On Analysis of some Nonlinear Systems of Partial Differential Equations of Continuum Mechanics

Dissertation
zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch–Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich–Wilhelms–Universität Bonn

vorgelegt von
Mark Steinhauer
aus
Bonn

Bonn 2003

Angefertigt mit Genehmigung der
Mathematisch–Naturwissenschaftlichen Fakultät der
Rheinischen Friedrich–Wilhelms–Universität Bonn

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Tag der Promotion:

This work is dedicated to four different
persons whom I hold in very deep regard:
my grandmothers Gertrud Dieck, Gertrud Voges,
my wife Franzis and the “flea”.

Contents

Abstract

In this work we consider systems of partial differential equations of continuum mechanics and analyze properties of their weak solutions, for instance their regularity properties.

We start in chapter 2 with the (local) regularity problem related to the equations modelling the mechanical behaviour of elasto-perfect plastic materials respectively to an elasto-viscoplastic approximation of these materials, i. e. we consider the Norton-Hoff approximation to Henckys law. These equations form a nonlinear systems of partial differential equations of second order and of elliptic type in the usual primal formulation, where one is interested in the displacement vector $u = u(x)$ respectively the strain tensor $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$. We study these systems via a dual approach which was developed by A. Bensoussan & J. Frehse. In this approach we look for the stress tensor $\sigma = \sigma(x)$ which solves the system of equations:

$$\begin{aligned} A\sigma + |\sigma^D|^{p-2}\sigma^D &= \varepsilon(u) \quad \text{in } \Omega \subset \mathbb{R}^d \\ \operatorname{div}\sigma + f &= 0 \end{aligned}$$

in the weak sense with mixed boundary conditions. We show local Hölder continuity of the stress tensor in two dimensions for the Norton-Hoff approximation of the Hencky law in plasticity theory and deduce also corresponding results for the strain tensor $\varepsilon(u)$.

The main tool to achieve this result is in the static case a logarithmic Morrey estimate, which was developed by J. Frehse together with A. Bensoussan and G. Seregin in the here considered context of the dual theory of elliptic systems. These logarithmic Morrey estimates combined with a suitable adapted estimate on higher integrability a la Meyers-Nečas-Gehring-Giaquinta-Modica give the final result.

We also deal with a system of partial differential equations describing a steady motion of an incompressible fluid with shear-dependent viscosity and

present a new global existence result for $p > \frac{2d}{d+2}$. Here p is the coercivity parameter of the nonlinear elliptic operator related to the stress tensor and d is the dimension of the space. Lipschitz test functions, a subtle splitting of the level sets of the maximal functions for the velocity gradients, and a decomposition of the pressure are incorporated to obtain almost everywhere convergence of the velocity gradients.

Finally we survey and improve some results concering uniqueness and regularity of solutions to the instationary Navier-Stokes equations in three (and higher) dimensions. In particular we shall show that the class of weak solutions which additionally belong to the space $L^2(0,T;BMO)$ guarantees uniqueness as well as regularity of the solution under consideration. We also discuss the related issue of controlling the blow-up phenomenon of smooth solutions to the Navier-Stokes equations. The method of proof which we present is elementary and depends deeply on the special structure of the nonlinear convective term $u \cdot \nabla u$ of the Navier-Stokes equations together with $\operatorname{div} u = 0$; namely the convective term is a "div-curl expression and according to Coifman, Lions, Meyer & Semmes it belongs to the Hardy space $\mathcal{H}^1$. This also shows that it is applicable to other equations in hydrodynamics as for example the Boussinesq equations, the equations of Magneto-Hydrodynamics and the equations of higher grade type fluids.

Chapter 1

Introduction

The subject of this work are local and/or global regularity properties of weak solutions of some elliptic respectively parabolic systems of partial differential equations from continuum mechanics.

In chapter 2 we consider the Norton-Hoff approximation of Hencky's law for an elasto-perfect plastic material in dual formulation:

$$A\sigma + |\sigma^D|^{p-2}\sigma^D = \varepsilon(u) \quad \text{in } \Omega\,, \tag{1}$$

$$\operatorname{div}\sigma + f = 0 \quad \text{in } \Omega \tag{2}$$

completed by boundary conditions of Neumann type for σ and of Dirichlet type for u. Here we already have used the following notation:

- $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$ in physical relevant situations) is a bounded domain with Lipschitz boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$ and $\mathcal{H}^{d-1}(\Gamma_D) > 0$;

- $\sigma = (\sigma_{ij})$ denotes the (symmetric) stress tensor: $\sigma_{ij} = \sigma_{ij}$;

- $\sigma^D := \sigma - \frac{1}{d}(\operatorname{trace}\sigma)Id$ is the deviator of σ, i. e. the trace free part of σ;

- $A = \big(A_{ij}^{hk}\big) \in L^\infty\big(\Omega; \mathbb{R}^{d^2\times d^2}\big)$ denotes the material depending elasticity tensor, for example the classical Lamé-Navier operator:

  $$(A\sigma)_{ij} := \frac{1}{2\mu}\sigma_{ij} - \frac{\lambda}{2\mu(2\mu+3\lambda)}(\operatorname{trace}\sigma)\delta_{ij}$$

  with given Lamé constants λ and μ;

- $\varepsilon(u) := \frac{1}{2}\big(\nabla u + (\nabla u)^T\big)$ is the linearized strain tensor and
- f stands for some given vector valued function representing some given volume force.

Equation (1) is the Euler-Lagrange equation of a variational problem where we minimize over a class of functions specified by equation (2) and the boundary conditions of the problem. So, the (unique) solution can be found by minimizing the corresponding variational integral which has p-growth as well as quadratic growth with respect to $|\sigma^D|$ and only quadratic growth with respect to trace $\sigma = \operatorname{tr}\sigma$ (for example in the case of the Lamé-Navier operator).

The just roughly described variational problem is the dual variational problem of the so called primal variational problem in which we look for a solution u respectively $\varepsilon(u)$ minimizing a corresponding variational integral instead of σ. Very roughly (and more incorrect than correct) we can view this as follows: Writing equation (1) in the form

$$\varepsilon(u) = \psi(\sigma)$$

and inverting the mapping ψ (provided that this is possible) we have:

$$\sigma = \psi^{-1}\big(\varepsilon(u)\big)$$

and putting this into equation (2) we obtain:

$$-\operatorname{div}\psi^{-1}\big(\varepsilon(u)\big) = f$$

which can (eventually) be recovered as Euler-Lagrange equation of a variational integral. How this is done precisely the reader can find in R. Temam [Tem85] or in [FS00] by using the Legendre transform instead of the mapping ψ, ψ^{-1}.

Even if the just described procedure is not completely correct we see that the primal formulation

$$-\operatorname{div}\psi^{-1}\big(\varepsilon(u)\big) = f$$

or the corresponding variational integral represents a nonlinear, “degenerate” elliptic system of second order with “anisotropic” growth condition for $|\varepsilon^D(u)|$:

- one ”part” has $\frac{p}{p-1}$-growth;

- the other one only **linear** growth.

In the limit case $p \uparrow +\infty$ which corresponds to the Hencky model we have to deal in the primal formulation with a variational problem with linear growth explaining partly the difficulty of the problem, because usually the underlying function space for variational problems with linear growth is not reflexive for example $W^{1,1}$, BV or here the space BD (see chapter 1). Another aspect of the problem which is more related to regularity is the fact that the elliptic system in the primal formulation is not of Uhlenbeck-Uralceva type, which means that the coupling (dependence on) in the gradient of the solution of the system is not given as a function of the modulus of the gradient, but only as a function of the modulus of $\varepsilon^D(u)$, i. e. of the deviator of $\varepsilon(u)$. This "explains" somehow that it will be not "easy" to obtain full regularity because up to now systems of Uhlenbeck-Uralceva type constitute the only "class" of systems for which full regularity of their solutions is known (see for example [Gia83] and [DiB93]). Moreover the different, anisotropic growth conditions for $|\varepsilon^D(u)|$ respectively σ^D make the problem even more complicated.

The first step uses as an essential tool a refined (inhomogeneous) hole-filling method developed by J. Frehse (and G. Seregin in the special form which we are going to use). The second step follows via more standard tools from the regularity theory of elliptic systems: Either by using the classical hole-filling method or by using the technique of reverse Hölder inequalities to obtain higher integrability of $\nabla\sigma$, i. e. $\nabla\sigma \in L^q_{\text{loc}}(\Omega)$ for some $q > 2 = d$. Since we believe that this inhomogenous hole-filling technique is well suited to treat regularity problems in limit cases where the integrability exponent is closed to the dimension d on one hand and on the other hand it seems to be very flexible with respect to different growth conditions, we formulate here a version of it:

Proposition 1-I *([SF99]) Suppose that $\Omega \subset \mathbb{R}^2$ is an open and bounded domain and that two functions $H \in L^2(\Omega)$ and $h \in W^{1,2}(\Omega)$ satisfy the estimate*

$$\int\limits_{B_R(x_0)} H^2(x)\,dx \leq C_1 \left\{ \int\limits_{T_R(x_0)} H^2(x)\,dx + R^\alpha \right\}^{1/2}$$
$$\frac{1}{R} \int\limits_{T_R(x_0)} |hH|\,dx\,.$$

for some positive α, any $x_0 \in \Omega$ and any $0 < R < R_0$ such that the ball $B_{2R}(x_0) \subset \Omega$. Here $T_R(x_0)$ denotes the annulus $B_{2R}(x_0) - B_R(x_0)$. Then, for any $q \geq 1$, there exists a positive constant C_2, depending on q, C_1, α, R_0, $\|h|W^{1,2}(\Omega)\|$ and $\|H|L^2(\Omega)\|$ such that

$$\int\limits_{B_R(x_0)} H^2(x)\,dx \leq \frac{C_2}{\left(\log_2 \frac{2R_0}{R}\right)^q}\,.$$

Let us remark right a way that a corresponding statement is true for $\Omega \subset \mathbb{R}^d$, $H \in L^{d/d-1}(\Omega)$ and $h \in W^{1,d}(\Omega)$ replacing the exponent 2 by $d/d-1$ for H and by d for h. Also $\frac{1}{R}$ has to be replaced by $\frac{1}{R^{d-1}}$.

In applications the function H will be usually "given" by the problem under consideration. The "flexibility" of the method, i. e. of the proposition, consists of a proper choice of the function h which one has to adapt to the problem one would like to deal with.

We also review some further regularity results for equations (1), (2) in chapter 2. Especially we discuss what is known concerning regularity up to the boundary.

It is well known that the unique solutions $(\sigma, u) = (\sigma^p, u^p)$ of equations (1), (2) converge for $p \uparrow +\infty$ to a solution $(\tilde{\sigma}, \tilde{u})$ of a limit problem which is the Hencky model, where the stresses $\tilde{\sigma}$ are unique (for $\tilde{u}$ uniqueness is not known) and satisfy the von Mises condition $|\sigma^D| \leq 1$.

Engineers expect continuity of stresses for both the approximation σ^p as well as their limit and they use it also in their numerical calculations. In contrast to this it is "known" that the displacement $\tilde{u}$ in the limit problem possesses discontinuities, which is another reason to work in the limit problem with the space $BD(\Omega)$. We refer to [Pan85, chap. 9] for more details. This fact also shows that it is more convenient to work with the dual variational problem and the stresses instead of the primal variational problem and the displacements.

As a first step towards regularity G. Seregin [Ser87] using an approximation of the primal variational problem and A. Bensoussan and J. Frehse [BF93] using the dual approach were able to establish $H^{1,2}_{\text{loc}}$-regularity for the approximation σ^p (p fixed) as well as for the limit as p tends to infinity. For this purpose they prove that the L^2-norm of $\nabla\sigma^p$ stays locally in Ω uniformly bounded as $p \uparrow +\infty$.

The main result of chapter 2 is concerned with the continuity of stresses and it reads:

Theorem 1-II *In space dimension $d = 2$ and for arbitrary, but fixed $p : 2 < p < +\infty$ the stress tensor $\sigma = \sigma^p$ is locally in Ω Hölder continuous:*

$$\sigma \in C^{0,\alpha}_{\text{loc}}(\Omega)$$

for some positive α. As a consequence of this result and equation (1) we also have:

$$\begin{aligned} \varepsilon(u) &\in C^{0,\beta}_{\text{loc}}(\Omega) \qquad \textit{and} \\ u &\in C^{1,\beta}_{\text{loc}}(\Omega) \end{aligned}$$

for some $\beta > 0$.

For the proof of the theorem we shall verify a **logarithmic** Morrey-condition for the quantity

$$H^2 := \nabla\varepsilon(u) : \nabla\sigma = D_k\varepsilon_{ij}(u)D_k\sigma_{ij}\,,$$

which estimates $|\nabla\sigma|^2$ and gives us therefore also a log-type Morrey condition for $\nabla\sigma$. This yields in a first step continuity and boundedness of σ and $\varepsilon(u)$ and in a second step by using these new information we obtain the Hölder continuity.

In chapter 3 we consider in an open bounded set Ω in $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$ the following system of partial differential equations:

$$-\text{div}\, T + \text{div}\,(v \otimes v) + \nabla P = f\,, \tag{1.1}$$

$$\text{div}\, v = 0\,, \tag{1.2}$$

$$v = 0 \text{ on } \partial\Omega\,. \tag{1.3}$$

Here $v = v(x) = \big(v^1(x)\,, \dots, v^d(x)\big)$ and $P = P(x)$ denote the unknown velocity and pressure fields at the point $x \in \Omega$, while $f = \big(f^1(x)\,, \dots, f^d(x)\big)$ is a given external force and T is the extra stress for which one has to specify the constitutive equation.

The system (1.1)–(1.3) describes steady motions of incompressible fluids. A typical example of considered models that are widely used in engineering practise is given by

$$T\big(D(v)\big) \equiv \nu_0|D(v)|^{p-2}D(v)\,, \tag{1.4}$$

where $D(v)$ denotes the symmetric part of the velocity gradient ∇v and $p \geq 1$. We, here, confine ourselves to a discussion of this typical example and refer the reader to chapter 3 for more examples, a more detailed discussion of the continuum mechanical background and more general assumptions on the extra stress sensor which are sufficient to obtain the desired existence result.

Our aim will be to prove the following existence result:

Theorem 1-III *Let $p > \frac{2d}{d+2}$ and assume $f \in W^{-1,p'}(\Omega)$. Then there exists a weak solution $v \in V_p \equiv \left\{ v \in \mathring{W}^{1,p}(\Omega) : \operatorname{div} v = 0 \right\}$ to (1.1)–(1.4) in the sense that*

$$\int_\Omega T\big(D(v)\big) : D(\Phi)\, dx = \langle f, \Phi \rangle_{1,p} + \int_\Omega (v \otimes v) : D(\Phi)\, dx \qquad \forall \Phi \in C^\infty_{0,\sigma}(\Omega)\,.$$

The notation we use here, will be explained in detail in chapter 3, section 2.

To prove existence of a weak solution to (1.1)–(1.3) two different methods were applied up to date. The first method is a combination of weak compactness for v respectively the Galerkin approximation v_m to v and monotonicity arguments (Minty's trick). This works for $p \geq \frac{3d}{d+2}$ and the reason for this comes from the convective term $v \cdot \nabla v \equiv \operatorname{div}(v \otimes v)$, which should be an element of such a Lebesgue space L^q that testing by some $\Phi \in V_p$ provides an L^1-function. Using Sobolev's inequality and Hölder's inequality one calculates easily the bound $p \geq \frac{3d}{d+2}$. This method was performed by J. L. Lions [Lio69] and O. A. Ladyzhenskaya [Lad67], [Lad68], [Lad69] in the late sixties.

The second method, which we call L^∞-truncation method, yields existence of a weak solution for $p \geq \frac{2d}{d+1}$. In this case the convective term is at least a L^1-function, explaining that the method is based on the construction of a special bounded test function by truncation. Since the truncation process destroys the solenoidal character of the test function one also needs a "good" characterisation of the pressure. The method also relies strongly on the strict monotonicity of T. This latter method was successfully applied to the steady problem in [FMS97] and [Růž97] (in [Růž97] the limiting case $p = \frac{2d}{d+1}$ is not included).

In chapter 3 we introduce yet another approach, which we would like to call Lipschitz truncation method, in order to prove the above stated theorem. For this purpose we construct a Lipschitz test function to show that for certain approximations v^n the tensors $D(v^n)$ converge almost everywhere to its weak limit $D(v)$, which is the crucial point in proving that v is a weak solution to (1.1)–(1.3). Because of earlier results we can restrict our consideration to the case $p \in \left(\frac{2d}{d+2}\, ,\, \frac{2d}{d+1} \right)$. Note that Lipschitz truncations of Sobolev functions were already successfully used in different context, see the references to chapter 3.

The novelty of our application of the Lipschitz approximation of Sobolev functions consists of the discovery of the mechanism of obtaining almost everywhere convergence of gradients for weakly convergent sequences. To discover this mechanism we have to refine substantially the properties of the Lipschitz approximation procedure (see Proposition 3-VII and 3-IX in chapter 3, section 5) and due to the fact that truncation destroys the solenoidal character we also need very precise control of the presssure P. This precise control is achieved in section 4 of chapter 3 by decomposing the pressure P into four parts P^{1_k}, P^{2_k}, P^{3_k} and P^{4_k} which are related to the nonlinear parts of the approximating system to (1.1)–(1.3) (see section 4, chapter 3) by four auxiliary Stokes problems. This leads finally to a (special) weak formulation of the approximating system for which we can use the Lipschitz truncation $(v^k - v)_\lambda$ of $(v^k - v)$ as test function and we obtain finally almost everywhere convergence of $D(v^k)$ to $D(v)$ and even stronger convergence properties (see chapter 3).

Lastly we mention that corresponding existence results (of weak solutions) for the time-dependent system are not known (especially in the case of the Dirichlet boundary condition $v = 0$ on $(0,T)\times\partial\Omega$) (see the cited references to chapter 3). However, we wish to emphasize, that we believe that a convenient probably not straightforward modification of the presented techniques can improve also the existence results for the evolutionary model.

The last chapter, chapter 4, is concerned with the initial value problem for the Navier-Stokes equation in $(0,T)\times\mathbb{R}^n$ with $0 < T < +\infty$ and $n \geq 3$:

$$\partial_t u^i - \Delta u^i + u^j D_j u^i + D_i \pi = f^i \quad \text{in } (0,T)\times\mathbb{R}^n, \tag{1.5}$$

$$\operatorname{div} u = 0 \quad \text{in } (0,T)\times\mathbb{R}^n, \tag{1.6}$$

$$u(0,x) = a(x)\,, \tag{1.7}$$

where $u = u(t,x) = \big(u^1(t,x),\dots,u^n(t,x)\big)$ and $\pi = \pi(t,x)$ denote the unknown velocity vector and pressure of the fluid at the point $(t,x) \in (0,T)\times\mathbb{R}^n$, while $a = a(x) = \big(a^1(x),\dots,a^n(x)\big)$ is the given initial velocity and $f = f(t,x) = \big(f^1(t,x),\dots,f^n(t,x)\big)$ is a given external force.

We are interested in the classical problem of finding sufficient conditions for weak solutions of (1.5)–(1.7) such that they become unique and/or regular. In section 1 of chapter 4 we introduce the classical Prodi-Serrin condition $PS(\alpha,\beta) \equiv \dfrac{2}{\alpha} + \dfrac{\beta}{n} = 1$, because it ensures the uniqueness and also the regularity of solutions besides the limit case $L^\infty(0,T;L^n)$ for which regularity is still not known. Then we try to give an up to date survey of what is known to these two problems and discuss "different" contributions and methods of proof to this circle of questions. Due to the fact that this field of research is presently "incredibly alive" (cited from the recent thesis of L. Berselli from the university of Pisa) and active and due to the limited knowledge of the author - as well as some personal taste - this survey is certainly not complete, probably even not comprehensive, but we try to give a somehow representative overview.

In section 2 we introduce the function spaces BMO of bounded mean oscillation and the Morrey space $L^{2,n-2}$ - besides some other standard spaces in this context - , recall some of their properties and introduce the notion of weak solution before stating our results.

The uniqueness theorem we shall prove in section 3 of chapter 4 reads as follows:

Theorem 1-IV *There exists at most one solution of (1.5)–(1.7) in the sense of Definition 4-I (see chapter 4) such that the solution u belongs to the class*

(i) $\quad u \in L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;\mathring{W}^{1,2}_\sigma) \cap L^2(0,T;BMO)$ *or*

(ii) $\quad u \in L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;\mathring{W}^{1,2}_\sigma)$ *and* $\nabla u \in L^2(0,T;L^{2,n-2})$.

Such a solution would be continuous from $[0,T]$ into L^2_σ and the usual energy inequality would turn into an identity.

The uniqueness classes specified in the theorem weaken respectively sharpen the assumption concerning the regularity with respect to the space variables of the following "known" uniqueness classes:

(i)) $\quad u \in L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;\mathring{W}^{1,2}_\sigma) \cap L^2(0,T;L^\infty)$,

(ii)) $\quad u \in L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;\mathring{W}^{1,2}_\sigma) \cap L^2(0,T;W^{1,n})$.

To illustrate how "large" our realized profit is we recall:

i) $L^\infty \subset BMO \subset L^p$ for any $p < +\infty$;

ii) $\log|x|$, $\log|P(x)| \in BMO$, P some homogeneous polynomial, so $L^\infty \subsetneqq BMO$;

iii) $\nabla u \in L^n \Longrightarrow \nabla u \in L^{2,n-2} \Longrightarrow u \in BMO$.

Concerning regularity we prove there the following Theorem:

Theorem 1-V *Suppose $a \in W^{1,2}_{\sigma}$ and u is a weak solution of (1.5)–(1.7) in the sense of Definition 4-I in $[0,T)$. If u satisfies*

i) $\nabla u \in L^2(0,T;L^{2,n-2})$ *or*

ii) $u \in L^2(0,T;BMO)$ *or*

iii) $\nabla^2 u \in L^1(0,T;L^{2,n-2})$ *or*

iv) $\nabla u \in L^1(0,T;BMO)$ *, then we have*

$u \in C(0,T;\mathring{W}^{1,2}_{\sigma} \cap L^2(0,T;W^{2,2}_{\sigma})$ and a corresponding estimate holds true. In particular u is a regular and unique solution in $[0,T]$.

The proofs of the theorems are based either on a Sobolev inequality for divergence free maps (in case of $\nabla u \in L^{2,n-2}$) which was proven by S. Chanillo in 1991 and was used by L. S. Evans to establish partial regularity of weakly harmonic stationary maps which are valued in spheres $S^{m-1} \subset \mathbb{R}^m$ or on a result from compensated integrability/compactness of Coifman, Lions, Meyer and Semmes telling us that the convective term $u^j D_j u^i$ belongs to $L^2(0,T;\mathcal{H}^1)$, where $\mathcal{H}^1 \subset L^1$ denotes the Hardy space, and the famous duality theorem of C. Fefferman asserting $(\mathcal{H}^1)^* = BMO$. After a discussion of the necessary tools – even with proof of what we think are the most important "points" – we provide the proofs of the theorems and discuss some slight generalisations respectively related versions of our theorems.

Let us remark that the results of chapter 4 which are described above were obtained by the author in 1996/97 and he gave a talk on this topic in spring 1997 in Lisbon on an international conference in honour of the seventieth birthday of Prof. Dr. J. Nečas. In the mean time appeared the paper "Bilinear Estimates in BMO and the Navier-Stokes Equations" of H. Kozono and Y. Taniuchi (Math. Zeitschr. 235 (1): 173–194, 2000) which contains more or less completely corresponding results, but using the machinery of "bilinear analysis" developed by Coifman and Meyer (see their book [MC97]), in their proofs. So we would like to point out that the results (and more important the proofs of them) are obtained really independently of the paper of Kozono and Taniuchi and we believe that the proofs we provide for them are "somehow" more elementary, but of course this is a question of personal taste.

Finally in section 4 of chapter 4 we try to argue in favour of regularity in the limit case $u \in L^\infty(0,T;L^n)$. We are not able to prove regularity in this case, but we can somehow unify some known results and give some supplements to what is known. For example we prove the following proposition:

Proposition 1-VI *If $v \in L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;W^{1,2}_\sigma) \cap L^\infty(0,T;L^n)$ is a weak solution of (1.5)–(1.7) under the assumption $a \in L^2_\sigma \cap L^n$, then there exists $\varepsilon = \varepsilon(\|v|L^\infty(0,T;L^n)\|) > 0$, such that $v \in L^\infty(0,T;L^{2+2\varepsilon}_\sigma)$ and $\nabla \frac{|v|^{1+\varepsilon}}{1+\varepsilon} \in L^2(0,T;L^2)$ with corresponding estimate.*

Using the result of the proposition together with some interpolation inequalities one can improve the known regularity properties for $\partial_t v$, $\nabla^2 v$ and $\nabla\pi$ via Solonnikov's estimates for the time-dependent Stokes equations. This and related remarks are also discussed in section 4, chapter 4.

Acknowledgements

First and foremost I would like to thank my advisor Prof. Dr. Jens Frehse for his unconditioned support and long-termed patience during the last years. I could always count on him and enjoyed discussing mathematical problems related to this work as well as discussing a lot of other mathematical subjects.

I am also very much obliged to Josef Málek not only for many fruitful and stimulating discussions on subjects related to this work, but also for collaboration and cooperation as well as (moral) support and friendship.

A large part of this thesis has been written at the Mathematical Seminar of the Agricultural Faculty of the University of Bonn headed by Prof. Dr. H. P. Helfrich. I wish to thank him for his continuous support as well as for the pleasant working atmosphere I found and still find there.

Of course I am also grateful to many of my colleagues at the Mathematical Seminar and the Department of Applied Analysis for moral support and help in several practicalities. Especially I wish to thank Carsten Ebmeyer, Moritz Kassmann and Gudrun Thäter for fruitful discussions and help in different situations and ways.

Last but not least I want to express my gratitude to Mrs. Inge Schmelzer for her excellent typing of my "sometimes" chaotic and horrible (concerning formulas) manuscript. I am also very grateful to Sonja Goj for reading several previous versions of the manuscript. The time she invested has certainly improved this final version. For help and advice concerning several problems with LaTeX and "the computer" I thank Claus Scholl, Moritz Kassmann and Alexander Steffens.

Still I would like to thank my wife Franzis for all her patience and support not only during the preparation of this work and I would like to thank everybody who supported me in one or the other way while preparing this work.

Finally I want to acknowledge the Department of Applied Analysis of the Institute of Applied Mathematics of the University of Bonn and the Math-

ematical Seminar of the Agricultural Faculty of the University of Bonn for providing me all the facilities for investigation and the SFB 256 "Nonlinear Partial Differential Equations" for financial support.

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Chapter 2

On Hölder-Continuity of Stresses for Two-Dimensional Problems of Plasticity

2.1 The Equations of Plasticity

In this section we recall the Hencky-law of plasticity (see [Hen24]) which describes the plastic behaviour of solids in the static case. We mainly follow in this introductory part the work of R. Temam [Tem86, Tem85].

2.1.1 The General Equations of Plasticity.

A solid body occupies at rest a region Ω of $\mathbb{R}^d$ ($d = 2, 3$ for the physical relevant situation) with boundary $\Gamma = \partial\Omega$. This solid undergoes deformations under the action of volume forces of density f inside Ω and surface forces of density g on some part Γ_N of Γ; f and g depend on $x = (x_1, \ldots, x_d) \in \Omega$ for$f$ and $x \in \Gamma_N$ for g.

The state of the deformed material is described by the tensor field $\sigma = \sigma(x)$ and the vector field $u = u(x)$. The tensor σ is the Cauchy stress tensor at point $x \in \Omega$, while u is the displacement of the material particle which is at point x when the body is at rest. Under the assumption of small displacements, the motion of the body is governed by the classical equation

$$\rho\ \partial_t^2 u = f + \operatorname{div}\ \sigma, \quad x \in \Omega,\, t > 0, \tag{2.1}$$

where ρ is the density of the solid and div σ is the vector with components

$$\text{div } \sigma := D_j\sigma_{ij} := \sum_{j=1}^{d} D_j\sigma_{ij} = \sum_{j=1}^{d} \frac{\partial \sigma_{ij}}{\partial x_j}, \quad i = 1, 2, \ldots, d; \tag{2.2}$$

σ_{ij} denoting the components of σ. Here we already used the Einstein index summation convention and $D_j\phi = \phi_{,j} = \frac{\partial \phi}{\partial x_j}$. Partial differentiation with respect to t is denoted by $\partial_t = \frac{\partial}{\partial t}$ or a dot, so that $\ddot{u} = \partial_t^2 u = \frac{\partial^2 u}{\partial t^2}$ for example.

As a consequence of the "principle of conservation of momentum" one obtains in particular the symmetry of the Cauchy stress tensor:

$$\sigma_{ij}(x) = \sigma_{ji}(x), \quad x \in \Omega. \tag{2.3}$$

In the static case, the evolution is slow, the acceleration term $\rho\ \ddot{u} = \rho\ \partial_t^2 u$ can be neglected and (2.1) is replaced by

$$\text{div } \sigma + f = 0. \tag{2.4}$$

Several other equations are necessary to describe the deformation of the body. First the continuity of forces on Γ_N, which implies

$$\sigma \cdot \nu := \sigma_{ij}\nu_j = g^i := g, \quad x \in \Gamma_N, \tag{2.5}$$

where $\nu = (\nu_1, \ldots, \nu_d)$ is the unit outward normal on Γ and $i = 1, \ldots, d$.

Then the displacement u is given on the complement Γ_D of Γ_N on Γ:

$$u = U, \quad x \in \Gamma_D. \tag{2.6}$$

2.1.2 The Constitutive Law

The general equations above are completed by the constitutive relation connecting the stresses σ to the strains $\varepsilon = \varepsilon(u)$,

$$\varepsilon_{ij}(u) = \frac{1}{2}(D_i u^j + D_j u^i)\ , \tag{2.7}$$

where we supposed that the deformation of the body – which in general can be described by the nonlinear deformation tensor

$$\mathbb{E}(u) = \frac{1}{2}(\nabla u + (\nabla u)^T + (\nabla u)^T \nabla u) \tag{2.8}$$

– is "small", so that we can neglect the quadratic term $(\nabla u)^T \nabla u$ and regard $\mathbb{E}(u)$ as the "linearized strain/ deformation tensor":

$$\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) \ . \tag{2.9}$$

Let $\mathbb{S}$ denote the space of all symmetric tensors of order 2 and let $\mathbb{S}^D$ be the subspace of symmetric tensors with vanishing trace. If $\xi = (\xi_{ij})$ belongs to $\mathbb{S}$ then its deviatoric $\xi^D \in \mathbb{S}^D$ is $\xi - \frac{1}{d}(\text{trace } \xi)I$ or in components

$$\xi_{ij}^D = \xi_{ij} - \frac{1}{d}\, \xi_{ll}\, \delta_{ij} \ . \tag{2.10}$$

If $\xi, \eta \in \mathbb{S}$, their scalar product in $\mathbb{S}$ is denoted

$$\xi : \eta := \xi_{ij}\eta_{ij} := \sum_{i,j=1}^{d} \xi_{ij}\eta_{ij} \tag{2.11}$$

and

$$|\xi| := \{\xi : \xi\}^{1/2} \tag{2.12}$$

denotes the Euclidean norm of $\xi \in \mathbb{S}$. Hence,

$$\xi : \eta := \xi^D : \eta^D + \frac{1}{d}\ (\text{trace } \xi)\ (\text{trace } \eta) \ . \tag{2.13}$$

In classical linear elasticity the relation between stresses and strains is linear at every point $x \in \Omega$:

$$\sigma_{ij} = 2\mu\varepsilon_{ij}(u) + \lambda\ \varepsilon_{kk}(u)\ \delta_{ij} \tag{2.14}$$

or

$$\begin{aligned} \sigma_{ij}^D &= 2\mu\varepsilon_{ij}^D \ , \\ \sigma_{kk} &= (2\mu + 3\lambda)\ \varepsilon_{kk}(u) = (2\mu + 3\lambda)\ \text{div}\ u \ , \end{aligned} \tag{2.15}$$

where λ, μ are the Lamé coefficients. Inverting these relations we can write

$$\varepsilon(u) = A\sigma \ , \tag{2.16}$$

where A is the invertible positive definite operator in $\mathbb{S}$ defined as

$$(2.17) \qquad (A\xi)_{ij} := \frac{1}{2\mu}\,\xi_{ij} - \frac{\lambda}{2\mu(2\mu+3\lambda)}\,\xi_{kk}\,\delta_{ij}\ , \quad \forall \xi \in \mathbb{S}\ .$$

More general A can be any positive definite operator from $\mathbb{S}$ into $\mathbb{S}$ which possibly can also depend on x meaning that the considered body is inhomogeneous.

For the Hencky-law we are given a convex set $K \subset \mathbb{S}$ in which the stress tensor must remain; K will be of the form

$$K^D \bigoplus \mathbb{R}\, I\ ,$$

where
(2.18)
$K^D \subset \mathbb{S}^D$ is a closed convex bounded set containing 0 in its interior.

For example K^D can be defined as

$$(2.19) \qquad K^D := \{\xi^D \in \mathbb{S}^D : \mathcal{F}(\xi^D) \le 0\},$$

where $\mathcal{F}$ – the so called flow function – is a given continuous and convex function. We will here deal with the special function

$$(2.20) \qquad \mathcal{F}_1(\xi^D) := \frac{1}{2}|\xi^D|^2 - k_*^2\ ,$$

which was introduced by R. von Mises to approximate the flow rule of H. Tresca (see for example [DL76] or [Zei88]).

For plastic behaviour it is first assumed that

$$(2.21) \qquad \sigma(x) \in K \quad \forall x$$

while the linear relation 2.16 between ε and σ is no longer valid, and instead we write

$$(2.22) \qquad \varepsilon_{ij}(u) = A_{ij}^{hk}\sigma_{hk} + \lambda_{ij}\ .$$

Then the Hencky-law give a set of conditions satisfied by λ when $\lambda \neq 0$, which according to J. J. Moreau [Mor68] are equivalent to

$$(2.23) \qquad \lambda_{ij}(\tau_{ij} - \sigma_{ij}) \le 0 \quad \forall \tau = (\tau_{ij}) \in K,$$

where (2.23) holds at every point $x \in \Omega$.

The relations (2.22) and (2.23) can also be written in alternate forms which are sometimes convenient. For instance if χ_K denotes the indicator function of the convex set K in $\mathbb{S}$, i. e.

$$\chi_K(\xi) := \begin{cases} 0 & \text{if} \quad \xi \in K \\ +\infty & \text{if} \quad \xi \in \mathbb{S} \setminus K \end{cases} \tag{2.24}$$

and $\partial\chi_K$ its subdifferential, cf. [ET99], then the material law (2.23) reads

$$\lambda = \varepsilon(u) - A\sigma \in \partial\chi_K \ . \tag{2.25}$$

Also, if $\tau \in K$, we denote by $C_K(\tau)$ the cone

$$C_K(\tau) := \{\xi \in \mathbb{S} : \xi = r(\eta - \tau), \eta \in K, r > 0\} \ .$$

If $\tau \in \overset{o}{K}$ (= the interior of K), then $C_K(\tau) = \mathbb{S}$, while if $\tau \in \partial K$, $C_K(\tau)$ is the cone of tangents to K at point τ. It is easy to see that $\partial\chi_K(\tau)$ is nothing else than the polar cone $C_K^0(\tau)$ of $C_K(\tau)$ and thus (2.25) is equivalent to the normality law:

$$\varepsilon(u) - A\sigma \in C_K^0(\tau) \ . \tag{2.26}$$

2.1.3 Notation and Function Spaces

We assume that Ω is an open connected, bounded subset of $\mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz boundary $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, $\Gamma_D \cap \Gamma_N = \emptyset$, $\mathcal{H}^{d-1}(\Gamma_D) > 0$, where $\mathcal{H}^{d-1}$ denotes the $(d-1)$–dimensional Hausdorff measure.
By ν we denote the unit outer normal to $\partial\Omega$.
We will use the following notation and function spaces:

$L^2(\Omega)$ = the space of L^2-functions from Ω to $\mathbb{R}$;

$L^2(\Omega; \mathbb{R}^d) = L^2(\Omega)^d$;

$L^2(\Omega; \mathbb{S})$ = the space of L^2-functions from Ω to $\mathbb{S}$;

$H^1(\Omega) = W^{1,2}(\Omega) := \{u \in L^2(\Omega) : D_i u \in L^2(\Omega) \ , \ i = 1, \ldots, d\}$
= Sobolev space of order 1 with exponent $p = 2$;

$H^1(\Omega; \mathbb{R}^d) := H^1(\Omega)^d$.

The scalar product and the norm on either $L^2(\Omega)$, $L^2(\Omega;\mathbb{R}^d)$ or $L^2(\Omega;\mathbb{S})$ are written $(\cdot,\cdot)$ and $|\cdot|$. The scalar product and the norm on $H^1(\Omega)$ or $H^1(\Omega;\mathbb{R})$ are written $((\cdot,\cdot))$ and $||\cdot||$.

Finally we denote by $L^p(\Omega)$ (respectively $L^p(\Omega;\mathbb{R}^d)$, $L^p(\Omega;\mathbb{S})$), $1 \leq p \leq +\infty$, the usual Lebesgue spaces of L^p-functions from Ω into $\mathbb{R}$ (respectively $\mathbb{R}^d, \mathbb{S}$), which are Banach spaces when endowed with their natural norm.

The space of real L^2-functions on $\Gamma = \partial\Omega$ for the $(d-1)$–dimensional Hausdorff measure $\mathcal{H}^{d-1}$ is denoted by $L^2(\Gamma) = L^2(\partial\Omega)$ and $L^2(\Gamma;\mathbb{R}^d) = L^2(\Gamma)^d$; $H^{\frac{1}{2}}(\Gamma) \subset L^2(\Gamma)$ is the space of traces on Γ of functions u in $H^1(\Omega)$, and $H^{-\frac{1}{2}}(\Gamma)$ the dual of $H^{\frac{1}{2}}(\Gamma)$ (cf. J. L. Lions & E. Magenes [LM72] or J. Nečas [Neč67]).

In order to relate surface forces on the boundary $\partial\Omega$ to the stress tensor we call the readers attention to the following result of R. Temam for domains with boundary of class C^1.

Lemma 2-I *If $\tau \in L^2(\Omega;\mathbb{S})$ and $\operatorname{div}\tau = (D_j\tau_{ij}) \in L^2(\Omega;\mathbb{R}^d)$, we can define the trace of $\tau\nu = (\tau_{ij}\nu_j)$ on $\partial\Omega$ as an element of $H^{-\frac{1}{2}}(\Gamma;\mathbb{R}^d)$, and obtain the following estimate with a constant C_1 depending only on Ω:*

$$||\tau\nu||_{H^{-\frac{1}{2}}(\Gamma;\mathbb{R}^d)} \leq C_1 \ \{|\tau|_{L^2(\Omega;\mathbb{S})} + |\operatorname{div}\tau|_{L^2(\Omega;\mathbb{R}^d)}\} , \tag{2.27}$$

for all $\tau \in L^2(\Omega;\mathbb{S})$, $\operatorname{div}\tau \in L^2(\Omega;\mathbb{R}^d)$. Furthermore, we have the generalized Green formula:

$$\begin{aligned} \int_\Omega \tau : \varepsilon(v)\,dx &= \int_\Omega \tau_{ij} : \varepsilon_{ij}(v)\,dx = \int_\Omega \tau_{ij} : D_j v^i\,dx \\ &= \int_\Gamma \tau_{ij} v^i \nu^j \,d\mathcal{H}^{d-1} - \int_\Omega D_j\tau_{ij} v^i\,dx \\ &= \langle \tau\nu, v\rangle_{H^{-\frac{1}{2}}(\Gamma)^d, H^{\frac{1}{2}}(\Gamma)^d} - \int_\Omega v\cdot\operatorname{div}\tau\,dx \end{aligned} \tag{2.28}$$

valid for every $v \in H^1(\Omega;\mathbb{R}^d)$ and $\tau \in L^2(\Omega;\mathbb{S})$ with $\operatorname{div}\tau \in L^2(\Omega;\mathbb{R}^d)$. Here the first term on the right hand side denotes the duality pairing between the spaces $H^{-\frac{1}{2}}(\Gamma;\mathbb{R}^d)$ and $H^{\frac{1}{2}}(\Gamma;\mathbb{R}^d)$.

For a proof of this result we refer to [Tem86, Tem85] or [CF88].

2.2 The Static Norton-Hoff Model as an Approximation of Hencky's Law for Elasto-Perfect Plastic Materials

We consider in our bounded domain $\Omega \subset \mathbb{R}^d$ the Norton-Hoff approximation of Hencky's law for elasto-perfect plastic materials in dual formulation:

$$A\sigma + |\sigma^D|^{p-2}\sigma^D = \varepsilon(u) \text{ in } \Omega, \tag{2.29}$$

$$\text{div } \sigma + f = 0 \text{ in } \Omega, \tag{2.30}$$

$$\sigma \cdot \nu = g \text{ on } \Gamma_N, \tag{2.31}$$

$$u = U \text{ on } \Gamma_D. \tag{2.32}$$

Here we used the following notation:

• $A = (A_{ij}^{hk}) \in L^\infty(\Omega; \mathbb{R}^{d^2 \times d^2})$ such that $A_{ij}^{hk} = A_{ji}^{hk} = A_{ij}^{kh} = A_{hk}^{ij}$ and $(A\tau, \tau) := A_{ij}^{hk}\tau_{hk}\tau_{ij} \geq \alpha|\tau|^2 = \alpha(\tau_{ij}, \tau_{ij})$ for all $\tau \in \mathbb{S}$ and some positive constant α, is the (material-dependent) inverse of the elasticity tensor (see the previous section for the classical Lamé-Navier Operator of linearized elasticity);

• f, g and U are given vector-valued functions which represent the given volume forces (f), surface forces (g) and the prescribed displacement (U) on Γ_D.

• $p \geq 2$ is a given real number which represents a penalty-parameter for penalising the constraint $|\sigma^D| \leq 1$ (note that to simplify we have "normalized" σ to have $k_* = \frac{\sqrt{2}}{2}$, see (2.20)), and which is intended to go to $+\infty$. The reason for this will be clear in the next subsection.

The equation (2.29) is the Euler-Lagrange equation of the following variational problem: Defining the functional $J_p(\sigma)$ as

$$\begin{aligned} J_p(\sigma) := &\frac{1}{2}\int_\Omega A_{ij}^{hk}(x)\sigma_{hk}(x)\sigma_{ij}(x)\, dx + \frac{1}{p}\int_\Omega |\sigma^D(x)|^p\, dx \\ &- \int_{\Gamma_N} \sigma_{ij}\nu_j g^i d\mathcal{H}^{d-1} \end{aligned} \tag{2.33}$$

and the set $\mathcal{K}_p$ as

$$\begin{aligned} \mathcal{K}_p := \big\{&\sigma \in L^2(\Omega; \mathbb{S}) : \text{div}\, \sigma + f = 0 \text{ in } \Omega\,, \\ &\sigma \cdot \nu = g \text{ on } \Gamma_N, \sigma^D \in L^p(\Omega; \mathbb{S}^D)\big\}\,, \end{aligned} \tag{2.34}$$

where we assumed that

$$f \in L^q(\Omega;\mathbb{R}^d) \text{ for every finite } q > 1 \,, \tag{2.35}$$

$$g \in H^{-\frac{1}{2}}(\Gamma_N;\mathbb{R}^d) \tag{2.36}$$

or in a better space as for example

$$L^2(\Gamma_N;\mathbb{R}^d)\,,$$

the Norton-Hoff model (with penalty-parameter p) corresponds to the variational problem:

$$J_p(\sigma) \longrightarrow \min \;\text{ over } \mathcal{K}_p \;! \tag{2.37}$$

Via the standard "Direct methods of the Calculus of Variations" it is not difficult to show that the problem (2.37) has a unique solution denoted by $\sigma = \sigma^p$. The uniqueness follows from the fact that the functional $J_p(\sigma)$ is strictly convex for $1 < p < +\infty$.
For later use we stress the following uniform estimate for a minimizing sequence $\{\sigma_k\}_{k\in\mathbb{N}}$:

$$\|\sigma_k\|_{0,2}^2 + \|\sigma_k^D\|_{0,p}^p \leq K \tag{2.38}$$

with a constant K independent of $k \in \mathbb{N}$ and $2 \leq p$.

A necessary and sufficient condition for $\sigma = \sigma^p$ to be a minimizer consists of

$$\int_\Omega A_{ij}^{hk}(x)\sigma_{hk}(x)\tau_{ij}(x) + |\sigma^D|^{p-2}\sigma_{ij}^D\tau_{ij}^D \, dx = 0 \tag{2.39}$$

$\forall \tau \in L^p(\Omega;\mathbb{S})$ such that $\operatorname{div}\tau = 0$, $\tau\nu = 0$.
By using this in conjunction with

Proposition 2-II *If* $\alpha \in L^p(\Omega;\mathbb{S})$ *satisfies*

$$\int_\Omega \alpha_{ij}(x)\beta_{ij}(x)\, dx = 0$$

$\forall \beta \in L^q(\Omega;\mathbb{S})$ *such that* $\operatorname{div}\beta = 0$*,* $\beta\nu = 0$ *on* Γ_N $(1 < p,q < +\infty, 1/p + 1/q = 1)$*, then there exists exactly one* $u \in W^{1,p}(\Omega;\mathbb{R}^d)$ *such that*

$$\alpha = \varepsilon(u) \quad \textit{and} \quad u = 0 \textit{ on } \Gamma_D\,.$$

We introduce the displacement:

$$u = u^p \in \mathring{W}^{1,\frac{p}{p-1}}_{\Gamma_D}(\Omega;\mathbb{R}^d) + U \quad , \quad \operatorname{div} u^p = \operatorname{trace}(A\sigma^p) \in L^2(\Omega)\ . \tag{2.40}$$

Under the following two different, additional hypotheses one can derive also uniform (with respect to $p \uparrow +\infty$) estimates on the displacement $u = u^p$:

$$\|\varepsilon(u^p)\|_{0,1;\Omega} \le K\,, \tag{2.41}$$

$$\|u^p\|_{0,\frac{d}{d-1};\Omega} \le K\,, \tag{2.42}$$

provided that

(2.43) a) the volume forces f are conservative:
$f = \nabla F$ with $F \in W^{1,p}(\Omega)$ for all $p \in (1,+\infty)$
and $F = 0$ on Γ_D or

(2.44) b) the safe load condition is satisfied:
$\exists \tau \in L^2(\Omega;\mathbb{S})$ such that $\operatorname{div}\tau + f = 0$ in $\Omega\,,\ \tau\nu = g$ on Γ_N
and $\exists \delta > 0$ such that $|\tau^D| \le 1 - \delta\,.$

The just mentioned uniform estimates (2.38), (2.41) and (2.42) lead finally to the following existence result for the constitutive law of the Hencky model:

Theorem 2-III *For $f \in L^{d+1}(\Omega;\mathbb{R}^d)$ and $g \in H^{-\frac{1}{2},2}(\Gamma_N)$ there exists exactly one $\sigma \in L^2(\Omega;\mathbb{S})$ such that*

$$\operatorname{div}\sigma + f = 0 \ \textit{in}\ \Omega\,,$$
$$\sigma\nu = g \ \textit{on}\ \Gamma_N\,,$$
$$|\sigma^D| \le 1 \ \textit{almost everywhere,}$$

and also (at least) one $u \in L^1(\Omega;\mathbb{R}^d)$ with $\varepsilon(u) \in \mathbb{M}(\Omega;\mathbb{S})$, $\operatorname{div} u \in L^2(\Omega)$ and $u = U$ on Γ_D in the sense that

$$\langle \varepsilon(u)\,\tau\ ,\ 1\rangle = 0$$

$\forall \tau \in L^2(\Omega;\mathbb{S})$, $\operatorname{div}\tau = 0$, $\tau\,\nu = 0$ *on* Γ_N, *such that*

$$\boxed{\langle \varepsilon(u) - A\sigma\ ,\ \tau - \sigma\rangle \le 0} \tag{2.45}$$

$\forall \tau \in L^2(\Omega,\mathbb{S})$, $\operatorname{div}\tau + f = 0$, $|\tau^D| \le 1$, $\tau\,\nu = g$ *on* Γ_N.

Here $\mathbb{M}(\Omega; \mathbb{S})$ denotes the space of Radon measures on Ω with values in $\mathbb{S}$ and the "angles" $\langle \cdot, \cdot \rangle$ represent the duality between $\mathbb{M}(\Omega)$ and $L^\infty(\Omega)$. The uniqueness of the displacement u is still not known (up to now).

A detailed proof of Theorem 2-III is given in Bensoussan-Frehse [BF93].

Next we review the H^1_{loc}-regularity for the stresses as proven in [BF93].

Theorem 2-IV *Assume that the volume forces f satisfy either (2.35) or (2.43) and in addition*

$$\Delta f \in L^d(\Omega; \mathbb{R}^d)\,, \tag{2.46}$$

then we have

(i) $\nabla\sigma = \nabla\sigma^p \in L^2_{\text{loc}}$ and $|\sigma^D|^{\frac{p-2}{2}} |\nabla\sigma^D| \in L^2_{\text{loc}}$, which means more precisely that for every $D \subset\subset \Omega$ there exists a positive constant $K = K(p, \operatorname{dist}(D, \partial\Omega), d)$ such that

$$\int_D |\nabla\sigma|^2\, dx \le K \tag{2.47}$$

and

$$\int_D |\sigma^D|^{p-2} |\nabla\sigma^D|^2\, dx \le K\ , \tag{2.48}$$

(ii) the estimates (2.47) and (2.48) hold with a constant $K = K(d, \operatorname{dist}(D, \partial\Omega))$ independent of p, i. e. we have uniformly with respect to p the statement

$$\nabla\sigma \in L^2_{\text{loc}} \tag{2.49}$$

and

$$\int_D |\nabla\sigma|^2\, dx \le K\,. \tag{2.50}$$

The proof of Theorem 2-IV is based on the following formal calculation, which we recall for later use and the convenience of the reader. We test equation (2.29) with $-D_\alpha(\eta^4 D_\alpha \sigma_{ij})$, where η is a suitable cut-off function for $D \subset\subset \Omega$:

$$\eta \in C_0^\infty(\Omega)\,,\ 0 \le \eta \le 1\,,\ \eta \equiv 1 \text{ on } D \subset\subset \Omega \tag{2.51}$$

and

$$|\nabla\eta| \le \frac{C(d)}{\operatorname{dist}(D,\partial\Omega)}\,. \tag{2.52}$$

Integration by parts yields on the left hand side:

$$\begin{aligned}(LHS) \equiv \int \eta^4 A_{ij}^{hk} D_\alpha\sigma_{hk} D_\alpha\sigma_{ij}\,dx &+ \int \eta^4|\sigma^D|^{p-2} D_\alpha\sigma_{ij}^D D_\alpha\sigma_{ij}^D\,dx\\ &+ (p-2)\int \eta^4|\sigma^D|^{p-4} D_\alpha\frac{|\sigma^D|^2}{2} D_\alpha\frac{|\sigma^D|^2}{2}\,dx\,.\end{aligned} \tag{2.53}$$

For simplicity we deal here only with the case of constant coefficients A_{ij}^{hk}; otherwise there would appear also a term like

$$\int \eta^4 (D_\alpha A_{ij}^{hk})\sigma_{hk} D_\alpha\sigma_{ij}\,dx\ ,$$

which does not create any trouble. Therefore we omit it here.
On the right hand side we obtain

$$\begin{aligned}\int D_\alpha\varepsilon_{ij}(u)\eta^4 D_\alpha\sigma_{ij}\,dx &= \frac{1}{2}\int (D_\alpha D_j u^i + D_\alpha D_i u^j)\eta^4 D_\alpha\sigma_{ij}\,dx\\ &= -\int D_\alpha u^j [D_\alpha\sigma_{ij} D_i\eta^4 + \eta^4 D_\alpha D_i\sigma_{ij}]\,dx\\ &= -\int D_\alpha u^j [D_\alpha\sigma_{ij} 4\eta^3 D_i\eta - \eta^4 D_\alpha f^j]\,dx\end{aligned} \tag{2.54}$$

by using equation (2.33) $D_i\sigma_{ij} = -f^j$ and the symmetry of σ. Next we integrate once more by parts the term with f:

$$\int \eta^4 D_\alpha u^j D_\alpha f^j\,dx = -\int u^j[\eta^4\Delta f^j + 4\eta^3 D_\alpha\eta f^j]\,dx \tag{2.55}$$

and write by using equation (2.29):

$$\begin{aligned}D_\alpha u^j &= 2\varepsilon_{\alpha j}(u) - D_j u^\alpha\\ &= 2(A_{\alpha j}^{hk}\sigma_{hk} + |\sigma^D|^{p-2}\sigma_{\alpha j}^D) - D_j u^\alpha\,.\end{aligned} \tag{2.56}$$

Therefore

$$\begin{aligned}-\int D_\alpha u^j D_\alpha\sigma_{ij} D_i\eta^4\,dx &= -\int u^\alpha[D_\alpha f^i D_i\eta^4 + D_\alpha\sigma_{ij} D_j D_i\eta^4]\,dx\\ &- 2\int A_{\alpha j}^{hk}\sigma_{hk} D_\alpha\sigma_{ij} D_i\eta^4\,dx - 2\int |\sigma^D|^{p-2}\sigma_{\alpha j}^D D_\alpha\sigma_{ij} 4\eta^3 D_i\eta\,dx\,.\end{aligned} \tag{2.57}$$

Finally we move D_α and observe that equation (2.29) implies

$$\operatorname{div} u = D_l u^l = \varepsilon_{ll}(u) = A_{ll}^{hk}\sigma_{hk} = \operatorname{tr}(A\sigma) \tag{2.58}$$

by applying the operator trace to equation (2.29). This yields

$$\begin{aligned}
&-\int u^\alpha D_\alpha \sigma_{ij}(4\eta^3 D_j D_i\eta + 12\eta^2 D_j\eta D_i\eta)\,dx \\
&\quad= \int (\operatorname{div} u)\sigma_{ij}(4\eta^3 D_j D_i\eta + 12\eta^2 D_j\eta D_i\eta)\,dx \\
&\quad+ \int u^\alpha \sigma_{ij}(4\eta^3 D_\alpha D_j D_i\eta + 12\eta^2 D_\alpha\eta D_j D_i\eta + 12\eta^2 D_\alpha D_j\eta D_i\eta \\
&\qquad+ 12\eta^2 D_j\eta D_\alpha D_i\eta + 24\eta D_\alpha\eta D_j\eta D_i\eta)\,dx\,.
\end{aligned} \tag{2.59}$$

Collecting our intermediate results we get

$$\begin{aligned}
&\int \eta^4 A_{ij}^{hk} D_\alpha\sigma_{hk} D_\alpha\sigma_{ij}\,dx + \int \eta^4|\sigma^D|^{p-2}|\nabla\sigma^D|^2\,dx \\
&\quad+ (p-2)\int \eta^4|\sigma^D|^{p-4}\left|\nabla\frac{|\sigma^D|^2}{2}\right|^2\,dx \\
&= \int u^\alpha\,\big\{\sigma_{ij}(4\eta^3 D_\alpha D_j D_i\eta + 12\eta^2 D_\alpha\eta D_j D_i\eta + 12\eta^2 D_\alpha D_j\eta D_i\eta \\
&\quad+ 12\eta^2 D_j\eta D_\alpha D_i\eta + 24\eta D_\alpha\eta D_j\eta D_i\eta) \\
&\quad-\eta^4\Delta f^\alpha - 4\eta^3 D_\beta\eta D_\beta f^\alpha - 4\eta^3 D_i\eta D_\alpha f^i\big\}\,dx \\
&\quad+ \int A_{ll}^{hk}\sigma_{hk}\sigma_{ij}(4\eta^3 D_j D_i\eta + 12\eta^2 D_j\eta D_i\eta)\,dx \\
&\quad- 2\int A_{\alpha j}^{hk}\sigma_{hk} D_\alpha\sigma_{ij} 4\eta^3 D_i\eta\,dx \\
&\quad- 2\int |\sigma^D|^{p-2}\sigma_{\alpha j}^D D_\alpha\sigma_{ij}^D 4\eta^3 D_i\eta\,dx \\
&\quad- \frac{2}{d}\int |\sigma^D|^{p-2}\sigma_{\alpha j}^D D_\alpha(\operatorname{tr}\sigma)4\eta^3 D_i\eta\,dx\;,
\end{aligned} \tag{2.60}$$

where we used $\sigma_{ij} = \sigma_{ij} - \frac{1}{d}(\operatorname{tr}\sigma)\delta_{ij} + \frac{1}{d}(\operatorname{tr}\sigma)\delta_{ij} = \sigma_{ij}^D + \frac{1}{d}(\operatorname{tr}\sigma)\delta_{ij}$ for the last two terms. From (2.60) one can see that one can establish the estimates (2.47), (2.48) of Theorem 2-IV for $d \le 4$ immediately via Sobolev's inequality even uniformly with respect to p thanks to the uniform estimates (2.41), (2.42) on the displacements. For the last term of (2.60) one takes into account the inequality

$$\begin{aligned}
&\int \eta^4 |\sigma^D|^{p-2} |\nabla \operatorname{tr} \sigma|^2 \, dx \\
&\le 2d^2 \int \eta^4 |\sigma^D|^{p-2} |\nabla \sigma^D|^2 \, dx + 2d^2 \int \eta^4 |\sigma^D|^{p-2} |f|^2 \, dx \,,
\end{aligned} \tag{2.61}$$

which follows from

$$\begin{aligned}
|\nabla \sigma^D|^2 &= D_k \sigma^D_{ij} D_k \sigma^D_{ij} \ge D_k \sigma^D_{kj} D_k \sigma^D_{kj} = |\operatorname{div} \sigma^D|^2 \\
&= \sum_j \Big(\sum_k D_k \sigma^D_{kj} \Big)^2 \\
&= \sum_j \Big(\sum_k D_k \big(\sigma_{kj} - \frac{1}{d} (\operatorname{tr} \sigma) \delta_{kj} \big) \Big)^2 \\
&= \sum_j \big(- f^j - \frac{1}{d} D_j (\operatorname{tr} \sigma) \big)^2 \\
&= \sum_j \big(f^j + \frac{1}{d} D_j \operatorname{tr} \sigma \big)^2 \\
&= |f|^2 + \frac{1}{d^2} |\nabla \operatorname{tr} \sigma|^2 + \frac{2}{d} f^j D_j \operatorname{tr} \sigma \,.
\end{aligned} \tag{2.62}$$

Remark 2-V 1) Theorem 2-IV in the above stated dual formulation was proven by Bensoussan-Frehse 1993 [BF93]. In the primal formulation a corresponding result for the Hencky model was obtained by G. Seregin in 1987 [Ser87] via uniform estimates for some approximation of the primal problem.
2) Due to the fact that the estimates of Theorem 2-IV are uniform with respect to p they apply also to the solution of the limit problem as $p \to +\infty$, i. e. to the solution of the Hencky model.
3) For fixed p we know from Theorem 2-IV

$$\int_D |\sigma^D|^{p-2} |\nabla \sigma^D|^2 \, dx \le \text{const}_p \,. \tag{2.63}$$

Observing that

$$\nabla \frac{|\sigma^D|^{p/2}}{p/2} = |\sigma^D|^{\frac{p}{2}-2} \sigma^D_{ij} \nabla \sigma^D_{ij} \le |\sigma^D|^{\frac{p}{2}-1} |\nabla \sigma^D|$$

we get from (2.63)

$$\frac{4}{p^2}\int\limits_D |\nabla|\sigma^D|^{p/2}|^2\,dx \le \mathrm{const}_p\,, \tag{2.64}$$

and provided D is regular enough to apply Sobolev's imbedding theorem we have from (2.64)

$$\sigma^D \in L^{\frac{dp}{d-2}}(D) \qquad \text{for } d \ge 3 \tag{+}$$

and

$$|\sigma^D|^{p/2} \in vmo(D) \qquad \text{for } d = 2\,. \tag{**}$$

From (+) respectively (**) and the equation $\operatorname{div}\sigma = -f$ we would like to conclude

$$\sigma \in L^{\frac{dp}{d-2}}(D) \quad \text{respective } \sigma \in L^q(D) \quad \forall q < +\infty\,, \tag{2.65}$$

so that equation (2.29) would give

$$\varepsilon(u) \in L^{\frac{dp}{(d-2)(p-1)}}(D) \quad \text{respective } \varepsilon(u) \in L^{\tilde q}(D) \quad \forall \tilde q < +\infty\,, \tag{2.66}$$

which with the aid of Korn's inequality and Sobolev's imbedding theorem could finally be turned into

$$\begin{cases} u & \in C^{0,\frac{1}{p}}(\bar D) \quad \text{for } d = 3 \quad \text{and finite } p\,,\\ u & \in C^{0,\alpha}(\bar D) \quad \text{for } d = 2 \quad \text{and for all } \alpha < 1\,,\\ u & \in L^{\frac{4p}{p-2}}(D) \quad \text{for } d = 4 \quad \text{and } 2 < p < +\infty\,. \end{cases} \tag{2.67}$$

The missing link to execute this procedure is contained in the following proposition, which we recall from Temam [Tem86]:

Proposition 2-VI *If $\sigma \in L^{p_1}(\Omega;\mathbb{S})$ with $\operatorname{div}\sigma \in L^r(\Omega;\mathbb{R}^d)$ and $\sigma^D \in L^s(\Omega;\mathbb{S})$, such that $1 < p_1$, $1 < r < d$, $s \ge \frac{rd}{d-r}$, then $\operatorname{tr}\sigma \in L^{\frac{dr}{d-r}}(\Omega)$, i. e. $\sigma \in L^{\frac{dr}{d-r}}(\Omega;\mathbb{S})$, and there is a constant $C = C(\Omega, r, s)$ such that*

$$\|\operatorname{tr}\sigma\|_{0,\frac{dr}{d-r}} \le C\left\{\|\sigma\|_{0,p_1} + \|\operatorname{div}\sigma\|_{0,r} + \|\sigma^D\|_{0,s}\right\}\,. \tag{2.68}$$

To obtain (2.65) we apply the proposition for $d = 3$ with $p_1 = 2$, $s = \frac{dp}{d-2} = 3p$ and $r = \frac{3p}{p+1}$; for $d = 2$ with $p_1 = 2$, s arbitrary, but finite and $r = \frac{2s}{2+s}$ and for $d = 4$ with $p_1 = 2$, $s = 2p$ and $r = \frac{4p}{p+2}$.

For the sake of completeness (at least to some extent) we review next some further known regularity results:

1) Concerning boundary regularity for the stresses Frehse-Málek obtained in [FM99] the following results for the case $g = 0$ on Γ_N and $U = 0$ on Γ_D:

- For arbitrary large, but fixed p the L^2-norm of the tangential derivatives $D_\tau \sigma$ can blow up at most as p for p tending to $+\infty$.
- In space dimension $d = 2$ one has for fixed p the inclusion $\nabla\sigma \in L^{2-\delta}(\Omega)$ for all $\delta > 0$ if $\Gamma_D \neq \emptyset$, and the blow-up rate is as before at most p.
- In space dimension $d \geq 3$ it holds for fixed p: $\nabla\sigma \in L^{\frac{d}{d-1}+\tilde{\delta}}(\Omega)$ for some $\tilde{\delta} > 0$.
- If Ω is a circle or a two-dimensional torus and the boundary condition is just given as Dirichlet boundary condition $u = 0$ on $\partial\Omega$, then the tangential derivatives $D_\tau \sigma$ belong to $L^2(\Omega)$ uniformly in p.

In all four cases an assumption like $f \in W^{2,d}(\Omega)$ for the force will be sufficient.

Let us also mention the following example of G. Seregin [Ser96]:
Using his approximations (σ^k, u^k) to the Hencky model, Seregin has constructed an example of solutions on a two-dimensional torus such that the boundary integral

$$\int\limits_{\partial\Omega} \varepsilon_{ij}(u^k) \frac{\partial \sigma_{ij}^k}{\partial \nu} d\mathcal{H}^1 \to +\infty \qquad \text{as } k \to +\infty\,.$$

In consequence he conjectures that either his approximation is not "good", or global $W^{1,2}$-regularity (for stresses) holds for convex domains only.

2) Hardt and Kinderlehrer proved 1983 in [HK83] a higher integrability result for the displacements u in Hencky's model; it reads as follows:
There exists some $q > \frac{d}{d-1}$, $q = q(D)$ for $D \subset\subset \Omega$ such that $u \in L^q(D; \mathbb{R}^d)$ with a corresponding estimate.
They deal (directly) with the primal formulation of Hencky's model and use the technique of reverse Hölder inequalities to obtain the result via a Gehring-Giaquinta-Modica-Stredulinsky argument.

3) For Hencky's model, i. e. the limit problem as p tends to $+\infty$ one obtains from Temam's result (Proposition 2-VI) or from Nečas' well-known theorem on equivalent norms (see [Neč66]):

$$\sigma \in L^q(\Omega; \mathbb{S}) \qquad \text{for any } q < +\infty\,.$$

The argument is as follows: Take in Proposition 2-VI $p_1 = 2$, $r = d - \varepsilon$ for some small positive ε and $s = +\infty \geq \frac{d(d-\varepsilon)}{\varepsilon} = \frac{d^2}{\varepsilon} - d$ to get $\operatorname{tr}\sigma$ and $\sigma \in L^{d^2/\varepsilon - d}$ with corresponding estimate. The choice $\varepsilon = \frac{d}{2}$ leads to $\sigma \in L^d$, $\varepsilon = \frac{d}{4}$ to $\sigma \in L^{3d}$, $\varepsilon = \frac{d}{8}$ to $\sigma \in L^{7d}$ and so on.

To go one step further it would be nice to have $\sigma \in L^\infty(\Omega; \mathbb{S})$. To conclude this from an estimate of $\|\sigma\|_{0,q}$ for arbitrary $q < +\infty$ it would be necessary to have $\|\sigma\|_{0,q} \leq$ const with a constant independent of q, but the constant $C = C(\Omega, r, s)$ in the estimate (2.68) of Proposition 2-VI depends on r and it blows up as $r \uparrow d$. Therefore we cannot conclude $\sigma \in L^\infty(Q; \mathbb{S})$. The best we can say is

$$\nabla \operatorname{tr}\sigma \in W^{-1,\infty}(\Omega; \mathbb{R}^d) = (\mathring{W}^{1,1}(\Omega; \mathbb{R}^d))' . \tag{2.69}$$

To see this remember that

$$\sigma_{ij} = \sigma_{ij}^D + \frac{1}{d}(\operatorname{tr}\sigma)\delta_{ij}$$

and therefore

$$\nabla \operatorname{tr}\sigma = \operatorname{div}\sigma - \operatorname{div}\sigma^D = -(f + \operatorname{div}\sigma^D)\,,$$

which shows (2.69) because of $\sigma^D \in L^\infty(\Omega; \mathbb{S})$ and $f \in L^d(\Omega; \mathbb{R}^d)$. From (2.69) one can deduce that

$$\|\sigma^p\|_{0,p} \leq Kp \tag{2.70}$$

with an absolute constant K (see [FM99, p.2] for details). (2.69) and (2.70) lead us to the following "guess":

$$\begin{aligned} &\sigma^D \in L^\infty(\Omega; \mathbb{S}), \operatorname{div}\sigma \in L^d(\Omega; \mathbb{R}^d) \quad \text{ and } \sigma \in L^2(\Omega; \mathbb{S}) \\ &\text{imply } \operatorname{tr}\sigma \quad \text{(and therefore also } \sigma) \in \mathit{bmo}(\Omega) \equiv L^{2,d}(\Omega)\,. \end{aligned} \tag{2.71}$$

The proof of (2.71) is "equivalent" to the proof of a version of Korn's (second) inequality in the Hardy space $\mathcal{H}^1$ (or some suitable local version of it). Due to the fact that we shall not use (2.71) later on, we do not prove (2.71) here, we only refer the interested reader to chapter 4, where the spaces $\mathit{bmo} = L^{2,d}$ (BMO there) and $\mathcal{H}^1$ are used and discussed at some length. For the fact that Korn's inequality does not hold for L^1 we refer to Temam's book [Tem85].

Let us now formulate our main result on Hölder-continuity for the stresses in two dimensions:

Theorem 2-VII *For space dimension $d = 2$ and arbitrary, but fixed p satisfying $2 < p < +\infty$, we have*

$$\sigma \in C^{0,\alpha}_{\text{loc}}(\Omega; \mathbb{S}) \text{ for some positive } \alpha\,, \tag{2.72}$$

i. e. $\sigma = \sigma^p$ is locally in Ω Hölder continuous and a corresponding estimate holds true, provided the volume forces f satisfy either

$$f \in W^{2,2}(\Omega; \mathbb{R}^2) \text{ and } \nabla^2 f \in L^{2,\lambda_0}_{\text{loc}}(\Omega; \mathbb{R}^8) \text{ for some } \lambda_0 > 0 \tag{2.73}$$

or

$$\begin{aligned} &f = \nabla F\,,\ F = 0 \text{ on } \Gamma_D \text{ and } F \in W^{3,2}(\Omega)\,,\\ &\nabla^3 F \in L^{2,\lambda_0}_{\text{loc}}(\Omega; \mathbb{R}^8) \text{ for some } \lambda_0 > 0\,. \end{aligned} \tag{2.74}$$

Corollary 2-VIII *In the situation of Theorem 2-VII we have*

$$u \in C^{1,\beta}_{\text{loc}}(\Omega) \text{ for some } \beta > 0\,. \tag{2.75}$$

Corollary 2-VIII is obtained immediately from equation (2.29) and a "suitable" version of Korn's inequality in Hölder spaces, which seems to be "mathematical folclore" in the sense that the author was and is not able to provide a nice reference for it, but everyone be asked for agreed that the result is true and known. Therefore here is a argument to justify this:

$$\varepsilon_{ij}(u) \in C^{0,\alpha}_{\text{loc}} \Rightarrow (\nabla u)_{ij} = (\partial_j u^i) \in C^{0,\alpha}_{\text{loc}}\,.$$

Let $F_{ij} \equiv 2\varepsilon_{ij}(u)$ and observe

$$\begin{aligned} -\partial_j \varepsilon_{ij}(u) &= -\frac{1}{2}(\partial_j\partial_j u^i + \partial_j\partial_i u^j)\\ &= -\frac{1}{2}(\Delta u^i - \partial_i \operatorname{div} u)\,, \end{aligned}$$

so we have

$$-(\Delta u + \nabla \operatorname{div} u) = -\operatorname{div} F = -\partial_j F_{ij}\,.$$

This is an elliptic system with constant coefficients and right hand side $-\operatorname{div} F$, $F \in C^{0,\alpha}_{\text{loc}} \cong \mathcal{L}^{2,n+2\alpha}_{\text{loc}}$, where $\mathcal{L}^{2,n+2\alpha}$ is a certain Campanato space, which is isomorphic to $C^{0,\alpha}$ (see [KJF77] or [Cam63]) and from Campanato's regularity theory for elliptic systems (see [Cam80]) we get: $\nabla u \in C^{0,\alpha}_{\text{loc}}$ plus estimate. Another possibility is to use potential theory to deduce the same conclusion, i. e. to use the so called “Giraud-Hölder-Korn-Lichtenstein inequality”, for which we refer to [Alt99] or [BJS79].

A possible idea to prove Theorem 2-VII consists of an application of the so called “hole-filling method” because in identity (2.62) all terms on the right hand side contain at least on derivative of the cut-off function η, which means that we have produced a hole there. On the other hand there are terms which contain second order derivatives, products of first order derivatives and even third order derivatives respectively products of first and/or second order derivatives, so it is not immediately clear that one can apply the “usual” hole-filling method. Another difficulty consists of the different growth behaviour of σ^D and $\mathrm{tr}(A\sigma)$:

- σ^D grows like $|\sigma^D|^{p-1}$, p “large”, but
- $\mathrm{tr}(A\sigma)$ grows only linearly.

Therefore the proof of Theorem 2-VII is based on a “refined inhomogeneous hole-filling method”. The special variant we are going to use was developed by J. Frehse and G. Seregin in [SF99] and seems to be well suited to handle situations like the above described one. We also would like to remark that inhomogeneous versions of the hole-filling method were “invented” and successfully used more than 20 years ago in several cases/situations by J. Frehse; see for example [Fre79], [Fre77], [Fre75]. The proof of Theorem 2-VII is based on the following Proposition:

Proposition 2-IX *Suppose that $H \in L^2(D)$ and $h \in W^{1,2}(D)$ satisfy the estimate*

$$(2.76) \qquad \int\limits_{B_R(x_0)} H^2(x)\,dx \le C_1 \left(\int\limits_{T_R(x_0)} H^2(x)\,dx + R^\alpha \right)^{1/2} \cdot \frac{1}{R} \int\limits_{T_R(x_0)} |hH|\,dx$$

for some positive α, any $x_0 \in D \subset\subset \mathbb{R}^2$ and any $0 < R < R_0$ such that $B_{2R_0}(x_0) \subset D$. Here $T_R(x_0)$ denotes the annulus $T_R(x_0) \equiv B_{2R}(x_0) - B_R(x_0)$. Then, for any real number $q \ge 1$, there is a positive constant C_2, depending on q, $\|h|W^{1,2}(D)\|$, C_1, $\|H|L^2(D)\|$, α and R_0, such that

$$(2.77) \qquad \int\limits_{B_R(x_0)} H^2\,dx \le \frac{C_2}{\left(\log_2 \frac{2R_0}{R}\right)^q}\,.$$

In other words: From estimate (2.76) one can conclude that the function H^2 satisfies a **logarithmic** *Morrey condition.*

A detailed proof of Proposition 2-IX is given in [SF99]. We just recall here the main steps to illustrate why we call it "inhomogeneous hole-filling method". The proof consists of two steps: In the first step one proves with the aid of Poincaré's inequality the following estimate:

(2.78)
$$\fint_{T_R(x_0)} |h(y) - \fint_{T_{R_0}(x_0)} h(x)\,dx|^2\,dy \le K\Big(\log_2 \frac{2R_0}{R}\Big) \int_{B_{2R_0}(x_0)} |\nabla h(x)|^2\,dx\,,$$

where K is an absolute constant and we make use of the notation for mean values
$$\fint_A f(x)\,dx = \frac{1}{|A|}\int_A f(x)\,dx\,.$$

The second step uses (2.78) to estimate (set $T = T_R(x_0)$ and $T_0 = T_{R_0}(x_0)$ for shorter notation)

(2.79)
$$\begin{aligned}\fint_T |hH|\,dx \le\ & \fint_T |h - \fint_{T_0} h\,dy|\ |H|\,dx + \Big(\fint_{T_0} |h|\,dx\Big)\cdot\Big(\fint_T |H|\,dx\Big)\\ \le\ & \bigg\{\Big(\fint_T |h - \fint_{T_0} h\,dy|^2\,dx\Big)^{1/2} + \Big(\fint_{T_0} |h|^2\,dy\Big)^{1/2}\bigg\}\cdot\Big(\fint_T H^2\,dx\Big)^{1/2}\\ \le\ & C\big(R_0\,,\ \|h|W^{1,2}(D)\|\big)\cdot\Big(\log_2\frac{2R_0}{R}\Big)^{1/2}\Big(\fint_T H^2\,dx\Big)^{1/2}.\end{aligned}$$

Taking into account the assumption (2.76) one arrives at the estimate

(2.80)
$$\begin{aligned}\int_{B_R(x)} H^2(x)\,dx \le C\big(C_1, R_0, \|h|W^{1,2}(D)\|\big)\cdot\Big(\log_2\frac{2R_0}{R}\Big)^{1/2}\\ \left\{\int_{T_R(x)} H^2(x)\,dx + R^\alpha\right\}\,,\end{aligned}$$

which is valid for $0 < R \le R_0$ and $B_{2R_0}(x_0) \subset D$. From (2.80) one fills the hole and obtains the assertion of the proposition via iteration as for the usual hole-filling technique (see [DHKW92] for a nice proof of this). Now we wish to apply Proposition 2-IX, where the function H is defined through

$$H^2 = D_k\varepsilon_{ij}(u)D_k\sigma_{ij}\,. \tag{2.81}$$

Let η be once more a cut-off function such that

$$\begin{aligned} &0 \le \eta \le 1 \quad \text{in } \Omega\,,\\ &\operatorname{supp}\eta \subset B_{2R}(x_0) \subset D = B_{2R_0}(x_0) \subset\subset \Omega\,,\\ &\eta \equiv 1 \quad \text{in } B_R(x_0) \quad \text{and}\\ &\max|\nabla^i\eta| \le \frac{\bar{C}}{R^i}\,, \quad i = 1,2,3\,. \end{aligned} \tag{2.82}$$

First we observe that $\eta^2 H$ belongs to $L^2(D)$ because of

$$\begin{aligned} \eta^4 H^2 &\equiv \eta^4 D_l\varepsilon_{ij}(u)D_l\sigma_{ij}\\ &= \eta^4\left\{A_{ij}^{hk}D_l\sigma_{hk}D_l\sigma_{ij} + |\sigma^D|^{p-2}D_l\sigma_{ij}^D D_l\sigma_{ij}^D\right.\\ &\qquad \left.+(p-2)|\sigma^D|^{p-4}\sigma_{mn}^D D_l\sigma_{mn}^D\sigma_{ij}^D D_l\sigma_{ij}^D\right\} \end{aligned} \tag{2.83}$$

by using equation (2.29). Taking into account the ellipticity of the coefficients A:

$$A_{ij}^{hk}\tau_{hk}\tau_{ij} \ge \alpha|\tau|^2 \quad \text{and} \quad A_{ij}^{hk}\tau_{hk}\tau_{ij} \le M|\tau|^2\,,$$

where M is a bound for the modulus of A_{ij}^{hk}, we have

$$\begin{aligned} &\eta^4\Big(\alpha|\nabla\sigma|^2 + |\sigma^D|^{p-2}|\nabla\sigma^D|^2 + (p-2)|\sigma^D|^{p-4}\left|\nabla\frac{|\sigma^D|}{2}\right|^2\Big)\\ &\le \eta^4 H^2\\ &\le \eta^4\Big(M|\nabla\sigma|^2 + |\sigma^D|^{p-2}|\nabla\sigma^D|^2 + (p-2)|\sigma^D|^{p-4}\left|\nabla\frac{|\sigma^D|}{2}\right|^2\Big) \end{aligned} \tag{2.84}$$

and therefore $H \in L^2(D)$ in view of Theorem 2-IV respectively (2.62).

Let us once more mention that we deal here only with the case of constant coefficients A_{ij}^{hk} for simplicity, otherwise one has to assume $A_{ij}^{hk} \in W^{1,\infty}_{\text{loc}}$ and to treat an additional term like

$$\eta^4\big(D_l A_{ij}^{hk}\big)\sigma_{hk}D_l\sigma_{ij}\,.$$

From (2.84) we see that, if we are able to show a logarithmic Morrey-condition for H^2, we have it also for $|\nabla\sigma|^2$! This will be our next aim. In a similar way as we derived the identity (2.62) we proceed for this purpose as follows: Let us introduce the tensor

$$\bar{\sigma} = \sigma - \sigma^0 \,, \tag{2.85}$$

where $\sigma^0 \in \mathbb{S}$ is a constant, symmetric matrix, and the vector field

$$\bar{u} = u - \kappa^0(x - x_0) - u_0 \,, \tag{2.86}$$

where $\kappa^0 \in \mathbb{S}$ is another constant, symmetric matrix, $x_0 \in \Omega$ is the center of our ball $B_R(x_0)$ and u_0 is a rigid deformation, i. e. u_0 satisfies $\varepsilon(u_0) \equiv 0$, which is equivalent to the statement, that u_0 is an affine (linear) map with a skew-symmetric coefficient matrix (see for example Ciarlet [Cia88] or Hlaváček and Nečas [NH80]). Then we have (obviously):

$$\begin{aligned} \int \eta^4 H^2 \, dx &= \int \eta^4 D_l \varepsilon_{ij}(u) D_l \sigma_{ij} \, dx \\ &= \int \eta^4 D_l \varepsilon_{ij}(\bar{u}) D_l \bar{\sigma}_{ij} \, dx \,. \end{aligned} \tag{2.87}$$

Now we proceed similarly as in deriving (2.62), i. e. we integrate suitably often by parts and use $\operatorname{div} \sigma = -f$:

$$\begin{aligned} &\int \eta^4 D_l \varepsilon_{ij}(\bar{u}) D_l \bar{\sigma}_{ij} \, dx \\ &= \int \eta^4 D_l \bar{u}^i D_l f^i \, dx - \int (D_j \eta^4) D_l \bar{u}^i D_l \bar{\sigma}_{ij} \, dx \\ &= - \int \eta^4 \bar{u}^i \Delta f^i \, dx - \int 4\eta^3 D_l \eta \; \bar{u}^i \; D_l f^i \, dx \\ &\quad - 2 \int (D_j \eta^4) \; \varepsilon_{il}(\bar{u}) \; D_l \bar{\sigma}_{ij} \, dx + \int (D_j \eta^4) \; D_i \bar{u}^l \; D_l \bar{\sigma}_{ij} \, dx \\ &= - \int \eta^4 \bar{u}^i \Delta f^i \, dx - \int 4\eta^3 D_l \eta \; \bar{u}^i \; D_l f^i \, dx \\ &\quad + \int (D_j \eta^4) \; \bar{u}^l \; D_l f^j \, dx - \int (D_i D_j \eta^4) \; \bar{u}^l \; D_l \bar{\sigma}_{ij} \, dx \\ &\quad - 2 \int (D_j \eta^4) \; \left[A_{il}^{hk} \sigma_{hk} + |\sigma^D|^{p-2} \sigma_{il}^D - \kappa_{il}^0 \right] \; D_l \bar{\sigma}_{ij} \, dx \,. \end{aligned} \tag{2.88}$$

Next we estimate all terms separately:

$$
\begin{aligned}
|\int \eta^4\, \bar u^i\, \Delta f^i\, dx| &\le \Big(\int\limits_{B_{2R}(x_0)} |\bar u|^2\, dx\Big)^{1/2} \Big(\int\limits_{B_{2R}(x_0)} |\Delta f|^2\, dx\Big)^{1/2} \\
&\le C_1(\|\Delta f|L^{2,\lambda_0}\|)\; R^{\lambda_0/2} \Big(\int\limits_{B_{2R}(x_0)} |\bar u|^2\, dx\Big)^{1/2}
\end{aligned} \tag{2.89}
$$

by using $\operatorname{supp}\eta \subset B_{2R}(x_0)$ and the assumed Morrey condition for $\nabla^2 f$.

$$
\begin{aligned}
|4\int \eta^3\, D_j\eta\, \bar u^l\, D_l f^j\, dx| &\le \frac{4\bar c}{R} \Big(\int\limits_{T_R(x_0)} |\bar u|^2\, dx\Big)^{1/2} \Big(\int\limits_{T_R(x_0)} |\nabla f|^2\, dx\Big)^{1/2} \\
&\le C(\|\nabla f|L^\infty(D)\|, \bar c) \Big(\int\limits_{T_R(x_0)} |\bar u|^2\, dx\Big)^{1/2} \\
&\le C(\|\nabla^2 f|L^{2,\lambda_0}(D)\|, \bar c) \Big(\int\limits_{T_R(x_0)} |\bar u|^2\, dx\Big)^{1/2} ,
\end{aligned} \tag{2.90}
$$

where we used $|T_R(x_0)| = CR^2$ and that ∇f belongs to $L^\infty(D)$ due to the "imbedding":

$$
\nabla^2 f \in L^{2,\lambda_0}(D) \Rightarrow \nabla f \in \mathcal{L}^{2,2+\lambda_0}(D) \cong C^{0,\lambda_0/2}(\bar D) ,
$$

which is a consequence of Morrey's Dirichlet-growth theorem or a consequence of Poincaré's inequality and Campanato's characterisation of Hölder continuous functions (see for example [Mor66] or [Cam80]).

$$
\begin{aligned}
&|\int (4\eta^3\, D_iD_j\eta + 12\eta^2\, D_i\eta\, D_j\eta)\, \bar u^l\, D_l\bar\sigma_{ij}\, dx| \\
&\le \frac{12\bar c}{R^2} \Big(\int\limits_{T_R(x_0)} |\bar u|^2\, dx\Big)^{1/2} \Big(\int\limits_{T_R(x_0)} \eta^4 |\nabla\bar\sigma|^2\, dx\Big)^{1/2}
\end{aligned} \tag{2.91}
$$

and finally

$$
\begin{aligned}
&|8\int \eta^3\, D_j\eta\, \varepsilon_{il}(\bar u)\, D_l\bar\sigma_{ij}\, dx| \\
&\le \frac{8\bar c}{R} \Big(\int\limits_{T_R(x_0)} |\varepsilon(\bar u)^2\, dx\Big)^{1/2} \Big(\int\limits_{T_R(x_0)} \eta^4 |\nabla\bar\sigma|^2\, dx\Big)^{1/2} .
\end{aligned} \tag{2.92}
$$

Observing that $\nabla\bar\sigma = \nabla\sigma$ and $|\nabla\sigma|^2 \le \frac{1}{\alpha} H^2$ we have

$$\Big(\int\limits_{T_R(x_0)} \eta^4 |\nabla\bar\sigma|^2 \, dx \Big)^{1/2} \le \frac{1}{\sqrt{\alpha}} \Big(\int\limits_{T_R(x_0)} \eta^4 H^2 \, dx \Big)^{1/2} . \tag{2.93}$$

By a "suitable version" of the Sobolev-Poincaré inequality we have

$$\Big(\int\limits_{T_R(x_0)} |\varepsilon(\bar u)|^2 \, dx \Big)^{1/2} \le C \int\limits_{T_R(x_0)} |\nabla \varepsilon(u)| \, dx \tag{2.94}$$

and

$$\begin{aligned} \Big(\int\limits_{T_R(x_0)} |\bar u|^2 \, dx \Big)^{1/2} &\le C\ R \Big(\int\limits_{T_R(x_0)} |\varepsilon(\bar u)|^2 \, dx \Big)^{1/2} \\ &\le C\ R \int\limits_{T_R(x_0)} |\nabla\varepsilon(u)| \, dx \end{aligned} \tag{2.95}$$

because of $\nabla\varepsilon(\bar u) = \nabla\varepsilon(u)$. Replacing $T_R(x_0)$ by $B_{2R}(x_0)$ in the last inequality gives an analogous estimate for $\big(\int\limits_{B_{2R}(x_0)} |\bar u|^2 \, dx \big)^{1/2}$.

Let us collect what we have:

$$\begin{aligned} \int\limits_{B_R(x_0)} H^2(x) \, dx \le\ & C_1(\|\Delta f | L^{2,\lambda_0}(D)\|) \ R^{1+\frac{\lambda_0}{2}} \int\limits_{B_{2R}(x_0)} |\nabla\varepsilon(u)| \, dx \\ & + C_2(\|\nabla^2 f | L^{2,\lambda_0}(D)\|, \bar c) \ R \int\limits_{T_R(x_0)} |\nabla\varepsilon(u)| \, dx \\ & + \frac{C_3}{\sqrt{\alpha} R} \Big(\int\limits_{T_R(x_0)} |\nabla\varepsilon(u)| \, dx \Big) \Big(\int\limits_{T_R(x_0)} H^2 \, dx \Big)^{1/2} . \end{aligned} \tag{2.96}$$

Next we would like to estimate $\nabla\varepsilon(u)$ in terms of H. For this purpose we define the function h as

$$h \equiv \frac{M}{\sqrt{\alpha}} + \big(1 + (p-2)^{1/2}\big) \ \max\big(1, |\sigma^D|^{\frac{p-2}{2}}\big) , \tag{2.97}$$

observe that equation (2.29) gives us (after differentiation):

(2.98)
$$
\begin{aligned}
|\nabla\varepsilon(u)| &= \left| A\nabla\sigma + |\sigma^D|^{p-2}\nabla\sigma^D + (p-2)|\sigma^D|^{p-4}\nabla\frac{|\sigma^D|^2}{2}\,\sigma^D \right| \\
&\le M\,|\nabla\sigma| + |\sigma^D|^{p-2}\,|\nabla\sigma^D| + (p-2)|\sigma^D|^{p-3}\,\left|\nabla\frac{|\sigma^D|^2}{2}\right| \\
&= M\,|\nabla\sigma| + |\sigma^D|^{\frac{p-2}{2}}\,|\sigma^D|^{\frac{p-2}{2}}\,|\nabla\sigma^D| \\
&\quad + (p-2)^{1/2}\,|\sigma^D|^{\frac{p-2}{2}}\,(p-2)^{1/2}\,|\sigma^D|^{\frac{p-4}{2}}\,\left|\nabla\frac{|\sigma^D|^2}{2}\right| \\
&\le \frac{M}{\sqrt{\alpha}}\,H + \max\big(1,|\sigma^D|^{\frac{p-2}{2}}\big)\,H + (p-2)^{1/2}\,\max\big(1,|\sigma^D|^{\frac{p-2}{2}}\big)\,H \\
&= \Big[\frac{M}{\sqrt{\alpha}} + \big(1+(p-2)^{1/2}\big)\,\max\big(1,|\sigma^D|^{\frac{p-2}{2}}\big)\Big]\,H = h\,H\,.
\end{aligned}
$$

For the auxiliary function h we have

$$
\begin{aligned}
|\nabla h|^2 &\le \big(1+(p-2)^{1/2}\big)^2\frac{(p-2)^2}{4}\,|\sigma^D|^{p-4}\,|\nabla\sigma^D|^2\,\chi_{\{|\sigma^D|>1\}} \\
&\le \big(1+(p-2)^{1/2}\big)^2\frac{(p-2)^2}{4}\,|\sigma^D|^{p-2}|\nabla\sigma^D|^2
\end{aligned}
\tag{2.99}
$$

and from Theorem 2-IV we know that $h \in W^{1,2}(D)$ (for fixed $p > 2$!!). Therefore (2.96) turns into

$$
\begin{aligned}
\int\limits_{B_R(x_0)} H^2(x)\,dx &\le C_1\big(\|\Delta f|L^{2,\lambda_0}(D)\|\big)\,R^{1+\frac{\lambda_0}{2}}\int\limits_{B_{2R}(x_0)}|hH|\,dx \\
&\quad + C_2\big(\|\nabla^2 f|L^{2,\lambda_0}(D)\|\big)\,R\int\limits_{T_R(x_0)}|hH|\,dx \\
&\quad + \frac{C_3(\sqrt{\alpha})}{R}\int\limits_{T_R(x_0)}|hH|\,dx\,\Big(\int\limits_{T_R(x_0)} H^2(x)\,dx\Big)^{1/2}.
\end{aligned}
\tag{2.100}
$$

The last two terms of this inequality are estimated by

$$
\big[C_2\big(\|\nabla^2 f|L^{2,\lambda_0}(D)\|\big) + C_3(\sqrt{\alpha})\big]\,\Big(\int\limits_{T_R(x_0)} H^2(x)\,dx + R^4\Big)^{1/2}\,\frac{1}{R}\int\limits_{T_R}|hH|\,dx\,,
$$

which is fine for applying Proposition 2-IX. The first will be "just" enlarged by Hölder's inequality:

$$
\begin{aligned}
&C_1\big(\|\Delta f|L^{2,\lambda_0}(D)\|\big)R^{1+\frac{\lambda_0}{2}}\int\limits_{B_{2R}(x_0)}|hH|\,dx\\
&\le C_1\big(\|\Delta f|L^{2,\lambda_0}(D)\|\big)R^{1+\frac{\lambda_0}{2}}\Big(\int\limits_{B_{2R}(x_0)}|h|^2\,dx\Big)^{1/2}\Big(\int\limits_{B_{2R}(x_0)}|H|^2\,dx\Big)^{1/2}\\
&\le C_1\big(\|\Delta f|L^{2,\lambda_0}(D)\|\big)R^{1+\frac{\lambda_0}{2}}\Big(\int\limits_{B_{2R_0}(x_0)}|h|^2\,dx\Big)^{1/2}\Big(\int\limits_{B_{2R_0}(x_0)}|H|^2\,dx\Big)^{1/2}\\
&= M\big(\|\Delta f|L^{2,\lambda_0}(D)\|,\|h\|_{0,2;B_{2R_0}(x_0)}\,,\ \|H\|_{0,2;B_{2R_0}(x_0)}\big)R^{1+\frac{\lambda_0}{2}}\,.
\end{aligned}
$$

So we have

$$
\begin{aligned}
&\int\limits_{B_R(x_0)}H^2(x)\,dx\le MR^{1+\frac{\lambda_0}{2}}\\
&\qquad+C_0\Big(\int\limits_{T_R(x_0)}H^2(x)\,dx+R^4\Big)^{1/2}\frac{1}{R}\int\limits_{T_R}|hH|\,dx\,,
\end{aligned}
\tag{2.101}
$$

where $M = M\big(\|\delta f|L^{2,\lambda_0}(D)\|,\|h\|_{0,2;B_{2R_0}(x_0)}\,,\ \|H\|_{0,2;B_{2R_0}(x_0)}\big)$ and $C_0 = C_2\big(\|\nabla^2 f|L^{2,\lambda_0}(D)\|\big)+C_3\big(\sqrt{\alpha}\big)$.

As we indicated above (see (2.79) and (2.80)) this can be transformed into

$$
\begin{aligned}
&\int\limits_{B_R(x)}H^2(x)\,dx\le C\big(C_0,R_0,\|h|W^{1,2}(D)\|\big)\\
&\quad\cdot\Big(\log_2\frac{2R_0}{R}\Big)^{1/2}\left\{\int\limits_{T_R(x_0)}H^2(x)\,dx+R^4\right\}+M\ R^{1+\frac{\lambda_0}{2}}\\
&= C\Big(\log_2\frac{2R_0}{R}\Big)^{1/2}\int\limits_{T_R(x_0)}H^2(x)\,dx+C\Big(\log_2\frac{2R_0}{R}\Big)^{1/2}R^4+M\ R^{1+\frac{\lambda_0}{2}}\,,
\end{aligned}
\tag{2.102}
$$

which is valid for $0 < R \le R_0$ and $B_{2R_0}(x_0) = D \subset\subset \Omega$. If we set

$$\varphi(R) := \int\limits_{B_R(x_0)} H^2 \, dx$$

and fill the hole by adding $C\big(\log_2 \frac{2R_0}{R}\big)^{1/2}\varphi(R)$ to both sides of (2.102) we obtain

$$\begin{aligned} \varphi(R) &\le \theta \left\{\varphi(2R) + R^4\right\} + \frac{M}{C\Big(\log_2\left(\frac{2R_0}{R}\right)\Big)^{1/2} + 1} R^{1+\frac{\lambda_0}{2}} \\ &= \theta \left\{ \varphi(2R) + R^4 + \frac{M}{C\Big(\log_2\left(\frac{2R_0}{R}\right)\Big)^{1/2}} \, R^{1+\frac{\lambda_0}{2}} \right\} \end{aligned} \tag{2.103}$$

with $\theta = \frac{C\log_2\left(\frac{2R_0}{R}\right)^{1/2}}{C\log_2\left(\frac{2R_0}{R}\right)^{1/2} + 1} < 1$.

Let us set

$$N := \left[\log_2 \frac{R_0}{R}\right]. \tag{2.104}$$

Then we have

$$2^{N+1} \ge \frac{R_0}{R} \ge 2^N \tag{2.105}$$

and for $i \le N$:

$$2^i R \le R_0 \, .$$

Therefore we obtain from (2.103) the next inequalities for $i = 0, 1, \dots, N-1$:

$$\begin{aligned} \varphi(2^i R) \le \frac{C\sqrt{N-i+2}}{C\sqrt{N-i+2}+1} \cdot \Big\{\varphi\big(2^{i+1}R\big) + R_0^4 2^{-4(N-i)} + \\ \frac{M}{C\sqrt{N-i+2}} \, R_0^{1+\frac{\lambda_0}{2}} \, 2^{-(1+\frac{\lambda_0}{2})\cdot(N-i)}\Big\} \, . \end{aligned} \tag{2.106}$$

Iteration with respect to i gives:

(2.107)

$$\varphi(R) \le \left\{\prod_{k=1}^{N} \frac{C\sqrt{k+2}}{C\sqrt{k+2}+1}\right\} \varphi(2^N R) + R_0^4 \sum_{i=1}^{N} \left\{\prod_{k=i}^{N} \frac{C\sqrt{k+2}}{C\sqrt{k+2}+1}\right\} 2^{-4i} +$$
$$M\ R_0^{1+\frac{\lambda_0}{2}} \sum_{i=1}^{N} \left\{\prod_{k=i}^{N} \frac{C\sqrt{k+2}}{C\sqrt{k+2}+1}\right\} 2^{-i(1+\frac{\lambda_0}{2})} \cdot \frac{1}{C\sqrt{i+2}}\,.$$

Now we take over from [SF99] the following inequalities:

$$\begin{aligned}\prod_{k=s}^{N} \frac{C\sqrt{k+2}}{C\sqrt{k+2}+1} &= \prod_{k=s}^{N}\Big(1 - \frac{1}{C\sqrt{k+2}+1}\Big)\\ &\le \exp\Big(-\sum_{k=s}^{N} \frac{1}{C\sqrt{k+2}+1}\Big)\,,\end{aligned} \tag{2.108}$$

$$\sum_{k=1}^{N} \frac{1}{C\sqrt{k+2}+1} \ge \frac{2}{(C+1)}\big(\sqrt{N+3} - \sqrt{s+2}\big)\,, \tag{2.109}$$

implying

$$\begin{aligned}\varphi(R) \le \exp\Big(-\frac{2\sqrt{N+1}}{C+1}\Big) \cdot \Big\{&\exp\Big(\frac{2\sqrt{3}}{C+1}\Big)\varphi(2R_0) + \\ &R_0^4 \sum_{i=1}^{N} 2^{-4i} \exp\Big(\frac{2\sqrt{i+1}}{C+1}\Big) + \frac{M}{C}\ R_0^{1+\frac{\lambda_0}{2}} \sum_{i=1}^{N} 2^{i(1+\frac{\lambda_0}{2})} \cdot \frac{1}{\sqrt{i+2}}\Big\}\,.\end{aligned} \tag{2.110}$$

Since, for any $q > 1$, there exists a positive constant $C(q)$ such that

$$\exp(-t) \le \frac{C(q)}{t^q} \qquad \text{for } t > 0$$

and

$$A \equiv \sum_{i=1}^{\infty} 2^{-4i} \exp\Big(\frac{2\sqrt{i+1}}{C+1}\Big) < +\infty\,,$$
$$B \equiv \sum_{i=1}^{\infty} 2^{-i(1+\frac{\lambda_0}{2})} \frac{1}{\sqrt{i+2}} < +\infty\,,$$

we get from (2.110) (and the definition of N!):

$$
\begin{aligned}
\varphi(R) &\le \frac{C(q,C)}{(N+1)^{q/2}}\left\{\exp\Big(\frac{2\sqrt{3}}{C+1}\Big)\varphi(2R_0)+A\,R_0^4+B\,\frac{M}{C}R_0^{1+\frac{\lambda_0}{2}}\right\} \\
&\le \frac{C(q,C)}{\Big(\log_2\big(\frac{2R_0}{R}\big)\Big)^{q/2}}\cdot C(A,B,C,R_0)\,.
\end{aligned} \tag{2.111}
$$

Especially for $q>2$ we have:

$$
\int\limits_{B_R(x_o)} H^2(x)\,dx \le \frac{C}{\log_2\big(\frac{2R_0}{R}\big)^{q/2}}\,, \qquad \frac{q}{2}>1\,. \tag{2.112}
$$

This is the logarithmic Morrey condition we wanted to establish.

In particular, we have by (2.112) and (2.84):

$$
\int\limits_{B_R(x_o)} |\nabla\sigma|^2\,dx \le \frac{\text{const}}{\big(\log_2\big(\frac{2R_0}{R}\big)\big)^{q}}\,, \qquad q>2,. \tag{2.113}
$$

This implies

$$
\sigma \in C^0_{\text{loc}}(\Omega) \tag{2.114}
$$

and, therefore,

$$
\varepsilon(u) \in C^0_{\text{loc}}(\Omega)\,, \tag{2.115}
$$

because of equation (2.29). These implications follow from Frehse's "logarithmic" analogue of the well known Dirichlet-growth theorem of Morrey (see [Mor68, Theorem 3.5.2] and [Fre77, Lemma 1.1]):

Lemma 2-X *Suppose $u \in W^{1,p}(B_R(x_0))$, $1\le p\le d$, and that there are constants $\mu>1$ and $L>0$ such that*

$$
\begin{gathered}
\int\limits_{B_r} |\nabla u|^p\,dx \le L^p\Big(\frac{r}{\delta}\Big)^{d-p}\left|\log\Big(\frac{r}{\delta}\Big)\right|^{-\mu p}\,, \\
0<r<\delta = R-|x-x_0|
\end{gathered} \tag{2.116}
$$

for every $x \in B_R(x_0)$. *Then* $u \in C^0(B_r(x_0))$ *for* $r < R$ *and*

$$|u(\xi) - u(x)| \le C\, L\delta^{1-\frac{d}{p}} \left|\log\Big(\frac{|x-\xi|}{\delta}\Big)\right|^{1-\mu} \tag{2.117}$$

for $|\xi - x| \le \frac{\delta}{2}$, *where*

$$C = 4\Gamma_d^{-1/p}(\mu-1)^{-1}\,,$$

Γ_d *being the volume of the unit ball in* $\mathbb{R}^d$.

To get (2.114) apply Lemma 2-X with $u = \sigma$, $p = 2 = d$, $L =$ const from (2.113) and $\mu = \frac{q}{2}$, q as in (2.113). Moreover, there is a constant $C = C(D)$ for $D \subset\subset \Omega$ such that

$$\|\sigma\|_{0,\infty;D}\,,\ \|\varepsilon(u)\|_{0,\infty;D} \le C(D)\,. \tag{2.118}$$

Due to this estimate and because of

$$\sigma = \sigma^D + \frac{1}{d}(\operatorname{tr}\sigma)I\,,\ |\sigma|^2 = |\sigma^D|^2 + \frac{1}{d}(\operatorname{tr}\sigma)^2$$

we also know

$$\sigma^D \in L^\infty(D) \quad \text{and} \quad \|\sigma^D\|_{0,\infty;D} \le \text{const}\,,$$

but this means that our auxiliary function h also belongs to $L^\infty(D)$ and is estimated correspondingly, therefore we can rewrite inequality (2.100) in the following way:

$$\begin{aligned}\int\limits_{B_R(x_0)} H^2(x)\,dx \le \tilde{C}_1\, R^{1+\frac{\lambda_0}{2}} \int\limits_{B_{2R}(x_0)} |H|\,dx + \tilde{C}_2\, R \int\limits_{T_R(x_0)} |H|\,dx \\ + \frac{\tilde{C}_3}{R} \int\limits_{T_R(x_0)} |H|\,dx \Big(\int\limits_{T_R(x_0)} H^2(x_0)\,dx\Big)^{1/2}\end{aligned} \tag{2.119}$$

for any $x_0 \in D \subset\subset \Omega$, $B_{2R}(x_0) \subset D$.

The last estimate immediately implies a usual Morrey condition for H^2 via the usual hole-filling (see for example [DHKW92] or [FR96]), from which we obtain

$$\nabla\sigma \in L^{2,2\gamma}_{\text{loc}}(\Omega) \quad \text{for some positive } \gamma \tag{2.120}$$

and

$$\sigma \in C^{0,\gamma}_{\mathrm{loc}}(\Omega) \tag{2.121}$$

either directly from (2.120) via Morrey's Dirichlet-growth theorem or via Poincaré's inequality and Campanato's characterisation of Hölder continuous functions: $\nabla\sigma \in L^{2,2\gamma}_{\mathrm{loc}} \Longrightarrow \sigma \in \mathcal{L}^{2,2+2\gamma}_{\mathrm{loc}} \cong C^{0,\gamma}_{\mathrm{loc}}$.

The Hölder continuity of σ implies then also the Hölder continuity of $\varepsilon(u)$ because of the equation

$$A\sigma + |\sigma^D|^{p-2}\sigma^D = \varepsilon(u)\ .$$

An even better information is available from (2.119), namely it implies immediately a higher integrability of H^2 because slightly rewritten (2.119) can be read as a "reverse Hölder inequality" which in conjunction with a well known result of Gehring-Giaquinto-Modica-Stredulinsky (see [Gia83]) implies the mentioned higher integrability of H^2, i. e. there are a constant $C = C(D)$ and a real number $t > 1$ such that

$$\int_D H^{2t}\,dx \le C(D)\,. \tag{2.122}$$

But, once again, we get from (2.122)

$$\int_D |\nabla\sigma|^{2t}\,dx \le C(D)\,, \tag{2.123}$$

which means, due to the Morrey-Sobolev imbedding $W^{1,2t} \hookrightarrow C^{0,1-\frac{1}{t}}$, $t > 1$ in two dimensions, that σ is a locally Hölder continuous function in Ω. Therefore Theorem 2-VII is (finally) proven.

Let us close this section with some remarks:

Remark 2-XI 1) Our proof as well as the result that $\varepsilon(u)$ is locally Hölder-continuous clearly show, that this method of proof cannot be used to show that the stresses are uniformly continuous with respect to p. So we still do not know whether the stresses σ_{ij} are continuous or not for the Hencky model, nevertheless it is good to know that at least the approximation of the Hencky model – namely our solutions to the Norton-Hoff-model – have (Hölder) continuous stresses.

On the other hand it might help that we know that the "solution-stresses" for the Norton-Hoff model are Hölder continuous, i. e. it might be possible to obtain a weaker – for example an L^∞-estimate, but uniformly with respect to p by using that the approximations are already Hölder continuous. Maybe we can clarify this in the future.

2) It would be good to have a physical reason/motivation to look on the function $H^2 = \nabla\varepsilon(u) : \nabla\sigma = D_k\varepsilon_{ij}(u)D_k\sigma_{ij}$, which somehow controls both quantities: $\nabla\sigma$ and $\nabla\varepsilon(u)$. In analogy to the quasistatic case which we are going to discuss in the next section we will call

$$B = \sigma_{ij}^D \cdot \varepsilon_{ij}(u) = \sigma_{ij}^D \varepsilon_{ij}^D(u)$$

the "loading function" (see P. Haupt [Hau00, discussion on p. 207-211] and the survey article of A. M. Freudenthal and H. Geininger: "Elasticity and Plasticity" in the Encyclopedia of Physics, vol. VI, [FG58, discussion on p. 278–283, esp. p. 280]).

So our regularity result(s) seem to indicate that a statement like: "if the loading function is *well-behaved* (whatever this means in some concrete situation), then stresses and/or strains are also *well-behaved*" is in some sense the physical explanation respective reason for regularity. In other words: A nicely controlled loading function implies that the "physical process" can be (somehow) nicely controlled, which means that we are able to control the relevant physical quantities with help of the loading function (and may be some other given, inherent in the problem functions ...).

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Chapter 3

On Existence of Steady Flows of Fluids with Shear-Dependent Viscosity

3.1 Introduction and Problem Formulation

We deal with a system of partial differential equations describing a steady motion of an incompressible fluid with shear-dependent viscosity and present a new global existence result for $p > \frac{2d}{d+2}$. Here p is the coercivity parameter of the nonlinear elliptic operator related to the stress tensor and d is the dimension of the space. Lipschitz test functions, a subtle splitting of the level sets of the maximal functions for the velocity gradients, and a decomposition of the pressure are incorporated to obtain almost everywhere convergence of the velocity gradients.

Let Ω be an open bounded set in $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. We consider the following system of partial differential equations in Ω

$$-\operatorname{div}\tilde{\mathbf{T}} + \operatorname{div}(\mathbf{v}\otimes\mathbf{v}) + \nabla P = \mathbf{f}\,, \tag{3.1}$$

$$\operatorname{div}\mathbf{v} = 0\,, \tag{3.2}$$

subject to the Dirichlet (no-slip) boundary conditions

$$\mathbf{v} = \mathbf{0} \quad \text{on} \quad \partial\Omega\,. \tag{3.3}$$

Here, $\mathbf{v} = \mathbf{v}(x) = (v^1(x), \dots, v^d(x))$ and $P = P(x)$ denote the unknown velocity and pressure fields at the point $x \in \Omega$, while $\mathbf{f} = \mathbf{f}(x) = (f^1(x), \ldots, f^d(x))$ is a given external force and $\tilde{\mathbf{T}}$ is the extra stress for which the constitutive equation will be provided below.

The system (3.1)–(3.2) describes steady motions of an incompressible fluid. If we suppose that $\tilde{\mathbf{T}}$ is a tensorial function of the velocity gradient $\nabla \mathbf{v}$, the principle of material frame indifference then implies that this dependence happens only through its symmetric part $\mathbf{D}(\mathbf{v})$. Thus,

$$\tilde{\mathbf{T}}(x) \equiv \mathbf{T}(x, \mathbf{D}(\mathbf{v}(x)))\,. \tag{3.4}$$

If in addition the fluid is homogeneous, then (3.4) reduces to

$$\tilde{\mathbf{T}}(x) \equiv \mathbf{T}(\mathbf{D}(\mathbf{v}(x)))\,. \tag{3.5}$$

Fluids with shear-dependent viscosity represent an important subclass of non-Newtonian fluids, consisting of fluids that have the ability to shear thin or shear thicken (see [Raj93]). The power-law fluids that enjoy significant attention among engineers and physicists fall into this category. Typical examples of considered models that are widely used in engineering practise are given by

$$\mathbf{T}_1(\mathbf{D}(\mathbf{v})) \equiv \nu_0 |\mathbf{D}(\mathbf{v})|^{p-2}\mathbf{D}(\mathbf{v}) + \mu_\infty \mathbf{D}(\mathbf{v})\,, \tag{3.6}$$

$$\mathbf{T}_2(\mathbf{D}(\mathbf{v})) \equiv \nu_0 (\mu_0 + |\mathbf{D}(\mathbf{v})|^2)^{\frac{p-2}{2}}\mathbf{D}(\mathbf{v}) + \mu_\infty \mathbf{D}(\mathbf{v})\,, \tag{3.7}$$

$$\mathbf{T}_3(\mathbf{D}(\mathbf{v})) \equiv \mu_\infty \mathbf{D}(\mathbf{v}) + \mu_1 \operatorname{arsinh}(|\mathbf{D}(\mathbf{v})|)\frac{\mathbf{D}(\mathbf{v})}{|\mathbf{D}(\mathbf{v})|}\,, \tag{3.8}$$

where μ_0, μ_1, μ_∞ and ν_0 are (at least) nonnegative constants and $p \geq 1$.

Note that if $p = 2$ and $\mu_\infty > 0$ in (3.6), the fluid is Newtonian and (3.1) reduces to the well known Navier–Stokes equations. For $1 \leq p < 2$, stresses given by (3.6), (3.7) model shear thinning fluids, while for $p > 2$ they model shear thickening phenomena in fluids. Since a lot of models can be assigned to the former case, there is a need to have an existence theory for $p \in \langle 1, 2)$. (We refer the interested reader to [MNRR96], [MRR95] and [Raj93] for a more detailed discussion of the continuum mechanical background to these fluids.)

In this chapter we present new results on the existence of weak solutions to (3.1)–(3.3).

To prove existence of a weak solution to (3.1)–(3.2) and (3.3) two different methods were applied before. The first method is a combination of (weak) compactness for $\mathbf{v}$ and monotonicity arguments, which reveals to be applicable to (3.1)–(3.2) and (3.3) if $p \geq \frac{3d}{d+2}$. It was performed by J. L. Lions [Lio69] and O.A. Ladyzhenskaya [Lad67], [Lad68], [Lad69] in the late sixties.

The second method, which we call the L^∞-truncation method, yields existence of a weak solution if $p \geq \frac{2d}{d+1}$. It is based on the construction of a special (bounded) test function, a precise characterisation of the pressure, and relies also strongly on the strict monotonicity of $\mathbf{T}$. This latter method was successfully applied to the steady problem in [FMS97] and [Růž97] (in [Růž97] the limiting case $p = \frac{2d}{d+1}$ is not included).

Here we introduce yet another approach, which we call the Lipschitz truncation method, in order to prove the existence of a weak solution for $p > \frac{2d}{d+2}$. We construct a Lipschitz test function to show that for certain approximations $\mathbf{v}^n$ the tensors $\mathbf{D}(\mathbf{v}^n)$ converge almost everywhere to its weak limit $\mathbf{D}(\mathbf{v})$, which is the crucial point in proving that $\mathbf{v}$ is a weak solution to (3.1), (3.2) and (3.3). Because of earlier results (mentioned above) we can restrict our consideration to the case $p \in (\frac{2d}{d+2}, \frac{2d}{d+1})$.

Note that Lipschitz truncations of Sobolev functions were already successfully used in different contexts, see [AF84], [AF88], [DHM97], [DHM00], [Iwa97], [Lan96], [Mül99], and [Zha88], [Zha90], [Zha92]. The novelty of our application of the Lipschitz approximation of Sobolev functions consists of discovering the mechanism of obtaining almost everywhere convergence of gradients for weakly convergent sequences.

If the convective term $\operatorname{div}(\mathbf{v} \otimes \mathbf{v})$ is neglected in (3.1) and the extra stress tensor $\mathbf{T} = (T_{ij})_{i,j=1}^d$ has a potential, i.e. $T_{ij} = \frac{\partial \Phi}{\partial D_{ij}}$, a variational approach can be used. We refer to the recent work of Fuchs and Seregin [Fuc96a], [Fuc96b], [FS97], [FS00], where in particular regularity questions for this kind of problems are discussed.

Finally we mention that the corresponding time-dependent system is treated in [BBN94], [FMS00], [Lad67], [Lad68], [Lad69], [Lio69], [MNR93], [MNR01], [MNRR96], [MRR95] and [Pok96]. We are not going to discuss the dependence of the known existence results on p. We wish to emphasize, however, that we believe that a convenient probably not straightforward modification of the techniques presented here can improve also the existence results for the evolutionary model.

The chapter is organized as follows: In Section 3.2 we fix our notations, provide the assumptions on the form of $\mathbf{T}$, show that they are satisfied by the examples given above, and formulate the main result. After that, in

Section 3.3, we introduce suitable approximations to (3.1) and study their basic properties. Then, in Section 3.4, we present a subtle decomposition of the pressure suitable to our analysis. The last Section 3.5 is devoted to the proof of the main theorem.

3.2 Notation and Main Theorem

First we fix our notation. Let $\mathbb{M}$ denote the space of all real $(d \times d)$ matrices $\mathbf{F} = (F_{ij})$, and $\mathbb{S}$ be its subspace consisting of all symmetric $(d \times d)$ matrices. Using the usual summation convention on repeated indices we set $\mathbf{a} \cdot \mathbf{b} \equiv a_i b_i$ for $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^d$ and $\mathbf{F} : \mathbf{H} \equiv F_{ij} H_{ij}$ for $\mathbf{F}$, $\mathbf{H} \in \mathbb{M}$. Also we set $|\mathbf{a}| \equiv (\mathbf{a} \cdot \mathbf{a})^{1/2}$ and $|\mathbf{F}| \equiv (\mathbf{F} : \mathbf{F})^{1/2}$.

For functions $\mathbf{v} : \Omega \to \mathbb{R}^d$ we introduce the differential operator

$$\mathbf{D}(\mathbf{v}) \equiv \frac{1}{2} \left(\nabla \mathbf{v} + \nabla \mathbf{v}^T \right) : \Omega \to \mathbb{S}, \qquad \text{where } D_{ij}(\mathbf{v}) \equiv \frac{1}{2} (D_i v^j + D_j v^i) .$$

We use standard notation of function spaces. If $1 \le q \le +\infty$, then $L^q(\Omega)$ and $W^{k,q}(\Omega)$ ($\mathring{W}^{k,q}(\Omega)$) denote the usual Lebesgue and Sobolev spaces of scalar-, vector- and tensor-valued functions (with zero traces at the boundary $\partial\Omega$). The norm of $u \in W^{k,q}(\Omega)$ is defined as $\|u\|^q_{k,q;\Omega} \equiv \sum_{|\alpha| \le k} \int_\Omega |D^\alpha u|^q \, dx$.

By $W^{-1,p'}(\Omega)$ we mean the dual space $(\mathring{W}^{1,p}(\Omega))'$ to $\mathring{W}^{1,p}(\Omega)$ with corresponding duality pairing $\langle ., . \rangle_{1,p,\Omega}$.

As usual, $C_0^\infty(\Omega)$ denotes the set of all C^∞-functions with compact support in Ω, while the space $C_{0,\sigma}^\infty(\Omega)$ consists of $\mathbf{\Phi} \in C_0^\infty(\Omega)$ such that $\operatorname{div} \mathbf{\Phi} = 0$. For $p, q \ge 1$ we set

$$\begin{aligned}
H_q &\equiv \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_{0,q}} = \{ \mathbf{v} \in L^q(\Omega) \, : \operatorname{div} \mathbf{v} = 0 \, , \mathbf{v} \cdot \mathbf{n} = 0 \text{ at } \partial\Omega \} \, , \\
V_p &\equiv \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\nabla \cdot\|_{0,p}} = \{ \mathbf{v} \in \mathring{W}^{1,p}(\Omega) \, : \operatorname{div} \mathbf{v} = 0 \} \, , \\
V_p^{'} &\equiv \text{ dual of } V_p \, .
\end{aligned}$$

The brackets $\langle . \, , \, . \rangle_{V_p}$ represent the last duality pairing.

If $\mathbf{g}, \mathbf{h}$ are vector-valued functions and $g_i h_i \in L^1(\Omega)$, then

$$(\mathbf{g}, \mathbf{h}) \equiv \int_\Omega \mathbf{g} \cdot \mathbf{h} \, dx \; .$$

Analogously, for tensor-valued functions $\boldsymbol{\eta}, \boldsymbol{\xi}$ satisfying $\eta_{ij}\xi_{ij} \in L^1(\Omega)$ we set

$$(\boldsymbol{\eta}, \boldsymbol{\xi}) \equiv \int_\Omega \boldsymbol{\eta} : \boldsymbol{\xi} \, dx \ ,$$

We will also use the Korn inequality (see [Neč66] for a proof) saying that for $1 < p < +\infty$ there exists a constant $K_p = K_p(\Omega)$ such that

$$\|\nabla \mathbf{v}\|_{0,p} \le K_p \, \|\mathbf{D}(\mathbf{v})\|_{0,p} \quad \text{for all } \mathbf{v} \in \mathring{W}^{1,p}(\Omega) \,. \tag{3.9}$$

Concerning the extra stress tensor $\mathbf{T} = (T_{ij}) \in \mathbb{S}$ we will assume that $\mathbf{T}$ is a Carathéodory-function (i.e. for each fixed $\mathbf{F} \in \mathbb{S}$ the function $x \mapsto \mathbf{T}(x, \mathbf{F})$ is (Lebesgue-) measurable in Ω and the function $\mathbf{F} \mapsto \mathbf{T}(x, \mathbf{F})$ is continuous in $\mathbb{S}$ for almost every $x \in \Omega$) and satisfies for some $p > 1$ the conditions of

- p-coercivity: there are $c_1 > 0$ and $\varphi_1 \in L^1(\Omega)$ such that

  $$\mathbf{T}(x, \boldsymbol{\eta}) : \boldsymbol{\eta} \ge c_1 |\boldsymbol{\eta}|^p - \varphi_1(x) \tag{3.10}$$

  for almost all $x \in \Omega$ and for all $\boldsymbol{\eta} \in \mathbb{S}$;

- polynomial growth of order $p - 1$: there are $c_2 > 0$ and $\varphi_2 \in L^{\frac{p}{p-1}}(\Omega)$ such that

  $$|\mathbf{T}(x, \boldsymbol{\eta})| \le c_2 |\boldsymbol{\eta}|^{p-1} + \varphi_2(x) \tag{3.11}$$

  for almost all $x \in \Omega$ and for all $\boldsymbol{\eta} \in \mathbb{S}$;

- strict monotonicity:

  $$(\mathbf{T}(x, \boldsymbol{\eta}) - \mathbf{T}(x, \boldsymbol{\xi})) : (\boldsymbol{\eta} - \boldsymbol{\xi}) > 0 \tag{3.12}$$

  for almost all $x \in \Omega$ and for all $\boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbb{S}$ such that $\boldsymbol{\eta} \neq \boldsymbol{\xi}$.

Definition 3-I *Assume that* $\mathbf{f} \in W^{-1,p'}(\Omega)$ *and (3.10)–(3.12) hold. We say that* $\mathbf{v} \in V_p$ *is a weak solution to problem (3.1)–(3.3) if*

$$\begin{aligned} \int_\Omega \mathbf{T}(x, \mathbf{D}(\mathbf{v})) : \mathbf{D}(\boldsymbol{\Phi}) \, dx &= \langle \mathbf{f}, \boldsymbol{\Phi} \rangle_{1,p} \\ &\quad + \int_\Omega (\mathbf{v} \otimes \mathbf{v}) : \mathbf{D}(\boldsymbol{\Phi}) \, dx \qquad \textit{for all } \boldsymbol{\Phi} \in C^\infty_{0,\sigma}(\Omega) \,. \end{aligned} \tag{3.13}$$

Note that $(\mathbf{v}\otimes\mathbf{v})_{ij} \equiv v^i v^j \in L^1(\Omega)$ for $p \geq \frac{2d}{d+2}$ due to Sobolev's imbedding theorem. Now we are ready to formulate our existence theorem.

Theorem 3-II *Let $p > \frac{2d}{d+2}$. Then there exists a weak solution $\mathbf{v} \in V_p$ to (3.1)–(3.3) in the sense of Definition 3-I.*

Let us finish this section by showing that the tensors given by formulas (3.6) and (3.7) in section 3.1 satisfy the hypotheses of Theorem 3-II if $\nu_0 > 0$ and μ_0, $\mu_\infty \geq 0$. (The third form (3.8) satisfies the assumption $p > \frac{2d}{d+2}$ only if $\mu_\infty > 0$.)

Example 3-III Consider $\mathbf{T}_1(\mathbf{F}) = \nu_0|\mathbf{F}|^{p-2}\mathbf{F} + \mu_\infty\mathbf{F}$ with constants $\nu_0 > 0$ and $\mu_\infty \geq 0$ and $\mathbf{F} \in \mathbb{S}$ (corresponding to $\mathbf{D}(\mathbf{v})$). Obviously $\mathbf{T}_1$ is continuous and satisfies both the growth condition

$$|\mathbf{T}(\mathbf{F})| \leq \nu_0|\mathbf{F}|^{p-1} + \mu_\infty|\mathbf{F}|$$

and the coercivity condition

$$\mathbf{T}(\mathbf{F}) : \mathbf{F} = \nu_0|\mathbf{F}|^p + \mu_\infty|\mathbf{F}|^2 \geq 2\nu_0|\mathbf{F}|^p\,.$$

For monotonicity we consider two cases.
In <u>Case 1</u> we assume $p \geq 2$. Then we have

$$(\mathbf{T}_1(\mathbf{F}_1) - \mathbf{T}_1(\mathbf{F}_2)) : (\mathbf{F}_1 - \mathbf{F}_2) \geq \nu_0\gamma_0(p,d)|\mathbf{F}_1 - \mathbf{F}_2|^p + \mu_\infty|\mathbf{F}_1 - \mathbf{F}_2|^2,$$

where we used Lemma 4.4 of [DiB93, page 13]. This inequality shows not only that $\mathbf{T}_1$ is strictly monotone, but also that $\mathbf{T}_1$ is uniformly monotone (see [Zei90, page 500ff., Def. 25.2]).
<u>Case 2</u> consists of $1 < p < 2$ and we are going to verify that

$$(\mathbf{T}_1(\mathbf{F}_1) - \mathbf{T}_1(\mathbf{F}_2)) : (\mathbf{F}_1 - \mathbf{F}_2) \geq \nu_0\gamma_1(p,d)\frac{|\mathbf{F}_1 - \mathbf{F}_2|^2}{(|\mathbf{F}_1| + |\mathbf{F}_2|)^{2-p}} + \mu_\infty|\mathbf{F}_1 - \mathbf{F}_2|^2.$$

To prove it, it is enough to show that

$$(|\mathbf{a}|^{p-2}\mathbf{a} - |\mathbf{b}|^{p-2}\mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \geq \gamma_1(p,d)\frac{|\mathbf{a} - \mathbf{b}|^2}{(|\mathbf{a}| + |\mathbf{b}|)^{2-p}} \quad \text{for } \mathbf{a}, \mathbf{b} \in \mathbb{R}^k\,,$$

which is due to the following computation

$$\begin{aligned}
&(|\mathbf{a}|^{p-2}\mathbf{a} - |\mathbf{b}|^{p-2}\mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\
&= \left(\int_0^1 \tfrac{d}{ds} |s\mathbf{a} + (1-s)\mathbf{b}|^{p-2} (s\mathbf{a} + (1-s)\mathbf{b})\, ds \right) \cdot (\mathbf{a} - \mathbf{b}) \\
&= \int_0^1 |s\mathbf{a} + (1-s)\mathbf{b}|^{p-2} |\mathbf{a} - \mathbf{b}|^2\, ds \\
&\qquad + \int_0^1 (p-2) |s\mathbf{a} + (1-s)\mathbf{b}|^{p-2} \left((\mathbf{a} - \mathbf{b}) \cdot \frac{s\mathbf{a} + (1-s)\mathbf{b}}{|s\mathbf{a} + (1-s)\mathbf{b}|} \right)^2 ds \\
&\geq (1 + \min(0, p-2)) \int_0^1 |s\mathbf{a} + (1-s)\mathbf{b}|^{p-2}\, ds\, |\mathbf{a} - \mathbf{b}|^2 \\
&\geq (p-1)\, \frac{|\mathbf{a} - \mathbf{b}|^2}{(|\mathbf{a}| + |\mathbf{b}|)^{2-p}}\,,
\end{aligned}$$

where we use the fact that $1 < p < 2$ and $|s\mathbf{a} + (1-s)\mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$ for all $s \in [0, 1]$. Thus, we conclude that $\mathbf{T}_1$ is strictly monotone, but in general not uniformly monotone.

Example 3-IV Consider $\mathbf{T}_2(\mathbf{F}) = \nu_0(\mu_0 + |\mathbf{F}|^2)^{\frac{p-2}{2}}\mathbf{F} + \mu_\infty \mathbf{F}$ with constants $\nu_0 > 0$ and $\mu_0, \mu_\infty \geq 0$. Similar considerations as in the previous example show that all hypotheses of Theorem 3-II are satisfied (see [MNRR96, chap. 5, pages 193–196, 198ff., Lemma 1.19]).

3.3 Approximations and Their Properties

Firstly we define an approximation to our problem: For $m = 1, 2, 3, \ldots, p > 1$ and $q \geq \frac{2p}{p-1} = 2p'$ we look for $(\mathbf{v}^m, P^m)$ solving in Ω

$$\begin{aligned}
-\mathrm{div}\, \mathbf{T}(\cdot, \mathbf{D}(\mathbf{v}^m)) + \mathrm{div}\, (\mathbf{v}^m \otimes \mathbf{v}^m) + \frac{1}{m} |\mathbf{v}^m|^{q-2} \mathbf{v}^m &= \mathbf{f} - \nabla P^m\,, \\
\mathrm{div}\, \mathbf{v}^m &= 0\,,
\end{aligned} \tag{3.14}$$

complemented by the boundary conditions

$$\mathbf{v}^m = \mathbf{0} \quad \text{on } \partial\Omega. \tag{3.15}$$

The following lemma can be proved by standard monotonicity arguments and the compact imbedding $\overset{\circ}{W}{}^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$ valid for $p > \frac{2d}{d+2}$.

Lemma 3-V *Let $p > \frac{2d}{d+2}$ and $q \geq \frac{2p}{p-1} = 2p'$. Suppose $\mathbf{f} \in W^{-1,p'}(\Omega)$. Then there exists $\mathbf{v}^m \in V_p \cap H_q$ satisfying*

$$\begin{aligned}\int_\Omega \mathbf{T}(\cdot, \mathbf{D}(\mathbf{v}^m)) : \mathbf{D}(\mathbf{\Phi})\, dx + \frac{1}{m} \int_\Omega |\mathbf{v}^m|^{q-2}\mathbf{v}^m \cdot \mathbf{\Phi}\, dx = \langle \mathbf{f}, \mathbf{\Phi} \rangle_{1,p} \\ + \int_\Omega (\mathbf{v}^m \otimes \mathbf{v}^m) : \mathbf{D}(\mathbf{\Phi})\, dx \quad \text{for all } \mathbf{\Phi} \in C^\infty_{0,\sigma}(\Omega)\,.\end{aligned} \tag{3.16}$$

Moreover, all $\mathbf{v}^m$ satisfy the following uniform estimate

$$\|\mathbf{D}(\mathbf{v}^m)\|^p_{0,p} + \|\nabla \mathbf{v}^m\|^p_{0,p} + \frac{1}{m}\|\mathbf{v}^m\|^q_{0,q} \leq K \tag{3.17}$$

and consequently, due to the growth condition (3.11) and Sobolev's imbedding theorem

$$\|\mathbf{T}(\cdot, \mathbf{D}(\mathbf{v}^m))\|_{0,p'} \leq K\,, \tag{3.18}$$

$$\|\mathbf{v}^m\|_{0,\frac{dp}{d-p}} \leq K\,, \tag{3.19}$$

$$\|\mathbf{v}^m \otimes \mathbf{v}^m\|_{0,\frac{dp}{2(d-p)}} \leq K\,. \tag{3.20}$$

Remark 3-VI (1) The constant K depends on $\|\mathbf{f}\|_{-1,p'}$, the constants in the Sobolev and Korn inequalities and d and maximizes all a priori estimates used in the text.

(2) For fixed m the existence result of Lemma 3-V is obvious for any $p > 1$ and $q \geq 2p'$. This follows from the compact imbedding $\mathring{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ (instead of $\mathring{W}^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$) together with the facts that $\|\mathbf{v}^m\|^q_{0,q} \leq Km$ and $\|\mathbf{v}\|_{0,2} \leq \|\mathbf{v}\|^\lambda_{0,p}\|\mathbf{v}\|^{1-\lambda}_{0,2p'}$ with $\lambda = \frac{1}{3-p}$ and $1 - \lambda = \frac{2-p}{3-p}$, and from the standard monotone operator theory (Minty's trick). ■

Next, we introduce the (approximative) pressures P^m observing that in (3.16) we can use test functions $\mathbf{\Phi}$ from $V_p \cap V_r = V_r$, where $\frac{1}{r} = 1 + \frac{2}{d} - \frac{2}{p} = \frac{(d+2)p-2d}{dp} < \frac{1}{d}$ because of $V_p \hookrightarrow\hookrightarrow L^{\frac{dp}{d-p}}$. Let us note that $V_r \hookrightarrow\hookrightarrow L^\infty$ for $\frac{2d}{d+2} < p < \frac{2d}{d+1}$. Defining the functional $\mathbf{F}^m$ as

$$\begin{aligned}\langle \mathbf{F}^m, \mathbf{\Phi} \rangle_{1,r,\Omega} \equiv \int_\Omega \mathbf{T}(\cdot, \mathbf{D}(\mathbf{v}^m)) : \mathbf{D}(\mathbf{\Phi})\, dx + \frac{1}{m} \int_\Omega |\mathbf{v}^m|^{q-2}\mathbf{v}^m \cdot \mathbf{\Phi}\, dx \\ - \int_\Omega (\mathbf{v}^m \otimes \mathbf{v}^m) : \mathbf{D}(\mathbf{\Phi})\, dx - \langle \mathbf{f}, \mathbf{\Phi} \rangle_{1,p,\Omega}\,,\end{aligned} \tag{3.21}$$

we see that $\langle \mathbf{F}^m, \mathbf{\Phi}\rangle_{1,r,\Omega} = 0$ for all $\mathbf{\Phi} \in C^\infty_{0,\sigma}(\Omega)$ due to (3.16). Moreover, $\mathbf{F}^m \in W^{-1,r'}(\Omega)$ and

$$\|\mathbf{F}^m\|_{-1,r'} \le K\ , \quad r' = \frac{r}{r-1} = \frac{dp}{dp-(d+2)p+2d} = \frac{dp}{2(d-p)}\ .$$

By a version of De Rham's theorem (see for example [AG94, Theorem 2.8, page 116 ff.]) there exists $P^m \in L^{r'}(\Omega)$ with zero mean value over each component of Ω such that

$$\langle \mathbf{F}^m, \mathbf{\Phi}\rangle_{1,r,\Omega} \equiv \langle -\nabla P^m, \mathbf{\Phi}\rangle_{1,r,\Omega} = \int_\Omega P^m \operatorname{div} \mathbf{\Phi}\, dx \tag{3.22}$$

and

$$\|P^m\|_{0,r'} \le C\ \|\nabla P^m\|_{-1,r'} \le C\ \|F^m\|_{-1,r'} \le K\,. \tag{3.23}$$

As a consequence of these observations we obtain the following equivalent weak formulation to (3.16):

$$\begin{aligned} &\int_\Omega \mathbf{T}(\cdot, \mathbf{D}(\mathbf{v}^m)) : \mathbf{D}(\mathbf{\Phi})\, dx + \frac{1}{m}\int_\Omega |\mathbf{v}^m|^{q-2}\mathbf{v}^m \cdot \mathbf{\Phi}\, dx \\ &= \langle \mathbf{f}, \mathbf{\Phi}\rangle_{1,p} + \int_\Omega (\mathbf{v}^m \otimes \mathbf{v}^m) : \mathbf{D}(\mathbf{\Phi})\, dx + \int_\Omega P^m \operatorname{div} \mathbf{\Phi}\, dx \end{aligned} \tag{3.24}$$

valid for all $m = 1, 2, 3, \dots$ and all $\mathbf{\Phi} \in \overset{\circ}{W}{}^{1,r}(\Omega)$ with $r = \frac{dp}{(d+2)p-2d}$. Note again that if $p \in (\frac{2d}{d+2}, \frac{2d}{d+1})$, then $r > d$.

The uniform estimates (3.17)–(3.20) and (3.23) imply the existence of a subsequence $\{(\mathbf{v}^k, P^k)\}_{k\in\mathbb{N}} = \{(\mathbf{v}^{m_k}, P^{m_k})\}_{k\in\mathbb{N}}$ of $\{\mathbf{v}^m\}_{m\in\mathbb{N}}$ and $(\mathbf{v}, P) \in V_p \times L^{r'}(\Omega)$ such that $(k \to \infty)$

$$\mathbf{D}(\mathbf{v}^k) \rightharpoonup \mathbf{D}(\mathbf{v}) \qquad \text{weakly in } L^p(\Omega)\,, \tag{3.25}$$

$$\nabla \mathbf{v}^k \rightharpoonup \nabla \mathbf{v} \qquad \text{weakly in } L^p(\Omega)\,, \tag{3.26}$$

$$\mathbf{v}^k \to \mathbf{v} \qquad \text{strongly in } L^s(\Omega) \text{ for all } s \in \langle 1, 2r')\,, \tag{3.27}$$

$$\mathbf{v}^k \to \mathbf{v} \qquad \text{almost everywhere in } \Omega\,, \tag{3.28}$$

$$\mathbf{T}(\cdot, \mathbf{D}(\mathbf{v}^k)) \rightharpoonup \boldsymbol{\chi} \qquad \text{weakly in } L^{p'}(\Omega)\,, \tag{3.29}$$

$$P^k \rightharpoonup P \qquad \text{weakly in } L^{r'}(\Omega)\,. \tag{3.30}$$

Now we want to pass to the limit in (3.24) as $k \to \infty$. In order to do so we first observe that (3.17) implies for every $\mathbf{\Phi} \in C_0^\infty(\Omega)$ and $k \to \infty$

$$\left| \frac{1}{k} \int_\Omega |\mathbf{v}^k|^{q-2} \mathbf{v}^k \cdot \mathbf{\Phi}\, dx \right| \le \frac{1}{k^{1/q}} \left(\frac{1}{k} \|\mathbf{v}^k\|_q^q \right)^{\frac{q-1}{q}} \|\mathbf{\Phi}\|_q \quad \to 0\,. \tag{3.31}$$

The convective term is treated with the aid of the compact imbedding $\mathring{W}^{1,p} \hookrightarrow\hookrightarrow L^2$, $p > \frac{2d}{d+2}$. Writing $\mathbf{v}^k = \mathbf{v} + \mathbf{v}^k - \mathbf{v}$, we have for every $\mathbf{\Phi} \in C_0^\infty(\Omega)$ and $k \to \infty$

$$\begin{aligned} \int_\Omega (\mathbf{v}^k \otimes \mathbf{v}^k) : \mathbf{D}(\mathbf{\Phi})\, dx &= \int_\Omega [(\mathbf{v}^k - \mathbf{v}) \otimes \mathbf{v}] : \mathbf{D}(\mathbf{\Phi})\, dx \\ &\quad + \int_\Omega [\mathbf{v} \otimes (\mathbf{v}^k - \mathbf{v})] : \mathbf{D}(\mathbf{\Phi})\, dx + \int_\Omega (\mathbf{v} \otimes \mathbf{v}) : \mathbf{D}(\mathbf{\Phi})\, dx \\ &\quad \to \int_\Omega (\mathbf{v} \otimes \mathbf{v}) : \mathbf{D}(\mathbf{\Phi})\, dx\ . \end{aligned} \tag{3.32}$$

Owing to (3.30) we also observe that for $\mathbf{\Phi} \in C_0^\infty(\Omega)$ and $k \to \infty$

$$\int_\Omega P^k \operatorname{div} \mathbf{\Phi}\, dx \quad \to \quad \int_\Omega P \operatorname{div} \mathbf{\Phi}\, dx\,. \tag{3.33}$$

Collecting our results we find that $\mathbf{v} \in V_p$ satisfies

$$\int_\Omega \boldsymbol{\chi} : \mathbf{D}(\mathbf{\Phi})\, dx = \langle \mathbf{f}, \mathbf{\Phi} \rangle_{1,p} + \int_\Omega (\mathbf{v} \otimes \mathbf{v}) : \mathbf{D}(\mathbf{\Phi})\, dx + \int_\Omega P \operatorname{div} \mathbf{\Phi}\, dx \tag{3.34}$$

for all $\mathbf{\Phi} \in C_0^\infty(\Omega)$ respective $\mathbf{\Phi} \in \mathring{W}^{1,r}(\Omega)$.

Our aim now is to demonstrate that $\boldsymbol{\chi} = \mathbf{T}(\cdot, \mathbf{D}(\mathbf{v}))$. For this purpose it suffices to show that

$$\mathbf{D}(\mathbf{v}^k) \to \mathbf{D}(\mathbf{v}) \qquad \text{in measure on } \Omega$$

or almost everywhere convergence on compact subsets of Ω. If this was true, we could find a further subsequence by a diagonal procedure (for simplicity we do not change notation) such that

$$\mathbf{D}(\mathbf{v}^k) \to \mathbf{D}(\mathbf{v}) \qquad \text{almost everywhere in } \Omega\,. \tag{3.35}$$

Then, by Vitali's theorem (with the aid of the growth condition (3.11)) we obtain

$$\int_\Omega \mathbf{T}(\cdot,\mathbf{D}(\mathbf{v}^k)) : \mathbf{D}(\mathbf{\Phi})\,dx \quad \to \quad \int_\Omega \mathbf{T}(\cdot,\mathbf{D}(\mathbf{v})) : \mathbf{D}(\mathbf{\Phi})\,dx \tag{3.36}$$

and we can finish the proof of Theorem 3-II.

Note also that once we have (3.35), we easily conclude from (3.17) respectively (3.25) using Vitali's theorem that

$$\mathbf{D}(\mathbf{v}^k) \to \mathbf{D}(\mathbf{v}) \qquad \text{strongly in } L^s(\Omega) \qquad \text{for all } s \in \langle 1, p),$$

which is due to (3.9) tantamount to

$$\mathbf{v}^k \to \mathbf{v} \qquad \text{strongly in } \mathring{W}^{1,s}(\Omega) \text{ for all } s \in \langle 1, p).$$

The missing proof of (3.35) will be given in the Section 3.5, while the next section is devoted to a decomposition of the pressure P^k.

3.4 Decomposition of the Pressure

We start with solving four auxiliary Stokes problems, $I = 1, 2, 3, 4$,

$$\begin{aligned} -\Delta \mathbf{u}^{I_k} + \nabla P^{I_k} &= \mathbf{H}^{I_k} && \text{in } \Omega, \\ \operatorname{div} \mathbf{u}^{I_k} &= 0 && \text{in } \Omega, \\ \mathbf{u}^{I_k} &= \mathbf{0} && \text{on } \partial\Omega, \end{aligned} \tag{3.37}$$

where

$$\begin{aligned} \mathbf{H}^{1_k} &= -\operatorname{div} \mathbf{T}(\cdot,\mathbf{D}(\mathbf{v}^k))) \in (\mathring{W}^{1,p}(\Omega))^*, \\ \mathbf{H}^{2_k} &= \operatorname{div}(\mathbf{v}^k \otimes (\mathbf{v}^k - \mathbf{v})) \in (\mathring{W}^{1,r}(\Omega))^*, \\ \mathbf{H}^{3_k} &= \operatorname{div}((\mathbf{v}^k - \mathbf{v}) \otimes \mathbf{v})) \in (\mathring{W}^{1,r}(\Omega))^*, \\ \mathbf{H}^{4_k} &= \frac{1}{k}|\mathbf{v}^k|^{q-2}\mathbf{v}^k \in (L^q(\Omega))^*. \end{aligned} \tag{3.38}$$

The classical theory for the Stokes system (cf. [AG94] for example) implies the existence of solutions $(\mathbf{u}^{I_k}, P^{I_k})$, $I = 1, 2, 3, 4$, with the following estimates on the pressures P^{I_k} having zero mean value over each connected component of Ω :

$$\|P^{1_k}\|_{0,p';\Omega} \le C\|\mathbf{H}^{1_k}\|_{(\mathring{W}^{1,p}(\Omega))^*} \le C\|\mathbf{T}(\cdot,\mathbf{D}(\mathbf{v}^k))\|_{0,p';\Omega}\,, \tag{3.39}$$

$$\begin{aligned}\|P^{2_k}\|_{0,r';\Omega} &\le C\|\mathbf{H}^{2_k}\|_{(\mathring{W}^{1,r}(\Omega))^*} \le C\|\mathbf{v}^k\otimes(\mathbf{v}^k-\mathbf{v})\|_{0,r';\Omega}\\ &\le \|\mathbf{v}^k\|_{0,2r';\Omega}\|\mathbf{v}^k-\mathbf{v}\|_{0,2r';\Omega}\,,\end{aligned} \tag{3.40}$$

$$\|P^{3_k}\|_{0,r';\Omega} \le C\|\mathbf{H}^{3_k}\|_{(\mathring{W}^{1,r}(\Omega))^*} \le \|\mathbf{v}^k-\mathbf{v}\|_{0,2r';\Omega}\|\mathbf{v}^k\|_{0,2r';\Omega}\,, \tag{3.41}$$

$$\begin{aligned}\|\nabla P^{4_k}\|_{0,q';\Omega} &\le C\|\mathbf{H}^{4_k}\|_{(L^q(\Omega))^*} \le C\frac{1}{k}\||\mathbf{v}^k|^{q-1}\|_{0,q';\Omega}\\ &\le \frac{1}{k^{1/q}}\left(\frac{1}{k^{1/q}}\|\mathbf{v}^k\|_{0,q;\Omega}\right)^{q-1}.\end{aligned} \tag{3.42}$$

As $2r' = \frac{dp}{d-p}$, it follows from (3.40), (3.41) and (3.27) that for $k\to\infty$ we have

$$P^{2_k}\to 0 \text{ and } P^{3_k}\to 0 \quad \text{strongly in } L^s(\Omega) \quad \text{for all } s\in\langle 1,r')\,. \tag{3.43}$$

Also, due to (3.17) we observe that for $k\to\infty$

$$\nabla P^{4_k}\to 0 \quad \text{strongly in } L^{q'}(\Omega)\,. \tag{3.44}$$

Of course, one has analogous estimates for $\mathbf{u}^{I_k}$. For our purpose, it is enough to know that

$$\mathbf{U}^k \equiv \mathbf{u}^{1_k}+\mathbf{u}^{2_k}+\mathbf{u}^{3_k}+\mathbf{u}^{4_k} \quad \text{with} \quad \operatorname{div}\mathbf{U}^k = 0$$

satisfy

$$\|\mathbf{U}^k\|_{1,r';\Omega} \le K\,. \tag{3.45}$$

Next, summing up the weak formulations of the problems $(3.37)_I$ over $I = 1,2,3,4$, and using (3.24) we obtain

$$\begin{aligned}\int_\Omega \nabla\mathbf{U}^k : \nabla\mathbf{\Phi}\,dx - \sum_{I=1}^{4}\int P^{I_k}\operatorname{div}\mathbf{\Phi}\,dx &= \langle\mathbf{f},\mathbf{\Phi}\rangle_{1,p} + \int P^k\operatorname{div}\mathbf{\Phi}\,dx\\ &+\int_\Omega(\mathbf{v}\otimes\mathbf{v}):\mathbf{D}(\mathbf{\Phi})\,dx \qquad \text{for all } \mathbf{\Phi}\in\mathring{W}^{1,r}(\Omega)\,.\end{aligned} \tag{3.46}$$

Taking $\mathbf{\Phi}$ from V_r in (3.46) (it means that $\operatorname{div}\mathbf{\Phi} = 0$) we conclude that

$$\int_\Omega \nabla \mathbf{U}^k : \nabla \mathbf{\Phi}\, dx = \langle \mathbf{f}, \mathbf{\Phi}\rangle_{1,p} + \int_\Omega (\mathbf{v}\otimes\mathbf{v}) : \mathbf{D}(\mathbf{\Phi})\, dx \quad \text{for all } \mathbf{\Phi}\in V_r. \tag{3.47}$$

This and (3.45) then imply

$$\mathbf{U}^k = \mathbf{U} \in \mathring{W}^{1,r'}(\Omega) \qquad \text{for all } k\in\mathbb{N}\,. \tag{3.48}$$

Indeed, it follows from (3.47) that for $k,\ell\in\mathbb{N}$

$$\int_\Omega \nabla(\mathbf{U}^k - \mathbf{U}^\ell) : \nabla\mathbf{\Phi}\, dx = 0 \qquad \text{for all } \mathbf{\Phi}\in V_r\,.$$

Choosing $\mathbf{\Phi}$ to be a solution of

$$\begin{aligned} -\Delta\mathbf{\Phi} + \nabla Q &= \frac{\mathbf{U}^k - \mathbf{U}^\ell}{|\mathbf{U}^k - \mathbf{U}^\ell|} && \text{in } \Omega\,,\\ \operatorname{div}\mathbf{\Phi} &= 0 && \text{in } \Omega\,,\\ \mathbf{\Phi} &= \mathbf{0} && \text{on } \partial\Omega\,, \end{aligned}$$

leads to (3.48).

Finally, taking again (3.24) into account and replacing $\int_\Omega P^k \operatorname{div}\mathbf{\Phi}\, dx$ with the aid of (3.46) and (3.48) we obtain

$$\begin{aligned} &\int_\Omega \mathbf{T}(\cdot,\mathbf{D}(\mathbf{v}^k)) : \mathbf{D}(\mathbf{\Phi})\, dx + \frac{1}{k}\int_\Omega |\mathbf{v}^k|^{q-2}\mathbf{v}^k\cdot\mathbf{\Phi}\, dx\\ &= \int_\Omega (\mathbf{v}^k\otimes\mathbf{v}^k) : \mathbf{D}(\mathbf{\Phi})\, dx - \sum_{I=1}^{4}\int_\Omega P^{I_k}\operatorname{div}\mathbf{\Phi}\, dx\\ &\quad + \int_\Omega \nabla\mathbf{U} : \nabla\mathbf{\Phi}\, dx - \int_\Omega (\mathbf{v}\otimes\mathbf{v}) : \mathbf{D}(\mathbf{\Phi})\, dx \qquad \text{for all } \mathbf{\Phi}\in \mathring{W}^{1,r}(\Omega)\,. \end{aligned} \tag{3.49}$$

The advantage of this formulation stems from more precise control of the particular pressures P^{1_k}, P^{2_k}, P^{3_k} and P^{4_k} owing to (3.39)–(3.44).

3.5 Almost Everywhere Convergence of $\mathbf{D}(\mathbf{v}^k)$ to $\mathbf{D}(\mathbf{v})$

The desired convergence of $\mathbf{D}(\mathbf{v}^k)$ to $\mathbf{D}(\mathbf{v})$ almost everywhere in Ω will certainly hold if one shows that for a given, but arbitrary $\eta > 0$ there is a subsequence $\{\mathbf{v}^\ell\}_{\ell\in\mathbb{N}} \subset \{\mathbf{v}^k\}_{k\in\mathbb{N}}$ such that (for some $\theta \in (0,1)$, say, $\theta = \frac{1}{2}$)

$$\lim_{\ell\to\infty} \int_\Omega \Big[(\mathbf{T}(\cdot,\mathbf{D}(\mathbf{v}^\ell)) - \mathbf{T}(\cdot,\mathbf{D}(\mathbf{v}))) : \mathbf{D}(\mathbf{v}^\ell - \mathbf{v})\Big]^\theta \, dx \le \eta \,. \tag{3.50}$$

To reach this goal it seems natural to consider

$$\mathbf{v}^k - \mathbf{v} \tag{3.51}$$

as a test function in (3.16), rewrite the left hand side of the obtained equality as in (3.50) with $\theta = 1$, and to show that the remaining terms are small as $k \to \infty$. Unfortunately, this idea works only for $p \ge \frac{3d}{d+2}$.

In [FMS97], the L^∞-truncation of (3.51), namely,

$$(\mathbf{v}^k - \mathbf{v})(1 - \min(\frac{|\mathbf{v}^k - \mathbf{v}|}{L}, 1)) \quad \text{with} \quad L > 0 \quad \text{small}\,, \tag{3.52}$$

has been successfully applied to deduce (3.50). The main difficulty is to show the smallness of the integral

$$\int_{\Omega_L^k} \mathbf{T}(\cdot,\mathbf{D}(\mathbf{v}^k) : \mathbf{D}((\mathbf{v}^k - \mathbf{v})(1 - \min(\frac{|\mathbf{v}^k - \mathbf{v}|}{L}, 1)))\, dx\,,$$

where $\Omega_L^k \equiv \{x \in \Omega; |\mathbf{v}^k(x) - \mathbf{v}(x)| < L\}$. The L^∞-truncation method works for $p \ge \frac{2d}{d+1}$; the bound is due to the required L^1-integrability of the convective term $v_k \frac{\partial \mathbf{v}}{\partial x_k}$.

Following the goal to prove Theorem 3-II we have observed that it is enough to restrict ourselves to the case $p \in (\frac{2d}{d+2}, \frac{2d}{d+1})$ and it is neccessary to use a smoother test function than in (3.52) in order to control the convective term; yet the test function should not differ from (3.51) too much.

For this purpose, we test (3.49) by

$$(\mathbf{v}^k - \mathbf{v})_\lambda \quad \text{with } \lambda > 0 \quad \text{large enough}\,, \tag{3.53}$$

using the notation z^k_λ to denote such a Lipschitz (i.e. $\mathring{W}^{1,\infty}$-) truncation of z^k so that z^k_λ coincides with z^k except for a small set A^k_λ.

Let us remark that the idea to approximate a given $\mathring{W}^{1,p}$-function $\mathbf{w}$ by a Lipschitz continuous function $\tilde{\mathbf{w}}$ which agrees with $\mathbf{w}$ on a "large" set has been developed earlier, see [AF88], [EG92], [GIS97], [Iwa97], [Lan96] and [DHM97], [DHM00] among others.

The proof of (3.50), and consequently of Theorem 3-II, is split into three steps. Firstly, in Proposition 3-VII, we study properties of $(\mathbf{v}^k - \mathbf{v})_\lambda$ for general λ. Then we cover the exceptional sets of non-coincidence A^k_λ by two sets F^k_λ and G^k_λ and show (see Proposition 3-VII and 3-IX) by fixing λ and taking a convenient subsequence $\{\mathbf{v}^\ell\}_{\ell\in\mathbb{N}}$ that certain quantities are small on these sets. Finally, we prove (3.50) in Proposition 3-X.

Proposition 3-VII *There is a constant $C = C(\Omega, d)$ such that whenever $\mathbf{w}^m \rightharpoonup 0$ weakly in $\mathring{W}^{1,p}(\Omega)$, then for all $\lambda > 0$ there is a sequence $\{\mathbf{w}^m_\lambda\}_{m\in\mathbb{N}} \subset \mathring{W}^{1,\infty}(\Omega)$ such that*

$$\|\mathbf{w}^m_\lambda\|_{1,\infty;\Omega} \le C\lambda\,. \tag{3.54}$$

Moreover, denoting $A^m_\lambda \equiv \{x \in \Omega; \mathbf{w}^m_\lambda(x) \ne \mathbf{w}^m(x)\}$ then

$$|A^m_\lambda| \le \frac{C}{\lambda^p}\|\nabla \mathbf{w}^m\|^p_{0,p;\Omega}\,, \tag{3.55}$$

Consequently,

$$\|\nabla \mathbf{w}^m_\lambda\|^p_{0,p;\Omega} \le C\|\nabla \mathbf{w}^m\|^p_{0,p;\Omega} \le K \tag{3.56}$$

and (as $m \to \infty$)

$$\begin{aligned} \mathbf{w}^m_\lambda &\to 0 \quad \text{strongly in } L^s(\Omega) \text{ for all } s \in \langle 1,\infty)\,,\\ \mathbf{w}^m_\lambda &\rightharpoonup 0 \quad \text{weakly in } \mathring{W}^{1,s}(\Omega) \text{ for all } s \in \langle 1,\infty)\,. \end{aligned} \tag{3.57}$$

In addition, we construct sets F^m_λ and G^m_λ such that

$$|A^m_\lambda| \le |F^m_\lambda| + |G^m_\lambda|\,, \tag{3.58}$$

$$|F^m_\lambda| \le \frac{C}{\lambda^p}\|\nabla \mathbf{w}^m\|^p_{0,p;\Omega}\,, \tag{3.59}$$

$$|G^m_\lambda| \le \frac{C}{\lambda^{2p}}\|\nabla \mathbf{w}^m\|^p_{0,p;\Omega}\,. \tag{3.60}$$

Before providing a proof of this proposition we recall Kirszbraun's Extension theorem (see [Lan96, Prop. 2.1, p. 708]).

Lemma 3-VIII *Let $\mathcal{M}$ be a metric space and $\mathcal{K}$ be a subset of $\mathcal{M}$ such that $u : \mathcal{K} \to \mathbb{R}$ is Lipschitz-continuous with Lipschitz constant L. Then there exists a continuation $\hat{u} : \mathcal{M} \to \mathbb{R}$ of u such that $\hat{u}$ is Lipschitz continuous with the same Lipschitz bound L and*

$$\sup_{x \in \mathcal{M}} |\hat{u}(x)| \leq \sup_{x \in \mathcal{K}} |u(x)| \, .$$

Proof of Proposition 3-VII: The proof is based on ideas from [Lan96, Prop. 2.2 p. 709] and [DHM00, Lemma 4.1 pp. 21/2]. Extending $\mathbf{w}^m$ by zero we obtain $\tilde{\mathbf{w}}^m \in \mathring{W}^{1,p}(\mathbb{R}^d) = W^{1,p}(\mathbb{R}^d)$ with $\tilde{\mathbf{w}}^m \rightharpoonup 0$ weakly in $W^{1,p} = W^{1,p}(\mathbb{R}^d)$. Recalling the definition of the Hardy-Littlewood maximal-function of $\nabla \tilde{\mathbf{w}}^m$:

$$M(\nabla \tilde{\mathbf{w}}^m)(x) \equiv \sup_{r>0} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} |\nabla \tilde{\mathbf{w}}^m(y)| \, dy \equiv \sup_{r>0} \quad \fint\limits_{B_r(x)} |\nabla \tilde{\mathbf{w}}^m(y)| \, dy \, ,$$

we define for $\lambda > 1$

$$R_\lambda^m \equiv F_\lambda^m \cup G_\lambda^m \cup \left\{ x \in \mathbb{R}^d : x \text{ is not a Lebesgue point of } \nabla \tilde{\mathbf{w}}^m \right\} \, , \tag{3.61}$$

where

$$\begin{aligned} F_\lambda^m &\equiv \left\{ x \in \mathbb{R}^d : \lambda < M(\nabla \tilde{\mathbf{w}}^m)(x) \leq \lambda^2 \right\} \, , \\ G_\lambda^m &\equiv \left\{ x \in \mathbb{R}^d : M(\nabla \tilde{\mathbf{w}}^m)(x) > \lambda^2 \right\} \, . \end{aligned} \tag{3.62}$$

Note that the Lebesgue measure of the last set in the definition of R_λ^m is zero.

Since $M : L^p \to L^p$ is a "bounded" operator (see for example [Ste70, pp. 4–12] or [Zie89, 2.8.2 Theorem p. 84f]), we obtain

$$\begin{aligned} \lambda |R_\lambda^m| &\leq \int_{R_\lambda^m} M(\nabla \tilde{\mathbf{w}}^m(x)) \, dx \leq \| M(\nabla \tilde{\mathbf{w}}^m) \|_{0,p} |R_\lambda^m|^{1-\frac{1}{p}} \\ &\leq C \| \nabla \tilde{\mathbf{w}}^m \|_{0,p} |R_\lambda^m|^{1-\frac{1}{p}} \, , \end{aligned}$$

which implies

$$|R_\lambda^m| \leq \frac{C}{\lambda^p} \| \nabla \tilde{\mathbf{w}}^m \|_{0,p}^p \quad \text{and} \quad |F_\lambda^m| \leq \frac{C}{\lambda^p} \| \nabla \tilde{\mathbf{w}}^m \|_{0,p}^p \, . \tag{3.63}$$

Analogously, one obtains

$$|G_\lambda^m| \le \frac{C}{\lambda^{2p}} \|\nabla \tilde{\mathbf{w}}^m\|_{0,p}^p \,. \tag{3.64}$$

Next, from Lemma 1 in [AF88] it follows that there is a constant $C(d)$ such that

$$|\tilde{\mathbf{w}}^m(x) - \tilde{\mathbf{w}}^m(y)| \le C(d)\,\lambda\,|x-y| \quad \text{on } \mathbb{R}^d \setminus R_m^\lambda \tag{3.65}$$

and

$$|\tilde{\mathbf{w}}^m(x) - (\tilde{\mathbf{w}}^m)_{x,r}| \le C(d)\,r\,\lambda \quad \text{on } \mathbb{R}^d \setminus R_m^\lambda \,.$$

Choosing $x \in \Omega \setminus R_m^\lambda$ and $r = 2\,\mathrm{dist}(x, \Omega^C)$, the Lipschitz regularity of the boundary implies the existence of A (independent of x) such that

$$|B_r(x) \cap \Omega^C| \ge A\,r^d \,.$$

Hence, Poincaré's inequality yields

$$|(\tilde{\mathbf{w}}^m)_{x,r}| \le C\,r \fint_{B_r(x)} |\nabla \tilde{\mathbf{w}}^m(y)|\,dy \le C\ \mathrm{dist}(x, \Omega^C)\,\lambda \,.$$

Thus,

$$|\tilde{\mathbf{w}}^m(x)| \le C\ \mathrm{dist}(x, \Omega^C)\,\lambda \quad \text{on } \mathbb{R}^d \setminus R_m^\lambda \,.$$

This implies that

$$\tilde{\mathbf{w}}_\lambda^m(x) \equiv \begin{cases} \mathbf{w}^m(x) & \text{on} \quad \Omega \setminus (R_m^\lambda) \\ \quad 0 & \text{on} \quad \mathbb{R}^d \setminus \Omega \end{cases}$$

is bounded and Lipschitz continuous on its domain of definition. Thus, by Lemma 3-VIII there exists an extension $\mathbf{w}_\lambda^m$ to $\mathbb{R}^d$ with Lipschitz constant $C(d)\,\lambda$ and L^∞–bound $C\rho\lambda$, where ρ denotes the diameter of Ω. The assertion (3.54) is proved.

Moreover, the set $A_\lambda^m = \{x \in \Omega; \mathbf{w}_\lambda^m(x) \ne \mathbf{w}^m(x)\}$ is a subset of R_λ^m and $|A_\lambda^m| \le |R_\lambda^m|$. This together with (3.61)–(3.64) yields (3.55) and (3.58)–(3.60). Further, by (3.63) we have

$$\begin{aligned} \|\nabla \mathbf{w}_\lambda^m\|_{0,p;\,\Omega} &= \|\nabla \mathbf{w}_\lambda^m\|_{0,p;\,\Omega\setminus R_\lambda^m} + \|\nabla \mathbf{w}_\lambda^m\|_{0,p;\,R_\lambda^m} \\ &\le \|\nabla \mathbf{w}^m\|_{0,p;\,\Omega\setminus R_\lambda^m} + C\lambda |R_\lambda^m|^{1/p} \\ &\le \|\nabla \mathbf{w}^m\|_{0,p;\,\Omega\setminus R_\lambda^m} + C\|\nabla \mathbf{w}^m\|_{0,p;\,\Omega} \le (C+1)\|\nabla \mathbf{w}^m\|_{0,p;\,\Omega} \,, \end{aligned}$$

which is (3.56). Since we also have (with the aid of $\|\mathbf{w}_\lambda^m\|_\infty \le C\rho\lambda$)

$$\|\mathbf{w}_\lambda^m\|_{0,p;\,\Omega} \le C\|\mathbf{w}^m\|_{0,p;\,\Omega}\,,$$

and $\mathbf{w}^m \rightharpoonup 0$ weakly in $\mathring{W}^{1,p}(\Omega)$ we use compact imbedding and interpolation to conclude $(3.57)_1$. From this and (3.54), $(3.57)_2$ follows easily. ■

Next, we consider $\{(\mathbf{v}^k, P^{I_k})\}_{k\in\mathbb{N}}^{I=1,2,3,4}$ and $\mathbf{v} \in V_p$ satisfying (3.17)–(3.20), (3.23), (3.25)–(3.30), (3.39)–(3.44) and (3.49), and set

$$g^k \equiv C\left(|\mathbf{D}(\mathbf{v}^k)|^p + |\mathbf{D}(\mathbf{v})|^p + |\varphi_2|^{\frac{p}{p-1}} + |P^{1_k}|^{\frac{p}{p-1}}\right)\,, \tag{3.66}$$

where φ_2 comes from (3.11).

Due to apriori estimates we see that g^k satisfy the uniform bound

$$\int_\Omega g^k\,dx \le K\,. \tag{3.67}$$

Proposition 3-IX *For a given $\varepsilon > 0$ there are a subsequence $\{\mathbf{v}^\ell\}_{\ell\in\mathbb{N}} \subset \{\mathbf{v}^k\}_{k\in\mathbb{N}}$ and $\lambda \ge \frac{1}{\varepsilon}$ independent of ℓ such that*

$$\int_{F_\lambda^\ell} g^\ell\,dx \le \varepsilon\,, \tag{3.68}$$

where

$$F_\lambda^\ell \equiv \{x\in\Omega; \lambda < M(\nabla(\mathbf{v}^\ell - \mathbf{v}))(x) \le \lambda^2\}\,. \tag{3.69}$$

Proof of Proposition 3-IX: For a given $\varepsilon \in (0,1)$ we find $N \in \mathbb{N}$ such that

$$N\varepsilon > K \qquad (K \text{ from } (3.67)) \tag{3.70}$$

and set

$$\lambda_0 = \frac{1}{\varepsilon}\,. \tag{3.71}$$

For each $k \in \mathbb{N}$ we introduce sets F_i^k, when $i = 0, 1, \ldots, N-1$

$$F_i^k \equiv \{x\in\Omega; \lambda_0^{2^i} < M(\nabla(\mathbf{v}^k - \mathbf{v}))(x) \le \lambda_0^{2^{i+1}}\}\,,$$

which are for fixed k mutually disjoint. Thus, due to (3.67)

$$\sum_{i=0}^{N-1} \int_{F_i^k} g^k \, dx \leq K \, .$$

Due to (3.70), however, for each k there is an index $i(k)$ such that

$$\int_{F_{i(k)}^k} g^k \, dx \leq \varepsilon \, .$$

As $i(k)$'s take values from the finite set $\{0, 1, \ldots, N-1\}$ there exists certainly a subsequence $\{\mathbf{v}^\ell\}_{\ell \in \mathbb{N}}$ of $\{\mathbf{v}^k\}_{k \in \mathbb{N}}$ and an index $i_0 \in \{0, 1, \ldots, N-1\}$ so that $i(\ell) = i_0$ for all $\ell \in \mathbb{N}$. Setting then $\lambda = \lambda_0^{2^{i_0}}$ and defining F_λ^ℓ as in (3.69) we observe that Proposition 3-IX is proved. ■

Proposition 3-X *Let $\theta \in (0,1)$ be chosen and $\eta > 0$ be arbitrary. Then the sequence $\{\mathbf{v}^\ell\}_{\ell \in \mathbb{N}}$ determined in Proposition 3-X satisfies (3.50).*

Proof of Proposition 3-X: We fix $p \in (\frac{2d}{d+2}, \frac{2d}{d+1})$ and recall that $r = \frac{(d+2)p-2d}{dp}$. Then we take $\varepsilon > 0$ so small that condition (3.87) specified at the end of the proof is fulfilled. To this ε, find $\{\mathbf{v}^\ell\}_{\ell \in \mathbb{N}} \subset \{\mathbf{v}^k\}_{k \in \mathbb{N}}$ and $\lambda \geq \frac{1}{\varepsilon}$ such that Proposition 3-IX holds. Now, we apply Proposition 3-VII to $(\mathbf{v}^\ell - \mathbf{v})$ and use the Lipschitz truncation $(\mathbf{v}^\ell - \mathbf{v})_\lambda$ as a test function in (3.49). We also subtract from both sides of the obtained equality the term

$$\int_\Omega \mathbf{T}(\cdot, \mathbf{D}(\mathbf{v}))) : \mathbf{D}((\mathbf{v}^\ell - \mathbf{v})_\lambda) \, dx \, .$$

Then we use the facts that $\mathbf{v}^\ell - \mathbf{v} = (\mathbf{v}^\ell - \mathbf{v})_\lambda$ on $\Omega \setminus A_\lambda^\ell$, and consequently $\operatorname{div}(\mathbf{v}^\ell - \mathbf{v})_\lambda = 0$ a.e. on $\Omega \setminus A_\lambda^\ell$. As a result of this consideration we obtain

$$\begin{aligned} J^\ell &\equiv \int\limits_{\Omega \setminus A_\lambda^\ell} \Big[\mathbf{T}(\cdot, \mathbf{D}(\mathbf{v}^\ell)) - \mathbf{T}(\cdot, \mathbf{D}(\mathbf{v}))\Big] : \mathbf{D}(\mathbf{v}^\ell - \mathbf{v}) \, dx \\ &\equiv I_1^\ell + I_2^\ell + I_3^\ell + I_4^\ell + I_5^\ell + I_6^\ell \, , \end{aligned} \tag{3.72}$$

where

$$I_1^\ell \equiv \int\limits_{A_\lambda^\ell} \Big[\mathbf{T}(\cdot, \mathbf{D}(\mathbf{v})) - \mathbf{T}(\cdot, \mathbf{D}(\mathbf{v}^\ell))\Big] : \mathbf{D}((\mathbf{v}^\ell - \mathbf{v})_\lambda) \, dx \ , \tag{3.73}$$

(3.74) $$I_2^\ell \equiv -\int\limits_{A_\lambda^\ell} P^{1_\ell}\mathrm{div}\,((\mathbf{v}^\ell - \mathbf{v})_\lambda)\,dx\ ,$$

(3.75) $$I_3^\ell \equiv -\int\limits_{A_\lambda^\ell} (P^{2_\ell} + P^{3_\ell})\mathrm{div}\,((\mathbf{v}^\ell - \mathbf{v})_\lambda)\,dx\ ,$$

(3.76) $$I_4^\ell \equiv \int\limits_\Omega \Big(\mathbf{v}^\ell \otimes (\mathbf{v}^\ell - \mathbf{v}) + (\mathbf{v}^\ell - \mathbf{v}) \otimes \mathbf{v}\Big) : \mathbf{D}((\mathbf{v}^\ell - \mathbf{v})_\lambda)\,dx\ ,$$

(3.77) $$I_5^\ell \equiv \int\limits_\Omega \Big(\nabla P^{4_\ell} - \frac{1}{\ell}|\mathbf{v}^\ell|^{q-2}\mathbf{v}^\ell\Big) \cdot (\mathbf{v}^\ell - \mathbf{v})_\lambda\,dx$$

and

(3.78) $$I_6^\ell \equiv \int_\Omega (\nabla \mathbf{U} - \mathbf{T}(\cdot, \mathbf{D}(\mathbf{v}))) : \nabla(\mathbf{v}^\ell - \mathbf{v})_\lambda\,dx\ .$$

We evaluate terms at the right hand side of (3.72) one after another. Note that λ is fixed and

(3.79) $$\|\nabla(\mathbf{v}^\ell - \mathbf{v})_\lambda\|_{0,\infty;\Omega} \le C\lambda\,.$$

We are interested in showing that all terms I_k^ℓ, $k = 1, 2, 3, 4, 5, 6$, are small for $\ell \to \infty$. First, using the compactness (3.27) and (3.43) together with (3.79) we observe that

(3.80) $$\lim_{\ell\to\infty} I_3^\ell + I_4^\ell = 0\,.$$

But the same is true for I_5^ℓ due to (3.44), (3.31) and (3.79). Thus

(3.81) $$\lim_{\ell\to\infty} I_5^\ell = 0\,.$$

When dealing with I_1^ℓ and I_2^ℓ, we use Proposition 3-VII and 3-IX, and the

Hölder inequality

(3.82)

$$
\begin{aligned}
|I_1^\ell + I_2^\ell| &= \Big| \int\limits_{F_\lambda^\ell \cup G_\lambda^\ell} \Big[\mathbf{T}(\cdot,\mathbf{D}(\mathbf{v})) - \mathbf{T}(\cdot,\mathbf{D}(\mathbf{v}^\ell)) - P^{1_\ell}\mathbf{I}\Big] : \nabla(\mathbf{v}^\ell - \mathbf{v})_\lambda \, dx \Big| \\
&\le \int\limits_{F_\lambda^\ell} |\ldots\ldots| \, dx + \int\limits_{G_\lambda^\ell} |\ldots\ldots| \, dx \\
&\le \Big(\int\limits_{F_\lambda^\ell} g^\ell \, dx \Big)^{\frac{p-1}{p}} \|\nabla(\mathbf{v}^\ell - \mathbf{v})_\lambda\|_{0,p,F_\lambda^\ell} + C\lambda \Big(\int\limits_{G_\lambda^\ell} g^\ell \, dx \Big)^{\frac{p-1}{p}} |G_\lambda^\ell|^{\frac{1}{p}} \\
&\le K(\varepsilon^{1-\frac{1}{p}} + \frac{C}{\lambda}) \le K\, C(\varepsilon^{1-\frac{1}{p}} + \varepsilon)\,.
\end{aligned}
$$

Further, from (3.57) applied to $(\mathbf{v}^\ell - \mathbf{v})_\lambda$ we know particularly that

$$
(\mathbf{v}^\ell - \mathbf{v})_\lambda \rightharpoonup 0 \qquad \text{weakly in } \mathring{W}^{1,r}(\Omega)\,. \tag{3.83}
$$

Since $\nabla \mathbf{U} \in L^{r'}(\Omega)$ and $\mathbf{T}(\cdot,\mathbf{D}(\mathbf{v})) \in L^{p'}(\Omega)$, we have

$$
\lim_{\ell \to \infty} I_6^\ell = 0\,. \tag{3.84}
$$

To summarize we have observed that

$$
\lim_{\ell \to \infty} J^\ell = KC(\varepsilon^{1-\frac{1}{p}} + \varepsilon)\,. \tag{3.85}
$$

Finally, fix $\theta \in (0,1)$ and denote the integral in (3.50) by Y^ℓ. Then we have

$$
\begin{aligned}
Y^\ell \le &\int_{\Omega \setminus A_\lambda^\ell} \Big[(\mathbf{T}(\cdot,\mathbf{D}(\mathbf{v}^\ell)) - \mathbf{T}(\cdot,\mathbf{D}(\mathbf{v}))) : \mathbf{D}(\mathbf{v}^\ell - \mathbf{v})\Big]^\theta \, dx \\
&+ \int_{A_\lambda^\ell} \Big[(\mathbf{T}(\cdot,\mathbf{D}(\mathbf{v}^\ell)) - \mathbf{T}(\cdot,\mathbf{D}(\mathbf{v}))) : \mathbf{D}(\mathbf{v}^\ell - \mathbf{v})\Big]^\theta \, dx\,.
\end{aligned} \tag{3.86}
$$

With the aid of the Hölder inequality and a priori estimates we obtain

$$
Y^\ell \le (J^\ell)^\theta |\Omega \setminus A_\lambda^\ell|^{1-\theta} + K|A_\lambda^\ell|^{1-\theta}\,.
$$

Using (3.86) and (3.58)–(3.60) we finally conclude

$$\begin{aligned}\lim_{\ell\to\infty} Y^\ell &\le |\Omega|^{1-\theta}(K\,C)^\theta(\varepsilon^{1-\frac{1}{p}}+\varepsilon)^\theta + K\left(\frac{C}{\lambda^p}\right)^{1-\theta}\\ &\le |\Omega|^{1-\theta}(K\,C)^\theta(\varepsilon^{1-\frac{1}{p}}+\varepsilon)^\theta + K\,(K\,C)^{1-\theta}\varepsilon^{p(1-\theta)}\,.\end{aligned}$$

If ε is taken at the beginning of the proof so that

$$|\Omega|^{1-\theta}(K\,C)^\theta(\varepsilon^{1-\frac{1}{p}}+\varepsilon)^\theta + K\,(K\,C)^{1-\theta}\varepsilon^{p(1-\theta)} < \eta\,, \tag{3.87}$$

then Proposition 3-X, and consequently Theorem 3-II are proved. ■

Let us finish with some final remarks.

Remark 3-XI (1) The proof of Theorem 3-II offers also another argument for the existence result in the case $p=\frac{2d}{d+1}$. This limiting case (for this p the convective term is "a priori" only in L^1) was included in our previous existence result in [FMS97] were we used the fact that the convective term $(\mathbf{v}\cdot\nabla)\mathbf{v}$ belongs locally to the Hardy space $\mathcal{H}^1$ (due to $\operatorname{div}\mathbf{v}=0$) and the duality of $\mathcal{H}^1$ and BMO (= the John-Nirenberg space of functions with bounded mean oscillation). The above given proof of Theorem 3-II works (of course) also in this case and therefore we do not need to use the above mentioned facts (in this case).

(2) On the other hand one can give an alternative proof of Theorem 3-II by using the following compensated integrability result: For $\frac{1}{r}=\frac{2}{p}-\frac{1}{d}=\frac{2d-p}{dp}$ and $\mathbf{w}\in V_p$ it holds

$$(\mathbf{w}\cdot\nabla)\mathbf{w}\in h^r(\Omega)\,,$$

where $h^r(\Omega)$ denotes the local Hardy space. Observe that $\frac{d}{d+1}<r<1$ is equivalent to $\frac{2d}{d+2}<p<\frac{2d}{d+1}$ (see [CLMS93], [Mül94] and [Nov98]).

Taking [Tri83, 2.11.3 pp. 180-182, 2.5.7 pp. 89-91 and 2.5.12 pp. 109-114] into account we can dispose of

$$h^r(\Omega)\equiv F^0_{r,2}(\Omega)\equiv \text{ Triebel-Lizorkin space for } 0<r\le 1$$

and

$$(h^r(\Omega))'\equiv(F^0_{r,2}(\Omega))'\equiv B^{d(\frac{1}{r}-1)}_{\infty,\infty}(\Omega)\equiv B^{\frac{2d-(d+1)p}{p}}_{\infty,\infty}(\Omega)\equiv C^{0,\alpha}(\overline{\Omega})\,,$$

where $\alpha=\frac{2d-(d+1)p}{p}\in(0,1)$ if and only if $\frac{2d}{d+2}<p<\frac{2d}{d+1}$. Noticing that our test function $\mathbf{\Phi}^k_\lambda=(\mathbf{v}^k-\mathbf{v})_\lambda$ belongs to $\mathring{W}^{1,\beta}(\Omega)$ for all finite β we certainly have

$\mathbf{\Phi}^k_\lambda \in C^{0,\alpha}(\overline{\Omega})$. The only difference to the above given proof appears now in dealing with the convective term: Instead of integrating by parts we keep it in the form

$$\int_\Omega (\mathbf{v}^k \cdot \nabla)\mathbf{v}^k \cdot \mathbf{\Phi}^k_\lambda \, dx \equiv \langle (\mathbf{v}^k \cdot \nabla)\mathbf{v}^k, \mathbf{\Phi}^k_\lambda \rangle$$

where the brackets now denote the duality between h^r and $C^{0,\alpha}$, use the uniform boundedness of $(\mathbf{v}^k \cdot \nabla)\mathbf{v}^k$ in h^r and have to ensure that it converges to zero for $k \to \infty$. This can be achieved by observing that

$$\mathbf{\Phi}^k_\lambda \to 0 \quad \text{strongly in } C^{0,\alpha} \quad \text{for suitable } \alpha \in (0,1)\,.$$

This however follows from $(3.57)_1$ and the interpolation inequalities

$$\begin{aligned} \|\mathbf{\Phi}^k_\lambda\|_{C^{0,\alpha}} &\le C\,\|\mathbf{\Phi}^k_\lambda\|^{\theta}_{\infty}\,\|\mathbf{\Phi}^k_\lambda\|^{1-\theta}_{C^{0,\beta}} \\ &\le C\|\mathbf{\Phi}^k_\lambda\|^{\frac{\theta}{2}}_{0,2d}\|\mathbf{\Phi}^k_\lambda\|^{\frac{\theta}{2}}_{1,2d}\,\|\mathbf{\Phi}^k_\lambda\|^{1-\theta}_{C^{0,\beta}} \le C\|\mathbf{\Phi}^k_\lambda\|^{\frac{\theta}{2}}_{0,2d}\|\mathbf{\Phi}^k_\lambda\|^{1-\frac{\theta}{2}}_{1,s}\,, \end{aligned}$$

valid for $0<\alpha<\beta<1$, $\theta = 1-\frac{\alpha}{\beta}$, $1-\theta = \frac{\alpha}{\beta}$ and $s \ge 2d$ so that $W^{1,s} \hookrightarrow C^{0,\beta}$.

The rest of the proof coincides with that before.

(3) Using the method of proof of our main theorem we can also generalize the result of Dal Maso and Murat [DMM98] to include “some” nonlinear terms on the right hand side satisfying “suitable” growth conditions, but we will here not follow these possibilities.

(4) Another possible use of our here developed scheme of proof would be in the theory of electrorheological fluids with shear-dependent viscosities (steady flows), but this will be a future project. The interested reader is referred to [Růž00]. ■

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Chapter 4

Some Remarks on Uniqueness and Regularity of Weak Solutions to the Navier-Stokes Equations

4.1 Introduction and Problem Formulation

Let us consider the initial value problem for the Navier-Stokes equations in $(0, T) \times \mathbb{R}^n$ with $0 < T < +\infty$ and $n \geq 3$:

$$\begin{aligned} \partial_t u^i - \Delta u^i + u^j D_j u^i + D_i \pi = f^i \quad & \text{in } (0,T) \times \mathbb{R}^n, \\ \operatorname{div} u = 0 \quad & \text{in } (0,T) \times \mathbb{R}^n, \\ u(0,x) = a(x), & \end{aligned} \tag{4.1}$$

where $u = u(t,x) = (u^1(t,x), \ldots, u^n(t,x))$ and $\pi = \pi(t,x)$ denote the unknown velocity vector and pressure of the fluid at the point $(t,x) \in (0,T) \times \mathbb{R}^n$, while $a = a(x) = (a^1(x), \ldots, a^n(x))$ is the given initial velocity vector and $f = f(t,x) = (f^1(t,x), \ldots, f^n(t,x))$ is a given external force.

We are interested in the classical problem of finding sufficient conditions for weak solutions of (4.1) such that they become unique and/or regular.

If $\gamma \in [1, +\infty]$, we denote the space $L^\gamma(\mathbb{R}^n)$ simply by L^γ and the canonical norm in this space by $||\cdot||_{0,\gamma}$. We use the same symbol to denote functional spaces consisting of scalar-, vector- or tensor functions. For example the space $L^\gamma \times \cdots \times L^\gamma$ (n times) is denoted simply by L^γ. This convention also applies to other symbols as, for instance, norms. Besides the Lebesgue spaces L^γ we shall use also the usual Sobolev spaces $W^{1,p} = W^{1,p}(\mathbb{R}^n) = H^{1,p}(\mathbb{R}^n)$ and their "solenoidal parts":

$$L^\gamma_\sigma := \{v \in L^\gamma : \operatorname{div} v = 0\}\,,$$
$$W^{1,p}_\sigma := \{v \in W^{1,p} : \operatorname{div} v = 0\}\,.$$

We know that for every $a \in L^2_\sigma$ and every $f \in L^1(0,T;L^2)$, there exists at least one weak solution u of (4.1) satisfying the energy inequality:

$$(4.2) \qquad ||u(t)||^2_{0,2} + 2\int_0^t ||\nabla u(s)||^2_{0,2}\,ds \le ||a||^2_{0,2} + 2\int_0^t (f(s),u(s))ds\,.$$

By a weak solution we mean a function u in $L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;H^{1,2}_\sigma)$ which satisfies (4.1) in the sense of distributions (for a more precise definition see section 5.2). This existence result was long ago proven by Leray in 1934 [Ler34] (in the case n = 3) and Hopf in 1951 [Hop51] (general case). Unfortunately we do not know up to now whether such weak solutions are unique and/or regular or not in the case $n \ge 3$. This is in contrast to the two dimensional case, in which we know that weak solutions are unique and regular (see for instance: Constantin, Foias [CF88], Ladyzhenskaya [Lad63], J. L. Lions [Lio69], P. L. Lions [Lio96], Temam [Tem95] or v. Wahl [vW82] and references cited therein). Therefore the question arises: under which additional assumption(s) on the weak solution are we able to deduce uniqueness and regularity?

Introducing the class $L^\alpha(0,T;L^\beta)$, Prodi [Pro59] showed 1959 that if u is a weak solution of (4.1) belonging to $L^\alpha(0,T;L^\beta)$ with α,β satisfying

$$(4.3) \qquad PS(\alpha,\beta) \equiv \frac{2}{\alpha} + \frac{n}{\beta} = 1\,,\ \beta > n$$

and $n = 3$, then u is unique. Two years later Foias [Foi61] proved the same result for general n but under the slightly more restrictive assumption

$$PS(\alpha,\beta) < 1\,.$$

For $n \le 4$ Serrin [Ser63] extended Prodi's and Foias' result to $\frac{2}{\alpha} + \frac{n}{\beta} = 1\,,\ \beta > n$ for bounded domains Ω in $\mathbb{R}^n$. More precisely, Serrin proved that if u is a weak solution belonging to $L^\alpha(0,T;L^\beta)$ for α and β as above and if v is another weak solution with the same data satisfying the energy inequality, then u and v coincide: $u \equiv v$. This criterion of Serrin has the advantage that it guarantees uniqueness of weak solutions where only one solution – say u – is required to be of class $L^\alpha(0,T;L^\beta)$ and the other one belongs to the larger class satisfying the usual energy inequality.

It is worth to remark that every weak solution of (4.1) in the class $L^4(0,T;L^4)$ fulfills the energy inequality, which turns in this case even into an identity. Therefore we can rephrase Serrin's result as follows: $L^\alpha(0,T;L^\beta)$ solutions are unique in the larger class of $L^4(0,T;L^4)$-solutions. One simply has to observe that functions which belong to $L^\infty(0,T;L^2) \cap L^2(0,T;H^{1,2}) \cap L^\alpha(0,T;L^\beta)$ are automatically also elements of $L^4(0,T;L^4)$, which follows from some interpolation inequalities (see also later section 5.2).

Another fact which follows from Serrin's criterion is the following: It does not matter which of the function classes $L^\alpha(0,T;L^\beta)$ satisfying (4.3) is used because they are all equivalent respectively they coincide: for example let u be a weak solution of (4.1) from $L^5(0,T;L^5)$ and v be one from $L^4(0,T;L^6)$ or $L^8(0,T;L^4)$ (here we take for simplicity $n=3$), then it follows: $u \equiv v$ from Serrin's result.

Further important contributions to this question were made by Masuda [Mas84], Sohr-von Wahl [SvW84] (see also von Wahl [vW82]) and finally by Kozono-Sohr [KS96]. Masuda extended Serrin's criterion to arbitrary domains $\Omega \subset \mathbb{R}^n$ for all $n \geq 2$. Especially he treated also the limit case $L^\infty(0,T;L^n)$, which was excluded before, and obtained uniqueness for the class of functions which belong to $L^\infty(0,T;L^n)$ and are continuous from the right with values in L^n. Sohr and von Wahl proved uniqueness in the critical case $\alpha = +\infty$, $\beta = n$ for bounded domains in $\mathbb{R}^n$ under the assumptions that the initial value a belongs to L^n_σ, that the outer force f is smooth in some sense and that v satisfies a certain stronger form of the energy inequality, but unfortunately one cannot guarantee this stronger energy inequality in general. After all Kozono and Sohr succeeded in proving in great generality a complete analogue of Serrin's result for $L^\infty(0,T;L^n)$. Although this result is absolutely satisfactory, the author believes that one can prove a slightly more general result, but this will be discussed elsewhere.

The reason for this comes from the scaling invariance properties of the Navier-Stokes equations. If the pair (u,p) solves (4.1), so does the family of pairs (u_λ, p_λ) defined by

$$u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x)\ , p_\lambda(t,x) = \lambda^2 p(\lambda^2 t, \lambda x)$$

(with force $f_\lambda = \lambda^3 f(\lambda^2 t, \lambda x)$). Scaling invariance now means that we would like to have the relation

$$\|u_\lambda | L^\alpha(0,T;L^\beta)\| = \|u|L^\alpha(0,T;L^\beta)\|$$

for all $\lambda > 0$ and this is exactly true only if the Prodi-Serrin condition $\frac{2}{\alpha}+\frac{n}{\beta} = 1$ is fulfilled. But indeed there are further function spaces which share this scaling property and are in fact larger than $L^\infty(0,T;L^n)$ for example: $L^\infty(0,T;L^n_w)$ ($L^n_w = L^{n,\infty}$ is the weak Lebesgue space or Lorentz space), $L^\infty(0,T;B^{-\frac{1}{2}}_{2n,n})$ ($B^{-\frac{1}{2}}_{2n,n}$ is a Besov space), $L^\infty(0,T;F^{-\frac{1}{2}}_{2n,1})$ ($F^{-\frac{1}{2}}_{2n,1}$ is a Triebel-Lizorkin space), $L^\infty(0,T;B^{-1}_{\infty,\infty})$ ($B^{-1}_{\infty,\infty}$ is another Besov space), ... have all the same scaling – or differential dimension (in the sense of Nikolskij-Triebel) namely $-1 = -\frac{n}{n} = -\frac{1}{2} - \frac{n}{2n} = -1 - \frac{n}{+\infty}$. (For PDE people: the differential dimension is nothing else than the Sobolev number!).

Let us finish our discussion of the uniqueness problem with the hint that one can find further details in Kozono-Sohr [KS96].

Concerning the regularity problem the starting paper is Serrin [Ser62]. There it is shown that a weak solution u is of class $C^\infty((0,T)\times\mathbb{R}^n)$ if it belongs to the space $L^\alpha(0,T;L^\beta)$ with exponents α, β satisfying $\frac{2}{\alpha}+\frac{n}{\beta} < 1$. In proving this Serrin considers the vorticity equation

$$\partial_t\omega^i - \Delta\omega^i + u^j\, D_j\omega^i - \omega^j\, D_j u^i = (\operatorname{curl} f)^i$$

where ω is the vorticity, i.e. $\omega \equiv \operatorname{curl} u$. Reading this equation as a heat equation and using the assumption on u he improved step by step the regularity of ω and using the fact that $\operatorname{div} u = 0$ also that of u.

Ten years later Fabes-Jones-Rivière [FJR72] treated the case of data in L^p and proved regularity and uniqueness under the same condition as Serrin supposed.

Sohr [Soh83] succeeded in proving that the class $L^\alpha(0,T;L^\beta)$ with $\frac{2}{\alpha}+\frac{n}{\beta} = 1$, $\beta > n$ is also a regularity class extending Serrin's result to the "limit case", in which the critical quantity $\frac{2}{\alpha}+\frac{n}{\beta}$ can be equal to 1, but (still) β must be strictly larger than $n =$ dimension. Essential tools in Sohr's approach are the use of the Yosida approximation and the potential theoretic estimates of Solonnikov [Sol68], [Sol77] and von Wahl [vW80]. He also investigated the limit case $L^\infty(0,T;L^n)$ and formulated a slightly more general criterion than von Wahl in [vW82].

Independently of each other Giga [Gig86b], Struwe [Str88] and Takahashi [Tak90], [Tak92] proved also the latter result with quite different methods.

In fact Giga applies "abstract semigroup-theory" (i.e. results like that of Fujita-Kato [FK64]), Struwe goes back to Serrin's idea of using the vorticity equation, but then he proceeds differently (employing test function techniques instead of integral representations). Takahashi finally refines results of Ladyženskaya-Solonnikov-Ural'ceva [LSU67] on parabolic systems via a cut-off technique.

The critical case $\alpha = +\infty$, $\beta = n$ was investigated by Giga [Gig86b], [Gig86a], von Wahl [vW86], Struwe [Str88] and Takahashi [Tak90], [Tak92]; von Wahl and Giga showed that u belongs to $C^\infty((0,T) \times \overline{\Omega})$ if u is a weak solution in $C^0([0,T); L^n(\Omega))$. Struwe and Takahashi proved the same claim, but under the assumption that $u \in L^\infty(0,T; L^n(\Omega))$ has a sufficiently small norm in that space. Continuing Giga's method H. Kato [Kat93] studied the Hausdorff dimension of the set E of possible time singularities of a weak solution u. Her result reads:

1) If u belongs to $L^q(0,T; D(A_p^\gamma))$, where A_p denotes the Stokes operator in L^p_σ and $0 \le \gamma_0 \equiv \frac{n}{2p} - \frac{1}{2} < \gamma < 1$, then with $k \equiv 1 - q(\gamma - \gamma_0) > 0$ the k-dimensional Hausdorff measure of the set E is zero and u is in $C^\infty(((0,T) \setminus E) \times \overline{\Omega})$.

2) If under the same assumptions as in 1) $k \le 0$, then E is empty, i.e. u is regular in $(0,T) \times \overline{\Omega}$.

She also showed the following uniqueness result: Let $\gamma_0 \equiv \frac{n}{4} - \frac{1}{2}$ and $0 < \gamma_0 < 1$. A weak solution belonging to $L^\infty(0,T; D(A^{\gamma_0}))$ is unique. This is clearly a "forerunner" of the uniqueness theorem of Kozono and Sohr.

Further regularity criteria were treated by Beirão Da Veiga [BadV95], [BaDV97]. Firstly he extended Serrin's regularity criterion to gradients showing that if

$$\nabla u \in L^\alpha(0,T; L^\beta) \quad \text{with} \quad \frac{2}{\alpha} + \frac{n}{\beta} = 2 \ ,$$

then

$$\nabla u \in C^0([0,T); L^{\frac{\alpha}{\alpha-1}}) \cap L^{\frac{\alpha}{\alpha-1}}(0,T; L^{\frac{2\beta}{n-2}}) \ ,$$

in particular u is a (strong and) regular solution.

Secondly he formulated a condition – which he calls Hypothesis A – that ensures that weak solutions u from $L^\infty(0,T; L^n)$ are strong ones and that his "Hypothesis A" is (much) weaker than continuity from the left, therefore improving the result of von Wahl [vW82].

The last contribution (to the knowledge of the author) to the regularity problem in the critical case $\alpha = +\infty$, $\beta = n$ was made by Kozono-Sohr

[KS97]. These authors showed that a weak solution u in $L^\infty(0,T;L^n(\Omega))$ is in $C^\infty((0,T)\times\overline{\Omega})$, if the difference between the left hand $\limsup_{t\uparrow t_*}\|u(t)\|_{0,n}$ and $\|u(t_*)\|_{0,n}$ is sufficiently small for every $t_*\in(0,T)$. The latter criterion covers especially the previous ones of Giga and von Wahl ($u\in C^0([0,T);L^n)$) and of Struwe respective Takahashi (smallness of the $L^\infty(0,T;L^n(\Omega))$-norm of u) and as an application Kozono and Sohr show that if the weak solution u in $L^\infty(0,T;L^n)$ possesses a left hand limit $\overline{u}=\lim_{t\uparrow t_*}u(t)$ in $L^n(\Omega)$ for every $t_*\in(0,T)$, then u is regular (in $C^\infty((0,T)\times\overline{\Omega})$). Moreover this yields uniqueness and regularity of weak solutions in the class $BV(0,T;L^n(\Omega))$, i.e. the class of all functions on (0,T) with values in $L^n(\Omega)$, which are of bounded variation (see Giusti [Giu84] or Federer [Fed69] for the definition and properties of the function space BV).

The purpose of this chapter is to study the following four limit cases of the above mentioned regularity classes:

$$
\begin{aligned}
&1)\quad u\in L^2(0,T;L^\infty)\ ,\\
&2)\quad \nabla u\in L^2(0,T;L^n)\ ,\\
&3)\quad \nabla u\in L^1(0,T;L^\infty)\ ,\\
&4)\quad \nabla^2 u\in L^1(0,T;L^n)\ ,
\end{aligned}
$$

and to weaken/sharpen the assumption concerning the regularity with respect to the space variables. Instead of 1) – 4) we will only assume

$$
\begin{aligned}
&1')\quad u\in L^2(0,T;BMO)\ ,\\
&2')\quad \nabla u\in L^2(0,T;L^{2,n-2})\ ,\\
&3')\quad \nabla u\in L^1(0,T;BMO)\ ,\\
&4')\quad \nabla^2 u\in L^1(0,T;L^{2,n-2})\ ,
\end{aligned}
$$

where BMO denotes the John-Nirenberg space of functions with bounded mean oscillation and $L^{2,n-2}$ denotes a certain Morrey space related to BMO via the Poincaré inequality (see section 4.2 for details). Our aim will be that under these weaker assumptions still the classical assertions are true, i.e. all four classes are uniqueness- and regularity classes.

4.2 Statements of the Results

Before stating our results, we introduce some function spaces, recall some of their properties and then we give the definition of a weak solution.

Let $C^\infty_{0,\sigma}$ denote the set of all C^∞ vector functions $\phi = (\phi^1, \dots, \phi^n)$ with compact support such that $\operatorname{div}\phi = 0$.
L^γ_σ is the closure of $C^\infty_{0,\sigma}$ with respect to the L^γ-norm $\|\cdot\|_{0,\gamma}$; $(\cdot,\cdot)$ denotes the duality pairing between L^γ and $L^{\gamma'}$, where $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$. $\overset{\circ}{W}{}^{1,\gamma}_\sigma$ denotes the closure of $C^\infty_{0,\sigma}$ with respect to the norm

$$\|\phi\|_{1,\gamma} \equiv \|\phi\|_{0,\gamma} + \|\nabla\phi\|_{0,\gamma}\,,$$

where $\nabla\phi = \left(\frac{\partial\phi^i}{\partial x_j}\right)$; $i,j = 1,\dots,n$.
For an interval I in $\mathbb{R}$ and a Banach space X, $L^p(I;X)$ and $C^m(I;X)$ denote the usual Banach spaces, where $1 \le p \le +\infty$, $m = 0,1,2,\dots$.
BMO denotes the John-Nirenberg space of functions with bounded mean oscillation i.e.

$$BMO \equiv \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \sup_B \frac{1}{|B|} \int_B |f(y) - f_B|\, dy < +\infty \right\}\,,$$

where $f_B \equiv \fint_B f(y)\, dy = \frac{1}{|B|}\int_B f(y)\, dy$ is the mean value of f over B and B denotes an arbitrary ball in $\mathbb{R}^n$. As a norm on BMO we take

$$\|f\|_{BMO} = \|f\|_* \equiv \sup_B \frac{1}{|B|} \int_B |f(y) - f_B|\, dy\,.$$

With this norm BMO becomes a Banach space, if we identify functions which differ by a constant:

$$\|f\|_{BMO} = 0 \Leftrightarrow f \equiv \text{const}\ .$$

So in fact BMO is a quotient space. To imagine how “large” the space BMO is we recall from the literature the following facts (see for example Stein [Ste70, Ste93] and Neri [Ner75]):

i) $L^\infty \subset BMO \subset L^p$ for any $p < +\infty$;

ii) $\log|x| \in BMO$, but $\log|x| \notin L^\infty$, so $L^\infty \subsetneqq BMO$;

iii) $\log|P(x)| \in BMO$, P some homogeneous polymonial.

$L^{2,n-2}$ denotes the Morrey space

$$L^{2,n-2} \equiv \left\{ f \in L^2(\mathbb{R}^n) : \|f \mid L^{2,n-2}\|^2 := \sup_{z,r} \frac{1}{r^{n-2}} \int_{B_r(z)} |f(y)|^2 \, dy < +\infty \right\} ,$$

where the supremum is taken over all $z \in \mathbb{R}^n$, $r > 0$. $L^{2,n-2}$ is a Banach space with respect to the norm $\|f|L^{2,n-2}\|$ and it is related to the space BMO via Poincaré's inequality:

$$u \in W^{1,2} \, , \; \nabla u \in L^{2,n-2} \Rightarrow u \in BMO$$

because of

$$\begin{aligned} \int_{B_r(z)} |u(y) - u_{B_r(z)}|^2 \, dy &\leq Cr^2 \int_{B_r(z)} |\nabla u(y)|^2 \, dy \\ &\leq Cr^n \|\nabla u|L^{2,n-2}\|^2 \end{aligned}$$

implying

$$\sup_{z,r} \frac{1}{|B_r(z)|} \int_{B_r(z)} |u(y) - u_{B_r(z)}|^2 \, dy \leq C \|\nabla u|L^{2,n-2}\|^2$$

which by means of Hölder's inequality gives

$$\|u|BMO\| \leq C \|\nabla u|L^{2,n-2}\| \, .$$

Our definition of weak solutions of (4.1) is as follows:

Definition 4-I *Let $a \in L^2_\sigma$ and $f \in L^1(0,T;L^2)$. A measurable function u is called a weak solution of (4.1) on $(0,T)$ if*

i) $u \in L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;\mathring{W}^{1,2}_\sigma)$;

ii)

$$\int_0^T \{ -(u, \partial_t \Phi) + (\nabla u, \nabla \Phi) + (u \cdot \nabla u, \Phi) \} \, dt = (a, \Phi(0)) + \int_0^T (f, \Phi) \, dt \tag{4.4}$$

for all $\Phi \in C^1([0,T);\Upsilon)$ with $\Phi(T) = 0$, where $\Upsilon \equiv \mathring{W}^{1,2}_\sigma \cap L^n$.

Remark 4-II
1. For u and Φ as above, the integral $\int\limits_0^T (u \cdot \nu, \Phi)\, dt$ is well defined since we have by the Sobolev inequality $\|u\|_{0,\frac{2n}{n-2}} \le C\|\nabla u\|_{0,2}$ (for $n \ge 3$) that

$$\int\limits_0^T |(u \cdot \nabla u, \Phi)|\, dt \le \int\limits_0^T \|u\|_{0,\frac{2n}{n-2}} \|\nabla u\|_{0,2} \|\Phi\|_{0,n}\, dt$$
$$\le C\big(\sup_{0<t<T} \|\Phi(t)\|_{0,n}\big) \int\limits_0^T \|\nabla u\|_{0,2}^2\, dt\,.$$

2. Υ is a Banach space with the norm $\|\Phi\|_\Upsilon \equiv \|\Phi\|_{1,2} + \|\Phi\|_{0,n}$. Under the assumption that the underlying domain Ω satisfies one of the following four hypotheses:

i) Ω is the whole space $\mathbb{R}^n (n \ge 3)$;

ii) Ω is the half spasce $\mathbb{R}^n_+ (n \ge 3)$;

iii) Ω is a bounded domain in $\mathbb{R}^n (n \ge 3)$ with $C^{2,\mu}$ boundary $\partial\Omega$ for some $0 < \mu < 1$;

iv) Ω is an exterior domain in $\mathbb{R}^n (n \ge 3)$, i.e. a domain having a compact complement $\mathbb{R}^n \backslash \Omega$ with $C^{2,\mu}$ boundary $\partial\Omega (0 < \mu < 1)$.

It is known that $C^\infty_{0,\sigma}$ is dense in Υ (see Masuda [Mas84, Proposition 1], Giga [Gig86b, Appendix] and Kozono-Sohr [KS96, Theorem 2]).
Hence we may take Φ as the test function of the above definition having the form $\Phi(t,x) = h(t)\phi(x)$, where $\phi \in C^\infty_{0,\sigma}$ and $h \in C^1([0,T))$ with $h(T) = 0$. See also Masuda [Mas84, Lemma 2.2].
3. After redefinition of $u(t)$ on a set of measure zero on $(0,T)$, we may assume that $u(t)$ is weakly continuous in L^2_σ (see Prodi [Pro59]).

Our theorem concerning uniqueness now reads:

Theorem 4-III *There is at most one solution of (4.1) in the sense of Definition 4-I such that*

i) $u \in L^\infty(0,T; L^2_\sigma) \cap L^2(0,T; \mathring{W}^{1,2}_\sigma) \cap L^2(0,T; BMO)$ *or*

ii) $u \in L^\infty(0,T; L^2_\sigma) \cap L^2(0,T; \mathring{W}^{1,2}_\sigma)$ *and* $\nabla u \in L^2(0,T; L^{2,n-2})$.

Such a solution would be continuous from $[0,T]$ *into* L^∞_σ.

Remark 4-IV

1. The above uniqueness classes weaken/sharpen the assumption concerning the regularity with respect to the space variables of the following “known” uniqueness-classes:

i) $u \in L^2(0, T; L^\infty)$,

ii) $\nabla u \in L^2(0, T; L^n)$,

iii) $\nabla u \in L^1(0, T; L^\infty)$,

iv) $\nabla^2 u \in L^1(0, T; L^n)$.

This will be clear from the proof of our theorem, but is mostly known to experts in this field of PDE theory. We refer to von Wahl [vW85], da Veiga [BadV95] for corresponding results in the literature.

Concerning regularity we prove in the sequel the following a priori estimates:

Theorem 4-V *Assume that u is a regular solution of (4.1) in some interval* $[0, T]$*. Then, if*

i) $\nabla u \in L^2(0, T; L^{2,n-2})$ *or*

ii) $u \in L^2(0, T; BMO)$ *or*

iii) $\nabla^2 u \in L^1(0, T; L^{2,n-2})$ *or*

iv) $\nabla u \in L^1(0, T; BMO)$,

one has

$$u \in C(0, T; \mathring{W}^{1,2}_\sigma) \cap L^2(0, T; \mathring{W}^{1,2}_\sigma \cap W^{2,2}) . \tag{4.5}$$

Moreover,

$$\begin{aligned} &\operatorname*{ess\,sup}_{0 \le t < T} \|\nabla u(t)\|^2_{0,2} + \int_0^T \|\nabla^2 u(\tau)\|^2_{0,2}\, d\tau \\ &\le C\ \|\nabla u(0)\|^2_{0,2} \left\{ 1 + \exp\left(C \int_0^T \|\nabla^{\alpha_i} u(\tau)|X_i\|^{\beta_i}\, d\tau \right) \right\} , \end{aligned} \tag{4.6}$$

where $i = 1, 2, 3, 4$ *and*

$$\alpha_1 = 1; \quad X_1 = L^{2,n-2}; \quad \beta_1 = 2; \tag{4.7}$$

$$\alpha_2 = 0; \quad X_2 = BMO; \quad \beta_2 = 2; \tag{4.8}$$

$$\alpha_3 = 2; \quad X_3 = L^{2,n-2}; \quad \beta_3 = 1; \tag{4.9}$$

$$\alpha_4 = 1; \quad X_4 = BMO; \quad \beta_4 = 1; \tag{4.10}$$

according to the cases (i)–(iv) above.

Here and in the sequel we denote by c (or $c_0, c_1, \ldots$) positive constants that depend, at most, on the dimension n and other absolute constants like the constant in the Sobolev inequality. The symbol c may be used, even in the same equation, to denote distinct constants.

In order to show how the above a priori estimates apply in the framework of classical (weak) Leray-Hopf solutions (see [Ler34] and [Hop51]), we state the following Theorem:

Theorem 4-VI *Suppose* $a \in W_\sigma^{1,2}$ *and* u *is a Leray-Hopf solution of (4.1) in the sense of Definition 4-I in* $[0, T)$*. If* u *belongs to one of the four classes specified in Theorem 4-V (i)–(iv), then*

$$u \in C(0, T; W_\sigma^{1,2}) \cap L^2(0, T; W_\sigma^{2,2}) \tag{4.11}$$

and

$$\begin{aligned} &\operatorname*{ess\,sup}_{0 \le t < T} \|\nabla u(t)\|_{0,2}^2 + \int_0^T \|\nabla^2 u(t)\|_{0,2}^2 \, dt \\ &\le C \, \|\nabla a\|_{0,2}^2 \left\{ 1 + \exp\left(c \int_0^T \|\nabla^{\alpha_i} u(\tau) | X_i\|^{\beta_i} \, d\tau \right) \right\} , \end{aligned} \tag{4.12}$$

where α_i*,* X_i *and* β_i *have the same meaning as in Theorem 4-V. In particular* u *is regular and unique solution in* $[0, T]$*.*

4.3 Proofs of the Theorems

We start by preparing the proof of Theorem 4-III. If u and v are weak solutions of (4.1) with the same exterior force f and the same initial velocity

a, which additionally belong to one of the classes specified there, we get for the difference $w := u - v$ the system of equations

$$
\text{(4.13)} \qquad \begin{cases} \partial_t w^i - \Delta w^i + D_i \pi_w = -w^j D_j v^i - u^j D_j w^i \,, \\ \operatorname{div} w = 0 \,, \\ w(0,x) = 0 \,. \end{cases}
$$

By testing with w^i we obtain formally the identity

$$
\text{(4.14)} \qquad \begin{aligned} \frac{1}{2}\frac{d}{dt}\|w(t)\|_{0,2}^2 + \|\nabla w(t)\|_{0,2}^2 &= -\int_{\mathbb{R}^n} w^j D_j v^i w^i \, dx \\ &= \int_{\mathbb{R}^n} w^j D_j w^i v^i \, dx \,, \end{aligned}
$$

because the pressure cancels due to $\operatorname{div} w = 0$ and the term

$$
\text{(4.15)} \qquad \begin{aligned} -\int_{\mathbb{R}^n} u^j D_j w^i w^i \, dx &= -\int_{\mathbb{R}^n} u^j D_j \frac{|w|^2}{2} \, dx = \\ &= \int_{\mathbb{R}^n} (\operatorname{div} u) \frac{|w|^2}{2} \, dx = 0 \end{aligned}
$$

vanishes due to $\operatorname{div} u = 0$. We also used integration by parts together with $\operatorname{div} w = 0$ to transform the right hand side in (4.14).
From (4.14) we would like to estimate the convective term $(w \cdot \nabla w, v) = \int_{\mathbb{R}^n} w^j D_j w^i v^i \, dx$ in such a way that on applying Gronwall's inequality we could finish the proof because of $w(0,x) = 0$ respective $\|w(0)\|_{0,2} = 0$. For this purpose we need several auxiliary results:

Proposition 4-VII *(S. Chanillo 1991)*
If $f \in W^{1,2}(\mathbb{R}^n)$, $g \in L^2_\sigma(\mathbb{R}^n)$ and $h \in W^{1,2}(\mathbb{R}^n)$ such that $\nabla h \in L^{2,n-2}(\mathbb{R}^n)$, then there exists a constant $C = C(n) > 0$ independent of f, g and h such that

$$
\text{(4.16)} \qquad \Big| \int_{\mathbb{R}^n} f g^j D_j h \, dx \Big| = \Big| \int_{\mathbb{R}^n} g^j D_j f h \, dx \Big| \le C \, \|\nabla h | L^{2,n-2}\| \, \|\nabla f\|_{0,2} \, \|g\|_{0,2} \,.
$$

We refer to [Cha91] for a proof of this proposition, but we shall make some comments to the proposition and its proof after completing the proof of Theorem 4-III in case of $\nabla v \in L^2(0,T;L^{2,n-2})$ (case (ii)). This will be an immediate consequence of Proposition (4-VII). Applying the proposition with $g^j = w^j$, $D_j f = D_j w^i$ and $h = v^i$ yields directly

$$\begin{aligned} \Big| \int_{\mathbb{R}^n} w^j D_j w^i v^i \, dx \Big| &\le C \|\nabla v | L^{2,n-2}\| \, \|\nabla w\|_{0,2} \, \|w\|_{0,2} \\ &\le \frac{1}{2} \, \|\nabla w\|_{0,2}^2 + \frac{C}{2} \, \|\nabla v | L^{2,n-2}\|^2 \, \|w\|_{0,2}^2 \end{aligned} \tag{4.17}$$

with little help of Young's inequality. Therefore (4.14) shows that

$$\frac{d}{dt}\|w(t)\|_{0,2}^2 + \|\nabla w(t)\|_{0,2}^2 \le C \, \|\nabla v(t) | L^{2,n-2}\|^2 \, \|w(t)\|_{0,2}^2 \tag{4.18}$$

and according to Gronwall's inequality in conjunction with $\|w(0)\|_{0,2} = 0$ we conclude by the assumption $\nabla v \in L^2(0,T;L^{2,n-2})$:

$$\|w(t)\|_{0,2}^2 = 0 \qquad \text{for } 0 \le t < T \, , \tag{4.19}$$

from which the desired uniqueness follows. Therefore Theorem 4-III (ii) is proven.

Remark 4-VIII
1. Chanillo's inequality (4.16) was used by L. C. Evans to establish partial regularity of weakly harmonic stationary maps which are valued in spheres $S^{m-1} \subset \mathbb{R}^m$. This result is a generalization of a regularity theorem of F. Hélein, who established that weakly harmonic mappings from two-dimensional surfaces into spheres are smooth. The interest in these results comes from the fact that the underlying system of partial differential equations is of critical growth:

$$-\Delta u^i = u^i |\nabla u|^2 \, , \quad |u|^2 = 1 \, .$$

For details see [Eva91].
2. It is tempting to prove Chanillo's inequality by employing the following scheme:

$$\Big| \int_{\mathbb{R}^n} f g \cdot \nabla h \, dx \Big| \le \Big(\int_{\mathbb{R}^n} |f|^2 |\nabla h|^2 \, dx \Big)^{1/2} \, \|g\|_{0,2}$$

using the Cauchy-Schwarz inequality. Now try to show

$$\Big(\int_{\mathbb{R}^n} |f|^2 |\nabla h|^2 \, dx \Big)^{1/2} \le C \Big(\int_{\mathbb{R}^n} |\nabla f|^2 \, dx \Big)^{1/2} \, ,$$

where the constant C depends only on "natural" quantities and on $\|\nabla h|L^{2,n-2}\| < +\infty$. This just about fails for general $h \in W^{1,2}$, $\nabla h \in L^{2,n-2}$, but it is in fact true for $h(x) = \log|x|$ because of $|\nabla h|^2 \leq \frac{1}{|x|^2}$ and the well known inequality

$$\int_{\mathbb{R}^n} \frac{|f|^2}{|x|^2}\, dx \leq K \int_{\mathbb{R}^n} |\nabla f|^2\, dx \qquad \forall f \in W^{1,2}$$

(see for example [Lad63] or [Gal94]). Remember: $h(x) = \log|x| \in BMO$! This strongly suggests that Chanillo's inequality will also be true for $h \in W^{1,2} \cap BMO$.
3. Another possibility to prove Chanillo's inequality consists of using the celebrated theorem of C. Fefferman about the duality between BMO and the Hardy space $\mathcal{H}^1$ in conjunction with a result on compensated integrability/compactness proven by R. Coifman, P. L. Lions, Y. Meyer and S. Semmes.[1] For details we refer to the paper of Evans ([Eva91]). However, Chanillo's derivation of inequality (4.16) is more elementary, if one is willing to accept some machinery from Harmonic Analysis. Keywords for his proof are: Harmonic extension to $\mathbb{R}^{n+1}_+$, Carleson cylinder, Carleson measures, non-tangential maximal function and their role in characterizing the space BMO.

The preceding remarks suggest that Chanillo's inequality will also be true by replacing the $L^{2,n-2}$-norm of ∇h with the BMO-norm of h. In fact we have

Proposition 4-IX *If $f \in W^{1,2}(\mathbb{R}^n)$, $g \in L^2(\mathbb{R}^n, \mathbb{R}^n)$, $\operatorname{div} g = 0$ and $h \in W^{1,2} \cap BMO(\mathbb{R}^n)$, then there exists a positive constant $C = C(n)$ independent of f, g and h such that*

$$\Big| \int_{\mathbb{R}^n} g^j D_j f h\, dx \Big| \leq C\, \|g\|_{0,2}\, \|\nabla f\|_{0,2}\, \|h|BMO\|\,. \tag{4.20}$$

Before proving Proposition 4-IX – the proof requires some preparation – let us quickly finish the proof of Theorem 4-III case (i): Setting $g^j = w^j$, $D_j f = D_j w^i$ and $h = v^i$ the proposition shows

$$\begin{aligned} \Big| \int_{\mathbb{R}^n} w^j D_j w^i v^i\, dx \Big| &\leq C\, \|w\|_{0,2}\, \|\nabla w\|_{0,2}\, \|v|BMO\| \\ &\leq \frac{1}{2}\, \|\nabla w\|_{0,2}^2 + \frac{C}{2}\, \|v|BMO\|^2\, \|w\|_{0,2}^2 \end{aligned} \tag{4.21}$$

[1]The terminology "compensated integrability" instead of compensated compactness was proposed by L. Tartar to the author.

and we can conclude in the same way as in case (ii) that $\|w(t)\|_{0,2}^2 = 0$ for $0 \le t < T$ (see (4.18), (4.19)).

Next we review the definition and some properties of the Hardy space $\mathcal{H}^1 = \mathcal{H}^1(\mathbb{R}^n)$. Assume that $g \in L^1(\mathbb{R}^n)$ and let φ be any smooth function with support in the unit ball:

$$\varphi \in C_0^\infty(B_1(0)), \quad \int\limits_{\mathbb{R}^n} \varphi(x)\,dx = 1. \tag{4.22}$$

We set

$$g^*(x) := \sup_{r>0} \Big| \frac{1}{r^n} \int g(y)\varphi\Big(\frac{x-y}{r}\Big)\,dy \Big| \tag{4.23}$$

and say that g belongs to the Hardy space $\mathcal{H}^1$ if $g^* \in L^1(\mathbb{R}^n)$. We write:

$$\|g|\mathcal{H}^1(\mathbb{R}^n)\| = \|g^*\|_{0,1}. \tag{4.24}$$

Observe that $g \in \mathcal{H}^1$ implies $\int\limits_{\mathbb{R}^n} g(x)\,dx = 0$. An equivalent definition of $\mathcal{H}^1$ can be given in terms of Riesz transforms

$$\mathcal{H}^1(\mathbb{R}^n) = \big\{ g \in L^1(\mathbb{R}^n) : R_j g \in L^1(\mathbb{R}^n) \quad \text{for } j = 1, \dots, n \big\} \tag{4.25}$$

with the norm

$$\|g|\mathcal{H}^1(\mathbb{R}^n)\|^\sim := \|g\|_{0,1} + \sum_j \|R_j g\|_{0,1}; \tag{4.26}$$

here R_j denotes the Riesz transform which is symbolically defined as

$$R_j := D_j(-\Delta)^{1/2} \qquad \text{for } j = 1, \dots, n \tag{4.27}$$

or in terms of Fourier multipliers

$$(R_j f)^\wedge = \mathrm{i}\frac{\xi_j}{|\xi|}\hat{f}, \tag{4.28}$$

where $\hat{f}$ denotes the Fourier transform of f. See Stein and Weiss [SW71] or Torchinsky [Tor86] for more detailed information. An easy consequence of these definitions is the fact that $\mathcal{H}^1(\mathbb{R}^n)$ is a separable Banach space. A fundamental theorem of C. Fefferman [Fef71], [FS72] asserts that

$$(\mathcal{H}^1)^* = BMO \tag{4.29}$$

and in particular provides the duality inequality

$$|\int_{\mathbb{R}^n} f(x)g(x)\,dx| \le C\|f|BMO\|\;\|g|\mathcal{H}^1(\mathbb{R}^n)\| \tag{4.30}$$

for example for $f \in L^\infty(\mathbb{R}^n)$, $g \in \mathcal{H}^1(\mathbb{R}^n)$ or $f \in BMO$, $g \in \mathcal{H}^1_a$, where $\mathcal{H}^1_a$ is the dense subspace of $\mathcal{H}^1$ consisting of finite linear combinations of $\mathcal{H}^1$ atoms (see Stein [Ste93], chapt. IV). The constant C in (4.30) depends only on the dimension n. It is also known that

$$\mathcal{H}^1 = (VMO)^*\,, \tag{4.31}$$

where VMO denotes the space of functions with vanishing mean oscillation. VMO was introduced by D. Sarason in 1974 and it is the completion of C_0^∞ with respect to the BMO-norm (see [SW71]).

Finally we reproduce for the reader's convenience a result of Coifman, Lions, Meyer and Semmes, based upon important contributions due to S. Müller (see [CLMS93] and [Mül90]).

Proposition 4-X *(i) Assume $E \in L^p(\mathbb{R}^n, \mathbb{R}^n)$, $B \in L^{p'}(\mathbb{R}^n, \mathbb{R}^n)$ with $1 < p < +\infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$ and*

$$\operatorname{div} B = 0\,, \qquad \operatorname{curl} E = W(E) = \nabla E - (\nabla E)^T = 0 \tag{4.32}$$

in the distribution sense. Then $B \cdot E = B_j E_j \in \mathcal{H}^1(\mathbb{R}^n)$, with the bound

$$\|B \cdot E\|_{\mathcal{H}^1(\mathbb{R}^n)} \le C\|B\|_{0,p'}\|E\|_{0,p}\,. \tag{4.33}$$

(ii) If $v \in L^{p'}_\sigma(\mathbb{R}^n)$, $u \in W^{1,p}(\mathbb{R}^n)$ with $1 < p < +\infty$, $\frac{1}{p'} + \frac{1}{p} = 1$, then $v \cdot \nabla u = v^j D_j u \in \mathcal{H}^1(\mathbb{R}^n)$ with the bound

$$\|v \cdot \nabla u\|_{\mathcal{H}^1(\mathbb{R}^n)} \le C\|v\|_{0,p'}\|\nabla u\|_{0,p'}\,. \tag{4.34}$$

Proof :

First observe that (ii) is the consequence of (i): Take $B = v$ and $E = \nabla u$. To prove (i) we introduce π such that $E = \nabla \pi \in L^p$, which is possible due to $\operatorname{curl} E = W(E) = 0$. Clearly $B \cdot E = B \cdot \nabla\pi \in L^1(\mathbb{R}^n)$. Now fix $\varphi \in C_0^\infty(B_1(0))$, $\int\limits_{\mathbb{R}^n} \varphi\, dx = 1$, choose $x \in \mathbb{R}^n$, $r > 0$, and set $\varphi_r(y) = \varphi\big(\frac{x-y}{r}\big)$.

$$\frac{1}{r^n}\int\limits_{\mathbb{R}^n} B\cdot\nabla\pi \;\; \varphi_r\, dy = -\frac{1}{r^n}\int\limits_{\mathbb{R}^n} (\pi - (\pi)_{x,r}) \;\; B\cdot\nabla\varphi_r\, dy$$

due to $\operatorname{div} B = 0$. Thus

$$|\frac{1}{r^n}\int\limits_{\mathbb{R}^n} B\cdot\nabla\pi \;\; \varphi_r\, dy| \le \frac{C}{r^{n+1}}\int\limits_{B_r(x)} |\pi - (\pi)_{x,r}| \;\; |B|\, dy\,. \tag{4.35}$$

Choose $p < \alpha < p^* = \frac{np}{n-p}$ in case of $1 < p < n$ and a finite $\alpha > p$ in case $p \ge n$ and let $\beta = \frac{\alpha}{\alpha-1}$, $1 < \beta < p' = \frac{p}{p-1}$. Then

$$\begin{aligned}
&|\frac{1}{r^n}\int\limits_{\mathbb{R}^n} B\cdot\nabla\pi\, \varphi_r\, dy| \\
&\le \frac{C}{r^{n+1}}\Big(\int\limits_{B_r(x)} |\pi - (\pi)_{x,r}|^\alpha\, dy\Big)^{1/\alpha} \;\Big(\int\limits_{B_r(x)} |B|^\beta\, dy\Big)^{1/\beta} \\
&\le \frac{C}{r^{1+n/\alpha}}\;\Big(\int\limits_{B_r(x)} |\pi - (\pi)_{x,r}|^\alpha\, dy\Big)^{1/\alpha} \Big(\fint\limits_{B_r(x)} |B|^\beta\, dy\Big)^{1/\beta} \\
&\le C\Big(\fint\limits_{B_r(x)} |\nabla\pi|^\gamma\, dy\Big)^{1/\gamma}\Big(\fint\limits_{B_r(x)} |B|^\beta\, dy\Big)^{1/\beta},
\end{aligned} \tag{4.36}$$

where γ is defined through $\alpha = \gamma^* = \frac{n\gamma}{n-\gamma}$, that is $\gamma = \frac{n\alpha}{n+\alpha} < p$ for $p < n$ and certainly $\gamma < n$ for $n \le p$. Consequently,

$$|\frac{1}{r^n}\int\limits_{\mathbb{R}^n} B\cdot\nabla\pi\, \varphi_r\, dy| \le C\; M(|\nabla\pi|^\gamma)^{1/\gamma} \;\; M(|B|^\beta)^{1/\beta}\,, \tag{4.37}$$

$M(\cdot)$ denoting the Hardy-Littlewood maximal function. Now $|\nabla\pi|^\gamma \in L^{p/\gamma}$, $\frac{p}{\gamma} > 1$. Thus

$$\|M(|\nabla\pi|)^\gamma)\|_{0,\frac{p}{\gamma}} \le C\|\; |\nabla\pi|^\gamma\|_{0,\frac{p}{\gamma}} \le C\|\nabla\pi\|_{0,p}^\gamma\,. \tag{4.38}$$

Similarly,

$$\|M(|B|^\beta)\|_{0,\frac{p'}{\beta}} \le C\| \, |B|^\beta\|_{0,\frac{p'}{\beta}} \le C\|B\|^\beta_{0,p'} \,. \tag{4.39}$$

Therefore we deduce

$$(4.40) \qquad \begin{aligned} (B\cdot E)^* = (B\cdot\nabla\pi)^* &= \sup_{r>0}\left|\frac{1}{r^n}\int_{\mathbb{R}^n} B\cdot\nabla\pi\varphi r\,dy\right| \in L^1\,, \\ \|(B\cdot E)^*\|_{0,1} &\le C\,\|M(|\nabla\pi|^\gamma)\|^{1/\gamma}_{0,\frac{p}{\gamma}}\;\;\|M(|B|^\beta)\|^{1/\beta}_{0,\frac{p'}{\beta}} \\ &\le C\|\nabla\pi\|_{0,p}\;\;\|B\|_{0,p'} \\ &= C\|E\|_{0,p}\;\;\|B\|_{0,p}\,. \end{aligned}$$

Now the proof of Proposition 4-IX is an immediate consequence of the last proposition and the duality inequality (4.30):

$$(4.41) \qquad \begin{aligned} |\int_{\mathbb{R}^n} gjD_jfh\,dx| &\le C\|g\cdot\nabla f|\mathcal{H}^1\|\;\;\|h|BMO\| \\ &\le C\|g\|_{0,2}\;\|\nabla f\|_{0,2}\;\;\|h|BMO\|\,. \end{aligned}$$

Let us interrupt our discussion for a short moment for inserting a comment on the duality inequality (4.30):

$$(*) \qquad |\int_{\mathbb{R}^n} f(x)g(x)\,dx| \le C\|f|BMO\|\;\;\|g|\mathcal{H}^1\|\,.$$

For general $f\in BMO$ and $g\in\mathcal{H}^1$, the integral $\int_{\mathbb{R}^n} fg\,dx$ does not converge absolutely, which means that fg is in general <u>not</u> in $L^1(\mathbb{R}^n)$. There is an "explicit" counterexample in Stein's book [Ste93, chapt. IV 6.2, p. 178]. Thus one needs to define this "integral" initially by taking g to be in an appropriate dense linear subspace of $\mathcal{H}^1$ or by taking f to be bounded, then the integrand belongs to L^1, the corresponding inequality (*) is true and one can extend this functional by continuity (see [Ste93, chapt. IV, p. 142 ff.]). More precisely, if $g\in\mathcal{H}^1_a$ and $f\in L^\infty$, then

$$(+) \qquad |\int_{\mathbb{R}^n} fg\,dx| \le C\|f|BMO\|\;\;\|g|\mathcal{H}^1\|\,.$$

For $f \in BMO$ we set

$$f^{(k)}(x) = \begin{cases} -k & \text{if } f(x) \le -k\ , \\ f(x) & \text{if } -k \le f(x) \le k\ , \\ k & \text{if } f(x) \ge k\ , \end{cases}$$

and observe that $f^{(k)} \to f$ almost everywhere as $k \to +\infty$ and

$$\|f^{(k)}|BMO\| \le C\|f|BMO\|$$

because $f^{(k)}(x) = \max(-k, \min(f(x), k))$ and BMO forms a lattice. So inequality (+) implies $|\int f^{(k)} g\,dx| \le C\|f\|_{BMO}\|g\|_{\mathcal{H}^1}$ for $g \in \mathcal{H}^1_a$, but any $g \in \mathcal{H}^1_a$ is bounded, has compact support and satisfies $\int\limits_{\mathbb{R}^n} g\,dx = 0$ (see [Ste93]), therefore we can pass to the limit as $k \to +\infty$ for such g because $f^{(k)}$ tends to f in $L^1(\mathbb{R}^n)$, if $f \in L^1(\mathbb{R}^n)$ due to the absolute continuity of the Lebesgue integral:

$$\begin{aligned} &E_k := \{x \in \mathbb{R}^n : |f(x)| \ge k\}\ , \\ &|E_k| \le \frac{\|f\|_{0,1}}{k}\ , \\ &\int\limits_{E_k} (|f(x)| - k)\,dx = \int\limits_{E_k} |f(x)|\Big(1 - \frac{k}{|f(x)|}\Big)\,dx \le \int\limits_{E_k} |f(x)|\,dx \to 0 \end{aligned}$$

because of $|E_k| \to 0$ for $k \to +\infty$. Thus (+) is established for $f \in BMO$ and $g \in \mathcal{H}^1_a$. On the other hand, (+) holds true for $f \in L^\infty$ and <u>all</u> $g \in \mathcal{H}^1$. From this we conclude that (+) also is satisfied for $f \in BMO$ and all $g \in \mathcal{H}^1 \cap L^p$ for some $p > 1$ because similar to the above considerations we have $f^{(k)} \to f$ in $L^{p'}$, $p' = \frac{p}{p-1}$, provided $f \in L^{p'}$. As a conclusion we write down that the duality inequality (4.30) is satisfied with fg even in L^1 for $f \in L^1 \cap BMO$ and $g \in \mathcal{H}^1 \cap L^p$ for some $p > 1$. This shows that the integrand $g^j D_j f h$ in (4.30) in fact belongs to L^1 (at least), if in addition to the assumptions of Proposition 4-IX g belongs to some L^p, $p > 2$, or ∇f belongs to some L^p with $p > 2$, otherwise the inequality has to be understood in the above "explained" sense as an inequality for a linear functional on $\mathcal{H}^1$. We will show now that in our applications this additional assumption is always satisfied:

Lemma 4-XI *(i) If* $u \in L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;\mathring{H}^1_\sigma \cap BMO)$, *then* $u \in L^4(0,T;L^4)$ *and*

$$\|u|L^4(0,T;L^4)\| \le C\|u|L^\infty(0,T;L^2_\sigma)\|^{1/2}\ \ \|u|L^2(0,T;BMO)\|^{1/2}\,. \tag{4.42}$$

(ii) If $u \in L^\infty(0,T;L^2) \cap L^2(0,T;W^{1,2})$ *and* $\nabla u \in L^1(0,T;BMO)$, *then* $\nabla u \in L^{4/3}(0,T;L^4)$ *and*

$$\|\nabla u|L^{4/3}(0,T;L^4)\| \le C\|\nabla u|L^2(0,T;L^2)\|^{1/2}\ \ \|\nabla u|L^1(0,T;BMO)\|^{1/2}\,. \tag{4.43}$$

Proof : (i) Let us introduce the **sharp function** of Fefferman and Stein. It is defined by

$$f^\#(x) := \sup_{x\in B}\frac{1}{|B|}\int\limits_B |f(y)-f_B|\,dy\,, \tag{4.44}$$

where the supremum is taken over all balls B containing x. Note that a function f is in BMO exactly when $f^\#$ is a bounded function, i. e.

$$f \in BMO \Leftrightarrow f^\# \in L^\infty\,. \tag{4.45}$$

Next we observe $f^\# \in L^\infty \cap L^2$ implies $f^\# \in L^4$:

$$\int\limits_{\mathbb{R}^n} |f^\#|^4\,dx \le \|f^\#\|^2_{0,\infty} \int\limits_{\mathbb{R}^n} |f^\#|^2\,dx \tag{4.46}$$

or expressed in norms

$$|f^\#\|_{0,4} \le \|f^\#\|^{1/2}_{0,\infty}\ \ \|f^\#\|^{1/2}_{0,2}\,. \tag{4.47}$$

As it is obvious that the sharp function is pointwise dominated by the (standard) Hardy-Littlewood maximal function, it follows that

$$\|f^\#\|_{0,2} \le 2\|M(f)\|_{0,2} \le C\|f\|_{0,2}\,. \tag{4.48}$$

So we arrive at

$$\|f^\#\|_{0,4} \le C\|f|BMO\|^{1/2}\ \ \|f\|^{1/2}_{0,2}\,, \tag{4.49}$$

saying $f \in BMO \cap L^2 \Rightarrow f^{\#} \in L^4$, but $f \in L^2$ and $f^{\#} \in L^4$ imply $f \in L^4$ and

$$\|f\|_{0,4} \le A_4 \|f^{\#}\|_{0,4} \le C\|f|BMO\|^{1/2}\|f\|_{0,2}^{1/2}\,. \tag{4.50}$$

For a proof of the last assertion we refer to Stein [Ste93, Theorem 2, p. 148 f.], Bennett & Sharpley [BS88, Corollary 7.5, p. 350], or [Ste94, chap. 5].

Applying these considerations – in particular (4.50) – with $f = u(t)$ or $f = \nabla u(t)$ raising both sides to the power 4 (for $f = u(t)$) respective to the power $4/3(f = \nabla u(t))$ and integrating with respect to time proves the lemma. ■

An immediate consequence of the last lemma is that integrals such as

$$\int u^j D_j u^i \varphi^i \, dx \quad \text{or} \quad \int w^j D_j v^i w^i \, dx$$

for $u \in L^2_\sigma \cap W^{1,2} \cap BMO$, $\varphi \in L^4$ respective $w \in L^2_\sigma \cap BMO$, $v \in W^{1,2}_\sigma$ are well defined in L^1 (as we wanted – see the above discussion). Another consequence consists in the fact that for weak solutions u of (4.1) in the class $u \in L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;W^{1,2} \cap BMO)$ the energy inequality is in fact an identity

$$\|u(t)\|_{0,2}^2 + 2\int_s^t \|\nabla u(\tau)\|_{0,2}^2 \, d\tau = \|u(s)\|_{0,2}^2 + 2\int_s^t \big(f(\tau), u(\tau)\big)\, d\tau \tag{4.51}$$

for all $0 < s \le t \le T$; in particular

$$u \in C^0(0,T;L^2_\sigma)\,. \tag{4.52}$$

This follows easily by testing the weak formulation of (4.1) by u, which is allowed in this case due to $u \in L^4(0,T;L^4)$ (see [vW85, chap. IV, p. 162 ff.] for more precise statements and details.)
This observation leads us to the following

Remark 4-XII The uniqueness criterion of Theorem 4-III can also be formulated in the following way:
If u and v are two weak solutions of (4.1) on $(0,T)$ with the same initial velocity $a \in L^2_\sigma$ such that u belongs additionally to $L^2(0,T;BMO)$ and v satisfies the energy inequality

$$\|v(t)\|_{0,2}^2 + 2\int_0^t \|\nabla v(\tau)\|_{0,2}^2 \, d\tau \le \|a\|_{0,2}^2$$

for $0 < t \leq T^2$, then $u(t) = v(t)$ for all $t \in [0, T]$.
For simplicity we took here $f \equiv 0$, but this is no serious restriction.

For two reasons we do not give a proof of this version of the uniqueness criterion here: Firstly, the above stated version coincides with Theorem 2(1) of a recent paper by Kozono-Taniuchi [KT00] and the reader can find there a detailed proof. Secondly, we shall not apply this version of the uniqueness criterion and we feel that it is not necessary to do this because the (formal) idea of the proof is clear: Testing the weak formulation for u by u provides the energy identity for u (see the discussion above), the energy inequality for v is fulfilled by assumption. On the other hand, testing formally the weak formulation for u by v and vice versa provides two other identities which after adding and combining with the two energy inequalities, gives (after some manipulations) an inequality which corresponds to (4.14) from which one can conclude the assertion. To justify this procedure one needs to use a "suitable approximation argument", but the rest works as just described. The approximation can be done (of course) in several ways, Kozono-Taniuchi use one which was developed by Masuda ([Mas84]), another possibility is to use the Yosida approximation; for this we refer to von Wahl [vW85] or the more recent book of H. Sohr [Soh01]. [2]

Let us finish this section on "uniqueness" with some comments on the marginal class $L^1(0, T; W^{1,\infty})$ respective $L^1(0, T; W^1_{BMO})$ where W^1_{BMO} denotes the space of functions u such that u and ∇u belong to BMO. Clearly $L^1(0, T; W^{1,\infty})$ is a uniqueness class for weak solutions of (4.1) as one can see from the identity (4.14) where we do not integrate by parts. From (4.14) we get for the difference $w = u - v$ the inequality:

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|_{0,2}^2 + \|\nabla w(t)\|_{0,2}^2 \leq \|\nabla v(t)\|_{0,\infty} \ \|w(t)\|_{0,2}^2 \ , \tag{4.53}$$

and immediately one obtains $w(t) \equiv 0$ from Gronwall's inequality.
Observe that exactly the same argument works for the Euler equations (drop the Laplacian in the Navier-Stokes equations). In fact, it is the sole class which is a uniqueness class for both equations (at least according to what is known to the author at present). In this regard we would like to mention that one can replace ∇v by $A(v) = \nabla v + (\nabla v)^T$ in all cases where one has found that it works under some additional assumptions for ∇v. The reason

[2]The results/remarks which are described in this chapter were obtained by the author in 1996/97 and he gave a talk on this topic in the spring 1997 on an international conference in honour of Prof. Dr. J. Nečas, which took place in Lisbon. So they are obtained independently from the paper of Kozono-Taniuchi.

is that

$$
w^j D_j v^i w^i = w^i D_i v^j w^j = \\
= \frac{1}{2} w^j (D_j v^i + D_i v^j) w^i = \frac{1}{2} w^j A_{ij}(v) w^i \,. \tag{4.54}
$$

Note that from $A(v) \in L^\infty$ it does in general not follow $\nabla v \in L^\infty$ because the Riesz transforms $R_j = D_j(-\Delta)^{-1/2}$ do not map L^∞ into itself, but only in $BMO \supset L^\infty$. Therefore the result for $A(v) \in L^1(0,T;L^\infty)$ is stronger than the one with $\nabla v \in L^1(0,T;L^\infty)$. Notice also that weak solutions which belong to $L^1(0,T;W^{1,\infty})$ satisfy the energy identity and therefore also the energy inequality because one can test under this assumption with the solution.

What about the assumption $\nabla v \in L^1(0,T;BMO)$? The bilinear estimates of Kozono-Taniuchi show that in this case we have

$$
\|v \cdot \nabla v\|_{0,2} \leq C\|v\|_{0,2} \ \|(-\Delta)^{1/2} v\|_{BMO} \leq C\|v\|_{0,2} \ \|\nabla v\|_{BMO} \,, \tag{4.55}
$$

here $(-\Delta)^{1/2}$ stands for a pseudo differential operator whose symbol on Fourier side is given by $|\xi|$, ξ denoting the variable on Fourier side. The fact that $\|(-\Delta)^{1/2}v\|_{BMO}$ and $\|\nabla v\|_{BMO}$ are comparable follows from a result on Fourier multipliers or equivalently from the boundedness of the Riesz transforms R_j on BMO (see also R. Strichartz [Str80]). The inequality (4.55) shows that one can use the solution v as test function and one can get the energy identity/inequality for $\nabla v \in L^1(0,T;BMO)$. On the other hand the bilinear estimates cannot be used to estimate the term $\int w^j D_j v^i w^i \, dx$ because there would also arise a term like $\|v\|_{0,2} \ \|\nabla w\|_{BMO}$, which we do not want to have. Precisely the bilinear estimate of Kozono-Taniuchi reads (see [KT00, Lemma 1 p.180]):

$$
\|w \cdot \nabla v\|_{0,2} \leq C\left\{\|w\|_{0,2} \ \|(-\Delta)^{1/2} v\|_{0,2} + \|v\|_{0,2} \ \|(-\Delta)^{1/2} w\|_{BMO}\right\} . \tag{4.56}
$$

Therefore we are not able to prove uniqueness under the assumption $\nabla v \in L^1(0,T;BMO)$ at the present. One could prove this, if one could justify an inequality like

$$
\|w \cdot \nabla v\|_{0,2} \leq C\|w\|_{0,2} \ \|\nabla v\|_{BMO} \,. \tag{4.57}
$$

We suspect that it is possible by taking into account the side conditions $\operatorname{div} w = 0 = \operatorname{div} v$ in our setting. In other words we guess that a div-curl expression like $w \cdot \nabla v$ belongs to L^p, $p \in (1,+\infty)$, if one of its members belongs to L^p and the other one to BMO, but up to now we are not able to prove it. Note, however, that such a result would be "somehow" a "dual" version of the $\mathcal{H}^1$-regularity for div-curl expressions of Coifman, Lions, Meyer and Semmes. An immediate consequence of this would be that also assumptions like $\nabla^2 v \in L^1(0,T;L^n)$ or even better $\nabla^2 v \in L^1(0,T;L^{2,n-2})$ would lead to uniqueness due to the imbedding $W^{1,n} \hookrightarrow BMO$ respective $W^{2,n} \hookrightarrow W^1_{BMO}$ or $H^{1,n-2} \hookrightarrow BMO$ respective $H^{2,n-2} \hookrightarrow W^1_{BMO}$.

Next we pass over to the proof of Theorem 4-VI, which we adapt to our setting and take it over from Beirão Da Veiga [BadV95]:

Proof of Theorem 4-VI: Since we assume $a \in W^{1,2}_\sigma$, one can prove the existence of a strong solution v on $[0,T_1]$ for some $T_1 > 0$. Here strong solution means that $v \in L^\infty(0,T_1;W^{1,2}_\sigma) \cap L^2(0,T_1;W^{2,2}_\sigma)$. This can be done either by a Galerkin approach (see Constantin-Foias [CF88] or Heywood [Hey80]) or by a fixed point argument (see Da Veiga [BaDV97]). This solution v is regular and unique (for instance, in the Leray-Hopf class) on $[0,T_1]$. Therefore $u \equiv v$ on $[0,T_1)$. By the a priori estimate(s) of Theorem 4-V, together with the assumption that u belongs to one of the four classes (i)–(iv) in Theorem 4-V, it follows that the estimates (4.12) hold in $[0,T_1]$ (together with the energy inequality, etc.). This argument shows that as long as one of the assumptions (i)–(iv) of Theorem 4-V holds, i. e. until the time T the regular (and unique) solution $v \equiv u$ satisfies (4.12), and can be extended by a continuation argument. In other words the regular solution v stays regular and does not loose regularity until time T (see for example [CF88] for more details).

Proof of Theorem 4-V: Since we assume that u is a regular solution (for instance a strong one) on $[0,T)$ we are allowed to test by $-\Delta u^i$ providing

(4.58)
$$\frac{1}{2}\frac{d}{dt}\,\|\nabla u(t)\|^2_{0,2} + \|\nabla^2 u(t)\|^2_{0,2} = \int\limits_{\mathbb{R}^n} D_k u^j D_j D_k u^i u^i \, dx - \int\limits_{\mathbb{R}^n} f^i \Delta u^i \, dx\,,$$

by using integration by parts

(4.59)
$$\begin{aligned}\int\limits_{\mathbb{R}^n} u^j D_j u^i \Delta u^i \, dx &= -\int\limits_{\mathbb{R}^n} D_k u^j D_j u^i D_k u^i \, dx - \int\limits_{\mathbb{R}^n} u^j D_j D_k u^i D_k u^i \, dx \\ &= \int\limits_{\mathbb{R}^n} D_k u^j D_j D_k u^i u^i \, dx + \int\limits_{\mathbb{R}^n} \Delta u^j D_j u^i u^i \, dx\,,\end{aligned}$$

observing that the pressure cancels due to $\operatorname{div} u = 0 = \operatorname{div} \Delta u$ and that

$$
\begin{aligned}
&\int_{\mathbb{R}^n} u^j D_j D_k u^i D_k u^i \, dx = \int_{\mathbb{R}^n} u^j D_j \frac{|\nabla u|^2}{2} \, dx \\
&= - \int_{\mathbb{R}^n} \operatorname{div} u \frac{|\nabla u|^2}{2} \, dx = 0 = - \int_{\mathbb{R}^n} \operatorname{div} \Delta u \frac{|u|^2}{2} \, dx \qquad (4.60) \\
&= \int_{\mathbb{R}^n} \Delta u^j D_j \frac{|u|^2}{2} \, dx = \int_{\mathbb{R}^n} \Delta u^j D_j u^i u^i \, dx \; .
\end{aligned}
$$

In case (i) we use Chanillo's inequality (4.16) of Proposition 4-VII to estimate:

$$
\left| \int_{\mathbb{R}^n} D_k u^j D_j D_k u^i u^i \, dx \right| \leq C \, \| \nabla u | L^{2,n-2} \| \, \| \nabla^2 u \|_{0,2} \, \| \nabla u \|_{0,2} \, . \qquad (4.61)
$$

Moving the L^2-norm of the second derivatives with Young's inequality to the left we obtain

$$
\begin{aligned}
&\frac{d}{dt} \| \nabla u(t) \|_{0,2}^2 + \| \nabla^2 u(t) \|_{0,2}^2 \qquad (4.62) \\
&\leq C \, \| \nabla u(t) | L^{2,n-2} \|^2 \, \| \nabla u(t) \|_{0,2}^2 + C \, \| f(t) \|_{0,2}^2
\end{aligned}
$$

and Gronwall's inequality delivers

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{t \in (0,T)} \| \nabla u(t) \|_{0,2}^2 \leq \exp \Big(C \int_0^T \| \nabla u(s) | L^{2,n-2} \|^2 \, ds \Big) \qquad (4.63) \\
&\left\{ \| \nabla a \|_{0,2}^2 + C \int_0^T \| f(s) \|_{0,2}^2 \, ds \right\} \; ,
\end{aligned}
$$

which by integrating (4.62) with respect to t finally leads to

(4.64)

$$
\begin{aligned}
&\operatorname*{ess\,sup}_{t \in (0,T)} \| \nabla u(t) \|_{0,2}^2 + \int_0^T \| \nabla^2 u(t) \|_{0,2}^2 \, dt \\
&\leq \Big(\| \nabla a \|_{0,2}^2 + C \int_0^T \| f(s) \|_{0,2}^2 \, ds \Big) \cdot \left\{ 1 + \exp \Big(C \int_0^T \| \nabla u(s) | L^{2,n-2} \|^2 \, ds \Big) \right\} \; .
\end{aligned}
$$

Case (i) is proven. To prove case (ii) we use instead of (4.16) (Proposition 4-VII) inequality (4.20) (Proposition 4-IX) and proceed in completely the same way as before, i. e. replacing in the formula above $\|\nabla u(t)|L^{2,n-2}\|$ by $\|u(t)|BMO\|$. Case (iii) and (iv) are proven in the same manner, but by starting to estimate the right hand side from the "stage" before:

$$|\int_{\mathbb{R}^n} D_k u^j D_j u^i D_k u^i \, dx| \le C \, \|\nabla^2 u|L^{2,n-2}\| \, \|\nabla u\|_{0,2}^2 \tag{4.65}$$

by Chanillo's inequality (4.16) ($\nabla f \sim D_j u^i, g \sim D_k u^j, h \sim D_k u^i$) or by (4.20)

$$|\int_{\mathbb{R}^n} D_k u^j D_j u^i D_k u^i \, dx| \le C \, \|\nabla u|BMO\| \, \|\nabla u\|_{0,2}^2 \,, \tag{4.66}$$

where we used the fact that $D_k u^j D_j u^i$ belongs to $\mathcal{H}^1$ and fulfills

$$\|D_k u^j D_j u^i|\mathcal{H}^1\| \le C \, \|\nabla u\|_{0,2}^2$$

for all $1 \le i, \, k \le n$ as a consequence of Proposition 4-X, so Theorem 4-V is proven.

Let us finish this section with some remarks:

Remark 4-XIII 1) The proof of Theorem 4-V in cases (iii) and (iv) shows that the same assertion holds for solutions of the Euler equations. This comes from the fact that in these cases we did not use the "dissipation-term" $-\Delta u$ during the proof. So we have proven also a regularity class criterion for Euler's equations which replaces the class $L^1(0,T;W^{1,\infty})$, which is known to be a regularity class for the Euler equations. In fact it is the sole class among the classes "$\nabla u \in L^\alpha(0,T;L^\beta)$ with $\frac{2}{\alpha} + \frac{n}{\beta} = 2$" to be a regularity class for these equations (at least according to what is known at present).

2) In all statements of Theorem 4-V one can replace ∇u by $W(u) = \nabla u - (\nabla u)^T$ or $A(u) = \nabla u + (\nabla u)^T$. This observation goes back to Kato-Ponce [KP88] and relies on the fact that the gradient (=Jacobian matrix here) can be recovered from the Spin tensor $W(u)$ or the deformation tensor $A(u)$ via some Riesz transforms provided $\operatorname{div} u = 0$:

$$D_j u^k = R_j R_i W_{ik}(u) = R_j R_i (D_k u^i - D_i u^k) \tag{4.67}$$

because of the following formal calculation:

$$\begin{aligned} R_i D_k u^i &= D_i(-\Delta)^{-1/2} D_k u^i = (-\Delta)^{-1/2} D_k \operatorname{div} u = 0 \,, \\ -R_i D_i u^k &= -D_i(-\Delta)^{-1/2} D_i u^k = (-\Delta)^{-1/2}(-\Delta) u^k = (-\Delta)^{1/2} u^k \,, \\ R_j(-\Delta)^{1/2} u^k &= D_j u^k \,. \end{aligned}$$

Similarly we have

$$D_j u^k = -R_j R_i A_{ik}(u) = -R_j R_i (D_k u^i + D_j u^k)\,. \tag{4.68}$$

Since all Riesz transforms are bounded operators on BMO, we see by (4.68), (4.67) that

$$\nabla u \in BMO \Leftrightarrow A(u) \in BMO \Leftrightarrow W(u) \in BMO \tag{4.69}$$

for solenoidal vector fields u.

3) The results of Theorem 4-V/4-VI can be reformulated in the spirit of the paper "Remarks on the break down of smooth solutions for the 3-D Euler equations" of Beale, Kato and Majda from 1984 (see [BKM84] or [DG95] and also [KP88]). Just to give the reader a "feeling" of this kind of result we formulate one version:

Theorem 4-XIV *If $u \in C^0([0,T); W^{s,p}_\sigma)$ is a solution to (4.1) with $u \notin C^0([0,T]; W^{s,p}_\sigma)$, then*

$$\int_0^T \|W(u)(t)|BMO\|\, dt = +\infty\,.$$

Here we supposed $1 < p < +\infty$ and $s > 1 + n/p$. This version corresponds to Theorem 4.7 in [KP88]. For similar statements (especially in the case $p = 2$) we refer also once more to Kozono-Taniuchi [KT00], see Corollary 1, Theorem 3 and Corollary 2 there. That the statements formulated there in a Hilbert space setting extend to the case of general p, $1 < p < +\infty$, is clear in view of [KP88] and what is written on p. 180–182 of [KT00]. Still the result can be generalized to a larger space than BMO, namely to the (homogeneous) Besov space $B^0_{\infty,\infty} \supset BMO$: see Bergh-Löfström [BL76], Peetre [Pee76] or the books by Triebel [Tri83],[Tri92] for a definition of this space and detailed study of Besov spaces. This generalization is proven by Kozono-Ogawa and Taniuchi (to appear in Math. Zeitschr., but see also Kozono's survey article "On Well-Posedness of the Navier-Stokes-Equations" in [Koz01]), and the proof is based on logarithmic Sobolev inequalities a lá Brezis-Wainger [BW80] in the framework of Besovspaces.

4) We dealt up to now only with the whole space case. Certainly the situation will be the same in a periodic setting, i. e. solving (4.1) on a cube or torus with periodic boundary conditions, but how to deal with the situation on a bounded domain Ω with Dirichlet boundary condition $u = 0$ on $(0,T) \times \partial\Omega$ seems to be not so obvious. Concerning uniqueness one can handle this case, too, because the necessary tools are mostly available in the literature: The spaces $\mathcal{H}^1$ and BMO belong to Triebel's

scale $\dot{F}^s_{p,q}$ (homogeneous space): $\mathcal{H}^1 = \dot{F}^0_{1,2}$ and $BMO = \dot{F}^0_{\infty,2}$ and have non-homogeneous counterparts in Triebel's theory: $F^0_{1/2} = h_1$, $F^0_{\infty,2} = bmo$ (see Triebel [Tri83],[Tri92]). Using these spaces it is possible to obtain a version of Proposition 4-X on bounded domains, so it seems to work.
For regularity the situation is not so nice on bounded domains because one needs to test by $-P\Delta u$, where P denotes Leray's projector on solenoidal vector fields, but due to the fact that P does not preserve boundary values it does not commute with the Laplacian and therefore we are not allowed to integrate by parts. This does certainly not mean that it is not possible to prove a corresponding result on bounded domains, but only that one has to look for "better" tools and methods to achieve something in this direction.

4.4 Some Remarks towards Regularity in the Limit Case $L^\infty(0,T;L^n)$

In this section we study the initial value problem for the Navier-Stokes equations in $(0,T)\times\mathbb{R}^n$, $n\geq 3$:

$$\left\{\begin{aligned} \partial_t v^i - \Delta v^i + v^j D_j v^i + D_i\pi &= 0\,,\\ \operatorname{div} v &= 0\,,\\ v(0,x) &= a(x) \end{aligned}\right. \tag{4.70}$$

under the assumption that v is a Leray-Hopf solution, i. e. $v\in L^\infty(0,T;L^2_\sigma)\cap L^2(0,T;W^{1,2}_\sigma)$, which in addition belongs to $L^\infty(0,T;L^n)$. (Sometimes we will change our point of view and assume that $v\in L^\infty(0,T;L^n)$ will be a suitable weak solution of (4.70) in the sense of Caffarelli-Kohn-Nirenberg respective Lin respective Ladyzhenskaya-Seregin (see [CKN82], [Lin98], [LS99]), but this will be pointed out and discussed when it is necessary.)
Up to now it is still open whether such solutions are regular or not despite all efforts of so many mathematicians. We would like to argue here in favour of regularity, but we are not able to prove it. Nevertheless we can unify some known results and give some supplements to what is known.
Good and nicely written reviews/surveys of the situation in this case can be found in:
1) Marco Cannone: "Viscous Flows in Besov Spaces", published in: Advances in Mathematical Fluid Mechanics (Lecture Notes of the Sixth Internat. School Math. Theory in Fluid Mechanics, Paseky, Cz, Sept. 1999, edited by J. Málek, J. Nečas, and M. Rokyta, Springer 2000).

2) G. P. Galdi: “An Introduction to the Navier-Stokes Initial-Boundary Value Problem”, published in: Fundamental Directions in Mathematic Fluid Mechanics (Advances in Mathematical Fluid Mechanics, Series published by Birkhäuser, edited by G. P. Galdi, J. G. Heywood and R. Rannacher, Birkhäuser 2000).
3) H. Kozono: ”On Well-Posedness of the Navier-Stokes Equations” and J. Neustupa and Patrick Penel: ”Anisotropic and Geometric Criteria for Interior Regularity of Weak Solutions to the 3 D Navier-Stokes Equations”. Both published in: Mathematical Fluid Mechanics, Recent Results and Open Questions (Advances in Math. Fluid Mechanics, Eds. J. Neustupa and P. Penel, Birkhäuser 2001).

These three survey articles by no means give the complete picture – there are certainly more contributions to this problem –, but they give a somehow “representative” overview of the different contributions to this question.

Let us start with a ”global” point of view:

Proposition 4-XV *If $v \in L^\infty(0,T;L^2_\sigma) \cap L^2(0,T;W^{1,2}_\sigma) \cap L^\infty(0,T;L^n)$ is a weak solution to (4.1) under the assumption $a \in L^2_\sigma \cap L^n$, then there exists $\varepsilon = \varepsilon(\|v|L^\infty(0,T;L^n)\|) > 0$, such that $v \in L^\infty(0,T;L^{2+2\varepsilon}_\sigma)$ and $\nabla|v|^{1+\varepsilon} \in L^2(0,T;L^2)$ with corresponding estimate.*

Proof 4-XV : We test (4.1) by $|v|^{2\varepsilon}v^i$ for some $\varepsilon > 0$ to be specified later and obtain:

$$(4.71)\qquad \begin{aligned} &\frac{1}{2+2\varepsilon}\frac{d}{dt}\|v\|^{2+2\varepsilon}_{0,2+2\varepsilon} + \int\limits_{\mathbb{R}^n} |v|^{2\varepsilon}|\nabla v|^2\,dx \\ &+ 2\varepsilon \int\limits_{\mathbb{R}^n} |v|^{2\varepsilon}\frac{|\nabla \frac{|v|^2}{2}|^2}{|v|^2}\,dx = 2\varepsilon \int\limits_{\mathbb{R}^n} \pi v^i |v|^{2\varepsilon-2} v^l D_i v^l\,dx \end{aligned}$$

by integrating the pressure term by parts and using $\operatorname{div} v = 0$. The convective term cancels due to $\operatorname{div} v = 0$:

$$(4.72)\qquad \begin{aligned} &\int\limits_{\mathbb{R}^n} v^j D_j v^i |v|^{2\varepsilon} v^i\,dx = \int\limits_{\mathbb{R}^n} v^j D_j \frac{|v|^{2+2\varepsilon}}{2+2\varepsilon}\,dx \\ &= -\int\limits_{\mathbb{R}^n} (\operatorname{div} v)\frac{|v|^{2+2\varepsilon}}{2+2\varepsilon}\,dx = 0\,. \end{aligned}$$

So we see that testing "globally" creates trouble with the pressure. We estimate the right hand side as follows:

$$
\begin{aligned}
&|\int \pi v^i |v|^{2\varepsilon-2} v^l D_i v^l \, dx| \\
&\le \Big(\int |v|^{2\varepsilon} \frac{|\nabla \frac{|v|^2}{2}|^2}{|v|^2} \, dx \Big)^{1/2} \Big(\int |\pi|^2 |v|^{2\varepsilon} \, dx \Big)^{1/2} \\
&\le \frac{1}{2} \int |v|^{2\varepsilon} \frac{|\nabla \frac{|v|^2}{2}|^2}{|v|^2} \, dx + \frac{1}{2} \int |\pi|^2 |v|^{2\varepsilon} \, dx
\end{aligned}
\tag{4.73}
$$

and get

$$
\begin{aligned}
&\frac{1}{2+2\varepsilon} \frac{d}{dt} \|v\|_{0,2+2\varepsilon}^{2+2\varepsilon} + \int_{\mathbb{R}^n} |v|^{2\varepsilon} |\nabla v|^2 \, dx \\
&\quad + \varepsilon \int_{\mathbb{R}^n} |v|^{2\varepsilon} \frac{|\nabla \frac{|v|^2}{2}|^2}{|v|^2} \, dx \le \varepsilon \int_{\mathbb{R}^n} |\pi|^2 |v|^{2\varepsilon} \, dx \, .
\end{aligned}
\tag{4.74}
$$

Next we observe

$$
\left| \nabla \frac{|v|^{1+\varepsilon}}{1+\varepsilon} \right|^2 = |v|^{2\varepsilon-2} \left| \nabla \frac{|v|^2}{2} \right|^2 \le |v|^{2\varepsilon} |\nabla v|^2 \tag{4.75}
$$

and apply the Sobolev imbedding

$$
\int_{\mathbb{R}^n} |\nabla f|^2 \, dx \ge C_0 \Big(\int_{\mathbb{R}^n} |f|^{\frac{2n}{n-2}} \, dx \Big)^{\frac{n-2}{n}}
$$

with $f = \frac{|v|^{1+\varepsilon}}{1+\varepsilon}$ in order to estimate the integral $\int |v|^{2\varepsilon} |\nabla v|^2 \, dx$ from below. This yields

$$
\begin{aligned}
&\frac{1}{2+2\varepsilon} \frac{d}{dt} \|v\|_{0,2+2\varepsilon}^{2+2\varepsilon} + C_0 \frac{1}{(1+\varepsilon)^2} \Big(\int_{\mathbb{R}^n} |v|^{\frac{2n(1+\varepsilon)}{n-2}} \, dx \Big)^{\frac{n-2}{n}} \\
&\quad + \varepsilon \int_{\mathbb{R}^n} |v|^{2\varepsilon-2} \left| \nabla \frac{|v|^2}{2} \right|^2 \, dx \le \varepsilon \int_{\mathbb{R}^n} |\pi|^2 |v|^{2\varepsilon} \, dx \, .
\end{aligned}
\tag{4.76}
$$

Applying Hölder's inequality with exponents $\frac{2n(1+\varepsilon)}{(n-2)2\varepsilon}$ for $|v|^{2\varepsilon}$ and $\frac{n(1+\varepsilon)}{n+2\varepsilon}$ for $|\pi|^2$ and then Young's inequality we arrive at

$$
\begin{aligned}
\int\limits_{\mathbb{R}^n} |\pi|^2|v|^{2\varepsilon}\,dx &\le \Big(\int\limits_{\mathbb{R}^n} |v|^{\frac{2n(1+\varepsilon)}{n-2}}\,dx\Big)^{\frac{n-2}{n}\frac{\varepsilon}{1+\varepsilon}} \Big(\int\limits_{\mathbb{R}^n} |\pi|^{\frac{2n(1+\varepsilon)}{n+2\varepsilon}}\,dx\Big)^{\frac{n+2\varepsilon}{2n}\cdot\frac{2}{1+\varepsilon}} \\
&\le \frac{\varepsilon}{1+\varepsilon}\Big(\int\limits_{\mathbb{R}^n} |v|^{\frac{2n(1+\varepsilon)}{n-2}}\,dx\Big)^{\frac{n-2}{n}} + \frac{1}{1+\varepsilon}\Big(\int\limits_{\mathbb{R}^n} |\pi|^{\frac{2n(1+\varepsilon)}{n+2\varepsilon}}\,dx\Big)^{\frac{n+2\varepsilon}{n}}.
\end{aligned}
\tag{4.77}
$$

Since v is divergence free, we get from (4.1) by applying the operator div to the equation:

$$-\Delta\pi = D_iv^jD_jv^i = D_iD_j(v^jv^i) \tag{4.78}$$

or equivalently

$$\pi = R_iR_j(v^jv^i) \tag{4.79}$$

with $R_i = D_i(-\Delta)^{-1/2}$ denoting the Riesz transform. Hence, by Calderon-Zygmund it follows that

$$\|\pi\|_{0,\frac{2n(1+\varepsilon)}{n+2\varepsilon}} \le C_1\,\|v\|^2_{0,\frac{4n(1+\varepsilon)}{n+2\varepsilon}}. \tag{4.80}$$

Observe that for $\varepsilon \downarrow 0$ the exponents for the pressure integrals have the behaviour $\frac{2n(1+\varepsilon)}{n+2\varepsilon} \downarrow 2$ for $n \ge 3$! So we have to estimate the quantity

$$\|v\|^{4(1+\varepsilon)}_{0,\frac{4n(1+\varepsilon)}{n+2\varepsilon}}$$

against $\|v\|^{2(1+\varepsilon)}_{0,\frac{2n(1+\varepsilon)}{n-2}}$ and $\|v\|_{0,n}$ in some power. Luckily as we are we find by interpolation respective Hölder

$$\|v\|^{4(1+\varepsilon)}_{0,\frac{4n(1+\varepsilon)}{n+2\varepsilon}} \le \|v\|^{2(1+\varepsilon)}_{0,\frac{2n(1+\varepsilon)}{n-2}}\|v\|^{2(1+\varepsilon)}_{0,n} \tag{4.81}$$

because of

$$\frac{1}{2}\cdot\frac{n-2}{2n(1+\varepsilon)} + \frac{1}{2}\cdot\frac{1}{n} = \frac{n-2}{4n(1+\varepsilon)} + \frac{2(1+\varepsilon)}{4n(1+\varepsilon)} = \frac{n+2\varepsilon}{4n(1+\varepsilon)}. \tag{4.82}$$

Therefore we finally get:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|v\|_{0,2+2\varepsilon}^{2+2\varepsilon} + C_0\frac{1}{1+\varepsilon}\|v\|_{0,\frac{2n(1+\varepsilon)}{n-2}}^{2(1+\varepsilon)} + \varepsilon(1+\varepsilon)\left\|\nabla\frac{|v|^{1+\varepsilon}}{1+\varepsilon}\right\|_{0,2}^2 \\
&\le \varepsilon^2\|v\|_{0,\frac{2n(1+\varepsilon)}{n-2}}^{2(1+\varepsilon)} + \varepsilon C_1\|v\|_{0,\frac{2n(1+\varepsilon)}{n-2}}^{2(1+\varepsilon)}\|v\|_{0,n}^{2(1+\varepsilon)} \\
&= \left(\varepsilon^2 + C_1\varepsilon\|v\|_{0,n}^{2(1+\varepsilon)}\right)\|v\|_{0,\frac{2n(1+\varepsilon)}{n-2}}^{2(1+\varepsilon)}\,.
\end{aligned}
\tag{4.83}
$$

From this inequality the assertion of the proposition follows by choosing ε so small that

$$
\varepsilon^2 + C_1\varepsilon\|v\|_{0,n}^{2(1+\varepsilon)} < \frac{C_0}{2} \le \frac{C_0}{1+\varepsilon} \qquad (\text{ for } 0 < \varepsilon \le 1)\,,
$$

which is possible due to $v \in L^\infty(0,T;L^n)$; moving the right hand side to the left and integrating with respect to t provides then a corresponding estimate.

Remark 4-XVI 1) The proposition as well as its proof are an output of several discussions with my advisor Prof. Dr. J. Frehse and I grateful acknowledge his "all the time readiness" to discuss all issues of this work with me.

2) The above given "proof" is of course formal, but it can be made rigorous by using a local-in-time existence result for strong solutions (in our setting here the most simple one is provided probably by Beirão da Veiga [BadV87]) and taking into account the uniqueness of $L^\infty(0,T;L^n)$-solutions.

3) The most important "contribution" of this result consists in the slightly improved space-regularity of the solution. As a consequence of this we can slightly improve the regularity of $\partial_t v$, $\nabla^2 v$ and $\nabla\pi$. Let us briefly sketch this: From $v \in L^\infty(0,T;L^n) \cap L^2(0,T;W^{1,2})$ we get via parabolic imbedding $v \in L^4(0,T;L^4)$ and then by a well known result of Solonnikov [Sol68]:

$$
\partial_t v, \nabla^2 v, \nabla\pi \in L^{4/3}(0,T;L^{4/3}) \tag{4.84}
$$

with corresponding estimate. Proposition 4-XV provides now the information

$$
v \in L^\infty(0,T;L^n) \cap L^{2(1+\varepsilon)}\big(0,T;L^{\frac{2n(1+\varepsilon)}{n-2}}\big)
$$

and by interpolation

$$
v \in L^{4(1+\varepsilon)}\big(0,T;L^{\frac{4n(1+\varepsilon)}{n+2\varepsilon}}\big)
$$

(see the proof). Therefore thanks to Solonnikov we get:

$$
\partial_t v, \nabla^2 v, \nabla\pi \in L^{\frac{4(1+\varepsilon)}{3+2\varepsilon}}\big(0,T;L^{\frac{4n(1+\varepsilon)}{3n+4\varepsilon}}\big) \tag{4.85}
$$

plus estimate. From $v \in L^{4(1+\varepsilon)}(0,T;L^{\frac{4n(1+\varepsilon)}{n+2\varepsilon}})$ and (4.85) one can also recover a better estimate for ∇v:

$$\nabla v \in L^{\frac{8n(1+\varepsilon)}{8+2\varepsilon}}\big(0,T;L^{\frac{8(1+\varepsilon)n}{4n+6\varepsilon}}\big)\,,$$

and by deriving the system one realizes that one can get also information on $\partial_t \nabla v$, $\nabla^2 v$, $\nabla^2 \pi$ (taking derivatives with respect to the space variables and realizing that the products arising from the convective term belong to some L^p-space with $p > 1$!). Nevertheless all this informations/ estimates are not strong enough to go further, in fact they are all weaker than the start information $v \in L^\infty(0,T;L^n)$, but of course they are slightly better than the usual regularity of any weak solution.
The improved regularity we just discussed can of course be used to study the "singular set" of a solution and the above discussed regularity properties can be "translated/transformed" into statements about the Hausdorff measure of the singular set. Once more one obtains slight improvements of what is known for any weak solution, but nothing which would help to go further.

4) The proof of Proposition 4-XV sheds also some light on the following question: Why is it not possible to use Moser's iteration scheme to get something better for the solution? As we saw during the proof of the proposition the convective term does not create troubles, but the pressure does. The reason for this seems to be that the pressure – which is a Lagrange multiplier due to the constraint of incompressibility $\operatorname{div} v = 0$ – is a "global" or better a non-local term and it seems to be rather difficult to overcome this problem ...

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