Cycle classes for algebraic De Rham cohomology and crystalline cohomology

Dissertation

zur

Erlangung des Doktorgrades (Dr. rer. nat.)

der

Mathematisch-Naturwissenschaftlichen Fakultät

der

Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

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Bonn 2002

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

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Tag der Promotion: 17.06.2002

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Introduction

The main part of this thesis comprising of the first two chapters is devoted to constructing cycle classes for algebraic De Rham cohomology. In order to give some motivation let us consider a connected scheme X which is smooth over a noetherian, regular and separated scheme S. Let $\mathrm{Vec}(X)$ denote the category of vector bundles on X (i.e. the category of locally free $\mathcal{O}_X$-modules of finite type), let $D(X)$ denote the derived category of the category of $\mathcal{O}_X$-modules and $D(X)_{\mathrm{perf}}$ the triangulated subcategory of $D(X)$ consisting of all perfect complexes. Then by [SGA 6] the canonical map between Grothendieck groups

$$K_0(\mathrm{Vec}(X)) \to K_0(D(X)_{\mathrm{perf}})$$

is an isomorphism making $K_0(D(X)_{\mathrm{perf}})$ into a λ-ring (augmented over $\mathbb{Z}$). Recall from [BI] that there exists a (uniquely determined) theory of Chern classes for the De Rham cohomology of X/S: For any object $E \in \mathrm{Vec}(X)$ there are Chern classes $c_i(E) \in H^{2i}_{\mathrm{DR}}(X/S)$ for $i \in \mathbb{N}$ satisfying functoriality, normalization and additivity. Moreover, the splitting principle holds and thus these Chern classes satisfy the usual formulas for exterior powers, symmetric powers and duals. Therefore we may apply the general theory of λ-rings ([SGA 6, V]), and if Ch( ) denotes the functor which assigns to a commutative and graded $\mathbb{Z}$-algebra the corresponding $\mathbb{Z}$-augmented λ-ring we obtain a total Chern class map

$$\tilde{c}\colon K_0(D(X)_{\mathrm{perf}}) \to \mathrm{Ch}(H^{2*}_{\mathrm{DR}}(X/S)).$$

Now suppose we are given $d \in \mathbb{N}$ and line bundles E_i with global sections s_i for $1 \le i \le d$. Let $K_i^{\cdot}$ denote the complex $\mathcal{O}_X \to E_i$ with $\mathcal{O}_X$ in degree -1 and where the map is induced by s_i. Forming $K^{\cdot} = \bigotimes_{i=1}^d K_i^{\cdot}$ one computes

$$\tilde{c}_d(K^{\cdot}) = (-1)^{d-1}\,(d-1)!\,\prod_{i=1}^{d} c_1(E_i).$$

If each E_i is the vector bundle induced by an effective Cartier divisor D_i on X which is smooth over S and if the scheme theoretical intersection $\bigcap_{i=1}^{j} D_i$ is a smooth S-scheme of codimension j in X for $j = 1, \ldots, d$ then the class $\prod_{i=1}^{d} c_1(E_i)$ coincides with the cycle class of the subscheme $\bigcap_{i=1}^{d} D_i$ which was constructed in [B, VI].

In view of the preceding discussion one may ask if, firstly, there exist cycle class maps yielding the correct cohomology classes without factors like $(d-1)!$, and secondly if it is possible to construct cycle classes for closed subschemes which are not necessarily smooth. These problems will be investigated in the first two chapters of this thesis. More specifically, let now S be a locally noetherian scheme, X a smooth regular scheme over S. Then for a closed subset Y of X and a natural number c such that $c \leq \operatorname{codim}(Y, X)$ where $\operatorname{codim}(Y, X)$ denotes the codimension of Y in X we construct cycle class maps

$$\mathrm{cl}_Y^c \colon K_0(D_Y(X)_{\mathrm{perf}}) \to H^{2c}_{\mathrm{DR}}(X/S).$$

Here $D_Y(X)_{\mathrm{perf}}$ denotes the thick triangulated subcategory of $D(X)$ consisting of all those perfect complexes on X which are acyclic outside Y with corresponding Grothendieck group $K_0(D_Y(X)_{\mathrm{perf}})$. Actually the maps cl_Y^c are supported on Y in the sense that they factor through the canonical morphism

$$H^{2c}_Y(X,\, \tau_{\geq c}(\Omega^{\cdot}_{X/S})) \to H^{2c}_{\mathrm{DR}}(\Omega^{\cdot}_{X/S})$$

where

$$\tau_{\geq c}(\Omega^{\cdot}_{X/S}) = \ldots \to 0 \to \Omega^{c}_{X/S} \to \Omega^{c+1}_{X/S} \to \ldots$$

is the brutal truncation of the De Rham complex in degree c. This truncated complex plays a crucial role because it is not hard to prove purity results like the vanishing of the relative hypercohomology sheaves $\underline{H}^i_Y(\tau_{\geq c}(\Omega^{\cdot}_{X/S}))$ for all $i < 2c$ which in turn make it possible to define these cycle maps locally. It is in fact the factorization

$$K_0(D_Y(X)_{\mathrm{perf}}) \to H^{2c}_Y(X,\, \tau_{\geq c}(\Omega^{\cdot}_{X/S}))$$

of cl_Y^c which is constructed explicitly. Of course since $H^i_Y(X,\, \tau_{\geq c}(\Omega^{\cdot}_{X/S})) = 0$ holds for all $i < 2 \cdot \operatorname{codim}(Y, X)$ the only case which is of interest here is when $c = \operatorname{codim}(Y, X)$.

It turns out that the morphisms cl_Y^c have all the properties one expects like the validity of intersection formulas and compatibility with flat morphisms, base change and Künneth morphisms. Furthermore, we use these cycle maps to define group homomorphisms

$$\eta_X^r \colon Z^r(X) \to H^{2r}_{\mathrm{DR}}(X/S)$$

where $Z^r(X)$ is the group of codimension r cycles on the scheme X. Then we show that η_X^r passes to rational equivalence and also that $\eta_X([Y])$ coincides with Berthelot's De Rham cohomology class ([B, VI, 3.1]) if Y is a closed subscheme of X which is smooth over S. Moreover, in the case where S is the spectrum of a field of characteristic zero the maps η_X^r agree with the cycle class maps obtained by Hartshorne in his paper [H2].

If X is in addition quasi-projective and S is the spectrum of a field we prove that the collection $(\eta^r_X)_{r\in\mathbb{N}}$ induces a ring homomorphism between the Chow group of X and the cohomology ring $H^{2*}_{\mathrm{DR}}(X/S)$. The reason why we need to restrict ourselves to the algebraic and quasi-projective case where we have the moving lemma at our disposal is due to the fact that the multiplicative structure of the Chow group is defined by making use of the refined Gysin morphisms of Fulton and MacPherson (see [F, Chap. 8]), and it is not at all well understood how De Rham cohomology behaves under these Gysin maps. However there is another way to define a product structure on the Chow group of a smooth variety over a field, namely by using the Bloch-Quillen isomorphism ([Q1]) and the product structure on K-theory. Although we make use of convenient K-theory spectra in the construction of the maps η^r_X it is highly unclear how to relate the K-theory product with the cup product on $H^{2*}_{\mathrm{DR}}(X/S)$ if X is not necessarily quasi-projective.

It is evident that the construction of the maps cl^c_Y and η^r_X fills out a non trivial gap in the theory of algebraic De Rham cohomology since before the existence of cycle classes without disturbing factors like $(c-1)!$ and $(r-1)!$ was only known if S was the spectrum of a perfect field (see [H2] for characteristic zero and [Gr] for characteristic $p>0$), if the cycles were induced by closed subschemes which were smooth over the base scheme S ([B]) or when torsion was neglected ([GM]).

Let us briefly indicate the construction of the cycle maps cl^c_Y. For that purpose let $E^\cdot = \tau_{\geq c}(\Omega^\cdot_{X/S})$. Since $\underline{H}^i_Y(E^\cdot)=0$ for all $i<2c$ it follows that the presheaf $U\mapsto H^{2c}_{Y\cap U}(U,\, E^\cdot|U)$ ($U\subseteq X$ open) is canonically isomorphic to the sheaf $\underline{H}^{2c}_Y(E^\cdot)$. Hence we are reduced to treating the case where X is affine. If d denotes the differential in degree c of $E^\cdot$ then from a hypercohomology spectral sequence we obtain an isomorphism

$$\mathrm{Ker}(H^c_Y(X,\, d)) \xrightarrow{\cong} H^{2c}_Y(X,\, E^\cdot).$$

Now we consider a strictly perfect complex $F^\cdot$ on X with a connection ∇, i.e. a connection on the associated graded module $F=\bigoplus_{k\in\mathbb{N}} F^k$ which preserves the grading. Then ∇ gives rise to a connection on $\mathcal{E}nd_{\mathcal{O}_X}(F)$ which we shall also denote by ∇ in what follows. If φ denotes the differential of $F^\cdot$ we can form the power $\nabla(\varphi)^c$ where the multiplication is given by the canonical algebra structure of $\Omega^\cdot_{X/S}\otimes_{\mathcal{O}_X}\mathcal{E}nd_{\mathcal{O}_X}(F)$. Then it can be shown that $\nabla(\varphi)^c$ is up to coboundaries divisible by $c!$ yielding a degree c cocycle $\frac{\nabla(\varphi)^c}{c!}$ of the complex $G^\cdot=\Omega^c_{X/S}\otimes_{\mathcal{O}_X}\mathcal{E}nd^\cdot_{\mathcal{O}_X}(F^\cdot)$. Applying the relative trace morphism Tr_Y of the strictly perfect complex $G^\cdot$ to $\frac{\nabla(\varphi)^c}{c!}$ we obtain an element of degree c of the complex $(H^c_Y(X,\, E^p))_{p\in\mathbb{Z}}$. In order to get a cohomology class in $H^{2c}_Y(X,\, E^\cdot)$ by using the above isomorphism we have to show that this element lies in the kernel of $H^c_Y(X,\, d)$. We reduce this problem to the case where X is a local regular ring and Y is the closed point of X. Then we prove that $K_0(D_Y(X)_{\mathrm{perf}})$ is generated by Koszul complexes for which the term $\mathrm{Tr}_Y(\frac{\nabla(\varphi)^c}{c!})$ can be easily computed, and from the computation we immediately see that it lies in the kernel of $H^c_Y(X,\, d)$. Having defined a cohomology class for a strictly perfect complex with connection we then establish that this yields a morphism $K_0(D_Y(X)_{\mathrm{perf}}) \to H^{2c}_Y(X,\, E^\cdot)$.

In chapter three we study the problem of constructing crystalline cycle classes for schemes $X/W(k)$ where k is a perfect field of characteristic $p > 0$, $W(k)$ is the ring of Witt vectors of k and X is a smooth k-scheme. This has been done by M. Gros in [Gr] using purity theorems for logarithmic Hodge-Witt cohomology. In order to sketch our approach let W_n denote the affine PD-scheme induced by the truncation $W(k)/p^nW(k)$ for $n \geq 1$. Let X_{Zar} denote the Zariski topos and let $W_n\Omega^\cdot_X$ denote the De Rham-Witt complex of order n for X. Then the complex $W_n\Omega^\cdot_X$ computes the crystalline cohomology of X/W_n in the sense that the hypercohomology $\mathbf{H}^*(X,\, W_n\Omega^\cdot_X)$ is canonically isomorphic to $H^*_{\mathrm{crys}}(X/W_n)$. Using the fact that each of the sheaves $W_n\Omega^p_X$ is Cohen-Macaulay with respect to the codimension filtration on X (i.e. the corresponding Cousin complex of $W_n\Omega^p_X$ is a resolution of $W_n\Omega^p_X$) we construct a canonical flabby resolution $\mathcal{C}(W_n\Omega^\cdot_X)$ of $W_n\Omega^\cdot_X$ whose components are certain direct sums of skyscraper sheaves induced by the "punctual" cohomology groups $H^*_x(W_n\Omega^p_X)$ for $x \in X$ and $p \in \mathbb{N}$.

Given a closed, integral subscheme V of X with $c = \mathrm{codim}(V,\, X)$ and generic point ζ we then construct for each $n \in \mathbb{N}$ an element $\mathrm{cl}_{X,\,n}(V)$ in $H^c_\zeta(W_n\Omega^c_X) \subseteq \Gamma(X,\, \mathcal{C}^{2c}(W_n\Omega^\cdot_X))$ and by studying the action of the Frobenius morphism of $(W_n\Omega^\cdot_X)_{n\geq 1}$ on the pro-objects $(H^d_x(W_n\Omega^\cdot_X))_{n\geq 1}$ for $x \in X$ of codimension $d \in \{c,\, c+1\}$ we show that $\mathrm{cl}_{X,\,n}(V)$ is a cocycle of $\mathcal{C}(W_n\Omega^\cdot_X)$ inducing therefore a class in $H^{2c}_{\mathrm{crys}}(X/W_n)$ which in fact agrees with the cycle class constructed for V in [Gr].

The construction of $\mathrm{cl}_{X,\,n}(V)$ is inspired by the following observation: Let $\mathfrak{U}$ be a smooth formal $\mathrm{Spf}(W(k))$-scheme which lifts an open neighbourhood of ζ in X, and denote by U_n the restriction of $\mathfrak{U}$ modulo p^n. Then the stalk $\mathcal{O}_{U_n,\,\zeta}$ is a local, c-dimensional Cohen-Macaulay ring, and any system of parameters for this ring induces via the morphism $d\log$ a class in $H^c_\zeta(\Omega^c_{U_n/W_n})$ which does not depend on the choice of this system and which gives rise to a class $\mathrm{cl}_\zeta(U_n/W_n)$ in the relative De Rham cohomology $H^{2c}_{X^c/X^{c+1}}(U_n,\, \Omega^{\cdot}_{U_n/W_n})$ where X^c (resp. X^{c+1}) is the set of points in X whose codimension is c (resp. $c+1$). In fact, the restriction of $\mathrm{cl}_{X,\,n}(V)$ to the open subset U_n of X actually yields a class lying in $H^{2c}_{X^c}(U_n,\, \Omega^{\cdot}_{U_n/W_n})$ which lifts $\mathrm{cl}_\zeta(U_n/W_n)$ with respect to the canonical map $H^{2c}_{X^c}(U_n,\, \Omega^{\cdot}_{U_n/W_n}) \to H^{2c}_{X^c/X^{c+1}}(U_n,\, \Omega^{\cdot}_{U_n/W_n})$.

My sincere thanks go to my advisor Prof. Dr. G. Faltings for all his help and encouragement, to my parents for constant support and to Andrea Miller for helpful conversations.

1 Preliminaries

1.1 Horizontality

Let S be a scheme. We shall say that a morphism $f\colon X \to S$ is weakly smooth if $\Omega^1_{X/S}$ is a vector bundle on X (i.e. a locally free $\mathcal{O}_X$-module of finite type). For a weakly smooth X/S we know by [EGA IV, 16.10.6] that for each $x \in X$ there exists an open neighbourhood U and a finite family $(x_\lambda)_{\lambda\in L}$ of elements in $\Gamma(U, \mathcal{O}_X)$ such that the collection $(d(x_\lambda))_{\lambda\in L}$ forms a base for the $\Gamma(U, \mathcal{O}_X)$-module $\Gamma(U, \Omega^1_{X/S})$.
In the following we fix such a weakly smooth scheme X/S. By an $\mathcal{O}_X$-module we shall mean a quasi-coherent $\mathcal{O}_X$-module throughout this section.

Let E be an $\mathcal{O}_X$-module. Recall that a connection on E is a homomorphism ∇ of abelian sheaves

$$\nabla\colon E \to \Omega^1_{X/S} \otimes_{\mathcal{O}_X} E$$

satisfying

$$\nabla(t \cdot e) = t \cdot \nabla(e) + d(t) \otimes e$$

where t and e are sections of $\mathcal{O}_X$ and E respectively over an open subset of X, and $d(t)$ denotes the image of t under the canonical exterior differentiation $d\colon \mathcal{O}_X \to \Omega^1_{X/S}$.

A connection ∇ may be extended to a homomorphism of abelian sheaves

$$\nabla^i\colon \Omega^i_{X/S} \otimes_{\mathcal{O}_X} E \to \Omega^{i+1}_{X/S} \otimes_{\mathcal{O}_X} E$$

via

$$\nabla^i(\omega \otimes e) = d(\omega) \otimes e + (-1)^i \omega \wedge \nabla(e)$$

where ω and e are sections of $\Omega^i_{X/S}$ and E respectively over an open subset of X, and where $\omega \wedge \nabla(e)$ denotes the image of $\omega \otimes \nabla(e)$ under the canonical map

$$\Omega^i_{X/S} \otimes_{\mathcal{O}_X} (\Omega^1_{X/S} \otimes_{\mathcal{O}_X} E) \to \Omega^{i+1}_{X/S} \otimes_{\mathcal{O}_X} E$$

which sends $\omega \otimes \tau \otimes e$ to $(\omega \wedge \tau) \otimes e$. Hence the collection $(\nabla^i)_{i\in\mathbb{N}}$ (with $\nabla^0 = \nabla$) defines an endomorphism on the graded $\mathcal{O}_X$-module $\Omega^{\cdot}_{X/S} \otimes_{\mathcal{O}_X} E$ of degree 1 which we denote by ∇ again. Recall that ∇ is integrable if $\nabla \circ \nabla = 0$ and in this case the resulting complex $(\Omega^{\cdot}_{X/S} \otimes_{\mathcal{O}_X} E, \nabla)$ defines the De Rham complex of (E, ∇).

If the $\mathcal{O}_X$-module $\Omega^1_{X/S}$ is free and we are given a base $\{\omega_1, \ldots, \omega_n\}$, we shall use the following notation: Let $\{\omega_1^*, \ldots, \omega_n^*\}$ denote the dual base.

Then for a section e of E over an open subset of X and $1 \leq p \leq n$ we set

$$\nabla_p(e) = \left(\omega_p^* \otimes \mathrm{id}_E\right)(\nabla(e)).$$

Let (E, ∇) and (F, ∇') be $\mathcal{O}_X$-modules with connections. Recall that an $\mathcal{O}_X$-linear map $s\colon E \to F$ is horizontal if the diagram

$$\begin{array}{ccc} E & \xrightarrow{s} & F \\ \nabla \big\downarrow & & \big\downarrow \nabla' \\ \Omega^1_{X/S} \otimes_{\mathcal{O}_X} E & \xrightarrow{\mathrm{id}_{\Omega^1_{X/S}} \otimes s} & \Omega^1_{X/S} \otimes_{\mathcal{O}_X} F \end{array}$$

is commutative.

Let (E, ∇) and (F, ∇') be $\mathcal{O}_X$-modules with connections and suppose that E is of finite presentation. Then the map

$$\nabla''\colon \mathcal{H}om_{\mathcal{O}_X}(E, F) \to \Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(E, F)$$

given by

$$\nabla''(\varphi) = \nabla' \circ \varphi - (\mathrm{id}_{\Omega^1_{X/S}} \otimes \varphi) \circ \nabla$$

defines a connection on $\mathcal{H}om_{\mathcal{O}_X}(E, F)$ which is integrable if both ∇ and ∇' are integrable. If $E = F$ and $\nabla = \nabla'$ we will often denote the induced connection on $\mathcal{E}nd_{\mathcal{O}_X}(E)$ by ∇ again. In the context of this description we can state our first result.

Lemma 1.1.1 *Let E be a vector bundle on X equipped with a connection ∇ and consider $\mathcal{O}_X$ with the canonical exterior differentiation $d\colon \mathcal{O}_X \to \Omega^1_{X/S}$ as its connection. Then the trace morphism*

$$\mathrm{tr}\colon \mathcal{E}\mathrm{nd}_{\mathcal{O}_\mathrm{X}}(\mathrm{E}) \to \mathcal{O}_\mathrm{X}$$

is horizontal.

Proof. We may assume that E and $\Omega^1_{X/S}$ are free with bases $e_1, \ldots, e_m$ and $dx_1, \ldots, dx_n$. Let $\varphi \in \mathrm{End}_{\mathcal{O}_X}(E)$. By what we have explained before the connection ∇ induces a connection on $\mathcal{E}nd_{\mathcal{O}_X}(E)$ which we will denote by the same symbol. We have to prove that

$$d(\mathrm{tr}(\varphi)) = (\mathrm{id}_{\Omega^1_{X/S}} \otimes \mathrm{tr})(\nabla(\varphi)).$$

As $\nabla(\varphi) = \sum_{p=1}^{n} dx_p \otimes \nabla_p(\varphi)$ it suffices to show that $d_p(\mathrm{tr}(\varphi)) = \mathrm{tr}(\nabla_p(\varphi))$ for each p. Writing $\varphi(e_q) = \sum_{r=1}^{m} \lambda_{q\,r} \cdot e_r$ and $\nabla(e_q) = \sum_{r=1}^{m} \tau_{q\,r} \otimes e_r$ with sections $\lambda_{q\,r}$ and $\tau_{q\,r}$ of $\mathcal{O}_X$ and $\Omega^1_{X/S}$ respectively we obtain for $1 \le q \le m$

$$\begin{aligned}(d_p \otimes \mathrm{id}_E)(\nabla \circ \varphi)(e_q) &= \sum_{r,\,s=1}^{m} d_p(\lambda_{qr}\tau_{rs}) \cdot e_s + \sum_{r=1}^{m} d_p(d\lambda_{qr}) \cdot e_r \\ (d_p \otimes \mathrm{id}_E)(\mathrm{id}_{\Omega^1_{X/S}} \otimes \varphi)(\nabla(e_q)) &= \sum_{r,\,s=1}^{m} d_p(\lambda_{rs}\tau_{qr}) \cdot e_s.\end{aligned}$$

Thus we conclude that

$$\mathrm{tr}(\nabla_p(\varphi)) = \sum_{r=1}^{m} d_p(d\lambda_{rr}) = d_p(\mathrm{tr}(\varphi)). \quad \square$$

Let $(E,\,\nabla)$, $(F,\,\nabla')$ be $\mathcal{O}_X$-modules with connections. Then we get a connection ∇'' on $E \otimes_{\mathcal{O}_X} F$ by the formula

$$\nabla''(e \otimes f) = \nabla(e) \otimes f + e \otimes \nabla'(f)$$

where e and f are sections of E and F respectively over an open subset of X. Again this connection turns out to be integrable if ∇ and ∇' are integrable.

Lemma 1.1.2 *Assume that E, F, G are $\mathcal{O}_X$-modules with connections and consider the morphism*

$$\sigma\colon \mathcal{H}om_{\mathcal{O}_X}(F,\,G) \otimes_{\mathcal{O}_X} \mathcal{H}om_{\mathcal{O}_X}(E,\,F) \to \mathcal{H}om_{\mathcal{O}_X}(E,\,G)$$

induced by composing $\mathcal{O}_X$-linear maps. Then σ is horizontal.

Proof. We denote the connections on the corresponding $\mathcal{O}_X$-modules by the same letter ∇. Then for sections φ and ψ of the modules $\mathcal{H}om_{\mathcal{O}_X}(E,\,F)$ and $\mathcal{H}om_{\mathcal{O}_X}(F,\,G)$ over an open subset of X it is easy to check that

$$\begin{aligned}(\mathrm{id}_{\Omega^1_{X/S}} \otimes \sigma)(\nabla(\psi) \otimes \varphi) &= \nabla(\psi) \circ \varphi \\ (\mathrm{id}_{\Omega^1_{X/S}} \otimes \sigma)(\psi \otimes \nabla(\varphi)) &= (\mathrm{id}_{\Omega^1_{X/S}} \otimes \psi) \circ \nabla(\varphi).\end{aligned}$$

As we obviously have $\nabla(\psi \circ \varphi) = \nabla(\psi) \circ \varphi + (\mathrm{id}_{\Omega^1_{X/S}} \otimes \psi) \circ \nabla(\varphi)$, the result follows. $\square$

Let E be an $\mathcal{O}_X$-algebra. A connection on E will be called compatible if the multiplication map $E \otimes_{\mathcal{O}_X} E \to E$ is horizontal.

Lemma 1.1.3 *Let E be an $\mathcal{O}_X$-algebra with a compatible connection ∇. Let X, Y be sections of the graded algebra $\Omega^{\cdot}_{X/S} \otimes_{\mathcal{O}_X} E$ and assume that X is homogeneous of degree n. Then we have*

$$\nabla(X \cdot Y) = \nabla(X) \cdot Y + (-1)^n X \cdot \nabla(Y).$$

Proof. Follows from a simple local calculation. $\square$

Corollary 1.1.4 *Under the hypotheses of* 1.1.3 *suppose in addition that the connection is integrable. Then*

$$\nabla(X \cdot \nabla(X)^{d-1}) = \nabla(X)^d$$

for every $d \geq 1$.

Let $E^{\cdot}$ be a complex of $\mathcal{O}_X$-modules. Then a connection on $E^{\cdot}$ shall be given by a collection $\nabla = (\nabla_{E^p})_{p \in \mathbb{Z}}$ where ∇_{E^p} is a connection on E^p for each $p \in \mathbb{Z}$. It follows that the connections on $E^{\cdot}$ are in bijective correspondence with those connections on its associated graded $\mathcal{O}_X$-module $\bigoplus_{n \in \mathbb{Z}} E^n$ which preserve the grading.
If $F^{\cdot}$ is another complex with a connection then by what we have pointed out above the complex $E^{\cdot} \otimes_{\mathcal{O}_X} F^{\cdot}$ (resp. $\mathcal{H}om^{\cdot}_{\mathcal{O}_X}(E^{\cdot}, F^{\cdot})$ if $E^{\cdot}$ and $F^{\cdot}$ are both bounded and each E^p is of finite presentation) comes along with a connection in a natural way.
We shall say that a chain map $f = (f^p)\colon E^{\cdot} \to F^{\cdot}$ is horizontal if f^p is horizontal for every p.

Proposition 1.1.5 *Let $E^{\cdot}$, $F^{\cdot}$, $G^{\cdot}$ be bounded complexes of $\mathcal{O}_X$-modules with connections. Then the following chain maps are horizontal:*

(a) *The evaluation map $\mathcal{H}om^{\cdot}_{\mathcal{O}_X}(E^{\cdot}, F^{\cdot}) \otimes_{\mathcal{O}_X} E^{\cdot} \to F^{\cdot}$.*

(b) *The canonical map $\mathcal{H}om^{\cdot}_{\mathcal{O}_X}(E^{\cdot}, F^{\cdot}) \otimes_{\mathcal{O}_X} G^{\cdot} \to \mathcal{H}om^{\cdot}_{\mathcal{O}_X}(E^{\cdot}, F^{\cdot} \otimes_{\mathcal{O}_X} G^{\cdot})$.*

(c) *The map $\mathcal{H}om^{\cdot}_{\mathcal{O}_X}(F^{\cdot}, G^{\cdot}) \otimes_{\mathcal{O}_X} \mathcal{H}om^{\cdot}_{\mathcal{O}_X}(E^{\cdot}, F^{\cdot}) \to \mathcal{H}om^{\cdot}_{\mathcal{O}_X}(E^{\cdot}, G^{\cdot})$ induced by composing $\mathcal{O}_X$-linear maps.*

Here each E^p (and each F^p for (c)*) is assumed to be of finite presentation.*

Proof. (c) follows from 1.1.2, for the rest we may assume that $\Omega^1_{X/S}$ is free with a base $\{dx_1, \ldots, dx_n\}$. Let (E, ∇), (F, ∇') be $\mathcal{O}_X$-modules with connections where E is of finite presentation. We denote by ∇'' the resulting connection on $\mathcal{H}om_{\mathcal{O}_X}(E, F)$. Then for $1 \leq p \leq n$, sections φ and e of $\mathcal{H}om_{\mathcal{O}_X}(E, F)$ and E respectively over an open subset of X we have

$$\nabla''_p(\varphi)(e) = \nabla'_p(\varphi(e)) - \varphi(\nabla_p(e)).$$

Using this formula the assertions (a) and (b) are then easy to prove. $\square$

1.2 Supercommutators and supertraces

R be a (commutative) ring, and let $S = \bigoplus_{n\in\mathbb{Z}} S^n$ be an associative, graded R-algebra. The degree of an element $s \in S^n$ will be denoted by $|s|$, i.e. $|s| = n$. Given homogeneous elements $x,\, y \in S$ we may form the supercommutator

$$[x,\, y] = x \cdot y - (-1)^{|x|\cdot|y|} y \cdot x.$$

Three basic properties of supercommutators which result directly from their definition are listed in the following lemma.

Lemma 1.2.1 *Let $x,\, y,\, z$ be homogeneous elements of S.*

(a) $[x,\, y] = (-1)^{|x|\cdot|y|+1}[y,\, x]$.

(b) $[x,\, y] \cdot z = [x,\, y \cdot z] - (-1)^{|x|\cdot|y|} y \cdot [x,\, z]$.

(c) $z \cdot [x,\, y] = [z \cdot x,\, y] - (-1)^{|x|\cdot|y|}[z,\, y] \cdot x$.

Our next result reflects a situation which will be relevant to us in Chapter 2 when we will be working with an explicit description of the (relative) trace map of a strictly perfect complex. The lemma is an immediate consequence of 1.2.1 (b)

Lemma 1.2.2 *Assume that the algebra S is unitary with $1_S \in S^0$. Let $x \in S^1$ and $y_0,\, \dots,\, y_n \in S^{-1}$. Assume that there exist elements $r_0,\, \dots,\, r_n$ of R satisfying $[x,\, y_i] = r_i \cdot 1_S$ for $i = 0,\, \dots,\, n$. Then*

$$[x,\, y_0 \cdot \ldots \cdot y_n] = \sum_{p=0}^{n} (-1)^p r_p \cdot \prod_{\substack{q=0,\\ q\neq p}}^{n} y_q.$$

Let $M = \bigoplus_{n\in\mathbb{Z}} M^n$ be a finitely generated, graded and projective R-module. We denote by $\mathrm{End}_R^n(M)$ the module of R-linear endomorphisms on M of degree n, i.e.

$$\mathrm{End}_R^n(M) = \{f \in \mathrm{End}_R(M);\, f(M^m) \subseteq M^{m+n} \,\text{for all}\, m \in \mathbb{Z}\}.$$

In particular the graded R-algebra

$$\mathrm{End}_R^{\cdot}(M) = \bigoplus_{n\in\mathbb{Z}} \mathrm{End}_R^n(M)$$

is a subalgebra of $\mathrm{End}_R(M)$. Let ϵ denote the involution on M given by

$$\epsilon = \bigoplus_{n\in\mathbb{Z}} (-1)^n \mathrm{id}_{M^n}.$$

Then for $f \in \mathrm{End}_R^n(M)$ we see that

$$\epsilon \circ f = \begin{cases} f \circ \epsilon, & n \text{ even} \\ -f \circ \epsilon, & n \text{ odd.} \end{cases}$$

Eventually we can define the supertrace

$$\mathrm{tr}_s \colon \mathrm{End}_R(M) \to R,\ \mathrm{tr}_s(\varphi) = \mathrm{tr}(\epsilon \circ \varphi).$$

Proposition 1.2.3

(a) tr_s *vanishes on endomorphisms of degree* $\neq 0$.

(b) *For homogeneous elements* φ, ψ *of* $\mathrm{End}_R^{\cdot}(M)$ *we have*

$$\mathrm{tr}_s[\varphi,\, \psi] = 0.$$

(c) *Given* $\varphi \in \mathrm{End}_R^0(M)$ *and* $n \in \mathbb{Z}$ *set* $\varphi^n = \varphi|M^n$ *which defines an endomorphism on* M^n. *Then*

$$\mathrm{tr}_s(\varphi) = \sum_{n \in \mathbb{Z}} (-1)^n \mathrm{tr}(\varphi^n).$$

Proof. (a) and (c) are evident. To prove statement (b) it is sufficient to treat the case where $|\varphi| + |\psi| = 0$. Then using the commutativity property of the trace map we deduce

$$\begin{aligned} \mathrm{tr}_s[\varphi,\, \psi] &= \mathrm{tr}(\epsilon\varphi\psi) + (-1)^{|\varphi|^2+1}\mathrm{tr}(\epsilon\psi\varphi) \\ &= (-1)^{|\varphi|^2}(\mathrm{tr}(\varphi\epsilon\psi) - \mathrm{tr}(\epsilon\psi\varphi)) \\ &= (-1)^{|\varphi|^2}(\mathrm{tr}(\epsilon\psi\varphi) - \mathrm{tr}(\epsilon\psi\varphi)) \\ &= 0. \quad \square \end{aligned}$$

Let X be a scheme. Let $E = \bigoplus_{n \in \mathbb{Z}} E^n$ be a graded, quasi-coherent $\mathcal{O}_X$-module of finite presentation and denote by $p^n \colon E \to E^n$ the projection onto its n-th component. For an open subset U of X and $n \in \mathbb{Z}$ consider the module of $\mathcal{O}_U$-linear endomorphisms on $E|U$ of degree n which is given by

$$\mathrm{End}_{\mathcal{O}_U}^n(E|U) = \{f \in \mathrm{End}_{\mathcal{O}_U}(E|U);\, p^k \circ (f|E^l) = 0 \text{ for } k \neq l + n\}.$$

This defines a sheaf of $\mathcal{O}_X$-modules $\mathcal{E}nd_{\mathcal{O}_X}^n(E)$ via

$$\mathcal{E}nd_{\mathcal{O}_X}^n(E)(U) = \mathrm{End}_{\mathcal{O}_U}^n(E|U)$$

and we get a graded algebra

$$\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E) = \bigoplus_{n\in\mathbb{Z}} \mathcal{E}nd^n_{\mathcal{O}_X}(E).$$

In a similar way we can define a graded algebra $\mathcal{E}nd^{\cdot}_{\mathbb{Z}}(\mathcal{F})$ for any sheaf of abelian groups $\mathcal{F}$. We shall make use of $\mathcal{E}nd^{\cdot}_{\mathbb{Z}}(\mathcal{F})$ in 1.3. Returning to the situation where E is a graded and quasi-coherent $\mathcal{O}_X$-module of finite presentation there is an obvious map

$$\mathcal{H}om^n_{\mathcal{O}_X}(E,\, E) = \prod_{k\in\mathbb{Z}} \mathcal{H}om_{\mathcal{O}_X}(E^k,\, E^{k+n}) \to \mathcal{E}nd^n_{\mathcal{O}_X}(E)$$

sending a section $(f_k)_{k\in\mathbb{Z}}$ to $\bigoplus_{k\in\mathbb{Z}} f_k$. It is easy to see that this map is an isomorphism. Thus we get an isomorphism

$$\mathcal{H}om^{\cdot}_{\mathcal{O}_X}(E,\, E) \xrightarrow{\cong} \mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E)$$

of graded $\mathcal{O}_X$-algebras. Henceforth we will not distinguish between these two graded algebras.
If $X = \mathrm{Spec}(R)$ is affine and $E \cong \widetilde{M}$ for a graded R-module $M = \bigoplus_{n\in\mathbb{Z}} M^n$ of finite presentation then the module $\mathcal{E}nd^n_{\mathcal{O}_X}(E)$ is obviously isomorphic to the sheafification of $\mathrm{End}^n_R(M)$. In particular we deduce that the canonical map

$$\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E) \to \mathcal{E}nd_{\mathcal{O}_X}(E)$$

is an isomorphism of $\mathcal{O}_X$-algebras if E is a graded vector bundle.
Now we consider the supertrace at the level of sheaves. Hence for any graded vector bundle $E = \bigoplus_{n\in\mathbb{Z}} E^n$ on the scheme X we obtain an $\mathcal{O}_X$-linear map

$$\mathrm{tr}_s \colon \mathcal{E}nd_{\mathcal{O}_X}(E) \to \mathcal{O}_X$$

such that the following properties hold.

Proposition 1.2.4

(a) tr_s *vanishes on sections of* $\mathcal{E}nd^n_{\mathcal{O}_X}(E)$ *for* $n \neq 0$.

(b) *If* φ *and* ψ *are homogeneous sections of* $\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E)$, *then*

$$\mathrm{tr}_s[\varphi,\, \psi] = 0.$$

(c) *Given a section* φ *of* $\mathcal{E}nd^0_{\mathcal{O}_X}(E)$ *over an open subset* U *of* X *let* φ^n *denote the restriction of* φ *to* $E^n|U$ *for* $n \in \mathbb{Z}$. *Then*

$$\mathrm{tr}_s(\varphi) = \sum_{n\in\mathbb{Z}} (-1)^n \mathrm{tr}(\varphi^n).$$

Proposition 1.2.5 *Let X/S be weakly smooth. Suppose that $E = \bigoplus_{n \in \mathbb{Z}} E^n$ is a graded vector bundle on X and we are given a connection ∇ on E preserving the grading. Consider the bundle $\mathcal{E}nd_{\mathcal{O}_X}(E)$ equipped with the connection induced by ∇. Then tr_s is horizontal.*

Proof. It is clear that the involution ϵ is horizontal. Together with 1.1.1 this implies the assertion. $\square$

Let $E^{\cdot}$ be a bounded complex of vector bundles on a scheme X. Using 1.2.3 (b) we get a chain map $\mathrm{Tr} = (\mathrm{Tr}^n)_{n \in \mathbb{Z}} \colon \mathcal{H}om_{\mathcal{O}_X}(E^{\cdot}, E^{\cdot}) \to \mathcal{O}_X$ where

$$\mathrm{Tr}^n = \begin{cases} \mathrm{tr}_s, & n = 0 \\ 0, & \text{otherwise.} \end{cases}$$

This chain map coincides with the trace map of [SGA 6, I, 8] at least if X is affine.

Corollary 1.2.6 *Let X/S be a weakly smooth scheme and $E^{\cdot}$ a bounded complex of vector bundles on X with a connection. Then the chain map* Tr *is horizontal.*

Let X/S be a weakly smooth scheme and $E = \bigoplus_{n \in \mathbb{Z}} E^n$ a graded $\mathcal{O}_X$-module of finite presentation. Then we may consider $\Omega^{\cdot}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E)$ not only as a subalgebra of $\Omega^{\cdot}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(E)$ but also as a graded algebra by forming the graded tensor product. In the sequel we will write $\hat{\cdot}$ for the multiplication which results from this graded tensor product structure. Thus for sections $\omega \otimes \varphi$ and $\tau \otimes \psi$ of $\Omega^k_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}nd^l_{\mathcal{O}_X}(E)$ and $\Omega^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}nd^n_{\mathcal{O}_X}(E)$ respectively we have

$$\begin{aligned} (\omega \otimes \varphi) \hat{\cdot} (\tau \otimes \psi) &= (-1)^{l \cdot m} (\omega \otimes \varphi) \cdot (\tau \otimes \psi) \\ &= (-1)^{l \cdot m} (\omega \wedge \tau) \otimes (\varphi \cdot \psi). \end{aligned}$$

Given a connection on E which preserves the grading it is clear that the restriction to $\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E)$ of the induced connection ∇ on $\mathcal{E}nd_{\mathcal{O}_X}(E)$ has the same property. For the rest of this section we fix a weakly smooth scheme X/S and a graded $\mathcal{O}_X$-module E of presentation equipped with a connection.∇.

Lemma 1.2.7 *With respect to the previous notation we have*

$$\nabla((\omega \otimes \varphi) \hat{\cdot} (\tau \otimes \psi)) = \nabla(\omega \otimes \varphi) \hat{\cdot} (\tau \otimes \psi) + (-1)^{l+k} (\omega \otimes \varphi) \hat{\cdot} \nabla(\tau \otimes \psi).$$

Proof. Follows easily from 1.1.3. $\square$

Corollary 1.2.8 *We have*

$$\nabla([\omega \otimes \varphi,\, \tau \otimes \psi]^\wedge) = [\nabla(\omega \otimes \varphi),\, \tau \otimes \psi]^\wedge \; + (-1)^{l+k}\, [\omega \otimes \varphi,\, \nabla(\tau \otimes \psi)]^\wedge$$

where $[*,\, *]^\wedge$ *denotes the supercommutator with respect to* $\hat{\cdot}$.

Lemma 1.2.9 *Let* U *be an open affine subset of* X *such that* $\Omega^1_{U/S}$ *is free with a base* $\omega_1, \ldots, \omega_n$. *Let* φ *be an odd degree section of* $\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E)$ *over* U *and suppose that* $[\varphi,\, \nabla_q(\varphi)] = 0$ *for every* $q \in \{1, \ldots, n\}$. *Then*

$$[\varphi,\, \nabla_q(\nabla_r(\varphi))] + [\nabla_q(\varphi),\, \nabla_r(\varphi)] = 0$$

for $q,\, r \in \{1, \ldots, n\}$.

Proof. Using 1.2.8 we observe that

$$0 = \nabla([\varphi,\, \nabla_r(\varphi)]) = [\nabla(\varphi),\, \nabla_r(\varphi)]^\wedge - [\varphi,\, \nabla(\nabla_r(\varphi))]^\wedge.$$

Now the assertion follows by expanding these supercommutators. $\square$

Corollary 1.2.10 *Under the hypotheses of* 1.2.9 *suppose in addition that the degree of* φ *is* 1 *and* $\varphi^2 = 0$, *i.e.* $(E,\, \varphi)$ *is a complex of* $\mathcal{O}_X$*-modules. Then for* $d \geq 1$ *the chain map*

$$\nabla(\varphi)^d \; - d! \cdot \sum_{1 \leq q_1 < \ldots < q_d \leq n} (\omega_{q_1} \wedge \cdots \wedge \omega_{q_d}) \otimes (\nabla_{q_1}(\varphi) \cdot \ldots \cdot \nabla_{q_d}(\varphi))$$

is null homotopic with respect to the complex $\Omega^d_{U/S} \otimes_{\mathcal{O}_U} \mathcal{E}nd^{\cdot}_{\mathcal{O}_U}(E|U)$ *(having the obvious differential). Here the power* $\nabla(\varphi)^d$ *is taken with respect to the multiplication of the algebra* $\Omega^{\cdot}_{U/S} \otimes_{\mathcal{O}_U} \mathcal{E}nd_{\mathcal{O}_U}(E|U)$.

Proof. Setting $N(d) = \{(q_1, \ldots, q_d) \in \mathbb{N}^d;\, q_j \leq d \;\; \forall\; 1 \leq j \leq d\}$ we get

$$\nabla(\varphi)^d = \sum_{(q_1, \ldots, q_d) \in N(d)} (\omega_{q_1} \wedge \cdots \wedge \omega_{q_d}) \otimes (\nabla_{q_1}(\varphi) \cdot \ldots \cdot \nabla_{q_d}(\varphi)).$$

Let $1 \leq j < d$ and $(q_1, \ldots, q_d) \in N(d)$. Then

$$[\nabla_{q_j}(\varphi),\, \nabla_{q_{j+1}}(\varphi)] = \nabla_{q_j}(\varphi) \cdot \nabla_{q_{j+1}}(\varphi) + \nabla_{q_{j+1}}(\varphi) \cdot \nabla_{q_j}(\varphi)$$

is a coboundary of $\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E)$ by 1.2.9. Using 1.2.1 it then follows that

$$\nabla_{q_1}(\varphi) \cdot \ldots \cdot \nabla_{q_{j-1}}(\varphi) \cdot [\nabla_{q_j}(\varphi),\, \nabla_{q_{j+1}}(\varphi)] \cdot \nabla_{q_{j+2}}(\varphi) \cdot \ldots \cdot \nabla_{q_d}(\varphi)$$

is a d-coboundary of that complex which easily implies the assertion. $\square$

To finish this section let E and F be graded vector bundles on X. Then for sections φ and ψ of $\mathcal{E}nd_{\mathcal{O}_X}(E)$ and $\mathcal{E}nd_{\mathcal{O}_X}(F)$ respectively over an open subset of X we may form the tensor product $\varphi \otimes \psi$ and obtain a section of $\mathcal{E}nd_{\mathcal{O}_X}(E \otimes_{\mathcal{O}_X} F)$. This induces a morphism of algebras

$$\mathcal{E}nd_{\mathcal{O}_X}(E) \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(F) \to \mathcal{E}nd_{\mathcal{O}_X}(E \otimes_{\mathcal{O}_X} F)$$

which maps $\mathcal{E}nd^m_{\mathcal{O}_X}(E) \otimes_{\mathcal{O}_X} \mathcal{E}nd^n_{\mathcal{O}_X}(F)$ into $\mathcal{E}nd^{m+n}_{\mathcal{O}_X}(E \otimes_{\mathcal{O}_X} F)$ for $m, n \in \mathbb{Z}$. This morphism is horizontal if E, F are equipped with connections and $E \otimes_{\mathcal{O}_X} F$ is considered with the induced connection. Now we obviously get

Lemma 1.2.11 *Let $m \in \mathbb{Z}$ and suppose that φ (resp. ψ) is an endomorphism of degree m (resp. $-m$) on E (resp. F). Then*

$$\operatorname{tr}_s(\varphi \otimes \psi) = \begin{cases} \operatorname{tr}_s(\varphi) \cdot \operatorname{tr}_s(\psi), & m = 0 \\ 0, & \text{otherwise.} \end{cases}$$

1.3 Superconnections

In this section we fix the following data: A scheme X, a graded, unitary and quasi-coherent $\mathcal{O}_X$-algebra $\mathcal{S} = \bigoplus_{n \in \mathbb{Z}} \mathcal{S}^n$ which is anti-commutative, flat and of finite presentation (as an $\mathcal{O}_X$-module), and a graded vector bundle $\mathcal{E} = \bigoplus_{n \in \mathbb{Z}} \mathcal{E}^n$ on X. Let

$$\mathcal{S}(\mathcal{E}) = \mathcal{S} \otimes_{\mathcal{O}_X} \mathcal{E}$$

which we regard as a graded $\mathcal{O}_X$-module with respect to the total grading induced by the bi-grading $(\mathcal{S}^p \otimes_{\mathcal{O}_X} \mathcal{E}^q)_{(p,q) \in \mathbb{Z}^2}$. Because $\mathcal{S}$ is anti-commutative we can view $\mathcal{S}(\mathcal{E})$ as both a left and right module over $\mathcal{S}$ where the relation between left and right multiplication is given by

$$\omega \cdot \alpha = (-1)^{|\omega||\alpha|}\, \alpha \cdot \omega$$

for homogeneous sections ω and α of $\mathcal{S}$ and $\mathcal{S}(\mathcal{E})$ respectively. In particular the sheaf $\mathcal{E}nd_{\mathcal{S}}(\mathcal{S}(\mathcal{E}))$ (resp. $\mathcal{E}nd(\mathcal{S}(\mathcal{E}))_{\mathcal{S}}$) of germs of left $\mathcal{S}$-linear (resp. right $\mathcal{S}$-linear) endomorphisms on $\mathcal{S}(\mathcal{E})$ carries a structure as a right (resp. left) $\mathcal{S}$-module. Just as in 1.3 we can define $\operatorname{End}^n_{\mathcal{S}}(\mathcal{S}(\mathcal{E}))$ and $\operatorname{End}^n(\mathcal{S}(\mathcal{E}))_{\mathcal{S}}$ as well as the sheaves $\mathcal{E}nd^n_{\mathcal{S}}(\mathcal{S}(\mathcal{E}))$ and $\mathcal{E}nd^n(\mathcal{S}(\mathcal{E}))_{\mathcal{S}}$ for every $n \in \mathbb{Z}$.
Let us assume for a moment that $X = \operatorname{Spec}(A)$ is affine so that $\mathcal{S} = \widetilde{S}$ and $\mathcal{E} = \widetilde{E}$ where $S = \bigoplus_{n \in \mathbb{Z}} S^n$ is a graded, unitary, anti-commutative and flat A-algebra of finite presentation (as an A-module), and $E = \bigoplus_{n \in \mathbb{Z}} E^n$ is a finitely generated projective A-module. For the following lemma we consider $S \otimes_A E$ as a graded A-module with respect to the (total) grading induced by the bi-grading $(S^p \otimes_A E^q)_{(p,q) \in \mathbb{Z}^2}$.

Lemma 1.3.1 *There exist canonical, functorial isomorphisms*

$$\mathcal{E}nd(\mathcal{S}(\mathcal{E}))_{\mathcal{S}} \xrightarrow{\cong} (\mathrm{End}(S \otimes_A E)_S)^{\sim}$$
$$\mathcal{E}nd^n(\mathcal{S}(\mathcal{E}))_{\mathcal{S}} \xrightarrow{\cong} (\mathrm{End}^n(S \otimes_A E)_S)^{\sim}$$

An analogous assertion holds for the sheaf of left $\mathcal{S}$-linear endomorphisms (of degree n) on $\mathcal{S}(\mathcal{E})$

Proof. We shall restrict ourselves to proving the statement concerning the sheaf of right $\mathcal{S}$-linear endomorphisms of degree n on $\mathcal{S}(\mathcal{E})$. Let M be a graded right S-module. Given $k \in \mathbb{Z}$ let $M(k)$ denote the graded right S-module given by $M(k)^l = M^{k+l}$ for $l \in \mathbb{Z}$. Recall that M is of finite presentation over S if there exists an exact sequence

$$P \xrightarrow{\varphi} Q \xrightarrow{\psi} M \to 0$$

where P, Q are finite direct sums of S-modules of the form $S(k)$ and φ, ψ are right S-linear of degree zero (cp. [EGA II, 2.1.1]). Then given a graded right S-module N the induced sequence

$$0 \to \mathrm{Hom}^n_S(M,\, N) \to \mathrm{Hom}^n_S(Q,\, N) \to \mathrm{Hom}^n_S(P,\, N)$$

is exact. Moreover, for $m \in \mathbb{N}$ and $i_1, \ldots, i_m \in \mathbb{Z}$ there is a functorial A-linear isomorphism

$$\mathrm{Hom}^n_S\left(\bigoplus_{k=1}^m S(i_k),\, N\right) \xrightarrow{\cong} \bigoplus_{k=1}^m \mathrm{Hom}^n_S(S(i_k),\, N) \xrightarrow{\cong} \bigoplus_{k=1}^m N_{n-i_k}.$$

Hence it follows that for $f \in A$ and a graded right S-module M of finite presentation the natural A-linear map

$$\mathrm{Hom}^n_S(M,\, N) \otimes_A A_f \to \mathrm{Hom}^n_{S_f}(M_f,\, N_f)$$

is an isomorphism. Noting that $S \otimes_A E$ is of finite presentation over S we can now argue as in [EGA I, 1.3.12 (ii)] to complete the proof. □

Thus we see that $\mathcal{E}nd_{\mathcal{S}}(\mathcal{S}(\mathcal{E}))$ and $\mathcal{E}nd(\mathcal{S}(\mathcal{E}))_{\mathcal{S}}$ are quasi-coherent subalgebras of $\mathcal{E}nd_{\mathcal{O}_X}(\mathcal{S}(\mathcal{E}))$. Setting

$$\mathcal{E}nd^{\cdot}_{\mathcal{S}}(\mathcal{S}(\mathcal{E})) = \bigoplus_{n\in\mathbb{Z}} \mathcal{E}nd^n_{\mathcal{S}}(\mathcal{S}(\mathcal{E})),$$
$$\mathcal{E}nd^{\cdot}(\mathcal{S}(\mathcal{E}))_{\mathcal{S}} = \bigoplus_{n\in\mathbb{Z}} \mathcal{E}nd^n(\mathcal{S}(\mathcal{E}))_{\mathcal{S}}$$

we obtain

Lemma 1.3.2 *The natural maps*

$$\mathcal{E}nd^{\cdot}_{\mathcal{S}}(\mathcal{S}(\mathcal{E})) \longrightarrow \mathcal{E}nd_{\mathcal{S}}(\mathcal{S}(\mathcal{E})) \quad \text{and} \quad \mathcal{E}nd^{\cdot}(\mathcal{S}(\mathcal{E}))_{\mathcal{S}} \longrightarrow \mathcal{E}nd(\mathcal{S}(\mathcal{E}))_{\mathcal{S}}$$

are isomorphisms.

For an integer n the sections of $\mathcal{E}nd^{n}_{\mathbb{Z}}(\mathcal{S}(\mathcal{E}))$ which are right $\mathcal{S}$-linear can be described as follows.

Lemma 1.3.3 *Let φ be a section of $\mathcal{E}nd^{n}_{\mathbb{Z}}(\mathcal{S}(\mathcal{E}))$. Then φ is right $\mathcal{S}$-linear if and only if*

$$\varphi(\omega \cdot \alpha) = (-1)^{|\omega|n}\, \omega \cdot \varphi(\alpha)$$

whenever ω and α are homogeneous sections of $\mathcal{S}$ and $\mathcal{S}(\mathcal{E})$.

Proof. Indeed, we have

$$\begin{aligned}
\varphi(\alpha \cdot \omega) = \varphi(\alpha) \cdot \omega \quad &\Leftrightarrow \quad (-1)^{|\omega||\alpha|}\, \varphi(\omega \cdot \alpha) = (-1)^{|\omega||\varphi(\alpha)|}\, \omega \cdot \varphi(\alpha) \\
&\Leftrightarrow \quad (-1)^{|\omega||\alpha|}\, \varphi(\omega \cdot \alpha) = (-1)^{|\omega|(n+|\alpha|)}\, \omega \cdot \varphi(\alpha) \\
&\Leftrightarrow \quad \varphi(\omega \cdot \alpha) = (-1)^{|\omega|n}\, \omega \cdot \varphi(\alpha) \quad \square
\end{aligned}$$

We shall also consider $\mathcal{S}(\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(\mathcal{E}))$ as a graded $\mathcal{O}_X$-algebra where its multiplication is given by

$$(\omega \otimes \varphi) \cdot (\tau \otimes \psi) = (-1)^{|\varphi||\tau|}\, \omega\tau \otimes \varphi\psi$$

for homogeneous sections ω, τ and φ, ψ of $\mathcal{S}$ and $\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(\mathcal{E})$. Let

$$F\colon \mathcal{S}(\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(\mathcal{E})) \to \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{S}(\mathcal{E}))$$

denote the uniquely determined morphism sending $\omega \otimes \varphi$ to

$$\tau \otimes e \mapsto (-1)^{|\varphi||\tau|}\, \omega\tau \otimes \varphi(e)$$

for homogeneous sections ω and φ (resp. τ and e) of $\mathcal{S}$ and $\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(\mathcal{E})$ (resp. $\mathcal{S}$ and $\mathcal{E}$).

Lemma 1.3.4 *The image of F is contained in $\mathcal{E}nd(\mathcal{S}(\mathcal{E}))_{\mathcal{S}}$.*

Proof. Let ω and φ be homogeneous sections of $\mathcal{S}$ and $\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(\mathcal{E})$ respectively. Then for homogeneous sections τ, τ' and e of $\mathcal{S}$ and $\mathcal{E}$ we deduce

$$F(\omega \otimes \varphi)(\tau' \cdot (\tau \otimes e)) = (-1)^{|\varphi|(|\tau'|+|\tau|)}\, \omega\tau'\tau \otimes \varphi(e)$$

and

$$
\begin{aligned}
(-1)^{|F(\omega\otimes\varphi)||\tau'|}\,\tau'\cdot F(\omega\otimes\varphi)(\tau\otimes e) &= (-1)^{(|\omega|+|\varphi|)|\tau'|+|\varphi||\tau|}\,\tau'\omega\tau\otimes\varphi(e)\\
&= (-1)^{|\varphi|(|\tau'|+|\tau|)}\,\omega\tau'\tau\otimes\varphi(e).
\end{aligned}
$$

Therefore by 1.3.3 we see that $F(\omega\otimes\varphi)$ is right $\mathcal{S}$-linear. $\square$

Hence according to 1.3.2 and 1.3.4 we may view F as a morphism

$$
\mathcal{S}(\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(\mathcal{E})) \longrightarrow \mathcal{E}nd^{\cdot}(\mathcal{S}(\mathcal{E}))_{\mathcal{S}}.
$$

This is what we shall do in the following.

Proposition 1.3.5 *F is a degree zero isomorphism of graded algebras.*

Proof. It is clear that F is of degree zero. Let ω, ω' and φ, φ' be homogeneous sections of $\mathcal{S}$ and $\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(\mathcal{E})$. Then for homogeneous sections τ and e of $\mathcal{S}$ and $\mathcal{E}$ we infer

$$
\begin{aligned}
F(\omega'\otimes\varphi'\cdot\omega\otimes\varphi)(\tau\otimes e) &= (-1)^{|\varphi'||\omega|}\,F(\omega'\omega\otimes\varphi'\varphi)(\tau\otimes e)\\
&= (-1)^{|\varphi'||\omega|+(|\varphi'|+|\varphi|)|\tau|}\,\omega'\omega\tau\otimes\varphi'(\varphi(e))\\
&= F(\omega'\otimes\varphi')(F(\omega\otimes\varphi)(\tau\otimes e))
\end{aligned}
$$

showing that F is a morphism of graded algebras. In order to prove the surjectivity of F we shall work with the supermodule structure of both $\mathcal{S}$ and $\mathcal{E}$, i.e. with their canonical $\mathbb{Z}_2$-gradings. Therefore we define $\mathcal{S}_0 = \bigoplus_{n\in\mathbb{Z}}\mathcal{S}^{2n}$, $\mathcal{S}_1 = \bigoplus_{n\in\mathbb{Z}}\mathcal{S}^{2n+1}$ and similarly $\mathcal{E}_0$, $\mathcal{E}_1$ for the vector bundle $\mathcal{E}$. We may assume that $X = \mathrm{Spec}(A)$ is an affine scheme and $\mathcal{E}_0$, $\mathcal{E}_1$ are free of finite rank with bases $(b_{0p})_{1\le p\le m_0}$, $(b_{1q})_{1\le q\le m_1}$ consisting of homogeneous elements of $\mathcal{E}$, where by the standard abuse we do not distinguish between quasi-coherent $\mathcal{O}_X$-modules and A-modules. We will only show that every even degree, right $\mathcal{S}$-linear endomorphism ψ on $\mathcal{S}(\mathcal{E})$ lies in the image of F, the case of an odd endomorphism being similar. Since ψ is of even degree it follows that $\psi(1\otimes b_{0p})\in\mathcal{S}_0\otimes\mathcal{E}_0\bigoplus\mathcal{S}_1\otimes\mathcal{E}_1$ and $\psi(1\otimes b_{1q})\in\mathcal{S}_0\otimes\mathcal{E}_1\bigoplus\mathcal{S}_1\otimes\mathcal{E}_0$ for every p, q. Let $(\omega_{0,p,0,r})_{1\le r\le m_0}\in\mathcal{S}_0^{m_0}$, $(\omega_{0,p,1,s})_{1\le s\le m_1}\in\mathcal{S}_1^{m_1}$, $(\omega_{1,q,0,r})_{1\le r\le m_0}\in\mathcal{S}_1^{m_0}$ and $(\omega_{1,q,1,s})_{1\le s\le m_1}\in\mathcal{S}_0^{m_1}$ be such that

$$
\begin{aligned}
\psi(1\otimes b_{0p}) &= \sum_{r=1}^{m_0}\omega_{0,p,0,r}\otimes b_{0r}+\sum_{s=1}^{m_1}\omega_{0,p,1,s}\otimes b_{1s},\\
\psi(1\otimes b_{1q}) &= \sum_{r=1}^{m_0}\omega_{1,q,0,r}\otimes b_{0r}+\sum_{s=1}^{m_1}\omega_{1,q,1,s}\otimes b_{1s}.
\end{aligned}
$$

Let $b^*_{\alpha,u,\beta,v}$ denote the endomorphism on $\mathcal{E}$ satisfying $b^*_{\alpha,u,\beta,v}(b_{\alpha u}) = b_{\beta v}$ and $b^*_{\alpha,u,\beta,v} = 0$ for the other base elements where $\alpha,\ \beta \in \{0,\ 1\}$ and $1 \le u \le m_0$ (resp. $1 \le u \le m_1$) for $\alpha = 0$ (resp. $\alpha = 1$), $1 \le v \le m_0$ (resp. $1 \le v \le m_1$) for $\beta = 0$ (resp. $\beta = 1$). Setting

$$\begin{aligned}\rho \ &= \ \sum_{p,\,r=1}^{m_0} \omega_{0,p,0,r} \otimes b^*_{0,p,0,r} + \sum_{p=1}^{m_0}\sum_{q=1}^{m_1} \omega_{0,p,1,q} \otimes b^*_{0,p,1,q} + \omega_{1,q,0,p} \otimes b^*_{1,q,0,p} \\ &\quad + \sum_{q,\,s=1}^{m_1} \omega_{1,q,1,s} \otimes b^*_{1,q,1,s}\end{aligned}$$

this gives us an even degree section of $\mathcal{S}(\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(\mathcal{E}))$. By construction we have

$$F(\rho)(1 \otimes b_{0p}) = \psi(1 \otimes b_{0p}) \quad \text{for} \quad 1 \le p \le m_0$$

and

$$F(\rho)(1 \otimes b_{1q}) = \psi(1 \otimes b_{1q}) \quad \text{for} \quad 1 \le q \le m_1.$$

Thus $F(\rho) = \psi$. It remains to show the injectivity of F. Suppose that σ lies in the kernel of F. We can write

$$\begin{aligned}\sigma \ &= \ \sum_{p,\,r=1}^{m_0} \tau_{0,p,0,r} \otimes b^*_{0,p,0,r} + \sum_{p=1}^{m_0}\sum_{q=1}^{m_1} \tau_{0,p,1,q} \otimes b^*_{0,p,1,q} + \tau_{1,q,0,p} \otimes b^*_{1,q,0,p} \\ &\quad + \sum_{q,\,s=1}^{m_1} \tau_{1,q,1,s} \otimes b^*_{1,q,1,s}\end{aligned}$$

for suitable coefficients $\tau_{0,p,0,r},\ \tau_{0,p,1,q},\ \tau_{1,q,0,p},\ \tau_{1,q,1,s}$ of $\mathcal{S}$. As

$$F(\sigma)(1 \otimes b_{0p}) = 0 = F(\sigma)(1 \otimes b_{1q})$$

it follows easily that all these coefficients must be zero. This completes the proof of the proposition. $\square$

From now on we shall identify the graded $\mathcal{O}_X$-algebra $\mathcal{S}(\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(\mathcal{E}))$ with $\mathcal{E}nd^{\cdot}(\mathcal{S}(\mathcal{E}))_{\mathcal{S}}$.
The supertrace $\mathrm{tr}_s \colon \mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(\mathcal{E}) \to \mathcal{O}_X$ introduced in 1.2 induces a left $\mathcal{S}$-linear map

$$\mathcal{S} \otimes_{\mathcal{O}_X} \mathrm{tr}_s \colon \mathcal{S}(\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(\mathcal{E})) \to \mathcal{S}.$$

For notational simplicity we shall also write tr_s instead of $\mathcal{S} \otimes_{\mathcal{O}_X} \mathrm{tr}_s$ in the sequel.

Lemma 1.3.6 *The supertrace* tr_s *is a degree zero morphism of graded* $\mathcal{O}_X$*-modules and kills supercommutators in* $\mathcal{S}(\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(\mathcal{E}))$*.*

Proof. By 1.2.4 (a) the morphism tr_s is of degree zero. To establish the second part of the assertion let ω, ω' and φ, φ' be homogeneous sections of $\mathcal{S}$ and $\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(\mathcal{E})$. Using 1.2.4 (b) we infer

$$\begin{aligned}\mathrm{tr}_s(\omega'\otimes\varphi'\cdot\omega\otimes\varphi) &= (-1)^{|\varphi'||\omega|}\,\omega'\omega\cdot\mathrm{tr}_s(\varphi'\varphi)\\ &= (-1)^{|\varphi'||\omega|+|\varphi'||\varphi|+|\omega'||\omega|}\,\omega\omega'\cdot\mathrm{tr}_s(\varphi\varphi')\\ &= (-1)^{|\omega'\otimes\varphi'||\omega\otimes\varphi|}\,\mathrm{tr}_s(\omega\otimes\varphi\cdot\omega'\otimes\varphi').\end{aligned}$$

Hence $\mathrm{tr}_s[\omega'\otimes\varphi',\,\omega\otimes\varphi]=0$. □

As a further consequence of 1.2.4 (b) we state

Lemma 1.3.7 *Suppose that* ψ *is a section of* $\mathcal{S}^n_{X/S}\otimes_{\mathcal{O}_X}\mathcal{E}nd^0_{\mathcal{O}_X}(\mathcal{E})$ *where* n *is odd. Then* $\mathrm{tr}_s(\psi^2)=0$.

Now we come to the main application of the theory we have developed so far in this section. Namley we assume that our scheme X is weakly smooth over some base scheme S with structure morphism $f\colon X\to S$ and let

$$\mathcal{S}=\Omega^{\cdot}_{X/S}=\bigoplus_{n\in\mathbb{N}}\Omega^n_{X/S}.$$

Following Quillen (cf. [Q2]) we define a superconnection on $\mathcal{E}$ to be an element D of $\mathrm{End}^1_{\mathbb{Z}}(\Omega^{\cdot}_{X/S}(\mathcal{E}))=\Gamma(X,\,\mathcal{E}nd^1_{\mathbb{Z}}(\Omega^{\cdot}_{X/S}(\mathcal{E})))$ having the property

$$D(\omega\cdot e)=d(\omega)\cdot e+(-1)^{|\omega|}\omega\cdot D(e)$$

for any homogeneous sections ω and e of $\Omega^{\cdot}_{X/S}$ and $\Omega^{\cdot}_{X/S}(\mathcal{E})$ respectively.

Remark 1.3.8 The following facts are easily verified.

(a) Any connection on $\mathcal{E}$ preserving the grading induces in the usual way a superconnection.

(b) Any superconnection is $f^{-1}(\mathcal{O}_S)$-linear.

(c) Let D, D' be two superconnections on $\mathcal{E}$. Then $D-D'$ is a (global) section of $\mathcal{E}nd^1(\Omega^{\cdot}_{X/S}(\mathcal{E}))_{\Omega^{\cdot}_{X/S}}$ by 1.3.3. On the other hand if ψ is a (global) section of $\mathcal{E}nd^{\cdot}(\Omega^{\cdot}_{X/S}(\mathcal{E}))_{\Omega^{\cdot}_{X/S}}$ then $D+\psi$ yields a superconnection.

Lemma 1.3.9 *Let D be a superconnection on $\mathcal{E}$ and φ a homogeneous section of $\mathcal{E}nd^{\cdot}(\Omega^{\cdot}_{X/S}(\mathcal{E}))_{\Omega^{\cdot}_{X/S}}$. Then $[D,\, \varphi]$ is a section of $\mathcal{E}nd^{\cdot}(\Omega^{\cdot}_{X/S}(\mathcal{E}))_{\Omega^{\cdot}_{X/S}}$.*

Proof. The assertion being local on X we may assume that X is affine. Then for homogeneous sections ω and e of $\Omega^{\cdot}_{X/S}$ and $\mathcal{E}$ we find

$$\begin{aligned}
[D,\, \varphi](\omega \cdot e) &= D(\varphi(\omega \cdot e)) - (-1)^{|\varphi|}\, \varphi(D(\omega \cdot e)) \\
&= D((-1)^{|\varphi||\omega|}\, \omega \cdot \varphi(e)) - (-1)^{|\varphi|}\, \varphi(d(\omega) \cdot e \\
&\quad +(-1)^{|\omega|}\, \omega \cdot D(e)) \\
&= (-1)^{|\varphi||\omega|}\, d(\omega) \cdot \varphi(e) + (-1)^{|\varphi||\omega|+|\omega|}\, \omega \cdot D(\varphi(e)) \\
&\quad -(-1)^{|\omega||\varphi|}\, d(\omega) \cdot \varphi(e) - (-1)^{|\varphi|+|\omega|+|\varphi||\omega|}\, \omega \cdot \varphi(D(e)) \\
&= (-1)^{(|\varphi|+1)|\omega|}\, \omega \cdot D(\varphi(e)) - (-1)^{(|\varphi|+1)|\omega|+|\varphi|}\, \omega \cdot \varphi(D(e)) \\
&= (-1)^{|[D,\, \varphi]||\omega|}\, \omega \cdot [D,\, \varphi](e)
\end{aligned}$$

and an application of 1.3.3 finishes the proof. $\square$

Thus we get an $f^{-1}(\mathcal{O}_S)$-linear operator

$$[D,\ \ ] \colon \mathcal{E}nd^{\cdot}(\Omega^{\cdot}_{X/S}(\mathcal{E}))_{\Omega^{\cdot}_{X/S}} \to \mathcal{E}nd^{\cdot}(\Omega^{\cdot}_{X/S}(\mathcal{E}))_{\Omega^{\cdot}_{X/S}}$$

of degree 1.

Lemma 1.3.10 *Let ∇ be a connection on $\mathcal{E}$ preserving the grading and denote by ∇' the induced connection on $\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(\mathcal{E})$. Then the diagram*

$$\begin{array}{ccc}
\Omega^{\cdot}_{X/S}(\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(\mathcal{E})) & \xrightarrow{F} & \mathcal{E}nd^{\cdot}(\Omega^{\cdot}_{X/S}(\mathcal{E}))_{\Omega^{\cdot}_{X/S}} \\
\nabla' \big\downarrow & & \big\downarrow [\nabla,\ ] \\
\Omega^{\cdot}_{X/S}(\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(\mathcal{E})) & \xrightarrow{F} & \mathcal{E}nd^{\cdot}(\Omega^{\cdot}_{X/S}(\mathcal{E}))_{\Omega^{\cdot}_{X/S}}
\end{array}$$

is commutative.

Proof. We may assume that X is affine and $\Omega^1_{X/S}$ is free with a base $dx_1,\, \ldots,\, dx_n$. Given global homogeneous sections ω and φ (resp. e) of $\Omega^{\cdot}_{X/S}$ and $\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(\mathcal{E})$ (resp. $\mathcal{E}$) we compute

$$\begin{aligned}
F(\nabla'(\omega \otimes \varphi))(e) &= F(d\omega \otimes \varphi + (-1)^{|\omega|}\, \omega \wedge \nabla'(\varphi))(e) \\
&= d\omega \otimes \varphi(e) + (-1)^{|\omega|} \sum_{p=1}^{n} F((\omega \wedge dx_p) \otimes \nabla'_p(\varphi))(e) \\
&= d\omega \otimes \varphi(e) + (-1)^{|\omega|}\, \omega \wedge \nabla(\varphi(e)) \\
&\qquad -(-1)^{|\omega|} \sum_{p=1}^{n} (\omega \wedge dx_p) \otimes \varphi(\nabla_p(e)).
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
[\nabla,\, F(\omega\otimes\varphi)](e) &= \nabla(F(\omega\otimes\varphi)(e)) - (-1)^{|\omega|+|\varphi|}\, F(\omega\otimes\varphi)(\nabla(e)) \\
&= \nabla(\omega\otimes\varphi(e)) - (-1)^{|\omega|+|\varphi|} \sum_{p=1}^{n} F(\omega\otimes\varphi)(dx_p\otimes\nabla_p(e)) \\
&= d\omega\otimes\varphi(e) + (-1)^{|\omega|}\omega\wedge\nabla(\varphi(e)) \\
&\qquad -(-1)^{|\omega|}\sum_{p=1}^{n}(\omega\wedge dx_p)\otimes\varphi(\nabla_p(e)).
\end{aligned}$$

Hence we conclude that $F(\nabla'(\omega\otimes\varphi)) = [\nabla,\, F(\omega\otimes\varphi)]$. □

Combining 1.2.5 and 1.3.10 we obtain

Corollary 1.3.11 *Under the hypotheses of* 1.3.10 *the diagram*

$$\begin{array}{ccc}
\mathcal{E}nd^{\cdot}(\Omega^{\cdot}_{X/S}(\mathcal{E}))_{\Omega^{\cdot}_{X/S}} & \xrightarrow{\mathrm{tr}_s} & \Omega^{\cdot}_{X/S} \\
{\scriptstyle [\nabla,\ ]}\downarrow & & \downarrow{\scriptstyle d} \\
\mathcal{E}nd^{\cdot}(\Omega^{\cdot}_{X/S}(\mathcal{E}))_{\Omega^{\cdot}_{X/S}} & \xrightarrow{\mathrm{tr}_s} & \Omega^{\cdot}_{X/S}
\end{array}$$

is commutative.

We can generalize the previous result as follows.

Proposition 1.3.12 *Let D be a superconnection on $\mathcal{E}$ and φ a section of the graded algebra $\mathcal{E}nd^{\cdot}(\Omega^{\cdot}_{X/S}(\mathcal{E}))_{\Omega^{\cdot}_{X/S}}$. Then*

$$\mathrm{tr}_s[D,\, \varphi] = d(\mathrm{tr}_s\varphi).$$

Proof. As the assertion is local on X we may assume that there exists a connection ∇ on $\mathcal{E}$ preserving the grading. Writing $D = \nabla + \psi$ where ψ is a section of $\mathcal{E}nd^1(\Omega^{\cdot}_{X/S}(\mathcal{E}))_{\Omega^{\cdot}_{X/S}}$ we infer

$$\mathrm{tr}_s[D,\, \varphi] = \mathrm{tr}_s[\nabla, \varphi] + \mathrm{tr}_s[\psi,\, \varphi] = \mathrm{tr}_s[\nabla,\, \varphi] = d(\mathrm{tr}_s\varphi)$$

using 1.3.6 and 1.3.10. □

We define the curvature of a superconnection D to be the morphism D^2 which is easily seen to be left $\Omega^{\cdot}_{X/S}$-linear and thus also right $\Omega^{\cdot}_{X/S}$-linear by 1.3.3, i.e. it is a (global) section of $\mathcal{E}nd^2(\Omega^{\cdot}_{X/S}(\mathcal{E}))_{\Omega^{\cdot}_{X/S}}$.

Corollary 1.3.13 *For $n \geq 1$ the section $\mathrm{tr}_s D^{2n}$ is a closed form.*

Proof. Using 1.3.12 we conclude

$$d(\mathrm{tr}_s D^{2n}) = \mathrm{tr}_s[D,\, D^{2n}] = 0$$

since $[D,\, D^{2n}] = 0$. $\square$

Proposition 1.3.14 *Let ∇ be a connection on $\mathcal{E}$ which preserves the grading. Let $\varphi \in \mathrm{End}^1_{\mathcal{O}_X}(\mathcal{E})$ such that $(\mathcal{E},\, \varphi)$ forms an acylic complex. Moreover, suppose that there exists an endomorphism $\psi \in \mathrm{End}^{-1}_{\mathcal{O}_X}(\mathcal{E})$ such that $[\varphi,\, \psi] = \mathrm{id}_{\mathcal{E}}$. Then*

$$\mathrm{tr}_s \nabla^2 = d(\mathrm{tr}_s(\psi \cdot \nabla(\varphi))).$$

Proof. Since the assertion is clearly local on X we may assume that there exists an integrable connection ∇' on $\mathcal{E}$ which preserves the grading and satisfies $\nabla'(\varphi) = 0$. Setting $A = \nabla - \nabla'$ which is a (global) section of the $\mathcal{O}_X$-module $\Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}nd^0_{\mathcal{O}_X}(\mathcal{E})$ we get $\nabla^2 = [\nabla',\, A] + A^2$ and therefore

$$\mathrm{tr}_s \nabla^2 = d(\mathrm{tr}_s A) + \mathrm{tr}_s A^2 = d(\mathrm{tr}_s A)$$

by 1.3.7 and 1.3.12. On the other hand we have

$$\mathrm{tr}_s(\psi \cdot \nabla(\varphi)) = \mathrm{tr}_s(\psi[\nabla' + A,\, \varphi]) = \mathrm{tr}_s([\varphi,\, A]\psi) = \mathrm{tr}_s(A)$$

by 1.2.1, 1.3.6 and 1.3.10. Eventually we deduce

$$\mathrm{tr}_s \nabla^2 = d(\mathrm{tr}_s A) = \mathrm{tr}_s(\psi \cdot \nabla(\varphi)). \quad \square$$

1.4 Relative Codimension

In this section all schemes are supposed to be locally noetherian.

Definition 1.4.1 Let $f\colon X \to S$ be a morphism of schemes and Y a closed subset of X. Then the codimension of Y in X with respect to f is given by

$$\mathrm{codim}_f(Y,\, X) := \inf_{s \in S}(\mathrm{codim}(Y_s,\, X_s)).$$

Since $\mathrm{codim}(Y_s,\, X_s) = \infty$ for $Y_s = \emptyset$ we see that

$$\mathrm{codim}_f(Y,\, X) = \inf_{y \in Y}(\mathrm{codim}(Y_{f(y)},\, X_{f(y)})).$$

Lemma 1.4.2 *With respect to the previous data we have*

$$\mathrm{codim}(Y,\, X) \geq \mathrm{codim}_f(Y,\, X).$$

Proof. Indeed, for $y \in Y$ using the formula

$$\operatorname{codim}(Y_{f(y)},\, X_{f(y)}) = \inf_{y' \in Y_{f(y)}} \big(\dim(\mathcal{O}_{X,\,y'} \otimes_{\mathcal{O}_{S,\,f(y)}} k(f(y)))\big)$$

we infer

$$\dim(\mathcal{O}_{X,\,y}) \geq \dim(\mathcal{O}_{X,\,y} \otimes_{\mathcal{O}_{S,\,f(y)}} k(f(y))) \geq \operatorname{codim}(Y_{f(y)},\, X_{f(y)})$$

which implies the assertion. $\square$

Suppose that $Y \neq \emptyset$ is a closed subset of X with $\operatorname{codim}(Y,\, X) > 0$. Taking $f = \mathrm{id}_X$ it follows that $\operatorname{codim}(Y_y,\, X_y) = 0$ for every $y \in Y$. This shows that $\operatorname{codim}(Y,\, X) = \operatorname{codim}_f(Y,\, X)$ is false in general.

Proposition 1.4.3

(a) *Consider a commutative diagram*

$$\begin{array}{ccc} X_0 & \xrightarrow{f'} & S_0 \\ \big\downarrow & & \big\downarrow \\ X & \xrightarrow{f} & S \end{array}$$

where X_0 (resp. S_0) is a subscheme of X (resp. S), and where the vertical maps are the immersions. Assume that X_0 contains $f^{-1}(f(Y))$. Then

$$\operatorname{codim}_f(Y,\, X) = \operatorname{codim}_{f'}(Y,\, X_0).$$

(b) *Let U be an open subset of X. Then*

$$\operatorname{codim}_{f|U}(Y \cap U,\, U) \geq \operatorname{codim}_f(Y,\, X).$$

(c) *Suppose that $f^{-1}(f(Y)) \subseteq \bigcup_{i \in I} U_i$ for a family $(U_i)_{i \in I}$ of open subsets of the scheme X. Then*

$$\operatorname{codim}_f(Y,\, X) = \inf_{i \in I}(\operatorname{codim}_{f|U_i}(Y \cap U_i,\, U_i)).$$

(d) *Suppose that $f(Y) \subseteq \bigcup_{j \in J} V_j$ for a family $(V_j)_{j \in J}$ of open subsets of the scheme S. Let $f_j \colon f^{-1}(V_j) \to V_j$ denote the base change of the map f by $V_j \hookrightarrow S$ for each $j \in J$. Then*

$$\operatorname{codim}_f(Y,\, X) = \inf_{j \in J}(\operatorname{codim}_{f_j}(Y \cap f^{-1}(V_j),\, f^{-1}(V_j))).$$

(e) *Consider a cartesian diagram*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & S' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & S. \end{array}$$

Then $\operatorname{codim}_{f'}(g'^{-1}(Y),\, X') \geq \operatorname{codim}_f(Y,\, X)$. *Moreover, both sides are equal if* $g'(g'^{-1}(Y)) = Y$.

Proof. The assertions (a)-(d) are clear. (e) follows from the transitivity of fibers and [EGA IV, 6.1.4]. $\square$

Corollary 1.4.4 *Let* $f\colon X \to S$ *be a smooth morphism and suppose we are given a closed immersion* $j\colon Y \hookrightarrow X$ *such that the composition* $f \circ j$ *is smooth. Then* $\operatorname{codim}_f(Y,\, X) = \operatorname{codim}(Y,\, X)$.

Proof. By 1.4.3 (c) we may assume that X and Y are smooth over S of relative dimension m and n respectively. Then we get

$$\operatorname{codim}(Y,\, X) = m - n = \operatorname{codim}(Y_{f(y)},\, X_{f(y)})$$

for $y \in Y$ using [EGA 0_{IV}, 14.2.2.2] and [EGA IV, 17.12.2]. $\square$

Lemma 1.4.5 *Let* $f\colon X \to S$ *be a flat morphism,* $Y \subseteq X$ *closed and suppose that* $\operatorname{depth}_{Y_s}(\mathcal{O}_{X_s}) \geq d$ *for all* $s \in S$. *Then* $\operatorname{depth}_Y(\mathcal{O}_X) \geq d$.

Proof. Let $y \in Y$, $s = f(y)$. Using [EGA IV, 6.3.1] we then deduce

$$\begin{aligned} \operatorname{depth}(\mathcal{O}_{X,\,y}) &= \operatorname{depth}(\mathcal{O}_{S,\,s}) + \operatorname{depth}(\mathcal{O}_{X,\,y} \otimes_{\mathcal{O}_{S,\,s}} k(s)) \\ &\geq \operatorname{depth}(\mathcal{O}_{S,\,s}) + \operatorname{depth}_{Y_s}(\mathcal{O}_{X_s}) \\ &\geq d. \end{aligned}$$

Hence $\operatorname{depth}_Y(\mathcal{O}_X) = \inf_{y \in Y}\big(\operatorname{depth}(\mathcal{O}_{X,\,y})\big) \geq d$. $\square$

Recall from [EGA IV, 6.8.1] that a morphism $f\colon X \to S$ is Cohen-Macaulay (CM) if it is flat and all fibers are Cohen-Macaulay schemes.

Proposition 1.4.6 *Let* $f\colon X \to S$ *be a morphism of schemes,* Y *a closed subset of* X *and* $d \in \mathbb{N}$. *Suppose that the following two conditions are satisfied:*

(a) f *is CM.*

(b) $\operatorname{codim}_f(Y, X) \geq d$.

Then $\operatorname{depth}_Y(\mathcal{O}_X) \geq d$. *In particular, for any vector bundle* $\mathcal{F}$ *on* X *we have* $\underline{H}^i_Y(\mathcal{F}) = 0$ *for all* $i < d$.

Proof. Let $y \in Y$. By (b) and the Cohen-Macaulayness of $X_{f(y)}$ we find that

$$\operatorname{depth}_{Y_{f(y)}}(\mathcal{O}_{X_{f(y)}}) = \operatorname{codim}(Y_{f(y)}, X_{f(y)}) \geq d$$

and therefore $\operatorname{depth}_Y(\mathcal{O}_X) \geq d$ by 1.4.5. The second part of the assertion follows from the first part and [G, 3.8]. □

Corollary 1.4.7 *Under the hypotheses of* 1.4.6 *there is a canonical functorial* $\Gamma(X, \mathcal{O}_X)$*-linear isomorphism*

$$\Gamma(X, \underline{H}^d_Y(\mathcal{F})) \xrightarrow{\cong} H^d_Y(X, \mathcal{F}).$$

In particular the presheaf $U \mapsto H^d_{Y\cap U}(U, \mathcal{F})$ $(U \subseteq X$ *open* $)$ *is a sheaf.*

Proof. This follows at once from 1.4.3 (b), 1.4.6 and the biregular spectral sequence

$$E_2^{pq} = H^p(X, \underline{H}^q_Y(\mathcal{F})) \;\Rightarrow\; H^n_Y(X, \mathcal{F}). \quad \square$$

Corollary 1.4.8 *Suppose that* $f\colon X \to S$ *is CM and* $Y \subseteq X$ *is closed with* $\operatorname{codim}_f(Y, X) = \operatorname{codim}(Y, X)$. *Then* $\operatorname{depth}_Y(\mathcal{O}_X) = \operatorname{codim}(Y, X)$.

The next two results which will be important to us later.

Proposition 1.4.9 *Let* $f\colon X \to S$ *be a smooth morphism,* $Y \subseteq X$ *closed and* $d \in \mathbb{N}$ *such that* $d \leq \operatorname{depth}_Y(\mathcal{O}_X)$. *Let* $\mathcal{F}$ *be a vector bundle on* X *with an integrable connection* ∇ *and corresponding De Rham complex* $\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^{\cdot}_{X/S}$. *Then there is natural functorial,* $f^{-1}\mathcal{O}_S$*-linear isomorphism*

$$\operatorname{Ker}(\underline{H}^d_Y(\nabla^d)) \xrightarrow{\cong} \underline{H}^{2d}_Y(\tau_{\geq d}(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^{\cdot}_{X/S})).$$

Proof. Set $K^{\cdot} = \tau_{\geq d}(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^{\cdot}_{X/S})$ and $L^{\cdot}_q = \underline{H}^q_Y(K^{\cdot})$ for $q \in \mathbb{Z}$. Consider the (first) spectral sequence of hypercohomology

$$E_2^{pq} = H^p(L^{\cdot}_q) \;\Rightarrow\; E^n = \underline{H}^n_Y(K^{\cdot})$$

which is clearly biregular. By hypothesis we have $E_2^{pq} = 0$ in case of $p < d$ and using [G, 3.8] this also holds for $q < d$. Therefore we deduce that

$\mathrm{Ker}(\underline{H}^d_Y(\nabla^d)) = E_2^{dd} = E_\infty^{dd}$. Moreover, if $(F^p(E^n))_p$ denotes the corresponding finite filtration of the abutment E^n we infer $F^d(E^n) = E^n$ and $F^{n-d+1}(E^n) = 0$ so that $\mathrm{gr}_d(E^{2d}) = E^{2d}$. Now our spectral sequence provides us with an isomorphism

$$\mathrm{Ker}(\underline{H}^d_Y(\nabla^d)) = E_\infty^{dd} \xrightarrow{\cong} \mathrm{gr}_d(E^{2d}) = E^{2d}. \quad \square$$

From the proof of 1.4.9 it follows that $\underline{H}^i_Y(\tau_{\geq d}(\mathcal{F} \otimes_{\mathcal{O}_X} \Omega^\cdot_{X/S})) = 0$ for all $i < 2d$.

Proposition 1.4.10 *Let f, Y and d be as in* 1.4.9. *Let $\mathcal{O}_T$ be a sheaf of rings on X, $K^\cdot$ a complex of $\mathcal{O}_T$-modules such that $K^i = 0$ for every $i < d$. Moreover, suppose that each K^n is a vector bundle on X. Then there is a natural functorial, $\mathcal{O}_T$-linear isomorphsm*

$$\Gamma(X, \underline{H}^{2d}_Y(K^\cdot)) \xrightarrow{\cong} H^{2d}_Y(X, K^\cdot).$$

In particular the presheaf $U \mapsto H^{2d}_{Y\cap U}(U, K^\cdot|U)$ ($U \subseteq X$ open) is a sheaf.

Proof. We only need to show the first assertion. Let $L^\cdot_q = (\underline{H}^q_Y(K^n))_{n\in\mathbb{Z}}$ for $q \in \mathbb{Z}$ and consider the biregular spectral sequence

$$E_2^{pq} = H^p(L^\cdot_q) \;\Rightarrow\; E^n = \underline{H}^n_Y(K^\cdot).$$

Then it follows that $E_2^{pq} = 0$ for $p < d$ or $q < d$ Thus $F^d(E^n) = E^n$ and $F^{n-d+1}(E^n) = 0$ for every $n \in \mathbb{Z}$. Hence $\underline{H}^n_Y(K^\cdot) = 0$ in case of $n < 2d$. Now we can use the biregular spectral sequence

$$E_2^{pq} = H^p(X, \underline{H}^q_Y(K^\cdot)) \;\Rightarrow\; E^n = H^n_Y(K^\cdot)$$

and infer that $F^0(E^{2d}) = E^{2d}$, $F^1(E^{2d}) = 0$ and $E_2^{0,2d} = E_3^{0,2d} = \ldots = E_\infty^{0,2d}$. This completes the proof of the proposition. $\square$

Let $f\colon X \to S$ and $g\colon X' \to X$ be morphisms of schemes, Y a closed subset of X and set $Y' = g^{-1}(X')$. Then in some cases, although not in general, a relation between $\mathrm{codim}_f(Y, X)$ and $\mathrm{codim}_{fg}(Y', X')$ does exist.

Lemma 1.4.11

(a) $\mathrm{codim}_f(Y, X) \geq \mathrm{codim}_{fg}(Y', X')$ *if g is surjective.*

(b) $\mathrm{codim}_f(Y, X) \leq \mathrm{codim}_{fg}(Y', X')$ *if f is CM and g is flat.*

(c) $\mathrm{codim}_f(Y, X) = \mathrm{codim}_{fg}(Y', X')$ *if f is CM and g is faithfully flat.*

Proof. (a) follows from [EGA IV, 6.1.4] and (c) is a trivial consequence of (a) and (b). To prove (b) let $y' \in Y'$, $y = g(y')$ and $s = f(y)$. Using the Cohen-Macaulayness of f and the obvious fact that $\mathrm{depth}(A) \leq \mathrm{depth}(B)$ for a flat homomorphism $A \to B$ of local noetherian rings we get

$$\mathrm{codim}(Y'_s, X'_s) \geq \mathrm{depth}_{Y'_s}(\mathcal{O}_{X'_s}) \geq \mathrm{depth}_{Y_s}(\mathcal{O}_{X_s}) = \mathrm{codim}(Y_s, X_s). \quad \square$$

Recall that two closed subsets Y and Z of a scheme X intersect properly if the following condition is satisfied:

- $\mathrm{codim}(V, X) = \mathrm{codim}(Y, X) + \mathrm{codim}(Z, X)$ whenever V is an irreducible component of $Y \cap Z$.

Lemma 1.4.12 *Let Y be an effective Cartier divisor on X and $Z \subseteq X$ a closed integral subscheme. Consider the following assertions:*

(a) *Y and Z intersect properly.*

(b) *Z is not contained in Y.*

(c) *$Y \cap Z$ is an effective Cartier divisor on Z.*

(d) *$\mathcal{T}or^i_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Z) = 0$ for all $i \neq 0$.*

Then we have (a) $\Rightarrow$ (b) *and* (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d). *All these assertions are equivalent in the following cases:*

(1) *X is irreducible and catenary.*

(2) *X is biequidimensional.*

Proof. (a) $\Rightarrow$ (b) If $Z \subseteq Y$ we would have

$$\mathrm{codim}(Z, X) = \mathrm{codim}(Y, X) + \mathrm{codim}(Z, X)$$

and thus $\mathrm{codim}(Y, X) = 0$ contradicting the hypothesis that Y is a purely 1-codimensional subscheme of X. For (b) $\Rightarrow$ (c) we may clearly assume that $X = \mathrm{Spec}(A)$ is affine, $Z = \mathrm{Spec}(A/\mathfrak{p})$ for a suitable $\mathfrak{p} \in \mathrm{Spec}(A)$ and Y is principal with equation a where $a \in A$ is A-regular. By assumption we have $a \notin \mathfrak{p}$ and hence $a + \mathfrak{p}$ is a non-zero divisor of the ring $A/\mathfrak{p}$. (c) $\Rightarrow$ (b) is trivial. (c) $\Rightarrow$ (d) The problem is local so we can use the previous notation. Denote also by a the endomorphism on $\mathcal{O}_X$ given by multiplication of a. Let $E^\cdot$ be the complex with $E^{-1} = E^0 = \mathcal{O}_X$, $E^p = 0$ otherwise and $d^{-1}_{E^\cdot} = a$. Then $E^\cdot$ gives us a free resolution of $\mathcal{O}_Y$. By assumption it follows that

$E^\cdot \otimes_{\mathcal{O}_X} \mathcal{O}_Z$ is a resolution of $\mathcal{O}_{Y\cap Z}$ and hence $H^i(E^\cdot \otimes_{\mathcal{O}_X} \mathcal{O}_Z) = 0$ for $i \neq 0$. (d) $\Rightarrow$ (c) we have $H^i(E^\cdot \otimes_{\mathcal{O}_X} \mathcal{O}_Z) = 0$ for $i \neq 0$ and thus $a + \mathfrak{p}$ is a non-zero divisor in $A/\mathfrak{p}$. Finally if condition (1) or (2) holds then (c) $\Rightarrow$ (a) results easily from [EGA 0_{IV}, 14.3.2.1, 14.3.3.2]. $\square$

To meet our purposes we shall say that two closed subsets Y and Z of X intersect nicely if

- $\mathrm{depth}_{Y\cap Z}(\mathcal{O}_X) = \mathrm{depth}_Y(\mathcal{O}_X) + \mathrm{depth}_Z(\mathcal{O}_X)$

Corollary 1.4.13 *Suppose that Y and Z intersect properly and one of the following conditions hold:*

(a) *X is Cohen-Macaulay.*

(b) *$f\colon X \to S$ is CM and $\mathrm{codim}_f(W, X) = \mathrm{codim}(W, X)$ for $W = Y, Z$.*

Then Y and Z intersect nicely.

Proof. (a) is evident, and (b) results from 1.4.8. $\square$

1.5 Strictly pseudo-coherent complexes on affine schemes

Let us introduce some terminology first. If $\mathcal{A}$ is an abelian category we will use the standard notation $C(\mathcal{A})$ (resp. $K(\mathcal{A})$, resp. $D(\mathcal{A})$) for the category of cochain complexes of $\mathcal{A}$-objects (resp. the homotopy category of cochain complexes of $\mathcal{A}$-objects, resp. the derived category of $\mathcal{A}$). Note that $D(\mathcal{A})$ exists if $\mathcal{A}$ is a Grothendieck category ([Fr]). In case T is a scheme and $\mathcal{A} = \mathrm{Mod}(\mathcal{O}_T)$ is the category of $\mathcal{O}_T$-modules we will write $C(T)$ (resp. $K(T)$, resp. $D(T)$) instead of $C(\mathcal{A})$ (resp. $K(\mathcal{A})$, resp. $D(\mathcal{A})$), and the full triangulated subcategory of $D(\mathcal{O}_T)$ consisting of those complexes of $\mathcal{O}_T$-modules having quasi-coherent cohomology will be denoted by $D_{\mathrm{qc}}(T)$. Furthermore, we denote by $D(T)_{\mathrm{perf}}$ the full triangulated subcategory of $D(T)$ consisting of all perfect complexes.
Throughout this section we fix an affine scheme $X = \mathrm{Spec}(R)$.

Lemma 1.5.1 *For an $\mathcal{O}_X$-module $\mathcal{F}$ the following assertions are equivalent:*

(a) *$\mathcal{F}$ is a vector bundle on X.*

(b) *There exists a finitely generated projective R-module P such that $\widetilde{P} \cong \mathcal{F}$.*

Proof. [EGA I, 1.4.4]. $\square$

Let $E^{\cdot}$ be a complex of $\mathcal{O}_X$-modules. Recall that $E^{\cdot}$ is strictly pseudo-coherent if it is a bounded above complex of vector bundles on X.

Proposition 1.5.2 *Let $E^{\cdot}$ be a strictly pseudo-coherent complex on X and $F^{\cdot}$ an object of $D_{\mathrm{qc}}(X)$. Then the canonical map*

$$\mathrm{Hom}_{K(X)}(E^{\cdot},\, F^{\cdot}) \to \mathrm{Hom}_{D(X)}(E^{\cdot},\, F^{\cdot})$$

is an isomorphism.

Proof. Letting $\mathcal{C} = \mathrm{Qco}(X)$ denote the category of quasi-coherent $\mathcal{O}_X$-modules we know by [BN, 5.1] that the canonical functor

$$D(\mathcal{C}) \to D_{\mathrm{qc}}(X)$$

is an equivalence. Thus we may assume that $F^{\cdot}$ is a complex of quasi-coherent $\mathcal{O}_X$-modules. Consider the canonical commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{K(\mathcal{C})}(E^{\cdot},\, F^{\cdot}) & \longrightarrow & \mathrm{Hom}_{D(\mathcal{C})}(E^{\cdot},\, F^{\cdot}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{K(X)}(E^{\cdot},\, F^{\cdot}) & \longrightarrow & \mathrm{Hom}_{D(X)}(E^{\cdot},\, F^{\cdot}) \end{array}$$

where the vertical maps are isomorphisms. In order to complete the proof it suffices to observe that the upper horizontal map is an isomorphism by 1.5.1, [EGA I, 1.4.2] and [Sp, 1.4]. $\square$

Corollary 1.5.3 *A strictly pseudo-coherent complex $E^{\cdot}$ is acyclic if and only if $\mathrm{id}_{E^{\cdot}}$ is null homotopic.*

Corollary 1.5.4 *Given objects $E^{\cdot}$, $F^{\cdot}$, $G^{\cdot}$ of the category $D_{\mathrm{qc}}(X)$ and chain maps $f\colon E^{\cdot} \to F^{\cdot}$, $g\colon F^{\cdot} \to G^{\cdot}$ suppose that $E^{\cdot}$ is strictly pseudo-coherent, g is a quasi-isomorphism and $g \circ f$ is null homotopic. Then f is null homotopic.*

2 Cycle classes for De Rham cohomology

2.1 Relative cohomology and a shifted mapping cone

Let X be a scheme and let $\mathcal{U} = (U_i)_{i \in I}$ be a covering of X by open subsets where I is a totally ordered index set. For each $(p+1)$-tuple $i_0 < \ldots < i_p$ of elements of I with $p \geq -1$ we set $U_{i_0,\ldots,i_p} = \bigcap_{q=0}^{p} U_{i_q}$ and denote by $j_{i_0,\ldots,i_p} \colon U_{i_0,\ldots,i_p} \hookrightarrow X$ the inclusion.
Given an $\mathcal{O}_X$-module M we denote as usual by $C^{\cdot}(\mathcal{U},\, M)$ the Čech complex of M with respect to $\mathcal{U}$, i.e.

$$C^p(\mathcal{U},\, M) = \begin{cases} \prod\limits_{i_0 < \ldots < i_p} (j_{i_0,\ldots,i_p})_*(M | U_{i_0,\ldots,i_p}), & p \geq 0 \\ 0, & p < 0. \end{cases}$$

Suppose that $E^{\cdot}$ is a complex of $\mathcal{O}_X$-modules. Then we can construct a double complex $C^{\cdot\cdot}(\mathcal{U},\, E^{\cdot})$ as follows. For $p,\, q \in \mathbb{Z}$ let $C^{p,q}(\mathcal{U},\, E^{\cdot}) = C^p(\mathcal{U},\, E^q)$ with $d^p_{C^{\cdot}(\mathcal{U},\, E^q)}$ (resp. $\prod_{i_0 < \ldots < i_p} (j_{i_0,\ldots,i_p})_*(d^q_{E^{\cdot}})$) being the first (resp. the second) differential. Then the Čech augmentation maps $\epsilon^p \colon E^p \to C^{0,p}(\mathcal{U},\, E^{\cdot})$ give rise to a chain map

$$\epsilon = (\epsilon^p)_p \colon E^{\cdot} \to \mathrm{Tot}(C^{\cdot\cdot}(\mathcal{U},\, E^{\cdot}))$$

where $\mathrm{Tot}(C^{\cdot\cdot}(\mathcal{U},\, E^{\cdot}))$ denotes the total direct sum complex associated with the double complex $C^{\cdot\cdot}(\mathcal{U},\, E^{\cdot})$.

Lemma 2.1.1 *The morphism ϵ is a quasi-isomorphism.*

Proof. This follows from the usual result on Čech resolutions for sheaves (see [Go, II, 5.2.1]) and the spectral sequence

$$''E_2^{pq} = H^p_{II}(H^q_I(C^{\cdot\cdot}(\mathcal{U},\, E^{\cdot}))) \;\Rightarrow\; H^n(\mathrm{Tot}(C^{\cdot\cdot}(\mathcal{U},\, E^{\cdot})))$$

which is regular by [EGA 0_{III}, 11.3.3 (iv)]. $\square$

Remark 2.1.2 We list some obvious properties which come along with the complex $\mathrm{Tot}(C^{\cdot\cdot}(\mathcal{U},\, E^{\cdot}))$.

(a) If I is finite, each U_i is retro-compact and each E^q is quasi-coherent, then $\mathrm{Tot}(C^{\cdot\cdot}(\mathcal{U},\, E^{\cdot}))$ is a complex of quasi-coherent $\mathcal{O}_X$-modules.

(b) If $E^{\cdot}$ is bounded below (resp. I is finite and $E^{\cdot}$ is bounded above), then the same holds for $\mathrm{Tot}(C^{\cdot\cdot}(\mathcal{U},\, E^{\cdot}))$.

(c) $E^{\cdot} \mapsto \mathrm{Tot}(C^{\cdot\cdot}(\mathcal{U},\, E^{\cdot}))$ yields an endofunctor on the category $C(X)$.

Of course the statements 2.1.1 and 2.1.2 (b), (c) are valid for any ringed space and not only for schemes. However for the next result we need the scheme structure of X.

Proposition 2.1.3 *Let $\mathcal{U} = (U_i)_{i\in I}$ be as before, and let $E^{\cdot}$ be a complex of quasi-coherent $\mathcal{O}_X$-modules. Consider a morphism $f\colon X \to Y$ and suppose that $f|U_i$ as well as the injection $U_i \hookrightarrow X$ are affine for each $i \in I$. Moreover, suppose that I is finite or $E^{\cdot}$ is bounded below. Then the $\mathcal{O}_X$-modules $\mathrm{Tot}^n(C^{\cdot\cdot}(\mathcal{U},\, E^{\cdot}))$ are f_*-acyclic.*

Proof. By assumption each of the sheaves $\mathrm{Tot}^n(C^{\cdot\cdot}(\mathcal{U},\, E^{\cdot}))$ is a finite direct sum of certain $C^{pq}(\mathcal{U},\, E^{\cdot})$. Thus it is sufficient to show that each $C^{pq}(\mathcal{U},\, E^{\cdot})$ is f_*-acyclic which in turn follows from the proof of [H1, III, 3.2]. □

Corollary 2.1.4 *Let $f\colon X \to Y$ be separated morphism of schemes such that $f|U_i$ is affine for each $i \in I$. Then for a bounded below complex $E^{\cdot}$ of quasi-coherent $\mathcal{O}_X$-modules the complex $f_*(\mathrm{Tot}(C^{\cdot\cdot}(\mathcal{U},\, E^{\cdot})))$ is deployed for computing $Rf_*(E^{\cdot})$.*

Given a scheme X, consider a family $\mathcal{U} = (U_i)_{i\in I}$ of open subsets of X. Let $U = \bigcup_{i\in I} U_i$, $Z = X - U$ and denote by $j\colon U \hookrightarrow X$ the inclusion. For a complex $E^{\cdot}$ of quasi-coherent $\mathcal{O}_X$-modules let $F^{\cdot} = j_*(\mathrm{Tot}(C^{\cdot\cdot}(\mathcal{U},\, E^{\cdot}|U)))$ and denote by τ the composition of the adjoint map $E^{\cdot} \to j_*(E^{\cdot}|U)$ with the chain map $j_*(\epsilon)$, where $\epsilon\colon E^{\cdot}|U \to \mathrm{Tot}(C^{\cdot\cdot}(\mathcal{U},\, E^{\cdot}|U))$ denotes the quasi-isomorphism we have introduced above. Then we get a canonical distinguished triangle

$$E^{\cdot} \xrightarrow{\tau} F^{\cdot} \xrightarrow{\mu} \mathrm{Cone}(\tau) \xrightarrow{\nu} E^{\cdot}[1].$$

Here we follow the conventions of [SGA $4\frac{1}{2}$, C. D.] so that μ and ν are given in degree n as follows: μ sends a section s of F^n to the section $(0,\, s)$ of $E^{n+1} \bigoplus F^n$, whereas ν maps a section $(t,\, u)$ of $E^{n+1} \bigoplus F^n$ to $-t$. By axiom (TR2) for triangulated categories it follows that the "shifted" triangle

$$\mathrm{Cone}(\tau)[-1] \xrightarrow{-\nu[-1]} E^{\cdot} \xrightarrow{\tau} F^{\cdot} \xrightarrow{\mu} \mathrm{Cone}(\tau)$$

is also distinguished.

Remark 2.1.5 Assume that X/S is weakly smooth and ∇ is a connection on $E^{\cdot}$. Then ∇ induces connections on $F^{\cdot}$ and $\mathrm{Cone}(\tau)$ in an evident way. It is clear that the chain maps τ, μ and ν are horizontal with respect to these connections.

Proposition 2.1.6 *With respect to the previous data assume that $j|U_i$ is affine for each $i \in I$ and $E^\cdot$ is bounded below. Then the distinguished triangles*

$$\mathrm{Cone}(\tau)[-1] \xrightarrow{-\nu[-1]} E^\cdot \xrightarrow{\tau} F^\cdot \xrightarrow{\mu} \mathrm{Cone}(\tau)$$

and

$$R\underline{\Gamma}_Z(E^\cdot) \longrightarrow E^\cdot \longrightarrow Rj_*(E^\cdot|U) \longrightarrow R\underline{\Gamma}_Z(E^\cdot)[1]$$

are naturally isomorphic in $D(X)$.

Proof. Pick a K-injective resolution (in the sense of [Sp]) $E^\cdot|U \xrightarrow{f} J^\cdot$ of $E^\cdot|U$. Then we can find a morphism of complexes $g\colon \mathrm{Tot}(C^{\cdot\cdot}(\mathcal{U}, E^\cdot|U)) \to J^\cdot$ such that $g \circ \epsilon = f$ up to homotopy. Hence the diagram

$$\begin{array}{ccc} E^\cdot & \xrightarrow{\tau} & F^\cdot \\ \downarrow & & \downarrow{\scriptstyle j_*(g)} \\ j_*(E^\cdot|U) & \xrightarrow{j_*(f)} & j_*(J^\cdot) \end{array}$$

is homotopy commutative where the vertical map on the left is the adjunction morphism. Now the assertion follows from 2.1.4 (which implies that the chain map $j_*(g)$ is a quasi-isomorphism), axiom (TR3) for triangulated categories and [H1, I, 1.1 (c)]. □

In what follows the complex $\mathrm{Cone}(\tau)[-1]$ will be denoted by $\mathcal{C}\mathrm{one}(\mathcal{U}, E^\cdot)$. Moreover, we define the complex $\mathrm{Cone}(\mathcal{U}, E^\cdot)$ to be $\Gamma(X, \mathcal{C}\mathrm{one}(\mathcal{U}, E^\cdot))$.

Corollary 2.1.7 *Under the hypotheses of* 2.1.6 *suppose in addition that X is an affine scheme and I is finite. Then $H^n(\mathrm{Cone}(\mathcal{U}, E^\cdot)) = H^n_Z(X, E^\cdot)$ for every $n \in \mathbb{Z}$.*

Proof. Note first that the morphism j is quasi-compact and separated. Moreover, the complex $\mathcal{C}\mathrm{one}(\mathcal{U}, E^\cdot)$ is bounded below by 2.1.2 (b). The n-th component of this complex is given by $E^{n+1} \bigoplus K^n$ where E^{n+1} is a quasi-coherent $\mathcal{O}_X$-module and K^n a finite direct product of quasi-coherent $\mathcal{O}_X$-modules. As quasi-coherent modules are acyclic for $\Gamma(X, *)$ and $R\Gamma(X, *)$ commutes with finite direct products we see that $E^{n+1} \bigoplus K^n$ is $\Gamma(X, *)$-acyclic for all $n \in \mathbb{Z}$. Hence $\mathcal{C}\mathrm{one}(\mathcal{U}, E^\cdot)$ is a bounded below complex of $\Gamma(X, *)$-acyclic modules. Thus the first hypercohomology spectral sequence of this complex (see [EGA 0_{III}, (11.4.3.1)]) degenerates showing that the canonical map

$$\mathrm{Cone}(\mathcal{U}, E^\cdot) \to R\Gamma(X, \mathcal{C}\mathrm{one}(\mathcal{U}, E^\cdot))$$

is an isomorphism in the derived category. $\square$

We want to take a closer look at the complex $\mathcal{C}\mathrm{one}(\mathcal{U},\, E^{\cdot})$. Unwinding the definitions we get for $n \in \mathbb{Z}$

$$\mathcal{C}\mathrm{one}^n(\mathcal{U},\, E^{\cdot}) = \bigoplus_{p \geq -1} \prod_{i_0 < \ldots < i_p} (j_{i_0,\,\ldots,\,i_p})_*(E^{n-1-p}|U_{i_0,\,\ldots,\,i_p}).$$

Thus a section of $\mathcal{C}\mathrm{one}^n(\mathcal{U},\, E^{\cdot})$ over a quasi-compact and quasi-separated open subscheme V of X may be described as a tuple $(u_{p,\,\lambda})$ where $p \geq -1$, where λ is running over all $(p+1)$-tuples $i_0 < \ldots < i_p$ of I and $u_{p,\,\lambda}$ is an element of $\Gamma(V \cap U_{i_0,\,\ldots,\,i_p},\, E^{n-1-p})$ with $(u_{p,\,\lambda})_\lambda \neq 0$ for only finitely many p. Writing $(u'_{p,\,\lambda}) = d^n_{\mathcal{C}\mathrm{one}(\mathcal{U},\, E^{\cdot})}((u_{p,\,\lambda}))$ and $\lambda = (i_0,\, \ldots,\, i_p)$ we see that

$$u'_{p,\,\lambda} = \begin{cases} d^n_{E^{\cdot}}(u_{-1,\,*}), & p = -1 \\ -u_{-1,\,*}|V \cap U_\lambda - d^{n-1}_{E^{\cdot}}(u_{0,\,\lambda}), & p = 0 \\ \sum\limits_{j=0}^{p} (-1)^{j+1} u_{p-1,\,(i_0,\,\ldots,\,\widehat{i_j},\,\ldots,\,i_p)}|V \cap U_\lambda - (-1)^p d^{n-1-p}_{E^{\cdot}}(u_{p,\,\lambda}), & p > 0. \end{cases}$$

Here we have used the asterisk $*$ to denote the empty tuple.

Remark 2.1.8 Let $\mathcal{O}_T$ be a commutative sheaf of rings on X and $E^{\cdot}$ a complex of $\mathcal{O}_T$-modules. Then as before we can form the complexes $C^{\cdot\cdot}(\mathcal{U},\, E^{\cdot})$ and $\mathrm{Tot}(C^{\cdot\cdot}(\mathcal{U},\, E^{\cdot}))$. If $E^{\cdot}$ is bounded below and each E^n is in addition a quasi-coherent $\mathcal{O}_X$-module then 2.1.3, 2.1.4 and 2.1.6 (with $D(X)$ being replaced by $D(\mathcal{O}_T)$) remain valid. Moreover, it is clear that the proof of 2.1.7 carries over to this situation.

Lemma 2.1.9 *Let $\mathcal{F}$ be a vector bundle on X. Moreover, suppose that for the collection of open subsets $\mathcal{U} = (U_i)_{i \in I}$ the index set I is finite.*

(a) *There is a natural functorial isomorphism of complexes*

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{C}\mathrm{one}(\mathcal{U},\, E^{\cdot}) \xrightarrow{\cong} \mathcal{C}\mathrm{one}(\mathcal{U},\, \mathcal{F} \otimes_{\mathcal{O}_X} E^{\cdot}).$$

(b) *Let $f\colon Y \to X$ be a morphism of schemes. Assume that $j|U_i$ is affine for each i or f is flat and each U_i is retro-compact. Then there is a natural functorial isomorphism of complexes*

$$f^*(\mathcal{C}\mathrm{one}(\mathcal{U},\, E^{\cdot})) \xrightarrow{\cong} \mathcal{C}\mathrm{one}(f^{-1}\mathcal{U},\, f^*E^{\cdot}).$$

Proof. (a) follows from [EGA 0_{I}, 5.4.8] and (b) from [EGA I, 9.3.3]. $\square$

Suppose that $\mathcal{U}'$ is another finite open cover of U. Setting $\mathcal{U}'' = \mathcal{U} \cup \mathcal{U}'$ we have canonical, functorial quasi-isomorphisms

$$\mathcal{C}\mathrm{one}(\mathcal{U}'',\, E^{\cdot}) \to \mathcal{C}\mathrm{one}(\mathcal{U},\, E^{\cdot})$$
$$\mathcal{C}\mathrm{one}(\mathcal{U}'',\, E^{\cdot}) \to \mathcal{C}\mathrm{one}(\mathcal{U}',\, E^{\cdot})$$

which by the calculus of fractions induce a functorial isomorphism

$$\lambda^{\mathcal{U}}_{\mathcal{U}'} \colon \mathcal{C}\mathrm{one}(\mathcal{U},\, E^{\cdot}) \xrightarrow{\cong} \mathcal{C}\mathrm{one}(\mathcal{U}',\, E^{\cdot})$$

in $D(X)$. The fact that the above chain maps are quasi-isomorphisms follows from [EGA 0_{III}, 11.1.5] and the spectral sequence we used in the proof of 2.1.1.

Lemma 2.1.10 *Let X be a scheme whose underlying topological space is locally noetherian. Let $Z' \subseteq Z$ be closed subsets of X and E a quasi-coherent $\mathcal{O}_X$-module. Then $\underline{H}^n_{Z/Z'}(E)$ is a quasi-coherent $\mathcal{O}_X$-module for all $n \in \mathbb{N}$.*

Proof. Consider the following part of the long exact cohomology sequence

$$\ldots \to \underline{H}^n_{Z'}(E) \to \underline{H}^n_{Z}(E) \to \underline{H}^n_{Z/Z'}(E) \to \underline{H}^{n+1}_{Z'}(E) \to \underline{H}^{n+1}_{Z}(E) \to \ldots$$

Now by [G, 2.2] the $\mathcal{O}_X$-modules $\underline{H}^k_{Z'}(E)$ and $\underline{H}^k_{Z}(E)$ are quasi-coherent for all $k \in \mathbb{N}$. Using the above exact sequence and [EGA I, 2.2.2 (iii)] we conclude that $\underline{H}^n_{Z/Z'}(E)$ must be quasi-coherent. $\square$

Corollary 2.1.11 *Fixing the previous data assume that $X = \mathrm{Spec}(A)$ is affine. Then $\underline{H}_{Z/Z'}(E)$ is the sheaf associated to the A-module $H^n_{Z/Z'}(X,\, E)$.*

Proof. Let $J^{\cdot}$ be a bounded below resolution of E such that each J^p is an injective $\mathcal{O}_X$-module and consider the second hypercohomology spectral sequence of $\Gamma(X,\, *)$ with respect to $J^{\cdot}$

$$E_2^{p\,q} = H^p(X,\, \underline{H}^q_{Z/Z'}(E)) \;\Rightarrow\; H^n_{Z/Z'}(X,\, E)$$

which is biregular. Since X is affine and by 2.1.10 we have $E_2^{p,\,q} = 0$ for all $p > 0$. Hence this spectral sequence degenerates giving us an A-linear isomorphism

$$\Gamma(X,\, \underline{H}^n_{Z/Z'}(E)) \cong H^n_{Z/Z'}(X,\, E).$$

In view of 2.1.10 and [EGA I, 1.4.2] this proves the assertion. $\square$

Let X be a scheme whose underlying topological space is locally noetherian. Given $x \in X$ we set $X(x) = \mathrm{Spec}(\mathcal{O}_{X,\,x})$ and denote by $i_x \colon X(x) \to X$

the canonical morphism of schemes which is a homeomorphism onto the subspace of X consisting of all generisations of x. Recall from [H1, VI] that there is a natural functor Γ_x going from the category $\mathrm{Ab}(X)$ of sheaves of abelian groups on X to the category of abelian groups sending an object E of $\mathrm{Ab}(X)$ to the subgroup of the stalk E_x consisting of all elements $\overline{s}$ which have a representative s in an open neighbourhood U of x such that the support of s is contained in $\overline{\{x\}} \cap U$.

Lemma 2.1.12 *For $E \in \mathrm{Ab}(X)$ we have $\Gamma_x(E) = \Gamma_{\{x\}}(X(x),\, i_x^{-1}E)$.*

Proof. Note that $\Gamma(X(x),\, i_x^{-1}E) = E_x$ and let us prove the inclusion "$\subseteq$" first. For $\overline{s} \in \Gamma_x(E)$ there exists an open neighbourhood U of x in X and an $s \in \Gamma(U,\, E)$ such that $s_x = \overline{s}$ and $|s| \subseteq \overline{\{x\}} \cap U$. Suppose that there exists a $y \in X(x) - \{x\}$ such that $y \in |\overline{s}|$. Then we get $y \in U$ and $y \notin \overline{\{x\}}$. Hence there exists an open neighbourhood $V \subseteq U$ of y such that $s|V = 0$. Now consider the commutative diagram

$$\begin{array}{ccccc}
\Gamma(U,\, E) & \longrightarrow & \Gamma(V,\, E) & & \\
\downarrow & & \downarrow & \searrow & \\
 & & & & E_y \\
\downarrow & & \downarrow & \nearrow & \\
\Gamma(X(x),\, i_x^{-1}E) & \longrightarrow & \Gamma(V \cap X(x),\, i_x^{-1}E) & &
\end{array}$$

with the evident maps. Then we deduce that $\overline{s}_y = s_y = 0$, a contradiction. It remains to prove "$\supseteq$". Let $0 \neq \overline{t} \in \Gamma_{\{x\}}(X(x),\, i_x^{-1}E) \subseteq E_x$ and choose an open and noetherian (as a topological space) neighbourhood U of x in X and $t \in \Gamma(U,\, E)$ such that $t_x = \overline{t}$. Suppose that we had $|t| \not\subseteq \overline{\{x\}} \cap U$. Shrinking U if necessary we may assume that each irreducible component of $|t|$ contains x. Let y be the generic point of such a component. Then $y \in X(x)$ and $t_y \neq 0$ by hypothesis. But using the above diagram (setting $V = U$) we see that t_y is the image of $\overline{t}$ under the natural map $\Gamma(X(x),\, i_x^{-1}E) \to E_y$ and thus $t_y = 0$, a contradiction. $\square$

Corollary 2.1.13 $R\Gamma_x(E^\cdot) = R\Gamma_{\{x\}}(X(x),\, i_x^{-1}(E^\cdot))$ *for* $E^\cdot \in D(\mathrm{Ab}(X))$.

Proof. Note first that by [Sp] both sides are really defined on the unbounded derived category. Then using [loc. cit.] we obtain

$$R\Gamma_x(E^\cdot) = R\Gamma_{\{x\}}(X(x),\, Ri_x^{-1}(E^\cdot)).$$

Now we complete the proof by observing that $Ri_x^{-1}(E^\cdot) = i_x^{-1}(E^\cdot)$ because the functor i_x^{-1} is exact. $\square$

Following the usual convention we set $H_x^i(F^\cdot) = R^i\Gamma_x(F^\cdot)$ for all $i \in \mathbb{N}$. For the rest of this section we shall assume that X is a noetherian affine scheme. Let $Z' \subseteq Z$ be two closed subsets of X which are stable under specialization. Recall that $x \in Z - Z'$ is said to be maximal in Z if $y \in Z'$ for every non-trivial specialization y of x.

Lemma 2.1.14 *Let $Z' \subseteq Z$ be as before and suppose that every $x \in Z - Z'$ is maximal in Z. Let E be a quasi-coherent $\mathcal{O}_X$-module. Then there is a canonical functorial isomorphism*

$$H^n_{Z/Z'}(E) \xrightarrow{\cong} \bigoplus_{x \in Z - Z'} H^n_x(E)$$

for all $n \in \mathbb{N}$.

Proof. Since the underlying topological space of X is noetherian it follows that its global section functor commutes with arbitrary direct sums. Together with 2.1.11 and [H1, IV, Variation 8, Motif F] this implies the assertion. $\square$

To give a concrete description of this isomorphism let $q\colon E \to J^\cdot$ be a quasi-isomorphism where $J^\cdot$ is a bounded below complex of quasi-coherent, injective $\mathcal{O}_X$-modules. Such a resolution of E exists by [H1, II, 7.18]. Given $x \in Z - Z'$ and forming $i_x^{-1}(q)$ we get a quasi-isomorphism $i_x^{-1}(E) \to i_x^{-1}(J^\cdot)$. Furthermore, $i_x^{-1}(J^\cdot)$ is a bounded below complex of quasi-coherent, flabby $\mathcal{O}_{X(x)}$-modules and thus it is deployed for computing $R\Gamma_{\{x\}}(X(x), i_x^{-1}(E))$. By [H1, IV, Variation 8, Motif F] there is a well defined chain isomorphism

$$\Gamma_{Z/Z'}(X, J^\cdot) \to \bigoplus_{x \in Z - Z'} \Gamma_x(J^\cdot) = \bigoplus_{x \in Z - Z'} \Gamma_{\{x\}}(X(x), i_x^{-1}(J^\cdot))$$

given in degree n as follows: If s is an element of $\Gamma_{Z/Z'}(X, J^n)$ which is represented by $s' \in \Gamma_Z(X, J^n)$ then s is mapped by this chain map to the collection of germs $(s'_x)_{x \in Z - Z'}$. Then the isomorphism of 2.1.14 is nothing but the map between cohomology groups induced by this chain map.
Now suppose that $y \in Z - Z'$ is a generic point of an irreducible component of Z. Then $i_y^{-1}(Z) = \{y\}$ and the natural map

$$H^n_Z(X, E) \to H^n_{\{y\}}(X(y), i_y^{-1}(E))$$

can be described as $H^n(\tau)$ where τ is the chain map

$$\Gamma_Z(X, J^\cdot) \to \Gamma_y(J^\cdot)$$

sending a section t of $\Gamma_Z(X,\, J^k)$ to its germ at y for $k \in \mathbb{N}$. Hence we deduce the following result

Lemma 2.1.15 *Let $n \in \mathbb{N}$. Then with respect to the previous hypotheses there is a commutative diagram*

$$\begin{array}{ccccc} H^n_Z(X,\, E) & \xrightarrow{f} & H^n_{Z/Z'}(X,\, E) & \xrightarrow{\quad g \quad} & \bigoplus_{x \in Z-Z'} H^n_x(E) \\ & \searrow^{h} & & & \downarrow \mathrm{pr} \\ & & & & H^n_{i_y^{-1}(Z)}(X(y),\, i_y^{-1}(E)) = H^n_y(E) \end{array}$$

where f and h are the canonical maps, g is the isomorphism given by 2.1.14 *and* pr *is the projection onto $H^n_y(E)$.*

2.2 Relative trace maps

Let X be an affine scheme, $E^\cdot$ a strictly perfect complex on X with differential φ, U a quasi-compact open subset of X and $\mathcal{U} = (U_i)_{i=0,\,\dots,\,n}$ a finite open cover of U by affine schemes. Let $Z = X - U$ and suppose that $E^\cdot$ is acyclic outside Z. It follows that $\mathcal{E}nd^\cdot_{\mathcal{O}_X}(E^\cdot)$ is acyclic outside Z and the canonical map

$$R\underline{\Gamma}_Z(\mathcal{E}nd^\cdot_{\mathcal{O}_X}(E^\cdot)) \to \mathcal{E}nd^\cdot_{\mathcal{O}_X}(E^\cdot)$$

is an isomorphism in the derived category $D(X)$ which implies that the chain map $-\nu[-1]$ of 2.1.6 is a quasi-isomorphism. Letting $I = \{0,\, \dots,\, n\}$ we find by 1.5.3 an endomorphism $\psi_i \in \mathrm{End}^{-1}_{\mathcal{O}_X}(E^\cdot_{U_i})$ such that

$$[\varphi|U_i,\, \psi_i] = \mathrm{id}_{E^\cdot|U_i}$$

for each $i \in I$. Given a chain map $f\colon E^\cdot \to E^\cdot$ and $p \in \mathbb{Z}$ we define an $\mathcal{O}_X$-linear map $\omega^p_f\colon \mathrm{End}^p_{\mathcal{O}_X}(E^\cdot) \to \mathrm{Cone}^p(\mathcal{U},\, \mathrm{End}^\cdot_{\mathcal{O}_X}(E^\cdot))$ by

$$\xi \mapsto \left(f \cdot \xi,\, \left((-1)^{\frac{q(q+3)}{2}+1}\, \psi_{i_0} \cdot \ldots \cdot \psi_{i_q} \cdot f \cdot \xi \right)_{\substack{0 \le q \le n, \\ i_0 < \ldots < i_q}} \right).$$

Proposition 2.2.1 *The collection $\omega_f = (\omega^p_f)_{p \in \mathbb{Z}}$ gives rise to a morphism of complexes $\mathrm{End}^\cdot_{\mathcal{O}_X}(E^\cdot) \to \mathrm{Cone}^\cdot(\mathcal{U},\, \mathrm{End}^\cdot_{\mathcal{O}_X}(E^\cdot))$. Up to homotopy it does not depend on the choice of the ψ_i.*

Proof. Let δ denote the differential of $\mathrm{Cone}^\cdot(\mathcal{U},\, \mathrm{End}^\cdot_{\mathcal{O}_X}(E^\cdot))$. Given $p \in \mathbb{Z}$ and $\xi \in \mathrm{End}^p_{\mathcal{O}_X}(E^\cdot)$ we set

$$(u_{q,\,\lambda}) = \delta^p\Big(f\xi,\, \big((-1)^{\frac{q(q+3)}{2}+1}\, \psi_{i_0} \cdot \ldots \cdot \psi_{i_q} \cdot f \cdot \xi \big) \Big).$$

Then we get $u_{-1,*} = [\varphi,\, f\xi]$ and

$$u_{0,\lambda} = -f\xi + [\varphi,\, \psi_\lambda f\xi] = -f\xi + \big(f\xi - \psi_\lambda[\varphi,\, f\xi]\big) = -\psi_\lambda[\varphi,\, f\xi]$$

by 1.2.1 (b). Using again 1.2.1 (b) and 1.2.2 we deduce for $0 < q \leq n$ and $\lambda = (i_0,\, \ldots,\, i_q)$ that

$$\begin{aligned} u_{q,\lambda} &= \sum_{r=0}^{q} (-1)^{\frac{(q-1)(q+2)}{2}+r}\, \psi_{i_0} \cdot \ldots \cdot \widehat{\psi_{i_r}} \cdot \ldots \cdot \psi_{i_q} \cdot f\xi \\ &\quad + (-1)^{\frac{q(q+3)}{2}+q}\, [\varphi,\, \psi_{i_0} \cdot \ldots \cdot \psi_{i_q}] f\xi \\ &\quad\quad + (-1)^{q\frac{q(q+3)}{2}+1}\, \psi_{i_0} \cdot \ldots \cdot \psi_{i_q} [\varphi,\, f\xi] \\ &= \sum_{r=0}^{q} (-1)^{\frac{(q-1)(q+2)}{2}+r}\, \psi_{i_0} \cdot \ldots \cdot \widehat{\psi_{i_r}} \cdot \ldots \cdot \psi_{i_q} \cdot f\xi \\ &\quad + \sum_{r=0}^{q} (-1)^{\frac{q(q+3)}{2}+q+r}\, \psi_{i_0} \cdot \ldots \cdot \widehat{\psi_{i_r}} \cdot \ldots \cdot \psi_{i_q} \cdot f\xi \\ &\quad\quad + (-1)^{\frac{q(q+3)}{2}+1}\, \psi_{i_0} \cdot \ldots \cdot \psi_{i_q} [\varphi,\, f\xi] \\ &= \quad (-1)^{\frac{q(q+3)}{2}+1}\, \psi_{i_0} \cdot \ldots \cdot \psi_{i_q} [\varphi,\, f\xi] \end{aligned}$$

because $(-1)^{q+1+\frac{q(q+3)}{2}} = (-1)^{\frac{(q-1)(q+2)}{2}}$. This completes the proof of the first part of the assertion. To show the second part let ω'_f denote the chain map obtained by a different choice of the ψ_i and consider the map $-\nu[-1]$ of 2.1.6. As $(-\nu[-1]) \circ \omega_f = (-\nu[-1]) \circ \omega'_f$ and $-\nu[-1]$ is a quasi-isomorphism it follows from 1.5.4 that ω and ω' must be homotopic. $\square$

In view of 2.1.6 we now obtain

Corollary 2.2.2 *The sheafification of ω_{id} yields the inverse of the natural isomorphism $R\underline{\Gamma}_Z(\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E^{\cdot})) \to \mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E^{\cdot})$ in $D(X)$.*

Suppose that $\mathcal{U}'$ is also a finite open cover of U by affines. Then it is clear that we get a commutative triangle

$$\begin{array}{ccc} \mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E^{\cdot}) & \xrightarrow{\omega_f} & \mathcal{C}\mathrm{one}(\mathcal{U},\, \mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E^{\cdot})) \\ & \searrow^{\omega_f} & \downarrow \lambda^{\mathcal{U}}_{\mathcal{U}'} \\ & & \mathcal{C}\mathrm{one}(\mathcal{U}',\, \mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E^{\cdot})) \end{array}$$

in $D(X)$. Here $\lambda^{\mathcal{U}}_{\mathcal{U}'}$ is the isomorphism we have considered at the end of the previous section.

Lemma 2.2.3 *Suppose that f, $g\colon E^{\cdot} \to E^{\cdot}$ are homotopic chain maps. Then ω_f and ω_g are homotopic.*

Proof. For a chain map $h\colon E^{\cdot} \to E^{\cdot}$ we denote by m_h the chain map on $\mathrm{End}^{\cdot}_{\mathcal{O}_X}(E^{\cdot})$ induced by left multiplication of h. By hypothesis there is an element $k \in \mathrm{End}^{-1}_{\mathcal{O}_X}(E^{\cdot})$ such that $f - g = [\varphi, k]$. Now we get

$$(-\nu[-1]) \circ (\omega_f - \omega_g) = m_f - m_g = m_{f-g} = m_{[\varphi,\, k]}$$

and $m_{[\varphi,\, k]}$ is easily seen to be null homotopic. Applying 1.5.4 we infer that $\omega_f - \omega_g$ is null homotopic. $\square$

For the rest of this section we shall write ω for ω_{id} as well as for its sheafification. Note that we simply have $\omega_f = \omega \circ m_f$. It follows from 2.1.6 that the relative trace map

$$\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E^{\cdot}) = R\underline{\Gamma}_Z(R\mathcal{H}om^{\cdot}_{\mathcal{O}_X}(E^{\cdot},\, E^{\cdot})) \xrightarrow{\mathrm{Tr}_Z} R\underline{\Gamma}_Z(\mathcal{O}_X) = \mathcal{C}\mathrm{one}(\mathcal{U},\, \mathcal{O}_X)$$

is given by the chain map $\mathcal{C}\mathrm{one}(\mathcal{U}, \mathrm{Tr}) \circ \omega$ where $\mathrm{Tr}\colon \mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E^{\cdot}) \to \mathcal{O}_X$ denotes the trace map we have considered in 1.2. Moreover, if $\mathcal{U}'$ is as before the triangle

$$\begin{array}{ccc} \mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E^{\cdot}) & \xrightarrow{\mathrm{Tr}_Z} & \mathcal{C}\mathrm{one}(\mathcal{U},\, \mathcal{O}_X) \\ & \searrow^{\mathrm{Tr}_Z} & \downarrow \lambda^{\mathcal{U}}_{\mathcal{U}'} \\ & & \mathcal{C}\mathrm{one}(\mathcal{U}',\, \mathcal{O}_X) \end{array}$$

is clearly commutative.

Remark 2.2.4 Given a connection ∇ on $E^{\cdot}$ consider $\mathcal{C}\mathrm{one}(\mathcal{U},\, \mathcal{O}_X)$ as a complex with connection induced by the canonical exterior differentiation $d\colon \mathcal{O}_X \to \Omega^1_{X/S}$. Then it does not necessarily follow that Tr_{Z} is horizontal because ω need not to be horizontal.

Given a cocycle ξ of degree d of $\mathrm{End}^{\cdot}_{\mathcal{O}_X}(E^{\cdot})$ and $p \in \mathbb{Z}$ let τ^p_ξ and $\tilde{\tau}^p_\xi$ denote the maps $\mathcal{E}nd^p_{\mathcal{O}_X}(E^{\cdot}) \to \mathcal{C}\mathrm{one}^{p+d}(\mathcal{U},\, \mathcal{O}_X)$ sending ζ to $\mathrm{Tr}^{p+d}_Z(\xi \cdot \zeta)$ and $(-1)^{d \cdot p}\mathrm{Tr}^{p+d}_Z(\zeta \cdot \xi)$ respectively. Then the collections $\tau_\xi = (\tau^p_\xi)_{p\in\mathbb{Z}}$ and $\tilde{\tau}_\xi = (\tilde{\tau}^p_\xi)_{p\in\mathbb{Z}}$ give rise to chain maps

$$\mathrm{End}^{\cdot}_{\mathcal{O}_X}(E^{\cdot}) \to \mathrm{Cone}^{\cdot}(\mathcal{U},\, \mathcal{O}_X)[d]$$

and we obtain

Lemma 2.2.5 *The chain maps τ_ξ and $\tilde{\tau}_\xi$ are homotopic.*

Proof. Define chain maps Tr_ξ and $\tilde{\mathrm{Tr}}_\xi$ going from $\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E^{\cdot})$ to $\mathcal{O}_X$ by sending a degree p section ζ of $\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E^{\cdot})$ to $\mathrm{Tr}(\xi \cdot \zeta)$ and $(-1)^{d \cdot p}\mathrm{Tr}(\zeta \cdot \xi)$ respectively. Then $\mathrm{Tr}_\xi = \tilde{\mathrm{Tr}}_\xi$ by 1.2.4 (b) and therefore

$$\tau_\xi = R\Gamma_Z(X, \mathrm{Tr}_\xi) = R\Gamma_Z(X, \tilde{\mathrm{Tr}}_\xi) = \tilde{\tau}_\xi$$

in $\mathrm{Hom}_{D(\Gamma(X, \mathcal{O}_X))}(\mathrm{End}^{\cdot}_{\mathcal{O}_X}(E^{\cdot}), \mathrm{Cone}(\mathcal{U}, \mathcal{O}_X)[d])$ which implies the assertion in view of 1.5.2 and [G, 2.2]. $\square$

To end this section we state a functoriality result.

Lemma 2.2.6 *Consider a morphism $f\colon X' \to X$ of affine schemes and let $Z' = f^{-1}(Z)$. Then the diagram*

$$\begin{array}{ccc} f^*(\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E^{\cdot})) & \xrightarrow{f^*(\mathrm{Tr}_Z)} & f^*(\mathcal{C}\mathrm{one}(\mathcal{U}, E^{\cdot})) \\ \downarrow & & \downarrow \\ \mathcal{E}nd_{\mathcal{O}_{X'}}(f^*E^{\cdot}) & \xrightarrow{\mathrm{Tr}_{Z'}} & \mathcal{C}\mathrm{one}(f^{-1}\mathcal{U}, f^*E^{\cdot}) \end{array}$$

is commutative in $C(X')$ where the vertical maps are the canonical chain isomorphisms.

Proof. Using 2.1.9 (b) this is evident. $\square$

2.3 Local construction of cohomology classes

Let $f\colon X \to S$ be a weakly smooth morphism of schemes with $X = \mathrm{Spec}(A)$ a noetherian affine scheme, let Z be a closed subset of X and $U = X - Z$. Suppose that $\Omega^1_{X/S}$ is free of rank r and we are given a base $\mathcal{B} = (\omega_i)_{1 \le i \le r}$ of $\Omega^1_{X/S}$, a finite open affine cover $\mathcal{U}$ of U and a strictly perfect complex $E^{\cdot}$ on X with differential φ which is acyclic on U, i.e. such that Z contains $|E^{\cdot}|$. For $m \in \mathbb{N}$ we shall consider the graded $\mathcal{O}_X$-module

$$\mathcal{E}^{\cdot}_m = \Omega^m_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E^{\cdot})$$

in the usual way as a complex with differential $\mathrm{id}_{\Omega^m_{X/S}} \otimes d^{\cdot}_{\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E^{\cdot})}$. Then for a section α of $\mathcal{E}^n_m$ which we may view as section of (total) degree $m+n$ of the graded algebras

$$\Omega^{\cdot}_{X/S}(\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E^{\cdot})) \cong \mathcal{E}nd^{\cdot}(\Omega^{\cdot}_{X/S}(E^{\cdot}))_{\Omega^{\cdot}_{X/S}}$$

we have introduced in 1.3, it is easily verified that

$$d^n_{\mathcal{E}^{\cdot}_m}(\alpha) = (-1)^m [\varphi, \alpha].$$

Tensoring $\mathrm{id}_{\Omega^m_{X/S}}$ with the relative trace map Tr_Z of $E^\cdot$ (cp. 2.2) we get by 2.1.9 a chain map

$$\mathcal{E}^\cdot_m \to \mathcal{C}\mathrm{one}(\mathcal{U},\, \Omega^m_{X/S})$$

which we shall also denote by Tr_Z in the following.
Let ∇ be a connection on $E^\cdot$ and $p \in \{1,\, \dots,\, r\}$. Then it is easy to see that

$$[\varphi,\, \nabla_p(\varphi)] = 0,$$

i.e. $\nabla_p(\varphi)$ is a 1-cocycle of $\mathcal{E}^\cdot_0$. Hence by 1.2.1 (b) it follows that for $k \in \mathbb{N}$ and $p_1,\, \dots,\, p_k \in \{1,\, \dots,\, r\}$ the element

$$(\omega_{p_1} \wedge \cdots \wedge \omega_{p_k}) \otimes \mathrm{Tr}_Z(\nabla_{p_1}(\varphi) \cdot \ldots \cdot \nabla_{p_k}(\varphi))$$

is a k-cocycle of the complex $\mathrm{Cone}(\mathcal{U},\, \Omega^k_{X/S})$.

Let $\nabla,\, \nabla'$ be connections on $E^\cdot$ and let $A = \nabla - \nabla'$ which is a section of $\Omega^1_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}nd^0_{\mathcal{O}_X}(E^\cdot)$. Writing $A_p = (\omega^*_p \otimes \mathrm{id}_{\mathcal{E}nd^0_{\mathcal{O}_X}(E^\cdot)}) \circ A$ where $(\omega^*_i)_{1\le i\le r}$ denotes the base which is dual to $(\omega_i)_{1\le i\le r}$ we get by 1.3.10

$$\nabla(\varphi) - \nabla'(\varphi) = [\nabla - \nabla',\, \varphi] = [A,\, \varphi] = \sum_{p=1}^{r} \omega_p \otimes [A_p,\, \varphi].$$

Thus $\nabla_p(\varphi) - \nabla'_p(\varphi)$ is a coboundary of the complex $\mathcal{E}^\cdot_0$ for $1 \le p \le r$ and using 1.2.1, 2.2.5 we obtain

Lemma 2.3.1 *Let $\nabla,\, \nabla'$ be connections on $E^\cdot$. Then*

$$(\omega_{p_1} \wedge \cdots \wedge \omega_{p_k}) \otimes \mathrm{Tr}_Z(\nabla_{p_1}(\varphi) \cdot \ldots \cdot \nabla_{p_k}(\varphi) - \nabla'_{p_1}(\varphi) \cdot \ldots \cdot \nabla'_{p_k}(\varphi))$$

is a coboundary of the complex $\mathrm{Cone}(\mathcal{U},\, \Omega^k_{X/S})$.

Setting $\omega_{p_1,\,\dots,\,p_k} = \omega_{p_1} \wedge \cdots \wedge \omega_{p_k}$ we deduce by 2.1.7, the commutative diagram just before 2.2.4 and 2.3.1

Corollary 2.3.2 *Let $k \ge 1$. Then*

$$\mathrm{cl}^k_{\mathcal{B},\,Z}(E^\cdot) = \sum_{1\le p_1<\ldots<p_k\le r} \mathrm{Tr}_Z(\omega_{p_1,\,\dots,\,p_k} \otimes (\nabla_{p_1}(\varphi) \cdot \ldots \cdot \nabla_{p_k}(\varphi)))$$

defines an element in $H^k_Z(X,\, \Omega^k_{X/S})$ which does neither depend on the choice of $\mathcal{U}$ nor on the choice of ∇.

For $k = 0$ we set $\mathrm{cl}^0_{\mathcal{B},Z}(E^{\cdot}) = \mathrm{Tr}_Z(\mathrm{id}_{E^{\cdot}})$. In order to give a conceptual description of $\mathrm{cl}^0_{\mathcal{B},Z}(E^{\cdot})$ we digress for a moment and introduce the Euler characteristic for perfect complexes. Let Y be any (not necessarily (locally) noetherian) scheme, and let $\mathbf{K}_0(D(Y)_{\mathrm{perf}})$ denote the presheaf on Y given by $V \mapsto K_0(D(V)_{\mathrm{perf}})$ ($V \subseteq Y$ open). If $\mathbf{Z}$ denotes the constant sheaf with values in $\mathbb{Z}$ then by [SGA 6, I, 6] the usual Euler characteristic induces a morphism of presheaves

$$\chi\colon \mathbf{K}_0(D(Y)_{\mathrm{perf}}) \to \mathbf{Z}.$$

We summarize some properties of this map which we shall need later.

Lemma 2.3.3

(a) *Let* $F^{\cdot} \in D(Y)_{\mathrm{perf}}$ *with* $\mathrm{codim}(|F^{\cdot}|, Y) > 0$. *Then* $\chi(F^{\cdot}) = 0$.

(b) *Let* $F^{\cdot}, G^{\cdot} \in D(Y)_{\mathrm{perf}}$. *Then* $\chi(F^{\cdot} \otimes^L_{\mathcal{O}_Y} G^{\cdot}) = \chi(F^{\cdot}) \cdot \chi(G^{\cdot})$.

(c) *Let* $f\colon Y' \to Y$ *be a morphism of schemes and* $F^{\cdot} \in D(Y)_{\mathrm{perf}}$. *Then* $\chi(Lf^*(F^{\cdot})) = f^*(\chi(F^{\cdot}))$.

Proof. For (b) and (c) see [SGA 6, I, 6.12]. To prove (a) note first that $|F^{\cdot}|$ is closed in X (indeed, this is a local question and hence we may assume that Y is affine. Then $|F^{\cdot}|$ is closed by [T, 3.3 (c)]). We may assume that $|F^{\cdot}| \neq \emptyset$. Clearly the restriction of $\chi(F^{\cdot})$ to the open set $Y - |F^{\cdot}|$ is zero. Let $x \in |F^{\cdot}|$ and choose an open affine neighbourhood U of x such that $F^{\cdot}|U$ is quasi-isomorphic to a bounded complex $G^{\cdot}$ where each G^n is a vector bundle of constant rank r_n. We claim that $U \neq |F^{\cdot}| \cap U$. If we had $U \subseteq |F^{\cdot}|$ it would follow that $\mathrm{codim}(|F^{\cdot}| \cap U, U) = 0$ and a fortiori $\mathrm{codim}(|F^{\cdot}|, Y) = 0$ which contradicts our hypothesis. Thus there exists a $y \in U - |F^{\cdot}|$ and we deduce

$$\chi(F^{\cdot}|U) = \sum_n (-1)^n\, r_n = \chi(F^{\cdot}_y) = 0. \quad \square$$

In what follows we will also write $\chi(F^{\cdot})$ for the image of $\chi(F^{\cdot})$ under the natural map $\mathbf{Z} \to \mathcal{O}_Y$. Returning to our situation an easy local consideration then shows that

$$\mathrm{cl}^0_{\mathcal{B},Z}(E^{\cdot}) = \mathrm{tr}_s(\mathrm{id}_{E^{\cdot}}) = \chi(E^{\cdot}).$$

An application of 1.2.10 reveals that $\mathrm{cl}^k_{\mathcal{B},Z}(E^{\cdot})$ was constructed by "dividing out $k!$", i.e.

Proposition 2.3.4 *We have* $\mathrm{Tr}_Z(\nabla(\varphi)^k) = k! \cdot \mathrm{cl}^k_{\mathcal{B},Z}(E^{\cdot})$ *in* $H^k_Z(X, \Omega^k_{X/S})$. *Here the power* $\nabla(\varphi)^k$ *is taken with respect to the multiplication which makes* $\Omega^{\cdot}_{X/S}(\mathcal{E}nd^{\cdot}_{\mathcal{O}_X}(E^{\cdot}))$ *a subalgebra of* $\Omega^{\cdot}_{X/S} \otimes_{\mathcal{O}_X} \mathcal{E}nd_{\mathcal{O}_X}(\bigoplus_{p\in\mathbb{Z}} E^p)$.

Suppose that $U' \subseteq U$ is open and let $Z' = X - U'$. Given $k \in \mathbb{N}$ we get a commutative triangle

$$\begin{array}{ccc}
\mathrm{Hom}_{D(X)}(E^{\cdot},\, E^{\cdot}[k]) = H^k(\mathrm{End}^{\cdot}_{\mathcal{O}_X}(E^{\cdot})) & \xrightarrow{\mathrm{Tr}_Z} & H^k_Z(X,\, \mathcal{O}_X) \\
& \searrow^{\mathrm{Tr}_{Z'}} & \downarrow \\
& & H^k_{Z'}(X,\, \mathcal{O}_X)
\end{array}$$

where the vertical map is the natural one. Noting that $X - |E^{\cdot}|$ is open we eventually obtain

Proposition 2.3.5 *Suppose that $E^{\cdot}$ is a strictly perfect complex on X having a connection, and Z is a closed subset of X such that $|E^{\cdot}| \subseteq Z$. Then for $k \in \mathbb{N}$ the class $\mathrm{cl}^k_{\mathcal{B},Z}(E^{\cdot})$ is the image of $\mathrm{cl}^k_{\mathcal{B},|E^{\cdot}|}(E^{\cdot})$ under the natural map $H^k_{|E^{\cdot}|}(X,\, \Omega^k_{X/S}) \to H^k_Z(X,\, \Omega^k_{X/S})$.*

The following property of the constructed cohomology class is evident.

Lemma 2.3.6 *The class $\mathrm{cl}^k_{\mathcal{B},Z}(E^{\cdot})$ commutes with restriction to open affine subschemes of X. More precisely, if V is an open affine subscheme of X and $\Omega^1_{V/S}$ is regarded as a free $\mathcal{O}_V$-module with base $\mathcal{B}' = (\omega_i|V)_{1\le i\le r}$ then the class $\mathrm{cl}^k_{\mathcal{B}',Z\cap V}(E^{\cdot}|V)$ is the image of $\mathrm{cl}^k_{\mathcal{B},Z}(E^{\cdot})$ under the natural morphism $H^k_Z(X,\, \Omega^k_{X/S}) \to H^k_{Z\cap V}(V,\, \Omega^k_{V/S})$.*

Let U, $\mathcal{U}$ and $E^{\cdot}$ as before. In addition let $V \subseteq X$ be open and quasi-compact, $\mathcal{V} = (V_i)_{0\le i\le m}$ a finite affine open cover of V, and let $E'^{\cdot}$ be a strictly perfect complex on X with differential φ' which is acyclic on V. Forming $E''^{\cdot} = E^{\cdot} \otimes_{\mathcal{O}_X} E'^{\cdot}$ we obtain a strictly perfect complex which is acyclic on $W = U \cup V$. Assume that $\mathcal{U} = (U_i)_{0\le i\le n}$ and let $U_{n+j+1} = V_j$ for $0 \le j \le m$ so that $(U_i)_{0\le i\le m+n+1}$ is a finite, open and affine cover of W. Let φ'' denote the differential of $E''^{\cdot}$ and let

$$\epsilon = \bigoplus_{p\in\mathbb{Z}} (-1)^p \cdot \mathrm{id}_{E^p} \in \mathrm{End}^0_{\mathcal{O}_X}(E^{\cdot}).$$

Then we have

$$\varphi'' = \varphi \otimes \mathrm{id}_{E'^{\cdot}} + \epsilon \otimes \varphi'.$$

Let $i \in \{0,\, \dots,\, n\}$. Then by 1.5.3 there exists a $\psi_i \in \mathrm{End}^{-1}_{\mathcal{O}_{U_i}}(E^{\cdot}|U_i)$ satisfying $[\varphi|U_i,\, \psi_i] = \mathrm{id}_{E^{\cdot}|U_i}$. Similarly for $j \in \{0,\, \dots,\, m\}$ there is an element $\psi'_j \in \mathrm{End}^{-1}_{\mathcal{O}_{V_j}}(E'^{\cdot}|V_j)$ such that $[\varphi'|V_j,\, \psi'_j] = \mathrm{id}_{E'^{\cdot}|V_j}$. Setting

$$\psi''_i = \begin{cases} \psi_i \otimes \mathrm{id}_{E'^{\cdot}|U_i}, & 0 \le i \le n \\ \epsilon|U_i \otimes \psi'_{i-n-1}, & i > n \end{cases}$$

it follows that $[\varphi''|U_i,\, \psi_i] = \mathrm{id}_{E''^{\cdot}|U_i}$ for $0 \le i \le m+n+1$. Suppose we are given connections ∇, ∇' on $E^{\cdot}$, $E'^{\cdot}$, and let ∇'' denote the induced "tensor product" connection on $E''^{\cdot}$. Let $1 \le k \le m+n+2$ and consider natural numbers $0 \le i_0 < \ldots < i_{k-1} \le m+n+1$. We are going to treat three cases:
Case 1: $i_{k-1} \le n$. Setting

$$\lambda_{p_1,\,\ldots,\,p_k} = \mathrm{tr}_s(\psi_{i_0} \cdot \ldots \cdot \psi_{i_{k-1}} \cdot \nabla_{p_1}(\varphi) \cdot \ldots \cdot \nabla_{p_k}(\varphi))$$

we deduce

$$\begin{aligned}\sum_{1 \le p_1 < \ldots < p_k \le r} & \omega_{p_1,\,\ldots,\,p_k} \otimes \mathrm{tr}_s(\psi''_{i_0} \cdot \ldots \cdot \psi''_{i_{k-1}} \cdot \nabla''_{p_1}(\varphi'') \cdot \ldots \cdot \nabla''_{p_k}(\varphi'')) \\ &= \sum_{1 \le p_1 < \ldots < p_k \le r} \omega_{p_1,\,\ldots,\,p_k} \otimes \mathrm{tr}_s(\lambda_{p_1,\,\ldots,\,p_k} \otimes \mathrm{id}_{U_{i_0,\,\ldots,\,i_{k-1}}}) \\ &= \chi(E'^{\cdot}|U_{i_0,\,\ldots,\,i_{k-1}}) \cdot \sum_{1 \le p_1 < \ldots < p_k \le r} \omega_{p_1,\,\ldots,\,p_k} \otimes \lambda_{p_1,\,\ldots,\,p_k}\end{aligned}$$

using 1.2.11 because $\mathrm{tr}_s(\mathrm{id}_{E'^{\cdot}|U_{i_0,\,\ldots,\,i_{k-1}}}) = \chi(E'^{\cdot}|U_{i_0,\,\ldots,\,i_{k-1}})$.
Case 2: $i_0 > n$. Setting

$$\lambda'_{p_1,\,\ldots,\,p_k} = \mathrm{tr}_s(\psi'_{i_0-n-1} \cdot \ldots \cdot \psi'_{i_{k-n-2}} \cdot \nabla_{p_1}(\varphi') \cdot \ldots \cdot \nabla_{p_k}(\varphi'))$$

we get in a similar way

$$\begin{aligned}\sum_{1 \le p_1 < \ldots < p_k \le r} & \omega_{p_1,\,\ldots,\,p_k} \otimes \mathrm{tr}_s(\psi''_{i_0} \cdot \ldots \cdot \psi''_{i_{k-1}} \cdot \nabla''_{p_1}(\varphi'') \cdot \ldots \cdot \nabla''_{p_k}(\varphi'')) \\ &= \chi(E^{\cdot}|U_{i_0,\,\ldots,\,i_{k-1}}) \cdot \sum_{1 \le p_1 < \ldots < p_k \le r} \omega_{p_1,\,\ldots,\,p_k} \otimes \lambda'_{p_1,\,\ldots,\,p_k}\end{aligned}$$

Case 3: $i_0 \le n$ and $i_{k-1} > n$. Let $h \in \{0,\, \ldots,\, k-2\}$ be maximal with $i_h \le n$. Using the fact that ϵ anti-commutes with odd-degree endomorphisms on $E^{\cdot}$ and setting

$$\kappa_{q_1,\,\ldots,\,q_{h+1}} = \mathrm{tr}_s(\psi_0 \cdot \ldots \cdot \psi_h \cdot \nabla_{q_1}(\varphi) \cdot \ldots \cdot \nabla_{q_{h+1}}(\varphi)),$$

$$\mu_{r_1,\,\ldots,\,r_{k-h-1}} = \mathrm{tr}_s(\psi'_0 \cdot \ldots \cdot \psi'_{k-h-2} \cdot \nabla_{r_1}(\varphi') \cdot \ldots \cdot \nabla_{r_{k-h-1}}(\varphi'))$$

we get

$$\begin{aligned}\sum_{1 \le p_1 < \ldots < p_k \le r} & \omega_{p_1,\,\ldots,\,p_k} \otimes \mathrm{tr}_s(\psi''_0 \cdot \ldots \cdot \psi''_{i_{k-1}} \cdot \nabla''_{p_1}(\varphi'') \cdot \ldots \cdot \nabla''_{p_k}(\varphi'')) \\ &= (-1)^{(k-h-1)(h+1)} \left(\sum_{1 \le q_1 < \ldots < q_{h+1} \le r} \omega_{q_1,\,\ldots,\,q_{h+1}} \otimes \kappa_{q_1,\,\ldots,\,q_{h+1}} \right) \\ &\qquad \cdot \left(\sum_{1 \le r_1 < \ldots < r_{k-h-1} \le r} \omega_{r_1,\,\ldots,\,r_{k-h-1}} \otimes \mu_{r_1,\,\ldots,\,r_{k-h-1}} \right)\end{aligned}$$

using 1.2.11 again. Setting $Z = X - U$ and $Z' = X - V$ we therefore have shown that

$$\mathrm{cl}^k_{\mathcal{B},\, Z\cap Z'}(E^\cdot \otimes_{\mathcal{O}_X} E'^\cdot) = \sum_{j=0}^{k} \mathrm{cl}^j_{\mathcal{B},\, Z}(E^\cdot) \cup \mathrm{cl}^{k-j}_{\mathcal{B},\, Z'}(E'^\cdot) \tag{1}$$

where $\cup$ denotes the corresponding cup products.

Proposition 2.3.7 *Given $f_1, \ldots, f_k \in A$ let $K^\cdot = K^\cdot(f_1, \ldots, f_k)$ denote the corresponding Koszul complex and suppose that Z is a closed subset of X containing the support of $K^\cdot$.*

(a) *We have*

$$\mathrm{cl}^k_{\mathcal{B},\, Z}(K^\cdot) = \frac{d(f_1) \wedge \cdots \wedge d(f_k)}{f_1 \cdot \ldots \cdot f_k}.$$

(b) *Given $m \in \mathbb{N}$ we have $H^m_Z(X, d)(\mathrm{cl}^m_{\mathcal{B},\, Z}(K^\cdot)) = 0$ in $H^m_Z(X, \Omega^{m+1}_{X/S})$.*

Proof. It is easily verified that $|K^\cdot| = \bigcap_{i=1}^k V(f_i)$ and thus we may assume that $Z = \bigcap_{i=1}^k V(f_i)$. As the collection $(D(f_i))_{i=1,\ldots,k}$ is a finite affine open cover of $X - Z$ we know by 2.1.7 that $H^k_Z(X, \Omega^k_{X/S})$ is a quotient of the localization of $\Gamma(X, \Omega^k_{X/S})$ at $f_1 \cdot \ldots \cdot f_k$ showing that the term on the right hand side of the formula in (a) makes sense and also that (b) is a consequence of (a). In order to show (a) let us first consider the case where $k = 1$. We write $f = f_1$ and denote by φ the differential of $K^\cdot$. According to the conventions in [C] the complex $K^\cdot$ is given by

$$\ldots \longrightarrow 0 \longrightarrow K^{-1} = A \xrightarrow{\varphi^{-1}} K^0 = A \longrightarrow 0 \longrightarrow \ldots$$

where φ^{-1} maps 1 to f. Define a section ψ of $\mathrm{End}^{-1}_{\mathcal{O}_X}(K^\cdot)$ over $D(f)$ by sending $1 \in K^0_f$ to $\frac{1}{f} \in K^{-1}_f$. Then we get $[\varphi, \psi] = \mathrm{id}_{K^\cdot|D(f)}$. Moreover, if ∇ denotes the connection on the complex $K^\cdot$ induced by the canonical exterior differentiation $d\colon \mathcal{O}_X \to \Omega^1_{X/S}$ we infer

$$\sum_{p=1}^{r} \omega_p \otimes \mathrm{tr}_s(\psi \nabla_p(\varphi)) = -\sum_{p=1}^{r} \omega_p \otimes \frac{d_p(f)}{f} = -\frac{d(f)}{f} \tag{2}$$

Thus (a) holds for $k = 1$. Now the general case follows by induction using 2.3.3 (b) (note that $\chi(K^\cdot) = 0$), the natural chain isomorphism

$$K^\cdot(f_1, \ldots, f_k) \cong K^\cdot(f_1, \ldots, f_{k-1}) \otimes_{\mathcal{O}_X} K^\cdot(f_k)$$

and the formulas (1), (2). $\square$

Let U, $\mathcal{U}$ and Z be as in the beginning of this section. Suppose we are given a chain map $f\colon E^{\cdot} \to F^{\cdot}$ between strictly perfect complexes on X which are both acyclic on U. Let $G^{\cdot} = \mathrm{Cone}(f)$ denote the mapping cone of f and write φ, φ', φ'' for the differentials of $E^{\cdot}$, $F^{\cdot}$, $G^{\cdot}$. For $k \in \mathbb{Z}$ the $\mathcal{O}_X$-module $\mathcal{E}nd^k_{\mathcal{O}_X}(G^{\cdot})$ is canonically isomorphic to

$$\mathcal{E}nd^k_{\mathcal{O}_X}(E^{\cdot}) \oplus \mathcal{H}om^k_{\mathcal{O}_X}(E^{\cdot}[1],\, F^{\cdot}) \oplus \mathcal{H}om^k_{\mathcal{O}_X}(F^{\cdot},\, E^{\cdot}[1]) \oplus \mathcal{E}nd^k_{\mathcal{O}_X}(F^{\cdot}).$$

Thus a section ξ of $\mathcal{E}nd^k_{\mathcal{O}_X}(G^{\cdot})$ over an open subset V of X may be regarded as a 2×2 matrix, i.e.

$$\xi = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

where α, β, γ and δ are uniquely determined sections over V of $\mathcal{E}nd^k_{\mathcal{O}_X}(E^{\cdot})$, $\mathcal{H}om^k_{\mathcal{O}_X}(E^{\cdot}[1],\, F^{\cdot})$, $\mathcal{H}om^k_{\mathcal{O}_X}(F^{\cdot},\, E^{\cdot}[1])$ and $\mathcal{E}nd^k_{\mathcal{O}_X}(F^{\cdot})$ respectively. Moreover, the multiplication of sections of $\mathcal{E}nd^k_{\mathcal{O}_X}(G^{\cdot})$ is given by multiplying the corresponding matrices. Fixing this notation we get

Lemma 2.3.8 *Assume that $k = 0$. Then $\mathrm{tr}_s(\xi) = \mathrm{tr}_s(\delta) - \mathrm{tr}_s(\alpha)$.*

Proof. Follows immmediately from 1.2.4 (c). $\square$

Lemma 2.3.9 *Assume that ∇, ∇' are connections on $E^{\cdot}$, $F^{\cdot}$ and denote by ∇'', $\nabla^\flat$, $\nabla^\sharp$ the induced connections on the complexes $G^{\cdot}$, $\mathcal{H}om^{\cdot}_{\mathcal{O}_X}(E^{\cdot}[1],\, F^{\cdot})$, $\mathcal{H}om^{\cdot}_{\mathcal{O}_X}(F^{\cdot},\, E^{\cdot}[1])$. Then for $p \in \{1,\, \ldots,\, r\}$ we have*

$$\nabla''_p(\xi) = \begin{pmatrix} \nabla_p(\alpha) & \nabla^\sharp_p(\gamma) \\ \nabla^\flat_p(\beta) & \nabla'_p(\delta) \end{pmatrix}.$$

Proof. This is clear. $\square$

Let I denote the (totally ordered) index set of $\mathcal{U}$. Given $i \in I$ we see by 1.5.2 that the chain map $f|U_i$ is null homotopic. Thus there exists a section h_i of $\mathcal{H}om^{-1}_{\mathcal{O}_X}(E^{\cdot},\, F^{\cdot})$ over U_i such that

$$f|U_i = \varphi'|U_i \circ h_i \,+\, h_i \circ \varphi|U_i.$$

Let ψ_i and ψ'_i be sections of $\mathcal{E}nd^{-1}_{\mathcal{O}_X}(E^{\cdot})$ and $\mathcal{E}nd^{-1}_{\mathcal{O}_X}(F^{\cdot})$ over U_i satisfying $[\varphi,\, \psi_i] = \mathrm{id}_{E^{\cdot}}$ and $[\varphi',\, \psi'_i] = \mathrm{id}_{F^{\cdot}}$ on U_i. Setting

$$\psi''_i = \begin{pmatrix} -\psi_i & 0 \\ h_i \circ \psi_i \,+\, \psi'_i \circ h_i & \psi'_i \end{pmatrix} \in \mathrm{End}^{-1}_{\mathcal{O}_X}(G^{\cdot}|U_i)$$

and using the fact that the differential of $G^\cdot$ is given by

$$\varphi'' = \begin{pmatrix} -\varphi & 0 \\ f & \varphi' \end{pmatrix}$$

a simple calculation shows that $[\varphi'', \psi_i''] = \mathrm{id}_{G^\cdot}$ on U_i. Hence by 2.3.8, 2.3.9 and 2.1.9, 2.2.1 we deduce for $k \in \mathbb{N}$ and $p_1, \ldots, p_k \in \{1, \ldots, r\}$ that

$$\mathrm{Tr}_Z \left(\prod_{j=1}^{k} \nabla_{p_j}(\varphi'') \right) = \mathrm{Tr}_Z \left(\prod_{j=1}^{k} \nabla'_{p_j}(\varphi') \right) - \mathrm{Tr}_Z \left(\prod_{j=1}^{k} \nabla_{p_j}(\varphi) \right)$$

and therefore

Lemma 2.3.10 *We have* $\mathrm{cl}^k_{\mathcal{B},Z}(F^\cdot) = \mathrm{cl}^k_{\mathcal{B},Z}(E^\cdot) + \mathrm{cl}^k_{\mathcal{B},Z}(G^\cdot)$.

Corollary 2.3.11 *If f is a quasi-isomorphism then* $\mathrm{cl}^k_{\mathcal{B},Z}(E^\cdot) = \mathrm{cl}^k_{\mathcal{B},Z}(F^\cdot)$ *for every* $k \in \mathbb{N}$.

We are now ready to establish

Proposition 2.3.12 *Let $Z \subseteq X$ be closed and denote by $D_Z(X)_{\mathrm{perf}}$ the thick triangulated subcategory of $D(X)_{\mathrm{perf}}$ consisting of those perfect complexes which are acyclic outside Z. Then for $k \in \mathbb{N}$ there is an uniquely determined cycle map*

$$\mathrm{cl}^k_{\mathcal{B},Z} \colon K_0(D_Z(X)_{\mathrm{perf}}) \to H^k_Z(X, \Omega^k_{X/S})$$

satisfying the following condition:

- *Whenever $E^\cdot$ is a strictly perfect complex on X having a connection then $\mathrm{cl}^k_{\mathcal{B},Z}([E])$ coincides with the class given by* 2.3.2.

Furthermore, this map commutes with restriction to open affine subschemes of X.

Proof. This follows from 2.3.2, 2.3.6, 2.3.10, the universal mapping property of Grothendieck groups and the following facts:

- Given a perfect complex $F^\cdot$ there exists a (finite) open affine cover $(U_i)_{i \in I}$ of X such that each $F^\cdot | U_i$ is quasi-isomorphic to a strictly perfect complex on U_i which has a connection.
- $\underline{H}^k_Z(X, \Omega^k_{X/S})$ is the sheaf associated to the A-module $H^k_Z(X, \Omega^k_{X/S})$ according to [G, 2.2].

- The intersection of two affine open subsets of X is affine. This holds by [EGA I, 5.3.10] because X is affine and a fortiori separated. $\square$

For the rest of this section we fix a closed subset Z of X.

Lemma 2.3.13 *Assume that $c = \operatorname{depth}_Z(\mathcal{O}_X)$ and let k be a natural number such that $k < c$. Then $\mathrm{cl}^k_{\mathcal{B},\,Z}(E^\cdot) = 0$.*

Proof. Indeed, if $k < c$ we have $H^k_Z(X,\, \Omega^k_{X/S}) = 0$ by [G, 2.2, 3.8] and a fortiori $\mathrm{cl}^k_{\mathcal{B},\,Z}(E^\cdot) = 0$. $\square$

Let $k \in \mathbb{N}$ and consider a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\ g\ } & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ S' & \longrightarrow & S \end{array}$$

where X and X' are noetherian affine schemes, f and f' are weakly smooth morphisms and $\Omega^1_{X/S}$, $\Omega^1_{X'/S'}$ are free of constant rank. Let $Z \subseteq X$ be closed, $Z' = g^{-1}(Z)$. The functor Lg^* maps $D_Z(X)_{\mathrm{perf}}$ into $D_{Z'}(X')_{\mathrm{perf}}$ and hence induces a map

$$Lg^*\colon K_0(D_Z(X)_{\mathrm{perf}}) \to K_0(D_{Z'}(X')_{\mathrm{perf}}).$$

Moreover, we have a natural map

$$H^k_Z(X,\, \Omega^k_{X/S}) \to H^k_{Z'}(X',\, \Omega^k_{X'/S'})$$

which we (abusively) denote by g^* in what follows. Suppose that the following condition is satisfied:

- There exist natural numbers $p \leq q$ and a base $\mathcal{B} = (\omega_i)_{1 \leq i \leq q}$ of $\Omega^1_{X/S}$ such that $\mathcal{B}' = (g^*(\omega_i))_{1 \leq i \leq p}$ is a base for $\Omega^1_{X'/S'}$ and in addition $g^*(\omega_i) = 0$ for $i > p$ (e.g. when the diagram is cartesian).

If $\mathrm{MC}(X/S)$ (resp. $\mathrm{MC}(X'/S')$) denotes the category of quasi-coherent $\mathcal{O}_X$-modules (resp. $\mathcal{O}_{X'}$-modules) with connections (and horizontal maps as morphisms) then there is a canonical pull back functor

$$\mathrm{MC}(X/S) \to \mathrm{MC}(X'/S')$$

(see [K, (1.1.4)] or [B, II, 1.2.5]). Using this functor and 2.2.6 we then obtain

Lemma 2.3.14 *With respect to the previous notation the diagram*

$$\begin{array}{ccc} K_0(D_Z(X)_{\mathrm{perf}}) & \xrightarrow{\mathrm{cl}^k_{\mathcal{B},\,Z}} & H^k_Z(X,\,\Omega^k_{X/S}) \\ {\scriptstyle Lg^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ K_0(D_{Z'}(X')_{\mathrm{perf}}) & \xrightarrow{\mathrm{cl}^k_{\mathcal{B}',\,Z'}} & H^k_{Z'}(X',\,\Omega^k_{X'/S'}) \end{array}$$

is commutative.

Next we turn to two questions, namely whether the cycle maps given by 2.3.12 really depend on the choice of a base for $\Omega^1_{X/S}$ and also whether we may pass from relative Hodge cohomology to De Rham cohomology.

Proposition 2.3.15 *Assume that X is a regular scheme and let k be a natural number such that $k \le \mathrm{codim}(Z,\,X) = \mathrm{depth}_Z(\mathcal{O}_X)$.*

(a) *Let $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ be two bases for the $\mathcal{O}_X$-module $\Omega^1_{X/S}$. Then we have $\mathrm{cl}^k_{\mathcal{B},\,Z} = \mathrm{cl}^k_{\widetilde{\mathcal{B}},\,Z}$.*

(b) *The morphism $K_0(D_Z(X)_{\mathrm{perf}}) \xrightarrow{\mathrm{cl}^k_{\mathcal{B},\,Z}} H^k_Z(X,\,\Omega^k_{X/S}) \xrightarrow{H^k_Z(X,\,d)} H^k_Z(X,\,\Omega^{k+1}_{X/S})$ vanishes.*

Before we can give a proof of 2.3.15 we need some preparation. Let Y be a noetherian scheme, $Y' \subseteq Y$ a closed subset and denote by $C_{Y'}(Y)$ the category of coherent $\mathcal{O}_Y$-modules whose support is contained in Y'. Clearly $C_{Y'}(Y)$ is an abelian category and if $Y'' \subseteq Y'$ is closed then $C_{Y''}(Y)$ is a strictly full abelian subcategory of $C_{Y'}(Y)$. We shall denote by $K_0(C_{Y'}(Y))$ the Grothendieck group of $C_{Y'}(Y)$ (cf. [SGA 5, VIII]) and by $[\mathcal{F}]$ the element in $K_0(C_{Y'}(Y))$ corresponding to an object $\mathcal{F}$ of $C_{Y'}(Y)$.

Lemma 2.3.16 *Let $(\mathcal{F}_i)_{i\in I}$ be a collection of objects of $C_{Y'}(Y)$ such that for every closed and irreducible subset Y'' of Y' with generic point ζ there exists an $i \in I$ such that $(\mathcal{F}_i)_\zeta$ is a $k(\zeta)$-vector space of dimension 1. Then $K_0(C_{Y'}(Y))$ is generated by the collection $([\mathcal{F}_i])_{i\in I}$*

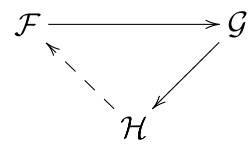
Proof. Let $\mathcal{C}$ denote the full subcategory of $C_{Y'}(Y)$ consisting of all those $\mathcal{F} \in C_{Y'}(Y)$ such that $[\mathcal{F}]$ lies in the subgroup of $K_0(C_{Y'}(Y))$ generated by the collection $(\mathcal{F}_i)_{i\in I}$. Then $\mathcal{C}$ is exact in the sense of [EGA III, 3.1.1] and by the dévissage lemma ([ibid., 3.1.2]) it follows that $\mathcal{C} = C_{Y'}(Y)$. □

Lemma 2.3.17 *Suppose that $C_{Y'}(Y) \subseteq D_{Y'}(Y)_{\mathrm{perf}}$ in $D(Y)$. Then the Grothendieck groups $K_0(C_{Y'}(Y))$ and $K_0(D_{Y'}(Y)_{\mathrm{perf}})$ are canonically isomorphic.*

Proof. As $C_{Y'}(Y) \subseteq D_{Y'}(Y)_{\mathrm{perf}}$ and every short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

of objects of $C_{Y'}(Y)$ induces an exact triangle



we have a canonical morphism $\mu\colon K_0(C_{Y'}(Y)) \to K_0(D_{Y'}(Y)_{\mathrm{perf}})$. Mapping the element $[E^{\cdot}]$ to

$$\sum_{i\in\mathbb{Y}'}(-1)^i \cdot [H^i(E^{\cdot})] \in K_0(C_{Y'}(Y))$$

for $E^{\cdot} \in D_{Y'}(Y)_{\mathrm{perf}}$ yields a morphism $\nu\colon K_0(D_{Y'}(Y)_{\mathrm{perf}}) \to K_0(C_{Y'}(Y))$, and it is easy to check that μ and ν are inverse to each other. $\square$

Corollary 2.3.18 *Suppose that $Y = \mathrm{Spec}(R)$ where R is a noetherian local ring with maximal ideal $\mathfrak{m}$. Let $Y' = \{\mathbf{m}\}$ and suppose that $\mathfrak{m}$ is generated by an R-regular sequence. Then $K_0(D_{Y'}(Y)_{\mathrm{perf}})$ is generated by a single Koszul complex.*

Proof. There exists an R-regular sequence $f_1, \ldots, f_n$ which generates $\mathfrak{m}$. Then the Koszul complex $K^{\cdot}(f_1, \ldots, f_n)$ gives us a projective resolution of $A/\mathfrak{m} = k(\mathfrak{m})$. Now an application of the dévissage lemma ([EGA III, 3.1.2]) shows that $C_{Y'}(Y) \subseteq D_{Y'}(Y)_{\mathrm{perf}}$. Hence by using 2.3.16 and 2.3.17 we see that $[K^{\cdot}(f_1, \ldots, f_n)]$ generates $K_0(D_{Y'}(Y)_{\mathrm{perf}})$. $\square$

Proof of 2.3.15. Recall that we have $X = \mathrm{Spec}(A)$. To prove (a) let $x \in X$, $S' = \mathrm{Spec}(\mathcal{O}_{S,\,f(x)})$, $X' = \mathrm{Spec}(\mathcal{O}_{X,\,x})$, $Z' = Z \cap X$. Then we get a canonical, commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & S' \\ {\scriptstyle g}\big\downarrow & & \big\downarrow \\ X & \longrightarrow & S \end{array}$$

where the morphisms are the evident ones. Let $\mathcal{B}_0 = (\omega_i)_{1\leq i\leq r}$ be any base of $\Omega^1_{X/S}$. Then the collection of germs $\mathcal{B}_0' = ((\omega_i)_x)_{1\leq i\leq r}$ gives us a base for the $\mathcal{O}_{X'/S'}$-module $\Omega^1_{X'/S'}$. Let $g^*\colon H^k_Z(X,\, \Omega^k_{X/S}) \to H^k_{Z'}(X',\, \Omega^k_{X'/S'})$ denote the induced map. By [G, 2.2], 2.1.7 and 2.1.9 we see that $H^k_{Z'}(X',\, \Omega^k_{X'/S'})$ is the stalk at x of the sheaf associated to the A-module $H^k_Z(X,\, \Omega^k_{X/S})$, and for

$\alpha \in K_0(D_Z(X)_{\mathrm{perf}})$ the element $g^*(\mathrm{cl}_{\mathcal{B}_0,\,Z}(\alpha))$ is the germ of $\mathrm{cl}_{\mathcal{B}_0,\,Z}(\alpha)$ at x. Hence using 2.3.14 we are reduced to proving (a) when A is a local regular ring which we shall assume henceforth. By 2.3.13 we can also assume that $k = \mathrm{codim}(Z,\, X)$. Let $Z'' = \{z \in Z;\, \dim(\mathcal{O}_{X,\,z}) \geq k+1\}$. Then every $x \in Z - Z'$ must be a generic point of an irreducible component of Z so that in particular $\mathrm{Spec}(\mathcal{O}_{X,\,x}) \cap Z = \{x\}$. Furthermore, the set Z'' is stable under specialization and every point in $Z - Z''$ is maximal in Z (in the sense of [H1, IV, Var.8, Motif F]). Let $\mathcal{V}$ denote the set of all those closed subsets of X which are finite unions of closures of points in Z''. Clearly $\mathcal{V}$ is directed with respect to $\subseteq$. Let $n \in \mathbb{N}$. As

$$H^k_{Z''}(X,\, \Omega^n_{X/S}) = \varinjlim_{V \in \mathcal{V}} H^k_V(X,\, \Omega^n_{X/S})$$

and $H^k_V(X,\, \Omega^n_{X/S})$ vanishes for each $V \in \mathcal{V}$ by [G, 2.2, 3.8] we infer that $H^k_{Z''}(X,\, \Omega^n_{X/S}) = 0$. Thus using the canonical exact sequence

$$H^k_{Z''}(X,\, \Omega^k_{X/S}) \to H^k_Z(X,\, \Omega^k_{X/S}) \to H^k_{Z/Z''}(X,\, \Omega^k_{X/S})$$

and 2.1.14, 2.1.15 as well as 2.3.14 applied to the canonical commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(\mathcal{O}_{X,\,x}) & \longrightarrow & \mathrm{Spec}(\mathcal{O}_{S,\,f(x)}) \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

for every $x \in Z - Z''$ we may finally assume that Z is the closed point of A. But in this case an application of 2.3.7 (a) and 2.3.18 establishes (a). For (b) making similar reduction steps as before we again arrive at the case where A is local and Z coincides with its closed point so that we may apply 2.3.7 (b) and 2.3.18 to complete the proof of the proposition. □

For the rest of this section we shall assume that X is a regular scheme and $k \leq \mathrm{codim}(Z,\, X)$. In the following we shall write cl^k_Z instead of $\mathrm{cl}^k_{\mathcal{B},\,Z}$ which is justified by 2.3.15 (a). Then by 2.3.15 (b) the image of cl^k_Z is contained in the group of degree k cocycles of the complex $L^{\cdot}_k = (H^k_Z(X,\, \Omega^p_{X/S}))_{p\in\mathbb{Z}}$. Recall that according to 1.4.9 and 1.4.10 we have a natural $\Gamma(S,\, \mathcal{O}_S)$-linear isomorphism

$$Z^k(L^{\cdot}_k) = \Gamma(X,\, \mathrm{Ker}(\underline{H}^k_Z(d))) \xrightarrow{\cong} H^{2k}_Z(X,\, \tau_{\geq k}(\Omega^{\cdot}_{X/S})).$$

Composing it with cl^k_Z we get a cycle map

$$\mathrm{cl}^k_Z \colon K_0(D_Z(X)_{\mathrm{perf}}) \to H^{2k}_Z(X,\, \tau_{\geq k}(\Omega^{\cdot}_{X/S})).$$

Note that $\mathrm{cl}_Z^k = 0$ when $k < \mathrm{codim}(Z, X)$.

Consider a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \longrightarrow & S \end{array}$$

where f, f' are weakly smooth, g is flat and X, X' are noetherian affine regular schemes (such that $\Omega^1_{X/S}$ and $\Omega^1_{X'/S'}$ are free of constant rank). Let $Z \subseteq X$ be closed, $Z' = g^{-1}(Z)$. As the morphism g is flat we infer that $\mathrm{depth}_Z(\mathcal{O}_X) \leq \mathrm{depth}_{Z'}(\mathcal{O}_{X'})$. Let Lg^* be as before but now we denote by g^* the natural map

$$H^{2k}_Z(X, \tau_{\geq k}(\Omega^{\cdot}_{X/S})) \to H^{2k}_{Z'}(X', \tau_{\geq k}(\Omega^{\cdot}_{X'/S'})).$$

Proposition 2.3.19 *With respect to the previous notation the diagram*

$$\begin{array}{ccc} K_0(D_Z(X)_{\mathrm{perf}}) & \xrightarrow{\mathrm{cl}_Z^k} & H^{2k}_Z(X, \tau_{\geq k}(\Omega^{\cdot}_{X/S})) \\ {\scriptstyle Lg^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ K_0(D_{Z'}(X')_{\mathrm{perf}}) & \xrightarrow{\mathrm{cl}_{Z'}^k} & H^{2k}_{Z'}(X', \tau_{\geq k}(\Omega^{\cdot}_{X'/S'})) \end{array}$$

is commutative.

Proof. It suffices to establish the commutativity of the diagram

$$\begin{array}{ccc} K_0(D_Z(X)_{\mathrm{perf}}) & \xrightarrow{\mathrm{cl}_Z^k} & H^k_Z(X, \Omega^k_{X/S}) \\ {\scriptstyle Lg^*}\downarrow & & \downarrow{\scriptstyle g^*} \\ K_0(D_{Z'}(X')_{\mathrm{perf}}) & \xrightarrow{\mathrm{cl}_{Z'}^k} & H^k_{Z'}(X', \Omega^k_{X'/S'}) \end{array}$$

Using the fact that g is "generisant" and arguing as in the proof of 2.3.15 we may assume that X, X' are local regular rings of the same dimension and Z, Z' are the closed points of X, X'. Then the assertion results from 2.2.6 and 2.3.7 (a). $\square$

Remark 2.3.20 (a) Suppose that in the previous diagram the morphism g is not necessarily flat but instead the hypotheses of 2.3.14 hold. Then by 2.3.14 we see that 2.3.19 remains valid.
(b) All the results in this section do also hold if X is Cohen-Macaulay and

we consider $\widetilde{K}_0(D_Z(X)_{\text{perf}})$ instead of $K_0(D_Z(X)_{\text{perf}})$, where $\widetilde{K}_0(D_Z(X)_{\text{perf}})$ denotes the Grothendieck group of the category consisting of all those complexes $E^{\cdot} \in D_Z(X_{\text{perf}})$ such that for every maximal point z of Z the class of $E^{\cdot}_z$ in $D_{\{z\}}(\operatorname{Spec}(\mathcal{O}_{Z,\,z}))$ lies in the subgroup of $K_0(D_{\{z\}}(\operatorname{Spec}(\mathcal{O}_{Z,\,z})))$ generated by all Koszul complexes which are acyclic off $\{z\}$.

2.4 De Rham cycle classes

All schemes we consider in this section are supposed to be regular. Combining the results of the previous section with 1.4.10 we now obtain

Theorem 2.4.1 *Let $f\colon X \to S$ be a smooth morphism. Let $Y \subseteq X$ be closed and $c \leq \operatorname{codim}(Y,\, X)$. Then there exists a unique homomorphism*

$$\operatorname{cl}_Y^c\colon K_0(D_Y(X)_{\text{perf}}) \to H_Y^{2c}(X,\, \tau_{\geq c}(\Omega^{\cdot}_{X/S}))$$

satisfying the following condition: Given an open affine cover $(U_i)_{i\in I}$ of X with $\Omega^1_{U_i/S}$ free of constant rank on a base $\mathcal{B}_i$ the diagram

$$\begin{array}{ccc}
K_0(D_Y(X)_{\text{perf}}) & \xrightarrow{\operatorname{cl}_Y^c} & H_Y^{2c}(X,\, \tau_{\geq c}(\Omega^{\cdot}_{X/S})) = \underline{H}_Y^{2c}(X,\, \tau_{\geq c}(\Omega^{\cdot}_{X/S})) \\
\downarrow & & \downarrow \\
K_0(D_{Y\cap U_i}(U_i)_{\text{perf}}) & \xrightarrow{\operatorname{cl}^c_{\mathcal{B}_i,\, Y\cap U_i}} & H^{2c}_{Y\cap U_i}(U_i,\, \tau_{\geq c}(\Omega^{\cdot}_{U_i/S})) = \underline{H}_Y^{2c}(U_i,\, \tau_{\geq c}(\Omega^{\cdot}_{X/S}))
\end{array}$$

is commutative for every $i \in I$. Here $\operatorname{cl}^c_{\mathcal{B}_i,\, Y\cap U_i}$ is the cycle map constructed in 2.3*, the vertical map on the left is the pull back induced by the open immersion $U_i \hookrightarrow X$, and the vertical map on the right is given by restricting sections of the sheaf $\underline{H}_Y^{2c}(\tau_{\geq c}(\Omega^{\cdot}_{X/S}))$.*

Given $\alpha \in K_0(D_Y(X)_{\text{perf}})$ we shall also write $\operatorname{cl}_Y^c(\alpha)$ for the corresponding class in $H^{2c}_{\text{DR}}(X/S)$, i.e for the image of $\operatorname{cl}_Y^c(\alpha)$ under the natural map

$$H_Y^{2c}(X,\, \tau_{\geq c}(\Omega^{\cdot}_{X/S})) \to H^{2c}_{\text{DR}}(X,\, \tau_{\geq c}(\Omega^{\cdot}_{X/S})) \to H^{2c}_{\text{DR}}(X/S).$$

Clearly we have $\operatorname{cl}_Y^c = 0$ when $c < \operatorname{codim}(Y,\, X)$. In case of $c = \operatorname{codim}(Y,\, X)$ we set $\operatorname{cl}_Y = \operatorname{cl}_Y^c$.

Corollary 2.4.2 *Let U be an open subscheme of X, $c \leq \operatorname{codim}(Y,\, X)$, and let $\rho\colon H_Y^{2c}(X,\, \tau_{\geq c}(\Omega^{\cdot}_{X/S})) \to H^{2c}_{Y\cap U}(U,\, \tau_{\geq c}(\Omega^{\cdot}_{U/S}))$ denote the natural map. Then we have $\rho \circ \operatorname{cl}_Y^c = \operatorname{cl}^c_{Y\cap U}$.*

Proof. This is clear by 1.4.9, 1.4.10 and 2.4.1. □

Consider a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ S' & \longrightarrow & S \end{array}$$

where f and f' are smooth. Let $Y \subseteq X$ be closed, $Y' = g^{-1}(Y)$, and let $c \in \mathbb{N}$ such that $c \leq \min(\operatorname{codim}(Y, X), \operatorname{codim}(Y', X'))$.

Proposition 2.4.3 *With respect to the previous data assume in addition that one of the following conditions holds:*

(a) *g is flat.*

(b) *$S = S'$ and g is a closed S-immersion.*

(c) *The diagram is cartesian.*

Then the diagram

$$\begin{array}{ccc} K_0(D_Y(X)_{\mathrm{perf}}) & \xrightarrow{\mathrm{cl}^c_Y} & H^{2c}_Y(X, \tau_{\geq c}(\Omega^{\cdot}_{X/S})) \\ {\scriptstyle Lg^*}\big\downarrow & & \big\downarrow{\scriptstyle g^*} \\ K_0(D_{Y'}(X')_{\mathrm{perf}}) & \xrightarrow{\mathrm{cl}^c_{Y'}} & H^{2c}_{Y'}(X', \tau_{\geq c}(\Omega^{\cdot}_{X'/S'})) \end{array}$$

is commutative.

Proof. (a) follows from 1.4.10, 2.3.20 and 2.4.1. To prove (b) we note first that the morphism g is a regular closed immersion ([EGA IV, 17.12.1]). Hence by [loc. cit., 17.12.2 (c)], [EGA 0_{I}, 5.2.2 (i)], 1.4.10 and 2.4.1 we may assume that X', X are affine and the hypotheses of 2.3.14 are satisfied. Now we conclude by 2.3.20. Finally (c) follows from 1.4.10, 2.3.20 and 2.4.1. □

For the rest of this section we fix a smooth morphism $f\colon X \to S$. Given $i = 1, 2$ let Y_i be a closed subscheme of X with $c_i \leq \operatorname{codim}(Y_i, X)$. As the derived tensor product $\otimes^L_{\mathcal{O}_X}$ maps $D_{Y_1}(X)_{\mathrm{perf}} \times D_{Y_2}(X)_{\mathrm{perf}}$ into $D_{Y_1 \cap Y_2}(X)_{\mathrm{perf}}$ we obtain a morphism

$$\otimes^L_{\mathcal{O}_X} : K_0(D_{Y_1}(X)_{\mathrm{perf}}) \times K_0(D_{Y_2}(X)_{\mathrm{perf}}) \to K_0(D_{Y_1 \cap Y_2}(X)_{\mathrm{perf}}).$$

Recall also that we have a cup product

$$H^{2c_1}_{Y_1}(X, \tau_{\geq c_1}(\Omega^{\cdot}_{X/S})) \times H^{2c_2}_{Y_2}(X, \tau_{\geq c_2}(\Omega^{\cdot}_{X/S})) \xrightarrow{\cup} H^{2(c_1+c_2)}_{Y_1 \cap Y_2}(X, \tau_{\geq c_1+c_2}(\Omega^{\cdot}_{X/S})).$$

Having introduced this terminology we are ready to state our next result which amounts to an intersection formula.

Proposition 2.4.4 *Suppose that* $c_1 + c_2 \leq \operatorname{codim}(Y, X)$ (*e.g. if* Y_1 *and* Y_2 *intersect nicely*). *Then for* $i = 1, 2$ *and* $\alpha_i \in K_0(D_{Y_i}(X)_{\text{perf}})$ *we have*

$$\text{cl}^{c_1+c_2}_{Y_1 \cap Y_2}(\alpha_1 \otimes^L_{\mathcal{O}_X} \alpha_2) = \text{cl}^{c_1}_{Y_1}(\alpha_1) \cup \text{cl}^{c_2}_{Y_2}(\alpha_2).$$

Proof. We may assume that $Y_1 \cap Y_2 \neq \emptyset$. Let $x \in Y_1 \cap Y_2$ and choose an open affine neighbourhood U of x with $\Omega^1_{U/S}$ free of constant rank and strictly perfect complexes $E_i^{\cdot}$ on U with connections such that $[E_i^{\cdot}] = \alpha_i|U$ for $i = 1, 2$. By the formula (1) on page 43, 2.3.13, 2.4.2 and the fact that $\text{depth}_{U \cap Y_i}(\mathcal{O}_U) \geq c_i$ we deduce

$$\begin{aligned}(\text{cl}^{c_1+c_2}_{Y_1 \cap Y_2}(\alpha_1 \otimes^L_{\mathcal{O}_X} \alpha_2))|U &= \text{cl}^{c_1+c_2}_{Y_1 \cap Y_2 \cap U}(E_1^{\cdot}|U \otimes_{\mathcal{O}_U} E_2^{\cdot}|U) \\ &= \text{cl}^{c_1}_{Y_1 \cap U}(E_1^{\cdot}|U) \cup \text{cl}^{c_2}_{Y_2 \cap U}(E_2^{\cdot}|U) \\ &= (\text{cl}^{c_1}_{Y_1}(\alpha_1) \cup \text{cl}^{c_2}_{Y_2}(\alpha_2))|U\end{aligned}$$

In view of 1.4.10 and 2.4.1 this implies the assertion. □

Proposition 2.4.5 *Let* $Y \subseteq X$ *be a closed subscheme of codimension* c *which is smooth over* S. *Then* $c = \operatorname{codim}(Y, X)$ *and* $\text{cl}_Y([\mathcal{O}_Y])$ *coincides with Berthelot's De Rham cohomology class* (*cp.* [B, VI, 3.1]).

Proof. Since $Y \hookrightarrow X$ is a regular closed immersion ([EGA IV, 17.12.1]) it follows that $\mathcal{O}_Y \in D_Y(X)_{\text{perf}}$. Moreover, we have $c = \operatorname{codim}(Y, X)$ by 1.4.8. As the second part of the assertion is of local nature (noting that the class of [loc. cit.] actually lives in $H^{2c}_Y(X, \tau_{\geq c}(\Omega^1_{X/S})))$ we may assume that X is affine with $\Omega^1_{X/S}$ free of constant rank, and the ideal which defines Y in X is generated by a regular sequence $f_1, \ldots, f_c$. As the Koszul complex $K^{\cdot}(f_1, \ldots, f_c)$ yields a resolution of $\mathcal{O}_Y$ we get

$$\text{cl}_Y([\mathcal{O}_Y]) = \frac{d(f_1) \wedge \cdots \wedge d(f_c)}{f_1 \cdot \ldots \cdot f_k}$$

by 2.3.7 (a). Using [loc. cit., 3.1.3, 3.1.7] this proves the assertion. □

Proposition 2.4.6 *Let* $Z \subseteq Y$ *be closed subsets of* X *and* $c \leq \operatorname{codim}(Y, X)$. *Then the diagram*

$$\begin{array}{ccc} K_0(D_Z(X)_{\text{perf}}) & \xrightarrow{\text{cl}^c_Z} & H^{2c}_Z(X, \tau_{\geq c}(\Omega^{\cdot}_{X/S})) \\ \downarrow & & \downarrow \\ K_0(D_Y(X)_{\text{perf}}) & \xrightarrow{\text{cl}^c_Y} & H^{2c}_Y(X, \tau_{\geq c}(\Omega^{\cdot}_{X/S})) \end{array}$$

is commutative where the vertical maps are the natural morphisms.

Proof. Follows from 1.4.9 and 2.3.5. □

Corollary 2.4.7 *Let Y be a closed subset of X and $c \leq \operatorname{codim}(Y, X)$. Let $\alpha \in K_0(D(X)_{\mathrm{perf}})$ and $\beta \in K_0(D_Y(X)_{\mathrm{perf}})$. Then we have*

$$\mathrm{cl}^c_Y(\alpha \otimes^L_{\mathcal{O}_X} \beta) = \chi(\alpha) \cup \mathrm{cl}^c_Y(\beta).$$

Proof. The problem being local in X we may assume that $X = \operatorname{Spec}(A)$ is affine with $\Omega^1_{X/S}$ free of constant rank, $c = \operatorname{depth}_Y(\mathcal{O}_X)$ and $\alpha = [E^{\cdot}]$, $\beta = [F^{\cdot}]$ where $E^{\cdot}$ and $F^{\cdot}$ are strictly perfect complexes of vector bundles of constant rank. Let J be an ideal of A such that $\operatorname{Spec}(A/J) = Y$. For $c = 0$ the assertion follows from 2.3.3 (b). Let us assume that $c > 0$ in what follows. By [G, 3.6] we find an A-regular sequence $a_1, \ldots, a_c$ in J. Let Y' denote the closed subscheme of X defined by the ideal $(a_1, \ldots, a_c)$, and let $U = X - Y$. Then $c = \operatorname{depth}_{Y'}(\mathcal{O}_X)$ and

$$\mathrm{cl}_{Y'}(\alpha \otimes^L_{\mathcal{O}_X} \beta) = \chi(E^{\cdot}) \cup \mathrm{cl}_{Y'}(\beta)$$

by the formula (1) on page 42 noting that $H^k(X, \Omega^k_{X/S}) = 0$ for all $k \geq 1$. Consider the natural exact sequence

$$\ldots H^{2c-1}_{U \cap Y'}(U, \tau_{\geq c}(\Omega^{\cdot}_{U/S})) \to H^{2c}_Y(X, \tau_{\geq c}(\Omega^{\cdot}_{X/S})) \to H^{2c}_{Y'}(X, \tau_{\geq c}(\Omega^{\cdot}_{X/S})) \ldots$$

From the proof of 1.4.9 it follows that $H^{2c-1}_{U \cap Y'}(U, \tau_{\geq c}(\Omega^{\cdot}_{U/S})) = 0$ because $\operatorname{depth}_{U \cap Y'}(\mathcal{O}_U) \geq c$. Now we may complete the proof by applying 2.4.6 and using the naturality of the cup product. □

Our next objective is to show that the cycle maps we have constructed are compatible with Künneth morphisms. For that purpose consider a cartesian diagram

$$\begin{array}{ccc} Z & \xrightarrow{p} & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & S \end{array}$$

where f and g are smooth morphisms. Suppose that X', Y' are closed subsets of X, Y and c, d are natural numbers satisfying $c \leq \operatorname{codim}(X', X)$ and $d \leq \operatorname{codim}(Y', Y)$. Setting $Z' = p^{-1}(Y') \cap q^{-1}(X')$ suppose in addition that $c + d \leq \operatorname{codim}(Z', Z)$. There is a natural Künneth map

$$\kappa \colon K_0(D_{X'}(X)_{\mathrm{perf}}) \otimes_{\mathbb{Z}} K_0(D_{Y'}(Y)_{\mathrm{perf}}) \to K_0(D_{Z'}(Z)_{\mathrm{perf}})$$

sending $[E^{\cdot}] \otimes [F^{\cdot}]$ to $q^*(E^{\cdot}) \otimes^L_{\mathcal{O}_Z} p^*(F^{\cdot})$. Recall that we have a Künneth map for De Rham cohomology

$$\tilde{\kappa} \colon H^{2c}_{X'}(X, \tau_{\geq c}(\Omega^{\cdot}_{X/S})) \otimes_{\mathbb{Z}} H^{2d}_{Y'}(Y, \tau_{\geq d}(\Omega^{\cdot}_{Y/S})) \to H^{2(c+d)}_{Z'}(Z, \tau_{\geq c+d}(\Omega^{\cdot}_{Z/S}))$$

which commutes with cup products.

Proposition 2.4.8 *With respect to the previous data suppose we are given* $\alpha \in K_0(D_{X'}(X)_{\mathrm{perf}})$ *and* $\beta \in K_0(D_{Y'}(Y)_{\mathrm{perf}})$*. Then*

$$\mathrm{cl}^{c+d}_{Z'}(\kappa(\alpha \otimes \beta)) = \tilde{\kappa}(\mathrm{cl}^c_{X'}(\alpha) \otimes \mathrm{cl}^d_{Y'}(\beta)).$$

Proof. As $\tilde{\kappa}$ commutes with cup products we get

$$\tilde{\kappa}(\mathrm{cl}^c_{X'}(\alpha) \otimes \mathrm{cl}^d_{Y'}(\beta)) = \tilde{\kappa}(\mathrm{cl}^c_{X'}(\alpha) \otimes 1) \cup \tilde{\kappa}(1 \otimes \mathrm{cl}^d_{Y'}(\beta)).$$

Now it is easily checked that the pull back

$$q^*\colon H^{2c}_{X'}(X, \tau_{\geq c}(\Omega^{\cdot}_{X/S})) \to H^{2c}_{q^{-1}(X')}(Z, \tau_{\geq c}(\Omega^{\cdot}_{Z/S}))$$

is nothing but $\tilde{\kappa} \circ (\mathrm{id}_{H^{2c}_{X'}(X, \tau_{\geq c}(\Omega^{\cdot}_{X/S}))} \otimes 1)$ and similarly for p^*. Thus we infer

$$\begin{aligned}\tilde{\kappa}(\mathrm{cl}^c_{X'}(\alpha) \otimes 1) \cup \tilde{\kappa}(1 \otimes \mathrm{cl}^d_{Y'}(\beta)) &= q^*(\mathrm{cl}^c_{X'}(\alpha)) \cup p^*(\mathrm{cl}^d_{Y'}(\beta)) \\ &= \mathrm{cl}^c_{q^{-1}(X')}(q^*(\alpha)) \cup \mathrm{cl}^d_{p^{-1}(Y')}(p^*(\beta)) \\ &= \mathrm{cl}^{c+d}_{Z'}(\kappa(\alpha \otimes \beta))\end{aligned}$$

using 2.4.3 (a) and 2.4.4. □

Next we examine whether our cycle map is homotopy invariant. As for affine bundles the situation is quite complicated we will restrict ourselves to regular schemes, i.e. we assume that X is a regular, separated and noetherian scheme. Let $\mathcal{F}$ be a vector bundle on X, $X' = \mathbb{A}(\mathcal{F})$ the corresponding affine bundle scheme with projection $p\colon X' \to X$. Suppose we are given two sections $s_0, s_1\colon X \hookrightarrow X'$ over X. Let Y be a closed subset of X and $Y' = f^{-1}(Y)$. In what follows we shall consider Thomason's K-theory spectra $K(\,)$ (see [TT, 3]) and likewise the Bass K-Theory spectra $K^B(X \,\mathrm{on}\, Y)$ and $K^B(X' \,\mathrm{on}\, Y')$ which were constructed in [TT, 6].

Lemma 2.4.9 *The map* p *induces a weak homotopy equivalence of* K*-theory spectra* $p^*\colon K^B(X \,\mathrm{on}\, Y) \longrightarrow K^B(X' \,\mathrm{on}\, Y')$.

Proof. For $Z = X$ the map $p^*\colon K(X) \to K(X')$ is a homotopy equivalence by [Q1, 7, 4.1] and [TT, 3.13, 3.21]. Consider the natural commutative diagram of maps of spectra

$$\begin{array}{ccc} K(X) & \xrightarrow{p^*} & K(X') \\ \downarrow & & \downarrow \\ K^B(X) & \xrightarrow{p^*} & K^B(X') \end{array}$$

where the vertical maps are the canonical morphisms (see [TT, 6.5]). As these vertical maps are homotopy equivalences by [TT, 6.8 (b)] the assertion holds for $Z = X$. The general case now follows from the 5-lemma and Thomason's localization theorem ([TT, 7.4]). $\square$

Passing to the first homotopy groups of these spectra we obtain

Corollary 2.4.10 *The map* $Lp^*\colon K_0(D_Y(X)_{\mathrm{perf}}) \to K_0(D_{Y'}(X')_{\mathrm{perf}})$ *is an isomorphism.*

Corollary 2.4.11 *We have* $Ls_0^* = Ls_1^*$ *where* Ls_0^* *and* Ls_1^* *are considered as maps* $K_0(D_{Y'}(X')_{\mathrm{perf}}) \to K_0(D_Y(X)_{\mathrm{perf}})$*.*

Now homotopy invariance is an immediate consequence of 2.4.3 (b) and 2.4.11

Corollary 2.4.12 *For* $c \leq \mathrm{codim}(Y, X)$ *we have* $s_0^* \circ \mathrm{cl}^c_{Y'} = s_1^* \circ \mathrm{cl}^c_{Y'}$*.*

Note that in our situation we always have $\mathrm{codim}(Y, X) = \mathrm{codim}(Y', X')$ which results from the following lemma.

Lemma 2.4.13 *Let* $g\colon W' \to W$ *be a faithfully flat morphism of Cohen-Macaulay schemes. Then* $\mathrm{codim}(V, W) = \mathrm{codim}(f^{-1}(V), W')$ *for every closed subset* V *of* W*.*

Proof. The proof is formally the same as in 1.4.11. $\square$

2.5 Cycle classes and intersection theory

Let X be a noetherian scheme. We shall consider a sequence of spectra induced by the notion of codimension. More precisely given $r \in \mathbb{N}$ let $\mathcal{C}^r(X)$ denote the full subcategory of $C(\mathcal{O}_X)$ consisting of all perfect complexes $E^{\cdot}$ which satisfy $\mathrm{codim}(|E^{\cdot}|, X) \geq r$, where $|E^{\cdot}|$ denotes the support of $E^{\cdot}$.

Lemma 2.5.1 *The category* $\mathcal{C}^r(X)$ *has the following properties:*

(a) *It is closed under finite degree shifts.*

(b) *It is closed under quasi-isomorphisms.*

(c) *Given an exact sequence* $0 \to E^{\cdot} \to F^{\cdot} \to G^{\cdot} \to 0$ *in* $C(\mathcal{O}_X)$ *such that two of the complexes* $E^{\cdot}$, $F^{\cdot}$, $G^{\cdot}$ *are in* $\mathcal{C}^r(X)$ *then the third one is also in* $\mathcal{C}^r(X)$*.*

Proof. Obvious. □

Combining 2.5.1 and [TT, 1.1.2, 1.3.6] we get

Corollary 2.5.2 *$\mathcal{C}^r(X)$ is a complicial biWaldhausen category with respect to the following two structures:*

(1) *The cofibrations are all monomorphisms, the quotient maps all epimorphisms (with respect to $\mathcal{C}(\mathcal{O}_X)$), and the weak equivalences are all quasi-isomorphisms.*

(2) *The cofibrations are all degree-wise split monomorphisms, the quotient maps all degree-wise split epimorphisms, and the weak equivalences are all quasi-isomorphisms.*

In both cases $\mathcal{C}^r(X)$ has a mapping cylinder and cocylinder satisfying the cylinder and cocylinder axioms (cp. [TT, 1.3.1]*).*

For $i = 1, 2$ let $\mathcal{C}^r_i(X)$ denote the category $\mathcal{C}^r(X)$ considered as complicial biWaldhausen with respect to the structure given by 2.5.2 (i). Then using [TT, 1.9.2] the complicial exact inclusion $\mathcal{C}^r_2(X) \to \mathcal{C}^r_1(X)$ induces a homotopy equivalence of K-theory spectra

$$K(\mathcal{C}^r_2(X)) \xrightarrow{\cong} K(\mathcal{C}^r_1(X)).$$

Henceforth we shall suppress the lower indices when we consider $\mathcal{C}^r(X)$ as complicial biWaldhausen. It is clear that the homotopy category $\mathrm{Ho}(\mathcal{C}^r(X))$ is nothing but the strictly full triangulated subcategory of $D(X)_{\mathrm{perf}}$ consisting of those $E^\cdot \in D(X)_{\mathrm{perf}}$ which satisfy $\mathrm{codim}(|E^\cdot|, X) \geq r$.

Recall from [TT, 1.5.6] the description of $K_0(\mathcal{C}^r(X))$. Let $F(\mathcal{C}^r(X))$ be the free abelian group on generators $[E^\cdot]$ where $E^\cdot \in \mathcal{C}^r(X)$, and let $G(\mathcal{C}^r(X))$ be the subgroup given by the following relations:

- $[E^\cdot] = [F^\cdot]$ if there is a quasi-isomorphism $E^\cdot \to F^\cdot$
- $[F^\cdot] = [E^\cdot] + [G^\cdot]$ for all cofibration sequences $E^\cdot \rightarrowtail F^\cdot \twoheadrightarrow G^\cdot$.

Since $\mathcal{C}^r(X)$ has a mapping cylinder satisfying the cylinder axiom every element of $K_0(\mathcal{C}^r(X))$ is given by $[H^\cdot]$ for some $H^\cdot$ in $\mathcal{C}^r(X)$.

Let X be a regular scheme. Sending $E^\cdot \in \mathcal{C}^r(X)$ to $\mathrm{cl}^r_{|E^\cdot|}(E^\cdot)$ yields a map $F(\mathcal{C}^r(X)) \to H^{2r}_{\mathrm{DR}}(X, \tau_{\geq r}(\Omega^\cdot_{X/S}))$, and using 2.4.1, 2.4.6 we see that $G(\mathcal{C}^r(X))$ is contained in its kernel. Thus we get a homomorphism

$$\mathbf{cl}^r_X \colon K_0(\mathcal{C}^r(X)) \to H^{2r}_{\mathrm{DR}}(X, \tau_{\geq r}(\Omega^\cdot_{X/S})).$$

It is clear that the image of the natural map $K_0(\mathcal{C}^{r+1}(X)) \to K_0(\mathcal{C}^r(X))$ which is induced by the complicial exact inclusion $\mathcal{C}^{r+1}(X) \hookrightarrow \mathcal{C}^r(X)$ is contained in the kernel of $\mathbf{cl}^r_X$. We denote this image by $\widetilde{K}_0(\mathcal{C}^r(X))$.

We can interpret $\mathbf{cl}^r_X$ as a direct limit of the cycle maps constructed in 2.4.1 as follows: Let $\mathcal{V}_X(r)$ denote the set of closed subsets Y of X satisfying $\operatorname{codim}(Y, X) \geq r$. Clearly $\mathcal{V}_X(r)$ is a directed set with respect to inclusion. Given $Y \in \mathcal{V}_X(r)$ let $\mathcal{C}_Y$ denote the full additive subcategory of $C(\mathcal{O}_X)$ consisting of all those perfect complexes whose support is contained in Y. Then the assertions 2.5.1 and 2.5.2 also hold for $\mathcal{C}_Y$, and the K-theory spectrum $K(\mathcal{C}_Y)$ is just the spectrum $K(X \,\mathrm{on}\, Y)$ of [TT]. We have canonical complicial exact inclusions $\mathcal{C}_Y \hookrightarrow \mathcal{C}^r(X)$ and $\mathcal{C}_Z \hookrightarrow \mathcal{C}_Y$ for $Z \subseteq Y$ closed. This yields a map of spectra

$$\tau^r \colon \varinjlim_{Y \in \mathcal{V}_X(r)} K(\mathcal{C}_Y) \to K(\mathcal{C}^r(X)).$$

Proposition 2.5.3

(a) τ^r *is a homotopy equivalence.*

(b) *With respect to the identification given by* (a) *we have* $\mathbf{cl}^r_X = \varinjlim_{Y \in \mathcal{V}_X(r)} \mathrm{cl}^r_Y$.

Proof. (b) follows from (a) and the construction of $\mathbf{cl}^r_X$. To prove (a) observe that we have

$$\varinjlim_{Y \in \mathcal{V}_X(r)} \mathcal{C}_Y = \mathcal{C}^r(X).$$

As the construction of $K(\ )$ preserves direct colimits of biWaldhausen categories we are done. $\square$

Our next objective is to show that $\mathbf{cl}^r_X$ is contravariant with respect to flat morphisms. More precisely consider a commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & S' \\ {\scriptstyle f}\big\downarrow & & \big\downarrow \\ X & \longrightarrow & S \end{array}$$

where the horizontal morphisms are smooth, f is flat and X' is a regular scheme. For $Y \in \mathcal{V}(r)$ we have $f^{-1}(Y) \in \mathcal{V}_{X'}(r)$, and moreover $f^*(E^\cdot) \in D(X')_{\mathrm{perf}}$ if $E^\cdot \in D(X)_{\mathrm{perf}}$. Thus the inverse image f^* induces exact functors

$\mathcal{C}_Y \to \mathcal{C}_{f^{-1}(Y)}$ and $\mathcal{C}^r(X) \to \mathcal{C}^r(X')$. Passing to the associated maps of spectra we obtain a commutative diagram

$$\begin{array}{ccc} K(\mathcal{C}_Y) & \xrightarrow{f^*} & K(\mathcal{C}_{f^{-1}(Y)}) \\ \downarrow & & \downarrow \\ K(\mathcal{C}^r(X)) & \xrightarrow{f^*} & K(\mathcal{C}^r(X')) \end{array}.$$

Proposition 2.5.4 *With respect to the previous data the diagram*

$$\begin{array}{ccc} K_0(\mathcal{C}^r(X)) & \xrightarrow{\mathbf{cl}^r_X} & H^{2r}_{\mathrm{DR}}(X,\, \tau_{\geq r}(\Omega^{\cdot}_{X/S})) \\ f^* \downarrow & & \downarrow f^* \\ K_0(\mathcal{C}^r(X')) & \xrightarrow{\mathbf{cl}^r_{X'}} & H^{2r}_{\mathrm{DR}}(X',\, \tau_{\geq r}(\Omega^{\cdot}_{X'/S'})) \end{array}$$

is commutative.

Proof. Let $E^{\cdot} \in \mathcal{C}^r(X)$, $Y = |E^{\cdot}|$ and $Y' = f^{-1}(Y)$. Then $Y' = |f^*(E^{\cdot})|$ by [T, 3.3 (b)] and

$$f^*(\mathrm{cl}^r_Y(E^{\cdot})) = \mathrm{cl}^r_{Y'}(f^*(E^{\cdot}))$$

by 2.4.3. □

For the rest of this section we fix an arbitrary regular base scheme S. All schemes will be of finite type and separated over S, the morphisms between schemes will be S-morphisms. Unless stated otherwise X will denote a smooth S-scheme. Let $\mathcal{M}^r(X)$ denote the Serre subcategory of the category of coherent $\mathcal{O}_X$-modules consisting of those coherent sheaves whose support is of codimension $\geq r$. We may consider $\mathcal{M}^r(X)$ in an evident way as an exact category and therefore as a biWaldhausen category ([TT, 1.9.2]). Using [SGA 6, IV, 2.5] we get an exact inclusion $\mathcal{M}^r(X) \hookrightarrow \mathcal{C}^r(X)$.

Proposition 2.5.5 *The inclusion $\mathcal{M}^r(X) \hookrightarrow \mathcal{C}^r(X)$ induces a homotopy equivalence of K-theory spectra $K(\mathcal{M}^r(X)) \xrightarrow{\cong} K(\mathcal{C}^r(X))$. In particular $K(\mathcal{C}^r(X))$ is naturally homotopy equivalent to Quillen's K-theory spectrum of $\mathcal{M}^r(X)$.*

Proof. The second part is clear by [W, 1.9]. The first part results from 2.5.3, [Q1, 5, (5.1)] and the relative version of [TT, 3.21]. □

Given $r \geq 0$ let $Z^r(X)$ denote the group of codimension r cycles on X. The group of cycles on X is then given by $Z(X) = \bigoplus_{r\in\mathbb{N}} Z^r(X)$. We have a canonical map $Z^r(X) \to K_0(\mathcal{M}^r(X))$ which sends a closed integral r-codimensional subscheme V of X to $[\mathcal{O}_V]$, the latter being the class of the sheaf $\mathcal{O}_V$ in $K_0(\mathcal{M}^r(X))$. Composing this map with the isomorphism $K_0(\mathcal{M}^r(X)) \xrightarrow{\cong} K_0(\mathcal{C}^r(X))$ given by 2.5.5 and $\mathbf{cl}^r_X$ we get a homomorphism

$$\eta^r_X \colon Z^r(X) \to H^{2r}_{\mathrm{DR}}(X, \tau_{\geq r}(\Omega^{\cdot}_{X/S})).$$

Proposition 2.5.6 *Assume that $\alpha \in Z^r(X)$ is rationally equivalent to zero. Then we have $\eta^r_X(\alpha) = 0$.*

Before we can prove this statement we need two auxiliary results.

Lemma 2.5.7 *Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module, $Y = \mathrm{Supp}(\mathcal{F})$. Suppose that $\mathrm{codim}(Y, X) \geq r$ and denote by $Y(r)$ the set of all those $y \in Y$ such that the closed integral subscheme $V(y)$ of X with generic point y is an irreducible component of Y satisfying $\mathrm{codim}(V(y), X) = r$. Then there exists an element $w \in \widetilde{K}_0(\mathcal{C}^r(X))$ such that*

$$[\mathcal{F}] = \sum_{y \in Y(r)} \mathrm{length}_{\mathcal{O}_{X,\,y}}(\mathcal{F}_y) \cdot [\mathcal{O}_{V(y)}] + w.$$

Proof. Using 2.5.5 this is proved as in [SGA 6, X, 1.1.2]. □

Corollary 2.5.8 *Let Y be a purely r-codimensional closed subscheme of X. Then we have $\eta^r_X([Y]) = \mathrm{cl}^r_X(\mathcal{O}_Y)$.*

Proof. As we have remarked before the subgroup $\widetilde{K}_0(\mathcal{C}^r(X))$ of $K_0(\mathcal{C}^r(X))$ is contained in the kernel of $\mathbf{cl}^r_X$. In view of 2.5.7 and [F, 1.5] this implies the assertion. □

Proof of 2.5.6 Let $P \in \{0, \infty\}$, $X_P = X \times_{\mathbb{Z}} P$ and denote by s_P the corresponding section $X \cong X_P \hookrightarrow \mathbb{P}^1_X = X \times_{\mathbb{Z}} \mathbb{P}^1_{\mathbb{Z}}$ over X. Suppose that W is a closed integral r-codimensional subscheme of $\mathbb{P}^1_X$ such that W and X_P intersect properly for $P = 0, \infty$. By 1.4.12 this implies $\mathcal{T}or^n_{\mathbb{P}^1_X}(\mathcal{O}_W, \mathcal{O}_{X_p}) = 0$ for all $n \neq 0$. Letting $W_p = X_p \cap W$ we may by [F, 1.6, 20.1] assume that $\alpha = [W_0] - [W_q\infty]$. Let $q \colon E^{\cdot} \to \mathcal{O}_W$ be a quasi-isomorphism where $E^{\cdot}$ is a bounded above complex of flat $\mathcal{O}_{\mathbb{P}^1_X}$-modules. By hypothesis $q \otimes_{\mathcal{O}_{\mathbb{P}^1_X}} \mathcal{O}_{X_P}$ is then a quasi-isomorphism $E^{\cdot} \otimes_{\mathcal{O}_{\mathbb{P}^1_X}} \mathcal{O}_{X_P} \to \mathcal{O}_{W_P}$ giving us a flat resolution of $\mathcal{O}_{W_P}$ on X. Using 2.4.3 we deduce

$$\begin{aligned} \eta^r_X([W_P]) &= \mathbf{cl}^r_X([E^{\cdot} \otimes_{\mathcal{O}_{\mathbb{P}^1_X}} \mathcal{O}_{X_P}]) \\ &= \mathbf{cl}^r_X(Ls^*_P(E^{\cdot})) \\ &= \mathrm{cl}^r_{W_P}(Ls^*_P(E^{\cdot})) \\ &= s^*_P(\mathrm{cl}^r_W(E^{\cdot})) \end{aligned}$$

Let D_P denote the effective Cartier divisor on $\mathbb{P}^1_X$ induced by s_P. Furthermore, let s_{P*} denote the Gysin map which corresponds to s_P ([B, VI, 3.1.2]). Then by [B, VI, 4.1.5] we get

$$s_{P*}(\eta^r_X([W_P])) = \mathrm{cl}^r_W(E^\cdot) \cup c_1(\mathcal{O}_{\mathbb{P}^1_X}(D_P))$$

where $c_1(\,)$ is the first Chern class. Since D_0 and D_∞ are linearly equivalent divisors we deduce that $s_{0*}(\eta^r_X([W_0])) = s_{\infty *}(\eta^r_X([W_\infty]))$ which proves the assertion because s_{0*}, $s_{\infty *}$ are injective and $s_{0*} = s_{\infty *}$. $\square$

Thus the various η^r_X induce a degree zero morphism of graded groups

$$\eta_X = (\eta^r_X)_{r\in\mathbb{N}} \colon CH^*(X) \to \bigoplus_{r\in\mathbb{N}} H^{2r}_{\mathrm{DR}}(X,\, \tau_{\geq r}(\Omega^\cdot_{X/S}))$$

where $CH^*(X) = \bigoplus_{r\in\mathbb{N}} CH^r(X)$ of course denotes the Chow group of X. Note that $H^0_{\mathrm{DR}}(X,\, \Omega^\cdot_{X/S}) = \Gamma(X,\, \mathcal{O}^d_X)$ where $\mathcal{O}^d_X$ is the sheaf of horizontal sections of the canonical exterior differentiation $d\colon \mathcal{O}_X \to \Omega^1_{X/S}$. Now we obtain the following normalization result.

Proposition 2.5.9 $\eta_X([X]) = 1$.

Proof. Indeed, by 2.5.8 and construction we have

$$\eta_X([X]) = \eta^0_X([X]) = \mathbf{cl}^0_X([\mathcal{O}_X]) = \chi(\mathcal{O}_X) = 1. \quad \square$$

Since we are working over an arbitrary regular base scheme S we have to use a variant of the ordinary dimension of schemes (which we have implicitly used in the proof of 2.5.6) due to Fulton (see [F, 20]). Let $f\colon Y \to S$ be an integral scheme and T the closure of $f(Y)$ in S. According to [loc. cit.] the S-dimension of Y is given by

$$\dim_S Y = \mathrm{trdg}(F(Y)/F(T)) - \mathrm{codim}(T,\, S).$$

Here $F(\,)$ denotes the corresponding function fields and $\mathrm{trdg}(\,)$ the degree of transcendence. If Y is not necessarily an integral scheme, we define its S-dimension to be the least upper bound of the $\dim_S V$ where V runs over all closed integral subschemes of Y. Note that $\dim_S Y = \dim(Y)$ if S is the spectrum of a field. We need the following result which is proved in [F, 20.1].

Lemma 2.5.10 *Let V be an integral scheme and W a closed integral subscheme of Y. Then*

$$\dim_S V = \dim_S W + \mathrm{codim}(W,\, V).$$

Recall that in our context a morphism $f\colon Y \to Z$ is of relative dimension d if the following condition is satisfied:

- If V is a closed integral subscheme of Z and V' an irreducible component of $f^{-1}(V)$, then $\dim_S V' = \dim_S V + d$.

Having defined this notion we can deduce a simple consequence.

Lemma 2.5.11 *Suppose that $f\colon Y \to Z$ is a flat morphism of some relative dimension d, V is a closed integral subscheme of Z whose codimension is k and $f^{-1}(V) \neq \emptyset$. Then $f^{-1}(V)$ is a purely k-codimensional closed subscheme of Y. In particular the induced pull back morphism on the corresponding Chow groups $f^*\colon CH^*(Z) \to CH^*(Y)$ is a graded map of degree zero.*

Proof. Let W be an irreducible component of Z containing V such that $\operatorname{codim}(V, W) = k$. Choose an irreducible component V' of $f^{-1}(V)$ and an irreducible component W' of $f^{-1}(W)$ which contains V'. Note that such a W' is an irreducible component of Y and also $\overline{f(V')} = V$ because f is "generisant". By hypothesis and 2.5.10 we then get

$$\begin{aligned}\operatorname{codim}(V', W') &= \dim_S W' - \dim_S V' \\ &= (\dim_S W + d) - (\dim_S V + d) \\ &= \operatorname{codim}(W, V) \\ &= k\end{aligned}$$

and therefore $\operatorname{codim}(f^{-1}(V), Y) \leq k$. Let now be U' any irreducible component of Y with generic point ξ such that $V' \subseteq U'$. If U denotes the closed integral subscheme of Z with generic point $f(\xi)$ then using again the fact that f is "generisant" we see that U is an irreducible component of Z, and U' is a fortiori an irreducible component of $f^{-1}(U)$. Noting that $V = \overline{f(V')} \subseteq \overline{f(U')} = U$ we get $\operatorname{codim}(V', U') = \operatorname{codim}(V, U)$ in a similar way as before and thus $\operatorname{codim}(f^{-1}(V), Y) \geq k$. $\square$

Using 2.5.4 and 2.5.11 we obtain

Proposition 2.5.12 *Let $f\colon Y \to X$ be a flat morphism of some relative dimension between smooth schemes. Then*

$$\eta_Y \circ f^* = f^* \circ \eta_X.$$

Recall our convention that X denotes a smooth scheme. We want to investigate whether the map η_X preserves intersection products. According to [F, Chap. 8, Chap. 20] an intersection product making $CH^*(X)$ into a commutative graded unitary ring is only defined in the following cases:

- $S = \mathrm{Spec}(k)$ where k is a field.
- S is a one-dimensional regular scheme.

Since both constructions make use of Gysin maps defined for Chow groups and the compatibility of De Rham cohomology with these morphisms is not well understood at all, we will restrict ourselves to the algebraic, quasi-projective case.

Proposition 2.5.13 *Suppose that X is smooth and quasi-projective over $S = \mathrm{Spec}(k)$ where k is a field. Then for $\alpha,\, \beta \in CH^*(X)$ we have*

$$\eta_X(\alpha \cdot \beta) = \eta_X(\alpha) \cup \eta_X(\beta).$$

Proof. By linearity we may assume that $\alpha = [V]$, $\beta = [W]$ for closed subschemes V, W of X of pure codimension p, q. Using the moving lemma ([Ro]) we may assume that V and W intersect properly. Thus

$$\alpha \cdot \beta = \sum_i (-1)^i [\mathcal{T}\mathrm{or}^i_{\mathcal{O}_X}(\mathcal{O}_V,\, \mathcal{O}_W)].$$

Let $E^\cdot \to \mathcal{O}_V$ and $F^\cdot \to \mathcal{O}_W$ be quasi-isomorphisms where $E^\cdot$ and $F^\cdot$ are bounded above complexes of flat $\mathcal{O}_X$-modules. For $r \in \mathbb{N}$ the inverse of the isomorphism $K_0(\mathcal{M}^r(X)) \xrightarrow{\cong} K_0(\mathcal{C}^r(X))$ induced by 2.5.5 is easily seen to be given by sending $G^\cdot \in \mathcal{C}^r(X)$ to

$$\sum_i (-1)^i\, [H^i(G^\cdot)].$$

Using 2.5.7 it follows that

$$\eta_X(\alpha \cdot \beta) = \mathbf{cl}^{p+q}_X(E^\cdot \otimes_{\mathcal{O}_X} F^\cdot).$$

Let $Y = \mathrm{Supp}(E^\cdot \otimes_{\mathcal{O}_X} F^\cdot)$. Then by construction of $\mathbf{cl}^{p+q}_X$, 1.4.13, 2.4.4, 2.4.6 and 2.5.8 we get

$$\begin{aligned} \mathbf{cl}^{p+q}_X(E^\cdot \otimes_{\mathcal{O}_X} F^\cdot) &= \mathrm{cl}^{p+q}_Y(E^\cdot \otimes_{\mathcal{O}_X} F^\cdot) \\ &= \mathrm{cl}^{p+q}_{V\cap W}(E^\cdot \otimes_{\mathcal{O}_X} F^\cdot) \\ &= \mathrm{cl}^p_V(E^\cdot) \cup \mathrm{cl}^q_W(F^\cdot) \\ &= \mathbf{cl}^p_X(E^\cdot) \cup \mathbf{cl}^q_X(F^\cdot) \\ &= \eta_X(\alpha) \cup \eta_X(\beta) \end{aligned}$$

which proves the assertion. □

Suppose that $S = \mathrm{Spec}(k)$ where k is a field of characteristic zero and Y is a closed integral subscheme of X. Then Hartshorne constructs in [H2] a cycle class for Y in $H^{2d}_{\mathrm{DR}}(X/S)$ where $d = \mathrm{codim}(Y, X)$. This class coincides with our $\eta_X([Y])$ since by the dimension theorem of [loc. cit.] we may assume that Y is smooth and in that case we may apply 2.4.5.

To end this section we want to construct Gysin maps for the spectra we have introduced at the beginning. In what follows X denotes an arbitrary scheme, not necessarily smooth over S. First we need another result whose proof may be found in [F, 20.1].

Lemma 2.5.14 *Let $f\colon X \to Y$ be a dominant morphism of integral schemes. Then we have* $\dim_S X = \dim_S Y + \mathrm{trdg}(F(X)/F(Y))$.

We use this statement to show

Lemma 2.5.15 *Let $f\colon X \to Y$ be a closed morphism of relative dimension d and V a closed subset of X. Then we have*

$$\mathrm{codim}(f(V), Y) \geq \mathrm{codim}(V, X) - d.$$

Proof. We may assume that V is a closed integral subscheme of X. Let W be an irreducible component of X with $V \subseteq W$. By [EGA IV, (14.2.2.3)] we have

$$\mathrm{codim}(f(V), Y) \geq \mathrm{codim}(f(V), f(W)) + \mathrm{codim}(f(W), Y)$$

where $f(W)$ is closed and irreducible because f is closed. Using this equation and 2.5.10, 2.5.14 we deduce

$$\begin{aligned} \mathrm{codim}(V, W) &= \dim_S W - \dim_S V \\ &\leq \dim_S W - \dim_S f(V) \\ &= \dim_S f(W) + d - \dim_S f(V) \\ &= \mathrm{codim}(f(V), f(W)) + d \\ &\leq \mathrm{codim}(f(V), Y) + d. \quad \square \end{aligned}$$

Proposition 2.5.16 *Suppose that $f\colon X \to Y$ is a proper and perfect morphism. Then for a closed subset Z of X the functor Rf_* maps $D_Z(X)$ into $D_{f(Z)}(Y)$.*

Proof. By [TT, 2.5.4] the functor Rf_* maps $D(X)_{\mathrm{perf}}$ into $D(Y)_{\mathrm{perf}}$. To complete the proof it suffices to show that for a perfect complex $E^{\cdot}$ on X

the inclusion $|Rf_*(E^\cdot)| \subseteq f_*(|E^\cdot|)$ holds. Assume that this is false and let $U = X - |E^\cdot|$ which is an open subset of X. Consider the fibre square

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{f'} & U \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

By assumption we have $Rf_*(E^\cdot)|U \neq 0$. But $E^\cdot|f^{-1}(U) = 0$ because of $f^{-1}(U) \subseteq X - |E^\cdot|$, and also $Rf_*(E^\cdot)|U = Rf'_*(E^\cdot|f^{-1}(U))$ by base change which yields a contradiction. $\square$

Let $f\colon X \to Y$ be a proper and perfect morphism between (not necessarily smooth) S-schemes. In the light of 2.5.15 and 2.5.16 we are going to show that the functor Rf_* induces a morphism of spectra $K(\mathcal{C}^r(X)) \to K(\mathcal{C}^{r-d}(X))$ for $r \in \mathbb{N}$ where we let $\mathcal{C}^{r-d}(X) = 0$ if $r - d < 0$. Here by a morphism we mean a morphism in the stable homotopy category of CW-spectra (see [A] or [Ma]) because we have to replace the codimension spectra by homotopy equivalent spectra. This is what we intend to do in the following. Given any scheme X let $\mathcal{C}^r_{\text{flab}}(X)$ denote the full additive subcategory of $\mathcal{C}^r(X)$ consisting of those $E^\cdot \in \mathcal{C}^r(X)$ such that $E^\cdot$ is bounded below and each E^n is a flabby $\mathcal{O}_X$-module. In particular each object of $\mathcal{C}^r_{\text{flab}}(X)$ is a K-flabby complex in the sense of [Sp] and is deployed for computing Rf_*. It is easily checked that this category is a complicial biWaldhausen category with respect to the default structure of [TT], i.e. the cofibrations are the degree-wise split monomorphisms whose cokernel is also an object of $\mathcal{C}^r_{\text{flab}}(X)$ and the weak equivalences are the quasi-isomorphisms. Clearly the inclusion $\mathcal{C}^r_{\text{flab}}(X) \hookrightarrow \mathcal{C}^r(X)$ is complicial exact. As $\mathcal{C}^r_{\text{flab}}(X)$ is closed under finite degree shifts and in addition under extensions in $C(\mathcal{O}_X)$ it results from [TT, 1.3.6] that this category has a mapping cylinder and cocylinder satisfying the cylinder and cocylinder axiom.

Lemma 2.5.17 *The canonical inclusion $\mathcal{C}^r_{\text{flab}}(X) \hookrightarrow \mathcal{C}^r(X)$ induces a homotopy equivalence of K-theory spectra $K(\mathcal{C}^r_{\text{flab}}(X)) \xrightarrow{\cong} K(\mathcal{C}^r(X))$.*

Proof. This follows from [TT, 1.9.8] because the induced functor between the corresponding homotopy categories $\text{Ho}(\mathcal{C}^r_{\text{flab}}(X)) \to \text{Ho}(\mathcal{C}^r(X))$ is easily seen to be an equivalence. $\square$

Proposition 2.5.18 *Suppose that $f\colon X \to Y$ is a proper and perfect morphism of relative dimension d. Then f_* maps $\mathcal{C}^r_{\text{flab}}(X)$ into $\mathcal{C}^{r-d}_{\text{flab}}(Y)$. Moreover, the resulting functor $f_!\colon \mathcal{C}^r_{\text{flab}}(X) \to \mathcal{C}^{r-d}_{\text{flab}}(Y)$ is complicial exact.*

Proof. For $E^{\cdot} \in \mathcal{C}^r_{\text{flab}}(X)$ we have $f_*(E^{\cdot}) \in \mathcal{C}^{r-d}_{\text{flab}}(Y)$ by 2.5.15, 2.5.16 and [Sp, 5.15 (b), 6.7 (a)]. To establish the second part of the assertion we only have to show that f_* preserves quasi-isomorphisms in $\mathcal{C}^r_{\text{flab}}(X)$ which holds e.g. by [Sp, 5.16]. $\square$

Passing to the stable homotopy category we thus obtain Gysin maps of spectra

$$f_!\colon K(\mathcal{C}^r(X)) \to K(\mathcal{C}^{r-d}(Y))$$

which are clearly functorial. Given a proper and perfect morphism $f\colon X \to Y$ of some relative dimension between arbitrary schemes then f_* induces a complicial exact functor $\mathcal{C}^0_{\text{flab}}(X) \to \mathcal{C}^0_{\text{flab}}(Y)$ which in turn gives us a morphism of spectra $f_*\colon K(X) \to K(Y)$. It is clear that the diagram

$$\begin{array}{ccc} K(\mathcal{C}^r(X)) & \xrightarrow{f_!} & K(\mathcal{C}^{r-d}(Y)) \\ \downarrow & & \downarrow \\ K(X) & \xrightarrow{f_*} & K(Y) \end{array}$$

is commutative where the vertical maps are induced by the corresponding complicial exact inclusions.
Given a regular scheme X we denote by $\mathrm{Fil}^r(X)$ the r-th component of the topological filtration on $K_0(X)$. Recall that $\mathrm{Fil}^r(X)$ is given by the image of the natural map $K_0(\mathcal{C}^r(X)) \to K_0(X)$. Let

$$\mathrm{gr}_{\mathrm{Top}}(X) = \bigoplus_r (\mathrm{Fil}^r(X)/\mathrm{Fil}^{r+1}(X))$$

denote the associated graded group. Using the above commutative diagram we see that for a proper and perfect morphism $f\colon X \to Y$ of some relative dimension there results a Gysin map

$$f_!\colon \mathrm{gr}_{\mathrm{Top}}(X) \to \mathrm{gr}_{\mathrm{Top}}(Y)$$

which is of degree $-d$. Recall that X satisfies Gersten's condition if the inclusions

$$\mathcal{C}^{r+1}(\mathrm{Spec}(\mathcal{O}_{X,x})) \hookrightarrow \mathcal{C}^r(\mathrm{Spec}(\mathcal{O}_{X,x}))$$

induce zero on K-groups for every $r \in \mathbb{N}$ and all $x \in X$. Quillen has proved that X satisfies Gersten's condition if it is (regular and) of finite type over a field ([Q1, 5.11]).
If X is regular and V is a r-codimensional closed integral subscheme of X then $\mathcal{O}_V \in \mathrm{Fil}^r(X)$ and thus

$$V \mapsto \text{class of } \mathcal{O}_V \bmod \mathrm{Fil}^{r+1}(X)$$

defines a degree zero map of graded groups $\eta_X : Z(X) \to \mathrm{gr}_{\mathrm{top}}(X)$. Moreover, if X satisfies Gersten's condition it was shown in [Gi] that

(a) the topological filtration is compatible with the ring stucture on $K_0(X)$ so that $\mathrm{gr}_{\mathrm{top}}(X)$ is a graded commutative unitary ring.

(b) the map η_X factors through rational equivalence, the induced map $\eta_X : CH^*(X) \to \mathrm{gr}_{\mathrm{top}}(X)$ is surjective and becomes an isomorphism after tensoring with $\mathbb{Q}$.

Recall that for a flat morphism $f: X \to Y$ we have constructed pull-backs $f^*: K(\mathcal{C}^r(Y)) \to K(\mathcal{C}^r(X))$ making the diagram

$$\begin{array}{ccc} K(\mathcal{C}^r(Y)) & \xrightarrow{f^*} & K(\mathcal{C}^r(X)) \\ \downarrow & & \downarrow \\ K(Y) & \xrightarrow{f^*} & K(X) \end{array}$$

commutative. Hence we get a degree zero map $f^*: \mathrm{gr}_{\mathrm{Top}}(Y) \to \mathrm{gr}_{\mathrm{Top}}(X)$ which is a ringhomomorphism if X and Y satisfy Gersten's condition.

Proposition 2.5.19 *(Projection Formula) Let $f: X \to Y$ be a morphism of relative dimension which is proper, perfect and flat. Suppose in addition that X and Y satisfy Gersten's condition. Then for elements $x \in \mathrm{gr}_{\mathrm{Top}}(X)$ and $y \in \mathrm{gr}_{\mathrm{Top}}(Y)$ we have*

$$f_!(x \cdot f^*(y)) = f_!(x) \cdot y.$$

Proof. In view of the previous discussion this results from an application of [SGA 6, III, 3.7]. □

3 Crystalline cycle classes

3.1 Some descent properties for nilpotent liftings

Lemma 3.1.1 *Let X be a scheme, Y a closed subscheme of X which is defined by a nilpotent sheaf of ideals. Then X is affine $\Leftrightarrow$ Y is affine.*

Proof. The direction "$\Rightarrow$" results from [EGA I, 9.1.16]. For "$\Leftarrow$" let $\mathcal{J}$ denote the ideal sheaf which defines Y in X. Then for $k \geq 0$ the $\mathcal{O}_X$-modules $\mathcal{J}^k/\mathcal{J}^{k+1}$ are quasi-coherent and thus they are quasi-coherent as $\mathcal{O}_Y$-modules ([EGA I, 2.2.4]). Hence we conclude by [EGA I, 2.3.5] that X is affine. □

For the next three results suppose we are given a commutative diagram of scheme morphisms

$$\begin{array}{ccc} Y_0 & \xrightarrow{F_0} & X_0 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ Y & \xrightarrow{F} & X. \end{array} \qquad (*)$$

where i and j are locally nilpotent closed immersions.

Lemma 3.1.2

(a) *F is affine $\Leftrightarrow$ F_0 is affine.*

(b) *F is separated $\Leftrightarrow$ F_0 is separated.*

(c) *F is quasi-separated $\Leftrightarrow$ F_0 is quasi-separated.*

(d) *F is quasi-compact $\Leftrightarrow$ F_0 is quasi-compact.*

Proof. (a) "$\Rightarrow$" holds by [EGA I, 9.1.16 (v)], whereas "$\Leftarrow$" follows easily from 3.1.1.
(b) "$\Rightarrow$" holds by [EGA I, 5.3.1 (v)]. Using the commutative diagram

$$\begin{array}{ccc} X_{\text{red}} & \xrightarrow{F_{\text{red}}} & X_{\text{red}} \\ \downarrow & & \downarrow \\ X_0 & \xrightarrow{F_0} & X_0 \end{array}$$

where the vertical maps are the canonical closed immersions, we see that the morphism F_{red} is separated ([EGA I, 5.3.1 (v)]) which in turn implies that F is separated ([loc. cit., 5.3.1 (vi)]).

(c) Follows from [EGA I, 6.1.9 (v), (vi)] because i is quasi-compact and surjective.
(d) is clear since i and j are homeomorphisms. □

Proposition 3.1.3

(a) *Assume that the ideal sheaf of Y_0 in Y is of finite type. Then F is (locally) of finite type $\Leftrightarrow$ F_0 is (locally) of finite type.*

(b) *Assume that the ideal sheaf of X_0 in X is of finite type and $(*)$ is cartesian. Then F is finite $\Leftrightarrow$ F_0 is finite.*

Proof. (a) We only have to show that F is locally of finite type if F_0 has this property. Let $y \in Y$, $x = F(y)$. Using 3.1.1 and [EGA I, 6.2.1.1] there exist affine open subsets U_0, V_0 of X_0, Y_0 such that $x \in U_0$, $y \in V_0$, $F_0(V_0) \subseteq U_0$ and the following conditions are satisfied:

(1) Let $A(U_0) = \Gamma(U_0, \mathcal{O}_{X_0})$ and $A(V_0) = \Gamma(V_0, \mathcal{O}_{Y_0})$. Then $A(V_0)$ is an $A(U_0)$-algebra of finite type.

(2) If U, V denote the open subsets of X, Y corresponding to U_0, V_0 then U and V are affine.

(3) The ideal J of $A(V)$ which defines $A(V_0)$ is nilpotent and finitely generated as an $A(V)$-module.

By assumption there exist $n \in \mathbb{N}$ and a surjective ringhomomorphism

$$\psi_0\colon A(U_0)[T_1, \ldots, T_n] \to A(V_0)$$

with indeterminates $T_1, \ldots, T_n$ such that $\psi_0|A(U_0)$ is the natural morphism $A(U_0) \to A(V_0)$. Hence there exists a ringhomomorphism

$$\psi\colon A(U)[T_1, \ldots, T_n] \to A(V)$$

such that $\psi|A(U)$ is the map $A(U) \to A(V)$ and the diagram

$$\begin{array}{ccc} A(U)[T_1, \ldots, T_n] & \xrightarrow{\psi} & A(V) \\ \downarrow & & \downarrow \\ A(U_0)[T_1, \ldots, T_n] & \xrightarrow{\psi_0} & A(V_0) = A(V)/J \cdot A(V) \end{array}$$

commutes, where the vertical morphisms are the canonical surjections. Let $\{s_1, \ldots, s_k\}$ be a set of generators for J. Then there exists a unique ringhomomorphism

$$\varphi\colon A(U)[T_1, \ldots, T_n, T_{n+1}, \ldots, T_{n+k}] \to A(V)$$

such that $\varphi|A(U)[T_1, \ldots, T_n] = \psi$ and $\varphi(T_{n+p}) = s_p$ for $1 \leq p \leq k$. Using (3) and the previous commutative diagram it is easily checked that φ is surjective. Hence F is locally of finite type.
(b) "$\Rightarrow$" holds by [EGA II, 6.1.5]. "$\Leftarrow$" By 3.1.2 (a) F is affine. Hence we are reduced to considering the case where $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, $X_0 = \mathrm{Spec}(A/I)$, $Y_0 = \mathrm{Spec}(B/J)$ for rings A, B and nilpotent, finitely generated ideals I, J of A, B. Let $\varphi\colon A \to B$ be the ringhomorphism which corresponds to F. Then by hypothesis we have $\varphi(I) \cdot B = J$ and therefore $\varphi(I^n) \cdot B = J^n$ for all $n \geq 1$. It suffices to prove that J^n is a finite A-module for all $n \geq 1$ which is clear in case of $n >> 0$ because J is nilpotent. Suppose that J^{n+1} is a finite A-module and let us prove that then J^n is finitely generated. Choose a finite set $\{e_1, \ldots, e_k\}$ generating the A-module I^n and a finite set of elements $\{f_1, \ldots, f_l\}$ of B whose residue classes generate B/J as an A-module. If we then pick generators $g_1, \ldots, g_m$ of the A-module J^{n+1} it is easily seen that the collection

$$\{\varphi(e_i) \cdot f_j \,;\, 1 \leq i \leq k,\, 1 \leq j \leq l\} \cup \{g_1, \ldots, g_m\}$$

generates J^n as an A-module. □

Combining 3.1.3 and [EGA I, 6.1.13 and 6.2.1.2] we obtain

Corollary 3.1.4 *With respect to the hypotheses of* 3.1.4 (a) *assume in addition that X is locally noetherian. Then F is (locally) of finite presentation $\Leftrightarrow$ F_0 is (locally) of finite presentation.*

3.2 The De Rham complex of a smooth formal scheme

All formal schemes we shall consider are supposed to be locally noetherian. Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be an adic morphism. We fix an ideal of definition $\mathcal{J}$ of $\mathfrak{Y}$ and set $Y_n = (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathcal{J}^{n+1})$, $X_n = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/f^*(\mathcal{J})^{n+1} \cdot \mathcal{O}_{\mathfrak{X}})$ for $n \geq 0$. Then X_n and Y_n are locally noetherian schemes. We shall say that f is (formally) smooth if the induced morphism $f_n\colon X_n \to Y_n$ is (formally) smooth for every n. Then we have the following characterization of formally smooth morphisms between formal schemes which shows that formal smoothness does not depend on the choice of $\mathcal{J}$.

Lemma 3.2.1 *For an adic morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ the following assertions are equivalent:*

(a) *f is formally smooth.*

(b) *For every $x \in \mathfrak{X}$ there exist open, formally affine noetherian neighbourhoods $U = \mathrm{Spf}(A)$, $V = \mathrm{Spf}(B)$ of x, $f(x)$ such that $f(U) \subseteq V$ and making A via the induced map $U \to V$ to a formally smooth B-algebra (in the sense of* [EGA 0_{IV}, 19.3.1]).

Proof. This follows from the following references: [EGA I, 10.12.3] and [EGA 0_{IV}, 19.4.2], [EGA IV, 17.1.2 (i), 17.1.6]. □

As a criterion for smoothness we state

Proposition 3.2.2 *Suppose that $f\colon \mathfrak{X} \to \mathfrak{Y}$ is an adic morphism such that each $f_n\colon X_n \to Y_n$ is flat. Then f is smooth $\Leftrightarrow$ f_0 is smooth.*

Proof. Taking into account that each f_n is locally of finite presentation by 3.1.4 the condition is sufficient by [EGA IV, 17.8.2]. □

Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a smooth morphism of formal schemes. Given $n \geq 0$ and $p \in \mathbb{N}$ we shall write $\Omega^p_{X_n}$ instead of $\Omega^p_{X_n/Y_n}$ in what follows. For $m \leq n$ consider the cartesian square

$$\begin{array}{ccc} X_m & \xrightarrow{f_m} & Y_m \\ {\scriptstyle u_{mn}}\downarrow & & \downarrow \\ X_n & \xrightarrow{f_n} & Y_n \end{array}$$

where the vertical maps are the natural transition morphisms. We obviously have $u^*_{mn}(\Omega^p_{X_n}) = \Omega^p_{X_m}$ and thus by [EGA I, 10.11.3] the sequence $(\Omega^p_{X_n})_{n\geq 0}$ gives rise to a coherent $\mathcal{O}_{\mathfrak{X}}$-module $\Omega^p_{\mathfrak{X}/\mathfrak{Y}}$ which is locally free of finite rank ([loc. cit., 10.11.10]). More precisely if $u_n\colon X_n \to X$ denotes the structure morphism then

$$\Omega^p_{\mathfrak{X}/\mathfrak{Y}} = \varprojlim_{n\geq 0} u_{n*}(\Omega^p_{X_n})$$

where the projective limit is taken in the category of sheaves on X with values in the category of topological spaces. Note that for an open and quasi-compact subset U of $\mathfrak{X}$ the space $u_{n*}(\Omega^p_{X_n})(U)$ carries the discrete topology ([EGA 0_{I}, 3.9.1]). For $m \leq n$ we have a commutative diagram

$$\begin{array}{ccc} \Omega^p_{X_n} & \longrightarrow & u_{mn*}(\Omega^p_{X_m}) \\ {\scriptstyle d^p_{\Omega^\cdot_{X_n}}}\downarrow & & \downarrow{\scriptstyle u_{mn*}(d^p_{\Omega^\cdot_{X_m}})} \\ \Omega^{p+1}_{X_n} & \longrightarrow & u_{mn*}(\Omega^{p+1}_{X_m}) \end{array}$$

where the horizontal maps are given by adjunction. Thus we deduce that the collection $(u_{n*}(d^p_{\Omega^{\cdot}_{X_n}}))_n$ yields a morphism between projective systems

$$(u_{n*}(\Omega^p_{X_n}))_n \longrightarrow (u_{n*}(\Omega^{p+1}_{X_n}))_n$$

of sheaves on $\mathfrak{X}$. Taking the projective limit we get a (continuous) $f^{-1}\mathcal{O}_{\mathfrak{Y}}$-linear homomorphism $d^p \colon \Omega^p_{\mathfrak{X}/\mathfrak{Y}} \to \Omega^{p+1}_{\mathfrak{X}/\mathfrak{Y}}$ which clearly satisfies $d^{p+1} \circ d^p = 0$. The resulting complex $(\Omega^{\cdot}_{\mathfrak{X}/\mathfrak{Y}}, d^{\cdot})$ defines the De Rham complex of $\mathfrak{X}/\mathfrak{Y}$.

3.3 Formal liftings

Let S be a locally noetherian scheme, $\mathcal{J} \subseteq \mathcal{O}_S$ a sheaf of ideals. Given $n \geq 0$ we set $S_n = (S, \mathcal{O}_S/\mathcal{J}^{n+1})$ and denote by $\widehat{S}$ the completion of S along $\mathcal{J}$ which is a locally noetherian formal scheme ([EGA I, 10.8.4]). The completion $\widehat{\mathcal{J}}$ of $\mathcal{J}$ gives us a natural ideal of definition. Moreover, the powers $\widehat{\mathcal{J}}$, $\widehat{\mathcal{J}}^{\,2}$, ... form a fundamental system of ideals of definition. We will assume throughout that all formal schemes over the completion $\widehat{S}$ are adic.
Given an S_0-scheme X_0 by a (formal) lifting of X over $\widehat{S}$ we shall mean a formal scheme $g \colon \mathfrak{X} \to \widehat{S}$ such that X_0 and $\mathfrak{X}_0 = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/g^*(\widehat{\mathcal{J}}) \cdot \mathcal{O}_{\mathfrak{X}})$ are isomorphic as S_0-schemes. In such a situation we will always identify X_0 and $\mathfrak{X}_0$. Similarly a (formal) lifting of a morphism $f \colon X_0 \to Y_0$ of S_0-schemes shall be given by an $\widehat{S}$-map $F \colon \mathfrak{X} \to \mathfrak{Y}$ such that $F_0 = f$ (where F_0 is taken with respect to $\widehat{\mathcal{J}}$). In particular $\mathfrak{X}/\widehat{S}$ and $\mathfrak{Y}/\widehat{S}$ are then formal liftings of X_0 and Y_0 respectively. Our first result deals with the existence of such liftings.

Proposition 3.3.1 *Suppose that X_0/S_0 is smooth and $x_0 \in X_0$. Then there exists an open neighbourhood U_0 of x_0 in X_0 and a smooth formal lifting $\mathfrak{U}/\widehat{S}$ of U_0/S_0.*

Proof. By [EGA IV, 18.1.1] there exists a smooth morphism $f \colon U \to S$ and an S_0-isomorphism $U \times_S S_0 \cong U_0$. Let $\mathfrak{X}$ be the completion of U along U_0. Then $\mathfrak{X}$ is a locally noetherian formal scheme by [EGA I, 10.8.4]. Furthermore, the structure morphism $\widehat{f} \colon \mathfrak{X} \to \widehat{S}$ is adic ([loc. cit., 10.9.5]) and evidently smooth. □

Proposition 3.3.2 *Let X_0/S_0 be smooth, and let $\mathfrak{X}/\widehat{S}$, $\mathfrak{X}'/\widehat{S}$ be two smooth liftings of X_0/S_0. Then $\mathfrak{X}$ and $\mathfrak{X}'$ are locally isomorphic over $\widehat{S}$.*

Proof. In fact by [B, VI, 3.3.4] a slightly more general result holds. □

Proposition 3.3.3 *Let $f\colon X_0 \to Y_0$ be a morphism of smooth S_0-schemes and $x \in X_0$, $y = f(x_0)$. Then there exist open neighbourhoods U_0 and V_0 of the points x_0 and y_0 respectively such that the following holds:*

(a) $f(U_0) \subseteq V_0$

(b) $f|U_0\colon U_0 \to V_0$ *has a (smooth) formal lifting.*

Moreover, if $X_0 = Y_0$ then the (smooth) formal lifting may be chosen in such a way that its source and target coincide.

Proof. By the smoothness assumption we find affine open neighbourhoods U_0, V_0 of x_0, y_0 such that $f(U_0) \subseteq V_0$ and

$$U_0 \hookrightarrow X_0 \to S_0 = U_0 \overset{g_0}{\to} S_0[T_1, \ldots, T_{k_0}] \overset{p_0}{\to} S_0$$
$$V_0 \hookrightarrow Y_0 \to S_0 = V_0 \overset{h_0}{\to} S_0[T_1, \ldots, T_{l_0}] \overset{q_0}{\to} S_0$$

where g_0, h_0 are étale, the T_i are indeterminates and p_0, q_0 denote the canonical projections. Using [EGA IV, 18.1.2] we deduce that there exist smooth S_1-schemes U_1, V_1 such that $U_0 = U_1 \times_{S_1} S_0$ and $V_0 = V_1 \times_{S_1} S_0$. Let $f_0 = f|U_0\colon U_0 \to V_0$. Because U_1 is affine (3.1.1) and V_1 is smooth over S_1 there exists an S_1-morphism $f_1\colon U_1 \to V_1$ such that the diagram

$$\begin{array}{ccc} V_0 & \longrightarrow & V_1 \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f_1} \\ U_0 & \longrightarrow & U_1 \end{array}$$

commutes where the horizontal maps are the injections. By [EGA 0_{I}, 1.2.9] we even know that this diagram is cartesian. Iterating this process we find adic, inductive (S_n)-systems (U_n), (V_n) (in the sense of [EGA I, 10.12.2]) and a morphism $(f_n)\colon (U_n) \to (V_n)$ between them. Thus by [loc. cit., 10.12.3] forming $\varinjlim_n f_n$ we get a morphism $\varinjlim_n U_n \to \varinjlim_n V_n$ of $\widehat{S}$-adic (smooth) formal schemes which lifts f_0. $\square$

Thus we see that any morphism of smooth S_0-schemes has a formal lifting at least locally.
Let us now consider (relative) formal liftings of $\mathcal{O}_{X_0}$-modules and their homomorphisms. Given a coherent $\mathcal{O}_{X_0}$-module $\mathcal{F}_0$ a formal lifting of $\mathcal{F}$ (relative to $\widehat{S}$) shall consist of the following data:

(a) A formal scheme $\mathfrak{X}$ which lifts X_0 over $\widehat{S}$.

(b) A coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{F}$ such that $u_0^*(\mathcal{F}) \cong \mathcal{F}$ where $u_0 \colon X_0 \to \mathfrak{X}$ denotes the canonical morphism.

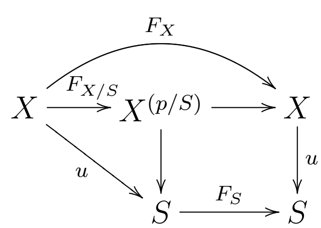
It should be clear what we shall mean by a formal lifting (relative to $\widehat{S}$) of a morphism of coherent $\mathcal{O}_{X_0}$-modules.

Proposition 3.3.4 *Let $\mathfrak{X}$ be a noetherian formal affine scheme over $\widehat{S}$. Let $\mathcal{F}$ and $\mathcal{G}$ be two vector bundles on $\mathfrak{X}$. Then any morphism of vector bundles $\mathcal{F}_0 \to \mathcal{G}_0$ on X_0 can be lifted to a morphism $\mathcal{F} \to \mathcal{G}$.*

Proof. Let R be a commutative ring, I a nilpotent ideal of R and denote by $\mathcal{P}(R)$ (resp. $\mathcal{P}(R/I)$) the category of finitely generated projective R-modules (resp. R/I-modules). Then it is well known that the functor $\mathcal{P}(R) \to \mathcal{P}(R/I)$ induced by the tensor product $* \otimes_R R/I$ is essentially surjective and full (e.g. by [SGA 5, XV, §3, no.3, Lemme 3]). Using this result, [EGA I, 10.11.3] and the construction we have used in the proof of 3.3.3 the assertion follows easily. $\square$

3.4 The relative Frobenius morphism

Given a scheme S of characteristic p (i.e. an $\mathbb{F}_p$-scheme) we denote by F_S its Frobenius endomorphism. For a morphism $u \colon X \to S$ of schemes we get the familiar commutative diagram



where the square is cartesian and $F_{X/S}$ denotes the relative Frobenius morphism which is an integral morphism and in addition an universal homeomorphism ([SGA 5, XV, §1, no. 2, Prop. 2]). In the following we will use the same notation $F_{X/S}$ for the ringhomomorphism $\mathcal{O}_{X^{(p/S)}} \to F_{X/S*}(\mathcal{O}_X)$ induced by the relative Frobenius morphism.

Lemma 3.4.1 *Consider the De Rham complex $(\Omega^{\cdot}_{X/S}, d)$ of X/S. Then the map $F_{X/S*}(d^n)$ is $\mathcal{O}_{X^{(p/S)}}$-linear for $n \geq 0$.*

Proof. We may assume that $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and u is induced by a homomorphism $\varphi \colon A \to B$ of rings. Then $F_{X/S}$ is given by the map

$B \otimes_A A \to B$ which sends $b \otimes a$ to $\varphi(a) \cdot b^p$. For a section x of $\Omega^n_{X/S}$ we then deduce

$$\begin{aligned} d^n(\varphi(a)b^p \cdot x) &= d^0(\varphi(a)b^p) \cdot x + \varphi(a)b^p \cdot d^n(x) \\ &= p\varphi(a)b^{p-1}d^0(b) \cdot x + \varphi(a)b^p \cdot d^n(x) \\ &= \varphi(a)b^p \cdot d^n(x). \quad \square \end{aligned}$$

Lemma 3.4.2 *Suppose that $u\colon X \to S$ is smooth of relative dimension r. Then $F_{X/S}$ is a finite morphism and $F_{X/S*}(\mathcal{O}_X)$ is a locally free $\mathcal{O}_{X^{(p/s)}}$-module of rank p^r.*

Proof. Using [SGA 5, XV, §1, no. 2, Prop. 1, Prop. 2 c)] we may assume that $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(A[T_1, \ldots, T_r])$ and u is induced by the canonical homomorphism $A \to A[T_1, \ldots, T_r]$ with indeterminates $T_1, \ldots, T_r$. Then by [loc. cit.] we have $X^{(p/S)} = X$ and $F_{X/S}$ is given by $T_i \mapsto T_i^p$ for $1 \le i \le r$. Now it is easy to see that the set

$$M = \{T_1^{i_1} \cdot \ldots \cdot T_r^{i_r};\ 0 \le i_j \le p-1,\ 1 \le j \le r\}$$

yields a basis for $A[T_1, \ldots, T_r]$ considered as an $A[T_1, \ldots, T_r]$-module via $F_{X/S}$. Since $\mathrm{card}(M) = p^r$ this finishes the proof. $\square$

In the situation of 4.2 it follows in addition that $F_{X/S}$ is flat (because it is an affine morphism and $F_{X/S*}(\mathcal{O}_X)$ is a flat $\mathcal{O}_{X^{(p/s)}}$-module).
Another tool for our further investigations will be the Cartier isomorphism which we are going to introduce now.

Theorem 3.4.3 (Cartier) Let $u\colon X \to S$ be a morphism of schemes where S is of characteristic p. Then there exists an unique morphism of graded $\mathcal{O}_{X^{(p/S)}}$-algebras

$$C^{-1}\colon \bigoplus_{i \in \mathbb{N}} \Omega^i_{X^{(p/S)}/S} \longrightarrow \bigoplus_{i \in \mathbb{N}} H^i(F_{X/S*}(\Omega^{\cdot}_{X/S}))$$

having the following properties:

(a) $C^{-1} = F_{X/S}$ in degree 0.

(b) $C^{-1}(d(x \otimes 1)) =$ the class of $x^{p-1}dx$ in $H^1(\Omega^{\cdot}_{X/S})$ for every local section x of $\mathcal{O}_X$.

Moreover, if u is smooth, then C^{-1} is an isomorphism.

Proof. [K, 7.2]. □

Suppose that X/S is smooth. Then we can give a local description of the inverse C of the Cartier isomorphism for closed 1-forms: Let U be an open subset of X and consider sections $x_1, \ldots, x_r$ of $\mathcal{O}_X$ over U which induce an étale S-morphism $U \to S[T_1, \ldots, T_r]$, i.e. such that $dx_1, \ldots, dx_r$ form a basis of $\Omega^1_{U/S}$. Setting $D_i = \frac{d}{dx_i}$ for $1 \leq i \leq r$ suppose we are given

$$\omega = \sum_{i=1}^{r} a_i \cdot dx_i \in \Omega^1_{X/S}(U)$$

such that $d\omega = 0$. Then we have

$$C\omega = \sum_{i=1}^{r} b_i \cdot (dx_i \otimes 1)$$

where the b_i are sections of $\mathcal{O}_{X^{(p/S)}}$ over U satisfying

$$F_{X/S}(b_i) = -D_i^{p-1} a_i.$$

See [Il, 0, 2.1] for a proof. With respect to these data we immediately deduce

Lemma 3.4.4 *Let k_i, $l_i \in \mathbb{N}$ with $0 \leq l_i \leq p-2$ and assume that $a_i = T_i^{k_i \cdot p + l_i}$ for $1 \leq i \leq r$. Then $C\omega = 0$.*

Let us now assume that X/S is smooth of relative dimension r. Then for $0 \leq i \leq r$ the wedge product induces a perfect pairing

$$\Omega^i_{X/S} \times \Omega^{r-i}_{X/S} \longrightarrow \Omega^r_{X/S}.$$

Applying the direct image $F_{X/S*}$ to this pairing and composing it with the canonical morphism $F_{X/S*}(\Omega^r_{X/S}) \to H^r(F_{X/S*}(\Omega^{\cdot}_{X/S}))$ and the inverse $H^r(F_{X/S*}(\Omega^{\cdot}_{X/S})) \to \Omega^r_{X^{(p/S)}/S}$ of C^{-1} in degree r we get a pairing

$$F_{X/S*}(\Omega^i_{X/S}) \times F_{X/S*}(\Omega^{r-i}_{X/S}) \longrightarrow \Omega^r_{X^{(p/S)}/S} \qquad (*).$$

Proposition 3.4.5 *The pairing $(*)$ is perfect.*

Proof. Using [SGA 5, XV, §1, no. 2, Prop. 1, Prop. 2 c)] and the obvious fact that C^{-1} is compatible with étale localisations we reduce to the case we have dealt with in the proof of 3.4.2 an whose notation we will use in the sequel. Then it follows that a basis for the $\mathcal{O}_{X^{(p/S)}}$-module $F_{X/S*}(\Omega^{r-i}_{X/S})$ is given by the set $\mathcal{B}$ of all elements of the form $T_1^{e_1} \cdot \ldots \cdot T_r^{e_r} \cdot dT_{f_1} \wedge \cdots \wedge dT_{f_{r-i}}$ where $1 \leq f_1 < \ldots < f_{r-i} \leq r$ and $0 \leq e_k \leq p-1$ for $1 \leq k \leq r$. Pick an element $\alpha = T_1^{e_1} \cdot \ldots \cdot T_r^{e_r} \cdot dT_{f_1} \wedge \cdots \wedge dT_{f_{r-i}}$ of $\mathcal{B}$. Let $\{f'_1, \ldots, f'_i\}$ be the set theoretical difference of $\{1, \ldots, r\}$ and $\{f_1, \ldots, f_{r-i}\}$ and define

$$\widehat{\alpha} = \left(\frac{1}{(p-1)!}\right)^r T_1^{(p-1)-e_1} \cdot \ldots \cdot T_r^{(p-1)-e_r} \cdot dT_{f'_1} \wedge \ldots \wedge dT_{f'_i}$$

which is a global section of $F_{X/S*}(\Omega^i_{X/S})$. If we also denote by $\widehat{\alpha}$ the morphism $F_{X/S*}(\Omega^{r-i}_{X/S}) \to \Omega^r_{X^{(p/S)}/S}$ induced by $\widehat{\alpha}$ via $(*)$ we get for $\beta \in \mathcal{B}$

$$\widehat{\alpha}(\beta) = \begin{cases} \pm(dT_1 \wedge \cdots \wedge dT_r) \otimes 1, & \beta = \alpha \\ 0, & \text{otherwise} \end{cases}$$

using 3.4.3 and 3.4.4. Hence it follows that the induced morphism

$$F_{X/S*}(\Omega^i_{X/S}) \to \mathcal{H}om_{\mathcal{O}_{X^{(p/S)}}}(F_{X/S*}(\Omega^{r-i}_{X/S}), \Omega^r_{X^{(p/S)}/S})$$

is surjective and therefore bijective because on both sides we have vector bundles of the same rank. $\square$

Before we proceed further we digress for a moment to establish two (easy) auxiliary results we will use in the sequel.

Lemma 3.4.6 *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be an adic morphism of formal schemes. Then f is flat if and only if f_n is flat for every $n \in \mathbb{N}$ (with respect to an ideal of definition $\mathcal{J} \subseteq \mathcal{O}_{\mathfrak{Y}}$).*

Proof. In view of [EGA 0_{I}, 7.6.12] and [EGA I, 10.1.5] the condition is necessary. In order to prove its sufficiency we may assume that $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{Y} = \mathrm{Spf}(B)$ for noetherian adic rings A, B and f is induced by a continuous ringhomomorphism $B \to A$. Let $J \subseteq B$ denote the unique ideal of definition such that $J^{\Delta} = \mathcal{J}$ (using the notation of [EGA I, 10.3]). By hypothesis the ideal $J \cdot A$ of A is an ideal of definition, and $A \otimes_B B/J^n = A/J^nA$ is a flat B/J^n-module for every n. Applying [EGA 0_{I}, 7.1.10] and [SGA 1, IV, 5.6] we deduce that A is B-flat. Using [EGA 0_{I}, 5.7.3] and [EGA I, 10.10.2, 10.10.5] this implies the flatness of f. $\square$

Corollary 3.4.7 *Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a smooth morphism of formal schemes and suppose that $\mathfrak{Y}$ has no p-torsion. Then $\mathfrak{X}$ has no p-torsion.*

Proof. Indeed, the morphism f is flat by 3.4.6. $\square$

Let p be a prime number. For the rest of this section we fix a noetherian ring A which is separated and complete for the p-adic topology and has no p-torsion. We write $T = \mathrm{Spec}(A)$, $\widehat{T} = \mathrm{Spf}(A)$, $T_n = \mathrm{Spec}(A/p^{n+1})$ for $n \in \mathbb{N}$ and $S = T_0$. Then

$$\widehat{T} = \varinjlim_n T_n$$

is a noetherian formal affine scheme without p-torsion. Moreover, suppose we are given a smooth morphism $u\colon X \to S$.

Proposition 3.4.8 *Let $F_{\mathfrak{X}/\widehat{T}}\colon \mathfrak{X} \to \mathfrak{Y}$ be a formal lifting of $F_{X/S}$ over $\widehat{T}$.*

(a) *$F_{\mathfrak{X}/\widehat{T}}$ is finite.*

(b) *If $F_{\mathfrak{X}/\widehat{T}}$ is flat, then $F_{\mathfrak{X}/\widehat{T}*}(\mathcal{O}_{\mathfrak{X}})$ is a locally free $\mathcal{O}_{\mathfrak{Y}}$-module of finite rank.*

(c) *If $\mathfrak{X}$ is smooth over $\widehat{T}$, then $F_{\mathfrak{X}/\widehat{T}}$ is flat and $\mathfrak{Y}$ is smooth over $\widehat{T}$.*

Proof. We may assume that the structure morphism u is smooth of some relative dimension r and write F instead of $F_{\mathfrak{X}/\widehat{T}}$ in what follows. Then F is finite by 3.4.2 and [EGA III, 4.8.1]. Suppose now that $F_{\mathfrak{X}/\widehat{T}}$ is flat. Given $n \in \mathbb{N}$ consider the cartesian diagram

$$\begin{array}{ccccc} X_0 & \xrightarrow{F_0} & Y_0 & \longrightarrow & T_0 \\ \downarrow & & \downarrow & & \downarrow \\ X_n & \xrightarrow{F_n} & Y_n & \longrightarrow & T_n \end{array}$$

where $F_0 = F_{X/S}$. The morphism F_n is finite by 3.1.3 and flat by 3.4.6. Since the vertical maps are closed nilpotent immersions and $F_{n*}(\mathcal{O}_{X_n})$ is a flat $\mathcal{O}_{Y_n}$-module it follows from [EGA IV, 11.4.9], [EGA I, 9.3.3] and 3.4.2 that $F_{n*}(\mathcal{O}_{X_n})$ is a locally free $\mathcal{O}_{Y_n}$-mdoule of rank p^r. Thus $F_*(\mathcal{O}_{\mathfrak{X}})$ is locally free of rank p^r by [EGA III, 4.8.6] and [EGA I, 10.11.10]. Finally let us assume that $\mathfrak{X}$ is $\widehat{T}$-smooth. As $X_n \to T_n$ is smooth, the structure map $Y_n \to T_n$ is locally of finite presentation (3.1.4), F_0 is flat and F_n is a homeomorphism we deduce by [EGA IV, 11.3.11] that both F_n and $Y_n \to T_n$ are flat. Hence F is flat (3.4.6) and $\mathfrak{Y}$ is $\widehat{T}$-smooth (3.2.2). $\square$

Let $F = F_{\mathfrak{X}/\widehat{T}}\colon \mathfrak{X} \to \mathfrak{Y}$ be a formal lifting of $F_{X/S}$ over $\widehat{T}$ such that $\mathfrak{X}$ is $\widehat{T}$-smooth. Note that according to (the proof of) 3.3.3 such a lifting exists if X is affine and the structure morphism $X \to S$ factors into

$$X \xrightarrow{g} S[T_1, \ldots, T_p] \xrightarrow{\pi} S$$

where the T_i are indeterminates, g is étale and π denotes the canonical projection.
Given k, $n \in \mathbb{N}$ consider the canonical $\mathcal{O}_{Y_n}$-linear map

$$F_n^k \colon \Omega^k_{Y_n/T_n} \to F_{n*}(\Omega^k_{X_n/T_n}).$$

If $m \leq n$ and $u_{mn} \colon Y_m \to Y_n$ denotes the transition map we obviously have $u^*_{mn}(F_n^k) = F_m^k$. Taking $F^k = \varprojlim_n F_n^k$ we get a (continuous) morphism of coherent $\mathcal{O}_{\mathfrak{Y}}$-modules

$$F^k_{\mathfrak{Y}/\widehat{T}} \xrightarrow{\omega^k} F_*(\Omega^k_{\mathfrak{X}/\widehat{T}}).$$

Moreover, the collection $(F^k)_{k\in\mathbb{N}}$ yields a morphism of complexes

$$F \colon \Omega^{\cdot}_{\mathfrak{Y}/\widehat{T}} \to F_*(\Omega^{\cdot}_{\mathfrak{X}/\widehat{T}}).$$

Lemma 3.4.9 *The image of F^k is contained in $p^k F_*(\Omega^k_{\mathfrak{X}/\widehat{T}})$.*

Proof. It suffices to treat the case where $k = 1$. Because the question is local we may assume that $S = \mathrm{Spec}(B)$, $X = \mathrm{Spec}(C)$ for suitable noetherian rings B, C and u is given by a ringhomomorphism $\varphi \colon B \to C$. For $b \in B$ and $c \in C$ we have $d(\varphi(b)c^p) = 0$ which implies that $\omega_0^1 = 0$. Thus the image of ω_n^1 is contained in $p\, F_{n*}(\Omega^k_{X_n/T_n})$ for every $n \in \mathbb{N}$. $\Box$

Hence as the formal scheme $\mathfrak{Y}$ has no p-torsion (3.4.7 and 3.4.8 (c)) we get a degree zero morphism of graded algebras

$$F_p \colon \Omega^{\cdot}_{\mathfrak{Y}/\widehat{T}} \to F_*(\Omega^{\cdot}_{\mathfrak{X}/\widehat{T}})$$

satisfying $p^k F_p^k = F^k$ for every $k \in \mathbb{N}$. Furthermore, the diagram

$$\begin{array}{ccc} \Omega^{\cdot}_{\mathfrak{Y}/\widehat{T}} & \xrightarrow{F_p} & F_*(\Omega^{\cdot}_{\mathfrak{X}/\widehat{T}}) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle F_*(d)} \\ \Omega^{\cdot}_{\mathfrak{Y}/\widehat{T}} & \xrightarrow{p\cdot F_p} & F_*(\Omega^{\cdot}_{\mathfrak{X}/\widehat{T}}) \end{array}$$

is commutative.
Thus if we compose the morphism $\Omega^k_{\mathfrak{Y}/\widehat{T}} \xrightarrow{F_p^k} F_*(\Omega^k_{\mathfrak{X}/\widehat{T}})$ with the projection $F_*(\Omega^k_{\mathfrak{X}/\widehat{T}}) \to F_{X/S*}(\Omega^k_{X/S})$ we get a map whose image is contained in the kernel of $F_{X/S*}(d^k_{\Omega^{\cdot}_{X/S}})$. This induces a degree zero morphism of graded algebras

$$\Omega^{\cdot}_{\mathfrak{Y}/\widehat{T}} \xrightarrow{\widetilde{F_p}} H^{\cdot}(F_{X/S*}(\Omega^{\cdot}_{X/S})).$$

Lemma 3.4.10 *Let* $\pi\colon \Omega^{\cdot}_{\mathfrak{Y}/\widehat{T}} \to \Omega^{\cdot}_{X^{(p/S)}/S}$ *denote the natural projection. Then the triangle*

$$\begin{array}{ccc} \Omega^{\cdot}_{\mathfrak{Y}/\widehat{T}} & \xrightarrow{\pi} & \Omega^{\cdot}_{X^{(p/S)}/S} \\ & \searrow^{\widetilde{F_p}} & \downarrow{\scriptstyle C^{-1}} \\ & & H^{\cdot}(F_{X/S*}\Omega^{\cdot}_{X/S}) \end{array}$$

is commutative, where C^{-1} *is the Cartier isomorphism.*

Proof. See [BO2, 1.4] or [Il, 0, 2.3.8]. □

3.5 Construction of cycle classes

Let X be a locally noetherian scheme and let $(X^d)_{d\in\mathbb{N}}$ denote the codimension filtration on X, i.e. $X^d = \{x \in X;\ \dim(\mathcal{O}_{X,x}) \geq d\}$. Recall from [H1, IV, §2] that an abelian sheaf E on X is Cohen-Macaulay (with respect to $(X^d)_{d\in\mathbb{N}}$) if the Cousin complex $\mathrm{Co}(E)$ of E is a (flabby) resolution of E.
Let $E^{\cdot}$ be a bounded below complex of abelian sheaves on X such that each E^p is Cohen-Macaulay. Then we can construct a canonical flabby resolution $\mathcal{C}(E^{\cdot})$ of $E^{\cdot}$. For that purpose let us first define a double complex $(F^{i\,j})_{i\,j}$ as follows: We set $F^{i\,j} = \mathrm{Co}^i(E^j)$. The first differential $d_{\mathrm{I}}^{i\,j}$ shall be given by the differential in degree i of the Cousin complex of E^j. The second differential $d_{\mathrm{II}}^{i\,j}$ shall be given by the map $\mathrm{Co}^i(E^j) \to \mathrm{Co}^i(E^{j+1})$ induced by the differential in degree j of $E^{\cdot}$ via the Cousin complex functor. Then it is easily verified that these data do indeed define a double complex and we define $\mathcal{C}(E^{\cdot})$ to be the total direct sum complex corresponding to that double complex. There is an evident chain map

$$E^{\cdot} \to \mathcal{C}(E^{\cdot})$$

and by a standard spectral sequence argument (which we have used in the proof of 2.1.1) it follows that this map is a quasi-isomorphism. Furthermore, $\mathcal{C}(E^{\cdot})$ is a bounded below complex of flabby sheaves (as X is locally noetherian every direct sum of flabby sheaves is flabby again ([Go, II, 3.1, 3.10])). By [H1, IV, 2.5] we have

$$\mathcal{C}^k(E^{\cdot}) = \bigoplus_{i+j=k} \bigoplus_{x\in X^i - X^{i+1}} \iota_x(H^i_x(E^j))$$

where ι_x is the functor which sends a group G to the sheaf on X which is the constant sheaf G on $\overline{\{x\}}$ and zero elsewhere. Notice that $\mathcal{C}(*)$ is functorial.

If the underlying topological space of X is noetherian and d, $k \in \mathbb{N}$ we have

$$\Gamma_{X^d}(X, \mathcal{C}^k(E^{\cdot})) = \bigoplus_{i=0}^{k-d} \bigoplus_{x \in X^{k-i} - X^{k-i+1}} H_x^{k-i}(E^i)$$

which follows from the above description of $\mathcal{C}^k(E^{\cdot})$ and 3.5.1 (a).

Lemma 3.5.1 *Let Y be a locally noetherian topological space and denote by* $\mathrm{Ab}(Y)$ *the category of abelian sheaves on Y.*

(a) *Suppose that Y is quasi-compact and let φ be a family of supports on Y. Then the functor $\Gamma_\varphi(Y, *)$ on* $\mathrm{Ab}(Y)$ *preserves arbitrary direct sums.*

(b) *Let $\underline{\varphi}$ be a sheaf of families of supports on Y. Then the functor $\underline{\Gamma}_{\underline{\varphi}}(*)$ on* $\mathrm{Ab}(Y)$ *preserves arbitrary direct sums.*

Proof. (a) By hypothesis it follows that Y is a noetherian topological space. We have

$$\Gamma_\varphi(Y, *) = \varinjlim_{V \in \varphi} \Gamma_V(Y, *)$$

and φ is directed with respect to $\subseteq$. Since direct limits preserve direct sums it suffices to show that $\Gamma_Z(Y, *)$ commutes with direct sums for any closed subset Z of Y. This is well known for $Z = Y$ (and holds more generally for any coherent topos, see [SGA 4, VI, 2.9.2, 5.2]) and follows easily from this case and the fact that the functor "restriction to open subschemes" commutes with direct sums.
(b) is a consequence of (a) and the fact that X has a base consisting of open, noetherian subspaces. □

As an example for Cohen-Macaulay sheaves we have

Proposition 3.5.2 *Let X be a (locally noetherian) Cohen-Macaulay scheme and let $\mathcal{F}$ be a vector bundle on X. Then $\mathcal{F}$ is Cohen-Macaulay (with respect to the codimension filtration).*

Proof. By [H1, IV, §1, Variation, Motif F] and [ibid., IV, 2.6] we have to show that $H_x^i(\mathcal{F}) = 0$ for all $x \in X^d - X^{d+1}$ and all i, d with $i \neq d$. Using 2.1.13 we may therefore assume that $X = \mathrm{Spec}(A)$ where A is a local Cohen-Macaulay ring. Let c be the dimension of A and denote by $\mathfrak{m}$ its closed point. To show that $H_{\mathfrak{m}}^i(\mathcal{F})$ for $i \neq c$ we can assume that $\mathcal{F} \neq 0$. Then $\mathrm{depth}_{\{\mathfrak{m}\}}(\mathcal{F}) = c$ which implies $H_{\mathfrak{m}}^i(\mathcal{F}) = 0$ for $i < c$ by 2.1.13 and [G, 3.8]. Let $f_1, \ldots, f_c$ be a regular system of parameters for the ring A.

Then $\mathcal{U} = (D(f_i))_{1 \leq i \leq c}$ is a finite open affine cover of $X - \{\mathfrak{m}\}$ and using the notation of 2.1 we deduce that $\mathrm{Cone}^i(\mathcal{U}, \mathcal{F}) = 0$ for $i > c$ and thus $H^i_{\mathfrak{m}}(X, \mathcal{F}) = 0$ for these i by 2.1.7 and 2.1.13. $\square$

We shall call a Cohen-Macaulay ring A special, if there exists a nilpotent ideal J of A such that A/J is regular. Moreover, a Cohen-Macaulay scheme Y is said to be special, if all of its stalks are special Cohen-Macaulay rings. Let $X = \mathrm{Spec}(A)$, where A is a local special Cohen-Macaulay ring of dimension c, and denote by $\mathfrak{m}$ its closed point. Furthermore, suppose that X is weakly smooth over a scheme S, i.e. X is an S-scheme such that the $\mathcal{O}_X$-module $\Omega^1_{X/S}$ is a vector bundle. Consider the complexes $L^{\cdot}_q = (H^q_{\mathfrak{m}}(X, \Omega^p_{X/S}))_{p \in \mathbb{Z}}$ having the obvious differentials and the (first) spectral sequence of hypercohomology

$$E_2^{p,\,q} = H^p(L^{\cdot}_q) \;\Rightarrow\; E^n = H^n_{\mathfrak{m}}(X, \Omega^{\cdot}_{X/S})$$

which is biregular. According to the proof of 3.5.2 we have $E_2^{p,\,q} = 0$ for $q \neq c$ and thus

$$\mathrm{gr}_{n-c}(E^n) = E^n = E_\infty^{n-c,\,c} = E_2^{n-c,\,c}$$

which amounts to $H^{n-c}(L^{\cdot}_c) = H^n_{\mathfrak{m}}(X, \Omega^{\cdot}_{X/S})$.
Unwinding the result 2.1.7 we can give an explicit description of the cohomology groups $H^c_{\mathfrak{m}}(X, \Omega^p_{X/S})$ for $p \in \mathbb{Z}$. Namely, choosing a system of parameters $f_1, \ldots, f_c \in \mathfrak{m}$ we get

$$H^c_{\mathfrak{m}}(X, \Omega^p_{X/S}) = \Omega^p_{X/S}[{f_1}^{-1}, \ldots, {f_c}^{-1}] / \sum_{i=1}^{c} \Omega^p_{X/S}[{f_1}^{-1}, \ldots, \widehat{{f_i}^{-1}}, \ldots, {f_c}^{-1}].$$

Given $n \in \mathbb{N}$ and $\omega \in \Omega^p_{X/S}$ the residue class of the element $\frac{\omega}{(f_1 \cdot \ldots \cdot f_c)^n}$ in $H^c_{\mathfrak{m}}(X, \Omega^p_{X/S})$ is mapped by the differential in degree p of the complex $L^{\cdot}_c$ to the residue class of

$$\frac{1}{(f_1 \cdot \ldots \cdot f_c)^{n+1}} \Big(f_1 \cdot \ldots \cdot f_c \cdot d\omega - n \cdot d(f_1 \cdot \ldots \cdot f_c) \wedge \omega \Big).$$

Proposition 3.5.3 *The residue class of the element $\frac{df_1 \wedge \cdots \wedge df_c}{f_1 \cdot \ldots \cdot f_c}$ in the group $H^c_{\mathfrak{m}}(X, \Omega^c_{X/S})$ does not depend on the choice of the system of parameters. Moreover, it induces a cohomology class in $H^{2c}_{\mathfrak{m}}(X, \Omega^{\cdot}_{X/S})$.*

Proof. By the explicit description of the differentials of the complex $L^{\cdot}_c$ the second part of the assertion is clear. Let $g_1, \ldots, g_d$ and $g'_1, \ldots, g'_d$ be two systems of parameters for A. Then the Koszul complexes $K^{\cdot}(g_1, \ldots, g_d)$ and $K^{\cdot}(g'_1, \ldots, g'_d)$ are quasi-isomorphic because they are quasi-isomorphic modulo a nilpotent ideal. In view of this observation the first part now results from 2.3.7, 2.3.12 and 2.3.20 (b). $\square$

Combining 2.1.13 and 3.5.3 we obtain

Corollary 3.5.4 *Let Y be a special Cohen-Macaulay scheme which is weakly smooth over some scheme S. Let $y \in Y$ and $c = \dim(\mathcal{O}_{Y,y})$. Then a system of parameters for $\mathcal{O}_{Y,y}$ induces in a natural way a class $\mathrm{cl}_y(Y/S)$ in $H^c_y(Y, \Omega^c_{Y/S})$ which does not depend on the choice of this system.*

Remark 3.5.5 Let us fix the notation of 3.5.4. Then by 3.5.2 the complex $\mathcal{C}(\Omega^{\cdot}_{Y/S})$ gives rise to a bounded below flabby resolution of the De Rham complex $\Omega^{\cdot}_{Y/S}$. Setting $G^{\cdot} = \Gamma(Y, \mathcal{C}(\Omega^{\cdot}_{Y/S}))/\Gamma(Y, \mathcal{C}(\Omega^{\cdot}_{Y/S}))$ we thus infer that $H^*_{Y^c/Y^{c+1}}(Y, \Omega^{\cdot}_{Y/S}) \cong H^*(G^{\cdot})$. Noting that

$$G^{2c} = \bigoplus_{x \in Y^c - Y^{c+1}} H^c_x(\Omega^c_{Y/S})$$

it follows from the second part of 3.5.3 that $\mathrm{cl}_y(Y/S)$ is a $2c$-cocycle of $G^{\cdot}$, i.e. it induces a cohomology class in $H^{2c}_{Y^c/Y^{c+1}}(Y, \Omega^{\cdot}_{Y/S})$.

Given an $\mathbb{F}_p$-scheme X we denote by $W(\mathcal{O}_X)$ (resp. $W_n(\mathcal{O}_X)$) the sheaves on X where $W(\mathcal{O}_X)(U)$ (resp. $W_n(\mathcal{O}_X)(U)$) is the ring of p-Witt vectors (resp. the ring of p-Witt vectors of length n) over $\Gamma(U, \mathcal{O}_X)$ for any open subset U of X (cf. [Il, 0, 1.5]). Moreover, the ringed spaces $(X, W(\mathcal{O}_X))$ and $(X, W_n(\mathcal{O}_X))$ will be denoted by $W(X)$ and $W_n(X)$ respectively. According to [Il, 0, 1.4, 1.5] the ringed space $W_n(X)$ is in fact a $(\mathbb{Z}/p^n)$-scheme and $W_n(\mathcal{O}_X)$ has a canonical PD-ring structure making $W_n(X)$ into a $(\mathbb{Z}/p^n)$-PD-thickening of X.
We shall also consider the pro-object $(W_n\Omega^{\cdot}_X)_{n\geq 1}$ where $W_n\Omega^{\cdot}_X$ denotes the De Rham-Witt complex of order n for X. The main reference for De Rham-Witt complexes which we shall use is [Il], see also [RI] for more applications of this theory. Recall that it comes along with a Frobenius morphism $F\colon W_{\cdot}\Omega^{\cdot}_X \to W_{\cdot-1}\Omega^{\cdot}_X$ and a morphism (Verschiebung) $V\colon W_{\cdot}\Omega^{\cdot}_X \to W_{\cdot+1}\Omega^{\cdot}_X$ satisfying

(a) $FV = VF = p$

(b) $dF = pFd$

(c) $Vd = pdV$

(d) $x \cdot V(y) = V(F(x) \cdot y)$ for $x \in W_n\Omega^i_X$, $y \in W_{n-1}\Omega^j_X$.

Recall that a scheme S is perfect of characteristic $p > 0$ if it is an $\mathbb{F}_p$-scheme such that the Frobenius map F_S (cf. 3.4) is an automorphism. We fix once and for all a locally noetherian, perfect Cohen-Macaulay scheme S of characteristic $p > 0$ and a scheme X which is smooth over S. In this case we know by [Il, 0, 1.5] and 3.4.2 that $W_n(X)$ is a locally noetherian $(\mathbb{Z}/p^n)$-scheme. Note that the PD-ideal of $W_n(S)$ is just $pW_n(S)$ and the natural map of schemes $W_n(X) \to W_n(S)$ is a PD-morphism. Using this morphism and the scheme map induced by the projection $W_n(\mathcal{O}_X) \to W_1(\mathcal{O}_X) = \mathcal{O}_X$ we shall consider X as a $W_n(S)$- scheme (to which the PD-structure of $W_n(S)$ extends by smoothness). In the following we shall simply write W_n instead of $W_n(S)$. Note also that $W_1\Omega^{\cdot}_X = \Omega^{\cdot}_{X/S}$.

Proposition 3.5.6 *The sheaves $W_n\Omega^p_X$ are Cohen-Macaulay for all n, p.*

Proof. By [Il, I, 3.1] we have a canonical, finite filtration $(\mathrm{Fil}^k W_n\Omega^p_X)_{k\in\mathbb{Z}}$ on $W_n\Omega^p_X$. Setting

$$\mathrm{gr}^k W_n\Omega^p_X = \mathrm{Fil}^k W_n\Omega^p_X / \mathrm{Fil}^{k+1} W_n\Omega^p_X$$

and using [H1, IV, 2.6] it is enough to show that $\mathrm{gr}^k W_n\Omega^p_X$ is Cohen-Macaulay for all k, n, p. For $n = 1$ this is clear by 3.5.2 because

$$\mathrm{gr}^k W_1\Omega^p_X = \begin{cases} W_1\Omega^p_X = \Omega^p_X, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

Let $n > 1$ in what follows. Since $\mathrm{gr}^k W_n\Omega^p_X = 0$ for $k \geq n$ we may assume that $n \geq k+1$. But then invoking [Il, I, 3.1.5, 3.9] we see that $\mathrm{gr}^k W_n\Omega^p_X$ has an $\mathcal{O}_X$-module structure making it to a vector bundle on X. Applying 3.5.2 we conclude that $\mathrm{gr}^k W_n\Omega^p_X$ is Cohen-Macaulay. □

Let $(X/W_n)_{\mathrm{crys}}$ be the crystalline topos of X/W_n with its canonical structure sheaf $\mathcal{O}_{X/W_n}$. Moreover, we denote by $u_{X/W_n} \colon (X/W_n)_{\mathrm{crys}} \to X_{\mathrm{Zar}}$ the natural morphism of topoi and by $f \colon X \to S$ the structure map. As before we denote by $(X^d)_{d\in\mathbb{N}}$ the codimension filtration on X.

Corollary 3.5.7 *Let $d \in \mathbb{N}$, $n \geq 1$ and $\mathcal{C}_n = \mathcal{C}(W_n\Omega^{\cdot}_X)$. Then there exist canonical isomorphisms*

(a) $H^*_{X^d}(X/W_n, \mathcal{O}_{X/W_n}) \cong H^*(\Gamma_{X^d}(X, \mathcal{C}_n))$

(b) $H^*_{X^d/X^{d+1}}(X/W_n, \mathcal{O}_{X/W_n}) \cong H^*(\Gamma_{X^d}(X, \mathcal{C}_n)/\Gamma_{X^{d+1}}(X, \mathcal{C}_n))$

Proof. By the theorem of comparison ([Il, II, 1.4]) there is a natural isomorphism

$$Ru_{X/W_n\,*}(\mathcal{O}_{X/W_n}) \xrightarrow{\cong} W_n\Omega^{\cdot}_X$$

in the derived category $D(f^{-1}\mathcal{O}_{W_n})$. Furthermore, by 3.5.6 and the previous discussion $\mathcal{C}_n$ is a (bounded below) flabby resolution of $W_n\Omega^{\cdot}_X$. Thus we deduce

$$\begin{aligned} R\Gamma_{X^d}(X/W_n,\, \mathcal{O}_{X/W_n}) &\cong R(\Gamma_{X^d}(X,\, *) \circ u_{X/W_n\,*})(\mathcal{O}_{X/W_n}) \\ &\cong R\Gamma_{X^d}(X,\, Ru_{X/W_n\,*}(\mathcal{O}_{X/W_n})) \\ &\cong \Gamma_{X^d}(X,\, \mathcal{C}_n) \end{aligned}$$

which establishes (a). (b) is proved similarly. □

Let $x \in X$ be a point of codimension d, $Y = \mathrm{Spec}(\mathcal{O}_{X,\,x})$ and denote by i_x the canonical morphism $Y \to X$. Given $i,\, n \in \mathbb{N}$ we want to describe the cohomology groups $H^d_x(W_n\Omega^i_X)$. By [Il, I, 1.12.4] we have

$$i_x^{-1}(W_n\Omega^i_X) = W_n\Omega^i_Y$$

and this sheaf is Cohen-Macaulay (with respect to the codimension filtration) because $W_n\Omega^i_X$ has this property. By [Il, 0, 1.5] the Witt scheme $W_n(Y)$ is a $(\mathbb{Z}/p^n)$-PD-thickening of Y and $A = \Gamma(Y,\ W_n(Y))$ is a local noetherian ring. Given a system of parameters $(f_j)_{1\le j\le d}$ for A it follows that $(D(\underline{f_j}))_{1\le j\le d}$ is a finite open affine cover of $W_n(Y) - \{x\}$ where $\underline{f_j}$ denotes the Teichmüller representative of f_j. Taking into account that $W_n\Omega^i_Y$ is a quasi-coherent $W_n\mathcal{O}_Y$-algebra ([Il, I, 1.13.1]) and using 2.1.7, 2.1.12 we deduce

$$H^d_x(W_n\Omega^i_X) = W_n\Omega^i_A[\underline{f_1}^{-1},\, \dots,\, \underline{f_d}^{-1}] / \sum_{j=1}^{d} W_n\Omega^i_A[\underline{f_1}^{-1},\, \dots,\, \widehat{\underline{f_j}^{-1}},\, \dots,\, \underline{f_d}^{-1}].$$

Let B be the localization of A at $f_1 \cdot \ldots \cdot f_d$ so that

$$W_n\Omega^i_A[\underline{f_1}^{-1},\, \dots,\, \underline{f_d}^{-1}] = W_n\Omega^i_B$$

by [Il, I, 1.11]. Setting $\widetilde{W_n}\Omega^i_B = \sum_{j=1}^{d} W_n\Omega^i_A[\underline{f_1}^{-1},\, \dots,\, \widehat{\underline{f_j}^{-1}},\, \dots,\, \underline{f_d}^{-1}]$ we then obtain

Lemma 3.5.8 *Let $i \in \mathbb{N}$ and consider the morphisms F, V on the pro-object $(W_n\Omega^{\cdot}_B)_{n\ge1}$.*

(a) *$F\colon W_{n+1}\Omega^i_B \to W_n\Omega^i_B$ maps $\widetilde{W}_{n+1}\Omega^i_B$ into $\widetilde{W}_n\Omega^i_B$ for all $n \ge 1$.*

(b) $V\colon W_n\Omega^i_B \to W_{n+1}\Omega^i_B$ *maps* $\widetilde{W_n}\Omega^i_B$ *into* $\widetilde{W_{n+1}}\Omega^i_B$ *for all* $n \geq 1$.

Proof. Let $\omega \in \Omega^i_A$, $1 \leq j \leq d$ and $m \in \mathbb{N}$. Then

$$F(\omega \cdot (\underline{f_1} \cdot \ldots \cdot \widehat{\underline{f_j}} \cdot \ldots \cdot \underline{f_c})^{-m}) = F(\omega) \cdot (\underline{f_1} \cdot \ldots \cdot \widehat{\underline{f_j}} \cdot \ldots \cdot \underline{f_c})^{-mp}$$

which proves (a). For an element $\tau \in \Omega^i_A[\underline{f_1}^{-1}, \ldots, \widehat{\underline{f_j}^{-1}}, \ldots, \underline{f_c}^{-1}]$ there exist $\tau' \in W_n\Omega^i_A$ and $m' \in \mathbb{N}$ such that $\tau = \tau' \cdot (\underline{f_1} \cdot \ldots \cdot \widehat{\underline{f_j}} \cdot \ldots \cdot \underline{f_c})^{-m'p}$. Thus

$$V(\tau) = V(F((\underline{f_1} \cdot \ldots \cdot \widehat{\underline{f_j}} \cdot \ldots \cdot \underline{f_c})^{-m'}) \cdot \tau') = (\underline{f_1} \cdot \ldots \cdot \widehat{\underline{f_j}} \cdot \ldots \cdot \underline{f_c})^{-m'}\, V(\tau')$$

lies in $\widetilde{W_{n+1}}\Omega^i_X$ which establishes (b). $\square$

Hence Frobenius and Verschiebung operate on the pro-object of differential graded $\mathbb{Z}$-algebras $(H^d_x(W_n\Omega^{\cdot}_X))_{n\geq 1}$. If y is a $(d+1)$-codimensional specialization of x in X then the differentials of the Cousin complex of each $W_n\Omega^i_X$ induce a morphism of pro-objects

$$(H^d_x(W_n\Omega^{\cdot}_X))_{n\geq 1} \to (H^{d+1}_y(W_n\Omega^{\cdot}_X))_{n\geq 1}$$

which commutes with Frobenius and Verschiebung.

Let $i \in \mathbb{N}$ and $(\omega_n)_{n\geq 1} \in \varprojlim_{n\geq 1} H^d_x(W_n\Omega^i_X)$. We shall say that $(\omega_n)_{n\geq 1}$ is F-invariant (resp. V-invariant) if $F(\omega_{n+1}) = \omega_n$ (resp. $V(\omega_n) = \omega_{n+1}$) for all $n \geq 1$.

For the rest of this section suppose that $S = \mathrm{Spec}(k)$ where k is a perfect field of characteristic $p > 0$ so that $W(k)$ is a complete discrete valuation ring. Consider a closed, integral subscheme V of X with generic point ζ and $c = \mathrm{codim}(V, X)$. By a formal lifting around ζ we shall mean a smooth formal $\mathrm{Spf}(W(k))$-scheme $\mathfrak{U}$ which lifts an open neighbourhood U of ζ in X over S. In this case we shall denote by U_n the restriction of $\mathfrak{U}$ modulo p^n for $n \geq 1$ and identify U with U_1.

Lemma 3.5.9

(a) *There exist formal liftings around* ζ.

(b) *Two formal liftings around* ζ *are locally isomorphic over* $\mathrm{Spf}(W(k))$.

(c) *If* ξ *is a specialization of* ζ *in* X *and* $\mathfrak{U}$ *a formal lifting around* ξ *then* $\mathfrak{U}$ *is also a formal lifting around* ζ.

Proof. (c) is trivial, (a) and (b) result from 3.3.1 and 3.3.2. □

By a formal Frobenius lifting around ζ we shall mean a tuple $(\mathfrak{U},\, F)$ where $\mathfrak{U}$ is a formal lifting around ζ and F is an $\mathrm{Spf}(W(k))$-endomorphism on $\mathfrak{U}$ which lifts the Frobenius morphism F_{U_1} (i.e. such that $F/p = F_{U_1}$. Using 3.3.3 (applied to F_X) we see that (a) and (c) of 3.5.9 are also valid for formal Frobenius liftings. For such a formal Frobenius lifting around ζ the Cartier morphism

$$\mathcal{O}_{\mathfrak{U}} \to W(\mathcal{O}_{\mathfrak{U}}/p \cdot \mathcal{O}_{\mathfrak{U}}) \cong W(\mathcal{O}_{U_1})$$

(see [Il, 0, 1.3.20]) gives rise to a PD-morphism

$$t_{\mathcal{F}_n} \colon \mathcal{O}_{U_n} \to W_n\mathcal{O}_{U_1} = W_n\mathcal{O}_X|U_1$$

noticing that the canonical PD-structure of $p \cdot \mathcal{O}_{W_n}$ extends to U_n by flatness. We define $\theta_{\mathcal{F}_n}$ to be the composition

$$\Omega^{\cdot}_{U_n/W_n} \to \Omega^{\cdot}_{W_n\mathcal{O}_{U_1}/W_n} \twoheadrightarrow W_n\Omega^{\cdot}_{U_1}$$

where the first map is the morphism between De Rham complexes induced by $t_{\mathcal{F}_n}$ and the second map is the canonical surjection ([Il, I, 1.3]). Thus we get a morphism

$$\theta_{\mathcal{F}_n} \colon \Omega^{\cdot}_{U_n/W_n} \to W_n\Omega^{\cdot}_{U_1}$$

between differential graded $(f|U_1)^{-1}\mathcal{O}_{W_n}$-algebras. This map is a quasi-isomorphism by [Il, II, 1.4].

Remark 3.5.10 (a) Let $\mathcal{F} = (\mathfrak{U},\, F)$ be a formal Frobenius lifting around ζ and assume that F' is another endomorphism on $\mathfrak{U}$ which lifts F_{U_1}. Then $\mathcal{F}' = (\mathfrak{U},\, F')$ is a formal Frobenius lifting around ζ, and if κ_{U_n} denotes the canonical isomorphism

$$Ru_{X/W_n\,*}(\mathcal{O}_{X/W_n}) \xrightarrow{\cong} \Omega^{\cdot}_{U_n/W_n}$$

in $D((f|U_1)^{-1}\mathcal{O}_{W_n})$ (cf. [B, V, 2.3.2]) we have $\theta_{\mathcal{F}_n} \circ \kappa_n = \theta_{\mathcal{F}'_n} \circ \kappa_n$ according to [Il, II, 1.1] and thus $\theta_{\mathcal{F}_n} = \theta_{\mathcal{F}'_n}$ in $D((f|U_1)^{-1}\mathcal{O}_{W_n})$.
(b) The map $\theta_{\mathcal{F}_n}$ can be defined in a more general context as follows: Let T be a perfect scheme of characteristic $p > 0$ and let X' be a T-scheme. Then for any PD-morphism $Y' \to W_n(X')$ over $W_n(T)$ where Y' is a $W_n(T)$-PD-thickening of X' there is a canonical map

$$\Omega^{\cdot}_{Y'/W_n(T)} \to W_n\Omega^{\cdot}_{X'}$$

between differential graded $\mathcal{O}_{W_n(T)}$-algebras. Furthermore, this map is functorial in an obvious sense.

Before we proceed further let us state two auxiliary results. Recall that for a locally noetherian scheme Y the regular locus of Y is given by

$$\mathrm{Reg}(Y) = \{y \in Y;\ \mathcal{O}_{Y,y} \text{ is regular}\}.$$

Then by [EGA IV, 6.12.8] we have

Proposition 3.5.11 *Let A be a local, noetherian and complete ring. Then the set $\mathrm{Reg}(Y)$ is open for any scheme Y which is locally of finite type over A.*

Corollary 3.5.12 *Let Y be an integral scheme which is locally of finite type over a perfect field k. Then $\mathrm{Reg}(Y)$ is a non-empty, open subscheme of Y which is smooth over k.*

Proof. Let ξ be the generic point of Y. Clearly we have $\xi \in \mathrm{Reg}(Y)$, and $\mathrm{Reg}(Y)$ is an open subset of Y by 3.5.11. The smoothness of $\mathrm{Reg}(Y)/k$ follows from [EGA IV, 17.15.1]. □

Let r denote the relative dimension of X/k in ζ. In the following we shall regard $W_n[T_{c+1}, \ldots, T_r]$ as a closed subscheme of $W_n[T_1, \ldots, T_r]$ via the canonical closed regular immersion

$$W_n[T_{c+1}, \ldots, T_r] \hookrightarrow W_n[T_1, \ldots, T_n]$$

which is defined by the ideal $(T_1, \ldots, T_c)$. Here of course the elements $T_1, \ldots, T_r$ are indeterminates. Furthermore, we denote by f_n the endomorphism on $W_n[T_1, \ldots, T_r]$ which is F_{W_n} on W_n and maps T_i to T_i^p for $1 \leq i \leq r$. By 3.5.12 and [EGA IV, 17.12.2] there exist an open affine neighbourhood U_1 of ζ in X and a factorization

$$U_1 \xrightarrow{u_1} W_1[T_1, \ldots, T_r] \xrightarrow{\pi} W_1 = \mathrm{Spec}(k)$$

of $f|U_1$ where u_1 is étale, π is the projection such that $V_1 = V \cap U_1$ is the scheme-theoretical inverse image of $W[T_{c+1}, \ldots, T_r]$ under u_1. Setting $F_1 = F_X|U_1$ it follows that $u_1 \circ F_1 = f_1 \circ u_1$. Using [EGA IV, 18.1.2] and proceeding inductively we can construct adic, inductive $(W_n)_{n\geq 1}$-systems (in the sense of [EGA I, 10.12.2]) $(V_n)_{n\geq 1}$, $(U_n)_{n\geq 1}$, $(W_n[T_1, \ldots, T_r])_{n\geq 1}$ and morphisms

$$(i_n)_n \colon (V_n)_n \to (U_n)_n$$
$$(u_n)_n \colon (U_n)_n \to (W_n[T_1, \ldots, T_r])_n$$
$$(F_n)_n \colon (U_n)_n \to (U_n)_n$$

between them such that the following conditions are satisfied:

(a) Each i_n is a closed immersion.

(b) Each u_n is an étale map.

(c) V_n is the scheme-theoretical inverse image of $W_n[T_{c+1}, \ldots, T_r]$ under u_n.

(d) $u_n \circ F_n = f_n \circ u_n$

Forming $\varinjlim_n i_n$ we thus obtain a closed immersion $\varinjlim_n V_n \to \varinjlim_n U_n$ between smooth formal $\mathrm{Spf}(W(k))$-schemes and

$$\mathcal{F} = \left(\varinjlim_n U_n, \varinjlim_n F_n\right)$$

gives rise to a formal Frobenius lifting around ζ. Note also that $\varinjlim_n u_n$ is a (formally étale) morphism

$$\varinjlim_n U_n \to W(k)\{T_1, \ldots, T_r\}$$

where $W(k)\{T_1, \ldots, T_r\}$ denotes the ring of restricted formal power series (cf. [EGA 0_{I}, 7.5]).
Recall that we have defined a morphism

$$\theta_{\mathcal{F}_n} : \Omega^{\cdot}_{U_n/W_n} \to W_n\Omega^{\cdot}_{U_1}$$

between differential graded $(f|U_1)^{-1}\mathcal{O}_{W_n}$-algebras. Setting

$$\mathrm{cl}_{X,n}(V) = H^c_\zeta(\theta^c_{\mathcal{F}_n})(\mathrm{cl}_\zeta(U_n/W_n))$$

we thus get an element in $H^c_\zeta(W_n\Omega^c_X) \subseteq \mathcal{C}^{2c}(X, W_n\Omega^{\cdot}_X)$.

Proposition 3.5.13 *The element* $\mathrm{cl}_{X,n}(V)$ *is a* $2c$*-cocycle of the complex* $\Gamma(X, \mathcal{C}(W_n\Omega^{\cdot}_X))$ *for every* $n \geq 1$*, i.e. it induces a class in* $H^{2c}_{\mathrm{crys}}(X/W_n)$ *which coincides with the cycle class constructed in* [*Gr*].

Proof. Note first that the collection $(\mathcal{C}(W_n\Omega^{\cdot}_X))_{n\geq 1}$ forms in an evident way a pro-complex and we obviously have

$$(\mathrm{cl}_{X,n}(V))_{n\geq 1} \in \varprojlim_{n\geq 1} \Gamma(X, \mathcal{C}^{2c}(W_n\Omega^{\cdot}_X)).$$

In order to establish that $\mathrm{cl}_{X,n}(V)$ is a cocycle we have to prove the following assertions:

(a) $\mathrm{cl}_{X,n}(V)$ lies in the kernel of the morphism $H^c_\zeta(W_n\Omega^c_X) \to H^c_\zeta(W_n\Omega^{c+1}_X)$ which is given by applying $H^c_\zeta(*)$ to the differential in degree c of $W_n\Omega^{\cdot}_X$.

(b) For every $(c+1)$-codimensional specialization ξ of ζ the element $\mathrm{cl}_{X,n}(V)$ lies in the kernel of the map $H^c_\zeta(W_n\Omega^c_X) \to H^{c+1}_\xi(W_n\Omega^c_X)$ which is induced by the differential in degree c of the Cousin complex of $W_n\Omega^c_X$.

To show (a) let $\mathcal{C}_n = \mathcal{C}(W_n\Omega^{\cdot}_X)$. Then we have

$$\mathrm{cl}_{X,n}(V) \in \Gamma_{X^c}(X,\, \mathcal{C}^{2c}_n)/\Gamma_{X^{c+1}}(X,\, \mathcal{C}^{2c}_n)$$

and it is sufficient to prove that $\mathrm{cl}_{X,n}(V)$ is a cocycle of the quotient complex $\Gamma_{X^c}(X,\, \mathcal{C}_n)/\Gamma_{X^{c+1}}(X,\, \mathcal{C}_n)$. To see this let us first observe that $\theta_{\mathcal{F}_n}$ induces by functoriality of $\mathcal{C}(*)$ a chain map (which is necessarily a quasi-isomorphism)

$$\mathcal{C}(\theta_{\mathcal{F}_n})\colon \mathcal{C}(\Omega^{\cdot}_{U_n/W_n}) \to \mathcal{C}(W_n\Omega^{\cdot}_U)$$

taking into account that each $\Omega^p_{U_n/W_n}$ is Cohen-Macaulay by 3.5.2. Then $\mathcal{C}(\theta_{\mathcal{F}_n})$ maps $\mathrm{cl}_\zeta(U_n/W_n)$ to $\mathrm{cl}_{X,n}(V)$ and it clearly suffices to establish that $\mathrm{cl}_\zeta(U_n/W_n)$ is a cocycle of $\Gamma_{X^c}(X,\, \mathcal{C}(\Omega^{\cdot}_{U_n/W_n}))/\Gamma_{X^{c+1}}(X,\, \mathcal{C}(\Omega^{\cdot}_{U_n/W_n}))$ which holds by 3.5.5.
To show (b) noting that each V_n is a smooth W_n-scheme the induced map given by 3.5.8

$$1 - F\colon H^{c+1}_\xi(W_{n+1}\Omega^k_X) \to H^{c+1}_\xi(W_n\Omega^k_X)$$

has a trivial kernel for $k < c+1$ ([Il, II, 5.7.2] and [Gr, II, 3.5.8]) where 1 denotes the morphism induced by the restriction $W_{n+1}\Omega^{\cdot}_X \to W_n\Omega^{\cdot}_X$ of the De Rham-Witt complex. Hence there are no non-trivial F-invariant elements in $\varprojlim_n H^{c+1}_\xi(W_n\Omega^c_X)$ and therefore it suffices to show that $(\mathrm{cl}_{X,n}(V))_n$ is F-invariant.
Let $t_i = u_n^*(T_i) \in \Gamma(U_n,\, \mathcal{O}_{U_n})$ for $1 \le i \le c$. By (c) it follows that the collection of germs $(t_{i,\zeta})_{1\le i\le c}$ forms a system of parameters for $\mathcal{O}_{U_n,\zeta}$. By (d) we have $F_n(t_i) = t_i^p$ and thus

$$\theta^0_{\mathcal{F}_n}(t_i) = t_{\mathcal{F}_n}(t_i) = \underline{t_i}$$

by [Il, 0, 1.3.18]. Therefore we get

$$\mathrm{cl}_{X,n}(V) = d\log \underline{t_{1,\zeta}} \cdot \ldots \cdot d\log \underline{t_{c,\zeta}} \qquad (*)$$

using 2.1.7 and 3.5.10 (b) (notice that the sheaf $i_\zeta^{-1}(W_n\Omega^c_X)$ is a quasi-coherent $W_n(\mathcal{O}_{U_n,\zeta})$-module where i_ζ denotes the canonical morphism between schemes $\mathrm{Spec}(\mathcal{O}_{U_n,\zeta}) \to U_n$). An application of [3.5.8] and [Il, I, 2.17]

now shows that $(1-F)(\mathrm{cl}_{X,n}(V))=0$ which proves (b).
Using 3.5.7 we see that $\mathrm{cl}_{X,n}(V)$ induces a class in $H^{2c}_{\mathrm{crys}}(X/W_n)$. From the explicit representation $(*)$ and [Gr, II, 3.5.6, 4.1.6] we conclude that this class coincides with the cycle class of V constructed in [loc. cit.]. □

Let us finally discuss the problem whether crystalline cycle classes for not necessarily closed subschemes of a smooth S-scheme X do exist when S is a reasonable base in characteristic p, say a perfect Cohen-Macaulay scheme. Let V, c and ζ be as before. Then for a Frobenius lifting $\mathcal{F}=(\mathfrak{U},\, F)$ around ζ the element

$$\mathrm{cl}_n(V)=H^c_\zeta(\theta^c_{\mathcal{F}_n})(\mathrm{cl}_\zeta(U_n/W_n))\subseteq \mathcal{C}^{2c}(X,\, W_n\Omega^{\cdot}_X)$$

seems to be a promising candidate for giving a cocycle of $\Gamma(X,\, \mathcal{C}(W_n\Omega^{\cdot}_X))$ and thus a class in $H^{2c}_{\mathrm{crys}}(X/W_n)$. In fact, it can be shown that $\mathrm{cl}_n(V)$ satisfies condition (a) of the proof of 3.5.13 and also that $(\mathrm{cl}_n(V))_{n\geq 1}$ is F-invariant. However it is not so clear whether there are no non-trivial F-invariant elements in $\varprojlim_n H^{c+1}_\xi(W_n\Omega^c_X)$ whenever ξ is a $(c+1)$-codimensional specialization of ζ.

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