Gluing Spaces and Analysis

Dissertation

zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen-Friedrich-Wilhelms-Universität Bonn

vorgelegt von Gustav Paulik
aus Göttingen

Bonn, Oktober 2004

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen-Friedrich-Wilhelms-Universität Bonn

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Tag der Promotion: 15. Dezember 2004

Diese Dissertation ist auf dem Hochschulschriftenserver der ULB Bonn http://hss.ulb.uni-bonn.de/diss_online elektronisch publiziert.

Contents

Introduction

The aim of this work is to study the gluing of several metric measure spaces (M_i, d_i, μ_i) for $i = 1, \ldots, k$ where on each of them a strongly local, regular Dirichlet form $(\mathcal{E}_i, D(\mathcal{E}_i))$ is defined. Additionally, each space satisfies a doubling property (or is Ahlfors-regular) and a strong scaling invariant Poincaré inequality for all balls holds. The glued space is denoted by (M, d, μ) and the new strongly local regular Dirichlet form by $(\mathcal{E}, D(\mathcal{E}))$. We start with $k = 2$ but all conditions and results are well suited so that the gluing can be extended to the case of gluing k metric measure spaces M_i along gluing sets A_i in an iterative procedure. Our main goal is to derive the doubling property for the measure μ and the metric

$$\rho(x, y) := \sup\{u(x) - u(y) : u \in D_{loc}(\mathcal{E}) \cap C(M),\ d\Gamma(u) \leq d\mu\}$$

and the scaling invariant Poincaré inequality on the glued space M, i.e. there exists a constant $c > 0$ such that for all balls $B(x, r) := \{y \in M : \rho(x, y) < r\}$ and for all $u \in D(\mathcal{E})$

$$\int_{B(x,r)} |u - u_{B(x,r)}|^2 d\mu \leq c \cdot r^2 \int_{B(x,r)} d\Gamma(u). \tag{1}$$

holds. Here $d\Gamma$ denotes the energy measure of the Dirichlet form $\mathcal{E}$. For that only assumptions on the Dirichlet forms $(\mathcal{E}_i, D(\mathcal{E}_i))$ and on the separate pieces M_i, $(i = 1, \ldots, k)$ shall be used.

The crucial motivation for this goal is a series of papers by K.-T. Sturm [St96], [St95b], M. Biroli, N.A. Tchou [BT97], M. Biroli, U. Mosco [BM95a], [BM95b] and a paper by Ramirez [Ra01] where many important applications for strongly local regular Dirichlet forms, the associated processes and the heat kernel are proved, provided the doubling property and a scale invariant Poincaré inequality hold true.

We succeeded to prove (1) provided a lower bound $c\frac{1}{r^2}$ on the "heat transmission coefficient"

$$\nu_i(B_i, N) := \inf\left\{ \frac{\int_{B_i} d\Gamma_i(u)}{\int_{B_i} |u|^2 d\mu_i} : u \in D(\mathcal{E}_i),\ \tilde{u}|_{N\cap B_i} = 0,\ u|_{B_i} \neq 0 \right\} \tag{2}$$

for certain sets B_i centered at A and certain sets $N \subset A \cap B_i$ holds true. Some other assumptions have to be made in order to get doubling which in turn we need to prove (1). Further conditions are discussed briefly below where we give an overview of the chapters.

In Chapter 1 an intrinsic metric d is constructed on $M = M_1 \cup_A M_2$ in a canonical way (cf. [BBI01]) as the length of the shortest continuous path between two points w.r.t. the local metrics d_1 and d_2 taking into account that the gluing sets $A_1 \subset M_1$ and $A_2 \subset M_2$ are identified via an equivalence relation R. Here the equivalence relation R comes from a bijective gluing map $\Phi : A_1 \mapsto A_2$, i.e.

$$x \sim_R y :\Leftrightarrow \Phi(x) = y,$$

and the gluing set is denoted by $A \subset M$. In order to avoid collapsing phenomena, for instance to keep the doubling property or to ensure that the new topology τ induced by d coincides with the topology τ^R coming from the topological identification, we choose Φ to be bilipschitz at least. With this we prove the comparability of the metrics d_i and d on the original pieces M_i in Section 1.2. The consistency of the new topology τ induced by d with the original topologies τ_i (i.e. $\forall\, O \in \tau : O \cap M_i \in \tau_i$) is proved in Section 1.4. In Section 1.5 we define the glued measure μ as

$$\mu(B) := \mu_1(B \cap M_1) + \mu_2(B \cap M_2) - \mu_1(B \cap A)$$

provided the measures μ_1 and μ_2 are consistent on the gluing set A. We prove that if the original measures are positive Radon measures μ is a positive Radon measure too. Given the doubling property on M_i we show in Section 1.6 that doubling holds on M if a "dimension homogeneity condition" is satisfied. This in particular is true for Ahlfors regular spaces. The extension to glue k spaces is discussed in more detail in Section 1.7. To illustrate our results we present several examples of gluing constructions in Section 1.8. Special cases of the gluing map Φ are treated briefly in Section 1.3.

In Chapter 2 we define the glued Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ given a consistency condition in a canonical way, i.e.

$$\mathcal{E}(u) := \int_{M_1} d\Gamma_1(u,u) + \int_{M_2} d\Gamma_2(u,u) - \int_A d\Gamma_1(u,u)$$

$\forall\, u \in C_0^{Lip}(M)$ while $d\Gamma_i$ is the energy measure of the Dirichlet form $\mathcal{E}_i$. We show in Section 2.1 that starting with two strongly local regular Dirichlet forms $(\mathcal{E}_1, D(\mathcal{E}_1))$ and $(\mathcal{E}_2, D(\mathcal{E}_2))$ on M_1 and M_2 we get a strongly local regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ as the closure of $C_0^{Lip}(M)$ w.r.t. $\mathcal{E}_1(\cdot) := \big(\mathcal{E}(\cdot) + ||\cdot||_{L^2(M,\mu)}\big)$ on the glued space M. This procedure can then be easily extended to glue k Dirichlet forms. In Section 2.2 we describe some possible gluing constructions of Dirichlet forms. In particular we show that glued spaces appear as the limit of converging spaces, as spiders for example in [Bo04]. Furthermore, the behavior of the associated diffusion X_t after hitting the gluing set A is discussed. Namely in glued graphs (or 2-dimensional Euclidean complexes) the process (X_t, P_x) with $x \in A$ will in some sense leave the set A in each direction with equal probability. Here A will be the set of vertices (or edges). In weighted graphs the process will leave A in each edge with the proportional probability of its weight. With Section 2.3 we close the chapter giving a proof of the comparability of the metrics d and ρ on M provided d_i and ρ_i are comparable on M_i.

The idea to prove the main result of this work in Chapter 3 is to reformulate the Poincaré inequality (1) as a lower bound for the spectral gap, i.e. to show that there exists a constant $c > 0$ such that

$$\frac{\int_{B(x,r)} d\Gamma(u)}{\int_{B(x,r)} |u|^2 d\mu} \geq \frac{c}{r^2} \tag{3}$$

holds for all functions $u \in D(\mathcal{E})$ with $u \neq 0$ and

$$u_{B(x,r)} = \frac{1}{B(x,r)} \int_{B(x,r)} u\, d\mu = 0.$$

This in turn can be reduced to a lower bound of the heat transmission coefficient (2) on the separated pieces M_i for $i = 1, \ldots, k$. For simplicity we start the technical proof with $k = 2$ and extend the result to general k in 3.1.3. In Section 3.2 we discuss two special cases of gluing which essentially simplifies the proof. Namely if the gluing set A is locally large enough, i.e. there exist constants $c_i > 0$ such that $\forall\, x \in A,\ r > 0$

$$\mu_i(B_i(x,r) \cap A) \geq c_i \mu_i(B_i(x,r))$$

holds on M_i we can prove the scale invariant Poincaré inequality without using (2). Further in the case of isometric gluing maps Φ our condition on the heat transmission coefficient simplifies significantly. Section 3.3 provides examples in the n-dimensional Euclidean setting, i.e. we check condition (2) for special gluing sets

$A \subset \mathbf{R}^n$. This lower bounds can be achieved by a rescaling argument with the results of Denzler [De99a], [De99b] who gives lower bounds for the spectral gap on domains with mixed Neumann-Dirichlet boundary condition.

For applications or consequences of this work we cite and discuss the results in [St95b], [St96] or [Ra01] in Chapter 4. First, our glued strongly local regular Dirichlet form $\mathcal{E}$ on M determines a diffusion (X_t, P_x). As mentioned above with the doubling property and the Poincaré inequality we get Harnack inequalites and with this by Moser iteration the Hölder continuity of solutions of $(L - \frac{\partial}{\partial t})u = 0$ while L is the associated operator to $\mathcal{E}$. A direct consequence is that (X_t, P_x) can be chosen to be a Feller process (cf. Section 4.4). Other consequences are upper and lower Gaussian estimates for the transition probabilities of the associated process which in turn implies that the diffusion X_t crosses the gluing set A in finite time with positive probability. This together with the results from [Ra01] is used to demonstrate in Section 4.5 that the short time asymptotic for the heat kernel, i.e.

$$\lim_{t \to 0} 2t \log p_t(x, y) = -\rho^2(x, y)$$

is true on our glued space provided one additional condition holds, namely that our Dirichlet form admits a carré du champ operator.

The last chapter treats a slightly different subject. There some generalizations of results by Amick [Am78] are derived. In [Am78] characterizations of the validity of the Poincaré inequality and of Rellichs compact embedding theorem on a domain $\Omega \subset \mathbf{R}^n$ in terms of the quantity

$$\Gamma_\Omega(\epsilon) := \sup_{u \in W_2^1(\Omega)} \frac{\int_{\Omega_\epsilon} |u|^2}{|u|^2_{W_2^1(\Omega)}}$$

with $\Omega_\epsilon := \{x \in \Omega : d(x, \partial\Omega) < \epsilon\}$ are given. Since $\Gamma_\Omega(\epsilon)$ is in $(0, 1]$ for all $\epsilon > 0$ and monotone in ϵ we can define

$$\Gamma_\Omega(0) := \lim_{\epsilon \to 0} \Gamma_\Omega(\epsilon).$$

Amick proved that $\Gamma_\Omega(0) = 0$ is equivalent with the compactness of the embedding $i_\Omega : W_2^1(\Omega) \hookrightarrow L^2(\Omega)$ and $\Gamma_\Omega(0) < 1$ is equivalent with the Poincaré inequality

$$\int_\Omega |u - u_\Omega|^2 \leq const. \int_\Omega |\nabla u|^2$$

for all $u \in W_2^1(\Omega)$. With help of an idea by Biroli and Tchou [BT97] we prove characterizations of this kind for strongly local regular Dirichlet forms on metric measure spaces which satisfy a scaling invariant Poincaré inequality for balls inside Ω.

Now we want to mention some similar results in the literature. In [EF01] Eells and Fuglede derive a scaling invariant Poincaré inequality on Riemannian polyhedra for the canonical Dirichlet form coming from the canonical Dirichlet form $\mathcal{E}(u,v) = \int \nabla u \nabla v dx$ on the single simplices. For that they make heavily use of the Euclidean structure. Contrary to [EF01] in this work a priori no Euclidean structure is required. Heinonen and Koskela [HeK98] prove a Poincaré inequality in the upper gradient framework (see for instance [He01], [HK00]) for glued spaces provided a Poincaré inequality holds on the original spaces M_i. They consider an isometric gluing map which makes the construction of the new metric d easier. We discuss this special case for our framework in Section 3.2.2. Further, their results require the stronger Ahlfors regularity of the original spaces, i.e. there exist constants $c_i > 0$ and $n \in \mathbf{N}$ such that for all balls $B_i(x,r)$ in M_i it holds that

$$c_i^{-1} r^n \leq \mu_i(B_i(x,r)) \leq c_i r^n$$

while for our proofs only the doubling property is necessary.

Remark:
Several metrics appear in this work, the basic intrinsic metrics d_i and d on the original spaces M_i and the glued space M as well as the intrinsic metrics ρ_i and ρ coming from the original Dirichlet forms $\mathcal{E}_i$ and the glued Dirichlet form $\mathcal{E}$. The gluing proceeds by the basic metrics but since we assume comparability of the metrics d_i and ρ_i and prove the comparability of d and ρ we can often switch between the two metrics. If not explicitly stated it should be clear from the context which metric is meant.

Chapter 1

Gluing of Metric Measure Spaces

In this chapter the basic notions are defined and a framework for gluing metric measure spaces will be developed. In particular the question of consistency for our gluing procedure will be treated. That means the comparability of the metric on the glued space with that of the original space. This, in consequence, will ensure that open, closed or compact sets on the glued space will be open, closed or compact when projected on the original spaces. This would still be true for the case of collapsing because only the fact that the new intrinsic metric on the glued space d becomes smaller compared with the old intrinsic metrics d_i on M_i is necessary. However we do not consider collapsing phenomena in this work. Our gluing conditions yield comparability of the metrics. To treat non-bilipschitz gluing maps, that means collapsing is allowed, one had to think of a proper definition of the new Dirichlet form and the measure defined on the glued space. This definition would not be unique. Further Lipschitz continuous functions on the glued space will be Lipschitz when considered on the original space with respect to the old metrics. For the gluing of positive Radon measures, provided they are consistent on the gluing set, we have mainly to check the inner regularity which is done in Theorem 1.28. At the end of this chapter we will state conditions to transfer a given doubling property of the original measures μ_1 and μ_2 on M_1 and M_2 to the glued measure μ on M and we will extend the gluing procedure in order to glue together k metric measure spaces $M_1, \ldots, M_k$. Some examples to illustrate the results will finish this chapter.

1.1 Gluing of Metric Spaces

In order to glue a finite number k of metric spaces together along a subset by certain equivalence relations one has to specify in which manner the equivalence relations and the new metric shall be defined. Before we come to the gluing procedure we need the following (cf. [BBI01]) :

Definition 1.1 (Induced Intrinsic Metric, Length Space) *Let (M,d) be a metric space and $\hat{d}$ the new metric, defined in the following way:*

$$\hat{d}(x,y) := \inf\{L(\gamma) : \gamma : [a,b] \to M, \gamma \in C([a,b]\,, M), \gamma(a) = x, \gamma(b) = y\}$$

while $L(\gamma)$ is the length of the continuous path γ w.r.t. the old metric d i.e.

$$L(\gamma) := \sup \sum_{i=1}^{N} d(\gamma(y_{i-1}), \gamma(y_i))$$

while the supremum is taken over all partitions of $[a,b]$, that is a finite collection of points $\{y_0, \dots, y_N\}$ such that $a = y_0 \leq y_1 \leq \ldots \leq y_N = b$. Then $\hat{d}$ is called the intrinsic metric or length metric w.r.t. the length structure on M given by the continuous paths on (M,d) and the new metric space $(M,\hat{d})$ is called a length space. If for each $x, y \in M$ there exists a shortest path γ connecting x and y the length space M is called strictly intrinsic.

An intrinsic metric on length spaces is generally defined w.r.t. a length structure, i.e. a set of admissible paths P in the set M with a given structure like closedness under restrictions, concatenations, reparametrizations and a map $L : P \mapsto \mathbf{R}_+ \cup \infty$ which gives the length of a path and satisfies certain properties like additivity, continuity and invariance under reparametrizations. Here the set of admissible paths will consist of all continuous paths in a given metric space (M,d) and the length measure $L : P \mapsto \mathbf{R}_+ \cup \infty$ will be defined as above.
One can imagine an animal living on the ground going from A to B and a bird moving in the air. The animal has another intrinsic metric then the bird since the bird can fly straight lines while the animal on the ground has to go round obstacles and therefore has not so many admissible paths. So the distance will be greater than that of the bird.

Remark 1.2 (Intrinsic Metric) *Since the operation $d \to \hat{d}$ is idempotent and the set of admissible paths is fixed there is only one intrinsic metric $\hat{d}$ on M w.r.t. d.*

In the following we start with $k = 2$ to simplify the setting and we will extend it later to a general $k \in \mathbf{N}$. So let (M_1, d_1) and (M_2, d_2) be two complete locally compact separable length spaces.

For gluing two different metric spaces we first need some kind of identification of the gluing parts. Since we want to prevent collapsing we have to use a bilipschitz bijection $\Phi : A_1 \mapsto A_2$ while $A_i \subset M_i$ for $i = 1, 2$, i.e.

$$\frac{1}{Lip\ \Phi}\ d_1(x, y) \leq d_2(\Phi(x), \Phi(y)) \leq Lip\ \Phi\ \ d_1(x, y),$$

with $Lip\ \Phi > 0$ the Lipschitz constant. There are several possibilities for such a map Φ which we will describe at the end of this section. To explain the gluing procedure we need one more definition:

Definition 1.3 (Quotient Semi-Metric) *Let (M, d) be a metric space and R an equivalence relation on M. Then the quotient semi-metric d_R is defined as:*

$$d_R(x, y) := \inf\{\sum_{i=1}^{k} d(p_i, q_i),\ p_1 = x,\ q_k = y,\ k \in \mathbf{N}\}$$

while the infimum is taken over all choices $\{p_i\}$, $\{q_i\}$ such that q_i is R-equivalent to p_{i+1}, for all $i = 1, \ldots, k - 1$.

The gluing procedure (see Fig.1.1) now divides into three steps (cf. [BBI01]):

Definition 1.4 (Gluing)

- *The first step is to take the disjoint union $M := M_1 \dot{\cup} M_2$. This is a metric space and the metric is defined in the following way:*

$$d(x, y) := \begin{cases} d_i(x, y) & \text{if } x, y \in M_i, \\ \infty & \text{otherwise.} \end{cases}$$

- *The second step is to define a semi metric d_R which uses the (bilipschitz) bijection $\Phi : A_1 \mapsto A_2$ on M to define an equivalence relation in the following way:*

$$x \sim_R y :\Leftrightarrow \Phi(x) = y.$$

- *In the end to get a real metric we have to pass from the semi-metric space (M, d_R) to the quotient metric space $(M/d_R, d_R)$ which is a metric space. One gets the resulting space by gluing along the relation R.*

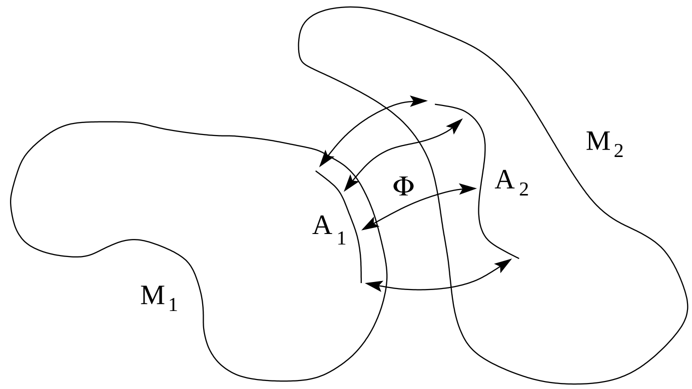

Figure 1.1: Bilipschitz gluing map $\Phi : A_1 \to A_2$

Remark 1.5

- $(M/d_R, d_R)$ *is a length space (cf. [BBI01]).*
- *The sets A_i will be closed because in the gluing procedure all points with zero distance in the new metric will be identified, s.t. there is no difference taking a set A_i or the closure of this set.*

In order to clarify the notations that we will use later on, we briefly give a formal explanation. For $i = 1, 2$ consider the canonical projection

$$\pi : M_1 \dot{\cup} M_2 \to M/d_R \quad \text{with} \quad \pi_(x) = R(x)$$

while $R(x)$ is the equivalence class of x in M/d_R. In the following we often use $A := \pi(A_1) = \pi(A_2)$ instead of A_i and we use M_i as the subset $\pi(M_i)$ in M/d_R. What exactly is meant should be obvious from the context. Consequently we denote our new glued space as $M_1 \cup_A M_2$ and say 'M_1 and M_2 are glued along the closed set A'. Further the new intrinsic metric d_R will be denoted by d.

Remark 1.6 *Note that d_i and d coincides locally on $M_i \setminus A$, that means for each $x \in M_i \setminus A_i \subset M_1 \cup_A M_2$ there exists an $r > 0$, s.t. $d_i|_{B_r(x)} = d|_{B_r(x)}$.*

The next lemma fixes what was laxly written above:

Lemma 1.7 *The gluing set $A \subset M$ is closed w.r.t. the topology induced by the new metric d on M.*

Proof: The set $M_i \setminus A = M_i \setminus A_i$ is open in the old topology of M_i. Therefore, for each $x \in M_i \setminus A_i$ there exist balls $B(x, \epsilon) \subset M_i \setminus A_i$. Since d and d_i coincide

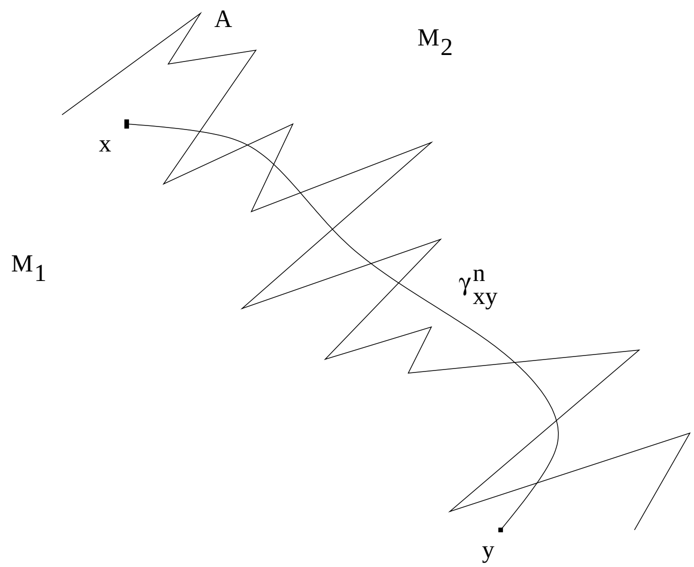

Figure 1.2: Possible distance approximating curves γ_{xy}^n between x and y

locally on $M_i \setminus A_i$ it holds that $B_i(x, \epsilon) = B(x, \epsilon)$. Hence $M_i \setminus A_i$ is open in M, s.t. $A = ((M_1 \cup M_2) \setminus A)^c = ((M_1 \setminus A) \cup (M_2 \setminus A))^c$ is closed.

■

1.2 Comparability

In the following section we will show that under our gluing condition, that Φ is bilipschitz, the metrics d_i and d are comparable on M_i. This is essential for our main results. In the following we mean by $d_1 \sim d_2$ on B that the distances d_1, d_2 are comparable on the set B, i.e.

$$\exists\, c > 0 : \forall x, y \in B : \frac{1}{c}\, d_1(x,y) \leq d_2(x,y) \leq c\, d_1(x,y)$$

holds. Intuitively this might be clear but by gluing not isometrically it can happen that the approximating curves in M cross the gluing set A several times or even infinitely often.

Lemma 1.8 (Comparability) *Let $M := M_1 \cup_A M_2$ be the metric space glued together by the two metric spaces M_1, M_2 along the closed subset A. If Φ is the bilipschitz gluing map between A_1 and A_2 then:*

$$d_i \sim d \;\; on \;\; M_i.$$

for $i = 1, 2$ holds.

Proof: We have to show the existence of $c > 0$, s.t. $\frac{1}{c}\ d_i(x,y) \le d(x,y) \le c\ d_i(x,y)$, $\forall\ x,y \in M_i$ and for $i = 1,2$. The second inequality is obvious, since by the construction of the new intrinsic metric, $d(x,y) \le d_i(x,y)$ holds on M_i for $i = 1,2$.

For the first inequality we switch shortly to the old notation, s.t. the glued metric d becomes d_R and d is the 'first step' metric of the gluing procedure. We consider the definition of the semi-metric

$$d_R(x,y) := \inf\{\sum_{i=1}^{k} d(p_i,q_i),\ p_1 = x,\ q_k = y,\ k \in \mathbf{N}\}$$

while

$$d(x,y) := \begin{cases} d_i(x,y) & \text{if } x,y \in M_i, \\ \infty & \text{otherwise,} \end{cases}$$

and q_i is R-equivalent to p_{i+1} with $x \sim_R y :\Leftrightarrow \Phi(x) = y$. Let $\{p_i^n\}_{i=1,\dots,k_n}$, $\{q_i^n\}_{i=1,\dots,k_n}$ be minimizing sequences for $k_n \in \mathbf{N}$ with $p_1^n = x$ and $q_{k_n}^n = y$ such that

$$\sum_{i=1}^{k_n} d(p_i^n, q_i^n) \to d_R(x,y) \quad \text{for} \quad n \to \infty.$$

To be precise, if x or y are in the set A we take the projection to M_1 or M_2, s.t. x and y are in the same M_i. W.l.o.g. let $x,y \in M_1$. Let $N \in \mathbf{N}$ be large enough, s.t. the sum is finite for all $n \ge N$. For all $n \ge N$ we have to show that $\exists c > 0$:

$$\sum_{i=1}^{k_n} d(p_i^n, q_i^n) \ge \frac{1}{c}\, d_1(x,y). \tag{1.1}$$

Then, by taking the limit, the proof is finished.

We consider two cases:

1. All q_i^n and p_i^n are in M_1. Then (1.1) is obvious since by definition $d(p_i^n, q_i^n) = d_1(p_i^n, q_i^n)$ holds and d_1 is intrinsic.

2. If some q_i^n, p_i^n are elements of M_2, we compare these excursions into M_2 with the metric d_1. Since we start in M_1 let $p_{i_*}^n \in M_2$ be the first element in M_2. Now define

$$\begin{aligned} S_1 &:= i_* \\ F_1 &:= \min\{i \ge S_1 : p_{i+1}^n \in M_1\}. \end{aligned}$$

This means that all elements between $p^n_{S_1}$ and $q^n_{F_1}$ are in M_2 because otherwise there would be a jump, s.t. $p^n_j \in M_2$ and $q^n_j \in M_1$ and therefore $d(p^n_j, q^n_j) = \infty$ which is a contradiction. Further define

$$\begin{aligned} S_{i+1} &:= \min\{j > F_i : p^n_j \in M_2\} \\ F_{i+1} &:= \min\{j \geq S_{i+1} : p^n_{j+1} \in M_1\}, \end{aligned}$$

until there is only an empty set to take the minimum of. Let S_{l_n}, F_{l_n} for $l_n \in \mathbf{N}$ be the last excursion into M_2. Since the sum is finite we know that $p^n_{S_i}, q^n_{F_i} \in A_2$ and all elements in between are in M_2. Therefore, we get the following estimate

$$\begin{aligned} \sum_{i=S_j}^{F_j} d(p^n_i, q^n_i) &= \sum_{i=S_j}^{F_j} d_2(p^n_i, q^n_i) \\ &\geq d_2(p^n_{S_j}, q^n_{F_j}) \end{aligned}$$

by the definition of d and the triangle inequality for d_2. Furthermore we know that $\Phi(q^n_{S_j-1}) = p^n_{S_j}$ and $\Phi(p^n_{F_j+1}) = q^n_{F_j}$ holds. This enables us to compare $d_2(p^n_{S_j}, q^n_{F_j})$ with $d_1(q^n_{S_i-1}, p^n_{F_j+1})$ via the bilipschitz gluing map Φ. This yields

$$d_2(p^n_{S_j}, q^n_{F_j}) \geq \frac{1}{Lip\ \Phi}\ d_1(q^n_{S_i-1}, p^n_{F_j+1}).$$

Hence we get

$$\begin{aligned} \sum_{i=1}^{k_n} d(p^n_i, q^n_i) &\geq \frac{1}{Lip\ \Phi} \sum_{j=1}^{l_n} d_1(q^n_{S_j-1}, p^n_{F_j+1}) \\ &\quad + \sum_{j=2}^{l_n} d_1(p^n_{F_{(j-1)}+1}, q^n_{S_j-1}) + d_1(p^n_1, q^n_{S_1}) + d_1(p^n_{F_{l_n}}, q^n_{k_n}) \\ &\geq \frac{1}{Lip\ \Phi}\ d_1(x,y) \end{aligned}$$

because of the triangle inequality for d_1 and $\frac{1}{Lip\ \Phi} < 1$.

■

As an important consequence of the comparability of the distances the balls in the new and the old metric are in some sense comparable too. Let

$$B(x,R) := \{y \in M : d(x,y) < R\}$$

and

$$B_i(x,R) := \{y \in M_i : d_i(x,y) < R\},$$

for $i = 1, 2$. Then the next lemma is true:

Lemma 1.9 *Together with the same assumptions as in the previous lemma the following relation holds* $\forall\, x \in M$*:*

$$(i)\;\; B_i(x,R) \subset B(x,R) \;\; \forall\, x \in M_i \;\; \forall i = 1,2$$

and there exists a constant $c > 0$ *s.t.:*

$$(ii)\;\; B(x,R) \subset B_i(x,cR) \cup B_j(z,2cR) \;\; \forall x \in M_i \;\; \forall i = 1,2 \;\; \textit{while} \;\; i \neq j$$

and $z \in B(x,R) \cap M_j$*. If* $B(x,R) \cap M_j = \emptyset$ *the last term vanishes.*

Proof: (i) The first inclusion is trivial since $d(x,y) \leq d_i(x,y)$, $\forall x,y \in M_i$, $\forall i = 1,2$ holds.
(ii) For the second one we need the last lemma. W.l.o.g. $x \in M_1 \backslash A$. If $B(x,R) \subset M_1$ then $B(x,R) = B_1(x,R)$ and the last term vanishes since the metrics d, d_1 coincide locally on M_1. If $B(x,R) \cap M_2 \neq \emptyset$ then:

$$B(x,R) \cap M_1 \subset B_1(x,cR)$$

since $d_i(x,y) \leq c\, d(x,y)$ and therefore:

$$\begin{aligned} B(x,R) \cap M_1 \quad &= \{y \in M_1 : d(x,y) < R\} \\ &\subset \{y \in M_1 : d_1(x,y) < cR\} \\ &= B_1(x,cR) \end{aligned}$$

For the set $B(x,R) \cap M_2$ just take a point $z \in B(x,R) \cap M_2$, then:

$$B(x,R) \cap M_2 \subset B_2(z,2cR)$$

since $d_2(x,y) \leq c\, d(x,y)$ and therefore:

$$\begin{aligned} B(x,R) \cap M_2 \quad &= \{y \in M_2 : d(x,y) < R\} \\ &\subset \{y \in M_2 : d_2(z,y) < 2cR\} \\ &= B_2(z,2cR) \end{aligned}$$

because $d(x,y) < R$ and $d(x,z) < R$ so that $d_2(z,y) \leq c\, d(z,y) \leq c\, d(z,x) + c\, d(x,y) < 2cR$.

■

The result of the last lemma is fundamental for our setting. Property 1.9 (i) is just a trivial consequence of the gluing procedure but implies that open, closed sets on M are open, closed when projected on the original spaces M_i. For property 1.9 (ii) the bilipschitz gluing map is necessary. It is needed to prove completeness in Section 1.4 or the Poincaré inequality in Chapter 3. If collapsing is allowed property 1.9 (ii) does not necessarily hold. More on that in Section 1.4.

Remark 1.10 *At the end of this section we want to demonstrate that in some cases it is also possible to start with a non-intrinsic metric. Let $G \subset \mathbf{R}^n$ be a Lipschitz domain cut out of $(\mathbf{R}^n, d^{eucl})$ while d^{eucl} is the Euclidean metric. Then in order to stay in our setting one has to ensure that the intrinsic metric d^G coming from the Euclidean metric d^{eucl} and all continuous paths lying in G is at least locally comparable to d^{eucl} in G. Since $d^{eucl}|_G \leq d^G$ holds, one has to take care for the other direction. But the Lipschitz boundary ∂G locally admits only shortest paths in G which are not longer than the length of paths in $\mathbf{R}^n$ with the same start- and endpoints times the Lipschitz constant L. Therefore, the other direction holds true locally.*

1.3 Particular Cases of Gluing

We will now briefly discuss two particular cases of gluing maps Φ where it is easy to verify that Φ is bilipschitz:

First let Φ be an isometry between (A_1, d_1) and (A_2, d_2). Then

$$d_i(x, y) = d(x, y) \tag{1.2}$$

holds true for all $x, y \in M_i$ and $i = 1, 2$. The reason is as follows. W.l.o.g. let $x, y \in M_1$. Take a shortest path γ w.r.t. the metric d which connects x and y lying in M (this shortest path exists as we will see in Section 1.4, since M is complete and locally compact and therefore strictly intrinsic, cf. [BBI01]). If γ lies completely in M_1 equality (1.2) holds clearly true since $d_i \geq d$ and the length of γ w.r.t. d_i is the same as the length w.r.t. d. If γ has excursions lying in $M \setminus M_1$, say $\gamma(t) \in M \setminus M_1$, let

$$p := \sup\{s < t : \gamma(s) \in M_1\}$$

and

$$q := \inf\{s > t : \gamma(s) \in M \setminus M_1\}.$$

Then $\gamma(p), \gamma(q) \in A$ because A is closed and $\gamma(]p, q[) \subset M \setminus M_1$. Now since γ restricted to the interval $[p, q]$ is a shortest path connecting $\gamma(p)$ and $\gamma(q)$ w.r.t. d_2 there exists a shortest path connecting the same points lying completely in M_1. Interchanging all excursions in this manner we end up with a new path γ^* lying in M_1 with the same length w.r.t. the metric d_1 as γ w.r.t. d. This yields $d_1(x, y) \leq d(x, y)$

and therefore the equality (1.2).

Another possibility to prove (1.2) is to imitate the proof of Lemma 1.8. In [HeK98], Heinonen and Koskela consider an isometric gluing map which avoids many of the difficulties arising in later proof for the Poincaré inequality for instance. We will discuss this briefly later in this work.

For the second one an additional condition on the geometry of A_1 and A_2 is needed:

Definition 1.11 (Bounded Geometry Condition) *We say a subset A of a metric space (M, d) satisfies the bounded geometry condition (BG) if:*

$$\exists\, c > 0 : \forall x, y \in A : d(x, y) \geq c \inf\{L(\gamma) : \gamma : [0, 1] \to A, \gamma(0) = x, \gamma(1) = y\}$$

while $L(\gamma)$ is the length of the path γ w.r.t. d.

Remark 1.12 *The (BG) condition implies that A is pathwise connected.*

Now let Φ be an isometry w.r.t. the induced length metrics, i.e. between $(A_1, d_1^{A_1})$ and $(A_2, d_2^{A_2})$ while $d_i^{A_i}$ comes from the operation $d_i \to \hat{d}_i =: d_i^{A_i}$ described in Definition 1.1. Then the (BG) condition can be written as:

$$\exists\, c > 0 : \; d_i(x, y) \geq c\, d_i^{A_i}(x, y) \quad \forall\, x, y \in A_i.$$

It is easy to see that the map $\Phi : A_1 \mapsto A_2$ is bilipschitz w.r.t. the original metrics d_1, d_2 on M_1, M_2:

W.l.o.g. let $x, y \in M_1$. Then

$$\begin{aligned} d_1(x, y) &\geq c\, d_1^{A_1}(x, y) \\ &= c\, d_2^{A_2}(\Phi(x), \Phi(y)) \\ &\geq c\, d_2(\Phi(x), \Phi(y)) \end{aligned}$$

holds while the last inequality comes from the fact that in our situation the operation $d \to \hat{d}$ enlarges the metric because there are less admissible paths. With such kind of gluing maps one can construct quite strange examples of glued spaces for instance two 2-dimensional spaces glued along curves of the same length but globally quite differently positioned.

1.4 Gluing and Topology

Now as we have defined what we mean by gluing we will figure out which properties of the old spaces will keep on the new one. The gluing procedure described above gives rise to a topology which in general coincides not necessarily with the topology coming from a topological identification. That means (back to the old notation $M = M_1 \dot{\cup} M_2$) the topology of the metric quotient M/d_R can be weaker than the topology of the topological quotient M/R even if they coincide as sets. This is not always the case as one can see for example if all rational points in the interval $[0,1]$ are glued together (all rational points in $[0,1]$ are R-equivalent). Then the topological quotient is very wild but the metric quotient is just a point.
But under certain additional conditions the topologies are the same. Suppose M/d_R and M/R coincides as sets and let

$$\tau^R := \{U \subset M_1 \cup_A M_2 : \pi^{-1}(U) \subset M_1 \dot{\cup} M_2 \text{ open}\}$$

while $\pi : M_1 \dot{\cup} M_2 \to M/R$ is the canonical projection so that τ^R is the finest topology for which π is continuous and

$$\tau^{d_R} := \{U \subset M_1 \cup_A M_2 : \forall\, x \in U\ \exists\, \epsilon > 0 : B(x,\epsilon) \subset U\}$$

is the topology induced by the new intrinsic metric d_R. Then we have:

Lemma 1.13 *If M/d_R and M/R coincide as sets, $\tau^{d_R} \subset \tau^R$ holds.*

Proof: Let $U \in \tau^{d_R}$ s.t. $\forall\, x \in U\ \exists\, \epsilon > 0 : B(x,\epsilon) \subset U$. Therefore, $\forall\, x \in \pi^{-1}(U) : \pi^{-1}(B(\pi(x),\epsilon)) \subset \pi^{-1}(U)$ and since $d < d_i$ it follows that $B_i(x,\epsilon) \subset \pi^{-1}(B(x,\epsilon))$ for $i = 1,2$ because $B_i(x,\epsilon) \subset B(x,\epsilon)$ holds. This implies that $\forall\, x \in \pi^{-1}(U)\ \exists\, \epsilon > 0 : B_i(x,\epsilon) \subset \pi^{-1}(U)$ so $\pi^{-1}(U)$ is open.

■

And with some additional conditions we get the same topologies:

Lemma 1.14 *If M/d_R and M/R coincide as sets and one of the following condition holds:*

(*i*) *$M_1 \dot{\cup} M_2$ is compact.*

(*ii*) *The equivalence relation R comes from a bilipschitz bijection $\Phi : A_1 \to A_2$.*

Then $\tau^{d_R} = \tau^R$ holds true.

Proof:

(i) Since $M_1 \dot{\cup} M_2$ is compact and the identity map $id : M/R \to M/d_R$ is continuous by the lemma above, one knows that id is a homeomorphism. That is true because M/R is compact too and M/d_R is a Hausdorff space. Therefore, id is a closed map because a closed set $X \subset M/R$ is compact and thus $id(X)$ is compact and also closed since M/d_R is Hausdorff. But if id is a closed map id^{-1} is continuous.

(ii) We have to show that $\tau^R \subset \tau^{d_R}$. Let $U \in \tau^R$ then $\pi^{-1}(U)$ open in $M_1 \dot{\cup} M_2$ and therefore $\pi^{-1}(U) \cap M_i$ is open in M_i, for $i = 1, 2$. This means for $i \neq j$ and $\forall\, x \in M \setminus M_j \cap U$:

$$\exists\, \epsilon > 0 : B_i(\pi^{-1}(x), \epsilon) \subset M_i \setminus A_i \cap \pi^{-1}(U),$$

s.t. $\pi(B_i(\pi^{-1}(x), \epsilon)) = B(x, \epsilon) \subset U$.

If $x \in A \cap U$ there exist two balls $B_i(\pi^{-1}(x), \epsilon_i) \subset M_i \cap \pi^{-1}(U)$ s.t. if one takes $\epsilon := \min\{\epsilon_1, \epsilon_2\}$ and $\delta := \frac{\epsilon}{2c}$ it follows that $B(x, \delta) \subset B_i(x, \epsilon) \cup B_j(x, \epsilon) \subset U$ hence $U \in \tau^{d_R}$ holds.

This finishes the proof.

■

In the next lemmata we collect some results which are important to transfer properties of continuous or measurable functions from the original spaces to the glued space. They will be used throughout this work, mostly without explicit statement. For these results it is not necessary that the gluing map Φ is bilipschitz. They would also hold for more general gluing maps which allow collapsing. This is because only part (i) of Lemma 1.9 is needed for which the inequality $d_i(x, y) \geq d(x, y)$ is responsible. But this inequality holds for all gluing maps by the definition of the glued intrinsic metric d. From now on, if not stated otherwise, M denotes again the glued space $M_1 \cup_A M_2$.

Lemma 1.15 (Open, Closed Sets)

(*i*) *If $B \subset M$ is open then $B \cap M_i$ is open in (M_i, d_i) for $i = 1, 2$.*

(*ii*) *If $B \subset M$ is closed then $B \cap M_i$ is closed in (M_i, d_i) for $i = 1, 2$.*

Proof: (i) For each $x \in B \cap M_i$ there exists a ball $B(x, r) \subset B$ and therefore $B_i(x, r) \subset B(x, r) \subset B \cap M_i$ by the Lemma 1.9 (i).

(ii) Follows from (i) by taking the complement.

■

Let $\mathcal{B}(M_i)$, $\mathcal{B}(M)$ be the Borel σ-fields of M_i, M, then we have the following

Corollary 1.16 (Measurable Sets) *If $B \in \mathcal{B}(M)$ then $B \cap M_i \in \mathcal{B}(M_i)$.*

Proof: For τ, τ_i the topologies on M, M_i induced by the metric d, d_i and $\sigma(\tau) = \mathcal{B}(M)$, $\sigma(\tau_i) = \mathcal{B}(M_i)$ the induced σ-fields the equality $\sigma(\tau) \cap M_i = \sigma(\tau \cap M_i)$ holds. Therefore, with Lemma 1.15 (i)

$$\mathcal{B}(M) \cap M_i = \sigma(\tau \cap M_i) \subset \sigma(\tau_i) = \mathcal{B}(M_i)$$

is true.

■

A simple but important consequence of the last corollary is that for a measurable function $f : M \mapsto \mathbf{R}$ the restriction $f|_{M_i}$ is measurable too.

Lemma 1.17 (Separable) *If the metric spaces (M_i, d_i) for $i = 1, 2$ are separable, then $M = M_1 \cup_A M_2$ is separable.*

Proof: For $i = 1, 2$ let N_i be a dense countable set in M_i. Then by Lemma 1.9 (i) the set $\pi(C_1) \cup \pi(C_2)$ is countable and dense in M.

■

Lemma 1.18 (Lipschitzfunctions) *Let $f : M \mapsto \mathbf{R}$ be a Lipschitzfunction w.r.t. d, then the restricted function $f|_{M_i}$ is Lipschitz w.r.t. d_i.*

Proof: This is an immediate consequence of the inequality $d_i(x, y) \geq d(x, y)$, $\forall\, x, y \in M_i$.

■

Under our gluing condition, namely that Φ is bilipschitz, we have the validity of $d_i(x, y) \leq c\, d(x, y)$ for a constant $c > 0$ and $\forall\, x, y \in M_i$ and therefore 1.9 (ii). This is needed for all proofs of the rest of this section:

Lemma 1.19 (Compact Sets) *If $B \subset M$ is compact then $B \cap M_i$ is compact in (M_i, d_i) for $i = 1, 2$.*

Proof: $B \cap M_i$ is closed and complete because M_i is complete. To show that $B \cap M_i$ is precompact we take the covering with ϵ-Balls of B in $M_1 \cup_A M_2$ and use Lemma 1.9 (ii).

■

Lemma 1.20 (Complete Metric Spaces) *Let (M_i, d_i) be complete metric spaces, then $M = M_1 \cup_A M_2$ is complete.*

Proof: Let $\{x_i\} \subset M$ be a Cauchy sequence in M w.r.t. the metric d. Then at least in one part of M, M_1 or M_2 there are infinite elements of $\{x_i\}$. They are a Cauchy sequence w.r.t. d too but since $c_i d_i(x, y) \leq d(x, y)$ holds for all $x, y \in M_i$, $i = 1, 2$ and M_i is complete, they are Cauchy for d_i as well and a limit $x \in M_i$ exists which finishes the proof by application of Lemma 1.8.

■

Lemma 1.21 (Locally compact) *If the metric spaces (M_i, d_i) for $i = 1, 2$ are locally compact $M = M_1 \cup_A M_2$ is locally compact.*

Proof: Local compactness is clear by using the comparability of balls (1.9 (ii)) to show that there exists a totally bounded neighbourhood. This is enough for compactness since M is complete by Lemma 1.20.

■

As a consequence we state the following remark:

Remark 1.22 (cf. [BBI01] Theorem 2.5.23) *If (M, d) is a complete locally compact length space then (M, d) is strictly intrinsic.*

At the end of this section we want to demonstrate that in avoiding collapsing phenomena by choosing Φ bilipschitz, all information in the sense of σ-fields are preserved by the gluing procedure. This in some sense means that the information about the Markov process which will be defined later via a Dirichlet form, is preserved when gluing Dirichlet spaces. This results are not used in the rest of this work but might be interesting for itself.

Lemma 1.23 (Open Sets) *Let $B_i \subset M_i$ be an open set in M_i w.r.t. d_i. Then there exists an open set $B \subset M$ w.r.t. to d such that*

$$B \cap A = B_i \cap A.$$

Proof: For all $y \in B_i \cap A$ take a ball $B_i(y, \epsilon_y) \subset B_i$ with $\epsilon_y > 0$. Then because of $d_i \sim d$ there exist balls $B(y, c\ \epsilon_y)$ for a universal constant $c > 0$, s.t. $B(y, c\ \epsilon_y) \cap A \subset B_i$. But then the union $B := B_i \cup \bigcup_{y \in A \cap B_i} B(y, c\epsilon_y)$ is open in M w.r.t. d and $B \cap A = B_i \cap A$ holds.

∎

Corollary 1.24 *The topologies τ_1, τ_2 of M_1, M_2 are subsets of the Borel σ-field $\mathcal{B}(M)$ on M.*

Proof: By the Lemma 1.23 for each open set $B_i \subset \tau_i$ for which $B_i \cap A \neq \emptyset$ holds there exists an open set B in (M, d) s.t. $B \cap A = B_i \cap A$ and therefore $B_i = (B \cap A) \cup (B \setminus A) \in \mathcal{B}(M)$.

∎

Corollary 1.25 *The Borel σ-fields $\mathcal{B}(M_1)$, $\mathcal{B}(M_2)$ of M_1, M_2 are subsets of the Borel σ-field $\mathcal{B}(M)$ on M.*

1.5 Gluing of Metric Measure Spaces

Let (M_i, d_i, μ_i) for $i = 1, 2$ be two metric measure spaces and μ_1, μ_2 positive Radon measures on $(M_1, d_1), (M_2, d_2)$ with $supp\,[\mu_i] = M_i$. Further let us assume that (M_i, d_i) are complete locally compact length spaces and (M, d) as usual the glued space via a bilipschitz gluing map Φ along a closed set A. Let $\mathcal{B}(M)$, $\mathcal{B}(M_i)$ be the Borel σ-field of M, M_i w.r.t. the topology induced by d, d_i. Then we glue measures in the following way:

Theorem 1.26 (Gluing of positive measures) *Let $\mu_1(B \cap A) = \mu_2(\Phi(B \cap A))$ $\forall B \in \mathcal{B}(M_1)$. Then the set function $\mu : \mathcal{B}(M) \to \mathbf{R}$ defined in the following way is a measure:*

$$\mu(B) := \mu_1(B \cap M_1) + \mu_2(B \cap M_2) - \mu_1(B \cap A)$$

and $\mu(B) = \infty$ if $\mu_i(B \cap M_i) = \infty$ for $i = 1$ or $i = 2$.

Remark 1.27

- *If in the first theorem the measure of the set A is zero in M_i for $i = 1$ and $i = 2$, i.e. $\mu_i(A) = 0$ then the assumption is trivially satisfied. For example by gluing two n-dimensional manifolds along an $(n-1)$-dimensional subset equipped with the n-dimensional Hausdorff measure on the manifolds.*

- *To demonstrate that the assumption in the definition of μ is somehow natural we consider the case of two Hausdorff measures μ_1 and μ_2. For Lipschitz maps $f : A_1 \mapsto A_2$ it holds that:*

$$\mu_2(f(B)) \leq C^d \mu_1(B) \quad \forall\, B \in \mathcal{B}(M_1) \cap A$$

while d is the dimension of the Hausdorff measure and C is the Lipschitz constant of f. If Φ is an isometric map we have $C = 1$ and therefore

$$\mu_2(\Phi(B \cap A_1)) = \mu_1(B \cap A_1) \;\; \forall\, B \in \mathcal{B}(M_1).$$

Proof: To proof the first theorem we have to check that the definition of μ together with the assumption satisfies the measure properties. By Lemma 1.16 it holds that

$$\forall B \in \mathcal{B}(M) : B \cap M_i \in \mathcal{B}(M_i) \text{ for } i = 1, 2.$$

Therefore, μ is well defined and only the σ-additivity is left to prove. But this comes from the measure properties of μ_i. Let (A_n) pairwise disjoint measurable sets in M. Then the following holds true:

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mu_1\left(\left[\bigcup_{n=1}^{\infty} A_n\right] \cap M_1\right) + \mu_2\left(\left[\bigcup_{n=1}^{\infty} A_n\right] \cap M_2\right) \\
&\quad -\mu_1\left(\left[\bigcup_{n=1}^{\infty} A_n\right] \cap A\right) \\
&= \mu_1\left(\bigcup_{n=1}^{\infty} [A_n \cap M_1]\right) + \mu_2\left(\bigcup_{n=1}^{\infty} [A_n \cap M_2]\right) \\
&\quad -\mu_1\left(\bigcup_{n=1}^{\infty} [A_n \cap A]\right) \\
&= \sum_{n=1}^{\infty} \mu_1(A_n \cap M_1) + \sum_{n=1}^{\infty} \mu_2(A_n \cap M_2) - \sum_{n=1}^{\infty} \mu_1(A_n \cap A)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} [\mu_1(A_n \cap M_1) + \mu_2(A_n \cap M_2) - \mu_1(A_n \cap A)] \\
&= \sum_{n=1}^{\infty} \mu(A_n),
\end{aligned}
$$

because of absolute convergence. In order to have a symmetric and consistent definition we have to ensure that μ coincides with the measures μ_1 and μ_2 on M_1 and M_2. This is the reason for the assumption, that means w.l.o.g. $\forall B \in \mathcal{B}(M_1)$:

$$
\begin{aligned}
\mu(B) &= \mu_1(B \cap M_1) + \mu_2(B \cap M_2) - \mu_1(B \cap A) \\
&= \mu_1(B) + \mu_2(B \cap A) - \mu_1(B \cap A) \\
&= \mu_1(B)
\end{aligned}
$$

because of $B \cap M_2 = B \cap A$ and in the same manner one gets

$$\forall\, B \in \mathcal{B}(M_2): \quad \mu(B) = \mu_2(B)$$

and the proof is finished.

■

A little more work has to be done in order to check the inner regularity for the glued measure μ:

Theorem 1.28 (Gluing of positive Radon measures) *Let μ_1, μ_2 be two positive Radon measures on M_1, M_2. Then the glued measure μ is a positive Radon measure.*

Proof: A Radon measure is per definition a measure defined on the Borel σ-field for a Hausdorff space which is locally finite and inner regular, i.e.

$$\forall B \in \mathcal{B}(M) : \mu(B) = \sup\{\mu(K) : K \subset B, K \text{ compact}\}.$$

By Theorem 1.26 we know that μ is a measure and still positive. Now to show the local finiteness just take $x \in M$. Then choose $\epsilon > 0$ small enough, s.t. $\mu_i(B_i(x,\epsilon)) < \infty$ and $\mu_j(B_j(x,\epsilon)) < \infty$ for $i \neq j$ as well if $x \in A$. Because of Lemma 1.9 (ii) there exists $\delta > 0$, s.t. $B(x,\delta) \subset B_i(x,\epsilon) \cup B_j(x,\epsilon)$ hence $\mu(B(x,\delta)) < \infty$.

To prove the inner regularity of μ take $B \subset \mathcal{B}(M)$ and choose K_n^i and K_n^j compact sets in M_i and M_j such that

$$\mu_i(K_n^i) \to \mu(B \cap M_i) \quad \text{for} \quad n \to \infty$$

and

$$\mu_j(K_n^j) \to \mu(B \cap M_j) \quad \text{for} \quad n \to \infty$$

because of the inner regularity of μ_i for $i = 1, 2$. The sets $\overline{K_n^i \cup K_n^j}$ are compact in M ($\forall\, n \in \mathbf{N}$) because they are closed, M is complete and they are precompact since K_n^i, K_n^j are precompact in M_i, M_j and Lemma 1.8 holds true. We have now:

$$\mu(K_n^i \cup K_n^j) \leq \mu(\overline{K_n^i \cup K_n^j}) \leq \mu(B)$$

while the second inequality comes from the following. Assume that $\overline{K_n^i \cup K_n^j} \subset B$ is wrong. Then there exists a sequence $\{x_n\}_{n \in \mathbf{N}} \subset K_n^i \cup K_n^j \subset B$ with $x = \lim_{n \to \infty} x_n \notin B$. But if $x \in M_i$ and there are infinitely many x_n in M_i too there is a subsequence $x_{n_k} \to x$ in (M_i, d_i) and this would mean: $x \in K_n^i \subset B$. On the other hand if infinitely many x_n are in M_j for $i \neq j$ there would be an $y \in K_n^j$ with $d(x, y) = 0$ s.t. $x_{n_k} \to y$ in (M_j, d_j) which contradicts $x \notin B$. Therefore, $\overline{K_n^i \cup K_n^j}$ is in B and the second inequality holds for all n.
To finish the proof we have to check that:

$$\mu(K_n^i \cup K_n^j) \to \mu(B) \text{ for } n \to \infty.$$

Assume that $\mu(K_n^i \cup K_n^j) \to c\ < \mu(B)$, then $\mu(K_n^i \cap K_n^j) \to c' > \mu(B \cap A)$ because of

$$\mu(K_n^i \cup K_n^j) = \mu(K_n^i) + \mu(K_n^j) - \mu(K_n^i \cap K_n^j),$$

$$\mu(B) = \mu(B \cap M_i) + \mu(B \cap M_j) - \mu(B \cap A)$$

for $i \neq j$ and

$$\mu(K_n^i) \to \mu(B \cap M_i)$$

for $i = 1, 2$. But this is a contradiction because of $K_n^i \cap K_n^j \subset B \cap A$ for all $n \in \mathbf{N}$.

$\blacksquare$

For the rest of this work we often use μ for the glued measure assuming the gluing condition is satisfied without explicitly stating it.

1.6 Gluing and Doubling

Now we want to glue metric measure spaces which satisfy the Ahlfors-regularity condition or at least a doubling condition and give a sufficient additional condition under which this still holds on the glued space. For Ahlfors-regularity we need no additional assumptions:

Theorem 1.29 (Ahlfors-regularity on glued spaces) *Let (M, d, μ) be a metric measure space, glued together along a set A via a bilipschitz gluing map Φ by two metric measure spaces (M_i, d_i, μ_i), $i = 1, 2$ which satisfy the Ahlfors-regularity condition for $x \in M_i$*

$$C_i^{-1} R^n \leq \mu_i(B_i(x, R)) \leq C_i R^n,$$

for universal constants $C_i > 0$. Then (M, d, μ) also satisfies the Ahlfors-regularity condition with a constant which only depends on C_1, C_2 and on the Lipschitz constant Lip Φ of Φ.

Proof: Let $C := \max\{C_1, C_2\}$. By Lemma 1.9 (i) we have for $x \in M_i$ that:

$$\begin{aligned} C^{-1} R^n &\leq \mu_i(B_i(x, R)) \\ &= \mu(B_i(x, R)) \\ &\leq \mu(B(x, R)). \end{aligned}$$

holds, since $\mu|_{M_i} = \mu_i$.
And for the second inequality we have $\forall\ x \in M_i,\quad R > 0,\quad i \neq j$ that because of $B(x, R) \subset B_i(x, cR) \cup B_j(z, 2cR)$ the following holds $\forall z \in B(x, R) \cap A$:

$$\begin{aligned} \mu(B(x, R)) &\leq \mu(B_i(x, R)) + \mu(B_j(z, 2cR)) \\ &= \mu_i(B_i(x, R)) + \mu_j(B_j(z, 2cR)) \\ &\leq CR^n + C(2cR)^n \\ &= (C + Cc^n 2^n) R^n. \end{aligned}$$

∎

For doubling we need an additional condition which in some sense reproduce the dimension homogeneity of the Ahlfors-regularity:

Theorem 1.30 (Doubling on glued spaces) *Let (M, d, μ) be the metric measure space, glued together along the set A by two metric measure spaces (M_i, d_i, μ_i), $i = 1, 2$ which satisfy the doubling condition*

$$\mu_i(B_i(x, 2R)) \leq C_i \mu_i(B_i(x, R)), \forall\, x \in M_i, R > 0,\ i = 1, 2$$

for constants $C_i > 0$. Then (M, d, μ) also satisfies the doubling condition in compact subsets if the following condition is satisfied: $\forall z \in A$, $r_n > 0$ and for all sequences $\{x_n\}_{n\in\mathbf{N}} \subset M_1, \{y_n\}_{n\in\mathbf{N}} \subset M_2$, s.t. $z = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$ and $\lim_{n\to\infty} r_n = 0$ there exist numbers $0 < k(z) < \infty$ and $N(z) \in \mathbf{N}$ such that

$$\forall\, n > N(z) : k(z)^{-1} \leq \frac{\mu(B_i(x_n, r_n))}{\mu(B_j(y_n, r_n))} \leq k(z).$$

Remark 1.31

- *In particular this situation is given by the Ahlfors-regularity.*
- *Another possibility is to have the interior of the gluing set $\mathring{A}$ nonempty. Then, by the gluing condition for measures, the dimension homogeneity is fulfilled too.*

Proof: Let $C := \max\{C_1, C_2\}$. The proof works by contradiction. Assume in a compact set that doubling does not hold. Then there exists a convergent subsequence of $r_n \to r$ and $x_n \to x$ and $C_n \to \infty$ for which the following inequality holds:

$$\mu(B(x_n, 2r_n)) > C_n \mu(B(x_n, r_n)).$$

We consider three cases:
1. The case $x_n \to x \in M_i \setminus A$ and $r_n \to 0$ is clear since there will be a N s.t. $\forall\, n > N : B(x_n, r_n) \subset M_i \setminus A$ and $C_n > C$ so there is a contradiction.
2. The case where $x_n \to x$ and $r_n \to r > 0$ is clear as well, because this would imply that $\mu(B(x, r)) = 0$.
3. So the last case where $x_n \to x \in A$ and $r_n \to 0$ is the one where the dimension homogeneity is needed. By the same arguments as in the proof for Ahlfors-regularity the following inequality holds $\forall\, y_n \in B(x_n, 2r_n) \cap A$:

$$\begin{aligned}\mu(B(x_n, 2r_n)) \quad &\leq \mu(B_i(x_n, 2r_n)) + \mu(B_j(y_n, 4cr_n)) \\ &\leq C\mu(B_i(x_n, r_n)) + C^2 \log_2(c)\mu(B_j(y_n, r_n))\end{aligned}$$

and by the dimension homogeneity condition there exists a number $N(x)$ s.t.

$$\forall\, n > N(x) : \mu(B_j(y_n, r_n)) \leq k(x)\mu(B_i(x_n, r_n))$$

and therefore

$$\begin{aligned}\forall\, n > N : \mu(B(x_n, 2r_n)) &\leq (C + k(x)C^2 \log_2(c))\mu(B_i(x_n, r_n)) \\ &\leq (C + k(x)C^2 \log_2(c))\mu(B(x_n, r_n))\end{aligned}$$

which is a contradiction because of $C_n \to \infty$.

$\blacksquare$

1.7 Gluing of k Spaces

As mentioned in the first chapter it is possible to extend the gluing procedure up to k metric spaces. Our framework is designed to glue together the glued space $M = M_1 \cup_A M_2$ with another space M_3 and then with M_4 and so on in order to get a complete locally compact length space

$$M' := (\ldots((M_1 \cup_A M_2) \cup_A M_3) \cup_A \ldots \cup_A M_k).$$

This is because all important properties are transported from the original spaces to the glued spaces, provided that the gluing conditions hold true.
Here we are interested in the special case to glue k spaces $\{M_i\}_{i=1,\ldots,k}$ along a 'common set' A. Formally this means that we consider bilipschitz gluing maps $\Phi_i : A_i \to A_{i+1}$, for $i = 1, \ldots, k-1$, with closed sets $A_i \subset M_i$. Now there are at least two possible gluing procedures:

The first one is to glue M_1, M_2 along A via Φ_1 and then glue $M_1 \cup_A M_2$ via $\tilde{\Phi}_2 : A \to A_3$ with M_3. Clearly $\tilde{\Phi}_2 : A \to A_3$ defined as $\tilde{\Phi}_2(x) := \Phi_2 \circ (\pi^{-1}(x) \cap A_2)$ is bilipschitz, because of Lemma 1.8. By this iterative procedure we get the glued space (M', d') as mentioned above.

The other way is to glue the k spaces simultaneously: For this purpose we need an equivalence relation R on the disjoint union $\dot{\bigcup}_{i=1}^k M_i$: $\forall\, x, y \in \dot{\bigcup}_{i=1}^k M_i$:

$$x \sim_R y :\Leftrightarrow \exists\, \Phi_{ij} := \Phi_i \circ \ldots \circ \Phi_j : A_i \to A_j : \Phi_{ij}(x) = y.$$

One can easily verify that this relation is an equivalence relation. Then consider the semi-metric d_R as defined in Definition 1.3 on $\dot{\bigcup}_{i=1}^k M_i$ w.r.t. the metric

$$d(x,y) := \begin{cases} d_i(x,y) & \text{if } x, y \in M_i, \\ \infty & \text{otherwise} \end{cases}$$

and proceed to the metric quotient $M := \dot{\bigcup}_{i=1}^{k} M_i/d_R$. That this gluing procedure yields the same complete locally compact length space as the first one is shown in the next proposition.

Proposition 1.32 *The length spaces (M', d') and (M, d_R) are isometrically equivalent.*

Proof: Define a bijective map $g : M' \to M$ as the identity on the embedded sets $M_i \hookrightarrow M'$ and $M_i \hookrightarrow M$. To show that g is isometric w.r.t. d' and d_R, first notice that $\forall\, x, y \in M'$ and $g(x), g(y) \in M$ we have

$$d'(x, y) \leq d_R(g(x), g(y)),$$

because all sequences $\{p_i\}$, $\{q_i\}$ in the definition of the semi-metric d_R are admissible sequences for the definition of the semi-metric d'_R which leads to d' and therefore it holds that $d'(p_i, q_i) \leq d_i(p_i, q_i)$ by the gluing procedure for d'. The proof for the other direction works by induction. The case $k = 2$ is clear. Assume equality holds for $k - 1$. Let $\{p_i\}$, $\{q_i\}$ be a sequence in the definition of the semi-metric d'_R which leads to d'. Then either $p_i, q_i \in M_k$ or $p_i, q_i \in (\ldots((M_1 \cup_A M_2) \cup_A M_3) \cup_A \ldots \cup_A M_{k-1})$. In the first case the contribution $d_k(p_i, q_i)$ for the approximating sum of d'_R and d_R is the same. Now let d^{k-1} be the glued metric of $(\ldots((M_1 \cup_A M_2) \cup_A M_3) \cup_A \ldots \cup_A M_{k-1})$ and d_R^{k-1} be the glued metric of $\dot{\bigcup}_{i=1}^{k-1} M_i/d_R$. Then by the assumption $d^{k-1}(p_i, q_i) = d_R^{k-1}(p_i, q_i)$ holds. Therefore, we can prove equality by using a contradiction argument because $d^{k-1}(p_i, q_i)$ can be approximated by a minimizing sequence for $d_R^{k-1}(p_i, q_i)$.

∎

The last proposition says that the succession of gluing is not important. Hence from now on we denote the glued space by $M = \bigcup_A^k M_i$, the intrinsic metric by d and the gluing set as $A = \bigcap^k M_i$. The sets M_i will be used in the same sense as for $k = 2$ as embedded subsets of M.

By Lemma 1.8 the metrics are comparable, i.e. $d_i \sim d$ on M_i, for all $i = 1, \ldots, k$. The results about measures and doubling transfers straightforward to the case of k-gluing. For examples we refer to the last section of this chapter.

1.8 Examples

The following examples shall illustrate previous definitions, partly prepare later examples and motivate our framework.

(i) **Bilipschitz gluing along curves**

The pictures show the idea of gluing two 2-dimensional Euclidean sets along bilipschitz curves. One simple example is to construct the boundary of a cube. Consider the following two copies of a subset of $\mathbf{R}^2$ equipped with the Euclidean metric:

$$H_i := \{x \in \mathbf{R}^2 : 0 \leq x_2 \leq 1, 0 \leq x_1 \leq 3\},$$

for $i = 1, 2$. Further define the boundary part:

$$A^{H_i} := \{x \in H_i : (x_1, x_2) \in \{0, 3\} \times \{0, 1\}\},$$

for $i = 1, 2$. Then glue H_1 and H_2 along A^{H_1} and A^{H_2} with the following gluing map $\Phi : A^{H_1} \to A^{H_2}$:

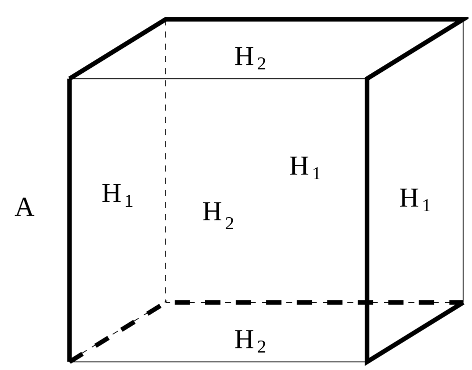

Figure 1.3: The cube

Since the sets A^{H_i} have both the same length 6 we can define Φ as an isometry w.r.t. $d^{A^{H_1}}$ and $d^{A^{H_2}}$, the restricted Euclidean metrics on A^{H_1} and A^{H_2}. We just fix the map Φ for two points, namely $\Phi((0,0)) = (1,1)$ and $\Phi((0,1)) = (2,1)$, and then extend Φ uniquely to A^{H_1}. Then Φ is bilipschitz w.r.t. the Euclidean metrics since the BG condition 1.11 is satisfied. This yields the boundary of a cube as a complete locally compact length space.

(ii) **Bilipschitz Transformations**

Another way to construct more general examples is to consider bilipschitz gluing maps $\Phi : M \to \Phi(M) \subset M'$ while M is a complete locally compact length space, M' is any metric space and A is any closed subset in M. Then

$$\Phi|_A : A \to \Phi(A) \subset M'$$

is bilipschitz and we can glue M with $\Phi(M)$ along A and $\Phi(A)$ while all our gluing conditions, even for doubling, are satisfied.

(iii) **K-Gluing**

Let $M_1, \ldots, M_k$ be k copies of a complete locally compact length space and let $A_i \subset M_i$ be the same closed subset in all these length spaces. Then there exist obviously isometric maps $\Phi_i : A_i \to A_{i+1}$ such that we can glue them all together along A to get the glued space $M = {}^A\bigcup_{i=1}^k M_i$.

Spiders, Trees, Graphs and Polyhedra

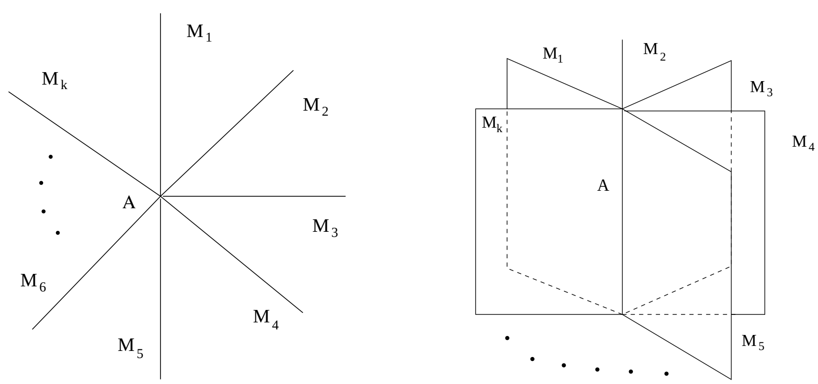

Figure 1.4: k-spider and k-sheet in dimension 2

Special examples of this k-gluing are spiders, trees, graphs or more general polyhedra. Since trees and graphs coincides locally with spiders we just give a brief description of spiders here.

Let $M_1, \ldots, M_k$ be k identical copies of R_+ and $A_i = \{0\} \subset \mathbf{R}_+$. Then we call $M = {}^A\bigcup_{i=1}^k M_i = M = {}^{\{0\}}\bigcup_{i=1}^k \mathbf{R}_+$ a k-spider.

Another way of constructing a k-spider is to consider $M_1, \ldots, M_{k-1}$ identical copies of $\mathbf{R}$ with $A_i = \mathbf{R}_- = \{x \in \mathbf{R} : x \leq 0\}$. By taking the identity on $\mathbf{R}_-$ as the gluing map we construct $M = \bigcup_{i=1}^{k-1} \mathbf{R}$ which is obviously a k-spider.

In higher dimensions we get what we call a k-sheet. Take k copies as $M_1, \ldots, M_k$ of $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n : x_1 \geq 0\}$ and $A_i = \{x \in \mathbf{R}^n : x_1 = 0\}$.

Analogous to the alternative construction of the k-spider one could take $(k-1)$ copies of $\mathbf{R}^n$ and $A_i = \{x \in \mathbf{R}^n : x_1 \leq 0\}$ to get a k-sheet.

Remark 1.33

- *More general one can construct nonlinear examples as Riemannian Polyhedra by using bilipschitz transformations for instance.*
- *With the method described above we can construct graphs of finite degree by gluing further $\mathbf{R}_+$ or $\mathbf{R}$ along one or more points.*

(iv) **Selfgluing**

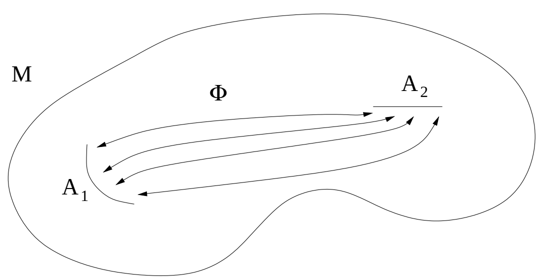

Figure 1.5: Selfgluing

By selfgluing we mean, for $A_1, A_2 \subset M$ closed sets and $\Phi : A_1 \to A_2$ a bilischitz map, the gluing of M with itself by identifying A_1 and A_2 via Φ. Here we just want to remark that we need a constant $c \geq 0$ such that $d(A_1, A_2) > c$. This is because then it fits in our setting. Other cases might work as well but one has to be careful in order to avoid collapsing phenomena. If $d(A_1, A_2) > 0$ holds, consider three copies of M and glue them successively along A_1 and A_2 so that we get a chain of glued spaces. Since all of our results are of local nature, this construction will yield the same properties.

To summarize the results of this chapter: if we glue k complete, locally compact, separable intrinsic metric measure spaces (M_i, d_i, μ_i) , $i = 1, \dots, k$ along closed sets A_i via bilipschitz maps $\Phi : A_i \to A_{i+1}$ the resulting space (M, d, μ) has the same properties and for $i = 1, \dots, k$ the distances d_i and d are comparable on M_i. Further, if doubling or Ahlfors regularity holds true on M_i together with dimension homogeneity, doubling or Ahlfors regularity holds on (M, d, μ).

Chapter 2

Gluing of Dirichlet Spaces

In this chapter the gluing of Dirichlet forms is defined. It is proved that the gluing of k strong local regular Dirichlet forms $\mathcal{E}_i$ on M_i yields a strong local regular Dirichlet form $\mathcal{E}$ on M. Besides the Lipschitz continuity only a consistency condition for the k Dirichlet forms on the gluing set A is necessary. Further a few examples of glued Dirichlet forms which are connected to other works (cf. [Bo04], [BK95] and [BK01]), are given and properties of the associated processes are described. At the end the intrinsic metric

$$\rho := \sup\{u(x) - u(y) : u \in D_{loc}(\mathcal{E}) \cap C(M), d\Gamma(u) \leq d\mu\}$$

with $d\Gamma$ the energy measure w.r.t $\mathcal{E}$, is studied. By assuming $d_i \sim \rho_i$ the comparability of the intrinsic metric d_i with the intrinsic metric ρ_i w.r.t. $\mathcal{E}_i$, the comparability of the glued metric d with ρ is shown. This is done in order to transfer the doubling property of (d_i, μ_i) from (ρ_i, μ_i) to (ρ, μ).

2.1 Gluing Dirichlet Spaces

We prepare the gluing of Dirichlet forms with a short lemma:

Lemma 2.1 *Let (M, d, μ) be a metric measure space while M is a locally compact, separable metric space and μ is a positive Radon measure. Then $C_0^{Lip}(M)$ is dense in $L^2(M, \mu)$.*

Proof: As a first step we show that each function $f \in E := \{1_A : A \in \mathcal{B}(M)\}$, while $\mathcal{B}(M)$ is the Borel σ-algebra in M, can be approximated by a Lipschitz function with compact support in the L^2-norm. Let $f := 1_A$. Since a separable metric space is polish we know by [Ba90] that the measure μ satisfies the outer regularity condition. Therefore, it exists an open set U s.t. $A \subset U$:

$$||1_U - f||_2 = ||1_{U \setminus A}||_2 = (\mu(U) - \mu(A))^{\frac{1}{2}} < \frac{\epsilon}{2} \Rightarrow \mu(U) < \infty.$$

Hence by inner regularity there exists a compact set $K \subset U$ s.t.:

$$\mu(U \setminus K) = \int 1_{U-K} d\mu \leq \left(\frac{\epsilon}{2}\right)^2 \Rightarrow ||1_U - 1_K||_2 < \frac{\epsilon}{2}.$$

Now with $h(x) := (1 - \frac{1}{\epsilon} d(K, x))_+$ and $\epsilon := \frac{1}{2} d(K, M \setminus U)$ there exists a function

$$h \in C_0^{Lip} : supp(h) \subset U : 1_K \leq h \leq 1_U$$

$$\Rightarrow 0 \leq 1_U - h \leq 1_U - 1_K \Rightarrow ||1_U - h||_2 < \frac{\epsilon}{2}$$

$$\Rightarrow ||f - h||_2 \leq ||f - 1_U||_2 + ||1_U - h||_2 < \epsilon.$$

This shows that $E \subset \overline{C_0^{Lip}(M)}^{||\cdot||_2}$. It is clear that $C_0^{Lip} \subset L^2(M)$. To end the proof one has to argue that the set E is dense in $L^2(M)$ since then the rest follows by the diagonal sequence argument. But for a function $g \in L^2(M)$ also g_- and g_+ lies in $L^2(M)$, so it can be assumed that $g \geq 0$. But then there exists a monotone sequence of elementary $\mathcal{B}(M)$-functions (g_n) s.t. $g_n \longrightarrow g$ and $0 \leq g_n \leq g$. So all g_n lie in $L^2(M)$. By the theorem of dominated convergence it follows that $g_n \longrightarrow g$ in the L^2-norm. But the elementary functions can also be approximated by simple indicator functions in the L^2-norm which finishes the proof.

■

Before stating the main result of this section just recall that by Lemma 1.18 and Lemma 1.19 the restrictions of Lipschitz functions $f \in \mathcal{C}_0^{Lip}(M)$ with compact support to one of the original spaces M_i is a Lipschitz function with compact support in M_i, i.e. $f|_{M_i} \in \mathcal{C}_0^{Lip}(M_i)$.

Theorem 2.2 *Let $M := M_1 \cup_A M_2$ be the glued metric measure space (M, d, μ) as above and $(\mathcal{E}_i, D(\mathcal{E}_i))$, $i = 1, 2$ two strongly local, regular Dirichlet forms on M_i, $i = 1, 2$. Further assume that $\mathcal{C}_0^{Lip}(M_i) \subset D(\mathcal{E}_i)$ densely and the Dirichlet forms are consistent on the gluing set, i.e.*

$$\int_{A_1} d\Gamma_1(u|_{M_1}, u|_{M_1}) = \int_{A_2} d\Gamma_2(u|_{M_2}, u|_{M_2})$$

$\forall\, u \in \mathcal{C}_0^{Lip}(M)$ while $d\Gamma_i$ for $i = 1, 2$ is the energy measure of the Dirichlet form $\mathcal{E}_i$. Then the new Form

$$\mathcal{E}(u) := \int_{M_1} d\Gamma_1(u|_{M_1}, u|_{M_1}) + \int_{M_2} d\Gamma_2(u|_{M_2}, u|_{M_2}) - \int_{A_1} d\Gamma_1(u|_{M_1}, u|_{M_1})$$

is a closable symmetric Markovian form on $L^2(M, \mu)$ and with

$$D(\mathcal{E}) := \overline{\mathcal{C}_0^{Lip}(M)}^{\sqrt{\mathcal{E}(\cdot) + ||\cdot||^2}}$$

the smallest closed extension of $\mathcal{E}$ which will be denoted also by $\mathcal{E}$, is a strongly local, regular Dirichlet form on M.

Proof: Since $\mathcal{C}_0^{Lip}(M_i) \subset D(\mathcal{E}_i)$ for $i = 1, 2$, $\mathcal{E}$ is well defined on $\mathcal{C}_0^{Lip}(M)$.

- $\mathcal{E}$ is symmetric:

  The properties of a symmetric form transfers directly to $\mathcal{E}$ since the energy measure is a bilinear form by polarization with values in the space of signed Radon measures.

- $\mathcal{E}$ is closable:

  The form defined on $\mathcal{C}_0^{Lip}(M)$ is closable. For a sequence $\{u_n\}_n \subset \mathcal{C}_0^{Lip}(M)$ with $\mathcal{E}[u_n - u_m] \to 0$ and $||u_n||_2 \to 0$ we take $u_n|_{M_i} \in \mathcal{C}_0^{Lip}(M_i)$. Then $\mathcal{E}[u_n|_{M_i}] \to 0$ since $(\mathcal{E}_i, D[\mathcal{E}_i])$ is closed. The rest follows by the definition of $\mathcal{E}$.

- $\mathcal{E}$ is Markovian:

  For $u \in \mathcal{C}_0^{Lip}(M)$ the function $v := (0 \vee u) \wedge 1$ is in $\mathcal{C}_0^{Lip}(M)$ too and one verifies $\mathcal{E}(v, v) \leq \mathcal{E}(u, u)$ with the truncation property of $\mathcal{E}_i$ on M_i for $i = 1, 2$.

Since $\mathcal{E}$ is a closable Markovian symmetric form by Theorem 3.1.1. in [Fot94] its smallest closed extension is again a symmetric Markovian form. As mention in the theorem we will denote this extension by $\mathcal{E}$ too. $D(\mathcal{E})$ is dense in $L^2(M, \mu)$ w.r.t. $||.||_2$ by Lemma 2.1, because $C_0^{Lip}(M) \subset D(\mathcal{E})$. Therefore, we have a Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on M. Further it holds that $\forall\, u \in D(\mathcal{E})$ the restricted function $u|_{M_i}$ is in the domain $D(\mathcal{E}_i)$ for $i = 1, 2$ which can be easily checked by the assumptions and the definition of $\mathcal{E}$.

Now only the strong local property and the regularity of $(\mathcal{E}, D(\mathcal{E}))$ is left to prove.

- $\mathcal{E}$ is regular:

  To prove regularity of $(\mathcal{E}, D(\mathcal{E}))$ it is enough to show that $\overline{\mathcal{C}_0^{Lip}(M)}^{||\cdot||_\infty} = \mathcal{C}_0(M)$ since $D(\mathcal{E}) := \overline{\mathcal{C}_0^{Lip}(M)}^{\sqrt{\mathcal{E}+||\cdot||^2}}$ and therefore $\mathcal{C}_0^{Lip}(M) \subset D(\mathcal{E})$ so $\mathcal{C}_0^{Lip}(M)$ would be a core. Take a function $f \in \mathcal{C}_0(M)$ and let $S := supp(f)$ be the compact support of f. Let

  $$B_\epsilon(S) := \{x \in M : d(x, S) \leq \epsilon\}$$

  and a compact neighborhood of S. Then by the Stone-Weierstrass theorem there is a sequence of Lipschitz functions $(f_n) \subset \mathcal{C}^{Lip}(B_\epsilon(S))$ s.t.

  $$f_n \to f$$

  w.r.t. $|| \cdot ||_\infty$-Norm on $B_\epsilon(S)$. To show this one just has to verify that $\mathcal{C}^{Lip}(B_\epsilon(S))$ is a subalgebra of $\mathcal{C}(B_\epsilon(S))$ and the constant functions are in $\mathcal{C}^{Lip}(B_\epsilon(S))$ as well as functions which separate points. The separation of points is done by the distance function which is clearly Lipschitz as a consequence of the triangle inequality. Since $B_\epsilon(S)$ is compact $\mathcal{C}^{Lip}(B_\epsilon(S))$ is a subalgebra. The function

  $$g(x) := (1 - \frac{1}{2\epsilon}\rho(S, x))_+$$

  is Lipschitz s.t. $f_n \cdot g \in \mathcal{C}^{Lip}(B_\epsilon(S))$ as well. Therefore,

  $$||f_n g - f||_\infty = max\{||f_n|_S - f|_S||_\infty, ||f_n g 1_{B_\epsilon(S)\setminus S}||_\infty\} \to 0,$$

  holds which finishes the proof because $f_n g \in \mathcal{C}_0^{Lip}(M)$.

- $\mathcal{E}$ is strong local:

  If $u, v \in D(\mathcal{E})$ with supp$[u]$ and supp$[v]$ compact s.t. v is constant on some open neighborhood U of supp$[u]$. Then $U \cap M_i$ is open in the old metric of M_i and an open neighborhood of supp$[u|_{M_i}]$ so that the problem can be reduced to the case of $\mathcal{E}_i$.

This finishes the proof.

$\blacksquare$

Remark 2.3

- *This definition of the domain is consistent with the definition of the forms* $\mathcal{E}_i$ *on* M_i*: If* $u \in D(\mathcal{E})$ *then* $u|_{M_i} \in D(\mathcal{E}_i)$ *and* $\mathcal{E}(u|_{M_i}) := \int_{M_i} d\Gamma(u,u) = \mathcal{E}_i(u|_{M_i})$*.*

- *For many of our examples the consistency condition*

$$\int_{A_1} d\Gamma_1(u|_{M_1}, u|_{M_1}) = \int_{A_2} d\Gamma_2(u|_{M_2}, u|_{M_2})$$

  $\forall\ u \in \mathcal{C}_0^{Lip}(M)$ *is trivially satisfied, because the gluing set* A *has often zero Lebesgue measure and the energy measure can be represented through the Lebesgue measure.*

Remark 2.4 (k-Gluing) *To glue* k *strongly local, regular Dirichlet forms* $(\mathcal{E}_i, D(\mathcal{E}_i))$ *one can do this successively by Theorem 2.2 to get a glued Dirchlet form* $(\mathcal{E}, D(\mathcal{E}))$ *which is strongly local, regular and* $\forall\ u \in D(\mathcal{E})$ *it holds that* $u|_{M_i} \in D(\mathcal{E}_i)$ *as well as*

$$\mathcal{E}(u|_{M_i}) := \int_{M_i} d\Gamma(u,u) = \mathcal{E}_i(u|_{M_i})$$

for $i = 1, \ldots, k$*.*

To shorten the notation in the following we will often use $d\Gamma(u)$ instead of $d\Gamma(u,u)$.

2.2 Examples

We will now give some examples which connect our framework with results in other works on Dirichlet forms and processes on singular spaces.

2.2.1 Converging Spaces

As a motivation we will show that our glued spaces play a role in other contexts as limits of converging spaces. In [Bo04] Dirichlet forms $\mathcal{E}$ on graphs, in particular the Dirichlet form coming from the canonical Laplacian on k-spiders, are approximated by Dirichlet forms $\mathcal{E}_n$ on tubes around the edges coming from the Laplace Beltrami operators on the tubes. In [Bo04] it is shown that under certain regularity conditions the sequence $\{\mathcal{E}_n\}_n$ is Γ-convergent to $\mathcal{E}$. According to [KS03], these results imply the convergence of the associated resolvents, semigroups and spectra.

We will prove now that the limit spaces of [Bo04] are arising naturally when gluing k spaces $\mathbf{R}_+$ with Dirichlet forms coming from the canonical Laplacian in our setting. In [Bo04] the limit space is defined in the following way: Let M be the k-spider defined in Section 1.8. Denoting the edges of the spider by $M_1, \dots, M_k$ with $M_i = \mathbf{R}_+$ for $i = 1, \dots, k$ the Dirichlet form on M is given by

$$\mathcal{E}(u) := \sum_{i=1}^{k} \int_{M_i} |u'(x)|^2 dx$$

defined on the closure of

$$D^C(\mathcal{E}) := \{u \in C(M) : u|_{M_i} \in H^{1,2}(M_i), i = 1, \dots, k\}$$

w.r.t. $\mathcal{E}_1(\cdot) = (\mathcal{E}(\cdot) + ||\cdot||^2)^{\frac{1}{2}}$. Our glued Dirichlet form $\mathcal{E}$ is defined in the same way but the domain is defined in a different way:

$$D^G := \overline{\mathcal{C}_0^{Lip}(M)}^{\mathcal{E}_1}$$

The next proposition will show that both domains actually coincide so that we can construct the limit space by gluing in our framework.

Proposition 2.5 *With the notations above it holds that:* $D^C(\mathcal{E}) = D^G(\mathcal{E})$.

Proof: That $D^G \subset D^C$ is clear by the definitions. For the other direction let $u \in C(M)$ with $u|_{M_i} \in H^{1,2}(\mathbf{R}_+)$ for $i = 1, \dots, k$. Then for each $i \in \{1, \dots, k\}$ there exists a sequence $\{u_j^i\}_j \subset C_0^{Lip}(\mathbf{R}_+)$ such that

$$\int_{\mathbf{R}_+} |(u|_{M_i} - u_j^i)'|^2 + \int_{\mathbf{R}_+} |u|_{M_i} - u_j^i|^2 \to 0 \quad \text{for} \quad j \to \infty. \tag{2.1}$$

The aim is now to construct a sequence $\{u_j\}_j \subset C_0^{Lip}(M)$ which converges to u w.r.t. $\mathcal{E}_1$. For this purpose fix an $\epsilon > 0$ and consider the following sequence of functions in each ray M_i:

$$\tilde{u}_j^i(x) := \begin{cases} u_j^i(x) & \text{if } x > \epsilon \\ \frac{\epsilon - x}{\epsilon}(u(0) - u_j^i(0)) + u_j^i(x) & \text{if } x \leq \epsilon. \end{cases}$$

On M consider the sequence of functions:

$$u_j(x) := \tilde{u}_j^i(x), \quad \text{if} \quad x \in M_i.$$

Then by definition $u_j \in C_0^{Lip}(M)$ holds. Since $u_j^i \to u|_{M_i}$ in L^2 for $j \to \infty$ there exists a subsequence which we will denote by u_j^i too, such that $u_j^i \to u|_{M_i}$ pointwise because u_j^i and u are continuous on M_i. Hence we have $u_j^i(0) \to u(0)$ and therefore by

$$\sum_{i=1}^k \int_{M_i} |\tilde{u}_j^i - u|_{M_i}|^2 = \sum_{i=1}^k \int_{[0,\epsilon]} |\tilde{u}_j^i - u|_{M_i}|^2 + \sum_{i=1}^k \int_{(\epsilon,\infty)} |\tilde{u}_j^i - u|_{M_i}|^2$$

and

$$\begin{aligned} \int_{[0,\epsilon]} |\tilde{u}_j^i - u|_{M_i}|^2 &= \int_{[0,\epsilon]} \left| \frac{\epsilon - x}{\epsilon}(u(0) - u_j'(0)) + u_j^i - u|_{M_i} \right|^2 \\ &\leq 2\int_{[0,\epsilon]} \left| \frac{\epsilon - x}{\epsilon}(u(0) - u_j'(0)) \right|^2 + 2\int_{[0,\epsilon]} \left| u_j^i - u|_{M_i} \right|^2 \end{aligned}$$

we have that $u_j \to u$ in L^2 because of $u_j^i \to u|_{M_i}$ in L^2. Further it holds that

$$\sum_{i=1}^k \int_{M_i} |(\tilde{u}_j^i - u|_{M_i})'|^2 = \sum_{i=1}^k \int_{[0,\epsilon]} |(\tilde{u}_j^i - u|_{M_i})'|^2 + \sum_{i=1}^k \int_{(\epsilon,\infty)} |(\tilde{u}_j^i - u|_{M_i})'|^2$$

and

$$\int_{[0,\epsilon]} |(\tilde{u}_j^i)' - (u|_{M_i})'|^2 \leq 2\int_{[0,\epsilon]} \frac{1}{\epsilon^2}(u(0) - u_j^i(0))^2 + 2\int_{[0,\epsilon]} |(u_j^i)' - (u|_{M_i})'|^2$$

which yields $u_j \to u$ for $j \to \infty$ w.r.t. $\mathcal{E}_1$, since $\int_{M_i} |(u_j^i - u|_{M_i})'|^2 \to 0$ and $u_j^i(0) \to u(0)$ holds for $j \to \infty$.

■

Remark 2.6 *An analogous result holds for graphs because the proof works locally around the vertices.*

2.2.2 Diffusions on Graphs and Euclidean Complexes

Here we want to show that the Markov process X_t associated to the glued Dirichlet form $\mathcal{E}$ behaves in same sense as one expects (for more details see chapter 4). Namely, we consider the Dirichlet form $\mathcal{E}$ on a k-spider M coming from the gluing of k Dirichlet forms

$$\mathcal{E}_i(u) := \frac{1}{2}\int_{\mathbf{R}_+} |u'(x)|^2 dx$$

on the single ray $M_i \sim \mathbf{R}_+$. The first step is to characterize the domain of the associated self-adjoint operator A on M:

Proposition 2.7 *If $\mathcal{E}$ is the glued Dirichlet form on a k-spider M coming from the canonical Laplacian on $\mathbf{R}_+$, then the domain of the associated operator is:*

$$D(A) := \{u \in D(\mathcal{E}) : u'' \in L^2(M), \sum_{i=1}^{k} (u|_{M_i})'(0) = 0\}. \tag{2.2}$$

Proof: By definition it holds that $D(A) \subset D(\mathcal{E})$ and $\mathcal{E}(u, v) = (-Au, v)\ \forall\ u \in D(A), \forall\ v \in D(\mathcal{E})$. Let $\mathcal{A} = \{u \in L^2(M) : (u|_{M_i})'' \in L^2(M_i) \quad \text{for} \quad i = 1, \ldots, k\}$. Then with $u \in D(A) \subset D(\mathcal{E})$ we have $\forall\ v \in D(\mathcal{E})$, $\mathcal{E}(u, v) = (-Au, v)$ and therefore $Au = \frac{1}{2}u''$ in $M_i \setminus \{0\}$ since $C_0^{Lip}(M) \subset D(\mathcal{E})$. Hence with $u_i = u|_{M_i}$, $v_i = v|_{M_i}$:

$$\mathcal{E}(u, v) = \frac{1}{2} \sum_{i=1}^{k} \int_{M_i} u_i' v_i' dx = -\frac{1}{2} \sum_{i=1}^{k} \int_{M_i} u_i'' v_i dx + \frac{1}{2} \sum_{i=1}^{k} u_i'(0) v_i(0)$$

by partial integration, which means that $\sum_{i=1}^{k} u_i'(0) = 0$.

For the other direction let $u \in \mathcal{A}$ and $\sum_{i=1}^{k} u_i'(0) = 0$. Then $u \in D(\mathcal{E})$ and integration by parts yields $\mathcal{E}(u, v) = -\frac{1}{2} \int_M u'' v d\mu$ for all $v \in D(\mathcal{E})$, while μ is the glued measure on M. Hence $u \in D(A)$.

∎

Remark 2.8

- *For a graph M and the corresponding glued Dirichlet form $\mathcal{E}$ coming from the canonical Laplacian on the edges one can characterize the domain $D(A)$ in an analogous way. Hence for all $u \in D(A)$ and x a vertex in M one has $\sum_{i=1}^{k} (u|_{M_i})'(x) = 0$, while $M_1, \ldots, M_k$ denote the edges, adjacent to x.*

- *By the Green formula:*

$$\int_G (v\Delta u - u\Delta v) dx = \int_{\partial G} (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}) dF$$

  instead of partial integration one can establish an analogous result for 2-dimensional Euclidean complexes when the glued Dirichlet form comes from the canonical Laplacian on the Euclidean simplices. Here n is the inward normal of G. Then all functions $u \in D(A)$ partially characterized by the following property:

$$\sum_{i=1}^{k} \frac{\partial u(x)}{\partial n_i(x)} = 0, \tag{2.3}$$

for all x in the interior of an edge e, while n_i is the inward normal for the i-th adjacent face of e.

- *In order to define the operator A pointwise on the edges of the Euclidean simplex or on the vertices of a graph one has to use a distributional argument to get*

$$Au(x) = \frac{1}{2}\left(\frac{\partial^2 u(x)}{\partial T_x^2} + \frac{1}{k}\sum_{i=1}^{k}\frac{\partial^2 u(x)}{\partial n_i^2(x)}\right)$$

for x inside an edge e, while T_x is the tangential vector along e. Similar for the generator A on a graph one gets

$$Au(x) = \frac{1}{2k}\sum_{i=1}^{k}(u|_{M_i})''(x)$$

for x a vertex of the graph.

These conditions for functions in the domain $D(A)$ coincide with the conditions given in [BK01], where a so called Brownian motion on 2-dimensional Euclidean complexes is constructed. This process is defined as a Brownian motion inside the faces and after it hits an edge it goes into one of the adjacent faces with equal probability. Now we will verify that these properties are adopted by our associated process X_t.

By properties 2.2 and 2.3 of functions in the domain $D(A)$ we can now calculate the probabilities of the associated Markov process X_t starting in a vertex of a graph or inside an edge of a 2 dimensional Euclidean complex to enter one of the adjacent faces.

Proposition 2.9 *Let z be a vertex of a graph M and M_i for $i = 1, \ldots, k$ the adjacent edges. Further let X_t be the corresponding Markov process to $\mathcal{E}$ with $X_0 = z$. Then for all $\epsilon > 0$ and $T = \inf\{t \geq 0 : d(X_t, z) \geq \epsilon\}$ we have $P(X_T \in M_i) = \frac{1}{k}$.*

Proof: Let $h^i \in D(A)$ be the following piecewise linear function inside $B_\epsilon(z)$:

$$h^i(x) := \begin{cases} \frac{1}{k} + \frac{k-1}{k\epsilon}d(z,x) & \text{if } x \in M_i \cap B_\epsilon(x) \\ \frac{1}{k} - \frac{1}{k\epsilon}d(z,x) & \text{if } x \in B_\epsilon(x) \setminus M_i. \end{cases}$$

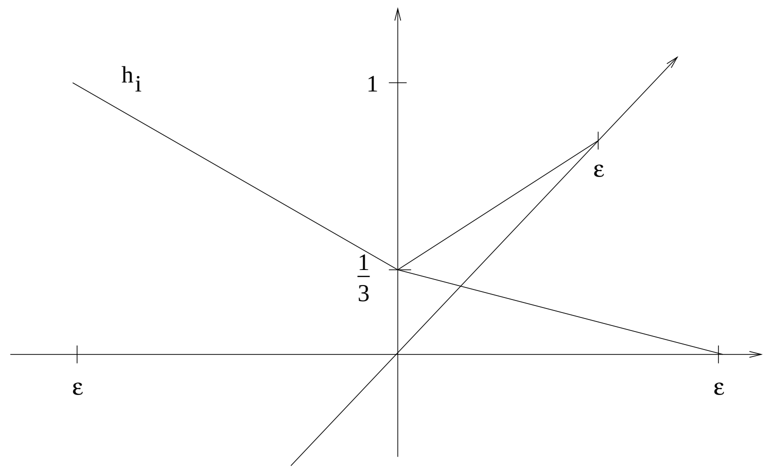

Figure 2.1: h^i on a 3-spider embedded in $\mathbf{R}^2$

Outside $B_\epsilon(z)$ h^i can be continued in an appropriate way, to ensure that $h^i \in D(A)$. By definition $h^i(z) = \frac{1}{k}$, $h^i(x) = 1$ for $x \in M_i$, $d(x,z) = \epsilon$ and $h^i(x) = 0$ for $x \notin M_i$, $d(x,z) = \epsilon$. Since

$$h^i(X_T) - h^i(X_0) - \int_0^T (Ah^i)(X_s)ds$$

is a martingale and $Ah^i = (h^i)'' = 0$ inside $B_\epsilon(z)$, we have

$$0 = E\left[h^i(X_T) - h^i(z)\right] = E\left[h^i(X_T)\right] - \frac{1}{k} = P(X_T \in M_i) \cdot 1 - \frac{1}{k}$$

which finishes the proof.

■

Remark 2.10 (Weighted Graphs) *We call a graph M a weighted graph if the glued measure μ comes from the gluing of k Lebesgue measures μ_i times a constant p_i. The associated process on the original spaces will behave like a Brownian motion but on the glued space the probabilities $P(X_T \in M_i)$ will not be the same for each i. To see this one imitates the proof of Proposition 2.7:*

$$\begin{aligned}
\mathcal{E}(u,v) &= \frac{1}{2}\sum_{i=1}^k \int_{M_i} u_i' v_i' p_i d\mu_i \\
&= -\frac{1}{2}\sum_{i=1}^k p_i \int_{M_i} u_i'' v_i d\mu_i + \frac{1}{2}\sum_{i=1}^k p_i u_i'(0) v_i(0),
\end{aligned}$$

which means that for all $u \in D(A)$ *it holds that*

$$\sum_{i=1}^{k} p_i u_i'(0) = 0. \tag{2.4}$$

Analogous to the proof of Proposition 2.7 but a little more tedious we have to define a function $h^i \in D(A)$ *which is harmonic in* $B_\epsilon(0)$*, satisfies 2.4 and has values* 1 *at* $M_i \cap \partial B_\epsilon(0)$ *and* 0 *at* $M_j \cap \partial B_\epsilon(0)$ *for* $i \neq j$*. By simple calculations we get such a function:*

$$h^i(x) := \begin{cases} b_i + \frac{1-b_i}{\epsilon} d(0,x) & \text{if } x \in M_i \\ b_i - \frac{b_i}{\epsilon} d(0,x) & \text{if } x \in B_\epsilon(0) \setminus M_i \end{cases}$$

while $b_i = \frac{p_i}{\sum_{j=1}^{k} p_j}$*. As in the proof of Proposition 2.9 we get* $P(X_T \in M_i) = \frac{p_i}{\sum_{j=1}^{k} p_j}$.

Remark 2.11 (Euclidean Complexes) *For a 2-dimensional Euclidean complex* M *one get similar results by constructing functions* $h^i \in D(A)$ *in the following way: Let* z *be a point inside an edge* e *of* M *with* k *adjacent faces* $M_1, \ldots, M_k$*. Consider the set*

$$A_\epsilon^c := \{x \in M : d(x,e) < \epsilon\} \cap \{x \in M : d(x,z) < c\}$$

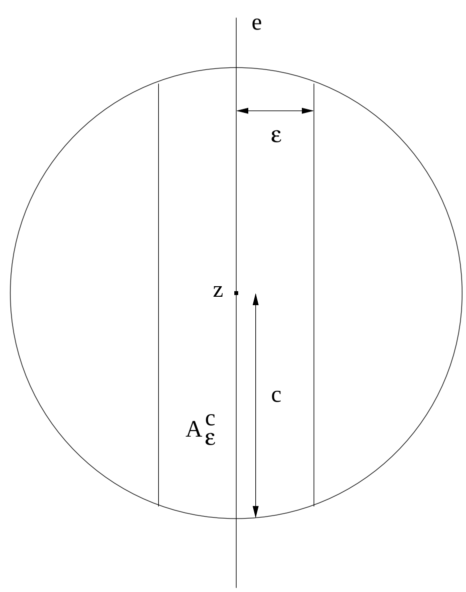

Figure 2.2: The set A_ϵ^c

and define h^i in the following way:

$$h^i(x) := \begin{cases} \frac{1}{k} + \frac{k-1}{k\epsilon} d(e,x) & \text{if } x \in M_i \cap B_c(z) \\ \frac{1}{k} - \frac{1}{k\epsilon} d(e,x) & \text{if } x \in B_c(z) \setminus M_i. \end{cases}$$

Further let $\tilde{h}^i$ be a function with $\tilde{h}|_{B_c(z)} = h^i|_{B_c(z)}$, so that $\tilde{h}^i \in D(A)$ and $T_c^\epsilon := \inf\{t \geq 0 : X_t \notin A_\epsilon^c\}$ with X_t the Markov process associated to $\mathcal{E}$ with $X_0 = z$. In the same manner as above we get

$$E\left[h^i(X_{T_c^\epsilon})\right] = \frac{1}{k}.$$

By splitting

$$\begin{aligned} \partial A_\epsilon^c &= \underbrace{\{x \in M : d(e,x) = \epsilon \text{ and } d(z,x) \geq c\}}_{=:B_1^\epsilon} \\ &\quad \dot{\cup} \underbrace{\{x \in M : d(z,x) = c \text{ and } d(e,x) < \epsilon\}}_{=:B_2^\epsilon} \end{aligned}$$

one can split up $E[h^i(X_{T_c^\epsilon})]$, so that for $c' \in [0,1]$:

$$\frac{1}{k} = E[h^i(X_{T_c^\epsilon})] = P(X_{T_c^\epsilon} \in B_1^\epsilon) + P(X_{T_c^\epsilon} \in B_2^\epsilon)c'$$

holds. If $c > 0$ is a constant small enough but fixed we choose $\epsilon_n \to 0$ for which $P(X_{T_c^{\epsilon_n}} \in B_2^{\epsilon_n}) \to 0$, otherwise there would be a contradiction. Hence this yields $\lim_{\epsilon \to 0} P(X_{T_c^\epsilon} \in B_1^\epsilon) = \frac{1}{k}$.

2.3 Intrinsic Metrics

In order to have doubling for the intrinsic metric induced by the Dirichlet form and to use comparability arguments in later proofs we have to assume

$$d_i \sim \rho_i \text{ on } M_i \text{ for } i = 1, 2$$

while

$$\rho_i(x,y) := \sup\{u(x) - u(y) : u \in D_{loc}(\mathcal{E}_i) \cap C(M_i), d\Gamma_i(u) \leq d\mu_i\}$$

In the following four lemmata we show that then $\rho \sim d$ holds on M while

$$\rho(x,y) := \sup\{u(x) - u(y) : u \in D_{loc}(\mathcal{E}) \cap C(M), d\Gamma(u) \leq d\mu\}.$$

Lemma 2.12 *With the assumptions above there exists a constant $c > 0$ such that*

$$d(x,y) \geq c\rho(x,y)$$

for all $x, y \in M$.

Proof: First we show this inequality on M_i for $i = 1,2$. Let $x, y \in M_i$ then there are constants $c', c'' > 0$ s.t. $d(x,y) \geq c'd_i(x,y) \geq c''\rho_i(x,y)$ holds. Since the restriction of functions $u \in D_{loc}(\mathcal{E}) \cap C(M)$ on M_i are in $D_{loc}(\mathcal{E}_i) \cap C(M_i)$ and if $d\Gamma(u) \leq d\mu$ the same holds for $u|_{M_i}$, i.e. $d\Gamma_i(u) \leq d\mu_i$ the metric ρ_i ist greater than ρ on M_i. Therefore, $d(x,y) \geq c\rho(x,y)$ on M_i.
Now let $x, y \in M$ be not in the same part M_i. Then take the shortest geodesic $\gamma_{x,y}$ w.r.t. d and any point $z \in \gamma_{x,y}[0,1] \cap A$. Then the following holds:

$$d(x,y) = d(x,z) + d(y,z) \geq c\rho(x,z) + c\rho(y,z) \geq c\rho(x,y),$$

by the triangle inequality and the fact that $\gamma_{x,y}$ is the shortest path w.r.t. d.

■

Lemma 2.13 *For all $x, y \in M_i$ for $i = 1,2$ there exists a constant $c > 0$, s.t.*

$$\rho(x,y) \geq cd(x,y).$$

Proof: The idea of the proof is to construct an admissible function $u \in \{u \in D_{loc}(\mathcal{E}) \cap C(M) : d\Gamma(u) \leq d\mu\}$ for each $x \in M_i$ s.t. $u(x) - u(y) \geq c\rho_i(x,y)$. Since $\rho_i(x,y) \geq c_1 d_i(x,y) \geq d(x,y)$ holds for $c_1 > 0$ on M_i this is sufficient.

There exists a function Ψ^0 in $D(\mathcal{E}) \cap C_0(M)$ with compact support $Y \subset M$ which is in $D(\mathcal{E}_2) \cap C_0(M_2)$ too with compact support $Y \cap M_2$. Further Ψ^0 satisfies $0 \leq \Psi^0 \leq 1$ on M and $\Psi^0 = 1$ on a relatively compact open set M_0 (cf. [F80], Lemma 1.4.2). By taking the minimum of the function and $K \cdot \Psi^0$ we assume that the functions have compact support to make the proof not even more technical.

The Construction:

W.l.o.g. let $x \in M_1$ and $\Psi(y) := \rho_1(x,y)$. Since we have the assumption that $\rho_i \sim d_i$ on M_i and $d_1 \sim d_2$ on A the following holds: $\exists\, c, C > 0 : \forall\, x, y \in A :$

$$c\rho_1(x,y) \leq \rho_2(x,y) \leq C\rho_1(x,y).$$

Therefore, $\Psi(\cdot)$ is Lipschitz continuous on A w.r.t. ρ_2 with minimal constant $\frac{1}{c}$:

$$|\Psi(y) - \Psi(z)| = |\rho_1(x,y) - \rho_1(x,z)| \leq \rho_1(y,z) \leq \frac{1}{c}\rho_2(y,z).$$

Now for every $n \in \mathbf{N}$ there exists a countable number of points $y_k = y_k^{(n)} \in A$, $k \in \mathbf{N}$, s.t. $\{\tilde{B}_2(\frac{1}{n}, y_k) : k \in \mathbf{N}\}$ is a covering of A with $\tilde{B}_2(r,x) := \{y \in \mathbf{M}_2 : \rho_2(x,y) < r\}$. We define a function

$$\phi_n^k(y) := (\Psi(y_k) - \frac{1}{c}\rho_2(y_k, y))_+$$

on M_2 for each k. This function has the following properties:

- ϕ_n^k is continuous on M_2 w.r.t. d
- $\phi_n^k \in D(\mathcal{E}_2) \cap C_0(M_2)$
- $d\Gamma(\phi_n^k) = \frac{1}{c^2} d\Gamma(\rho_2(y_k, \cdot)) \leq \frac{1}{c^2} d\mu$
- $\phi_n^k(y) \leq \Psi(y)$ on A

The reason for the last property is:

$$\begin{aligned}
\phi_n^k(y) \quad &= (\Psi(y_k) - \frac{1}{c}\rho_2(y_k, y))_+ \\
&= (\Psi(y_k) - \Psi(y) + \Psi(y) - \frac{1}{c}\rho_2(y_k, y))_+ \\
&\leq (\frac{1}{c}\rho_2(y_k, y) + \Psi(y) - \frac{1}{c}\rho_2(y_k, y))_+ \\
&= \Psi(y).
\end{aligned}$$

Since $\Psi(y) \geq 0$ everywhere on A.

Now define

$$\overline{\phi}_n(y) := \sup_k \phi_n^k(y)$$

and

$$\Psi_n(y) := \sup_{l \leq n} \overline{\phi}_l(y).$$

Because $\Psi_n(\cdot)$ is monotone increasing in n and bounded by above, there exists a limit $\overline{\Psi} := \lim_{n\to\infty} \Psi_n(y)$. Even more since Ψ_n is continuous in d and $\{\Psi_n\}_{n\in\mathbf{N}}$ is a

Cauchy sequence w.r.t. $||\cdot||_\infty$, $\overline{\Psi}$ is continuous in d. That is for $n, m \geq \mathbf{N}$:

$$\begin{aligned}
||\Psi_n - \Psi_m||_\infty \quad &= \sup_{M_2} |\Psi_n(y) - \Psi_m(y)| \\
&= \sup_k \sup_{y \in B_2(\frac{1}{N}, y_k^{(N)})} |\Psi_n(y) - \Psi_m(y)| \\
&\leq \sup_k \sup_{y \in B_2(\frac{1}{N}, y_k^{(N)})} |\Psi_n(y) - \Psi_n(y_k^{(N)})| + |\Psi_m(y_k^{(N)}) - \Psi_m(y)| \\
&\leq \sup_k (\frac{1}{c}\frac{1}{N} + \frac{1}{c}\frac{1}{N}) \\
&= const. \cdot \frac{1}{N},
\end{aligned}$$

since $\Psi_n(y_k^{(N)}) = \Psi_m(y_k^{(N)})$. On the gluing set A the functions Ψ and $\overline{\Psi}$ coincide, since

$$\begin{aligned}
|\Psi(y) - \overline{\Psi}(y)| \quad &\leq |\Psi(y) - \Psi_n(y)| + |\overline{\Psi}(y) - \Psi_n(y)| \\
&\leq \frac{2}{c}\frac{1}{n} + \frac{2}{c}\frac{1}{n}
\end{aligned}$$

with an analogous argument as above. So we can stick together the two continuous functions Ψ and $\overline{\Psi}$ to get one continuous function

$$u(y) := \begin{cases} \Psi(y) & \text{if } y \in M_1 \\ \overline{\Psi}(y) & \text{if } y \in M_2 \end{cases}$$

w.r.t. d.
In the last part we show that $d\Gamma(u) \leq \max\{1, \frac{1}{c^2}\} d\mu$ and $u \in D_{loc}(\mathcal{E}) \cap C(M)$, s.t. for $c < 1$ u is an admissible function in $\{u \in D_{loc}(\mathcal{E}) \cap C(M), d\Gamma(c \cdot u) \leq d\mu\}$ and therefore:

$$\begin{aligned}
c\rho_1(x, y) \quad &= c\rho_1(x, y) - c\rho_1(x, x) \\
&= c\Psi(y) - c\Psi(x) \\
&= cu(y) - cu(x) \\
&\leq \sup\{v(x) - v(y) : v \in D_{loc}(\mathcal{E}) \cap C(M), d\Gamma(v) \leq d\mu\} \\
&= \rho(x, y),
\end{aligned}$$

$\forall\ y \in M_1$ which will finish the proof because the proof for M_2 is the same.

At first the functions $\overline{\phi}_n$ and so the functions Ψ_n satisfy the property:

$$d\Gamma(\overline{\phi}_n) \leq \frac{1}{c^2} d\mu$$

and

$$d\Gamma(\Psi_n) \leq \frac{1}{c^2} d\mu$$

on M_2 since ϕ_n^k do. By the theorem of Banach-Saks there exists a weak convergent subsequence of $\{\Psi_n\}_{n\in N}$ which we denote by $\{\Psi_k\}_{k\in N}$ such that

$$\Psi_n^* := \sum_{k=1}^{n} \Psi_k$$

converges in the Dirichlet norm $\sqrt{\mathcal{E}_1(\cdot) + ||\cdot||_2}$ on M_2. The important properties directly transfer to Ψ_n^*. Further one identifies the limits of Ψ_n^* and Ψ_n in $L^2(M_2,\mu)$, since Ψ_n converges in L^2. By the strong convergence of Ψ_n^* we now have $\forall\ A \subset M_2$

$$\begin{aligned}\left|\int_A d\Gamma(\overline{\Psi}) - \int_A d\Gamma(\Psi_n^*)\right| &\leq \left|\int_A d\Gamma(\overline{\Psi} - \Psi_n^*)\right| \\ &\leq \int_{M_2} d\Gamma(\overline{\Psi} - \Psi_n^*) \to 0, \quad \text{for} \quad n \to \infty,\end{aligned}$$

and therefore $d\Gamma(\overline{\Psi}) \leq \frac{1}{c^2} d\mu$ holds for $\overline{\Psi}$ too. We now know that $d\Gamma(u) \leq \max\{1, \frac{1}{c^2}\} d\mu$ and $u \in D_{loc}(\mathcal{E}) \cap C(M)$ and that finishes the proof.

■

To see that the metric ρ is intrinsic in the sense of Definition 1.1 (cf. [St95a]) we prove in the following lemma:

Lemma 2.14 *The Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on (M, d, μ) is strongly regular, i.e. the topology induced by the intrinsic metric:*

$$\rho(x,y) := \sup\{u(x) - u(y) : u \in D_{loc}(\mathcal{E})\} \cap C(X),\ \ d\Gamma(u) \leq d\mu\}$$

coincides with the topology induced by the intrinsic metric d which comes from the gluing procedure. Then (M, ρ) is a length space.

Proof: Let τ be the topology induced by the metric d and $\tilde{\tau}$ the topology induced by ρ.
We first show that $\tilde{\tau} \subset \tau$: Since $\rho(x,y) \leq cd(x,y)$ with fixed $c > 0$ holds for all $x, y \in M$ we have for a ball $\tilde{B}_r(x) := \{y \in M : \rho(x,y) < r\}$ that for each $z \in \tilde{B}_r(x)$ there exists a ball $\tilde{B}_\epsilon(z) \subset \tilde{B}_r(x)$ and the ball $B_{\frac{\epsilon}{c}}(z) \subset \tilde{B}_\epsilon(z) \subset \tilde{B}_r(x)$, s.t. $\tilde{B}_r(x) \in \tau$.
For the other direction $\tau \subset \tilde{\tau}$ take a ball $B_r(x) := \{y \in M : d(x,y) < r\}$. Then for all $z \in B_r(x) \cap (M_i - A)$ there exists a ball $B_\epsilon(z) \subset B_r(x)$ since $M_i - A$ is open.

Because of $d(x,y) \leq c'\rho(x,y)$ for a fixed constant $c' > 0$ and $x, y \in M_i$ it holds that $\tilde{B}_{\frac{\epsilon}{c'}}(z) \subset B_\epsilon(z) \subset B_r(x)$. If $z \in B_r(x) \cap A$ and $B_\epsilon(z) \subset B_r(x)$ we have to show that $\tilde{B}_{\frac{\epsilon}{c'}}(z) \subset B_\epsilon(z)$ to finish the proof. But since $d(z,y) \leq c'\rho(z,y) \leq \epsilon$ for all $y \in M_i$ this is true. By [St95a] then (M, ρ) is a length space.

■

Lemma 2.15 *If there exists a $c > 0$ s.t. $d(x,y) \leq c\rho(x,y)$ for all $x, y \in M_i$ the same holds true for all $x, y \in M$.*

Proof: Since with $\rho(x,y) \leq const.d(x,y)$ the Dirichlet space $(\mathcal{E}, D(\mathcal{E}))$ is strongly regular and (M, ρ) is a length space. If $x, y \in M$ are not in the same part M_i take the shortest geodesic $\gamma_{x,y}$ w.r.t. ρ and a point $z \in A \cap \gamma_{x,y}[0,1]$. Then it holds that:

$$c\rho(x,y) = c\rho(x,z) + c\rho(y,z) \geq d(x,z) + d(y,z) \geq d(x,y)$$

because $x, z \in M_i$ and $y, z \in M_j$ which finishes the proof.

■

Corollary 2.16 *An analogous result for the comparability of balls w.r.t. d_i and d holds for ρ_i and ρ.*

Proof: This is a direct consequence of $\rho \sim d$ and $\rho_i \sim d_i$.

■

Corollary 2.17 *Doubling holds for (ρ, μ) if doubling holds for (d, μ).*

Remark 2.18 *The last corollaries and lemmata and in particular Lemma 2.14 will be used frequently throughout the next chapters without mentioning it explicitly. One reason is to use the framework and results of [St95b], [St96] where the strong regularity of the Dirichlet form is one of the three key assumptions beside doubling for (ρ, μ) and the scaling invariant Poincaré inequality.*

Example 2.19 *For our examples in $\mathbf{R}^n$ or domains $\Omega \subset \mathbf{R}^n$ with the canonical Dirichlet form $\mathcal{E}(u,v) = \int \nabla u \cdot \nabla v dx$ we briefly demonstrate that the Euclidean metric d coincides with ρ or the restricted intrinsic metric d_Ω coming from the*

Euclidean metric on $\mathbf{R}^n$ *coincides with* ρ *coming from the Dirchlet form* $\mathcal{E}$ *on* $H_0^{1,2}(\Omega)$ *resp. Also for* $\mathcal{E}$ *on* $H^{1,2}$ *considered as a Dirichlet form on* $L^2(\overline{\Omega}, dx)$ *this is true: For the direction* $d \leq \rho$ *or* $d_\Omega \leq \rho$ *one just has to recognize that the map* $u : x \mapsto d(x, y)$ *(resp.* $u : x \mapsto d_\Omega(x, y)$*) lies in* $H^{1,2}(\mathbf{R}^n)$ *(resp.* $H_0^{1,2}(\Omega)$*) and that* $d\Gamma(u) = |\nabla u|^2 \leq 1$ *holds true. For the other direction consider admissible functions* u *with* $|\nabla u|^2 \leq 1$ *so that clearly* $|u(x) - u(y)| \leq d(x, y)$ *(resp.* $|u(x) - u(y)| \leq d_\Omega(x, y)$*) holds.*

Chapter 3

Poincaré Inequality or Spectral Gap on Glued Spaces

In this chapter the main intention of this work will be discussed. Namely, to give conditions in the gluing set A so that given that the Poincaré inequality on balls inside M_i holds, the Poincaré inequality on balls in M holds. The name of this chapter indicates that we are using spectral gap techniques to prove our theorems. After gluing two spaces we treat the case of k-gluing which is not successively done but simultaneously. There are two difficulties which make the conditions and the proof a little tedious. The first one is that the measure of the gluing set A might be too small or even zero and the second one that the gluing map Φ is bilipschitz. In the second section we study the case that $\mu_i(A_i \cap B_i(x,r))$ is large enough for all $x \in A_i$. Further we discuss the simplifications which occur if Φ is isometric. To finish this chapter, examples of gluing in the 1-dimensional case and in the n-dimensional case for $n \geq 2$ are given.

3.1 The Poincaré Inequality on Glued Spaces

Now as we have clarified what is understood by gluing of metric measure spaces and of Dirichlet spaces we will investigate under what conditions one can glue spaces in order to get Poincaré inequality on the resulting glued spaces. One natural assumption is the Poincaré inequality for balls in the original spaces. Another assumption has to be made on the boundary. This boundary condition is a scaling invariant lower bound for the spectral gap with mixed (Neumann-Dirichlet) values on the gluing set A.

From now let (M, d, μ) be the glued metric measure space, locally compact and separable, coming from gluing (M_i, d_i, μ_i) along A by bilipschitz maps and $(\mathcal{E}, D(\mathcal{E}))$ the strong local regular Dirichlet form coming from gluing the original Dirichlet forms $(\mathcal{E}_i, D(\mathcal{E}_i))$ on M_i. Further the doubling property for the measure μ w.r.t. ρ holds.

Now we fix some notations which are frequently used in the sequel:

- By
$$I_B(u) := \frac{\int_B d\Gamma(u)}{\int_B |u|^2 d\mu}$$
we denote the Rayleigh quotient on the set $B \subset M$ w.r.t. the energy measure $d\Gamma$ of $\mathcal{E}$.

- $D(\mathcal{E}, B)$ denotes the completion of $D_B := \{u|_B : u \in D(\mathcal{E})\}$ w.r.t.
$$\mathcal{E}^B(u) := \left(\int_B |u|^2 d\mu + \int_B d\Gamma(u) \right)^{\frac{1}{2}}.$$

- We will use the notation
$$u_B := \frac{1}{\mu(B)} \int_B u \, d\mu$$
for the mean value of u w.r.t. a set $B \subset M$ and the measure μ.

- For $u \in D(\mathcal{E})$ let $\tilde{u}$ be the quasicontinuous version of u.

- With $B_i(x, r)$ and $B(x, r)$ we denote open balls w.r.t. ρ_i and ρ.

In particular $D(\mathcal{E}, B)$ is a Hilbert space and $\mathcal{E}|_B$ is defined on it as a positive definite symmetric bilinear form.

Further we need some definitions to formulate our theorems. Most of them are stated in dependence on $i \in \{1, \ldots, k\}$, i.e. are related to the i-th original space M_i:

Definition 3.1

- *We say that the (P) condition is satisfied inside $M_i \setminus A$ if the (strong) Poincaré inequality holds for all balls $B_r \subset M_i \setminus A$ with radius $r > 0$. This means, there exists a constant $c_p^i > 0$ such that for all balls $B_r \subset M_i \setminus A$ and for all functions u in the domain $D(\mathcal{E}_i)$ of the Dirichlet form*
$$\int_{B_r} |u - u_{B_r}|^2 d\mu \leq c_p^i r^2 \int_{B_r} d\Gamma(u)$$
holds true.

- *In the following we call* $\{B_i^c(r,x) \subset M_i : x \in A,\, r > 0\}$ *for* $i = 1, \dots, k$ *a comparable system of measurable sets in* M_i *if there exists a constant* $c_i^c > 0$*, s.t.* $\forall\, x \in A$ *and* $\forall\, r > 0$

$$\frac{1}{c_i^c} B_i^c(r,x) \subset B_i(x,r) \subset c_i^c B_i^c(r,x)$$

holds while $B_i(x,r) = \{y \in M_i : \rho_i(x,y) < r\}$ *and* $cB_i^c(r,x) = B_i^c(cr,x)$*. In particular the balls* $B_i(x,r)$ *for* $x \in A$ *w.r.t.* ρ_i *provide obviously a comparable system of sets.*

- *We say that the Rellich condition is fulfilled if for each set of functions* $\{u_n\}_n$ *which are uniformly bounded in* $\mathcal{E}^B$ *there exists a strong convergent subsequence* $\{u_{n_k}\}_k$ *in* $L^2(B,\mu)$*.*

- *Let*

$$\nu_i(B_i, N) := \inf \left\{ \frac{\int_{B_i} d\Gamma_i(u)}{\int_{B_i} |u|^2 d\mu_i} : u \in D(\mathcal{E}_i),\, \tilde{u}|_{N \cap B_i} = 0,\, u|_{B_i} \neq 0 \right\}$$

for $i = 1, \dots, k$*. Then the HT (Heat Transmission) condition is fulfilled if the (P) condition holds for balls inside* $M_i \setminus A$ *and there exists a comparable system of sets* $\{B_i^c(r,x) \subset M_i : x \in A,\, r > 0\}$ *for* $i = 1, \dots, k$*, s.t. the Rellich condition holds on* $B_i^c := B_i^c(r,x)$*. Further the following inequality is satisfied:*

$$\nu_i^*(B_i^c) \geq c_{ht}^i \frac{1}{r^2} \tag{3.1}$$

for a constant $c_{ht}^i > 0$ *and all* $0 < r \leq R$ *with a fixed* $R > 0$ *while*

$$\nu_i^*(B_i^c) := \inf_{\substack{N \subset B_i^c \cap A \\ m(N) \geq \alpha \cdot m(B_i^c \cap A)}} \nu_i(B_i^c, N)$$

is called the heat transmission coefficient with m *an arbitrary measure on* A *with constants* $c_i > 0$*, s.t.* $m(B_i(\frac{r}{2}, x) \cap A) \geq c_i \cdot m(B_i(r,x) \cap A)$ *for all* $r > 0$*,* $x \in A$*. Here* $\alpha > 0$ *is a universal constant dependent on* c_i*, the Lipschitz constants* $Lip\ \Phi_1, \dots, Lip\ \Phi_{k-1}$ *and the constant* c_i^c *coming from the comparable system of sets.*

Remark 3.2 • *By the comparability of the metrics it is enough to find such a measure for one* A_i*. The doubling property will then hold w.r.t. the other metrics.*

- *One example for the measure m is the $(n-1)$ dimensional Hausdorff measure in the examples in Section 3.3.2.*

- *Since α depends on all gluing maps and all constants c_i^c one has to be careful that α changes if the number of glued spaces changes. For our examples in 3.3.2 it is enough to have any but fixed α to get constants c_{ht}^i such that (3.1) holds. For isometric gluing, i.e. the gluing maps are isometries, the constant α is equal $\frac{1}{2}$ (see Section 3.2.2).*

3.1.1 Preparatory Lemmata

The next lemma shows that a minimizing element of $I_B(\cdot)$ over all $u \in D(\mathcal{E})$ with $u_B = 0$ exists in $D(\mathcal{E}, B)$ if the Rellich condition holds for the sets $B \cap M_i$ on the original spaces M_i.

Lemma 3.3 *Let*

$$\nu := \inf \left\{ I_B(u) : \int_B u \, d\mu = 0, \ u \in D(\mathcal{E}) \right\}$$

while $B := B_1 \cup B_2$ is the union of $B_1 \subset M_1$ and $B_2 \subset M_2$. Assume that the Rellich condition is satisfied on B_i for $i = 1, 2$. Then the following holds:

$$(i) \qquad \exists u \in D(\mathcal{E}, B) : I_B(u) = \nu \ \text{ and } \ u_B = 0,$$

$$(ii) \qquad \exists \{u_k\}_k \subset D(\mathcal{E}), \, (u_k)_B = 0 : \mathcal{E}^B(u_k - u) \to 0.$$

Proof: **(i)** Take the minimizing sequence $\{u_n\}_n \subset D(\mathcal{E})$ with $(u_n)_B = 0$, s.t. $I_B(u_n) \to \nu$ for $n \to \infty$. Since the sequence is $\mathcal{E}^B$-bounded and $(D(\mathcal{E}, B), \mathcal{E}^B)$ is a Hilbert space we can find a weakly convergent subspace $\{u_{n_k}\}_k$ with $u_{n_k} \rightharpoonup u \in D(\mathcal{E}, B)$. It is obvious that $u_B = 0$ because of weak convergence it follows that

$$\int_B u \, d\mu = \lim_{n \to \infty} \int_B u_n \cdot 1 \, d\mu = 0.$$

Now it is enough to show that $I_B(u) \le \nu$ holds since $I_B(u) < \nu$ would lead to a contradiction. Therefore, we have to show that I_B is lower semicontinuous w.r.t. weak convergence in $D(\mathcal{E}, B)$ because then the following holds:

$$I_B(u) \le \liminf_{k \to \infty} I_B(u_{n_k}) \le \lim_{k \to \infty} I_B(u_{n_k}) = \nu.$$

Since we know that $\int_B |\cdot|^2 d\mu$ is continuous w.r.t. weak $\mathcal{E}^B$-convergence because of the Rellich condition it suffices to prove that $\int_B d\Gamma(\cdot)$ is lower semicontinuous w.r.t. weak $\mathcal{E}^B$-convergence:

$$\begin{aligned} I_B(u) \quad &\leq \frac{\liminf_{k\to\infty}\int_B d\Gamma(u_k)}{\int_B u^2 d\mu} \\ &= \liminf_{k\to\infty, j\geq k} \frac{\int_B d\Gamma(u_j)}{\lim_{m\to\infty}\inf_{i\geq m}\int_B u_i^2 d\mu} \\ &\leq \lim_{k\to\infty}\left(\inf_{j\geq k}\frac{\int_B d\Gamma(u_j)}{\int_B u_j^2 d\mu}\right) \\ &= \liminf_{k\to\infty}\frac{\int_B d\Gamma(u_k)}{\int_B u_k^2 d\mu}. \end{aligned}$$

Clearly $\int_B d\Gamma(\cdot)$ is lower semicontinuous w.r.t. $\mathcal{E}^B$-convergence. Because $\int_B d\Gamma(\cdot)$ is convex we can use the theorem of Banach-Saks to get lower semicontinuity w.r.t. weak $\mathcal{E}^B$-convergence. Let $\{u_n\}_n \subset D(\mathcal{E})$ be the weakly convergent subsequence in $D(\mathcal{E}, B)$. Then we may assume that

$$\omega := \lim_{n\to\infty}\int_B d\Gamma(u_n) = \liminf_{n\to\infty}\int_B d\Gamma(u_n)$$

exists for a subsequence. If we take a further subsequence which will be denoted again by $\{u_n\}_n$ we have $\frac{1}{k}\sum_{n=1}^k u_n \to u$ strongly in $\mathcal{E}^B$, since $\mathcal{E}^B(u_n) \leq c$ for a constant $c > 0$ and $\forall n \in \mathbf{N}$ by the Banach-Saks theorem.
Now let $g_k^N := \frac{1}{k}\sum_{n=1}^k u_{N+n}$, then $g_k^N \to u$ for $k \to \infty$ and $\forall N \in \mathbf{N}$. Then we get $\forall N \in \mathbf{N}$:

$$\int_B d\Gamma(g_k) \leq \frac{1}{k}\sum_{n=1}^k \int_B d\Gamma(u_{N+n})$$

by the convexity of $\int_B d\Gamma(\cdot)$. Now choose $\epsilon > 0$ and $N \in \mathbf{N}$ large enough, s.t. $\forall n \in \mathbf{N}$: $\int_B d\Gamma(u_{N+n}) < \omega + \epsilon$. This gives us $\limsup_{k\to\infty}\int_B d\Gamma(g_k) \leq \omega$ and therefore

$$\begin{aligned} \int_B d\Gamma(u) \quad &\leq \liminf_{k\to\infty}\int_B d\Gamma(g_k) \\ &\leq \limsup_{k\to\infty}\int_B d\Gamma(g_k) \\ &\leq \omega \\ &= \liminf_{n\to\infty}\int_B d\Gamma(u_n) \end{aligned}$$

holds by the lower semicontinuity of $\int_B d\Gamma(\cdot)$ w.r.t. strong convergence which finishes the proof.

(ii) In Hilbert spaces weak convergence together with convergence of the norms implies strong convergence. Hence with $u_n \rightharpoonup u$, in $\mathcal{E}^B$,

$$\int_B d\Gamma(u_n) \to \int_B d\Gamma(u)$$

follows by the proof above and

$$\int_B |u_n|^2 d\mu \to \int_B |u|^2 d\mu$$

follows by Rellich so we have $u_n \to u$ strongly in $D(\mathcal{E}, B)$.

∎

Remark 3.4 *Since $D(\mathcal{E}) = \overline{\mathcal{C}_0^{Lip}(M)}^{\sqrt{\mathcal{E}(\cdot)+\|\cdot\|^2}}$ there exists a minimizing sequence $\{u_k\}_k \subset \mathcal{C}_0^{Lip}(M)$ with $\mathcal{E}^B(u_k - u) \to 0$ and $(u_k)_B = 0$. Just take $u_k \to u$ and then relabel $u_k := u_k - (u_k)_B$. Since we have $(u_k)_B = \frac{1}{\mu(B)} \int_B u_k d\mu \to \frac{1}{\mu(B)} \int_B u d\mu = 0$ (strong L^2-convergence implies weak convergence) it holds that $\mathcal{E}^B(u_k - u) \to 0$ for $k \to \infty$.*

For the proof of our main result we need an approximation lemma so that it suffices to establish estimates only for $u \in D(\mathcal{E})$ instead of $u \in D(\mathcal{E}, B)$.

Lemma 3.5 *Let $\{u_n\}_n \subset D(\mathcal{E})$ be a minimizing sequence such that $u_n|_B \to u$ strongly in $(D(\mathcal{E}, B), \mathcal{E}^B)$ as in Lemma 3.3. Let $\nu := I_B(u)$ and $\nu_n := I_B(u_n)$. Then there exists an $\epsilon > 0$ and an $N \in \mathbf{N}$ such that for all $n \geq N$ and for all $\phi \in D(\mathcal{E}, B)$ with $\mathcal{E}^B(\phi) < c$ while $c > 0$ is a fixed constant*

$$\left| \int_B d\Gamma(u_n, \phi) - \nu_n \int_B u_n \phi d\mu \right| \leq \epsilon.$$

and $|\nu_n - \nu| \leq \epsilon$ holds.

Proof: For all v in $D(\mathcal{E}, B)$ the following holds:

$$\begin{aligned}
\frac{d}{d\alpha}\bigg|_{\alpha=0} [I_B(u + \alpha v)] &= \frac{d}{d\alpha}\bigg|_{\alpha=0} \left[\frac{\int_B d\Gamma(u + \alpha v)}{\int_B |u + \alpha v|^2 d\mu} \right] \\
&= \frac{d}{d\alpha}\bigg|_{\alpha=0} \left[\frac{\int_B d\Gamma(u) + \alpha^2 \int_B d\Gamma(v) + \alpha \int_B d\Gamma(u, v)}{\int_B |u|^2 d\mu + \alpha^2 \int_B |v|^2 d\mu + \int_B uv d\mu} \right] \\
&= \left[\frac{\int_B d\Gamma(u, v) \int_B |u|^2 d\mu - \int_B d\Gamma(u) \int_B uv d\mu}{\int_B |u|^2 d\mu \int_B |u|^2 d\mu} \right] \\
&= \frac{\int_B d\Gamma(u, v)}{\int_B |u|^2 d\mu} - I_B(u) \frac{\int_B uv d\mu}{\int_B |u|^2 d\mu}.
\end{aligned}$$

Since $D(\mathcal{E}, B)$ is the closure of all functions in $D(\mathcal{E})$ restricted on B w.r.t. $\mathcal{E}^B$ the infimum does not change if one takes the infimum over all functions in $D(\mathcal{E}, B)$ instead of $D(\mathcal{E})$. Therefore, with $v_B = \frac{1}{\mu(B)} \int_B v \, d\mu$:

$$0 = \frac{d}{d\alpha}\bigg|_{\alpha=0} [I_B(u - \alpha v + \alpha v_B)] = \frac{\int_B d\Gamma(u,v)}{\int_B |u|^2 d\mu} - I_B(u) \frac{\int_B uv d\mu}{\int_B |u|^2 d\mu}$$

because of $\int_B u \, d\mu = 0$. By the calculations above, $v = \phi$ and because of Rellich w.l.o.g. $\int_B |u|^2 \, d\mu = 1$ it follows that:

$$\begin{aligned}
&\left| \int_B d\Gamma(u_n, \phi) - \nu_n \int_B u_n \phi d\mu \right| \\
&= \left| \int_B d\Gamma(u_n, \phi) - \nu_n \int_B u_n \phi d\mu - \int_B d\Gamma(u, \phi) + \nu \int_B u \phi d\mu \right| \\
&= \left| \int_B d\Gamma(u_n - u, \phi) + \int_B (\mu_n u_n - \nu u) \phi d\mu \right| \\
&\leq \left| \int_B d\Gamma(u_n - u, \phi) \right| + \left| \int_B (\nu_n u_n - \nu u) \phi d\mu \right| \\
&\leq \left(\int_B d\Gamma(u_n - u) \int_B d\Gamma(\phi) \right)^{\frac{1}{2}} + \left(\int_B (\nu_n u_n - \nu u)^2 d\mu \int_B \phi^2 d\mu \right)^{\frac{1}{2}} \\
&\leq c^{\frac{1}{2}} \left[\left(\int_B d\Gamma(u_n - u) \right)^{\frac{1}{2}} + \left(\int_B (\nu_n u_n - \nu_n u + \nu_n u - \nu u)^2 d\mu \right)^{\frac{1}{2}} \right] \\
&\leq c^{\frac{1}{2}} \left[\left(\int_B d\Gamma(u_n - u) \right)^{\frac{1}{2}} + \left(2 \left(\int_B \nu_n^2 (u_n - u)^2 d\mu + \int_B (\nu_n - \nu)^2 u^2 d\mu \right) \right)^{\frac{1}{2}} \right].
\end{aligned}$$

Here every term goes to zero for $n \to \infty$ because of the strong convergence in $\mathcal{E}^B$.

■

Remark 3.6 *The lemma is just a special case of the fact that the derivative of $I_B(\cdot)$ at u in the direction v can be approximated by the derivatives of I_B at u_n in the direction v if u_n converges strongly in $\mathcal{E}^B$ against u.*

One last lemma is necessary before proving our main result:

Lemma 3.7 *Let μ be a finite measure on a measurable set B with $\mu(B) \leq \infty$ and $\{u_n\}_n \subset L^2(B, \mu)$ a sequence of functions converging to u in L^2, i.e.*

$$\int_B |u_n - u|^2 d\mu \to 0 \ \text{ for } \ n \to \infty.$$

If $\int_{\{u \gtrless 0\}} |u|^2 d\mu = c > 0$ then $\int_{\{u_n \gtrless 0\}} |u_n|^2 d\mu \to c$ for $n \to \infty$.

Proof: We only consider the case '>' and assume

$$\int_{\{u>0\}} |u|^2 d\mu = c > 0.$$

Since $\int_B |u_n - u|^2 d\mu \to 0$ for $n \to \infty$ we have stochastic μ-convergence, i.e. $\forall \delta > 0$:

$$\mu(\{|u_n - u| \geq \delta\}) \to 0.$$

This holds because of the Tschebyscheff inequality in the form:

$$\mu(\{|u_n - u| \geq \delta\} \cap B) \leq \delta^{-2} \int_B |u_n - u|^2 d\mu.$$

Therefore, with

$$f_r(x) := \begin{cases} x^2 \wedge r & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

the following holds:

$$\begin{aligned}
\left| \int_{\{u_n>0\}} |u_n|^2 d\mu - \int_{\{u>0\}} |u|^2 d\mu \right| \quad &= \left| \int_B f_\infty \circ u_n d\mu - \int_B f_\infty \circ u d\mu \right| \\
&\leq \left| \int_B f_\infty \circ u_n d\mu - \int_B f_r \circ u_n d\mu \right| \\
&+ \left| \int_B f_r \circ u d\mu - \int_B f_r \circ u d\mu \right| \\
&+ \left| \int_B f_r \circ u d\mu - \int_B f_\infty \circ u d\mu \right| \\
&\leq \int_{\{|u_n|>\sqrt{r}\}} |u_n|^2 d\mu + \int_{\{|u|>\sqrt{r}\}} |u|^2 d\mu \\
&+ \left| \int_B f_r \circ u_n d\mu - \int_B f_r \circ u d\mu \right|.
\end{aligned}$$

Since for the first two terms one can choose $r > 0$ large enough, s.t.

$$\int_{\{|u_n|>\sqrt{r}\}} |u_n|^2 d\mu + \int_{\{|u|>\sqrt{r}\}} |u|^2 d\mu < \frac{\epsilon}{2}$$

for arbitrary small $\epsilon > 0$ and independently from n. This is because $u_n \to u$ in L^2 and therefore $\{u_n^2\}_{n \in \mathbf{N}}$ is uniformly integrable, s.t. for the finite measure μ the following holds: Let $g \geq 0$ an $\frac{\epsilon}{8}$-bound for $\{u_n^2\}_{n \in \mathbf{N}}$ then $\forall n \in N$ $\exists \delta > 0$:

$$\begin{aligned}
\int_{\{|u_n|^2>\delta\}} |u_n|^2 d\mu \quad &= \int_{\{|u_n|^2>\delta\} \cap \{|u_n|^2 \geq g\}} |u_n|^2 d\mu + \int_{\{|u_n|^2>\delta\} \cap \{|u_n|^2 < g\}} |u_n|^2 d\mu \\
&\leq \int_{\{|u_n|^2 \geq g\}} |u_n|^2 d\mu + \int_{\{g>\delta\}} g d\mu \\
&\leq \frac{\epsilon}{8} + \int_{\{g>\delta\}} g d\mu,
\end{aligned}$$

while $\int_{\{g>\delta\}} g d\mu < \frac{\epsilon}{8}$ for δ large enough. If we choose δ even larger we get

$$\int_{\{|u|>\sqrt{r}\}} |u|^2 d\mu < \frac{\epsilon}{4}$$

For the last term we have to choose at first δ small enough, s.t. for $A_n := \{|u_n - u| \geq \delta\}$, $2r\delta\mu(B) < \frac{\epsilon}{4}$ and then n large enough, s.t. $2\mu(A_n)||f_r||_\infty < \frac{\epsilon}{4}$. Because then we have:

$$\begin{aligned}\left|\int_B f_r \circ u_n d\mu - \int_B f_r \circ u d\mu\right| &\leq \int_B |f_r \circ u_n - f_r \circ u| d\mu \\ &= \int_B 1_{A_n} |f_r \circ u_n - f_r \circ u| d\mu \\ &+ \int_B 1_{A_n^c} |f_r \circ u_n - f_r \circ u| d\mu \\ &\leq 2\mu(A_n)||f_r||_\infty + 2r\delta\mu(A_n^c),\end{aligned}$$

and that finishes the proof because of $\mu(A_n^c) \leq \mu(B)$.

■

3.1.2 Gluing of two Spaces

The idea of the main result is to prove a weak Poincaré inequality. Then one can use an argument by Jerison [Je86] and Sturm [St96] to derive the strong Poincaré inequality if doubling still holds on the glued space M. By weak Poincaré inequality we mean the following: For fixed constants $0 < c < 1$, $C > 0$ and $\forall\, u \in D(\mathcal{E})$, $r > 0$:

$$\int_{B_{cr}(x)} |u - u_{B_{cr}(x)}|^2 d\mu \leq Cr^2 \int_{B_r(x)} d\Gamma(u)$$

holds true while $B_r(x) := \{y \in M : \rho(x, y) < r\}$ is the ball w.r.t. the intrinsic metric ρ coming from the Dirichlet form $\mathcal{E}$.

Remark 3.8 *In Sturm [St96] the constant c is equal $\frac{1}{2}$. But this can be deduced for arbitrary $c > 0$ by a simple covering argument and the doubling property (similar to the chaining argument in the proof of 5.6).*

Theorem 3.9 *Let $k = 2$ and assume the HT condition holds. For $x \in A$ let $Q_r(x) := B_1^c(x, r) \cup B_2^c(x, r)$, with $\{B_i^c(x, r)\}$ the comparable sytems of sets in M_i, for $i = 1, 2$. Then the following inequality holds:*

$$\nu \geq \frac{1}{2} \min_{i \in \{1,2\}} \nu_i^*(B_i^c)$$

where

$$\nu := \inf \left\{ \frac{\int_{Q_r(x)} d\Gamma(u)}{\int_{Q_r(x)} |u|^2 d\mu} : u \in D(\mathcal{E}),\ \int_{Q_r(x)} u\, d\mu = 0,\ u|_{Q_r(x)} \neq 0 \right\}.$$

Proof: Denote $B_i := B_i^c(x,r)$ and let $\Psi \in D(\mathcal{E}, Q_r(x))$ be the minimizer of $I_{Q_r(x)}$ over all $u|_{Q_r(x)}$, s.t. $u \in D(\mathcal{E})$ and $\int_{Q_r(x)} u d\mu = 0$, i.e. $\nu = I_{Q_r(x)}(\Psi)$. By Lemma 3.3 there exists a minimizing sequence $\{\Psi^n\}_n \subset D(\mathcal{E})$ with $\Psi^n|_{Q_r(x)} \to \Psi$ in $(D(\mathcal{E}, Q_r(x)), \mathcal{E}^{Q_r(x)})$, s.t. $\int_{Q_r(x)} \Psi^n d\mu = 0$, $\int_{Q_r(x)} |\Psi|^2 d\mu = 1$ and $\int_{Q_r(x)} \Psi d\mu = 0$. By Remark 3.4 one can replace $\{\Psi^n\}_n$ with a sequence in $\mathcal{C}_0^{Lip}(M)$. As a minimizing sequence and with $\nu_n = I_{Q_r(x)}(\Psi^n)$ it holds that $\nu_n \to \nu$ for $n \to \infty$. By Lemma 3.5 we have for $\phi = \Psi_+^n$ that Δ_n tends to zero for $n \to \infty$ while

$$\Delta_n := \int_{Q_r(x)} d\Gamma(\Psi^n, \Psi_+^n) - \nu_n \int_{Q_r(x)} \Psi^n \Psi_+^n d\mu.$$

Therefore, we get the following:

$$\nu_n \sum_{i=1}^{2} \int_{B_i^{+,n}} |\Psi^n|^2 d\mu + \Delta_n \geq \frac{1}{2} \sum_{i=1}^{2} \int_{B_i^{+,n}} d\Gamma(\Psi^n) \tag{3.2}$$

while $B_i^{+,n} := \{y \in B_i : \Psi^n(y) > 0\}$.

We want to exclude that

$$\int_{Q_r^{+,n}(x)} |\Psi^n|^2 d\mu \to 0$$

for $n \to \infty$ while $Q_r^{+,n}(x) := \{y \in Q_r(x) : \Psi^n(y) > 0\}$. Note that $\int_{Q_r(x)} |\Psi|^2 = 1$ and $\int_{Q_r(x)} \Psi d\mu = 0$, hence $\int_{Q_r^+(x)} |\Psi|^2 d\mu > 0$ while $Q_r^+(x) := \{y \in Q_r(x) : \Psi(y) > 0\}$. Since $\mu|_{Q_r(x)}$ is a finite measure, we can use Lemma 3.7 to show that $\lim_{n\to\infty} \int_{Q_r^{+,n}} |\Psi^n|^2 d\mu > 0$ holds.

This means that the first case

$$\int_{B_i^{+,n}} |\Psi^n|^2 d\mu \to 0 \ \text{ for } \ n \to \infty$$

is possible for only one $i \in \{1,2\}$. Therefore, with (3.2) we get for $i \neq j$:

$$\begin{aligned} \nu_n + \nu_n \frac{\int_{B_i^{+,n}} |\Psi^n|^2 d\mu}{\int_{B_j^{+,n}} |\Psi^n|^2 d\mu} + \frac{\Delta_n}{\int_{B_j^{+,n}} |\Psi^n|^2 d\mu} \quad &\geq \frac{1}{2} \frac{\sum_{i=1}^2 \int_{B_i^{+,n}} d\Gamma(\Psi^n)}{\int_{B_j^{+,n}} |\Psi^n|^2 d\mu} \\ &\geq \frac{1}{2} \frac{\int_{B_j^{x,n}} d\Gamma(\Psi^n)}{\int_{B_j^{+,n}} |\Psi^n|^2 d\mu} = \frac{1}{2} I_{B_j^{+,n}}(\Psi^n), \end{aligned}$$

and

$$\delta_n := \nu_n \frac{\int_{B_i^{+,n}} |\Psi^n|^2 d\mu}{\int_{B_j^{+,n}} |\Psi^n|^2 d\mu} + \frac{\Delta_n}{\int_{B_j^{+,n}} |\Psi^n|^2 d\mu} \text{ goes to zero for } n \to \infty.$$

In the second case if

$$\lim_{n\to\infty} \int_{B_i^{+,n}} |\Psi^n|^2 d\mu > 0 \text{ for } i = 1,2$$

there exists an $j \in \{1,2\}$ such that

$$\nu_n \int_{B_j^{+,n}} |\Psi^n|^2 d\mu + \frac{\Delta_n}{2} \geq \frac{1}{2} \int_{B_j^{+,n}} d\Gamma(\Psi^n)$$

and divided by $\int_{B_j^{+,n}} |\Psi^n|^2 \, d\mu$ as above we get:

$$\nu_n + \frac{\Delta_n}{2\int_{B_j^{+,n}} |\Psi^n|^2 d\mu} \geq \frac{1}{2} I_{B_j^{+,n}}(\Psi^n)$$

while

$$\delta_n' := \frac{\Delta_n}{2\int_{B_j^{+,n}} |\Psi^n|^2 d\,\mu} \text{ goes to zero for } n \to \infty.$$

Since with $\Psi^n \in D(\mathcal{E})$ the positive part Ψ_+^n is in $D(\mathcal{E})$ and the quasicontinuous version $\tilde{\Psi}_+^n|_{A\cap\{\Psi^n\leq 0\}} = 0$ the following holds for $\delta_n'' = \delta_n$ or $\delta_n'' = \delta_n'$ depending on which case we are in:

$$\begin{aligned} \nu_n + \delta_n'' \quad & \geq \frac{1}{2} I_{B_j^{+,n}}(\Psi^n) = \frac{1}{2} I_{B_j^n}(\Psi_+^n) \\ & \geq \frac{1}{2} \inf \left\{ \frac{\int_{B_j} d\Gamma(u)}{\int_{B_j} |u|^2 d\mu} : u \in D(\mathcal{E}) : \tilde{u}|_{A\cap\{\Psi^n\leq 0\}} = 0 \right\} \\ & \geq \frac{1}{2} \nu_j(B_j, A \cap \{\Psi^n \leq 0\}). \end{aligned}$$

By analogous calculations for the negative part Ψ_-^n we get for a sequence $\delta_n \to 0$:

$$\begin{aligned} \nu_n + \delta_n \quad & \geq \quad \frac{1}{2} \min_i \nu_i(B_i, A \cap \{\Psi^n \leq 0\}) \vee \frac{1}{2} \min \nu_i(B_i, A \cap \{\Psi^n \geq 0\}) \\ & \geq \quad \frac{1}{2} \min_i \nu_i(B_i, B_1 \cap B_2 \cap \{\Psi^n \leq 0\}) \vee \frac{1}{2} \min \nu_i(B_i, B_1 \cap B_2 \cap \{\Psi^n \geq 0\}) \\ & \geq \quad \frac{1}{2} \inf_{N \subset B_1 \cap B_2} \{ \min_i \nu_i(B_i, N) \vee \min \nu_i(B_i, (B_1 \cap B_2) \setminus N) \}, \end{aligned}$$

because of $B_1 \cap B_2 \subset A$ and one can choose $N = B_2 \cap B_2 \cap \{\Psi^n \leq 0\}$ or $N = B_2 \cap B_2 \cap \{\Psi^n \geq 0\}$. Since for $N \subset B_1 \cap B_2$ for an arbitrary measure m on A we have either $m(N) \geq \frac{1}{2}m(B_1 \cap B_2)$ or $m((B_1 \cap B_2) \setminus N) \geq \frac{1}{2}m(B_1 \cap B_2)$ it follows:

$$\begin{aligned} \nu_n + \delta_n \quad & \geq \frac{1}{2} \inf_{N \subset B_1 \cap B_2, m(N) \geq \frac{1}{2}m(B_1 \cap B_2)} \min_i \nu_i(B_i, N) \\ & = \frac{1}{2} \min_i \inf_{N \subset B_1 \cap B_2, m(N) \geq \frac{1}{2}m(B_1 \cap B_2)} \nu_i(B_i, N). \end{aligned}$$

Now B_1 and B_2 are centered in A and ρ_1, ρ_2 and ρ are comparable, so there exists a constant $c' > 0$ such that w.l.o.g.:

$$\begin{aligned} \frac{1}{c_1^c} B_1 \cap A \quad & = \quad \frac{1}{c_1^c} B_1^c \cap A \subset B_1(x,r) \cap A \subset B(x,r) \cap A \subset c' B_2(x,r) \cap A \\ & \subset c_2^c c' B_2^c(x,r) \cap A \subset c_2^c c' B_2 \cap A. \end{aligned}$$

Therefore, there exists a constant $c^* := \frac{1}{c_1^c c_2^c c'} > 0$ s.t. $c^* B_i \cap A \subset B_1 \cap B_2$ for $i = 1, 2$ and by the doubling property for m we get for a constant $\alpha > 0$

$$\frac{1}{2} m(B_1 \cap B_2) \geq \frac{1}{2} m(c^* B_i \cap A) \geq \alpha m(B_i \cap A).$$

Hence

$$\begin{aligned} \nu_n + \delta_n \quad & \geq \frac{1}{2} \min_i \inf_{N \subset B_1 \cap B_2, m(N) \geq \alpha m(B_i \cap A)} \nu_i(B_i, N) \\ & \geq \frac{1}{2} \min_i \inf_{N \subset B_i \cap A, m(N) \geq \alpha m(B_i \cap A)} \nu_i(B_i, N) \end{aligned}$$

while the last inequality holds just because of $B_1 \cap B_2 \subset A \cap B_i$.
Since $\nu_n \to \nu$ and $\delta_n \to 0$ for $n \to \infty$ we have:

$$\nu \geq \frac{1}{2} \min_i \inf_{\substack{N \subset B_i \cap A \\ m(N) \geq \alpha m(B_i \cap A)}} \nu_i(B_i, N) = \frac{1}{2} \min_{i \in \{1,2\}} \nu_i^*(B_i)$$

while $B_i = B_i^c(x,r)$ which finishes the proof.

■

The last theorem shows that under the HT condition with $\nu_i^*(B_i^c(r,x)) \geq c_{ht}^i \frac{1}{r^2}$ we have for all sets $Q_r(x) := B_1^c(x,r) \cup B_2^c(x,r)$ that

$$\int_{Q_r(x)} |u - u_{Q_r(x)}|^2 d\mu \leq \frac{2r^2}{\min_i c_{ht}^i} \int_{Q_r(x)} d\Gamma(u)$$

holds $\forall\, u \in D(\mathcal{E})$ with universal constant $\frac{2}{\min_i c_{ht}^i} > 0$.
The next theorem states the main result that a strong Poincaré inequality holds on M given the HT condition holds:

Theorem 3.10 *Suppose the HT condition holds. Then the strong Poincaré inequality holds on M, i.e.: $\exists C > 0$*

$$\forall\, r > 0,\, x \in M: \quad \int_{B(r,x)} |u - u_{B(r,x)}|^2 d\mu \le Cr^2 \int_{B(r,x)} d\Gamma(u) \quad \forall\, u \in D(\mathcal{E}).$$

Proof: First we will show that a 'weak Poincaré inequality' holds on M and then by the doubling property on M w.r.t. (ρ, μ) we can deduce by the argument of Jerison ([Je86]) which can be used on metric spaces as well (cf. Sturm [St96]) that a strong Poincaré inequality will hold on M. Let $B(r,x) := \{y \in M : \rho(x,y) < r\}$ be an open ball in M w.r.t. to the intrinsic metric ρ. We consider two cases:

1.Case $B(r,x) \cap A = \emptyset$:

Then $B(r,x) \subset M_i$ for one $i \in \{1,2\}$ and there exists a ball $B_i(cr,x)$, s.t. $B(r,x) \subset B_i(cr,x) \subset M_i$. Further it exists a ball $B(c'r,x)$, s.t. $B(r,x) \subset B_i(cr,x) \subset B(c'r,x)$. Then with the (P) condition the following holds $\forall\, u \in D(\mathcal{E})$:

$$\begin{aligned}
\int_{B(r,x)} |u - u_{B(r,x)}|^2 d\mu &\le \int_{B(r,x)} |u - u_{B_i(cr,x)}|^2 d\mu \\
&\le \int_{B_i(cr,x)} |u - u_{B_i(cr,x)}|^2 d\mu \\
&\le c_p^i r^2 \int_{B_i(cr,x)} d\Gamma(u) \\
&\le c_p^i r^2 \int_{B(c'r,x)} d\Gamma(u).
\end{aligned}$$

2.Case $B(r,x) \cap A \neq \emptyset$:

Then we can find two comparable sets $B_1^c := B_1^c(cr,z)$, $B_2^c := B_2^c(cr,z)$ with a fixed constant $c > 0$ in M_1, M_2 w.r.t. ρ_1, ρ_2, s.t. $z \in B(r,x) \cap A$ and $B(r,x) \subset B_1^c \cup B_2^c := Q_{cr}(z)$ and a ball $B(c'r,x)$ with a fixed constant $c' > 0$, s.t. $Q_{cr}(z) \subset B(c'r,x)$. By Theorem 3.9 we get $\forall\, r > 0$, $x \in M$:

$$\begin{aligned}
\int_{B(r,x)} |u - u_{B(r,x)}|^2 d\mu &\le \int_{B(r,x)} |u - u_{Q_{cr}(z)}|^2 d\mu \\
&\le \int_{Q_{cr}(z)} |u - u_{Q_{cr}(z)}|^2 d\mu \\
&\le \text{const} \cdot r^2 \int_{Q_{cr}(z)} d\Gamma(u) \\
&\le \text{const} \cdot r^2 \int_{B(c'r,x)} d\Gamma(u)
\end{aligned}$$

which finishes the proof since we have shown that for universal constants $c', c'' > 0$ and $\forall\, r > 0$, $x \in M$:

$$\int_{B(r,x)} |u - u_{B(r,x)}|^2 d\mu \leq c'' r^2 \int_{B(c'r,x)} d\Gamma(u)$$

holds $\forall\, u \in D(\mathcal{E})$. Now with Remark 3.8 and doubling for (μ, ρ) we get the strong Poincaré inequality on M by [St96].

■

3.1.3 Gluing of k Spaces

By iteration of the gluing procedure for metric spaces we get the resulting glued space $M := \bigcup_{i=1}^k M_i$ with the common gluing set $A := \bigcap_{i=1}^k M_i$. This procedure coincides with the simultaneous gluing procedure of all M_i, i.e. the two resulting intrinsic metrics coincides (see Proposition 1.32). By the results of the first chapters together with the comparability of d_i and ρ_i one gets the comparability of the resulting intrinsic metric d on M and the original metrics d_i on M_i. If the measures μ_i are doubling and compatible on A and the energy measures $d\Gamma_i$ are compatible on A too as above one gets the comparability $\rho \sim d$ with ρ the intrinsic metric coming from the new strong local regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ on M.
The next theorem shows that if the HT condition is fulfilled the Poincaré inequality holds on the glued space M:

Theorem 3.11 *Let $M = \bigcup_{i=1}^k M_i$ be the glued metric space, μ a doubling measure and $(\mathcal{E}, D(\mathcal{E}))$ the glued Dirichlet form on M. If the HT condition is fulfilled the strong Poincaré inequality holds on M for balls w.r.t. the metric ρ.*

Proof: Let $Q_r(x) := \bigcup_{i=1}^k B_i^c(x, r)$ for $x \in A$. The idea is to prove an analogous lower bound for

$$\nu := \inf \left\{ \frac{\int_{Q_r(x)} d\Gamma(u)}{\int_{Q_r(x)} |u|^2 d\mu} : u \in D(\mathcal{E}),\ \int_{Q_r(x)} u\, d\mu = 0,\ u|_{Q_r(x)} \neq 0 \right\}$$

as in Theorem 3.9. Namely, we will show that

$$\nu \geq \frac{1}{k^2} \min_{i \in \{1,\ldots,k\}} \nu_i^*(B_i^c)$$

holds. Then with the HT condition we have

$$\nu \geq \frac{1}{k^2 r^2} \min_{i \in \{1,\ldots,k\}} c_{ht}^i$$

and the rest of the proof is entirely analogous to the proof of Theorem 3.10.

Clearly the preparatory Lemmata 3.3 and 3.5 still hold true under the HT-condition. In order to prove the lower bound only the part of the proof of Theorem 3.9 which has to be slightly modified will be discussed here. By Lemma 3.5 we have that if $\{\Psi_n\}$ is the minimizing sequence

$$\Delta_n := \int_{Q_r(x)} d\Gamma(\Psi^n, \Psi^n_+) - \nu_n \int_{Q_r(x)} \Psi^n \Psi^n_+ d\mu$$

goes to zero for $n \to \infty$ while $\nu_n \to \nu$. With the same notations $B_i^{+,n}$ and $Q_r^{+,n}(x)$ as in Theorem 3.9 this yields

$$\nu_n \sum_{i=1}^{k} \int_{B_i^{+,n}} |\Psi^n|^2 d\mu + \Delta_n \geq \frac{1}{k} \sum_{i=1}^{k} \int_{B_i^{+,n}} d\Gamma(\Psi^n). \tag{3.3}$$

and Lemma 3.7 gives us

$$\lim_{n\to\infty} \int_{Q_r^{+,n}(x)} |\Psi^n|^2 d\mu > 0.$$

Now two cases are possible. In the first case there are up to $(k-1)$ components on which Ψ^n goes to zero in L^2, i.e. $\exists\, 1 \leq i_1 < i_2 < \ldots < i_l \leq k$ for $l \leq k-1$, s.t.

$$\int_{B_{i_m}^{+,n}} |\Psi^n|^2 d\mu \to 0 \;\text{ for }\; n \to \infty,\; m = 1, \ldots, l,$$

and complementary $\exists\, 1 \leq j_1 < j_2 < \ldots < j_{k-l} \leq k$, s.t.

$$\lim_{n\to\infty} \int_{B_{j_m}^{+,n}} |\Psi^n|^2 d\mu > 0 \;\text{ for }\; m = 1, \ldots, k-l.$$

Now dividing (3.3) through $\sum_{m=1}^{k-l} \int_{B_{j_m}^{+,n}} |\Psi^n|^2 d\mu$ yields

$$\begin{aligned}\nu_n + \nu_n \frac{\sum_{m=1}^{l} \int_{B_{i_m}^{+,n}} |\Psi^n|^2 d\mu}{\sum_{m=1}^{k-l} \int_{B_{j_m}^{+,n}} |\Psi^n|^2 d\mu} &+ \frac{\Delta_n}{\sum_{m=1}^{k-l} \int_{B_{j_m}^{+,n}} |\Psi^n|^2 d\mu} \\ &\geq \frac{1}{k} \frac{\sum_{i=1}^{k} \int_{B_i^{+,n}} d\Gamma(\Psi^n)}{\sum_{m=1}^{k-l} \int_{B_{j_m}^{+,n}} |\Psi^n|^2 d\mu}.\end{aligned}$$

Here

$$\delta_n := \nu_n \frac{\sum_{m=1}^{l} \int_{B_{i_m}^{+,n}} |\Psi^n|^2 d\mu}{\sum_{m=1}^{k-l} \int_{B_{j_m}^{+,n}} |\Psi^n|^2 d\mu} + \frac{\Delta_n}{\sum_{m=1}^{k-l} \int_{B_{j_m}^{+,n}} |\Psi^n|^2 d\mu}$$

goes to zero for $n \to \infty$. Taking the maximum $j* \in \{j_1, \dots, j_{k-l}\}$, s.t.

$$\lim_{n\to\infty} \int_{B_{j^*}^{+,n}} |\Psi^n|^2 d\mu = \max_{j\in\{j_1,\dots,j_{k-l}\}} \lim_{n\to\infty} \int_{B_j^{+,n}} |\Psi^n|^2 d\mu$$

one gets for n large enough

$$\begin{aligned} \nu_n + \delta_n &\geq \frac{1}{k} \frac{\sum_{i=1}^k \int_{B_i^{+,n}} d\Gamma(\Psi^n)}{k \int_{B_{j_m}^{+,n}} |\Psi^n|^2 d\mu} \\ &\geq \frac{1}{k^2} \frac{\int_{B_{j^*}^{+,n}} d\Gamma(\Psi^n)}{\int_{B_{j^*}^{+,n}} |\Psi^n|^2 d\mu} \\ &= \frac{1}{k^2} I_{B_{j^*}^{+,n}}(\Psi^n). \end{aligned}$$

In the second case all limits are not equal to zero so there exists $j \in \{1, \dots, k\}$, s.t. by (3.3)

$$\nu_n + \frac{\Delta_n}{k \int_{B_j^{+,n}} |\Psi^n|^2 d\mu} \geq \frac{1}{k^2} I_{B_j^{+,n}}(\Psi^n)$$

while

$$\delta_n' := \frac{\Delta_n}{k \int_{B_j^{+,n}} |\Psi^n|^2 d\mu}$$

goes to zero for $n \to \infty$. As in the proof for $k = 2$ we get for a new sequence $\delta_n^* \to 0$

$$\nu_n + \delta_n^* \geq \min_i \frac{1}{k^2} \nu_i(B_i^c, A \cap \{\Psi^n \leq 0\}) \vee \min_i \frac{1}{k^2} \nu_i(B_i^c, A \cap \{\Psi^n \geq 0\}).$$

Because of $\bigcap_{i=1}^k B_i^c \subset A$ and since one can choose

$$N = \bigcap_{i=1}^k B_i^c \cap \{\Psi^n \leq 0\} \quad \text{or} \quad N = \bigcap_{i=1}^k B_i^c \cap \{\Psi^n \geq 0\}$$

one gets

$$\begin{aligned} \nu_n + \delta_n^* &\geq \min_i \frac{1}{k^2} \nu_i(B_i^c, \bigcap_{i=1}^k B_i^c \cap \{\Psi^n \leq 0\}) \vee \min_i \frac{1}{k^2} \nu_i(B_i^c, \bigcap_{i=1}^k B_i^c \cap \{\Psi^n \geq 0\}) \\ &\geq \inf_{N \subset \bigcap_{i=1}^k B_i^c} \{\min_i \frac{1}{k^2} \nu_i(B_i^c, N) \vee \min_i \frac{1}{k^2} \nu_i(B_i^c, \bigcap_{i=1}^k B_i^c \setminus N)\}. \end{aligned}$$

As $N \subset \bigcap_{i=1}^k B_i^c$ we have either $m(N) \geq \frac{1}{2}m(\bigcap_{i=1}^k B_i^c)$ or $m(\bigcap_{i=1}^k B_i^c \setminus N) \geq \frac{1}{2}m(\bigcap_{i=1}^k B_i^c)$, s.t.

$$\begin{aligned}
\nu_n + \delta_n^* &\geq \frac{1}{k^2} \inf_{\substack{N \subset \bigcap_i B_i^c \\ m(N) \geq \frac{1}{2}m(\bigcap_i B_i^c)}} \min_i \nu_i(B_i^c, N) \\
&= \frac{1}{k^2} \min_i \inf_{\substack{N \subset \bigcap_i B_i^c \\ m(N) \geq \frac{1}{2}m(\bigcap_i B_i^c)}} \nu_i(B_i^c, N)
\end{aligned}$$

holds. Since the sets B_i^c are centered in A and the metrics ρ_i, ρ are comparable there exists a constant $c' > 0$ s.t. $c'B_j^c \cap A \subset \bigcap_{i=1}^k B_i^c$ for all $j \in \{1, \ldots, k\}$ and by the doubling property for m we get for a constant $\alpha > 0$ analogous to the proof of Theorem 3.9

$$\begin{aligned}
\nu_n + \delta_n^* &\geq \frac{1}{k^2} \min_i \inf_{\substack{N \subset \bigcap_i B_i^c \\ m(N) \geq \alpha m(B_i^c \cap A)}} \nu_i(B_i^c, N) \\
&\geq \frac{1}{k^2} \min_i \inf_{\substack{N \subset B_i^c \cap A \\ m(N) \geq \alpha m(B_i^c \cap A)}} \nu_i(B_i^c, N) \\
&= \frac{1}{k^2} \min_{i \in \{1,\ldots,k\}} \nu_i^*(B_i^c) \\
&\geq \frac{1}{k^2 r^2} \min_{i \in \{1,\ldots,k\}} c_{ht}^i
\end{aligned}$$

while the second inequality holds just because $\bigcap_{i=1}^k B_i^c \subset A \cap B_i^c$. By taking the limits this finishes the proof.

■

3.2 Special Cases

Two special cases are treated here which simplifies the situation significantly but naturally restricts the class of examples. The first case requires that the gluing set has not measure zero and therefore intensifies the gluing conditions for the measures and the Dirichlet forms. In the second case the gluing map Φ is isometric which simplifies many proofs in this work as well as it simplifies the HT condition.

3.2.1 Conditions on A

We will now give some conditions on the measures μ_i and the gluing sets A_i in order to get the strong Poincaré inequality.

Lemma 3.12 *Let $B \in \mathcal{B}(M)$ be a set in the Borel σ-field of M and $S \subset B$ a subset of B with $S \subset \mathcal{B}(M)$ and $\mu(S) > 0$, then $\forall u \in L^2(M,\mu)$ and $\forall c \in \mathbf{R}$:*

$$\int_B |u - u_S|^2 d\mu \leq 4\frac{\mu(B)}{\mu(S)} \int_B |u - c|^2 d\mu$$

holds true.

Proof: By the triangle inequality and the Hölder inequality with $p = q = 2$:

$$\begin{aligned}
\int_B |u - u_S|^2 d\mu &\leq 2\int_B |u - c|^2 d\mu + 2\int_B |c - u_S|^2 d\mu \\
&= 2\int_B |u - c|^2 d\mu + 2\mu(B)\left|c - \frac{1}{\mu(S)}\int_S u d\mu\right|^2 \\
&= 2\int_B |u - c|^2 d\mu + 2\frac{\mu(B)}{\mu(S)^2}\left|\int_S (c - u) d\mu\right|^2 \\
&\leq 2\int_B |u - c|^2 d\mu + 2\frac{\mu(B)}{\mu(S)}\int_S |c - u|^2 d\mu \\
&\leq 4\frac{\mu(B)}{\mu(S)}\int_B |u - c|^2 d\mu
\end{aligned}$$

because $\mu(S) \leq \mu(B)$ holds.

■

Theorem 3.13 *Let $M = M_1 \bigcup_A M_2$ be the glued metric measure space and $(\mathcal{E}, D(\mathcal{E})$ the regular strong local Dirichlet form as above. Assume that doubling holds for (ρ, μ) and for the closed gluing sets A_i and the measures μ_i it holds that $\exists\, R_i > 0, c_i > 0$:*

$$\forall\, x \in A,\ 0 < r \leq R_i:\ \mu_i(B_i(x,r) \cap A) \geq c_i \mu_i(B_i(x,r)). \tag{3.4}$$

Further assume that the strong scaling invariant Poincaré inequality holds on M_i, i.e. $\exists\, c_p^i > 0 : \forall\, x \in M_i, r > 0$:

$$\int_{B_i(x,r)} |u - u_{B_i(x,r)}|^2 d\mu_i \leq c_p^i r^2 \int_{B_i(x,r)} d\Gamma(u)$$

$\forall\, u \in D(\mathcal{E}_i)$, then M satisfies a strong scaling invariant Poincaré inequality.

Proof: First we show that property (3.4) transfers directly to M. Let $R := \min(R_1, R_2)$ and $c := \min(c_1, c_2)$ then $\forall\, x \in A,\ 0 < r \leq R$ with $B_i(x,r) \subset B(x,r)$

$$\begin{aligned}
\mu(B(x,r) \cap A) &\geq \frac{1}{2}\mu_1(B_1(x,r) \cap A) + \frac{1}{2}\mu_2(B_2(x,r) \cap A) \\
&\geq \frac{c_1}{2}\mu_1(B_1(x,r)) + \frac{c_2}{2}\mu_2(B_2(x,r)) \\
&\geq \frac{c}{2}(\mu_1(B_1(x,r)) + \mu_2(B_2(x,r))) \\
&\geq \frac{c}{2}(\mu(B(x,c'r)))
\end{aligned}$$

while the last inequality comes from the comparability of ρ_i and ρ on M_i and therefore $\exists\ c' > 0 : B(x, c'r) \subset B_1(x, r) \cup B_2(x, r)$. Then the doubling property yields (3.4) for μ on M.

Now by Lemma 3.12 with $c := u_{B_i(x,c'r)}$ and c' the comparison constant, s.t. $\frac{1}{c'}\rho_i \leq \rho \leq c'\rho_i$ we get for all functions $u \in D(\mathcal{E})$ and $x \in A,\ 0 < r \leq R$:

$$\begin{aligned}
\int_{B(x,r)} |u - u_{B(x,r)}|^2 d\mu &\leq \int_{B(x,r)} |u - u_{B(x,r)\cap A}|^2 d\mu \\
&\leq \int_{B(x,r)\cap M_1} |u - u_{B(x,r)\cap A}|^2 d\mu \\
&+ \int_{B(x,r)\cap M_2} |u - u_{B(x,r)\cap A}|^2 d\mu \\
&\leq 4\frac{\mu(B(x,r)\cap M_1)}{\mu(B(x,r)\cap A)} \int_{B(x,r)\cap M_1} |u - u_{B_1(x,c'r)}|^2 d\mu \\
&+ 4\frac{\mu(B(x,r)\cap M_2)}{\mu(B(x,r)\cap A)} \int_{B(x,r)\cap M_2} |u - u_{B_2(x,c'r)}|^2 d\mu \\
&\leq c'' \int_{B_1(x,c'r)} |u - u_{B_1(x,c'r)}|^2 d\mu \\
&+ c'' \int_{B_2(x,c'r)} |u - u_{B_2(x,c'r)}|^2 d\mu \\
&\leq c'' c_p^1 r^2 \int_{B_1(x,c'r)} d\Gamma(u) + c'' c_p^2 r^2 \int_{B_2(x,c'r)} d\Gamma(u) \\
&\leq 2c'' r^2 \max_i \{c_p^i\} \int_{B_1(x,c'r)\cup B_2(x,c'r)} d\Gamma(u) \\
&\leq 2c'' r^2 \max_i \{c_p^i\} \int_{B(x,{c'}^2 r)} d\Gamma(u)
\end{aligned}$$

with constant $c'' > 0$. Hence the weak Poincaré inequality on M holds. This is clear for $B(x, r) \subset M_i \setminus A$ and for $B(x, r) \cap A \neq 0$ we take $B(z, 2r)$ for $z \in B(x, r) \cap A$, s.t. $B(x, r) \subset B(z, 2r)$:

$$\begin{aligned}
&\int_{B(x,r)} |u - u_{B(x,r)}|^2 d\mu \leq \int_{B(x,r)} |u - u_{B(z,2r)}|^2 d\mu \\
\leq &\int_{B(x,2r)} |u - u_{B(z,2r)}|^2 d\mu \leq 8c'' r^2 \max_i \{c_p^i\} \int_{B(z,2{c'}^2 r)} d\Gamma(u).
\end{aligned}$$

To finish the proof we use a chaining argument by Jerison [Je86] which was extended to metric spaces by Sturm [St96]. This argument derives the strong Poincaré inequality if the weak Poincaré inequality is given and doubling holds for μ w.r.t. the intrinsic metric ρ.

■

Remark 3.14 (k-Gluing) *To glue k spaces one can generalize Theorem 3.13 in a straightforward way.*

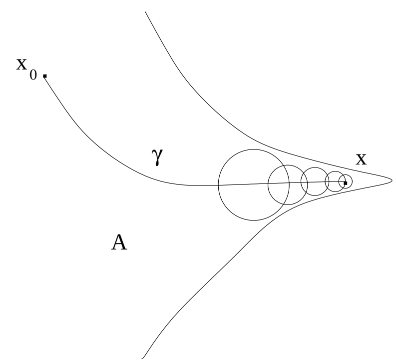

Figure 3.1: Gluing set A with cone condition 3.4

Remark 3.15 (Cone Condition for A) *Here a condition on the gluing set A will be given, s.t. property (3.4) is fulfilled. Assume A satisfies the following property:*

There exist a constant $0 < c \leq 1$ and a point $x_0 \in \overset{\circ}{A}$, s.t. each $x \in A$ can be joined to x_0 by a curve $\gamma : [0, l] \to A$ parametrized by arc length with $\gamma(l) = x_0$ and $B(\gamma(t), ct) \subset A$.

Because $\overset{\circ}{A} \neq 0$ there exists an $R > 0$, s.t. $B(x_0, R) \subset A$. Let $x \in A$ and $0 < r \leq R$, then there exists a ball $B(z, \frac{cr}{2}) \subset B(x, r) \cap A$. This is true because either $z = x_0$ is in $B(x, \frac{r}{2})$ or if $x_0 \notin B(x, \frac{r}{2})$ take $z = \gamma(\frac{r}{2})$. Since by the doubling property and $B(x, r) \subset B(z, 2r)$

$$\mu(B(x,r)) \leq \mu(B(z,2r)) \leq c'\mu(B(z, \frac{cr}{2}))$$

holds for a constant $c' > 0$. Therefore,

$$\mu(B(x,r)) \leq c'\mu(B(z, \frac{cr}{2})) = c'\mu(B(z, \frac{cr}{2}) \cap A) \leq c'\mu(B(x,r) \cap A)$$

yields property (3.4).

Example 3.16 *Simple examples for the Theorem 3.13 are k-pods or k-sheets. For the k-pods one can check the Poincaré inequality quite easily by direct calculations but also by verifying our gluing conditions (see Section 3.3.1). Let* $\mathbf{R}^n_{(i)}$ *for* $i = 1, \ldots, k$ *be k copies of* $\mathbf{R}^n$ *equipped with the usual Euclidean metric, Lebesgue measure* λ^n *and the canonical Dirichlet form* $\mathcal{E}(u) := \int |\nabla u|^2 d\lambda^n$*. Now one can glue these spaces via* $(k-1)$ *isometric maps* $\Phi_i : \mathbf{R}^{n,+}_{(i)} \to \mathbf{R}^{n,+}_{(i+1)}$, $i = 1, \ldots, k-1$ *along* $\mathbf{R}^{n,+}_{(i)} := \{x \in \mathbf{R}^n_{(i)} : x_1 \geq 0\}$ *as described in Section 1.7. This yields the k-sheet* $M^k := \bigcup^k_{\mathbf{R}^{n,+}} \mathbf{R}^n_{(i)}$*. The Poincaré inequality holds for balls on* $\mathbf{R}^n$*. Property (3.4) holds clearly for* $\mathbf{R}^{n,+}$ *by the Remark (3.15). Therefore, the strong Poincaré inequality holds on k-sheets.*

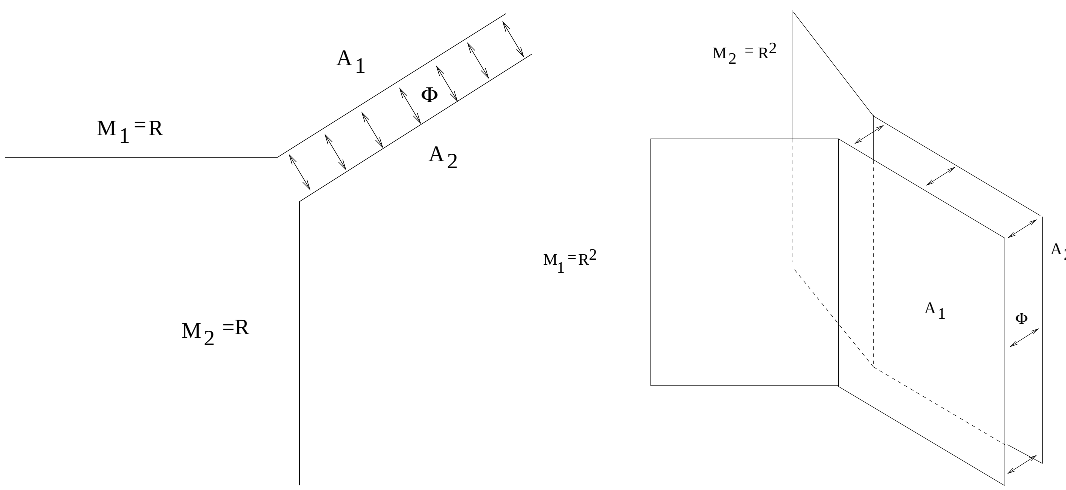

Figure 3.2: Alternative gluing of k-spiders and k-sheets

3.2.2 Isometric Gluing

We will now use the technique by Jerison [Je86] and Sturm [St96] to show that if two metrics like ρ_i and d_i are comparable it does not matter if one states that the strong Poincaré inequality holds for one or for the other metric because one implicates the other and vice versa. The only additional condition on d_i is to be compatible with the topology and the existence of geodesics. But these conditions hold for all metrics in this work.

Proposition 3.17 *Let* d_1, d_2 *be two comparable metrics in* (M, μ) *such that* μ *is a doubling measure w.r.t.* d_1 *or* d_2*. If the strong Poincaré inequality holds for balls w.r.t.* d_1 *then the strong Poincaré inequality holds for balls w.r.t.* d_2 *too.*

Proof: By the comparability of d_1 and d_2 one can easily see that doubling for μ holds w.r.t. d_1 and d_2. Further there exists a constant $c \geq 1$ s.t. $B_1(x,r) \subset B_2(x,cr)$ and $B_2(x,r) \subset B_1(x,cr)$ while $B_i(x,r) := \{y \in M : d_i(y,x) < r\}$. This together with the Poincaré inequality for balls w.r.t. d_1 yields with $C > 0$ the Poincaré constant:

$$\begin{aligned}
\int_{B_2(x,r)} |u - u_{B_2(x,r)}|^2 d\mu &\leq \int_{B_2(r,x)} |u - u_{B_1(cr,x)}|^2 d\mu \\
&\leq \int_{B_1(cr,x)} |u - u_{B_1(cr,x)}|^2 d\mu \\
&\leq Cr^2 \int_{B_1(cr,x)} d\Gamma(u) \\
&\leq Cr^2 \int_{B_2(c^2r,x)} d\Gamma(u)
\end{aligned}$$

while the first inequality comes from $\min_{a \in \mathbf{R}} \int_B |u - a|^2 d\mu = \int_B |u - u_B|^2 d\mu$ for any measurable set $B \in \mathcal{B}(M)$. So the weak Poincaré inequality holds for balls w.r.t. d_2. Together with the doubling property we now use the result of Jerison [Je86] and Sturm [St96] to get the strong version which finishes the proof.

■

We now consider an isometric gluing map Φ in which case the proofs become easier. The distances in M_i will not be changed by the gluing procedure, i.e. $d_i(x,y) = d(x,y)$, $\forall x, y \in M_i$ and therefore a ball in the new metric $B(x,r) := \{y \in M : d(x,y) < r\}$ with $x \in A$ is the union of balls in M_i, i.e. $B(x,r) = \bigcup_{i=1}^k B_i(x,r)$ with the same radius r. If we now check the proofs above and take the balls w.r.t. d_i as the comparable systems of sets we get the Poincaré inequality for balls w.r.t. the metric d on M and this yields the Poincaré inequality for balls w.r.t. ρ if doubling holds and the metrics d, ρ are comparable as we know from Corollary 3.17. Then the HT condition simplifies, i.e.

$$\inf_{\substack{N \subset B_i(r,x) \cap A \\ m(N) \geq \frac{1}{2} m(B_i \cap A)}} \nu_i(B_i, N) \geq c_{ht} \frac{1}{r^2}, \tag{3.5}$$

($\alpha = \frac{1}{2}$) and the measure m has not to be a doubling measure on A (see the end of the proofs of Theorem 3.9 and 3.11).

3.3 Examples

Now we will discuss examples in the one-dimensional and the n-dimensional Euclidean setting with $n \geq 2$.

3.3.1 Spiders and Graphs

For dimension one, i.e. spiders, trees or graphs, there are two alternative constructions as described in Section 1.8. If the gluing set A is $\mathbf{R}_+$ one can use Theorem 3.13 as described in Example 3.16. If the gluing set is just one point and the Dirichlet form comes from the canonical Laplacian we can verify the HT condition in a direct way:

Let $f \in C^1([0,r])$ with $f(0) = 0$. Then with $f(t) = \int_0^t f'(s)ds$ we can apply the Cauchy-Schwarz inequality to get

$$f^2(t) = \left(\int_0^t f'(s)ds\right)^2 \leq t\int_0^t f'(s)^2 ds \leq t\int_0^r f'(s)^2 ds$$

for $t \in [0,r]$. Hence

$$\int_0^r f^2(t)dt \leq \int_0^r tdt \int_0^r f'(s)^2 ds \leq \frac{r^2}{2}\int_0^r f'(s)^2 ds$$

holds true.

3.3.2 Examples in $\mathbf{R}^n$ for $n \geq 2$

In the following section two classes of nontrivial examples for applications of the main theorem are presented. Throughout this section λ^n denotes the n-dimensional Lebesgue or Hausdorff measure while by λ^{n-1} the $(n-1)$-dimensional Hausdorff measure is meant. Let $\mathbf{R}^n_+ := \{x \in \mathbf{R}^n : x_1 \geq 0\}$ and $n \geq 2$ for the rest of this section.

The first class of metric measure spaces which shall be glued together here are open subsets Ω of the Euclidean space $\mathbf{R}^n$ along closed subset $A \subset \Omega$ with dimension $n-1$ while the Dirichlet space defined on Ω is given by the canonical example

$$\mathcal{E}(u,v) := \frac{1}{2}\sum_{i=1}^n \int \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} d\lambda^n$$

of a strong local regular Dirichlet form on $H_0^{1,2}(\Omega)$.

The second class consists of closed bounded subsets Ω of the Euclidean space $\mathbf{R}^n$ with Lipschitz boundary $\partial\Omega$ along the closed subset $A = \partial\Omega$ while the Dirichlet space defined on Ω is given by the canonical example $(\mathcal{E}, D(\mathcal{E}))$ with $D(\mathcal{E}) = H^{1,2}(\Omega)$ as the maximum Markovian extension of the form $(\mathcal{E}, \mathcal{C}_0^\infty(\Omega))$. This form is then a regular strong local Dirichlet form on $L^2(\Omega, \lambda^n)$.

What is left is the description of a possible gluing set $A \subset \Omega$ in the first class and the proof for the HT condition on A. For this purpose we recall the following nontrivial result by Denzler [De99a]:

Theorem 3.18 *Let $B \subset \mathbf{R}^n$ be a bounded convex domain, $n \geq 3$. Then for any $D \subset \partial B$ of area $\lambda^{n-1}(D) = c_1$ for $c_1 > 0$ one has the estimate:*

$$\nu_i(B, D) \geq \min \left\{ c_2(B) \frac{c_1^{\left(\frac{n-2}{n-1}\right)}}{\lambda^n(B)}, \frac{\nu(B)}{2} \right\}$$

for a constant $c_2(B) > 0$ and $\nu(B)$ the first Neumann eigenvalue on B.

Remark 3.19 *In this estimate $c_2(B)$ depends only on a lower bound for the first Neumann eigenvalue $\nu(B)$ of B and some geometric condition on B. A similar bound holds true for $n = 2$ (cf. [De99a]).*

This theorem will now serve as a starting point for a class of examples where the gluing set A lies in a $(n-1)$-dimensional hyperplane intersecting Ω:

Theorem 3.20 *Let $A \subset \Omega$ be a closed set lying in a $(n-1)$-dimensional hyperplane $H \subset \mathbf{R}^n$ intersecting Ω, s.t. $dist(A, \partial\Omega) > R > 0$. Further for all balls $B_r(x)$ with $x \in A$ and $r \leq R$ it holds that*

$$\lambda^{n-1}(A \cap B_r(x)) \geq r^{n-1} c_0$$

for a constant $c_0 > 0$. Then a scaling invariant Poincaré inequality for mixed boundary value functions holds or in other terms the HT condition is satisfied, i.e. for $c_1 > 0$ there exists a constant $c_2 > 0$ such that for all $N \subset B_r(x) \cap A$ and $\lambda^{n-1}(N) \geq c_1 \lambda^{n-1}(A \cap B_r(x))$:

$$\nu(B_r(x), N) \geq c_2 \frac{1}{r^2}$$

while

$$\nu(B_r(x), N) := \inf \left\{ \frac{\int_{B_r(x)} |\nabla u|^2 d\lambda^n}{\int_{B_r(x)} |u|^2 d\lambda^n}, u|_N = 0, u \in H_0^{1,2}(\Omega) \right\}$$

is the lowest Neumann-Dirichlet eigenvalue.

Proof: By the invariance under translation and rotation one can consider the problem for the hyperplane

$$\partial\mathbf{R}^n_+ := \{x \in \mathbf{R}^n : x_1 = 0\}$$

while $A \subset \partial\mathbf{R}^n_+$. Denzler's theorem [De99a] tells us that for the intersection $B_1(x) \cap \mathbf{R}^n_+$, $x \in \partial\mathbf{R}^n_+$ the following holds:

$$\inf_{\substack{N \subset B_1(x) \cap \partial\mathbf{R}^n_+ \\ \lambda^{n-1}(N) \geq c_1 c_0}} \nu(B_1(x) \cap \mathbf{R}^n_+, N) \geq c(B_1(x) \cap \mathbf{R}^n_+, c_1 c_0). \tag{3.6}$$

W.l.o.g. we can choose $x = 0$. Take a function $u \in H^{1,2}_0(\Omega)$ which is zero on a set $N \subset B_r(x) \cap A$ and $\lambda^{n-1}(N) \geq c_1 \lambda^{n-1}(A \cap B_r(x))$ holds. The set $\frac{1}{r}N$ is defined in the following way:

$$\frac{1}{r}N := \{\frac{1}{r}y : y \in N\}.$$

Then

$$\begin{aligned}
\lambda^{n-1}(\frac{1}{r}N) &\geq \frac{1}{r^{n-1}}\lambda^{n-1}(N) \\
&\geq \frac{1}{r^{n-1}}c_1\lambda^{n-1}(A \cap B_r(x)) \\
&\geq \frac{1}{r^{n-1}}c_1 r^{n-1} c_0 = c_1 c_0
\end{aligned}$$

holds. The first inequality holds because the map $y \mapsto \frac{1}{r}y$ is a Lipschitz map with Lipschitz constant $\frac{1}{r}$ and the second inequality holds because λ^{n-1} is the $n-1$ dimensional Hausdorff measure. By defining

$$v(y) := u(ry)$$

one gets a function $v \in H^{1,2}_0(\frac{1}{r}\Omega)$ extending with zero or not respectively (depending on $r > 1$ or $r < 1$). It holds that $v|_{(\frac{1}{r}N)} = 0$ because u is zero on N. Therefore, one gets:

$$\begin{aligned}
\int_{B_r \cap \mathbf{R}^n_+} |u(x)|^2 d\lambda^n(x) &= \int_{B_1 \cap \mathbf{R}^n_+} |u(ry)|^2 r^n d\lambda^n(y) \\
&= r^n \int_{B_1 \cap \mathbf{R}^n_+} |v(y)|^2 d\lambda^n(y) \\
&\leq r^n c \int_{B_1 \cap \mathbf{R}^n_+} |\nabla v(y)|^2 d\lambda^n(y) \\
&= r^n c \int_{B_1 \cap \mathbf{R}^n_+} |\nabla u(ry)|^2 d\lambda^n(y)
\end{aligned}$$

$$= r^{n+2}c\int_{B_1\cap\mathbf{R}^n_+}|(\nabla u)(ry)|^2 d\lambda^n(y)$$
$$= r^2 c\int_{B_r\cap\mathbf{R}^n_+}|\nabla u(x)|^2 d\lambda^n(x).$$

Here the transformation rule is applied in the first and in the last equality and the chain rule in the fourth equality while the inequality comes from 3.6. Because of symmetry the same holds true for $B_r\cap\mathbf{R}^n_-$ with $\mathbf{R}^n_- := \{x\in\mathbf{R}^n : x\le 0\}$. Sticking together both inequalities for u one gets the following:

$$\int_{B_r}|u(x)|^2 d\lambda^n(x)\le r^2 c\int_{B_r}|\nabla u(x)|^2 d\lambda^n(x).$$

Therefore, it holds for all $u\in H_0^{1,2}(\Omega)$ with $u|_N = 0$ for a set $N\subset B_r(x)\cap A$ and $\lambda^{n-1}(N)\ge c_1\lambda^{n-1}(A\cap B_r(x))$ that with $c_2 := \frac{1}{c}$:

$$\nu(B_r(x),N)\ge c_2\frac{1}{r^2}$$

which finishes the proof.

■

The first theorem gives the answer to the question whether the HT condition holds for simple examples of linear glued spaces.
In order to extend the results above to the case of nonlinear gluing sets a bilipschitz transformation can be used. Let $\eta:\mathbf{R}^n\to\mathbf{R}^n$ be a bilipschitz map with constant L, i.e.:

$$\frac{1}{L}|x-y|\le|\eta(x)-\eta(y)|\le L|x-y|.$$

If A and Ω have the same properties as in Theorem 3.20 one can prove the HT condition for $\eta(A)$ and $\eta(\Omega)$.

Theorem 3.21 *The HT condition is fulfilled for $\eta(A)$ and $\eta(\Omega)$, i.e. for $x\in\eta(A)$ and for all $N\subset\eta(A)\cap Q_r(x)$ with $\lambda^{n-1}(N)\ge c_1\lambda^{n-1}(\eta(A)\cap Q_r(x))$:*

$$\nu(Q_r(x),N)\ge\frac{c_2}{L^{2+n}}\frac{1}{r^2}$$

while $Q_r(x)=\eta(B_r(\eta^{-1}(x)))$.

Proof: Let $u\in H_0^{1,2}(\eta(\Omega))\cap C^1(\eta(\Omega))$ be a function which is zero on the set $N\subset Q_r(x)\cap\eta(A)$ with $\lambda^{n-1}(N)\ge c_1\lambda^{n-1}(\eta(A)\cap Q_r(x))$. Then the function v

defined as $v(y) := u(\eta(y))$ is absolutely continuous on lines and therefore $v \in H_0^{1,2}(\Omega)$ if $|v|_{H_0^{1,2}(\Omega)} < \infty$ is shown.
The Jacobian of η at x is bounded from below through L^{-n} :

$$|\det D\eta(x)| \geq L^{-n}.$$

Hence one obtains:

$$\begin{aligned}
\int_{B_r(\eta^{-1}(x))} |v(y)|^2 d\lambda^n(y) &\leq L^n \int_{B_r(\eta^{-1}(x))} |u(\eta(y))|^2 |\det D\eta(y)| d\lambda^n(y) \\
&= L^n \int_{Q_r(x)} |u(z)^2 d\lambda^n(z).
\end{aligned}$$

To keep the derivative part bounded one computes:

$$\begin{aligned}
\int_{B_r(\eta^{-1}(x))} |\nabla v(y)|^2 d\lambda^n(y) &\leq \int_{B_r(\eta^{-1}(x))} |(D\eta)^* \nabla u(\eta(y))|^2 d\lambda^n(y) \\
&\leq L^2 \int_{B_r(\eta^{-1}(x))} |\nabla u(\eta(y))|^2 d\lambda^n(y) \\
&\leq L^{2+n} \int_{B_r(\eta^{-1}(x))} |\nabla u(\eta(y))|^2 |\det D\eta(y)| d\lambda^n(y) \\
&= L^{2+n} \int_{Q_r(x)} |\nabla u(z)|^2 d\lambda^n(z)
\end{aligned}$$

while besides the chain rule and the transformation rule (see [EG92] or [Fed69]) for integrals the estimates $|(D\eta)^*(y)|^2 < L^2$ and $|\det D\eta(x)| > L^{-n}$ were used.
Since u is zero on N the function v is zero on $\eta^{-1}(N)$ and for $\eta^{-1}(N)$ the following holds:

$$\begin{aligned}
\lambda^{n-1}(\eta^{-1}(N)) &\geq \frac{1}{L^{n-1}} \lambda^{n-1}(N) \\
&\geq \frac{1}{L^{n-1}} c_1 \lambda^{n-1}(\eta(A) \cap Q_r(x)) \\
&\geq \frac{1}{L^{n-1}} c_1 L^{n-1} \lambda^{n-1}(A \cap B_r(\eta^{-1}(x))) \\
&= c_1 \lambda^{n-1}(A \cap B_r(\eta^{-1}(x))).
\end{aligned}$$

So one has to apply the Theorem 3.20 with constant c_1. That $Q_r(x)$ satisfies the HT condition comes from the fact that η is bilipschitz. Hence it follows for $Q_r(x)$ with $x \in A$ and $r \leq \frac{R}{2}$:

$$\int_{Q_r(x)} |u(y)|^2 d\lambda^n(y) = \int_{Q_r(x)} |v(\eta^{-1}(y))|^2 d\lambda^n(y)$$

$$
\begin{aligned}
&\leq L^n \int_{Q_r(x)} |v(\eta^{-1}(y))|^2 |\det D\eta^{-1}(y)| d\lambda^n(y) \\
&= L^n \int_{B_r(\eta^{-1}(x))} |v(z)|^2 d\lambda^n(z) \\
&\leq \frac{r^2}{c_2} L^n \int_{B_r(\eta^{-1}(x))} |\nabla v(z)|^2 d\lambda^n(z) \\
&= \frac{r^2}{c_2} L^n \int_{B_r(\eta^{-1}(x))} |\nabla (u \circ \eta)(z)|^2 d\lambda^n(z) \\
&= \frac{r^2}{c_2} L^n \int_{B_r(\eta^{-1}(x))} |(D\eta)^*(z) \nabla u(\eta(z))|^2 d\lambda^n(z) \\
&\leq \frac{r^2}{c_2} L^2 L^{2n} \int_{B_r(\eta^{-1}(x))} |\nabla u(\eta(z))|^2 |\det D\eta(z)| d\lambda^n(z) \\
&= \frac{r^2}{c_2} L^{2+2n} \int_{Q_r(x)} |\nabla u(y)|^2 d\lambda^n(z)
\end{aligned}
$$

while we have used that $|\det D\eta^{-1}(x)| > L^{-n}$ a.e. because η is bilipschitz and the same tools as in the calculation above. Now by division and the fact that this holds for all functions u with $u|_N = 0$ and the properties for N one gets the same lower bound. This finishes the proof for $\frac{c_2}{L^{2+2n}}$ instead of c_2.

■

Now to check the HT condition for the second class of examples one has to model the boundary. A part of this has already been proved in Theorem 3.20.
Since $\partial\Omega$ is Lipschitz continuous there exists a finite cover of open sets $U_1, \ldots, U_m$ of $\partial\Omega$ and $\partial\Omega \cap U_i$ is the graph of a Lipschitz function $f : \mathbf{R}^{n-1} \mapsto \mathbf{R}$ such that Ω is locally the set above the graph, i.e.

$$\{(x_1, \ldots, x_n) : x_1 > f(x_2, \ldots, x_n)\} \cap U_i = \Omega \cap U_i.$$

Therefore, there exist constants $R, L > 0$ such that for all $x \in \partial\Omega$ there exist a bilipschitz map $\eta_i^x : \mathbf{R}^n \mapsto \mathbf{R}^n$ with Lipschitz constant L and $\eta_i^x(B(r,0)) \subset U_i$, $\eta_i^x(0) = x$ for all $r < R$. Further $\eta_i^x(B(r,0) \cap \{x \in \mathbf{R}^n : x_1 = 0\}) = \eta_i^x(B(r,0)) \cap \partial\Omega$ while $B(r,0)$ is a ball w.r.t. the Euclidean metric. For instance define $\eta_i^x(x_1, \ldots, x_n) = (-f(x_2, \ldots, x_n), x_2, \ldots, x_n)$. Hence by the images of the sets $B(r,0) \cap \mathbf{R}_+^n$ w.r.t. the maps η_i^x for all $x \in \partial\Omega$, $r < R$ and $i \in \{1, \ldots, m\}$ we get the comparable systems of sets. This a direct consequence of Theorem 3.20 and Theorem 3.21 if one replaces B_r by $B_r \cap \mathbf{R}_+^n$ and $Q_r(x) = \eta(B_r(\eta^{-1}(x)))$ by $Q_r(x) = \eta_i^x(B(r,0) \cap R_+^n)$. That the Poincaré inequality holds for balls in the interior of Ω is clear. Further the intrinsic metric d_Ω on Ω coincides with the intrinsic metric ρ coming from the Dirichlet form (see Example 2.19).

Remark 3.22 *The example of the cube (cf. Section 1.8) fits into our second class so that Poincaré holds for the cube. More generally one can construct Euclidean complexes by iteration of the gluing procedure.*

Chapter 4

Applications for Markov Processes

In this chapter we summarize some applications of our results. Mainly we consider the diffusion process (X_t, P_x) which is properly associated with the strongly local Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$. Provided all our gluing conditions hold we have the existence of the heat kernel $p_t(x, y)$ as well as lower and upper Gaussian bounds for $p_t(x, y)$ by the results in [St95b] and [St96]. Furthermore Hölder continuity holds for harmonic functions on M and for local solutions of $(L - \frac{\partial}{\partial t})u = 0$ while L is the associated operator to $(\mathcal{E}, D(\mathcal{E}))$. They come from the Harnack inequality described in [St96] which also implicates that the process (X_t, P_x) can be chosen to be strong Feller. Finally we demonstrate that with an additional assumption on $(\mathcal{E}, D(\mathcal{E}))$ a short-time asymptotic result for the heat kernel $p_t(x, y)$ can be proved for our glued space. Further applications by the results of M. Biroli, N.A. Tchou [BT97] and M. Biroli, U. Mosco [BM95a], [BM95b] on homogenous spaces w.r.t. the intrinsic metric ρ shall only be mentioned here and not be discussed in the sequel.

4.1 Markov Processes

Since $(\mathcal{E}, D(\mathcal{E}))$ is a strongly local regular Dirichlet form on $L^2(M, \mu)$ one can construct a μ-symmetric Markov process (X_t, P_x) whose transition semigroup $(P_t)_{t>0}$ is properly associated with the contraction semigroup $(T_t)_{t>0}$ of $(\mathcal{E}, D(\mathcal{E}))$ in the following sense

$$P_t u \quad \text{is a quasicontinuous version of} \quad T_t u$$

for all $u \in L^2(M, \mu)$ and $t > 0$ (cf. [Fot94], Theorem 7.2.1). Moreover, since $(\mathcal{E}, D(\mathcal{E}))$ is strongly local the process (X_t, P_x) can be chosen to be a diffusion process, i.e. X_t has continuous paths P_x-a.e. (cf. [Fot94], Theorem 7.2.2).

The uniqueness of the attachment of a diffusion process (X_t, P_x) to the regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ is given in the following sense (cf. [Fot94], Theorem 4.2.7): Let (X_t^1, P_x^1), (X_t^2, P_x^2) be two m-symmetric diffusion processes. If their transition semigroups $(P_t^1)_{t>0}$, $(P_t^2)_{t>0}$ are properly associated with $(\mathcal{E}, D(\mathcal{E}))$ there exists a set $N \subset M$ of capacity zero such that

$$P_t^1(x, B) = P_t^2(x, B)$$

$\forall\, x \in M \setminus N$ and $\forall\, B \in \mathcal{B}(M)$ with $B \subset M \setminus N$. Therefore, statements about the process associated with $(\mathcal{E}, D(\mathcal{E}))$ holds usually P_x-a.s. for q.e. $x \in M$.

We presented some examples of diffusion processes in Section 2.2.2 in particular to discuss the behavior of the process when hitting the gluing set A. But it is a priori not clear wether the process ever hits the gluing set. The set A could be of capacity zero. For instance if one glues higher dimensional spaces along a finite number of points. In this case there is no connection of the original spaces from the diffusions perspective. We discuss this topic in Section 4.3. Of course there are more examples of strong local regular Dirichlet forms which can be glued together and give rise to other diffusion processes than the so called Brownian motion treated in Section 2.2.2.

4.2 The Heat Kernel

Up to now we have ignored the results about the Poincaré inequality on glued spaces. In [St95b] and [St96] a series of results based on the validity of the scale invariant Poincaré inequality are presented. The framework is quite general namely the basic space X is assumed locally compact, separable, Hausdorff and the Dirichlet form $\mathcal{E}$ on $L^2(X, m)$ is strongly local and regular while m is a Radon measure with support X. Therefore, our setting fits perfectly into this framework with $M = X$ and $\mu = m$. In [St95b] and [St96] time-dependent Dirichlet forms $\mathcal{E}_t$ for $t \in \mathbf{R}$ one a common domain $\mathcal{F} \subset L^2(X, m)$ are considered in order to study solutions of parabolic equations $L_t u = \frac{\partial}{\partial t} u$ while L_t are the operators on $L^2(X, m)$ associated with $\mathcal{E}_t$. In our setting $\mathcal{E}_t = \mathcal{E}$ is time-independent so that some assumptions in [St95b] and [St96] become trivial and we end up with four additional assumptions we have to check before adopting the results:

(i) Strong regularity: The topology induced by ρ is the same as the original one.

(ii) Doubling property: There exists a constant $c > 0$ such that

$$\mu(B_{2r}(x)) \leq c\mu(B_r(x))$$

for all $r > 0$, $x \in M$ and $B_{2r}(x) \subset M$.

(iii) Poincaré inequality: There exists a constant c_p such that for all balls $B_r(x) \subset M$ we have

$$\int_{B_r(x)} |u - u_{B_r(x)}|^2 d\mu \leq c_p r^2 \int_{B_r(x)} d\Gamma(u)$$

for all $u \in D(\mathcal{E})$.

(iv) Balls are relative compact in M.

By our gluing conditions we have the comparability of ρ and d and therefore (i) is fulfilled. Condition (iv) holds because we start with complete spaces and therefore our glued space is complete w.r.t. ρ which is equivalent to (iv) (cf. [St96], Lemma 1.1). The conditions (ii) and (iii) hold true if the corresponding gluing conditions are satisfied which is in particular true for our examples in Section 3.3.1 and 3.3.2.

If (i)-(iv) are valid the fundamental solution (or heat kernel) of the parabolic operator $L - \frac{\partial}{\partial t}$ (L the associated operator to $(\mathcal{E}, D(\mathcal{E}))$) exits, (cf. [St95b], Prop. 2.3), with

$$T_t u(y) = \int_M p(t,x,y)u(x)\mu(dx)$$

for all $u \in L^1(M,\mu) \cup L^\infty(M,\mu)$ while $(T_t)_{t>0}$ is the associated contraction semigroup to $\mathcal{E}$.

4.3 Estimates for the Transition Probabilities

From now on we assume that (i)-(iv) hold true for our glued space (M, d, μ) with the strong local regular Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ with intrinsic metric ρ. Then a series of upper and lower bounds for the heat kernel (or transition probabilities for (X_t, P_x)) are given which we summarize in the following:

Upper bounds:

Theorem 4.1 (cf. [St95b], Corollary 2.7, [St96], Corollary 4.2) *For all $x, y \in M$ and $t > 0$:*

$$p(t,x,y) \leq C\mu^{-\frac{1}{2}}(B_{\sqrt{t}}(x))\mu^{-\frac{1}{2}}(B_{\sqrt{t}}(y)) \exp\left(-\frac{\rho^2(x,y)}{4t}\right)\left(1+\frac{\rho^2(x,y)}{t}\right)^{\frac{N}{2}}$$

with a constant C only depending on N, the doubling constant in the sense $\mu(B_{2r}(x)) \leq 2^N\mu(B_r(x))$.

More sophisticated estimates with bounds depending on the spectral gap or estimates for the derivatives $\frac{\partial}{\partial t}^j p(t,x,y)$ for $j \in \mathbf{N}$ (cf. [St95b], Theorem 2.6, Corollary 2.7) are possible.

Lower bounds:

Theorem 4.2 (cf. [St96]) *For all $x, y \in M$ and $t > 0$:*

$$p(t,x,y) \geq \frac{1}{C}\mu^{-1}(B_{\sqrt{t}}(x)) \exp\left(-C\frac{\rho^2(x,y)}{t}\right)$$

holds while the constant C only depends on the doubling constant and the Poincaré constant c_p.

Remark 4.3

- *The lower bound gives an answer to the question if the process hits or transverse the gluing set A, whenever condition (i)-(iv) hold true for the glued space for instance for our examples in Section 3.3.1 and 3.3.2.*
- *In Section 4.5 another lower bound will be discussed in order to study the short-time asymptotic of the heat kernel.*

4.4 Hölder Continuity and Strong Feller Processes

In [St96] the equivalence of (ii), (iii), (iv) under (i) with a Sobolev inequality on balls and with this the equivalence of (ii), (iii) under (i) and (iv) with a parabolic Harnack inequality for the operator $L - \frac{\partial}{\partial t}$ is proved ([St96], Theorem 2.6. and Theorem 3.5.). As a consequence it is deduced that (i)-(iv) implies Hölder continuity of local solutions of the parabolic equation $(L - \frac{\partial}{\partial t})u = 0$ and of the elliptic equation $Lu = 0$ by the iteration technique of J.Moser. Therefore, we have for our glued space M:

Proposition 4.4 (cf. [St96], Cor.3.3.) *Let u be a harmonic function on M, i.e. $Lu = 0$ on M. Then u is Hölder continuous, i.e. there exist constants $\alpha \in]0,1[$ and C such that for all balls $B_{2r}(x)$ and $y, z \in B_r(x)$*

$$|u(y) - u(z)| \leq C \sup_{B_{2r}(x)} |u| \left(\frac{|y-z|}{r} \right)^{\alpha} \tag{4.1}$$

holds.

For the validity of (4.1) it is enough that $Lu = 0$ holds on $B_{2r}(x)$. As another corollary of the Harnack inequality one gets the strong Liouville property:

Corollary 4.5 (cf. [St96], Cor.3.4.) *All nonnegative local solutions of $Lu = 0$ on M are constant on X.*

Another consequence of (i)-(iv) is the following:

Proposition 4.6 *The diffusion process (X_t, P_x) properly associated with $(\mathcal{E}, D(\mathcal{E}))$ can be chosen to be a strong Feller process.*

Proof: This is a consequence of the Harnack inequality since

$$T_t u(x) = \int_M p_t(x,y) u(y) d\mu(y)$$

is a local solution of $(L - \frac{\partial}{\partial t})u = 0$ and therefore it is Hölder continuous.

∎

4.5 Short-Time Asymptotic of the Heat Kernel

The short-time asymptotic of the heat kernel has been proved by Varadhan [Va67] for Riemannian manifolds and has been generalized by Norris [No97] to Lipschitz manifolds. In [Ra01] with (i)-(iv) and some additional assumptions a further generalization to Dirichlet forms on locally compact spaces has been proved. The additional assumption for our setting is:

(v) Carré du champ: The Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ admits a carré du champ operator, i.e. a nonnegative definite, symmetric continuous bilinear form

$$\Gamma : D(\mathcal{E}) \times D(\mathcal{E}) \to L^1(M, \mu)$$

such that $\mathcal{E}(u,v) = \frac{1}{2} \int_M \Gamma(u,v) d\mu$.

Remark 4.7 *If one assumes (v) for the original spaces one gets (v) for the glued space by the definition of the glued form $\mathcal{E}$, i.e.*

$$\Gamma(u,v)(x) = \Gamma_i(u|_{M_i}, v|_{M_i}) \quad \textit{for} \quad x \in M_i.$$

For our examples in in Section 3.3.1 and 3.3.2 $\Gamma(u,v)(x) = \nabla u(x)\nabla v(x)$ holds.

In [Ra01] with (i)-(v) it is shown that

$$\liminf_{t\to 0} 2t \log P_t(x,y) \geq -\rho^2(x,y)$$

holds. The upper bound described in Theorem 4.1 can be rewritten so that for every $\epsilon > 0$ with a constant $C = C(\epsilon, N)$ it holds that

$$p(t,x,y) \leq C\mu^{-\frac{1}{2}}(B_{\sqrt{t}}(x))\mu^{-\frac{1}{2}}(B_{\sqrt{t}}(y)) \exp\left(-\frac{\rho^2(x,y)}{(4+\epsilon)t}\right)$$

(cf. [St95b]) by absorbing the polynomial term into the exponential term. Then by the doubling property this yields for every $\epsilon > 0$

$$\limsup_{t\to 0} 2t \log p_t(x,y) \leq -\frac{1}{2+\epsilon}\rho^2(x,y).$$

Here ρ is the intrinsic metric w.r.t. the energy measure $d\Gamma$. By defining $\mathcal{E}(u,v) := \frac{1}{2}\int_M d\Gamma(u,v)$ and letting $\epsilon \to 0$ we get

$$\limsup_{t\to 0} 2t \log p_t(x,y) \leq -\rho^2(x,y).$$

Hence we have the following classical short-time asymptotic result for heat kernels on our glued spaces:

Theorem 4.8 *Assume that we glue spaces as described above. If the gluing conditions are satisfied so that (i)-(v) is satisfied on the glued space, then*

$$\lim_{t\to 0} 2t \log p_t(x,y) = -\rho^2(x,y).$$

holds true for the heat kernel $p_t(x,y)$ on the glued space M.

Chapter 5

Some Remarks on Rellich's Compact Embedding and the Poincaré Inequality

The intention of this chapter is to give a generalization for a result by Amick [Am78] to metric measure spaces instead of the Euclidean space $\mathbf{R}^n$ and more general Dirichlet forms instead of the canonical form $\mathcal{E}(u,u) := \int_\Omega |\nabla u|^2$. In [Am78] the author gives a characterization for the validity of Rellichs compact embedding and for the Poincaré inequality on a bounded domain Ω in $\mathbf{R}^n$ in terms of conditions on the boundary. Namely he defines a quantity

$$\Gamma_\Omega(\epsilon) := \sup_{u \in W_2^1(\Omega)} \frac{\int_{\Omega_\epsilon} |u|^2}{|u|^2_{W_2^1(\Omega)}}$$

while $\Omega_\epsilon := \{x \in \Omega : d(x, \partial\Omega) < \epsilon\}$ with d the Euclidean metric on $\mathbf{R}^n$. Since $\Gamma_\Omega(\epsilon) \in (0,1]$, $\forall \epsilon > 0$ and $\Gamma_\Omega(\epsilon)$ is monotone in ϵ one can define

$$\Gamma_\Omega(0) := \lim_{\epsilon \to 0} \Gamma_\Omega(\epsilon).$$

Then the embedding $i_\Omega : W_2^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact if and only if $\Gamma_\Omega(0) = 0$. Further the Poincaré inequality on Ω holds, i.e. $\exists\, c > 0 : \forall\, u \in W_2^1(\Omega)$:

$$\int_\Omega |u - u_\Omega|^2 \le c \int_\Omega |\nabla u|^2 \tag{5.1}$$

if and only if $\Gamma_\Omega(0) < 1$. In [Am78] the inequality 5.1 is written in the form

$$\int_\Omega |u|^2 \le const. \left\{ \left| \int_\Omega u \right|^2 + \int_\Omega |\nabla u|^2 \right\}$$

$\forall\ u \in W_2^1(\Omega)$ which for fixed Ω is the same as 5.1 as one can easily deduce. These characterizations represent the fact that with the usual contradiction argument one can get the Poincaré inequality out of Rellichs compact embedding but not vice versa. Amick gives examples of domains that satisfy a Poincaré inequality but fail to have a compact embedding. Therefore, he uses the well known technique of rooms and passages to construct domains Ω with $\Gamma_\Omega(0) = 1$ or $0 < \Gamma_\Omega(0) < 1$. Domains Ω with $\Gamma_\Omega(0) = 0$ are easy to find, just take $\partial\Omega$ to be differentiable.
The idea in [Am78] was to use the following lemma:

Lemma: *Let $\Omega \subset R^n$ be a bounded domain. If U is an open set with $\overline{U} \subset \Omega$ then there exists a domain V such that $\overline{U} \subset V \subset \overline{V} \subset \Omega$ and ∂V is analytic.*

From the lemma only ∂V to be differentiable was required because with this the author could use a kind of weak Rellich embedding and a weak Poincaré inequality to shift the problem to the boundary, i.e. to prove the equivalences for Rellich he needs for $\Omega \setminus \overline{\Omega_\epsilon} \subset U \subset \overline{U} \subset \Omega$ that $W_2^1(\Omega) \hookrightarrow W_2^1(U) \hookrightarrow L^2(U)$ is compact and to prove the equivalence for the Poincaré inequality he needs that

$$\int_U |u - u_U|^2 \leq const. \int_\Omega |\nabla u|^2$$

holds for all $u \in W_2^1(\Omega)$. To circumvent these problems in metric spaces we use an idea of Biroli and Tchou [BT97] who prove a compact embedding theorem for functions with Dirichlet boundary condition in Ω and for the 'weak Poincaré' inequality we use a chaining argument similar to the technique in Jerison [Je86] but without the difficult counting of the Whitney balls.

Throughout this chapter let (M, d, μ) be a metric measure space on which the doubling property for μ holds and the metric d is intrinsic. Let $(\mathcal{E}, D(\mathcal{E}))$ be a strongly local regular Dirichlet form on M and $\Omega \subset M$ a bounded open subset(in Section 5.2 it is also connected). Further let $H \subset L^2(\Omega, \mu)$ be a subspace of functions such that the scaling invariant Poincaré inequality holds for all balls $B(x, r) \subset \Omega$, i.e. $\exists\ c > 0$: $\forall\ B(x, r) \subset \Omega$:

$$\int_{B(x,r)} |u - u_{B(x,r)}|^2 d\mu \leq cr^2 \int_{B(x,r)} d\Gamma(u)$$

holds. For instance $H = \overline{H_\Omega}^{\mathcal{E}_1|_\Omega}$ with $H_\Omega = \{u|_\Omega : u \in D(\mathcal{E})\}$ and $\mathcal{E}_1|_\Omega(\cdot) := \left(\int_\Omega |\cdot|^2 d\mu + \int_\Omega d\Gamma(\cdot)\right)^{\frac{1}{2}}$ are possible function spaces. We define $\Gamma_\Omega(\epsilon)$ as

$$\Gamma_\Omega(\epsilon) := \sup_{u \in H} \frac{\int_{\Omega_\epsilon} |u|^2 d\mu}{\mathcal{E}_1|_\Omega(u)},$$

for which $\Gamma_\Omega(0) := \lim_{\epsilon\to 0}\Gamma_\Omega(\epsilon)$ exists as above. Here Ω_ϵ is defined in an analogous way w.r.t. the metric d.

At first we need some consequences from the doubling property which are proved in the following lemmata.

Lemma 5.1 *Let $\Omega \subset M$ be a bounded subset. Then for each $r > 0$ there exists a cover $\{B(x_i, r)\}_{i=1,\ldots,q}$ for Ω with the following properties:*

(*i*) $x_i \in \Omega$

(*ii*) $\Omega \subset \bigcup_{i=1}^{q} B(x_i, r)$

(*iii*) $d(x_i, x_j) \geq r$ *for* $i \neq j$

Proof: Take $B(x_1, \frac{r}{2})$ for any $x_1 \in \Omega$ and $r > 0$ fixed. Then chose the x_n for $n = 2, 3, \ldots$ in the following way:

$$x_n \in \Omega_n := \Omega \setminus \bigcup_{i=1}^{n-1} B(x_i, \frac{r}{2}),$$

s.t.

$$B(x_n, \frac{r}{2}) \cap B(x_i, \frac{r}{2}) = \emptyset \quad \forall i = 1, \ldots, n-1$$

until there is no x_{q+1} in such a manner. This procedure is finite because the doubling property yields the following:

$$\exists N \in \mathbf{N} : \forall x \in M, R > 0 : B(x, R) \text{ contains max. } N \text{ points } x_i : d(x_j, x_i) \geq r$$

(see [CW71]). By this definition $\Omega \subset \bigcup_{i=1}^{q} B(x_i, r)$ holds true, since if $x \in \Omega_q$, it follows $d(x, x_i) < \frac{r}{2} + \frac{r}{2} = r$ and therefore $x \in B(x_i, r)$. If $x \in \Omega \setminus \Omega_q$ then there exists an x_i s.t. $x \in B(x_i, r)$. The covering was chosen in a way that 1 and 3 holds clearly.

■

Lemma 5.2 *If $c > 0$ is the doubling constant for μ let $\nu := \log_2 c$. Further let $\{B(x_i, r)\}_{i=1,\ldots,q}$ be a cover of Ω for fixed $r > 0$ as in Lemma 5.1 and N the maximum number of balls $B_i := B(x_i, r)$ that covers a point $x \in E$ which is equal to the number of points x_i in $B(x, r)$. Then the following estimates holds:*

(*i*) $q \leq c\left(\frac{2(diam(\Omega)+r)}{r}\right)^{\nu}$, *(ii)* $\mu(B(x,r) \geq \frac{1}{c}\left(\frac{r}{R}\right)^{\nu} \mu(B(x,R))$ *for any* $R \geq r$,

(iii) $N \leq 2^{4\nu}$.

Proof: By $c\,\mu(B(x,r)) \geq \mu(B(x,2r))$ one gets via iteration:

$$\mu(B(x,r)) \geq \frac{1}{c} \cdot \frac{1}{c^{\log_2\left(\frac{R}{r}\right)}} \mu(B(x,R)).$$

Now (ii) holds because of $c^{\log_2\left(\frac{R}{r}\right)} = \left(\frac{R}{r}\right)^{\log_2 c}$. With (ii) one gets

$$\mu(B(x_i,\frac{r}{2})) \geq 2^{-4\nu}\mu(B(x,2r)) \text{ for } x \in B(x_i,r)$$

because of

$$\mu(B(x_i,\frac{r}{2})) \geq \frac{1}{c}\left(\frac{1}{8}\right)^{\nu} \mu(B(x_i,4r)) \geq 2^{-\nu} \cdot 2^{-3\nu} \cdot \mu(B(x,2r)).$$

Together with

$$\begin{aligned} N2^{-4\nu}\mu(B(x,2r)) &\leq N \min_{x_i \in B(x,r)} \mu(B(x_i,\frac{r}{2})) \\ &\leq \mu(\bigcup_{i=1}^{N} B(x_i,\frac{r}{2})) \\ &\leq \mu(B(x,2r)) \end{aligned}$$

(iii) follows. To prove (i) let $R = diam\Omega + \frac{r}{2}$ then

$$\sum_{i=1}^{q} \mu(B(x_i,\frac{r}{2})) \leq \mu(B(x,R))$$

holds. By (ii) one gets for $x \in B(x_i,\frac{r}{2})$ and $R' := R + \frac{r}{2} = diam\Omega + r$

$$\mu(B(x,R)) \leq \mu(B(x_i,R')) \leq c\left(\frac{2(diam\Omega + r)}{r}\right)^{\nu} \mu(B(x_i,\frac{r}{2}))$$

for all $i \in \{1,\ldots,q\}$. Now take the minimum of $\mu(B(x_i,\frac{r}{2}))$ for $i = 1,\ldots,q$ and divide by $\mu(B(x_i,\frac{r}{2}))$ then one has:

$$q \leq \frac{\mu(B(x,R))}{\mu(B(x_i,\frac{r}{2}))} \leq c\left(\frac{2(diam\Omega + r)}{r}\right)^{\nu}.$$

This finishes the proof of the lemma.

∎

5.1 Rellich Embedding

In order to prove the other implication a weaker form of the Rellich embedding theorem is needed. That means in a doubling metric measure space where the Poincaré inequality holds for all balls one has the following:

Lemma 5.3 *Let $\{u_n\} \subset H$ be a sequence of functions with*

$$\int_\Omega |u_n|^2 d\mu + \int_\Omega d\Gamma(u_n) \leq C < \infty$$

uniformly and Ω open, bounded subset in M. Then there exists an $u \in L^2(\Omega, \mu)$ s.t. for $\epsilon > 0$: there exists a subsequence $\{u_{n_k}\}_k$ which we relabel as $\{u_n\}_n$ and $u_n \to u$ in $L^2(\Omega \setminus \Omega_\epsilon, \mu)$ while $\Omega_\epsilon := \{x \in \Omega : d(x, \partial\Omega) < \epsilon\ \}$.

This lemma is based on a technique used in [BT97].

Proof: Since u_n is bounded in $L^2(\Omega)$ it has a weakly convergent sequence u_{n_k} (where u_n is used for u_{n_k}) in $L^2(\Omega)$ towards $u \in L^2(\Omega)$. Now take a cover $\{B(x_i, r)\}_{i=1,\dots,q}$ for $\Omega \setminus \Omega_\epsilon$ with $r < \epsilon$ as in the Lemma 5.1. Therefore, $\Omega \setminus \Omega_\epsilon \subset \bigcup_{i=1}^q B(x_i, r) \subset \Omega$ holds. Denote $B_i := B(x_i, r)$ and $\omega_{n,m} := u_n - u_m$ then one has to show that $\int_{\Omega\setminus\Omega_\epsilon} \omega_{n,m}^2 d\mu \to 0$ for $n, m \to \infty$ in order to get the strong L^2 convergence of u_n:

$$\begin{aligned}
\int_{\Omega\setminus\Omega_\epsilon} \omega_{n,m}^2 d\mu &\leq \sum_{i=1}^q \int_{B_i} \omega_{n,m}^2 d\mu \\
&= \sum_{i=1}^q \int_{B_i} |\omega_{n,m} - (\omega_{n,m})_i + (\omega_{n,m})_i|^2 d\mu \\
&\leq \underbrace{2\sum_{i=1}^q \int_{B_i} |\omega_{n,m} - (\omega_{n,m})_i|^2 d\mu}_{I_1:=} + \underbrace{2\sum_{i=1}^q \int_{B_i} |(\omega_{n,m})_i|^2 d\mu}_{I_2:=}
\end{aligned}$$

while $(\omega_{n,m})_i := \frac{1}{\mu(B_i)} \int_{B_i} \omega_{n,m} d\mu$. Now one has to control I_1 and I_2. For I_2 the following holds

$$\begin{aligned}
I_2 &\leq 2\sum_{i=1}^q \mu(B_i) \left(\frac{1}{\mu(B_i)} \int_{B_i} \omega_{n,m} d\mu \right)^2 \\
&\leq 2q \sup_{i\in\{1,\dots,q\}} \left[\frac{1}{\mu(B_i)} \left(\int_{B_i} \omega_{n,m} d\mu \right)^2 \right]
\end{aligned}$$

$$
\begin{aligned}
&\le 2c\left(\frac{2(diam\Omega+r)}{r}\right)^{\nu} c\left(\frac{diam\Omega}{r}\right)^{\nu}\cdot\ldots \\
&\qquad\qquad \ldots\cdot\frac{1}{\mu(B(x_i,diam\Omega))}\sup_{i\in\{1,\ldots,q\}}\left(\int_{B_i}\omega_{n,m}d\mu\right)^2 \\
&\le c'\sup_{i\in\{1,\ldots,q\}}\left(\int_{B_i}\omega_{n,m}d\mu\right)^2
\end{aligned}
$$

while $c' := 2c^2\left(\frac{2(diam\Omega+r)}{r}\right)^{\nu}\left(\frac{diam\Omega}{r}\right)^{\nu}\frac{1}{\mu(\Omega)}$. To estimate I_1 one needs the Poincaré inequality for balls and the result of Lemma 5.1 that for each point $x\in\Omega$ there are at maximum N balls B_i with $x\in B_i$:

$$
\begin{aligned}
I_1 &\le 2\sum_{i=1}^{q} c_p r^2\int_{B_i} d\Gamma(\omega_{n,m}) \\
&\le 2c_p r^2 N\int_{\Omega} d\Gamma(\omega_{n,m}) \\
&\le 2c_p r^2 NC.
\end{aligned}
$$

Therefore, one has

$$
\int_{\Omega\setminus\Omega_\epsilon}\omega_{n,m}^2 d\mu \le 2c_p r^2 NC + c'\sup_{i\in\{1,\ldots,q\}}\left(\int_{B_i}\omega_{n,m}d\mu\right)^2.
$$

Now first choose $r=\left(\frac{\epsilon}{4c_pNC}\right)^{\frac{1}{2}}$ and second choose n and m large enough, s.t.

$$
\sup_{i\in\{1,\ldots,q\}}\left(\int_{B_i}\omega_{n,m}d\mu\right)^2 \le \frac{\epsilon}{2c'}.
$$

This is possible because u_n is weakly convergent in $L^2(\Omega,\mu)$. Hence

$$
\int_{\Omega\setminus\Omega_\epsilon}|\omega_{n,m}|^2 d\mu \le \epsilon
$$

holds for n,m large enough.

∎

We will now prove the equivalence of $\Gamma_\Omega(0)=0$ and the validity of a kind of compact embedding theorem.

Theorem 5.4 *Let $\{u_n\}\subset H$ be a sequence of functions and C a constant with $C>0$ such that for all n*

$$
\int_\Omega |u_n|^2 d\mu + \int_\Omega d\Gamma(u_n) \le C < \infty
$$

holds. Then the following statements are equivalent:

(*i*) $\Gamma_\Omega(0) = 0$

(*ii*) *There exists an* $u \in L^2(\Omega)$ *such that for a subsequence* $\{u_{n_k}\}_k$ *which we relabel as* $\{u_n\}_n$ *it holds that* $u_n \to u$ *strongly in* $L^2(\Omega)$

Proof: (ii) $\Rightarrow$ (i) : First assume that $\Gamma_\Omega(0) > 0$ and denote $A := \Gamma_\Omega(0)$. Therefore, there exist sequences $\{\epsilon_n\}$ and $\{u_n\}$ with $\epsilon_n \to 0$ as $n \to \infty$ and

$$\left(\int_\Omega |u_n|^2 d\mu + \int_\Omega d\Gamma(u_n)\right)^{\frac{1}{2}} = 1,$$

$n = 1, 2, \ldots$ such that

$$\int_{\Omega_{\epsilon_n}} |u_n|^2 > \frac{A}{2} \quad n = 1, 2, \ldots.$$

Because of the hypothesis one knows that there exits a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \to u$ in $L^2(\Omega, \mu)$. By the triangle inequality it follows that:

$$\left(\int_{\Omega_{\epsilon_n}} |u|^2 d\mu\right)^{\frac{1}{2}} \geq \left(\int_{\Omega_{\epsilon_n}} |u_{n_k}|^2 d\mu\right)^{\frac{1}{2}} - \left(\int_{\Omega_{\epsilon_n}} |u_{n_k} - u|^2 d\mu\right)^{\frac{1}{2}} \tag{5.2}$$

$$\geq \frac{\sqrt{A}}{2\sqrt{2}} > 0 \tag{5.3}$$

while the second inequality holds for sufficiently large n_k because of the L^2-convergence of u_{n_k} towards u. Since $\Omega_{\epsilon_n} \to \emptyset$ gives $\mu(\Omega_{\epsilon_n}) \to 0$ it follows that $\int_{\Omega_{\epsilon_n}} |u|^2 d\mu \to 0$ as $n \to \infty$, so this contradicts 5.2.

(i) $\Rightarrow$ (ii) : For the other direction let $\Gamma_\Omega(0) = 0$, i.e. that $\Gamma_\Omega(\epsilon) \to 0$ for $\epsilon \to 0$. Let $\{u_n\}$ be any sequence in H such that

$$\int_\Omega |u_n|^2 d\mu + \int_\Omega d\Gamma(u_n) = 1. \tag{5.4}$$

Then there exists an $u \in L^2(\Omega)$ and a subsequence relabeled as $\{u_n\}$ such that

$$u_n \to u \quad \text{weakly in} \quad L^2(\Omega) \tag{5.5}$$

as $n \to \infty$. Now let $\delta > 0$ and $\epsilon > 0$ be small enough such that $\Gamma_\Omega(\epsilon) < \delta$. By Lemma 5.3 we know that $u_n \to u$ strongly in $L^2(\Omega \setminus \Omega_\epsilon, \mu)$. Therefore,

$$|u - u_n|^2_{L^2(\Omega \setminus \Omega_\epsilon)} \leq \delta \tag{5.6}$$

for n sufficiently large and from the definition of $\Gamma_\Omega(\epsilon)$ one gets

$$|u - u_n|^2_{L^2(\Omega_\epsilon)} \leq \delta \left(|u - u_n|^2_{L^2(\Omega)} + \int_\Omega d\Gamma(u - u_n)\right).$$

Together with 5.4 and 5.5 it follows that

$$\lim_{n\to\infty} |u-u_n|^2_{L^2(\Omega_\epsilon)} \leq \delta\left(1-|u|^2_{L^2(\Omega)} - \int_\Omega d\Gamma(u)\right) \leq \delta. \tag{5.7}$$

Finally 5.6 and 5.7 yields

$$|u-u_n|^2_{L^2(\Omega)} = |u-u_n|^2_{L^2(\Omega\setminus\Omega_\epsilon)} + |u-u_n|^2_{L^2(\Omega_\epsilon)} \leq 2\delta$$

for n sufficiently large. Since δ was chosen arbitrarily $u_n \to u$ in $L^2(\Omega)$ and the proof is finished.

■

Remark 5.5 *If $\mathcal{E}|_\Omega$ defined on the set of restricted functions H_Ω (via its energy measure) is closable w.r.t the norm $\mathcal{E}_1|_\Omega$ one can consider the closure $\overline{H}_\Omega := \overline{H_\Omega}^{\mathcal{E}_1|_\Omega}$ and reformulate Theorem 5.4 in the following way:*

$$\Gamma_\Omega(0) = 0 \quad \Leftrightarrow \quad \textit{The embedding} \;\; i_\Omega : \overline{H}_\Omega \to L^2(\Omega) \;\; \textit{is compact.}$$

5.2 Poincaré Inequality

The following lemmata prepare the proof of the equivalence and fix the gap in the proof of Amick to extend them to metric measure spaces.

Lemma 5.6 *Let Ω be an open bounded connected set in M. Then there exist a constant $c > 0$ and an $\epsilon > 0$ small enough, s.t.*

$$\int_{\Omega\setminus\Omega_\epsilon} |u-u_{\Omega\setminus\Omega_\epsilon}|^2 d\mu \leq c\cdot\int_\Omega d\Gamma(u)$$

for all $u \in H$ while $\Omega_\epsilon := \{x \in \Omega : d(x,\partial\Omega) < \epsilon\}$.

Proof: First take a finite cover of balls $\{B(x_i,\frac{\epsilon}{2})\}_{i=1,\dots,q}$ of $\Omega\setminus\Omega_\epsilon$ as in the Lemma 5.1 while the properties of the cover that comes from the doubling condition are not necessary here. Since M is a length space it is locally arcwise connected, s.t. Ω is arcwise connected too. Further for $I := [0,1]$ the length of a path $\gamma : I \to \Omega$ between $x, y \in \Omega$ is finite as it is clear by compactness of $\gamma(I)$ and the fact that M is a length space. Denote $B_i := B(x_i,\frac{\epsilon}{2})$ and fix a set $B_k \in \{B_i\}_{i=1,\dots,q}$. Let $\gamma_{x_k,x_i} : I \to \Omega$ for $i \neq k$ be a set of finite length curves inside Ω which connect the

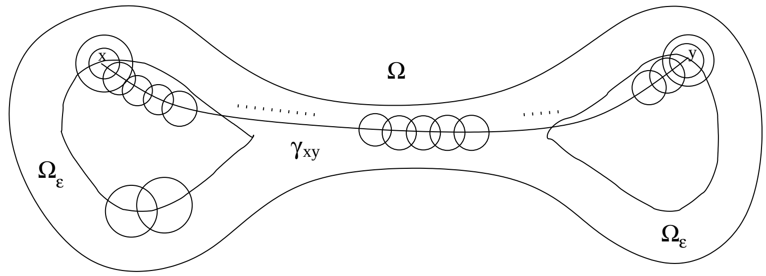

Figure 5.1: Proof of Lemma 5.6

midpoints of B_k with the midpoint of B_i for $i \in \{1, ...q\} \setminus \{k\}$ respectively. Now consider the distances $d(\gamma_{x_k,x_i}(I), \partial\Omega)$ and take the minimum

$$\delta := \min_{i \in \{1,...q\} \setminus \{k\}} d(\gamma_{x_k,x_i}(I), \partial\Omega).$$

This minimum δ exists and is nonzero because of the compactness of $\gamma_{x_k,x_i}(I)$. Consider a finite cover of $\gamma_{x_k,x_i}(I)$ with balls B_j^{ki} for $j = 1, \ldots, n_{ki}$ with radius $\rho(B_j^{ki}) = \frac{\delta}{4}$ while $\rho(B)$ denotes the radius of a ball B. Then enlarge the radius of these balls by the factor two so that $\rho(B_j^{ki}) = \frac{\delta}{2}$ and still $B_j^{ki} \subset \Omega$ holds.
Now w.l.o.g. $B_1^{ki} \subset B_i$, $B_{n_{ki}}^{ki} \subset B_k$, s.t. $\mu(B_i) \leq c_d \mu(B_i \cap B_1^{ki})$, $\mu(B_i) \leq c_d \mu(B_k \cap B_{n_{ki}}^{ki})$ and $\mu(B_i) \leq c_d \mu(B_j^{ki} \cap B_{j+1}^{ki})$ for $j = 1, \ldots, n_{ki} - 1$ holds for a universal constant c_d because of the doubling property and the boundedness of Ω. If c_p is the constant for the Poincaré inequality for balls then with the following calculation: $\forall\ u \in H$

$$\begin{aligned}
\int_B |u_{B_1} - u_{B_2}|^2 d\mu &= \frac{\mu(B)}{\mu(B_1 \cap B_2)} \int_{B_1 \cap B_2} |u_{B_1} - u_{B_2}|^2 d\mu \\
&\leq \frac{2\mu(B)}{\mu(B_1 \cap B_2)} \left[\int_{B_1 \cap B_2} |u - u_{B_1}|^2 d\mu + \int_{B_1 \cap B_2} |u - u_{B_2}|^2 d\mu \right] \\
&\leq \frac{2\mu(B)}{\mu(B_1 \cap B_2)} \left[\int_{B_1} |u - u_{B_1}|^2 d\mu + \int_{B_2} |u - u_{B_2}|^2 d\mu \right]
\end{aligned}$$

while $B_1 \cap B_2 \neq \emptyset$, one gets

$$\begin{aligned}
\int_{B_i} |u - u_{B_k}|^2 d\mu &= \int_{B_i} |u - u_{B_i} + u_{B_i} - u_{B_1^{ki}} \\
&\quad + \sum_{j=1}^{n_{ki}-1} (u_{B_j^{ki}} - u_{B_{j+1}^{ki}}) + u_{B_{n_{ki}}^{ki}} - u_{B_k}|^2 d\mu \\
&\leq 2n_{ki} \left[\int_{B_i} |u - u_{B_i}|^2 d\mu + \int_{B_i} |u_{B_i} - u_{B_1^{ki}}|^2 d\mu \right. \\
&\quad \left. + \sum_{i=1}^{n_{ki}-1} \int_{B_i} |u_{B_j^{ki}} - u_{B_{j+1}^{ki}}|^2 d\mu + \int_{B_i} |u_{B_{n_{ki}}^{ki}} - u_{B_k}|^2 d\mu \right]
\end{aligned}$$

$$\begin{aligned}
&\leq 10 \cdot n_{ki} \cdot c_p \cdot \epsilon^2 \cdot c_d \cdot \Bigg[\int_{B_i} d\Gamma(u) + \int_{B_1^{ki}} d\Gamma(u) \\
&\quad + \sum_{i=1}^{n_{ki}-1} \left(\int_{B_j^{ki}} d\Gamma(u) + \int_{B_{j+1}^{ki}} d\Gamma(u) \right) \\
&\quad + \int_{B_{n_{ki}}^{ki}} d\Gamma(u) + \int_{B_k} d\Gamma(u) \Bigg] \\
&\leq N \cdot 10 \cdot n_{ki} \cdot c_p \cdot \epsilon^2 \cdot c_d \cdot \int_\Omega d\Gamma(u),
\end{aligned}$$

using that $\sum_{k=1}^{q} n_{ki} < N$ for N large enough. Summing over all B_i one gets

$$\begin{aligned}
\int_{\Omega \setminus \Omega_\epsilon} |u - u_{B_k}|^2 d\mu &\leq \sum_{i=1}^{q} \int_{B_i} |u - u_{B_k}|^2 d\mu \\
&\leq q \cdot N \cdot 10 \cdot n_{ki} \cdot c_p \cdot \epsilon^2 \cdot c_d \cdot \int_\Omega d\Gamma(u).
\end{aligned}$$

Because of

$$\min_{a \in \mathbf{R}} \int_{\Omega \setminus \Omega_\epsilon} |u - a|^2 d\mu = \int_{\Omega \setminus \Omega_\epsilon} |u - u_{\Omega \setminus \Omega_\epsilon}|^2 d\mu$$

this finishes the proof.

■

The proof of Lemma 5.6 is similar to the proof of the Poincaré inequality by Jerison [Je86] but there is no counting of 'Whitney-balls' nessecary because the irregularity is enclosed in the boundary which is cut out by Ω_ϵ.
The next theorem will give the characterization for the validity of a Poincaré inequality on an open, bounded and connected subset $\Omega \subset M$.

Theorem 5.7 *For $\Omega \subset M$ open, bounded and connected the following statements are equivalent*

(*i*) $\Gamma_\Omega(0) < 1$

(*ii*) $\exists\, c > 0 : \forall\, u \in H\ :\ \int_\Omega |u - u_\Omega|^2 d\mu \leq c \int_\Omega d\Gamma(u)$

Proof: (ii) $\Rightarrow$ (i) : Assume that $\Gamma_\Omega(0) = 1$. Then there exists a sequence of positive numbers $\{\epsilon_n\}$ with $\epsilon_n \to 0$ and a sequence $\{u_n\}$ of functions such that:

$$\lim_{n\to\infty} \int_{\Omega_{\epsilon_n}} |u_n|^2 d\mu = 1$$

and

$$\left(\int_\Omega |u_n|^2 d\mu + \int_\Omega d\Gamma(u_n)\right)^{\frac{1}{2}} = 1,\ n = 1, 2, \dots$$

holds. Therefore, one gets

$$\left.\begin{array}{c} \int_\Omega d\Gamma(u_n) \\ \int_{\Omega\setminus\Omega_{\epsilon_n}} |u_n|^2 d\mu \end{array}\right\} \to 0 \ \text{ as } \ n \to \infty. \tag{5.8}$$

By the Poincaré inequality

$$\int_\Omega |u_n - (u_n)_\Omega|^2 d\mu \leq c(\Omega) \int_\Omega d\Gamma(u_n)$$

while $c(\Omega)$ depends on Ω only, it follows that

$$\lim_{n\to\infty} \int_\Omega |u_n - (u_n)_\Omega|^2 d\mu = 0. \tag{5.9}$$

The triangle inequality yields

$$|(u_n)_\Omega| \cdot \mu(\Omega \setminus \Omega_{\epsilon_n})^{\frac{1}{2}} \leq \left(\int_{\Omega\setminus\Omega_{\epsilon_n}} |u_n - (u_n)_\Omega|^2 d\mu\right)^{\frac{1}{2}} + \left(\int_{\Omega\setminus\Omega_{\epsilon_n}} |u_n|^2 d\mu\right)^{\frac{1}{2}}.$$

Now together with 5.8 and 5.9 this implies:

$$\lim_{n\to\infty} (u_n)_\Omega = 0. \tag{5.10}$$

But 5.9 and 5.10 give the following contradiction

$$\begin{aligned} \lim_{n\to\infty} \int_\Omega |u_n - (u_n)_\Omega|^2 d\mu &= \lim_{n\to\infty} \int_\Omega |u_n|^2 d\mu \\ &= \lim_{n\to\infty} \left(\int_\Omega |u_n|^2 d\mu + \int_\Omega d\Gamma(u_n)\right)^{\frac{1}{2}} = 1 \end{aligned}$$

which implies that $\Gamma_\Omega(0) < 1$.

(i) $\Rightarrow$ (ii) : For the reverse direction assume that $\Gamma_\Omega(0) < 1$ and let $\epsilon > 0$ be a fixed number small enough that Lemma 5.6 holds and so that $\Gamma_\Omega(\epsilon) \leq \alpha < 1$. For any $u \in H$ we have

$$\int_\Omega |u - u_{\Omega\setminus\Omega_\epsilon}|^2 d\mu \leq \int_{\Omega_\epsilon} |u - u_{\Omega\setminus\Omega_\epsilon}|^2 d\mu + \int_{\Omega\setminus\Omega_\epsilon} |u - u_{\Omega\setminus\Omega_\epsilon}|^2 d\mu. \tag{5.11}$$

Because of $\Gamma_\Omega(\epsilon) \leq \alpha$ it follows that

$$\int_{\Omega_\epsilon} |u - u_{\Omega\setminus\Omega_\epsilon}|^2 d\mu \leq \alpha \int_\Omega |u - u_{\Omega\setminus\Omega_\epsilon}|^2 d\mu + \alpha \int_\Omega d\Gamma(u). \tag{5.12}$$

Now 5.11 and 5.12 yield

$$\int_\Omega |u - u_{\Omega\setminus\Omega_\epsilon}|^2 d\mu \leq \alpha \int_\Omega |u - u_{\Omega\setminus\Omega_\epsilon}|^2 d\mu + \alpha \int_\Omega d\Gamma(u) + \int_{\Omega\setminus\Omega_\epsilon} |u - u_{\Omega\setminus\Omega_\epsilon}|^2 d\mu$$

and therefore

$$\int_\Omega |u - u_{\Omega\setminus\Omega_\epsilon}|^2 d\mu \leq \left(\frac{\alpha}{1-\alpha}\right) \int_\Omega d\Gamma(u) + \left(\frac{1}{1-\alpha}\right) \int_{\Omega\setminus\Omega_\epsilon} |u - u_{\Omega\setminus\Omega_\epsilon}|^2 d\mu \quad (5.13)$$

follows. Since by Lemma 5.6 it holds that

$$\int_{\Omega\setminus\Omega_\epsilon} |u - u_{\Omega\setminus\Omega_\epsilon}|^2 d\mu \leq c \int_\Omega d\Gamma(u)$$

and so one gets with

$$\int_\Omega |u - u_{\Omega\setminus\Omega_\epsilon}|^2 d\mu \leq \left(\frac{\alpha}{1-\alpha}\right) \int_\Omega d\Gamma(u) + \left(\frac{1}{1-\alpha}\right) c \int_\Omega d\Gamma(u)$$

for all $u \in H$ and because of

$$\min_{a\in\mathbf{R}} \int_\Omega |u - a|^2 d\mu = \int_\Omega |u - u_\Omega|^2 d\mu$$

the proof is finished.

■

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Danksagung

Zuerst möchte ich meinem Betreuer Karl-Theodor Sturm für die Anregung und Betreuung dieser Arbeit danken. Die hilfreichen Gespräche mit ihm haben entscheidend zum Entstehen dieser Arbeit beigetragen. Dem Warwick Mathematics Institute, insbesondere David Elworthy, danke ich für seine Gastfreundschaft während meines Aufenthalts in Warwick.

Desweiteren gilt mein Dank der Arbeitsgruppe sowie dem ganzen Intstitut für die angenehme Atmosphäre während meiner Promotion, insbesondere Anca Bonciocat, Tom Christiansen, Martin Grothaus, Natalie Grunewald, Atle Hahn, Martin Hesse und Kostja von Renesse für viele anregende Diskussionen und Ermutigungen.

Nicht zuletzt danke ich meinen Eltern für ihre Unterstützung und meiner Freundin Hannah sowie meiner Schwester Julia für ihre Aufmunterungen und ihre Geduld.