega^{(k-1)} + \mathrm{d}_{\mathcal{N}}(\rho^{(k-1)})$$

with

$$\omega^{(0)} = \omega.$$

Similarly, $\rho_{\pm 0}$ is defined by terms of the form

$$(\rho_{\pm 0})^{(k)} = (-1)^{p+1+k}(s_j)^*(\overline{D}^{j+1}_{R,q})^*(\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega)\right)^{(k)}),$$

which are analogously defined.

We can now iteratively apply Lemma 5.1.3 and the fact that d_{dR} commutes with pull-backs to show that

$$d_{dR}^{\pm 0}(\omega^{(k)}) = \left(d_{dR}^{\pm 0}(\omega)\right)^{(k)}$$

and

$$d_{dR}^{\pm 0}(\rho^{(k)}) = -(\rho_{\pm 0})^{(k)}.$$

For this, we can use anti-commutativity of $d_{\mathcal{N}}$ and d_{dR} as well as the fact that $d_{\mathcal{N}}$ retains the splitting induced by E, such that the anti-commutativity is retained on each component when decomposing d_{dR}. Similarly, d_{dR} commutes with pull-backs and simplicial maps retain the splitting by Lemma 2.1.32, such that the components of d_{dR} also commute with the pull-back $(s_j)^*$ in each step. This then immediately implies that

$$d_{dR}^{\pm 0}(\rho) = -\rho_{\pm 0}.$$

Note that the analogous argument is applicable to d_{dR}^{+1}.

For the implication mentioned in the statement, note that $\rho_{\pm 0}$ depends linearly on $d_{dR}^{\pm 0}(\omega)$ by Lemma 2.3.11, such that in particular $d_{dR}^{\pm 0}(\omega) = 0$ implies that

$$d_{dR}^{\pm 0}(\rho) = -\rho_{\pm 0} = 0.$$

□

Lemma 5.1.5. *Let G be a proper and regular Lie groupoid and E a multiplicative Ehresmann connection with respect to the isotropy algebroid. Assume there is an admissible embedded submanifold R of G_0 with $Sat(R) = G_0$. Consider a $J^E G$-invariant form $\omega \in \Omega^p(G_q)$ of E-vertical degree p_1 and $\sigma \in \Omega^p(G_{q-1})$ chosen according to the assignment of Lemma 3.1.24, where σ is of E-vertical degree p_1.*

Further consider $d_{dR}^{\pm 0}(\omega)$ and also chose $\sigma_{\pm 0} \in \Omega^{p+1}(G_{q-1})$ according to the assignment of Lemma 3.1.24, where $\sigma_{\pm 0}$ is of E-vertical degree p_1. Then we obtain that

$$d_{dR}^{\pm 0}(\sigma) = -\sigma_{\pm 0}.$$

In particular, if $d_{dR}(\omega)$ has a vanishing component of E-vertical degree p_1, then $d_{dR}(\sigma)$ also has a vanishing component of E-vertical degree p_1.

Remark. The reason that we can only note this result for $d_{dR}^{\pm 0}$ this time, is that σ is chosen according to three different methods according to Corollary 3.1.17, Lemma 3.1.14 and Corollary 3.1.18, and which method applies on which component might be altered by d_{dR}^{+1}. This is why the explicit form of $d_{dR}^{\pm 0}$ as $\widehat{d_{\mathrm{Cart}}}$ is a key ingredient for the result.

Proof. For the proof of this claim, we will decompose ω into its component $\omega_{(p_0,1,1,\ldots,1;q)}$ and the remainder

$$\omega - \omega_{(p_0,1,1,\ldots,1;q)}.$$

Following the proof of Lemma 3.1.24, we will choose the preimage of the remainder according to Corollary 3.1.17 and Lemma 3.1.14 and then use Corollary 3.1.18 to find an $J^E G$-isotropy Cartan form equivalent to $\omega_{(p_0,1,1,\ldots,1;q)}$.

If we now consider $d_{dR}^{\pm 0}$, we find that as it is given by $\widehat{d_{\mathrm{Cart}}}$ it retains the E-vertical multidegree of any $J^E G$-invariant form. So it is sufficient to show, that the methods of Corollary 3.1.17, Lemma 3.1.14 and Corollary 3.1.18 each anti-commute with $d_{dR}^{\pm 0}$ on the relevant forms.

If we consider the proof of Lemma 3.1.16, we find that the preimages are chosen based on two formulas: One, which is described in A.4 that only consists of sums of terms constructed from identities and pull-backs by simplicial maps and one very explicit one described in the proof of the lemma. All components of d_{dR} commute with pull-backs by simplicial maps and the explicit formula given is linear in all components of $(A^*_{\mathrm{iso}})^{\oplus q}$ and therefore commutes with the derivation C of Lemma 4.1.15 for the connection of the form $\nabla = (\nabla^E)^{\oplus q}$ which gives an alternate computation for $\mathrm{d}_{\mathrm{Cart}} = \mathrm{d}^{\pm 0}_{\mathrm{dR}}$. The precise argument for the commutativity uses the product structure of the connection and is analogue to the proof of Lemma 4.1.16 or parts of the proof of Lemma 4.3.15. That the construction anti-commutes with $\mathrm{d}^{\pm 0}_{\mathrm{dR}}$ now simply follows from the fact that as we have described in Corollary 3.1.17, the preimages are scaled each with $(-1)^p$ and $(-1)^{p+1}$ respectively.

If we consider the proof of Lemma 3.1.14 and 3.1.18, we see that it is sufficient to prove that contraction of two adjacent vertical components commutes with $\mathrm{d}^{\pm 0}_{\mathrm{dR}}$, as the construction of the preimage contains the sign $(-1)^{p+i}$, which will turn commutativity into anti-commutativity. However, we know that on $J^E G$-invariant forms $\mathrm{d}^{\pm 0}_{\mathrm{dR}}$ can alternatively be computed by C, which is a derivative. As the contraction is given by the multiplication of the two components it hence must commute. □

Lemma 5.1.6. *Let G be a proper and regular Lie groupoid and E a multiplicative Ehresmann connection with respect to the isotropy algebroid. Assume there is an admissible embedded submanifold R of G_0 with $\mathrm{Sat}(R) = G_0$. Assume that there is a form $\omega_{(p,q)} \in \Omega^p(G_q)$ with $q > 0$ such that $\mathrm{d}_{\mathcal{N}}(\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})) = 0$ and all of the components of ω of E-vertical degree p_1 with either $p_1 < q$ or $p_1 = q+1$ vanish. If furthermore the component of $\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})$ in E-vertical degree q vanishes, we can find a form $\omega_{(p+1,q-1)} \in \Omega^{p+1}(G_{q-1})$ with the following properties:*

- *We have that*
$$\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)}) = -\,\mathrm{d}_{\mathcal{N}}(\omega_{(p+1,q-1)}).$$
- *in E-vertical degree $p_1 < q-1$ and $p_1 = q$ the components of $\omega_{(p+1,q-1)}$ vanish.*
- *the component in E-vertical degree $q-1$ of $\mathrm{d}_{\mathrm{dR}}(\omega_{(p+1,q-1)})$ vanishes.*

Remark. Note that if $q = 1$, we have that $\mathrm{d}_{\mathrm{dR}}(\omega_{(p+1,q-1)})$ itself has E-vertical degree 0, meaning that $\omega_{(p+1,q-1)}$ is closed under d_{dR}.

Proof. The idea of this proof is mainly to apply Lemma 5.1.4 and Lemma 5.1.5, which imply that we can apply our construction of choosing preimages under $\mathrm{d}_{\mathcal{N}}$ in a way that certain components in certain degrees vanish. In the following, we will mostly spell out these decompositions explicitly.

By Corollary 3.1.26, we know that $\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})$ is cohomologically equivalent to its projection to its component in E-vertical degree q. As this vanishes, we know that $\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})$ is exact in the cohomology of $\mathfrak{C}_p$. Hence there is an $\omega_{(p+1,q-1)}$ which is a preimage of $-\,\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})$ under $\mathrm{d}_{\mathcal{N}}$.

We can choose it by the assignment of Corollary 3.1.25: First, we choose $\rho_{(p+1,q-1)}$ as in the assignment of Lemma 2.3.11, with respect to the trivial cover, i.e. the cover given by the open $U = G_0 = \mathrm{Sat}(R)$ together with the admissible embedded manifold R. Note that this construction retains the E-vertical degree, so we may equivalently compute $\rho_{(p+1,q-1)}$ component-wise. Then we can choose the components of $\sigma_{(p+1,q-1)}$ in each E-vertical degree according to the assignment of Lemma 3.1.24. Now it remains to set $\omega_{(p+1,q-1)} = \rho_{(p+1,q-1)}) + \sigma_{(p+1,q-1)}$.

We now aim to show that this choice of $\omega_{(p+1,q-1)}$ satisfies all properties of the lemma. While property that
$$\mathrm{d}_{\mathcal{N}}(\omega_{(p+1,q-1)}) = -\,\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})$$

is immediate, we will now need to study its components of homogeneous E-vertical degrees. In particular, we need to understand its component of E-vertical degree $q-1$.

We may note, that $\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})$ only has components of E-vertical degree $p_1 = q-1$ and $p_1 \geq q+1$ according to the decomposition of d_{dR} of Lemma 4.1.1 and the assumption that the component of E-vertical degree $p_1 = q$ vanishes. As the construction of Lemma 2.3.11 retains the E-vertical degree of a form, we have that $\rho_{(p+1,q-1)}$ also only consists of elements of E-vertical degrees $p_1 = q-1$ and $p_1 \geq q+1$. The same holds for $\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)}) + \mathrm{d}_{\mathcal{N}}(\rho_{(p+1,q-1)})$ as $\mathrm{d}_{\mathcal{N}}$ retains the splitting of E by Corollary 2.1.33. As the construction of Lemma 3.1.24 also retains E-vertical degrees, we obtain the same result for $\sigma_{(p+1,q-1)}$. (Note that $\sigma_{(p+1,q-1)}$ arises from $\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)}) + \mathrm{d}_{\mathcal{N}}(\rho_{(p+1,q-1)})$ and not $\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})$ itself.)

Furthermore, we have that the component $\omega^{(q)}_{(p,q)}$ of E-vertical degree q of ω determines

$$(\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)}))^{(q-1)} = \mathrm{d}^{-1}_{\mathrm{dR}}(\omega^{(q)}_{(p,q)})$$

and

$$0 = (\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)}))^{(q)} = \mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega^{(q)}_{(p,q)})$$

using the analogue notation for the respective E-homogeneous components. This also follows from the decomposition of d_{dR} of Lemma 4.1.1 and the assumption that $\omega^{(q+1)}_{(p,q)} = 0$. We also know that $\mathrm{d}_{\mathrm{dR}}(\mathrm{d}_{\mathrm{dR}}(\omega^{(q)}_{(p,q)})) = 0$, so in particular the component in E-vertical degree $q-1$ of this term vanishes. But this component is for degree reasons given by

$$0 = \mathrm{d}^{\pm 0}_{\mathrm{dR}}\left(\mathrm{d}^{-1}_{\mathrm{dR}}\left((\omega_{(p,q)})^{(q)}\right)\right) + \mathrm{d}^{-1}_{\mathrm{dR}}\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}\left((\omega_{(p,q)})^{(q)}\right)\right) = \mathrm{d}^{\pm 0}_{\mathrm{dR}}\left((\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)}))^{(q-1)}\right).$$

By Lemma 5.1.4, we therefore have that $\rho^{(q-1)}_{(p+1,q-1)}$ satisfies

$$\mathrm{d}^{\pm 0}_{\mathrm{dR}}\left((\rho_{(p,q)})^{(q-1)}\right) = 0.$$

Again, as $\mathrm{d}_{\mathcal{N}}$ retains the splitting of E by Corollary 2.1.33, this implies that

$$\mathrm{d}^{\pm 0}_{\mathrm{dR}}\left((\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)}))^{(q-1)} + \mathrm{d}_{\mathcal{N}}\left((\rho_{(p,q)})^{(q-1)}\right)\right) = 0$$

satisfies the analogous property and we can apply Lemma 5.1.5 to obtain that

$$\mathrm{d}^{\pm 0}_{\mathrm{dR}}\left((\sigma_{(p,q)})^{(q-1)}\right) = 0.$$

This proves the last assertion of the lemma, as for degree reasons

$$\left(\mathrm{d}_{\mathrm{dR}}(\omega_{(p+1,q-1)})\right)^{(q-1)} = \mathrm{d}^{\pm 0}_{\mathrm{dR}}\left(\left(\omega_{(p+1,q-1)}\right)^{(q-1)}\right),$$

where

$$\left(\omega_{(p+1,q-1)}\right)^{(q-1)} = \left(\rho_{(p+1,q-1)}\right)^{(q-1)} + \left(\sigma_{(p+1,q-1)}\right)^{(q-1)}$$

by definition. □

Proposition 5.1.7. *Let G be a proper and regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Assume there is an admissible embedded submanifold R of G_0 with $Sat(R) = G_0$. A $J^E G$-isotropy Cartan form $\omega_{(p,q)} \neq 0$ has a closed homogeneous descending extension ω if and only if it is* *closed under the J^EG-isotropy Cartan differential. The assignment of closed descending homogeneous extensions to J^EG-isotropy Cartan forms*

$$\omega_{(p,q)} \in \ker(\widehat{\mathrm{d}^E_{\mathrm{Cart}}}) \subset \mathcal{R}_E^{(p,q)} \mapsto \omega \in \mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$$

can be expressed explicitly and the assignment is $\mathbb{R}$-linear in $\omega_{(p,q)}$. Furthermore, the extension is natural with respect to the inclusion of $G|_R \subset G$, in the sense that the extension of $\omega_{(p,q)}|_R$ as a form on $(G|_R)_q$ precisely agrees with $\omega|_R$.

Proof. We have already shown that any element admitting a homogeneous extension is closed under $\mathrm{d}_{\mathrm{Cart}}$ in Lemma 4.2.6.

To show the converse, consider a J^EG-isotropy Cartan form $\omega_{(p,q)} \neq 0$ which is closed under the Cartan differential. Firstly, we may note, that $\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})$ has a vanishing component of E-vertical degree q according to Proposition 4.1.20. If we assume $q = 0$, we have that $\widehat{\mathrm{d}_{\mathrm{Cart}}}$ and d_{dR} agree (as on $\Omega(G_0)$, all forms have E-vertical degree 0), such that

$$\mathrm{d}_{\mathrm{dR}}(\omega_{(p,0)}) = 0$$

and $\omega_{(p,0)}$ is a closed descending homogeneous extension of itself. Therefore, we may without loss of generality assume that $q \geq 1$.

In this case, note that J^EG-isotropy Cartan forms are homogeneous of E-vertical degree q and closed under $\mathrm{d}_{\mathcal{N}}$, so in particular $\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})$ is also closed under $\mathrm{d}_{\mathcal{N}}$. We can now inductively choose $\omega_{(p+k,q-k)}$ for $k \leq q$ by Lemma 5.1.6, as by the properties we obtain from the statement we have that

$$\mathrm{d}_{\mathcal{N}}(\mathrm{d}_{\mathrm{dR}}(\omega_{(p+k,q-k)})) = -\,\mathrm{d}_{\mathrm{dR}}(\mathrm{d}_{\mathcal{N}}(\omega_{(p+k,q-k)})) = \mathrm{d}_{\mathrm{dR}}(\mathrm{d}_{\mathrm{dR}}(\omega_{(p+k-1,q-k+1)})) = 0,$$

which is the only condition necessary to apply the lemma iteratively that is not immediate. Furthermore, for $\omega_{(p+q,0)}$, we have that by Lemma 5.1.6 holds that $\mathrm{d}_{\mathrm{dR}}(\omega_{(p+q,0)})$ vanishes in E-vertical degree 0, but analogous to a previous argument this means that automatically

$$\mathrm{d}_{\mathrm{dR}}(\omega_{(p+q,0)}) = 0.$$

This implies, that indeed

$$\omega = \sum_{k=0}^{q} \omega_{(p+k,q-k)}$$

is a closed descending homogeneous extension of $\omega_{(p,q)}$.

The fact, that this construction satisfies that the extension of $\omega_{(p,q)}|_R$ agrees with $\omega|_R$ is due to the fact that Lemma 5.1.6 uses the assignment of Corollary 3.1.25 to construct the preimage, which satisfies that the restriction of the preimage only depends on the restriction of the $\mathrm{d}_{\mathcal{N}}$-closed form.

□

Corollary 5.1.8. *There exists an isomorphism of graded vector spaces*

$$\Phi : H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))) \to H^\bullet(\mathrm{Tot}(\mathcal{R}_E)) \cong H^\bullet(\mathrm{Tot}(\mathfrak{R}_E))$$

Proof. By Proposition 5.1.7, we have a linear assignment between J^EG-isotropy Cartan forms and their closed descending homogeneous extensions. However, this assignment does not automatically map cohomologically equivalent forms to cohomologically equivalent extensions. By the axiom of choice we can alter the given assignment to satisfy this condition. Then, we can apply Theorem 4.2.16. □

Lemma 5.1.9. *Consider a regular and proper Lie groupoid G together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid and an admissible embedded manifold R such that $\mathrm{Sat}(R) = G_0$. Assume that we have a closed descending homogeneous extension ω of some $\omega_{(p,q)}$ in bidegree (p, q) where $\omega|_R = 0$. Then ω is exact and we may choose the preimage ρ satisfying $\rho|_R = 0$.*

Proof. This follows directly from Lemma 2.3.11, as we can construct a preimage of ω under $\mathbf{d}$ iteratively: Start with a preimage of $\omega_{(p,q)}$ under $\mathrm{d}_{\mathcal{N}}$, which by the aforementioned result can be chosen satisfying $\rho_{(p,q-1)}|_R = 0$. Consider

$$\omega - \mathbf{d}(\rho_{(p,q-1)}),$$

which is a closed descending homogeneous extension of some element in bidegree $(p+k, q-k)$ for $k > 0$ and satisfies (as d_{dR} restricts to submanifolds) that

$$\left(\omega - \mathbf{d}(\rho_{(p,q-1)})\right)|_R = 0$$

as well. We can proceed analogously and construct $\rho_{(p+k,q-k-1)}$ and iterate until

$$\mathbf{d}(\rho_{(p,q-1)} + \rho_{(p+k,q-k-1)} + \dots) = \omega,$$

where by construction

$$\rho = \rho_{(p,q-1)} + \rho_{(p+k,q-k-1)} + \dots$$

satisfies $\rho|_R = 0$. □

Lemma 5.1.10. *Consider a regular and proper Lie groupoid G with a multiplicative Ehresmann connection E with respect to the isotropy and an admissible embedded manifold R such that $\mathrm{Sat}(R) = G_0$. Assume that $E|_R$ is flat. Further assume we have a closed descending homogeneous extension ω of some $\omega_{(p,q)}$ in bidegree (p, q) where $\omega|_R$ vanishes in $E|_R$-vertical degrees up to q. Then ω is exact and we may choose the preimage ν satisfying that $\nu|_R$ also vanishes in E-vertical degrees up to q.*

Proof. The proof in this case is analogous to the proof of Lemma 5.1.9. The only difference is that the preimage $\nu_{(p,q-1)}$ is now chosen by considering both $\rho_{(p,q)}$ and $\sigma_{(p,q)}$. We have that $\rho_{(p,q)}$ by Lemma 2.3.11 satisfies that $\rho_{(p,q)}|_R$ only depends on $\omega_{(p,q)}|_R$ and by construction and Proposition 2.3.10 this means it vanishes in $E|_R$-vertical degrees up to q. Similarly by Corollary 2.3.12, we have that the same holds for $\omega_{(p,q)} + \mathrm{d}_{\mathcal{N}}(\rho_{(p,q)})$ in its entirety. Following the construction of $\sigma_{(p,q)}$ according to Lemma 3.1.24 retains E-vertical degree as well. So

$$(\rho_{(p,q-1)} + \sigma_{(p,q-1)})|_R$$

vanishes in E-vertical degrees up to q and by Lemma 4.3.3 this is retained under d_{dR} due to flatness of $E|_R$.

For this reason, the iterative method of Lemma 5.1.9 of constructing preimages for the components of ω can be applied here as well. □

Lemma 5.1.11. *Consider a proper and regular Lie groupoid G equipped with a multiplicative Ehresmann connection E (with respect to the isotropy) and furthermore let $R \subset G_0$ be an admissible embedded manifold such that $\mathrm{Sat}(R) = G_0$. Assume that $E|_R$ is a flat multiplicative Ehresmann connection on $G|_R$. Let $\omega_{(p,q)} \in \Omega^p(G_q)$ be an form satisfying that $\mathrm{d}_{\mathcal{N}}(\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})) = 0$ and all of the components of ω of E-vertical degree p_1 with either $p_1 < q$ or $p_1 = q + 1$ vanish. Furthermore assume that the component of $\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})$ in E-vertical degree q vanishes.*

Then the form $\omega_{(p+1,q-1)}$ described in Lemma 5.1.6 additionally has the property that if $\omega_{(p,q)}|_R$ satisfies that its E-vertical components vanish for all degrees up to q', the same holds for

$$\omega_{(p+1,q-1)}|_R.$$

Additionally, if $q' \geq q-1$, then $\omega_{(p+1,q-1)}|_R$ only has non-vanishing components in E-vertical degrees for which $p_1 > q$.

Proof. By Lemma 2.2.7, we have that the values of $((\overline{D}^i_R)^*\alpha)$ on $G|_R$ only depend on the values of α on $G|_R$ and the dependence is linear. The same property holds for pull-backs along simplicial maps such as $(s_j)^*$ and their sum such as $\mathrm{d}_{\mathcal{N}}$, as by definition the maps itself restrict to the simplicial structure of $(G|_R)_\bullet$. In the construction of $\omega_{(p+1,q-1)}$ of Lemma 5.1.6, we can now recall that for each E-vertical degree k we define the corresponding component of $\omega_{(p+1,q-1)}$ as

$$\omega^{(k)}_{(p+1,q-1)} = \rho^{(k)}_{(p+1,q-1)} + \sigma^{(k)}_{(p+1,q-1)},$$

where $\rho^{(k)}_{(p+1,q-1)}$ arises from $\big(\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})\big)^{(k)}$ under the assignment of Lemma 2.3.11. This assignment is for general ω defined by an inductive procedure in which we iteratively define $\omega^{(l)}$ and $\rho^{(l)}$ along formulas only using terms constructed from $(\overline{D}^i_{R,q})^*$, $(s_j)^*$ and $\mathrm{d}_{\mathcal{N}}$. The previous argument together with the fact that d_{dR} and decomposition into E-homogeneous components commutes with restriction to $G|_R$ implies that $\rho^{(k)}_{(p+1,q-1)}|_R$ arises precisely from the construction of Lemma 5.1.6 applied to $\omega_{(p,q)}|_R$ on $G|_R$. In particular, if

$$\big(\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)}|_R)\big)^{(k)} = \big(\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})\big)^{(k)}|_R = 0$$

then also

$$\rho^{(k)}_{(p+1,q-1)}|_R = 0.$$

Similarly, the term $\sigma^{(k)}_{(p+1,q-1)}$ arises from $\big(\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})\big)^{(k)} + \mathrm{d}_{\mathcal{N}}(\rho^{(k)}_{(p+1,q-1)})$ under the assignment of Lemma 3.1.24. Here, we can check that all constructions used to construct preimages are induced by fibre-wise operations on bundles over G_0. In particular, if we have that $\omega_{1_{G_0}}(m) = 0$ for a general $\omega_{1_{G_0}}$, then we immediately find that the associated $\sigma_{1_{G_0}}(m) = 0$. Furthermore, we can recover the values of $\omega_{1_{G_0}}$ from $\omega_{1_{G_0}}|_R$ due to the explicit form of Proposition 2.1.41 (which implies we can recover the value at 1_r for any $r \in R$) and Lemma 2.1.38 (using that $\mathrm{Sat}(R) = G_0$). So in this case we also have that we can recover $\sigma^{(k)}_{(p+1,q-1)}|_R$ if we apply the construction of Lemma 3.1.24 to $\big(\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)}|_R)\big)^{(k)} + \mathrm{d}_{\mathcal{N}}(\rho^{(k)}_{(p+1,q-1)}|_R)$. In particular, if

$$\big(\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})\big)^{(k)}|_R = 0$$

then also

$$\big(\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})\big)^{(k)} + \mathrm{d}_{\mathcal{N}}(\rho^{(k)}_{(p+1,q-1)})|_R = 0$$

and therefore

$$\sigma^{(k)}_{(p+1,q-1)}|_R = 0.$$

We can also note that recalling the definitions and methods behind these arguments reveals that the term $\omega^{(k)}_{(p+1,q-1)}|_R$ we have constructed this way precisely agrees with the term constructed from $\omega^{(k)}_{(p,q)}|_R$ as an element of $\Omega_{\mathrm{BSS}}(G|_R)$ using the method of Lemma

5.1.6. Now let us apply this to our setting: By assumption, $E|_R$ is flat on $G|_R$. In particular, we can see that

$$\mathrm{d}_{\mathrm{dR}}^{-1}\left(\omega_{(p,q)}\right)|_R = \mathrm{d}_{\mathrm{dR}}^{-1}\left(\omega_{(p,q)}|_R\right) = 0$$

due to $\mathrm{d}_{\mathrm{dR}}^{-1} \equiv 0$ on any form on $(G|_R)_q$ according to Lemma 4.3.3. This means that if all E-vertical components up to degree q' of $\omega_{(p,q)}|_R$ vanish, the same holds for $\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})|_R$ and therefore $\omega_{(p+1,q-1)}|_R$. Similarly, as the E-vertical component of degree q of $\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})|_R$ vanishes, the same also holds for $\omega_{(p+1,q-1)}|_R$. In particular, we have that if $q' \geq q$ we obtain that $\omega_{(p+1,q-1)}|_R$ vanishes in E-vertical degrees $p_1 \leq q$. We can also note that the construction of Lemma 5.1.6 of $\omega_{(p+1,q-1)}|_R$ from $\omega_{(p,q)}|_R$ agrees with the construction of Lemma 4.3.4 for the trivial admissible cover $\{R\}$ of $(G|_R)_0 = R$ in this case.

□

Lemma 5.1.12. *Consider a regular and proper Lie groupoid G with a multiplicative Ehresmann connection E with respect to the isotropy algebroid and an admissible embedded manifold R such that $\mathrm{Sat}(R) = G_0$ and $E|_R$ is a flat multiplicative Ehresmann connection on $G|_R$. Let $\omega_{(p,q)} \in \Omega^p(G_q)$ be a $J^E G$-isotropy Cartan form closed under the Cartan differential $\widehat{\mathrm{d}_{\mathrm{Cart}}}$. Then its closed descending homogeneous extension ω constructed in Proposition 5.1.7 satisfies that for $k > 0$ the term*

$$\omega_{(p+k,q-k)}|_R$$

only consists of non-vanishing components of E-vertical degree at least $q+1$.

This property defines ω uniquely up to an exact term. If ω' is another form with the same properties, we may choose the preimage ρ mapping to $\omega - \omega'$ under $\mathrm{d}_{\mathcal{N}}$ to satisfy that in all components $\rho_{(r,s)} \in \Omega^r(G_s)$ holds that $\rho_{(r,s)}|_R$ vanishes in E-vertical degrees for which $p_1 \leq q$.

Proof. The construction of ω of Proposition 5.1.7 uses the result of Lemma 5.1.6 inductively. We have seen in Lemma 5.1.11, that if the term $\omega_{(p+l,q-l)}|_R$ satisfies, that its E-vertical components of degree at most q vanish, then the term $\omega_{(p+l+1,q-l-1)}|_R$ satisfies the same condition. The lemma also states that the fact that $\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega_{(p,q)})|_R$ vanishes in E-vertical degree q is sufficient to start the induction.

Furthermore, for any two choices ω and ω', we have that $\omega - \omega'$ only has non-vanishing components in bidegrees $(p+k, q-k)$ for $k \leq 0$ by definition. In particular, we have that by our assumption the expression

$$(\omega - \omega')|_R$$

only has non-vanishing components in E-vertical degrees satisfying $p_1 \leq q+1$. In particular, we can apply Lemma 5.1.10 to obtain the statement of the lemma.

□

Lemma 5.1.13. *Let G be a proper and regular Lie groupoid and E a flat multiplicative Ehresmann connection with respect to the isotropy algebroid. Assume there is an admissible embedded submanifold R of G_0 with $Sat(R) = G_0$.*

Consider the closed descending homogeneous extension ν according to Proposition 5.1.7 of a $J^E G$-isotropy Cartan form $\nu_{(r,s)}$ closed under $\mathrm{d}_{\mathrm{Cart}}$. Assume that there is a form $\omega_{(p,q)} \in \Omega^p(G_q)$ with $p+q+1 = r+s$ and $q > 0$ such that $\mathrm{d}_{\mathcal{N}}(\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)}) - \nu_{(p+1,q)}) = 0$. Also assume that all of the components of $\omega_{(p,q)}$ of E-vertical degree $p_1 < q$ and $p_1 = q+1$ vanish. If furthermore the component in E-vertical degree q of $\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)}) - \nu_{(p+1,q)}$ vanishes, we can find a form $\omega_{(p+1,q-1)} \in \Omega^{p+1}(G_{q-1})$ with the following properties:

a) We have that

$$\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)}) = -\,\mathrm{d}_{\mathcal{N}}(\omega_{(p+1,q-1)}) + \nu_{(p+1,q)}.$$

b) $\omega_{(p+1,q-1)}$ *has vanishing components in E-vertical degree* $p_1 < q-1$ *and* $p_1 = q$.

c) The component in E-vertical degree $q-1$ *of* $\mathrm{d}_{\mathrm{dR}}(\omega_{(p+1,q-1)})$ *and* $\nu_{(p+2,q-1)}$ *agree.*

Proof. We will follow a similar strategy as in Lemma 5.1.6. For this, let us consider $\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)}) - \nu_{(p+1,q)}$. It is closed under the vertical differential, so it represents a cohomology class, but as it is cohomologically equivalent to its projection of E-vertical degree q, we find that it is exact and we may choose a preimage. This can be done according to Corollary 3.1.25 by choosing $\rho_{(p+1,q-1)}$ and $\sigma_{(p+1,q-1)}$ (where the choice is detailed in Lemma 2.3.11 and Lemma 3.1.24). As both of these terms retain the E-vertical degree, they vanish in degree q and in degrees $p_1 < q-1$, as $\nu_{(p+1,q)}$ stems from the construction of Lemma 5.1.6 and hence vanishes in these components and $\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)})$ does so by assumption and the fact of Lemma 4.1.1 that d_{dR} can at most reduce the E-vertical degree of a homogeneous form by one.

Lastly, we will prove the claim that the component in E-vertical degree $q-1$ of $\mathrm{d}_{\mathrm{dR}}(\omega_{(p+1,q-1)})$ agrees with $\nu_{(p+2,q-1)}$. Let us first analyse $\mathrm{d}_{\mathrm{dR}}(\omega_{(p+1,q-1)})$: We know that $\omega_{(p+1,q-1)}$ is chosen as the $\mathrm{d}_{\mathcal{N}}$-preimage of a term that only has components (as argued above) of E-vertical degree $q-1$ and $p_1 > q$. We have already noted that d_{dR} can reduce the E-vertical degree of any homogeneous component by at most one, so that

$$\big(\mathrm{d}_{\mathrm{dR}}(\omega_{(p+1,q-1)})\big)^{(q-1)} = \mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega_{(p+1,q-1)}^{(q-1)})$$

where $\big(\mathrm{d}_{\mathrm{dR}}(\omega_{(p+1,q-1)})\big)^{(q-1)}$ denotes the component of E-vertical degree $q-1$ of $\mathrm{d}_{\mathrm{dR}}(\omega_{(p+1,q-1)})$ and the analogous notation is used for $\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega_{(p+1,q-1)}^{(q-1)})$.

As we choose the preimages degree-wise, the component of E-vertical degree $q-1$ of $\omega_{(p+1,q-1)}$ is only dependent of the component of E-vertical degree $q-1$ of that term

$$\nu_{(p+1,q)} - \mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)}).$$

However, by construction, this component is given by

$$0 - \mathrm{d}_{\mathrm{dR}}^{-1}(\omega_{(p,q)}^{(q)}) = -\,\mathrm{d}_{\mathrm{dR}}^{-1}(\omega_{(p,q)}^{(q)}),$$

where $\omega_{(p,q)}^{(q)}$ denotes the component of $\omega_{(p,q)}$ of E-vertical degree q. So we obtain by Lemma 5.1.4 and Lemma 5.1.5 that

$$\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\omega_{(p+1,q-1)}^{(q-1)})$$

is actually precisely the preimage of

$$\mathrm{d}_{\mathrm{dR}}^{\pm 0}\left(\mathrm{d}_{\mathrm{dR}}^{-1}(\omega_{(p,q)}^{(q)})\right)$$

under the analogous construction.

We have that $\nu_{(p+2,q-1)}$ is constructed as the $\mathrm{d}_{\mathcal{N}}$-preimage of $-\,\mathrm{d}_{\mathrm{dR}}(\nu_{(p+1,q)})$ by choosing $\sigma'_{(p+2,q-1)}$ and $\rho'_{(p+2,q-1)}$ according to Corollary 3.1.25. The component of E-vertical degree $q-1$ is determined by $-\,\mathrm{d}_{\mathrm{dR}}^{-1}(\nu_{(p+1,q)}^{(q)})$, where $\nu_{(p+1,q)}^{(q)}$ denotes the component of $\nu_{(p+1,q)}$ of E-vertical degree q. However, we have assumed that

$$\nu_{(p+1,q)}^{(q)} = (\mathrm{d}_{\mathrm{dR}}(\omega_{(p,q)}))^{(q)}$$

denoting the component of $\mathrm{d_{dR}}(\omega_{(p,q)})$ in E-vertical degree q, which for degree reasons implies

$$\nu^{(q)}_{(p+1,q)} = \mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega^{(q)}_{(p,q)}).$$

So we find that

$$\nu^{(q-1)}_{(p+2,q-1)}$$

is the preimage of

$$-\,\mathrm{d}^{-1}_{\mathrm{dR}}\left(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega^{(q)}_{(p,q)})\right).$$

However, $\mathrm{d}^{\pm 0}_{\mathrm{dR}}$ and $\mathrm{d}^{-1}_{\mathrm{dR}}$ anti-commute by Lemma 4.1.2, such that

$$-\,\mathrm{d}^{-1}_{\mathrm{dR}}(\nu^{(q)}_{(p+1,q)}) = -\,\mathrm{d}^{-1}_{\mathrm{dR}}(\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega^{(q)}_{(p,q)})) = \mathrm{d}^{\pm 0}_{\mathrm{dR}}(\mathrm{d}^{-1}_{\mathrm{dR}}(\omega^{(q)}_{(p,q)})).$$

So we find that the E-vertical degree $q-1$ of $\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\omega_{(p+1,q-1)})$ and $\nu_{(p+2,q-1)}$ agree. Therefore we have shown that all properties of the statement hold. □

Proposition 5.1.14. *Let G be a proper and regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. We can extend the construction of Proposition 5.1.7 to a linear injective chain complex map*

$$\Psi : \omega_{(p,q)} \in \mathrm{Tot}(\mathcal{R}_E) \mapsto \omega \in \mathrm{Tot}(\Omega_{\mathrm{BSS}}(G)).$$

In particular, Ψ is a quasi-isomorphism of chain complexes and the construction of Φ of Corollary 5.1.8 recovers $H(\Psi)^{-1}$, where Φ is constructed from the restriction of Ψ to $\widehat{\mathrm{d}^{E}_{\mathrm{Cart}}}$-closed $J^E G$-isotropy Cartan forms.

Proof. We have already constructed an assignment of $J^E G$-isotropy Cartan forms closed under $\widehat{\mathrm{d}_{\mathrm{Cart}}}$ to their closed descending homogeneous extensions in Proposition 5.1.7. This was based in the construction of Lemma 5.1.6, which can be generalised by the construction of Lemma 5.1.13 for $\nu = 0$. Now for a general $J^E G$-isotropy Cartan form ω, we can choose $\nu = \widehat{\mathrm{d}_{\mathrm{Cart}}}(\omega)$. This is closed under $\widehat{\mathrm{d}_{\mathrm{Cart}}}$ as the differential squares to 0. We now claim that in analogy to Proposition 5.1.7 that the construction of Lemma 5.1.13 can be iterated to construct

$$\omega = \sum_{k=0}^{q} \omega_{(p+k,q-k)}$$

with the desired properties.

For this, we have to check that we can start the induction for $\omega_{(p,q)}$. In this case, we have that $\nu_{(p+1,q)} = \widehat{\mathrm{d}_{\mathrm{Cart}}}(\omega_{(p,q)})$ such that automatically $\mathrm{d}_{\mathcal{N}}(\nu_{(p,q+1)}) = 0$ and $\mathrm{d}_{\mathcal{N}}(\omega_{(p,q)}) = 0$ as both forms are $J^E G$-isotropy Cartan. By anti-commutativity of the differentials we obtain

$$\mathrm{d}_{\mathcal{N}}(\mathrm{d_{dR}}(\omega_{(p,q)}) - \nu_{(p+1,q)}) = 0.$$

This, together with the fact that $\widehat{\mathrm{d}_{\mathrm{Cart}}} \equiv \mathrm{d}^{\pm 0}_{\mathrm{dR}}$ by Proposition 4.1.20 proves that indeed Lemma 5.1.13 is applicable.

Now for the induction step, we need to check that the result obtained by applying Lemma 5.1.13 $\omega_{(p+k+1,q-k-1)}$ is constructed satisfying the conditions of Lemma 5.1.13 itself, i.e. that

$$\mathrm{d}_{\mathcal{N}}\left(\mathrm{d_{dR}}(\omega_{(p+k+1,q-k-1)}) - \nu_{(p+k+2,q-k-1)}\right) = 0$$

as well as that

$$\omega_{(p+k+1,q-k-1)}$$

only has non-vanishing E-vertical components in degrees $p_1 = q-k-1$ and $p_1 \geq q-k+1$ and that

$$\mathrm{d_{dR}}(\omega_{(p+k+1,q-k-1)}) - \nu_{(p+k+2,q-k-1)}$$

vanishes in E-vertical degree $q-k-1$.

For the first of these assertions, we have that by Lemma 5.1.6 that $\omega_{(p+k+1,q-k-1)}$ is constructed satisfying

$$\mathrm{d}_{\mathcal{N}}(\omega_{(p+k+1,q-k-1)}) = \nu_{(p+k+1,q-k)} - \mathrm{d_{dR}}(\omega_{(p+k,q-k)})$$

which implies

$$\mathrm{d}_{\mathcal{N}}(\mathrm{d_{dR}}(\omega_{(p+k+1,q-k-1)})) = -\,\mathrm{d_{dR}}(\mathrm{d}_{\mathcal{N}}(\omega_{(p+k+1,q-k-1)})) = -\,\mathrm{d_{dR}}(\nu_{(p+k+1,q-k)}).$$

Therefore in particular

$$\begin{aligned}\mathrm{d}_{\mathcal{N}}\left(\mathrm{d_{dR}}(\omega_{(p+k+1,q-k-1)}) - \nu_{(p+k+2,q-k-1)}\right) &= -(\mathrm{d_{dR}}(\nu_{(p+k+1,q-k)}) + \mathrm{d}_{\mathcal{N}}(\nu_{(p+k+2,q-k-1)}))\\ &= 0\end{aligned}$$

as ν is $\mathbf{d}$-closed. The second and third assertion are contained directly in the statement of Lemma 5.1.13.

This concludes the proof of the induction step. So we obtain a collection of $\omega_{(p+k,q-k)}$ for any $0 \leq k \leq q$ where the fact that all forms on G_0 are of E-vertical degree 0 immediately implies that

$$\mathrm{d_{dR}}(\omega_{(p+q,0)}) = \nu_{(p+q+1,0)}.$$

Together with all previous equations, we obtain that

$$\omega = \sum_{k=0}^{q} \omega_{(p+k,q-k)}$$

defines a form satisfying that

$$\mathbf{d}(\omega) = \nu.$$

The map constructed this way is linear (as the construction of every preimage is) and injective, as any non-zero $J^E G$-isotropy Cartan form is mapped to an extension of itself, which is in particular non-zero again. As a chain complex map, the restriction of Ψ to $J^E G$-isotropy Cartan forms closed under $\mathrm{d_{Cart}}$ yields an assignment that maps cohomologically equivalent $J^E G$-isotropy Cartan forms to cohomologically equivalent elements of $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$. Therefore we can apply Theorem 4.2.16 to obtain Φ, which by construction is an isomorphism inverse to $H(\Psi)$. This implies that Ψ is a quasi-isomorphism. □

Corollary 5.1.15. *Let G be a proper and regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. If we consider the image of Ψ of Proposition 5.1.14, it is closed under* $\mathbf{d}$ *(i.e. the total differential of the Bott–Shulman–Stasheff complex). If we consider the corresponding subcomplex* $\Psi(\mathrm{Tot}(\mathcal{R}_E)) \subset \mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$*, the inclusion induces an isomorphism on cohomology.*

Proof. We will apply Proposition 5.1.14. As Ψ is a chain complex map, $\mathrm{Im}(\Psi)$ is closed under $\mathbf{d}$ and therefore a subcomplex of $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$. We further have that $H(\Psi)$ is an isomorphism that factors as

$$H(\mathrm{Tot}(\mathcal{R}_E)) \to H(\mathrm{Im}(\Psi)) \to H(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))).$$

By construction, Ψ is injective and by definition also surjective onto its image, and therefore an isomorphism, which projects onto an isomorphism in cohomology. So the right arrow above, which is precisely the projection of the inclusion map onto the cohomology, is an isomorphism as well. $\square$

Remark. Note that as Ψ is an isomorphism onto its image, we immediately have an inverse Ψ^{-1} defined on $\mathrm{Im}(\Psi)$ which lifts Φ.

Lemma 5.1.16. *Consider a regular and proper Lie groupoid G together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid and an admissible embedded manifold $R \subset G_0$.*

Let ω be a closed descending homogeneous extension of $\omega_{(p,q)}$, where $\omega_{(p,q)}$ is homogeneous of E-vertical degree q and all terms $\omega_{(p+k,q-k)}|_R$ for $k > 0$ only have E-homogeneous components of E-vertical degree at least $q+1$. Let ν be a closed descending homogeneous extension of $\nu_{(p',q')}$, where $\nu_{(p',q')}$ is homogeneous of E-vertical degree q' and all terms $\nu_{(p'+k,q'-k)}|_R$ for $k > 0$ only have E-homogeneous components of E-vertical degree at least $q'+1$.

Then $\omega \cup \nu$ is a closed descending homogeneous extension of

$$\mathrm{pr}^*_{1,\dots,q}\,\omega_{(p,q)} \wedge \mathrm{pr}^*_{p+1,\dots,q+q'}\,\nu_{(p',q')}.$$

where all terms $(\omega \cup \nu)_{(p+p'+l,q+q'-l)}|_S$ for $l > 0$ only have E-homogeneous components of E-vertical degree at least $q + q' + 1$.

Proof. By Lemma 4.2.18, we immediately obtain that $\omega \cup \nu$ is a closed descending homogeneous extension of

$$\mathrm{pr}^*_{1,\dots,q}\,\omega_{(p,q)} \wedge \mathrm{pr}^*_{p+1,\dots,q+q'}\,\nu_{(p',q')}.$$

Additionally, we have that the cup product is defined naturally using the simplicial structure of G, such that

$$(\omega \cup \nu)|_R,$$

i.e. the restriction of $\omega \cup \nu$ to $(G|_R)_{q+q'} \subset G_{q+q'}$, only depends on $\omega|_R$ and $\nu|_R$. However, we can apply Lemma 4.2.19 on these restrictions and their respective homogeneous components, which then yields the statement. $\square$

Lemma 5.1.17. *Consider a regular and proper Lie groupoid G together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid and an admissible embedded manifold R such that $\mathrm{Sat}(R) = G_0$. Assume that $E|_R$ is a flat multiplicative Ehresmann connection on $G|_R$.*

Let $\omega_{(p,q)} \in \Omega^p(G_q)$ and $\nu_{(p',q')} \in \Omega^{p'}(G_{q'})$ be $J^E G$-isotropy Cartan forms. We then have that

$$\Psi(\omega_{(p,q)}) \cup \Psi(\nu_{(p',q')})$$

is cohomologically equivalent to $\Psi(\omega_{(p,q)} \breve{\cdot} \nu_{(p',q')})$, where Ψ denotes the map of Proposition 5.1.14.

Proof. By Lemma 5.1.16, we have that

$$\Psi(\omega_{(p,q)}) \cup \Psi(\nu_{(p',q')})$$

is a closed descending homogeneous extension of

$$\mathrm{pr}^*_{1,\dots,q}\,\omega_{(p,q)} \wedge \mathrm{pr}^*_{p+1,\dots,q+q'}\,\nu_{(p',q')}$$

where all terms $(\omega \cup \nu)_{(p+p'+l,q+q'-l)}|_R$ for $l > 0$ only have E-homogeneous components of E-vertical degree at least $q + q' + 1$. By Lemma 4.2.20, we know that there exists a $\sigma \in \Omega^{(p+p')}(G_{(q+q'-1)})$ such that

$$\Psi(\omega_{(p,q)}) \cup \Psi(\nu_{(p',q')}) + \mathbf{d}(\sigma)$$

is a closed descending homogeneous extension of

$$\omega_{(p,q)} \check{\cdot} \nu_{(p',q')}.$$

Considering the proof of Lemma 4.2.20, the construction of σ can be traced back to Lemma 3.1.15 which is applied to obtain Lemma 3.1.18 (which then is used in the proof of Lemma 3.1.19). Doing so describes the construction of σ from $\mathrm{pr}^*_{1,\dots,q}\,\omega_{(p,q)} \wedge \mathrm{pr}^*_{p+1,\dots,q+q'}\,\nu_{(p',q')}$ by contractions of adjacent copies of A^*_{iso} and the permutation action of $\Sigma_{(q+q')}$. This immediately implies that σ has E-vertical degree $q + q'$. As by Proposition 4.1.18 and Lemma 4.1.15 we have that $\mathrm{d}^{\pm 0}_{\mathrm{dR}}$ corresponds to a derivation with respect to the algebra structure on $\Gamma(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \wedge^{p_1}(A^*_{\mathrm{iso}})^{\oplus q})$, where the expression on each copy of A^*_{iso} agrees, we find that $\mathrm{d}^{\pm 0}_{\mathrm{dR}}$ commutes with these operations. Therefore it also follows that

$$\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\sigma)|_R$$

can be analogously be constructed from

$$\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\mathrm{pr}^*_{1,\dots,q}\,\omega_{(p,q)} \wedge \mathrm{pr}^*_{p+1,\dots,q+q'}\,\nu_{(p',q')})|_R = 0,$$

implying that $\mathrm{d}^{\pm 0}_{\mathrm{dR}}(\sigma)|_R = 0$ as well. Note that by Lemma 4.3.3, $\mathrm{d}^{-1}_{\mathrm{dR}} \equiv 0$ on $G|_R$. Therefore,

$$\Psi(\omega_{(p,q)}) \cup \Psi(\nu_{(p',q')}) + \mathbf{d}(\sigma)$$

restricts on R to a form where all components in some bidegree $(p + p' + l, q + q' - l)$ for $l > 0$ only have E-homogeneous components of E-vertical degree at least $q + q' + 1$.

This directly implies that

$$\Psi(\omega_{(p,q)} \check{\cdot} \nu_{(p',q')}) - \big(\Psi(\omega_{(p,q)}) \cup \Psi(\nu_{(p',q')}) + \mathbf{d}(\sigma)\big)$$

is exact by Lemma 5.1.12. The statement of the lemma follows. □

Proposition 5.1.18. *Consider a regular and proper Lie groupoid G together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Consider the chain complex map*

$$\Psi : \omega_{(p,q)} \in \mathrm{Tot}(\mathcal{R}_E) \mapsto \omega \in \mathrm{Tot}(\Omega_{\mathrm{BSS}}(G)),$$

established in Proposition 5.1.14 as a quasi-isomorphism with inverse $H(\Psi)^{-1} = \Phi$. If $E|_R$ is flat, we have that both Φ and $H(\Psi)$ are isomorphisms of $\mathbb{R}$-algebras.

Proof. We have by Lemma 4.2.22 that Φ as constructed from Ψ retains the $\mathbb{R}$-algebra structure if and only if

$$\Psi(\omega_{(p,q)} \check{\cdot} \nu_{(r,s)})$$

is cohomologically equivalent to

$$\Psi(\omega_{(p,q)}) \cup \Psi(\nu_{(r,s)})$$

for any $\widehat{\mathrm{d}_{\mathrm{Cart}}}$-closed $J^E G$-isotropy Cartan forms $\omega_{(p,q)}$ and $\nu_{(r,s)}$. This is precisely the statement of Lemma 5.1.17 using the condition that $E|_R$ is flat. □

Theorem 5.1.19. *Consider a regular and proper Lie groupoid G with a multiplicative Ehresmann connection E and an admissible embedded manifold R such that $\mathrm{Sat}(R) = G_0$. Then the associated differentiable stack cohomology groups can be computed as*

$$H^\bullet_{\mathrm{DiffStack}}([G_0//G_1]) \cong H^\bullet(\mathrm{Tot}(\mathcal{R}_E)) \cong H^\bullet(\mathrm{Tot}(\mathfrak{R}_E)).$$

If furthermore $E|_R$ is flat, we have that the isomorphisms additionally preserve the ring structure of $H^\bullet_{\mathrm{DiffStack}}([G_0//G_1])$.

Proof. By Theorem 0.2.32, we have that $H^\bullet_{\mathrm{DiffStack}}([G_0//G_1])$ agrees with the cohomology ring of $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$. By Proposition 5.1.14, we have that Ψ is a quasi-isomorphism, such that indeed

$$H^\bullet_{\mathrm{DiffStack}}([G_0//G_1]) \cong H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))) \cong H^\bullet(\mathrm{Tot}(\mathcal{R}_E))$$

as groups. Furthermore, we have by Proposition 5.1.18 that $H(\Psi)$ preserves the ring structure if $E|_R$ is flat. Lastly, we have that $\mathcal{R}_E$ and $\mathfrak{R}_E$ are isomorphic by Proposition 2.1.41, such that both bicomplexes induce models for the differentiable stack cohomology. □

5.2 Eliminating the necessity for glueing

Aim of this section: Theorem 5.2.31

Let G be a proper and regular Lie groupoid and E be a multiplicative Ehresmann connection on G. Then we have a graded isomorphism of $\mathbb{R}$-algebras

$$\Phi : H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))) \to H^\bullet(\mathrm{Tot}(\mathcal{R}_E)) \cong H^\bullet(\mathrm{Tot}(\mathfrak{R}_E)).$$

In particular, both $\mathrm{Tot}(\mathcal{R}_E)$ and $\mathrm{Tot}(\mathfrak{R}_E)$ are models that can be used to compute the cohomology of the differentiable stack associated to G as

$$H^\bullet_{\mathrm{DiffStack}}([G_0//G_1]) \cong H^\bullet(\mathrm{Tot}(\mathcal{R}_E)) \cong H^\bullet(\mathrm{Tot}(\mathfrak{R}_E)).$$

In the previous section, we have constructed an assignment from J^EG-isotropy Cartan form to their descending homogeneous extensions for an arbitrary multiplicative Ehresmann connection. It was defined in a way that $\widehat{\mathrm{d}^E_{\mathrm{Cart}}}$-closed J^EG-isotropy Cartan forms are mapped to closed descending homogeneous extensions. This gave us a lot of results that are surprisingly similar to those we obtained in the case that E was flat. A key assumption for the construction to work, however, was that there is an admissible embedded manifold $R \subset G_0$ with $\mathrm{Sat}(R) = G_0$. This is also not the case generally.

Unfortunately, the assignment we obtain can not be glued in case we need more than one admissible embedded manifold to cover G_0, as d_{dR} is not $C^\infty_{\mathrm{bas}}(G_0)$-linear. We will however, in this section, aim to find a multiplicative Ehresmann connection for which glueing is not necessary. In particular, we will be able to prove a statement that recoves the case where E is flat as well as the case where $R \subset G_0$ is admissible with $\mathrm{Sat}(R) = G_0$ and $E|_R$ is flat relatively quickly at the beginning of this section. There will be also many more cases covered, where similarly, the property of E being flat will suffice on restrictions to much smaller submanifolds. These more relaxed properties will still be too strict to give a general construction. However, their local existence will already give rise to an explicit construction of an assignment between J^EG-isotropy Cartan forms and their descending homogeneous extensions. We will conclude the section with the proof that for a proper and regular Lie groupoid G the necessary choices of an admissible cover, locally defined connections and a specific globally defined connection can always be made.

The lemmas and propositions of this section are, again, mostly technical. Some will be referring to properties of earlier definitions and constructions, others regarding more fundamental properties of flat multiplicative Ehresmann connections on Lie group bundles which can be nicely classified or constructions like glueing multiplicative Ehresmann connections, which is a notion we had not previously explored yet.

Lemma 5.2.1. *Consider a regular and proper Lie groupoid G together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Furthermore consider R and R' admissible embedded submanifolds of G_0. Assume we are given a form ω defined on the open subset $\left(G|_{\mathrm{Sat}(R)\cap\mathrm{Sat}(R')}\right)_q \subset G_q$. Also consider any $r \in R$ and $r' \in R'$ within the same orbit and any jet*

$$V_x \in J^1(s : G_1 \to G_0)$$

such that

- *$s(x) = r$ and $t(x) = r'$*
- *the image of $V_x \cap E_x$ under Ts contains T_rR (i.e. the lift of T_rR is E-horizontal)*
- *the image $Tt(V_x \cap (Ts)^{-1}(T_rR))$ is contained in $T_{r'}R'$.*

Then we have that for any z with $p^l_q(z)$ in the orbit of r and r' holds

$$\begin{aligned}&\left(\int_{a\in s^{-1}(p^l_q(z))\cap t^{-1}(\{r\})} \left((\dot{D}^l_{V_a,q})^*((\dot{D}^l_{V_x,q})^*\omega)\right)_z \mathrm{d}\mu_{\mathrm{Haar}}\right)\\ =&c\cdot\left(\int_{b\in s^{-1}(p^l_q(z))\cap t^{-1}(\{r'\})} \left((\dot{D}^l_{V_b,q})^*(\omega)\right)_z \mathrm{d}\mu_{\mathrm{Haar}}\right),\end{aligned}$$

where $c \in \mathbb{Q}$ is the fraction given by

$$c = \frac{|O_{r'} \cap R'|}{|O_r \cap R|}.$$

Remark. Note that all the conditions on V_x only concern the subspace that projects to T_rR under Ts. Hence if a subspace $W_x \subset T_xG_1$ exists that maps isomorphically to T_rR under Ts and satisfies the respective conditions analogous to those for V_x in the lemma, it can always be extended to some V_x as required.

Proof. As an extended induced action, we have by Lemma 1.2.26 that

$$(\dot{D}^l_{V_a,q})^*(\dot{D}^l_{V_x,q})^* \equiv (\dot{D}^l_{\widehat{\mu}(V_x,V_a),q})^*,$$

where $\widehat{\mu(V_x, V_a)}$ denotes the composition of jets as defined in 1.2.2. This, in turn, is defined as

$$\{T\mu(X, Y) \text{ for } X \in V_x, Y \in V_a \text{ with } Ts(X) = Tt(Y)\}.$$

From this definition, we already have that $\mu(V_x, V_a)$ only depends on $V_x \cap (Ts)^{-1}(T_rR)$, as $T_rR = Tt(V_a)$ for all $a \in s^{-1}(p^l_q(z)) \cap t^{-1}(\{r\})$. In particular, we have that the image of $\mu(V_x, V_a)$ under Tt is contained in $T_{r'}R'$, but also, as any element in $V_x \cap (Ts)^{-1}(T_rR)$ is E-horizontal, we have that so is $\mu(V_x, V_a)$.

This means that $\mu(V_x, V_a) = V_b$, for $b = \mu(x, a)$. So we obtain

$$\left(\int_{a\in s^{-1}(p_q^l(z))\cap t^{-1}(\{r\})} \left((\dot{D}^l_{V_{\mu(x,a)},q})^*(\omega)\right)_z \mathrm{d}\mu_{\mathrm{Haar}}\right)$$

and as left multiplication with x induces an isomorphism on the respective index sets, this agrees with

$$\left(\int_{b\in s^{-1}(p_q^l(z))\cap t^{-1}(\{r'\})} \left((\dot{D}^l_{V_b,q})^*(\omega)\right)_z \mathrm{d}\mu'\right),$$

where μ' is the induced measure. However, this induced measure is different from the one occurring in the integral of the statement.

Generally, we have that the Haar measure on $s^{-1}(p_q^l(z)) \cap t^{-1}(\{r\})$ is induced by the Haar measure on $s^{-1}(\{r\}) \cap t^{-1}(\{r\})$ and the Haar measure on $s^{-1}(p_q^l(z)) \cap t^{-1}(\{r'\})$ is induced by the Haar measure on $s^{-1}(\{r'\}) \cap t^{-1}(\{r'\})$. These two groups are related by taking the adjoint with x and hence the induced Haar measures on these groups agree. What does not agree is the rescaling constant, which on R is given by $|\mathcal{O}_r \cap R|$ and on R' is given by $|\mathcal{O}_{r'} \cap R'|$.

Thus we obtain that the Haar measure on $s^{-1}(p_q^l(z)) \cap t^{-1}(\{r\})$ induces a measure on $s^{-1}(p_q^l(z)) \cap t^{-1}(\{r'\})$ which is given by the Haar measure scaled by the constant c. □

Lemma 5.2.2. *Consider a regular and proper Lie groupoid G together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Furthermore consider R and R' admissible embedded submanifolds of G_0. Assume we are given a form ω defined on the open subset $\left(G|_{Sat(R)\cap Sat(R')}\right)_q \subset G_q$. Also assume that for any $r \in R$ and $r' \in R'$ within the same orbit there exists a jet $V_x \in J^1(s : G_1 \to G_0)$ such that*

- *$s(x) = r$ and $t(x) = r'$*
- *the image of $V_x \cap E_x$ under Ts contains T_rR (i.e. the lift of T_rR is E-horizontal)*
- *the image $Tt(V_x \cap (Ts)^{-1}(T_rR))$ is contained in $T_{r'}R'$ and*
- *we have*
$$\iota^*\omega \equiv \iota^*(\dot{D}^l_{V_x,q})^*\omega$$
at any z with $p_q^l(z) = r$, where ι^* denotes the inclusion $(p_q^l)^{-1}(R') \subset G_q$.

Then we have that
$$(\dot{D}^l_{R',q})^*\omega = (\dot{D}^l_{R,q})^*\omega$$

Remark. By concatenating, this means that ω also needs to be invariant under certain jets at morphisms from r to other elements of R.

Proof. We have by definition that

$$\left((\overline{D}^l_{R,q})^*\omega\right)_z = \sum_{r\in R\cap\mathcal{O}_{p_q^l(z)}} \left(\int_{a\in s^{-1}(p_q^l(z))\cap t^{-1}(\{r\})} \left((\dot{D}^l_{V_a,q})^*(\omega)\right)_z \mathrm{d}\mu_{\mathrm{Haar}}\right).$$

and similarly

$$\left((\overline{D}^l_{R',q})^*\omega\right)_z = \sum_{r'\in R'\cap\mathcal{O}_{p_q^l(z)}} \left(\int_{b\in s^{-1}(p_q^l(z))\cap t^{-1}(\{r'\})} \left((\dot{D}^l_{V_b,q})^*(\omega)\right)_z \mathrm{d}\mu_{\mathrm{Haar}}\right).$$

By Lemma 5.2.1, we have that for any fixed $r' \in R' \cap \mathcal{O}_{p_q^l(z)}$ holds

$$\sum_{r \in R \cap \mathcal{O}_{p_q^l(z)}} \left(\int_{a \in s^{-1}(p_q^l(z)) \cap t^{-1}(\{r\})} \left((\dot{D}^l_{V_a,q})^* ((\dot{D}^l_{V_x,q})^*(\omega)) \right)_z \mathrm{d}\mu_{\mathrm{Haar}} \right)$$
$$= \sum_{r \in R \cap \mathcal{O}_{p_q^l(z)}} c \cdot \left(\int_{b \in s^{-1}(p_q^l(z)) \cap t^{-1}(\{r'\})} \left((\dot{D}^l_{V_b,q})^*(\omega) \right)_z \mathrm{d}\mu_{\mathrm{Haar}} \right),$$

where the E-jets V_x depend on r and r'. We may note that technically, dependence is only on

$$\iota^* (\dot{D}^l_{V_x,q})^* \omega$$

instead of $(\dot{D}^l_{V_x,q})^* \omega$ as $(\dot{D}^l_{V_a,q})_*$ maps $TG_q|_{(p_q^l)^{-1}(s(a))}$ to $(Tp_q^l)^{-1}(T_{r'}R')$ by definition.

By our assumption, this means that we can replace $(\dot{D}^l_{V_x,q})^* \omega$ by ω and obtain

$$\sum_{r \in R \cap \mathcal{O}_{p_q^l(z)}} \left(\int_{a \in s^{-1}(p_q^l(z)) \cap t^{-1}(\{r\})} \left((\dot{D}^l_{V_a,q})^*(\omega) \right)_z \mathrm{d}\mu_{\mathrm{Haar}} \right)$$
$$= \sum_{r \in R \cap \mathcal{O}_{p_q^l(z)}} c \cdot \left(\int_{b \in s^{-1}(p_q^l(z)) \cap t^{-1}(\{r'\})} \left((\dot{D}^l_{V_b,q})^*(\omega) \right)_z \mathrm{d}\mu_{\mathrm{Haar}} \right),$$

We can write c independently of r and r' as

$$c = \frac{|\mathcal{O}_{p_q^l(z)} \cap R'|}{|\mathcal{O}_{p_q^l(z)} \cap R|}$$

and summing over all possible choices of $r' \in R' \cap \mathcal{O}_{p_q^l(z)}$ we obtain

$$\sum_{r \in R \cap \mathcal{O}_{p_q^l(z)}, r' \in R' \cap \mathcal{O}_{p_q^l(z)}} \left(\int_{a \in s^{-1}(p_q^l(z)) \cap t^{-1}(\{r\})} \left((\dot{D}^l_{V_a,q})^*(\omega) \right)_z \mathrm{d}\mu_{\mathrm{Haar}} \right)$$
$$= \sum_{r \in R \cap \mathcal{O}_{p_q^l(z)}, r' \in R' \cap \mathcal{O}_{p_q^l(z)}} c \cdot \left(\int_{b \in s^{-1}(p_q^l(z)) \cap t^{-1}(\{r'\})} \left((\dot{D}^l_{V_b,q})^*(\omega) \right)_z \mathrm{d}\mu_{\mathrm{Haar}} \right)$$

or equivalently

$$|\mathcal{O}_{p_q^l(z)} \cap R'| \cdot (D^l_R)^* \omega \equiv |\mathcal{O}_{p_q^l(z)} \cap R| \cdot c \cdot (D^l_{R'})^* \omega,$$

at $z \in (G|_{\mathrm{Sat}(R) \cap \mathrm{Sat}(R')})_q$, which can be rescaled to

$$\left((\overline{D}^l_{R,q})^* \omega \right)_z = \left((\overline{D}^l_{R',q})^* \omega \right)_z .$$

As z was chosen arbitrary, this yields the statement on the entire space. □

Lemma 5.2.3. *Consider a regular and proper Lie groupoid G together with a multiplicative Ehresmann connection E with respect to the isotropy algebroid. Furthermore consider R and R' admissible embedded submanifolds of G_0. Assume we are given a form ω defined on the open subset $\left(G|_{Sat(R) \cap Sat(R')}\right)_q \subset G_q$. Also assume that for any $r \in R$ and $r' \in R'$ within the same orbit there exists a section $\gamma : U \subset R \to G_1$ of s on an open U containing r such that*

- $t(\gamma(r)) = r'$
- *γ is horizontal in the sense that for any $n \in U$ (not just for r itself) we have*
$$T_n\gamma(T_nR) \subset E$$
- *$t \circ \gamma$ is a map from R to R'*
- *we have*
$$\iota^*\omega \equiv (\widetilde{D}^i_{\gamma,q})^*\omega$$
at any z with $p^i_q(z) = r$, where ι^* denotes the inclusion $(p^i_q)^{-1}(R) \subset G_q$.

Then we have that both
$$(\overline{D}^i_{R,q})^*(\omega) = (\overline{D}^i_{R',q})^*(\omega)$$
and
$$(\overline{D}^i_{R,q})^* \, \mathrm{d_{dR}}(\omega) = (\overline{D}^i_{R',q})^* \, \mathrm{d_{dR}}(\omega)$$
meaning that in particular
$$(\overline{D}^i_{R,q})^* \, \mathrm{d}^{-1}_{\mathrm{dR}}(\omega) = (\overline{D}^i_{R',q})^* \, \mathrm{d}^{-1}_{\mathrm{dR}}(\omega).$$

Proof. First off, note that we may assume that ω is E-homogeneous without loss of generality - because the only condition on ω is that
$$\iota^*\omega \equiv (\widetilde{D}^i_{\gamma,q})^*\omega,$$
and as γ is E-horizontal it is equivalent for this condition to hold for ω and to hold for all E-homogeneous components of ω.

This result is a direct application of Lemma 5.2.2. We have that the collection
$$V_x = T_r\gamma(T_rR) \oplus (\ker(Tt) \cap E_x)$$
for all $r \in R$ and $r' \in R'$ defines jets with the required property. It remains to show that
$$\iota^*\omega \equiv \iota^*(\dot{D}^i_{V_x,q})^*\omega$$
and
$$\iota^* \, \mathrm{d_{dR}}(\omega) \equiv \iota^*(\dot{D}^i_{V_x,q})^* \, \mathrm{d_{dR}}(\omega)$$
on $(p^i_q)^{-1}(R')$.

However, we have that
$$\iota^*(\dot{D}^i_{V_x,q})^*\omega \equiv (\widetilde{D}^i_{\gamma,q})^*\omega,$$
as by definition $(\dot{D}^i_{V_x,q})_*$ when restricted to $(Tp^i_q)^{-1}(T_rR)$ only depends on
$$V_x \cap (Ts)^{-1}(T_rR) = T_r\gamma$$
and hence agrees with the notion of $(\widetilde{D}^i_{\gamma,q})_*$.

Similarly, we have that
$$\iota^*(\dot{D}^i_{V_x,q})^* \, \mathrm{d_{dR}}(\omega) \equiv (\widetilde{D}^i_{\gamma,q})^* \, \mathrm{d_{dR}}(\omega),$$
where by assumption
$$(\widetilde{D}^i_{\gamma,q})^* \, \mathrm{d_{dR}}(\omega) = \mathrm{d_{dR}}\left((\widetilde{D}^i_{\gamma,q})^*(\omega)\right) = \mathrm{d_{dR}}\left(\iota^*(\omega)\right) = \iota^*(\mathrm{d_{dR}}(\omega)).$$

So indeed

$$(\overline{D}^i_{R,q})^* \, \mathrm{d}_{\mathrm{dR}}(\omega) = (\overline{D}^i_{R',q})^* \, \mathrm{d}_{\mathrm{dR}}(\omega)$$

and because $(\overline{D}^i_{R,q})^*$ and $(\overline{D}^i_{R',q})^*$ retain the E-horizontal degree

$$(\overline{D}^i_{R,q})^* \, \mathrm{d}^{-1}_{\mathrm{dR}}(\omega) = (\overline{D}^i_{R',q})^* \, \mathrm{d}^{-1}_{\mathrm{dR}}(\omega).$$

□

Definition 5.2.4. Let G be a proper and regular Lie groupoid together with an admissible cover where the admissible embedded manifolds are given by $\{R_n\}_{0\leq n\leq f}$ (or $\{R_n\}_{n\in\mathbb{N}_0}$ in case of a finite cover). Consider a multiplicative Ehresmann connection E (with respect to the isotropy algebroid). Then E **is flatly compatible with the cover** if for every pair n and n' and every $r \in R_n$ and $r' \in R_{n'}$ in the same orbit we have that there exists a section $\gamma : U \subset R_n \to G_1$ of s on an open U containing r such that

- $t(\gamma(r)) = r'$
- $t \circ \gamma$ is a map from R_n to $R_{n'}$.
- γ is horizontal in the sense that for any $u \in U$ (not just for r itself) we have

$$T_u\gamma(T_uR) \subset E.$$

Lemma 5.2.5. *Consider a proper and regular Lie groupoid G together with an admissible cover and a multiplicative Ehresmann connection E which are flatly compatible. Then any J^EG-isotropy Cartan form $\omega_{(p,q)} \in \Omega^p(G_q)$ has a descending homogeneous extension ω, such that the assignment*

$$\Psi : \mathrm{Tot}(\mathcal{R}_E) \to \mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$$
$$\omega_{(p,q)} \mapsto \omega$$

is a map of chain complexes. In particular, if $\omega_{(p,q)}$ is a J^EG-isotropy Cartan form closed under $\widehat{\mathrm{d}_{\mathrm{Cart}}}$, then $\Psi(\omega_{(p,q)})$ is a closed descending homogeneous extension of $\omega_{(p,q)}$.

Proof. We apply the same construction as in Proposition 5.1.7 and Proposition 5.1.14 to all Lie groupoids of the form $G|_{\mathrm{Sat}(R_n)}$ with the respective admissible embedded R_n for $n \in \mathbb{N}_0$ (or $0 \leq n \leq f$).

We will need to prove that two things remain true throughout the iterative procedures simultaneously: That the $\omega_{(p+k,q-k)}$ are (on appropriate domains) independent of the choice of R_n and hence describe the restrictions of a unique global form and also that they satisfy that

$$\iota^*(\omega_{(p+k,q-k)}) \equiv (\widetilde{D}^i_{\gamma,q})^*(\omega_{(p+k,q-k)})$$

for any sections γ of the form as described in the definition of flat compatibility in Definition 5.2.4. Note that this in particular true if $k = 0$ and $\omega_{(p,q)}$ is J^EG-isotropy Cartan as all γ as described are E-horizontal at any point, such that both claims are true for $k = 0$.

Let us first consider the case of Proposition 5.1.7, i.e. where $\omega_{(p,q)}$ is $\widehat{\mathrm{d}_{\mathrm{Cart}}}$-closed. If we now intend to prove the induction step from k to $k+1$, we need to show that $\omega_{(p+k+1,q-k-1)}$ satisfies the same properties by analysing the how it was constructed in Lemma 5.1.6. We find that it consists out of two terms, the first one is $\rho_{(p+k+1,q-k-1)}$, which was constructed from

$$\mathrm{d}_{\mathrm{dR}}(\omega_{(p+k,q-k)})$$

and the second one is $\sigma_{(p+k+1,q-k-1)}$, which was constructed from

$$\mathrm{d}_{\mathrm{dR}}(\omega_{(p+k,q-k)}) + \mathrm{d}_{\mathcal{N}}(\rho_{(p+k,q-k)}).$$

We further know know that $\rho_{(p+k+1,q-k-1)}$ is given by an iterated procedure where by Proposition 2.3.10 we have that

$$\rho^{(l)} = (-1)^{p+l}(s_l)^*(\overline{D}^{l+1}_{R,q-k})^*(\nu^{(l)})$$

and

$$\nu^{(l)} = \nu^{(l-1)} + \mathrm{d}_{\mathcal{N}}(\rho^{(l-1)})$$

where $\nu^{(0)} = \mathrm{d}_{\mathrm{dR}}(\omega_{(p+k,q-k)})$. We therefore find that we can apply Lemma 5.2.3, such that $\rho^{(0)}$, while a priori defined with respect to each R_n on $\mathrm{Sat}(R_n)$ separately can be glued uniquely to a globally defined form.

We can note that $(\widetilde{D}^i_{\gamma,q-k-1})^*$ of the invariance condition of Lemma 5.2.3 is an honest pull-back and commutes with d_{dR} and can be interchanged with the simplicial maps according to Corollary 1.2.34 and $(\overline{D}^{l+1}_{R,q-k})^*$ according to Lemma 2.2.9 and Lemma 2.2.10. Due to this fact, we have that $\rho^{(0)}$ also satisfies the invariance condition and so does $\omega^{(1)}$. We can now iteratively apply Lemma 5.2.3 and the previous arguments, where the induced $\nu^{(l)}$ and $\rho^{(l)}$ respectively all satisfy the invariance condition.

So not only have we found now, that $\rho_{(p+k+1,q-k-1)}$ is well-defined, we have also found that it satisfies the invariance condition given by

$$\iota^*(\rho_{(p+k+1,q-k-1)}) \equiv (\widetilde{D}^i_{\gamma,q})^*(\rho_{(p+k+1,q-k-1)})$$

for all sections γ as described in the definition of flatly compatible in Definition 5.2.4.

The same is true for $\sigma_{(p+k+1,q-k-1)}$ which we choose according to Lemma 3.1.24, as this is by definition a J^EG-invariant form. Also, its construction is independent from any choice of cover.

This concludes the proof in case of a $\widehat{\mathrm{d}_{\mathrm{Cart}}}$-closed J^EG-isotropy Cartan form as in Proposition 5.1.7. However, in the more general case we construct preimages of terms of the form

$$\nu_{(p+k+1,q-k)} - \mathrm{d}_{\mathrm{dR}}(\omega_{(p+k,q-k)}),$$

where $\nu_{(p+k+1,q-k)}$ is precisely constructed in the manner described above. In this case, the analogous arguments hold to show that both

$$\nu_{(p+k+1,q-k)}$$

and

$$\mathrm{d}_{\mathrm{dR}}(\omega_{(p+k,q-k)})$$

satisfy the invariance condition. Then, the analogous arguments imply that in the induction step in Proposition 5.1.14, we again obtain $\omega_{(p+k+1,q-k-1)}$ independent of the choice of any R_n and satisfying

$$\iota^*(\omega_{(p+k+1,q-k-1)}) \equiv (\widetilde{D}^i_{\gamma,q})^*(\omega_{(p+k+1,q-k-1)}).$$

Therefore the descending homogeneous extensions of J^EG-isotropy Cartan forms are generally independent of the choice of R_n for a flatly compatible cover and connection and there in particular exists a uniquely defined global extension of the locally defined descending homogeneous extensions to a global one. □

Example 5.2.6. A flat multiplicative Ehresmann connection E is flatly compatible with every cover. So for any given cover we obtain that the construction of Lemma 5.2.5 yields a closed homogeneous descending extension of any $J^E G$-isotropy Cartan form closed under $\widehat{\mathrm{d}_{\mathrm{Cart}}}$. However, by the same argument used to justify that glueing is not necessary as the local choice of the extension is independent of the chosen admissible manifold, we can indeed find that the entire construction also agrees with that of Lemma 4.3.5.

Lemma 5.2.7. *Given a proper and regular Lie groupoid G and a locally finite saturated cover $\{O_n\}_{n\in\mathbb{N}}$ as well as a family $\{R_n\}_{n\in\mathbb{N}}$ of admissible embedded manifolds generating an admissible cover. Let $\{U_n \subset O_n\}$ be a saturated subordinate refinement together with a family of cut-off-functions $\{\xi_n\}_{n\in\mathbb{N}}$. Then consider a $\mathbf{d}$-closed element ω for which the restrictions $\omega|_{O_n}$ for all $n \in \mathbb{N}$ can respectively be written as a locally finite sum*

$$\omega|_{O_n} = \sum_{k\in\mathbb{N}} \omega_k^{(n)}$$

such that each $\omega_k^{(n)}$ is $\mathbf{d}$-closed, satisfies $\omega_k^{(n)}|_{R_k} = 0$ and is supported on a subset of a saturated closed set $C_k^{(n)} \subset \mathrm{Sat}(R_k) \cap O_n$. Then ω is exact.

Furthermore, assume for any $k \in \mathbb{N}$ we have a flat multiplicative Ehresmann connection E_k on $G|_{R_k}$ and for any pair of $k \in \mathbb{N}$ and $n \in \mathbb{N}$, we have a multiplicative Ehresmann connection $E_k^{(n)}$ on $\mathrm{Sat}(R_k) \cap O_n$, such that $E_k^{(n)}|_{R_k\cap O_n} = E_k|_{R_k\cap O_n}$. Then consider a $\mathbf{d}$-closed element ω which is a closed descending homogeneous extension of an element in bidegree (p, q) for which the restrictions $\omega|_{O_n}$ for all $n \in \mathbb{N}$ can respectively be written as a locally finite sum

$$\omega|_{O_n} = \sum_{k\in\mathbb{N}} \omega_k^{(n)}$$

such that each $\omega_k^{(n)}$ is $\mathbf{d}$-closed, satisfies that all components $\alpha_{(p+k,q-k)}$ of $\omega_k^{(n)}|_{R_k}$ for $k \in \mathbb{N}_0$ vanish in $(E_k|_{O_n\cap R_k})$-vertical degrees $p_1 < q + 1$ and is supported on a subset of a saturated closed set $C_k^{(n)} \subset \mathrm{Sat}(R_k) \cap O_n$. Then ω is exact.

Proof. We will proceed iteratively with the construction of a preimage. Consider some $\omega_k^{(0)}$ and choose a preimage $\nu_k^{(0)}$ with respect to R_k and, in the first case considered by the Lemma, any multiplicative Ehresmann connection, where the preimage exists according to Lemma 5.1.9. In the second case, use $E_k^{(0)}$, where the preimage exists according to Lemma 5.1.10. Note that a priori, $\nu_k^{(0)}$ is only well-defined on $\mathrm{Sat}(R_k) \cap O_0$, but by construction, whenever $\omega_k^{(0)}$ vanishes on a saturated open (e.g. $\mathrm{Sat}(R_k) \setminus C_k^{(0)}$), so does $\nu_k^{(0)}$ as the construction of the components of the preimage (based on Lemma 2.3.11 and Lemma 3.1.24) is $C^\infty_{\mathrm{bas}}(G_0)$-linear and d_{dR} is local. So we can extend $\nu_k^{(0)}$ to all of O_0. We can consider

$$\omega' = \omega - \mathbf{d}(\xi_0 \cdot \sum_{k\in\mathbb{N}_0} \nu_k^{(0)}),$$

where we understand $\xi_0 \cdot \sum_{k\in\mathbb{N}} \nu_k^{(0)}$ as a form smoothly extended by 0. Note that the sum is well-defined as the cover induced by the $\{R_k\}_{k\in\mathbb{N}_0}$ is locally finite and that ω' is clearly cohomologically equivalent to ω. Furthermore, we have in the first case considered in the lemma that as $\nu_k^{(0)}|_{R_k}$ vanishes, so does $\mathbf{d}(\xi_1 \cdot \nu_k^{(0)})$ and in the second case, we obtain that $\mathbf{d}(\xi_1 \cdot \nu_k^{(0)})$ is only supported in E-vertical degrees $p_1 \geq q+1$, which additionally uses that $E_k^{(n)}$ is flat such that d_{dR} can only increase the E-vertical degree by Lemma 4.3.3. As the support of ξ_0 is closed and saturated as a subset of G_0, the same holds for $\mathrm{supp}(\xi_0) \cap C_k^{(0)}$, meaning that $\mathbf{d}(\xi_0 \cdot \nu_k^{(0)})$ is supported on a subset of a closed saturated set in G_q.

So ω' also satisfies that it decomposes into a sum of the form

$$\omega'|_{O_n} = \sum_{k\in\mathbb{N}} (\omega')_k^{(n)} = \sum_{k\in\mathbb{N}} \omega_k^{(n)} - \mathbf{d}(\xi_0 \cdot \nu_k^{(0)})|_{O_n},$$

where each $(\omega')_k^{(n)}$ is again $\mathbf{d}$-closed, supported on a subset of a saturated closed set $C_k^{(n)} \subset \mathrm{Sat}(R_k) \cap O_n$ and satisfies either $(\omega')_k^{(n)}|_{R_k} = 0$ or $(\omega')_k^{(n)}|_{R_k}$ only has non-vanishing components of E_k-vertical degree $p_1 \geq q+1$. Furthermore we have by construction that $\omega' \equiv 0$ on $G|_{U_0}$.

If we iterate this procedure, we obtain in each step that ω is cohomologically equivalent to a form vanishing on $\cup_{0\leq i\leq n} G|_{U_i}$. At the same time, we only add exact terms coming from terms supported only on O_n, meaning that as $\{O_n\}_{n\in\mathbb{N}}$ is locally finite, there is a well-defined locally finite sum of all the exact terms added in the iterated procedure, as well as their preimages. This sum of the preimages ν satisfies that $\omega + \mathbf{d}(\nu)$ vanishes on all $G|_{U_i}$, and therefore on all of G. In particular, ω is exact. □

Lemma 5.2.8. *Consider a proper and regular groupoid G together with an admissible cover induced by $\{R_n\}_{n\in\mathbb{N}}$ (or $\{R_n\}_{0\leq n\leq f}$) and a multiplicative Ehresmann connection E with respect the the isotropy. Now consider a saturated open cover $\{O_k\}_{k\in\mathbb{N}}$ and a family of multiplicative Ehresmann connections $\{E^{(k)}\}_{k\in\mathbb{N}}$ where each $E^{(k)}$ is defined on $G|_{O_k}$. Furthermore assume that this data satisfies:*

- *The open cover $\{O_k\}_{k\in\mathbb{N}_0}$ admits a subordinate refinement $\{U_k\}_{k\in\mathbb{N}_0}$ such that*

$$\overline{U}_k \subset O_k$$

  for any $k \in \mathbb{N}_0$ with admissible cut-off-functions and an admissible partition of unity.
- *We have that $E^{(k)}|_{R_n\cap O_k} = E|_{R_n\cap O_k}$ for each $n \in \mathbb{N}$ and $k \in \mathbb{N}$.*
- *On $G|_{O_k}$ we have that $E^{(k)}$ is flatly compatible with the cover induced by the admissible embedded manifolds $\{R_n \cap O_k\}_{n\in\mathbb{N}}$.*

Then we have that any $J^E G$-isotropy Cartan form $\omega_{(p,q)} \in \Omega^p(G_q)$ closed under $\widehat{\mathrm{d}_{\mathrm{Cart}}}$ has a closed descending homogeneous extension ω. This form can be chosen to satisfy that

$$[\omega|_{O_k}] = [\omega^{(k)}] \in H(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G|_{O_k})))$$

where $\omega^{(k)}$ is the closed descending homogeneous extension of $\omega_{(p,q)}|_{O_k}$ as constructed in Lemma 5.2.5 from $E^{(k)}$ and the flatly compatible cover.

Proof. Consider a $J^E G$-isotropy Cartan form $\omega_{(p,q)}$ closed under $\widehat{\mathrm{d}_{\mathrm{Cart}}}$. We can, on each O_k consider the unique $J^{E^{(k)}} G$-isotropy Cartan form $\omega_{(p,q)}^{(k)}$ satisfying that

$$[(\omega_{(p,q)})|_{O_k}] = [\omega_{(p,q)}^{(k)}] \in H^q((\mathfrak{C}_p)|_{O_k}).$$

We know by Lemma 3.1.23, that these two forms correspond to identical expressions at the unit section. As for any R_n we have $E^{(k)}|_{R_n\cap O_k} = E|_{R_n\cap O_k}$, we can note that this implies

$$(\omega_{(p,q)})|_{R_n\cap O_k} = (\omega_{(p,q)}^{(k)})|_{R_n\cap O_k}.$$

Now, by Lemma 4.3.5, we have that there is a closed descending homogeneous extension of $\omega^{(k)}_{(p,q)}$, which we will call $\omega^{(k)}$. Note that for any R_n, we have that

$$\omega^{(k)}|_{\mathrm{Sat}(R_n \cap O_k)}$$

does by construction and flatness agree with the extension as in Proposition 5.1.7. In particular, we have that

$$\omega^{(k)}|_{R_n \cap O_k}$$

only depends on the values of $\omega^{(k)}_{(p,q)}|_{R_n \cap O_k}$ and $E^{(k)}|_{R_n \cap O_k}$.

This means that any $\omega^{(k)}|_{R_n \cap O_k}$ only depends on $\omega_{(p,q)}|_{R_n \cap O_k}$ and $E|_{R_n \cap O_k}$, and that in particular, we have

$$\omega^{(k)}|_{R_n \cap O_k} \equiv \omega^{(l)}|_{R_n \cap O_l}$$

whenever there is an intersection on which both are defined. This means that

$$(\omega^{(k)} - \omega^{(l)})|_{O_k \cap O_l}$$

is a closed form vanishing on any $R_n \cap O_k \cap O_l$. In particular, we can consider a partition of unity underlying the $\{\mathrm{Sat}(R_n)\}_{n \in \mathbb{N}}$ and write

$$(\omega^{(k)} - \omega^{(l)})|_{O_k \cap O_l} = \sum_{n \in \mathbb{N}} \chi_n \cdot (\omega^{(k)} - \omega^{(l)})|_{O_k \cap O_l}$$

to which we then can apply Lemma 5.2.7 to find that $(\omega^{(k)} - \omega^{(l)})|_{O_k \cap O_l}$ is actually exact. Note that this is the step of the proof where we need the refinement of the cover and the existence of cut-off functions.

Now we will construct a series of closed forms ω_a which each is defined on $\cup_{0 \le k \le a} O_k$. In each step we will have that $\omega_a|_{O_k}$ is given by an expression of the form

$$\omega^{(k)} + \sum (\omega_a)^{(k)}_n,$$

where we have that $(\omega_a)^{(k)}_n$ is $\mathbf{d}$-closed, supported on a subset of a saturated closed subset of $\mathrm{Sat}(R_k) \cap O_n$ and satisfies $(\omega_a)^{(k)}_n|_{R_n} = 0$. Therefore in particular $\omega_a|_{O_k}$ is cohomologically equivalent to $\omega^{(k)}$. The proof will be by induction, with start as $\omega_0 = \omega^{(0)}$.

Now assume, that we have constructed ω_i on $\cup_{0 \le k \le i} O_i$. We will find that when restricted to $O_j \cap O_{i+1}$ for any $0 \le j \le i$, the term

$$(\omega_i - \omega^{(i+1)})|_{O_j \cap O_{i+1}}$$

is given by

$$\sum_{n \in \mathbb{N}} (\chi_n \cdot (\omega^{(j)} - \omega^{(i+1)})) + \sum_{n \in \mathbb{N}} (\omega_i)^{(k)}_n.$$

So we may apply Lemma 5.2.7 to $(\omega_i - \omega^{(i+1)})|_{(\cup_{0 \le j \le i}) \cap O_{i+1}}$ on $G|_{(\cup_{0 \le j \le i}) \cap O_{i+1}}$ with the cover by $\{O_j \cap O_{i+1}\}_{0 \le j \le i}$ and find that the term $(\omega_i - \omega^{(i+1)})|_{O_j \cap O_{i+1}}$ is actually exact. Let ν be a preimage. From the proof of Lemma 5.2.7 it is immediately clear that $\nu|_{O_j \cap O_{i+1}}$ also admits a decomposition into terms each vanishing on some R_n.

We can now apply Corollary 1.5.6 to obtain a pair of basic functions $f_1^{(n)}$ and $f_2^{(n)}$ on $(\cup_{0 \le j \le i} O_j) \cup \mathrm{Sat}(R_n \cap O_{i+1})$ satisfying $f_1^{(n)} + f_2^{(n)} = 1$ each vanishing on either an open neighbourhood around $\mathrm{Sat}(R_n \cap O_{i+1}) \setminus (\cup_{0 \le j \le i} O_j)$ or an open neighbourhood around

$(\cup_{0\leq j\leq i}O_j)\setminus \mathrm{Sat}(R_n\cap O_{i+1})$. Using the partition of unity underlying the admissible cover, we obtain that

$$f_1 = \sum_{n\in\mathbb{N}} \chi_n \cdot f_1^{(n)}$$

similarly satisfies that $f_1 \equiv 0$ on an open neighbourhood of $O_{i+1} \setminus (\cup_{0\leq j\leq i}O_j)$ and $f_1 \equiv 1$ on an open neighbourhood of $(\cup_{0\leq j\leq i}) \setminus O_{i+1}$ such that together with

$$f_2 = 1 - f_1$$

we obtain two functions defined on all of $(\cup_{0\leq j\leq i}O_j) \cup O_{i+1}$.

If we now consider $f_2 \cdot \nu$, it is well-defined on all of $(\cup_{0\leq j\leq i}O_j)$, where we have that

$$\omega_i - \mathbf{d}(f_2 \cdot \nu)$$

only differs from ω_i by an exact term and furthermore satisfies that

$$\omega_i - \mathbf{d}(f_2 \cdot \nu) \equiv \omega^{(i+1)}$$

on any open on which $f_2 \equiv 1$. In particular, this is true for an open neighbourhood of $O_{i+1}\setminus(\cup_{0\leq j\leq i}O_j)$, meaning that we can smoothly extend ω_i by $\omega^{(i+1)}$ to a form defined on all of $(\cup_{0\leq j\leq i}O_j) \cup O_{i+1}$. By the assumption on the decomposition of ω_a into terms vanishing on some R_n and the previous remark on ν, we have that this new form immediately satisfies the same property on any O_j for $0 \leq j \leq i$. On O_{i+1}, the analogous argument holds when applied to

$$\omega^{(i+1)} + \mathbf{d}(f_1 \cdot \nu) \equiv \omega_i + \mathbf{d}(f_2 \cdot \nu),$$

where the first expression is actually well-defined on all of O_{i+1} and precisely agrees with the extension of the form by $\omega^{(i+1)}$ on $O_{i+1} \setminus (\cup_{0\leq j\leq i}O_j)$.

So as the cover of the O_n is locally finite and by construction, the values of any ω_i and ω_{i+1} agree outside of O_{i+1}, we obtain a well-defined limit ω' of the family $\{\omega_i\}_{i\in\mathbb{N}}$. This limit is not an extension of $\omega_{(p,q)}$, however (using that ν by construction is only supported in bidegrees (p', q') for which $q' < q$), we obtain that

$$[(\omega'_{(p,q)})|_{O_k}] = [\omega^{(k)}_{(p,q)}] \in H^q(\mathfrak{C}_p)$$

for any $k \in \mathbb{N}$. So by our previous considerations we have that also

$$[(\omega'_{(p,q)})|_{O_k}] = [(\omega_{(p,q)})|_{O_k}] \in H^q(\mathfrak{C}_p).$$

This immediately implies that when employing the method of Lemma 4.2.4 to find a form ω cohomologically equivalent to ω' such that its degree (p, q) component is $J^E G$-isotropy Cartan, we have that ω is indeed a closed descending homogeneous extension of $\omega_{(p,q)}$. As the cover furthermore admits an admissible partition of unity, we can use that $\mathrm{d}_{\mathcal{N}}$ is $C^\infty_{\mathrm{bas}}(G_0)$-linear to choose local preimages under $\mathrm{d}_{\mathcal{N}}$ on all of the O_k of

$$(\omega'_{(p,q)})|_{O_k} - (\omega_{(p,q)})|_{O_k}$$

respectively. Here, we can even use the fact that $\omega'_{(p,q)}$ decomposes into $\omega^{(k)}_{(p,q)}$ and a sum of terms vanishing on some respective R_n and the fact that $\omega^{(k)}_{(p,q)}|_{R_n} = \omega_{(p,q)}|_{R_n}$ for any R_n to construct the preimage of $(\omega'_{(p,q)})|_{O_k} - (\omega_{(p,q)})|_{O_k}$ as a sum of terms which each vanish on $R_n \cap O_k$ (and are supported on a closed saturated subset of $\mathrm{Sat}(R_n) \cap O_k$). As this is simultaneously the preimage of $(\omega')|_{O_k} - (\omega)|_{O_k}$ this term can be written as a sum of an analogous form, and similarly, so does

$$\omega|_{O_k} - \omega^{(k)}.$$

□

Lemma 5.2.9. *Let G be a proper and regular groupoid together with an admissible cover induced by $\{R_n\}_{n\in\mathbb{N}}$ (or $\{R_n\}_{0\leq n\leq f}$) and let E be a multiplicative Ehresmann connection with respect to the isotropy algebroid. The assignment between $J^E G$-isotropy Cartan forms closed under $\widehat{\mathrm{d}^E_{\mathrm{Cart}}}$ and their closed descending homogeneous extensions of Lemma 5.2.8 satisfies that it maps cohomologically equivalent forms of* $\mathrm{Tot}(\mathcal{R}_E)$ *to cohomologically equivalent forms of* $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$.

Proof. First of all, let us note that the method of Lemma 5.2.8 is linear, i.e. that the statement of the claim is equivalent to showing that if $\omega_{(p,q)}$ is exact with respect to $\mathrm{d}_{\mathrm{Cart}}$, then ω is exact.

Now note that any of the $\omega^{(k)}$ in the construction by Lemma 5.2.8 satisfies that $\omega^{(k)}|_{R_n\cap O_k}$ and $\omega|_{R_n\cap O_k}$ agree for any $n\in\mathbb{N}_0$ based on the fact that $E^{(k)}|_{R_n\cap O_k}$ is flatly compatible and agrees with $E|_{R_n\cap O_k}$. So if we choose a preimage $\nu^{(k)}$ of $\omega^{(k)}$ according to the method of Lemma 5.2.7, we have that the construction of $\nu^{(k)}|_{R_n\cap O_k}$ agrees with the construction of a preimage of $\omega|_{R_n\cap O_k}$ on $G|_{R_n\cap O_k}$. In particular, it only depends on $E|_{R_n\cap O_k}$, such that

$$\nu^{(k)} - \nu^{(l)}$$

vanishes on R_n for any $n\in\mathbb{N}$.

Let us now consider the construction of ω' of Lemma 5.2.8 by defining a series of ω_i. We aim to iteratively define a preimage of each ω_i which we will denote ν_i, starting with $\nu_0 = \nu^{(0)}$. Now note that ω_{i+1} was constructed from ω_i by the formula

$$\omega_i + \mathbf{d}(f_2\cdot\alpha),$$

where α is chosen as a preimage under $\mathbf{d}$ of

$$\omega_i - \omega^{(i+1)}.$$

In particular, we have that

$$\nu_i - \nu^{(i+1)} - \alpha$$

is $\mathbf{d}$-closed. By construction, we will have that on any O_j for $0\leq j\leq i$, the restriction

$$\nu_i|_{O_j}$$

will only differ from $\nu^{(i)}$ by a sum of terms $\beta^{(j)}_n$ satisfying that

$$\beta^{(j)}_n|_{R_n\cap O_j} = 0.$$

This immediately implies that

$$\nu_i - \nu^{(i+1)} - \alpha|_{O_j}$$

is of a similar form (when considering the proof of Lemma 5.2.8 on the analogous property for α).

In particular, we have that

$$\nu_i - \nu^{(i+1)} - \alpha|_{O_j}$$

is exact with preimage ρ. So

$$\nu_i - f_2 \cdot \alpha - \mathbf{d}(f_2 \cdot \rho)$$

is now a well-defined form on $\cup_{0\leq j\leq i} O_j$ satisfying that on an open neighbourhood of $O_{i+1} \setminus (\cup_{0\leq j\leq i} O_)$ (i.e. where $f_2 \equiv 1$) we have that it agrees with $\nu^{(i+1)}$. Furthermore

$$\mathbf{d}\left(\nu_i - f_2 \cdot \alpha - \mathbf{d}(f_2 \cdot \rho)\right) = \omega_i - \mathbf{d}(f_2 \cdot \alpha),$$

which agrees with ω_{i+1}. So the extension of this form by $\omega^{(i+1)}$ on $O_{i+1} \setminus (\cup_{0\leq j\leq i} O_j)$ is smooth and a preimage of ω_{i+1}. The argument that

$$\nu_{i+1}|_{O_j} - \nu^{(j)}$$

decomposes as claimed previously is analogous to the proof of Lemma 5.2.8. So is the existence of a limit ν'. We have that $\omega - \omega'$ consists of an exact term, where the construction of the preimage θ has already been explicitly described in Lemma 5.2.8. So now, we have that

$$\nu' + \theta$$

is a preimage of ω, and in particular ω is $\mathbf{d}$-exact.

We can analogously analyse the cohomology class of the leading component. For this, it is relevant that both θ and all of the terms arising as α in the previous construction are supported in bidegrees $(p+1, q-1)$ by construction (as the leading component of ω is in degree $(p, q+1)$). Therefore, on O_k, we have that

$$\nu' + \theta|_{O_k}$$

has its leading component given by $\nu^{(k)}_{(p,q)}$ up to some $\mathrm{d}_{\mathcal{N}}$-exact terms and in particular, the $\mathrm{d}_{\mathcal{N}}$-cohomology class precisely agrees with

$$[\nu_{(p,q)}|_{O_k}].$$

So adding an $\mathbf{d}$-exact term corresponding to the preimage under $\mathrm{d}_{\mathcal{N}}$ of the difference between $\nu' + \theta$ and $\nu_{(p,q)}$ in bidegree (p, q) will precisely yield a descending homogeneous extension of $\nu_{(p,q)}$, which we can denote as ν, with the property that

$$\mathbf{d}(\nu) = \omega.$$

We can explicitly check that this method recovers the construction of Lemma 5.2.8 in the case that $\omega = 0$.

□

Lemma 5.2.10. *Consider a proper and regular groupoid G together with an admissible cover induced by $\{R_n\}_{n\in\mathbb{N}}$ (or $\{R_n\}_{0\leq n\leq f}$) and a multiplicative Ehresmann connection E with respect the the isotropy. Now consider a saturated open cover $\{O_k\}_{k\in\mathbb{N}}$ and a family of multiplicative Ehresmann connections $\{E^{(k)}\}_{k\in\mathbb{N}}$ where each $E^{(k)}$ is defined on $G|_{O_k}$. Assume that these objects satisfy the conditions of Lemma 5.2.8 and consider the corresponding construction for any $J^E G$-isotropy Cartan forms $\omega_{(p,q)}$ and $\nu_{(r,s)}$ supported in bidegrees (p, q) and (r, s) respectively. Assume that there is a family of flat connections E_n on any $G|_{R_n}$ and the connections $E^{(k)}$ used in the construction satisfy that for any $n \in \mathbb{N}$ we have $E^{(k)}|_{R_n\cap O_k} = E_n|_{R_n\cap O_k}$. Then, we obtain that the associated closed descending homogeneous extension*

$$\Psi(\omega_{(p,q)} \check{\cdot} \nu_{(r,s)})$$

is cohomologically equivalent to

$$\Psi(\omega_{(p,q)}) \cup \Psi(\nu_{(r,s)})$$

Proof. We can review the construction of $\omega = \Psi(\omega_{(p,q)})$ to note that apart from bidegree (p,q), we have that all components $\omega_{(p+k,q-k)}$ of bidegree $(p+k,q-k)$ for $k>0$ on any of the O_k decompose into sums of terms which for some R_n restrict to a form only supported in components of E_n-vertical degree at least $q+1$. We can see this by the decomposition of

$$\omega|_{O_k} = \omega^{(k)} + \sum_{n\in\mathbb{N}} \omega_n^{(k)} = \omega_{(p,q)} + \sum_{n\in\mathbb{N}} \chi_n \cdot (\omega^{(k)} - \omega_{(p,q)}) + \omega_n^{(k)}$$

where the terms we are summing over are given each as the sum of a term that vanishes on the corresponding R_n and a term coming from $\omega^{(k)}$ which restricts to a form only supported in components of E_n-vertical degree at least $q+1$. Similarly, we can decompose $\nu = \Psi(\nu_{(r,s)})$ into its component $\nu_{(r,s)}$ and a sum of terms which restrict on some R_m to a form supported only in the components of E_m-vertical degree at least s.

This also means that the product

$$\omega \cup \nu$$

can be written as the sum of

$$\omega_{(p,q)} \cup \nu_{(r,s)}$$

and a sum of terms which for a corresponding R_n restrict to an expression supported in E_n-vertical degrees at least $q+s+1$. Similarly, we have that

$$\Psi(\omega_{(p,q)} \breve{\cdot} \nu_{(r,s)})$$

decomposes into $\omega_{(p,q)} \breve{\cdot} \nu_{(r,s)}$ and a sum of the analogous form. Now it only remains to note that as $\omega_{(p,q)}$ and $\nu_{(r,s)}$ are $\mathrm{d}_{\mathcal{N}}$-closed, so are both $\omega_{(p,q)} \cup \nu_{(r,s)}$ and $\omega_{(p,q)} \breve{\cdot} \nu_{(r,s)}$ and by Lemma 4.2.20 they are cohomologically equivalent. Furthermore, we have that their preimage can be constructed explicitly by only using the contraction of two adjacent factors of $(A^*_{\mathrm{iso}})^{\otimes q}$ and considering the action of Σ_{q+s} by Lemma 3.1.18 and Lemma 3.1.15. However, as by Proposition 4.1.18 and Lemma 4.1.15 we have that $\mathrm{d}_{\mathrm{dR}}^{\pm 0}$ corresponds to a derivation with respect to the algebra structure on $\Gamma(G_0, \wedge^{p_0}(TG_0/\mathcal{F}_0)^* \otimes \wedge^{p_1}(A^*_{\mathrm{iso}})^{\oplus q})$, where the expression on each copy of A^*_{iso} agrees, we have that the permutation action and contraction both commute with $\mathrm{d}_{\mathrm{dR}}^{\pm 0}$. So the preimage σ of

$$\omega_{(p,q)} \cup \nu_{(r,s)} - \omega_{(p,q)} \breve{\cdot} \nu_{(r,s)}$$

is a $J^E G$-invariant form for which $\mathrm{d}_{\mathrm{dR}}^{\pm 0}(\sigma)$ vanishes, especially on any restriction to any R_n, such that we have that as a whole

$$\Psi(\omega_{(p,q)}) \cup \Psi(\nu_{(r,s)}) - \Psi(\omega_{(p,q)} \breve{\cdot} \nu_{(r,s)}) - \mathbf{d}(\sigma),$$

decomposes into a sum of terms where each restricts on a specific R_n to a form supported only in E_n-vertical degrees at least $q+1$. By Lemma 5.2.7, this implies that the term is exact. As so is $\mathbf{d}(\sigma)$ we directly obtain that therefore $\Psi(\omega_{(p,q)}) \cup \Psi(\nu_{(r,s)})$ and $\Psi(\omega_{(p,q)} \breve{\cdot} \nu_{(r,s)})$ are cohomologically equivalent. □

Proposition 5.2.11. *Consider a proper and regular Lie groupoid G together with an admissible cover induced by $\{R_n\}_{n\in\mathbb{N}_0}$ (or $\{R_n\}_{0\le n\le f}$) and an open saturated cover $\{O_k\}_{k\in\mathbb{N}_0}$ with an admissible partition of unity with respect to the latter. Further consider a multiplicative Ehresmann connection E with respect to the isotropy Lie algebroid and a family of multiplicative Ehresmann connections $\{E^{(k)}\}_{k\in\mathbb{N}_0}$, where each $E^{(k)}$ is a multiplicative Ehresmann connection with respect to the isotropy on $G|_{O_k}$, as well as a family of multiplicative Ehresmann connections $\{E_n\}_{n\in\mathbb{N}_0}$, where each E_n is a multiplicative Ehresmann connection with respect to the isotropy on $G|_{R_n}$. Assume that the connections satisfy that*

- *E_n is a flat multiplicative Ehresmann connection for any $n \in \mathbb{N}_0$.*
- *We have that $E^{(k)}|_{R_n \cap O_k} = E|_{R_n \cap O_k}$ for each $n \in \mathbb{N}$ and $k \in \mathbb{N}$.*
- *On $G|_{O_k}$ we have that $E^{(k)}$ is flatly compatible with the cover induced by the admissible embedded manifolds $\{R_n \cap O_k\}_{n \in \mathbb{N}}$.*

Then the map

$$\Phi : H(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))) \to H(\mathrm{Tot}(\mathcal{R}_E))$$

as constructed in Theorem 4.2.16 from the assignment of Lemma 5.2.8 is an isomorphism of $\mathbb{R}$-algebras.

Proof. By Lemma 5.2.9 we have that the assignment of Lemma 5.2.8 satisfies the requirements to apply Theorem 4.2.16 constructing Φ. By Lemma 4.2.22 we have that Φ induced by some linear assignment Ψ is a morphism of $\mathbb{R}$-algebras if and only if for any two $\widehat{\mathrm{d}_{\mathrm{Cart}}}$-closed $J^E G$-isotropy Cartan forms $\omega_{(p,q)}$ and $\nu_{(r,s)}$ holds that

$$\Psi(\omega_{(p,q)}) \cup \Psi(\nu_{(r,s)})$$

is cohomologically equivalent to $\Psi(\omega_{(p,q)} \,\breve{\cdot}\, \nu_{(r,s)})$. This is the case by Lemma 5.2.10. $\square$

From here on out, we will show that indeed, for any Lie groupoid G there exists an admissible cover together with a family of locally defined multiplicative Ehresmann connection that are flatly compatible and agree on restrictions with a globally defined E. It will turn out, that the construction of Proposition 1.5.9 is actually almost the right choice (up to a refinement) for the admissible manifolds $\{R_n\}_{n \in \mathbb{N}}$ and we can actually choose the induced cover $O_k = \mathrm{Sat}(R_k)$.

Obtaining the multiplicative Ehresmann connection (and proving that it is flat) is more difficult. While we know (due to way we chose R_n with respect to the local form of $G|_{R_n}$), that there are flat Ehresmann connections on the restrictions of the groupoid, we don't know if they are compatible with any extension to $G|_{\cup R_n}$ (this will turn out to be true locally) or to the entirety of G (this will turn out to be true locally on some refinement).

Before we turn to proving this, we will however start by laying some groundwork first. We will analyse how to extend multiplicative Ehresmann connections from $G|_R$ to $G|_{\mathrm{Sat}(R)}$ and how to glue multiplicative Ehresmann connections, which is a surprisingly non-canonical notion.

Lemma 5.2.12. *Consider a proper and regular Lie groupoid G together with an admissible embedded manifold R such that $\mathrm{Sat}(R) = G_0$. Then a multiplicative Ehresmann connection E with respect to the isotropy is uniquely determined by the data of $E|_R$ the Ehresmann connection on $G|_R$ and the restriction $E|_{t^{-1}(R)}$ of E to*

$$s : t^{-1}(R) \to G_0$$

as a connection with respect to the surjective submersion.

Furthermore, the compatibility condition necessary for $E|_R$ and $E|_{t^{-1}(R)}$ to arise from a multiplicative Ehresmann connection E (with respect to the isotropy) is that $E|_{t^{-1}(R)} \equiv E|_R$ on $G|_R$ and

$$T\mu : E|_R \times_{TG_0} E|_{t^{-1}(R)} \subset TG_1 \times_{TG_0} TG_1 \to TG_1$$

only takes values in $E|_{t^{-1}(R)}$.

Proof. The fact that any Ehresmann connection E induces $E|_R$ and $E|_{t^{-1}(R)}$ with the given property is immediate.

For the converse, assume we are given $E|_R$ and $E|_{t^{-1}(R)}$, let us show there is a unique extension E.

We will fist check well-definedness. For this, consider $x \in G_1$. By assumption, there is some $r \in R$ in the same orbit as $s(x)$ and $t(x)$. We can choose $x_s \in G_1$ and $x_t \in G_1$ with target r and source $s(x)$ and $t(x)$ respectively. Furthermore, we can compute

$$g = \mu\left(x_t, x, (x_s)^{-1}\right) \in G|_R,$$

which implies that

$$\mu\left((x_t)^{-1}, g, x_s\right) = x.$$

It is sufficient for the definition of E to describe the projection of any $X' \in T_xG_1$ to its horizontal component. For this, consider $Ts(X')$, which, as

$$s : t^{-1}(R) \to G_0$$

is a surjective submersion with a connection, can be lifted at x_s to $X_s \in E|_{t^{-1}(R)}$ uniquely. Similarly, $Tt(X')$ can be lifted uniquely at x_t to $X_t \in E|_{t^{-1}(R)}$.

Now we can compose

$$T\mu(X_t, X', (T(-)^{-1})(X_s)) := X'_r.$$

We know that X'_r is a tangent vector at $g \in G|_R$ with $Ts(X'_r) \in T_rR$ and $Tt(X'_r) \in T_rR$, implying that $X'_r \in T(G|_R)$. This means that we can project it to its $E|_R$-horizontal component X_r.

We can then compute

$$T\mu((T(-)^{-1})(X_t), X_r, X_s) = X$$

and obtain a tangent vector at x for which $Ts(X) = Ts(X')$ and $Tt(X) = Tt(X')$, such that X and X' only differ by a vector in the isotropy bundle. We define X to be the horizontal projection of X'. Note that this indeed satisfies the necessary condition for a horizontal projection, as whenever $X' \in IG_x$, we have $X_s = 0$ and $X_t = 0$ as both lift the 0 vector, implying that $X'_r \in IG_1$ and $X_r = 0$ as well.

We now need to check if this definition depends on the choice of x_s or x_t, as all other constructions were uniquely described. So consider different choices y_s and y_t with the implied vectors Y_s, Y_t, Y'_r and Y_r. We can consider $g_s \in G|_R$ with

$$y_s = \mu(g_s, x_s)$$

and define $Z_s \in E|_R$ as the unique lift of $Tt(X_s)$ at g_s. Then we obtain that

$$T\mu(Z_s, X_s)$$

is $E|_{t^{-1}(R)}$-horizontal and lifts $Ts(X_s)$ to $T(t^{-1}(R))$ at y_s. Hence we obtain that it agrees with Y_s.

We can look at similar constructions for Y_t and obtain that Y'_r and X'_r can be related by left and right multiplication with $E|_R$-horizontal vectors. However, as $E|_R$ is a multiplicative Ehresmann connection, we know that the horizontal projections of

$$T\mu(X_t, X', (T(-)^{-1})(X_s)) \in T(G|_R)$$

and

$$T\mu(Y_t, X', (T(-)^{-1})(Y_s)) \in T(G|_R)$$

are then related by left and right multiplication with both tangent vectors simultaneously, as they are $E|_R$ horizontal such that they are projected onto themselves. Inserting this in all the formulas yields that both definitions of X agree.

Next, we need to argue, that the horizontal projection described this way indeed defines a multiplicative Ehresmann connection. For this, it is important, that at units $x = 1_m$ the connection contains the subspace tangent to the unit section. However, in the case that $X' = Tu_1(Z)$ for $Z \in T_m G_0$, we can choose $x_s = x_t$ and $X_s = X_t$, such that X'_r is also given by a vector tangent to the unit section. As $E|_R$ was assumed to be a multiplicative Ehresmann connection, this implies $X_r = X'_r$ and $X = X'$.

Furthermore, we need to check that multiplying two E-horizontal vectors X and Y yields an E-horizontal result. However, by definition, to multiply the vectors, we know that $Ts(X) = Tt(Y)$, meaning that if we choose $x_s = y_t$ we can choose $X_s = Y_t$. Unravelling the definitions, we then find that when we replace X and Y by the terms of their E-horizontal projections the term $\mu(X, Y)$ agrees with the term defining the E-horizontal projection of $\mu(X, Y)$ itself.

In a similar manner, we have that if X is E-horizontal, we can choose any X_s and X_t and for $Y' = (T(\bullet)^{-1})(X)$ choose $Y_s = X_t$ and $Y_t = X_s$. Then the term Y'_r in the given construction is given precisely by the inverse to the term X'_r. But as by construction, X is E-horizontal if and only if X'_r is $E|_R$-horizontal, we find that Y'_r is $E|_R$-horizontal and therefore Y is E-horizontal.

Lastly, it remains to check that E described this way indeed restricts to $E|_R$ and $E|_{t^{-1}(R)}$. For this, note that if $x \in t^{-1}(R)$, we can choose $r = t(x)$ as well as $x_t = 1_{t(x)}$ and $x_s = x$. Then we have that $g = 1_{t(x)}$ and for any $E|_{t^{-1}(R)}$-horizontal vector X' we can use that $E|_R$ agrees with $E|_{t^{-1}(R)}$ to lift it at x_s to X' itself and at x_t to $(Tu_1)(Tt(X'))$. However, this implies that $X'_r = (Tu_1)(Tt(X'))$, such that the $E|_R$-horizontal projection is given by the identity, implying that the E-horizontal projection of X' is given by the identity as well and therefore that X' is E-horizontal. As X' was chosen as an arbitrary $E|_{t^{-1}(R)}$-horizontal vector, this means that $E \equiv E|_{t^{-1}(R)}$ on $t^{-1}(R)$. In particular, E restricts to $E|_R$. □

Lemma 5.2.13. *Consider a proper and regular Lie groupoid G together with an admissible embedded manifold R such that* $\mathrm{Sat}(R) = G_0$. *Consider a family of multiplicative Ehresmann connections $\{E^{(n)}\}_{n\in\mathbb{N}_0}$ with respect to the isotropy on a saturated open cover $\{U_n\}_{n\in\mathbb{N}_0}$ such that the family of Ehresmann connections $E^{(n)}|_{R\cap U_n}$ satisfies that for any two $k, l \in \mathbb{N}_0$, we have*

$$E^{(k)}|_{R\cap U_k} \equiv E^{(l)}|_{R\cap U_l}.$$

Then given any partition of unity $\{\chi_n \in C^\infty_{\mathrm{bas}}(G_0)\}_{n\in\mathbb{N}_0}$ underlying the $\{U_n\}_{n\in\mathbb{N}_0}$ we can glue the connections $E^{(n)}|_{t^{-1}(R)}$ on

$$s : t^{-1}(R \cap U_n) \to G_0$$

and $E^{(n)}|_{R\cap U_n}$ on $G|_R$ to the data as given in Lemma 5.2.12 encoding a multiplicative Ehresmann connection E with respect to the isotropy.

Proof. It is immediately clear that we obtain a connection $E|_{t^{-1}(R)}$ on

$$s : t^{-1}(R) \to G_0$$

and a connection $E|_R$ on $G|_R$ which splits TG_1 with respect to IG_1. We still need to show that the result is multiplicative and that $E|_{t^{-1}(R)}$ restricts to $E|_R$.

However, we have that on $G|_R$, the Ehresmann connections $E^{(n)}|_R$ agree wherever two of them are defined. Therefore the result $E|_R$ extends any given $E^{(n)}|_R$. As the multiplicativity can be checked locally on saturated sets, we have that all connections $E|_{t^{-1}(R\cap U_n)}$ on

$$s : t^{-1}(R \cap U_n) \to G_0$$

are invariant under left-multiplication with $E|_R$ and hence so is the result of glueing. Also, as all $E|_{t^{-1}(R\cap U_n)}$ agree with $E|_R$ on the appropriate restriction, so does the result after glueing.

By Lemma 5.2.12, this compatibility implies that there is a multiplicative Ehresmann connection E that restricts to $E|_R$ and $E|_{t^{-1}(R)}$. □

Unfortunately, this method of glueing multiplicative Ehresmann connections does not only depend on the partition of unity, but also on R.

Lemma 5.2.14. *Consider a proper and regular Lie groupoid G together with two admissible embedded manifolds R_1 and R_2 such that* $\mathrm{Sat}(R_1) = \mathrm{Sat}(R_2) = G_0$. *Consider a family of multiplicative Ehresmann connections $\{E^{(n)}\}_{n\in\mathbb{N}}$ with respect to the isotropy on a saturated open cover $\{U_n\}_{n\in\mathbb{N}}$ with an underlying partition of unity $\{\chi_n\}_{n\in\mathbb{N}}$. Further assume that the family of Ehresmann connections $E^{(n)}|_{R_1\cap U_n}$ satisfies that for any two $k, l \in \mathbb{N}$, we have*

$$E^{(k)}|_{R_1\cap U_k} \equiv E^{(l)}|_{R_1\cap U_l}$$

on $R_1 \cap \mathrm{supp}(\chi_l) \cap \mathrm{supp}(\chi_k)$ and the family of Ehresmann connections $E^{(n)}|_{R_2\cap U_n}$ satisfies that for any two $k, l \in \mathbb{N}$, we have

$$E^{(k)}|_{R_2\cap U_k} \equiv E^{(l)}|_{R_2\cap U_l}$$

on $R_2 \cap \mathrm{supp}(\chi_l) \cap \mathrm{supp}(\chi_k)$.

Then, if $R_1 \cup R_2$ is an embedded manifold of G_0, the two multiplicative Ehresmann connections E_1 and E_2 obtained by the construction of Lemma 5.2.13 with respect to R_1 or R_2 satisfy that

$$E_1 \equiv E_2$$

on $G|_{R_1\cup R_2}$, extending each of the restrictions $\{E^{(n)}|_{R_1\cap U_n}\}_{n\in\mathbb{N}_0}$ on $G|_{R_1\cap U_n}$ and each of the $\{E^{(n)}|_{R_2\cap U_n}\}_{n\in\mathbb{N}_0}$ on $G|_{R_2\cap U_n}$.

If furthermore $E^{(l)} \equiv E^{(k)}$ wherever defined on $G|_{R_1\cup R_2}$, we find that

$$E_1 \equiv E_2$$

on all of G.

Remark. This lemma has implications in more general settings as well. If R_1 and R_2 are any admissible embedded manifolds, we can consider $U = \mathrm{Sat}(R_1) \cap \mathrm{Sat}(R_2)$ which is open saturated. In this setting $R_1 \cap U$ and $R_2 \cap U$ satisfy the required property to apply the previous statement to the groupoid $G|_U$.

Proof. Let us denote the two results of the construction of Lemma 5.2.13 by E_1 and E_2.

Consider $t^{-1}(R_1)\cap s^{-1}(R_2)$. The multiplicative Ehresmann connection E_1 restricted to this set is determined by $E_1|_{t^{-1}(R_1)}$ and hence is constructed by glueing the $E^{(n)}$ according to the partition of unity. On $t^{-1}(R_2) \cap s^{-1}(R_1)$ we have that similarly, E_2 is determined by $E_2|_{t^{-1}(R_2)}$ and hence is constructed by glueing the $E^{(n)}$, meaning that as all the $E^{(n)}$ and E_1 and E_2 is preserved under taking inverses and the partition of unity consists of basic functions, we have that

$$E_1|_{t^{-1}(R_1)\cap s^{-1}(R_2)} = E_2|_{t^{-1}(R_1)\cap s^{-1}(R_2)}.$$

The same argument holds for $t^{-1}(R_2) \cap s^{-1}(R_1)$ and the fact that all $E^{(n)}$ and hence any result of glueing them agrees on $G|_{R_1}$ and $G|_{R_2}$ is by assumption.

Furthermore, we have that if on $G|_{R_1\cup R_2}$ all $E^{(n)}$ agree, that

$$E_1|_{t^{-1}(R_1)\cap s^{-1}(R_2)} = E_2|_{t^{-1}(R_1)\cap s^{-1}(R_2)} \equiv E^{(n)}|_{t^{-1}(R_1)\cap s^{-1}(R_2)},$$

wherever the latter is defined.

Now note that we are generally defining $E_1|_{t^{-1}(R_1)}$ and $E_2|_{t^{-1}(R_2)}$ by a similar process: For either $x_1 \in t^{-1}(R_1)$ or $x_2 \in t^{-1}(R_2)$ we define the respective multiplicative Ehresmann connection by glueing the horizontal projections with respect to the partition of unity. However, if x_1 and x_2 are in the same orbit we can relate those projections according to $E^{(n)}$ by left multiplying with a horizontal jet in $t^{-1}(R_1) \cap s^{-1}(R_2)$ or $t^{-1}(R_2) \cap s^{-1}(R_1)$.

So if horizontality in $t^{-1}(R_1)\cap s^{-1}(R_2)$ is a uniquely defined notion (as all multiplicative Ehresmann connections agree on the corresponding groupoid) this implies that the result of $E_1|_{t^{-1}(R_1)}$ at x_1 and $E_2|_{t^{-1}(R_2)}$ at x_2 are also related by left multiplying with a horizontal jet in $t^{-1}(R_1) \cap s^{-1}(R_2)$ or $t^{-1}(R_2) \cap s^{-1}(R_1)$. (as this is a linear map on the tangent space).

However, when reconstructing E_1 and E_2 from $E_1|_{t^{-1}(R_1)}$ and $E_2|_{t^{-1}(R_2)}$, we can use precisely left multiplication and adjoint action with elements of $t^{-1}(R_1)\cap s^{-1}(R_2)$ to relate the construction of a general horizontal vector. The adjoint is acting on $E_1|_{R_1}$ and $E_2|_{R_2}$ respectively, but as we have shown that $E_1 \equiv E_2$ on both R_1 and R_2, we find that indeed the result agrees. □

Corollary 5.2.15. *Consider a proper and regular Lie groupoid G, a saturated open subset $U \subset G_0$ and an admissible embedded manifold R together with a multiplicative Ehresmann connection E_1 on $G|_U$ and E_2 on $G|_{\mathrm{Sat}(R)}$. Then there exists a multiplicative Ehresmann connection E on $G|_{U\cup \mathrm{Sat}(R)}$ which extends E_1 on $U \setminus \mathrm{Sat}(R)$ and E_2 on $\mathrm{Sat}(R) \setminus U$.*

Proof. Without loss of generality, assume that $G_0 = U \cup \mathrm{Sat}(R)$ as we can otherwise restrict the groupoid to this saturated open. We have that the subsets $\mathrm{Sat}(R) \setminus U$ and $U \setminus \mathrm{Sat}(R)$ are both saturated and closed and by definition disjoint. We may, by Corollary 1.5.6 assume that we can separate them by basic functions χ_1 and χ_2 with $\chi_1 + \chi_2 = 1$, such that $\chi_1 \equiv 1$ on an open neighbourhood of $\mathrm{Sat}(R) \setminus U$ and $\chi_2 \equiv 1$ on an open neighbourhood of $U \setminus \mathrm{Sat}(R)$.

On $U \cap \mathrm{Sat}(R)$ we now may apply Lemma 5.2.13 and glue the connections with respect to $R \cap U$ and this partition of unity (where $R \cap U$ is admissible by Lemma 1.4.31). On any saturated open on which $\chi_1 \equiv 1$ we recover E_1 and similar for E_2. By construction, this is the case for open collars of $U \setminus \mathrm{Sat}(R)$ and $\mathrm{Sat}(R) \setminus U$, meaning the result can be extended to $U \cup \mathrm{Sat}(R)$ by the original connections. □

The first step that we might want to tackle now is describing on how to proceed when given two admissible embedded manifolds $R_1 \subset G_0$ and $R_2 \subset G_0$ for which the associated groupoids $G|_{R_1}$ and $G|_{R_2}$ are trivialisable in the sense of the local form of an s-proper groupoid. Assuming that $\overline{R_1} \cap \overline{R_2} = \emptyset$, we want to find a flat multiplicative Ehresmann connection on $G|_{(R_1\cup R_2)}$.

This construction will use facts about the automorphism groups of a Lie group. However, while the local form of $G|_{R_1}$ and $G|_{R_2}$ is in both cases given by an action Lie groupoid (because we are looking at the neighbourhood of some isotropy group acting), the corresponding group in both cases is not necessarily identical. This complicates the proceedings.

So again, we will postpone the proof and consider some auxiliary definitions first.

Lemma 5.2.16. *Consider a proper and regular Lie groupoid G. Then the subset $H_1 \subset G_1$ given by*

$$\begin{aligned} H_1 = \{x \in G \text{ s.t. } s(x) = t(x) \text{ and} \\ \text{for any } X \in T_{s(x)}G_0 \text{ there is } \widehat{X} \in T_xG_1 \text{ with } Tt(\widehat{X}) = Ts(\widehat{X}) = X\} \end{aligned}$$

defines a proper and regular Lie subgroupoid. It admits an adjoint action in the sense of Example 1.1.30. The construction furthermore commutes with the restriction to admissible embedded submanifolds and induces a quotient groupoid G/H. In case $l_0 = 0$, we have that G/H is étale.

Remark. We can note that for a foliation of dimension $l_0 = 0$ the given set agrees with the description

$$\{x \in G \text{ s.t. } s(x) = t(x) \text{ and any } X \in T_xG_1 \text{ satisfies } Tt(X) = Ts(X)\}.$$

However, this definition would yield the empty set in case $l_0 \neq 0$.

Proof. In this proof, we will derive the statement for increasingly general settings. Let us first consider the case of an action groupoid where a compact Lie group $\mathcal{G}$ is acting on a manifold M with a $\mathcal{G}$-invariant Riemannian metric (which then always exists), where the orbits are discrete. Then geodesics are mapped to geodesics. In particular, if $m \in M$ is a fixed point of some $g \in G$ and $L_g : M \to M$ satisfies that

$$T_mL_g : T_mM \to T_mM$$

is given by the identity, then also

$$L_g \equiv \text{id}$$

on any exponential coordinate chart given by an open ball $U_m \subset M$ open around $m \in M$. Furthermore, as the orbits are discrete, we find that the entire connected component of G containing g must satisfy that $L_{g'} \equiv \text{id}$ on U_m, as any such $L_{g'}(r)$ for any $r \in U_m$ must be contained in the same constant leaf of the foliation as $L_g(r) = r$. Similarly, if

$$T_mL_g : T_mM \to T_mM$$

is not given by the identity, there is an open neighbourhood around m on which the entire connected component of g induces a fixed non-trivial diffeomorphism, which, even at any fixed point, induces a non-trivial map on the tangent spaces. In particular, any lift of any $X \in T_{s(x)}G_0$ (whether it is a lift with respect to the product structure or not) will have an image under Tt according to this diffeomorphism, such that it is distinct from X itself. Hence for the associated action groupoid G, any element $(g, m) \in H$ admits an open neighbourhood $s^{-1}(U_m) \cap t^{-1}(U_m)$ on which the inclusion $H \subset G$ agrees with the inclusion of connected components. So H is an embedded submanifold of G.

Now let us cover the case where we do not assume G to be an action groupoid, but we assume that G is proper and regular and the orbits are discrete. If we combine the previous fact with Lemma 1.4.37, we obtain that for any point $m \in G_0$ there is again an open neighbourhood U_m such that any of the connected components of $G|_{U_m}$ can either induce the identity on TG_0 at all points or at no point at all. In particular, the subgroupoid H is well-defined.

Now let us return to the general case. Here, we know by Corollary 1.4.38 that for any $m \in G_0$ we can find an admissible embedded manifold R through m, and by Lemma 1.4.14 there exists a nice compatible coordinate chart, which we may without loss of generality assume to take values in a cube. Analogous to the proof of Lemma 1.4.32 we can integrate some lift of $\mathcal{F}_0$ to A (i.e. it maps under t to R while retaining the local leaves). This results in a family of local sections inducing bijections between $\{0\} \times V$ and $\{b\} \times V$ where $\{0\} \times V$ is precisely the set corresponding to the intersection of the domain with R. Instead of understanding the family sections as such, we can instead consider the inverses, which on the open subset of $\mathbb{R}^{l_0} \times V$ contained in the cube (i.e. the codomain of the coordinate chart) define an s-section γ with image in R. Let us denote this open as U_m. We obtain an isomorphism

$$\begin{aligned} G|_{U_m} &\cong G|_V \times (U \subset \mathbb{R}^{l_0}) \times (U \subset \mathbb{R}^{l_0}) \\ x &\mapsto (\mu(\gamma(t(x)), x, \gamma(s(x))^{-1}), \mathrm{pr}_{\mathbb{R}^{l_0}}(t(x)), \mathrm{pr}_{\mathbb{R}^{l_0}}(s(x))) \end{aligned}$$

It is immediate from the description that on the right-hand side the induced groupoid structure corresponds under the identification $\mathbb{R}^{l_0} \times V \cong V \times \mathbb{R}^{l_0}$ to

$$\begin{aligned} s' : G|_V \times U \times U &\to V \times U \cong U_m \\ (x, a, b) &\mapsto (x, b) \end{aligned}$$

and

$$\begin{aligned} t' : G|_V \times U \times U &\to V \times U \cong U_m \\ (x, a, b) &\mapsto (x, a). \end{aligned}$$

We can directly notice, that the diagonal map

$$\Delta : G|_R \times \mathbb{R}^{l_0} \subset G|_R \times \mathbb{R}^{l_0} \times \mathbb{R}^{l_0}$$

describes an embedding, and by our previous arguments, so does

$$H|_R \times \mathbb{R}^{l_0} \subset G|_R \times \mathbb{R}^{l_0}.$$

We now claim that this embedding, when intersected with the codomain of the constructed isomorphism, precisely describes H locally. For this, firstly notice that as $x \in H$ implies $s(x) = t(x)$, we have that H can indeed can indeed only contain elements of the diagonal. Also, we have that the definition of H is local, meaning that $H \cap G|_{U_m} = H|_{U_m}$ and $H|_R \cap G|_V = H|_V$. Therefore, we now need to show that any element of $H|_{U_m}$ corresponds to an element of $H|_V \times U \times U$ and also the converse.

We have that at any point of

$$G|_V \times U \times U,$$

any lift of some $(X, 0) \in TR \times T\mathbb{R}^{l_0}$ with respect to Ts' (i.e. in its first and third component) corresponds in $G|_{U_m}$ to a lift of X when interpreted as an element of TG_0. Therefore, if at some point $x \in H|_{U_m}$ there exist a lift $\widehat{X}$ of some $X \in TR$ with $Tt(\widehat{X}) = Ts(\widehat{X}) = X$, it will correspond to a vector $(\widehat{X'}, 0, 0)$ in $G|_V \times U \times U$ where its image under Tt' is given by $(X, 0)$ as well. In particular $\widehat{X'} \in TG|_V$ satisfies that $Tt(\widehat{X'}) = Ts(\widehat{X'}) = X$. Therefore, any point of $H|_{U_m}$ has its first component contained in $H|_V$. At the same time, we may simply define the lift of any (X, Y) at some $x \in H|_V \times \Delta(U)$ as $(\widehat{X'}, Y, Y)$, where $\widehat{X'}$ is chosen as previously described for the lift of $(X, 0)$. We can explicitly compute that it is indeed a lift with respect to Ts' and Tt'. So indeed, all elements of the embedded manifold $H|_V \times \Delta(U)$ correspond to elements of $H|_{U_m}$ and the two descriptions agree. In particular, $H|_{U_m}$ has the structure of an embedded submanifold. Furthermore, we find that the construction of H from G commutes with the restriction to R around m. But as R and m were arbitrary, this holds for every admissible embedded submanifold.

We now aim to check that H is normal, i.e. that the equivalence class of x up to left multiplication with elements of H agrees with the equivalence class of x up to right multiplication with elements of H. As H is a subgroupoid of G, we have that the quotient G/H is the quotient of a free and proper groupoid action, which is always well-defined and if H is normal, it immediately inherits a groupoid structure. It is also immediately clear from the formulation that H is a normal subgroupoid if and only if for any $x \in G_1$ holds that

$$\mu(x, h, x^{-1}) \in H$$

for any $h \in G|_{\{s(x)\}} \cap H$. This is also precisely the condition necessary to define an adjoint action as in Example 1.1.30.

Now by Corollary 1.2.9 we know that we can extend x to a local bisection γ, where we will denote its domain by O and its codomain by W, where γ covers the diffeomorphism

$$\phi : O \to W.$$

We then find that

$$\begin{aligned} \widetilde{D}^0_{\gamma,1} \circ \widetilde{D}^1_{\gamma,1} : G|_W &\to G|_O \\ y &\mapsto \mu(\gamma(t(y)), y, (\gamma(s(y)))^{-1}) \end{aligned}$$

is a bijection with inverse $\widetilde{D}^0_{\gamma^{-1},1} \circ \widetilde{D}^1_{\gamma^{-1},1}$, even though it is not a groupoid morphism. Also, by assumption, it maps $h \in G|_{\{s(x)\}} \cap H$ precisely to $\mu(x, h, x^{-1})$. We can now for any $X \in T_{t(x)}W$ consider $Y = T\phi^{-1}(X) \in T_{s(x)}O$. By assumption, we can find a lift $\widehat{Y} \in T_h G|_O$, such that $Tt(\widehat{Y}) = Ts(\widehat{Y}) = Y$. We can now explicitly compute that

$$Tt((\widetilde{D}^0_{\gamma^{-1},1} \circ \widetilde{D}^1_{\gamma^{-1},1})_* \widehat{Y}) = Tt(T\widetilde{D}^0_{\gamma^{-1},1}(\widehat{Y})) = T\phi(Tt(\widehat{Y})) = X$$

and analogously for Ts. So indeed, any $X \in T_{t(x)}G_0$ we can define a lift $\widehat{X}$ as required in the definition of H. So indeed, H is normal and therefore the quotient G/H is well-defined.

Furthermore, we have that the dimension of H is precisely given by $k_1 - l_0$, as we can see from the explicit form of the embedding $H \subset G$ and the dimension of its s-fibres is therefore $k_1 - k_0 - l_0$. This implies that the quotient is of dimension $k_0 + l_0$ and if $l_0 = 0$, that the groupoid (G/H) is étale. □

Definition 5.2.17. Consider a proper and regular Lie groupoid G. Then the subgroupoid H defined in Lemma 5.2.16 will be referred to as the **rigid isotropy subgroupoid**.

Lemma 5.2.18. *Consider a proper and regular Lie groupoid G and assume that $R_1 \subset G_0$ and $R_2 \subset G_0$ are admissible embedded manifolds. Consider some connected component*

$$i \subset \underline{s}^{-1}(R_1) \cap \underline{t}^{-1}(R_2) \cap G/H$$

where $\underline{s}$ and $\underline{t}$ denote source and target maps of the groupoid G/H. The component i has a preimage $\widehat{i}$ under the projection $G \to G/H$.

If there is a section $\gamma : U \subset R_1 \to G_1$ of s such that

$$\mathrm{pr}_{G/H} \circ \gamma : U \subset R_1 \to (G/H)_1$$

is an isomorphism onto i, we have

$$\widehat{i} \cong H|_U,$$

where the isomorphism is equivariant under the action of $H|_U$ on both spaces by right multiplication.

Furthermore, if γ induces a diffeomorphism between U and $V \subset R_2$, we have additionally that

$$\widehat{i} \cong H|_V$$

where the isomorphism is equivariant under the action of $H|_V$ on both spaces by left multiplication. The concatenated isomorphism

$$H|_U \cong H|_V$$

agrees with $\widetilde{\mathrm{Ad}}^H_\gamma$.

Proof. The isomorphism can be explicitly written down as

$$\begin{aligned} H|_U &\to \widehat{i} \\ x &\mapsto \mu(\gamma(s(x)), x) \end{aligned}$$

and

$$\begin{aligned} \widehat{i} &\to H|_U \\ x &\mapsto \mu((\gamma(t(x)))^{-1}, x), \end{aligned}$$

where we can now check that the expressions are well-defined. For the first map, this follows as we can note that $s \equiv t$ on $H|_U$ and γ is a section of s. The result is an element of $\widehat{i}$, as x projects to $1_{\underline{t}(x)} \in G/H$, such that the composition projects to $\mathrm{pr}(\gamma(s(x)))$, which is in i by definition.

Now consider the second map. Here, we need to check that $t(x) = s((\gamma(t(x)))^{-1})$ and that the composition is actually an element of $H|_U$. By assumption, any $x \in \widehat{i}$ projects to $\underline{x} \in i$ with the same image under the respective source and target maps. In particular as $\mathrm{pr} \circ \gamma$ is an isomorphism and $\underline{s}$ is a left inverse by definition, we have that the restriction $\underline{s}|_i : i \to U$ is well-defined with target space U instead of G_0 and describes the unique left and right inverse. This implies $s(x) = \underline{s}(\underline{x}) \in U$, such that the term $(\gamma(s(x)))^{-1}$ exists. Furthermore, it implies that $\mathrm{pr}(\gamma(\underline{s}(\underline{x}))) = \underline{x}$ and we have that

$$s((\gamma(s(x)))^{-1}) = t(\gamma(s(x))) = t(\gamma(\underline{s}(\underline{x}))) = \underline{t}(\underline{x}) = t(x).$$

So now it only remains to be seen that

$$\mu((\gamma(t(x)))^{-1}, x) \in H|_U.$$

However, this is by definition, as we have that $x \in \widehat{i}$ projects to $\underline{x}$, while $\gamma(s(x)) \in \widehat{i}$ projects to the unique element $x' \in i$ with $\underline{s}(x') = s(x) = \underline{s}(\underline{x})$, implying that $x' = \underline{x}$. So they lie in the same equivalence class of G/H and in particular, the expression we are analysing is an element of H. That it is contained in $H|_U$ follows immediately from the consideration that $s(x) \in U$ and left multiplication does not alter the source.

The fact that the two maps between $\widehat{i}$ and $H|_U$ are inverses is immediate. They also are equivariant under the action by right multiplication.

In case that γ induces a diffeomorphism between U and V, recall that we have defined a section $\gamma^{-1} : V \to G_1$ which maps precisely to the values inverse to those in the graph of γ. We can now analogously check that

$$\begin{aligned} H|_V &\to \widehat{i} \\ x &\mapsto \mu(x, (\gamma^{-1}(t(x)))^{-1}) \end{aligned}$$

and

$$\begin{aligned} \widehat{i} &\to H|_V \\ x &\mapsto \mu(x, \gamma^{-1}(t(x))), \end{aligned}$$

are isomorphisms analogously, where the only additional argument needed is the fact that by construction, $\mathrm{pr} \circ \gamma^{-1} = (\mathrm{pr} \circ \gamma)^{-1}$ such that

$$t(\gamma^{-1}(t(x))) = \underline{t}((\mathrm{pr} \circ \gamma)^{-1}(\underline{t}(\underline{x}))) = \underline{t}(\underline{x}^{-1}) = \underline{s}(\underline{x}) = s(x).$$

By writing out the concatenation of both morphisms, we obtain

$$\begin{array}{ccccc} H|_U & \longrightarrow & \widehat{i} & \longrightarrow & H|_V \\ x & \longmapsto & \mu(\gamma(s(x)), x) & & \\ & & x & \longmapsto & \mu(x, \gamma^{-1}(t(x))) \\ x & \longmapsto & \mu(\gamma(s(x)), x) & \longrightarrow & \mu(\gamma(s(x)), x, \gamma^{-1}(t(\gamma(s(x))))), \end{array}$$

where we can note that $(\gamma^{-1} \circ t \circ \gamma)(u) = (\gamma(u))^{-1}$ for any $u \in U$, such that this notion indeed agrees with $\widetilde{\mathrm{Ad}}^H_{\gamma,q}$. □

The groupoid H is very well-behaved. For example, because it is a proper Lie group bundle we have that for any two points in a connected component of $H_0 = G_0$, their fibres are (non-canonically) isomorphic groups. By construction, the same is also true for any two points in the same orbit of G_0. This means that while it is difficult to compare $G|_{R_1}$ and $G|_{R_2}$ for two admissible manifolds with the same saturation even when they admit local trivialisations, it will be feasible to compare $H|_{R_1}$ and $H|_{R_2}$.

This comparison will be based on a well-known result by Iwasawa that is not about Lie groupoids, but Lie groups. On first glance, the relation is somewhat hidden, but we will uncover it in the following.

Theorem 5.2.19 ([27], Lemma 1.1)**.** *If $\mathcal{G}$ is a compact Lie group, then*

$$\mathrm{Aut}(\mathcal{G}) / \mathrm{Inn}(\mathcal{G})$$

is discrete.

However, this statement implies direct consequences for the connections on Lie group bundles:

Lemma 5.2.20. *Let M be a connected and contractible manifold and $\mathcal{G}$ a compact Lie group. Assume there is an automorphism*

$$\begin{tikzcd} \mathcal{G} \times M \arrow[r, "\phi"] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & \mathcal{G} \times M \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ M \arrow[r, "\mathrm{id}"] & M \end{tikzcd}$$

of the trivial action groupoid which restricts to id *on* $\mathcal{G} \times \{m\}$*. Then there exists*

$$\gamma \in C^\infty(M, \mathcal{G}) = \Gamma(M, \mathcal{G} \times M),$$

such that $\phi = \widetilde{\mathrm{Ad}}^G_{\gamma,1}$.

Furthermore, the choice of γ *is unique up to*

$$\varepsilon \in C^\infty(M, \mathcal{C}) = \Gamma(M, \mathcal{C} \times M),$$

where $\mathcal{C} \subset \mathcal{G}$ *is the center subgroup of* $\mathcal{G}$*.*

Remark. If we ask that $\gamma(m) = 1_m$, we obtain that γ is unique up to ε also satisfying $\varepsilon(m) = 1_m$. This would for example imply that ε only takes values in the connected component of $\mathcal{C}$ containing 1_m.

Proof. We have that for each $n \in M$ the groupoid morphism induces an automorphism on $\mathcal{G} \times \{n\}$. As ϕ is continuous, the induced map $A_\phi \in C^0(M, \mathrm{Aut}(\mathcal{G}))$ is continuous as well. As M is connected, so is the image of A_ϕ, meaning that at each $n \in M$ the automorphism $A_\phi(n)$ is in the same connected component of $\mathrm{Aut}(\mathcal{G})$. As this component must contain id (which is the image at m), we know that it is a subset of $\mathrm{Inn}(\mathcal{G})$ by Theorem 5.2.19. So we can write

$$A_\phi \in C^0(M, \mathrm{Inn}(\mathcal{G})).$$

Furthermore, we have that $\mathrm{Inn}(\mathcal{G})$ carries a canonical manifold structure (as a submanifold of $\mathrm{Aut}(\mathcal{G})$). As ϕ is a smooth groupoid morphism, we find that $A_\phi \in C^0(M, \mathrm{Inn}(\mathcal{G}))$ must actually be smooth with respect to this structure as well. Note that there is a canonical projection

$$\mathcal{G} \to \mathrm{Inn}(\mathcal{G})$$

which is a smooth Lie group homomorphism and establishes

$$\mathrm{Inn}(\mathcal{G}) \cong \mathcal{G}/\mathcal{C}.$$

Now, because M is contractible and $\mathcal{G}$ is compact, we may lift A_ϕ to $\gamma \in C^\infty(M, \mathcal{G})$: A smooth contraction f_c of M to a point induces $A_\phi \circ f_c \in C^\infty(M \times [0,1], \mathcal{G}/\mathcal{C})$ (deforming A_ϕ to a constant, which can clearly be lifted) and any connection of $\mathcal{G}$ as a bundle over $\mathcal{G}/\mathcal{C}$. Choose a lift of $A_\phi \circ f_c$ to $C^\infty(M \times [1-t, 1], \mathcal{G})$ along this connection for the largest t possible. Well-definedness for $t = 1$ follows from compactness of $\mathcal{G}$. By construction of the map from $\mathcal{G}$ to $\mathrm{Inn}(\mathcal{G})$, we precisely have that the obtained section γ satisfies that $\widetilde{\mathrm{Ad}}^0_\gamma$ (which acts by $\mathrm{Ad}_{\gamma(n)}$ on each fibre) agrees with ϕ.

For the second claim, we only need to apply the previous result to the groupoid morphism $\phi = \mathrm{id}$. We have that in this case $\mathrm{Ad}_{\gamma(n)} : \mathcal{G} \to \mathcal{G}$ must be trivial for any $n \in M$. This however is equivalent to $\gamma(n) \in \mathcal{C}$ for any $n \in M$. □

Lemma 5.2.21. *Let* G *be a proper and regular Lie groupoid. Assume that there are two connected and contractible admissible embedded manifolds* $R_1 \subset G_0$ *and* $R_2 \subset G_0$ *with two* *flat multiplicative Ehresmann connections E_1 and E_2 on $H|_{R_1}$ and $H|_{R_2}$ and a bisection $\gamma : R_1 \to G_1$ which induces an isomorphism*

$$\mathrm{pr}_{G/H} \circ \gamma : R_1 \to i \subset \underline{t}^{-1}(R_2) \cap \underline{s}^{-1}(R_1) \subset G/H$$

to a connected component i of $\underline{t}^{-1}(R_2) \cap \underline{s}^{-1}(R_1)$.

Then there is a construction for a flat connection E (equivalently with respect to s or t) on the component $\widehat{i}$ which is the preimage of the connected component i under $\mathrm{pr}_{G/H}$. It is compatible with E_1 and E_2 in the sense that $T\mu$ restricts to the spaces

$$T\mu : E \times_{TR_1} E_1 \to E$$

and

$$T\mu : E_2 \times_{TR_2} E \to E.$$

Further, it admits an E-horizontal bisection $\gamma' : R_1 \to G_1$, which is unique up to left-multiplication with

$$n \in R_2 \mapsto \mu(h(n), \varepsilon(n)) = \mu(\varepsilon(n), h(n))$$

where $h : R_2 \to (H|_{R_2})_1$ is an E_2-horizontal section and

$$\varepsilon : R_2 \to \mathcal{C}_1 \subset H_1,$$

is a section where $\mathcal{C}$ denotes the center subgroupoid of H. It is also unique up to right multiplication with

$$n \in R_1 \mapsto \mu(h(n), \varepsilon(n)) = \mu(\varepsilon(n), h(n))$$

where $h : R_1 \to (H|_{R_1})_1$ is an E_1-horizontal section and

$$\varepsilon : R_1 \to \mathcal{C}_1 \subset H_1$$

is a section.

Remark. If we consider the path representing a non-zero tangent vector at $m \in G_0$ as an embedding $(-\varepsilon, \varepsilon) \subset G_0$ then we may consider $H|_{(-\varepsilon,\varepsilon)}$. As this is one-dimensional, any multiplicative Ehresmann connection is flat and we can apply this statement to some $R_1 = R_2$ admissible around m. We obtain that while the multiplicative Ehresmann connection on $H|_{(-\varepsilon,\varepsilon)}$ is not unique, it is unique up to some adjoint action. This means that for the infinitesimal ∇^E on $A_{\mathrm{iso}}|_{(-\varepsilon,\varepsilon)}$ and the associated connection on $S^\bullet(A_{\mathrm{iso}})|_{(-\varepsilon,\varepsilon)}$ the same must hold true. In particular, its induced connection on $(S^\bullet(A_{\mathrm{iso}}))^G|_{(-\varepsilon,\varepsilon)}$ is unique and as the path was chosen arbitrarily, this generalises to $(S^\bullet(A_{\mathrm{iso}}))^G$ entirely. This in particular motivates, as we have already remarked on page 213, that the Cartan differential squares to 0.

Proof. According to Lemma 5.2.18 there is an isomorphism between the described component of $s^{-1}(R_1) \cap t^{-1}(R_2)$ and $H|_{R_1}$ and $H|_{R_2}$ respectively induced by γ and γ^{-1}. So both E_1 and E_2 induce a connection on $\widehat{i}$. Unfortunately, using either of these isomorphisms would not yield the desired result. We will, however, employ a similar method using a different isomorphism.

For this, we first need to alter γ by another section. For this consider

$$\widetilde{\mathrm{Ad}}^H_{\gamma,q} : H|_{R_2} \to H|_{R_1}.$$

We can pull-back E_1 via this map. As R_2 is contractible and both E_1 and E_2 are flat both yield unique trivialisations

$$\begin{array}{ccc} H|_{R_2} & \cong & \mathcal{G}_m \times R_2 \\ {\scriptstyle t}\downarrow\downarrow{\scriptstyle s} & & {\scriptstyle t}\downarrow\downarrow{\scriptstyle s} \\ R_2 & \cong & R_2 \end{array}$$

where $\mathcal{G}$ describes the group structure of the fibre of $H|_{R_2}$ at some $m \in R_2$. The trivialisations agree at $\mathcal{G}_m \times \{m\}$.

By Lemma 5.2.20, we have that there is then a section $\overline{\gamma} : R_2 \to (H|_{R_2})_1$ such that

$$\widetilde{\mathrm{Ad}}^H_{\overline{\gamma},q} : H|_{R_2} \to H|_{R_2}$$

identifies E_2 (on the domain) with the pull-back of E_1 to $H|_{R_2}$ (on the target) and thus

$$\widetilde{\mathrm{Ad}}^H_{\overline{\gamma}\circ\gamma,q} : H|_{R_2} \to H|_{R_1}$$

identifies E_1 and E_2. Let us denote $\gamma' = \overline{\gamma} \circ \gamma$.

If we now use γ' to identify the described component of $s^{-1}(R_1) \cap t^{-1}(R_2)$ with $H|_{R_1}$ and $H|_{R_2}$ respectively, we have that using either E_1 or E_2 will define the same connection by Lemma 5.2.18. Using the identification with $H|_{R_1}$ will by definition make the connection compatible with multiplication from the right (as the isomorphism is defined by multiplying on the left, which commutes with right multiplication) and the analogous statement holds for $H|_{R_2}$, meaning that indeed the new connection is compatible with either.

Lastly note that γ' is by definition a flat bisection, as it corresponds to the identity section in $(H|_{R_1})_1$ (and similarly for $(H|_{R_2})_1$ when using $(\gamma')^{-1}$). Also note that any such flat bisection γ'' satisfies that

$$\widetilde{\mathrm{Ad}}^H_{\gamma''\circ(\gamma')^{-1}} : H|_{R_2} \to H|_{R_2}$$

is compatible with E_2. If we consider the value at m, the result is some element of H which can be extended to a flat section h, corresponding to a constant section of $\mathcal{G}_m \times R_2$. The map $\widetilde{\mathrm{Ad}}^H_h$ is also compatible with E_2 and furthermore agrees with $\widetilde{\mathrm{Ad}}^H_{\gamma''\circ(\gamma')^{-1}}$ at m. This implies by Lemma 5.2.20 that their difference can be expressed by a section contained in the center of $\mathcal{G}_m$, which under the identification with $H|_{R_2}$ corresponds to

$$\varepsilon : R_2 \to C_1.$$

□

Corollary 5.2.22. *Let G be a proper and regular Lie groupoid. Assume that there are two admissible embedded manifolds $R_1 \subset G_0$ and $R_2 \subset G_0$ with two flat multiplicative Ehresmann connections E_1 and E_2 on $H|_{R_1}$ and $H|_{R_2}$ and a bisection $\gamma : R_1 \to G_1$. Further assume that γ induces an isomorphism*

$$\mathrm{pr}_{G/H} \circ\gamma : R_1 \to i \subset \underline{t}^{-1}(R_2) \cap \underline{s}^{-1}(R_1) \subset G/H$$

to a connected component i of $\underline{t}^{-1}(R_2) \cap \underline{s}^{-1}(R_1)$.

Then there is a construction for a flat connection E (equivalently with respect to s or t) on the component $\widehat{i}$ of $G|_{R_1\cup R_2}$ which is the preimage of the connected component i under $\mathrm{pr}_{G/H}$. It is compatible with E_1 and E_2 in the sense that $T\mu$ restricts to the spaces

$$T\mu : E \times_{TR_1} E_1 \to E$$

and

$$T\mu : E_2 \times_{TR_2} E \to E.$$

Further, the section γ is E-horizontal.

Proof. We can review the proof of Lemma 5.2.21 and see that the contractibility was only necessary to replace γ by γ'. This step is no longer necessary due to the additional assumption on γ. That the connection induced on $\widehat{i}$ satisfies all necessary properties can be checked locally on connected and contractible admissible subsets of R_1 which exist by Lemma 1.4.37. □

So we have constructed a method to extend two flat multiplicative Ehresmann connections (in particular when on two properly disjoint embedded manifolds) in a particular way - not necessarily to a multiplicative Ehresmann connection, but to some sort of connection. We will later apply it to recover an actual multiplicative Ehresmann connection.

Lemma 5.2.23. *Let G be a proper and regular Lie groupoid. Consider two admissible embedded manifolds $R_1 \subset G_0$ and $R_2 \subset G_0$ with two flat multiplicative Ehresmann connections E_1 and E_2 on $H|_{R_1}$ and $H|_{R_2}$. Assume that $R_1 \cup R_2$ is an embedded manifold of G_0 (i.e. R_1 and R_2 are disjoint or intersect on an open). Let i be a connected component of $G/H|_{R_1 \cup R_2}$ that is not an identity component. Denote by $\widehat{i}$ the component of $G|_{R_1 \cup R_2}$ that projects to i. Assume that there is a bisection $\gamma : R_1 \to G_1$ that induces a diffeomorphism between R_1 and R_2 which furthermore is an isomorphism onto i under $\mathrm{pr}_{G/H} \circ \gamma$. Further assume that $\widetilde{\mathrm{Ad}}^0_{\gamma,1}$ is a diffeomorphism of $H|_{R_1}$ and $H|_{R_2}$ which relates the two Ehresmann connections.*

Also consider any number of local bisections $\{\gamma_k : R_1 \to G_1\}_{0 \leq k \leq f}$ with the following properties:

- *all $\mathrm{pr}_{G/H} \circ \gamma_k$ agree with $\mathrm{pr}_{G/H} \circ \gamma$ as maps to G/H*
- *all γ_k induce diffeomorphisms of the corresponding open subsets of $H|_{R_1}$ and $H|_{R_2}$ respecting the chosen Ehresmann connections.*

Then for any weighting $\{\lambda_k \in \mathbb{R}\}$ there is an averaging construction resulting in a flat connection on $\widehat{i}$ which is invariant under left multiplication by E_2 and right multiplication by E_1.

Proof. Consider the given construction of Corollary 5.2.22 to use γ_k to construct a connection $E^{(k)}$ on the component $\widehat{i}$ of $G|_{R_1 \cup R_2}$ corresponding to the connected component i of $(G/H)|_{R_1 \cup R_2}$ corresponding to γ (and therefore also γ_k).

Then we average the resulting connections $E^{(k)}$ (i.e. their vertical projection maps) with respect to the weighting. This is trivially still a connection with respect to s and t. The retained invariance under E_1 and E_2 is also immediately inherited.

Now, we still need to argue that the resulting connection is flat. Note that as flatness is a local property, we may check this construction on any restriction to an open subset of R_1. So by Lemma 1.4.37 we may without loss of generality assume R_1 to be connected and contractible. If we now consider the Lemma 5.2.21, we have that any bisection γ defining a flat connection on $\widehat{i}$ compatible with E_1 and E_2 is unique up to an E_1-horizontal section

$$h : R_1 \to G|_{R_1}$$

and a section

$$\epsilon : R_1 \to \mathcal{C},$$

where $\mathcal{C}$ is the center of H. This means that all $\mu \circ (\gamma_k, h_k)$ agree with γ up to ϵ_k for any γ_k. Now we can note that as $\mathcal{C}$ is an abelian Lie group bundle, the exponential map

$$\exp : A_{\mathcal{C}} \to \mathcal{C}$$

is a fibre-wise group homomorphism (and therefore universal covering map) with respect to the vector bundle structure of the Lie algebroid $A_{\mathcal{C}}$ associated to $\mathcal{C}$. As R_1 was assumed contractible, we can now choose lifts $\widehat{\epsilon_k}$ and average these with respect to the vector bundle structure. Projecting to $\mathcal{C}$ returns a section ε. We claim that $\mu(\gamma, \varepsilon)$ is E-horizontal, which, in combination with the flatness of E_1, yields that the connection on $\widehat{\imath}$ is flat as well.

For this, let us start with the observation that as E_1 is flat and $H|_{R_1}$ is a Lie group bundle, the trivialisation induced by E_1 also restricts to $\mathcal{C}$ in the sense that any element of c defines an E_1-flat section of $H|_{R_1}$ contained in $\mathcal{C}$. This means that for any $r \in R_1$ we can consider $\epsilon_k(r)$ and $\epsilon(r)$ as a E_1-horizontal section to $\mathcal{C}$. So by construction the vertical projection of the connection induced by γ will agree with the vertical projection of the connection induced by $\mu \circ (\gamma, \epsilon(r))$, in particular at r itself and a similar statement holds for $\mu \circ (\gamma_k, \mu \circ (\epsilon_k(r)^{-1}, \epsilon(r)))$. So we have that the difference between these vertical projections at $\mu(\gamma(r), \epsilon(r))$ is given by the left translation of

$$T_r\mu \circ (\epsilon_k, \epsilon_k(r)^{-1}).$$

Subsequently, the difference between the vertical projection of the connection induced by γ and the vertical projection of E is given at $\mu(\gamma(r), \epsilon(r))$ by the left translation of

$$\sum_{0 \le k \le f} \lambda_k \cdot (T_r\mu \circ (\epsilon_k, \epsilon_k(r)^{-1})).$$

Now, it only remains to notice, that as exp is a local diffeomorphism, the vertical projections are uniquely determined by their pull-backs to $A_{\mathcal{C}}$ and furthermore, that due to exp being a diffeomorphism these are given by

$$\sum_{0 \le k \le f} \lambda_k \cdot (T_r\widehat{\epsilon_k} - \widehat{\epsilon_k}(r))).$$

On this expression we can now apply $\mathbb{R}$-linearity of the tangent map as well as the definition of ϵ to obtain that indeed

$$\sum_{0 \le k \le f} \lambda_k \cdot (T_r\widehat{\epsilon_k} - \widehat{\epsilon_k}(r))) = T_r\widehat{\epsilon} - \widehat{\epsilon}(r)),$$

meaning that at r the connection on $\widehat{\imath}$ induced by $\mu \circ (\gamma, \epsilon)$ agrees with E, and in particular, $\mu \circ (\gamma, \epsilon)$ is E-horizontal at r. As r was chosen arbitrarily, this holds for all of R_1 and in particular, E is flat. $\square$

Corollary 5.2.24. *Let G be a proper and regular Lie groupoid. Consider two admissible embedded manifolds $R_1 \subset G_0$ and $R_2 \subset G_0$. Assume that $R_1 \cup R_2$ is an embedded manifold of G_0 (i.e. R_1 and R_2 are disjoint or intersect on an open). Let $\{E^{(k)}\}_{0 \le k \le f}$ be a family of multiplicative Ehresmann connections on $G|_{R_1 \cup R_2}$ extending the two flat connections E_1 on $G|_{R_1}$ and E_2 on $G|_{R_2}$, such that their restriction to $t^{-1}(R_1) \cap s^{-1}(R_2)$ is flat as well. Then the same properties hold for the result E of averaging over the $E^{(k)}$ in the sense of Lemma 5.2.13 using any weighting $\{\lambda_k \in \mathbb{R}\}$ as a partition of unity.*

Proof. As any $E^{(k)}$ is flat, we have that any $\widehat{i}$ admits $E^{(k)}$-horizontal sections locally. Restricting to sufficiently small admissible open subsets of R_1 (which we can do according to Lemma 1.4.37) shows that the construction of E on $\widehat{i}$ according to Lemma 5.2.13 actually agrees with the construction of E on $\widehat{i}$ according to Lemma 5.2.23, which is indeed flat. □

We can now relate this fact to the situation we are actually interested in: The cover constructed in Chapter 1. Our goal in this section is to construct a multiplicative Ehresmann connection on the entire groupoid, but our intermediate goal will be to construct a flat multiplicative Ehresmann connection on $G|_{R_n}$.

Lemma 5.2.25. *Let G be a proper and regular groupoid. Consider two admissible embedded manifolds R_1 and R_2 as well as a connected components i of*

$$\underline{t}^{-1}(R_2) \cap \underline{s}^{-1}(R_1).$$

Then there is a unique connected component i^{-1} of

$$\underline{t}^{-1}(R_1) \cap \underline{s}^{-1}(R_2).$$

consisting of precisely

$$i^{-1} := \{x^{-1} \text{ for } x \in i\}.$$

Proof. We have that $(\bullet)^{-1}$ is its self-inverse and exchanges $\underline{s}$ and $\underline{t}$, such that it induces an isomorphism

$$\underline{t}^{-1}(R_2) \cap \underline{s}^{-1}(R_1) \cong \underline{t}^{-1}(R_1) \cap \underline{s}^{-1}(R_2),$$

which in particular maps connected components to connected components, which yields the statement. □

Lemma 5.2.26. *Let G be a proper and regular groupoid. Consider three admissible embedded manifolds R_1, R_2 and R_3 as well as a connected components i of*

$$\underline{t}^{-1}(R_3) \cap \underline{s}^{-1}(R_2)$$

and a connected component i'

$$\underline{t}^{-1}(R_2) \cap \underline{s}^{-1}(R_1).$$

Assume that $\underline{s}$ restricts to an isomorphism on i and furthermore both $\underline{s}$ and $\underline{t}$ restrict to an isomorphism on i'. Then the inclusion of sets

$$\mu(i, i') := \{\mu(x, y) \text{ for } x \in i \text{ and } y \in i' \text{ with } \underline{s}(x) = \underline{t}(y)\} \subset \underline{t}^{-1}(R_3) \cap \underline{s}^{-1}(R_1)$$

is an inclusion of an open subset on which $\underline{s}$ restricts to an isomorphism. In particular, any connected subset of $\underline{s}(\mu(i, i'))$ is associated to a unique connected component $i'' \in \underline{t}^{-1}(R_3) \cap \underline{s}^{-1}(R_1)$. Furthermore, if $R_1 = R_3$ and $i' = i^{-1}$ as defined in Lemma 5.2.25, we obtain that $\mu(i, i') = u_1(\underline{t}(i))$.

Proof. First off, consider $V = \underline{s}(i) \cap \underline{t}(i')$. As both maps are diffeomorphisms onto their images, we have that the images respectively are open, so V is open as well. Now consider $U = \underline{s}(\underline{t}^{-1}(V))$, which by the analogous argument is open as well. We claim that $\mu(i, i')$ is diffeomorphic to U through the restriction of $\underline{s}$. As $\underline{s}$ is a smooth map, it suffices to show that $\underline{s}$ is an isomorphism of sets as well as that there are locally defined smooth inverses. The existence of these inverses also proves that their joint image, i.e. $\mu(i, i')$ is open as well.

Let us start with the fact that $\underline{s}$ is an isomorphism of sets. By construction, we have $\underline{s}(\mu(x,y)) = \underline{s}(y)$. This uniquely determines y as well as $\underline{t}(y)$ by the assumption that $\underline{s}$ and $\underline{t}$ are isomorphisms on i'. However, this in turn determines $\underline{s}(x)$ and therefore x, so we can recover $\mu(x,y)$ from its image under $\underline{s}$ proving injectivity. Surjectivity follows by the analogous argument, as by construction, we have that both $y \in i' \cap \underline{s}^{-1}(U) = i' \cap \underline{t}^{-1}(V)$ and $x \in i \cap \underline{s}^{-1}(V)$ are well-defined for any point of U.

Now to construct the local inverses, consider $\mu(x,y) \in \underline{t}^{-1}(R_3) \cap \underline{s}^{-1}(R_1)$. We have that as $\underline{s}$ restricts to an isomorphism on both i and i', any local sections γ_x and γ_y of $\underline{s}$ respectively around x and y defines an inverse of $\underline{s}$. We can now consider

$$\mu \circ (\gamma_x, \gamma_y).$$

By construction, this is a section of $\underline{s}$ with image in $\mu(i,i')$, so it must automatically be a local inverse to $\underline{s}$.

In the case where $i' = i^{-1}$, we can review the procedures of this proof to find that $V = \underline{s}(i) \cap \underline{t}(i^{-1}) = \underline{t}(i^{-1})$ and

$$U = \underline{s}(\underline{t}^{-1}(\underline{t}(i^{-1}))) = \underline{s}(i^{-1}) = \underline{t}(i).$$

We also know that for any $x \in i$ we have $x^{-1} \in i^{-1}$ and therefore $1_{\underline{t}(x)} \in \mu(i,i^{-1})$. So as $\underline{s}$ restricts to an isomorphism on $\mu(i,i^{-1})$ and $u_1(\underline{t}(i))$ is a subset on which $\underline{s}$ is already surjective, we must have

$$\mu(i,i^{-1}) = u_1(\underline{t}(i)).$$

□

Lemma 5.2.27. *Let G be a proper and regular Lie groupoid and H the rigid isotropy subgroupoid as in Definition 5.2.17 and H/G the associated groupoid as constructed in Lemma 5.2.16. Consider the construction of Proposition 1.5.9. For the admissible cover constructed that way there exists*

- *a family of flat multiplicative Ehresmann connections $E'|_{R_n}$ on $H|_{R_n}$.*
- *a family of local s-sections $\{\gamma_{(n,n',i)} : U \subset R_n \to s^{-1}(R_n) \cap t^{-1}(R_{n'})\}_{(n,n',i)\in I}$ defined on the index set I of all tuples (n,n',i) where i denotes a connected component of the intersection $\underline{s}^{-1}(R_n) \cap \underline{t}^{-1}(R_{n'})$ in (G/H).*

These families may be chosen such that

- *We have*
$$\gamma_{(n,n,1_{R_n})} = 1|_{R_n}.$$
- *We have*
$$\gamma_{(n,n,i)} : R_n \to G_1.$$
- *All maps $\mathrm{pr}_{(G/H)} \circ \gamma_{(n,n',i)}$ induce an isomorphism between their domain U and i.*
- *All maps $t \circ \gamma_{(n,n',i)}$ are diffeomorphisms between their domain and image.*
- *The induced isomorphism*
$$\widetilde{\mathrm{Ad}}^H_{\gamma_{(n,n',i)}} : H|_{V\subset R_{n'}} \to H|_{U\subset R_n}$$
maps $E'|_{R_{n'}}$ to $E'|_{R_n}$.

- *We have that*
$$\mu \circ \left(\gamma_{(n,n',i)}, \gamma_{(n',n,i^{-1})}\right)$$
is horizontal with respect to $E'|_{R_n}$.

Proof. First of all, note that by construction, we have that each R_n comes with a trivialisation of $G|_{R_n}$ compatible with the multiplicative structure, as this is true for every $\widehat{R}_m$ when constructing the cover, so also for subsets and by Lemma 1.4.32 also for the actual R_n (this last isomorphism between $G|_{R_n}$ and $G|_{R'_n}$ is not canonical, but for every $n \in \mathbb{N}$ we can fix one for the remainder of this proof). This in particular implies a trivialisation of $H|_{R_n}$ as a Lie group bundle, which induces a flat multiplicative Ehresmann connection $E'|_{R_n}$. Also we can choose constant sections for all $(n, n, i) \in I$ on all of R_n according to the trivialisation, which are compatible with $H|_{R_n}$ in the required way, where we choose $\gamma_{(n,n,1_{R_n})} = 1|_{R_n}$ which is always trivial independently of the connection. Concatenating sections on inverse components is then also $E'|_{R_n}$-horizontal and so is the composition

$$\mu \circ \left(\gamma_{(n,n,i')} \circ \gamma_{(n,n,i)}, \left(\gamma_{(n,n,\mu(i',i))}\right)^{-1}\right).$$

We will now cover the case of a component where $n \neq n'$. We will distinguish between two cases: R_n and $R_{n'}$ are added simultaneously to the cover (i.e. both are constructed from the same $\widehat{R_m}$ and U_m) or R_n and R_m arise from different $\widehat{R_m}$ and U_m.

Consider n and $n' \neq n$ for which the associated R_n and $R_{n'}$ are added simultaneously to the cover. In the construction, the sets are indexed by points of $\widehat{R_k}$ which we call m and m' respectively. We may construct the associated $\gamma_{(n,n',i)}$ for every (n, n', i) by considering the isomorphism

$$G|_{R_m} \cong G|_{R'_m} \subset G|_{\widehat{R_k}}$$

where we, as previously detailed, have a trivialisation and in particular a flat connection on $G|_{\widehat{R_k}}$, for which any element can be extended to a flat s-section on all of $\widehat{R_k}$. Note that as $(G/H)|_{\widehat{R_k}}$ is trivial as well, s restricts to an isomorphism onto $\widehat{R_k}$ on any connected component and any s-section therefore also induces an (inverse) isomorphism onto its corresponding component. In particular, the same is true for the open subset $i' \in \underline{s}^{-1}(R'_m) \cap \underline{t}^{-1}(R'_{m'}) \subset (G/H)|_{\widehat{R_k}}$, where we choose i' corresponding to i under the previous isomorphisms. This means that indeed, when extending an element of $\widehat{i'}$ to a horizontal s-section, we obtain that the restriction to $s(\widehat{i})$ induces an isomorphism onto i'.

We have that these $\gamma_{(n,n',i)}$ automatically satisfy almost all the properties we require in the lemma, as the trivialisations of $G|_{R_n}$ and $G|_{R_{n'}}$ defining $E'|_{R_n}$ and $E'|_{R_{n'}}$ are precisely chosen to correspond to the trivialisation of $G|_{\widehat{R_k}}$.

For every n and n' where n is added later than n', we have that R_n was chosen such that $\overline{R_n}$ was either disjoint to $\mathrm{Sat}(\overline{R_{n'}})$ or contained in $\mathrm{Sat}(\widehat{R}_k)$, where $\widehat{R}_k$ is the admissible manifold from which $R_{n'}$ was constructed. However, in the first of these cases there is no $(n, n', i) \in I$.

In the second case, we claim that $\underline{s}^{-1}(R_n) \cap \underline{t}^{-1}(R_{n'}) \subset G/H$ has a useful structure induced by the inclusion

$$\underline{s}^{-1}(R_n) \cap \underline{t}^{-1}(R_{n'}) \subset \underline{s}^{-1}(R_n) \cap \underline{t}^{-1}(\widehat{R}_k).$$

We firstly can use the assumption that there is an s-section γ'_m inducing a diffeomorphism of Q'_m with an open subset of $\widehat{R}_k$ to construct an s-section inducing a diffeomorphism of R_n with the corresponding $O \subset \widehat{R}_k$. This immediately implies that

$$s^{-1}(R_n) \cap t^{-1}(R_{n'}) \cong s^{-1}(O) \cap t^{-1}(R_{n'})$$

and

$$s^{-1}(R_n) \cap t^{-1}(\widehat{R}_k) \cong s^{-1}(O) \cap t^{-1}(\widehat{R}_k),$$

where the above inclusion corresponds to

$$s^{-1}(O) \cap t^{-1}(R_{n'}) \subset s^{-1}(O) \cap t^{-1}(\widehat{R}_k).$$

Due to the trivialisation of $G|_{\widehat{R}_k}$ and therefore $G/H|_{\widehat{R}_k}$ we have that under projection to G/H both $\underline{s}^{-1}(O) \cap \underline{t}^{-1}(\widehat{R}_k)$ and $\underline{s}^{-1}(R_n) \cap \underline{t}^{-1}(\widehat{R}_k)$ are trivial discrete covers of O and R_n with respect to $\underline{s}$. Here, we are using that the isomorphism is H-equivariant under the right action of H, as left and right multiplication commute. The component containing the identities in $\underline{s}^{-1}(O) \cap \underline{t}^{-1}(\widehat{R}_k)$ corresponds to the image component of γ'_m in $\underline{s}^{-1}(R_n) \cap \underline{t}^{-1}(\widehat{R}_k)$, showing in particular that $\mathrm{pr}_{G/H} \circ \gamma'_m$ is an isomorphism onto its image.

However, we have that this isomorphism is not necessarily compatible with the flat multiplicative Ehresmann connection of $H|_{R_n}$ and $H|_O$. Here, the connection on $H|_{R_n}$ is induced by the restriction of the trivialisation of

$$G|_{R'_n} \subset G|_{\widehat{R}_k}$$

and the isomorphism $G|_{R_n} \cong G|_{R'_n}$ as in Lemma 1.4.32 and the flat multiplicative Ehresmann connection on $H|_O$ is induced by restricting the trivialisation of $G|_{\widehat{R}_k}$. However, as R_n is simply connected, we can apply the method of Lemma 5.2.21 to replace γ'_m with an s-section γ_m for which $\widetilde{\mathrm{Ad}}^H_{\gamma_m,1}$ does retain the Ehresmann connections (and therefore also the trivialisations) on $H|_{R_n}$ and $H|_O$.

We have already noted that $G|_{\widehat{R}_k}$ is a trivial bundle with respect to s, where each connected component induces a unique diffeomorphism of $\widehat{R}_k$ onto itself. This implies that $s^{-1}(O) \cap t^{-1}(\widehat{R}_k)$ has the same structure as a trivial bundle with respect to s and furthermore, each component induces a diffeomorphism of O onto another open of $\widehat{R}_k$. The same holds true in G/H. This implies that for every (n, n', i) we can understand i as a connected open subset of a connected component of

$$\underline{s}^{-1}(R_n) \cap \underline{t}^{-1}(\widehat{R}_k),$$

and in particular both s and t are diffeomorphisms onto their image. We may choose an s-section to some corresponding component of $s^{-1}(R_n) \cap t^{-1}(\widehat{R}_k)$ that is constant with respect to the previously described trivialisation, and as its projection to G/H is an inverse to $\underline{s}$ (both left and right inverse), its restriction to $U = \underline{s}(i)$ must automatically project to an inverse to the restriction of $\underline{s}$ to i. We will define it to be $\gamma_{(n,n',i)} : U \subset R_n \to G_1$. It is immediate that its image under t agrees with the image of its projection under $\underline{t}$ and hence is contained in $R_{n'}$. Note, that it induces a diffeomorphism onto the image of i under $\underline{t}$ as the restriction of a diffeomorphism onto its image.

As all such sections $\gamma_{(n,n',i)} : U \subset R_n \to G_1$ are bisections by our previous considerations we have that their inverses are well-defined sections that can be chosen as $\gamma_{(n',n,i^{-1})}$. This construction immediately implies that the concatenation

$$\mu \circ \left(\gamma_{(n,n',i)}, \gamma_{(n',n,i^{-1})}\right) \equiv u_1$$

is horizontal with respect to any multiplicative Ehresmann connection (analogously for the reverse order). Through this process, we can furthermore also cover the case where n is added before n' from the previous construction.

□

Lemma 5.2.28. *Let G be a proper and regular Lie groupoid and consider $\{R_n\}_{n\in\mathbb{N}_0}$ (or $\{R_n\}_{0\leq n\leq f}$) an admissible cover by connected and contractible admissible embedded manifolds satisfying that the sets $\overline{R_n}$ are pairwise disjoint. Denote by H the rigid isotropy subgroupoid of G of Definition 5.2.17. Assume that there exists a family $\{E'_n\}_{n\in\mathbb{N}_0}$ where E'_n is a flat multiplicative Ehresmann connection on $H|_{R_n}$ (with respect to the isotropy).*

Furthermore assume that there exists a family

$$\{\gamma_{(n,n',i)} : U \subset R_n \to s^{-1}(R_n) \cap t^{-1}(R_{n'})\}_{(n,n',i)\in I},$$

over the index set I over all tuples (n, n', i) where i denotes a connected component of the intersection $\underline{s}^{-1}(R_n)\cap\underline{t}^{-1}(R_{n'})$ in (G/H) such that the following assumptions are satisfied:

- *for every $\gamma_{(n,n',i)}$ we have that*
$$\mathrm{pr}\circ\gamma_{(n,n',i)} : U \to \underline{s}^{-1}(R_n) \cap \underline{t}^{-1}(R_{n'}) \subset (G/H)$$
is an isomorphism to i
- *we have*
$$\gamma_{(n,n,1_{R_n})} = 1|_{R_n}$$
- *all maps $t\circ\gamma_{(n,n',i)}$ are diffeomorphisms and the respective induced isomorphism onto its image given by*
$$\widetilde{\mathrm{Ad}}^{H}_{\gamma_{(n,n',i)}} : H|_U \to H|_{R_{n'}}$$
pulls back $E'_{n'}$ to $E'_n|_U$
- *both*
$$\mu\left(\gamma_{(n,n',i)}, \left(\gamma_{(n',n,i^{-1})}\right)^{-1}\right)$$
and
$$\mu\left(\gamma_{(n,n,i')}\circ\gamma_{(n,n,i)}, \left(\gamma_{(n,n,i'')}\right)^{-1}\right)$$
are horizontal with respect to $E'|_{R_n}$ for any i and i' and $i'' = \mu(i, i')$, assuming this is well-defined.

Then there exists a multiplicative Ehresmann connection E on G with respect to the isotropy and an admissible refinement $\{S_n \subset R_n\}_{n\in\mathbb{N}_0}$ such that $E|_{S_n}$ is flat on $G|_{S_n}$ and agrees with $E'_n|_{S_n}$ on $H|_{S_n}$. Furthermore, there is a family $\{E^{(k)}\}_{k\in\mathbb{N}}$ of multiplicative Ehresmann connections defined on $G|_{\mathrm{Sat}(S_k)}$ respectively, where each is flatly compatible with the admissible cover generated by the $S_n \cap \mathrm{Sat}(S_k)$. Furthermore, they satisfy that $E^{(k)}|_{S_n} \equiv E|_{S_n}$ for any $n \in \mathbb{N}$.

Proof. This proof will be quite long and technical. It can be subdivided into the following steps:

1. Construct a family of flat multiplicative Ehresmann connections E_n on $G|_{R_n}$ extending E'_n on $H|_{R_n}$.
2. Construct a family of flat multiplicative Ehresmann connections $\widetilde{E}^{(k)}$ on $G|_{\cup R_n\cap\mathrm{Sat}(R_k)}$ extending $E_n|_{R_n\cap\mathrm{Sat}(R_k)}$ on $G|_{R_n\cap\mathrm{Sat}(R_k)}$.
3. Construct an admissible cover by admissible embedded manifolds $\{S_n \subset R_n\}_{n\in\mathbb{N}_0}$ and a family of multiplicative Ehresmann connections $E^{(k)}$ on $G|_{\mathrm{Sat}(R_k)}$, which on $G|_{\cup S_n\cap\mathrm{Sat}(R_k)}$ extend $\widetilde{E}^{(k)}$.

4. Construct a multiplicative Ehresmann connection E on G such that E agrees with E_n on $G|_{S_n}$ for any $n \in \mathbb{N}$.

Step 1:

We aim to construct a flat multiplicative Ehresmann connection E_n on $G|_{R_n}$ extending E'_n on $H|_{R_n}$. For this, consider some connected component i of $(G/H)|_{R_n}$ and its preimage $\widehat{i}$ under the projection $G \to G/H$. Note that as each R_n is admissible the map $\underline{s} : (G/H)|_{R_n} \to R_n$ is a cover (as $(G/H)|_{R_n}$ is étale by Lemma 5.2.16 and $\underline{s}$-proper by construction), and as R_n is connected and contractible, the cover is trivial and each i and hence also each $\gamma_{(n,n,i)}$ will induce a diffeomorphism of R_n.

By Lemma 5.2.18, we have then that the $\gamma_{(n,n,i)}$ identifies $\widehat{i}$ with $H|_{R_n}$, meaning we can define the continuation of E'_n to $\widehat{i}$, which is the unique connection on $\widehat{i}$ compatible with the action induced by left multiplication with elements of E'_n where $\gamma_{(n,n,i)}$ is horizontal. By the invariance of E'_n under the adjoint action of $\gamma_{(n,n,i)}$ we find that the connection is also compatible with the action induced by right multiplication with elements of E'_n. We can do this for every connected component i and every induced $\widehat{i}$ to obtain a connection with respect to the isotropy on all of $G|_{R_n}$.

As we have that both results of multiplication and inverses of the $\gamma_{(n,n,i)}$ only differ from the corresponding element of the family by a section that is E'_n-horizontal, we have that the connection on $G|_{R_n}$ is actually a multiplicative Ehresmann connection E_n with respect to the isotropy.

Step 2:

Again, we can follow the use the induced identification of Lemma 5.2.18 to transfer the connection $E'_{n'}$ on $H|_{U_{(n',n,i)}}$ to $\widehat{i}$. Equivalently, we could also transfer the connection E'_n on $H|_W$ to $\widehat{i}$, where $W \subset R_n$ is the open given by the image of $t \circ \gamma_{(n',n,i)}$. This is the construction analogous to Corollary 5.2.22, as $\gamma_{(n',n,i)}$ is compatible with E'_n and $E'_{n'}$. The result is the unique E'_n-equivariant connection for which $\gamma_{(n',n,i)}$ is horizontal.

We can furthermore extend the isomorphism

$$H|_W \cong \widehat{i}$$

of Lemma 5.2.18 to an isomorphism

$$G|_{R_n} \cap s^{-1}(W) \cong t^{-1}(R_n) \cap s^{-1}(U_{(n',n,i)}).$$

This isomorphism induces a unique E_n-equivariant connection $E_{(n',n,i)}$ on the entire space $t^{-1}(R_n) \cap s^{-1}(U_{(n',n,i)})$ in a similar way, which is characterised by the fact that $\gamma_{(n',n,i)}$ is horizontal. It can equivalently be characterised by the fact that $\gamma_{(n,n,i')} \circ \gamma_{(n',n,i)}$ is horizontal for any $i' \in (G/H)|_{R_n}$, (where we implicitly use the fact that $(G/H)|_{R_n}$ is a trivial cover of R_n with respect to s and the domain of $\gamma_{(n,n,i')}$ is therefore all of R_n). We will denote the connection it as $E_{(n',n,i)}$.

Furthermore we have that if

$$U_{(n',n,i)} \cap U_{(n',n,i')} \neq \emptyset \subset R_{n'}$$

for any other i', then the intersection of $\mu(i, (i')^{-1})$ with the preimage of any connected component of $U_{(n',n,i)} \cap U_{(n',n,i')}$ under $\underline{s}$ gives some component i'' of $(G/H)|_{R_n}$. That $(G/H)|_{R_n}$ is a trivial cover of R_n via $\underline{s}$ implies that this component must admit a section $\gamma_{(n,n,i'')}$ and that this section induces a diffeomorphism on

$$\underline{t}^{-1}(R_n) \cap \underline{s}^{-1}(R_{n'}).$$

As we know that any diffeomorphism maps connected components isomorphically to connected components, we have that i and i' now correspond to each other. However, as this isomorphism is induced by left multiplication, it leaves the source of an element unchanged and we subsequently find that

$$U_{(n',n,i)} = U_{(n',n,i')}.$$

In particular, the spaces $t^{-1}(R_n) \cap s^{-1}(U_{(n',n,i)})$ and $t^{-1}(R_n) \cap s^{-1}(U_{(n',n,i')})$ also agree.

However, choosing $\gamma_{(n',n,i')}$ instead of $\gamma_{(n',n,i)}$ to construct a connection on the space $t^{-1}(R_n) \cap s^{-1}(U_{(n',n,i)})$, will not yield the same connection as the previously described $E_{(n',n,i)}$. Recall that the connection $E_{(n',n,i)}$ could be characterised by the fact that $\gamma_{(n',n,i)}$ was a horizontal map. Instead, the construction will now yield a connection $E_{(n',n,i')}$, where $\gamma_{(n,n,\mu(i,(i')^{-1}))} \circ \gamma_{(n',n,i')}$ is a horizontal section from $U_{(n',n,i)}$ to $\widehat{i}$.

We know by Lemma 5.2.23 that averaging over the connections arising for the different choices while retaining flatness is possible if they induce the same adjoint action between $H|_{U_{(n',n,i)}}$ and $H|_{R_n}$, but this condition is one of our assumptions on the family of sections. So we may average over all the connections $E_{(n',n,i')}$ we obtain on $t^{-1}(R_n) \cap s^{-1}(U_{(n',n,i)})$, as there are only finitely many components of $(G/H)|_{R_n}$ due to s-properness. The result is not only E'_n-invariant, but also E_n-invariant: Firstly, all the connections we consider satisfy by assumption that they are E'_n-invariant. The result is E_n-invariant by construction if it is also retained by left-multiplication with any $\gamma_{(n,n,j)}$. However, this will only permute the indices of the averaging procedure, as we can by associativity first compose

$$\mu \circ \left(\gamma_{(n,n,j)}, \gamma_{(n,n,i')}\right)$$

which by assumption agrees with $\gamma_{(n,n,\mu(j,i'))}$ up to a section horizontal in E'_n, meaning the bisections induce the same E'_n-invariant connection.

As we have seen that the connected components of $\underline{t}^{-1}(R_n) \cap \underline{s}^{-1}(R_{n'})$ can be all associated by their image under the source to some $\underline{t}^{-1}(R_n) \cap \underline{s}^{-1}(U_{(n',n,i)})$ for a disjoint collection of $U_{(n',n,i)}$, we find that the collection of all these results of glueing the $E_{(n',n,i')}$ for all components i of $\underline{t}^{-1}(R_n) \cap \underline{s}^{-1}(R_{n'}) \subset (G/H)_1$ already induces a connection on $\underline{t}^{-1}(R_n) \cap \underline{s}^{-1}(R_{n'})$. However, we want to incorporate one additional structure.

Similarly to previous considerations for $(G/H)|_{R_n}$, we have that the components of $(G/H)|_{R_{n'}}$ induce a group action on the components of $t^{-1}(R_n) \cap s^{-1}(R_{n'})$ in G/H. In particular, composition with $\gamma_{(n',n',i'')}$ induces an isomorphism on $\underline{t}^{-1}(R_n) \cap \underline{s}^{-1}(R_{n'})$, but it does not necessarily retain the Ehresmann connection resulting from the previous construction. Instead, if i is a connected component of $\underline{t}^{-1}(R_n) \cap \underline{s}^{-1}(R_{n'}) \subset (G/H)_1$ the connection on its preimage $\widehat{i}$ is per that construction given by averaging (indexed by i') over all those E'_n-invariant connections for which $\mu \circ (\gamma_{(n,n,i')}, \gamma_{(n',n,\mu((i')^{-1},i))})$ is horizontal. Under the isomorphism induced by $\gamma_{(n',n',i'')}$ we have that $\widehat{i}$ has its preimage given by $\widehat{\mu(i,(i'')^{-1})} \subset t^{-1}(R_n) \cap s^{-1}(U_{(n',n,i)})$ (which is in this case well-defined) and we obtain the push-forward of the connection to $\widehat{i}$ which is given by averaging (indexed over i') over all those E'_n-invariant connections for which $\mu \circ (\gamma_{(n,n,i')}, \gamma_{(n',n,\mu((i')^{-1},i,(i'')^{-1}))}, \gamma_{(n',n',i'')})$ is horizontal.

We can take this into consideration by averaging a second time over the finitely many components of $(G/H)|_{R_{n'}}$ (indexed by i'') or equivalently averaging not over the connections $E_{(n',n,i')}$ (i.e. where $\mu \circ \left(\gamma_{(n,n,i')}, \gamma_{(n',n,\mu((i')^{-1},i)}\right)$ is horizontal), but instead over connections induced by horizontal sections of the form

$$\mu \circ \left(\gamma_{(n,n,i')}, \gamma_{(n',n,\mu((i')^{-1},i,(i'')^{-1}))}, \gamma_{(n',n',i'')}\right).$$

We will denote these as $E_{(n',n)}$. Again, the averaging is well-defined and its result is flat by Lemma 5.2.23. The chosen averaging procedure will automatically imply that the connections $E_{(n',n)}$ on $t^{-1}(R_n) \cap s^{-1}(R_{n'})$ will be both E_n-invariant and $E_{n'}$-invariant, in analogy to E_n-invariance earlier.

Now we have obtained connections $E_{(n',n)}$ on each $t^{-1}(R_n) \cap s^{-1}(R_{n'})$ compatible with E_n and $E_{n'}$. We can for fixed $k \in \mathbb{N}$ consider

$$t^{-1}(R_k) \cap G|_{\cup R_n} = \cup_{n \in \mathbb{N}_0} t^{-1}(R_k) \cap s^{-1}(R_n)$$

and as the components are disjoint, we obtain a connection $\widehat{E}^{(k)}|_{t^{-1}(R_k)}$. E_k-invariance allows us to apply Lemma 5.2.12 to obtain a connection on all of $G|_{\cup R_n \cap \mathrm{Sat}(R_k)}$, which we will denote as $\widetilde{E}^{(k)}$.

On R_n, we have that $\widetilde{E}^{(k)}|_{R_n \cap \mathrm{Sat}(R_k)}$ agrees with E_n, as the connection $\widetilde{E}^{(k)}|_{R_n \cap \mathrm{Sat}(R_k)}$ is precisely constructed from the connection on $t^{-1}(R_k) \cap s^{-1}(R_n)$, which was E_n-invariant.

Step 3:

Again, we will aim to use Lemma 5.2.12 to construct a multiplicative Ehresmann connection on $G|_{\mathrm{Sat}(R_k)}$. The connection on $G|_{R_k}$ is given by E_k, the connection on $t^{-1}(R_k)$ with respect to s has yet to be constructed. From the previous step, we have however obtained flat connections on each $t^{-1}(R_k) \cap s^{-1}(R_n)$ with respect to s for each $n \in \mathbb{N}_0$, so it would be desirable to extend these to a connection on all of $t^{-1}(R_k)$. Unfortunately, there is no general construction for this. However, we will describe a construction, which for any closed subset $C \subset R_n$ will describe a E_k-invariant connection E_C on $t^{-1}(R_k)$ that agrees with $\widetilde{E}^{(k)}|_{t^{-1}(R_k) \cap s^{-1}(R_n)}$ on an open around $t^{-1}(R_k) \cap s^{-1}(C)$. We will further describe how we can even alter this construction to with respect to an entire collection $\{C_n \subset R_n\}$ of closed subsets.

The construction mentioned works in two steps: First, we will consider only one closed set $C \subset R_n$ and describe how to construct a connection on an open subset of $t^{-1}(R_k)$ containing $t^{-1}(R_k) \cap s^{-1}(C)$ closely related to $\widetilde{E}^{(k)}$. Afterwards, we will use this connection to construct an extension to all of $t^{-1}(R_k)$. This second step (a simple glueing procedure) can be easily adjusted to take multiple connections for a specific collection of $\{C_n \subset R_n\}$ into consideration when extending to a connection $E^{(k)}|_{t^{-1}(R_k)}$ on $t^{-1}(R_k)$.

We will start by describing the first of these two steps: Given a $\widetilde{E}^{(k)}$-horizontal bisection $\gamma : U \to t^{-1}(R_k) \cap s^{-1}(R_n)$ defined on an open set U, we can extend it on an open set in $W \subset \mathrm{Sat}(R_k)$ with $U = W \cap R_n$ to a s-section with image in $t^{-1}(R_k)$ as s is a submersion and $R_n \cap \mathrm{Sat}(R_k)$ is an embedded manifold. So if we consider $\widetilde{E}^{(k)}$ on $t^{-1}(R_k) \cap s^{-1}(R_n)$, we can cover $R_n \cap \mathrm{Sat}(R_k)$ by finitely many open sets U_l small enough to admit such a bisection. From this procedure we obtain s-sections on some open sets $W_l \subset \mathrm{Sat}(R_k)$ and associated E_k-invariant connections on $t^{-1}(R_k) \cap s^{-1}(W_l)$ extending $\widetilde{E}$ on $t^{-1}(R_k) \cap s^{-1}(U_l)$. Without loss of generality we may assume that the cover $\{W_l\}$ (of $V = \cup W_l$, which is open in $\mathrm{Sat}(R_k)$) is locally finite, as we can otherwise choose a refinement. Choosing a partition of unity underlying the $\{W_l\}$ (as a cover of their union) always restricts to a partition of unity underlying the $\{U_l\}$, implying that glueing the E_k-invariant connections on $t^{-1}(R_k) \cap s^{-1}(W_l)$ defined by the extended sections results in an E_n-invariant connection on $t^{-1}(R_k) \cap s^{-1}(\cup W_l)$. Let us denote it by E_C. We have that it extends $\widetilde{E}^{(k)}$ on $t^{-1}(R_k) \cap s^{-1}(\cup U_l)$, i.e on $t^{-1}(R_k) \cap s^{-1}(R_n)$.

Now let us describe the second step, first in the case where we are only considering one $C \subset R_n$. In this case, we have just described a procedure to obtain a connection on some $t^{-1}(R_k) \cap s^{-1}(V)$, where V is an open containing R_n (specifically $V = \cup W_k$). To extend it to all of $t^{-1}(R_k)$, it only remains to choose an arbitrary E_n-invariant connection on $t^{-1}(R_k) \cap s^{-1}(G_0 \setminus C)$, which is open and then glue according to a partition of unity

underlying
$$t^{-1}(R_k) = t^{-1}(R_k) \cap s^{-1}(G_0 \setminus C) \cup t^{-1}(R_k) \cap s^{-1}(V).$$

Note that choosing an E_k-invariant connection on $t^{-1}(R_k) \cap s^{-1}(G_0 \setminus C)$ is possible similar to previous arguments for example by choosing local sections which define unique E_k-invariant connections locally and glueing or alternatively choosing an arbitrary connection and then averaging (over the fibres of H and the components of G/H).

The fact that the resulting connection on $t^{-1}(R_k) \cap s^{-1}(C)$ agrees with $\widetilde{E}|_{t^{-1}(R_k)\cap s^{-1}(R_n)}$ implies that the induced connection via the construction of Lemma 5.2.12 yields $\widetilde{E}^{(k)}$ on an open around
$$\mathrm{Sat}(R_k) \cap G|_C \subset \mathrm{Sat}(R_k) \cap G|_{R_n}$$
and analogously around $t^{-1}(C) \cap s^{-1}(R_k) \subset t^{-1}(R_n) \cap s^{-1}(R_k)$, i.e. we recover $\widetilde{E}^{(k)}$ on an open around
$$\mathrm{Sat}(R_k) \cap G|_{R_k \cup C} \subset \mathrm{Sat}(R_k) \cap G|_{R_k \cup R_n}.$$

We have now concluded in this second step, that for a closed subset $C \subset R_n$ we can extend $\widetilde{E}^{(k)}|_{t^{-1}(R_k)\cap s^{-1}(R_n)}$ to all of $t^{-1}(R_k)$ retaining the connection around $C \subset R_n$. This particular holds for the closed set
$$C_n = \mathrm{supp}(\chi_n) \cap R_n.$$

We can now describe the same procedure taking the collection of all the closed sets $\{C_l\}$ into account simultaneously: For each R_l we will obtain some V_l and a connection on $t^{-1}(R_k) \cap s^{-1}(V_l)$. Note that as the collection of $\{\mathrm{Sat}(R_l)\}$ is locally finite, we have that $\cup C_l$ is a locally finite union and hence still a closed set. So $\mathrm{Sat}(R_k) \setminus (\cup C_l)$ is open and together with the sets $V_l \setminus (\cup_{l' \neq l} C_{l'})$ covers $\mathrm{Sat}(R_k)$ entirely. For the latter sets we have constructed a connection on $t^{-1}(R_k) \cap s^{-1}(V_l)$ (which can simply be restricted), while for $t^{-1}(R_k) \cap s^{-1}(\mathrm{Sat}(R_k) \setminus (\cup_l C_l))$ we can again choose an E_k-invariant connection arbitrarily. By construction, the result when glueing with respect to a partition of unity underlying this cover will be an E_k-invariant connection which we denote as $E^{(k)}|_{t^{-1}(R_k)}$.

We will now apply Lemma 5.2.12 to E_k and $E^{(k)}|_{t^{-1}(R_k)}$, which as we have seen recovers $\widetilde{E}^{(k)}$ on each $G|_{R_k \cup (C_l \cap \mathrm{Sat}(R_k))}$, so in particular on
$$G|_{\cup S_n \cap \mathrm{Sat}(R_k)}$$
for
$$S_n = \{r \in R_n \text{ s.t. } \chi_n(r) > 0\}.$$

Now it remains to note that $\mathrm{Sat}(S_n)$ also defines an admissible cover of G_0. That it is indeed a cover is immediate: As χ_n is basic and $\mathrm{supp}(\chi_n) \subset \mathrm{Sat}(R_n)$, we have that $\mathrm{Sat}(S_n) = \{m \in G_0 \text{ s.t. } \chi_n(m) > 0\}$ and $\{\chi_n\}$ is a partition of unity, so at each point there must be one strictly positive function. As it is a subordinate subcover of $\{\mathrm{Sat}(R_n)\}_{n \in \mathbb{N}_0}$, we find that it is also locally finite. Furthermore, we have that by Lemma 1.3.4 $S_n \subset R_n$ is a saturated subset, meaning that by Lemma 1.4.31 the S_n are indeed admissible.

Step 4:

Lastly, it only remains to glue the $E^{(k)}$ according to the method of Corollary 5.2.15. We will do so inductively: Starting with $E^{(0)}$ on $U_0 = \mathrm{Sat}(S_0)$ and $E^{(1)}$ on $\mathrm{Sat}(S_1)$ we will apply Corollary 5.2.15 for a multiplicative Ehresmann connection $\overline{E}_1$ on $U_1 = U_0 \cup \mathrm{Sat}(S_1)$. When considering the proof of the statement, we obtain that the multiplicative Ehresmann connection arises from a glueing procedure as described in Lemma 5.2.13 with respect to S_1, however we obtain from Lemma 5.2.14 that on $S_n \cap \mathrm{Sat}(S_1)$ the resulting connection extends E_n on $G|_{S_n}$ for any $n \in \mathbb{N}$. On $S_n \cap (U_0 \cap \operatorname{Sat}(S_1))$, the same statement holds immediately as it was a prerequisite for $\overline{E}_0 = E^{(0)}$.

For the induction step, we analogously consider $\overline{E}_k$ on $U_k = \cup_{0 \le l \le k} \operatorname{Sat}(S_l)$ and $E^{(k+1)}$ on $\operatorname{Sat}(S_{k+1})$. Again, we will apply Corollary 5.2.15 for a multiplicative Ehresmann connection $\overline{E}_{k+1}$ on $U_{k+1} = U_k \cup \operatorname{Sat}(S_{k+1})$. We analogously obtain from Lemma 5.2.14 that on $S_n \cap \operatorname{Sat}(S_{k+1})$ the resulting connection extends E_n on $G|_{S_n}$ for any $n \in \mathbb{N}$ and by the induction hypothesis, we obtain the same outside of $\operatorname{Sat}(S_{k+1})$.

As the cover is locally finite and in each step we only change the multiplicative Ehresmann connection on $\operatorname{Sat}(S_k)$, we obtain a multiplicative Ehresmann connection E with respect to the isotropy as the limit of this construction. As in each step

$$\overline{E}_k|_{S_n} \equiv E_n$$

on the open $S_n \cap U_k$ where the left-hand expression is defined, the limit will indeed satisfy that

$$E|_{S_n} \equiv E_n.$$

□

Corollary 5.2.29. *Let G be a proper and regular Lie groupoid where the dimension of the orbit foliation $\mathcal{F}_0$ on G_0 is given by $l_0 \neq 0$. Then there exists*

- *an admissible cover induced by $\{R_n\}_{n\in\mathbb{N}_0}$ (or $\{R_n\}_{0\le n\le f}$)*
- *an induced open saturated cover $\{O_k = \operatorname{Sat}(R_k)\}_{k\in\mathbb{N}_0}$*
- *a multiplicative Ehresmann connection E on G with respect to the isotropy Lie algebroid*
- *a family of multiplicative Ehresmann connections $\{E^{(k)}\}_{k\in\mathbb{N}_0}$, where each $E^{(k)}$ is a multiplicative Ehresmann connection with respect to the isotropy on $G|_{O_k}$*
- *a family of multiplicative Ehresmann connections $\{E_n\}_{n\in\mathbb{N}_0}$, where each E_n is a multiplicative Ehresmann connection with respect to the isotropy on $G|_{R_n}$*

such that these objects satisfy the following:

- *E_n is a flat multiplicative Ehresmann connection for any $n \in \mathbb{N}_0$.*
- *We have that $E^{(k)}|_{R_n\cap O_k} = E|_{R_n\cap O_k}$ for each $n \in \mathbb{N}$ and $k \in \mathbb{N}$.*
- *On $G|_{O_k}$ we have that $E^{(k)}$ is flatly compatible with the cover induced by the admissible embedded manifolds $\{R_n \cap O_k\}_{n\in\mathbb{N}}$.*

In particular, the map

$$\Phi : H(\operatorname{Tot}(\Omega_{\mathrm{BSS}}(G))) \to H(\operatorname{Tot}(\mathcal{R}_E))$$

as constructed in Theorem 4.2.16 from the assignment of Lemma 5.2.8 is an isomorphism of $\mathbb{R}$-algebras.

Proof. Consider the cover constructed in Proposition 1.5.9. As $l_0 \neq 0$, we find that we may choose the R_n such that their closures are pairwise disjoint. By Lemma 5.2.27, we know that we may now assume the conditions of Lemma 5.2.28, which we can apply to construct the cover by the S_n and the flatly compatible Ehresmann connection E. Note that by Lemma 1.5.8, there is an admissible partition of unity underlying the cover $\{O_k\}_{k\in\mathbb{N}_0}$. By Proposition 5.2.11, this implies precisely that we obtain Φ, which is an $\mathbb{R}$-algebra isomorphism. □

Lemma 5.2.30. *Let G and be a proper and regular Lie groupoid where the orbits are discrete, i.e. the dimension l_0 of the orbit foliation $\mathcal{F}_0$ is given by $l_0 = 0$.*

Then there is a proper and regular Lie groupoid G' for which there is an admissible cover $\{R'_n\}_{n\in\mathbb{N}_0}$ (or $\{R'_n\}_{0\leq n\leq f}$) such that the $\overline{R'_n}$ are pairwise disjoint and satisfy there exists the data as constructed in Lemma 5.2.27.

Furthermore, there is a groupoid map $p : G' \to G$ consisting of local isomorphisms and $G \subset G'$ an open inclusion of groupoids, such that it is a section of p and the image consists of connected components.

The map p satisfies that for every multiplicative Ehresmann connection E on G there exists a multiplicative Ehresmann connection E' on G', such that the pull-back of a $J^E G$-invariant form is $J^{E'}G$-invariant and the induced map $p^* : (\Omega(G_q))^{J^E G} \to (\Omega(G'_q))^{J^{E'}G}$ is an isomorphism. Furthermore, this isomorphism maps $J^E G$-isotropy Cartan forms to $J^{E'}G$-isotropy Cartan forms and the induced map

$$p^* : \mathcal{R}_E \to \mathcal{R}_{E'}$$

is a morphism of dg-$\mathbb{R}$-algebras. The inverse map is given by the restriction of forms to $G \subset G'$.

In particular, if every $J^{E'}G'$-isotropy Cartan form closed under $\mathrm{d}_{\mathrm{Cart}}$ has a closed descending homogeneous extension, the same holds for every $J^E G$-isotropy Cartan form. If there is an assignment

$$\Psi' : \ker(\widehat{\mathrm{d}^{E'}_{\mathrm{Cart}}}) \subset \mathrm{Tot}(\mathcal{R}_{E'}) \to \mathrm{Tot}(\Omega_{\mathrm{BSS}}(G'))$$

mapping $J^E G$-Cartan forms to their closed descending homogeneous extensions it induces an assignment of the form

$$\Psi : \ker(\widehat{\mathrm{d}^{E}_{\mathrm{Cart}}}) \subset \mathrm{Tot}(\mathcal{R}_{E}) \to \mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$$

where if Ψ' is compatible with multiplication in the sense of Lemma 4.2.22, so is Ψ.

Proof. Let us first define G'. By Proposition 1.5.9, we can choose an admissible cover of G_0 with respect to a family $\{R_n\}_{n\in\mathbb{N}_0}$ (or $\{R_n\}_{0\leq n\leq f}$) of admissible embedded manifolds. By Lemma 5.2.27, it satisfies all the conditions of Lemma 5.2.28 up to the pairwise disjointness of the $\overline{R_n}$.

Now define G'_0 as the totally disjoint union

$$G_0 \cup (\cup_{n\in\mathbb{N}_0} R_n)\,.$$

This comes with a natural surjective submersion to G_0 along which we can pull back the groupoid structure. To be explicit, this yields G'_1 as the disjoint union

$$G_1 \cup \left(\cup_{n\in\mathbb{N}_0} s^{-1}(R_n) \cup t^{-1}(R_n)\right) \cup (\cup_{n,n'\in\mathbb{N}_0} s^{-1}(R_n) \cap t^{-1}(R'_n)).$$

Source and target maps, which we will denote by s' and t' are induced by the maps on G, where in the decomposition of G'_1 as described above the first component G_1 maps to G_0 under s' and t', the last component $(G|_{\cup R_n})_1$ maps to $\cup_{n\in\mathbb{N}_0} R_n$ under s' and t' and in the middle component $s^{-1}(R_n)$ maps to R_n under s' and to G_0 under t' while $t^{-1}(R_n)$ maps to R_n under t' and to G_0 under s'. The multiplication is induced by the one on G analogously.

The induced collection $\{R'_n\}_{n\in\mathbb{N}_0}$ where each R'_n is given by the connected component of G'_0 that contains the disjoint copy of R_n still satisfies the admissibility conditions, as

$G'|_{R'_n} \cong G|_{R_n}$. Furthermore, it is still a cover, as the saturation $\mathrm{Sat}(R'_n)$ contains R'_n itself (i.e. the component that is a disjoint copy of R_n) as well as $\mathrm{Sat}(R_n) \subset G_0$ by construction. Units are given by those in G_1 and those in $G|_{R_n} = s^{-1}(R_n) \cap t^{-1}(R_n)$.

From this construction we directly obtain that the canonical maps $G'_0 \to G_0$ and $G'_1 \to G_1$ assemble into a groupoid morphism $p : G' \to G$. On each connected component we can see that p is actually given by a map corresponding to the inclusion of an open subset and in particular a local isomorphism. The inclusion of $G_0 \subset G'_0$ similarly induces an inclusion $G \subset G'$ which is a right-inverse to p. The fact that its image consists of connected components follows immediately from the decomposition of G'_1 as described above.

As p is given by a local isomorphism on G'_1, we can define E' as the local pull-back of E. As the multiplication on G' is induced by that on G, we directly obtain that E' is multiplicative and we can similarly check that IG'_1 corresponds precisely to the pull-back of IG_1. Assume we have a $J^E G$-invariant form $\omega_{(p,q)} \in \Omega^p(G_q)$. Then $p^*\omega_{(p,q)} \in \Omega^p(G'_q)$ is a $J^{E'}G'$-invariant form, as the composition and hence the action of E' are compatible with p. The restriction of forms to G defines a left inverse to p^* and we claim that it is not only surjective but bijective, which would in particular imply that p^* is an isomorphism as well.

So assume that we have a $J^{E'}G'$-invariant form such that it restricts to 0 on $G \subset G'$. We know that the value at any point $z \in G_q$ of any $J^{E'}G'$-invariant form is determined by the value of the form at any other point in the same leaf as z by Lemma 2.1.38. However, $\mathrm{Sat}(G_0) = G'_0$ by construction, so indeed any leaf with respect to the foliation F_q on G'_q intersects G. So indeed, if a $J^{E'}G'$-invariant form vanishes on its restriction to the open $G_q \subset G'_q$, it vanishes everywhere. Hence both the restriction of $J^{E'}G'$-invariant forms to G and p^* actually describe inverse isomorphisms.

This immediately implies that if p^* and restriction induce isomorphisms between $J^E G'$-isotropy Cartan forms and $J^{E'}G'$-isotropy Cartan forms they retain the dg-$\mathbb{R}$-algebra structure - as multiplication is defined point-wise on the values at the units and the Cartan differential can be computed locally. Hence the result of either operation at a point $z \in G_q$ can be determined by its open neighbourhood $G_q \subset G'_q$. The fact that p^* actually does restrict follows as by construction E corresponds to E' locally under p, so multidegrees but also the action of the permutation group are retained by the pull-back.

So if we now assume that E' satisfies that all $J^{E'}G'$-isotropy Cartan forms closed under $\mathrm{d}_{\mathrm{Cart}}$ has a closed descending homogeneous extension and $\omega_{(p,q)}$ is a $J^E G$-isotropy Cartan form closed under $\widehat{\mathrm{d}^E_{\mathrm{Cart}}}$, we can by assumption choose a closed descending homogeneous extension for $p^*\omega_{(p,q)}$. If we denote it by ω, we find that $\omega|_G$ is a closed descending homogeneous extension of $\omega_{(p,q)}$. As we have previously seen that p^* and restriction retain the differential and algebra structure, it immediately follows that if Ψ' is an assignment describing the extension of $J^{E'}G'$-isotropy Cartan forms, then the induced map Ψ on $J^E G$-isotropy Cartan forms retains in particular the following properties:

- If Ψ' is linear, so is Ψ.
- If Ψ' maps cohomologically equivalent elements of $\mathrm{Tot}(\mathcal{R}_{E'})$ to cohomologically equivalent elements of $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G'))$, then Ψ maps cohomologically equivalent elements of $\mathrm{Tot}(\mathcal{R}_E)$ to cohomologically equivalent elements of $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$.
- If Ψ' satisfies that

$$\Psi'(\omega_{(p,q)} \,\check{\cdot}\, \nu_{(r,s)})$$

is cohomologically equivalent in $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$ to

$$\Psi'(\omega_{(p,q)}) \cup \Psi'(\nu_{(r,s)})$$

for any two $J^{E'}G$-isotropy Cartan forms $\omega_{(p,q)}$ and $\nu_{(r,s)}$, then Ψ satisfies that

$$\Psi(\omega_{(p,q)} \,\check{\cdot}\, \nu_{(r,s)})$$

is cohomologically equivalent in $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$ to

$$\Psi(\omega_{(p,q)}) \cup \Psi(\nu_{(r,s)})$$

for any two $J^{E}G$-isotropy Cartan forms $\omega_{(p,q)}$ and $\nu_{(r,s)}$.

□

Theorem 5.2.31. *Let G be a proper and regular Lie groupoid and E be a multiplicative Ehresmann connection on G. Then we have a graded isomorphism of $\mathbb{R}$-algebras*

$$\Phi : H^\bullet(\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))) \to H^\bullet(\mathrm{Tot}(\mathcal{R}_E)) \cong H^\bullet(\mathrm{Tot}(\mathfrak{R}_E)).$$

In particular, both $\mathrm{Tot}(\mathcal{R}_E)$ and $\mathrm{Tot}(\mathfrak{R}_E)$ are models that can be used to compute the cohomology of the differentiable stack associated to G as

$$H^\bullet_{\mathrm{DiffStack}}([G_0//G_1]) \cong H^\bullet(\mathrm{Tot}(\mathcal{R}_E)) \cong H^\bullet(\mathrm{Tot}(\mathfrak{R}_E)).$$

Proof. In Corollary 5.2.29 we have already shown that the claim holds if the dimension of the orbit foliation $\mathcal{F}_0$ on G_0 is given by $l_0 \neq 0$. So assume that $l_0 = 0$, then we can construct the Lie groupoid G' of Lemma 5.2.30 together with a cover which satisfies the conditions of Lemma 5.2.28. Applying this yields a subcover, a multiplicative Ehresmann connection E defined on G and the families of multiplicative Ehresmann connections $\{E^{(k)}\}$ (each defined on $G|_{\mathrm{Sat}(S_k)}$) and $\{E_n\}$ (each defined on $G|_{R_n}$). Note that by Lemma 1.5.8, there is an admissible partition of unity underlying the cover $\{\mathrm{Sat}(S_k)\}_{k\in\mathbb{N}_0}$. So this is precisely the data required in Proposition 5.2.11, meaning we obtain an assignment

$$\Psi^E : \ker(\widehat{\mathrm{d}^E_{\mathrm{Cart}}}) \subset \mathrm{Tot}(\mathcal{R}_E) \to \mathrm{Tot}(\Omega_{\mathrm{BSS}}(G'))$$

satisfying by Lemma 4.2.22 that

$$\Psi^E(\omega_{(p,q)} \,\check{\cdot}\, \nu_{(r,s)})$$

is cohomologically equivalent in $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G'))$ to

$$\Psi^E(\omega_{(p,q)}) \cup \Psi^E(\nu_{(r,s)})$$

for any two J^EG'-isotropy Cartan forms $\omega_{(p,q)}$ and $\nu_{(r,s)}$.

We can now choose any multiplicative Ehresmann connection F on G and apply 5.2.30 to construct a multiplicative Ehresmann connection F' with respect to the isotropy on G'. We have by Corollary 4.2.24 that Ψ^E induces an assignment

$$\Psi' : \ker(\widehat{\mathrm{d}^{F'}_{\mathrm{Cart}}}) \subset \mathrm{Tot}(\mathcal{R}_{F'}) \to \mathrm{Tot}(\Omega_{\mathrm{BSS}}(G'))$$

also satisfying the compatibility with multiplication. It then follows from Lemma 5.2.30, that we obtain an induced assignment

$$\Psi : \ker(\widehat{\mathrm{d}^{F}_{\mathrm{Cart}}}) \subset \mathrm{Tot}(\mathcal{R}_{F'}) \to \mathrm{Tot}(\Omega_{\mathrm{BSS}}(G'))$$

which furthermore satisfies

$$\Psi\big(\omega_{(p,q)} \breve{\cdot} \nu_{(r,s)}\big)$$

is cohomologically equivalent in $\mathrm{Tot}(\Omega_{\mathrm{BSS}}(G))$ to

$$\Psi\big(\omega_{(p,q)}\big) \cup \Psi\big(\nu_{(r,s)}\big)$$

for any two $J^F G$-isotropy Cartan forms $\omega_{(p,q)}$ and $\nu_{(r,s)}$ implying by Lemma 4.2.22 that the map Φ constructed from Ψ as in Proposition 4.2.16 is indeed an $\mathbb{R}$-algebra morphism. $\square$

Appendix A

Various References

A.1 Interchanging Rules

The following is a collection of all the computation rules (there are a lot of redundancies). Let us start by recalling the simplicial identities given by

$$d_i \circ s_j = \begin{cases} s_j \circ d_{i-1} & \text{if } i > j+1 \\ s_{j-1} \circ d_i & \text{if } i+1 \leq j \\ \mathrm{id} & \text{else.} \end{cases}$$

and

$$s_j \circ d_i = \begin{cases} d_{i+1} \circ s_j & \text{if } i > j \\ d_i \circ s_{j+1} & \text{if } i \leq j. \end{cases}$$

The identities between local multiplication and face maps are given by

$$d_i \circ D^l_{\gamma,q+1} = \begin{cases} D^l_{\gamma,q} \circ d_i & \text{if } i > l \\ D^{(l-1)}_{\gamma,q} \circ d_i & \text{if } i < l \\ d_i & \text{if } i = l, \end{cases}$$

and

$$D^l_{\gamma,q} \circ d_i = \begin{cases} d_i \circ D^l_{\gamma,q+1} & \text{if } i > l \\ d_i \circ D^{(l+1)}_{\gamma,q+1} & \text{if } i \leq l. \end{cases}$$

The identities with respect to the degeneracies are

$$s_j \circ D^l_{\gamma,q} = \begin{cases} D^l_{\gamma,q+1} \circ s_j & \text{if } j > l \\ D^{l+1}_{\gamma,q+1} \circ D^l_{\gamma,q+1} \circ s_j & \text{if } j = l \\ D^{(l+1)}_{\gamma,q+1} \circ s_j & \text{if } j < l, \end{cases}$$

and

$$D^l_{\gamma,q} \circ s_j = \begin{cases} s_j \circ D^l_{\gamma,q} & \text{if } j > l \\ s_j \circ D^{(l-1)}_{\gamma,q} & \text{if } j+1 < l. \end{cases}$$

These, together with various explicit calculations on $D^i_{x,q}$ and Ad^H_x will be used frequently in the thesis - however, many of the proofs are quite straightforward.

Corollary A.1.1 (previously: Corollary 1.2.32)**.** *Let G be a Lie groupoid. We have that*

$$(\dot{D}^0_{V,0})_* \circ (d_0)_* = (d_0)_* \circ (\dot{D}^1_{V,1})_*$$

on T_zG_1 whenever the base-point x of V satisfies $s(x) = s(z)$ and

$$(\dot{D}^0_{V,0})_* \circ (d_1)_* = (d_1)_* \circ (\dot{D}^0_{V,1})_*.$$

on T_zG_1 whenever the base-point x of V satisfies $s(x) = t(z)$.

Proof. We have noted in Lemma 1.1.29 that these equations hold for the corresponding Lie groupoid actions defining $D^i_{x,q}$, which we will denote by L^i_q. As by definition

$$(\dot{D}^i_{V,q})_*(X) = (TL^i_q)(Y, X)$$

and we have that

$$(TL^0_0)(Y, Td_0(X)) = Td_0((TL^1_1)(Y, X)),$$

we find that V acts via Y on both X (when considering L^1_1 with respect to the bundle $s : G_1 \to G_0$) and $Td_0(X)$ (considering L^0_0 with respect to the bundle $\mathrm{id} : G_0 \to G_0$.) □

Lemma A.1.2 (previously: Lemma 1.2.33)**.** *Let G be a Lie groupoid and $x \in G_1$. For any $q \in \mathbb{N}_0$ and $0 \leq l \leq q + 1$ as well as $0 \leq i \leq q + 1$ we have that on $(p^l_{q+1})^{-1}(\{s(x)\})$ holds that*

$$d_i \circ D^l_{x,q+1} = \begin{cases} D^l_{x,q} \circ d_i & \text{if } i > l \\ D^{(l-1)}_{x,q} \circ d_i & \text{if } i < l \\ d_i & \text{if } i = l. \end{cases}$$

Furthermore, for any $q \in \mathbb{N}_0$ with $q \geq 1$ and $0 \leq l \leq q$ as well as $0 \leq i \leq q + 1$ we have that on $(p^l_{q+1} \circ d_i)^{-1}(\{s(x)\})$ holds that

$$D^l_{x,q} \circ d_i = \begin{cases} d_i \circ D^l_{x,q+1} & \text{if } i > l \\ d_i \circ D^{(l+1)}_{x,q+1} & \text{if } i \leq l. \end{cases}$$

Also, for any $q \in \mathbb{N}_0$ and $0 \leq l \leq q$ as well as $0 \leq j \leq q$ we have that on $(p^l_q)^{-1}(\{s(x)\})$ holds that

$$s_j \circ D^l_{x,q} = \begin{cases} D^l_{x,q+1} \circ s_j & \text{if } j > l \\ D^{l+1}_{x,q+1} \circ D^l_{\gamma,q+1} \circ s_j & \text{if } j = l \\ D^{(l+1)}_{x,q+1} \circ s_j & \text{if } j < l. \end{cases}$$

Lastly, for any $q \in \mathbb{N}_0$ and $0 \leq l \leq q + 1$ as well as $0 \leq j \leq q$ we have that on $(p^l_q \circ s_j)^{-1}(\{s(x)\})$ holds that

$$D^l_{x,q+1} \circ s_j = \begin{cases} s_j \circ D^l_{x,q} & \text{if } j > l \\ s_j \circ D^{(l-1)}_{x,q} & \text{if } j + 1 < l. \end{cases}$$

Proof. Let us first consider the case $q \neq 0$. The face maps are then (mostly) given by multiplication of two adjacent elements of $G_q = G_1 \times_{G_0} \cdots \times_{G_0} G_1$. If the index of the face map agrees with the index of the action, we can compute the composition $d_i \circ D^i_{x,q+1}$ explicitly as

$$\left(z_1, \ldots, \mu(\mu(z_i, (x)^{-1}), \mu(x, z_{i+1})), \ldots, z_q\right) = \left(\ldots, \mu(z_i, z_{i+1}), \ldots\right).$$

If the indices i and l differ, it is immediately clear that the diagonal actions commute with the multiplication of adjacent elements by analogously writing out the formulas, but as the face maps decrease the number of factors of G_1, the index of the diagonal action will shift if $i < l$. Spelling this out also for the cases of d_0 and d_q, which are the cases we have not covered yet, follows the same procedure.

Similarly, as face maps correspond to an insertion of a unit, we have

$$s_j \circ D^l_{x,q} = \begin{cases} D^l_{x,q+1} \circ s_j & \text{if } j > l \\ D^{l+1}_{x,q+1} \circ D^l_{\gamma,q+1} \circ s_j & \text{if } j = l \\ D^{(l+1)}_{x,q+1} \circ s_j & \text{if } j < l, \end{cases}$$

where the description in case of $j = l$ comes from the fact that

$$\begin{aligned} &\left(\dots, \mu(z_i, (x)^{-1}), 1_{t(x)}, \mu\left(x, z_{i+1}\right), \dots\right) \\ =&(\mu(z_i, (x)^{-1}), \mu(x, 1_{s(x)}, (x)^{-1}), \mu(x, z_{i+1})) \end{aligned}$$

The remaining statements on $D^l_{x,q} \circ s_j$ can be obtained similarly. However, there are cases ($j = l$ and $j = l - 1$) where the expression can not be written down in a similar way as before. □

Corollary A.1.3 (previously: Corollary 1.2.34)**.** *Let G be a Lie groupoid and $\gamma \in \Gamma_{s,\mathrm{loc}}(G_0, G_1)$. For any $q \in \mathbb{N}_0$ and $0 \le l \le q+1$ as well as $0 \le i \le q+1$ we have that on the respective domains of definition it holds that*

$$d_i \circ D^l_{\gamma,q+1} \equiv \begin{cases} D^l_{\gamma,q} \circ d_i & \text{if } i > l \\ D^{(l-1)}_{\gamma,q} \circ d_i & \text{if } i < l \\ d_i & \text{if } i = l. \end{cases}$$

and on the respective domains of definition it holds that

$$D^l_{\gamma,q} \circ d_i \equiv \begin{cases} d_i \circ D^l_{\gamma,q+1} & \text{if } i > l \\ d_i \circ D^{(l+1)}_{\gamma,q+1} & \text{if } i \le l. \end{cases}$$

The statements on the relations between $\widetilde{D}^l_{\gamma,q}$ and s_j are analogous to those of Lemma 1.2.33.

Proof. As we have that for any point $z \in G_q$ in the domain of definition that $\widetilde{D}^l_{\gamma,q}(z) = D^l_{x,q}$ for the appropriate $x = \gamma(p^l_q(z))$, the relations of Lemma 1.2.33 are immediately transferable. □

Corollary A.1.4 (previously: Corollary 1.2.35)**.** *Let G be a Lie groupoid and $V \in J^1(s : G_1 \to G_0)$ with base-point $x \in G_1$. For any $q \in \mathbb{N}_0$ and $0 \le l \le q+1$ as well as $0 \le i \le q+1$ we have that on T_zG_q for z satisfying $p^j_q(z) = s(x)$ holds that*

$$(d_i)_* \circ (\dot{D}^l_{V,q+1})_* = \begin{cases} (D^l_{V,q})_* \circ (d_i)_* & \text{if } i > l \\ (D^{(l-1)}_{V,q})_* \circ (d_i)_* & \text{if } i < l \\ (d_i)_* & \text{if } i = l, \end{cases}$$

and on T_zG_q for z satisfying $p^j_q(d_i(z)) = s(x)$ holds that

$$(\dot{D}^l_{V,q})_* \circ (d_i)_* = \begin{cases} (d_i)_* \circ (D^l_{V,q+1})_* & \text{if } i > l \\ (d_i)_* \circ (D^{(l+1)}_{V,q+1})_* & \text{if } i \le l. \end{cases}$$

The statements on the relations between $(\dot{D}^l_{V,q})_*$ and s_j are analogous to those of Lemma 1.2.33.

Proof. We have that for any point $z \in G_q$ as described in the statement $\dot{D}^l_{V,q}(X)$ is given by

$$TL(Y, X)$$

where L describes the action defining $D^l_{x,q}$ and $Y \in V$. As the tangent functor is indeed a functor, we can therefore apply the results of Lemma 1.2.33. It only remains to note that in the process, we have not altered the acting element x, such that also Y remains unchanged when interchanging the maps. In particular, the property $Y \in V$ is retained such that we indeed can recover the expressions on the right hand side. □

Lemma A.1.5 (previously: Lemma 1.2.36)**.** *Let G be a Lie groupoid and $x \in G_1$. For any $q \in \mathbb{N}$ and $0 \leq i \leq q$ we have that on $(p^i_q)^{-1}(\{s(x)\})$ holds that*

$$p^i_q \circ D^i_{x,q} = D^0_{x,0} \circ p^i_q.$$

The analogous statements

$$p^i_q \circ \widetilde{D}^i_{\gamma,q} \equiv \widetilde{D}^0_{\gamma,0} \circ p^i_q$$

and

$$(p^i_q) \circ (\dot{D}^i_{V,q})_* = (\dot{D}^0_{V,0})_* \circ (p^i_q)_*$$

hold as well on the appropriate spaces.

Proof. If we consider any $z \in (p^i_q)^{-1}(\{s(x)\})$ and write

$$z = (z_1, \dots, z_q)$$

we can explicitly compute

$$D^i_{x,q}(z) = (z_1, \dots, \mu(z_i, x^{-1}), \mu(x, z_{i+1}), \dots)$$

and

$$p^i_q \left(D^i_{x,q}(z)\right) = t(x),$$

while by assumption $p^i_q(z) = s(x)$ and

$$D^0_{x,0}(s(x)) = t(x).$$

That the statements for the induced actions hold can be obtained in analogy to Corollary 1.2.34 and Corollary 1.2.35. □

Lemma A.1.6 (previously: Lemma 1.2.37)**.** *Let G be a Lie groupoid and $x \in G_1$ as well as $y \in G_1$. For any $q \in \mathbb{N}$ and $0 \leq i \leq q$ and for any $0 \leq j \leq q$ with $i \neq j$ we have that on $(p^i_q)^{-1}(\{s(x)\}) \cap (p^j_q)^{-1}(\{s(y)\})$ holds that*

$$D^j_{y,q} \circ D^i_{x,q} = D^i_{x,q} \circ D^j_{y,q}.$$

The analogue statements

$$\widetilde{D}^j_{\beta,q} \circ \widetilde{D}^i_{\gamma,q} \equiv \widetilde{D}^i_{\gamma,q} \circ \widetilde{D}^j_{\beta,q}.$$

and

$$(\dot{D}^j_{W,q})_* \circ (\dot{D}^i_{V,q})_* = (\dot{D}^i_{V,q})_* \circ (\dot{D}^j_{W,q})_*.$$

hold as well on the appropriate spaces.

Proof. If we consider any $z \in (p_q^i)^{-1}(\{s(x)\}) \cap (p_q^j)^{-1}(\{s(y)\})$ and write

$$z = (z_1, \dots, z_q)$$

we can explicitly compute both expressions as

$$(z_1, \dots, \mu(z_j, y^{-1}), \mu(y, z_{j+1}), \dots, \mu(z_i, x^{-1}), \mu(x, z_{i+1}), \dots)$$

in case $j < i - 1$, and

$$(z_1, \dots, \mu(z_j, y^{-1}), \mu(y, z_i, x^{-1}), \mu(x, z_{i+1}), \dots)$$

in case $j = i - 1$, and

$$(z_1, \dots, \mu(z_i, x^{-1}), \mu(x, z_{i+1}, y^{-1}), \mu(y, z_{j+1}), \dots)$$

in case $j = i + 1$, and

$$(z_1, \dots, \mu(z_i, x^{-1}), \mu(x, z_{i+1}), \dots, \mu(z_j, y^{-1}), \mu(y, z_{j+1}), \dots)$$

in case $j > i + 1$.

That the statements for the induced actions hold can be obtained in analogy to Corollary 1.2.34 and Corollary 1.2.35. □

Lemma A.1.7 (previously: Lemma 1.2.38)**.** *Let G be a Lie groupoid, $H \subset G$ a Lie subgroupoid as in Example 1.1.30 and $x \in G_1$. For any $q \in \mathbb{N}$ we have that on $H_q \cap (p_q^i)^{-1}(\{s(x)\})$ holds that*

$$\mathrm{Ad}^H_{x,q} \equiv D^q_{x,q} \circ \dots \circ D^1_{x,q} \circ D^0_{x,q}.$$

The analogue statements

$$\widetilde{\mathrm{Ad}}^H_{\gamma,q} \equiv \widetilde{D}^q_{\gamma,q} \circ \dots \circ \widetilde{D}^1_{\gamma,q} \circ \widetilde{D}^0_{\gamma,q}.$$

and

$$\widehat{\mathrm{Ad}}^H_{V,q} \equiv \dot{D}^q_{x,q} \circ \dots \circ \dot{D}^1_{x,q} \circ \dot{D}^0_{x,q}.$$

hold as well on the appropriate spaces.

Proof. We can explicitly compute for any $z \in H_q \cap (p_q^i)^{-1}(\{s(x)\})$ that

$$\mathrm{Ad}^H_{x,q}(z) = \big(\mu(x, z_1, x^{-1}), \dots, \mu(x, z_q, x^{-1})\big)$$

while

$$D^0_{x,q}(z) = (\mu(x, z_1), z_2, \dots, z_q)$$

and therefore

$$D^1_{x,q}(D^0_{x,q}(z)) = \big(\mu(x, z_1, x^{-1}), \mu(x, z_2), z_3, \dots, z_q\big)$$

and iterating yields

$$D^q_{x,q}(\dots(D^1_{x,q}(D^0_{x,q}(z)))) = \big(\mu(x, z_1, x^{-1}), \dots, \mu(x, z_q, x^{-1})\big)$$

That the statements for the induced actions hold can be obtained in analogy to Corollary 1.2.34 and Corollary 1.2.35. □

Lemma A.1.8 (previously: Lemma 1.2.39)**.** *Let G be a Lie groupoid and $x \in G_1$. For any $q \in \mathbb{N}$ we have that on $(\mathrm{Iso}^0(G))_0 \cap (p_0^0)^{-1}(\{s(x)\})$ holds that*

$$u_q \circ D^0_{x,0} = D^q_{x,q} \circ \cdots \circ D^1_{x,q} \circ D^0_{x,q} \circ u_q.$$

The analogue statements

$$u_q \circ \widetilde{D}^0_{\gamma,0} = \widetilde{D}^q_{\gamma,q} \circ \cdots \circ \widetilde{D}^1_{\gamma,q} \circ \widetilde{D}^0_{\gamma,q} \circ u_q$$

and

$$(u_q)_* \circ (\dot{D}^0_{V,0})_* = (\dot{D}^q_{V,q})_* \circ \cdots \circ (\dot{D}^1_{V,q})_* \circ (\dot{D}^0_{V,q})_* \circ (u_q)_*$$

hold as well on the appropriate spaces

Proof. We have that $(\mathrm{Iso}^0(G))_0 \cap (p_0^0)^{-1}(\{s(x)\})$ only consists of a single point $\{m\}$ for $m \in G_0$ for which

$$D^0_{x,0}(m) = t(x)$$

and in coordinates

$$u_q(m) = (1_m, \ldots, 1_m),$$

such that we can explicitly compute in analogy to Lemma 1.2.37 that

$$D^q_{x,q}(\ldots(D^1_{x,q}(D^0_{x,q}(u_q(m))))) = (\mu(x, 1_m, x^{-1}), \ldots, \mu(x, 1_m, x^{-1})) = (1_{t(x)}, \ldots, 1_{t(x)}).$$

That the statements for the induced actions hold can be obtained in analogy to Corollary 1.2.34 and Corollary 1.2.35. □

A.2 Topology of object spaces and Saturated Sets

At the beginning of Section 1.4, we have listed a few definitions and properties related to the saturation of subsets of the object space G_0 of a Lie groupoid. For ease of reading, we have not included all proofs or remarks directly, but instead referred for some of them to this appendix.

Lemma A.2.1 (previously: Lemma 1.4.2)**.** *Let G be a groupoid (not necessarily a Lie groupoid). The following are equivalent:*

- *The set S is saturated.*
- *The set S satisfies $t^{-1}(S) = s^{-1}(S)$.*

Proof. By definition, we have

$$\mathrm{Sat}(S) = t(s^{-1}(S)),$$

but as s and t in the definition are symmetrical via taking the inverse on G_1, we find that also

$$\mathrm{Sat}(S) = s(t^{-1}(S)).$$

So if we assume that $\mathrm{Sat}(S) = S$, we have that $t^{-1}(S) = t^{-1}(t(s^{-1}(S))) \supset s^{-1}(S)$ while we analogously find that $s^{-1}(S) \supset t^{-1}(S)$, meaning

$$t^{-1}(S) = s^{-1}(S).$$

Similarly, if we assume that $t^{-1}(S) = s^{-1}(S)$, we find that

$$S \subset \mathrm{Sat}(S) = s(t^{-1}(S)) = s(s^{-1}(S)) \subset S$$

such that $S = \mathrm{Sat}(S)$, i.e. S is saturated. □

Lemma A.2.2 (previously: Lemma 1.4.5). *Consider a Lie groupoid G and a saturated embedded submanifold $U \subset G_0$. Then the groupoid $G|_U$ defined in Definition 1.4.4, i.e. given by*

- *object space $(G|_U)_0 = U$*
- *morphism space $(G|_U)_0 = s^{-1}(U) = t^{-1}(U)$*

is a Lie groupoid.

Proof. This follows immediately, as $t^{-1}(U) \subset G_1$ is open, and therefore defines an open embedded submanifold and smoothness of the structure maps is a local property. □

Lemma A.2.3 (previously: Lemma 1.4.6). *Let G be a Lie groupoid. The saturation of an open subset of G_0 is open.*

Proof. The saturation can be expressed as $t(s^{-1}(S))$ and if S is open, so is $s^{-1}(S)$. However, submersions are open maps (which can be seen for example using the standard form of a submersion), such that the saturation is indeed an open set. □

Lemma A.2.4 (previously: Lemma 1.4.7). *Let G be a proper Lie groupoid. The saturation of any compact set $C \subset G_0$ is closed.*

Proof. Consider a sequence of points $\{a_n \in \mathrm{Sat}(C)\}_{n\in\mathbb{N}_0}$ which converges in G_0. By definition, for each a_n there is an $x_n \in G_1$ with $t(x_n) = a_n$ and $s(x_n) \in C$. By compactness, these two sequences contains corresponding sub-sequences where $s(x_n)$ is converging with limit in C (without loss of generality we may assume this to hold for the sequences a_n and x_n itself).

Therefore by properness, the sequence $\{x_n\}_{n\in\mathbb{N}_0}$ has a convergent subsequence with limit $x \in G_1$, where $s(x)$ and $t(x)$ are given by the limit of the $s(x_n)$ and $t(x_n)$ respectively. Hence, we have $s(x) \in C$ and hence $t(x) \in \mathrm{Sat}(C)$. However, $t(x)$ is the limit of the a_n, such that $\mathrm{Sat}(C)$ must be closed. □

Proposition A.2.5 (previously: Proposition 1.4.10). *Consider a proper Lie groupoid G. Any two closed saturated subsets $C \subset G_0$ and $D \subset G_0$ which are disjoint can be separated by open saturated sets. This means that there exist open sets $N \subset G_0$ and $O \subset G_0$ with $C \subset N$ and $D \subset O$ which also satisfy*

$$N \cap O = \emptyset.$$

Proof. We know that the statement is true for N and O ordinary open sets, and it remains to show that they can be chosen saturated. The idea of the proof is quite similar to the ordinary one.

We will first consider C' and D' compact with $\mathrm{Sat}(C') \cap \mathrm{Sat}(D') = \emptyset$ and prove the analogous result. Then we will cover the case where C' is compact and D is closed and saturated with $\mathrm{Sat}(C') \cap D = \emptyset$. Lastly, we will obtain the result in the case described in the proposition.

Pick any complete metric on G_0. For C' and D' compact, we can then define

$$N_\varepsilon = \{m \in G_0 \text{ with } |m - C'| < \varepsilon\}$$

and

$$O_\delta = \{m \in G_0 \text{ with } |m - D'| < \delta\}.$$

The term $|m - C'|$ denotes the minimum value of $|m - c|$ for $c \in C'$, which exists as C' is compact. That N_ε and O_δ are open follows from the triangle inequality in a straightforward way.

If we now let the parameters ε and δ converge to 0, we obtain that either

$$\mathrm{Sat}(N_\varepsilon) \cap \mathrm{Sat}(O_\delta) = \emptyset$$

for sufficiently small ε and δ or that there is a sequence $\{x_n\}_{n \in \mathbb{N}_{\geq 1}}$ with $s(x_n) \in N_{\frac{1}{n}}$ and $t(x_n) \in O_{\frac{1}{n}}$. We have that for any such sequence both the sequence $s(x_n)$ and the sequence $t(x_n)$ have a converging subsequence with limit in C' and D' respectively (determined each by the sequences of the closest points to $s(x_n)$ or $t(x_n)$ in C' or D'). We may hence without loss of generality assume that both $s(x_n)$ and $t(x_n)$ themselves converge, implying due to properness that a subsequence of the x_n converges to some $x \in G_1$ with $s(x) \in C'$ and $t(x) \in D'$. However, this contradicts that $\mathrm{Sat}(C') \cap \mathrm{Sat}(D') = \emptyset$, thus for small enough ε and δ

$$\mathrm{Sat}(N_\varepsilon) \cap \mathrm{Sat}(O_\delta) = \emptyset.$$

Note that if we consider the same argument for a fixed N_ε and only let δ go to 0, we obtain that either

$$\mathrm{Sat}(N_\varepsilon) \cap \mathrm{Sat}(O_\delta) = \emptyset.$$

for small enough δ or $\mathrm{Sat}(\overline{N_\varepsilon}) \cap \mathrm{Sat}(D') \neq \emptyset$. The argument in that case relies on the fact that $\overline{N_\varepsilon}$ is compact, which holds independently of ε as the metric was chosen complete. If the metric was not chosen complete, we could still apply the argument for ε small enough (such that all closed ε-balls centered in C' are compact, this value can be constructed through a corresponding lower semi-continuous function on C').

Now, let us consider the case where C' is compact and D is closed and saturated with $\mathrm{Sat}(C') \cap D = \emptyset$. First off, we aim to show that there exists an ε such that $\mathrm{Sat}(\overline{N_\varepsilon}) \cap D = \emptyset$. However, as D itself is saturated, this is equivalent to finding ε such that

$$\overline{N_\varepsilon} \cap D = \emptyset.$$

As $\overline{N_\varepsilon} \subset N_{2\varepsilon}$, we can just choose this as

$$\varepsilon < \frac{1}{2}|C' - D|,$$

where the latter value might only exist as an infimum, but is non-zero as C' is compact.

As already remarked in the previous case, this implies that for a compact D' with $\mathrm{Sat}(D') \subset D$ there must be some $O_\delta \supset D'$ with

$$\mathrm{Sat}(N_\varepsilon) \cap \mathrm{Sat}(O_\delta) = \emptyset.$$

We can exhaust D by compact subsets D'_n, cover each with some $O^{(n)}$ of the form O_δ for the respective D'_n and then take the union over the corresponding sets of the form $\mathrm{Sat}(O^{(n)})$. This will in particular cover the union of all D'_n, which is D. The union of open saturated sets is itself open and saturated, and by construction it will be disjoint from $\mathrm{Sat}(N_\varepsilon)$.

Now consider the case originally proposed, where C and D are closed and saturated. We can now exhaust G_0 by compact subsets, which simultaneously gives us an exhaustion of C and D. In each step, we will consider the compact subsets $C'_n \subset C$ and $D'_n \subset D$ and choose $N^{(n)}$ and $O^{(n)}$ such that

a) $N^{(n)}$ covers C_n'

b) $O^{(n)}$ covers D_n'

c) $\overline{N^{(n)}}$ does not intersect D

d) $\overline{O^{(n)}}$ does not intersect C

e) $\mathrm{Sat}(N^{(n)}) \cap \mathrm{Sat}(O^{(n)}) = \emptyset$

f) $\mathrm{Sat}(N^{(n)}) \cap \mathrm{Sat}(O^{(k)}) = \emptyset$ for all $k < n$

g) $\mathrm{Sat}(N^{(k)}) \cap \mathrm{Sat}(O^{(n)}) = \emptyset$ for all $k < n$.

This is possible for sets of the form N_ε and O_δ with respect to C_n' and D_n' for small enough values of ε and δ. This is because these constraints are all satisfied for small enough values of either ε or δ or both (which we have seen for each item at some previous step in the proof) and the number of constraints is finite for each construction step.

Now if we consider

$$N' = \bigcup_{n\in\mathbb{N}_0} N^{(n)}$$

and

$$O' = \bigcup_{n\in\mathbb{N}_0} O^{(n)}$$

we have that

a) N' covers C

b) O' covers D

c) $\mathrm{Sat}(N) \cap \mathrm{Sat}(O) = \emptyset$.

So if we now set $N = \mathrm{Sat}(N')$ and $O = \mathrm{Sat}(O')$ we obtain the result of the proposition. □

Corollary A.2.6 (previously: Corollary 1.4.11)**.** *Consider a proper Lie groupoid G. Any two closed saturated subsets $C \subset G_0$ and $D \subset G_0$ which are disjoint can be separated by open saturated sets N and O which satisfy*

$$\overline{N} \cap \overline{O} = \emptyset.$$

Proof. As a first step, we can apply Proposition 1.4.10 to obtain N' and O'. We have that $G_0 \backslash N' = D'$ is saturated and closed. We can reapply Proposition 1.4.10 to C and D' to obtain N'' and O''. Now we have that for $N = N''$ and $O = O'$ holds

$$\overline{N} \cap \overline{O} \subset (G_0 \backslash D') \cap D' = \emptyset.$$

□

Corollary A.2.7 (previously: Corollary 1.4.12)**.** *Let G be a proper Lie groupoid. Consider the space*

$$Q = G_0/F_0 = G_0/\sim,$$

where $\sim$ denotes the equivalence relation of Lemma 1.1.11. Then the quotient topology on Q is T_4 (i.e. normal and T_1).

Proof. By definition, a set $O \subset Q$ is open, if its preimage in G_0 is and analogously for closed sets. Also by definition, the preimage of any set is saturated, as the saturation is precisely generated by the same relation $\sim$. By Corollary 1.4.9, we have that the preimage of a point $q \in Q$ (i.e. an orbit) is closed, meaning that Q is T_1. The statement of Proposition 1.4.10 is precisely equivalent to separating closed sets in Q, meaning that Q is T_4. □

Lemma A.2.8 (previously: Lemma 1.4.14)**.** *Consider a manifold G_0 with a regular foliation F_0 of rank l_0, i.e. where we can locally around any $m \in G_0$ choose coordinates such that the chosen coordinate chart is given by a cube*

$$U \subset \mathbb{R}^{k_0} = \mathbb{R}^{l_0} \times \mathbb{R}^{k_0 - l_0}$$

and the leaves of the foliation are given by

$$x^{l_0+1} = c^{l_0+1}, \quad \dots, \quad x^{k_0} = c^{k_0}$$

and its associated involutive distribution by

$$\Big(T(\mathbb{R}^{l_0}) \times \{0_y\}_{y \in \mathbb{R}^{k_0 - l_0}}\Big)|_U.$$

Then for any embedded manifold R complementary to the orbits of codimension l_0 (i.e. dimension $k_0 - l_0$) we can choose a coordinate chart

$$\phi : V \subset G_0 \to U \subset \mathbb{R}^{k_0}$$

of the form described above with the additional property that

$$\phi(R \cap V) = (\{0\}^{l_0} \times \mathbb{R}^{k_0 - l_0}) \cap U.$$

Proof. For any point $r \in R$ we may choose a coordinate chart

$$\phi : V \to U$$

compatible with the foliation as described. We can consider the function

$$\mathrm{pr}_{\mathbb{R}^{k_0 - l_0}} \circ \phi : V \to \mathbb{R}^{k_0 - l_0}$$

which on $R \cap V$ restricts to

$$f := (\mathrm{pr}_{\mathbb{R}^{k_0 - l_0}} \circ \phi)|_{R \cap V}$$

and analogously consider the function

$$g := (\mathrm{pr}_{\mathbb{R}^{l_0}} \circ \phi)|_{R \cap V}$$

We claim that f is a local diffeomorphism. For this, it is sufficient to show that $T_r f$ is an isomorphism e.g. by the inverse function theorem ([30], Theorem 4.5). As the dimensions of R and $\mathbb{R}^{k_0 - l_0}$ agree, we may check injectivity: Assume there is some $X \in T_r R$ such that $Tf(X) = 0$. This implies that $T\phi(X) \in \ker(T\,\mathrm{pr}_{\mathbb{R}^{l_0}})$, which can be formulated as $T\phi(X) \in T(\mathbb{R}^{l_0}) \times \{0_y\}_{y \in \mathbb{R}^{k_0 - l_0}}$. However, this means that as ϕ is a diffeomorphism, $X \in \mathcal{F}_0$ and by our assumption of $X \in T_r R$ we obtain that $X = 0$.

Hence we can find an inverse function f^{-1} locally around $f(r)$ on $U' \subset U$. We can now consider $V' = \phi^{-1}(U')$ and define the coordinate chart

$$\begin{aligned} \phi' &: V' \to \mathbb{R}^{k_0} \\ &x \mapsto \phi(x) - (g(f^{-1}(((\phi(x))^{l_0+1}, \dots, (\phi(x))^{k_0}))), 0 \in \mathbb{R}^{k_0 - l_0}). \end{aligned}$$

As we can explicitly express the inverse as

$$y \mapsto \phi^{-1}\left(y + (g(f^{-1}((y^{l_0+1}, \dots, y^{k_0}))), 0 \in \mathbb{R}^{k_0-l_0})\right),$$

which is well-defined on an open set containing the image, ϕ' is indeed a local diffeomorphism. It is clear by construction, that elements of $R \cap V'$ are getting mapped to $\{0\} \times \mathbb{R}^{k_0-l_0}$. It only remains to restrict domain and codomain in a suitable way such that the codomain is given by a cube. □

Lemma A.2.9 (previously: Lemma 1.4.15). *Let G be a regular Lie groupoid where the rank of the orbit foliation is l_0. The saturation of a embedded manifold R of codimension l_0 complementary to the orbits is an open subset of G_0 and in particular an embedded manifold.*

Proof. Consider $r \in R$. Locally around r we may by Lemma 1.4.14 choose coordinates compatible with the foliation and R, i.e. a coordinate chart

$$U \subset \mathbb{R}^{k_0} = \mathbb{R}^{l_0} \times \mathbb{R}^{k_0-l_0}$$

on which the foliation corresponds to

$$\left(T(\mathbb{R}^{l_0}) \times \{0_y\}_{y \in \mathbb{R}^{k_0-l_0}}\right)|_U$$

and R locally corresponds to

$$U \cap \{0\}^{l_0} \times \mathbb{R}^{k_0-l_0}.$$

As U is open and contains $r \in R$, we will find that for some $\varepsilon > 0$ a neighbourhood

$$r + (-\varepsilon, \varepsilon)^{k_0}$$

is contained in U. As this is an open set, its saturation V_r is also open by Lemma 1.4.6, and as it contains r it also contains $\mathrm{Sat}(\{r\})$. If we can also show that $r + (-\varepsilon, \varepsilon)^{k_0} \subset \mathrm{Sat}(R)$ it would immediately follow that $V_r \subset \mathrm{Sat}(R)$ and furthermore

$$\bigcup_{r \in R} V_r \subseteq \mathrm{Sat}(R) = \bigcup_{r \in R} \mathrm{Sat}(\{r\}) \subseteq \bigcup_{r \in R} V_r,$$

meaning that $\mathrm{Sat}(R) = \cup_{r \in R} V_r$ is an open subset of G_0.

For this, we consider that the orbit foliation $\mathcal{F}_0$ is precisely the infinitesimal counterpart of the saturation procedure and all paths contained in a leaf can be lifted along the target map to paths of morphisms with constant source map. In particular this means that for any point $m \in U$, we have that its entire connected component in

$$U \cap m + \mathbb{R}^{l_0} \times \{0\}$$

is contained in $\mathrm{Sat}(\{m\})$. In particular, if $m \in R$, we have that

$$U \cap m + \mathbb{R}^{l_0} \times \{0\} \subset \mathrm{Sat}(R)$$

However, by construction we have that

$$r + (\{0\}^{l_0} \times (-\varepsilon, \varepsilon)^{k_0-l_0}) \subset R$$

which implies that

$$r + (\mathbb{R}^{l_0} \times (-\varepsilon, \varepsilon)^{k_0-l_0}) \cap U \subset \mathrm{Sat}(R)$$

and in particular

$$r + (-\epsilon, \epsilon)^{k_0} \subset \mathrm{Sat}(R).$$

□

Lemma A.2.10 (previously: Lemma 1.4.16)**.** *For a Lie groupoid G, consider $R \subset G_0$ an embedded manifold complementary to the orbit foliation $\mathcal{F}_0$, such that* $\mathrm{Sat}(R)$ *is an embedded manifold. Then for any $r \in R$ we have $T_r \mathrm{Sat}(R) = T_r R \oplus \mathcal{F}_0$ and*

$$s|_{t^{-1}(R)} : t^{-1}(R) \to \mathrm{Sat}(R)$$

is a submersion.

Proof. As R is an embedded manifold and t is a surjective submersion, we have that $t^{-1}(R)$ is a submanifold of G_1. The map s is still surjective onto $\mathrm{Sat}(R)$, but we do not know if it is a submersion.

However, as the map is still smooth, we can consider

$$Ts : T(t^{-1}(R)) \to T\,\mathrm{Sat}(R).$$

At each point $x \in t^{-1}(R)$ we have by Lemma 1.2.8 that there exists a jet $V \in (J^1G)_1$ with $V \subset T_x G_1$. As Tt is an isomorphism on V, we find that $V' = V \cap T_x(t^{-1}(R))$ has the same dimension as $T_r R$, i.e. the dimension of R and Tt restricts to an isomorphism. This in particular means that $T_x(t^{-1}(R))$ splits into a direct sum

$$T_x(t^{-1}(R)) = V' \oplus \ker(T_x t) = V' \oplus (\mathcal{F}_1 \cap T_x t^{-1}(R)),$$

where we have used that the orbit foliation $\mathcal{F}_0$ vanishes when restricted to R, such that the image of $\mathcal{F}_1$ under $T_x t$ (which is usually contained in $\mathcal{F}_0$) must also vanish. We also know for the same reason that the image of V' under Ts (which is still injective) does not contain elements of $\mathcal{F}_0$, while the entirety of $\mathcal{F}_1$ maps to $\mathcal{F}_0$ surjectively. So indeed, the map Ts has constant rank, which is at each point given by $\dim(V') + \dim(\mathcal{F}_0) = \dim(R) + \dim(\mathcal{F}_0)$. In particular, if $\dim(\mathrm{Sat}(R)) = \dim(R) + \dim(\mathcal{F}_0)$, we know that Ts is surjective at each point, i.e. that s is a submersion. We can also note that $\dim(\mathrm{Sat}(R)) \geq \dim(R) + \dim(\mathcal{F}_0)$, as at $r \in R$ both $T_r R$ and $\mathcal{F}_0$ are disjoint subspaces of $T_r \mathrm{Sat}(R)$.

So we either have that this condition is satisfied, or that each point is a critical point and by surjectivity the image of the critical points consists of the entirety of $\mathrm{Sat}(R)$. However, by Sard's theorem, we have that the critical values must have measure 0 with respect to the outer Lebesgue measure on any coordinate chart on $\mathrm{Sat}(R)$ ([39], Theorem 7.2.). This means that there can not be any critical values and s is a surjective submersion. ☐

Corollary A.2.11 (previously: Corollary 1.4.17)**.** *For a Lie groupoid G, consider $R \subset G_0$ an embedded manifold complementary to the orbit foliation $\mathcal{F}_0$, such that* $\mathrm{Sat}(R)$ *is an embedded manifold.*

Then we have that $t^{-1}(R) \cap s^{-1}(R)$ is a manifold embedded in G_1. If G was proper, then the map

$$(t, s) : t^{-1}(R) \cap s^{-1}(R) \to R \times R$$

is proper as well.

Proof. Firstly note, that in analogy to the argument of Lemma 1.4.16, $t^{-1}(R)$ has a manifold structure and $t^{-1}(R) \subset G_1$ is an embedding, as t is a surjective submersion (and hence locally is given by a projection) and R is an embedded manifold. Furthermore, the statement of Lemma 1.4.16 implies that $s|_{t^{-1}(R)}$ is a surjective submersion as well, and hence

$$(s|_{t^{-1}(R)})^{-1} = t^{-1}(R) \cap s^{-1}(R)$$

is an embedded submanifold of $t^{-1}(R)$ and therefore G_1. If we now consider a sequence of $x_i \in s^{-1}(R) \cap t^{-1}(R)$ such that their source and target maps converge in R, we find that as G is proper, this implies that there is a sub-sequence with a limit $x \in G$. But as we assumed that the $s(x_i)$ and $t(x_i)$ converge in R we find that $x \in t^{-1}(R) \cap s^{-1}(R)$ as well. □

Corollary A.2.12 (previously: Corollary 1.4.19)**.** *If G is a regular Lie groupoid with an orbit foliation $\mathcal{F}_0$ of dimension l_0 and further $R \subset G_0$ is an embedded manifold of codimension l_0 complementary to the orbits, then*

$$s|_{t^{-1}(R)} : t^{-1}(R) \to \mathrm{Sat}(R)$$

is a submersion and $t^{-1}(R) \cap s^{-1}(R)$ is a manifold embedded in G_1 defining a subgroupoid of G which we will denote $G|_R$. If G was proper, then so is $G|_R$.

Proof. By Lemma 1.4.15, we have that $\mathrm{Sat}(R)$ is an embedded manifold, meaning that we can apply Corollary 1.4.17 which directly translates to the given result. □

Corollary A.2.13 (previously: Corollary 1.4.20)**.** *If G is a regular Lie groupoid with an orbit foliation $\mathcal{F}_0$ of dimension l_0 and further $R_1 \subset G_0$ and $R_2 \subset G_0$ are embedded manifolds of codimension l_0 complementary to the orbits, then for any $m \in R_1 \cap \mathrm{Sat}(R_2)$ there is a local s-section defined on $U \subset R_1$ around m inducing a diffeomorphism between U and an open submanifold of R_2.*

Proof. As

$$s|_{t^{-1}(R_2)} : t^{-1}(R_2) \to \mathrm{Sat}(R_2)$$

is a submersion by Corollary 1.4.19, we can find a local s-section $\gamma : V \to t^{-1}(R_2)$, which then by definition has its image contained in R_2. Considering $T_m(t \circ \gamma)$, we have that its kernel is contained in $\mathcal{F}_0$ by definition and the fact that γ is an s-section. So locally on $U \subset V$ we have that γ induces a diffeomorphism. □

A.3 Constructions on jet bundles

In the thesis, we have sometimes omitted proofs of statements concerning jet bundles as these arguments are standard but technical. For completeness, we have included them in this section.

Lemma A.3.1 (cf. [34], p. 33, previously: Lemma 1.2.1)**.** *Let G be a Lie groupoid. The space consisting of bisection jets*

$$J^1(s,t) := \left\{ V_x \subset T_x G_1 \ \textit{s.t.}\ T_x s : V_x \to T_{s(x)} G_0 \ \textit{and}\ T_x t \ \textit{are isos} \right\}$$

is an open submanifold of

$$J^1(s : G_1 \to G_0) = \left\{ V_x \subset T_x G_1 \ \textit{s.t.}\ T_x s : V_x \to T_{s(x)} G_0 \ \textit{is an iso} \right\}.$$

Proof. From general theory about jet manifolds we know that

$$J^1(s : G_1 \to G_0) \to G_1$$

is an affine vector bundle which comes with a canonical smooth map to

$$J^1(\mathrm{pr}_2 : G_0 \times G_0 \to G_0) \to G_0 \times G_0$$

induced by (t, s).

For any $x \in G_1$ and any choice of local coordinates around $s(x)$ and $t(x)$, we can consider the corresponding open of

$$J^1(\mathrm{pr}_2 : G_0 \times G_0 \to G_0) \to G_0 \times G_0,$$

where we can express the jets as matrices and compute their determinants.

The determinant is a smooth function, such that the set where it remains non-zero is open as well. However, we know that the condition of a non-zero determinant is equivalent to $T\,\mathrm{pr}_1$ inducing an isomorphism between the jet and $T_m G_0$ for the corresponding $m \in G_0$. This also shows that the condition is actually independent of the choice of coordinates.

This implies that we have constructed an open subset of $J^1(\mathrm{pr}_2 : G_0 \times G_0 \to G_0)$, which, when we take the preimage in $J^1(s : G_1 \to G_0)$, yields precisely those jets $V_x \subset T_x G_1$ for which $T_x t$ is an isomorphism. □

Lemma A.3.2 (previously: Lemma 2.1.5)**.** *Consider a Lie groupoid G together with a multiplicative Ehresmann connection E with respect to a bundle of ideals B. Then the subspace*

$$J^E(s : G_1 \to G_0) \subset J^1(s : G_1 \to G_0)$$

has the structure of a closed embedded submanifold.

Proof. For this, we first need to recall the manifold structure of $J^1(s : G_1 \to G_0)$: We can use the local form of a submersion, in our case s, as a projection from $\mathbb{R}^{k_1}$ to its first k_0 coordinates (cf. [30] 4.12.), where k_0 and k_1 are the dimensions of G_0 and G_1 respectively. Thus, we know that locally the subspaces W_x at x that project isomorphically to $T_{s(x)}G_0$ can be expressed by the lift of the standard basis vectors of $T_{s(x)}G_0 \cong T_{s(x)}\mathbb{R}^{k_0}$. The collection of these, expressed in coordinates of the standard basis at $T_x G_1 \cong T_x\mathbb{R}^{k_1}$, define a $k_0 \times k_1$-matrix associated to each W_x, satisfying that the first k_0 entries of each row are according to the identity matrix. The matrix may be seen as the section of Ts at $s(x)$ associated to W_x and expressed in coordinates.

To make the vector bundle modelling the affine structure of $J^1(s : G_1 \to G_0)$ (as a bundle over G_1) explicit, this is the same as choosing $k_0 \times (k_1 - k_0)$-matrices, or equivalently, maps from $T_{s(x)}G_0$ to $\ker(Ts) \subset T_x G_1$. When considered as matrices, we now express only the last $(k_1 - k_0)$ components of the preimages of the standard basis vectors of $T_{s(x)}\mathbb{R}^{k_0} \simeq \mathbb{R}^{k_0}$ in $T_x\mathbb{R}^{k_1} \simeq \mathbb{R}^{k_1}$. The affine structure has the base-point given by the identity matrix extended by zeroes and the associated jet.

Given a multiplicative Ehresmann connection, we can project each vector of W_x to its vertical and horizontal component according to the splitting $TG_1 = E \oplus V$. Because the vertical projection is linear, it induces an affine map between $J^1(s : G_1 \to G_0)$ to the bundle (which is actually a vector bundle) $s^*(T^*G_0) \otimes V$ where the fibre over each $y \in G_1$ is given by the space of linear maps from $T_{s(y)}G_0$ to V_y.

Furthermore, as V vanishes under Ts, we have that the target space is actually a subspace of the vector space modelling the affine structure. Furthermore, when considering the map induced by vertical projection, we find that it is given by the identity on this vector bundle model, meaning in particular that it is surjective. This means that no matter the image of the base-point of the affine structure, the induced map between the bundles is surjective as well. Furthermore, subtracting the image under this map from any local section of $J^1(s : G_1 \to G_0) \to G_1$ will yield a local section in the kernel of the vertical projection map.

We have that $J^E(s : G_1 \to G_0)$ is precisely the kernel of the affine vertical projection map that we have described. Additionally, any local section in the kernel of this map will yield a local vector bundle model such that the map on the models is linear. As the kernel of a vector bundle map of constant rank is a vector bundle, we have that surjectivity implies that the kernel $J^E(s : G_1 \to G_0)$ is an affine bundle modelled on a subvector bundle of the vector bundle modelling $J^1(s : G_1 \to G_0)$. In particular, it is a closed and embedded manifold. □

Lemma A.3.3 (previously: Lemma 2.1.9). *Given a Lie groupoid G together with a multiplicative Ehresmann connection E with respect to a bundle of ideals B. There is a canonical inclusion $J^E G \to J^1 G$ and a canonical projection $J^1 G \to J^E G$. The latter establishes $J^1 G$ as a VB-groupoid over $J^E G$.*

Proof. By construction $J^E G$ is a subgroupoid of $J^1 G$, it is well defined by Lemma 2.1.7. However, as analogous to the proof of Lemma 2.1.5, there is also a canonical map induced by the horizontal projection instead of the vertical one, which automatically maps elements of $J^1(s : G_1 \to G_0)$ to elements of $J^E(s : G_1 \to G_0)$. The map only changes the vectors up to components in V (i.e. the vertical bundle as induced by B). When recalling the proof of Lemma 1.2.1, this implies that the map from $J^1(s : G_1 \to G_0)$ to $J^1(s : G_0 \times G_0 \to G_0)$ factors through $J^E(s : G_1 \to G_0)$, such that the image of $(J^1 G)_1$ must be contained in $(J^1 G)_1 \cap J^E(s : G_1 \to G_0) = (J^E G)_1$.

It still remains to check that the groupoid structure is compatible: W_x and its projection W'_x differ precisely by an element in $s^*(TG_0) \otimes V$ as we have seen in the proof of Lemma 2.1.5. Expressing any $X \in W_x$ as $X' + Z_X$ for $X' \in W'_x$ and $Z_X \in V$ yields that for any pair of composable vectors X and Y contained in $\widehat{\mu}(W_x, U_y)$ that

$$(X, Y) = (X' + Z_X, Y' + Z_Y) = (X', Y') + (Z_X, Z_Y)$$

as $V \subset \ker(Ts) \cap \ker(Tt)$ and hence Z_X and Z_Y are always composable. Using that V is closed under the groupoid structure maps, we can check explicitly that the projection $(J^1 G)_1 \to (J^E G)_1$ is compatible with multiplication and inverses. The fact that V is compatible with the groupoid structure follows by an analogous argument as above, as any two composable elements (Z_1, Z_2) can be written as

$$(Z_1, Z_2) = (0, Z_2) + (Z_1, 0)$$

and therefore their image under $T\mu$ corresponds to the sum of the left translation of Z_2 by the base-point of Z_1 and the right translation of Z_1 under the base-point of Z_2. However, V is by Lemma 2.1.2 invariant under both left and right translation. That the unit map is compatible follows as the horizontal projection of a vector tangent to the unit section is given by the vector itself, implying the same for the units of $J^1 G$. □

Lemma A.3.4 (previously: Lemma 2.1.25). *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy. Consider any $x \in G_1$ and a linear isomorphism $\psi : T_{s(x)} G_0 \to T_{t(x)} G_0$ induced by some $V_x \in (J^E G)_1$.*

Then for any $\lambda \in \mathbb{R}$ there is a $V_x^\lambda \in (J^E G)_1$ inducing an isomorphism ψ^λ with

$$\psi^\lambda(X) = \lambda \cdot (\psi(X))$$

for any $X \in \mathcal{F}_0$. They can be constructed in a way that the limit $\lim_{\lambda \to 0} \psi^\lambda$ is well-defined and the limit $\lim_{\lambda \to 0} V_x^\lambda \in J^E(s : G_0 \to G_1)$ is well-defined.

Proof. We can choose any (non-canonical) decomposition of V_x as

$$V_x = (V_x \cap \mathcal{F}_1) \oplus W_x$$

which via the isomorphisms Ts and Tt induces splittings

$$T_{s(x)}G_0 = (\mathcal{F}_0)_{s(x)} \oplus Ts(W_x)$$

and

$$T_{t(x)}G_0 = (\mathcal{F}_0)_{t(x)} \oplus Tt(W_x).$$

By construction, the splittings are retained by φ. Now we will aim to define V_x^λ as a superspace of W_x by altering the direct summand $(V_x \cap \mathcal{F}_1)$.

For every $X \in (V_x \cap \mathcal{F}_1)$, we define X^λ as the unique E-horizontal vector satisfying

$$Ts(X^\lambda) = Ts(X)$$

and

$$Tt(X^\lambda) = \lambda \cdot Tt(X).$$

The vector X^λ indeed exists, as both $Ts(X)$ and $\lambda \cdot Tt(X)$ are elements of $\mathcal{F}_0$ with base-points in the same orbit by assumption. Furthermore as the construction is linear and Ts is retained (in particular $X^\lambda \in \mathcal{F}_1$), the collection

$$V_x^\lambda = \{X^\lambda \text{ for } X \in (V_x \cap \mathcal{F}_1)\} \oplus W_x$$

is canonically a E-horizontal subspace of T_xG_1 on which Ts is an isomorphism to $T_{s(x)}G_0$.

By construction we have that Tt retains the splitting and maps W_x isomorphically to $Tt(W_x)$. On the other summand, Tt is given by the concatenation of an isomorphism (Tt evaluated on X) by multiplication with λ, but as $\lambda \neq 0$, this is also an isomorphism.

This concludes that $V_x^\lambda \in (J^EG)_1$ and by construction the induced map φ^λ satisfies the required equation. For the claim on the limits, note that by construction, we have that

$$\psi^\lambda|_{Ts(W_x)} = \psi|_{Ts(W_x)}$$

such that convergence is trivial and the series of ψ^λ on $(\mathcal{F}_0)_{s(x)}$ depends linearly on λ and hence converges to 0. Similarly, we find that the limit of the V_x^λ retains the splitting, where the second component W_x agrees for all λ and the first component converges to the space containing a E-horizontal vector X^0 uniquely associated to any $X \in (\mathcal{F}_1 \cap V_x)$ defined by the properties that

$$Ts(X^0) = Ts(X)$$

and

$$Tt(X^0) = 0.$$

□

Lemma A.3.5 (previously: Lemma 2.1.27). *Let G be a regular Lie groupoid together with a multiplicative Ehresmann connection E with respect to the isotropy. When considered as subsets of*

$$J^1(s : G_1 \to G_0),$$

we have that

$$J^E(s : G_1 \to G_0)$$

is the closure of

$$(J^EG)_1 \subset (J^1G)_1 \subset J^1(s : G_1 \to G_0).$$

Proof. In Lemma 2.1.5, we have already seen that $J^E(s : G_1 \to G_0)$ is a closed subset of $J^1(s : G_1 \to G_0)$. In the proof of Lemma 2.1.25, we have now explicitly constructed certain elements V'_x as limits of a sequence $V^\lambda_x \in (J^E G)_1$. However, that construction can easily be amended to construct any V'_x as a limit: For this choose a (non-canonical) decomposition of

$$V'_x = (V'_x \cap \mathcal{F}_1) \oplus W'_x$$

which via Ts induces a splitting on

$$T_{s(x)}G_0 = (\mathcal{F}_0)_{s(x)} \oplus Ts(W'_x).$$

We have that $V'_x \cap \mathcal{F}_1$ induces a linear map

$$\psi'|_{(\mathcal{F}_0)_{s(x)}} : (\mathcal{F}_0)_{s(x)} \to (\mathcal{F}_0)_{t(x)}.$$

This map is not an isomorphism. However, as the subspace of $(l \times l)$-matrices given by invertible ones $GL(l \times l) \subset \mathrm{Mat}(l \times l)$ is dense, we may simply choose

$$\psi^\lambda|_{(\mathcal{F}_0)_{s(x)}} : (\mathcal{F}_0)_{s(x)} \to (\mathcal{F}_0)_{t(x)}.$$

as a converging sequence of isomorphisms with limit $\psi'|_{(\mathcal{F}_0)_{s(x)}}$.

Now for every $X \in (V'_x \cap \mathcal{F}_1)$, we define X^λ as the unique E-horizontal vector satisfying

$$Ts(X^\lambda) = Ts(X)$$

and

$$Tt(X^\lambda) = \psi^\lambda|_{(\mathcal{F}_0)_{s(x)}}(Ts(X)).$$

This vector indeed exists, as both expressions are contained in $\mathcal{F}_0$.

Now consider the collection

$$V^\lambda_x = \{X^\lambda \text{ for } X \in (V'_x \cap \mathcal{F}_1)\} \oplus W'_x.$$

We claim that $V^\lambda_x \in (J^E G)_1$ for all λ of the sequence. For this, let us note that $Ts(X^\lambda) = Ts(X)$ for the corresponding $X \in (V'_x \cap \mathcal{F}_1)$, such that Ts retains the splitting of the vector spaces. Furthermore, $Ts(X)$ uniquely identifies X and the associated X^λ, meaning that $Ts : V^\lambda_x \to T_{s(x)}G_0$ is still an isomorphism. To see the same for Tt, assume some $Y \in V^\lambda_x$ satisfies $Tt(Y) = 0$. Then we must have that $Y \in \mathcal{F}_1$. By definition, this means that $Y = X^\lambda$ for some $X \in (V'_x \cap \mathcal{F}_1)$. Therefore

$$0 = Tt(Y) = Tt(X^\lambda) = \psi^\lambda|_{(\mathcal{F}_0)_{s(x)}}(Ts(X^\lambda)).$$

However, we assumed that $\psi^\lambda|_{(\mathcal{F}_0)_{s(x)}}$ was an isomorphism, meaning that $Ts(X^\lambda) = 0$ as well. But X^λ is E-horizontal, meaning that $Ts(X^\lambda) = Tt(X^\lambda) = 0$ implies $X^\lambda = 0$ by Corollary 2.1.18. So $Tt : V^\lambda_x \to T_{t(x)}G_0$ is injective and for dimension reasons this concludes that Tt is an isomorphism. □

A.4 The generalised Eilenberg–Zilber Theorem

As this is a very important part of Chapter 3 (and in particular Lemma 3.1.16), I would like to explore the generalisation of the Eilenberg–Zilber theorem that I use. The name Eilenberg–Zilber–Cartier theorem was used by Bott in [8] (cf. p. 300) as well as the name

"Cartier extension of the Eilenberg–Zilber theorem" (p. 298). It references [16] by Dold and Puppe for the result, which also accredit Cartier in their work.

The paper of Dold and Puppe is in German, which unfortunate for the non-German speaking members of the research community (although luckily for me it is a language I speak). It is stated somewhat as follows:

Theorem A.4.1 ([16], p. 213)**.** *There is a universal (chain) homotopy equivalence between the total and the diagonal complex of the double object V. More generally holds: Let $\mathfrak{X} = \mathfrak{X}(V)$, $\mathfrak{N} = \mathfrak{N}(V)$ the total or diagonal complex of V respectively; in particular holds $\mathfrak{X}_0 = \mathfrak{N}_0 = V_{0,0}$.*

(a) Every universal morphism $\mathfrak{X}_0 \to \mathfrak{N}_0$ (in particular the identity one) is extendable to a universal chain morphism

(b) Any two universal chain morphisms, which agree in dimension 0 (i.e. $\mathfrak{X}_0 \to \mathfrak{N}_0$) are universally homotopic.

The notion of "double object" used by Dold and Puppe would today be referred to as a bisimplicial object in an abelian category and the notions of the total complex can be formulated with the following definition.

Definition A.4.2 ([21], p. 205)**.** The **Moore bicomplex for a bisimplicial abelian group** A is the bicomplex having (p, q)-chains $A(p, q)$, horizontal boundary

$$\partial_{\text{hor}} = \sum_{i=0}^{p} (-1)^i d_i^{\text{hor}} : A(p, q) \to A(p-1, q)$$

and vertical boundary

$$\partial_{\text{vert}} = \sum_{i=0}^{q} (-1)^{p+i} d_i^{\text{vert}} : A(p, q) \to A(p, q-1).$$

Here, the index hor denotes the maps corresponding to the structure maps of the simplicial structure denoted in the first component and the index vert denotes the maps corresponding to the structure maps of the simplicial structure denoted in the second component.

Of course, this notion does not only make sense for bisimplicial abelian groups, but can be generalised to recover the setting of the previous theorem. It transforms any bisimplicial object into a double complex in the respective category, for which the associated notion of the associated total chain complexes is a standard construction.

However, for our purposes, we need the dual of the given objects, in the sense that the constructions that we are working with are bi-cosimplicial. Here, the analogue definitions take similar form.

Definition A.4.3 (cf. [23] pp. 732-733)**.** Let X be a bi-cosimplicial object in an abelian category $\mathcal{A}$, i.e. an object of $\mathcal{A}^{\Delta\times\Delta}$. The **associated (unnormalised) cochain bicomplex** $C(X)$ is given by

$$C(X)^{(p,q)} = X^{(p,q)}$$

and horizontal and vertical differentials

$$\partial_{\text{hor}} = \sum_{i=0}^{p+1} (-1)^i d_i : X^{(p,q)} \to X^{(p+1,q)}$$

and

$$\partial_{\mathrm{vert}} = \sum_{j=0}^{q+1} (-1)^{p+j} d_i : X^{(p,q)} \to X^{(p,q+1)}.$$

Here, the index hor denotes the maps corresponding to the structure maps of the simplicial structure denoted in the first component (or equivalently induced by the left-hand copy of Δ) and the index vert denotes the maps corresponding to the structure maps of the simplicial structure denoted in the second component.

The other concept of the diagonal complex can be described as follows:

Example A.4.4 (cf. [23] p. 733)**.** Let X be a bi-cosimplicial object in an abelian category $\mathcal{A}$, i.e. an object of $\mathcal{A}^{\Delta\times\Delta}$. The image of X under the diagonalisation functor Diag can be described as the cosimplicial object

$$\mathrm{Diag}(X)^p = X^{(p,p)}$$

with coface maps

$$d^i := d^i_{\mathrm{hor}} \circ d^i_{\mathrm{vert}} : X^{p,p} \to X^{p+1,p+1}$$

and codegeneracy maps

$$s^j := s^j_{\mathrm{hor}} \circ s^j_{\mathrm{vert}} : X^{p+1,p+1} \to X^{p,p}$$

for $0 \leq i \leq p+1$ and $0 \leq j \leq p$. Here, the index hor denotes the maps corresponding to the structure maps of the simplicial structure denoted in the first component (or equivalently induced by the left-hand copy of Δ) and the index vert denotes the maps corresponding to the structure maps of the simplicial structure denoted in the second component.

The associated (unnormalised) chain complex $C(\mathrm{Diag}(X))$ can be similarly described as given by

$$\mathrm{Diag}(X)^p = X^{(p,p)}$$

with differential

$$\partial = \sum_{i=0}^{p+1} (-1)^i d^i_{\mathrm{hor}} \circ d^i_{\mathrm{vert}} : X^{p,p} \to X^{p+1,p+1}.$$

The version of the Eilenberg–Zilber theorem that is used for the proof earlier in this thesis is the following cosimplicial version.

Theorem A.4.5 ([23], Theorem A.3.)**.** *Let A be an abelian category with enough injectives. There is a natural isomorphism*

$$\pi^\bullet(\mathrm{Diag}(X)) = H^\bullet(C(\mathrm{Diag}(X))) = H^\bullet(\mathrm{Tot}(C(X))),$$

where $\mathrm{Tot}(C(X))$ *denotes the total complex associated to the double cochain complex* $C(X)$ *as in Definition A.4.3.*

As we aim to construct the preimages of exact elements explicitly, the proof of this is also of interest. Already in [16], the shuffle map and Alexander-Whitney map are explicitly mentioned to be universal homotopy equivalences (cf. p. 217). In our covariant setting, this will take the following form:

Definition A.4.6 ([23], p. 734)**.** The **cosimplicial Alexander-Whitney map** is a map of bicomplexes

$$g^{p,q} : C(X)^{p,q} \to C(X)^{p+q,p+q}$$

defined when denoting $p+q=n$ as

$$g^{p,q} := d^n_{\text{hor}} \circ d^{n-1}_{\text{hor}} \cdots \circ d^{p+1}_{\text{hor}} \circ d^0_{\text{vert}} \circ d^0_{\text{vert}} \cdots \circ d^0_{\text{vert}}.$$

It induces a map

$$g^n : \mathrm{Tot}(C(X))^n \to \mathrm{Diag}(C(X))^n$$

given on each component of $\mathrm{Tot}(C(X))^n$ by the respective $g^{p,q}$.

Definition A.4.7 ([23], p. 735)**.** The **shuffle coproduct formula** is given by a family of linear maps

$$f^{p,q} : C(X)^{p+q,p+q} \to C(X)^{p,q}$$

defined when denoting $p+q=n$ as

$$f^{p,q} := \sum_{\sigma \in \{(p,q)\text{-shuffles}\}} \mathrm{sgn}(\sigma) s^{\sigma(n)}_{\text{hor}} \circ s^{\sigma(n-1)}_{\text{hor}} \cdots \circ s^{\sigma(p+1)}_{\text{hor}} \circ s^{\sigma(p)}_{\text{vert}} \cdots \circ s^{\sigma(1)}_{\text{vert}}$$

The proof that the Alexander-Whitney map and the shuffle coproduct formula are actually homotopy inverses is more often found in the simplicial instead of cosimplicial formulations. Of course, the statement itself does not depend on this formulation, as a cosimplicial object is simplicial in the opposite category. Heuristically, it must be true that any homotopy equivalence in this setting must be constructed from the bisimplicial and abelian structure alone, as this is the only required structure in our setting. It can be explicitly constructed in a series of steps.

Theorem A.4.8 (cf. [21], Chapter III, Theorem 2.4., pp. 145, 149-150)**.** *The inclusion*

$$\iota : N(A) \to C(A)$$

of the normalised chain complex

$$N(A)_n := \bigcap_{i=0}^{n-1} \ker(d_i)$$

in the Moore complex of a simplicial abelian group A is a chain homotopy equivalence.

Its homotopy inverse f satisfies

$$f \circ \iota = \mathrm{id}$$

and

$$\iota \circ f - \mathrm{id} = dT + Td$$

for a homotopy T defined in degree n by

$$T_n := i_0 \dots i_{n-2} t_{n-1} f_{n-2} \dots f_0 + i_0 \dots i_{n-3} t_{n-2} f_{n-3} \dots f_0 + \cdots + i_0 t_1 f_0 + t_0,$$

where

$$f_a(x) = x - s_{j+1} d_{j+1}(x)$$

for $n \geq a+2$ and

$$f = f_{n-2} \dots f_0$$

and

$$t_b(x) = (-1)^b s_{b+1}(x)$$

for $n \geq b+1$ and all the i_c refer to inclusions of different subcomplexes.

However, in many cases the normalised chain complex is not treated as a subcomplex, but rather a quotient. This is according to the following fact.

Theorem A.4.9 (cf. [21], p. 146). *Let A be a simplicial abelian group. There is an isomorphism of chain complexes*

$$N(A) \cong A/D(A).$$

The reason why we are considering these normalised complexes is the fact that in the special case of the product of two simplicial abelian groups there is a very explicit proof for the Eilenberg–Zilber theorem.

Definition A.4.10 (cf. [17], p. 63). Let K and L simplicial abelian groups. Then their **product** is defined as

$$(K \times L)_q = K_q \otimes L_q,$$

where face and degeneracy maps correspond to the tensor of the respective face and degeneracy maps of K and L.

Theorem A.4.11 (cf. [18], Theorem 2.1a, p. 53). *Let K and L be simplicial abelian groups. Then there is a chain homotopy equivalence*

$$N(K \times L) \underset{\underline{g}}{\overset{\underline{f}}{\rightleftarrows}} N(K) \otimes N(L),$$

where $\underline{f}$ is induced by the Alexander-Whitney map and $\underline{g}$ is induced by the shuffle coproduct formula. Explicitly,

$$\underline{fg} = \mathrm{id}$$

and there is a homotopy $\underline{\Phi}$ such that

$$\underline{gf} - \mathrm{id} = d\underline{\Phi} + \underline{\Phi}d.$$

The homotopy can be extended to a map Φ on the unnormalised complexes that is determined by the equations

$$\Phi_0 = 0$$

where Φ_0 denotes the map in degree 0 and

$$\Phi_q = -(\Phi_{q-1})' + (gf)'s_0.$$

Together with the earlier fact about the normalised complexes being homotopic to the unnormalised ones

Theorem A.4.12 (cf. [18], Theorem 2.1). *Let K and L be simplicial abelian groups. Then the maps f and g induce a chain homotopy equivalence*

$$C(K \times L) \underset{g}{\overset{f}{\rightleftarrows}} C(K) \otimes C(L).$$

The theorem in this form is what is originally referred to as the Eilenberg–Zilber theorem. It relates to the generalised version in the sense that the product can also be used to define a bisimplicial abelian group, given at the index (p, q) by

$$K_p \otimes L_q.$$

By definition, its diagonal simplicial group precisely agrees with $K \times L$. We will now see that from this result, one can obtain the generalised version directly, at least for simplicial abelian groups.

Corollary A.4.13 (cf. [21], pp. 205-206)**.** *For $K = \mathbb{Z}\Delta^p$ and $L = \mathbb{Z}\Delta^q$, we obtain that*

$$K \times L = \mathrm{Diag}(\mathbb{Z}\Delta^{p,q})$$

and

$$C(\mathbb{Z}\Delta^p) \otimes C(\mathbb{Z}\Delta^q) = \mathrm{Tot}(C(\mathbb{Z}\Delta^{p,q}))$$

up to natural isomorphism, such that we obtain a homotopy equivalence

$$C(\mathrm{Diag}(\mathbb{Z}\Delta^{p,q})) \underset{g}{\overset{f}{\rightleftarrows}} \mathrm{Tot}(C(\mathbb{Z}\Delta^{p,q})).$$

Every bisimplicial abelian group A sits in a functorial exact sequence

$$\bigoplus_{\tau\to\sigma} \mathbb{Z}\Delta^{r,s} \to \bigoplus_{\sigma:\Delta^{p,q}\to A} \mathbb{Z}\Delta^{p,q} \to A \to 0.$$

The functors Tot *and* Diag *are both right exact and preserve direct sums, so the chain maps*

$$f : C(\mathrm{Diag}(\mathbb{Z}\Delta^{p,q})) \to \mathrm{Tot}(C(\mathbb{Z}\Delta^{p,q}))$$

uniquely extend to a natural chain map

$$f : C(\mathrm{Diag}(A)) \to \mathrm{Tot}(C(A))$$

which is natural in bisimplicial abelian groups A. Similarly, the chain maps

$$g : \mathrm{Tot}(C(\mathbb{Z}\Delta^{p,q})) \to C(\mathrm{Diag}(\mathbb{Z}\Delta^{p,q}))$$

induce a natural chain map

$$g : \mathrm{Tot}(C(A)) \to C(\mathrm{Diag}(A)).$$

The same argument implies that the bicosimplicial chain homotopies $fg \simeq \mathrm{id}$ and $gf \simeq \mathrm{id}$ extend uniquely to chain homotopies which are natural in bisimplicial abelian groups.

This description of how to extend the Eilenberg–Zilber theorem as in Theorem A.4.12 to a more general version also has the practical implication that the homotopy in the generalised version can be constructed on the level of bisimplices and that whenever we have a formula describing the homotopy as a sum of terms constructed from identity, face and degeneracy maps, the same formulas can be transferred to any bisimplicial abelian group. Currently, we have that the homotopy can be expressed in these terms and Φ which appears in Theorem A.4.11. Luckily, there is indeed a formula for this term. It is attributed to Rubio, who published it in [38] in 1991 (cf. pp. 41-42).

Theorem A.4.14 (cf. [36], Theorem 6.2)**.** *The map Φ defined as in Theorem A.4.11 is equivalently characterised by the formula*

$$\begin{aligned}\Phi_n = \sum (-1)^{n-p-q}\,\mathrm{sgn}((\alpha,\beta))\big(&s^{\mathrm{hor}}_{\beta(q)+n-p-q}\cdots s^{\mathrm{hor}}_{\beta(1)+n-p-q}s^{\mathrm{hor}}_{n-p-q-1}d^{\mathrm{hor}}_{n-q+1}\cdots d^{\mathrm{hor}}_{n}\\ &s^{\mathrm{vert}}_{\alpha(p+1)+n-p-q}\cdots s^{\mathrm{vert}}_{\alpha(1)+n-p-q}d^{\mathrm{vert}}_{n-p-q}\cdots d^{\mathrm{hor}}_{n-q-1}\big)\end{aligned}$$

where the sum is over all indices (p, q, α, β) where $0 \le q \le n-1$ and $0 \le p \le n-q-1$ and (α, β) is a $(p+1, q)$-shuffle.

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