Interpolation categories for homology theories

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Abstract

For a given homological functor, that satisfies some technical assumptions, from a triangulated category, that has an underlying model category, to an abelian category with enough injectives we construct a tower interpolation categories. These are categories over which the functor factorizes and that capture more and more information according to the injective dimension of the images of the functor. The categories are obtained by proving the existence of certain model structures on cosimplicial objects over the initial model category. These model structures are truncated versions of previously known resolution model structures. Examples of functors fitting in our framework are given by every generalized homology theory represented by a ring spectrum satisfying the Adams-Atiyah condition. The constructions are closely related to the modified Adams spectral sequence and give a very conceptual approach to the moduli problem and the associated obstruction theory of the functor. We can easily reprove statements about moduli spaces known in the literature and we hope that our interpolation categories reveal more information about classification problems at least when the injective dimension of the target category is finite and probably small.

Introduction

Algebraic topology, or more precisely homotopy theory, is the study of geometric objects up to some equivalence, known as weak homotopy equivalence, by translating the geometrical or homotopical information into algebraic data. The mathematical device to do this are functors from the homotopy category of some geometric category to an algebraic category, and a first example is given by the homotopy groups of a topological space. The rather mysterious term "homotopy category" will be explained below.

It was realized soon that homotopy groups are very hard to compute and that there exist functors with properties that make them easier to compute although, of course, they might not contain as much information as the homotopy groups. These functors are called homological functors or homology theories and a long era of struggling for their precise definition and their general properties was brought to an end with the foundational work of Eilenberg and Steenrod in 1952. Here the functors were considered as being defined on the category of spaces with values in abelian groups together with the property of homotopy invariance. The axioms given there, known as the Eilenberg-Steenrod axioms, uniquely characterize a functor called singular homology theory which was invented by Poincaré around 1900. In the list of axioms there is one, called the dimension axiom, which prescribes the values of the functor on spheres. If we omit this axiom we suddenly get a big variety of other interesting functors, called generalized homology theories, the first one of which was discovered by Thom in 1958 who named it "cobordism". Almost at the same time Atiyah, Grothendieck and Hirzebruch constructed a functor that they called "K-theory" which was soon, around 1960, realized to be a generalized cohomology theory. There is an associated covariant theory to which we also refer as (topological) K-theory which is a generalized homology theory. Since then many more examples of generalized homology theories have been discovered and their importance has been increasing in the last 50 years and is felt throughout mathematics. From now on we will simply drop the adjective "generalized" from the notion.

This thesis contributes to the general theory a calculus of interpolation categories. These categories depend on a given homology theory and are intended to give a very conceptual approach to realizations and moduli problems of this functor and to the associated obstruction theory. They interpolate in a precise sense between the geometric source and the algebraic target category, see section 5.4.

Before we proceed we will outline what is meant with the phrase "homotopy category". In the particular case above we mean the category of topological spaces where we keep the class of objects fixed and formally invert the weak equivalences which are maps that induce isomorphisms on all homotopy groups including π_0; in general we mean a category which is obtained from some other category by formally inverting a class of morphisms. This process is called localization and we interchangingly use the words "homotopy category" and "localized category".

The concept of localization arose in several different areas of mathematics as for example in homological algebra. The basic notion here is that of a resolution. Resolutions are used to define and compute derived functors which are functors that satisfy an analogous invariance property as topological homology theories above. Changing the resolution does not alter the derived functor. We can restate this by saying that derived functors are genuinely defined on a localized category. We refer the reader to [DS95] or [Hov99] for a precise definition of a localization of a category. It is given by a certain universal property.

There are set theoretical problems with this construction – we have to leave our universe sometimes – as well as there are enormous difficulties in getting a useful grip on the morphisms between two objects. The standard way to overcome these problems is to prove the existence of a model structure on the category we want to localize. We call a category with a model structure a model category. This machinery was invented by Quillen in [Qui67] and is sometimes referred to as homotopical algebra. It encompasses homological algebra as, what one might call, abelian subtheory. There is a list of axioms for a model category $\mathcal{M}$ including the datum of a subcategory $\mathcal{W}$, whose morphisms are called weak equivalences, which are the morphism we intend to invert. The rough idea is to replace the category $\mathcal{M}$ by a full subcategory of nice objects, called the fibrant and cofibrant objects. On this subcategory the axioms suffice to give an equivalence relation on the sets of morphisms between two objects, called the homotopy relation, and we can simply take the quotient by this relation to obtain the morphism sets in our homotopy category. This description does not involve objects other than the source and the target. Finally it is proved that this homotopy category satisfies the universal property required for a localization. The merit of this procedure is that we have a clean construction of the homotopy category and we get reasonably good description of the set of morphisms between two objects, in other words the homotopy classes of maps between two homotopy types. Moreover it becomes conceptionally easy to define derived functors. The way to do that is the standard way from homological algebra. First we resolve an object, which means that we approximate it fibrantly or cofibrantly, and then we apply the functor we want to derive. To define our interpolation categories we want to apply the concept of resolution, but first we have to explain some properties we have to assume on our homological functor and what is the realization or moduli problem.

One essential feature of homological functors is that the axioms force them to commute in an appropriate way with suspensions. Speaking in technical terms they are defined, or factorize over a triangulated category. We suppose therefore that our homological functor F is defined on a category $\mathcal{M}$ carrying a stable model structure which ensures that the homotopy category $\mathcal{T} = Ho(\mathcal{M})$, over which F necessarily factors, is triangulated. Finally we assume that the target category $\mathcal{A}$ is abelian with enough injectives. Since we will have to make additional assumptions we will give examples of functors that fit in our framework at the end of the introduction.

Our ultimate goal now is to answer the question whether there exists to a given object A in the abelian target category $\mathcal{A}$ an object X in the triangulated source category $\mathcal{T}$ together with an isomorphism $FX \cong A$ and if yes, how many different objects exist. We also would like to describe liftings of morphisms. This is the realization or moduli problem for the homological functor F.

In general we can give no immediate answer to this question and we have to restrict ourselves to developing an obstruction calculus leading to an often infinite series of obstructions to existence of realizations which is according to Haynes Miller "the poor man's approach" to the moduli problem. However, the problem is difficult. Obstructions are elements of abelian groups that vanish if and only if we can push some realization process one step further. If all obstructions vanish we have found a realization.

Before we outline the essential idea of making use of injective resolutions we have to make one starting step: We want to restrict to a source category whose objects are really distinguishable by F. Since we will never be able to tell two objects apart by F in case there is a morphism in $\mathcal{T}$ between them that induces an isomorphism via F, it makes no sense to keep them separate. We want to identify them. In homotopy theory this process is known as Bousfield localization and is available for all homology theories, see [Bou79] and [Hir03]. This process supplies a new stable model structure on $\mathcal{M}$ whose weak equivalences are exactly those morphisms that induce isomorphisms via F. We replace $\mathcal{T}$ by the homotopy category with respect to this new F-local model structure and we will assume from the beginning on that our functor F detects isomorphisms.

Now we want to exploit the already mentioned bridge that model category theory supplies between (stable) homotopy theory and homological algebra. Given an object A we consider an injective resolution of it. This corresponds via the Dold-Kan correspondence to a fibrant approximation in the category of cosimplicial objects over $\mathcal{A}$ where we view A as a constant cosimplicial object. The last assumption we put on our functor F is that there exist enough F-injective objects, or sometimes called Eilenberg-MacLane objects. These are objects E in $\mathcal{T}$ such that $F(E)$ is injective in $\mathcal{A}$ and for every object X in $\mathcal{T}$ there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{T}}(X, E) \cong \mathrm{Hom}_{\mathcal{A}}(F(X), F(E))$$

This enables us to lift the injective resolution of A to a cosimplicial object over $\mathcal{T} = Ho(\mathcal{M})$. But in general cosimplicial objects over $\mathcal{T}$ cannot be replaced by a cosimplicial object over the underlying model category $\mathcal{M}$ since a diagram that commutes up to homotopy cannot always be changed to a diagram that commutes strictly. Obstructions against this in special cases are sometimes referred to as Massey products or Toda brackets. There is an easy account of such an obstruction calculus for realizing objects using just the triangulated structure of $\mathcal{T}$ in the appendix of [BKS04]. The homotopy cosimplicial objects can be replaced by strict ones over $\mathcal{M}$ exactly for those objects in $\mathcal{A}$ that possess a realization in $\mathcal{T}$ via F. This is proved in 5.5.3. Thus carrying out our realization process means that we start to strictify our up-to-homotopy-cosimplicial objects. Here we view cosimplicial objects as resolutions of objects in the underlying category which are like objects "concentrated in degree zero". The category of cosimplicial objects over a category $\mathcal{C}$ is denoted by $c\mathcal{C}$, objects are denoted by $X^\bullet$.

The requirements on our functor F suffice to construct a spectral sequence which is known in the literature as the modified Adams spectral sequence. In contrast to the original Adams spectral sequence it is constructed using absolute injective resolutions while the classical one uses relative injective resolutions. We shortly explain the connection with the theory of interpolation categories in 4.2.35 and 6.2. Accounts of the modified Adams spectral sequence are given in [Bri68], [Bou85], [Dev97] and [Fra96].

Returning to our realization process it is conceptually and technically easier to start with already strict cosimplicial objects over $\mathcal{M}$ and search for those that look like a constant cosimplicial object up to some appropriate notion of homotopy on $c\mathcal{M}$. More precisely we want to invert maps $X^\bullet \to Y^\bullet$ that induce quasi-isomorphisms $NF(X^\bullet) \to NF(Y^\bullet)$ of cochain complexes over $\mathcal{A}$. Here N is the Dold-Kan normalization. After having come to this point the reader may already guess that we are looking for a model structure on $c\mathcal{M}$ with precisely these maps as weak equivalences. Luckily this model structure was found just in time for us in the very elegant paper [Bou03].

The model structures constructed there have the very suggestive name "resolution model structures" and, indeed, they perfectly serve for our purpose. The first resolution model structure appeared in [DKS93] and were used to study the realization problem for the homotopy group functor on spaces. This was pursued further in [DKS95] and [BDG01]. Another resolution model structure was considered in [GH04] to study the existence of A_∞- or E_∞-structures on ring spectra. All these model structures are exhibited as a special case of the general one in [Bou03]. While Bousfield in his paper was more interested in defining certain completion functors the other sources directly address some moduli problem in the vein we tried to explain here. They were able to set up an obstruction calculus and give a description of the moduli space of the intermediate steps of realizations of an objects. Such an intermediate step is called a potential n-stage. It is characterized by some equations, see 5.3.1, that come out of the process of "adding" step by step the injective terms given by a resolution of the object in $\mathcal{A}$ to the already existing resolution in $\mathcal{M}$. The adjective "potential" should remind of the possibility that the object looks like an honest n-stage, but might not be realizable in some future step.

Our new idea is to define the notion of n-equivalence 3.1.4 and 3.2.1 and to show that this is part of a model structure called the n-truncated resolution model structure. This is proved in 3 and is the technical heart of the thesis. The maps that are inverted are maps $X^\bullet \to Y^\bullet$ in $c\mathcal{M}$ that induce isomorphisms on some cosimplicial homotopy groups up to degree n, see 3.2.2. If $\mathcal{M}$ carries the discrete model structure, for example if $\mathcal{M} = \mathcal{A}$, then this just corresponds to morphisms that induce isomorphisms on cohomology groups $H^sNF(X^\bullet) \to H^sNF(Y^\bullet)$ for $0 \le s \le n$. Potential n-stages are now objects that are n-equivalent to some constant cosimplicial object. If the object from $\mathcal{A}$ is indeed realizable, these n-stages are exactly the cofibrant approximations of the constant cosimplicial object given by the realization. We are now able to define the category of n-stages in 5.1.1 called the n-th interpolation category for F. This itself is interesting and is an even more conceptual approach to the moduli problem than that taken in [BDG01]. With our tools at hand we are for instance able to prove the analogous results about moduli spaces of [BDG01] in our case of a homological functor in a straightforward manner reducing everything to the black box theorem 1.4.7. We think, the methods should generalize to the unstable or non-linear setting, but we have not checked that, yet.

Our framework supplies whole categories, and not just obstruction elements or moduli spaces, and it is our hope that this reveals more information about homological functors, at least when the injective dimension of the target category is finite and probably small. There is a similar theory of interpolation categories in [Bau99], but at the moment we do not understand the relationship between these theories.

We remark that our results dualize. This means that the whole theory of truncated model structures also works in the simplicial case where we have a class of projective models instead of injective ones. In fact all the other articles considering resolution model structures except of [Bou03] were written simplicially. Compare [Jar04].

A future project will be to study complex K-theory localized at a prime $p > 2$, since the injective dimension of the target category is then 2. This is shown in [Bou85] and an algebraic classification of the category of $KU_{(p)}$-local spectra has been obtained among other things in [Fra96]. In general almost all homology theories that are used by topologists qualify as a homological functor with enough injectives, see 4.1.7. More precisely every ring spectrum satisfying an Atiyah-Adams condition given in [Ada74, 15.1.] induces such a homological functor. This includes spectra like $H\mathbb{F}_p, KO, KU, MO, MU, MSp$ and S and many theories derived from them. An important exception is $H\mathbb{Z}$, integral singular homology. But there are also algebraic examples like taking cohomology groups viewed as a functor from the derived category of $\mathcal{A}$ to the category of graded objects over $\mathcal{A}$.

Another question is if the truncated model structures can also be given for an unstable model category or a target category which is exact in the sense of Quillen. It might also be interesting to develop interpolation categories in a setting using relative homological algebra. This would directly correspond to the usual Adams spectral sequence.

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Dedication

I want to dedicate my thesis to my friend Ralf Fütterer. Next to my parents he was the one who influenced my life and my view of the world most profoundly. I remember him as an unbreakable strong personality with sharp intelligence, an unbeatable sense of humor and an unbelievable ability of giving people the feeling that they do the right thing. He died unexpectedly on July 23rd 2002 during our joint holidays.

Contents

1 Cosimplicial objects and the Reedy structure

This first section is just a recollection of well known definitions and results. It is given to fix notation and to present facts we are going to need later. In 1.2 various decomposition functors for simplicial and cosimplicial objects are explained. In 1.3 we describe the Reedy structure on cosimplicial objects and the internal simplicial structure, with which the Reedy structure is compatible, as well as the Tot-functor. In 1.4 we give the definition of the moduli space of an object in a model category together with the main theorem 1.4.7 about these spaces.

1.1 Model categories

The notion of a model category was introduced in [Qui67]. Today there are several axiom systems, but they differ just in minor technical details. In general we refer the reader to [DS95], [GJ99], [Hov99] and [Hir03] for the theory of model categories and for simplicial techniques in homotopy theory.

Definition 1.1.1 When we speak of a **model category** we mean a category together with three subcategories whose morphisms are called weak equivalences, cofibrations and fibrations satisfying a list of axioms to which we refer by **MC1** to **MC5** and that are given for example in [GJ99, p. 1], [Hov99, Definition 1.1.3.] or [DS95, 3.3. Def.] with the exception that we always assume that there exists arbitrary small limits and colimits.

We do not insist on the existence of functorial factorization as it is axiomatized in [Hov99]. Usually model structures admit functorial factorization, and this will also be true here. So they satisfy the stronger definition of Hovey.

Let $\mathcal{M}$ be a category with arbitrary small limits and colimits. Thus $\mathcal{M}$ satisfies **MC1**. The data and conditions required by the remaining axioms is called a **model structure** on $\mathcal{M}$.

A model category is called **simplicial** if it satisfies the extra axioms **SM7** and **SM7'** which are equivalent to each other and are given in [GJ99, II.3.1 Axiom] and [GJ99, II.3.11.].

We call a model structure **stable** if it is pointed and a commutative square is a homotopy pullback square if and only if it is a homotopy pushout square. This property ensures that the associated homotopy category is triangulated.

Definition 1.1.2 Let $G : \mathcal{C} \to \mathcal{B}$ be a functor from some model category $\mathcal{C}$ to an arbitrary category $\mathcal{B}$ that maps weak equivalences between fibrant objects to isomorphisms. Then we denote by $\boldsymbol{RG}{:}Ho(\mathcal{C}) \to \mathcal{B}$ its **right derived functor** obtained by applying G to a fibrant approximation. The observation that this works is known as Quillen's total derived functor theorem and is displayed in [GJ99, II.7.] or [Hov99, §1].

Analogously we denote a **left derived functor**, if it exists, by $\boldsymbol{LG}$. It is constructed in the dual way.

Definition 1.1.3 Let $f : X \to Y$ be a map in a model category. We follow [Hir03] by calling a **cofibrant approximation of $\boldsymbol{X}$** a cofibrant object A together with a weak equivalence $A \to X$, while a **cofibrant replacement** is the result of the functorial factorization applied to the unique map $* \to X$. A **cofibrant approximation of $\boldsymbol{f}$** is a commutative diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \widetilde{Y} \\ {\scriptstyle\simeq}\big\downarrow & & \big\downarrow{\scriptstyle\simeq} \\ X & \xrightarrow[f]{} & Y \end{array}$$

where $\widetilde{f}$ is a cofibration between cofibrant objects and the vertical maps are weak equivalences. We usually just refer to it by $\widetilde{f}$ leaving the diagram understood. We define **fibrant approximations** to objects and maps and fibrant replacements by dualizing the previous notions.

1.2 Latching and matching objects

Definition 1.2.1 Let Δ be the category of finite ordinal numbers. Here we mean the category with the natural numbers as objects and we identify $n \in \mathcal{N}$ with the ordered set $\{0, ..., n\}$. The morphisms are the order preserving functions between these sets.

Let $\mathcal{C}$ be an arbitrary category. Then a **simplicial object** over $\mathcal{C}$ is a contravariant functor from Δ to $\mathcal{C}$. A **cosimplicial object** over $\mathcal{C}$ is a covariant functor from Δ to $\mathcal{C}$. We denote the corresponding categories by $\boldsymbol{s\mathcal{C}}$ and $\boldsymbol{c\mathcal{C}}$.

Let Δ_n be the full subcategory of Δ consisting of all objects $\leq n$. Let

$$j_n : \Delta_n \hookrightarrow \Delta$$

be the **inclusion functor**. Functors from Δ_n to a category $\mathcal{C}$ are called $\boldsymbol{n}$**-truncated** simplicial and cosimplicial objects over $\mathcal{C}$, their categories are denoted by $\boldsymbol{s_n\mathcal{C}}$ and $\boldsymbol{c_n\mathcal{C}}$.

Now we assume that $\mathcal{C}$ possesses all limits and colimits. The pullback functor $(j_n)^* : c\mathcal{C} \to c_n\mathcal{C}$ is called the **cosimplicial truncation functor** that possesses a left adjoint

$$\ell^n := \operatorname*{LKan}_{j_n} : c_n\mathcal{C} \to c\mathcal{C}$$

given by a left Kan extension. Dually the functor $(j_n)^*$ also possesses a right adjoint

$$r^n := \operatorname*{RKan}_{j_n} : c_n\mathcal{C} \to c\mathcal{C}$$

given by right Kan extension. These functors exist since $\mathcal{C}$ has all limits and colimits and a description of them is given in 1.2.2.

Let $(j_n)^* : s\mathcal{C} \to s_n\mathcal{C}$ also denote the **simplicial truncation functor**. It possesses a right adjoint

$$r_n := \operatorname*{RKan}_{j_n} : s_n\mathcal{C} \to s\mathcal{C}$$

given by a right Kan extension. Dually $(j_n)^*$ possesses a left adjoint

$$\ell_n := \operatorname*{LKan}_{j_n} : s_n\mathcal{C} \to c\mathcal{C}$$

given by left Kan extension. For existence here we have the same remark as for the cosimplicial case above.

Note the sub- versus the superscripts to distinguish the co- and the contravariant case. Note also that $\ell_0 : \mathcal{C} \to s\mathcal{C}$ and $r^0 : \mathcal{C} \to c\mathcal{C}$ are the embedding functors as constant (co-)simplicial objects. The dual of the following two lemmas is proved in [Hir03, Prop. 16.2.6.].

Lemma 1.2.2 *The left adjoint $\ell^n : c_n\mathcal{C} \to c\mathcal{C}$ of the cosimplicial truncation functor $(j_n)^*$ can be described by the following isomorphisms:*

$$(\ell^n x^\bullet)^m \cong \operatorname*{colim}_{\substack{k \hookrightarrow m \\ k \leq n}} x^k$$

There exists also the following coequalizer diagram:

$$\coprod_{(n-1)\hookrightarrow m} x^{n-1} \rightrightarrows \coprod_{n \hookrightarrow m} x^n \to (\ell^n x^\bullet)^m,$$

where the two maps are induced by the relation $d^j d^i = d^i d^{j-1}$ for each $0 \leq i < j \leq n$.

Lemma 1.2.3 *The right adjoint $r_n : s_n\mathcal{C} \to s\mathcal{C}$ of the simplicial truncation functor $(j_n)^*$ can be described by the following isomorphisms:*

$$(r_n x_\bullet)^m \cong \lim_{\substack{k \hookrightarrow m \\ k \leq n}} x_k$$

There exists also the following equalizer diagram:

$$(r_n x_\bullet)^m \to \prod_{n \hookrightarrow m} x_n \rightrightarrows \prod_{(n-1)\hookrightarrow m} x_{n-1}$$

where the two maps are induced by the relation $d_j d_i = d_i d_{j-1}$ for each $0 \leq i < j \leq n$.

Definition 1.2.4 We define the notion of a **skeleton** and **coskeleton**. No matter if we are in the simplicial or cosimplicial setting, an n-skeleton will be n-truncation followed by left Kan extension, while an n-coskeleton will be n-truncation followed by right Kan extension. So let $X^\bullet$ be in $c\mathcal{C}$ and $Y_\bullet$ in $s\mathcal{C}$. Then we define:

$$\begin{array}{rclcrcl} \operatorname{sk}_n X^\bullet & := & \ell^n (j_n)^* X^\bullet, & \qquad & \operatorname{cosk}_n X^\bullet & := & r^n (j_n)^* X^\bullet, \\ \operatorname{sk}_n Y_\bullet & := & \ell_n (j_n)^* Y_\bullet, & \qquad & \operatorname{cosk}_n Y_\bullet & := & r_n (j_n)^* Y_\bullet \end{array}$$

Definition 1.2.5 For a simplicial object $X_\bullet$ over $\mathcal{C}$ we define its ***n*-th matching object** by

$$M_n X_\bullet := \lim_{\substack{k \hookrightarrow n \\ k < n}} X_k \text{ in } \mathcal{C}$$

and its ***n*-th latching object** by

$$L_n X_\bullet := \operatorname*{colim}_{\substack{n \twoheadrightarrow k \\ k < n}} X_k \text{ in } \mathcal{C}.$$

For a cosimplicial object $X^\bullet$ over $\mathcal{C}$ we define its ***n*-th matching object** by

$$M^n X^\bullet := \lim_{\substack{n \twoheadrightarrow k \\ k < n}} X^k \text{ in } \mathcal{C}$$

and its ***n*-th latching object** by

$$L^n X^\bullet := \operatorname*{colim}_{\substack{k \hookrightarrow n \\ k < n}} X^k \text{ in } \mathcal{C}.$$

Beware of an index shift! In [GJ99] our $M^n X^\bullet$ and $L^n X^\bullet$ are denoted by $M^{n-1} X^\bullet$ and $L^{n-1} X^\bullet$. Note also again the subscript versus the superscript.

Remark 1.2.6 From 1.2.2 and 1.2.3 we get descriptions of the matching and latching objects in terms of (co-)equalizer diagrams. By formal nonsense there are also the following isomorphisms:

$$L^{n+1}X^\bullet = (\ell^n (j_n)^* X^\bullet)^{n+1} = (\mathrm{sk}_n\, X^\bullet)^{n+1}$$
$$M_{n+1}X_\bullet = (r_n (j_n)^* X_\bullet)_{n+1} = (\mathrm{cosk}_n\, X_\bullet)_{n+1}$$

We will also consider partial matching and latching objects. These will be defined in 2.1.17, where we will describe them more conveniently using the action of simplicial sets on $c\mathcal{C}$.

1.3 The Reedy structure on cosimplicial objects

Let $\mathcal{M}$ be a simplicial model category. Let $c\mathcal{M}$ be the category of cosimplicial objects over $\mathcal{M}$.

Definition 1.3.1 We define the following classes of morphisms that will constitute the **Reedy structure** on $c\mathcal{M}$. Let $X^\bullet \to Y^\bullet$ be a morphism in $c\mathcal{M}$. It is called

(i) a **Reedy equivalence** if for every $s \in \mathbb{N}$ the maps $X^s \to Y^s$ are weak equivalences in $\mathcal{M}$.

(ii) a **Reedy cofibration** if for every $s \in \mathbb{N}$ the induced maps

$$X^s \sqcup_{L^s X^\bullet} L^s Y^\bullet \to Y^s$$

are cofibrations in $\mathcal{M}$.

(iii) a **Reedy fibration** if for every $s \in \mathbb{N}$ the induced maps

$$X^s \to M^s X^\bullet \times_{M^s Y^\bullet} Y^s$$

are fibrations in $\mathcal{M}$.

Remark 1.3.2 In 2.1.9 we will define the so called external or canonical simplicial structure on $c\mathcal{M}$. This structure does not refer to an underlying simplicial structure on $\mathcal{M}$, but it is not compatible with the Reedy structure. It just satisfies a weakened version of SM7 (see [Bou03, 2.10.]) or remark 2.1.10. For the Reedy structure over a simplicial model category $\mathcal{M}$ there is a compatible simplicial structure that we will describe in the following definition.

Definition 1.3.3 If $\mathcal{M}$ is a simplicial model category, we can provide the category $c\mathcal{M}$ with a simplicial structure called the **internal (simplicial) structure**. For $K \in \mathcal{S}$ and $X^\bullet, Y^\bullet \in c\mathcal{M}$ we set

$$(X^\bullet \otimes^{\mathrm{int}} K)^n := X^n \otimes_{\mathcal{M}} K$$

Therefore we get by adjointness

$$\mathrm{map}^{\mathrm{int}}(X^\bullet, Y^\bullet)_n := \mathrm{Hom}_{c\mathcal{M}}(X^\bullet \otimes^{\mathrm{int}} \Delta^n, Y^\bullet)$$

as well as

$$\mathrm{hom}^{\mathrm{int}}(K, X^\bullet)^n := \mathrm{hom}_{\mathcal{M}}(K, X^n).$$

The following theorem was proved in [Ree74], see also [GJ99, VII.2.12.] and [Hir03, 16.3.4.].

Theorem 1.3.4 *The category $c\mathcal{M}$ together with the Reedy structure becomes a model category. It becomes a simplicial model structure if we provide it with the internal simplicial structure.*

Definition 1.3.5 If $\mathcal{M}$ is a bicomplete simplicial model category then we can prolong the tensor, cotensor and mapping space functor to cosimplicial spaces. Let X be in $\mathcal{M}$, $Y^\bullet$ in $c\mathcal{M}$ and $K^\bullet$ in $c\mathcal{S}$. Define $\mathrm{map}^{\mathrm{pro}}(X, Y^\bullet)$ in $c\mathcal{S}$ and $X \otimes^{\mathrm{pro}} K^\bullet$ in $c\mathcal{M}$ termwise, and let the cotensor $\hom^{\mathrm{pro}}(K^\bullet, \_) : c\mathcal{M} \to \mathcal{M}$ be given by the right adjoint to $\_ \otimes^{\mathrm{pro}} K^\bullet : \mathcal{M} \to c\mathcal{M}$. We will call this the **prolonged internal structure**. To give an explicit description and to show existence of the right adjoint, observe that it can be written as an end over the category Δ of finite ordinal numbers

$$\hom^{\mathrm{pro}}(K^\bullet, Y^\bullet) = \int_\Delta \hom_{\mathcal{M}}(K^\bullet, Y^\bullet) \in \mathcal{M},$$

where $\hom_{\mathcal{M}}(K^\bullet, Y^\bullet)$ is viewed as a functor $\Delta^{\mathrm{op}} \times \Delta \to \mathcal{M}$, compare lemma 1.3.7. These functors satisfy the analogues of **SM7'** and **SM7**.

Definition 1.3.6 For an object $X^\bullet$ in $c\mathcal{M}$ we define its **total object** by

$$\mathrm{Tot}\, X^\bullet := \hom^{\mathrm{pro}}(\Delta^\bullet, X^\bullet) \in \mathcal{M},$$

where $\Delta^\bullet$ denotes the collection of standard-n-simplices viewed as a cosimplicial space. We also define

$$\mathrm{Tot}_s\, X^\bullet := \hom^{\mathrm{pro}}(\mathrm{sk}_s \Delta^\bullet, X^\bullet),$$

where $\mathrm{sk}_s\Delta^\bullet$ is the cosimplicial space that consists of $\mathrm{sk}_s\Delta^n$ in cosimplicial degree n.

Lemma 1.3.7 $\mathrm{Tot}\, X^\bullet$ *fits into the following equalizer diagram*

$$\mathrm{Tot}\, X^\bullet \longrightarrow \prod_{n \geq 0} \hom_{\mathcal{M}}(\Delta^n, X^n) \rightrightarrows \prod_{n \to m} \hom_{\mathcal{M}}(\Delta^n, X^m),$$

using the cotensor functor $\hom_{\mathcal{M}}$ *of the underlying category* $\mathcal{M}$ *and the obvious maps induced by $n \to m$. This description with the obvious modifications holds also for* Tot_s.

Proof: By definition of an end over the category Δ.

□

Lemma 1.3.8 *The functor* Tot *is right adjoint to the functor* $\_ \otimes^{\mathrm{pro}} \Delta^\bullet : \mathcal{M} \to c\mathcal{M}$.

Proof: By definition of $\mathrm{Tot}\,(X^\bullet) = \hom^{\mathrm{pro}}(\Delta^\bullet, X^\bullet)$ in remark 1.3.5.

□

Lemma 1.3.9 *The functor* $\mathrm{Tot}_{n+1} : c\mathcal{M} \to \mathcal{M}$ *is given by the composition of* $\mathrm{cosk}_{n+1} : c\mathcal{M} \to c\mathcal{M}$ *with* $\mathrm{Tot} : c\mathcal{M} \to \mathcal{M}$.

Proof: For every Y in $\mathcal{M}$ and for every $n \geq 0$ the following isomorphism

$$Y \otimes^{\mathrm{pro}} \mathrm{sk}_n\, \Delta^\bullet \cong \mathrm{sk}_n\, (Y \otimes^{\mathrm{pro}} \Delta^\bullet)$$

holds in $c\mathcal{M}$. This can be shown by a degreewise computation using the Yoneda lemma, the universal property of colimits and the fact that $Y \otimes_{\mathcal{M}} \_$ is left adjoint to $\mathrm{map}_{\mathcal{M}}(Y, \_)$. Having established this we can use the adjunction from 1.3.8 between Tot and $\_ \otimes^{\mathrm{pro}} \Delta^\bullet$

on one side and Tot_{n+1} and $\_ \otimes^{\mathrm{pro}} \mathrm{sk}_n \Delta^\bullet$ on the other side to prove for every Y in $\mathcal{M}$ that we have the natural isomorphism

$$\mathrm{Hom}_{\mathcal{M}}(Y, \mathrm{Tot}\, X^\bullet) \cong \mathrm{Hom}_{\mathcal{M}}(Y, \mathrm{Tot}_n\, X^\bullet),$$

which finishes the proof of the lemma again by referring to Yoneda.

□

Lemma 1.3.10 *If $g : X^\bullet \to Y^\bullet$ is a Reedy equivalence between Reedy fibrant objects then the induced map $g_* : \mathrm{Tot}\, X^\bullet \to \mathrm{Tot}\, Y^\bullet$ is a weak equivalence of fibrant objects in $\mathcal{M}$.*

Proof: [Hir03, 19.5.6.]

□

Definition 1.3.11 For every $X^\bullet$ in $c\mathcal{M}$ the fiber of the canonical map $X^n \to M^n X^\bullet$ in $\mathcal{M}$ is called the **geometrical normalization** and is denoted by $\boldsymbol{N^n X^\bullet}$.

Lemma 1.3.12 *For Reedy fibrant $X^\bullet$ the fiber of the map $\mathrm{Tot}_n\, X^\bullet \to \mathrm{Tot}_{n-1}\, X^\bullet$ is given by $\Omega^n N^n X^\bullet$.*

Proof: [BK72, p. 282] or [GJ99, p. 391].

□

1.4 Moduli spaces in model categories

The following definitions could be given for simplicial categories with a suitable subcategory of weak equivalences. But we will not insist on full generality here.

Definition 1.4.1 Let $\mathcal{M}$ be a simplicial model category and let $\mathcal{W}$ be its subcategory of weak equivalences. For a cofibrant and fibrant object X we define the **simplicial monoid of self equivalences** denoted by **haut($\boldsymbol{X}$)** by setting

$$\mathrm{haut}(X)_n := \mathrm{Hom}_{\mathcal{W}}(X \otimes \Delta^n, X).$$

This is indeed a monoid, since we can compose two maps $f, g : X \otimes \Delta^n \to X$ by setting $g \circ f$ equal to

$$X \otimes \Delta^n \xrightarrow{\mathrm{id} \otimes \Delta} X \otimes (\Delta^n \times \Delta^n) \cong (X \otimes \Delta^n) \otimes \Delta^n \xrightarrow{f \otimes \mathrm{id}} X \otimes \Delta^n \xrightarrow{g} X,$$

where $\Delta : \Delta^n \to \Delta^n \times \Delta^n$ is the diagonal. If we need to specify a model structure on $\mathcal{M}$, because there are several possible choices, we write the name of the structure as an index, so e.g. $\mathrm{haut}_{\mathrm{Reedy}}(X^\bullet)$ denotes the simplicial monoid in the Reedy structure on $c\mathcal{M}$.

Remark 1.4.2 It is an easy, but quite important observation that the space $\mathrm{haut}(X)$ consists of those connected components of the space $\mathrm{map}(X, X)$ stemming from the simplicial structure of $\mathcal{M}$ that are given by the vertices corresponding to weak self equivalences.

Definition 1.4.3 Let $\mathcal{M}$ be a model category. We define the **moduli space of an object** X in $\mathcal{M}$ to be the nerve of the following category:

$$\begin{aligned} \text{Objects} &= \text{ Objects of } \mathcal{M} \text{ that are weakly equivalent to } X \\ \text{Morphisms} &= \text{ weak equivalences} \end{aligned}$$

It is denoted by $\boldsymbol{\mathcal{M}(X)}$. Note that for each X in $\mathcal{M}$ this moduli space is non-empty and connected. If S is a set of objects in $\mathcal{M}$, we define $\boldsymbol{\mathcal{M}(S)}$ to be the nerve of the full subcategory of $\mathcal{W}$, whose objects are weakly equivalent to an element of S.

Definition 1.4.4 Let $\underline{1}$ be the category consisting of two objects, their identities and one morphism between them. A $\underline{1}$-diagram in a category $\mathcal{C}$ is just a morphism in $\mathcal{C}$.

Remark 1.4.5 The category $\mathcal{M}^{\underline{1}}$ of morphisms in $\mathcal{M}$ inherits a model structure, where the weak equivalences are given by pointwise weak equivalences in $\mathcal{M}$. Actually, there are two different ways to construct such a structure, one uses pointwise fibrations and the other one uses pointwise cofibrations. For details we refer to [DS95, 10.13.].

Definition 1.4.6 We define the **moduli space of a morphism** in $\mathcal{M}$ in the same way as for objects: Let f be a morphism in $\mathcal{M}$. Its moduli space is the space $\mathcal{M}(f)$ from definition 1.4.3 where we view f as an object in $\mathcal{M}^{\underline{1}}$ supplied with the model structure from 1.4.5.

The important theorem about moduli spaces in model categories is the following one.

Theorem 1.4.7 *Let $\mathcal{M}$ be a simplicial cofibrantly generated model category and let X be an object of $\mathcal{M}$. Then the moduli space $\mathcal{M}(X)$ is weakly equivalent to the space $B\mathrm{haut}(X)$.*

Proof: [DK84, Prop. 2.3.]

□

2 Resolution model structures

The goal of this section is to describe a model structure on the category $c\mathcal{M}$ of cosimplicial objects over certain model categories $\mathcal{M}$. It is called resolution model structure or E_2-model structure. The reason for the first name is that we think of cosimplicial objects as analogues of cochain complexes. The resolution model structure provides fibrant approximations that are analogues of injective resolutions. The reason for the second name is that there is a spectral sequence (2.1) and the weak equivalences here are exactly those maps that induce an isomorphism on the E_2-term of this spectral sequence. A special resolution model structure was first introduced in [DKS93] and was later studied in [DKS95] and [BDG01] to attack the realization problem for Π-algebras. Following these tracks a similar resolution model structures was developed in [GH] and used in [GH04] to study A_∞- and E_∞-structures on ring spectra. In [Bou03] a very general and elegant treatment of resolution model structures is given that exhibits the previous ones as special cases. Bousfield in his paper calls these structures the $\mathcal{G}$-resolution structures because there is the freedom of choosing an appropriate class of injective models $\mathcal{G}$ which will be explained in subsection 2.1. We will abbreviate this and call the resolution model structures simply $\mathcal{G}$-structures.

The whole subsection 2.1 is a very brief review of [Bou03]. We reproduce most of the facts, we need from it, although not all, but we will omit the proofs. We will also describe the external simplicial structure which is compatible with the $\mathcal{G}$-structure and we will supply the simplicial structure for the truncated versions of the $\mathcal{G}$-structure in 3.2. The reader who is familiar with [Bou03] can skip this subsection.

In subsection 2.2 we reformulate the $\mathcal{G}$-structure described in 2.1 in the style as they are presented in [BDG01] and [GH04] using the natural homotopy groups from 2.2.4. This is necessary in order to be able to truncate the $\mathcal{G}$-model structure, as we will do in 3.2.

2.1 The $\mathcal{G}$-structure on cosimplicial objects

This subsection recapitulates the results of [Bou03]. Let $\mathcal{M}$ be a simplicial left proper pointed model category, and let $\mathcal{G}$ be a class of group objects in the homotopy category $Ho(\mathcal{M})$ that is closed under the loop and suspension functor. In [Bou03] this closure property was not assumed and it was observed that the class of $\mathcal{G}$-injectives to be defined below contains all objects of the form ΩG for $G \in \mathcal{G}$. In our intended applications the model structure on $\mathcal{M}$ is stable, so like in [GH04, Def. 3.2] it is convenient for us the assume from the beginning that $\mathcal{G}$ contains all suspensions and desuspensions. It is also useful when considering the spiral exact sequence, see 2.2.7. As a result of this we do not need to consider graded morphism sets $[X, \Omega^n Y]$ for $n \in \mathbb{N}$ in definitions 2.1.2 and 2.1.6 and we can replace the class of $\mathcal{G}$-injectives by $\mathcal{G}$ in lemma 2.1.3(iii).

Definition 2.1.1 For objects X and Y in $\mathcal{M}$ we will denote the set of morphisms in the homotopy category $Ho(\mathcal{M})$ of $\mathcal{M}$ by $[\boldsymbol{X}, \boldsymbol{Y}]$. Compare the corresponding definition 4.1.1.

Definition 2.1.2 A map $i : A \to B$ in $Ho(\mathcal{M})$ is called **$\mathcal{G}$-monic** when $i^* : [B, G] \to [A, G]$ is surjective for each $G \in \mathcal{G}$. We say that a map in $\mathcal{M}$ is $\mathcal{G}$-monic if it induces a $\mathcal{G}$-monic map in $Ho(\mathcal{M})$.

An object I is called **$\mathcal{G}$-injective** when $i^* : [B, I] \to [A, I]$ is surjective for each $\mathcal{G}$-monic map $i : A \to B$.

We call a fibration in $\mathcal{M}$ a **$\mathcal{G}$-injective fibration** if it has the right lifting property with respect to every $\mathcal{G}$-monic cofibration.

We say that $Ho(\mathcal{M})$ **has enough $\mathcal{G}$-injectives** if each object in $Ho(\mathcal{M})$ is the source of a $\mathcal{G}$-monic map to a $\mathcal{G}$-injective target. We say that $\mathcal{G}$ is **functorial**, if these maps can be chosen functorially.

Lemma 2.1.3 (i) *A fibration in $\mathcal{M}$ is $\mathcal{G}$-injective if and only if it has the right lifting property for each $\mathcal{G}$-monic cofibration between cofibrant objects.*

(ii) *A cofibration in $\mathcal{M}$ is $\mathcal{G}$-monic if and only if it has the left lifting property for each $\mathcal{G}$-injective fibration.*

(iii) *A cofibration in $\mathcal{M}$ is $\mathcal{G}$-monic if and only if it has the left lifting property for each map $G \to *$ with fibrant $G \in \mathcal{G}$.*

(iv) *A fibrant object G is $\mathcal{G}$-injective if and only the map $G \to *$ is a $\mathcal{G}$-injective fibration.*

Proof: (i) is stated as [Bou03, lemma 3.6.], (ii) and (iii) follow from [Bou03, lemma 3.8.] and its proof. (iv) is [Bou03, 3.7].

□

Lemma 2.1.4 *Every map in $\mathcal{M}$ can be factored into a $\mathcal{G}$-monic cofibration followed by a $\mathcal{G}$-injective fibration.*

Proof: [Bou03, lemma 3.9]

□

Remark 2.1.5 From lemma 2.1.3 it follows that the class of $\mathcal{G}$-monic cofibrations and the class of $\mathcal{G}$-injective fibrations or the class of $\mathcal{G}$-injective fibrant objects mutually determine each other via lifting properties. This fact and the observation of lemma 2.1.4 will be necessary to start a Reedy type induction on the cosimplicial degree in the proof of the lifting axioms for the $\mathcal{G}$-structure that we are going to describe now, and its derivative, the n-$\mathcal{G}$-structure (Def. 3.2.1), in the next section.

Definition 2.1.6 Let $c\mathcal{M}$ be the category of cosimplicial objects over $\mathcal{M}$ and let $\mathcal{G}$ be a class of group objects in $Ho(\mathcal{M})$. We call a map $f : X^\bullet \to Y^\bullet$ a

(i) **$\mathcal{G}$-equivalence** if $f_* : [Y^\bullet, G] \to [X^\bullet, G]$ is a weak equivalence of simplicial groups for each $G \in \mathcal{G}$.

(ii) **$\mathcal{G}$-cofibration** if f is a Reedy cofibration and $f^* : [Y^\bullet, G] \to [X^\bullet, G]$ is a fibration of simplicial groups for each $G \in \mathcal{G}$.

(iii) **$\mathcal{G}$-fibration** if $f : X^n \to Y^n \times_{M^n Y^\bullet} M^n X^\bullet$ is a $\mathcal{G}$-injective fibration for $n \geq 0$.

These three classes of maps will be called the ***$\mathcal{G}$*-structure on *$c\mathcal{M}$***.

Remark 2.1.7 Note that a map $X^\bullet \to Y^\bullet$ is a $\mathcal{G}$-equivalence if and only if the induced maps

$$\pi_s[Y^\bullet, G] \to \pi_s[X^\bullet, G]$$

are isomorphisms for all $G \in \mathcal{G}$ and all $s \geq 0$, where it suffices to consider the constant map $X^0 \to G$ as basepoint. By 2.1.19 we can replace the set $\mathcal{G}$ by all $\mathcal{G}$-injective objects.

Remark 2.1.8 Now that we have given the definition of the $\mathcal{G}$-structure we want to upgrade it to a simplicial structure. Before doing that we need to provide a short explanation. For $X^\bullet$ in $c\mathcal{M}$ and L in $\mathcal{S}$ we can perform the following coend-construction: Let $\coprod_{L_\ell} X^m$ be the coproduct in $\mathcal{M}$ of copies of X^m indexed by the set L_ℓ, and view this as a functor $\Delta^{\mathrm{op}} \times \Delta \to \mathcal{M}$. Then we can take the coend

$$X^\bullet \otimes_\Delta L := \int^\Delta \coprod_{L_\ell} X^m \in \mathcal{M}.$$

Explicitly this is given by the coequalizer

$$\coprod_{\ell \to m} \coprod_{L_\ell} X^m \rightrightarrows \coprod_{\ell \geq 0} \coprod_{L_\ell} X^\ell \to X^\bullet \otimes_\Delta L$$

using the obvious maps induced by $\ell \to m$.

We are now ready to describe the functors that will enrich all our model structures to simplicial model categories.

Definition 2.1.9 We define a simplicial structure on $c\mathcal{M}$. Let K be in $\mathcal{S}$ and $X^\bullet$ and $Y^\bullet$ in $c\mathcal{M}$, then set

$$(X^\bullet \otimes^{\mathrm{ext}} K)^n := X^\bullet \otimes_\Delta (K \times \Delta^n),$$

where $\times$ denotes the usual product of simplicial sets and Δ^n is the standard n-simplex,

$$\hom^{\mathrm{ext}}(K, X^\bullet)^n := \prod_{K_n} X^n,$$

where the product is taken over the set of n-simplices of K, and finally

$$\mathrm{map}^{\mathrm{ext}}(X^\bullet, Y^\bullet)_n := \mathrm{Hom}_{c\mathcal{M}}(X^\bullet \otimes^{\mathrm{ext}} \Delta^n, Y^\bullet).$$

We call this the **external (simplicial) structure** on $c\mathcal{M}$. Note that we do not refer to any simplicial structure of $\mathcal{M}$. From now on we will usually drop the superscript $\mathrm{map}^{\mathrm{ext}}$, see the end of 2.2.4. It is easy to see that $\mathrm{map}(X^\bullet, Y^\bullet)$ can be written as an end, that is as the equalizer of the following maps induced by $m \to n$:

$$\mathrm{map}^{\mathrm{ext}}(X^\bullet, Y^\bullet)_0 \to \prod_n \mathrm{Hom}_{\mathcal{M}}(X^n, Y^n) \rightrightarrows \prod_{m \to n} \mathrm{Hom}_{\mathcal{M}}(X^m, Y^n)$$

Remark 2.1.10 In contrast to the $\mathcal{G}$-structure the Reedy structure is not compatible with the external simplicial structure just defined. It will satisfy just the weaker version of (SM7'), where we drop the condition that a trivial cofibration of simplicial sets makes the pushout map into a trivial cofibration (see the formulation of 3.2.11). Despite that fact the next result shows that the Tot-functor transforms external homotopies in $c\mathcal{M}$ into homotopies in $\mathcal{M}$. The following lemma is [Bou03, 2.11.] where [Mey90] is quoted.

Lemma 2.1.11 *For every $Y^\bullet$ in $c\mathcal{M}$ and every K in $\mathcal{S}$ there is a natural isomorphism*

$$\operatorname{Tot}\hom^{\text{pro}}(K, Y^\bullet) \cong \hom_{\mathcal{M}}(K, \operatorname{Tot} Y^\bullet).$$

Proof: By adjointness we can also show for every X in $\mathcal{M}$ and every K in $\mathcal{S}$ that there is a natural isomorphism

$$(A \otimes^{\text{pro}} \Delta^\bullet) \otimes^{\text{ext}} K \cong (A \otimes_{\mathcal{M}} K) \otimes^{\text{pro}} \Delta^\bullet.$$

But this follows from the isomorphism in $\mathcal{M}$

$$(A \otimes^{\text{pro}} \Delta^\bullet) \otimes_\Delta (K \times \Delta^n) \cong A \otimes_{\mathcal{M}} (K \times \Delta^n)$$

for every $n \geq 0$, obtained by applying $A \otimes_{\mathcal{M}} \_$ to $\Delta^\bullet \otimes_\Delta (K \times \Delta^n) \cong K \times \Delta^n$ in $\mathcal{S}$.

□

Lemma 2.1.12 *Let f and g be maps in $c\mathcal{M}$ between Reedy fibrant objects $X^\bullet$ and $Y^\bullet$ that are externally homotopic, which means:*

$$[f] = [g] \in \pi_0 \mathrm{map}^{\text{ext}}(X^\bullet, Y^\bullet)$$

Then $\operatorname{Tot} f$ *equals* $\operatorname{Tot} g$ *in* $\pi_0 \mathrm{map}_{\mathcal{M}}(\operatorname{Tot} X^\bullet, \operatorname{Tot} Y^\bullet)$.

Proof: This is [Bou03, 2.13.] and follows from 2.1.11.

□

Here is the main theorem 3.3 from [Bou03].

Theorem 2.1.13 *The category $c\mathcal{M}$ of cosimplicial objects over a simplicial left proper pointed model category $\mathcal{M}$ with a class $\mathcal{G}$ of homotopy group objects and enough $\mathcal{G}$-injectives together with the $\mathcal{G}$-structure and the external simplicial structure becomes a simplicial left proper pointed model category. If the model structure on $\mathcal{M}$ has functorial factorizations, so has the $\mathcal{G}$-structure on $c\mathcal{M}$.*

Remark 2.1.14 Of course, every Reedy equivalence is a $\mathcal{G}$-equivalence. Note also that the cofibrant objects in this $\mathcal{G}$-structure are exactly the Reedy cofibrant ones.

Remark 2.1.15 $\mathcal{G}$-fibrant objects are Reedy fibrant, but not conversely. If X is an object in $\mathcal{M}$, then the constant cosimplicial object $r^0 X$ is Reedy fibrant by [Hir03, 16.9.5]. If G is a $\mathcal{G}$-injective fibrant object, then $r^0 G$ is $\mathcal{G}$-fibrant.

Remark 2.1.16 In order to state some results on this structure we have to define partial latching objects. Using the Δ-coend from remark 2.1.8 we observe that we have isomorphisms:

$$\begin{aligned} X^n &\cong X^\bullet \otimes_\Delta \Delta^n \\ L^n X^\bullet &\cong X^\bullet \otimes_\Delta \partial\Delta^n \end{aligned}$$

Partial latching objects are now obtained by substituting various subcomplexes of Δ^n. See the next definition.

Definition 2.1.17 Let ι_n be the unique non-degenerate n-simplex of Δ^n. For $0 \le k \le n$ let $\Lambda^n_k \subseteq \Delta^n$ be the subcomplex spanned by all $d_i \iota_n$ for $i \neq k$. We call this the ***k*-horn** on Δ^n. Let $X^\bullet$ be an object of $c\mathcal{M}$. Set

$$L^n_k X^\bullet := X^\bullet \otimes_\Delta \Lambda^n_k.$$

These objects are called the **partial latching objects** of $X^\bullet$.

Now we can cite a different characterization of (trivial) $\mathcal{G}$-cofibrations.

Lemma 2.1.18 *Let $X^\bullet \to Y^\bullet$ be a Reedy cofibration in $c\mathcal{M}$.*

(i) *It is a trivial $\mathcal{G}$-cofibration if and only if the induced maps*

$$X^n \sqcup_{L^n X^\bullet} L^n Y^\bullet \to Y^n$$

are $\mathcal{G}$-monic for all $n \ge 0$.

(ii) *It is a $\mathcal{G}$-cofibration if and only if the induced maps*

$$X^n \sqcup_{L^n_k X^\bullet} L^n_k Y^\bullet \to Y^n$$

are $\mathcal{G}$-monic for all $n \ge k \ge 0$ and $n \ge 1$.

Proof: [Bou03, Prop. 3.13.]

□

Remark 2.1.19 The previous lemma implies that the $\mathcal{G}$-structure is already determined by the $\mathcal{G}$-injective objects. In particular we obtain the characterization of $\mathcal{G}$-equivalences in 2.1.7.

Lemma 2.1.20 *The functor* $\mathrm{sk}_n : c\mathcal{M} \to c\mathcal{M}$ *from* 1.2.4 *maps Reedy cofibrations to Reedy cofibrations.*

Proof: This fact follows from the following equations:

$$(\mathrm{sk}_n A^\bullet)^s = \begin{cases} A^s & , \text{ for } 0 \le s \le n \\ L^{n+1} A^\bullet & , \text{ for } s = n+1 \\ L^s (\mathrm{sk}_n A^\bullet) & , \text{ for } s \ge n+2 \end{cases}$$

□

Remark 2.1.21 The corresponding statement of 2.1.20 is not true for $\mathcal{G}$-cofibrations. Compare 3.2.14(ii).

The analogue of the following statement for the Reedy structure is well known.

Lemma 2.1.22 *The functor* $\_ \otimes^{\mathrm{pro}} \Delta^\bullet : \mathcal{M} \to c\mathcal{M}$ *maps cofibrations resp. trivial cofibrations to $\mathcal{G}$-cofibrations resp. $\mathcal{G}$-trivial cofibrations. In particular* $\_ \otimes^{\mathrm{pro}} \Delta^\bullet$ *and* Tot *form a Quillen pair.*

Note that $X \otimes^{\text{pro}} \Delta^\bullet \to r^0 X$ is a Reedy cofibrant replacement, if X is cofibrant. Hence $L(\_ \otimes^{\text{pro}} \Delta^\bullet) = Lr^0$ is the left derived functor of r^0 in the sense of 1.1.2.

Proof: By the same arguments as in the proof of 2.1.11 there are natural isomorphisms

$$\begin{aligned} &L^s(A \otimes^{\text{pro}} \Delta^\bullet) \cong (A \otimes^{\text{pro}} \Delta^\bullet) \otimes_\Delta \partial\Delta^s \cong A \otimes_{\mathcal{M}} \partial\Delta^s \text{ and} \\ &L^s_k(A \otimes^{\text{pro}} \Delta^\bullet) \cong A \otimes_{\mathcal{M}} \Lambda^s_k \end{aligned}$$

for every A in $\mathcal{M}$ and all $s \geq k \geq 0$. $\mathcal{M}$ is a simplicial model category, hence if $A \to B$ is a cofibration resp. a trivial cofibration in $\mathcal{M}$, then by the upper resp. the lower isomorphisms all the different latching maps from 2.1.18 are trivial cofibrations in $\mathcal{M}$ and therefore $\mathcal{G}$-monic. We have proved that $\_ \otimes^{\text{pro}} \Delta^\bullet$ maps cofibrations resp. trivial cofibrations to $\mathcal{G}$-cofibrations resp. $\mathcal{G}$-trivial cofibrations.

□

2.2 The spiral exact sequence and the natural homotopy groups

In the previous sections we studied for an object $X^\bullet$ in $c\mathcal{M}$ the simplicial groups $[X^\bullet, G]$, where G came from an class of injective models $\mathcal{G}$. The $\mathcal{G}$-equivalences were defined via these objects, we had to look at homotopy groups $\pi_s[X^\bullet, G]$. We now define another concept of homotopy groups, the natural homotopy groups of a cosimplicial object introduced in the simplicial setting in [DKS95] and further studied in [BDG01] and [GH04]. This reformulation, which is taken from [GH04], is needed to truncate the $\mathcal{G}$-structure, as we will do in section 3.2.

To obtain nice descriptions of the objects defined in 2.2.1 and 2.2.2 we use a convenient model for Δ^s as explained in [BDG01, 1.5.]. When we write Δ^s_c we mean Δ^s/Λ^s_0 where we collapsed the 0-horn on Δ^s defined in 2.1.17.

Definition 2.2.1 For an object $X^\bullet$ in $c\mathcal{M}$ we define its ***s*-th external suspension** $\Sigma_{\text{ext}} X^\bullet =: X^\bullet \wedge^{\text{ext}} \Delta^s/\partial\Delta^s$ by the pushout diagram:

$$\begin{array}{ccc} X^\bullet = X^\bullet \otimes^{\text{ext}} * & \longrightarrow & X^\bullet \otimes^{\text{ext}} \Delta^s_c/\partial\Delta^s_c \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & X^\bullet \wedge^{\text{ext}} \Delta^s/\partial\Delta^s =\!=: \Sigma^s_{\text{ext}} X^\bullet \end{array}$$

In [GH04] this is called the n-th $X^\bullet$-sphere.

Definition 2.2.2 For an object $X^\bullet$ in $c\mathcal{M}$ we define its ***s*-th external loop object** $\Omega^s_{\text{ext}} X^\bullet$ by the pullback diagram:

$$\begin{array}{ccc} \Omega^s_{\text{ext}} X^\bullet & \longrightarrow & \hom^{\text{ext}}(\Delta^s_c/\partial\Delta^s_c, X^\bullet) \\ \downarrow & \lrcorner & \downarrow \\ * & \longrightarrow & X^\bullet = \hom^{\text{ext}}(*, X^\bullet) \end{array}$$

Remark 2.2.3 For an object G in $\mathcal{M}$ the object $\Omega^s_{\text{ext}} r^0 G$ in $c\mathcal{M}$ is given by

$$(\Omega^s_{\text{ext}} r^0 G)^t = \begin{cases} * & , \text{ for } 0 \leq t < s \\ G & , \text{ for } t = s \\ \prod\limits_{s \hookrightarrow t} G & , \text{ for } t > s. \end{cases}$$

If G is $\mathcal{G}$-injective and fibrant, $\Omega^s_{\text{ext}} r^0 G$ will be $\mathcal{G}$-fibrant for all $s \geq 0$ by remark 2.1.15. $\Omega^s_{\text{ext}} r^0 G$ can also be described as $r^s G[s]$, where $G[s]$ in $c_s\mathcal{M}$ is the s-truncated cosimplicial object given by

$$G[s]^t = \begin{cases} * & , \text{ for } 0 \leq t \leq s-1 \\ G & , \text{ for } t = s \end{cases}$$

Definition 2.2.4 Let $X^\bullet$ be Reedy cofibrant in $c\mathcal{M}$ and $G \in \mathcal{G}$. We let $r^0 G$ be the constant cosimplicial object with the notation from 1.2.1. Then the mapping space functor from the external structure in 2.1.9 provides us with a fibrant H-space $\text{map}^{\text{ext}}(X^\bullet, r^0 G)$ with homotopy inverse since G is a homotopy group object. This mapping space has homotopy meaning for the $\mathcal{G}$-structure by 2.2.3. The **natural homotopy groups of a cosimplicial object with coefficients in $G \in \mathcal{G}$** are given by

$$\begin{aligned} \pi_s^\natural(X^\bullet, G) &:= [X^\bullet, \Omega^s_{\text{ext}} r^0 G]_{\mathcal{G}} \\ &\cong \pi_s \text{map}^{\text{ext}}(X^\bullet, r^0 G), \end{aligned}$$

where for the second isomorphism we take the constant map as basepoint. For a non-Reedy cofibrant $X^\bullet$ the first term still makes good sense, but for the second isomorphism we would have to replace $X^\bullet$ functorially Reedy cofibrantly.

These groups were first defined in the context of simplicial spaces in [DKS95]. We took the name from [GH04]. These groups correspond to the groups $\widehat{\pi}_i X$ of [BDG01].

From now on we will sometimes drop the superscript map^{ext} or $\otimes^{\text{ext}}$ since unless otherwise stated we will always refer to the external structure from 2.1.9.

Remark 2.2.5 Note that suspending externally shifts these homotopy groups by -1, for $n \geq 0$ we have:

$$\begin{aligned} \pi_n^\natural(\Sigma_{\text{ext}} X^\bullet, G) &\cong \pi_{n+1}^\natural(X^\bullet, G) \\ \pi_n[\Sigma_{\text{ext}} X^\bullet, G] &\cong \pi_{n+1}[X^\bullet, G] \end{aligned}$$

Remark 2.2.6 The two types of homotopy groups are connected by a Hurewicz homomorphism

$$\pi_s^\natural(X^\bullet, G) \to \pi_s[X^\bullet, G]$$

for each $s \geq 0$ constructed in [DKS95, 7.1]. One of the main results of that paper is [DKS95, 8.1] (also [GH04, 3.8] and [GH]) that this Hurewicz homomorphism for each $G \in \mathcal{G}$ fits into a long exact sequence, the so-called **spiral exact sequence**

$$\begin{aligned} &... \to \pi_{s-1}^\natural(X^\bullet, \Omega_{\text{int}} G) \to \pi_s^\natural(X^\bullet, G) \to \pi_s[X^\bullet, G] \to \pi_{s-2}^\natural(X^\bullet, \Omega_{\text{int}} G) \to ... \\ &... \to \pi_2[X^\bullet, G] \to \pi_0^\natural(X^\bullet, \Omega_{\text{int}} G) \to \pi_1^\natural(X^\bullet, G) \to \pi_1[X^\bullet, G] \to 0, \end{aligned}$$

where Ω_{int} is the usual (internal) loop space functor on $\mathcal{M}$, plus an isomorphism

$$\pi_0^\natural(X^\bullet, G) \cong \pi_0[X^\bullet, G].$$

As explained in [DKS95, 8.3.] or [GH04, (3.1)] these long exact sequences can be spliced together to give an exact couple and an associated spectral sequence

$$\pi_p[X^\bullet, \Omega^q G] \implies \pi_0^\natural(X^\bullet, \Omega^{p+q} G) \tag{2.1}$$

Compare 4.2.31 or [GH04, 3.9] for a relation between the target of this spectral sequence and $[\,\text{Tot}\, X^\bullet, \Omega^{p+q} G]$.

We will now reformulate the $\mathcal{G}$-structure in terms of this natural point of view. The result will be theorem 2.2.9. The moral of this theorem is that we get the same model category no matter whether we use the simplicial groups $[X^\bullet, G]$ or the H-spaces $\mathrm{map}^{\mathrm{ext}}(X^\bullet, G)$ to define our $\mathcal{G}$-structure.

Lemma 2.2.7 *A map $X^\bullet \to Y^\bullet$ in $c\mathcal{M}$ is a $\mathcal{G}$-equivalence if and only if it induces isomorphisms*

$$\pi_s^\natural(\widetilde{Y}^\bullet, G) \to \pi_s^\natural(\widetilde{X}^\bullet, G)$$

for all $s \geq 0$ and all $G \in \mathcal{G}$ and some Reedy cofibrant approximation $\widetilde{X}^\bullet \to \widetilde{Y}^\bullet$.

Proof: This follows immediately from 2.1.7 and the spiral exact sequence by simultaneous induction over the whole class $\mathcal{G}$ and the five-lemma. Remember that $\mathcal{G}$ is closed under loops by assumption.

□

Lemma 2.2.8 *A Reedy cofibration $A^\bullet \to B^\bullet$ is a $\mathcal{G}$-cofibration if and only if for all fibrant $G \in \mathcal{G}$ the maps*

$$\mathrm{map}(B^\bullet, r^0 G) \to \mathrm{map}(A^\bullet, r^0 G)$$

are fibrations of simplicial sets.

Proof: Let $A^\bullet \to B^\bullet$ be a Reedy cofibration. By lemma 2.1.18 it is a $\mathcal{G}$-cofibration if and only if the maps

$$A^s \sqcup_{L_k^s A^\bullet} L_k^s B^\bullet \to B^s \tag{2.2}$$

are $\mathcal{G}$-monic cofibrations for all $s \geq k \geq 0$. On the other hand the map

$$\mathrm{map}(B^\bullet, r^0 G) \to \mathrm{map}(A^\bullet, r^0 G)$$

is a fibration of simplicial sets if and only if there exists a lifting in all diagrams

$$\begin{array}{ccc} \Lambda_k^s & \longrightarrow & \mathrm{map}(B^\bullet, r^0 G) \\ \downarrow & \nearrow & \downarrow \\ \Delta^s & \longrightarrow & \mathrm{map}(A^\bullet, r^0 G) \end{array}$$

for all $s \geq k \geq 0$. These diagrams are adjoint to the following diagrams:

$$\begin{array}{ccc} (A^\bullet \otimes_\Delta \Delta^s) \sqcup_{(A^\bullet \otimes_\Delta \Lambda_k^s)} (B^\bullet \otimes_\Delta \Lambda_k^s) & \longrightarrow & G \\ \downarrow & \nearrow & \downarrow \\ B^\bullet \otimes_\Delta \Delta^s & \longrightarrow & * \end{array}$$

We see that the dotted lifting exists if and only if (2.2) is $\mathcal{G}$-monic.

□

Theorem 2.2.9 *The $\mathcal{G}$-structure on $c\mathcal{M}$ can be described in the following way: A map $X^\bullet \to Y^\bullet$ is a*

(i) *$\mathcal{G}$-equivalence if and only if for all fibrant $G \in \mathcal{G}$ the induced map*

$$\mathrm{map}(\widetilde{Y}^\bullet, r^0 G) \to \mathrm{map}(\widetilde{X}^\bullet, r^0 G)$$

is a weak equivalence of simplicial sets, where $\widetilde{X}^\bullet \to \widetilde{Y}^\bullet$ is a Reedy cofibrant approximation of $X^\bullet \to Y^\bullet$ in the sense of 1.1.3.

(ii) *$\mathcal{G}$-cofibration if and only if it is a Reedy cofibration and for all fibrant $G \in \mathcal{G}$ the induced map*

$$\mathrm{map}(Y^\bullet, r^0G) \to \mathrm{map}(X^\bullet, r^0G)$$

is a fibration of simplicial sets.

(iii) *$\mathcal{G}$-fibration if and only if the induced maps*

$$X^s \to M^s X^\bullet \times_{M^s Y^\bullet} Y^s$$

are $\mathcal{G}$-injective fibrations in $\mathcal{M}$ for all $s \geq 0$.

Proof: This follows from 2.2.7 and 2.2.8. We just have to check for one basepoint in 2.2.7 since the involved spaces are H-spaces.

□

3 Truncated resolution model structures

We have already explained that we intend to truncate Bousfield's $\mathcal{G}$-structure. The first step is subsection 3.1, where we describe a left Bousfield localization of the usual model category of simplicial sets. Weak equivalences are now maps that induce isomorphisms on homotopy groups up to dimension n, and fibrant replacements are Postnikov sections, therefore the name Postnikov-n-structure.

In subsection 3.2 we use this localized version together with the natural formulation of the $\mathcal{G}$-structure to construct the n-$\mathcal{G}$-structure on $c\mathcal{M}$.

3.1 Postnikov localization of simplicial sets

We want to use the concept of localization with respect to a map to introduce a model structure on the category $\mathcal{S}$ of simplicial sets, such that the weak equivalences are the maps that induce isomorphisms on homotopy up to degree n. We will call this structure the Postnikov-n structure. Fibrant replacements will be n-th Postnikov sections. This technique is known as left Bousfield localization. For a comprehensive and more general treatment see [Hir03]. Although this structure is mentioned in [Hir03], this seems to be the first time that the Postnikov-n-structure is studied in greater detail. The characterization of Postnikov-n-fibrations in 3.1.6 and 3.1.9 are new as well as the determination of a set of generating trivial cofibrations in 3.1.8.

The main existence result is 3.1.12. The characterization of fibrations in corollary 3.1.9 will be used in the proof of 3.2.8.

Let $|\_| : \mathcal{S} \to \mathrm{Top}$ be geometric realization. Let $\mathrm{Sing} : \mathrm{Top} \to \mathcal{S}$ be its right adjoint, the singular functor.

Definition 3.1.1 For $n \geq 0$ let $\boldsymbol{f_n}$: $\partial\Delta^{n+2} \to \Delta^{n+2}$ be the standard inclusion in $\mathcal{S}$. A simplicial set W is called $\boldsymbol{f_n}$**-local** if it is fibrant and if the induced map

$$\mathrm{map}(\Delta^{n+2}, W) \xrightarrow{(f_n)^*} \mathrm{map}(\partial\Delta^{n+2}, W)$$

is a weak equivalence. A map $g : X \to Y$ is called an $\boldsymbol{f_n}$**-local equivalence** if for every f_n-local space W the induced map

$$\mathrm{map}(Y, W) \xrightarrow{g^*} \mathrm{map}(X, W)$$

is a weak equivalence. It follows from [Hir03, 10.5.2] that this definition is independent of the choice of a cofibrant approximation of g.

Lemma 3.1.2 *An object W is f_n-local if and only if W is fibrant, i.e. a Kan complex, and $\pi_s(|W|, w) \cong 0$ for all $s > n$ and all $w \in W_0$.*

Proof: This is [Hir03, Prop. 1.5.1., Prop. 1.3.3.]. Since $\partial\Delta^s \to \Delta^s$ is an inclusion of cell complexes, an object W is f_n-local if and only if the simplicial set is fibrant and the map $W \to 0$ has the right lifting property with respect to the maps $\partial\Delta^s \to \Delta^s$ for all $s \geq n+1$. But this is equivalent to $\pi_s W \cong 0$ for all $s \geq n+1$.

□

Lemma 3.1.3 *A map $A \to B$ is an f_n-local equivalence if and only if it induces isomorphisms $\pi_s|A| \to \pi_s|B|$ for $0 \leq s \leq n$ and every vertex of A.*

Proof: This is proved in [Hir03, 1.5.2., 1.5.4.].

□

Definition 3.1.4 We construct the **Postnikov-n-structure** on $\mathcal{S}$ by defining the following three classes of maps: We call a morphism $A \to B$ in $\mathcal{S}$

(i) a **Postnikov-n-equivalence** if it is an f_n-local equivalence (see lemma 3.1.3).

(ii) a **Postnikov-n-cofibration** if it is a cofibration of simplicial sets.

(iii) a **Postnikov-n-fibration** if it has the right lifting property with respect to all Postnikov-n-trivial cofibrations.

Remark 3.1.5 The existence of this model structure is shown by the general theorem [Hir03, 4.1.1] on the existence of left Bousfield localizations. Next we want to get a grip on the Postnikov-n-fibrations.

Lemma 3.1.6 *A map $X \to Y$ has the right lifting property with respect to all Postnikov-n-trivial cofibrations if and only if it is a fibration and the induced maps*

$$\pi_s|X| \to \pi_s|Y|$$

are isomorphisms for $s > n$ and for every vertex of X.

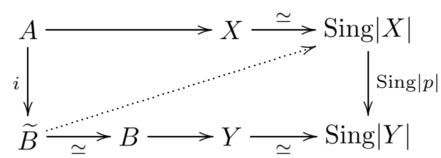
Proof: Let $p : X \to Y$ be a fibration with the asserted properties. To show that it is a Postnikov-n-fibration, we need to find a dotted lifting in the diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

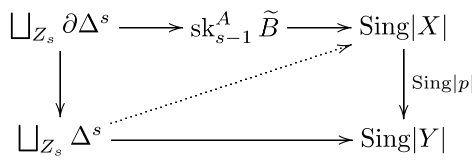
where i is a cofibration that induces isomorphisms $\pi_s|A| \to \pi_s|B|$ for all $0 \leq s \leq n$ and all vertices of A. We can replace i by a cofibration $\widetilde{i} : A \to \widetilde{B}$, where $\widetilde{B}$ is obtained from A by attaching cells of dimension $> n$. Using the concepts of relative skeleta and degeneracy-free diagrams from [GJ99, VII.1.], we can describe $\widetilde{B}$ by the following data: $\operatorname{sk}_n^A \widetilde{B} = A$ and for each $s > n$ there exists a set Z_s, such that the diagram

$$\begin{array}{ccc} \coprod_{Z_s} \partial\Delta^s & \longrightarrow & \operatorname{sk}_{s-1}^A \widetilde{B} \\ \downarrow & \ulcorner & \downarrow \\ \coprod_{Z_s} \Delta^s & \longrightarrow & \operatorname{sk}_s^A \widetilde{B} \end{array}$$

is a pushout square. Because the usual model structure on $\mathcal{S}$ is proper, it suffices to construct a lifting in the diagram

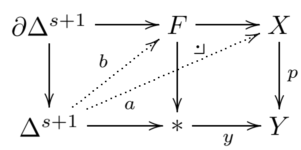


where Sing is the singular functor. We do this skeleton by skeleton. We solve the lifting problem



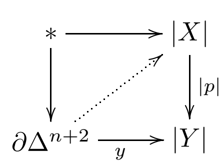
inductively for $s > n$. The start $\mathrm{sk}_n^A \widetilde{B} = A$ poses no conditions, and for $s > n$ the lifting exists by the assumptions on $X \to Y$.

On the other hand let $X \to Y$ be a Postnikov-n-fibration. We have to show that all maps $\pi_s|X| \to \pi_s|Y|$ are isomorphisms for $s > n$ and all vertices of X. Let F be the fiber over some vertex $y \in p(X)$. Since F is fibrant, every element in $\pi_s F$ is represented by a map $\partial\Delta^{s+1} \to F$. Consider for $s \geq n$ the following diagram:



For $s > n$ $\partial\Delta^{s+1} \to \Delta^{s+1}$ is a Postnikov-n-trivial cofibration. So we get lifting a by the defining property of p, and b, because the right hand square is a pullback. This shows that $\pi_s F = 0$ for all $s > n$.

The lemma will follow from the long exact homotopy sequence of $|p| : |X| \to |Y|$, after we have shown that $\pi_{n+1}|X| \to \pi_{n+1}|Y|$ is surjective for each vertex of X as a basepoint. First we make the remark that a map p in $\mathcal{S}$ has the right lifting property with respect to a class $\mathcal{C}$ of maps if and only if $|p|$ has the left lifting property with respect to the class of geometric realizations of $\mathcal{C}$. This follows since the trivial fibration $X \to \mathrm{Sing}|X|$ has a section. Now let an element in $\pi_{n+1}|Y|$ be represented by



Since $* \to \partial\Delta^{n+2}$ is a Postnikov-n-trivial cofibration and $|p|$ inherits the lifting properties of p, this element y has a preimage in $\pi_{n+1}|X|$, and we are done.

□

Remark 3.1.7 We define the following list of sets of maps in $\mathcal{S}$:

$$\begin{array}{rcl} I & := & \{\ \partial\Delta^{s+1} \to \Delta^{s+1} \mid s \geq 0\ \} \cup \{\varnothing \to \Delta^0\} \\ J & := & \{\ \Lambda_k^{s+1} \to \Delta^{s+1} \mid s \geq 0,\ s+1 \geq k \geq 0\} \\ J_n & := & \{\ \partial\Delta^s \to \Delta^s \mid s \geq n+2\} \end{array}$$

Then J is a set of generating trivial cofibrations for the usual model structure of simplicial sets, while I is a set of generating cofibrations for both the usual and the Postnikov-n-structure.

Corollary 3.1.8 *The set J_n forms a set of generating trivial cofibrations for the Postnikov-n-structure.*

Proof: All the maps in J_n are Postnikov-n-trivial cofibrations, so all Postnikov-n-fibrations have the right lifting property with respect to J_n. Conversely a map with the right lifting property with respect to J_n is a fibration that induces isomorphisms on the homotopy groups of the geometrical realization in dimensions above n. So it is a Postnikov-n-fibration by lemma 3.1.6.

□

Corollary 3.1.9 *Let $X \to Y$ be a fibration between simplicial sets. It is a Postnikov-n-fibration if and only if it is a fibration, such that the induced maps*

$$X_s \to M_s X \times_{M_s Y} Y_s$$

for $s \geq n+2$ are surjective.

Proof: This follows from lemma 3.1.8 by adjointness.

□

Reading the proof carefully we could slightly generalize the next statement.

Lemma 3.1.10 *Let $f : X \to Y$ be a fibration between fibrant simplicial sets that induces a surjection on π_{n+1} for every vertex of X. Then the map*

$$\operatorname{cosk}_{n+1} X \to \operatorname{cosk}_{n+1} Y$$

is a fibration. In particular the functor $\operatorname{cosk}_{n+1} : \mathcal{S} \to \mathcal{S}$ maps Postnikov-n-fibrations between fibrant simplicial sets to fibrations.

Proof: Let $f : X \to Y$ be a map as required in the lemma. Then we see that f has the right lifting property with respect to the set $J \cup \{* \to \partial\Delta^{n+2}\}$. We observe that Postnikov-n-fibration have this property.

We have to show that for all $s \geq 0$ and $0 \leq k \leq s+1$ there exist dotted liftings in the following diagram:

$$\begin{array}{ccc} \Lambda_k^{s+1} & \longrightarrow & \operatorname{cosk}_{n+1} X \\ \downarrow & \nearrow & \downarrow \\ \Delta^{s+1} & \longrightarrow & \operatorname{cosk}_{n+1} Y \end{array}$$

By adjointness we can delete the $\operatorname{cosk}_{n+1}$ on the right by adding a sk_{n+1} on the left. Then we immediately see that for $0 \leq s \leq n$ and all admitted k the liftings exist since $X \to Y$ is a fibration. Even more easily we find liftings for $s \geq n+2$ by observing:

$$(s+1) - (n+1) \geq 2 \;\Rightarrow\; \operatorname{sk}_{n+1} \Lambda_k^{s+1} = \operatorname{sk}_{n+1} \Delta^{s+1}$$

We are left with the step for $s = n+1$ and by inserting this we obtain the following diagram:

$$\begin{array}{ccc} \Lambda_k^{n+2} & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \partial\Delta^{n+2} & \longrightarrow & Y \end{array}$$

Here the lifting exists because of the surjectivity of $\pi_{n+1}X \to \pi_{n+1}Y$ for every basepoint of X.

□

The proof of the next lemma was outlined to me by Hirschhorn.

Lemma 3.1.11 *The Postnikov-n-structure is proper.*

Proof: Since left Bousfield localizations of left proper model categories are always left proper by [Hir03, 4.1.1.], it suffices to show right properness. Given a pullback square

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ A & \xrightarrow[h]{} & Y \end{array}$$

where f is a Postnikov-n-fibration and h is a Postnikov-n-equivalence, factor h into

$$A \xrightarrow{j} B \xrightarrow{k} Y$$

where j is a trivial cofibration and k is a fibration. We then have that k is also a Postnikov-n-equivalence. Thus let b be some vertex of B and let F be the fiber of k at $k(b)$, then $\pi_i F \cong 0$ for $i \leq n-1$ and $\pi_n F \to \pi_n B$ is the zero map, i.e., $\pi_{n+1}Y \to \pi_n F$ is surjective.

If Q is the pullback

$$\begin{array}{ccc} Q & \xrightarrow{r} & X \\ {\scriptstyle g}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ B & \xrightarrow[k]{} & Y \end{array}$$

then F is also the fiber of r. We then have the commutative square

$$\begin{array}{ccc} \pi_n X & \longrightarrow & \pi_{n-1}Q \\ \downarrow & & \downarrow \\ \pi_n Y & \longrightarrow & \pi_{n-1}F \end{array}$$

in which the two vertical homomorphisms are isomorphisms by 3.1.6. Since the bottom horizontal map is surjective, this implies that the top horizontal map is also surjective. Since F is $(n-1)$-connected, this implies that the map $r : Q \to X$ is a Postnikov-n-equivalence.

Since the usual model structure on simplicial sets is right proper, we know that $P \to Q$ is an ordinary weak equivalence. Of course, then $P \to Q \to X$ is a Postnikov-n-equivalence, which proves the lemma.

□

Theorem 3.1.12 *On the category $\mathcal{S}$ of simplicial sets the Postnikov-n-structure is a simplicial proper cofibrantly generated model structure.*

Proof: Follows from [Hir03, 4.1.1.] and lemma 3.1.11.

□

3.2 The n-$\mathcal{G}$-structure on $c\mathcal{M}$

In this subsection we set up the main technical machinery of our work. We use the Postnikov-n-structure on simplicial sets to truncate the $\mathcal{G}$-structure on cosimplicial objects over $\mathcal{M}$ in the sense that a weak equivalence will be a map that induces isomorphisms on the natural homotopy groups (Def. 2.2.4) up to degree n. This model structure enables us to define in subsection 5.1 interpolation categories for homology theories and develop an obstruction theory.

This localization is similar to a right Bousfield localization in the sense that we retain the same fibrations. But we are not able to apply the general machinery of [Hir03], because for example we do not know, whether the Bousfield-$\mathcal{G}$-structure on $c\mathcal{M}$ is cofibrantly generated or not. As in Bousfield's paper we do not use small object arguments.

It is a subtle point that we use the natural formulation of the $\mathcal{G}$-structure to produce our n-$\mathcal{G}$-structure on $c\mathcal{M}$, and not the one with the simplicial abelian groups $[X^\bullet, G]$. The reason is the construction of a cofibrant approximation functor (see 3.2.12 and 3.2.13). Such a functor would have to give a good truncation of the chain complex associated to $[X^\bullet, G]$ or equivalently a Postnikov section of this simplicial group by manipulating just $X^\bullet$. Such a functor does not exist. One might add that the reason for the non-existence is that colimits in a model category do not have to have homotopy meaning in general.

Before describing the model structure we explain some terminology. Remember also that unless otherwise stated we always refer to the external simplicial structure from 2.1.9.

Definition 3.2.1 For $G \in \mathcal{G}$ we denoted by $r^0 G$ the constant cosimplicial object over G. On the category $c\mathcal{M}$ we define for $n \geq 0$ the ***n*-$\mathcal{G}$-structure**. We call a map $f : X^\bullet \to Y^\bullet$

(i) an ***n*-$\mathcal{G}$-equivalence** if it induces a Postnikov-n-equivalence

$$\mathrm{map}(\widetilde{Y}^\bullet, r^0 G) \to \mathrm{map}(\widetilde{X}^\bullet, r^0 G)$$

for all fibrant $G \in \mathcal{G}$, where $\widetilde{X}^\bullet \to \widetilde{Y}^\bullet$ is a Reedy cofibrant approximation to f, see 1.1.3. It follows from [Hir03, 10.5.2] that this definition is is independent of the choice of a cofibrant approximation.

(ii) an ***n*-$\mathcal{G}$-cofibration** if it is a Reedy cofibration and the induced map

$$\mathrm{map}(Y^\bullet, r^0 G) \to \mathrm{map}(X^\bullet, r^0 G)$$

is a Postnikov-n-fibration for every fibrant $G \in \mathcal{G}$.

(iii) an ***n*-$\mathcal{G}$-fibration** if it has the right lifting property with respect to all trivial n-$\mathcal{G}$-cofibrations. (See 3.2.6(ii).)

We will sometimes refer to the $\mathcal{G}$-structure as the **∞-$\mathcal{G}$-structure**.

Remark 3.2.2 By definition of a Postnikov-n-equivalence (see 3.1.4) a map $X^\bullet \to Y^\bullet$ is an n-$\mathcal{G}$-equivalence if and only if it induces isomorphisms

$$\pi_s^\natural(Y^\bullet, G) \to \pi_s^\natural(X^\bullet, G)$$

for all $G \in \mathcal{G}$ and $0 \leq s \leq n$. For the definition of the natural homotopy groups see 2.2.4.

Remark 3.2.3 Because $r^0 G$ is a constant cosimplicial object we have a natural isomorphism

$$\mathrm{map}(X^\bullet, r^0 G) \cong \mathrm{Hom}_{\mathcal{M}}(X^\bullet, G)$$

of simplicial sets, where on the right side the functor $\mathrm{Hom}_{\mathcal{M}}(\_, G)$ is applied degreewise.

The rest of this section is devoted to the proof that this is really a model structure. We begin with the obvious things.

Lemma 3.2.4 *The axioms* **MC1***,* **MC2** *and* **MC3** *are satisfied.*

Proof: Obvious.

□

Definition 3.2.5 We say that a map is an **n-$\mathcal{G}$-trivial cofibration**, if it is an n-$\mathcal{G}$-cofibration as well as an n-$\mathcal{G}$-equivalence. We also say that a map is an **n-$\mathcal{G}$-trivial fibration**, if it is an n-$\mathcal{G}$-fibration as well as an n-$\mathcal{G}$-equivalence.

Lemma 3.2.6 *We have the following immediate characterizations:*

(i) *A map is an n-$\mathcal{G}$-trivial cofibration if and only if it is an $\mathcal{G}$-trivial cofibration.*

(ii) *A map is an n-$\mathcal{G}$-fibration if and only if it is a $\mathcal{G}$-fibration.*

Proof: Part (ii) follows from (i) by the right lifting property. Part (i) follows, since a Reedy cofibration $X^\bullet \to Y^\bullet$ is an n-$\mathcal{G}$-trivial cofibration if and only if it induces Postnikov-n-trivial fibrations $\mathrm{map}(Y^\bullet, r^0G) \to \mathrm{map}(X^\bullet, r^0G)$ for all fibrant $G \in \mathcal{G}$. Postnikov-n-trivial fibrations are the same as usual trivial fibrations, because we did not change cofibrations, when we localized to the Postnikov-n-structure. But a map $X^\bullet \to Y^\bullet$ that induces a usual trivial fibration of simplicial groups, is exactly a $\mathcal{G}$-trivial cofibration.

□

Corollary 3.2.7 *The axioms* **MC4(ii)** *and* **MC5(ii)** *hold:*

(i) *Every map can be factored into an n-$\mathcal{G}$-trivial cofibration and an n-$\mathcal{G}$-fibration.*

(ii) *The n-$\mathcal{G}$-trivial cofibrations have the left lifting property with respect to the n-$\mathcal{G}$-fibrations.*

Proof: Since the classes of maps in question coincide with those classes in the $\mathcal{G}$-structure, the lemma follows from the corresponding statements in [Bou03, 3.16.].

□

Next we prove a characterization of n-$\mathcal{G}$-cofibrations analogous to the characterization of $\mathcal{G}$-cofibrations in lemma 2.1.18.

Lemma 3.2.8 *For a Reedy cofibration $A^\bullet \to B^\bullet$ the following are equivalent:*

(i) *$A^\bullet \to B^\bullet$ is an n-$\mathcal{G}$-cofibration.*

(ii) *$A^\bullet \to B^\bullet$ is a $\mathcal{G}$-cofibration and the maps*

$$A^s \sqcup_{L^s A^\bullet} L^s B^\bullet \to B^s$$

for all $s \geq n+2$ are $\mathcal{G}$-monic.

Proof: If we start with condition (i), then by definition the map

$$\mathrm{map}(B^\bullet, r^0G) \to \mathrm{map}(A^\bullet, r^0G)$$

is a Postnikov-n-fibration of simplicial sets. So by 3.1.9 it is a fibration with the additional property that for all fibrant $G \in \mathcal{G}$ the maps

$$\mathrm{map}(B^\bullet, r^0G)_s \to M_s\mathrm{map}(B^\bullet, r^0G) \times_{M_s\mathrm{map}(A^\bullet, r^0G)} \mathrm{map}(A^\bullet, r^0G)_s \qquad (3.1)$$

for $s \geq n+2$ are surjective. By definition of the mapping space functor the map (3.1) is isomorphic to:

$$\begin{aligned}&\mathrm{Hom}_{c\mathcal{M}}(B^\bullet \otimes \Delta^s, r^0 G) \to \\ &\qquad \mathrm{Hom}_{c\mathcal{M}}(B^\bullet \otimes \partial\Delta^s, r^0 G) \times_{\mathrm{Hom}_{c\mathcal{M}}(A^\bullet \otimes \partial\Delta^s, r^0 G)} \mathrm{Hom}_{c\mathcal{M}}(A^\bullet \otimes \Delta^s, r^0 G)\end{aligned}$$

Using the adjunction relation between r^0 and the degree-zero-functor we deduce that for $s \geq n+2$ the map (3.1) induces a dotted lifting in the diagram:

$$\begin{array}{ccc} A^s \sqcup_{L^s A^\bullet} L^s B^\bullet & \longrightarrow & G \\ \downarrow & \nearrow & \\ B^s & & \end{array}$$

This proves that $A^s \sqcup_{L^s A^\bullet} L^s B^\bullet \to B^s$ is $\mathcal{G}$-monic for $s \geq n+2$ is $\mathcal{G}$-monic. This is condition (ii), and, of course, we can go backwards in all adjunctions proving the equivalence between the two statements.

□

Lemma 3.2.9 *Let $i : A^\bullet \to B^\bullet$ be a Reedy cofibration. The map i is an n-$\mathcal{G}$-cofibration if and only if the map*

$$\mathrm{Hom}_{\mathcal{M}}(B^\bullet, G) \to \mathrm{Hom}_{\mathcal{M}}(A^\bullet, G)$$

is a Postnikov-n-fibration in $\mathcal{S}$ for each fibrant object $G \in \mathcal{G}$ in $\mathcal{M}$.

Proof: This follows readily from remark 3.2.3.

□

Lemma 3.2.10 *The n-$\mathcal{G}$-cofibrations as well as the n-$\mathcal{G}$-trivial cofibrations are closed under pushouts.*

Proof: This follows directly from lemma 3.2.9.

□

Lemma 3.2.11 *The n-$\mathcal{G}$-structure satisfies* **(SM7')**. *Explicitly: Let $i : A^\bullet \to B^\bullet$ be an n-$\mathcal{G}$-cofibration and let $j : J \to K$ be a cofibration of finite simplicial sets. Consider the pushout map:*

$$(A^\bullet \otimes K) \sqcup_{(A^\bullet \otimes J)} (B^\bullet \otimes J) \ \to \ (B^\bullet \otimes K)$$

The pushout map is an n-$\mathcal{G}$-cofibration which is n-G-trivial, or equivalently $\mathcal{G}$-trivial, if either i or j is trivial.

Proof: This proof is an adaption of [Bou03, Prop. 3.17]. The only statement, which is not already covered by that proposition is the fact that the pushout map is an n-$\mathcal{G}$-cofibration and not merely a $\mathcal{G}$-cofibration.

We use the adjunction isomorphism $\mathrm{Hom}_{\mathcal{M}}(A^\bullet \otimes K, G) \cong \mathrm{map}_{\mathcal{S}}(K, \mathrm{Hom}_{\mathcal{M}}(A^\bullet, G))$ to convert the conditions on the pushout map to be an n-$\mathcal{G}$-cofibration into the condition that the map

$$\begin{array}{c} \mathrm{map}_{\mathcal{S}}(K, \mathrm{Hom}_{\mathcal{M}}(B^\bullet, G)) \\ \downarrow \\ \mathrm{map}_{\mathcal{S}}(K, \mathrm{Hom}_{\mathcal{M}}(A^\bullet, G)) \underset{\mathrm{map}_{\mathcal{S}}(J, \mathrm{Hom}_{\mathcal{M}}(A^\bullet, G))}{\times} \mathrm{map}_{\mathcal{S}}(J, \mathrm{Hom}_{\mathcal{M}}(B^\bullet, G)) \end{array}$$

be a Postnikov-n-fibration for each fibrant $\mathcal{G}$-injective G. By [Bou03, Prop. 3.17] this map is a fibration. Again by adjunction we translate the extra conditions for being a Postnikov-n-fibration into the existence of liftings in the diagrams

$$\begin{array}{ccc} (J\times\Delta^{s+1})\sqcup_{(J\times\partial\Delta^{s+1})}(K\times\partial\Delta^{s+1}) & \longrightarrow & \mathrm{Hom}_{\mathcal{M}}(B^\bullet,G) \\ \downarrow & \nearrow & \downarrow \\ K\times\Delta^{s+1} & \longrightarrow & \mathrm{Hom}_{\mathcal{M}}(A^\bullet,G) \end{array} \tag{3.2}$$

for all $s>n$. Except for $s=n=0$ the Blakers-Massey theorem or equivalently the left properness of the Postnikov-n-structure shows that the right hand vertical maps in (3.2) are n-connected and in particular Postnikov-n-trivial cofibrations. The result then follows from 3.2.9 and 3.1.12.

For $s=n=0$ a Postnikov-0-fibration is a trivial fibration onto the components in the image. The liftings exist now, because j induces a bijection of the connected components of the source and the target of the right hand vertical maps in (3.2).

□

To prove the remaining axioms we begin by noting in 3.2.13 that there exist functorial cofibrant approximations. It is no more true that the class of n-$\mathcal{G}$-cofibrant objects coincides with the Reedy cofibrant ones as it was the case for the $\mathcal{G}$-structure and which was an important technical point there. It is true that the n-$\mathcal{G}$-cofibrant objects are also $\mathcal{G}$-cofibrant. We first determine n-$\mathcal{G}$-cofibrant objects.

Lemma 3.2.12 *An object $A^\bullet$ in $c\mathcal{M}$ is n-$\mathcal{G}$-cofibrant if and only if it is Reedy cofibrant and if $\pi_s^\natural(A^\bullet,G)=0$ for all $G\in\mathcal{G}$ and $s>n$.*

Proof: Obvious.

□

Remark 3.2.13 We are going to construct a cofibrant approximation to an object $X^\bullet$. First take a functorial Reedy cofibrant replacement $A^\bullet\to X^\bullet$. Observe now that for $n\geq 0$ we have the following isomorphisms since r^0G is constant:

$$\begin{aligned}\mathrm{map}(\mathrm{sk}_{n+1}A^\bullet,r^0G) &= \mathrm{Hom}_{\mathcal{M}}(\ell^{n+1}(j_{n+1})^*A^\bullet,G)\\ &= r_{n+1}(j_{n+1})^*\mathrm{Hom}_{\mathcal{M}}(A^\bullet,G)\\ &= \mathrm{cosk}_{n+1}\,\mathrm{map}(A^\bullet,r^0G)\end{aligned}$$

We also note that for a Reedy cofibrant $A^\bullet$ the simplicial set $\mathrm{map}(A^\bullet,r^0G)$ is fibrant by lemma 2.1.3. The reason is that $L^sA^\bullet\to A^s$ is a $\mathcal{G}$-monic cofibration between cofibrant objects, while r^0G is $\mathcal{G}$-fibrant by 2.1.15.

The functor cosk_{n+1} produces a model for the n-th Postnikov section of a simplicial set, if it is fibrant, and hence the canonical map $\mathrm{sk}_{n+1}A^\bullet\to A^\bullet\to X^\bullet$ is an n-$\mathcal{G}$-cofibrant approximation. This argument also yields functoriality of this approximation.

Corollary 3.2.14 *Let $n\geq 0$.*

(i) *$sk_{n+1}:c\mathcal{M}\to c\mathcal{M}$ maps n-$\mathcal{G}$-equivalences to $\mathcal{G}$-equivalences.*

(ii) *$sk_{n+1}:c\mathcal{M}\to c\mathcal{M}$ maps n-$\mathcal{G}$-cofibrations between Reedy cofibrant objects to $\mathcal{G}$-cofibrations.*

Proof: (i) is obvious. For (ii) we know by 2.1.20 that the resulting map is again a Reedy cofibration. The claim follows now from 3.1.10 and the definition of n-$\mathcal{G}$-cofibrations.

□

The next lemma is an easy, but central observations for the proof of MC5(i). It tells us that on n-$\mathcal{G}$-cofibrant objects the n-$\mathcal{G}$-structure agrees with the $\mathcal{G}$-structure.

Lemma 3.2.15 (i) *A map between n-$\mathcal{G}$-cofibrant objects is an n-$\mathcal{G}$-equivalence if and only if it is a $\mathcal{G}$-equivalence.*

(ii) *A map between n-$\mathcal{G}$-cofibrant objects is an n-$\mathcal{G}$-cofibration if and only if it is a $\mathcal{G}$-cofibration.*

Proof: (i) is obvious. With (ii) the only-if-part is also obvious.

According to lemma 3.2.12 an n-$\mathcal{G}$-cofibrant object $A^\bullet$ has $\pi_s^\natural(A^\bullet, G) = 0$ for every $G \in \mathcal{G}$ and every $s > n$. Thus the extra requirement for $\mathcal{G}$-cofibrations to be an n-$\mathcal{G}$-cofibration given by 3.2.1(ii) and 3.1.6 is trivially met. This proves the if-part of (ii).

□

Lemma 3.2.16 *A map between n-$\mathcal{G}$-cofibrant objects can be factored into an n-$\mathcal{G}$-cofibration followed by an n-$\mathcal{G}$-equivalence.*

Proof: Each map $X^\bullet \to Y^\bullet$ can be factored into a $\mathcal{G}$-cofibration $X^\bullet \to I^\bullet$ followed by a $\mathcal{G}$-equivalence $I^\bullet \to Y^\bullet$ by [Bou03, Prop. 3.20.]. $I^\bullet$ is Reedy cofibrant, since $X^\bullet$ is, and it is $\mathcal{G}$-equivalent to $Y^\bullet$. Therefore it is n-$\mathcal{G}$-cofibrant by the characterization 3.2.12. The claim follows now from 3.2.15.

□

Lemma 3.2.17 *Each map in $c\mathcal{M}$ can be factored into an n-$\mathcal{G}$-cofibration followed by an n-$\mathcal{G}$-trivial fibration. This is* **MC5(i)**.

Proof: Let $X^\bullet \to Y^\bullet$ be an arbitrary map. Use a Reedy cofibrant replacement $A^\bullet \to B^\bullet$ and then approximate this with the map $\mathrm{sk}_{n+1} A^\bullet \to \mathrm{sk}_{n+1} B^\bullet$, where the objects are n-$\mathcal{G}$-cofibrant by lemma 3.2.13. This approximation possesses a factorization $\mathrm{sk}_{n+1} A^\bullet \to I^\bullet \xrightarrow{\simeq} \mathrm{sk}_{n+1} B^\bullet$ into an n-$\mathcal{G}$-cofibration followed by an n-$\mathcal{G}$-equivalence by lemma 3.2.16. Let $E^\bullet$ be the pushout of $\mathrm{sk}_{n+1} A^\bullet \to X^\bullet$ along $\mathrm{sk}_{n+1} A^\bullet \to I^\bullet$. Consider the diagram:

$$\begin{array}{ccccc}
\mathrm{sk}_{n+1} A^\bullet & \longrightarrow & I^\bullet & \xrightarrow{\mathcal{G}-\simeq} & \mathrm{sk}_{n+1} B^\bullet \\
{\scriptstyle (j_{n+1})^*-=}\downarrow & & \downarrow & & \downarrow{\scriptstyle (j_{n+1})^*-=} \\
A^\bullet & \longrightarrow & \widetilde{I}^\bullet & \longrightarrow & B^\bullet \\
{\scriptstyle \text{Reedy}-\simeq}\downarrow & & \downarrow & & \downarrow{\scriptstyle \text{Reedy}-\simeq} \\
X^\bullet & \longrightarrow & E^\bullet & \longrightarrow & Y^\bullet
\end{array}$$

Here both squares on the left are pushout diagrams. Remember that the maps $\mathrm{sk}_{n+1} A^\bullet \to A^\bullet$ and $\mathrm{sk}_{n+1} B^\bullet \to B^\bullet$ are special n-$\mathcal{G}$-equivalences, since they are equalities up to cosimplicial degree $n+1$. This is depicted in the diagram by the label "$(j_{n+1})^* - =$".

The map $A^\bullet \to \widetilde{I}^\bullet$ is an n-$\mathcal{G}$-cofibration by 3.2.10. $I^\bullet \to \widetilde{I}^\bullet$ is an n-$\mathcal{G}$-equivalence, because in fact $(j_{n+1})^* I^\bullet = (j_{n+1})^* \widetilde{I}^\bullet$. Then $\widetilde{I}^\bullet \to B^\bullet$ is an n-$\mathcal{G}$-equivalence by the two-out-of-three axiom. We can extend all the statements to the lower row by 3.2.10 and by the left properness of the Reedy structure. We deduce that $X^\bullet \to E^\bullet$ is an n-$\mathcal{G}$-cofibration and that $\widetilde{I}^\bullet \to E^\bullet$ is a Reedy equivalence. It follows that $E^\bullet \to Y^\bullet$ is an n-$\mathcal{G}$-equivalence. This proves that every map can be factored into an n-$\mathcal{G}$-cofibration followed by an n-$\mathcal{G}$-equivalence.

To prove the original factorization we use lemma 3.2.7 and factor $E^\bullet \to Y^\bullet$ further into an n-$\mathcal{G}$-trivial cofibration and an n-$\mathcal{G}$-fibration which has to be an n-$\mathcal{G}$-trivial fibration by two-out-of-three. This establishes the lemma.

□

Lemma 3.2.18 *The n-$\mathcal{G}$-trivial fibrations have the right lifting property with respect to n-$\mathcal{G}$-cofibrations. This is* **MC4(i)**.

Proof: Suppose we are given a diagram

$$\begin{array}{ccc} A^\bullet & \longrightarrow & X^\bullet \\ {\scriptstyle n-\mathcal{G}-\mathrm{cof.}}\ \downarrow i & & p \downarrow\ {\scriptstyle n-\mathcal{G}-\simeq,\ n-\mathcal{G}-\mathrm{fib.}} \\ B^\bullet & \longrightarrow & Y^\bullet \end{array} \tag{3.3}$$

where i is an n-$\mathcal{G}$-cofibration and p is an n-$\mathcal{G}$-trivial fibration. Since the Reedy structure on $c\mathcal{M}$ is left proper, we can assume without loss of generality that all objects are Reedy cofibrant (see [Bou03, lemma 3.5.]). By applying the n-$\mathcal{G}$-cofibrant approximation functor sk_{n+1} we obtain a diagram:

$$\begin{array}{ccc} \mathrm{sk}_{n+1} A^\bullet & \longrightarrow & \mathrm{sk}_{n+1} X^\bullet \\ {\scriptstyle \mathrm{sk}_{n+1} i} \downarrow & & {\scriptstyle \mathrm{sk}_{n+1} p} \downarrow\ {\scriptstyle \mathcal{G}-\simeq} \\ \mathrm{sk}_{n+1} B^\bullet & \longrightarrow & \mathrm{sk}_{n+1} Y^\bullet \end{array} \tag{3.4}$$

The map $\mathrm{sk}_{n+1} i$ is a $\mathcal{G}$-cofibration by 3.2.14(ii). But $\mathrm{sk}_{n+1} p$ is not a $\mathcal{G}$-fibration, although it is a $\mathcal{G}$-equivalence by 3.2.14(i). We must repair this in order to get a lifting for the lower part up to degree $n+1$. We will construct diagram (3.7) below, where there is a $\mathcal{G}$-trivial fibration $\widetilde{X}^\bullet \to \mathrm{sk}_{n+1} Y^\bullet$, whose $(n+1)$-truncation equals the $(n+1)$-truncation of $X^\bullet \to Y^\bullet$. Hence up to degree $n+1$ diagram (3.7) coincides with diagram (3.4) or respectively (3.3) and admits a lifting. We will now explain this procedure.

Starting with (3.4) we construct a factorization $\mathrm{sk}_{n+1} X^\bullet \to \widetilde{X}^\bullet \to \mathrm{sk}_{n+1} Y^\bullet$ into a $\mathcal{G}$-trivial cofibration followed by a $\mathcal{G}$-fibration. This is done by an application of MC5(ii) for the $\mathcal{G}$-structure. $\widetilde{X}^\bullet \to \mathrm{sk}_{n+1} Y^\bullet$ will be a $\mathcal{G}$-trivial fibration by 2-out-of-3. But even more we claim that we can produce such a factorization with $(j_{n+1})^* X^\bullet = (j_{n+1})^* \widetilde{X}^\bullet$. We have to delve into the factorization process more closely.

We remind the reader that for an arbitrary morphism $I^\bullet \to J^\bullet$ in $c\mathcal{M}$ MC5(ii) for the $\mathcal{G}$-structure is obtained by an inductive factorization

$$I^s \sqcup_{L^s I^\bullet} L^s Z^\bullet \xrightarrow[\text{cofibration}]{\mathcal{G}-\text{monic}} Z^s \xrightarrow[\text{fibration}]{\mathcal{G}-\text{injective}} M^s Z^\bullet \times_{M^s J^\bullet} J^s \tag{3.5}$$

into a $\mathcal{G}$-monic cofibration and a $\mathcal{G}$-injective fibration, which is possible by lemma 2.1.4. This is an analogous process as for the Reedy structure.

Since n-$\mathcal{G}$-fibrations and $\mathcal{G}$-fibrations coincide, p is a $\mathcal{G}$-fibration. Hence we know that the maps $X^s \to M^s X^\bullet \times_{M^s Y^\bullet} Y^s$ are $\mathcal{G}$-injective fibrations. Applying the inductive factorization (3.5) to $\mathrm{sk}_{n+1} X^\bullet \to \mathrm{sk}_{n+1} Y^\bullet$ we can take Z^s to be $X^s = X^s \sqcup_{L^s X^\bullet} L^s Z^\bullet$ as long as $s \le n+1$. This proves that we can choose $\widetilde{X}^s := (\mathrm{sk}_{n+1} X^\bullet)^s = X^s$ for $0 \le s \le n+1$, or in other words:

$$(j_{n+1})^* \widetilde{X}^\bullet = (j_{n+1})^* X^\bullet \tag{3.6}$$

Just as a remark MC5(i) would not have worked, even if at first sight the outcome had been the same. MC5(i) for the $\mathcal{G}$-structure is not obtained with this Reedy type inductive argument and thus is not appropriate for the previous step. Also a potentially easier argument by applying cosk_{n+1} to $X^\bullet \to Y^\bullet$ in (3.3) fails, because cosk_{n+1} of an n-$\mathcal{G}$-trivial fibration does not have to be a $\mathcal{G}$-trivial fibration.

As we already noticed $\widetilde{X}^\bullet \to \mathrm{sk}_{n+1} Y^\bullet$ is a $\mathcal{G}$-trivial fibration by 2-out-of-3. So we have succeeded in constructing the promised diagram

$$\begin{array}{ccc} \mathrm{sk}_{n+1} A^\bullet & \longrightarrow & \widetilde{X}^\bullet \\ {\scriptstyle \mathcal{G}-\mathrm{cof.}}\downarrow & \nearrow & \downarrow {\scriptstyle \mathcal{G}-\simeq,\ \mathcal{G}-\mathrm{fib.}} \\ \mathrm{sk}_{n+1} B^\bullet & \longrightarrow & \mathrm{sk}_{n+1} Y^\bullet, \end{array} \tag{3.7}$$

where of course $\mathrm{sk}_{n+1} A^\bullet \to \widetilde{X}^\bullet$ is the composition $\mathrm{sk}_{n+1} A^\bullet \to \mathrm{sk}_{n+1} X^\bullet \to \widetilde{X}^\bullet$. We already observed that $\mathrm{sk}_{n+1} A^\bullet \to \mathrm{sk}_{n+1} B^\bullet$ is a $\mathcal{G}$-cofibration by 3.2.14(ii). Therefore we get the dotted lifting in (3.7).

We want to extend this lifting to the remaining degrees $s > n+1$. We are entitled to write down the next diagram:

$$\begin{array}{ccc} A^{n+2} \sqcup_{L^{n+2}A^\bullet} L^{n+2}B^\bullet & \longrightarrow & X^{n+2} \\ \downarrow & \nearrow & \downarrow \\ B^{n+2} & \longrightarrow & M^{n+2}X^\bullet \times_{M^{n+2}Y^\bullet} Y^{n+2} \end{array}$$

Again by 3.2.8 the left arrow is a $\mathcal{G}$-monic cofibration, and we get a lifting $B^{n+2} \to X^{n+2}$. But from now on we can proceed by induction just like in the $\mathcal{G}$-structure or the Reedy case, since for $s \ge n+2$ all maps $A^s \sqcup_{L^sA^\bullet} L^sB^\bullet \to B^s$ are $\mathcal{G}$-monic cofibrations by lemma 3.2.8 and all maps $X^s \to M^sX^\bullet \times_{M^sY^\bullet} Y^s$ are $\mathcal{G}$-injective fibrations by lemma 3.2.6 and Definition 2.1.6.

□

Lemma 3.2.19 *Given a pushout diagram*

$$\begin{array}{ccc} A^\bullet & \xrightarrow{\alpha} & X^\bullet \\ {\scriptstyle i}\downarrow & \ulcorner & \downarrow \\ B^\bullet & \xrightarrow[\beta]{} & P^\bullet \end{array}$$

where α is an n-$\mathcal{G}$-equivalence and i is an n-$\mathcal{G}$-cofibration then β is an n-$\mathcal{G}$-equivalence.

Proof: We apply the functor $\mathrm{map}(\_, r^0G)$ and the resulting square is a pullback square by 3.2.3 or 3.2.9. The assertion is now implied by lemma 3.1.11, which states the right properness of the Postnikov-n-structure.

□

Summarizing this subsection we have proved the following theorem.

Theorem 3.2.20 *For a pointed left proper simplicial model category $\mathcal{M}$ the n-$\mathcal{G}$-structure on the category $c\mathcal{M}$ of cosimplicial objects over $\mathcal{M}$ is a pointed simplicial left proper model structure. If $\mathcal{M}$ possesses functorial factorization then the n-$\mathcal{G}$-structure has functorial factorizations.*

Remark 3.2.21 Let $X^\bullet$ be an object in $c\mathcal{M}$. The skeletal filtration of a Reedy cofibrant approximation to $X^\bullet$ consists of n-$\mathcal{G}$-cofibrant approximations $A_n^\bullet$ to $X^\bullet$ for the various n, and these assemble into a sequence

$$A_0^\bullet \to A_1^\bullet \to A_2^\bullet \to ... \to X^\bullet$$

which captures higher and higher natural homotopy groups. So this can be viewed as a co-Postnikov-tower for $X^\bullet$.

3.3 The tower of truncated homotopy categories

Definition 3.3.1 The identity of $c\mathcal{M}$ induces a functor

$$c\mathcal{M}^{(n+1)-\mathcal{G}} \to c\mathcal{M}^{n-\mathcal{G}}$$

from $c\mathcal{M}$ equipped with the $(n+1)$-$\mathcal{G}$-structure to $c\mathcal{M}$ with the n-$\mathcal{G}$-structure, which preserves weak equivalences and fibrations. Likewise the cofibrant approximation functor sk_{n+1} induces a functor in the same direction. It preserves weak equivalences by 3.2.14(i). Both are connected by a natural transformation

$$\mathrm{sk}_{n+1} \to \mathrm{id},$$

which is an n-$\mathcal{G}$-equivalence. We observe that these functors and the natural transformation descend to the respective homotopy categories. These functors become naturally isomorphic and we obtain a tower

$$... \to Ho(c\mathcal{M}^{(n+1)-\mathcal{G}}) \stackrel{\sigma_n}{\to} Ho(c\mathcal{M}^{n-\mathcal{G}}) \to ... \to Ho(c\mathcal{M}^{1-\mathcal{G}}) \stackrel{\sigma_0}{\to} Ho(c\mathcal{M}^{0-\mathcal{G}})$$

of categories. We call this **the tower of truncated homotopy categories** associated to $\mathcal{M}$ and $\mathcal{G}$. In fact sk_{n+1} is the left derived functor of the identity functor in this case.

Remark 3.3.2 Note that both functors sk_{n+1} and id possess right adjoints, but they are not left Quillen functors since they do not preserve cofibrations.

Remark 3.3.3 From 1.3.8 and lemma 2.1.22 we get a Quillen pair

$$\mathcal{M} \underset{\mathrm{Tot}}{\overset{\_\otimes^{\mathrm{pro}}\Delta^\bullet}{\rightleftarrows}} c\mathcal{M}^{\mathcal{G}} .$$

Note that the natural transformation $\_\otimes^{\mathrm{pro}}\Delta^\bullet \to r^0$ gives a Reedy cofibrant replacement by [Hir03, 17.1.4.] and hence a $\mathcal{G}$-cofibrant replacement. It follows that both induce the same left derived functor:

$$Ho(\mathcal{M}) = \mathcal{T} \underset{R\,\mathrm{Tot}}{\overset{Lr^0}{\rightleftarrows}} Ho(c\mathcal{M}^{\mathcal{G}}) .$$

Definition 3.3.4 The identity functor and sk_{n+1} also induce functors

$$\mathcal{M} \stackrel{r^0}{\to} c\mathcal{M}^{\mathcal{G}} \to c\mathcal{M}^{n-\mathcal{G}}$$

and for $Ho(\mathcal{M}) =: \mathcal{T}$

$$\mathcal{T} \to Ho(c\mathcal{M}^{\mathcal{G}}) \to Ho(c\mathcal{M}^{n-\mathcal{G}}).$$

We will denote this composition of functors by $\boldsymbol{\theta_n}$. We arrive at the following diagram:

$$\begin{array}{c}
\mathcal{T} = Ho(\mathcal{M}) \\
\theta_{n+1} \swarrow \quad \downarrow \theta_n \quad \searrow \theta_1 \quad \searrow \theta_0 \\
\cdots \longrightarrow Ho(c\mathcal{M}^{(n+1)-\mathcal{G}}) \xrightarrow[\sigma_n]{} Ho(c\mathcal{M}^{n-\mathcal{G}}) \longrightarrow \cdots \longrightarrow Ho(c\mathcal{M}^{1-\mathcal{G}}) \xrightarrow[\sigma_0]{} Ho(c\mathcal{M}^{0-\mathcal{G}})
\end{array}$$

This diagram is a 2-commuting diagram of functors. For details we refer to [Hov99]. 2-commutativity it provided by the relation $\mathrm{sk}_n \, \mathrm{sk}_{n+1} = \mathrm{sk}_n$.

3.4 Abelian example: truncated derived categories

Let $\mathcal{A}$ be an abelian category with enough injective objects. Let $\mathcal{I}$ be the class of injective objects. The following obvious observations are central to our constructions. In our applications $\mathcal{A}$ is graded, but the results here can be given in the ungraded situation as well.

Lemma 3.4.1 *A map $M \to N$ in $\mathcal{A}$ is a monomorphism resp. isomorphism if and only if it induces a surjection resp. bijection*

$$\mathrm{Hom}_{\mathcal{A}}(N, I) \to \mathrm{Hom}_{\mathcal{A}}(M, I)$$

for all injective objects I in $\mathcal{A}$.

Lemma 3.4.2 *Let C^* be an arbitrary cochain complex over $\mathcal{A}$ and let I be in $\mathcal{I}$. Then there is a canonical isomorphism*

$$H_s \mathrm{Hom}_{\mathcal{A}}(C^*, I) \cong \mathrm{Hom}_{\mathcal{A}}(H^s C^*, I).$$

Remark 3.4.3 If we view $\mathcal{A}$ as a discrete model category, we can equip the category $c\mathcal{A}$ of cosimplicial objects over $\mathcal{A}$ with the $\mathcal{I}$-injective model structure derived from 2.1.13. It follows from [Bou03, 4.4] that this model structure corresponds to the classical model structure from [Qui67] for the nonnegative cochain complexes $\mathrm{CoCh}^{\geq 0}(\mathcal{A})$ via the Dold-Kan-correspondence.

Using the cogenerating property of the class of injective objects of lemma 3.4.1 and the dualization statement of 3.4.2 we deduce the following characterizations for maps of complexes:

The $\mathcal{I}$-equivalences are the cohomology equivalences, the $\mathcal{I}$-cofibrations are the maps that are monomorphisms in positive degrees, and the $\mathcal{I}$-fibrations are those that are (split) surjective with injective kernel in all degrees. The fibrant objects are the degreewise injective objects, while all objects are cofibrant.

Definition 3.4.4 We denote the homotopy category associated to the $\mathcal{I}$-structure on $c\mathcal{A}$ by $\boldsymbol{D^{\geq 0}(\mathcal{A})}$. It is equivalent to the full subcategory of the derived category $D(\mathcal{A})$ of $\mathcal{A}$ consisting of nonnegative cochain complexes.

We can now consider the n-$\mathcal{I}$-structure from 3.2.20. We will call its associated homotopy category $Ho(c\mathcal{A}^{n-\mathcal{I}})$ the ***n*-truncated derived category** and denote it by $\boldsymbol{D_n^{\geq 0}(\mathcal{A})}$. From 3.3.1 we get a tower of categories

$$D^{\geq 0}(\mathcal{A}) \to \ldots \to D_n^{\geq 0}(\mathcal{A}) \to D_{n-1}^{\geq 0}(\mathcal{A}) \to \ldots \to D_0^{\geq 0}(\mathcal{A}) \xrightarrow{H^0} \mathcal{A}$$

induced by the identity functor and where the last functor is just ordinary zeroth cohomology. This functor is an equivalence, which can easily be seen directly or will follow from 5.4.2. Note also that H^0 is covariant and to be taken with an internal grading if $\mathcal{A}$ is graded.

Remark 3.4.5 We are now going to describe the n-$\mathcal{I}$-equivalences. As $\mathcal{A}$ is given the discrete model structure, loop objects ΩI vanish and there are isomorphisms

$$[X, I] \cong \mathrm{Hom}_{\mathcal{A}}(X, I).$$

Therefore the spiral exact sequence collapses to isomorphisms

$$\pi_s^{\natural}(X^\bullet, I) \cong \pi_s[X^\bullet, I] \cong H_s N\mathrm{Hom}_{\mathcal{A}}(X^\bullet, I).$$

Since I is injective we also have the following isomorphism:

$$H_s N\mathrm{Hom}_{\mathcal{A}}(X^\bullet, I) \cong \mathrm{Hom}_{\mathcal{A}}(H^s N X^\bullet, I)$$

We conclude that a map $X^\bullet \to Y^\bullet$ in $c\mathcal{A}$ is an n-$\mathcal{I}$-equivalence if and only if it induces isomorphisms

$$H^s N X^\bullet \to H^s N Y^\bullet$$

for all $0 \leq s \leq n$. Here $N : c\mathcal{A} \to \mathrm{CoCh}^{\geq 0}(\mathcal{A})$ denotes the normalization functor.

4 Homological functors

In subsection 4.1 we are going to describe the class of functors for which we will construct interpolation categories and set up an obstruction calculus for the realization problem these functors give rise to. In subsection 4.2 we will exploit the general theory of $\mathcal{G}$-structures to obtain model structures related to the realization problem of such functors by choosing a well suited class $\mathcal{G}$.

4.1 Homological functors with enough injectives

Definition 4.1.1 From now on let $\boldsymbol{\mathcal{T}}$ always denote a triangulated category. The set of morphisms for X and Y in $\mathcal{T}$ will be denoted by $[\boldsymbol{X}, \boldsymbol{Y}]$. The shift functor or **suspension** of $\mathcal{T}$ will be denoted by $\boldsymbol{\Sigma}$. It is, of course, an equivalence of categories.

Let $\boldsymbol{\mathcal{A}}$ always be a graded abelian category, which means that we require that $\mathcal{A}$ possesses a **shift functor** denoted by [1] which is an equivalence of categories. Let $[n]$ denote the n-fold iteration of [1].

Definition 4.1.2 Let $F_* : \mathcal{T} \to \mathcal{A}$ be a covariant functor, where the star stands for the grading of $\mathcal{A}$. We say that F_* is **homological**, if it satisfies the following conditions:

(i) F_* is a graded functor, in other words, it commutes with suspensions, so there are a natural equivalences

$$F_* \Sigma X \cong F_{*-1} X \cong (F_* X)[1],$$

which are part of the structure.

(ii) F_* is additive saying that it commutes with arbitrary coproducts.

(iii) F_* converts distinguished triangles into long exact sequences.

Remark 4.1.3 Later we will assume that $\mathcal{T}$ has an underlying model category $\mathcal{M}$. The suspension functor Σ here is internal to the model structure on $\mathcal{M}$. The reader should be aware that this has nothing to do with the external construction Σ_{ext} from 2.2.1 which is derived from the external simplicial structure on the cosimplicial objects $c\mathcal{M}$ over $\mathcal{M}$.

Definition 4.1.4 We say that $\boldsymbol{F : \mathcal{T} \to \mathcal{A}}$ **detects isomorphisms** or equivalently that $\boldsymbol{\mathcal{T}}$ **is** $\boldsymbol{F}$**-local** if a map $X \to Y$ in $\mathcal{T}$ is an isomorphism if and only if the induced map $F_*X \to F_*Y$ in $\mathcal{A}$ is an isomorphism.

Definition 4.1.5 Let $F_*:\mathcal{T} \to \mathcal{A}$ be a homological functor from a triangulated category to a graded abelian category and let I be an injective object in $\mathcal{A}$. Consider the following functor:

$$X \mapsto \operatorname{Hom}_{\mathcal{A}}(F_*X, I)$$

By Brown representability this functor is representable by an object $E(I)$ of $\mathcal{T}$. If the canonical morphism $F_*E(I) \to I$ induced by

$$\mathrm{id}_{E(I)} \in [E(I), E(I)] \cong \operatorname{Hom}_{\mathcal{A}}(F_*E(I), I)$$

is an isomorphism, then we call $E(I)$ an $\boldsymbol{(F, I)}$**-Eilenberg-MacLane object**. Usually we will just say that $E(I)$ is $\boldsymbol{F}$**-injective**.

Definition 4.1.6 We will say that the functor F_* **possesses enough injectives**, if every object in $\mathcal{T}$ admits a morphism to an F-injective object that induces a monomorphism in $\mathcal{A}$.

Remark 4.1.7 There are a lot of examples of functors with enough injectives that detect isomorphisms. E.g. it is proved in [Hov04] that every topologically flat ring spectrum E, where E_*E is commutative, induces a homological functor

$$E_* : Ho(\text{Spectra})_E \to E_*E - \text{comod}$$

from the E-local stable homotopy category to the category of E_*E-comodules, which possesses enough injectives. Note also that this functor detects isomorphism, since we localized at E. From this data we can construct a spectral sequence, which is known as the E-based modified Adams-spectral sequence.

Remark 4.1.8 For our functor F to possess enough injectives it is a necessary condition that the target category $\mathcal{A}$ possesses enough injectives. In the applications we have in mind this is always the case, see remark 4.1.7.

There are two convenient facts about finding F-injectives that are derived from [Fra96, 2.1.1.] or [Dev97, Thm. 1.5.]:

(i) If F detects isomorphisms, then every representing object X in $\mathcal{T}$, whose image $F_*X \cong I$ is injective in $\mathcal{A}$, is an (F, I)-Eilenberg-MacLane-object.

(ii) Retracts of F-injective objects are again F-injective.

Remark 4.1.9 If F_* is a homological functor that possesses enough injectives and detects isomorphism, then there is the following observation taken from [Fra96, 2.1. Lemma 1]: Given another representing object $\widetilde{E}(I)$, there is a unique morphism

$$\widetilde{E}(I) \to E(I)$$

in $\mathcal{T}$ lifting the identity of I, and this is an isomorphism. This can be reformulated in the following way: Let $\mathcal{T}_{F-\text{inj}}$ denote the full subcategory of $\mathcal{T}$ consisting of the F-injective objects. Let $\mathcal{A}_{\text{inj}}$ denote the full subcategory of $\mathcal{A}$ consisting of the injective objects. Then the functor F induces an equivalence $\mathcal{T}_{F-\text{inj}} \to \mathcal{A}_{\text{inj}}$.

4.2 The F-injective model structure

We will now apply the machinery from sections 2 and 3 to the realization problem for a homological functor F with enough injectives. We will construct model structures, called the F-injective and the n-F-injective model structure, which turn out to be very useful for our setting. We will list the important properties and characterizations of weak equivalences, fibrations and cofibrations in terms of F. Finally in lemma 4.2.32 we will prove a recognition principle of realizations of objects of the target category, and in 4.2.35 we will explain the connection to the modified Adams spectral sequence. For the F-structure the parallel results were already obtained in [BDG01] and [GH04], but the truncated notions are new. First we will recapitulate the assumptions on F that will be valid for the rest of the work.

Assumptions 4.2.1 From now on let $\mathcal{T}$ be the homotopy category of a simplicial left proper stable model category $\mathcal{M}$. Let $F_* : \mathcal{T} \to \mathcal{A}$ be a homological functor with enough F-injectives that detects isomorphisms in $\mathcal{T}$ as explained in 4.1.2, 4.1.4 and 4.1.6. We will call the composition $\mathcal{M} \to \mathcal{T} \to \mathcal{A}$ of F_* with the canonical functor from $\mathcal{M}$ to its homotopy category also F_*. By applying it levelwise we can prolong it to a functor $c\mathcal{M} \to c\mathcal{A}$ that we will again call F_*. Note that these assumptions mean that a map $X \to Y$ in $\mathcal{M}$ induces an isomorphism $F_*X \to F_*Y$ if and only if it was a weak equivalence.

Remember that we put in 3.4.3 a model structure on $c\mathcal{A}$. Now we want to set up a corresponding model structure on $c\mathcal{M}$. This will be the $\mathcal{G}$-model structure from 2.1.13 with the following choice of $\mathcal{G}$.

Definition 4.2.2 We take as our class of injective models $\mathcal{G}$ the class of all F-injective objects in $\mathcal{M}$ which were defined in 4.1.5 . This class $\mathcal{G}$ will be fixed for the rest of this work. We denote our special choice of the class of injective models in $\mathcal{M}$ by

$$\{F\text{-injectives}\} =: \boldsymbol{\{F\text{-Inj}\}}.$$

We will rename the classes of maps in definition 2.1.6 and call them ***F*-injective equivalences**, ***F*-injective fibrations** and **cofibrations**. Sometimes we will abbreviate even this and simply say F-equivalent or F-fibrant and so on. We will call this model structure on $c\mathcal{M}$ the ***F*-injective model structure**. The truncated model structure from 3.2.20 will be called ***n*-*F*-injective structure** or just n-F-structure.

Remark 4.2.3 Note that the choice of $\mathcal{G} = \{F\text{-injectives}\}$ is really admissible. First of all $\mathcal{M}$ is stable, so all objects are homotopy group objects. And next it is easy to check that this class is closed under (de-)suspensions and that there are enough F-injectives.

Definition 4.2.4 Definition 2.1.2 supplies the notions of a $\mathcal{G}$-monic map and a $\mathcal{G}$-injective object. A map will be called ***F*-monic** if it is $\mathcal{G}$-monic with $\mathcal{G} = \{F\text{-Inj}\}$.

Note that in the term "$\{F$-Inj$\}$-injective object" the inner "injective" refers to 4.1.5, while the outer "injective" refers to 2.1.2. Anyway this distinction is not too important by lemma 4.2.6.

Lemma 4.2.5 *A map is F-monic if and only if it induces a monomorphism under F.*

Proof: By definition an map $i : A \to B$ is F-monic if and only if it induces a surjection

$$[B, E] \to [A, E]$$

for all F-injective objects E. By 4.1.8(ii) this is equivalent to

$$\mathrm{Hom}_{\mathcal{A}}(F_*B, I) \to \mathrm{Hom}_{\mathcal{A}}(F_*A, I)$$

being a surjection for all injective objects I in $\mathcal{A}$. The fact follows now from lemma 3.4.1.

□

The class $\{F-\mathrm{Inj}\}$ is saturated in the following sense.

Lemma 4.2.6 *The two classes $\{F-\mathrm{Inj}\}$ and $\{\{F-\mathrm{Inj}\}$-injectives$\}$ coincide.*

Proof: We have to prove that an object E that has the right lifting property with respect to every F-monic cofibration is already F-injective as defined in 4.1.5. Because there are enough F-injective there exists an F-monic map $E \to I$ into an F-injective object. By the lifting property it follows that E is an retract of I and therefore F-injective by 4.1.8(ii).

□

As a consequence of theorems 2.1.13 and 3.2.20 we have the following model structures at hand.

Theorem 4.2.7 *Let $\mathcal{M}$ be a pointed simplicial left proper stable model category und set $Ho(\mathcal{M}) =: \mathcal{T}$ and let $\mathcal{A}$ be an abelian category. Let $F : \mathcal{T} \to \mathcal{A}$ be a homological functor that possesses enough injectives and that detects isomorphisms. On $c\mathcal{M}$ there is a pointed simplicial left proper model structure given by the (n-)F-injective equivalences, the (n-)F-injective cofibrations and the (n-)F-injective fibrations. The simplicial structure is always the external one.*

In fact $Ho(c\mathcal{M})^F$ behaves like the category of non-negative cochain complexes inside the full derived category of an abelian category with enough injectives. See the discussion between 4.2.19 and 4.2.25.

We will now list explicit characterizations of F-injective equivalences, F-injective cofibrations and F-injective fibrations. The next statements are both direct consequences of the cogenerating properties of the chosen classes of injective models. Here N^* always denotes the Dold-Kan-normalization functor $c\mathcal{A} \to \mathrm{CoCh}^{\geq 0}(\mathcal{A})$.

Lemma 4.2.8 *Fix one $s \geq 0$. For $X^\bullet \to Y^\bullet$ the following statements are equivalent:*

(i) *The induced map $\pi_s[Y^\bullet, G] \to \pi_s[X^\bullet, G]$ is an isomorphism for all F-injective G.*

(ii) *The induced map $\pi^s F_* X^\bullet \to \pi^s F_* Y^\bullet$ is an isomorphism.*

Proof: First of all we have the isomorphism

$$\pi_s[X^\bullet, G] \cong H^s N\mathrm{Hom}_{\mathcal{A}}(F_* X^\bullet, F_* G)$$

if G is F-injective. Then the lemma follows from 3.4.2 and the fact mentioned in 4.1.9, that if G runs through all F-injectives then F_*G ranges over all injectives in $\mathcal{A}$.

□

Corollary 4.2.9 *A map $X^\bullet \to Y^\bullet$ in $c\mathcal{M}$ is an F-injective equivalence if and only if for some F-fibrant approximation $\widetilde{X}^\bullet \to \widetilde{Y}^\bullet$ the induced maps*

$$H^s NF\widetilde{X}^\bullet \to H^s NF\widetilde{Y}^\bullet$$

are isomorphisms for all $s \geq 0$, in other words, if and only if it induces an $\mathcal{I}$-equivalence, which means that it induces a quasi-isomorphism $NF\widetilde{X}^\bullet \to NF\widetilde{Y}^\bullet$.

Proof: This follows readily from 4.2.8.

□

Lemma 4.2.10 *A map $i : A^\bullet \to B^\bullet$ is an F-injective cofibration if and only if it is a Reedy-cofibration and if it induces monomorphisms*

$$N^k F A^\bullet \to N^k F B^\bullet$$

for all $k \geq 1$.

Proof: The map i is an F-cofibration if and only if the induced map

$$[B^\bullet, G] \to [A^\bullet, G]$$

is a fibration of simplicial sets for all $G \in \mathcal{G}$. The result now follows from 3.4.1, 3.4.2 and 4.1.9 and the fact that for simplicial abelian groups a map is a fibration if and only if it induces a surjection of the normalizations in positive degrees.

□

Remark 4.2.11 There is no obvious way to characterize n-F-equivalences in terms of $\pi^s F_*(\_)$. By the dualization procedure from 3.4.2 these groups correspond to the terms $\pi_s[\_, G]$ in the spiral exact sequence from 2.2.6. After all, the n-F-equivalences were defined in terms of the groups $\pi_s^\natural(\_, G)$ and the induction used to prove 2.2.7 crawling up the spiral exact sequence does not yield anything useful if it stops at some finite stage. Still, of course, we have the consequence of 3.2.14 that a map $X^\bullet \to Y^\bullet$ between n-F-cofibrant objects is an n-F-equivalence if and only if it is an F-equivalence, which is equivalent to inducing isomorphisms on $\pi^s F_*(\_)$ for all $s \geq 0$.

The following result will not be needed later.

Lemma 4.2.12 *Let $A^\bullet \to B^\bullet$ be an F-cofibration with cofiber $C^\bullet$ that induces a monomorphism $N^0 F_* A^\bullet \to N^0 F_* B^\bullet$. Then there is a long exact sequence*

$$0 \to H^0 N F_* A^\bullet \to H^0 N F_* B^\bullet \to H^0 N F_* C^\bullet \to H^1 N F_* A^\bullet \to \ldots$$
$$\ldots \to H^s N F_* A^\bullet \to H^s N F_* B^\bullet \to H^s N F_* C^\bullet \to H^{s+1} N F_* A^\bullet \to \ldots$$

Proof: This can be proved by 4.2.10.

□

We need a little bit more care to describe F-injective fibrations. First of all we remind the reader that F-injective fibrations and n-F-injective fibrations coincide by lemma 3.2.6. By definition a map $X^\bullet \to Y^\bullet$ is an F-injective fibration if and only if all the maps $X^s \to M^s X^\bullet \times_{M^s Y^\bullet} Y^s$ for $s \geq 0$ are $\mathcal{G}$-injective fibrations in $\mathcal{M}$ in the sense of definition 2.1.3 with $\mathcal{G} = \{F\text{-Inj}\}$. Thus we characterize $\{F\text{-Inj}\}$-injective fibrations in $\mathcal{M}$.

Lemma 4.2.13 *A map in $\mathcal{M}$ is an $\{F\text{-Inj}\}$-injective fibration if and only if it is a fibration, whose fiber is F-injective and that induces an epimorphism under F.*

Proof: By [Bou03, 3.10.] a map $X \to Y$ in $\mathcal{M}$ is a $\mathcal{G}$-injective fibration if and only if it is a retract of a $\mathcal{G}$-cofree map $X' \to Y'$. A $\mathcal{G}$-cofree map is a map that can be expressed as a composition $X' \to Y' \times E \to Y'$, where $X' \to Y' \times E$ is a trivial fibration in $\mathcal{M}$, $Y' \times E \to Y'$ is the projection onto Y' and E is $\mathcal{G}$-injective.

The assertion is true for $\{F\text{-Inj}\}$-cofree maps. Here we use the fact that $\mathcal{T}$ is F-local (see 4.1.4), so weak equivalences in $\mathcal{M}$ induce isomorphisms under F. But the claim is also true for retracts. This is obvious for surjectivity. The fiber condition follows from 4.1.8, since the fiber of $X \to Y$ is a retract of the fiber of $X' \to Y'$ which is weakly equivalent to E and therefore itself F-injective.

Conversely let $X \to Y$ be a fibration that has an F-injective fiber E and that induces a surjection under F. $\mathcal{M}$ is stable, hence we get a long exact F-sequence for $X \to Y$ and it follows $X \simeq E \times Y$. We deduce that $X \to Y$ has the right lifting property with respect to every $\{F\text{-Inj}\}$-monic cofibration. So it is an $\{F\text{-Inj}\}$-injective fibration.

□

Definition 4.2.14 Let $\mathcal{N}$ be a simplicial model category. Let $(\Delta^s_k)_+$ be the category with objects

$$s \twoheadrightarrow l, l < s, l \neq k,$$

where $s \twoheadrightarrow l$ is a surjection in Δ and with morphisms given by the morphisms in Δ under s. For an object $X^\bullet$ in $c\mathcal{N}$ define the functor $F_{X^\bullet} : (\Delta^s_k)_+ \to \mathcal{N}$ by sending $s \twoheadrightarrow l$ to X^l. We define its **partial matching object $M^s_k X^\bullet$** whenever $0 \leq k \leq s-1$ by

$$M^s_k X^\bullet := \lim_{(\Delta^s_k)_+} F_{X^\bullet}.$$

We also set $M^s_s X^\bullet := M^s X^\bullet$.

Let $X^\bullet \to Y^\bullet$ be an F-fibration. Then $X^s \to M^s X^\bullet \times_{M^s Y^\bullet} Y^s$ is an F-injective fibration in $\mathcal{M}$ for all $s \geq 0$. As mentioned in [Bou03, 5.3] it follows from [GJ99, VII.2.6.] that $X^s \to Y^s$ is an F-injective fibration for all $s \geq 0$, too. But it also follows that for an F-fibrant $X^\bullet$ the maps $X^s \to M^s X^\bullet$ for all $s \geq 0$ and $X^s \to M^s_k X^\bullet$ for all $s \geq 0$ and $0 \leq k \leq s$ are F-injective fibrations in $\mathcal{M}$. Consider the following pullback square

$$\begin{array}{ccc} M^s_{k+1} X^\bullet & \longrightarrow & X^{s-1} \\ \downarrow & & \downarrow \\ M^s_k & \longrightarrow & M^{s-1}_k X^\bullet \end{array}$$

which is taken from [GJ99, p. 392]. We remind the reader of the index shift explained in 1.2.5. For F-fibrant $X^\bullet$ this square is also a homotopy pullback square and by the stability of $\mathcal{M}$ we obtain a long exact F-sequence, which collapses to the short exact sequence

$$0 \to F_* M^s_{k+1} X^\bullet \to F_* X^{s-1} \oplus F_* M^s_k X^\bullet \to F_* M^{s-1}_k X^\bullet \to 0, \tag{4.3}$$

since $F_* X^{s-1} \to F_* M^{s-1}_k X^\bullet$ is surjective by 4.2.13. This is the argument needed to make the inductive step in the proof of [GJ99, VIII.1.8.] work and we can translate it to show the following result. Recall that N^s was defined in 1.3.11.

Lemma 4.2.15 *For an F-fibrant $X^\bullet$ and all $s \geq 0$ we have a natural isomorphism*

$$F_* N^s X^\bullet \cong N^s F_* X^\bullet.$$

Proof: Insert F_* for π_t in the proof of [GJ99, VIII.1.8., p. 392].

□

Corollary 4.2.16 *A Reedy fibration $X^\bullet \to Y^\bullet$ between F-fibrant objects is an F-fibration if and only if it induces a fibration $F_* X^\bullet \to F_* Y^\bullet$ in the $\mathcal{I}$-structure of $c\mathcal{A}$.*

Proof: First of all we observe that for an F-fibrant $Y^\bullet$ and all $s \geq 0$ we have the isomorphism

$$F_*(Y^s \times_{M^sY^\bullet} M^sX^\bullet) \cong F_*Y^s \times_{F_*M^sY^\bullet} F_*M^sX^\bullet =: P^s.$$

This follows like in (4.3) because the pullback square

$$\begin{array}{ccc} Y^s \times_{M^sY^\bullet} M^sX^\bullet & \longrightarrow & M^sX^\bullet \\ \downarrow & & \downarrow \\ Y^s & \longrightarrow & M^sY^\bullet \end{array}$$

has homotopy meaning in $\mathcal{M}$ and $F_*Y^s \to F_*M^sY^\bullet$ is surjective. We look at the following diagram that exists by the previous isomorphism and that has exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_*N^sX^\bullet & \longrightarrow & F_*X^s & \longrightarrow & F_*M^sX^\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & F_*N^sY^\bullet & \longrightarrow & P^s & \longrightarrow & F_*M^sX^\bullet & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_*N^sY^\bullet & \longrightarrow & F_*Y^s & \longrightarrow & F_*M^sY^\bullet & \longrightarrow & 0 \end{array} \tag{4.4}$$

Note that the kernel of $F_*X^s \to P^s$ is isomorphic to the kernel of $F_*N^sX^\bullet \to F_*N^sY^\bullet$. Now let $X^\bullet \to Y^\bullet$ be an F-fibration between F-fibrant objects. Then $F_*X^s \to P^s$ is surjective with injective kernel. Hence $F_*N^sX^\bullet \to F_*N^sY^\bullet$ is surjective with injective kernel. Using 4.2.15 we deduce that $F_*X^\bullet \to F_*Y^\bullet$ is an $\mathcal{I}$-fibration in $c\mathcal{A}$. Conversely, if $F_*X^\bullet \to F_*Y^\bullet$ is an $\mathcal{I}$-fibration then $N^sF_*X^\bullet \to N^sF_*Y^\bullet$ is surjective with injective kernel. Using 4.2.15 and (4.4) we see that all maps $F_*X^s \to F_*(M^sX^\bullet \times_{M^sY^\bullet} Y^s)$ are surjective with injective kernel I^s. It also follows that the fiber R^s of $X^s \to M^sX^\bullet \times_{M^sY^\bullet} Y^s$ satisfies $F_*R^s \cong I^s$ by the long, or better short exact F-sequence. Hence R^s is F-injective by 4.1.8(i). This shows that for $s \geq 0$ the maps $X^s \to M^sX^\bullet \times_{M^sY^\bullet} Y^s$ are F-injective fibrations by their characterization in 4.2.13.

□

Lemma 4.2.17 *The functor $F_* : c\mathcal{M} \to c\mathcal{A}$ maps F-homotopy pullbacks to $\mathcal{I}$-homotopy pullbacks.*

Proof: It is sufficient to prove that F_* preserves pullbacks after F-fibrant replacement. Let $X^\bullet \to Z^\bullet \leftarrow Y^\bullet$ be F-fibrations between F-fibrant objects. It follows by [Bou03, 5.3.] that all maps $X^s \to Z^s \leftarrow Y^s$ are F-injective fibrations in $\mathcal{M}$, in particular they are fibrations and induce surjection under F_*. The pullback square

$$\begin{array}{ccc} X^s \times_{Z^s} Y^s & \longrightarrow & X^s \\ \downarrow & & \downarrow \\ Y^s & \longrightarrow & Z^s \end{array}$$

is also a homotopy pullback square in $\mathcal{M}$, and hence a homotopy pushout. The long exact F-sequence resulting from this collapses to the short exact sequences

$$0 \to F_*(X^s \times_{Z^s} Y^s) \to F_*X^s \oplus F_*Y^s \to F_*Z^s \to 0,$$

which proves the lemma.

□

Lemma 4.2.18 *An object $X^\bullet$ is F-fibrant if and only if it is Reedy fibrant such that for all $s \geq 0$ the objects X^s and $N^s X^\bullet$ in $\mathcal{M}$ are fibrant and F-injective and the map $F_* N^s X^\bullet \to F_* X^s$ is a monomorphism.*

Proof: Follows from 4.2.16 and its proof.

□

The category $c\mathcal{M}$ equipped with the F-structure behaves very much like the full subcategory $\mathrm{CoCh}^{\geq 0}(\mathcal{A})$ of nonnegative cochain complexes within the derived category $D(\mathcal{A})$ does. This is displayed by the statements 4.2.24 and 4.2.25. We are going to need a dual version of the functor $\overline{W} : s\mathrm{Ab} \to s\mathrm{Ab}$ which is sometimes called the Eilenberg-MacLane-functor or the Kan-suspension.

Definition 4.2.19 Let $\mathcal{N}$ be a pointed model category. We define a functor $W : c\mathcal{N} \to c\mathcal{N}$. Let $X^\bullet$ be a cosimplicial object. Let $\boldsymbol{WX^\bullet}$ be defined by the following equations:

$$(WX^\bullet)^s := \prod_{i=0}^{s} X^i$$

The structural maps of a cosimplicial object are constructed by the process dual to the one described in [GJ99, p. 192]. There is a map $WX^\bullet \to X^\bullet$ given by projection

$$\prod_{i=0}^{s} X^i \to X^s.$$

Let $\boldsymbol{\overline{W}X^\bullet}$ be the fiber of $WX^\bullet \to X^\bullet$.

Remark 4.2.20 Let $X^\bullet$ be in $c\mathcal{M}$. The map $WX^\bullet \to X^\bullet$ is a Reedy-fibration if and only if every $X^\bullet$ is Reedy fibrant. It is a $\mathcal{G}$-fibration for some general $\mathcal{G}$ if and only if in addition all X^s are $\mathcal{G}$-injective. In both cases $\overline{W}X^\bullet$ has homotopy meaning, see 4.2.22.

Lemma 4.2.21 *If we take $\mathcal{G} = \{F\mathrm{-Inj}\}$ then $WX^\bullet$ is F-equivalent to $*$.*

Proof: Since $FWX^\bullet \cong WFX^\bullet$ it suffices by 4.2.9 to show that $WA^\bullet$ is $\mathcal{I}$-equivalent to $*$ for arbitrary $A^\bullet$ in $c\mathcal{A}$. This follows by dualizing [GJ99, III.5.].

□

Remark 4.2.22 Hence $\overline{W}X^\bullet$ is a model for the loop object $\Omega_{\mathrm{ext}} X^\bullet$, but the model differs from the one obtained by the simplicial structure in 2.2.2 on point set level. Anyway they are weakly equivalent. If $A^\bullet$ is in $c\mathcal{A}$ this object can also be obtained in the following way:

$$\begin{array}{ccc} c\mathcal{A} & \xrightarrow{\overline{W}} & c\mathcal{A} \\ {\scriptstyle N}\downarrow & & \uparrow{\scriptstyle \Gamma} \\ \mathrm{CoCh}^{\geq 0}(\mathcal{A}) & \xrightarrow[{[1]_{\mathrm{ext}}}]{} & \mathrm{CoCh}^{\geq 0}(\mathcal{A}) \end{array}$$

where $(A^*[1]_{\mathrm{ext}})^s = A^{s+1}$ is the external shift functor of cochain complexes (which should not be confused with the internal shift $[1]$ from 4.1.1), N is normalization and Γ is the Dold-Kan-functor. In particular if $A^\bullet$ is in $c\mathcal{A}$ we have:

$$H^s N\overline{W}A^\bullet = \begin{cases} 0 & , \text{ for } s = 0 \\ H^{s-1} N A^\bullet & , \text{ for } s \geq 1 \end{cases} \tag{4.5}$$

For every F-fibrant $X^\bullet$ we get a map

$$\Sigma_{\mathrm{ext}}\overline{W}X^\bullet \to X^\bullet \tag{4.6}$$

in $c\mathcal{M}$ which descends to a natural transformation $\Sigma_{\mathrm{ext}}\Omega_{\mathrm{ext}} \to \mathrm{Id}$ of endofunctors of $Ho(c\mathcal{M}^{\mathcal{G}})$.

Lemma 4.2.23 *For every F-fibrant object $X^\bullet$ in $c\mathcal{M}$ the map $\Sigma_{\mathrm{ext}}\overline{W}X^\bullet \to X^\bullet$ is an F-equivalence.*

Proof: We note that $F\Sigma_{\mathrm{ext}}\overline{W}X^\bullet = \Sigma_{\mathrm{ext}}\overline{W}FX^\bullet$ because F is applied levelwise and commutes with finite products. Now the fact follows from 4.2.9, 2.2.5 and (4.5).

□

Corollary 4.2.24 *The map (4.6) induces a natural equivalence $\Sigma_{\mathrm{ext}}\Omega_{\mathrm{ext}} \cong \mathrm{Id}$ of endofunctors of $Ho(c\mathcal{M}^F)$.*

Proof: This is just the recollection of what was proved in 4.2.20, 4.2.22 and 4.2.23.

□

Furthermore we have an isomorphism $\Omega_{\mathrm{ext}}\Sigma_{\mathrm{ext}}X^\bullet \cong X^\bullet$ in $Ho(c\mathcal{M}^{\mathcal{G}})$ as long as the objects in question are "connected".

Lemma 4.2.25 *Let $X^\bullet$ be an F-fibrant object such that $\pi^0F_*X^\bullet = 0$. Then the canonical map $X^\bullet \to \overline{W}\Sigma_{\mathrm{ext}}X^\bullet$ is an F-equivalence.*

Proof: The condition $\pi^0F_*X^\bullet = 0$ is equivalent to $0 = \pi_0[X^\bullet, G] \cong \pi_0^\natural(X^\bullet, G)$ by 3.4.2 and 2.2.6. Hence the map $X^\bullet \to \overline{W}\Sigma_{\mathrm{ext}}X^\bullet$ induces isomorphisms on $H^sNF_*(\_)$ for all $s \geq 0$, so it is an F-equivalence.

□

Remark 4.2.26 In a stable model category $\mathcal{M}$ finite products and finite coproducts are weakly equivalent. It follows that for the Reedy structure and in particular for every $\mathcal{G}$-structure on $c\mathcal{M}$ and their truncated versions finite products and coproducts are weakly equivalent.

Corollary 4.2.27 *For every $0 \leq n \leq \infty$ the category $Ho(c\mathcal{M}^{n-\mathcal{G}})$ is additive and the functors $\sigma_n : Ho(c\mathcal{M}^{(n+1)-F}) \to Ho(c\mathcal{M}^{n-F})$ and $\theta_n : \mathcal{T} \to Ho(c\mathcal{M}^{n-F})$ are additive.*

Proof: By 4.2.24 every object in $Ho(c\mathcal{M}^{n-F})$ for $0 \leq n \leq \infty$ is isomorphic to a double suspension, hence every object is an abelian cogroup object in the homotopy category. The restriction functor σ_n is induced by the functor $\mathrm{sk}_{n+1} : c\mathcal{M} \to c\mathcal{M}$. Colimits commute with each other, so for a Reedy cofibrant $X^\bullet$ we have:

$$\Sigma_{\mathrm{ext}}\,\mathrm{sk}_{n+1}\,X^\bullet \simeq \mathrm{sk}_{n+1}\,\Sigma_{\mathrm{ext}}X^\bullet$$

This proves the additivity on homotopy level. The argument for θ_n is similar.

□

Definition 4.2.28 We will denote the biproduct of a pair of objects $X^\bullet$ and $Y^\bullet$ in $Ho(c\mathcal{M}^{n-F})$ for $0 \leq n \leq \infty$ by $\boldsymbol{X^\bullet \oplus Y^\bullet}$.

Remark 4.2.29 We can construct fold and diagonal maps for the biproducts $X^\bullet \oplus X^\bullet$ if $X^\bullet$ is fibrant and cofibrant. Their homotopy class is uniquely determined.

Finally we have to recognize the total object of special F-fibrant objects. We will use this in 5.5.2 and 5.5.3 to show that we have found realizations of an object in $\mathcal{A}$.

Definition 4.2.30 For an object $Y^\bullet$ in $c\mathcal{M}$ let $\mathbf{Fib}_s\boldsymbol{Y}^\bullet$ denote the fiber of $\operatorname{Tot}_s Y^\bullet \to \operatorname{Tot}_{s-1} Y^\bullet$.

Remark 4.2.31 In 2.2.6 we mentioned that the spiral exact sequence can be spliced together to an exact couple giving the spectral sequence (2.1):

$$\pi_p[X^\bullet, \Omega^q G] \implies \pi_0^\natural(X^\bullet, \Omega^{p+q}G)$$

Like in [GH04, 3.9] it should follow that there is an isomorphism of the target group of this spectral sequence and $[\operatorname{Tot} X^\bullet, \Omega^{p+q}G]$, but we have not checked that. We consider another spectral sequence which conjecturally is isomorphic to the first one and which is obtained by considering the G-cohomology spectral sequence of the total tower $\{\operatorname{Tot}_s Y^\bullet\}$ for every $G \in \mathcal{G}$. Its E_1-term consists of

$$E_1^s = G^*\mathrm{Fib}_s Y^\bullet = [\mathrm{Fib}_s Y^\bullet, G]_*.$$

Since $Y^\bullet$ is Reedy-fibrant, there is an isomorphism $\mathrm{Fib}_s Y^\bullet \cong \Omega^s N^s Y^\bullet$ by 1.3.12, where $N^s Y^\bullet := \mathrm{fiber}(Y^s \to M^s Y^\bullet)$ is the geometric normalization of $Y^\bullet$. Moreover it is true that there is an isomorphism

$$G^*(\mathrm{Fib}_s Y^\bullet) = G^*(\Omega^s N^s Y^\bullet) \cong N^s(G^{*+s} Y^\bullet),$$

where on the right hand side N^s denotes the normalization of complexes. Also the spectral sequence differential $d_1 : G^*(\mathrm{Fib}_{s+1} Y^\bullet) \to G^*(\mathrm{Fib}_s Y^\bullet)$ coincides up to sign with the boundary of the normalized cochain complex $N^\bullet(G^* Y^\bullet)$. Hence:

$$E_2^s = \pi_s[Y^\bullet, G]$$

From [Boa99, theorem 6.1(a)] it follows that this spectral sequences converges strongly to $\operatorname{colim}_s [\operatorname{Tot}_s X^\bullet, G]$.

Theorem 4.2.32 *Let $Y^\bullet$ be an F-fibrant object with the property that $\pi_s[Y^\bullet, G] = 0$ for all $s > 0$ and every $G \in F\text{-Inj}$. Then there is an isomorphism*

$$\mathrm{Hom}_{\mathcal{A}}(\pi^0 F_* Y^\bullet, F_* G) \cong [\operatorname{Tot} Y^\bullet, G]$$

for every $G \in F\text{-Inj}$. In particular there is a natural isomorphism

$$F_* \operatorname{Tot} Y^\bullet \cong \pi^0 F_* Y^\bullet.$$

Proof: Let G be an F-injective object. By assumption the G^*-spectral sequence of $Y^\bullet$ collapses with vanishing E_2-term except for $\pi^0[Y^\bullet, G]$ in degree 0. We claim further that there is an isomorphism

$$\operatorname*{colim}_s [\operatorname{Tot}_s Y^\bullet, G] \cong [\operatorname{Tot} Y^\bullet, G].$$

The edge homomorphism of the spectral sequence yields the isomorphism we are looking for, once we have shown our claim. To prove it we define another tower $\{\widetilde{Y}^s\}$. We let $\widetilde{Y}^s$ be the fiber of $\operatorname{Tot} Y^\bullet \to \operatorname{Tot}_{s-1} Y^\bullet$ for $s \geq 1$ and $\widetilde{Y}^0 = \operatorname{Tot} Y^\bullet$. If we apply the functor $[\_, G]$ and then pass to the colimit we obtain the following long exact sequence:

$$\ldots \to \operatorname*{colim}_s [\widetilde{Y}^s, G]_* \to \operatorname*{colim}_s [\operatorname{Tot}_s Y^\bullet, G]_* \to [\operatorname{Tot} Y^\bullet, G]_* \to \operatorname*{colim}_s [\widetilde{Y}^s, G]_{*-1} \to \ldots$$

We see that the claim follows if we are able to prove that the maps $[\widetilde{Y}^s, G] \to [\widetilde{Y}^{s+1}, G]$ are zero. Going back to the definition of $\widetilde{Y}^s$ it follows that there is a homotopy fiber sequence

$$\widetilde{Y}^{s+1} \to \widetilde{Y}^s \to \mathrm{Fib}_s Y^\bullet,$$

and we deduce that we can equally prove that the map $[\mathrm{Fib}_s Y^\bullet, G] \to [\widetilde{Y}^s, G]$ is surjective for every $G \in F\text{-Inj}$. This is equivalent to the condition that $F_*\widetilde{Y}^s \to F_*\mathrm{Fib}_s Y^\bullet$ is injective. For $s = 0$ this is obvious because here this map is just the inclusion of $\pi^0 F_* \operatorname{Tot} Y^\bullet$ into $F_* Y^0$. From $\pi^s F_* Y^\bullet = 0$ for $s \geq 1$ it follows that the sequence

$$F_*Y^0 = F_*\mathrm{Fib}_0 Y^\bullet \to F_*\mathrm{Fib}_1 Y^\bullet \to ... \to F_*\mathrm{Fib}_{s-1} Y^\bullet \to F_*\mathrm{Fib}_s Y^\bullet \to ...$$

is exact since it is isomorphic to the normalized complex $NF_*Y^\bullet$ by 1.3.12. The objects $\widetilde{Y}^s$ correspond to the syzygies of this resolution. So we see inductively

$$F_*\widetilde{Y}^s \cong \operatorname{coker}\,[F_*\mathrm{Fib}_{s-2} Y^\bullet \to F_*\mathrm{Fib}_{s-1} Y^\bullet] \cong \ker\,[F_*\mathrm{Fib}_s Y^\bullet \to F_*\mathrm{Fib}_{s+1} Y^\bullet],$$

which proves that $F_*\widetilde{Y}^s \to F_*\mathrm{Fib}_s Y^\bullet$ is injective.

□

Corollary 4.2.33 *For every Y in $\mathcal{M}$ let $r^0 Y \to Y^\bullet$ be an F-fibrant approximation. Then the canonical map $Y \to \operatorname{Tot} Y^\bullet$ is an isomorphism in $\mathcal{T}$.*

Proof: We immediately derive this result from 4.2.32.

□

Remark 4.2.34 Let $Y^\bullet$ be an F-fibrant object. If we apply F_* to the total tower of $Y^\bullet$ we obtain a spectral sequence whose first term is given by

$$E_1^{s,t} = F_{t-s}\Omega^s N^s X^\bullet \cong F_t N^s X^\bullet$$

using 1.3.12. Since $Y^\bullet$ is F-fibrant it follows by 4.2.15:

$$F_* N^s Y^\bullet \cong \ker\,[F_* Y^s \to F_* M^s Y^\bullet] \cong N^s F_* Y^\bullet$$

Hence we get a spectral sequence

$$E_2^{s,t} = \pi^s F_t X^\bullet \implies \lim_k F_{t-s} \operatorname{Tot}_k X^\bullet,$$

which converges strongly by [Boa99, 7.4] if ${\lim^1}_r E_r^{*,*} = 0$. The differentials go like

$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}.$$

In the situation of 4.2.32 this spectral sequence equally collapses at the E_2-term and with 4.2.32 there is an isomorphism $\lim_k F_* \operatorname{Tot}_k X^\bullet \cong F_* \operatorname{Tot} X^\bullet$. Again the edge homomorphism gives $F_* \operatorname{Tot} Y^\bullet \cong \pi^0 F_* Y^\bullet$.

Remark 4.2.35 Finally we remark that we get back to the modified Adams spectral sequence if we apply the functor $[X, \_]$ to the total tower of an F-fibrant approximation $Y^\bullet$ of an object Y from $\mathcal{T}$. The modified Adams spectral sequence is constructed in the same way as the original Adams spectral sequence, but it uses absolute injective resolutions instead of relative ones. It was introduced in [Bri68]. Other accounts are given in [Bou85], [Dev97] and [Fra96]. The E_1-term is given by

$$E_1^{s,t} = [X, N^s Y^\bullet]_t.$$

Since $Y^\bullet$ is an F-fibrant approximation to r^0Y it follows that the E_2-term takes the following form

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(F_*X, F_*Y),$$

which is independent of the choice of the F-fibrant approximation and functorial in X and Y. The differentials are maps

$$d_r : E_r^{s,t} \to E_r^{s+r,t+r-1}.$$

For convergence results we have to take the usual precautions. It is shown in [Bie02] that it converges strongly to $[X, F^\wedge Y]$ if $\lim^1{}_r E_r^{*,*} = 0$. Here F is a topologically flat ring spectrum with F_*F commutative (see [Hov04]) and $F^\wedge Y$ is the F-nilpotent completion of Y.

5 Interpolation categories for homological functors

In this chapter we are finally able to put the pieces together and construct a tower of interpolation categories for a homological functor with enough injectives. In paragraph 5.1 we give the definition of the interpolation categories. In paragraph 5.2 an axiomatic description taken from [Bau99, VI.5] is cited. In 5.3 we follow [BDG01] and [GH04] and develop an obstruction calculus for the realization problem of F. Summarizing this in 5.4.1 shows that we have constructed a tower of interpolation categories in the sense of 5.2.4. Here in paragraph 5.4 we state other properties to convince the reader that the interpolation categories really deserve their name. In paragraph 5.5 we reprove the analogous statements about moduli spaces from [BDG01], but with considerably less effort. The whole moduli space problem is reduced to the black box theorem 1.4.7.

5.1 The tower of interpolation categories

We are now heading towards the fact that the tower of truncated homotopy category in 3.3.1 , where we plug in the n-F-structures from theorem 4.2.7, supply a tower of interpolation categories for the functor F.

Recall that $F : \mathcal{T} \to \mathcal{A}$ is a homological functor with enough F-injectives and $\mathcal{T} = Ho(\mathcal{M})$, where $\mathcal{M}$ is a pointed left proper simplicial stable model category. In 4.2.2 we took $\mathcal{G}$ to be the class F-Inj of F-injective objects in $\mathcal{M}$ and $\mathcal{I}$ to be the class of injective objects in $\mathcal{A}$.

Definition 5.1.1 Let $n \geq 0$. Let $IM_n(F)$ be the full subcategory of $c\mathcal{M}$ that consists of those objects $X^\bullet$, such that $F_*X^\bullet$ is $(n+1)$-$\mathcal{I}$-equivalent to a constant object in $c\mathcal{A}$. We call this category the ***n*-th interpolation model of *F***.

Let $IP_n(F)$ be the image of $IM_n(F)$ in $Ho(c\mathcal{M}^{n-\mathcal{G}})$. We call this category the ***n*-th interpolation category of the functor *F***.

Remark 5.1.2 Note that the $(n+1)$ in the definition of the n-th interpolation model is not a misprint. Let $X^\bullet$ be an object in $IP_n(F)$. We assume without loss of generality that it is n-F-cofibrant. Then we know that $F_*X^\bullet$ is $(n+1)$-$\mathcal{I}$-equivalent to $*$, which is equivalent to $\pi_s[X^\bullet, G] = 0$ for $1 \leq s \leq n+1$ and $\pi_s^\natural(X^\bullet, G) = 0$ for $s \geq n+1$ and all F-injective G respectively. From the spiral exact sequence it follows for all $G \in \{F\text{-Inj}\}$ that

$$\pi_s^\natural(X^\bullet, \Omega G) \cong \pi_{s+1}^\natural(X^\bullet, G) \text{ for } 0 \leq s \leq n$$

where
$$\pi_0^\natural(X^\bullet, G) \cong \pi_0[X^\bullet, G] \cong \mathrm{Hom}_{\mathcal{A}}(\pi^0 F_* X^\bullet, G)$$
and
$$\pi_n^\natural(X^\bullet, \Omega G) \cong \pi_{n+2}[X^\bullet, G],$$
and all other groups vanish, which is exactly what we want for a potential n-stage to be defined in 5.3.1.

The notion of equivalence in $Ho(c\mathcal{M}^{0-\mathcal{G}})$ is rather coarse, hence a lot of objects become identified. As n grows, fewer and fewer objects qualify for $IP_n(F)$, while the equivalences get finer and finer.

Remark 5.1.3 From 3.3.1 we get a tower of truncated homotopy categories and we can restrict the functors σ_n defined in 3.3.1 and 3.3.4 to our interpolation categories $IP_n(F)$. Observe also that $\theta_n X$ for some X in $\mathcal{T}$ lands in $IP_n(F)$ for each $n \geq 0$.

There is also an additional functor $\pi^0 F_* \cong H^0 N F_* : IP_n(F) \to \mathcal{A}$, which is derived from the functor $c\mathcal{M} \to \mathcal{A}$ given by
$$X^\bullet \mapsto \pi^0 F_* X^\bullet.$$

We arrive at the following tower:

$$\begin{array}{ccccccccc} \mathcal{T} & & & \xrightarrow{\quad F \quad} & & & & & \mathcal{A} \\ \downarrow & & & & & & & & \uparrow{\scriptstyle H^0NF=\pi^0F_*} \\ IP_n(F) & \longrightarrow & IP_{n-1}(F) & \longrightarrow & \cdots & \longrightarrow & IP_1(F) & \longrightarrow & IP_0(F) \end{array} \tag{5.1}$$

As mentioned in 3.3.4 this diagram commutes in the 2-category of categories. The commutativity relations are provided by the equation $\mathrm{sk}_n \, \mathrm{sk}_{n+1} = \mathrm{sk}_n$ and the fact that n-F-equivalences always induce isomorphisms on $\pi^0 F_* X^\bullet$ for each $n \geq 0$.

5.2 Extension of categories

The definitions in this paragraph are taken from [Bau99].

Definition 5.2.1 Let $\mathcal{C}$ be a category. Let $\mathrm{Fac}\,\mathcal{C}$ be the **category of factorizations** of $\mathcal{C}$. It is the Grothendieck construction on $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$ with respect to the functor $\mathrm{Hom}_{\mathcal{C}}(\,\_\,,\,\_\,)$. Explicitly it has the morphisms of $\mathcal{C}$ as objects, and a morphism $f \to g$ is given by a commutative diagram:

$$\begin{array}{ccc} \cdot & \longleftarrow & \cdot \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ \cdot & \longrightarrow & \cdot \end{array}$$

Definition 5.2.2 A **natural system of abelian groups** on a category $\mathcal{C}$ is a functor from $\mathrm{Fac}\,\mathcal{C}$ to the category Ab of abelian groups.

Remark 5.2.3 There is a canonical functor $\mathrm{Fac}\,\mathcal{C} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, sending a morphism to its source and target. Hence each bifunctor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Ab}$ induces a natural system on $\mathcal{C}$.

The following definition is taken from [Bau99, VI(5.4)], where we have included a freeness assumption in (i).

Definition 5.2.4 Let $\sigma : \mathcal{C} \to \mathcal{D}$ be a functor, and let G and H be natural systems on $\mathcal{D}$. We call the diagram

$$G \stackrel{\gamma}{\to} \mathcal{C} \stackrel{\sigma}{\to} \mathcal{D} \stackrel{\text{ob}}{\to} H$$

an **exact sequence of categories**, if the following conditions are satisfied:

(i) For all objects A and B in $\mathcal{C}$ and for each morphism $f \in \text{Hom}_{\mathcal{D}}(\sigma(A), \sigma(B))$ there is a free and transitive action γ of the group $G(f)$ on the set $\sigma^{-1}(f) \subset \text{Hom}_{\mathcal{C}}(A, B)$. We will write $\alpha f := \gamma(\alpha, f)$. This action satisfies the **linear distributivity law**:

$$(\alpha \widetilde{f})(\beta \widetilde{g}) = (f_* \alpha + g^* \beta)(\widetilde{f}\widetilde{g})$$

for all $\widetilde{f} \in p^{-1}(f), \widetilde{g} \in p^{-1}(g), \alpha \in G(f)$ and $\beta \in G(g)$.

(ii) For all objects A and B in $\mathcal{C}$ and all morphisms $f : \sigma(A) \to \sigma(B)$ in $\mathcal{D}$ there is an obstruction element $\text{ob}(f) \in H(f)$ given, such that

$$\text{ob}(f) = 0$$

if and only if there exists a morphism $\widetilde{f} : A \to B$ with $\sigma(\widetilde{f}) = f$.

(iii) For all $f : \sigma(A) \to \sigma(B)$ and $g : \sigma(B) \to \sigma(C)$ we have the following equation:

$$\text{ob}(gf) = g_* \text{ob}(f) + f^* \text{ob}(g)$$

(iv) For all objects A in $\mathcal{C}$ and for all $\alpha \in H(\text{id}_{\sigma(A)})$ there is an object Y in $\mathcal{C}$ with the property that $\sigma(A) = \sigma(Y)$ and $\text{ob}(\text{id}_{\sigma(A)}) = \alpha$.

5.3 Realizations and obstructions calculus

In this subsection we develop an obstruction calculus for realizing objects and morphism along a homological functor F_* with enough injectives. In general we follow [BDG01] and [GH04]. See also [Bau99]. Since we are in a completely linear or stable situation the theory required to set up the obstruction calculus simplifies compared to the other settings. Nevertheless the simplifications in paragraph 5.5 compared to [BDG01] result from the use of truncated resolution model structures.

Our task was to look out for realizations in $Ho(\mathcal{M}) = \mathcal{T}$ of objects in $\mathcal{A}$. To motivate our next definition, let X be an object in $\mathcal{M}$ and let $X^\bullet \to r^0 X$ be an n-$\mathcal{G}$-cofibrant approximation. We know:

$$\pi_s[r^0 X, G] = \begin{cases} [X, G] & , \text{ if } s = 0 \\ 0 & , \text{ else} \end{cases}$$

With the spiral exact sequence we can calculate:

$$\pi_s^\natural(X^\bullet, G) = \begin{cases} [X, \Omega^s G] & , \text{ if } 0 \le s \le n \\ 0 & , \text{ for } s > n \end{cases}$$

And respectively:

$$\pi_s[X^\bullet, G] = \begin{cases} [X, G] & , \text{ if } s = 0 \\ [X, \Omega^{n+1} G] & , \text{ if } s = n+2 \\ 0 & , \text{ else} \end{cases}$$

All these sets of isomorphisms determine each other vice versa. Of course, this is not the way we will encounter such spaces, since we are seeking for realization and not starting with them. Instead we will take these equations as the defining conditions of our successive realizations.

Definition 5.3.1 Let A be an object in the abelian target category $\mathcal{A}$. We will call a Reedy cofibrant object $X^\bullet$ in $\mathcal{M}$ a **potential n-stage for A** following [BDG01] and [GH04], if it satisfies the following properties:

$$\pi_s^\natural(X^\bullet, G) \cong \begin{cases} \operatorname{Hom}_{\mathcal{A}}(A, F_{*+s}G) & , \text{ if } 0 \le s \le n \\ 0 & , \text{ for } s > n \end{cases}$$

Note that this also makes sense for $n = \infty$. In this case an object satisfying these equations is simply called an **∞-stage**. The reason is that by 5.5.3 it is not "potential" any more.

Remark 5.3.2 If $X^\bullet$ is a potential n-stage for an object A in $\mathcal{A}$ then $\operatorname{sk}_n X^\bullet$ is a potential $(n-1)$-stage for A.

Remark 5.3.3 Since $\mathcal{G}$ was the class of F-injectives, the class $\{F_* G | G \in \mathcal{G}\}$ is cogenerating the category $\mathcal{A}$, and we derive for a potential n-stage $X^\bullet$ from the previous properties and the spiral exact sequence the following equations:

$$\pi^s F_* X^\bullet \cong H^s N F_* X^\bullet = \begin{cases} A & , \text{ if } s = 0 \\ A[n+1] & , \text{ if } s = n+2 \\ 0 & , \text{ else} \end{cases}$$

The shift functor $[\_]$ is the internal shift from 4.1.1. Note in particular that here $F_* X^\bullet$ is $(n+1)$-$\mathcal{I}$-equivalent to $*$ in accordance with remark 5.1.2.

Following the outlined philosophy we start the process of realizing an object A in $\mathcal{A}$ with a potential 0-stage. Later we will study the difference between n-stages and $(n+1)$-stages. The difference or layers are of a special type and we start be defining these layers in 5.3.6 and 5.3.11. These layers have a certain representation property, see 5.3.15 and 5.3.19. This turns out to be extremely useful when we describe the obstruction calculus. To prove this property we have to consider algebraic analogs of theses layers defined in 5.3.4 and 5.3.9 and they should not be confused with each other. As to be expected, 0-stages and 0-layers will coincide.

Definition 5.3.4 Let A be an object of $\mathcal{A}$. We call an object $I^\bullet$ in $c\mathcal{A}$ an **object of type $K(A,0)$** if it is weakly equivalent to $r^0 A$ with respect to the $\mathcal{I}$-structure. The next remark justifies that we denote such objects simply by $K(A,0)$.

Remark 5.3.5 If $I^\bullet$ is an object of type $K(A,0)$, then there is a weak equivalence $I^\bullet \to r^0(\pi^0 I^\bullet) = r^0 A$. It follows that the moduli space $\mathcal{M}_{\mathcal{I}}(K(A,0))$ of objects of type $K(A,0)$ is connected. It is weakly equivalent to $B\operatorname{haut}(K(A,0))$ by 1.4.7. We easily see that $\operatorname{Aut}(A) \simeq \operatorname{haut}(K(A,0))$ where $\operatorname{Aut}(A)$ is discrete. It follows that the moduli space weakly equivalent to $B\operatorname{Aut}(A)$.

Definition 5.3.6 Let A be an object in $\mathcal{A}$. We call an object $X^\bullet$ in $c\mathcal{M}$ an **object of type $L(A,0)$**, if it is F-equivalent to a potential 0-stage for A. Thus $X^\bullet$ has to satisfy the following equations:

$$\pi_s^\natural(X^\bullet, G) = \begin{cases} \operatorname{Hom}_{\mathcal{A}}(A, F_* G) & , \text{ for } s = 0 \\ 0 & , \text{ else} \end{cases}$$

Remark 5.3.7 Objects of type $L(A,0)$ are essentially unique as we will prove in 5.3.18. Their existence is also easily established. We choose an exact sequence

$$0 \to A \to I^0 \xrightarrow{d} I^1$$

with I^0 and I^1 injective. The map d is induced by a map $E(I^0) \to E(I^1)$ in $\mathcal{T} = Ho(\mathcal{M})$ between F-injective objects by 4.1.5 that we will also call d. This d again is represented by a map d in $\mathcal{M}$ if we choose the models for $E(I^0)$ and $E(I^1)$ to be fibrant and cofibrant. Now define a 1-truncated cosimplicial object

$$E(I^0) \rightleftarrows E(I^0) \times E(I^1)$$

with

$$d^0 = \begin{pmatrix} 1 \\ d \end{pmatrix}, \; d^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and } s^0 = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

By applying a Reedy cofibrant approximation and the left Kan extension l^1 from 1.2.1 we get the result.

Remark 5.3.8 From the spiral exact sequence or from 5.3.1 and 5.3.3 it follows for an object of type $L(A,0)$ that we have for every $G \in \mathcal{G}$:

$$\pi_s[L(A,0), G] = \begin{cases} \mathrm{Hom}_{\mathcal{A}}(A, G) & , \text{ if } s = 0 \\ \mathrm{Hom}_{\mathcal{A}}(A, \Omega G) & , \text{ if } s = 2 \\ 0 & , \text{ else} \end{cases}$$

or equivalently:

$$H^s N F_* L(A,0) = \begin{cases} A & , \text{ if } s = 0 \\ A[1] & , \text{ if } s = 2 \\ 0 & , \text{ else} \end{cases}$$

We emphasize again that an object of type $L(A,0)$ the image $F_*L(A,0)$ is 1-$\mathcal{I}$-equivalent to $r^0 A$, as we can see with these equations.

Definition 5.3.9 Let N be an object of $\mathcal{A}$ and let $n \geq 1$. We call an object $J^\bullet$ in $c\mathcal{A}$ an **object of type $\boldsymbol{K(N,n)}$**, if the following conditions are satisfied:

$$\pi^s F_* J^\bullet \cong \begin{cases} N & , \text{ if } s = n \\ 0 & , \text{ else} \end{cases}$$

We denote $J^\bullet$ by $K(N,n)$. These objects are essentially unique by the following remark.

Remark 5.3.10 Objects of type $K(N,n)$ exist, for example is $\Omega^n_{\mathrm{ext}} r^0 N$ or equivalently $\overline{W}^n r^0 N$ such an object. The moduli space is given $B\mathrm{Aut}(N)$, since the functor Ω_{ext} induces an obvious equivalence $\mathcal{M}(K(N,n)) \to \mathcal{M}(K(N,n+1))$ for $n \geq 0$. In particular this space is connected.

Definition 5.3.11 Let N be an object in $\mathcal{A}$ and $n \geq 1$. We call an object $Y^\bullet$ an **object of type $\boldsymbol{L(N,n)}$**, if the following conditions are satisfied:

$$\pi^\natural_s(Y^\bullet, G) = \begin{cases} \mathrm{Hom}_{\mathcal{A}}(N, F_*G) & , \text{ if } s = n \\ 0 & , \text{ else} \end{cases}$$

Existence and uniqueness of these objects are explained in 5.3.13. We denote them by $L(N,n)$. Do not confuse these objects with objects of type $K(N,n)$ in $c\mathcal{A}$, see 5.3.9, the end of the remark 5.3.12 and 5.3.14.

Remark 5.3.12 By the spiral exact sequence we compute from 5.3.11:

$$\pi_s[L(N,n), G] = \begin{cases} \mathrm{Hom}_{\mathcal{A}}(N, F_*G) & , \text{ if } s = n \\ \mathrm{Hom}_{\mathcal{A}}(N, F_{*+1}G) & , \text{ if } s = n+2 \\ 0 & , \text{ else} \end{cases}$$

By the defining property of the F-injective objects in $\mathcal{G}$ we get:

$$\pi^s F_* L(N,n) = \begin{cases} N & , \text{ if } s = n \\ N[1] & , \text{ if } s = n+2 \\ 0 & , \text{ else} \end{cases}$$

Both sets of data are equivalent to the defining equations of an object of type $L(N,n)$ in Definition 5.3.11. In particular it follows that $F_* L(N,n)$ is not an object of type $K(N,n)$.

Remark 5.3.13 Objects of type (N,n) exist in $c\mathcal{M}$, they can be given by setting:

$$L(N,n) := \Omega^n_{\text{ext}} L(N,0) \quad \text{or} \quad L(N,n) := \overline{W}^n L(N,0)$$

The moduli space of all objects of type $L(N,n)$ is given by $B\text{Aut}(N)$. This is proved by observing that it follows from 4.2.23 and 4.2.25 that Σ_{ext} and $\overline{W}$ induce mutually inverse homotopy equivalences of $\mathcal{M}(L(N,n))$ and $\mathcal{M}(L(N,n+1))$ for $n \geq 0$. Note that we still have to determine the moduli space of objects of type $L(N,0)$ in 5.3.18. From lemma 4.2.25 we also get that $\Sigma_{\text{ext}} L(N,n+1) \cong L(N,n)$.

Remark 5.3.14 Despite of the fact that $F_* L(N,n)$ is not of type $K(N,n)$ in $c\mathcal{A}$, there is a close connection explained in the following lemma. First we have to prepare ourselves. Let $n \geq 1$ and let N be an object in $\mathcal{A}$. By remark 2.2.3 the isomorphism of N and $\pi^n F_* L(N,n)$ defines a map $K(N,n) \to F_* L(N,n)$. So by first applying F_* and then pulling back along this arrow we obtain a map

$$\phi_n(Y^\bullet) : \text{map}(L(N,n), Y^\bullet) \to \text{map}(K(N,n), F_* Y^\bullet). \tag{5.2}$$

Here we assume that $L(N,n)$ and $K(N,n)$ are Reedy cofibrant.

The proof of the next statement is exactly parallel to the proof of [BDG01, Prop. 8.7.], although in our case linearity assures the result also for $n = 0, 1$. This lemma is one of the central ingredients in the obstruction calculus as well as for the proof of 5.4.2.

Lemma 5.3.15 *For every $Y^\bullet$ in $c\mathcal{M}$ and Reedy cofibrant objects of type $L(N,n)$ and $K(N,n)$ the map $\phi_n(Y^\bullet)$ from* (5.2) *is a natural weak equivalence.*

Proof: It suffices to prove the result for the case, where $Y^\bullet$ is of the form $\Omega^s_{\text{ext}} r^0 G$ for some $G \in \{F\text{-Inj}\}$ and $s \geq 0$. This follows with the following argument. The source and target of the natural transformation ϕ_n map homotopy pullbacks to homotopy pullbacks, because by 4.2.17 F_* preserves homotopy pullbacks. So we can extend the result to objects that are obtained by finite homotopy pullbacks from objects of the form $\Omega^s_{\text{ext}} r^0 G$. If $Y^\bullet$ is F-fibrant the fibers of $\text{cosk}_n Y^\bullet \to \text{cosk}_{n-1} Y^\bullet$ are all of the form $\Omega^s_{\text{ext}} r^0 G$ for some $G \in \{F\text{-Inj}\}$ and $s \geq 0$ by the dual version [GJ99, VII 1.7.] and $Y^\bullet$ is the limit of its coskeleta. So we can prove the lemma for all F-fibrant objects, but replacing F-fibrantly induces weak equivalences of the mapping spaces if the first variables are Reedy cofibrant.

Let G be arbitrary in F-Inj. The statement is non-trivial only in the cases for $0 \leq s < n$ because for $s = n$ we compute by 4.2.23:

$$\text{map}(K(N,n), F_* \Omega^n_{\text{ext}} r^0 G) \simeq \text{map}(K(N,0), r^0 F_* G) \simeq \ell_0 \text{Hom}_{\mathcal{A}}(N, F_* G)$$

Here $\ell_0(\_)$ is the constant simplicial object. Hence π_0 is given by $\text{Hom}_{\mathcal{A}}(N, F_* G)$ and $\pi_i = 0$ for $i > 0$. With the same observation we can show that for $s > n$ the space $\text{map}(K(N,n), \Omega^s_{\text{ext}} r^0 G)$ is contractible. We check the case $\text{map}(L(N,n), \Omega^s_{\text{ext}} r^0 G)$ by inspecting the definition 5.3.11 and we get the same results on π_i for $s = n$ as well as the contractibility for $s > n$. Furthermore it is clear that for $s = n$ the map $\pi_0\phi_n(\Omega^n_{\mathrm{ext}} r^0 G)$ is an automorphism of $\mathrm{Hom}_{\mathcal{A}}(N, F_* G)$ by construction. Now $\phi_n(\Omega^n_{\mathrm{ext}} r^0 G)$ is already a weak equivalence, while for $s > n$ there is nothing to prove. By downward induction we show that $\phi_n(\Omega^s G)$ is a weak equivalence for $0 \le s < n$, because we have

$$\phi_n(\Omega^s_{\mathrm{ext}} r^0 G) \simeq \Omega\, \phi_n(\Omega^{s-1}_{\mathrm{ext}} r^0 G),$$

and source and target of (5.2) are connected for the simple reason that by 3.2.3 there is only one vertex in

$$\begin{aligned} \mathrm{map}(K(N, n-s), r^0 G) &= \mathrm{Hom}_{\mathcal{A}}(K(N, n-s), G) \text{ and} \\ \mathrm{map}(L(N, n-s), r^0 G) &= \mathrm{Hom}_{\mathcal{M}}(L(N, n-s), G), \end{aligned}$$

because for $n - s > 0$ we have $(\Omega^{n-s} K(N,0))_0 = 0$ and $(\Omega^{n-s} L(N,0))_0 = *$.

□

Remark 5.3.16 Observe here that a priori we have two abelian group structures on $[L(N,n), Y^\bullet]_F$, but they coincide since for instance the fold map induces a commutative diagram

$$\begin{array}{ccc} K(N,n) \oplus K(N,n) & \longrightarrow & F_* L(N,n) \oplus F_* L(N,n) \\ \downarrow & & \downarrow \\ K(N,n) & \longrightarrow & F_* L(N,n) \end{array}$$

in $Ho(c\mathcal{A})$.

Corollary 5.3.17 *An object $X^\bullet$ is of type $L(A,0)$ if and only if it satisfies the following conditions:*

(i) *There is an isomorphism $\pi^0 F_* X^\bullet \cong A$ in $\mathcal{A}$.*

(ii) *For every $Y^\bullet$ in $c\mathcal{M}$ the natural map*

$$[X^\bullet, Y^\bullet] \to \mathrm{Hom}_{\mathcal{A}}(A, \pi^0 F_* Y^\bullet)$$

is an isomorphism.

Proof: Objects that satisfy (i) and (ii) are of type $L(A,0)$ because we can calculate:

$$\pi^\natural_s(X^\bullet, G) \cong [X^\bullet, \Omega^s_{\mathrm{ext}} r^0 G]_F \cong \begin{cases} \mathrm{Hom}_{\mathcal{A}}(A, F_* G) & \text{, for } s = 0 \\ 0 & \text{, else} \end{cases}$$

The other direction follows from 5.3.15.

□

Finally we can determine the moduli space of all objects of type $L(A,0)$.

Corollary 5.3.18 *The moduli space of all objects of type $L(A,0)$ is connected and we have the following weak equivalence:*

$$\mathcal{M}_F(L(A,0)) \simeq B\mathrm{Aut}(A)$$

Proof: The moduli space is connected: Let $L(A,0)$ be some reference object and let $X^\bullet$ be another object of type $L(A,0)$. By pulling back id_A along the isomorphism of

5.3.17(ii) we obtain a map $X^\bullet \to L(A,0)$ which induces an isomorphism on $\pi_0^\natural(\_, G)$ for every $G \in \{F\text{-Inj}\}$. Both are potential 0-stages, so this is the only group to check.

Now we will prove that the moduli space is weakly equivalent to the moduli space of objects of type $K(A,0)$. Then the result will follow from 5.3.5. By 1.4.7 there are canonical weak equivalences

$$\mathcal{M}_F(L(A,0)) \simeq B\mathrm{haut}_F(L(A,0)) \text{ and } \mathcal{M}_{\mathcal{I}}(K(A,0)) \simeq B\mathrm{haut}_{\mathcal{I}}(K(A,0)).$$

It suffices to prove that $\mathrm{haut}_F(L(A,0)) \simeq \mathrm{haut}(K(A,0))$, because both objects are fibrant grouplike simplicial monoids and B preserves weak equivalences between fibrant simplicial sets. By 5.3.15 we have the following weak equivalences:

$$\mathrm{map}(L(A,0), L(A,0)) \simeq \mathrm{map}(K(A,0), F_*L(A,0)) \simeq \ell_0\mathrm{Hom}_{\mathcal{A}}(A,A)$$

By passing to the appropriate components we see that $\mathrm{haut}_F(L(A,0)) \simeq \ell_0\mathrm{End}_{\mathcal{A}}(A)$ which finishes the proof. Here $\ell_0(\_)$ denotes the constant simplicial object.

□

Definition 5.3.19 Consider $K(N,n)$ in $c\mathcal{A}$ for $n \geq 0$. We assume that $K(N,n)$ is Reedy cofibrant. Let $\Lambda^\bullet$ be an object in $c\mathcal{A}$. Then we define

$$\mathrm{map}(K(N,n), \widetilde{\Lambda}^\bullet) =: \mathcal{H}^n(\Lambda^\bullet, N),$$

where $\Lambda^\bullet \to \widetilde{\Lambda}^\bullet$ is a fibrant approximation, to be the ***n*-th cohomology space** of $\Lambda^\bullet$ with coefficients in N. We define the ***n*-th cohomology** of $\Lambda^\bullet$ by

$$\pi_0\mathcal{H}^n(\Lambda^\bullet, N) =: H^n(\Lambda^\bullet, N).$$

In the next lemma we will give an interpretation of these cohomology groups.

Remark 5.3.20 It follows for any $\Lambda^\bullet$ in $c\mathcal{A}$:

$$\Omega\mathcal{H}^n(\Lambda^\bullet, N) \simeq \mathcal{H}^{n-1}(\Lambda^\bullet, N)$$

Lemma 5.3.21 *Let $\Lambda^\bullet$ in $c\mathcal{A}$ be $\mathcal{I}$-fibrant and n-$\mathcal{I}$-equivalent to $r^0\pi^0\Lambda^\bullet$. Then there is a natural isomorphism*

$$H^n(\Lambda^\bullet, N[k]) \cong \mathrm{Ext}^{n,k}(N, \pi^0\Lambda^\bullet)$$

of abelian groups.

Proof: The canonical map $r^0\pi^0\Lambda^\bullet \to \Lambda^\bullet$ obtained by adjunction factors into the composition $r^0\pi^0\Lambda^\bullet \to \mathrm{sk}_{n+1}\Lambda^\bullet \to \Lambda^\bullet$ of n-$\mathcal{I}$-equivalences and we can approximate $\mathrm{sk}_{n+1}\Lambda^\bullet$ $\mathcal{I}$-fibrantly by $I^\bullet$ which yields an injective resolution of $\pi^0\Lambda^\bullet$ after normalization. Now $K(N[k],n)$ is an $(n+1)$-skeleton and we compute:

$$\begin{aligned} H^n(\Lambda^\bullet, N[k]) &= \pi_0\mathrm{map}(K(N[k],n), \Lambda^\bullet) \cong \pi_0\mathrm{map}(K(N[k],n), I^\bullet) \\ &\cong \mathrm{Ext}^{n,k}(N, \pi^0\Lambda^\bullet) \end{aligned}$$

□

Remark 5.3.22 Let $Y^\bullet$ be F-fibrant, such that $F_*Y^\bullet$ is n-$\mathcal{I}$-equivalent to $r^0\pi^0F_*Y^\bullet$. Altogether lemma 5.3.15 and lemma 5.3.21 yield the following isomorphism of abelian groups:

$$\pi_0\mathrm{map}(L(A[k],n), Y^\bullet) \cong \mathrm{Ext}^{n,k}_{\mathcal{A}}(A, \pi^0F_*Y^\bullet)$$

Here we assume $L(A[k],n)$ and $Y^\bullet)$ to be both Reedy cofibrant. This is functorial in $Y^\bullet$. It is not quite functorial in A, but for a morphism $A \to B$ after having chosen two objects $L(A[k],n)$ and $L(B[k],n)$ there is a uniquely determined homotopy class $L(A[k],n) \to L(B[k],n)$ inducing $A \to B$. The result tells us that an object $L(N,n)$ represents the cohomology functor $H^n(F_*(\_),N)$ in the homotopy category $Ho(c\mathcal{M}^F)$. Note that the isomorphism is in particular valid if $Y^\bullet$ is an F-fibrant n-stage.

We want to construct an obstruction calculus for lifting things from an interpolation category to the next one. In order to carry this out, we study the difference between potential $(n-1)$-stages and potential n-stages. In 5.3.24 and 5.3.25 we will prove the existence of certain homotopy pushout diagrams, where the difference between two stages is recognized as objects of type $L(N,n)$ for suitable N and n. This construction can be viewed as a (potential) co-Postnikov-tower, compare 5.3.27.

The following two lemmas are an example for the simplifications we get for the stable case. The next lemma is the collapsed version of the so-called difference construction in [BDG01, 8.4.].

Lemma 5.3.23 *Let $n \geq 1$ and provide $c\mathcal{M}$ with the F-structure. Let $f: X^\bullet \to Y^\bullet$ be a map in $c\mathcal{M}$, which induces an isomorphism on $\pi^0 F_*$ and whose homotopy cofiber $C^\bullet$ has the property that $\pi^s F_* C^\bullet = 0$ for $0 \leq s \leq n-1$. Let $P^\bullet$ be the homotopy fiber of f. Then there are isomorphisms $P^\bullet \cong \Omega_{\text{ext}} C^\bullet$ and $\Sigma_{\text{ext}} P^\bullet \cong C^\bullet$ in $Ho(c\mathcal{M}^F)$ and $\text{sk}_{n+2} P^\bullet$ is an object of type $L(\pi^n F_* C^\bullet, n+1)$.*

Proof: This follows directly from 4.2.24 and 4.2.25 and the long exact $\pi^\natural_*$-sequence.

□

Lemma 5.3.24 *Let $X_n^\bullet$ be a potential n-stage for A. Then $\text{sk}_n X_n^\bullet =: X_{n-1}^\bullet$ is a potential $(n-1)$-stage for A, and there is a homotopy cofiber sequence in $c\mathcal{M}^F$:*

$$L(A[n], n+1) \to X_{n-1}^\bullet \to X_n^\bullet$$

This sequence is also a homotopy fiber sequence $c\mathcal{M}^F$.

Proof: Call $C_n = \text{hocofib}(X_{n-1}^\bullet \to X_n^\bullet)$. We know that $\pi^\natural_s$ of the homotopy cofiber of C_n vanishes except in dimension n. Hence $\text{sk}_{n+2} C_n$ is F-equivalent to C_n. From 5.3.23 that $\Sigma_{\text{ext}} \Omega_{\text{ext}} C_n \simeq C_n \simeq \Omega_{\text{ext}} \Sigma_{\text{ext}} C_n$ and that $\Omega_{\text{ext}} C_n$ is an object of type $L(A[n], n+1)$. We also see that the sequence is a homotopy cofiber sequence as well as a homotopy fiber sequence.

□

Lemma 5.3.25 *Let there be given a homotopy cofiber sequence in $c\mathcal{M}^F$:*

$$L(A[n], n+1) \xrightarrow{w_{n-1}} X_{n-1}^\bullet \longrightarrow X_n^\bullet$$

Let $X_{n-1}^\bullet$ be a potential $(n-1)$-stage for A. A Reedy cofibrant approximation to $X_n^\bullet$ is a potential n-stage for A if and only if the map w_{n-1} induces an isomorphism $A[n] \cong \pi^{n+1} F_* X_{n-1}^\bullet$.

Proof: It follows from 4.2.12 that there is an exact sequence

$$0 \to \pi^n F_* X_n^\bullet \to \pi^{n+1} F_* L(A[n], n+1) \xrightarrow{\cong} \pi^{n+1} F_* X_{n-1}^\bullet \to \pi^{n+1} F_* X_n^\bullet \to 0$$

and an isomorphism $\pi^{n+2}F_*X_n^\bullet \cong \pi^{n+3}F_*L(A[n], n+1) \cong A[n+1]$. All other groups of the form $\pi^s F_* X_n^\bullet$ for $s > 0$ vanish, hence $X_n^\bullet$ has the right homotopy groups for a potential n-stage for A, we just need to approximate it Reedy cofibrantly.

□

Now we start the obstruction against the existence of realizations of objects in $IP_{n-1}(F)$.

Definition 5.3.26 Let $X_{n-1}^\bullet$ be a potential n-stage for an object A. We call an object $X^\bullet$ a **potential n-stage over $X_{n-1}^\bullet$** if $X^\bullet$ is a potential n-stage and $\mathrm{sk}_n X^\bullet$ is F-equivalent to $X_{n-1}^\bullet$. By 3.2.15(i) this is equivalent to $X_{n-1}^\bullet$ being n-F-equivalent to $\mathrm{sk}_n X^\bullet$.

The obstruction against the existence of an n-stage over a given $(n-1)$-stage is the existence of a map w_{n-1} like in 5.3.25. We are now going to reformulate this in algebraic terms. We already know from remark 5.3.3 that for an $(n-1)$-stage $X_{n-1}^\bullet$ its image $F_*X_{n-1}^\bullet$ has the same cohomology groups as an object of type $K(A,0) \oplus K(A[n], n+1)$. Without loss of generality we assume $X_{n-1}^\bullet$ to be F-fibrant. Hence we know that such a w_{n-1} exists if and only if we are able to construct a map

$$\omega_{n-1} : K(A[n], n+1) \to F_*X_{n-1}^\bullet$$

inducing an isomorphism on $\pi^{n+1}F_*(\_)$, because by the representing property 5.3.15 it follows that we were then able to choose w_{n-1} such that

$$\pi_0[\phi(X_{n-1}^\bullet)(w_{n-1})] = \pi_0[\omega_{n-1}].$$

From 5.3.3 we have the homotopy cofiber sequence

$$K(A[n], n+2) \to \mathrm{sk}_{n+1} F_*X_{n-1}^\bullet \to F_*X_{n-1}^\bullet$$

and we can consider the following diagram:

$$
\begin{array}{ccccc}
 & & \mathrm{sk}_1 F_*X^\bullet_{n-1} & \xrightarrow{\simeq} & K(A,0)\simeq r^0A \\
 & & \downarrow \cong & & \\
K(A[n],n+2) & \xrightarrow{\beta_{n-1}} & \mathrm{sk}_{n+1} F_*X^\bullet_{n-1} & & \\
\downarrow & * \quad \text{ho-}\ulcorner & \downarrow & & \\
* & \longrightarrow & \mathrm{sk}_{n+2} F_*X^\bullet_{n-1} & \xrightarrow{\cong} & F_*X^\bullet_{n-1} \\
\downarrow & & & \nearrow & \\
K(A[n],n+1) & & \cdots\cdots \omega_{n-1} \cdots\cdots & &
\end{array} \qquad (5.3)
$$

Observe also that we have isomorphisms

$$H^{n+2}(\mathrm{sk}_{n+1} F_*X_{n-1}^\bullet, A[n]) \xrightarrow{\cong} H^{n+2}(r^0A, A[n]) = \mathrm{Ext}_{\mathcal{A}}^{n+2,n}(A, A)$$

of abelian groups by lemma 5.3.21 or remark 5.3.22.

Definition 5.3.27 The homotopy class b_{n-1} of the map β_{n-1} in

$$\pi_0\mathrm{map}(K(A[n], n+2), r^0A) = H^{n+2}(r^0A, A[n]) \cong \mathrm{Ext}_{\mathcal{A}}^{n+2,n}(A, A)$$

will be called the **obstruction class** or the **co-k-invariant** of the potential $(n-1)$-stage $X_{n-1}^\bullet$. Diagram (5.3) is dual to the concept of k-invariants of a Postnikov-tower.

Lemma 5.3.28 *In* (5.3) *the map* ω_{n-1} *inducing an isomorphism on* $\pi^{n+1}F_*(\_)$ *exists if and only if* β_{n-1} *is nullhomotopic.*

Proof: Obvious.

□

Theorem 5.3.29 *Let* $n \geq 1$ *and* A *be an object of* $\mathcal{A}$. *Let* $X^\bullet_{n-1}$ *be a potential* $(n-1)$*-stage of* A. *There exists a potential* n*-stage* $X^\bullet_n$ *over* $X^\bullet_{n-1}$ *if and only if the co-k-invariant* b_{n-1} *from definition* 5.3.27 *in* $\mathrm{Ext}^{n+2,n}(A,A)$ *vanishes.*

Proof: The theorem is deduced from the discussion starting at 5.3.25 and using 5.3.21.

□

Now we are concerned with telling apart different realizations.

Definition 5.3.30 Let $X^\bullet_n$ and $Y^\bullet_n$ be potential n-stages for an object A with $\mathrm{sk}_n X^\bullet_n \simeq X^\bullet_{n-1} \simeq \mathrm{sk}_n Y^\bullet_n$. The homotopy fiber of the canonical maps from $X^\bullet_{n-1}$ to $X^\bullet_n$ and $Y^\bullet_n$ is $L(A[n], n+1)$ by 5.3.24. We obtain two maps

$$v_{X^\bullet_n} \text{ and } v_{Y^\bullet_n} : L(A[n], n+1) \to X^\bullet_{n-1},$$

The **difference class** of the objects $X^\bullet_n$ and $Y^\bullet_n$ is defined to be the class

$$\delta(X^\bullet_n, Y^\bullet_n) := \pi_0(v_{X^\bullet_n}) - \pi_0(v_{Y^\bullet_n}) \in \pi_0\mathcal{H}^{n+1}(F_*X^\bullet_{n-1}, A[n]) \cong \mathrm{Ext}^{n+1,n}(A,A).$$

Remark 5.3.31 The proof of the next theorem shows that this defines an action of $\mathrm{Ext}^{n+1,n}_{\mathcal{A}}(A,A)$ on the class of F-equivalence classes of potential n-stages over a given potential $(n-1)$-stage. It is obviously transitive. This proves first of all that there is just a set of such equivalence classes or, what is the same, of realizations in $IP_n(F)$ of a given object in $IP_{n-1}(F)$.

Theorem 5.3.32 *Let* $n \geq 1$. *There is an action of* $\mathrm{Ext}^{n+1,n}(A,A)$ *on the set of* F*-equivalence classes of potential* n*-stages of* A *over a given potential* $(n-1)$*-stages which is transitive and free.*

Proof: Let $X^\bullet_n$ be a potential n-stage with $X^\bullet_{n-1} := \mathrm{sk}_n X^\bullet_n$. Let $[\kappa] \in \mathrm{Ext}^{n+1,n}_{\mathcal{A}}(A,A)$. Take a representative κ and call the following composition c:

$$K(A[n], n+1) \xrightarrow{\kappa} K(A,0) \longrightarrow K(A,0) \oplus K(A[n], n+1) \cong F_*X^\bullet_{n-1}$$

Note that this c induces the zero map on $\pi^{n+1}F_*(\_)$. Take a map $\omega_{n-1} : K(A[n], n+1) \to F_*X^\bullet_{n-1}$ from (5.3) representing the homotopy class of $w_{n-1} : L(A[n], n+1) \to X^\bullet_{n-1}$ associated to $X^\bullet$ by 5.3.24 and add it to c. The resulting map $\omega_{n-1} + c$ will still induce an isomorphism on π^{n+1}. So we can form the cofiber $Y^\bullet_n$ of the corresponding map $L(A[n], n+1) \to X^\bullet_{n-1}$ and we observe that it is a potential n-stage and that it realizes the given difference class $\kappa = \delta(X^\bullet_n, Y^\bullet_n)$. This process is obviously additive in $[\kappa]$, therefore we have a group action. It is also clear that $X^\bullet_n \cong Y^\bullet_n$ in $IP_n(F)$ if and only if $\kappa = 0$.

□

Now we are going to describe the obstruction for lifting maps from $IP_{n-1}(F)$ to $IP_n(F)$.

Definition 5.3.33 Let $n \geq 1$. Let $X^\bullet$ and $Y^\bullet$ be objects in $IP_n(F)$ and let $\varphi : \sigma_n X^\bullet \to \sigma_n Y^\bullet$ be a map in $IP_{n-1}(F)$. We say that $\boldsymbol{\varphi}$ **lifts** if there is a map $\Phi : X^\bullet \to Y^\bullet$ such that $\sigma_n \Phi = \varphi$. In this case we call Φ a **lifting** of φ.

Remark 5.3.34 By definition every object $W^\bullet$ in $IP_n(F)$ can be approximated by a potential n-stage, which corresponds to n-F-cofibrant approximation, or it can be approximated F-fibrantly such that $F_*W^\bullet$ is $(n+1)$-$\mathcal{I}$-equivalent to $r^0\pi^0F_*W^\bullet$.

Between a potential n-stage $X^\bullet$ and an F-fibrant $Y^\bullet$ where $F_*Y^\bullet$ is $(n+1)$-$\mathcal{I}$-equivalent to $\pi^0F_*Y^\bullet$ every morphism in $IP_n(F)$ can be represented by a map $f : X^\bullet \to Y^\bullet$ in $c\mathcal{M}$.

Assume that we are given a morphism from $X^\bullet$ to $Y^\bullet$ in $IP_{n-1}(F)$, then this can be represented by a map $f : \mathrm{sk}_n X^\bullet \to Y^\bullet$. Now f lifts if and only if there is a map $\widetilde{f} : X^\bullet \to Y^\bullet$ such that

$$\mathrm{sk}_n X^\bullet \longrightarrow X^\bullet \longrightarrow Y^\bullet$$

is homotopic to f in $c\mathcal{M}^F$.

Theorem 5.3.35 *A morphism $\sigma_n X^\bullet \to \sigma_n Y^\bullet$ in $IP_{n-1}(F)$ lifts to a morphism $X^\bullet \to Y^\bullet$ in $IP_n(F)$ if and only if $\mathrm{ob}_n(f)$ in $\mathrm{Ext}^{n+1,n}_{\mathcal{A}}(\pi^0F_*X^\bullet, \pi^0F_*Y^\bullet)$ defined in (5.5) vanishes.*

Proof: We assume without loss of generality that $X^\bullet$ is a potential n-stage for an object A and that $Y^\bullet$ is F-fibrant such that $F_*Y^\bullet$ is $(n+1)$-$\mathcal{I}$-equivalent to r^0B in $c\mathcal{A}$. We can achieve this by approximations in the n-F-structure. Also without loss of generality we can replace to homotopy cofiber sequence

$$L(A[n], n+1) \xrightarrow{w_{n-1}} X^\bullet_{n-1} \longrightarrow X^\bullet_n$$

in $c\mathcal{M}^F$ of 5.3.25 by an actual cofiber sequence using factorizations in the F-structure. This means that from the data we have constructed the following solid arrow diagram

$$\begin{array}{ccccc} L(A[n], n+1) & \xrightarrow{w_{n-1}} & \mathrm{sk}_n X^\bullet & \xrightarrow{f} & Y^\bullet \\ \downarrow & & \downarrow & \nearrow & \\ PL & \longrightarrow & X^\bullet & & \end{array} \tag{5.4}$$

where $PL \overset{F}{\simeq} *$ is a path object in the F-structure for $L(A[n], n+1)$. We conclude that the existence of the dotted liftings in diagram (5.4) are equivalent to each other. By 5.3.15 we deduce that an extension of f to $X^\bullet$ exists if and only if the map

$$K(A[n], n+1) \xrightarrow{\phi(fw_{n-1})} F_*Y^\bullet$$

is null homotopic, where ϕ is the map from (5.2). $\phi(fw_{n-1})$ defines an obstruction element

$$\mathrm{ob}_n(f) := [\phi(fw_{n-1})] \in H^{n+1}(F_*Y^\bullet, A[n]) = \mathrm{Ext}^{n+1,n}_{\mathcal{A}}(\pi^0F_*X^\bullet, \pi^0F_*Y^\bullet) \tag{5.5}$$

by 5.3.21. Recall that $F_*Y^\bullet$ is $(n+1)$-$\mathcal{I}$-equivalent to $r^0\pi^0F_*Y^\bullet$. So remark 5.3.22 applies.

□

Before we proceed to the next theorem we have to reformulate the obstruction defined in (5.5). Here we use the "almost stability" of $Ho(c\mathcal{M}^F)$ that is displayed in 4.2.24, 4.2.25 and 5.3.24. In the situation of (5.4) let

$$\mathrm{sk}_n Y^\bullet \xrightarrow{\upsilon} \widetilde{Y}^\bullet_{n-1} \xrightarrow{\widetilde{\upsilon}} Y^\bullet$$

be a factorization of the canonical inclusion map into an F-trivial cofibration followed by an $(n-1)$-F-trivial fibration. Obviously $\widetilde{Y}^\bullet_{n-1}$ is a potential $(n-1)$-stage and the co-k-invariant of $Y^\bullet$ is represented by $\upsilon \circ w_{n-1}^{Y^\bullet}$. We allow ourselves to further denote this as $w_{n-1}^{Y^\bullet}$. Then the map $f : \mathrm{sk}_n X^\bullet \to Y^\bullet$ lifts to a map $\widetilde{f} : \mathrm{sk}_n X^\bullet \to \widetilde{Y}_n^\bullet$ whose homotopy class is uniquely determined. We choose Reedy cofibrant objects $L(A[n], n+1)$ and $L(B[n], n+1)$ and cofibrations representing $w_{n-1}^{X^\bullet}$ and $w_{n-1}^{Y^\bullet}$ and consider the following diagram:

$$\begin{array}{ccc} L(A[n], n+1) & \xrightarrow{w_{n-1}^{X^\bullet}} & \mathrm{sk}_n X^\bullet \\ {\scriptstyle \overline{f}}\downarrow & & \downarrow{\scriptstyle \widetilde{f}} \\ L(B[n], n+1) & \xrightarrow[w_{n-1}^{Y^\bullet}]{} & \widetilde{Y}_{n-1}^\bullet \end{array} \tag{5.6}$$

Here $\overline{f}$ is induced by f in the following sense: It represents the uniquely determined homotopy class that induces the map $\pi^0 F_*(f) : A \to B$, compare 5.3.22.

We observe that, if we are given a diagram like (5.4), we get a diagram (5.6) and we have the following equation

$$[\phi(f w_{n-1}^{X^\bullet})] = (\widetilde{\upsilon})_* \left([\widetilde{f} w_{n-1}^{X^\bullet}] - [\widetilde{w}_{n-1}^{Y^\bullet} \overline{f}] \right) \in \mathrm{Ext}_{\mathcal{A}}^{n+1,n}(A, B), \tag{5.7}$$

since $L(B[n], n+1) \xrightarrow{w_{n-1}^{Y^\bullet}} \widetilde{Y}_{n-1}^\bullet \xrightarrow{\widetilde{\upsilon}} Y^\bullet$ is a homotopy cofiber sequence.

Lemma 5.3.36 *The obstruction* $\mathrm{ob}_n(f)$ *from* (5.5) *vanishes if and only if the diagram* (5.6) *commutes in* $Ho(c\mathcal{M}^F)$.

Proof: If the square commutes up to homotopy we can strictify it by changing $\overline{f}$ and $\widetilde{f}$ within their homotopy class. Then we can apply the pushout functor and obtain a map $X^\bullet \to Y^\bullet$. We can turn this process around if we remember that the homotopy cofiber sequence in 5.3.24 is also a homotopy fiber sequence. So if a lifting exists, which is equivalent to $\mathrm{ob}_n(f) = 0$, then this diagram commutes.

□

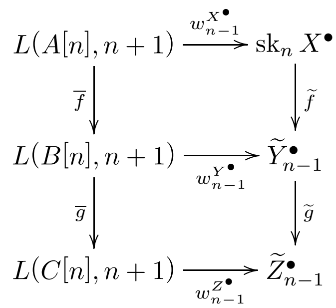
Theorem 5.3.37 *Let* $X^\bullet$ *and* $Y^\bullet$ *be objects in* $IP_n(F)$ *such that* $\pi^0 F_* X^\bullet = A$ *and* $\pi^0 F_* Y^\bullet = B$. *Then the homomorphism*

$$\mathrm{Hom}_{IP_{n-1}(F)}(\sigma_n X^\bullet, \sigma_n Y^\bullet) \to \mathrm{Ext}_{\mathcal{A}}^{n+1,n}(A, B)$$

satisfies property (iii) *of* 5.2.4.

Proof: There is a map of sets $\mathrm{Hom}_{IP_{n-1}(F)}(\sigma_n X^\bullet, \sigma_n Y^\bullet) \to \mathrm{Ext}_{\mathcal{A}}^{n+1,n}(A, B)$, where we map an f like in (5.4) to $\mathrm{ob}_n(f)$ as defined in (5.5). This is well defined by 5.3.15. We easily see that it is a homomorphism of abelian groups when we put the fold map $Y^\bullet \oplus Y^\bullet \to Y^\bullet$ in the place of $Y^\bullet$ in diagram 5.4.

Property (iii) of 5.2.4 follows immediately by 5.3.36 by considering two squares like (5.6) for $[f] : \sigma_n X^\bullet \to \sigma_n Y^\bullet$ and $[g] : \sigma_n Y^\bullet \to \sigma_n Z^\bullet$ respectively.

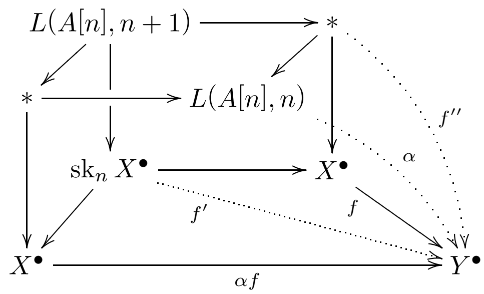


Now the statement follows directly from 5.7 and 5.6. The homotopy classes involving the term $w_{n-1}^{Y^\bullet}$ cancel out.

□

From diagram (5.4) we are now going to derive the obstruction against the uniqueness of the realization of f.

Definition 5.3.38 Let $f : X^\bullet \to Y^\bullet$ be a map of potential n-stages for A and B respectively and for $n \geq 1$. Let $\alpha \in \mathrm{Ext}_{\mathcal{A}}^{n,n}(A, B)$. We define a new map $\alpha f : X^\bullet \to Y^\bullet$ by the following diagram:



(5.8)

This diagram commutes up to homotopy, it can be strictified by choosing appropriate replacements for $*$ by path objects. The top square, the square on the left and the square in the back part of (5.8) are homotopy pushout squares. The datum of a map f is equivalent to giving maps f' and f'' making the obvious square (homotopy) commutative. Prescribing the homotopy class of α is equivalent to the existence of a map αf whose homotopy class is, like that of f, a lifting of the homotopy class of f' in the sense of 5.3.33. Of course, the homotopy class of αf is uniquely determined by the homotopy class of α (and of f).

Theorem 5.3.39 *The construction in definition* 5.3.38 *defines an action of* $\mathrm{Ext}_{\mathcal{A}}^{n,n}(A, B)$ *on* $\mathrm{Hom}_{IP_n(F)}(X^\bullet, Y^\bullet)$. *Two morphisms agree on the n-skeleton if and only if they are in the same orbit. The restriction of the action to the set of realizations in* $IP_n(F)$ *of a given morphism in* $IP_{n-1}(F)$ *is transitive and free. The restricted action satisfies the linear distributivity law from* 5.2.4.

Proof: The fact that 5.8 defines an action is easily seen by considering the fold map $Y^\bullet \oplus Y^\bullet \to Y^\bullet$ in the place of $Y^\bullet$. Also obvious is the assertion that two morphisms are in the same orbit if and only if they realize the same morphism in $IP_{n-1}(F)$ as well as the fact that the action is transitive and free, if we restrict to realizations of a given morphism. To prove that the linear distributivity law holds we observe that the data to construct $(\beta g)(\alpha f)$ is contained in the map

$$\begin{array}{ccccccc} & & L(A[n],n) & \xrightarrow{\quad\alpha\quad} & & Y^\bullet & \\ L(A[n],n) & \xrightarrow{\text{diag}} & \oplus & & L(B[n],n) & & Y^\bullet \xrightarrow{g} Z^\bullet, \\ & & L(A[n],n) & \xrightarrow{\overline{f}} & L(B[n],n) & \xrightarrow{\beta} & Z^\bullet \end{array}$$

where the homotopy class of $\overline{f}$ is induced by $f : X^\bullet \to Y^\bullet$. The induced map $X^\bullet \to Z^\bullet$ is $(g_*\alpha + f^*\beta)(gf)$.

□

Corollary 5.3.40 *Let $n \geq 1$. Let $X^\bullet$ and $Y^\bullet$ be potential n-stages for objects A and B in $\mathcal{A}$. We have an exact sequence:*

$$0 \to \operatorname{Ext}^{n,n}_{\mathcal{A}}(A,B) \to \operatorname{Hom}_{IP_n(F)}(X^\bullet, Y^\bullet) \to \operatorname{Hom}_{IP_{n-1}(F)}(\operatorname{sk}_n X^\bullet, \operatorname{sk}_n Y^\bullet)$$
$$\to \operatorname{Ext}^{n+1,n}_{\mathcal{A}}(A,B)$$

Proof: On the right exactness follows from 5.3.37. In the middle it follows from the transitivity of the action of $\operatorname{Ext}^{n,n}_{\mathcal{A}}(A,B)$ proved in 5.3.39. On the left it follows if we remember that the action of $\operatorname{Ext}^{n,n}_{\mathcal{A}}(A,B)$ on the liftings of 0 is free.

□

In the previous theorems we determined the obstructions for the realization problem. In the next section we will be able to give a description of the moduli spaces of realizations, see 5.5.10 and 5.5.11. An obstruction calculus for realizing objects using only the triangulated structure is among other things described in [BKS04].

5.4 Properties of interpolation categories

In this paragraph we prove some results about interpolation categories. Let $F : \mathcal{T} \to \mathcal{A}$ be a homological functor with enough injectives that detects isomorphism as in 4.2.1.

Theorem 5.4.1 *Let F be a homological functor as in* 4.2.1 *and $n \geq 1$. The following diagram*

$$\operatorname{Ext}^{n,n}_{\mathcal{A}}(\pi^0 F_*(\_), \pi^0 F_*(\_)) \to IP_n(F) \to IP_{n-1}(F) \to \operatorname{Ext}^{n+1,n}_{\mathcal{A}}(\pi^0 F_*(\_), \pi^0 F_*(\_))$$

is an exact sequence of categories in the sense of 5.2.4.

Proof: We have to check the various points in definition 5.2.4. That the Ext-terms here define natural systems of abelian groups is clear. Property (i) of 5.2.4 is proved in 5.3.39. (ii) follows from 5.3.35. Point (iii) is shown in 5.3.37 and (iv) follows from the proof of theorem 5.3.32.

□

Theorem 5.4.2 *The functor $\pi^0 F_* : IP_0(F) \to \mathcal{A}$ is an equivalence of categories.*

Proof: We will prove that $\pi^0 F$ is essentially surjective and induces a bijection

$$\operatorname{Hom}_{IP_0(F)}(X^\bullet, Y^\bullet) \to \operatorname{Hom}_{\mathcal{A}}(\pi^0 F_* X^\bullet, \pi^0 F_* Y^\bullet) \tag{5.9}$$

for arbitrary objects $X^\bullet$ and $Y^\bullet$ in $IP_0(F)$.

Let A be in $\mathcal{A}$. Choose an injective resolution $A \to I^\bullet$. Using remark 4.1.9 we can realize $I^\bullet$ as a diagram in $\mathcal{T} = Ho(\mathcal{M})$. It was shown in 5.3.7 that we can realize the beginning part of this resolution by a 1-truncated cosimplicial object $E^\bullet$ in $c_1\mathcal{M}$. Let $X^\bullet$ in $c\mathcal{M}$ be $l^1 E^\bullet$, where l^1 is the left Kan-extension from 1.2.1. Such an object is in $IM_0(F)$ because it is an object of the form $L(A, 0)$. By construction we have

$$\pi^0 F_* X^\bullet \cong \ker[F_* E^0 \xrightarrow{d^1 - d^0} F_* E^1] \cong \ker[I^0 \to I^1] \cong A,$$

which proves that $\pi^0 F_*$ is essentially surjective. Now let $X^\bullet$ and $Y^\bullet$ be objects in $IP_0(F)$. Suppose we are given a map $A \to B$ in $\mathcal{A}$ with $A = \pi^0 F_* X^\bullet$ and $B = \pi^0 F_* B^\bullet$. We can assume that $X^\bullet$ is 0-F-cofibrant and $Y^\bullet$ is F-fibrant. Then $X^\bullet$ is of type $L(A, 0)$. A map from A to B can be extended to a map

$$K(A, 0) \to F_* Y^\bullet,$$

since r^0 is left adjoint to taking the maximal augmentation $\pi^0(\_)$. Now 5.3.15 delivers us a map $L(A, 0) = X^\bullet \to Y^\bullet$ in $c\mathcal{M}$ inducing $A \to B$. Hence the functor is full.

Finally let $X^\bullet \to Y^\bullet$ be a morphism in $IP_0(F)$ that is in the kernel of the map (5.9). Again we assume, that $X^\bullet$ is 0-F-cofibrant and $Y^\bullet$ is F-fibrant. This implies that the morphism is represented by a map $X^\bullet \to Y^\bullet$ in $c\mathcal{M}$, but also that $X^\bullet$ is of type $L(A, 0)$. The induced map

$$K(A, 0) \to F_* X^\bullet \to F_* Y^\bullet$$

is null homotopic by assumption. But then the fact that $L(A, 0) = X^\bullet \to Y^\bullet$ is null homotopic follows again from 5.3.15. This proves that $\pi^0 F_*(\_)$ is faithful.

□

Definition 5.4.3 Recall that $\mathcal{A}$ has enough injectives by assumption. We consider the full subcategory $\boldsymbol{\mathcal{T}_n}$ of $\mathcal{T}$ consisting of those objects X such that $F_* X$ has injective dimension $\leq n$. This defines an increasing filtration of $\mathcal{T}$ with $\mathcal{T}_0$ equal to the full subcategory of F-injective objects $\mathcal{T}_{\text{inj}}$. The inclusion functors $\mathcal{T}_n \hookrightarrow \mathcal{T}$ will be called $\boldsymbol{i_n}$.

Theorem 5.4.4 *Let A be an object in $\mathcal{A}$ of injective dimension $\leq n+2$ for $n \geq 0$. If there exists an object $X^\bullet$ in $IP_n(F)$ with $\pi^0 F_* X^\bullet \cong A$ or equivalently a potential n-stage for A, then A is realizable in $\mathcal{T}$.*

Proof: Obvious since the obstructions against the existence of a realization as an ∞-stage $\widetilde{X}^\bullet$ lie in $\mathrm{Ext}_{\mathcal{A}}^{n+3+s, n+1+s}(A, A)$ for $s \geq 0$ and these groups vanish by assumption. Now $\mathrm{Tot}\, \widetilde{X}^\bullet$ is a realization of A by 4.2.32.

□

Recall that $\theta_n : \mathcal{T} \to IP_n(F)$ was defined in 3.3.4 and maps to $IP_n(F)$ by 5.1.3.

Theorem 5.4.5 *The functors $\theta_k i_n : \mathcal{T}_n \to IP_k(F)$ are full for $k \geq n-1$. The functors $\theta_k i_n : \mathcal{T}_n \to IP_k(F)$ are faithful for $k \geq n$.*

Proof: If X is an object of $\mathcal{T}$ then its image $\theta_k X$ in $IP_k(F)$ is the k-F-equivalence class of $r^0 X$. Let X and Y be in $\mathcal{T}_n$ where we assume from the beginning on that both are fibrant and cofibrant. We have to show that the map

$$\mathrm{Hom}_{\mathcal{T}_n}(X, Y) \to \mathrm{Hom}_{IP_k(F)}(\theta_k X, \theta_k Y) \tag{5.10}$$

is a bijection for $k \geq n-1$. To prove surjectivity we take F-fibrant replacements $\widetilde{X}^\bullet$ and $Y^\bullet$ of $\theta_k X = r^0 X$ and $\theta_k Y = r^0 Y$ respectively and then we replace $\widetilde{X}^\bullet$ Reedy cofibrantly by $X^\bullet$. Now each morphism $[f]$ in $\mathrm{Hom}_{IP_k(F)}(r^0X, r^0Y) \cong \mathrm{Hom}_{IP_k(F)}(X^\bullet, Y^\bullet)$ is represented by a map

$$f : \mathrm{sk}_{k+1} X^\bullet \to Y^\bullet$$

in $c\mathcal{M}$. The obstructions against extending this map to higher skeleta of $X^\bullet$ lie in $\mathrm{Ext}_{\mathcal{A}}^{k+2+s,k+1+s}(F_*X, F_*Y)$ for $s \geq 0$ by 5.3.35. All these groups vanish for $k \geq n-1$ because the injective dimension is smaller than or equal to $n < k+2$. We end up with a map $f_\infty : X^\bullet \to Y^\bullet$. Now we get a morphism

$$\widetilde{f} : X \cong \mathrm{Tot}\, X^\bullet \xrightarrow{\mathrm{Tot}\, f_\infty} \mathrm{Tot}\, Y^\bullet \cong Y$$

in $\mathcal{T}$, where the isomorphisms are the canonical maps from 4.2.33. By lemma 2.1.22 or remark 3.3.3 $R\,\mathrm{Tot}$ and Lr^0 are a Quillen pair, and so $\sigma_n \widetilde{f}$ corresponds to $[f]$ via the isomorphism

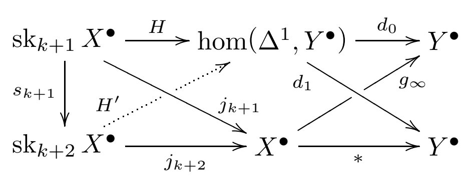
$$\mathrm{Hom}_{IP_k(F)}(r^0X, r^0Y) \cong \mathrm{Hom}_{IP_k(F)}(X^\bullet, Y^\bullet)$$

induced by the various replacements. So we have shown that θ_n is full.

The second part of the theorem amounts to prove the injectivity of the map

$$\mathrm{Hom}_{\mathcal{T}_k}(X, Y) \to \mathrm{Hom}_{IP_k(F)}(\theta_k X, \theta_k Y) \tag{5.11}$$

for $k \geq n$. This map is a homomorphism of abelian groups since θ_k is additive. Let $g : X \to Y$ represent a morphism that is mapped to zero. Again we pick replacements $X^\bullet$ and $Y^\bullet$ of r^0X and r^0Y as above. We find a map $g_\infty : X^\bullet \to Y^\bullet$ whose homotopy class is uniquely determined by $r^0g : r^0X \to r^0Y$ and which is nullhomotopic in $c\mathcal{M}^F$ when we restrict it to the $(k+1)$-skeleton of $X^\bullet$. This is displayed in the following solid arrow diagram which strictly commutes:



The evaluation maps d_0 and $d_1 : \mathrm{hom}(\Delta^1, Y^\bullet) \to Y^\bullet$ are F-equivalences, so for both objects their $\pi^0 F_*$-term is isomorphic to F_*Y in $\mathcal{T}_n$. In particular it follows from the first part of the theorem that H' exists with $H's_{k+1} \simeq H$ in the F-structure. Actually the proof of 5.3.35 shows that we can arrange this to be strictly equal. It tells us that d_0H' and $g_\infty j_{k+2}$ are both extensions of the map $g_\infty j_{k+1} = d_0H$. The obstructions against uniqueness of liftings, which are the homotopy classes of these extensions, lie in $\mathrm{Ext}_{\mathcal{A}}^{k+1,k+1}(F_*X, F_*Y)$ and this group vanishes since the injective dimension is smaller than or equal to $n < k+1$ by assumption. It follows that $g_\infty j_{k+2}$ is F-homotopic to d_0H'. The same argument works with the other evaluation map d_1 and shows that $g_\infty j_{k+2}$ is nullhomotopic. By induction we can extend this over all skeleta since all higher obstruction groups also vanish. The skeletal tower of $X^\bullet$ is a tower of F-cofibrations between Reedy cofibrant (aka. F-cofibrant) objects by 2.2.8 since $X^\bullet$ is Reedy cofibrant, therefore we have

$$X^\bullet \cong \operatorname*{colim}_k \mathrm{sk}_k X^\bullet \cong \operatorname*{hocolim}_k \mathrm{sk}_k X^\bullet.$$

Hence the successive extensions give us a map $g'_\infty : X^\bullet \to Y^\bullet$ which on one side is homotopic to g_∞ and on the other to $*$. Because the homotopy class of g_∞ or g'_∞ corresponds under the isomorphism

$$\pi_0 \mathrm{map}(X^\bullet, Y^\bullet) \cong \pi_0 \mathrm{map}(r^0X, r^0Y)$$

to $r^0 g$, this shows that our original map $r^0 g$ is nullhomotopic in $c\mathcal{M}$. Finally constant cosimplicial objects over fibrant objects are Reedy fibrant, so lemma 2.1.12 applies and we can conclude:

$$[g] = 0 \in \pi_0 \mathrm{map}_{\mathcal{M}}(X, Y)$$

□

Actually the previous statement can be strengthened since for both assertion only the fact that Y is in $\mathcal{T}_n$ was needed.

5.5 Moduli spaces of realizations

Definition 5.5.1 Let A be an object in $\mathcal{A}$. We define the **space of realizations** of A to be the moduli space of all objects X in $\mathcal{M}$, such that their image $F_* X$ is isomorphic to A (see Def. 1.4.3). We will write $\mathbf{Real}(\boldsymbol{A})$.

We define the **space of $\boldsymbol{n}$-th partial realizations** of A to be the moduli space of all objects $X^\bullet$ in $c\mathcal{M}$ that are potential n-stages for A (see Def. 5.3.1). We will write $\mathbf{Real}_{\boldsymbol{n}}(\boldsymbol{A})$. Everything makes also sense for $n = \infty$ and hence we define in the same way the **space of $\boldsymbol{\infty}$-stages** of A and denote it by $\mathbf{Real}_{\boldsymbol{\infty}}(\boldsymbol{A})$.

Recall that ∞-$\mathcal{G}$-structure is just another name for the $\mathcal{G}$-structure. The first theorem we are heading for is 5.5.3 which tells us that ∞-stages are the same as actual realizations in $\mathcal{T} = Ho(\mathcal{M})$. The next step is theorem 5.5.4 which relates the moduli space of ∞-stages to the spaces $\mathrm{Real}_n(A)$ of potential n-stages. Finally we establish in 5.5.10 a fiber sequence involving $\mathrm{Real}_{n-1}(A)$ and $\mathrm{Real}_n(A)$.

Remark 5.5.2 To relate an ∞-stage of an object in $\mathcal{A}$ to an actual realization we use the functor $\mathrm{Tot} : c\mathcal{M} \to \mathcal{M}$. To compute $F_* \mathrm{Tot}\, X^\bullet$ we use a cohomology spectral sequence associated to the total tower of $X^\bullet$. So, if $X^\bullet$ is Reedy fibrant, there is a spectral sequence with

$$E_2^s = \pi_s[X^\bullet, G].$$

From lemma 4.2.32 and its proof we can read off that for an ∞-stage $X^\bullet$ of an object A the spectral sequence collapses and its edge homomorphism gives the following isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(A, F_* G) \cong \pi_0[X^\bullet, G] \cong \operatorname*{colim}_s [\,\mathrm{Tot}_s\, X^\bullet, G] \cong [\,\mathrm{Tot}\, X^\bullet, G]$$

for every $G \in \mathcal{G}$ or equivalently an isomorphism

$$F_* \,\mathrm{Tot}\, X^\bullet \cong A.$$

We see that the functor Tot induces a natural map

$$\mathrm{Real}_\infty(A) \to \mathrm{Real}(A). \tag{5.12}$$

Theorem 5.5.3 *The map* (5.12) *is a weak equivalence of spaces.*

Proof: Let X be a realization of A in $\mathcal{M}$. Then the canonical map $X \to \mathrm{Tot}\, r^0 X = \mathrm{Tot}_0\, r^0 X = X$ is even an isomorphism in $\mathcal{M}$.

Let $X^\bullet$ be a vertex in $\mathrm{Real}_\infty(A)$, in other words an ∞-stage of A. Without loss of generality we assume that $X^\bullet$ is F-fibrant and Reedy cofibrant, because these manipulations induce self equivalences of the moduli space $\mathrm{Real}_\infty(A)$. But now the map

$$r^0\, \mathrm{Tot}\, X^\bullet \to X^\bullet$$

is an F-equivalence by construction. This shows that the maps induced by Tot and r^0 are mutually inverse homotopy equivalences.

□

Theorem 5.5.4 *The canonical map*

$$\mathrm{Real}_\infty(A) \to \operatorname*{holim}_n \mathrm{Real}_n(A)$$

is a weak equivalence.

We prove this theorem after having established two lemmas.

Definition 5.5.5 Let $\mathbf{weak}_{\boldsymbol{S}}\boldsymbol{(A^\bullet, B^\bullet)}$ denote the simplicial set given by

$$\mathrm{weak}_S(A^\bullet, B^\bullet)_n := \mathrm{Hom}_{\mathcal{W}_S}(A^\bullet \otimes \Delta^n, B^\bullet),$$

where $\mathcal{W}_S$ is the subcategory of weak equivalences in some simplicial model structure S on $c\mathcal{M}$. If $A^\bullet$ is fibrant and cofibrant in S then

$$\mathrm{weak}_S(A^\bullet, A^\bullet) = \mathrm{haut}_S(A^\bullet)$$

by definition 1.4.1. Analogously to remark 1.4.2 we observe that $\mathrm{weak}_S(A^\bullet, B^\bullet)$ is a union of connected components of $\mathrm{map}_S(A^\bullet, B^\bullet)$.

Lemma 5.5.6 *Let $\mathcal{G}$ be a class of injective models for $\mathcal{M}$. Let $X^\bullet$ be a Reedy cofibrant object and $Y^\bullet$ be a $\mathcal{G}$-fibrant object in $c\mathcal{M}$. Then there is a canonical map*

$$\operatorname*{holim}_n \mathrm{weak}_{n-\mathcal{G}}(\mathrm{sk}_{n+1} X^\bullet, Y^\bullet) \xrightarrow{\simeq} \lim_n \mathrm{weak}_{n-\mathcal{G}}(\mathrm{sk}_{n+1} X^\bullet, Y^\bullet) \cong \mathrm{weak}_{\mathcal{G}}(X^\bullet, Y^\bullet)$$

where the first one is a weak equivalence and the second one is an isomorphism which are natural in both variables for $\mathcal{G}$-equivalences.

Proof: First we observe that the corresponding statement for the functor $\mathrm{map}(\_,\_)$ is true. Here $\mathrm{map}(\_,\_)$ which is the external mapping space from 2.1.9 always has homotopy meaning since $\mathrm{sk}_{n+1} X^\bullet \to X^\bullet$ is an n-$\mathcal{G}$-cofibrant approximation. Also the tower maps are fibrations by (SM7') because they are induced by the $\mathcal{G}$-cofibration $\mathrm{sk}_n X^\bullet \to \mathrm{sk}_{n+1} X^\bullet$. Finally $\mathrm{map}(\_,\_)$ turns colimits in the first variable into limits and $\mathrm{colim}_n \mathrm{sk}_{n+1} X^\bullet \cong X^\bullet$.

The proof is finished by the above remark that $\mathrm{weak}_S(X^\bullet, Y^\bullet)$ is a union of components in $\mathrm{map}_S(X^\bullet, Y^\bullet)$ and that these components form a tower because n-$\mathcal{G}$-equivalences are mapped to $(n-1)$-equivalences by the restriction of the upper maps.

□

Lemma 5.5.7 *Let $X^\bullet$ be F-fibrant and Reedy cofibrant, then the canonical map*

$$\mathrm{haut}_F(X^\bullet) \to \operatorname*{holim}_n \mathrm{haut}_{n-F}(\mathrm{sk}_{n+1} X^\bullet)$$

is a weak equivalence.

Remark 5.5.8 Note that the homotopy self equivalences on the right hand side can also be taken in the F-structure since n-F-equivalences and F-equivalences agree on n-F-cofibrant objects by lemma 3.2.15.

Proof of 5.5.7: The inclusions of the skeletons into $X^\bullet$ induce the following commutative diagram:

$$\begin{array}{ccc} \operatorname{haut}_{n-F}(\operatorname{sk}_{n+1} X^\bullet) & \longrightarrow & \operatorname{haut}_{(n-1)-F}(\operatorname{sk}_n X^\bullet) \\ \simeq\downarrow & & \downarrow\simeq \\ \operatorname{weak}_{n-F}(\operatorname{sk}_{n+1} X^\bullet, X^\bullet) & \xrightarrow{\omega_n} & \operatorname{weak}_{(n-1)-F}(\operatorname{sk}_n X^\bullet, X^\bullet) \end{array}$$

Both horizontal maps fit into a tower of maps when we vary n. We want to compute the homotopy limit of the upper tower. To do this we have to replace this tower by an objectwise weakly equivalent one in which the tower maps are fibrations. This is provided by the lower tower as we proved in 5.5.6. The vertical maps are homotopy equivalences because $\operatorname{sk}_{n+1} X^\bullet \to X^\bullet$ is a cofibrant approximation in the n-F-structure by 3.2.13. The result follows now from 5.5.6.

□

Proof of 5.5.4: By theorem 1.4.7 we have the following weak equivalences

$$\operatorname{Real}_\infty(A) \simeq \coprod_{\langle X^\bullet\rangle_F} B\operatorname{haut}_F(X^\bullet)$$

where the coproduct is taken over all F-equivalence classes $\langle X^\bullet\rangle$ of ∞-stages $X^\bullet$ of A. By the same theorem we obtain the first of the next two weak equivalences

$$\operatorname{Real}_n(A) \simeq \coprod_{\langle X_n^\bullet\rangle_F} B\operatorname{haut}_F(X_n^\bullet) \simeq \coprod_{\langle X_n^\bullet\rangle_{n-F}} B\operatorname{haut}_{n-F}(X_n^\bullet),$$

where the coproduct is taken over all F-equivalence classes $\langle X_n^\bullet\rangle$ of potential n-stages $X_n^\bullet$ of A. By lemma 3.2.15 there is a one-to-one correspondence between equivalence classes of potential n-stages in the F-structure and in the n-F-structure and there is a weak equivalence $\operatorname{haut}_F(X^\bullet) \simeq \operatorname{haut}_{n-F}(X^\bullet)$. Hence we get the second weak equivalence where the coproduct is taken over weak equivalence classes in the n-F-structure. The theorem follows now from lemma 5.5.7 and the fact that the classifying space functor B from simplicial monoids to $\mathcal{S}$ preserves weak equivalences, fibrations and limits.

□

Remark 5.5.9 Let $X^\bullet$ be an F-fibrant object with $\pi^s F_* X^\bullet \cong 0$ for $s>0$ and $X^\bullet = \operatorname{cosk}_n X^\bullet$. This means that the normalized complex $NF_*X^\bullet$ is an injective resolution of $F_* \operatorname{Tot} X^\bullet$ of length n or equivalently that $X^\bullet$ is an object of $\mathcal{T}_n(F)$. It follows that $\operatorname{Tot}_s X^\bullet \cong \operatorname{Tot} X^\bullet$ for $s \geq n$, hence we have equivalences

$$\operatorname{Real}(F_* \operatorname{Tot} X^\bullet) \xleftarrow{\simeq} \operatorname{Real}_\infty(F_* \operatorname{Tot} X^\bullet) \xrightarrow{\simeq} \operatorname{Real}_n(F_* \operatorname{Tot} X^\bullet).$$

Theorem 5.5.10 *Let $X_{n-1}^\bullet$ be a potential $(n-1)$-stage for an object A in $\mathcal{A}$. Then there is a fiber sequence*

$$\mathcal{H}^{n+1}(r^0 A, A[n]) \to \operatorname{Real}_n(A)_{X_{n-1}^\bullet} \to \mathcal{M}(X_{n-1}^\bullet),$$

where $\operatorname{Real}_n(A)_{X_{n-1}^\bullet}$ are those components of $\operatorname{Real}_n(A)$ that correspond to objects $X^\bullet$ with $\operatorname{sk}_n X^\bullet \simeq X_{n-1}^\bullet$.

Proof: By 5.3.24 there is a cofiber sequence

$$X_{n-1}^\bullet \to X_n^\bullet \to L(A[n], n)$$

in $c\mathcal{M}^F$ inducing the following fiber sequence in $\mathcal{S}$:

$$\operatorname{map}(L(A[n], n), X_n^\bullet) \to \operatorname{map}(X_n^\bullet, X_n^\bullet) \to \operatorname{map}(X_{n-1}^\bullet, X_n^\bullet)$$

Passing to appropriate components gives a fiber sequence

$$\operatorname{map}(L(A[n], n), X_n^\bullet) \to \operatorname{weak}_{n-F}(X_n^\bullet, X_n^\bullet) \to \operatorname{weak}_{(n-1)-F}(X_{n-1}^\bullet, X_n^\bullet)$$

of grouplike simplicial monoids. Applying the classifying space functor B to this sequence yields a fiber sequence

$$B \operatorname{map}(L(A[n], n), X_n^\bullet) \to \mathcal{M}(X_n^\bullet)_{X_{n-1}^\bullet} \to \mathcal{M}(X_{n-1}^\bullet).$$

Let $\Gamma : \operatorname{CoCh}^{\geq 0}(\mathcal{A}) \to c\mathcal{A}$ be the Dold-Kan-functor. We compute finally using 5.3.15:

$$\begin{aligned} B \operatorname{map}(L(A[n], n), X_n^\bullet) &\simeq B \operatorname{map}(K(A[n], n), \operatorname{sk}_{n+1} F_* X_n^\bullet) \\ &\simeq B \operatorname{map}(K(A[n], n), r^0 A) \\ &\simeq B\, \Gamma(\operatorname{Hom}_{\mathcal{A}}(N, A)[n]_{\text{ext}}) \\ &\simeq \Gamma(\operatorname{Hom}_{\mathcal{A}}(N, A)[n+1]_{\text{ext}}) \\ &\simeq \mathcal{H}^{n+1}(A, A[n]), \end{aligned}$$

where $\operatorname{Hom}_{\mathcal{A}}(N, A)[n]_{\text{ext}}$ is viewed as a cochain complex concentrated in degree n. Here $[1]_{\text{ext}}$ is the external shift from 4.2.22.

□

Theorem 5.5.11 *Let $f : X_n^\bullet \to Y_n^\bullet$ be a map of potential n-stages for objects A and B respectively in $\mathcal{A}$. Then there is a fiber sequence*

$$\mathcal{H}^n(A[n], B) \to \mathcal{M}(f)_{\operatorname{sk}_n f} \to \mathcal{M}(\operatorname{sk}_n f)$$

Proof: We can assume without loss of generality that $X_n^\bullet$ and $Y_n^\bullet$ are Reedy cofibrant and F-fibrant. As in the proof of 5.5.10 we obtain a fiber sequence

$$\operatorname{map}(L(A[n], n), Y_n^\bullet) \to \operatorname{map}(X_n^\bullet, Y_n^\bullet) \to \operatorname{map}(\operatorname{sk}_n X_n^\bullet, Y_n^\bullet).$$

Proceeding like in the previous proof we arrive at the conclusion.

□

6 Examples and applications

6.1 Very low dimensions

Example 6.1.1 If the injective dimension of the target category is 0 then the tower of interpolation categories simply collapses to the equivalences:

$$\begin{array}{ccc} IP_0(F) & \xrightarrow{\cong} & \mathcal{A} \\ \cong \uparrow & & \uparrow \cong \\ \mathcal{T} = \mathcal{T}_0 & \xrightarrow[\cong]{F} & \mathcal{A}_{\text{inj}} \end{array}$$

Here the lower equivalence was already stated in 4.1.9. This happens for instance when we look at the functor $F = H^*$ from the derived category of the category of vector spaces over a field to the category of graded vector spaces. The $*$ in H^* corresponds to the usual dimension or degree of the cohomology of a cochain complex, and H^* is to be interpreted as the graded version of H^0 from 3.4.4.

Considering $\mathbb{Z}$ instead of a field in the last example takes us to the case of injective dimension 1.

Example 6.1.2 If the injective dimension of $\mathcal{A}$ is 1 then the tower of interpolation categories has one non-trivial step:

$$\begin{array}{ccccc} \mathcal{T} = \mathcal{T}_1 & \xrightarrow{\cong} & IP_1(F) & & \\ & & \downarrow & & \\ & & IP_0(F) & \xrightarrow{\cong} & \mathcal{A} \\ & & \uparrow & & \uparrow \\ & & \mathcal{T}_0 & \xrightarrow[\cong]{} & \mathcal{A}_{\text{inj}} \end{array}$$

We can express this using 5.3.39 or 5.4.1 by saying that

$$\operatorname{Ext}^{1,1}_{\mathcal{A}}(F_*(\_), F_*(\_)) \to \mathcal{T} \to \mathcal{A}$$

is a linear extension of categories which is defined in [Bau99, VI.5].

6.2 $KU_{(p)}$-local spectra

Already the case of injective dimension 2 is a very interesting example, since Bousfield has shown in [Bou85] that the category of $KU_{(p)*}KU_{(p)}$-comodules which is the natural target category for complex K-theory localized at p denoted by $KU_{(p)}$ has injective dimension 2 for primes $p > 2$.

In case of dimension 2 we have two steps:

$$\mathcal{T} = \mathcal{T}_2 = IP_2(F) \longrightarrow IP_1(F) \longrightarrow IP_0(F) \xrightarrow{\cong} \mathcal{A},$$

It follows from the obstruction calculus that the functor $\mathcal{T} \to \mathcal{A}$ is essentially surjective since the first obstruction against realizing objects lives in $\operatorname{Ext}^{3,1}_{\mathcal{A}}(F_*X, F_*X) = 0$. The obstruction against uniqueness of realizations of objects from $IP_0(F)$ in $IP_1(F)$ are given by 5.3.32, they live in $\operatorname{Ext}^{2,1}_{\mathcal{A}}(F_*X, F_*X)$ and all of the elements correspond to different realizations. We do not have any obstruction for lifting from $IP_1(F)$ to $IP_2(F)$. So the isomorphism classes in $\mathcal{T}$ correspond bijectively to pairs (A, κ), where A is an object from $\mathcal{A}$ and $\kappa \in \operatorname{Ext}^{2,1}_{\mathcal{A}}(A, A)$. In the case of $F_* = KU_{(p)*}$ the category $\mathcal{A}$ is the category of $KU_{(p)*}KU_{(p)}$-comodules denoted by $KU_{(p)*}KU_{(p)} - \text{comod}$ and $\mathcal{T}$ is the stable homotopy category of spectra that are at the same time KU-local and p-local for $p > 2$. We have reproved Bousfield's theorem [Bou85, theorem 9.1] which states this classification in the case of $KU_{(p)}$.

Let us examine the situation of theorem 5.3.37 more closely. We note that for $n = 0$ the term $\operatorname{Hom}_{IP_0(F)}(\sigma_0 X^\bullet, \sigma_0 Y^\bullet)$ is isomorphic to $\operatorname{Hom}_{\mathcal{A}}(\pi^0 F_* X^\bullet, \pi^0 F_* Y^\bullet)$ via the equivalence of categories proved in 5.4.2. Now we assume that the cosimplicial objects come from objects X and Y in $\mathcal{T}$. Then the map in theorem 5.3.37 has the same form as the differential

$$d_2 : \operatorname{Hom}^t_{\mathcal{A}}(F_*X, F_*Y) \to \operatorname{Ext}^{2,t+1}_{\mathcal{A}}(F_*X, F_*Y)$$

of the modified Adams spectral sequence. Indeed, it should follow from [DKS95, §8] or [GH04, 3.9], although we have not checked the details for our dual case, that the exact couple that we get by applying the functor $[\_, Y^\bullet]_F$ to the skeletal tower of $X^\bullet$ is isomorphic to the derived couple of the exact couple

$$\begin{array}{ccccccccc} \cdots \longleftarrow & [X, \mathrm{Tot}_{s-1} Y^\bullet] & \longleftarrow & [X, \mathrm{Tot}_s Y^\bullet] & \longleftarrow & [X, \mathrm{Tot}_{s+1} Y^\bullet] & \longleftarrow \cdots \\ & \downarrow & \nearrow & \downarrow & \nearrow & & \\ & [X, N^s \Omega^s Y^\bullet] & & [X, N^s \Omega^s Y^\bullet] & & & \end{array}$$

used to construct the modified Adams spectral sequence, compare 4.2.35. Then equation (5.7) corresponds to the fact [Bou85, Prop. 8.10.] that for an $f \in \mathrm{Hom}^t_{\mathcal{A}}(F_*X, F_*Y)$ there is the equation

$$d_2 f = k_Y f + (-1)^{t+1} f k_X.$$

Here k_X and k_Y were called the $E(1)_*$-k-invariant and are the elements associated to X and Y in their $\mathrm{Ext}^{2,1}$-term via the classification above. The sign comes from plugging in $\Omega^t Y$ instead of Y. From 5.3.35 and 5.3.36 it follows that an $f : F_*X \to F_*Y$ lifts to $IP_1(F)$ if and only if $d_2 f = 0$. Every map in $IP_1(F)$ lifts further since there are no higher obstructions, so we derive [Bou85, Cor. 8.11.], which states that a map f is induced by an element of $[X, Y]$ if and only if $d_2 f$ vanishes.

We hope to reprove other statements from [Bou85] and [Fra96] in future work.

References

[Ada74] J. F. Adams. *Stable homotopy and generalised homology.* The University of Chicago Press, Chicago, 1974.

[Bau99] H.J. Baues. *Combinatorial Foundations of Homology and Homotopy.* Monographs in Mathematics. Springer, 1999.

[BDG01] D. Blanc, W.G. Dwyer, and P.G. Goerss. The realization space of a Π-algebra: a moduli problem in algebraic topology. http://www.nd.edu/ wgd, 2001.

[Bie02] Georg Biedermann. Injective completion with respect to homology. unpublished, 2002.

[BK72] A.K. Bousfield and D.M. Kan. *Homotopy limits, completions and localizations.* Number 304 in Lecture Notes in Mathematics. Springer-Verlag, 1972.

[BKS04] D. Benson, H. Krause, and S. Schwede. Realizability of modules over Tate cohomology. *to appear: Transactions of the American Mathematical Society*, 2004.

[Boa99] J.M. Boardman. Conditionally convergent spectral sequences. In J.-P. Meyer, J. Morava, and W.S. Wilson, editors, *Homotopy Invariant Algebraic Structures: A Conference in Honor of J. Michael Boardman*, volume 239 of *Contemporary Mathematics*, pages 49–84. AMS, 1999.

[Bou79] A.K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18:257–281, 1979.

[Bou85] A.K. Bousfield. On the homotopy theory of k-local spectra at an odd prime. *Amer. J. Math.*, 107(4):895–932, 1985.

[Bou03] A.K. Bousfield. Cosimplicial resolutions and homotopy spectral sequences in model categories. *Geomtry and Topology*, 7:1001–1053, 2003.

[Bri68] Hans-Berndt Brinkmann. Relative homological algebra and the Adams spectral sequence. *Arch. Math. (Basel)*, 19:137–155, 1968.

[Dev97] Ethan S. Devinatz. Morava modules and Brown-Comenetz duality. *Amer. J. Math.*, 119(4):741–770, 1997.

[DK84] W.G. Dwyer and D.M. Kan. A classification for diagrams of simplicial sets. *Topology*, 23(2):139–155, 1984.

[DKS93] W. G. Dwyer, D. M. Kan, and C. R. Stover. An E^2 model category structure for pointed simplicial spaces. *J. Pure Appl. Algebra*, 90(2):137–152, 1993.

[DKS95] W.G. Dwyer, D.M. Kan, and C.R. Stover. The bigraded homotopy groups $\pi_{i,j}X$ of a pointed simplicial space. *Journal of Pure and Applied Algebra*, 103:167–188, 1995.

[DS95] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995.

[Fra96] J. Franke. Uniqueness theorems for certain triangulated categories possessing an Adams spectral sequence. K-Theory Archives, http://www.math.uiuc.edu/K-Theory, 139, 1996.

[GH] P.G. Goerss and M.J. Hopkins. Resolutions in Model Categories. unpublished.

[GH04] P.G. Goerss and M.J. Hopkins. Moduli spaces of commutative ring spectra. http://www.math.nwu.edu/ pgoerss, to appear in: London Mathematical Society, Lecture Note Series, Cambridge University Press, 2004.

[GJ99] Paul G. Goerss and John F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[Hir03] Philip S. Hirschhorn. *Model categories and their localizations*, volume 99 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2003.

[Hov99] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[Hov04] Mark Hovey. Homotopy theory of comodules over a Hopf algebroid. In *Homotopy Theory: Relations with Algebraic Geomtry, Group Cohomology, and Algebraic K-Theory*, volume 346 of *Contemporary Mathematics*, pages 261–304. American Mathematical Society, 2004.

[Jar04] J.F. Jardine. Bousfield's E_2 model theory for simplicial objects. In *Homotopy Theory: Relations with Algebraic Geomtry, Group Cohomology, and Algebraic K-Theory*, volume 346 of *Contemporary Mathematics*, pages 261–304. American Mathematical Society, 2004.

[Mey90] Jean-Pierre Meyer. Cosimplicial homotopies. *Proc. Amer. Math. Soc.*, 108(1):9–17, 1990.

[Qui67] Daniel G. Quillen. *Homotopical algebra*. Lecture Notes in Mathematics, No. 43. Springer-Verlag, Berlin, 1967.

[Ree74] C.L. Reedy. Homotopy theory of model categories. unpublished, http://www-math.mit.edu/∼psh, 1974.