Harmonic Maps into Trees and Graphs - Analytical and Numerical Aspects

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Introduction

Abstract

The main topic of this work is the definition and investigation of a nonlinear energy for maps with values in trees and graphs and the analysis of the corresponding nonlinear Dirichlet problem. The nonlinear energy is defined using a semigroup approach based on Markov kernels and the nonlinear Dirichlet problem is given as a minimizing problem of the nonlinear energy. Conditions for the existence and uniqueness of a solution to the nonlinear Dirichlet problem are presented.
A numerical algorithm is developed to solve the nonlinear Dirichlet problem for maps from a two dimensional Euclidean domain into trees. The problem is discretized using a suitable finite element approach and convergence of a corresponding iterative numerical method is proven.
Furthermore, for graph targets homotopy problems are analyzed. For particular domain spaces the existence of a minimizer of the nonlinear energy in a given homotopy class is shown.

A smooth map $f : M \to N$ between Riemannian manifolds is called harmonic if its tension field $\tau(f) := trace \nabla(df)$ vanishes [Jos95]. Well known examples are harmonic functions ($N = \mathbb{R}$), geodesics ($M \subset \mathbb{R}$) and minimal surfaces. Harmonic maps play an important role in many areas of mathematics, see [EL78], [EL88] for a survey. In the last decade, the study of maps into more general target spaces was developed, e.g. [GS92], [JY93].
Korevaar/Schoen ([KS93], [KS97]) and Jost ([Jos94], [Jos97b]) independently began to develop a theory of harmonic maps into metric spaces of nonpositive curvature in the sense of Alexandrov (briefly: NPC spaces). These developments are based on the fact that a canonical extension of the energy functional can be defined for maps with values in NPC spaces. In the approach by Korevaar/Schoen, the domain space is still a Riemannian manifold. In Jost's approach, the domain space is a locally compact metric space equipped with an abstract Dirichlet form, replacing the Riemannian manifold equipped with the classical Dirichlet form. Eells/Fuglede study harmonic maps between Riemannian polyhedra in [EF01]. For recent proceedings in the more specific case of maps into Riemannian polyhedra we refer to [Fug01], [Fug03a], [Fug03b]. Picard has investigated harmonic maps into trees [Pic04].

Besides Riemannian manifolds the most simple NPC spaces are metric trees and in particular trees with only one branchpoint ("spiders"). To study and understand the nonlinear effects (e.g. on regularity and stability of harmonic maps) arising from negative curvature one may restrict oneself to these prototypes of NPC spaces.

In the first two parts of this work we will study the nonlinear Dirichlet problem for harmonic maps with values in spiders and trees. These studies yield the main module for the analysis of the nonlinear Dirichlet problem for maps into graphs which is done in the last part of this work.

In the first chapter, we analyze the nonlinear Dirichlet problem for harmonic maps $v : M \to N$ from a measure space (M, m) with a local regular Dirichlet form on it into a spider N. Spiders are the simplest examples of trees, they consist of one branchpoint and a finite number of edges.
Let (M, m) be a measure space with a local regular Dirichlet form $\mathcal{E}$ on it with generator A and semigroup e^{At} given by a semigroup of Markov kernels p_t. We will define a canonical extension $\mathcal{E}_N$ of the energy $\mathcal{E}$ for maps $v : M \to N$ using the semigroup p_t by

$$\mathcal{E}_N(v) := \limsup_{t\to 0} \frac{1}{2t} \int_M \int_M d^2(v(x), v(y)) p_t(x, dy) m(dx).$$

This definition yields the identity

$$\sum \mathcal{E}(v_i) = \mathcal{E}_N(v), \tag{1}$$

whereby $v_i : M \to \mathbb{R}$ is the projection of v on the i-th edge of the spider N. If the operator A is the Laplace operator Δ on $\mathbb{R}^k$ then one has

$$\mathcal{E}_N(v) = \sum \int_{\mathbb{R}^k} |\nabla v_i|^2.$$

The nonlinear Dirichlet problem for a given map g with $\mathcal{E}_N(g) < \infty$ and a subset $D \subset M$ is to find a map u with $u = g$ on $M \backslash D$ which minimizes the nonlinear energy $\mathcal{E}_N$ (either on M or, equivalently, on D). Such a map is called harmonic on D. Conditions for the existence and uniqueness of a solution to the nonlinear Dirichlet problem will be given.

In the special case $M = \mathbb{R}^2$, $\mathcal{E}$ being the classical Dirichlet form on $\mathbb{R}^2$, D being a polygonal set we will define a numerical algorithm to solve the nonlinear Dirichlet problem.
Within this case, we fix suitable triangulations $\mathcal{T}_h$ of D and define a discrete nonlinear energy $\mathcal{E}_N^h$ for maps $\bar{v}_h : \mathcal{N}_h \to N$, whereby $\mathcal{N}_h$ denotes the set of vertices of the triangulation $\mathcal{T}_h$. This yields a discrete nonlinear Dirichlet problem, i.e., for a map $g : \mathbb{R}^2 \to N$ with $\mathcal{E}_N(g) < \infty$ one searches a map $\bar{u}_h : \mathcal{N}_h \to N$ with $\bar{u}_h = g$ on $\partial D \cap \mathcal{N}_h$ minimizing the discrete nonlinear energy $\mathcal{E}_N^h$. For the construction of the algorithm solving this problem we mainly use the fact that the maps which minimize the discrete energy can be obtained by

iteration of nonlinear Markov operators. The latter are defined as barycenters of discrete probability distributions on the spider.
Furthermore, we define a prolongation operator J_h which extends maps defined on the vertices to maps defined on the whole domain D in such a way that

$$\mathcal{E}_N(J_h(\bar{u}_h)) \leq \mathcal{E}_N^h(\bar{u}_h) + R_{g,D} \to \mathcal{E}_N(u) \qquad h \to 0,$$

with a nonnegative constant $R_{g,D}$ only depending on the polygonal domain D, the regularity of the triangulation $\mathcal{T}_h$, and the map g. From this the L^2-convergence of $J_h(\bar{u}_h)$ to the solution u of the nonlinear Dirichlet problem follows as a straightforward consequence.
In addition, we discuss a generalization of the nonlinear energy for maps with values in a spider with a countable number of edges.

In the second chapter, we will study the nonlinear Dirichlet problem for harmonic maps $v : M \to N$ from a measure space (M, m) with a local regular Dirichlet form on it into finite trees.
Given a measure space (M, m) with a local regular Dirichlet form $\mathcal{E}$ on it with generator A and semigroup e^{At} given by a semigroup of Markov kernels p_t, we will define a canonical extension $\mathcal{E}_N$ of the energy $\mathcal{E}$ for maps $v : M \to N$ by

$$\mathcal{E}_N(v) := \sup_{\substack{\varphi \in \mathcal{C}_c(M) \\ 0 \leq \varphi \leq 1}} \limsup_{t \to 0} \frac{1}{2t} \int_M \int_M \varphi(x) d^2(v(x), v(y)) p_t(x, dy) m(dx) \tag{2}$$

with $\mathcal{C}_c(M)$ being the set of all continuous functions on M with compact support. We will prove

$$\mathcal{E}_N(v) = \sum \mu_{<v_i>}(M),$$

whereby $v_i : M \to \mathbb{R}_+$ is the projection of v on the i-th edge of the tree N and $\mu_{<v_i>}$ is the energy measure of v_i. Note that hence definition (2) is consistent with the previous definition (1) for the case of a spider N. Again, if the operator A is the Laplace operator Δ on $\mathbb{R}^k$ one has

$$\mathcal{E}_N(v) = \sum \int_{\mathbb{R}^k} |\nabla v_i|^2.$$

To study harmonic maps into trees, Picard (cf. [Pic04]) presented another definition of nonlinear energy:

$$\tilde{\mathcal{E}}_N(v) = \sup\{\mathcal{E}(\phi \circ v) : \phi \text{ non expanding}\}.$$

We will prove that our definition of nonlinear energy coincides with the definition of Picard. Furthermore, we will show in the special case $M = \mathbb{R}^k$ or M a Riemannian manifold and $\mathcal{E}$ the classical Dirichlet form the equivalence of our nonlinear energy to the nonlinear energy given by Korevaar/Schoen in [KS93].

The nonlinear Dirichlet problem for a given map g with $\mathcal{E}_N(g) < \infty$ and a subset $D \subset M$ is to find a map u with $\tilde{u} = \tilde{g}$ quasi everywhere on $M \backslash D$ where $\tilde{u}, \tilde{g}$ denote quasi continuous versions of u and g, resp., which minimizes the nonlinear energy $\mathcal{E}_N$. We will present conditions for the existence and uniqueness of a solution to the nonlinear Dirichlet problem.

In the special case $M = \mathbb{R}^2$, $\mathcal{E}$ being the classical Dirichlet form on $\mathbb{R}^2$, D being a polygonal set, we will extend the numerical algorithm from the first chapter to solve the nonlinear Dirichlet problem for maps with values in finite trees.
Finally, we discuss a generalization of the nonlinear energy for maps with values in trees with a countable number of edges.

In the last chapter, we will study graph targets. Let (M, m) be a compact measure space with universal cover $\tilde{M}$ and with a local regular Dirichlet form $\mathcal{E}$ on $L^2(\tilde{M}, \tilde{m})$ given by a semigroup of Markov kernels p_t. In addition, let (N, d) be a graph with a finite number of edges.
Before we define the nonlinear energy for maps $v : M \to N$ we will study equivariant mapping problems. This is motivated by the fact that any continuous map $v : M \to N$ lifts to an equivariant map $\tilde{v} : \tilde{M} \to \tilde{N}$, whereby the universal cover $\tilde{N}$ of the graph N is a tree with a countable number of edges.
Given an equivariant map $\tilde{v} : \tilde{M} \to \tilde{N}$ we say that two projections $\tilde{v}_i$ and $\tilde{v}_j$ are equivalent ($\tilde{v}_i \sim \tilde{v}_j$) if there is an element γ of the group of covering transformations of $\tilde{M}$ such that $\tilde{v}_i = \tilde{v}_j \circ \gamma$. This yields an equivalence relation on the set of all projections $\tilde{v}_i, i \in \mathbb{N}$, and if there is a projection $\tilde{v}_i \in \mathcal{D}_{loc}(\mathcal{E})$ we will prove for all projections $\tilde{v}_j$ with $\tilde{v}_i \sim \tilde{v}_j$ that $\tilde{v}_j \in \mathcal{D}_{loc}(\mathcal{E})$ and

$$\mu_{<\tilde{v}_i>}(\tilde{M}) = \mu_{<\tilde{v}_j>}(\tilde{M}).$$

Therefore, we define the nonlinear energy function $\mathcal{E}_{\tilde{N}}$ for an equivariant map $\tilde{v} : \tilde{M} \to \tilde{N}$ by

$$\mathcal{E}_{\tilde{N}}(\tilde{v}) := \sum_{\tilde{v}_i \in \mathbb{F}(\tilde{v})/\sim} \mu_{<\tilde{v}_i>}(\tilde{M}),$$

whereby $\mathbb{F}(\tilde{v})$ denotes the set of all projections of $\tilde{v}$. We will show that for any fundamental domain M_0 for M, in $\tilde{M}$, such that $\bar{M}_0$ is compact and ∂M_0 has measure zero one has

$$\mathcal{E}_{\tilde{N}}(\tilde{v}) := \sum_{i \in \mathbb{N}} \mu_{<\tilde{v}_i>}(M_0).$$

Furthermore, in this context the nonlinear Dirichlet problem for a given map $\tilde{g}$ with $\mathcal{E}_{\tilde{N}}(\tilde{g}) < \infty$ and a subset $D \subset M_0$ is to find a map $\tilde{u}$ which minimizes the nonlinear energy $\mathcal{E}_{\tilde{N}}$. We will present conditions for the existence and uniqueness of a solution to the nonlinear Dirichlet problem using results from the second chapter.

In the next step, we define the nonlinear energy function $\mathcal{E}_N$ for a map $v : M \to N$ which is the projection of an equivariant map $\tilde{v} : \tilde{M} \to \tilde{N}$ by

$$\mathcal{E}_N(v) := \mathcal{E}_{\tilde{N}}(\tilde{v}).$$

In addition, we define the nonlinear Dirichlet problem for graph-valued maps and we obtain conditions for the existence and uniqueness of a solution.

Finally, we will analyze homotopy problems. Given a continuous map $g : M \to N$ denote the homotopy class of g by $Hom(g)$. Now, one looks for a map $u \in Hom(g)$ which minimizes the nonlinear energy function $\mathcal{E}_N$ in this class, i.e.

$$\mathcal{E}_N(u) = \min_{v \in Hom(g)} \mathcal{E}_N(v).$$

In the special case that M is a compact manifold with $\partial M = \emptyset$ and that p_t is the heat semigroup on $\tilde{M}$ we will show for any given continuous map $g : M \to N$ the existence of such a minimizer in $Hom(g)$. For the proof, we will show that for any map $v \in Hom(g)$ our definition of nonlinear energy coincides with the energy definition introduced by Korevaar/Schoen in [KS93]. Similar results will be obtained in the case that M is a Riemannian polyhedron.

In the appendix of this work, we will discuss the equivalence of various locality properties for regular Dirichlet forms, e.g. in the sense of Fukushima (cf. [FOT94]) and in the sense of Bouleau/Hirsch (cf. [BH91]).

Overview

The major points of this work are

- the definition of the nonlinear energy for maps with values in trees and graphs as a canonical extension of a given energy,
- the "energy decomposition" of the nonlinear energy,
- the comparison of our nonlinear energy with other possible definitions of nonlinear energy,
- the analysis of the corresponding nonlinear Dirichlet problem,
- the construction of a numerical algorithm to solve the nonlinear Dirichlet problem,
- the proof of convergence of this numerical method,
- implementation of the algorithm, visualization of the resulting maps, and
- the analysis of homotopy problems.

These points will be presented in different generality (related to the target).

Nonlinear Energy: The definition of the nonlinear energy for maps into spiders and trees is given in Section 1.1 and Section 2.1, resp. In Definition 3.10 we define a nonlinear energy function for equivariant maps with values in the universal cover of a graph and we prove that this energy function is equivariant (cf. Theorem 3.11). The nonlinear energy of a map with values in a graph is given by the energy of the equivariant lift of this map (see Section 3.2).

Energy Decomposition: In Theorem 1.3 and Theorem 2.7 the energy decomposition for maps into spiders and trees, resp. is given.

Comparison: For maps into trees, we show that our definition of nonlinear energy coincides with the nonlinear energy defined by Picard (cf. Proposition 2.12). Further comparison results for tree and graph targets with the nonlinear energy defined by Korevaar/Schoen are given in Proposition 1.5, Subsection 2.1.1 and Theorem 3.21.

Nonlinear Dirichlet Problem: For spider, tree and graph targets we consider the nonlinear Dirichlet problem and we present conditions for the existence and uniqueness of a solution (cf. Section 1.2, Section 2.2 and Subsection 3.2.1).

Algorithm: To solve the nonlinear Dirichlet problem for maps from a two dimensional Euclidean domain into spiders and trees numerical algorithms are developed in the Subsections 1.3.1 – 1.3.2 and the Subsections 2.3.1 – 2.3.3, resp.

Convergence: For both numerical methods the convergence is proven in Subsection 1.3.3 and Subsection 2.3.4.

Implementation: In Subsection 1.3.4 we discuss the expected order of convergence of the numerical algorithm in the case of a spider target. Furthermore, for spider and tree targets we present visualizations of solutions to the nonlinear Dirichlet problem in Subsection 1.3.4 and Subsection 2.3.4.

Homotopy problems: For graph targets homotopy problems are analyzed in Subsection 3.2.2. For particular domain spaces the existence of a minimizer of the nonlinear energy in a given homotopy class is proven.

Chapter 1

Spiders

In this chapter, we analyze harmonic maps $v : M \to N$ from a measure space (M, m) with a local regular Dirichlet form $\mathcal{E}$ on it into a spider (N, d). Let A be the generator of $\mathcal{E}$ and let the semigroup e^{At} be given by a semigroup of Markov kernels p_t. We define a canonical extension $\mathcal{E}_N$ of the energy $\mathcal{E}$ for maps $v : M \to N$ by

$$\mathcal{E}_N(v) := \limsup_{t \to 0} \frac{1}{2t} \int_M \int_M d^2(v(x), v(y)) p_t(x, dy) m(dx).$$

One of the main issues is the following "energy decomposition"

$$\sum \mathcal{E}(v_i) = \mathcal{E}_N(v),$$

whereby $v_i : M \to \mathbb{R}$ is the projection of v on the i-th edge of the spider N.
Defining the nonlinear Dirichlet problem as a minimizing problem of the nonlinear energy we present conditions for the existence and uniqueness of a solution to the nonlinear Dirichlet problem.
Another important point is the development of a numerical algorithm to solve the nonlinear Dirichlet problem for maps from a two dimensional Euclidean domain into a spider. For this we discretize the problem using a suitable finite element approach and an iterative numerical method to solve the discrete problem is constructed. Furthermore, we define a prolongation operator which extends the discrete maps to maps on the whole domain and we prove the L^2-convergence of the extended discrete solutions to the solution to the nonlinear Dirichlet problem using finite element projection techniques.

Throughout this chapter, we fix a σ-finite measure space (M, m) and a regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(M, m)$. Moreover, we assume

(A1) $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is local, that is, $v, w \in \mathcal{D}(\mathcal{E})$, $\operatorname{supp}[v]$ and $\operatorname{supp}[w]$ are compact, $v \equiv 0$ on a neighbourhood of $\operatorname{supp}[w] \quad \Rightarrow \quad \mathcal{E}(v, w) = 0$.

(A2) The semigroup $(T_t)_{t \geq 0}$ corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is given by a semigroup of Markov kernels $p_t(x, dy)$.

Remark 1.1

(i) Assumption (A2) is always fulfilled if M is a locally compact separable metric space, and the regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is conservative. In particular, this assumption is fulfilled for $M = \mathbb{R}^k$ with m being the Lebesgue measure λ on $\mathbb{R}^k$, and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ being the classical Dirichlet form, i.e. $\mathcal{E}(u) = \int_{\mathbb{R}^k} |\nabla u|^2 d\lambda$.

(ii) The assumptions (A1) and (A2) yield that for functions $v, w \in \mathcal{D}(\mathcal{E})$ with $v \cdot w = 0$ a.e. it holds $\mathcal{E}(v, w) = 0$ (cf. Appendix A.1).

Throughout this chapter, fix $n \in \mathbb{N}$ and denote the set $\{1, \ldots, n\}$ by I. We define the n-spider as the metric space (N, d) where

$$N := \{(i, t) : i \in I, t \in \mathbb{R}_+\} / \sim$$

with $(i, 0) \sim (j, 0)$ for every $i, j \in I$. A distance d is defined on N by

$$d((i, s), (j, t)) = \begin{cases} |s - t|, & \text{if } i = j \\ s + t, & \text{otherwise.} \end{cases}$$

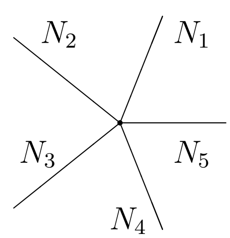

Figure 1.1: The 5-spider

Additionally, we consider the following functions defined on N by

$$c: \ N \to I \cup \{0\}, \qquad (i, t) \ \mapsto \ \begin{cases} i, & \text{if } t \neq 0 \\ 0, & \text{otherwise,} \end{cases}$$

$$\pi: \ N \to \mathbb{R}_+, \qquad (i, t) \ \mapsto \ t$$

and

$$\pi_j: \ N \to \mathbb{R}_+, \qquad (i, t) \ \mapsto \ \delta_{ij} \cdot t.$$

In the sequel, we use the decomposition $\bigcup_{i \in I \cup \{0\}} N_i$ of N, with $N_0 := o := \{(1, 0)\}$ and $N_i := \{(i, t) : t \in \mathbb{R}_+\}, i \in I$. In this way, to each measurable map $v : M \to N$ one may associate a family of functions $v_i : M \to \mathbb{R}, i \in I$, defined by

$$v_i := \pi_i \circ v.$$

The number $\pi(x)$ plays the role of the modulus of x and $c(x)$ is a generalization of $\mathrm{sgn}(x)$ and interpreted as colour of x.

Remark 1.2 *If $n = 2$ then N, N_1 and N_2 can be identified with $\mathbb{R}, \mathbb{R}_+$ and $\mathbb{R}_-$, resp. Then the functions $c(x), \pi(x), \pi_1(x), \pi_2(x)$ coincide with $\operatorname{sgn}(x), |x|, x_+, x_-$, resp. and $v_1(x), v_2(x)$ coincide with $v_+(x), v_-(x)$.*

1.1 Nonlinear Energy

In this section, we define the nonlinear energy for maps with values in an n-spider using the semigroup p_t.

Given a measurable map $v : M \to N$ we define the nonlinear energy function $\mathcal{E}_N$ by

$$\mathcal{E}_N(v) := \limsup_{t\to 0} \frac{1}{2t} \int_M \int_M d^2(v(x), v(y)) p_t(x, dy) m(dx) \tag{1.1}$$

with $\mathcal{D}(\mathcal{E}_N) := \{v : M \to N \text{ measurable: } \mathcal{E}_N(v) < \infty \text{ and } v_i \in L^2(M, m),\ \forall i \in I\}$.

Theorem 1.3 *For each map $v : M \to N$ the condition $v \in \mathcal{D}(\mathcal{E}_N)$ is equivalent to*

$$v_i \in \mathcal{D}(\mathcal{E}),\ \forall i \in I \quad \textit{and} \quad \sum_{i\in I} \mathcal{E}(v_i) < \infty.$$

In this situation, for each $v \in \mathcal{D}(\mathcal{E}_N)$ the following equalities hold

$$\mathcal{E}_N(v) = \lim_{t\to 0} \frac{1}{2t} \int_M \int_M d^2(v(x), v(y)) p_t(x, dy) m(dx) \tag{1.2}$$

$$= \sum_{i\in I} \mathcal{E}(v_i) \tag{1.3}$$

with

$$\mathcal{E}(v_i) = \lim_{t\to 0} \frac{1}{2t} \int_M \int_M |v_i(x) - v_i(y)|^2 p_t(x, dy) m(dx).$$

For a detailed proof see Section 1.4.

Corollary 1.4 *On $\mathbb{R}^k$ with the Lebesgue measure λ, let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the classical Dirichlet form. For all $v \in \mathcal{D}(\mathcal{E}_N)$ one has*

$$\mathcal{E}_N(v) = \sum_{i\in I} \int_{\mathbb{R}^k} |\nabla v_i|^2 d\lambda. \tag{1.4}$$

In the next proposition, we will show that our notion of nonlinear energy coincides with the notion of nonlinear energy introduced by Korevaar/Schoen.

Proposition 1.5 *In the situation of Corollary 1.4, our definition of the nonlinear energy $\mathcal{E}_N$ coincides with the definition of energy introduced in [KS93]. That is, for all measurable $v : \mathbb{R}^k \to N$ one has*

$$\mathcal{E}_N(v) = \lim_{r\to 0} \frac{c_k}{r^{k+1}} \int_{\mathbb{R}^k} \int_{\partial B_r(x)} d^2(v(x), v(y))\sigma_{r,x}(dy)\lambda(dx)$$

where

$$c_k = \frac{k}{4\pi^{k/2}} \cdot \Gamma(k/2) = \frac{k}{4\pi^{k/2}} \int_0^\infty x^{k/2-1} \exp(-x)dx$$

and $\sigma_{r,x}$ denotes the surface measure on the sphere $\partial B_r(x)$.

Proof: Let us define for $t > 0$ and measurable maps $v : \mathbb{R}^k \to N$

$$\mathcal{E}_N^t(v) := \frac{1}{2t} \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} d^2(v(x), v(y))p_t(x, dy)\lambda(dx).$$

Using the definitions and notations of [KS93] it holds

$$\begin{aligned}
\mathcal{E}_N^t(v) &= \frac{1}{2t} \int_{\mathbb{R}^k} \int_{\mathbb{R}^k} d^2(v(x), v(y))(2\pi t)^{-k/2} \exp\left(-\frac{|x-y|^2}{2t}\right) dydx \\
&= \int_{\mathbb{R}^k} \left[\int_0^\infty \frac{1}{2t \cdot (2\pi t)^{k/2}} \exp\left(-\frac{r^2}{2t}\right) \cdot \underbrace{\left(\int_{\partial B_r(x)} d^2(v(x), v(y))\sigma_{r,x}(dy)\right)}_{r^{k+1}e_r(x)} dr \right] dx \\
&= \int_{\mathbb{R}^k} \left[\int_0^\infty \left(\frac{r^2}{2t}\right)^{\frac{k}{2}+1} \cdot \frac{1}{\pi^{k/2}} \exp\left(-\frac{r^2}{2t}\right) \cdot e_r(x) \cdot \frac{1}{r}\, dr \right] dx \\
&= \int_{\mathbb{R}^k} \left[\int_0^\infty \left(r^2\right)^{\frac{k}{2}+1} \cdot \frac{1}{\pi^{k/2}} \exp\left(-r^2\right) \cdot e_{\sqrt{2t}\cdot r}(x) \cdot \frac{1}{r}\, dr \right] dx \\
&= c_k \int_{\mathbb{R}^k} \left[\int_0^\infty e_{\sqrt{2t}\cdot r}(x)\nu(dr) \right] dx
\end{aligned}$$

with

$$\nu(dr) := \frac{1}{c_k} \cdot r^{k+1} \frac{1}{\pi^{k/2}} \exp\left(-r^2\right) dr$$

and

$$c_k = \int_0^\infty r^{k+1} \frac{1}{\pi^{k/2}} \exp\left(-r^2\right) dr.$$

The measure ν is a probability measure on $\mathbb{R}_+$. Furthermore, using substitution and partial integration one can show

$$c_k = \frac{k}{4\pi^{k/2}} \cdot \Gamma(k/2).$$

Now, we define for a sequence $(\sigma_n)_n \searrow 0$ the probability measures

$$\nu_n(dr) := \frac{1}{c_{k,n}} \cdot \left(\left(\frac{r}{\sigma_n}\right)^{k+1} \frac{1}{\pi^{k/2}} \exp\left(-\frac{r^2}{\sigma_n^2}\right) - \left(\frac{2}{\sigma_n}\right)^{k+1} \frac{1}{\pi^{k/2}} \exp\left(-\frac{4}{\sigma_n^2}\right)\right)^+ \frac{1}{\sigma_n}\, dr$$

with

$$c_{k,n} := \int_0^\infty \left(\left(\frac{r}{\sigma_n}\right)^{k+1} \frac{1}{\pi^{k/2}} \exp\left(-\frac{r^2}{\sigma_n^2}\right) - \left(\frac{2}{\sigma_n}\right)^{k+1} \frac{1}{\pi^{k/2}} \exp\left(-\frac{4}{\sigma_n^2}\right)\right)^+ \frac{1}{\sigma_n}\, dr.$$

One can assure

$$\operatorname{supp}[\nu_n] \subset [0,2]$$

by choosing σ_1 sufficiently small. Moreover, by monotone convergence it follows

$$\int_{\mathbb{R}^k} \left[\int_0^\infty e_{\frac{\sqrt{2t}}{\sigma_n}\cdot r}(x)\nu_n(dr)\right] \stackrel{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^k} \left[\int_0^\infty e_{\sqrt{2t}\cdot r}(x)\nu(dr)\right]. \tag{1.5}$$

In addition, by Theorem 1.5.1 in [KS93] the limit

$$\lim_{t\to 0} \int_{\mathbb{R}^k} \left[\int_0^\infty e_{\frac{\sqrt{2t}}{\sigma_n}\cdot r}(x)\nu_n(dr)\right]$$

exists for all n and coincides with

$$\lim_{r\to 0} \frac{1}{r^{k+1}} \int_{\mathbb{R}^k} \int_{\partial B_r(x)} d^2(v(x), v(y))\sigma_{r,x}(dy)\lambda(dx).$$

Hence, (1.5) yields the claim. □

1.2 Nonlinear Dirichlet Problem

The nonlinear Dirichlet problem for a given map g with $\mathcal{E}_N(g) < \infty$ and a subset $D \subset M$ is to find a map u with $u = g$ on $M\backslash D$ which minimizes the nonlinear energy $\mathcal{E}_N$.

Definition 1.6 (Nonlinear Dirichlet problem) *Given a map $g \in \mathcal{D}(\mathcal{E}_N)$ and a set $D \subset M$, let us define the class of maps*

$$V_N(g) := \{v \in \mathcal{D}(\mathcal{E}_N) : v = g \ m\text{-a.e. on } M\backslash D\}.$$

A map $u \in V_N(g)$ is called a solution to the nonlinear Dirichlet problem for g whenever

$$\mathcal{E}_N(u) = \min_{v\in V_N(g)} \mathcal{E}_N(v).$$

Remark: A refined definition of the Dirichlet problem would require to replace the class $V_N(g)$ by $\tilde{V}_N(g) := \{v \in \mathcal{D}(\mathcal{E}_N) : \tilde{v} = \tilde{g}$ quasi everywhere on $M\backslash D\}$ where $\tilde{v}, \tilde{g}$ denote quasi-continuous versions of v and g, resp. However, in the next sections in our application both classes coincide since D always will have a "nice" boundary.

The next result states a sufficient condition for the existence (and uniqueness) of a solution to the nonlinear Dirichlet problem in terms of the so-called linear spectral bound λ_D of an open set $D \subset M$, that is,

$$\lambda_D := \inf \left\{ \mathcal{E}(v) : v \in L_0^2(D), \int_M v^2 dm = 1 \right\} \tag{1.6}$$

where $L_0^2(D) := \{v \in L^2(M) : v = 0$ m-a.e. on $M\backslash D\}$ and $\mathcal{E}(v) := +\infty$ if $v \notin \mathcal{D}(\mathcal{E})$.

Theorem 1.7 *Given an open set $D \subset M$ such that $\lambda_D > 0$, there exists a unique solution to the nonlinear Dirichlet problem for any $g \in \mathcal{D}(\mathcal{E}_N)$.*

Proof: Let $L^2(M, \mathcal{M}, m)$ denote the space of all square integrable functions $v : M \to \bar{\mathbb{R}}$ with the usual Hilbertian norm $|| \cdot ||_{L^2}$. For $D \in \mathcal{M}$ we put $L_0^2(D) := \{v \in L^2(M) : v = 0$ m-a.e. on $M\backslash D\}$ regarding as a subspace of $L^2(M)$.
For measurable maps $v, \tilde{v} : M \to N$ we define the (pseudo) distance $d_2(v, \tilde{v}) := ||d(v, \tilde{v})||_{L^2}$, where $d(v, \tilde{v})(x) := d(v(x), \tilde{v}(x))$, and for a fixed measurable map $g : M \to N$ the space of maps $L^2(D, N, g)$ by

$$L^2(D, N, g) := \{f : M \to N \text{ measurable} : d(v, g) \in L_0^2(D)\}.$$

It holds $V_N(g) \subset L^2(D, N, g)$. For all $v \in L^2(D, N, g)\backslash V_N(g)$ we put $\mathcal{E}_N(v) := \infty$.

The metric space (N, d) has nonpositive curvature in the sense of A. D. Alexandrov, that is, for any two points $\gamma_0, \gamma_1 \in N$ and any $t \in [0, 1]$ there exists a point $\gamma_t \in N$ such that for all $z \in N$

$$d^2(z, \gamma_t) \leq (1-t)d^2(z, \gamma_0) + td^2(z, \gamma_1) - (1-t)td^2(\gamma_0, \gamma_1).$$

For any two geodesics $\gamma, \varphi : [0, 1] \mapsto N$ and any $t \in [0, 1]$, the previous inequality leads to

$$d^2(\gamma_t, \varphi_t) \leq (1-t)d^2(\gamma_0, \varphi_0) + td^2(\gamma_1, \varphi_1) - t(1-t)[d(\gamma_0, \gamma_1) - d(\varphi_0, \varphi_1)]^2 \tag{1.7}$$

(cf. Korevaar/Schoen [KS93], Jost [Jos94]).
The set of maps $V_N(g)$ is convex, whereby the geodesic v_t connecting two maps $v_0, v_1 \in V_N(g)$ is defined pointwise as follows: for each $x \in M$, $t \mapsto v_t(x)$ is the (unique) geodesic (parameterized by arc length) connecting $v_0(x), v_1(x) \in N$.
To prove the existence of a unique minimizer u of the energy $\mathcal{E}_N$ on $V_N(g)$, first we show that $\mathcal{E}_N$ is lower semicontinuous on $L^2(D, N, g)$ and strictly convex on $V_N(g)$.

Given $v_0, v_1 \in V_N(g)$ let v_t be the geodesic connecting v_0 and v_1. Inequality (1.7) with $\gamma_t = v_t(x)$ and $\varphi_t = v_t(y)$ yields

$$\begin{aligned} d^2(v_t(x), v_t(y)) \quad \leq \quad & (1-t)d^2(v_0(x), v_0(y)) + td^2(v_1(x), v_1(y)) \\ & -t(1-t)[d(v_0(x), v_1(x)) - d(v_0(y), v_1(y))]^2. \end{aligned}$$

Integrating both sides w.r.t. $p_s(x, dy)m(dx)$ gives

$$\mathcal{E}_N^s(v_t) \leq (1-t)\mathcal{E}_N^s(v_0) + t\mathcal{E}_N^s(v_1) - (1-t)t\mathcal{E}^s(d(v_0, v_1)), \tag{1.8}$$

whereby for each $s > 0$

$$\mathcal{E}_N^s(v) := \frac{1}{2s}\int_M \int_M d^2(v(x), v(y))p_s(x, dy)m(dx)$$

and

$$\mathcal{E}^s(f) := \frac{1}{2s}\int_M \int_M |f(x) - f(y)|^2 p_s(x, dy)m(dx).$$

Furthermore, $v, \tilde{v} \in V_N(g)$ implies $d(v, \tilde{v}) \in \mathcal{D}(\mathcal{E})$. Indeed,

$$\mathcal{E}(d(v, \tilde{v})) \leq 2\mathcal{E}_N(v) + 2\mathcal{E}_N(\tilde{v})$$

since

$$|d(v(x), \tilde{v}(x)) - d(v(y), \tilde{v}(y))| \leq d(v(x), v(y)) + d(\tilde{v}(x), \tilde{v}(y)).$$

Taking $\limsup_{s\to 0}$ in (1.8) yields

$$\mathcal{E}_N(v_t) \leq (1-t)\mathcal{E}_N(v_0) + t\mathcal{E}_N(v_1) - (1-t)t\mathcal{E}(d(v_0, v_1)), \tag{1.9}$$

because $\mathcal{E}(d(v_0, v_1)) = \lim_{s\to 0} \mathcal{E}^s(d(v_0, v_1))$.
On the other hand, by spectral theory, one has

$$\mathcal{E}(d(v, \tilde{v})) \geq \lambda \cdot \int_M d^2(v(x), \tilde{v}(x))m(dx)$$

where $\lambda := \lambda_D > 0$ by assumption. Thus inequality (1.9) implies

$$\mathcal{E}_N(v_t) \leq (1-t)\mathcal{E}_N(v_0) + t\mathcal{E}_N(v_1) - (1-t)t\lambda \cdot d_2^2(v, \tilde{v}) \tag{1.10}$$

showing that $\mathcal{E}_N$ is strictly convex on $V_N(g)$. To prove that $\mathcal{E}_N$ is lower semicontinuous, let us define for all $v \in L^2(D, N, g)$ and $t > 0$

$$\tilde{\mathcal{E}}_N^t(v) := \frac{1}{2t}\int_M \int_M \sum_{i\in I} |v_i(x) - v_i(y)|^2 p_t(x, dy)m(dx).$$

For each fixed $t > 0$, $\tilde{\mathcal{E}}^t_N : L^2(D, N, g) \to \mathbb{R}_+$ is continuous. Indeed, by the triangle inequality, for every $v, \tilde{v} \in L^2(D, N, g)$ and $\delta > 0$ we have

$$\begin{aligned}\tilde{\mathcal{E}}^t_N(v) &\leq \frac{1}{2t}\sum_{i\in I}\int_M\int_M |v_i(x) - \tilde{v}_i(x) + \tilde{v}_i(x) - \tilde{v}_i(y) + \tilde{v}_i(y) - v_i(y)|^2 p_t(x, dy)m(dx) \\ &\leq (1+\delta)\tilde{\mathcal{E}}^t_N(\tilde{v}) + \frac{2}{t}(1+\frac{1}{\delta})\sum_{i\in I}\int_M |v_i(x) - \tilde{v}_i(x)|^2 m(dx) \\ &\leq (1+\delta)\tilde{\mathcal{E}}^t_N(\tilde{v}) + \frac{6}{t}(1+\frac{1}{\delta})d_2^2(v, \tilde{v}).\end{aligned}$$

Furthermore, $\tilde{\mathcal{E}}^t_N(v)$ is non-decreasing as t decreases (see e.g. [FOT94]).
Hence,

$$\tilde{\mathcal{E}}_N := \lim_{t\to 0}\tilde{\mathcal{E}}^t_N$$

is lower semicontinuous on $L^2(D, N, g)$ and due to Theorem 1.3, $\tilde{\mathcal{E}}_N$ coincides with $\mathcal{E}_N$ on $L^2(D, N, g)$.

Now let $(v_n)_n$ be a sequence in $V_N(g)$ with $\lim_{n\to\infty}\mathcal{E}_N(v_n) = \inf_{v\in V_N(g)}\mathcal{E}_N(v) =: \alpha$. Then for $n, k \to \infty$ (see (1.10))

$$\alpha \leq \frac{1}{2}\underbrace{\mathcal{E}_N(v_k)}_{\to\alpha} + \frac{1}{2}\underbrace{\mathcal{E}_N(v_n)}_{\to\alpha} - \frac{1}{4}\lambda d_2^2(v_n, v_k).$$

Consequently, $d_2^2(v_n, v_k) \to 0$ for $n, k \to \infty$, i.e., $(v_n)_n$ is a Cauchy sequence in $L^2(D, N, g)$. Therefore, there exists $u = \lim_{n\to\infty} v_n \in L^2(D, N, g)$. Moreover, $\liminf_{n\to\infty}\mathcal{E}_N(v_n) \geq \mathcal{E}_N(u)$ by the lower semicontinuity of $\mathcal{E}_N$ on $L^2(D, N, g)$.
Hence, $u \in V_N(g)$ and u is the minimizer of $\mathcal{E}_N$ on $V_N(g)$.
Uniqueness: Assume that $\mathcal{E}_N(u_0) = \mathcal{E}_N(u_1) = \inf_{v\in V_N(g)}\mathcal{E}_N(v) = \alpha$. Inequality (1.10) yields

$$\alpha \leq \mathcal{E}_N(u_{1/2}) \leq \frac{1}{2}\alpha + \frac{1}{2}\alpha - \frac{1}{4}\lambda d_2^2(u_0, u_1)$$

implying $d_2^2(u_0, u_1) = 0$. □

1.3 Nonlinear Dirichlet Problem for Polygonal Domains in $\mathbb{R}^2$

In the special case $(M, m) = (\mathbb{R}^2, \lambda)$ with the corresponding classical Dirichlet form $\mathcal{E}$ and $D \subset \mathbb{R}^2$ being a polygonal set we will define a numerical algorithm to solve the nonlinear Dirichlet problem.
For this, we fix suitable triangulations $\mathcal{T}_h$ of D and define a discrete nonlinear energy $\mathcal{E}^h_N$ for maps $\bar{v}_h : \mathcal{N}_h \to N$, whereby $\mathcal{N}_h$ denotes the set of vertices of the triangulation $\mathcal{T}_h$. This

yields a discrete nonlinear Dirichlet problem, i.e., for a map $g : \mathbb{R}^2 \to N$ with $\mathcal{E}_N(g) < \infty$ one searches a map $\bar{u}_h : \mathcal{N}_h \to N$ with $\bar{u}_h = g$ on $\partial D \cap \mathcal{N}_h$ minimizing the discrete nonlinear energy $\mathcal{E}_N^h$. We construct an iterative numerical method to solve this problem. Furthermore, we define a prolongation operator J_h which extends maps defined on the vertices to maps defined on the whole domain D in such a way that

$$\mathcal{E}_N(J_h(\bar{u}_h)) \leq \mathcal{E}_N^h(\bar{u}_h) + R_{g,D} \to \mathcal{E}_N(u) \qquad h \to 0,$$

with a nonnegative constant $R_{g,D}$ only depending on the polygonal domain D, the regularity of the triangulation $\mathcal{T}_h$, and the map g. From this, the L^2-convergence of $J_h(\bar{u}_h)$ to the solution u of the nonlinear Dirichlet problem follows as a straightforward consequence.

1.3.1 Discrete Nonlinear Dirichlet Problem

In the sequel, let us suppose that an admissible and regular triangulation $\mathcal{T}_h$ of the polygonal D in the sense of [Cia78] is given. In addition, we suppose the triangles to be "acute". This, means that all interior angles of all triangles of $\mathcal{T}_h$ are less than or equal to $\frac{\pi}{2}$. Finally, we assume that for the map $g \in \mathcal{D}(\mathcal{E}_N)$, specifying the boundary values for the nonlinear Dirichlet problem, $\pi \circ g$ is the modulus of a linear function on the boundary faces of $\mathcal{T}_h$. For this situation we define, a discrete nonlinear Dirichlet problem which unique solution is used to approximate the solution of the "continuous" nonlinear Dirichlet problem.

However, before we start to discuss the nonlinear case, we will have a closer look on the linear case.

In the sequel, $\mathcal{N}_h = \{x_1, \ldots, x_l\}$ denotes the set of all vertices of the triangulation $\mathcal{T}_h$. We divide $\mathcal{N}_h$ into two disjoint sets

$$\mathring{\mathcal{N}}_h := \mathcal{N}_h \backslash \partial D \quad \text{and} \quad \mathcal{N}_h^{\partial} := \mathcal{N}_h \cap \partial D.$$

Definition 1.8 *We denote by V^h the standard space of piecewise affine finite elements on $\mathcal{T}_h$ and by $\{\phi_h^i, 1 \leq i \leq l\}$ the corresponding nodal basis of V^h, see [Cia78]. Furthermore, we define a Markov kernel p on $\mathcal{N}_h$ by*

$$\forall x_i, x_j \in \mathcal{N}_h : \qquad p(x_i, x_j) := \begin{cases} -\frac{(\nabla\phi_h^i, \nabla\phi_h^j)}{(\nabla\phi_h^i, \nabla\phi_h^i)}, & \text{if } x_i \sim x_j, \\ 0, & \text{otherwise}, \end{cases}$$

where $x_i \sim x_j$ means that there is an edge connecting x_i and x_j and we define a measure μ on $\mathcal{N}_h$ by

$$\forall x_i \in \mathcal{N}_h : \qquad \mu(x_i) := (\nabla\phi_h^i, \nabla\phi_h^i).$$

Remark: Due to the assumptions on the triangulations $\mathcal{T}_h$ one has $(\nabla\phi_h^i, \nabla\phi_h^j) \le 0$ (cf. [Tho97]). Furthermore, it holds $\sum_{j=1}^l \phi_h^j = 1$ and for $i \in \{1, \dots, l\}$ one has

$$0 = (\nabla 1, \nabla\phi_h^i) = \sum_{j=1}^{l} (\nabla\phi_h^j, \nabla\phi_h^i)$$

which yields

$$1 = \sum_{\substack{j=1\\ j\neq i}}^{l} -\frac{(\nabla\phi_h^j, \nabla\phi_h^i)}{(\nabla\phi_h^i, \nabla\phi_h^i)} = \sum_{x_j \in \mathcal{N}_h} p(x_i, x_j).$$

Lemma 1.9 *Given a function $v_h \in V^h$, for all $1 \le i \le l$ define $v_h^i := v_h(x_i)$. Then*

$$\int_D |\nabla v_h|^2 \, d\lambda = \frac{1}{2} \sum_{i=1}^{l} \sum_{j=1}^{l} (v_h^i - v_h^j)^2 p(x_i, x_j) \mu(x_i) \tag{1.11}$$

and, moreover,

$$\int_D |\nabla v_h|^2 d\lambda = -\sum_{T \in \mathcal{T}_h} \sum_{\substack{i,j=0\\ i<j}}^{2} (v_h(x_i^T) - v_h(x_j^T))^2 \int_T \nabla\phi_h^{i,T} \nabla\phi_h^{j,T} \, d\lambda, \tag{1.12}$$

whereby $x_0^T, x_1^T, x_2^T \in \mathcal{N}_h$ denote the vertices of a triangle $T \in \mathcal{T}_h$ and $\phi_h^{i,T}$ denote the corresponding elements of the standard basis.

The difference between formulas (1.11) and (1.12) is that in (1.11) we sum over all vertices of the triangulation and in (1.12) we sum over all triangles.

Proof: The identity $v_h(x) = \sum_{i=1}^l v_h^i \phi_h^i(x)$ leads to

$$\begin{aligned}
&\frac{1}{2} \sum_{i=1}^{l} \sum_{j=1}^{l} (v_h^i - v_h^j)^2 p(x_i, x_j) \mu(x_i) \\
=\ & \frac{1}{2} \left(2 \sum_{i=1}^{l} (v_h^i)^2 \sum_{\substack{j=1\\ j\neq i}}^{l} [-(\nabla\phi_h^i, \nabla\phi_h^j)] + 2 \sum_{i=1}^{l} \sum_{\substack{j=1\\ j\neq i}}^{l} v_h^i v_h^j (\nabla\phi_h^i, \nabla\phi_h^j) \right) \\
=\ & \sum_{i=1}^{l} \sum_{j=1}^{l} v_h^i v_h^j (\nabla\phi_h^i, \nabla\phi_h^j) \\
=\ & \int_D |\nabla v_h|^2 d\lambda.
\end{aligned}$$

A similar procedure shows equation (1.12). $\square$

Now, we are going to extend our frame from functions $v : M \to \mathbb{R}$ to maps $v : M \to N$ where N is the n-spider.

Definition 1.10 (Discrete nonlinear Dirichlet problem) *Given a map $g : \partial D \to N$, let us define*

$$\bar{V}_N^h(g) := \left\{\bar{v}_h : \mathcal{N}_h \to N : \bar{v}_h(x) = \bar{g}_h(x) \quad \forall x \in \mathcal{N}_h^\partial\right\}$$

with $\bar{g}_h(x) := g(x), \forall x \in \mathcal{N}_h^\partial$. A map $\bar{u}_h : \mathcal{N}_h \to N$ is called a solution to the discrete nonlinear Dirichlet problem for g whenever $\bar{u}_h$ fulfills the following two conditions:

1. $\bar{u}_h \in \bar{V}_N^h(g)$
2. $\mathcal{E}_N^h(\bar{u}_h) = \min_{\bar{v}_h \in \bar{V}_N^h(g)} \mathcal{E}_N^h(\bar{v}_h)$, *where*

$$\mathcal{E}_N^h(\bar{v}_h) := \frac{1}{2} \sum_{x_i, x_j \in \mathcal{N}_h} d^2(\bar{v}_h(x_i), \bar{v}_h(x_j)) p(x_i, x_j) \mu(x_i) \tag{1.13}$$

is called the discrete energy corresponding to $\mathcal{T}_h$.

According to [Stu01] we have the following result.

Proposition 1.11 *For each $g : \partial D \to N$ there is a unique solution to the discrete nonlinear Dirichlet problem for g.*

Given the Markov operator p from Definition 1.8 we define another Markov operator $p_{\mathcal{N}_h}$ on $\mathcal{N}_h$ by

$$p_{\mathcal{N}_h}(x, y) \quad := \quad \mathbb{1}_{\mathring{\mathcal{N}}_h}(x) p(x, y) + \mathbb{1}_{\mathcal{N}_h^\partial}(x) \delta_{\{x\}}(y),$$

where $\mathbb{1}.$ denotes the indicator function of a set and $\delta_{\{x\}}$ is the Dirac measure with mass at x.

In the sequel, for a given Markov operator q on $\mathcal{N}_h$, we denote by q^N the associated nonlinear Markov operator acting on each map $\bar{v} : \mathcal{N}_h \to N$ by

$$q^N \bar{v}(x) = \operatorname*{argmin}_{z \in N} \sum_{y \in \mathcal{N}_h} d^2(z, \bar{v}(y)) q(x, y),$$

see [Stu01]. In other words, if $(X_n, \mathbb{P}_x)$ is a random walk with transition probability q then

$$q^N \bar{v}(x) = \operatorname*{argmin}_{z \in N} \mathbb{E}_x d^2(z, \bar{v}(X_1)).$$

Proposition 1.12 *For each $\bar{v}_h \in \bar{V}^h_N(g)$ the following two conditions are equivalent:*

1. $p^N_{\mathcal{N}_h} \bar{v}_h = \bar{v}_h$
2. *$\bar{v}_h$ is a solution to the discrete nonlinear Dirichlet problem for g.*

The proof follows closely the arguments used in [Stu01].

Remark:

1. In the linear case (i.e. $N = \mathbb{R}$), the matrix A with components $A_{ij} = \mu(x_i)(\delta_{ij} - p(x_i, x_j))$ is the well–known stiffness matrix and $\bar{u}_h$ solves a corresponding linear system of equations. Furthermore, the matrix Q with entries $Q_{ij} = p(x_i, x_j)$ is the iteration matrix of the Jacobi algorithm. Thus, the algorithm itself coincides with the corresponding Markov process (see below).
2. If $\bar{v}_h : \mathcal{N}_h \to N$ is a map such that $\bar{v}_h = p^N_{\mathcal{N}_h} \bar{v}_h$, then on $\mathring{\mathcal{N}}_h$ the map $\bar{v}_h$ is given by
$$\bar{v}_h(x) = \operatorname*{argmin}_{z \in N} \left\{ \sum_{y \in \mathcal{N}_h} d^2(z, \bar{v}_h(y)) p(x, y) \right\}, \qquad x \in \mathring{\mathcal{N}}_h.$$

To solve the discrete nonlinear Dirichlet problem, we construct a nonlinear Markov operator Q in such a way that for each $\bar{v}_h \in \bar{V}^h_N(g)$ one has
$$\lim_{n \to \infty} Q^n \bar{v}_h = \bar{u}_h.$$

In order to define this nonlinear Markov operator Q, let us first define the following Markov operators $p_1, \ldots, p_k, k := \#\mathring{\mathcal{N}}_h$, and q:
$$p_i(x, y) \;\; := \;\; \begin{cases} p(x, y), & \text{if } x = x_i \text{ and } x \sim y \\ 1, & \text{if } x \neq x_i \text{ and } x = y \\ 0, & \text{otherwise} \end{cases} \qquad i = 1, \ldots, k$$
$$q(x, y) \;\; := \;\; p_k \circ \cdots \circ p_1(x, y).$$

Lemma 1.13 *There exists an exponent $r \in \mathbb{N}$ such that*
$$||q^r||_{\infty,\infty} := \sup \left\{ ||q^r v||_\infty : ||v||_\infty = 1, v = 0 \text{ on } \mathcal{N}^\partial_h \right\} < 1.$$

Proof: At first, consider $v(x_i) = v^+(x_i) = 1$ for every interior nodes x_i. In each step at least one nodal value of an interior node decreases. Indeed, this is due to the averaging effect of the application of $p_i(\cdot, \cdot)$ over neighbouring nodes. But there is only a finite number of nodes. Hence, there exists a number of iterations $r \leq k$ after which the initial value 1 on every node has been decreased. Furthermore, we observe that $v \leq v^+$ implies $q^r v \leq q^r v^+$. Hence, we are done. □

Remark: Based on an ordering of the nodes $x \in \overset{\circ}{\mathcal{N}}_h$ with increasing graph distance from the boundary nodes on the edge graph of the triangulation we can achieve $r = 1$ in Lemma 1.13.

Definition 1.14 *To each $i = 1, \ldots, k$ let p_i^N be the nonlinear Markov operator associated to p_i. We define the nonlinear Markov operator Q by*

$$Q := p_k^N \circ \cdots \circ p_1^N.$$

Proposition 1.15 *For each map $\bar{v}_h \in \bar{V}_N^h(g)$ such that $\bar{v}_h = Q\bar{v}_h$, one has*

$$\bar{v}_h(x) = \operatorname*{argmin}_{z \in N} \left\{ \sum_{y \in \mathcal{N}_h} d^2(z, \bar{v}_h(y)) p(x,y) \right\}, \qquad \forall x \in \overset{\circ}{\mathcal{N}}_h.$$

Proof: By construction of each p_i, it follows that

$$p_1^N \bar{v}_h(x) = \begin{cases} \operatorname{argmin}_{z \in N} \{\sum_{y \in \mathcal{N}_h} d^2(z, \bar{v}_h(y)) p(x,y)\}, & \text{if } x = x_1 \\ \bar{v}_h(x), & \text{if } x \neq x_1. \end{cases}$$

and

$$p_i^N \bar{v}_h(x_1) = \bar{v}_h(x_1) \qquad i = 2, \ldots, k$$

for all $\bar{v}_h : \mathcal{N}_h \to N$. The equation $Q\bar{v}_h = \bar{v}_h$ leads to

$$p_1^N \bar{v}_h(x_1) = \bar{v}_h(x_1)$$

and the assertion follows for $x_1 \in \overset{\circ}{\mathcal{N}}_h$. For $x_i \in \overset{\circ}{\mathcal{N}}_h, i > 1$, the proof is analogue. □

Proposition 1.16 *Let $\bar{u}_h$ be the solution to the discrete nonlinear Dirichlet problem for g. Then for each $\bar{v}_h \in \bar{V}_N^h(g)$ one has*

$$\lim_{n \to \infty} d_\infty(Q^n \bar{v}_h, \bar{u}_h) = 0, \quad \textit{where } d_\infty(\bar{v}_h, \bar{w}_h) := \sup_{x \in \mathcal{N}_h} d(\bar{v}_h(x), \bar{w}_h(x)).$$

Proof: According to Theorem 5.2 in [Stu01] and Lemma 1.13

$$d_\infty(Q^r \bar{v}_h, Q^r \bar{w}_h) \leq ||q^r(d(\bar{v}_h, \bar{w}_h))||_\infty \leq ||q^r||_{\infty,\infty} \cdot d_\infty(\bar{v}_h, \bar{w}_h)$$

for all $\bar{v}_h, \bar{w}_h \in \bar{V}_N^h(g)$. Hence, there exists a map $\bar{w}_h \in \bar{V}_N^h(g)$ such that $\bar{w}_h = Q\bar{w}_h$ and for all $\bar{v}_h \in \bar{V}_N^h(g)$ it holds

$$d_\infty(Q^n \bar{v}_h, \bar{w}_h) \to 0 \qquad n \to \infty$$

(cf. proof of Theorem 6.4 in [Stu01]). Therefore, by Propositions 1.11, 1.12, and 1.15, one obtains $\bar{w}_h = \bar{u}_h$. □

Remark:

1. The previous construction combined with Proposition 1.16 yields the following algorithm:

   $\bar{v}_h = g|_{\mathcal{N}_h}$
   do
   $\quad \bar{w}_h = \bar{v}_h$
   $\quad$ for $j = 1$ to k
   $\qquad \bar{v}_h(x_j) = p_j^N \bar{v}_h(x_j) = \text{argmin}_{z\in N}\{\sum_{y\in\mathcal{N}_h} d^2(z, \bar{v}_h(y))p(x_j, y)\}$
   until $(\max_{x_j\in\mathcal{N}_h} d(\bar{v}_h(x_j), \bar{w}_h(x_j)) \leq EPS)$.

   Here EPS is a user prescribed threshold value. This algorithm provides an approximation to the exact solution $\bar{u}_h$ of the discrete nonlinear Dirichlet problem for the boundary value map g.

2. There is an easy way to calculate

$$\operatorname*{argmin}_{z\in N} \sum_{x\in N} d^2(z, x)q(x),$$

   whereby $q(x)$ is a discrete probability distribution on N with finite support.
   For each $i \in I = \{1, \ldots, n\}$ define the numbers

$$r_i(q) := \sum_{x\in N_i} d(o, x)q(x) \quad \text{and} \quad b_i(q) := r_i(q) - \sum_{\substack{j\in I\\ j\neq i}} r_j(q).$$

   It holds $b_i(q) > 0$ for at most one $i \in I$. If $b_i(q) > 0$ for any $i \in I$ one has

$$\operatorname*{argmin}_{z\in N} \sum_{x\in N} d^2(z, x)q(x) = (i, b_i(q)).$$

   On the other hand, if $b_i(q) \leq 0$ for all $i \in I$ it holds

$$\operatorname*{argmin}_{z\in N} \sum_{x\in N} d^2(z, x)q(x) = o.$$

1.3.2 Extending Maps on Vertices to Maps on the Domain

By means of a proper prolongation procedure, to each map in $\bar{V}_N^h(g)$ we are going to associate a map in $V_N(g)$. In other words, each map $\bar{v}_h$ which is defined on the vertices of the triangulation $\mathcal{T}_h$ will be extended to a map v_h, defined on the whole domain D, with almost the same energy, i.e., for each $\bar{v}_h \in \bar{V}_N^h(g)$ we will verify that

$$\mathcal{E}_N(v_h) \leq \mathcal{E}_N^h(\bar{v}_h) + R_{g,D},$$

with a nonnegative constant $R_{g,D}$ only depending on the polygonal domain D, the regularity of the triangulation $\mathcal{T}_h$, and the map g.

As before, let us consider the sets $D, \mathcal{T}_h, \mathcal{N}_h = \{x_1, \dots, x_l\}$, and a map $g \in \mathcal{D}(\mathcal{E}_N)$. Given a vector $\bar{v}_h \in N^l$ our aim is to construct a continuous map $v_h : \bar{D} \to N$, affine on each triangle $T \in \mathcal{T}_h$ (or better affine on appropriate subtriangles of each triangle T), such that $v_h^i := v_h(x_i) = \bar{v}_h(x_i)$ for all $i = 1, \dots, l$. Hence, we will define v_h on each triangle $T \in \mathcal{T}_h$ separately. Let $T \in \mathcal{T}_h$ be given with vertices a_0, a_1, a_2. To define $v_h|_T$ we have to distinguish the following cases:

(i) $\#(\{c(\bar{v}_h(a_j))\}_{j\in\{0,1,2\}}) = 1$

(ii) $\#(\{c(\bar{v}_h(a_j))\}_{j\in\{0,1,2\}}) = 2 \quad \text{and} \quad \exists j \in \{0,1,2\} : c(\bar{v}_h(a_j)) = 0$

(iii) $\#(\{c(\bar{v}_h(a_j))\}_{j\in\{0,1,2\}}) = 2 \quad \text{and} \quad \forall j \in \{0,1,2\} : c(\bar{v}_h(a_j)) > 0$

(iv) $\#(\{c(\bar{v}_h(a_j))\}_{j\in\{0,1,2\}}) = 3 \quad \text{and} \quad \exists j \in \{0,1,2\} : c(\bar{v}_h(a_j)) = 0$

(v) $\#(\{c(\bar{v}_h(a_j))\}_{j\in\{0,1,2\}}) = 3 \quad \text{and} \quad \forall j \in \{0,1,2\} : c(\bar{v}_h(a_j)) > 0$

case (i):
We define an affine function $l : T \to \mathbb{R}$ with $l(a_j) = \pi(\bar{v}_h(a_j)), j = 0,1,2$ and for each $x \in T$ we set $v_h|_T(x) := (c(\bar{v}_h(a_0)), l(x))$.

case (ii):
Without loss of generality, we may assume that $c(\bar{v}_h(a_0)) > 0$. Then we define an affine function $l : T \to \mathbb{R}$ by $l(a_j) := \pi(\bar{v}_h(a_j)), j = 0,1,2$ and for each $x \in T$ we set $v_h|_T(x) := (c(\bar{v}_h(a_0)), l(x))$.

case (iii):
Without loss of generality, we may assume that $c(\bar{v}_h(a_0)) = c(\bar{v}_h(a_2))$. Then we define the points $a_{0,1}$ and $a_{1,2}$ by

$$a_{i-1,i} = \gamma_{i-1,i} a_i + (1 - \gamma_{i-1,i}) a_{i-1},$$

where

$$\gamma_{i-1,i} = \frac{\pi(\bar{v}_h(a_{i-1}))}{\pi(\bar{v}_h(a_i)) + \pi(\bar{v}_h(a_{i-1}))} \qquad i \in \{1,2\}$$

In addition, on the triangle $T_1 := \Delta a_{0,1} a_1 a_{1,2}$ we define an affine function $l : T_1 \to \mathbb{R}$ by $l(a_1) := \pi(\bar{v}_h(a_1)), l(a_{0,1}) := l(a_{1,2}) := 0$ and on $R_{0,2} := T \backslash T_1$ we define a bilinear function $b : R_{0,2} \to \mathbb{R}$ by $b(a_0) := \pi(\bar{v}_h(a_0)), b(a_2) := \pi(\bar{v}_h(a_2)), b(a_{0,1}) := b(a_{1,2}) := 0$. Then we set

$$v_h|_T(x) := \begin{cases} (c(\bar{v}_h(a_1)), l(x)), & \text{if } x \in T_1 \\ (c(\bar{v}_h(a_0)), b(x)), & \text{if } x \in R_{0,2}. \end{cases}$$

case (iv):
Without loss of generality, we may assume that $c(\bar{v}_h(a_1)) = 0$. We define the point $a_{0,2}$ by

$$a_{0,2} = \gamma_{0,2} a_0 + (1 - \gamma_{0,2}) a_2, \qquad \text{where} \qquad \gamma_{0,2} = \frac{\pi(\bar{v}_h(a_2))}{\pi(\bar{v}_h(a_0)) + \pi(\bar{v}_h(a_2))}$$

and we construct on the triangles $T_0 := \Delta a_0 a_1 a_{0,2}$ and $T_2 := \Delta a_{0,2} a_1 a_2$ two affine functions $l_0 : T_0 \to \mathbb{R}$ by $l(a_0) := \pi(\bar{v}_h(a_0)), l(a_1) := l(a_{0,2}) := 0$ and $l_2 : T_2 \to \mathbb{R}$ by $l(a_2) := \pi(\bar{v}_h(a_2)), l(a_1) := l(a_{0,2}) := 0$. Then we define

$$v_h|_T(x) := \begin{cases} (c(\bar{v}_h(a_0)), l_0(x)), & \text{if } x \in T_0 \\ (c(\bar{v}_h(a_2)), l_2(x)), & \text{if } x \in T_2. \end{cases}$$

case (v):
In the sequel, we interpret all the indices i as i mod (3).
We define the points $a_{i,i+1}, i \in \{0,1,2\}$ by

$$a_{i,i+1} = \gamma_{i,i+1} a_i + (1 - \gamma_{i,i+1}) a_{i+1},$$

where

$$\gamma_{i,i+1} = \frac{\pi(\bar{v}_h(a_{i+1}))}{\pi(\bar{v}_h(a_i)) + \pi(\bar{v}_h(a_{i+1}))} \qquad i \in \{0,1,2\}$$

and on the triangles $T_i := \Delta a_i a_{i,i+1} a_{i,i+2}, i \in \{0,1,2\}$ we define the affine functions $l_i : T_i \to \mathbb{R}$, $l_i(a_i) := \pi(\bar{v}_h(a_i)), l_j(a_{i,i+1}) := l_j(a_{i,i+2}) := 0$, for $i \in \{0,1,2\}$.
Moreover we define $T_{0,1,2} := \Delta a_{0,1} a_{0,2} a_{1,2}$ and we set

$$v_h|_T(x) := \begin{cases} (c(\bar{v}_h(a_i)), l_i(x)), & \text{if } x \in T_i \quad i \in \{0,1,2\} \\ (1,0), & \text{if } x \in T_{0,1,2}. \end{cases}$$

The five cases described above are graphically summarized in the following figures. In all these cases, points of the spider are described by a colour ($\triangleq$ axis) and a height ($\triangleq$ distance from origin). The black colour describes the origin.

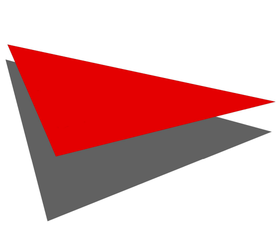

Figure 1.2: case (i)

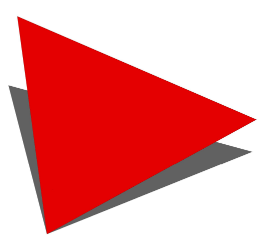

Figure 1.3: case (ii)

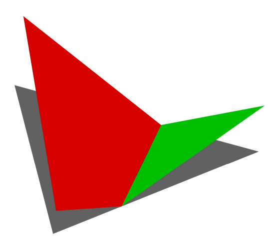

Figure 1.4: case (iii)

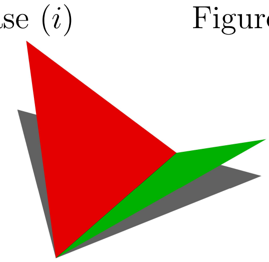

Figure 1.5: case (iv)

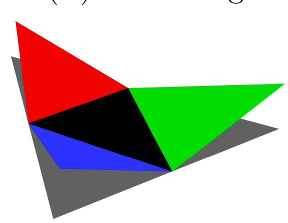

Figure 1.6: case (v)

Definition 1.17 *We define an injective mapping* $J_h : \bar{V}_N^h(g) \to V_N(g)$ *by*

$$J_h(\bar{v}_h)(x) := \begin{cases} v_h(x), & \text{if } x \in D \\ g(x), & \text{otherwise,} \end{cases}$$

for $\bar{v}_h \in \bar{V}_N^h(g)$. *In the sequel, we will denote the prolongation* $J_h(\bar{v}_h)$ *of* $\bar{v}_h$ *just by* v_h.

Remark: Note that for each $\bar{v}_h \in \bar{V}_N^h(g)$ one has

$$\int_D |\nabla(\pi_i(v_h))|^2 d\lambda < \infty, \qquad \forall i \in \{1, \ldots, n\}$$

and

$$v_h(x) = g(x), \qquad \forall x \in \mathbb{R}^2 \setminus D.$$

Therefore, v_h is well defined as an element of the space $V_N(g)$. In fact, according to Corollary 1.4 one has

$$\mathcal{E}_N(v_h) = \sum_{j=1}^n \left[\int_D |\nabla(\pi_j(v_h))|^2 d\lambda + \int_{\mathbb{R}^2 \setminus D} |\nabla(\pi_j(g))|^2 d\lambda \right]$$

Proposition 1.18 *For every* $\bar{v}_h \in V_N^h(g)$ *one has*

$$\mathcal{E}_N(v_h) \leq \mathcal{E}_N^h(\bar{v}_h) + R_{g,D}, \tag{1.14}$$

where

$$R_{g,D} := \sum_{i=1}^n \int_{\mathbb{R}^2 \setminus D} |\nabla(\pi_i(g))|^2 d\lambda. \tag{1.15}$$

Proof: Observe that due to (1.12) the discrete nonlinear energy $\mathcal{E}_N^h(\bar{v}_h)$ may be rewritten as

$$\mathcal{E}_N^h(\bar{v}_h) = -\frac{1}{2} \sum_{T \in \mathcal{T}_h} \sum_{x_i, x_j \in \mathcal{N}_h} d^2(\bar{v}_h(x_i), \bar{v}_h(x_j)) \int_T \nabla \phi_h^{i,T} \nabla \phi_h^{j,T} \, d\lambda.$$

By the definition of J_h and Corollary 1.4,

$$\begin{aligned} \mathcal{E}_N(v_h) &= \sum_{i=1}^n \left[\int_{\mathbb{R}^2 \setminus D} |\nabla(\pi_i(v_h))|^2 + \sum_{T \in \mathcal{T}_h} \int_T |\nabla(\pi_i(v_h))|^2 \right] \\ &= R_{g,D} + \sum_{T \in \mathcal{T}_h} \sum_{i=1}^n \int_T |\nabla(\pi_i(v_h))|^2. \end{aligned}$$

Thus, the rest of the proof amounts to show that for each $T \in \mathcal{T}_h$ with vertices a_0, a_1, a_2 with $v_h^i := v_h(a_i), i \in \{0, 1, 2\}$, the following inequality holds:

$$\begin{aligned} \sum_{j=1}^n \int_T |\nabla \pi_j(v_h)|^2 \, d\lambda \;\; \leq \;\; & -d^2(v_h^0, v_h^1) \int_T \nabla \phi_h^{0,T} \nabla \phi_h^{1,T} d\lambda \\ & -d^2(v_h^1, v_h^2) \int_T \nabla \phi_h^{1,T} \nabla \phi_h^{2,T} d\lambda \\ & -d^2(v_h^0, v_h^2) \int_T \nabla \phi_h^{0,T} \nabla \phi_h^{2,T} d\lambda. \end{aligned} \tag{1.16}$$

By the definition of J_h, to each $\bar{v}_h \in \bar{V}_N^h$ one has to prove (1.16) for the five different cases described at the beginning of this section. The cases $(i) - (iv)$ can be reduced to the well known linear case, holding the equality in (1.16). Indeed, if at most two colours are involved we can apply the identification discussed in Remark 1.2. To treat the case (v), let us introduce the notation $\alpha_i = c(v_h^i), i \in \{0, 1, 2\}$. We obtain

$$\sum_{j=1}^{n} \int_T |\nabla \pi_j(v_h)|^2 \, d\lambda = \sum_{i=0}^{2} \int_{T_i} |\nabla \pi_{\alpha_i}(v_h)|^2 \, d\lambda.$$

For $i = 0, 1, 2$ one obtains $\nabla \pi_{\alpha_i}(v_h) \equiv \beta_i$ for some constant β_i. Hence,

$$\int_{T_i} |\nabla \pi_{\alpha_i}(v_h)|^2 \, d\lambda = \frac{\lambda(T_i)}{\lambda(T)} \int_T \beta_i^2.$$

Furthermore, $\beta_i = \nabla w_h^i$, where w_h^i is affine on T with nodal values $w_h^i(a_i) = \pi_{\alpha_i}(v_h^i)$ and $w_h^i(a_{i\pm1}) = -\pi_{\alpha_{i\pm1}}(v_h^{i\pm1})$, again due to the identification in Remark 1.2 on distinct edges. Hence, by formula (1.12) we obtain

$$\begin{aligned}
\int_{T_i} |\nabla \pi_{\alpha_i}(v_h)|^2 \, d\lambda &= \frac{\lambda(T_i)}{\lambda(T)} \int_T \left|\nabla w_h^i\right|^2 d\lambda \\
= - \Bigg[d^2(v_h^i, v_h^{i+1}) &\int_T \nabla \phi_h^{i,T} \nabla \phi_h^{i+1,T} d\lambda \; + \; d^2(v_h^{i+1}, v_h^{i+2}) \int_T \nabla \phi_h^{i+1,T} \nabla \phi_h^{i+2,T} d\lambda \\
&+ d^2(v_h^i, v_h^{i+2}) \int_T \nabla \phi_h^{i,T} \nabla \phi_h^{i+2,T} d\lambda \Bigg] \cdot \lambda(T_i)/\lambda(T), \qquad i \in \{0, 1, 2\},
\end{aligned}$$

which completes the proof, since $\lambda(T_0 \cup T_1 \cup T_2) \leq \lambda(T)$. □

1.3.3 Convergence

In what follows, we will consider a sequence of successively refined, regular triangulations $\mathcal{T}_h$ and ask for the convergence of the resulting discrete harmonic maps $u_h \in V_N(g)$ to the solution u of the continuous problem for $h \to 0$. For the ease of presentation, we here restrict to homogeneously refined meshes, i.e. we assume

$$\min_{T \in \mathcal{T}_h} h(T) \geq c \max_{T \in \mathcal{T}_h} h(T)$$

with $h(T) = \mathrm{diam}(T)$. In our applications we generate the sequence of triangulation applying an iterative subdivision of triangles into four congruent triangles [Bra92]. In the sequel p resp. μ denote the Markov kernel resp. the measure defined in Subsection 1.3.1 corresponding to the given triangulation $\mathcal{T}_h$. Furthermore, we will use a generic constant C.

Theorem 1.19 *Let $\bar{u}_h$ be the solution to the discrete nonlinear Dirichlet problem for a map g as described above and let $J_h : \bar{V}_N^h(g) \to V_N(g)$ be the mapping defined in Subsection 1.3.2. Then*

$$\lim_{h \to 0} \mathcal{E}_N(u_h) = \mathcal{E}_N(u). \tag{1.17}$$

For the proof of Theorem 1.19 we need a couple of preliminary definitions and lemmata.

Definition 1.20 *For a triangulation $\mathcal{T}_h$ we define the set*

$$S_i := \cup\{T \in \mathcal{T}_h : x_i \in T\}, \quad x_i \in \mathcal{N}_h$$

called the patch for the vertex x_i.

Definition 1.21 *Given a function $v \in H^{1,2}(D)$, let p_i be the local L^2-projection of $v|_{S_i}$ to the set $\mathcal{P}_1(S_i)$ of all affine functions on S_i. The corresponding Clement interpolation operator $\mathcal{I}_h$ is defined by*

$$\mathcal{I}_h v := \sum_{i=1}^{l} p_i(x_i)\phi_h^i \,.$$

In [Cle75] this interpolation operator is discussed and interpolation error estimates are proven in Sobolev norms. In what follows, we require interpolation error estimates in Hölder norms given in the following Lemma.

Lemma 1.22 *Suppose v is a Hölder continuous function on $\bar{D}$, i.e. for some $0 < \alpha < 1$ the estimate $|v(x) - v(y)| \le C_\alpha |x-y|^\alpha$ holds for all $x, y \in \bar{D}$, then there is a constant $C_I > 0$ independent of h such that*

$$|\mathcal{I}_h v(x) - v(x)| \le C_I \cdot h^\alpha, \qquad \forall x \in \bar{D}.$$

Proof: At first we show that for every S_i the local L^2 projection p_i defined above is Hölder continuous with respect to the Hölder exponent α. Indeed, let us first fix a set S_i and consider candidates $q \in \mathcal{P}_1$ for the best L^2 projection p_i on S_i. We observe that if $\|\nabla q\| \ge C \max_{x,y \in S_i} |v(x) - v(y)|$ for C large enough, then the constant function $\tilde{q} := |S_i|^{-1} \int_{S_i} v$ leads to a smaller projection error. Hence, we immediately observe that $\|\nabla p_i\| \le C h^\alpha$. Due to the regularity of the triangulation the constant C can be chosen independent of S_i and i. Next, we observe that by the mean value theorem there is a point $y_i \in S_i$ such that $p_i(y_i) = v(y_i)$. Thus, we get

$$|p_i(x) - v(x)| \le |p_i(x) - p_i(y_i)| + |v(y_i) - v(x)| \le C\,|x - y_i|^\alpha \le C\,h^\alpha \,.$$

Finally, on each triangle $T \in \mathcal{T}_h$ the operator $\mathcal{I}_h$ is a convex combination of p_i values. Thus, we obtain the desired result. □

Due to our homogeneity assumption we obtain

Lemma 1.23 *The total number n_h of triangles $T \in \mathcal{T}_h$ with $T \cap \partial D \neq \emptyset$ may be bounded by*

$$n_h \le ch^{-1}$$

with a constant c independent of the triangulations.

Proof of Theorem 1.19:
Since g is Lipschitz continuous on ∂D one has that the solution to the nonlinear Dirichlet problem u is Hölder continuous with $\alpha > \log_4 3$ (cf. [Ser94] and Proposition 1.5). In the following, we will denote the Hölder constant of the map u by C_α. Now we define

$$N_0 := \{x \in D : u(x) = o\}$$

and

$$N_0^h := \{y \in D : \mathrm{dist}(y, N_0) \leq \gamma \cdot h\}$$

for a constant $\gamma > 0$. Then

$$(\pi_i(u) - \delta_h)^+(x) = 0 \qquad \forall x \in N_0^h$$

holds for all $i \in \{1, \ldots, n\}$ with $\delta_h := C_\alpha \gamma^\alpha \cdot h^\alpha$.
By this construction we ensure that the black region ($\pi \equiv 0$) is a fat strip which is of the minimal width $2\gamma \cdot h$. Hence, choosing γ large enough we are able to avoid an interference of the involved local L^2 projections in the construction of a comparison function.
For each $i \in \{1, \ldots, n\}$ we define $\mathcal{I}^\delta_{h,i}(u) := \mathcal{I}_h((\pi_i(u) - \delta_h)^+)$. It holds

$$||\mathcal{I}^\delta_{h,i}(u) - (\pi_i(u) - \delta_h)^+||_{1,2} = \nu(h) \overset{h\to 0}{\longrightarrow} 0 \quad \forall i \in \{1, \ldots, \}$$

(cf. [Cle75] and Corollary 1.4). Moreover, one has

$$\left| \int_D |\nabla((\pi_i(u) - \delta_h)^+)|^2 d\lambda - \int_D |\nabla(\pi_i(u))|^2 d\lambda \right| \to 0 \qquad h \to 0.$$

Thus, it follows

$$\int_D |\nabla(\mathcal{I}^\delta_{h,i}(u))|^2 d\lambda \leq \int_D |\nabla(\pi_i(u))|^2 d\lambda + \beta(h), \tag{1.18}$$

where $\beta(h)$ is converging to 0 for $h \to 0$.

Observe that the functions $(\pi_i(u) - \delta_h)^+, 1 \leq i \leq n$, are Hölder-continuous with the same constants α and C_α as u. Hence, according to Lemma 1.22, the following inequalities hold for each $i \in \{1, \ldots, n\}$, $x, y \in T$:

$$\begin{aligned} |\mathcal{I}^\delta_{h,i}(u)(x) - \mathcal{I}^\delta_{h,i}(u)(y)| &\leq |\mathcal{I}^\delta_{h,i}(u)(x) - (\pi_i(u) - \delta_h)^+(x)| \\ &\quad + |(\pi_i(u) - \delta_h)^+(x) - (\pi_i(u) - \delta_h)^+(y)| \\ &\quad + |(\pi_i(u) - \delta_h)^+(y) - \mathcal{I}^\delta_{h,i}(u)(y)| \\ &\leq (2\,C_I + C_\alpha) \cdot h^\alpha \end{aligned}$$

and

$$\begin{aligned} |\mathcal{I}^\delta_{h,i}(u)(x) - (\pi_i(u))(y)| &\leq |\mathcal{I}^\delta_{h,i}(u)(x) - \mathcal{I}^\delta_{h,i}(u)(y)| + |\mathcal{I}^\delta_{h,i}(u)(y) - (\pi_i(u) - \delta_h)^+(y)| \\ &\quad + |(\pi_i(u) - \delta_h)^+(y) - (\pi_i(u))(y)| \\ &\leq |\mathcal{I}^\delta_{h,i}(u)(x) - \mathcal{I}^\delta_{h,i}(u)(y)| + (C_I + C_\alpha \gamma^\alpha) h^\alpha \end{aligned}$$

as well as

$$\begin{aligned}|(\pi_i(u))(x)-(\pi_i(u))(y)| &\leq |(\pi_i(u))(x)-\mathcal{I}^\delta_{h,i}(u)(x)|+|\mathcal{I}^\delta_{h,i}(u)(x)-\mathcal{I}^\delta_{h,i}(u)(y)|\\ &\quad +|\mathcal{I}^\delta_{h,i}(u)(y)-(\pi_i(u))(y)|\\ &\leq |\mathcal{I}^\delta_{h,i}(u)(x)-\mathcal{I}^\delta_{h,i}(u)(y)|+C\cdot h^\alpha.\end{aligned}$$

By means of $\mathcal{I}^\delta_{h,i}(u)$ one can now introduce a piecewise affine function ξ_i^h on $\bar{D}$, which obeys the imposed boundary conditions on the nodes. Thus, we define its nodal values:

$$\xi_i^h(x_j):=\begin{cases}\mathcal{I}^\delta_{h,i}(u)(x_j), & \text{if } x_j\notin\partial D\\ (\pi_i(u))(x_j), & \text{if } x_j\in\partial D\end{cases}$$

for all $x_j\in\mathcal{N}_h$.

On any triangle $T\in\mathcal{T}_h$ with $T\cap\partial D=\emptyset$ one has $\xi_i^h\equiv\mathcal{I}^\delta_{h,i}(u)$. Thus, to compare the energy of ξ_i^h with the energy of $\mathcal{I}^\delta_{h,i}(u)$ it is sufficient to analyze the differences on "boundary triangles". For a given triangle $T\in\mathcal{T}_h$ with $T\cap\partial D\neq\emptyset$, with vertices a_0,a_1,a_2, and $i\in\{1,\dots,n\}$, we obtain

$$\begin{aligned}\int_T|\nabla\xi_i^h|^2d\lambda &\overset{(1.12)}{=} \sum_{\substack{s,t=0\\ s<t}}^{2}-|\xi_i^h(a_s)-\xi_i^h(a_t)|^2\cdot\int_T\nabla\phi_h^{s,T}\nabla\phi_h^{t,T}d\lambda\\ &\leq \sum_{\substack{s,t=0\\ s<t}}^{2}-(|\mathcal{I}^\delta_{h,i}(u)(a_s)-\mathcal{I}^\delta_{h,i}(u)(a_t)|+C\cdot h^\alpha)^2\cdot\int_T\nabla\phi_h^{s,T}\nabla\phi_h^{t,T}d\lambda\\ &\leq \sum_{\substack{s,t=0\\ s<t}}^{2}\Big[-|\mathcal{I}^\delta_{h,i}(u)(a_s)-\mathcal{I}^\delta_{h,i}(u)(a_t)|^2\cdot\int_T\nabla\phi_h^{s,T}\nabla\phi_h^{t,T}d\lambda\\ &\qquad +2\,|\mathcal{I}^\delta_{h,i}(u)(a_s)-\mathcal{I}^\delta_{h,i}(u)(a_t)|C\cdot h^\alpha+(C\cdot h^\alpha)^2\Big]\\ &\leq \int_T|\nabla\mathcal{I}^\delta_{h,i}(u)|^2+C\cdot h^{2\alpha}\end{aligned}$$

where we have the scaling behavior of the local stiffness matrix in two dimensions

$$-\int_T\nabla\phi_h^{i,T}\nabla\phi_h^{j,T}\leq C,$$

for all triangles $T\in\mathcal{T}_h$ and nodes $x_i,x_j\in\mathcal{N}_h$. According to Lemma 1.23 we obtain

$$\int_D|\nabla\xi_i^h|^2d\lambda=\sum_{T\in\mathcal{T}_h}\int_T|\nabla\xi_i^h|^2d\lambda\leq\sum_{T\in\mathcal{T}_h}\int_T|\nabla(\mathcal{I}^\delta_{h,i}(u))|^2d\lambda+n_h\cdot C\cdot h^{2\alpha}\tag{1.19}$$

for all $i \in \{1, \dots, n\}$. Furthermore, we can estimate $n_h \le ch^{-1}$ and, hence, $n_h \cdot C \cdot h^{2\alpha} \le Ch^{2\alpha-1}$. Finally, we verify that $2\alpha - 1 > 2\log_4 3 - 1 \ge 0.5849..$. Hence, the effect of our correction in the neighbourhood of the boundary ∂D on the energy tends to zero as $h \to 0$. Using the functions ξ_i^h our aim is now to construct a map $\bar{v}_h \in \bar{V}_N^h(g)$. For this purpose we will use the fact that the functions $\mathcal{I}_{h,i}^\delta(u)$ are not interfering with each other and that $\xi_i^h(x) = (\pi_i(g))(x)$ for all $x \in \mathcal{N}_h^\partial$. We define the map $\bar{v}_h \in \bar{V}_N^h(g)$ by

$$\bar{v}_h(x) := \begin{cases} (j, \xi_j^h(x)), & \text{if } \exists\, j \in \{1, \dots, n\} : \xi_j^h(x) \neq 0 \\ o, & \text{otherwise} \end{cases}$$

for all $x \in \mathcal{N}_h$. We observe that this definition is not ambiguous. Indeed, by construction there is at most one j with $\xi_j^h(x) \neq 0$.
Due to (1.12), the discrete nonlinear energy $\mathcal{E}_N^h(\bar{w}_h)$ of a map $\bar{w}_h \in \bar{V}_N^h(g)$ can be written as

$$\mathcal{E}_N^h(\bar{w}_h) = \sum_{T \in \mathcal{T}_h} \underbrace{-\frac{1}{2} \sum_{x_i, x_j \in \mathcal{N}_h} d^2(\bar{w}_h(x_i), \bar{w}_h(x_j)) \int_T \nabla\phi_h^{i,T} \nabla\phi_h^{j,T} d\lambda}_{:= E_T^h(\bar{w}_h)}.$$

To obtain an estimate of the discrete nonlinear energy of $\bar{v}_h$ we have to investigate $E_T^h(\bar{v}_h)$ for all $T \in \mathcal{T}_h$. Let us denote by $\mathcal{H}_h$ the set of all triangles $T \in \mathcal{T}_h$ such that $T \cap \partial D \neq \emptyset$ and there exist two vertices x, y of the triangle T with $x, y \in \partial D$ such that $0 \neq c(g(x)) \neq c(g(y)) \neq 0$. Due to our assumption on g, we know that $\#\mathcal{H}_h \le C$ independent of h. We observe

$$E_T^h(\bar{v}_h) \le \begin{cases} \sum_{i=1}^n \int_T |\nabla\xi_i^h|^2 d\lambda, & \text{if } T \in \mathcal{T}_h \backslash \mathcal{H}_h \\ 2 \cdot \sum_{i=1}^n \int_T |\nabla\xi_i^h|^2 d\lambda, & \text{if } T \in \mathcal{H}_h, \end{cases}$$

leading to

$$\mathcal{E}_N^h(\bar{v}_h) \;\le\; \sum_{i=1}^n \sum_{T \in \mathcal{T}_h} \int_T |\nabla\xi_i^h|^2 d\lambda + \sum_{i=1}^n \sum_{T \in \mathcal{H}_h} \int_T |\nabla\xi_i^h|^2 d\lambda. \tag{1.20}$$

Furthermore, we observe that $\mathcal{E}_N^h(\bar{u}_h) \le \mathcal{E}_N^h(\bar{v}_h)$ because $\bar{u}_h$ is the minimizer of the discrete nonlinear energy $\mathcal{E}_N^h$. Hence, it follows

$$\begin{aligned} \mathcal{E}_N(u_h) &\overset{(1.14)}{\le} \mathcal{E}_N^h(\bar{u}_h) + R_{g,D} \\ &\le \mathcal{E}_N^h(\bar{v}_h) + R_{g,D} \\ &\overset{(1.20)}{\le} \sum_{i=1}^n \int_D |\nabla\xi_i^h|^2 d\lambda + \sum_{i=1}^n \sum_{T \in \mathcal{H}_h} \int_T |\nabla\xi_i^h|^2 d\lambda + R_{g,D} \\ &\overset{(1.19)}{\le} \sum_{i=1}^n \int_D |\nabla(\mathcal{I}_{h,i}^\delta(u))|^2 d\lambda + \sum_{i=1}^n \sum_{T \in \mathcal{H}_h} \int_T |\nabla\xi_i^h|^2 d\lambda + C \cdot h^{2\alpha-1} + R_{g,D} \\ &\overset{(1.18)}{\le} \sum_{i=1}^n \int_{\mathbb{R}^2} |\nabla(\pi_i(u))|^2 d\lambda + \theta(h) \\ &\overset{(1.4)}{=} \mathcal{E}_N(u) + \theta(h) \end{aligned}$$

where

$$\theta(h) := \sum_{i=1}^{n} \sum_{T \in \mathcal{H}_h} \int_T |\nabla \xi_i^h|^2 d\lambda + C \cdot h^{2\alpha - 1} + \beta(h).$$

Obviously, $\theta(h) \to 0$ as $h \to 0$. This yields the desired result $\lim_{h\to 0} \mathcal{E}_N(u_h) = \mathcal{E}_N(u)$. □

Corollary 1.24 *For $h \to 0$ the discrete finite element solutions u_h converge in L^2 to the solution u of the continuous nonlinear Dirichlet problem.*

Proof: Given $v_0, v_1 \in V_N(g)$ let v_t be the geodesic connecting v_0 and v_1. Then inequality (1.10) in the proof of Theorem 1.7 yields

$$\mathcal{E}_N(v_t) \leq (1-t)\mathcal{E}_N(v_0) + t\mathcal{E}_N(v_1) - (1-t)t\lambda_D \cdot d_2^2(v, \tilde{v}) \tag{1.21}$$

with $\lambda_D > 0$ which shows that $\mathcal{E}_N$ is strictly convex on $V_N(g)$.
Now, let $u_{h,t}$ be the geodesic connecting u and u_h. Then the last inequality yields

$$\mathcal{E}_N(u) \leq \mathcal{E}_N(u_{h,\frac{1}{2}}) \leq \frac{1}{2}\mathcal{E}_N(u) + \frac{1}{2}\mathcal{E}_N(u_h) - \frac{1}{4}\lambda_D d_2^2(u, u_h),$$

and, thus,

$$\frac{1}{2}\lambda_D d_2^2(u, u_h) \leq \mathcal{E}_N(u_h) - \mathcal{E}_N(u).$$

Hence, the claimed convergence follows from Theorem 1.19. □

1.3.4 Numerical Results

Before we present a couple of numerical results for different boundary data, let us discuss the expected order of convergence of the numerical method. Let us consider the following explicit harmonic map. Let (N, d) be a 3-spider and $D := [-2, 2]^2 \subset \mathbb{R}^2$. Then the map $u : D \to N$ given by

$$u(x,y) = \begin{cases} (1, |x^3 - 3xy^2|/10), & \text{if } -\pi \leq \arctan(x,y) < -4\pi/6 \\ (1, |x^3 - 3xy^2|/10), & \text{if } 0 \leq \arctan(x,y) < 2\pi/6 \\ (2, |x^3 - 3xy^2|/10), & \text{if } -4\pi/6 \leq \arctan(x,y) < 0 \\ (2, |x^3 - 3xy^2|/10), & \text{if } 2\pi/6 \leq \arctan(x,y) < 4\pi/6 \\ (3, |x^3 - 3xy^2|/10), & \text{otherwise} \end{cases}$$

is a harmonic function on D. Now, we define the boundary data g as a Lagrangian interpolation of $u|_{\partial D}$ onto the piecewise linear and continuous functions on ∂D. In particular, we interpolate u at boundary nodes of the triangulations $\mathcal{T}_h$. Next, we have numerically solved the corresponding discrete nonlinear Dirichlet problem and computed the norm of the error

$u_h - u$ for a sequence of successively refined grids, with grid sizes $h_k = 0.21,\ 0.10,\ 0.06,\ 0.03$. Finally, we evaluate the experimental order of convergence

$$EOC = \frac{\log \|\pi(u_{h_{k+1}}) - \pi(u)\| - \log \|\pi(u_{h_k}) - \pi(u)\|}{\log h_{k+1} - \log h_k},$$

where we either consider the L^2 or the $H^{1,2}$ norm evaluated via numerical quadrature. The following tables lists the corresponding results

h	$\|\|u-u_h\|\|_{L^2}$	EOC	$\|\|u-u_h\|\|_{H^{1,2}}$	EOC
0.21	6.838e-3	2.0071	3.119e-1	0.5665
0.10	1.620e-4	2.0023	1.066e-2	1.4924
0.06	5.171e-4	2.0004	6.011e-2	1.0049
0.03	1.611e-4	1.9877	3.336e-2	1.0043

Obviously, the EOC reflects a second order convergence in the L^2 norm and a first order convergence in the $H^{1,2}$ norm and thus equals the expected convergence rate of the pure interpolation error. Hence, we observe optimal convergence in the class of piecewise linear approximations.
Figure 1.7 now shows the numerical results for different boundary data and Figure 1.8 depicts a couple of intermediate results corresponding to different iteration steps of our numerical method.

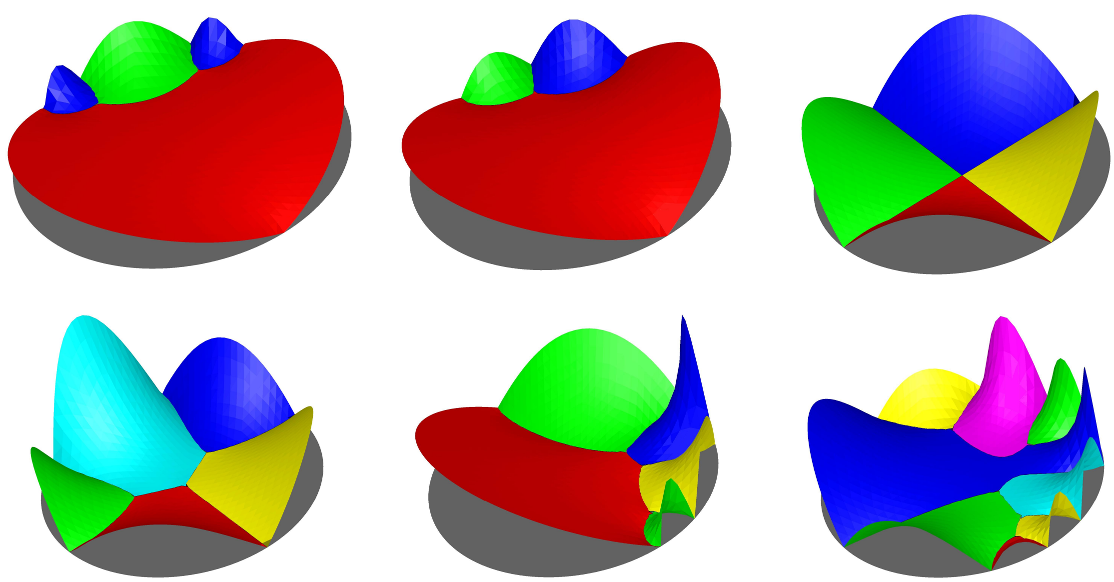

Figure 1.7: We depict various discrete harmonic maps $v_h \in V_N(g)$ for different boundary data g

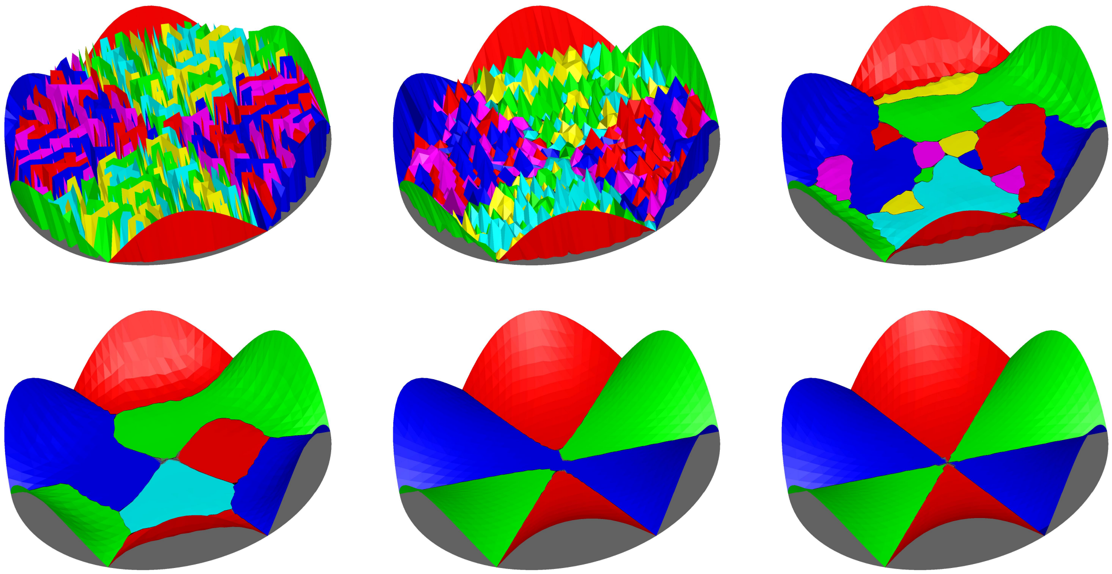

Figure 1.8: For different steps of our relaxation scheme we show intermediate results (from left to right and from top to bottom the steps 0, 1, 5, 10, 50, 250 are displayed)

1.4 Proof of Theorem 1.3

For the proof of Theorem 1.3 we need a couple of preliminary definitions and lemmata. In the sequel, we assume that assumptions (A1) and (A2) are fulfilled.

Lemma 1.25 *Assume that $n = 2$. In this case N is equivalent to $\mathbb{R}$ and $(\mathcal{E}_N, \mathcal{D}(\mathcal{E}_N))$ coincides with the given Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. In other words one has*

$$\mathcal{E}(u) = \lim_{t\to 0} \frac{1}{2t} \int_M \int_M |u(x) - u(y)|^2 \, p_t(x, dy) m(dx) \quad \textit{for each } u \in \mathcal{D}(\mathcal{E}). \tag{1.22}$$

Proof. The Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ may be characterized by

$$\mathcal{D}(\mathcal{E}) = \left\{ u \in L^2(M, m) : \lim_{t\to 0} \frac{1}{t}(u - T_t u, u) < \infty \right\}$$

and

$$\mathcal{E}(u, v) = \lim_{t\to 0} \frac{1}{t}(u - T_t u, v), \quad u, v \in \mathcal{D}(\mathcal{E}),$$

see e.g. [FOT94], [MR92]. Assumption (A2) and the symmetry of the measure $p_t(x,dy)m(dx)$ imply

$$\begin{aligned}\frac{1}{t}(u-T_tu,u) &= \frac{1}{t}\int_M\left(u(x)-\int_M u(y)p_t(x,dy)\right)\cdot u(x)m(dx)\\ &= \frac{1}{t}\int_M\left(\int_M u^2(x)p_t(x,dy)-\int_M u(x)u(y)p_t(x,dy)\right)m(dx)\\ &= \frac{1}{2t}\int_M\int_M u^2(x)-2u(x)u(y)+u^2(y)p_t(x,dy)m(dx)\\ &= \frac{1}{2t}\int_M\int_M |u(x)-u(y)|^2\,p_t(x,dy)m(dx).\end{aligned}$$

This is enough to prove this lemma, because for each $u\in L^2(M,m)$ the inner product $\frac{1}{t}(u-T_tu,u)$ is non-decreasing as t decreases. □

Lemma 1.26 *For all $u,v\in\mathcal{D}(\mathcal{E})$ such that $u\cdot v=0$ a.e. one has*

$$\lim_{t\to 0}\frac{2}{t}\int_M\int_M u(x)v(y)p_t(x,dy)m(dx)=0. \tag{1.23}$$

Proof. Assumption (A1) and (A2) yield

$$\mathcal{E}(u+v)=\mathcal{E}(u)+2\mathcal{E}(u,v)+\mathcal{E}(v)=\mathcal{E}(u)+\mathcal{E}(v).$$

A direct application of the previous Lemma leads to the required result. □

Definition 1.27 *For all $u,v\in\mathcal{D}(\mathcal{E})$ such that $u\cdot v=0$ a.e. we define the family of sets*

$$\begin{aligned}D_u &:= \{x\in M: u(x)\neq 0\},\quad D_v := \{x\in M: v(x)\neq 0\},\\ D_0^{v,u} &:= M\backslash(D_v\cup D_u).\end{aligned}$$

Lemma 1.28 *For all $u,v\in\mathcal{D}(\mathcal{E})$ such that $u\cdot v=0$ a.e., the following equalities hold*

$$\begin{aligned}\mathcal{E}(u) &= \lim_{t\to 0}\frac{1}{2t}\left[\int_{D_u}\int_{D_u}|u(x)-u(y)|^2p_t(x,dy)m(dx)\right.\\ &\quad\left.+2\int_{D_u}\int_{M\backslash D_u}u^2(x)p_t(x,dy)m(dx)\right]\end{aligned} \tag{1.24}$$

and

$$\lim_{t\to 0}\frac{2}{t}\int_{D_u}\int_{D_v}u(x)v(y)p_t(x,dy)m(dx)=0. \tag{1.25}$$

Proof. Let $u \in \mathcal{D}(\mathcal{E})$ be given. Then equality (1.22) yields

$$\begin{aligned}
\mathcal{E}(u) &= \lim_{t\to 0} \frac{1}{2t} \int_M \int_M |u(x)-u(y)|^2 p_t(x,dy)m(dx) \\
&= \lim_{t\to 0} \frac{1}{2t} \Bigg[\int_{D_u} \int_{D_u} |u(x)-u(y)|^2 p_t(x,dy)m(dx) \\
&\quad + \int_{D_u} \int_{M\setminus D_u} u^2(x) p_t(x,dy)m(dx) + \int_{M\setminus D_u} \int_{D_u} u^2(y) p_t(x,dy)m(dx) \\
&\quad + \int_{M\setminus D_u} \int_{M\setminus D_u} 0 \; p_t(x,dy)m(dx) \Bigg] \\
&= \lim_{t\to 0} \frac{1}{2t} \Bigg[\int_{D_u} \int_{D_u} |u(x)-u(y)|^2 p_t(x,dy)m(dx) \\
&\quad + 2 \int_{D_u} \int_{M\setminus D_u} u^2(x) p_t(x,dy)m(dx) \Bigg].
\end{aligned}$$

This proves the first assertion. The second one is a consequence of Lemma 1.26. Indeed, we obtain

$$\lim_{t\to 0} \frac{2}{t} \int_{D_u} \int_{D_v} u(x)v(y)p_t(x,dy)m(dx) = \lim_{t\to 0} \frac{2}{t} \int_M \int_M u(x)v(y)p_t(x,dy)m(dx) = 0.$$

□

Remark 1.29 *For all $u \in L^2(M,m)$ the function*

$$\frac{1}{2t} \Bigg[\int_{D_u} \int_{D_u} |u(x)-u(y)|^2 p_t(x,dy)m(dx) + 2 \int_{D_u} \int_{M\setminus D_u} u^2(x) p_t(x,dy)m(dx) \Bigg]$$

is non-decreasing as t decreases. Moreover, the condition

$$u \in \mathcal{D}(\mathcal{E})$$

is equivalent to the condition

$$\lim_{t\to 0} \frac{1}{2t} \Bigg[\int_{D_u} \int_{D_u} |u(x)-u(y)|^2 p_t(x,dy)m(dx) + 2 \int_{D_u} \int_{M\setminus D_u} u^2(x) p_t(x,dy)m(dx) \Bigg] < \infty.$$

Proof of Theorem 1.3:

Given a measurable map $v : M \to N$, let us define the sets

$$D_i \;:=\; \{x \in M : v_i(x) > 0\}, \quad i \in I, \quad \text{and} \quad D_0 := M\setminus(\cup_{i\in I} D_i).$$

The definition of the metric d and the symmetry of the measure $p_t(x,dy)m(dx)$ yield

$$\begin{aligned}
&\int_M \int_M d^2(v(x),v(y))p_t(x,dy)m(dx) \\
= &\sum_{i\in I}\Bigg[\int_{D_i}\int_{D_i}(v_i(x)-v_i(y))^2 p_t(x,dy)m(dx) \\
&+\sum_{\substack{j\in I\\ j\neq i}}\int_{D_i}\int_{D_j}(v_i(x)+v_j(y))^2 p_t(x,dy)m(dx)+\int_{D_i}\int_{D_0} v_i^2(x)p_t(x,dy)m(dx)\Bigg] \\
&+\sum_{j\in I}\left[\int_{D_0}\int_{D_j} v_j^2(y)p_t(x,dy)m(dx)\right].
\end{aligned}$$

Concerning the last three terms, note that for each $i,j\in I, i\neq j$, one may write

$$\begin{aligned}
&\int_{D_i}\int_{D_j}(v_i(x)+v_j(y))^2 p_t(x,dy)m(dx) \\
= &\int_{D_i}\int_{D_j} v_i^2(x)p_t(x,dy)m(dx)+\int_{D_i}\int_{D_j} v_j^2(y)p_t(x,dy)m(dx) \\
&+2\int_{D_i}\int_{D_j} v_i(x)v_j(y)p_t(x,dy)m(dx).
\end{aligned}$$

Thus, the sum

$$\begin{aligned}
&\sum_{i\in I}\left[\sum_{\substack{j\in I\\ j\neq i}}\int_{D_i}\int_{D_j}(v_i(x)+v_j(y))^2 p_t(x,dy)m(dx)\right] \\
&+2\sum_{j\in I}\left[\int_{D_0}\int_{D_j} v_j^2(y)p_t(x,dy)m(dx)\right]
\end{aligned}$$

is equal to

$$\begin{aligned}
&2\sum_{i\in I}\sum_{\substack{j\in I\cup\{0\}\\ j\neq i}}\left[\int_{D_i}\int_{D_j} v_i^2(x)p_t(x,dy)m(dx)\right] \\
&+2\sum_{\substack{i,j\in I\\ i\neq j}}\int_{D_i}\int_{D_j} v_i(x)v_j(y)p_t(x,dy)m(dx) \\
= &\ 2\sum_{i\in I}\int_{D_i}\int_{M\setminus D_i} v_i^2(x)p_t(x,dy)m(dx) \\
&+4\sum_{\substack{i,j\in I\\ i<j}}\int_{D_i}\int_{D_j} v_i(x)v_j(y)p_t(x,dy)m(dx).
\end{aligned}$$

In this way, we obtain

$$\begin{aligned}
&\frac{1}{2t}\int_M\int_M d^2(v(x),v(y))p_t(x,dy)m(dx)\\
= \;&\frac{1}{2t}\sum_{i\in I}\left[\int_{D_i}\int_{D_i}(v_i(x)-v_i(y))^2p_t(x,dy)m(dx)\right.\\
&\left.+\,2\int_{D_i}\int_{M\setminus D_i}v_i^2(x)p_t(x,dy)m(dx)\right]\\
&+\frac{2}{t}\sum_{\substack{i,j\in I\\ i<j}}\left[\int_{D_i}\int_{D_j}v_i(x)v_j(y)p_t(x,dy)m(dx)\right].
\end{aligned}$$

Now, assume that $v_i \in \mathcal{D}(\mathcal{E})$ for each $i \in I$. For all $i \in I$ let us consider the non-decreasing function as t decreases

$$\begin{aligned}
&\frac{1}{2t}\left[\int_{D_i}\int_{D_i}(v_i(x)-v_i(y))^2p_t(x,dy)m(dx)\right.\\
&\left.+\,2\int_{D_i}\int_{M\setminus D_i}v_i^2(x)p_t(x,dy)m(dx)\right]
\end{aligned}$$

(see Remark 1.29).

By equality (1.24) in Lemma 1.28 we obtain

$$\begin{aligned}
&\lim_{t\to 0}\sum_{i\in I}\frac{1}{2t}\left[\int_{D_i}\int_{D_i}(v_i(x)-v_i(y))^2p_t(x,dy)m(dx)\right.\\
&\qquad\left.+\,2\int_{D_i}\int_{M\setminus D_i}v_i^2(x)p_t(x,dy)m(dx)\right]\\
= \;&\sum_{i\in I}\lim_{t\to 0}\frac{1}{2t}\left[\int_{D_i}\int_{D_i}(v_i(x)-v_i(y))^2p_t(x,dy)m(dx)\right.\\
&\qquad\left.+\,2\int_{D_i}\int_{M\setminus D_i}v_i^2(x)p_t(x,dy)m(dx)\right]\\
= \;&\sum_{i\in I}\mathcal{E}(v_i).
\end{aligned}$$

On the other hand, one has

$$\begin{aligned}
&\limsup_{t\to 0}\sum_{\substack{i,j\in I\\ i<j}}\frac{2}{t}\left[\int_{D_i}\int_{D_j}v_i(x)v_j(y)p_t(x,dy)m(dx)\right]\\
\le \;&\sum_{\substack{i,j\in I\\ i<j}}\limsup_{t\to 0}\frac{2}{t}\left[\int_{D_i}\int_{D_j}v_i(x)v_j(y)p_t(x,dy)m(dx)\right]=0.
\end{aligned}$$

Hence, one can conclude that

$$\begin{aligned} & \limsup_{t\to 0} \frac{1}{2t} \int_M \int_M d^2(v(x), v(y)) p_t(x, dy) m(dx) \\ = \ & \lim_{t\to 0} \frac{1}{2t} \int_M \int_M d^2(v(x), v(y)) p_t(x, dy) m(dx) \\ = \ & \sum_{i\in I} \mathcal{E}(v_i) < \infty. \end{aligned}$$

In this way, we have proven the necessary condition. To show the inverse implication let us consider a function $v \in \mathcal{D}(\mathcal{E}_N)$. Then,

$$\begin{aligned} & \sum_{i\in I} \lim_{t\to 0} \frac{1}{2t} \left[\int_{D_i} \int_{D_i} (v_i(x) - v_i(y))^2 p_t(x, dy) m(dx) \right. \\ & \qquad \left. + 2 \int_{D_i} \int_{M\setminus D_i} v_i^2(x) p_t(x, dy) m(dx) \right] \\ \leq \ & \limsup_{t\to 0} \frac{1}{2t} \int_M \int_M d^2(v(x), v(y)) p_t(x, dy) m(dx) < \infty \end{aligned}$$

and the proof follows by Remark 1.29. □

1.5 Generalizations to Spiders with a Countable Number of Edges

In this section, we will generalize the definition of the nonlinear energy to the case where (N, d) is a spider with a countable number of edges

$$N := \{(i, t) : i \in \mathbb{N}, t \in \mathbb{R}_+\}/\sim$$

with $(i, 0) \sim (j, 0)$ for every $i, j \in \mathbb{N}$. The distance d is defined on N by

$$d((i, s), (j, t)) = \begin{cases} |s - t|, & \text{if } i = j \\ s + t, & \text{otherwise.} \end{cases}$$

As before, we define for $j \in \mathbb{N}$ projections

$$\pi_j : N \to \mathbb{R}_+, \qquad (i, t) \mapsto \delta_{ij} \cdot t$$

such that to each measurable map $v : M \to N$ one may associate a family of functions $v_i : M \to \mathbb{R}, i \in \mathbb{N}$, defined by

$$v_i := \pi_i \circ v.$$

Throughout this section, for $k \geq 1$ we will denote by N_k the "subspiders" of N with k edges such that $N_k \subset N_{k+1}$.

Given a measurable map $v : M \to N$, we define maps $v_k : M \to N_k, k \geq 1$, by

$$v_k(x) := \begin{cases} v(x), & \text{if } v(x) \in N_k \\ o, & \text{otherwise} \end{cases}$$

with $o := \{(1,0)\}$ and denoting for $k \geq 1$ the projections of N_k by $\pi_{k,i}$ we define

$$v_{k,i} := \pi_{k,i} \circ v_k \qquad 1 \leq i \leq k.$$

Then one obtains

Lemma 1.30 *For each $k \geq 1$ and for each measurable map $v : M \to N$ it holds $v_{k,i} = v_i, \forall i \in \{1, \ldots, k\}$.*

This leads to the following definition of the nonlinear energy function $\mathcal{E}_N$:

Definition 1.31 *Denoting the nonlinear energy function for maps with values in N_k by $\mathcal{E}_N^k$ we define for measurable maps $v : M \to N$ the nonlinear energy function $\mathcal{E}_N$ by*

$$\mathcal{E}_N(v) := \lim_{k \to \infty} \mathcal{E}_N^k(v_k) \tag{1.26}$$

with $\mathcal{D}(\mathcal{E}_N) := \{v : M \to N \text{ measurable} : v_k \in \mathcal{D}(\mathcal{E}_N^k), \forall k \geq 1, \text{ and } \mathcal{E}_N(v) < \infty\}$.

Also in this context, we can prove an "energy decomposition" as we did before for the n-spider.

Theorem 1.32 *For each map $v : M \to N$ the condition $v \in \mathcal{D}(\mathcal{E}_N)$ is equivalent to*

$$v_i \in \mathcal{D}(\mathcal{E}), \forall i \in \mathbb{N} \quad \text{and} \quad \sum_{i \in \mathbb{N}} \mathcal{E}(v_i) < \infty.$$

In this situation, for each $v \in \mathcal{D}(\mathcal{E}_N)$ the following equality holds

$$\mathcal{E}_N(v) = \sum_{i \in \mathbb{N}} \mathcal{E}(v_i).$$

Proof: Let $v \in \mathcal{D}(\mathcal{E}_N)$ by given. Then it follows from Theorem 1.3 for all $k \geq 1$ that $v_i = v_{k,i} \in \mathcal{D}(\mathcal{E}), \forall i \leq k$. Furthermore one has

$$\sum_{i \in \mathbb{N}} \mathcal{E}(v_i) \leq \mathcal{E}_N(v).$$

Given a measurable map $v : M \to N$ such that $v_i \in \mathcal{D}(\mathcal{E}), \forall i \in \mathbb{N}$, it holds for all $k \geq 1$

$$\mathcal{E}_N^k(v_k) = \sum_{i=1}^{k} \mathcal{E}(v_i)$$

and by Definition 1.31 one obtains

$$\mathcal{E}_N(v) = \sum_{i \in \mathbb{N}} \mathcal{E}(v_i).$$

□

Chapter 2

Trees

In this chapter, we study harmonic maps $v : M \to N$ form a measure space (M, m) with a local regular Dirichlet form $\mathcal{E}$ on it into a finite tree (N, d). Let A be the generator of $\mathcal{E}$ and let the semigroup e^{At} be given by a semigroup of Markov kernels p_t. We define the nonlinear energy $\mathcal{E}_N$ function for maps $v : M \to N$ by

$$\mathcal{E}_N(v) := \sup_{\substack{\varphi \in \mathcal{C}_c(M) \\ 0 \le \varphi \le 1}} \limsup_{t \to 0} \frac{1}{2t} \int_M \int_M \varphi(x) d^2(v(x), v(y)) p_t(x, dy) m(dx)$$

with $\mathcal{C}_c(M)$ being the set of all continuous functions on M with compact support. One of the main issues is an "energy decomposition" of the following form:

$$\mathcal{E}_N(v) = \sum \mu_{<v_i>}(M)$$

whereby $v_i : M \to \mathbb{R}_+$ is the projection of v on the i-th edge of the tree N and $\mu_{<v_i>}$ is the energy measure of v_i. Using this result we prove that our nonlinear energy coincides with the nonlinear energy defined by Picard.

Furthermore, conditions for the existence and uniqueness of a solution to the corresponding nonlinear Dirichlet problem are presented.

Another important point is the extension of the numerical algorithm from the first chapter to solve the nonlinear Dirichlet problem for maps from a two dimensional Euclidean domain into a finite tree. The problem is discretized in the same way as in chapter one and the convergence of the corresponding numerical method is proven. In addition, we show the Hölder continuity of the solution to the nonlinear Dirichlet problem.

Finally, a generalization to trees with a countable number of edges is discussed.

Throughout this chapter, we fix a locally compact separable measure space (M, m) with m being a Radon measure and a regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(M, m)$. Moreover, we assume

(A1) $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is local, that is, $v, w \in \mathcal{D}(\mathcal{E})$, $\text{supp}[v]$ and $\text{supp}[w]$ are compact, $v \equiv 0$ on a neighbourhood of $\text{supp}[w] \quad \Rightarrow \quad \mathcal{E}(v, w) = 0$.

(A2) The semigroup $(T_t)_{t\geq 0}$ corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is given by a semigroup of Markov kernels $p_t(x, dy)$.

(A3) It holds $p_t(x, dy)m(dx) \ll m(dy)m(dx) \qquad \forall t > 0$.

Remark 2.1

(i) Assumption (A2) is always fulfilled if M is a locally compact separable metric space, and the regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is conservative. In particular, these assumptions are fulfilled for $M = \mathbb{R}^k$ with m being the Lebesgue measure λ on $\mathbb{R}^k$, and $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ being the classical Dirichlet form, i.e. $\mathcal{E}(u) = \int_{\mathbb{R}^k} |\nabla u|^2 d\lambda$.

(ii) The assumptions (A1) and (A2) yield that the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is also strongly local (cf. Appendix A.1).

In the sequel, we denote by $\mathcal{D}_{loc}(\mathcal{E})$ the set of functions v which are locally in $\mathcal{D}(\mathcal{E})$ (i.e. on each relatively compact open subset of M there exists a function of $\mathcal{D}(\mathcal{E})$ which coincides with v.) In $\mathcal{D}(\mathcal{E})$ one can consider energy measures $\mu_{<v,u>}$ and $\mu_{<v>} = \mu_{<v,v>}$ such that $\mathcal{E}(v)$ is the total mass $\mu_{<v>}(M)$. These measures can also be defined for functions in $\mathcal{D}_{loc}(\mathcal{E})$. Thus, one can define the energy for functions in $\mathcal{D}_{loc}(\mathcal{E})$ by $\mathcal{E}(v) = \mu_{<v>}(M)$ (finite or infinite). In addition, we denote by $\mathcal{D}^b_{loc}(\mathcal{E})$ the subspace of $\mathcal{D}_{loc}(\mathcal{E})$ consisting of bounded functions with finite energy. For details see e.g. [BM95].

Furthermore, let (N, d) be a finite tree. This means N consists of a finite number of edges, which are isometric to closed intervals of $\mathbb{R}$, glued together at some endpoints such that N is a connected and simply connected space without loops. The distance d between two points $x, y \in N$ is given by the length of the unique injective path connecting x and y.
The endpoints of the edges are the vertices of the tree. Vertices which belong to only one edge are the leaves of the tree and we will call a vertex which belongs to more than one edge branchpoint.
In the following, $\mathcal{A}$ and $\mathcal{V}$ denote the set of edges and vertices, resp., of the finite tree (N, d) and we fix a leave as the root o of the tree.

Additionally, we consider the following function defined on the set of edges $\mathcal{A}$ by

$$d_o : \mathcal{A} \to \mathbb{R}, \qquad a \mapsto \inf_{x \in a} d(o, x)$$

which describes the distance between the edge a and the root o. Since N is a finite tree, d_o is a function with values in a discrete set $\{\zeta_0 = 0, \zeta_1, \ldots, \zeta_m\} \subset \mathbb{R}_+$.
Let us denote by $\mathcal{A}_i, i \in \{0, \ldots, m\}$ the set of all edges $a \in \mathcal{A}$ with $d_o(a) = \zeta_i$ and by $n_i := \#\mathcal{A}_i$, $n := \#\mathcal{A}$ the number of edges in $\mathcal{A}_i$, $\mathcal{A}$, resp. To simplify the proofs of the major

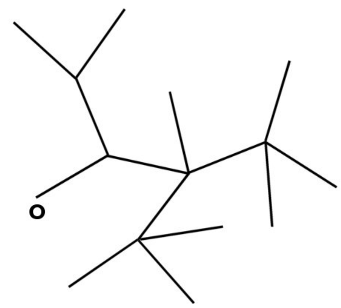

Figure 2.1: Example of a Finite Tree

results, we will renumber the edges in $\mathcal{A}$ such that

$$\begin{array}{rcl} a_1 & \in & \mathcal{A}_0 \\ a_2, \dots, a_{n_1+1} & \in & \mathcal{A}_1 \\ a_{n_1+2}, \dots, a_{n_1+n_2+1} & \in & \mathcal{A}_2 \\ & \vdots & \\ a_{n_1+\dots+n_{m-1}+2}, \dots, a_{n_1+\dots+n_m+1} & \in & A_m. \end{array}$$

From now on, we will denote the geodesic between two points $x, y \in N$ by $\gamma_{x,y}$.

Definition 2.2 *Given a tree (N, d) we define the function*

$$\begin{array}{rcl} \xi : \mathcal{A} \times \mathcal{A} & \rightarrow & \mathbb{N} \\ (a_i, a_j) & \mapsto & \xi(a_i, a_j) \end{array}$$

where $\xi(a_i, a_j)$ is the number of edges $a \in \mathcal{A}, a \neq a_i$ and $a \neq a_j$ with $a \subset \gamma_{x,y}$ for some (hence all) $x \in a_i$ and for some (hence all) $y \in a_j$.

In the sequel, we will denote the endpoints of an edge a_i by e_i^- and e_i^+, whereby we assume $d(o, e_i^-) < d(o, e_i^+)$ and we will use a disjoint decomposition $\bigcup_{i=1}^n \tilde{a}_i$ of N, with $\tilde{a}_1 := a_1$ and $\tilde{a}_i := a_i \backslash \{e_i^-\}, 2 \leq i \leq n$. In this way, each point $x \in N$ can be described by a tuple $(c, h) \in \{1, \dots, n\} \times \mathbb{R}_+$. Let us assume that x is element of $\tilde{a}_i$, then $x \triangleq (i, d(e_i^-, x))$.

Given two edges a_i, a_j we write $a_i \sim a_j$ if $\tilde{a}_i \cap \gamma_{o,x} \neq \emptyset$ for some (hence all) $x \in a_j$ or $\tilde{a}_j \cap \gamma_{o,y} \neq \emptyset$ for some (hence all) $y \in a_i$ and $a_i \not\sim a_j$ otherwise.

Definition 2.3 *For $i \in \{1, \dots, n\}$ we define projections $\pi_i : N \rightarrow [0, d(e_i^-, e_i^+)], 1 \leq i \leq n$, by*

$$\pi_i(x) := \begin{cases} d(e_i^-, x), & \textit{if } x \in \tilde{a}_i \\ d(e_i^-, e_i^+), & \textit{if } x \in \tilde{a}_j, j \neq i, a_i \subset \gamma_{o,x} \\ 0, & \textit{otherwise.} \end{cases}$$

Furthermore, we define $\pi : N \to \mathbb{R}_+$ *by*

$$\pi(x) := d(o, x).$$

In this way, to each measurable map $v : M \to N$ one may associate a family of functions $v_i : M \to \mathbb{R}_+, i \in \{1, \ldots, n\}$, defined by

$$v_i := \pi_i \circ v.$$

We will call these functions v_i projections of v.

2.1 Nonlinear Energy

In this section, we define a nonlinear energy for maps with values in a finite tree using the semigroup p_t.

Given a measurable map $v : M \to N$ we define the energy function $\mathcal{E}_N$ by

$$\mathcal{E}_N(v) := \sup_{\substack{\varphi \in \mathcal{C}_c(M)\\ 0 \leq \varphi \leq 1}} \limsup_{t \to 0} \frac{1}{2t} \int_M \int_M \varphi(x) d^2(v(x), v(y)) p_t(x, dy) m(dx) \tag{2.1}$$

with $\mathcal{D}(\mathcal{E}_N) := \{v : M \to N \text{ measurable} : \mathcal{E}_N(v) < \infty\}$ and $\mathcal{C}_c(M)$ being the set of all continuous functions on M with compact support.

Before we formulate the main theorem of this section, let us deduce some properties of the maps of $\mathcal{D}(\mathcal{E}_N)$.

Proposition 2.4 *Let* $v \in \mathcal{D}(\mathcal{E}_N)$ *be given. Then one has*

$$v_i \in \mathcal{D}^b_{loc}(\mathcal{E}) \quad \forall i \in \{1, \ldots, n\}.$$

In addition, it holds

$$\int_M \varphi(x) \mu_{<v_i>}(dx) = \lim_{t \to 0} \frac{1}{2t} \int_M \int_M \varphi(x) |v_i(x) - v_i(y)|^2 p_t(x, dy) m(dx) \qquad \forall \varphi \in \mathcal{C}_c(M).$$

Proof: For all $i \in \{1, \ldots, n\}$ one has

$$|v_i(x) - v_i(y)| \leq d(v(x), v(y)) \qquad \forall x, y \in M.$$

Hence,

$$\limsup_{t \to 0} \frac{1}{2t} \int_M \int_M \varphi(x) |v_i(x) - v_i(y)|^2 p_t(x, dy) m(dx) < \infty \qquad \forall \varphi \in \mathcal{C}_c(M), 0 \leq \varphi \leq 1.$$

Since the Dirichlet form $\mathcal{E}$ is regular, for every relatively compact open subset A of M there is a function $\psi \in \mathcal{C}_c(M), 0 \leq \psi \leq 1$ with $\psi \equiv 1$ on $\bar{A}$ and $\psi \in \mathcal{D}(\mathcal{E})$. Defining $w := \psi \cdot v_i$ it holds $w \in L^2(M,m)$, $w \equiv v_i$ on A, $\operatorname{supp}[w] \subseteq \operatorname{supp}[\psi]$ and w is bounded.
Choosing a compact subset K of M with $\operatorname{supp}[\psi] \subset K$ and $dist(\operatorname{supp}[\psi], \partial K) > 0$ it follows from the symmetry of $p_t(x,dy)m(dx)$ that

$$\begin{aligned}
&\lim_{t\to 0} \frac{1}{2t} \int_M \int_M |w(x)-w(y)|^2 p_t(x,dy)m(dx) \\
= \; & \lim_{t\to 0} \frac{1}{2t} \left[\int_K \int_K |w(x)-w(y)|^2 p_t(x,dy)m(dx) + \int_K \int_{M\setminus K} |w(x)-w(y)|^2 p_t(x,dy)m(dx) \right. \\
& \left. \quad + \int_{M\setminus K} \int_K |w(x)-w(y)|^2 p_t(x,dy)m(dx) \right] \\
= \; & \lim_{t\to 0} \frac{1}{2t} \left[\int_K \int_K |w(x)-w(y)|^2 p_t(x,dy)m(dx) + 2 \int_{\operatorname{supp}[\psi]} \int_{M\setminus K} w^2(x) p_t(x,dy)m(dx) \right].
\end{aligned}$$

Since the Dirichlet form $\mathcal{E}$ is local, it holds

$$\lim_{t\to 0} \frac{1}{2t} \int_{\operatorname{supp}[\psi]} \int_{M\setminus K} w^2(x) p_t(x,dy)m(dx) = 0$$

which yields

$$\begin{aligned}
&\lim_{t\to 0} \frac{1}{2t} \int_M \int_M |w(x)-w(y)|^2 p_t(x,dy)m(dx) \\
= \; & \lim_{t\to 0} \frac{1}{2t} \int_K \int_K |w(x)-w(y)|^2 p_t(x,dy)m(dx).
\end{aligned}$$

Furthermore, one has

$$\begin{aligned}
&\lim_{t\to 0} \frac{1}{2t} \int_K \int_K |(\psi\cdot v_i)(x) - (\psi \cdot v_i)(y)|^2 p_t(x,dy)m(dx) \\
\leq \; & \limsup_{t\to 0} \frac{1}{2t} \cdot C \left[\int_K \int_K |\psi(x)-\psi(y)|^2 p_t(x,dy)m(dx) \right. \\
& \left. \quad + \int_K \int_K |v_i(x)-v_i(y)|^2 p_t(x,dy)m(dx) \right] \\
\leq \; & C \cdot \mathcal{E}(\psi) + C \cdot \limsup_{t\to 0} \frac{1}{2t} \int_M \int_M \phi(x) |v_i(x)-v_i(y)|^2 p_t(x,dy)m(dx)
\end{aligned}$$

with $\phi \in \mathcal{C}_c(M), 0 \leq \phi \leq 1$ and $\phi \equiv 1$ on K. Hence, $w \in \mathcal{D}(\mathcal{E})$ and $v_i \in \mathcal{D}_{loc}(\mathcal{E}) \cap L^\infty(M,m)$.
For all $u \in \mathcal{D}(\mathcal{E}) \cap L^\infty(M,m)$ it holds (see [BM95])

$$\int_M \varphi(x) \mu_{<u>}(dx) = \lim_{t\to 0} \frac{1}{2t} \int_M \int_M \varphi(x) |\tilde{u}(x) - \tilde{u}(y)|^2 p_t(x,dy)m(dx) \qquad \forall \varphi \in \mathcal{C}_c(M),$$

where $\tilde{u}$ is the quasi-continuous modification of u. From assumption (A3) and by localization one obtains

$$\int_M \varphi(x) \mu_{<v_i>}(dx) = \lim_{t\to 0} \frac{1}{2t} \int_M \int_M \varphi(x) |v_i(x) - v_i(y)|^2 p_t(x,dy)m(dx) \qquad \forall \varphi \in \mathcal{C}_c(M).$$

In addition, one has

$$\mu_{<v_i>}(M) = \sup_{\substack{\varphi\in\mathcal{C}_c(M)\\ 0\le\varphi\le 1}} \lim_{t\to 0}\frac{1}{2t}\int_M\int_M \varphi(x)|v_i(x)-v_i(y)|^2 p_t(x,dy)m(dx) < \infty$$

which yields $v_i \in \mathcal{D}^b_{loc}(\mathcal{E})$. □

For the proof of the next proposition we need the following lemma.

Lemma 2.5 *Given two functions $u, v \in \mathcal{D}^b_{loc}(\mathcal{E})$ with $u \equiv c$ on supp$[v]$ it holds*

$$\lim_{t\to 0}\frac{1}{t}\int_M\int_M \varphi(x)(u(x)-u(y))\cdot(v(x)-v(y))p_t(x,dy)m(dx) = 0 \qquad \forall\varphi\in\mathcal{C}_c(M). \quad (2.2)$$

Proof: Since the Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is also strongly local one has (cf. [Pic04])

$$\mu_{<u+v>} = \mu_{<u>} + \mu_{<v>}$$

and it holds

$$\int_M \varphi(x)\mu_{<u>}(dx) = \lim_{t\to 0}\frac{1}{2t}\int_M\int_M \varphi(x)|u(x)-u(y)|^2 p_t(x,dy)m(dx) \quad (2.3)$$

for all $\varphi\in\mathcal{C}_c(M)$ (see [BM95] and Proof of Proposition 2.4). Therefore, for any $\varphi\in\mathcal{C}_c(M)$ and sufficient small $t>0$ the equations

$$\begin{aligned} & \frac{1}{2t}\int_M\int_M \varphi(x)|(u+v)(x)-(u+v)(y)|^2 p_t(x,dy)m(dx) \\ = \quad & \frac{1}{2t}\int_M\int_M \varphi(x)|(u(x)-u(y))+(v(x)-v(y))|^2 p_t(x,dy)m(dx) \\ = \quad & \frac{1}{2t}\int_M\int_M \varphi(x)|u(x)-v(y)|^2 p_t(x,dy)m(dx) \\ & +\frac{1}{2t}\int_M\int_M \varphi(x)|v(x)-v(y)|^2 p_t(x,dy)m(dx) \\ & +\frac{1}{2t}\int_M\int_M 2\cdot\varphi(x)(u(x)-u(y))\cdot(v(x)-v(y))p_t(x,dy)m(dx) \end{aligned}$$

lead to

$$\lim_{t\to 0}\frac{1}{t}\int_M\int_M \varphi(x)(u(x)-u(y))\cdot(v(x)-v(y))p_t(x,dy)m(dx) = 0.$$

□

Proposition 2.6 *Let $v\in\mathcal{D}(\mathcal{E}_N)$ be given. Defining $D_i = v^{-1}(\tilde{a}_i), i=1,\dots,n$, it holds*

$$\xi(a_i,a_j) > 0 \quad \Rightarrow \quad \lim_{t\to 0}\frac{1}{t}\int_{D_i}\int_{D_j}\varphi(x)p_t(x,dy)m(dx) = 0 \qquad \forall\varphi\in\mathcal{C}_c(M).$$

Proof: Let $a_i, a_j \in \mathcal{A}$ with $\xi(a_i, a_j) > 0$ be given. By definition of the function ξ there is an vertex $\hat{a} := a_k \in \mathcal{A}$ with $\hat{a} \subset \gamma_{x,y}$ for some $x \in a_i$ and for some $y \in a_j$ and we denote the midpoint of this vertex $\hat{a}$ by $\hat{m}$. Given a point $x \in \hat{a}$ with $d(o, \hat{m}) < d(o, x)$ we may assume without restrictions that $\inf_{y \in a_i} d(\hat{m}, y) < \inf_{y \in a_i} d(x, y)$. Now, we define two bounded functions $\phi_i, \phi_j : N \to \mathbb{R}_+$ by

$$\phi_j(x) := \begin{cases} d(\hat{m}, x), & \text{if } x \in \hat{a} \text{ and } d(o, \hat{m}) \leq d(o, x) \\ d(\hat{m}, x), & \text{if } x \in \tilde{a}_l, l > k, a_k \sim a_l \\ 0, & \text{otherwise} \end{cases}$$

and

$$\phi_i(x) := \begin{cases} d(\hat{m}, x), & \text{if } x \in \hat{a} \text{ and } d(o, x) \leq d(o, \hat{m}) \\ 0, & \text{if } x \in \hat{a} \text{ and } d(o, \hat{m}) \leq d(o, x) \\ 0, & \text{if } x \in \tilde{a}_l, l > k, a_k \sim a_l \\ d(\hat{m}, x), & \text{otherwise.} \end{cases}$$

Defining $v_i^\phi := \phi_i \circ v$, $v_j^\phi := \phi_j \circ v$ one has $v_i^\phi \geq 0, v_j^\phi \geq 0$ and

$$v_i^\phi, v_j^\phi \in \mathcal{D}^b_{loc}(\mathcal{E}),$$

because of $|v_i^\phi(x) - v_i^\phi(y)| \leq d(v(x), v(y))$, $|v_j^\phi(x) - v_j^\phi(y)| \leq d(v(x), v(y))$ for all $x, y \in M$ (see proof of Proposition 2.4). Furthermore, by construction of ϕ_i and ϕ_j it holds $v_i^\phi \cdot v_j^\phi = 0$ and, thus, (cf. Lemma 2.5)

$$\lim_{t \to 0} \frac{1}{t} \int_M \int_M |\varphi(x)| [v_i^\phi(x) v_j^\phi(y) + v_i^\phi(y) v_j^\phi(x)] p_t(x, dy) m(dx) = 0 \qquad \forall \varphi \in \mathcal{C}_c(M).$$

Hence, it follows

$$\begin{aligned} 0 &= \lim_{t \to 0} \frac{1}{t} \int_M \int_M |\varphi(x)| \cdot [v_i^\phi(x) v_j^\phi(y) + v_i^\phi(y) v_j^\phi(x)] p_t(x, dy) m(dx) \\ &\geq \lim_{t \to 0} \frac{1}{t} \int_{D_i} \int_{D_j} |\varphi(x)| \cdot v_i^\phi(x) \cdot v_j^\phi(y) p_t(x, dy) m(dx) \\ &\geq \lim_{t \to 0} \frac{c}{t} \int_{D_i} \int_{D_j} |\varphi(x)| p_t(x, dy) m(dx) \\ &\geq \lim_{t \to 0} \frac{c}{t} \left| \int_{D_i} \int_{D_j} \varphi(x) p_t(x, dy) m(dx) \right| \geq 0 \qquad \forall \varphi \in \mathcal{C}_c(M), \end{aligned}$$

because of $v_i^\phi \geq d(\hat{m}, e_k^-)$ on D_i and $v_j^\phi \geq d(\hat{m}, e_k^+)$ on D_j. Therefore, it holds

$$\lim_{t \to 0} \frac{1}{t} \int_{D_i} \int_{D_j} \varphi(x) p_t(x, dy) m(dx) = 0 \qquad \forall \varphi \in \mathcal{C}_c(M).$$

□

Theorem 2.7 *For each map $v : M \to N$ the condition $v \in \mathcal{D}(\mathcal{E}_N)$ is equivalent to*

$$v_i \in \mathcal{D}^b_{loc}(\mathcal{E}), \forall i \in \{1, \dots, n\}$$

and

$$\lim_{t \to 0} \frac{1}{t} \int_{v^{-1}(\tilde{a}_i)} \int_{v^{-1}(\tilde{a}_j)} \varphi(x) p_t(x, dy) m(dx) = 0 \tag{2.4}$$

for all $a_i, a_j \in \mathcal{A}$ with $\xi(a_i, a_j) > 0$ and for all $\varphi \in \mathcal{C}_c(M)$. In this situation, for each $v \in \mathcal{D}(\mathcal{E}_N)$ the following equalities hold

$$\begin{aligned} \mathcal{E}_N(v) &= \sup_{\substack{\varphi \in \mathcal{C}_c(M) \\ 0 \le \varphi \le 1}} \lim_{t \to 0} \frac{1}{2t} \int_M \int_M \varphi(x) d^2(v(x), v(y)) p_t(x, dy) m(dx) \\ &= \sum_{i=1}^n \mu_{<v_i>}(M). \end{aligned} \tag{2.5}$$

For a detailed proof see Section 2.4.

Remark 2.8 *Given a map $v : M \to N$ with $v_i \in \mathcal{D}^b_{loc}(\mathcal{E})$, $i \in \{1, \dots, n\}$, and two points $x, y \in N$ denote the projection of v on the geodesic $\gamma_{x,y}$ by $v_{x,y}$. Then equation (2.4) ensures that*

$$v_{x,y} \in \mathcal{D}^b_{loc}(\mathcal{E}).$$

A descriptive interpretation is that condition (2.4) prevents jumps of the map v from one edge to another edge which are not adjacent to each other.

2.1.1 Other Definitions of Nonlinear Energy

Now, we will compare our definition of nonlinear energy with the definitions given by Korevaar/Schoen (cf. [KS93]) and Picard (cf. [Pic04]).

Korevaar/Schoen developed a theory of harmonic maps into NPC spaces by defining a canonical extension E_N of the energy functional for maps with values in an NPC space (X, d). In this approach, the domain space M is a k-dimensional Riemannian manifold and

$$E_N(v) = \sup_{\substack{\varphi \in \mathcal{C}_c(M) \\ 0 \le \varphi \le 1}} \lim_{r \to 0} \frac{c_k}{r^{k+1}} \int_M \int_{\partial B_r(x)} \varphi(x) d^2(v(x), v(y)) \sigma_{r,x}(dy) m(dx) \tag{2.6}$$

where

$$c_k = \frac{k}{4\pi^{k/2}} \cdot \Gamma(k/2) = \frac{k}{4\pi^{k/2}} \int_0^\infty x^{k/2-1} \exp(-x) dx$$

and $\sigma_{r,x}$ denotes the surface measure on the sphere $\partial B_r(x)$.

Proposition 2.9 *On* $\mathbb{R}^k$ *with the Lebesgue measure* λ*, let* $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ *be the classical Dirichlet form and let* (N,d) *be a finite tree. For all* $v\in\mathcal{D}(\mathcal{E}_N)$ *one has*

$$\begin{aligned}\mathcal{E}_N(v) &= \sum_{i=1}^n \int_{\mathbb{R}^k} |\nabla v_i|^2 d\lambda \\ &= \sup_{\substack{\varphi\in\mathcal{C}_c(\mathbb{R}^k)\\ 0\le\varphi\le 1}} \lim_{r\to 0}\frac{c_k}{r^{k+1}}\int_M\int_{\partial B_r(x)}\varphi(x)d^2(v(x),v(y))\sigma_{r,x}(dy)m(dx) \\ &= E_N(v)\end{aligned}$$

Proof: The first equation follows easily from Theorem 2.7 and the proof of the second equation works out in a similar way as the proof of Proposition 1.5. □

Proposition 2.10 *Let* M *be a connected compact Riemannian manifold without boundary and let* $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ *be the classical Dirichlet form on* M *given by the Laplace-Beltrami operator. Then, our definition of the nonlinear energy* $\mathcal{E}_N$ *coincides with definition of energy introduced by Korevaar/Schoen in [KS93].*

For the proof it is sufficient to show that for any map $v: M\to N$ with finite energy in the sense of Korevaar/Schoen one has

$$E_N(v)=\sum_{i=1}^n\int_M |\nabla v_i(x)|^2 d\mu(x)$$

where μ is the Riemannian volume measure on M. For details see Subsection 2.1.2.

Another possible definition of nonlinear energy for maps with values in a finite tree N is given by Picard (see [Pic04]).

Definition 2.11 *The set* $\mathcal{D}_N^b$ *is the space of* N*-valued maps such that* $\varphi\circ v$ *is in* $\mathcal{D}_{loc}^b(\mathcal{E})$ *for any Lipschitz function* $\varphi: N\to\mathbb{R}$*. For* v *in this space define*

$$\tilde{\mathcal{E}}_N(v) := \sup\left\{\mathcal{E}(\varphi\circ v) : \varphi \textit{ non expanding }\right\}.$$

In the next proposition, we will prove that this definition coincides with our definition of nonlinear energy.

Proposition 2.12 *It holds*

$$\mathcal{D}(\mathcal{E}_N)=\mathcal{D}_N^b$$

and

$$\mathcal{E}_N(v)=\tilde{\mathcal{E}}_N(v)\qquad \forall v\in\mathcal{D}(\mathcal{E}_N).$$

Proof: Let $v \in \mathcal{D}(\mathcal{E}_N)$ be given. It follows from Proposition 2.4 that $v_i \in \mathcal{D}^b_{loc}(\mathcal{E}), 0 \leq 1 \leq n$. Lemma 5.1.10 in [Pic04] yields $v \in \mathcal{D}^b_N$ and

$$\tilde{\mathcal{E}}_N(v) = \sum_{i=1}^{n} \mu_{<v_i>}(M) \overset{(2.5)}{=} \mathcal{E}_N(v).$$

Given any map $v \in \mathcal{D}^b_N$ one has $v_i \in \mathcal{D}^b_{loc}(\mathcal{E}), 0 \leq 1 \leq n$. Furthermore, the functions ϕ_j and ϕ_i defined in the proof of Proposition 2.6 are Lipschitz functions. Hence $\phi_i \circ v$ and $\phi_j \circ v$ are in $\mathcal{D}^b_{loc}(\mathcal{E})$ and with the same arguments as in the proof of Proposition 2.6 one can show that (2.4) holds for any map in $\mathcal{D}^b_N$. Then Theorem 2.7 yields $v \in \mathcal{D}(\mathcal{E}_N)$ and again by Lemma 5.1.10 in [Pic04] one obtains $\mathcal{E}_N(v) = \tilde{\mathcal{E}}_N(v)$. □

2.1.2 Decomposition of the Energy from Korevaar/Schoen

In this section, we will prove the energy decomposition of the nonlinear energy given by Korevaar/Schoen for tree-valued maps.

In the sequel, let Ω be a Riemannian domain, that is, Ω is a connected, open subset of a k-dimensional Riemannian manifold M having the property that its metric completion $\bar{\Omega}$ is a compact subset of M and let μ be the Riemannian volume measure on M. Without restrictions, we assume that the length of all edges of the finite tree N is equal to one.

For the readers convenience let us repeat some notations and definitions from the work [KS93] of Korevaar/Schoen.

Definition 2.13 *We define the space $L^2(\Omega, N)$ as the set of Borel-measurable maps $v : \Omega \to N$ for which*

$$\int_\Omega d^2(u(x), q) d\mu(x) < \infty$$

for some $q \in N$.

The space $L^2(\Omega, N)$ is a complete metric space, with distance function D defined by

$$D^2(u, v) = \int_\Omega d^2(u(x), v(x)) d\mu(x).$$

Definition 2.14 (Nonlinear Energy of Korevaar/Schoen) *Let $v \in L^2(\Omega, N)$ be given and let $\mathcal{C}_c(\Omega)$ be the set of all continuous functions on Ω with compact support. Then, for $\epsilon > 0$ and $f \in \mathcal{C}_c(\Omega)$ define*

$$E_{\epsilon,f}(v) = c_k \cdot \int_{\Omega_\epsilon} f(x) \underbrace{\int_{S(x,\epsilon)} \epsilon^{-k-1} d^2(v(x), v(y)) \, d\sigma_{x,\epsilon}(y)}_{e_\epsilon(x)} \, d\mu(x)$$

with $\Omega_\epsilon = \{x \in \Omega : dist(x, \partial\Omega) > \epsilon\}$, $S(x,\epsilon) = \{y \in \Omega : |y - x| = \epsilon\}$, $d\sigma_{x,\epsilon}(y)$ *the* $(k-1)$*-dimensional surface measure on* $S(x,\epsilon)$ *and*

$$c_k = \int_{S^{k-1}} |x^1|^2 d\sigma(x)$$

(where $x = (x^1, \dots, x^k) \in \mathrm{I\!R}^k$ *and* $S^{k-1} = \{|x| = 1\}$*).*
The map v *has finite (nonlinear) energy (and one writes* $v \in W^{1,2}(\Omega, N)$*), if*

$$\sup_{\substack{f \in \mathcal{C}_c(\Omega) \\ 0 \le f \le 1}} \left(\limsup_{\epsilon \to 0} E_{\epsilon,f}(v) \right) \equiv E(v) < \infty.$$

Remark 2.15 *For the constant* c_k *from the previous definition one has*

$$\begin{aligned} c_k &= \frac{k}{4\pi^{k/2}} \cdot \Gamma(k/2) = \frac{k}{4\pi^{k/2}} \int_0^\infty x^{k/2-1} \exp(-x) dx \\ &= \frac{1}{\pi^{k/2}} \int_0^\infty x^{k+1} \exp\left(-x^2\right) dx. \end{aligned}$$

Now, we will show that for any map $v \in W^{1,2}(\Omega, N)$ one has

$$E(v) = \sum_{i=1}^{n} \int_\Omega |\nabla v_i(x)|^2 d\mu(x) \tag{2.7}$$

with $v_i, 1 \le i \le n$, being the projections of v on the i-th edge of the tree N.

For this, we need a couple of preliminary results.

Lemma 2.16 *Let* $x, y \in N$ *be given. For the distance* d *between* x *and* y *it holds*

$$d(x,y) = \begin{cases} |\pi_i(x) - \pi_i(y)|, & \text{if } x, y \in \tilde{a}_i \\ 1 - \pi_i(x) + \pi_j(y) + \xi(a_i, a_j), & \text{if } x \in \tilde{a}_i, y \in \tilde{a}_j, j \sim i, j > i \\ 1 - \pi_j(y) + \pi_i(x) + \xi(a_i, a_j), & \text{if } x \in \tilde{a}_i, y \in \tilde{a}_j, j \sim i, j < i \\ \pi_i(x) + \pi_j(y) + \xi(a_i, a_j), & \text{if } x \in \tilde{a}_i, y \in \tilde{a}_j, j \not\sim i \end{cases}$$

Proof: This follows by the definition of the metric d, the Definitions 2.2 and 2.3 and by the assumption of the length of the edges. □

Proposition 2.17 *It holds*

$$W^{1,2}(\Omega, \mathrm{I\!R}) = W^{1,2}(\Omega).$$

Furthermore, the energy densities coincide. Thus, given $v \in L^2(\Omega, \mathrm{I\!R})$ *with finite energy* $E(v)$ *the measure* $c_k\, e_\epsilon(x) d\lambda(x)$ *converges weakly to* $|\nabla v(x)|^2 d\lambda(x)$*.*

Proof: See Theorem 1.5.1, 1.6.2 in [KS93].

Lemma 2.18 *Given $v \in W^{1,2}(\Omega, N)$ one has $v_i \in W^{1,2}(\Omega)$ for all $i \in \{1, \ldots, n\}$.*

Proof: This follows easily from $|v_i(x) - v_i(y)| \leq d(v(x), v(y))$. □

Lemma 2.19 *Given two functions in $u, v \in W^{1,2}(\Omega)$ with $u \equiv c$ on supp$[v]$ one has*

$$\lim_{\epsilon \to 0} \int_{\Omega_\epsilon} \int_{S(x,\epsilon)} f(x) \cdot \epsilon^{-k-1}(u(x) - u(y)) \cdot (v(x) - v(y)) \, d\sigma_{x,\epsilon}(y) \, d\mu(x) = 0$$

for all $f \in \mathcal{C}_c(\Omega)$.

Proof: It holds

$$|\nabla(u+v)|^2 = |\nabla u|^2 + |\nabla v|^2.$$

Therefore, one has for any $f \in \mathcal{C}_c(\Omega)$

$$\begin{aligned} & \int_{\Omega_\epsilon} \int_{S(x,\epsilon)} f(x) \cdot \epsilon^{-k-1} |(u+v)(x) - (u+v)(y)|^2 \, d\sigma_{x,\epsilon}(y) \, d\mu(x) \\ = & \int_{\Omega_\epsilon} \int_{S(x,\epsilon)} f(x) \cdot \epsilon^{-k-1} |u(x) - u(y)|^2 \, d\sigma_{x,\epsilon}(y) \, d\mu(x) \\ & \quad + \int_{\Omega_\epsilon} \int_{S(x,\epsilon)} f(x) \cdot \epsilon^{-k-1} |v(x) - v(y)|^2 \, d\sigma_{x,\epsilon}(y) \, d\mu(x) \\ & \quad + \int_{\Omega_\epsilon} \int_{S(x,\epsilon)} 2 \cdot f(x) \cdot \epsilon^{-k-1}(u(x) - u(y)) \cdot (v(x) - v(y)) \, d\sigma_{x,\epsilon}(y) \, d\mu(x). \end{aligned}$$

With Proposition 2.17 one obtains

$$\lim_{\epsilon \to 0} \int_{\Omega_\epsilon} \int_{S(x,\epsilon)} f(x) \cdot \epsilon^{-k-1}(u(x) - u(y)) \cdot (v(x) - v(y)) \, d\sigma_{x,\epsilon}(y) \, d\mu(x) = 0.$$

□

Lemma 2.20 *Given $u, v \in W^{1,2}(\Omega)$ such that $u \cdot v = 0$ one has*

$$\begin{aligned} \lim_{\epsilon \to 0} \Bigg[& \int_{\Omega_u \cap \Omega_\epsilon} \int_{S(x,\epsilon) \cap \Omega_v} f(x) \epsilon^{-k-1} u(x) v(y) \, d\sigma_{x,\epsilon}(y) \, d\mu(x) \\ & + \int_{\Omega_v \cap \Omega_\epsilon} \int_{S(x,\epsilon) \cap \Omega_u} f(x) \epsilon^{-k-1} u(y) v(x) \, d\sigma_{x,\epsilon}(y) \, d\mu(x) \Bigg] = 0 \qquad \forall f \in \mathcal{C}_c(\Omega) \end{aligned}$$

with $\Omega_u := \{x \in \Omega : u(x) \neq 0\}$ and $\Omega_v := \{x \in \Omega : v(x) \neq 0\}$.

Proof: Lemma 2.19 leads to

$$\lim_{\epsilon\to 0}\int_{\Omega_\epsilon}\int_{S(x,\epsilon)} f(x)\cdot\epsilon^{-k-1}(u(x)-u(y))\cdot(v(x)-v(y))\,d\sigma_{x,\epsilon}(y)\,d\mu(x)=0 \qquad \forall f\in\mathcal{C}_c(\Omega)$$

or, equivalently,

$$\lim_{\epsilon\to 0}\Big[-\int_{\Omega_u\cap\Omega_\epsilon}\int_{S(x,\epsilon)\cap\Omega_v} f(x)\epsilon^{-k-1}u(x)v(y)\,d\sigma_{x,\epsilon}(y)\,d\mu(x) - \int_{\Omega_v\cap\Omega_\epsilon}\int_{S(x,\epsilon)\cap\Omega_u} f(x)\epsilon^{-k-1}u(y)v(x)\,d\sigma_{x,\epsilon}(y)\,d\mu(x)\Big]=0 \qquad \forall f\in\mathcal{C}_c(\Omega).$$

□

Lemma 2.21 *Let $v\in W^{1,2}(\Omega,N)$ be given. Defining $\Omega_i=v^{-1}(\tilde a_i), i=1,\dots,n$, it holds*

$$\xi(a_i,a_j)>0 \quad\Rightarrow\quad \lim_{\epsilon\to 0}\int_{\Omega_\epsilon\cap\Omega_i}\int_{S(x,\epsilon)\cap\Omega_j} f(x)\epsilon^{-k-1}\,d\sigma_{x,\epsilon}(y)\,d\mu(x)=0 \qquad \forall f\in\mathcal{C}_c(\Omega).$$

Proof: Let $a_i,a_j\in\mathcal{A}$ with $\xi(a_i,a_j)>0$ be given. By definition of the function ξ there is an vertex $\hat a:=a_k\in\mathcal{A}$ with $\hat a\subset\gamma_{x,y}$ for some $x\in a_i$ and for some $y\in a_j$ and we denote the midpoint of this vertex $\hat a$ by $\hat m$. Given a point $x\in\hat a$ with $d(o,\hat m)<d(o,x)$ we may assume without restrictions that $\inf_{y\in a_i}d(\hat m,y)<\inf_{y\in a_i}d(x,y)$. Now, we define two bounded functions $\phi_i,\phi_j:N\to\mathbb{R}_+$ by

$$\phi_j(x):=\begin{cases} d(\hat m,x), & \text{if } x\in\hat a \text{ and } d(o,\hat m)\le d(o,x)\\ d(\hat m,x), & \text{if } x\in\tilde a_l, l>k, a_k\sim a_l\\ 0, & \text{otherwise}\end{cases}$$

and

$$\phi_i(x):=\begin{cases} d(\hat m,x), & \text{if } x\in\hat a \text{ and } d(o,x)\le d(o,\hat m)\\ 0, & \text{if } x\in\hat a \text{ and } d(o,\hat m)\le d(o,x)\\ 0, & \text{if } x\in\tilde a_l, l>k, a_k\sim a_l\\ d(\hat m,x), & \text{otherwise.}\end{cases}$$

Defining $v_i^\phi:=\phi_i\circ v$, $v_j^\phi:=\phi_j\circ v$ one has $v_i^\phi\ge 0, v_j^\phi\ge 0$ and

$$v_i^\phi, v_j^\phi\in W^{1,2}(\Omega),$$

because of $|v_i^\phi(x)-v_i^\phi(y)|\le d(v(x),v(y))$, $|v_j^\phi(x)-v_j^\phi(y)|\le d(v(x),v(y))$ for all $x,y\in\Omega$ (cf. Lemma 2.18). Furthermore, by construction of ϕ_i and ϕ_j it holds $v_i^\phi\cdot v_j^\phi=0$ and, thus, (cf. Lemma 2.19)

$$\lim_{\epsilon\to 0}\int_{\Omega_\epsilon}\int_{S(x,\epsilon)} f(x)\cdot\epsilon^{-k-1}(v_i^\phi(x)-v_i^\phi(y))\cdot(v_j^\phi(x)-v_j^\phi(y))\,d\sigma_{x,\epsilon}(y)\,d\mu(x)=0.$$

Hence, it follows

$$\begin{aligned}
0 &= \lim_{\epsilon\to 0} \epsilon^{-k-1} \int_{\Omega_\epsilon} \int_{S(x,\epsilon)} |f(x)| \cdot |v_i^\phi(x) v_j^\phi(y) + v_i^\phi(y) v_j^\phi(x)| \; d\sigma_{x,\epsilon}(y)\; d\mu(x) \\
&\geq \lim_{\epsilon\to 0} \epsilon^{-k-1} \int_{\Omega_\epsilon\cap\Omega_i} \int_{S(x,\epsilon)\cap\Omega_j} |f(x)| \cdot v_i^\phi(x) \cdot v_j^\phi(y)\; d\sigma_{x,\epsilon}(y)\; d\mu(x) \\
&\geq \lim_{\epsilon\to 0} c\cdot \epsilon^{-k-1} \int_{\Omega_\epsilon\cap\Omega_i} \int_{S(x,\epsilon)\cap\Omega_j} |f(x)|\; d\sigma_{x,\epsilon}(y)\; d\mu(x) \\
&\geq \lim_{\epsilon\to 0} c\cdot \epsilon^{-k-1} \left| \int_{\Omega_\epsilon\cap\Omega_i} \int_{S(x,\epsilon)\cap\Omega_j} f(x)\; d\sigma_{x,\epsilon}(y)\; d\mu(x) \right| \geq 0 \qquad \forall f \in \mathcal{C}_c(\Omega),
\end{aligned}$$

because of $v_i^\phi \geq d(\hat{m}, e_k^-)$ on Ω_i and $v_j^\phi \geq d(\hat{m}, e_k^+)$ on Ω_j. Therefore, it holds

$$\lim_{\epsilon\to 0} \epsilon^{-k-1} \int_{\Omega_\epsilon\cap\Omega_i} \int_{S(x,\epsilon)\cap\Omega_j} f(x)\; d\sigma_{x,\epsilon}(y)\; d\mu(x) = 0 \qquad \forall f \in \mathcal{C}_c(\Omega).$$

□

Lemma 2.22 *Let $v \in W^{1,2}(\Omega, N)$ be given. Defining $\Omega_i = v^{-1}(\tilde{a}_i), i = 1, \ldots, n$, it holds for any measurable bounded function u*

$$\xi(a_i, a_j) > 0 \quad \Rightarrow \quad \lim_{\epsilon\to 0} \int_{\Omega_\epsilon\cap\Omega_i} \int_{S(x,\epsilon)\cap\Omega_j} f(x) u(x) \epsilon^{-k-1}\; d\sigma_{x,\epsilon}(y)\; d\mu(x) = 0 \qquad \forall f \in \mathcal{C}_c(\Omega).$$

Lemma 2.23 *Let $v \in W^{1,2}(\Omega, N)$ be given. For two projections v_i, v_j of v such that $i \sim j, i < j$ one has*

$$\begin{aligned}
\lim_{\epsilon\to 0} \Bigg[&\int_{\Omega_i\cap\Omega_\epsilon} \int_{\Omega_j\cap S(x,\epsilon)} f(x) \epsilon^{-k-1} (1 - v_i(x)) v_j(y)\; d\sigma_{x,\epsilon}(y)\; d\mu(x) \\
&+ \int_{\Omega_j\cap\Omega_\epsilon} \int_{\Omega_i\cap S(x,\epsilon)} f(x) \epsilon^{-k-1} (1 - v_i(y)) v_j(x)\; d\sigma_{x,\epsilon}(y)\; d\mu(x) \Bigg] = 0 \qquad \forall f \in \mathcal{C}_c(\Omega)
\end{aligned}$$

with $\Omega_i := v^{-1}(\tilde{a}_i)$ and $\Omega_j := v^{-1}(\tilde{a}_j)$.

Proof: Lemma 2.19 and Lemma 2.22 yield

$$\begin{aligned}
0 &= \lim_{\epsilon\to 0} \int_{\Omega_\epsilon} \int_{S(x,\epsilon)} f(x) \cdot \epsilon^{-k-1} (v_i(x) - v_i(y)) \cdot (v_j(x) - v_j(y))\; d\sigma_{x,\epsilon}(y)\; d\mu(x) \\
&= \lim_{\epsilon\to 0} \Bigg[\int_{\Omega_i\cap\Omega_\epsilon} \int_{\Omega_j\cap S(x,\epsilon)} f(x) \epsilon^{-k-1} (1 - v_i(x)) v_j(y)\; d\sigma_{x,\epsilon}(y)\; d\mu(x) \\
&\qquad + \int_{\Omega_j\cap\Omega_\epsilon} \int_{\Omega_i\cap S(x,\epsilon)} f(x) \epsilon^{-k-1} (1 - v_i(y)) v_j(x)\; d\sigma_{x,\epsilon}(y)\; d\mu(x) \Bigg]
\end{aligned}$$

for all $f \in \mathcal{C}_c(\Omega)$. □

Lemma 2.24 *Let $v \in W^{1,2}(\Omega, N)$ be given. For two projections v_i, v_j of v such that $i \not\sim j$ one has*

$$\lim_{\epsilon\to 0}\left[\int_{\Omega_i\cap\Omega_\epsilon}\int_{\Omega_j\cap S(x,\epsilon)} f(x)\epsilon^{-k-1}v_i(x)v_j(y)\, d\sigma_{x,\epsilon}(y)\, d\mu(x) + \int_{\Omega_j\cap\Omega_\epsilon}\int_{\Omega_i\cap S(x,\epsilon)} f(x)\epsilon^{-k-1}v_i(y)v_j(x)\, d\sigma_{x,\epsilon}(y)\, d\mu(x)\right] = 0$$

for all $f \in \mathcal{C}_c(\Omega)$ with $\Omega_i := v^{-1}(\tilde{a}_i)$ and $\Omega_j := v^{-1}(\tilde{a}_j)$.

Proof: This follows from Lemma 2.20, $\Omega_i \subset \Omega_{v_i}, \Omega_j \subset \Omega_{v_j}$ and $v_i \geq 0, v_j \geq 0$. □

Proof of Equation(2.7):

Lemma 2.16 yields for a measurable map $v : \Omega \to N$

$$d(v(x), v(y)) = \begin{cases} |v_i(x) - v_i(y)|, & \text{if } x, y \in \Omega_i \\ 1 - v_i(x) + v_j(y) + \xi(a_i, a_j), & \text{if } x \in \Omega_i, y \in \Omega_j, j \sim i, j > i \\ 1 - v_j(y) + v_i(x) + \xi(a_i, a_j), & \text{if } x \in \Omega_i, y \in \Omega_j, j \sim i, j < i \\ v_i(x) + v_j(y) + \xi(a_i, a_j), & \text{if } x \in \Omega_i, y \in \Omega_j, j \not\sim i \end{cases}$$

with $\Omega_i := v^{-1}(\tilde{a}_i), i \in \{1, \ldots, n\}$.

Thus, one has

$$\begin{aligned} E_{\epsilon,f}(v) &= \epsilon^{-k-1}\int_{\Omega_\epsilon} f(x)\int_{S(x,\epsilon)} d^2(v(x), v(y))\, d\sigma_{x,\epsilon}(y)\, d\mu(x) \\ &= \epsilon^{-k-1}\sum_{i=1}^{n}\Big[\int_{\Omega_i\cap\Omega_\epsilon}\int_{\Omega_i\cap S(x,\epsilon)} f(x)|v_i(x) - v_i(y)|^2\, d\sigma_{x,\epsilon}(y)\, d\mu(x) \\ &\quad + \sum_{i\not\sim j}\int_{\Omega_i\cap\Omega_\epsilon}\int_{\Omega_j\cap S(x,\epsilon)} f(x)|\xi(a_i, a_j) + v_j(x) + v_i(x)|^2\, d\sigma_{x,\epsilon}(y)\, d\mu(x) \\ &\quad + \sum_{\substack{j<i\\ i\sim j}}\int_{\Omega_i\cap\Omega_\epsilon}\int_{\Omega_j\cap S(x,\epsilon)} f(x)|\xi(a_i, a_j) + 1 - v_j(x) + v_i(x)|^2\, d\sigma_{x,\epsilon}(y)\, d\mu(x) \\ &\quad + \sum_{\substack{j>i\\ i\sim j}}\int_{\Omega_i\cap\Omega_\epsilon}\int_{\Omega_j\cap S(x,\epsilon)} f(x)|\xi(a_i, a_j) + 1 - v_i(x) + v_j(x)|^2\, d\sigma_{x,\epsilon}(y)\, d\mu(x)\Big] \\ &= \epsilon^{-k-1}\Big[\sum_{i=1}^{n}\int_{\Omega_i\cap\Omega_\epsilon}\int_{\Omega_i\cap S(x,\epsilon)} f(x)|v_i(x) - v_i(y)|^2\, d\sigma_{x,\epsilon}(y)\, d\mu(x) \\ &\quad + \sum_{i=1}^{n}\sum_{i\not\sim j}\int_{\Omega_i\cap\Omega_\epsilon}\int_{\Omega_j\cap S(x,\epsilon)} f(x)[v_i^2(x) + v_j^2(y) + \xi^2(a_i, a_j) + 2v_i(x)\xi(a_i, a_j) \\ &\qquad + 2v_j(y)\xi(a_i, a_j) + 2v_i(x)v_j(y)]\, d\sigma_{x,\epsilon}(y)\, d\mu(x) \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\sum_{\substack{j<i\\ i\sim j}}\int_{\Omega_i\cap\Omega_\epsilon}\int_{\Omega_j\cap S(x,\epsilon)} f(x)[1+\xi^2(a_i,a_j)+v_j^2(y)+v_i^2(x)\\
&\qquad -2v_j(y)+2v_i(x)+2\xi(a_i,a_j)-2v_j(y)v_i(x)\\
&\qquad -2v_j(y)\xi(a_i,a_j)+2v_i(x)\xi(a_i,a_j)]\ d\sigma_{x,\epsilon}(y)\ d\mu(x)\\
&+\sum_{i=1}^{n}\sum_{\substack{j>i\\ i\sim j}}\int_{\Omega_i\cap\Omega_\epsilon}\int_{\Omega_j\cap S(x,\epsilon)} f(x)[1+\xi^2(a_i,a_j)+v_j^2(y)+v_i^2(x)\\
&\qquad +2v_j(y)-2v_i(x)+2\xi(a_i,a_j)-2v_j(y)v_i(x)\\
&\qquad +2v_j(y)\xi(a_i,a_j)-2v_i(x)\xi(a_i,a_j)]\ d\sigma_{x,\epsilon}(y)\ d\mu(x)\Bigg]
\end{aligned}
$$

Using Lemma 2.21 it holds

$$
\begin{aligned}
\lim_{\epsilon\to 0} E_{\epsilon,f}(v) &= \lim_{\epsilon\to 0}\epsilon^{-k-1}\Bigg[\sum_{i=1}^{n}\int_{\Omega_i\cap\Omega_\epsilon}\int_{\Omega_i\cap S(x,\epsilon)} f(x)|v_i(x)-v_i(y)|^2\ d\sigma_{x,\epsilon}(y)\ d\mu(x)\\
&\quad+\sum_{i=1}^{n}\sum_{i\not\sim j}\int_{\Omega_i\cap\Omega_\epsilon}\int_{\Omega_j\cap S(x,\epsilon)} f(x)[v_i^2(x)+v_j^2(y)+2v_i(x)v_j(y)]\ d\sigma_{x,\epsilon}(y)\ d\mu(x)\\
&\quad+\sum_{i=1}^{n}\sum_{\substack{j<i\\ i\sim j}}\int_{\Omega_i\cap\Omega_\epsilon}\int_{\Omega_j\cap S(x,\epsilon)} f(x)[1+v_j^2(y)+v_i^2(x)\\
&\qquad -2v_j(y)+2v_i(x)(1-2v_j(y))]\ d\sigma_{x,\epsilon}(y)\ d\mu(x)\\
&\quad+\sum_{i=1}^{n}\sum_{\substack{j>i\\ i\sim j}}\int_{\Omega_i\cap\Omega_\epsilon}\int_{\Omega_j\cap S(x,\epsilon)} f(x)[1+v_j^2(y)+v_i^2(x)\\
&\qquad -2v_i(x)+2v_j(y)(1-v_i(x))]\ d\sigma_{x,\epsilon}(y)\ d\mu(x)\Bigg]
\end{aligned}
$$

and with Lemma 2.23 and Lemma 2.24 one obtains

$$
\begin{aligned}
\lim_{\epsilon\to 0} E_{\epsilon,f}(v) &= \lim_{\epsilon\to 0}\epsilon^{-k-1}\Bigg[\sum_{i=1}^{n}\int_{\Omega_i\cap\Omega_\epsilon}\int_{\Omega_i\cap S(x,\epsilon)} f(x)|v_i(x)-v_i(y)|^2\ d\sigma_{x,\epsilon}(y)\ d\mu(x)\\
&\quad+\sum_{i=1}^{n}\sum_{i\not\sim j}\int_{\Omega_i\cap\Omega_\epsilon}\int_{\Omega_j\cap S(x,\epsilon)} f(x)[v_i^2(x)+v_j^2(y)]\ d\sigma_{x,\epsilon}(y)\ d\mu(x)\\
&\quad+\sum_{i=1}^{n}\sum_{\substack{j<i\\ i\sim j}}\int_{\Omega_i\cap\Omega_\epsilon}\int_{\Omega_j\cap S(x,\epsilon)} f(x)[|1-v_j(y)|^2+v_i^2(x)]\ d\sigma_{x,\epsilon}(y)\ d\mu(x)\\
&\quad+\sum_{i=1}^{n}\sum_{\substack{j>i\\ i\sim j}}\int_{\Omega_i\cap\Omega_\epsilon}\int_{\Omega_j\cap S(x,\epsilon)} f(x)[|1-v_i(x)|^2+v_j^2(y)]\ d\sigma_{x,\epsilon}(y)\ d\mu(x)\Bigg].
\end{aligned}
$$

Furthermore, for a projection v_i of a map $v \in W^{1,2}(\Omega, N)$ one has

$$\begin{aligned} E_{\epsilon,f}(v_i) \;=\; & \epsilon^{-k-1} \int_{\Omega_i \cap \Omega_\epsilon} \int_{\Omega_i \cap S(x,\epsilon)} f(x)|v_i(x) - v_i(y)|^2 \; d\sigma_{x,\epsilon}(y) \; d\mu(x) \\ & + \epsilon^{-k-1} \Bigg[\sum_{i \not\sim k} \int_{\Omega_i \cap \Omega_\epsilon} \int_{\Omega_k \cap S(x,\epsilon)} f(x) v_i^2(x) \; d\sigma_{x,\epsilon}(y) \; d\mu(x) \\ & \quad + \sum_{\substack{k<i \\ i\sim k}} \int_{\Omega_i \cap \Omega_\epsilon} \int_{\Omega_k \cap S(x,\epsilon)} f(x) v_i^2(x) \; d\sigma_{x,\epsilon}(y) \; d\mu(x) \\ & \quad + \sum_{\substack{k>i \\ i\sim k}} \int_{\Omega_i \cap \Omega_\epsilon} \int_{\Omega_k \cap S(x,\epsilon)} f(x)|1 - v_i(x)|^2 \; d\sigma_{x,\epsilon}(y) \; d\mu(x) \\ & \quad + \sum_{i \not\sim k} \int_{\Omega_k \cap \Omega_\epsilon} \int_{\Omega_i \cap S(x,\epsilon)} f(x) v_i^2(y) \; d\sigma_{x,\epsilon}(y) \; d\mu(x) \\ & \quad + \sum_{\substack{k<i \\ i\sim k}} \int_{\Omega_k \cap \Omega_\epsilon} \int_{\Omega_i \cap S(x,\epsilon)} f(x) v_i^2(y) \; d\sigma_{x,\epsilon}(y) \; d\mu(x) \\ & \quad + \sum_{\substack{k>i \\ i\sim k}} \int_{\Omega_k \cap \Omega_\epsilon} \int_{\Omega_i \cap S(x,\epsilon)} f(x)|1 - v_i(y)|^2 \; d\sigma_{x,\epsilon}(y) \; d\mu(x) \Bigg]. \end{aligned}$$

Thus,

$$\sum_{i=1}^{n} \lim_{\epsilon \to 0} E_{\epsilon,f}(v_i) = \lim_{\epsilon \to 0} E_{\epsilon,f}(v)$$

which yields (cf. Proposition 2.17)

$$E(v) = \sum_{i=1}^{n} \int_{\Omega} |\nabla v_i(x)|^2 d\mu(x). \tag{2.8}$$

Hence, if M is a connected compact Riemannian manifold without boundary and if $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is the classical Dirichlet form on M given by the Laplace-Beltrami operator our definition of the nonlinear energy $\mathcal{E}_N$ coincides with definition of energy introduced by Korevaar/Schoen. This follows easily from equation (2.8) (with $\Omega = M$) and Theorem 2.7.

2.1.3 Special Case that m is a Finite Measure

Throughout this subsection, let m be a finite measure on M. In this special case, one can simplify the definition of the nonlinear energy function.

Given a measurable map $v : M \to N$ we define the energy function $\hat{\mathcal{E}}_N$ by

$$\hat{\mathcal{E}}_N(v) := \limsup_{t \to 0} \frac{1}{2t} \int_M \int_M d^2(v(x), v(y)) p_t(x, dy) m(dx) \tag{2.9}$$

with $\mathcal{D}(\hat{\mathcal{E}}_N) := \{v : M \to N \text{ measurable} : \hat{\mathcal{E}}_N(v) < \infty\}$.

In this more simple case, all measurable bounded functions are in $L^2(M,m)$ and one can deduce that the projections v_i of a map $v \in \mathcal{D}(\hat{\mathcal{E}}_N)$ are elements of $\mathcal{D}(\mathcal{E})$.

Proposition 2.25 *Let $v \in \mathcal{D}(\hat{\mathcal{E}}_N)$ be given. Then one has*

$$v_i \in \mathcal{D}(\mathcal{E}), \quad \forall i \in \{1,\dots,n\}.$$

Proof: For all $i \in \{1,\dots,n\}$ it holds

$$|v_i(x) - v_i(y)| \leq d(v(x),v(y)) \quad \forall x,y \in M.$$

Hence,

$$\limsup_{t\to 0} \frac{1}{2t} \int_M \int_M |v_i(x) - v_i(y)|^2 p_t(x,dy)m(dx) < \infty \qquad \forall i \in \{1,\dots,n\}$$

which yields together with $v_i \in L^2(M,m)$ (cf. [FOT94]) the claim. □

Similar to Theorem 2.7 one obtains the following result.

Theorem 2.26 *For each map $v : M \to N$ the condition $v \in \mathcal{D}(\hat{\mathcal{E}}_N)$ is equivalent to*

$$v_i \in \mathcal{D}(\mathcal{E}), \forall i \in \{1,\dots,n\}$$

and

$$\lim_{t\to 0} \frac{1}{t} \int_{v^{-1}(\tilde{a}_i)} \int_{v^{-1}(\tilde{a}_j)} p_t(x,dy)m(dx) = 0 \quad \forall a_i, a_j \in \mathcal{A} \text{ with } \xi(a_i,a_j) > 0. \tag{2.10}$$

In this situation, for each $v \in \mathcal{D}(\hat{\mathcal{E}}_N)$ the following equalities hold

$$\begin{aligned} \hat{\mathcal{E}}_N(v) &= \lim_{t\to 0} \frac{1}{2t} \int_M \int_M d^2(v(x),v(y)) p_t(x,dy)m(dx) \\ &= \sum_{i=1}^n \mathcal{E}(v_i). \end{aligned} \tag{2.11}$$

with

$$\mathcal{E}(v_i) = \lim_{t\to 0} \frac{1}{2t} \int_M \int_M |v_i(x) - v_i(y)|^2 p_t(x,dy)m(dx).$$

Proof: The proof works out in the same way as the proof of Theorem 2.7. □

Corollary 2.27 *Let M be a compact subset of $\mathbb{R}^k$ with smooth boundary, let p_t^M be the heat semigroup on M reflected on ∂M and let $\mathcal{E}$ be the corresponding Dirichlet form. For all $v \in \mathcal{D}(\hat{\mathcal{E}}_N)$ one has*

$$\hat{\mathcal{E}}_N(v) = \sum_{i=1}^n \int_M |\nabla v_i|^2 d\lambda. \tag{2.12}$$

Remark 2.28 *In the situation that m is a finite measure the nonlinear energy definition given in (2.9) coincides with the definition of nonlinear energy given by Picard.*
Furthermore, it holds

$$\hat{\mathcal{E}}_N = \mathcal{E}_N \qquad \text{and} \qquad \mathcal{D}(\hat{\mathcal{E}}_N) = \mathcal{D}(\mathcal{E}_N).$$

2.2 Nonlinear Dirichlet Problem

The nonlinear Dirichlet problem for a given map g with $\mathcal{E}_N(g) < \infty$ and a subset $D \subset M$ is to find a map u with $\tilde{u} = \tilde{g}$ quasi everywhere on $M \backslash D$ where $\tilde{u}, \tilde{g}$ denote quasi-continuous versions of u and g, resp., which minimizes the nonlinear energy $\mathcal{E}_N$.

Definition 2.29 (Nonlinear Dirichlet problem) *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a regular Dirichlet form on $L^2(M, m)$ which fulfills the conditions (A1) - (A3). Given a map $g \in \mathcal{D}(\mathcal{E}_N)$ and a set $D \subset M$, let us define the class of maps*

$$\tilde{V}_N(g) := \{v \in \mathcal{D}(\mathcal{E}_N) : \tilde{v} = \tilde{g} \text{ quasi everywhere on } M \backslash D\}$$

where $\tilde{v}, \tilde{g}$ denotes quasi-continuous versions of v and g, resp. A map $u \in \tilde{V}_N(g)$ is called solution to the nonlinear Dirichlet problem for g whenever

$$\mathcal{E}_N(u) = \min_{v \in \tilde{V}_N(g)} \mathcal{E}_N(v).$$

The next result (based on a result in [Pic04]) states a sufficient condition for the existence and uniqueness of a solution to the nonlinear Dirichlet problem.

Theorem 2.30 *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a regular Dirichlet form on $L^2(M, m)$ which fulfills the conditions (A1) - (A3) with diffusion X_t. Given a relatively compact open subset D such that X_t quits D during its lifetime and a quasi-continuous map $g \in \mathcal{D}(\mathcal{E}_N)$ there exists a unique (up to modifications) map $u \in \mathcal{D}(\mathcal{E}_N)$ which solves the nonlinear Dirichlet problem for g and D.*

Proof: Proposition 2.12 yields $(\mathcal{E}_N, \mathcal{D}(\mathcal{E}_N)) = (\tilde{\mathcal{E}}_N, \mathcal{D}^b_N)$ and, thus, the claim is a consequence of Proposition 5.2.6 in [Pic04]. □

2.3 Nonlinear Dirichlet Problem for Polygonal Domains in $\mathbb{R}^2$

In this section, let D be a polygonal subset of $\mathbb{R}^2$ and let $M \subset \mathbb{R}^2$ as described in Corollary 2.27 with $D \subset M$ and $dist(D, \partial M) > 0$ be given. Furthermore, let p_t^M be the heat semigroup reflected on ∂M, let $\mathcal{E}$ be the corresponding Dirichlet form on (M, λ) and let us define the nonlinear energy function $\mathcal{E}_N$ as in (2.9) (see Remark 2.28).
The nonlinear Dirichlet problem we will analyze is the following:

Definition 2.31 *Given the set D and a map $g \in \mathcal{D}(\mathcal{E}_N)$, let us define the class of maps*

$$V_N(g) := \{v \in \mathcal{D}(\mathcal{E}_N) : v = g \text{ } m\text{-a.e. on } M \backslash D\}.$$

A map $u \in V_N(g)$ is called a solution to the nonlinear Dirichlet problem for g whenever

$$\mathcal{E}_N(u) = \min_{v \in V_N(g)} \mathcal{E}_N(v).$$

Remark:

1. The classes $V_N(g)$ and $\tilde{V}_N(g)$ (cf. Definition 2.29) coincide, because D has a "nice" boundary.

2. There exists a unique solution to the nonlinear Dirichlet problem.

To solve the nonlinear Dirichlet problem we discretize the problem and construct an iterative numerical method in the same way as described in Subsection 1.3.1. Then, we define a prolongation operator which extends discrete maps to maps defined on the whole domain D similar as in Subsection 1.3.2. Furthermore, we will prove the L^2-convergence of the extended discrete solutions to the solution to the nonlinear Dirichlet problem.

2.3.1 Discrete Nonlinear Dirichlet Problem

In the sequel, let us suppose that an admissible and regular triangulation $\mathcal{T}_h$ of D in the sense of [Cia78] is given. In addition, we suppose the triangles to be "acute". This, means that all interior angles of all triangles of $\mathcal{T}_h$ are less than or equal to $\frac{\pi}{2}$. Finally, we assume that for the map $g \in \mathcal{D}(\mathcal{E}_N)$, specifying the boundary values for the nonlinear Dirichlet problem, $\pi \circ g$ is the modulus of a linear function on the boundary faces of $\mathcal{T}_h$ and that for each triangle T with at least two vertices $x, y \in \partial D$ it holds $\xi(g(x), g(y)) = 0$. (One can assure this by choosing the triangulation $\mathcal{T}_h$ fine enough.)

Having a closer look on Subsection 1.3.1 one will notice that all definitions and results in this section can be transferred one by one to the case where (N, d) is a finite tree instead of an n-spider. Due to this, we will repeat only the main definitions and results using the notations of Subsection 1.3.1.

Definition 2.32 (Discrete nonlinear Dirichlet problem) *Given a map $g : \partial D \to N$ let us define*

$$\bar{V}_N^h(g) := \left\{ \bar{v}_h : \mathcal{N}_h \to N : \bar{v}_h(x) = \bar{g}_h(x) \quad \forall x \in \mathcal{N}_h^\partial \right\}$$

with $\bar{g}_h(x) := g(x), \forall x \in \mathcal{N}_h^\partial$. A map $\bar{u}_h : \mathcal{N}_h \to N$ is called a solution to the discrete nonlinear Dirichlet problem for g whenever $\bar{u}_h$ fulfills the following two conditions:

1. $\bar{u}_h \in \bar{V}_N^h(g)$

2. $\mathcal{E}_N^h(\bar{u}_h) = \min_{\bar{v}_h \in \bar{V}_N^h(g)} \mathcal{E}_N^h(\bar{v}_h)$, *where*

$$\mathcal{E}_N^h(\bar{v}_h) := \frac{1}{2} \sum_{x_i, x_j \in \mathcal{N}_h} d^2(\bar{v}_h(x_i), \bar{v}_h(x_j)) p(x_i, x_j) \mu(x_i) \tag{2.13}$$

is called the discrete energy corresponding to $\mathcal{T}_h$.

According to [Stu01] we have the following result.

Proposition 2.33 *For each $g : \partial D \to N$ there is a unique solution to the discrete nonlinear Dirichlet problem for g.*

Definition 2.34 *We define Markov operators $p_1, \dots, p_k, k := \#\mathring{\mathcal{N}}_h$ by*

$$p_i(x, y) := \begin{cases} p(x, y), & \text{if } x = x_i \text{ and } x \sim y \\ 1, & \text{if } x \neq x_i \text{ and } x = y \\ 0, & \text{otherwise} \end{cases} \qquad i = 1, \dots, k$$

and a nonlinear Markov operator Q by

$$Q := p_k^N \circ \cdots \circ p_1^N .$$

Proposition 2.35 *Let $\bar{u}_h$ be the solution to the discrete nonlinear Dirichlet problem for g. Then for each $\bar{v}_h \in \bar{V}_N^h(g)$ one has*

$$\lim_{n \to \infty} d_\infty(Q^n \bar{v}_h, \bar{u}_h) = 0, \quad \text{where } d_\infty(\bar{v}_h, \bar{w}_h) := \sup_{x \in M} d(\bar{v}_h(x), \bar{w}_h(x)).$$

Also the algorithm to get an approximation to the exact solution $\bar{u}_h$ to the discrete nonlinear Dirichlet problem for the boundary value map g is the same:

$\bar{v}_h = g|_{\mathcal{N}_h}$
do
    $\bar{w}_h = \bar{v}_h$
    for $j = 1$ to k
        $\bar{v}_h(x_j) = p_j^N \bar{v}_h(x_j) = \text{argmin}_{z \in N} \{\sum_{y \in \mathcal{N}_h} d^2(z, \bar{v}_h(y)) p(x_j, y)\}$
until $(\max_{x_j \in \mathcal{N}_h} d(\bar{v}_h(x_j), \bar{w}_h(x_j)) \leq EPS)$.

Here EPS is a user prescribed threshold value.

Obviously, it is more complicated to determine the barycenter of a discrete probability distribution on a tree than on a spider.
An algorithm to calculate $\text{argmin}_{z \in N} \{\sum_{y \in \mathcal{N}_h} d^2(z, \bar{v}_h(y)) p(x_j, y)\}$ is given in the next subsection.

2.3.2 Barycenters on Finite Trees

Without restrictions, we will assume that the length of all edges of the finite tree (N, d) is equal to one. Furthermore, let q be a discrete probability distribution on N with finite support.

Just to remember, the point

$$b(q) := \operatorname*{argmin}_{z\in N} \sum_{x\in N} d^2(z,x)q(x) \tag{2.14}$$

is called barycenter of q. For details see [Stu01].

To develop the algorithm to calculate the barycenter of a discrete probability distribution on N with finite support we will present an identification of this barycenter.

Let z be a vertex of the tree N. The set $N\backslash\{z\}$ decomposes into a finite disjoint family K_z of connected components $N_{z,i}, i \in K_z$, and for each $i \in K_z$ we define the numbers

$$r_{z,i}(q) := \sum_{x\in N_{z,i}} d(z,x)q(x), \qquad b_{z,i}(q) := r_{z,i}(q) - \sum_{\substack{j\in K_z\\ j\neq i}} r_{z,j}(q).$$

Note that $b_{z,i}(q) > 0$ for at most one $i \in K_z$.

Proposition 2.36 *If $b_{z,i}(q) \in]0,\frac{1}{2}]$ for some $z \in \mathcal{V}$ and for some $i \in K_z$ then $b(q)$ is the unique point $x \in N$ with $x \in N_{z,i}$ and $d(z,x) = b_{z,i}(q)$. If $b_{z,i}(q) \in]0,\frac{1}{2}[$ one has $b_{z,i}(q) < \max_{j\in K_{\tilde{z}}} b_{\tilde{z},j}(q)$ for all other points $\tilde{z} \in \mathcal{V}, \tilde{z} \neq z$. If $b_{z,i}(q) = \frac{1}{2}$ there is exactly one other point $\tilde{z} \in \mathcal{V}$ with $b_{\tilde{z},j}(q) = \frac{1}{2}, j \in K_{\tilde{z}}$ and for all other $\hat{z} \in \mathcal{V}, \hat{z} \neq z, \hat{z} \neq \tilde{z}$ it holds $b_{z,i}(q) < \max_{j\in K_{\hat{z}}} b_{\hat{z},j}(q)$. If $b_{z,i}(q) \leq 0$ for some $z \in \mathcal{V}$ and for all $i \in K_z$ then $b(q) = z$.*

Proof: Let $b(q) \notin \mathcal{V}$ and let us denote by z_q the point in $\mathcal{V}$ with $d(z_q, b(q)) \in\,]0,\frac{1}{2}]$ minimal. Without restrictions, we assume that $b(q) \in N_{z_q,1}$. Then

$$\begin{aligned}
&\sum_{x\in N} d^2(b(q),x)q(x) \\
= &\sum_{x\in N_{z_q,1}} (d(z_q,b(q)) - d(z_q,x))^2 q(x) + \sum_{\substack{j\in K_{z_q}\\ j\neq 1}} \sum_{x\in N_{z_q,j}} (d(z_q,b(q)) + d(z_q,x))^2 q(x)
\end{aligned}$$

is minimal. Thus, $r \mapsto F(r)$, where

$$F(r) := \sum_{x\in N_{z_q,1}} (r - d(z_q,x))^2 q(x) + \sum_{\substack{j\in K_{z_q}\\ j\neq 1}} \sum_{x\in N_{z_q,j}} (r + d(z_q,x))^2 q(x),$$

attains its minimum on $]0,1[$ in $r = d(z_q, b(q))$. This implies

$$\begin{aligned}
0 = \frac{1}{2}F'(d(z_q,b(q))) &= d(z_q,b(q)) - r_{z_q,1}(q) + \sum_{\substack{j\in K_{z_q}\\ j\neq 1}} r_{z_q,j}(q) \\
&= d(z_q,b(q)) - b_{z_q,1}(q)
\end{aligned}$$

and, thus, $d(z_q, b(q)) = b_{z_q,1}$.

If $b(q)$ is not the midpoint of an edge then $b_{z_q,1}(q) \in]0, \frac{1}{2}[$. Now, let $z \in \mathcal{V} \cap N_{z_q,i}, i \neq 1$, be given and let us assume without restrictions that $z_q \in N_{z,1}$. Then one has

$$\begin{aligned}
b_{z,1}(q) &= \sum_{x \in N_{z,1}} d(z,x)q(x) - \sum_{\substack{j \in K_z \\ j \neq 1}} \sum_{x \in N_{z,j}} d(z,x)q(x) \\
&\geq \sum_{\substack{j \in K_{z_q} \\ j \neq i}} \sum_{x \in N_{z_q,j}} (d(z_q,z) + d(z_q,x))q(x) + \sum_{x \in N_{z_q,j} \cap N_{z,1}} (d(z_q,z) - d(z_q,x))q(x) \\
&\quad - \sum_{x \in N_{z_q,j} \setminus N_{z,1}} (d(z_q,x) - d(z_q,z))q(x) \\
&= d(z_q,z) + \sum_{\substack{j \in K_{z_q} \\ j \neq i}} \sum_{x \in N_{z_q,j}} d(z_q,x)q(x) - \sum_{x \in N_{z_q,i}} d(z,x)q(x) \\
&\geq d(z_q,z) + b_{z_q,1}(q) \geq 1 + b_{z_q,1}(q).
\end{aligned}$$

Given $z \in \mathcal{V} \cap N_{z_q,1}$, let us assume without restrictions $z_q \in N_{z,1}$. Then

$$\begin{aligned}
b_{z,1}(q) &= \sum_{x \in N_{z,1}} d(z,x)q(x) - \sum_{\substack{j \in K_z \\ j \neq 1}} \sum_{x \in N_{z,j}} d(z,x)q(x) \\
&\geq \sum_{\substack{i \in K_{z_q} \\ i \neq 1}} \sum_{x \in N_{z_q,i}} (d(z_q,z) + d(z_q,x))q(x) + \sum_{x \in N_{z_q,1} \cap N_{z,1}} (d(z_q,z) - d(z_q,x))q(x) \\
&\quad - \sum_{x \in N_{z_q,1} \setminus N_{z,1}} (d(z_q,x) - d(z_q,z))q(x) \\
&= d(z_q,z) + \sum_{\substack{i \in K_{z_q} \\ i \neq 1}} \sum_{x \in N_{z_q,i}} d(z_q,x)q(x) - \sum_{x \in N_{z_q,1}} d(z_q,x)q(x) \\
&= d(z_q,z) + \sum_{\substack{i \in K_{z_q} \\ i \neq 1}} r_{z_q,i}(q) - r_{z_q,1}(q) \\
&= d(z_q,z) - b_{z_q,1}(q) \\
&> \underbrace{d(z_q,z)}_{\geq 1} \underbrace{- b_{z_q,1}(q) - \frac{1}{2}}_{> -1} + b_{z_q,1}(q) > b_{z_q,1}(q).
\end{aligned}$$

If $b(q)$ is the midpoint of an edge of the tree it follows, that there are two points $z_q, \tilde{z}_q \in \mathcal{V}$ with $d(z_q, b(q)) = d(\tilde{z}_q, b(q)) = \frac{1}{2}$. Thus, one has $\max_{i \in K_{z_q}} b_{z_q,i}(q) = \max_{j \in K_{\tilde{z}_q}} b_{\tilde{z}_q,j}(q) = \frac{1}{2}$ and with the same arguments as before one can show $\max_{i \in K_{z_q}} b_{z_q,i}(q) < \max_{j \in K_z} b_{z,j}(q), z \in \mathcal{V}, z \neq z_q, z \neq \tilde{z}_q$.

Similarly, $b(q) \in \mathcal{V}$, implies $F'(0) \geq 0$ with

$$F(r) := \sum_{x \in N} (r + d(z_q,x))^2 q(x)$$

and, thus, $0 \geq b_{z_q,i}(q)$. □

The previous proposition yields the following algorithm to calculate the barycenter of a discrete probability distribution q on a tree N with finite support:

$\mathcal{V}_c := \emptyset$
do
    $z \in \mathcal{V} \backslash \mathcal{V}_c$
    $b = \max_{i \in K_z} b_{z,i}(q)$
    $cc = \operatorname{argmax}_{i \in K_z} b_{z,i}(q)$
    $\mathcal{V}_c = \mathcal{V}_c \cup \{z\}$
until $(b \leq \frac{1}{2})$
if $b \leq 0$
    $b(q) = z$
else if
    $b(q) = x$ with $x \in N_{z,cc}$ and $d(x, z) = b$

2.3.3 Extending Maps on Vertices to Maps on the Domain

By means of a proper prolongation procedure, to each map in $\bar{V}_N^h(g)$ we are going to associate a map in $V_N(g)$. In other words, each map $\bar{v}_h$ which is defined on the vertices of the triangulation $\mathcal{T}_h$ will be extended to a map v_h, defined on the whole domain D, with almost the same energy, i.e. for each $\bar{v}_h \in \bar{V}_N^h(g)$ one has

$$\mathcal{E}_N(v_h) \leq \mathcal{E}_N^h(\bar{v}_h) + R_{g,D},$$

with a nonnegative constant $R_{g,D}$ only depending on the polygonal domain D, the regularity of the triangulation $\mathcal{T}_h$, and the map g.

Let us consider the set D, the triangulation $\mathcal{T}_h$, the set of all vertices of the triangulation $\mathcal{N}_h = \{x_1, \ldots, x_l\}$, and a map $g \in \mathcal{D}(\mathcal{E}_N)$ as described above. For simplicity, we will assume that the length of of all edges of the tree is equal to one.
Given a vector $\bar{v}_h \in N^l$ our aim is to construct a continuous map $v_h : \bar{D} \to N$, affine on each triangle $T \in \mathcal{T}_h$ (or better affine on appropriate subtriangles of each triangle T), such that $v_h^i := v_h(x_i) = \bar{v}_h(x_i)$ for all $i = 1, \ldots, l$. For this purpose it is enough to define v_h on each triangle $T \in \mathcal{T}_h$. Let $T \in \mathcal{T}_h$ be given with vertices x_0, x_1, x_2 and let $\bar{v}_h(x_0) \in \tilde{a}_i, \bar{v}_h(x_1) \in \tilde{a}_j, \bar{v}_h(x_2) \in \tilde{a}_k$. To define $v_h|_T$ we have to distinguish the following cases:

(i) $i \sim j, i \sim k$ and $j \sim k$

(ii) $i \sim j, i \not\sim k$ and $j \not\sim k$

(iii) $i \sim j, i \sim k$ and $j \not\sim k$

(iv) $i \not\sim j, i \not\sim k$ and $j \not\sim k$

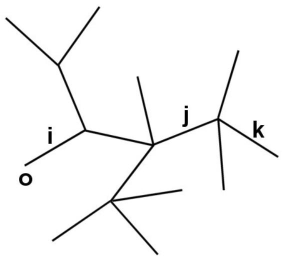

Figure 2.2: case (i)

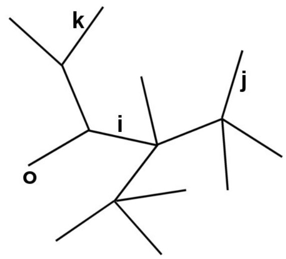

Figure 2.3: case (ii)

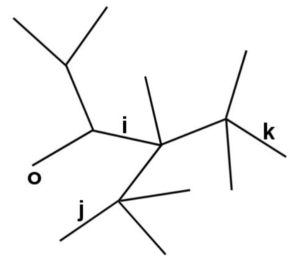

Figure 2.4: case (iii)

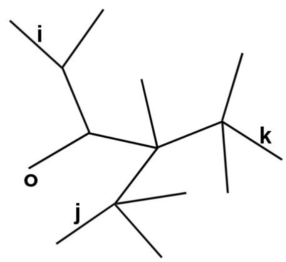

Figure 2.5: case (iv)

Before defining the prolongation, let us introduce the following definitions:

- Given an edge $a \in \mathcal{A}$ and a point $x \in N$ we define $d(x,a) := \inf_{y\in a} d(x,y)$.
- Given k edges $a_{i_1}, \ldots, a_{i_k} \in \mathcal{A}$ we define the branchpoints $\tilde{b}_{i_1\ldots i_k} \in \mathcal{V}$ and $\hat{b}_{i_1\ldots i_k} \in \mathcal{V}$ by

$$\begin{aligned}\tilde{b}_{i_1\ldots i_k} &:= \underset{z\in\mathcal{V}\cap\tilde{S}_{i_1\ldots i_k}}{\operatorname{argmax}} \{d(o,z)\}\\ \hat{b}_{i_1\ldots i_k} &:= \underset{z\in\mathcal{V}\cap\hat{S}_{i_1\ldots i_k}}{\operatorname{argmax}} \{d(o,z)\}\end{aligned}$$

with

$$\tilde{S}_{i_1\ldots i_k} := \bigcap_{j=i_1}^{i_k} \gamma_{o,e_j^-} \quad \text{and} \quad \hat{S}_{i_1\ldots i_k} := \bigcap_{\substack{j,l=i_1\\ j>l}}^{i_k} \gamma_{x_j,x_l}, \quad x_r \in a_r, r \in \{i_1,\ldots,i_k\}.$$

case(i): Without restriction, we may assume that $d(o,a_i) \le d(o,a_j) \le d(o,a_k)$ and we set $n := \xi(a_i,a_k)$. Furthermore, let $a_{l_1},\ldots,a_{l_n}$ be the edges $a \in \mathcal{A}$ with $a \subset \gamma_{x,y}, x \in a_i, y \in a_k$, whereby we assume that $d(x,a_{l_1}) \le \ldots d(x,a_{l_n})$.
Now, we define an affine function $\beta : T \to \mathbb{R}_+$ with $\beta(x_j) = \pi(\bar{v}_h(x_j)), j = 0,1,2$.
Let us assume that $\bar{v}_h(x_0) \in a_i$. Then we set for each $x \in T$

$$v_h|_T(x) := \begin{cases} (i,\beta(x)), & \text{if } \beta(x_0) \le \beta(x) \le [\beta(x_0)+1]\\ (l_1,\beta(x)), & \text{if } [\beta(x_0)+1] < \beta(x) \le [\beta(x_0)+2]\\ \vdots & \\ (l_n,\beta(x)), & \text{if } [\beta(x_0)+n] < \beta(x) \le [\beta(x_0)+(n+1)]\\ (k,\beta(x)), & \text{if } [\beta(x_0)+(n+1)] < \beta(x)\end{cases}$$

with $[z] \stackrel{\triangle}{=}$ largest integer less or equal than z.

In the case $\bar{v}_h(x_0) \in \mathcal{N} \backslash \{o\}$ we set for each $x \in T$

$$v_h|_T(x) := \begin{cases} e_i^+, & \text{if } \beta(x) = \beta(x_0) \\ (l_1, \beta(x)), & \text{if } \beta(x_0) < \beta(x) \leq \beta(x_0) + 1 \\ \vdots & \\ (l_n, \beta(x)), & \text{if } \beta(x_0) + n < \beta(x) \leq \beta(x_0) + (n+1) \\ (k, \beta(x)), & \text{if } \beta(x_0) + (n+1) < \beta(x) \end{cases}$$

and if $\bar{v}_h(x_0) = o$ we set for each $x \in T$

$$v_h|_T(x) := \begin{cases} o, & \text{if } \beta(x) = 0 \\ (i, \beta(x)), & \text{if } 0 < \beta(x) \leq 1 \\ (l_1, \beta(x)), & \text{if } 1 < \beta(x) \leq 2 \\ \vdots & \\ (l_n, \beta(x)), & \text{if } n < \beta(x) \leq n+1 \\ (k, \beta(x)), & \text{if } n+1 < \beta(x) \end{cases}$$

case(ii): Without restriction, we may assume $d(o, a_i) \leq d(o, a_j)$. Furthermore, let $a_{l_1}, \ldots, a_{l_n}$ be the edges $a \in \mathcal{A}$ with $a \cap \gamma_{\tilde{b}_{ijk}, y} \neq \emptyset, y \in a_k$, whereby we assume that $d(x, a_{l_1}) \leq \cdots \leq d(x, a_{l_n})$ and let $a_{k_1}, \ldots, a_{k_m}$ be the edges $a \in \mathcal{A}$ with $a \cap \gamma_{\tilde{b}_{ijk}, y} \neq \emptyset, y \in a_j$, and $d(x, a_{k_1}) \leq \cdots \leq d(x, a_{k_m})$.
Now, we define an affine function $\beta : T \to \mathbb{R}$ with $\beta(x_r) = \pi(\bar{v}_h(x_r)) - \pi(\tilde{b}_{ijk}), r = 0, 1$ and $\beta(x_2) = -(\pi(\bar{v}_h(x_2)) - \pi(\tilde{b}_{ijk}))$ and for each $x \in T$ we set

$$v_h|_T(x) := \begin{cases} \tilde{b}_{ijk} & \text{if } \beta(x) = 0 \\ (l_1, -\beta(x) + \pi(\tilde{b}_{ijk})), & \text{if } -1 \leq \beta(x) < 0 \\ \vdots & \\ (l_n, -\beta(x) + \pi(\tilde{b}_{ijk})) & \text{if } \beta(x) < -(n-1) \\ (k_1, \beta(x) + \pi(\tilde{b}_{ijk})) & \text{if } 0 < \beta(x) \leq 1 \\ \vdots & \\ (k_m, \beta(x) + \pi(\tilde{b}_{ijk})) & \text{if } (m-1) < \beta(x) \end{cases}$$

case(iii): In the sequel, we interpret all the indices i as i mod (3).
We define the points $x_{i,i+1}, i \in \{0, 1, 2\}$ by

$$x_{i,i+1} = \gamma_{i,i+1} x_i + (1 - \gamma_{i,i+1}) x_{i+1},$$

where

$$\gamma_{i,i+1} = \frac{\pi(\bar{v}_h(x_{i+1})) - \pi(b_{jk})}{\pi(\bar{v}_h(x_i)) + \pi(\bar{v}_h(x_{i+1})) - 2\pi(b_{jk})} \qquad i \in \{0, 1, 2\}$$

and on the triangles $T_i := \Delta x_i x_{i,i+1} x_{i,i+2}, i \in \{0,1,2\}$ we define the maps $w_i : T_i \to N$, as described in case(i), with $w(x_i) := \bar{v}_h(x_i), w(x_{i,i+1}) := w_i(x_{i,i+2}) := \tilde{b}_{jk}$, for $i \in \{0,1,2\}$. Moreover, we define $T_{0,1,2} := \Delta x_{0,1} x_{0,2} x_{1,2}$ and we set

$$v_h|_T(x) := \begin{cases} w_i(x), & \text{if } x \in T_i \quad i \in \{0,1,2\} \\ \tilde{b}_{jk}, & \text{if } x \in T_{0,1,2}. \end{cases}$$

case(iv): Without restriction, we may assume, that $\hat{b}_{ijk} \notin \gamma_{o,x}$ and that $\hat{b}_{ijk} \in \gamma_{o,y} \cap \gamma_{o,z}$. We define the points $x_{i,i+1}, i \in \{0,1,2\}$ by

$$x_{i,i+1} = \gamma_{i,i+1} x_i + (1 - \gamma_{i,i+1}) x_{i+1},$$

where

$$\gamma_{i,i+1} = \frac{\pi(\bar{v}_h(x_{i+1})) - \pi(\hat{b}_{ijk})}{\pi(\bar{v}_h(x_i)) + \pi(\bar{v}_h(x_{i+1})) - 2\pi(\hat{b}_{ijk})} \qquad i \in \{0,1,2\}$$

and on the triangles $T_i := \Delta x_i x_{i,i+1} x_{i,i+2}, i \in \{1,2\}$ we define the maps $w_i : T_i \to N$, as described in case(i), with $w_i(x_i) := \bar{v}_h(x_i), w_i(x_{i,i+1}) := w_i(x_{i,i+2}) := \hat{b}_{ijk}$, for $i \in \{1,2\}$. Moreover, we define on the triangle $T_0 := \Delta x_0 x_{0,1} x_{0,2}$ a map $w_0 : T_0 \to N$, as described in case(ii), with $w_0(x_0) := \bar{v}_h(x_0), w_0(x_{0,1}) := w_0(x_{0,2}) := \hat{b}_{ijk}$, and we set

$$v_h|_T(x) := \begin{cases} w_i(x), & \text{if } x \in T_i \quad i \in \{0,1,2\} \\ \hat{b}_{ijk}, & \text{if } x \in T_{0,1,2} \end{cases}$$

with $T_{0,1,2} := \Delta x_{0,1} x_{0,2} x_{1,2}$.

The four cases described above are graphically summarized in the following figures. In all these cases, points of the tree are described by a colour ($\hat{=}$ edge) and a height ($\hat{=}$ distance from root). The black colour for different heights describes the branchpoints.

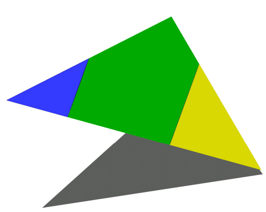

Figure 2.6: case (i)

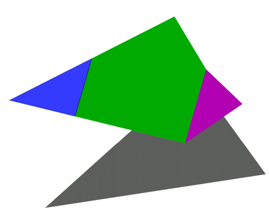

Figure 2.7: case (ii)

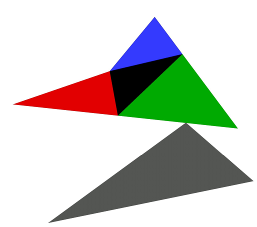

Figure 2.8: case (iii)

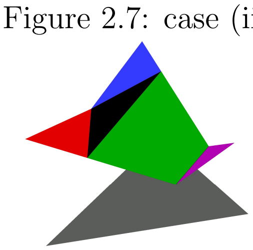

Figure 2.9: case (iv)

Definition 2.37 *We define an injective mapping* $J_h : \bar{V}_N^h(g) \to V_N(g)$ *by*

$$J_h(\bar{v}_h)(x) := \begin{cases} v_h(x), & \text{if } x \in D \\ g(x), & \text{otherwise,} \end{cases}$$

for $\bar{v}_h \in \bar{V}_N^h(g)$. *In the sequel, we will denote the prolongation* $J_h(\bar{v}_h)$ *of* $\bar{v}_h$ *just by* v_h.

Remark: Note that for each $\bar{v}_h \in \bar{V}_N^h(g)$ one has

$$\int_D |\nabla(\pi_i(v_h))|^2 d\lambda < \infty \qquad \forall i \in \{1, \dots, n\}$$

and

$$v_h(x) = g(x) \qquad \forall x \in M \backslash D.$$

Therefore, v_h is well defined as an element of the space $V_N(g)$. In fact, according to Corollary 2.27 one has

$$\mathcal{E}_N(v_h) = \sum_{i=1}^n \left[\int_D |\nabla(\pi_i(v_h))|^2 d\lambda + \int_{M\backslash D} |\nabla(\pi_i(g))|^2 d\lambda \right].$$

Proposition 2.38 *For every* $\bar{v}_h \in \bar{V}_N^h(g)$ *one has*

$$\mathcal{E}_N(v_h) \leq \mathcal{E}_N^h(\bar{v}_h) + R_{g,D}. \tag{2.15}$$

where

$$R_{g,D} := \sum_{i=1}^n \int_{M\backslash D} |\nabla(\pi_i(g))|^2 d\lambda. \tag{2.16}$$

Proof: Observe that due to (1.12) the discrete nonlinear energy $\mathcal{E}_N^h(\bar{v}_h)$ may be rewritten as

$$\mathcal{E}_N^h(\bar{v}_h) = -\frac{1}{2} \sum_{T\in\mathcal{T}_h} \sum_{x_i, x_j \in \mathcal{N}_h} d^2(\bar{v}_h(x_i), \bar{v}_h(x_j)) \int_T \nabla\phi_h^{i,T} \nabla\phi_h^{j,T} d\lambda.$$

Furthermore, by definition of J_h and Corollary 2.27,

$$\mathcal{E}_N(v_h) = \sum_{T\in\mathcal{T}_h} \sum_{i=1}^n \int_T |\nabla(\pi_i(v_h))|^2 d\lambda + R_{g,D}.$$

Thus, the rest of the proof amounts to show that for each $T \in \mathcal{T}_h$ with vertices a_0, a_1, a_2 and with $v_h^i := v_h(a_i), i \in \{0, 1, 2\}$, the following inequality holds:

$$\begin{aligned} \sum_{j=1}^n \int_T |\nabla\pi_j(v_h)|^2 \, d\lambda \;\; &\leq \;\; -d^2(v_h^0, v_h^1) \int_T \nabla\phi_h^{0,T} \nabla\phi_h^{1,T} d\lambda \\ &\quad -d^2(v_h^1, v_h^2) \int_T \nabla\phi_h^{1,T} \nabla\phi_h^{2,T} d\lambda \; - \; d^2(v_h^0, v_h^2) \int_T \nabla\phi_h^{0,T} \nabla\phi_h^{2,T} d\lambda. \end{aligned} \tag{2.17}$$

By the definition of J_h, to each $\bar{v}_h \in \bar{V}_N^h$ one has to prove (2.17) for the four different cases described at the beginning of this subsection. The cases (i) and (ii) can be reduced to the well known linear case, holding the equality in (2.17). The proofs in the cases (iii) and (iv) are similar such that we will only treat case (iv).
By construction of v_h, it holds for $i = \{0, 1, 2\}$ that

$$\sum_{j=1}^{n} \int_{T_i} |\nabla(\pi_j(v_h))|^2 d\lambda = \int_{T_i} \beta_i^2 d\lambda = \frac{\lambda(T_i)}{\lambda(T)} \int_T \beta_i^2 d\lambda$$

for some constant β_i.
Furthermore, $\beta_i = \nabla w_h^i$, where w_h^i is affine on T with nodal values $w_h^i(a_i) = d(v_h^i, \hat{b}_{ijk})$ and $w_h^i(a_{i\pm1}) = -d(v_h^{i\pm1}, \hat{b}_{ijk})$.
Given a function w affine on T it holds (cf. (1.12))

$$\int_T |\nabla w|^2 d\lambda = -\sum_{\substack{i,j=0 \\ i<j}} (w(a_i) - w(a_j))^2 \int_T \nabla\phi_h^{i,T} \nabla\phi_h^{j,T} d\lambda \tag{2.18}$$

Hence, one obtains

$$\begin{aligned}
&\int_{T_i} \beta_i^2 d\lambda = \frac{\lambda(T_i)}{\lambda(T)} \int_T |\nabla w_h^i|^2 d\lambda \\
&= -\left[d^2(v_h^i, v_h^{i+1}) \int_T \nabla\phi_h^{i,T} \nabla\phi_h^{i+1,T} d\lambda + d^2(v_h^{i+1}, v_h^{i+2}) \int_T \nabla\phi_h^{i+1,T} \nabla\phi_h^{i+2,T} d\lambda \right. \\
&\qquad \left. + d^2(v_h^i, v_h^{i+2}) \int_T \nabla\phi_h^{i,T} \nabla\phi_h^{i+2,T} d\lambda \right] \cdot \lambda(T_i)/\lambda(T), \qquad i \in \{0, 1, 2\},
\end{aligned}$$

which completes the proof, since $\lambda(T_0 \cup T_1 \cup T_2) \leq \lambda(T)$. □

2.3.4 Convergence

In what follows, we will consider a sequence of successively refined, regular triangulations $\mathcal{T}_h$ and ask for the convergence of the resulting discrete harmonic maps $u_h \in V_N(g)$ to the solution u of the continuous problem for $h \to 0$. For simplicity we here restrict to homogeneously refined meshes, i.e. we assume

$$\min_{T \in \mathcal{T}_h} h(T) \geq c \max_{T \in \mathcal{T}_h} h(T)$$

with $h(T) = \operatorname{diam}(T)$. As in Subsection 1.3.3 we generate the sequence of triangulation applying an iterative subdivision of triangles into four congruent triangles and we will use a generic constant C.

Theorem 2.39 *Let $\bar{u}_h$ be the solution to the discrete nonlinear Dirichlet problem for a map g as described above and let $J_h : \bar{V}_N^h(g) \to V_N(g)$ be the mapping defined in Section 2.3.3. Then*

$$\lim_{h \to 0} \mathcal{E}_N(u_h) = \mathcal{E}_N(u). \tag{2.19}$$

The proof works out in a similar way as the proof of Theorem 1.19. For the readers convenience let us recall that S_i is the patch for the vertex x_i (cf. Definition 1.20) and that $\mathcal{I}_h$ denotes the Clement operator (cf. Definition 1.21). Furthermore, for the proof of Theorem 2.39 we need the following lemma.

Lemma 2.40 *Let v be a Hölder continuous function on $\bar{D}$ and let $\delta_h < 1$ with $\delta_h \searrow 0$ for $h \to 0$ be given. Then there is a constant $C_I > 0$ independent of h such that*

$$|\mathcal{I}_h(\frac{1}{1-\delta_h}v(x)) - \frac{1}{1-\delta_h}v(x)| \leq C_I \cdot h^\alpha, \qquad \forall x \in \bar{D}.$$

Proof: This result is a consequence of Lemma 1.22 using the fact that the local L^2-projection p_i^h of $\frac{1}{1-\delta_h}v|_{S_i}$ on S_i is $\frac{1}{1-\delta_h} \cdot p_i$ and that $\frac{1}{1-\delta_h}$ is bounded from above for all h by a constant c. □

Proof of Theorem 2.39:
For simplicity, we will assume that the length of all edges of the tree is equal to one.
With similar arguments as used in Section 2.1.2 one can show that for any map $v : D \to N$ with finite energy in the sense of Korevaar/Schoen (cf. [KS93]) it holds

$$E_N(v) = \sum_{i=1}^n \int_D |\nabla v_i(x)|^2 d\lambda(x)$$

with v_i being the projections of v. Thus, from Corollary 2.27, from the Lipschitz continuity of g on ∂D and from [Ser94] it follows that the solution to the nonlinear Dirichlet problem u is Hölder continuous on $\bar{D}$ with $\alpha > \log_4 3$.
In the sequel, we will denote the Hölder constant of the map u by C_α. Now, we define

$$N_0^i := \{x \in D : u(x) = e_i^-\}$$

and

$$N_0^{i,h} := \{y \in D : \operatorname{dist}(y, N_0^i) \leq \gamma \cdot h\}$$

for a constant $\gamma > 0$. Then (for sufficiently small h)

$$\frac{1}{1-\delta_h}\left((\pi_i(u) - \delta_h)^+\right)(x) = 0 \qquad \forall x \in N_0^{i,h}$$

holds for all $i \in \{1, \ldots, n\}$ with $\delta_h := C_\alpha \gamma^\alpha \cdot h^\alpha$.
By this construction, we ensure that the regions where $u \equiv e_i^-$ are fat strips which are of the minimal width $\gamma \cdot h$. Hence, choosing γ large enough we are able to avoid an interference of the involved local L^2 projections in the construction of a comparison function.
For each $i \in \{1, \ldots, n\}$ we define $\mathcal{I}_{h,i}^\delta(u) := \mathcal{I}_h(\frac{1}{1-\delta_h}((\pi_i(u) - \delta_h)^+))$. It holds

$$||\mathcal{I}_{h,i}^\delta(u) - \frac{1}{1-\delta_h}((\pi_i(u) - \delta_h)^+)||_{1,2} = \nu(h) \stackrel{h \to 0}{\longrightarrow} 0 \quad \forall i \in \{1, \ldots, n\}$$

(cf. [Cle75] and Corollary 2.27). Moreover, one has

$$\left|\int_D |\nabla(\frac{1}{1-\delta_h}((\pi_i(u)-\delta_h)^+))|^2 d\lambda - \int_D |\nabla(\pi_i(u))|^2 d\lambda\right| \to 0 \qquad h\to 0.$$

Thus, it follows

$$\int_D |\nabla(\mathcal{I}^\delta_{h,i}(u))|^2 d\lambda \leq \int_D |\nabla(\pi_i(u))|^2 d\lambda + \beta(h) \tag{2.20}$$

where $\beta(h)$ is converging to 0 for $h\to 0$.

Observe that the functions $\frac{1}{1-\delta_h}((\pi_i(u)-\delta_h)^+), 1\leq i\leq n$, are Hölder continuous with the same Hölder coefficient α as u and the Hölder constant $\frac{1}{1-\delta_h}\cdot C_\alpha$. Hence, according to Lemma 2.40, the following inequalities hold for each $i\in\{1,\dots,n\}, x,y\in T$:

$$\begin{aligned}
|\mathcal{I}^\delta_{h,i}(u)(x)-\mathcal{I}^\delta_{h,i}(u)(y)| &\leq |\mathcal{I}^\delta_{h,i}(u)(x)-\frac{1}{1-\delta_h}((\pi_i(u)-\delta_h)^+)(x)| \\
&\quad +|\frac{1}{1-\delta_h}((\pi_i(u)-\delta_h)^+)(x)-\frac{1}{1-\delta_h}((\pi_i(u)-\delta_h)^+)(y)| \\
&\quad +|\frac{1}{1-\delta_h}((\pi_i(u)-\delta_h)^+)(y)-\mathcal{I}^\delta_{h,i}(u)(y)| \\
&\leq (2\,C_I+\frac{1}{1-\delta_h}\cdot C_\alpha)\cdot h^\alpha
\end{aligned}$$

and

$$\begin{aligned}
|\mathcal{I}^\delta_{h,i}(u)(x)-(\pi_i(u))(y)| &\leq |\mathcal{I}^\delta_{h,i}(u)(x)-\mathcal{I}^\delta_{h,i}(u)(y)| \\
&\quad +|\mathcal{I}^\delta_{h,i}(u)(y)-\frac{1}{1-\delta_h}((\pi_i(u)-\delta_h)^+)(y)| \\
&\quad +|\frac{1}{1-\delta_h}((\pi_i(u)-\delta_h)^+)(y)-(\pi_i(u))(y)| \\
&\leq |\mathcal{I}^\delta_{h,i}(u)(x)-\mathcal{I}^\delta_{h,i}(u)(y)|+(C_I+\frac{1}{1-\delta_h}\cdot C_\alpha\gamma^\alpha)h^\alpha
\end{aligned}$$

as well as

$$\begin{aligned}
|(\pi_i(u))(x)-(\pi_i(u))(y)| &\leq |(\pi_i(u))(x)-\mathcal{I}^\delta_{h,i}(u)(x)|+|\mathcal{I}^\delta_{h,i}(u)(x)-\mathcal{I}^\delta_{h,i}(u)(y)| \\
&\quad +|\mathcal{I}^\delta_{h,i}(u)(y)-(\pi_i(u))(y)| \\
&\leq |\mathcal{I}^\delta_{h,i}(u)(x)-\mathcal{I}^\delta_{h,i}(u)(y)|+C\cdot h^\alpha.
\end{aligned}$$

(Notice that $\frac{1}{1-\delta_h}$ is bounded from above for all h sufficiently small).

By means of $\mathcal{I}^\delta_{h,i}(u)$ one can now introduce a piecewise affine function ξ^h_i on $\bar{D}$, which obeys the imposed boundary conditions on the nodes. Thus, we define its nodal values:

$$\xi^h_i(x_j) := \begin{cases} \mathcal{I}^\delta_{h,i}(u)(x_j), & \text{if } x_j\notin\partial D \\ (\pi_i(u))(x_j), & \text{if } x_j\in\partial D \end{cases}$$

for all $x_j \in \mathcal{N}_h$.

With similar arguments as in the proof of Theorem 1.19 we can show

$$\int_D |\nabla \xi_i^h|^2 d\lambda = \sum_{T\in\mathcal{T}_h} \int_T |\nabla \xi_i^h|^2 d\lambda \le \sum_{T\in\mathcal{T}_h} \int_T |\nabla (\mathcal{I}_{h,i}^{\delta}(u))|^2 d\lambda + \eta(h) \tag{2.21}$$

for all $i \in \{1, \ldots, n\}$ with $\eta(h) \to 0$ as $h \to 0$.

Using the functions ξ_i^h our aim is now to construct a map $\bar{v}_h \in \bar{V}_N^h(g)$. For this purpose, we will use the fact that the functions $\mathcal{I}_{h,i}^{\delta}(u)$ are not interfering with each other and that $\xi_i^h(x) = (\pi_i(g))(x)$ for all $x \in \mathcal{N}_h^{\partial}$. We define the map $\bar{v}_h \in \bar{V}_N^h(g)$ by

$$\bar{v}_h(x) := \begin{cases} (j, \xi_j^h(x)), & \text{if } \exists\, j \in \{1, \ldots, n\} : 0 < \xi_j^h(x) < 1 \\ e_j^+, & \text{if } \xi_k^h(x) \in \{0,1\}\ \forall k, \xi_k^h(x) = 1\ \forall k < j, k \sim j \\ & \quad \text{and } \xi_k^h(x) = 0\ \forall k > j, k \sim j \\ o, & \text{otherwise} \end{cases}$$

for all $x \in \mathcal{N}_h$. We observe that this definition is not ambiguous. Indeed, by construction there is at most one j with $0 < \xi_j^h(x) < 1$.
Due to (1.12), the discrete nonlinear energy $\mathcal{E}_N^h(\bar{w}_h)$ of a map $\bar{w}_h \in \bar{V}_N^h(g)$ can be written as

$$\mathcal{E}_N^h(\bar{w}_h) = \sum_{T\in\mathcal{T}_h} \underbrace{-\frac{1}{2} \sum_{x_i, x_j \in \mathcal{N}_h} d^2(\bar{w}_h(x_i), \bar{w}_h(x_j)) \int_T \nabla \phi_h^{i,T} \nabla \phi_h^{j,T} d\lambda}_{:= E_T^h(\bar{w}_h)}.$$

To obtain an estimate of the discrete nonlinear energy of $\bar{v}_h$ we have to investigate $E_T^h(\bar{v}_h)$ for all $T \in \mathcal{T}_h$. Let us denote by $\mathcal{H}_h$ the set of all triangles $T \in \mathcal{T}_h$ which have two vertices x, y of T such that $\bar{v}_h(x) \in \tilde{a}_i$ and $\bar{v}_h(y) \in \tilde{a}_j$ for any $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Due to our assumption on g, the chosen iterative subdivision, and γ large enough we know that $\#\mathcal{H}_h \le C$ independent of h. We observe

$$E_T^h(\bar{v}_h) \le \begin{cases} \sum_{i=1}^n \int_T |\nabla \xi_i^h|^2 d\lambda, & \text{if } T \in \mathcal{T}_h \backslash \mathcal{H}_h \\ 2 \cdot \sum_{i=1}^n \int_T |\nabla \xi_i^h|^2 d\lambda, & \text{if } T \in \mathcal{H}_h, \end{cases}$$

leading to

$$\mathcal{E}_N^h(\bar{v}_h) \le \sum_{i=1}^n \sum_{T\in\mathcal{T}_h} \int_T |\nabla \xi_i^h|^2 d\lambda + \sum_{i=1}^n \sum_{T\in\mathcal{H}_h} \int_T |\nabla \xi_i^h|^2 d\lambda. \tag{2.22}$$

Furthermore, we observe that $\mathcal{E}_N^h(\bar{u}_h) \leq \mathcal{E}_N^h(\bar{v}_h)$ because $\bar{u}_h$ is the minimizer of the discrete nonlinear energy $\mathcal{E}_N^h$. Hence, it follows

$$\begin{aligned}
\mathcal{E}_N(u_h) &\overset{(2.15)}{\leq} \mathcal{E}_N^h(\bar{u}_h) + R_{g,D} \\
&\leq \mathcal{E}_N^h(\bar{v}_h) + R_{g,D} \\
&\overset{(2.22)}{\leq} \sum_{i=1}^n \int_D |\nabla \xi_i^h|^2 d\lambda + \sum_{i=1}^n \sum_{T\in\mathcal{H}_h} \int_T |\nabla \xi_i^h|^2 d\lambda + R_{g,D} \\
&\overset{(2.21)}{\leq} \sum_{i=1}^n \int_D |\nabla(\mathcal{I}_{h,i}^\delta(u))|^2 d\lambda + \sum_{i=1}^n \sum_{T\in\mathcal{H}_h} \int_T |\nabla \xi_i^h|^2 d\lambda + \eta(h) + R_{g,D} \\
&\overset{(2.20)}{\leq} \sum_{i=1}^n \int_M |\nabla(\pi_i(u))|^2 d\lambda + \theta(h) \\
&\overset{(2.12)}{=} \mathcal{E}_N(u) + \theta(h)
\end{aligned}$$

where

$$\theta(h) := \sum_{i=1}^n \sum_{T\in\mathcal{H}_h} \int_T |\nabla \xi_i^h|^2 d\lambda + \eta(h) + \beta(h).$$

Obviously, $\theta(h) \to 0$ as $h \to 0$. This yields the desired result $\lim_{h\to 0} \mathcal{E}_N(u_h) = \mathcal{E}_N(u)$. □

Corollary 2.41 *For $h \to 0$ the discrete finite element solutions u_h converge in L^2 to the solution u of the continuous nonlinear Dirichlet problem.*

Proof: Given $v_0, v_1 \in V_N(g)$ let v_t be the geodesic connecting v_0 and v_1. Using the arguments from the proof Theorem 1.7 one can show

$$\mathcal{E}_N(v_t) \leq (1-t)\mathcal{E}_N(v_0) + t\mathcal{E}_N(v_1) - (1-t)t\lambda_D \cdot d_2^2(v,\tilde{v}) \tag{2.23}$$

with $\lambda_D > 0$. Now, let $u_{h,t}$ be the geodesic connecting u and u_h. Then the last inequality yields

$$\mathcal{E}_N(u) \leq \mathcal{E}_N(u_{h,\frac{1}{2}}) \leq \frac{1}{2}\mathcal{E}_N(u) + \frac{1}{2}\mathcal{E}_N(u_h) - \frac{1}{4}\lambda_D d_2^2(u,u_h),$$

and, thus,

$$\frac{1}{2}\lambda_D d_2^2(u,u_h) \leq \mathcal{E}_N(u_h) - \mathcal{E}_N(u).$$

Hence, the claimed convergence follows from Theorem 2.39. □

Now, we will present a couple of numerical results for the following tree (see Figure 2.10).

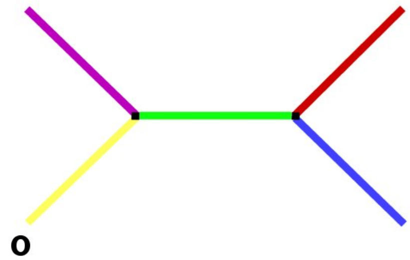

Figure 2.10: Finite Tree with 5 Edges and and 2 Branchpoints

Let us repeat that a point of the tree is described by a colour which represents an edge and a height which is the distance from the root. The vertices of the tree are described by the black colour.

Figure 2.11 shows the numerical results for different boundary data and Figure 2.12 depicts a couple of intermediate results corresponding to different iteration steps of our numerical method.

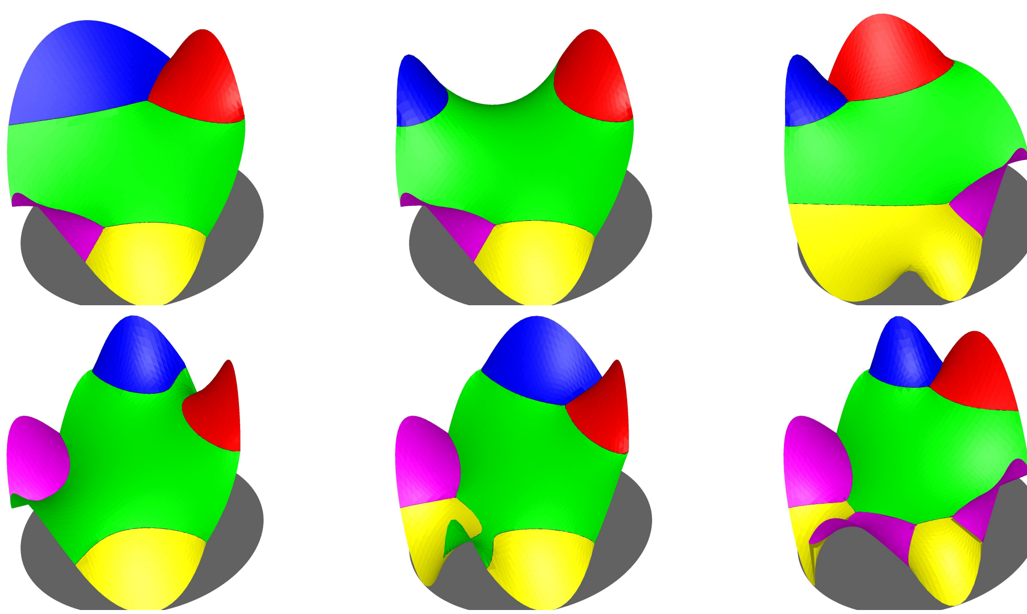

Figure 2.11: We depict various discrete harmonic maps $v_h \in V_N(g)$ for different boundary data g

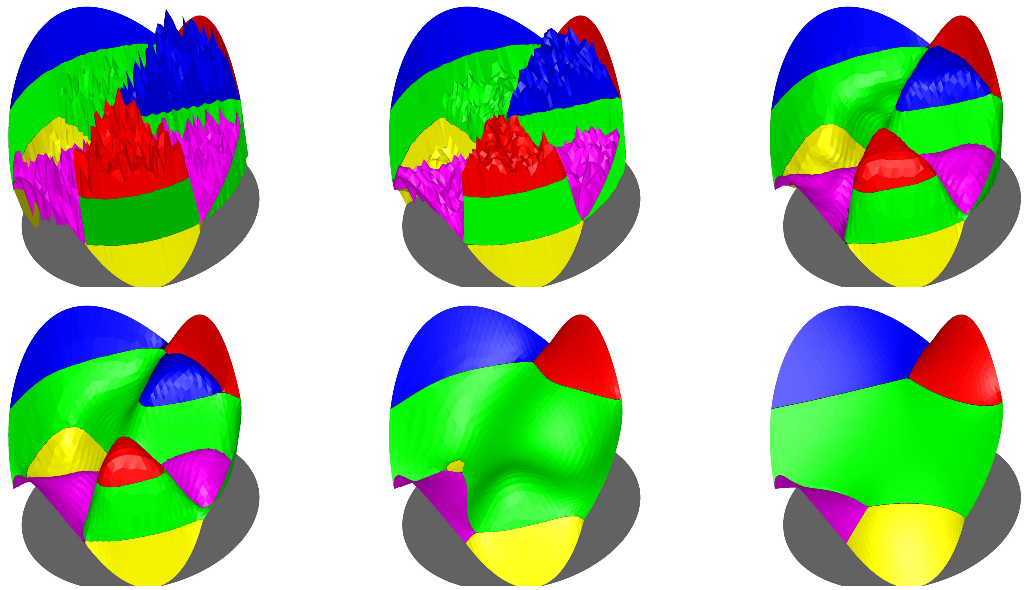

Figure 2.12: For different steps of our relaxation scheme we show intermediate results (from left to right and from top to bottom the steps 0, 1, 5, 10, 50, 250 are displayed)

2.4 Proof of Theorem 2.7

For the proof of Theorem 2.7 we need a couple of preliminary lemmata. Without restrictions, we assume that the length of all edges of the tree N is equal to one.

Lemma 2.42 *For all $u, v \in \mathcal{D}^b_{loc}(\mathcal{E})$ such that $u \cdot v = 0$ one has*

$$\lim_{t\to 0} \frac{1}{t} \left[\int_{D_u} \int_{D_v} \varphi(x)u(x)v(y)p_t(x,dy)m(dx) + \int_{D_v} \int_{D_u} \varphi(x)u(y)v(x)p_t(x,dy)m(dx) \right] = 0 \qquad \forall \varphi \in \mathcal{C}_c(M) \tag{2.24}$$

with $D_u := \{x \in M : u(x) \neq 0\}$ and $D_v := \{x \in M : v(x) \neq 0\}$.

Proof: Lemma 2.5 leads to

$$\lim_{t\to 0} \frac{1}{t} \int_M \int_M \varphi(x)[-u(y)v(x) - u(x)v(y)]p_t(x,dy)m(dx) = 0 \qquad \forall \varphi \in \mathcal{C}_c(M)$$

or, equivalently,

$$\lim_{t\to 0} \frac{1}{t} \left[-\int_{D_u} \int_{D_v} \varphi(x)u(x)v(y)p_t(x,dy)m(dx) - \int_{D_v} \int_{D_u} \varphi(x)u(y)v(x)p_t(x,dy)m(dx) \right] = 0 \qquad \forall \varphi \in \mathcal{C}_c(M) \tag{2.25}$$

□

Lemma 2.43 *Let $v \in \mathcal{D}(\mathcal{E}_N)$ be given. Defining $D_i = v^{-1}(\tilde{a}_i), i = 1, \ldots, n$, it holds for any measurable bounded function u*

$$\xi(a_i, a_j) > 0 \quad \Rightarrow \quad \lim_{t\to 0} \frac{1}{t} \int_{D_i} \int_{D_j} \varphi(x)u(x)p_t(x, dy)m(dx) = 0. \tag{2.26}$$

Proof: This follows from Proposition 2.6. □

Lemma 2.44 *Let $v \in \mathcal{D}(\mathcal{E}_N)$ be given. For two projections v_i, v_j of v such that $i \sim j$, $i < j$ one has*

$$\lim_{t\to 0} \frac{1}{t} \Bigg[\int_{D_i} \int_{D_j} \varphi(x)(1 - v_i(x))v_j(y)p_t(x, dy)m(dx) + \int_{D_j} \int_{D_i} \varphi(x)(1 - v_i(y))v_j(x)p_t(x, dy)m(dx) \Bigg] = 0 \qquad \forall \varphi \in \mathcal{C}_c(M)$$

with $D_i := v^{-1}(\tilde{a}_i), D_j := v^{-1}(\tilde{a}_j)$.

Proof: With $D_l := v^{-1}(\tilde{a}_l), l \in \{1, \ldots, n\}$ we define

$$\begin{aligned} A &:= \cup D_l \quad \text{with } l \not\sim i \text{ or } l \sim i, l < i \\ B &:= \cup D_l \quad \text{with } i < l < j, l \sim j \text{ or } l \sim i, l \not\sim j \\ C &:= \cup D_l \quad \text{with } j < l, l \sim j. \end{aligned}$$

Then $A \cup D_i \cup B \cup D_j \cup C$ is a disjoint decomposition of M and one obtains for any $\varphi \in \mathcal{C}_c(M)$

$$\begin{aligned} &\frac{1}{t} \int_M \int_M \varphi(x)(v_i(x) - v_i(y)) \cdot (v_j(x) - v_j(y))p_t(x, dy)m(dx) \\ = \; &\frac{1}{t} \int_A \int_{D_j} \varphi(x)v_j(y)p_t(x, dy)m(dx) + \frac{1}{t} \int_A \int_C \varphi(x)p_t(x, dy)m(dx) \\ &+ \frac{1}{t} \int_{D_i} \int_{D_j} \varphi(x)(1 - v_i(x))v_j(y)p_t(x, dy)m(dx) \\ &+ \frac{1}{t} \int_{D_i} \int_C \varphi(x)(1 - v_i(x))p_t(x, dy)m(dx) \\ &+ \frac{1}{t} \int_{D_j} \int_A \varphi(x)v_j(x)p_t(x, dy)m(dx) + \frac{1}{t} \int_C \int_A \varphi(x)p_t(x, dy)m(dx) \\ &+ \frac{1}{t} \int_{D_j} \int_{D_i} \varphi(x)(1 - v_i(y))v_j(x)p_t(x, dy)m(dx) \\ &+ \frac{1}{t} \int_C \int_{D_i} \varphi(x)(1 - v_i(y))p_t(x, dy)m(dx). \end{aligned}$$

Lemma 2.43 and Lemma 2.5 yield

$$\begin{aligned} 0 &= \lim_{t\to 0}\frac{1}{t}\int_M\int_M \varphi(x)(v_i(x)-v_i(y))\cdot(v_j(x)-v_j(y))p_t(x,dy)m(dx) \\ &= \lim_{t\to 0}\frac{1}{t}\Bigg[\int_{D_i}\int_{D_j}\varphi(x)(1-v_i(x))v_j(y)p_t(x,dy)m(dx) \\ &\qquad + \int_{D_j}\int_{D_i}\varphi(x)(1-v_i(y))v_j(x)p_t(x,dy)m(dx)\Bigg]. \end{aligned}$$

□

Lemma 2.45 *Let $v \in \mathcal{D}(\mathcal{E}_N)$ be given. For two projections v_i, v_j of v such that $i \not\sim j$ one has*

$$\lim_{t\to 0}\frac{1}{t}\left[\int_{D_i}\int_{D_j}\varphi(x)v_i(x)v_j(y)p_t(x,dy)m(dx) + \int_{D_j}\int_{D_i}\varphi(x)v_i(y)v_j(x)p_t(x,dy)m(dx)\right] = 0$$

for all $\varphi \in \mathcal{C}_c(M)$ with $D_i := v^{-1}(\tilde a_i), D_j := v^{-1}(\tilde a_j)$.

Proof: This follows from Lemma 2.42, $D_i \subset D_{v_i}$, $D_j \subset D_{v_j}$ and $v_i \geq 0$, $v_j \geq 0$. □

Proof of Theorem 2.7:

In the sequel, we define for measurable maps $v : M \to N$ resp. measurable functions $v : M \to \mathbb{R}$ for all $t > 0$ and for any $\varphi \in \mathcal{C}_c(M)$

$$E_t^\varphi(v) = \frac{1}{2t}\int_M\int_M \varphi(x)d^2(v(x),v(y))p_t(x,dy)m(dx)$$

resp.

$$E_t^\varphi(v) = \frac{1}{2t}\int_M\int_M \varphi(x)|v(x)-v(y)|^2 p_t(x,dy)m(dx).$$

Lemma 2.16 yields for a measurable map $v : M \to N$

$$d(v(x),v(y)) = \begin{cases} |v_i(x)-v_i(y)|, & \text{if } x,y \in D_i \\ 1-v_i(x)+v_j(y)+\xi(a_i,a_j), & \text{if } x \in D_i, y \in D_j, j\sim i, j>i \\ 1-v_j(y)+v_i(x)+\xi(a_i,a_j), & \text{if } x \in D_i, y \in D_j, j\sim i, j<i \\ v_i(x)+v_j(y)+\xi(a_i,a_j), & \text{if } x \in D_i, y\in D_j, j\not\sim i \end{cases}$$

with $D_i := v^{-1}(\tilde a_i), i \in \{1,\dots,n\}$.

Thus, for a measurable map $v \in \mathcal{D}(\mathcal{E}_N)$ one has

$$\begin{aligned}
E_t^{\varphi}(v) &= \frac{1}{2t}\sum_{i=1}^{n}\Big[\int_{D_i}\int_{D_i}\varphi(x)|v_i(x)-v_i(y)|^2 p_t(x,dy)m(dx) \\
&\quad + \sum_{i\not\sim j}\int_{D_i}\int_{D_j}\varphi(x)|\xi(a_i,a_j)+v_j(y)+v_i(x)|^2 p_t(x,dy)m(dx) \\
&\quad + \sum_{\substack{j<i\\ i\sim j}}\int_{D_i}\int_{D_j}\varphi(x)|\xi(a_i,a_j)+1-v_j(y)+v_i(x)|^2 p_t(x,dy)m(dx) \\
&\quad + \sum_{\substack{j>i\\ i\sim j}}\int_{D_i}\int_{D_j}\varphi(x)|\xi(a_i,a_j)+1-v_i(x)+v_j(y)|^2 p_t(x,dy)m(dx)\Big] \\
&= \frac{1}{2t}\Big[\sum_{i=1}^{n}\int_{D_i}\int_{D_i}\varphi(x)|v_i(x)-v_i(y)|^2 p_t(x,dy)m(dx) \\
&\quad + \sum_{i=1}^{n}\sum_{i\not\sim j}\int_{D_i}\int_{D_j}\varphi(x)[v_i^2(x)+v_j^2(y)+\xi^2(a_i,a_j)+2v_i(x)\xi(a_i,a_j) \\
&\qquad +2v_j(y)\xi(a_i,a_j)+2v_i(x)v_j(y)]p_t(x,dy)m(dx) \\
&\quad + \sum_{i=1}^{n}\sum_{\substack{j<i\\ i\sim j}}\int_{D_i}\int_{D_j}\varphi(x)[1+\xi^2(a_i,a_j)+v_j^2(y)+v_i^2(x) \\
&\qquad -2v_j(y)+2v_i(x)+2\xi(a_i,a_j)-2v_j(y)v_i(x) \\
&\qquad -2v_j(y)\xi(a_i,a_j)+2v_i(x)\xi(a_i,a_j)]p_t(x,dy)m(dx) \\
&\quad + \sum_{i=1}^{n}\sum_{\substack{j>i\\ i\sim j}}\int_{D_i}\int_{D_j}\varphi(x)[1+\xi^2(a_i,a_j)+v_j^2(y)+v_i^2(x) \\
&\qquad +2v_j(y)-2v_i(x)+2\xi(a_i,a_j)-2v_j(y)v_i(x) \\
&\qquad +2v_j(y)\xi(a_i,a_j)-2v_i(x)\xi(a_i,a_j)]p_t(x,dy)m(dx)\Big].
\end{aligned}$$

Using Lemma 2.43 it follows

$$\begin{aligned}
\limsup_{t\to 0} E_t^{\varphi}(v) &= \limsup_{t\to 0}\frac{1}{2t}\Big[\sum_{i=1}^{n}\int_{D_i}\int_{D_i}\varphi(x)|v_i(x)-v_i(y)|^2 p_t(x,dy)m(dx) \\
&\quad + \sum_{i=1}^{n}\sum_{i\not\sim j}\int_{D_i}\int_{D_j}\varphi(x)[v_i^2(x)+v_j^2(y)+2v_i(x)v_j(y)]p_t(x,dy)m(dx) \\
&\quad + \sum_{i=1}^{n}\sum_{\substack{j<i\\ i\sim j}}\int_{D_i}\int_{D_j}\varphi(x)[1+v_j^2(y)+v_i^2(x) \\
&\qquad -2v_j(y)+2v_i(x)(1-v_j(y))]p_t(x,dy)m(dx)
\end{aligned}$$

$$+\sum_{i=1}^{n}\sum_{\substack{j>i\\ i\sim j}}\int_{D_i}\int_{D_j}\varphi(x)[1+v_j^2(y)+v_i^2(x)$$
$$\left.-2v_i(x)+2v_j(y)(1-v_i(x))]p_t(x,dy)m(dx)\right]$$

and with Lemma 2.44 and Lemma 2.45 one obtains

$$\begin{aligned}\limsup_{t\to 0} E_t^{\varphi}(v) &= \limsup_{t\to 0}\frac{1}{2t}\left[\sum_{i=1}^{n}\int_{D_i}\int_{D_i}\varphi(x)|v_i(x)-v_i(y)|^2 p_t(x,dy)m(dx)\right.\\ &\quad +\sum_{i=1}^{n}\sum_{i\not\sim j}\int_{D_i}\int_{D_j}\varphi(x)[v_i^2(x)+v_j^2(y)]p_t(x,dy)m(dx)\\ &\quad +\sum_{i=1}^{n}\sum_{\substack{j<i\\ i\sim j}}\int_{D_i}\int_{D_j}\varphi(x)[|1-v_j(y)|^2+v_i^2(x)]p_t(x,dy)m(dx)\\ &\quad \left.+\sum_{i=1}^{n}\sum_{\substack{j>i\\ i\sim j}}\int_{D_i}\int_{D_j}\varphi(x)[|1-v_i(x)|^2+v_j^2(y)]p_t(x,dy)m(dx)\right].\end{aligned}$$

Furthermore, for a projection v_i of a map $v\in\mathcal{D}(\mathcal{E}_N)$ one has

$$\begin{aligned}E_t^{\varphi}(v_i) &= \frac{1}{2t}\int_{D_i}\int_{D_i}\varphi(x)|v_i(x)-v_i(y)|^2 p_t(x,dy)m(dx)\\ &\quad +\frac{1}{2t}\left[\sum_{i\not\sim k}\int_{D_i}\int_{D_k}\varphi(x)v_i^2(x)p_t(x,dy)m(dx)\right.\\ &\quad +\sum_{\substack{k<i\\ i\sim k}}\int_{D_i}\int_{D_k}\varphi(x)v_i^2(x)p_t(x,dy)m(dx)\\ &\quad +\sum_{\substack{k>i\\ i\sim k}}\int_{D_i}\int_{D_k}\varphi(x)|1-v_i(x)|^2 p_t(x,dy)m(dx)\\ &\quad +\sum_{i\not\sim k}\int_{D_k}\int_{D_i}\varphi(x)v_i^2(y)p_t(x,dy)m(dx)\\ &\quad +\sum_{\substack{k<i\\ i\sim k}}\int_{D_k}\int_{D_i}\varphi(x)v_i^2(y)p_t(x,dy)m(dx)\\ &\quad \left.+\sum_{\substack{k>i\\ i\sim k}}\int_{D_k}\int_{D_i}\varphi(x)|1-v_i(y)|^2 p_t(x,dy)m(dx)\right].\end{aligned}$$

Hence,

$$\sum_{i=1}^{n} \lim_{t\to 0} E_t^{\varphi}(v_i) \quad = \quad \lim_{t\to 0} E_t^{\varphi}(v)$$

which yields

$$\begin{aligned}
\mathcal{E}_N(v) \quad &= \quad \sup_{\substack{\varphi\in\mathcal{C}_c(M)\\ 0\le\varphi\le 1}} \limsup_{t\to 0} E_t^{\varphi}(v) = \sup_{\substack{\varphi\in\mathcal{C}_c(M)\\ 0\le\varphi\le 1}} \lim_{t\to 0} E_t^{\varphi}(v) \\
&= \quad \sup_{\substack{\varphi\in\mathcal{C}_c(M)\\ 0\le\varphi\le 1}} \sum_{i=1}^{n} \lim_{t\to 0} E_t^{\varphi}(v_i) = \sup_{\substack{\varphi\in\mathcal{C}_c(M)\\ 0\le\varphi\le 1}} \sum_{i=1}^{n} \int_M \varphi(x)\mu_{\langle v_i\rangle}(dx).
\end{aligned}$$

Assuming that $v_i \in \mathcal{D}^b_{loc}(\mathcal{E}), \forall i \in \{1,\dots,n\}$, and that condition (2.4) holds one can prove the inverse implication in the same way (because of Lemma 2.5 and Lemma 2.42 and because of the fact that Lemma 2.44 and 2.45 still hold). □

2.5 Generalizations to Trees with a Countable Number of Edges

In this section, we will generalize the definition of the nonlinear energy to the case where (N,d) is a metric tree with a countable number of edges which are all isometric to closed intervals of $\mathbb{R}$. Let us denote by $\mathcal{A} = \{a_i, i \in \mathbb{N}\}$ the set of edges of N. As in Definition 2.3 we define projections $\pi_i : N \to [0, d(e_i^-, e_i^+)], i \in \mathbb{N}$, and to each measurable map $v : M \to N$ we associate a family of functions $v_i : M \to [0, d(e_i^-, e_i^+)], i \in \mathbb{N}$, defined by

$$v_i := \pi_i \circ v \qquad \forall i \in \mathbb{N}.$$

There exists a sequence $(N_k, d)_{k\ge 0}$ of connected finite subtrees of (N,d) with $N_k \subset N_{k+1}, \forall k \ge 0$, such that $N_k \to N, k \to \infty$. We denote the set of edges of N_k by $\mathcal{A}_k = \{a_i, i \in I_k \subset \mathbb{N}\}$. Given a measurable map $v : M \to N$ we define maps $v_{N_k} : M \to N_k, k \ge 0$, by

$$v_{N_k}(x) := \begin{cases} v(x), & \text{if } v(x) \in N_k \\ \mathrm{argmin}_{y\in N_k} d(y, v(x)), & \text{if } v(x) \notin N_k. \end{cases}$$

For each $k \ge 0$ let us denote the projections of N_k by $\pi_{k,i}, i \in I_k$, and for each measurable map $v : M \to N$ we define for all $i \in I_k$

$$v_{k,i} := \pi_{k,i} \circ v_{N_k}.$$

Lemma 2.46 *For each $k \ge 0$ and for each measurable map $v : M \to N$ it holds $v_{k,i} = v_i, \forall i \in I_k$.*

Definition 2.47 *Denoting the nonlinear energy function for maps with values in N_k by $\mathcal{E}_{N_k}$ we define for measurable maps $v : M \to N$ the energy function $\mathcal{E}_N$ by*

$$\mathcal{E}_N(v) := \lim_{k\to\infty} \mathcal{E}_{N_k}(v_{N_k})$$

with $\mathcal{D}(\mathcal{E}_N) := \{v : M \to N \text{ measurable} : v_{N_k} \in \mathcal{D}(\mathcal{E}_{N_k}), \forall k \ge 0, \text{ and } \mathcal{E}_N(v) < \infty\}$.

Theorem 2.48 *For each map $v : M \to N$ the condition $v \in \mathcal{D}(\mathcal{E}_N)$ is equivalent to*

$$v_i \in \mathcal{D}^b_{loc}(\mathcal{E}), \forall i \in \mathbb{N}, \qquad \sum_{i\in\mathbb{N}} \mu_{<v_i>}(M) < \infty \tag{2.27}$$

and

$$\lim_{t\to 0} \frac{1}{t} \int_{v^{-1}(\tilde{a}_i)} \int_{v^{-1}(\tilde{a}_j)} \varphi(x) p_t(x, dy) m(dx) = 0 \tag{2.28}$$

for all $a_i, a_j \in \mathcal{A}$ with $\xi(a_i, a_j) > 0$ and for all $\varphi \in \mathcal{C}_c(M)$. In this situation, for each $v \in \mathcal{D}(\mathcal{E}_N)$ the following equality hold

$$\mathcal{E}_N(v) \;\; = \;\; \sum_{i\in\mathbb{N}} \mu_{<v_i>}(M).$$

Proof: Let $v \in \mathcal{D}(\mathcal{E}_N)$ be given. Then it follows from Theorem 2.7 for all $k \geq 0$ that $v_i = v_{k,i} \in \mathcal{D}^b_{loc}(\mathcal{E})$ and (2.4) holds for all vertices in $\mathcal{A}_k$. Furthermore, one has

$$\sum_{i\in\mathbb{N}} \mu_{<v_i>}(M) \leq \mathcal{E}_N(v).$$

Given a map $v : M \to N$ such that (2.27) and (2.28) are fulfilled for all $k \geq 0$ it holds $v_{N_k} \in \mathcal{D}(\mathcal{E}_{N_k})$ and

$$\mathcal{E}_{N_k}(v_{N_k}) = \sum_{i\in I_k} \mu_{<v_i>}(M)$$

and by Definition 2.47 one obtains

$$\mathcal{E}_N(v) = \sum_{i\in\mathbb{N}} \mu_{<v_i>}(M).$$

□

Remark 2.49 *As before, one can show that our definition of nonlinear energy coincides with the energy introduced in [Pic04]. Hence, one can define again the corresponding nonlinear Dirichlet problem and transfer Theorem 2.30.*

Chapter 3

Graphs

In this chapter, we study graph targets. Let (M, m) be a compact measure space with universal cover $\tilde{M}$ and with a local regular Dirichlet form $\mathcal{E}$ on $L^2(\tilde{M}, \tilde{m})$ given by a semigroup of Markov kernels p_t. Furthermore, let (N, d) be a graph with a finite number of edges.
Before we define a nonlinear energy for measurable maps $v : M \to N$ equivariant mapping problems are studied. This is motivated by the fact, that any continuous map $v : M \to N$ lifts to a equivariant map $\tilde{v} : \tilde{M} \to \tilde{N}$, whereby the universal cover $\tilde{N}$ of the graph N is a tree with a countable number of edges.
Given an equivariant map $\tilde{v} : \tilde{M} \to \tilde{N}$ we say that two projections $\tilde{v}_i$ and $\tilde{v}_j$ are equivalent ($\tilde{v}_i \sim \tilde{v}_j$) if there is an element γ of the group of covering transformations of $\tilde{M}$ such that $\tilde{v}_i = \tilde{v}_j \circ \gamma$. This yields an equivalence relation on the set of all projections $\mathbb{F}(\tilde{v})$ and the nonlinear energy is defined by

$$\mathcal{E}_{\tilde{N}}(\tilde{v}) := \sum_{\tilde{v}_i \in \mathbb{F}(\tilde{v})/\sim} \mu_{<\tilde{v}_i>}(\tilde{M})$$

whereby $\mu_{<\tilde{v}_i>}$ is the energy measure of $\tilde{v}_i$.
One of the important issues is that for any fundamental domain M_0 of M, in $\tilde{M}$, such that $\bar{M}_0$ is compact and ∂M_0 has measure zero it holds

$$\mathcal{E}_{\tilde{N}}(\tilde{v}) = \sum_{i \in \mathbb{N}} \mu_{<\tilde{v}_i>}(M_0).$$

Therefore, the nonlinear energy function $\mathcal{E}_N$ for a map $v : M \to N$ which is the projection of an equivariant map $\tilde{v} : \tilde{M} \to \tilde{N}$ is defined by

$$\mathcal{E}_N(v) := \mathcal{E}_{\tilde{N}}(\tilde{v}).$$

Furthermore, conditions for the existence and uniqueness of a solution to the corresponding nonlinear Dirichlet problem are given.
The analysis of homotopy problems is another major point of this chapter. For particular domain spaces M the existence of a minimizer of the nonlinear energy in a given homotopy class is shown.

Throughout this chapter, M will be a path connected and locally simply connected topological compact space with universal cover $\tilde{M}$. Let $\mathcal{M}$ and $\tilde{\mathcal{M}}$ be the Borel σ-fields on M and $\tilde{M}$, resp., and let m be a finite measure on $(M, \mathcal{M})$. In the sequel, we denote by $\tilde{m}$ the lift of the measure m to $(\tilde{M}, \tilde{\mathcal{M}})$, by G_M the group of covering transformations and by $\varphi_M : \tilde{M} \to M$ the covering map. We choose, once for all, base points $\tilde{z}$ in $\tilde{M}$ and $z \in \varphi_M(\tilde{z})$ in M. The fundamental group $\pi_1(M, z)$ of M is canonically isomorphic to G_M and the sets in $\tilde{M}$ can be identified with the G_M-invariant sets in $\tilde{M}$.
We assume that there is a fundamental domain M_0 for M, in $\tilde{M}$, such that $\bar{M}_0$ is compact and ∂M_0 has measure zero. Moreover, let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a regular Dirichlet form on $L^2(\tilde{M}, \tilde{m})$ with

(A1) $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is local, that is, $v, w \in \mathcal{D}(\mathcal{E})$, $\mathrm{supp}[v]$ and $\mathrm{supp}[w]$ are compact, $v \equiv 0$ on a neighbourhood of $\mathrm{supp}[w] \quad \Rightarrow \quad \mathcal{E}(v, w) = 0$.

(A2) The semigroup $(T_t)_{t \geq 0}$ corresponding to the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is given by a semigroup of Markov kernels $p_t(x, dy)$ on $\tilde{M}$.

(A3) It holds $p_t(x, dy)\tilde{m}(dx) \ll \tilde{m}(dy)\tilde{m}(dx) \qquad \forall t > 0$
and $p_t(\gamma x, A) = p_t(x, \gamma^{-1}A) \qquad \forall x \in \tilde{M}, A \in \tilde{\mathcal{M}}, \gamma \in G_M$.

(A4) For all $u \in \mathcal{D}_{loc}(\mathcal{E})$ the energy measure $\mu_{<u>}$ has a density $\Gamma(u)$.

Remark 3.1

(i) The conditions (A1) and (A2) yield that the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is also strongly local (cf. Appendix A.1).

(ii) Condition (A3) yields for the measure $p_t \times \tilde{m} := p_t(x, dy)\tilde{m}(dx)$ defined on $\tilde{M} \times \tilde{M}$

$$p_t \times \tilde{m} = \gamma(p_t \times \tilde{m}) \quad \forall \gamma \in G_M. \tag{3.1}$$

Furthermore, let (N, d) be a finite graph. This means N consists of a finite number of edges, which are isometric to closed intervals of $\mathbb{R}$, glued together at some endpoints, such that N is a connected space, possibly with loops (see e.g. Figure 3.1).

We have the following proposition:

Proposition 3.2 *Let (N, d) be a finite graph. Then its universal cover $(\tilde{N}, \tilde{d})$ is a tree with a countable number of edges and each edge is isometric to a closed interval of $\mathbb{R}$.*
The fundamental group $\pi_1(N)$ of (N, d) is canonically isomorphic to the group of covering transformations G_N which acts properly discontinuously on $\tilde{N}$, i.e. each point $z \in \tilde{N}$ has a neighborhood $\tilde{N}_0$ such that $\tilde{N}_0 \cap \eta \tilde{N}_0 = \emptyset$ for all $\eta \in G_N \backslash \{1\}$. The space N can be identified with $\tilde{N}/G_N$. Then the covering map $\varphi : \tilde{N} \to N$ is given by $\varphi_N(x) = G_N x$.
The group G_N is a subgroup of $isom(\tilde{N})$, i.e. $\tilde{d}(\eta x, \eta y) = \tilde{d}(x, y)$ for all $x, y \in \tilde{N}$ and $\eta \in G_N$.

Proof: For the first assertion, see Corollary 2.3.2 in [Jos97a]. For the topological results, we refer to [SZ88]. □

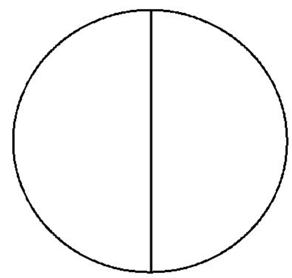

Figure 3.1: Example of a Finite Graph

3.1 Nonlinear Energy for Equivariant Maps

Before we define the nonlinear energy for maps with values in a finite graph let us discuss a nonlinear energy function for equivariant maps with values in the universal cover of a finite graph.

Throughout this section, let (N, d) be the universal cover of a finite graph (cf. Figure 3.2).

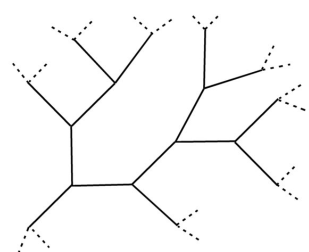

Figure 3.2: Subset of the Universal Cover of the Graph from Figure 3.1

In the following, we denote the set of edges $\{a_i, i \in \mathbb{N}\}$ of N by $\mathcal{A}$ and the midpoint of an edge a_i by z_i.
We need to define for all edges consistently which is the "left" vertex e_i^- and which is the "right" vertex e_i^+ of any edge $a_i, i \in \mathbb{N}$. This in some sense arbitrary, because the tree N has an infinite number of edges and there is no canonical orientation. Therefore, we choose any edge a_r as reference edge and denote the corresponding two vertices arbitrarily by e_r^- and e_r^+. We define the following sets

$$\begin{aligned} L_r &:= \{a \in \mathcal{A} : e_r^+ \notin \gamma_{x,z_r} \text{ for any } x \in a\} \\ R_r &:= \{a \in \mathcal{A} : e_r^- \notin \gamma_{x,z_r} \text{ for any } x \in a\} \end{aligned}$$

with $\gamma_{x,y}$ being the geodesic between the points x and y.
Now, for any edge $a_i \in L_r$ we denote the corresponding vertices by e_i^- and e_i^+ in such a way that

$$d(e_i^-, z_r) > d(e_i^+, z_r)$$

and for any edge $a_j \in R_r$ such that

$$d(e_j^-, z_r) < d(e_j^+, z_r).$$

Definition 3.3 *Given (N, d) we define the function*

$$\begin{array}{rcl} \xi : \mathcal{A} \times \mathcal{A} & \to & \mathbb{N} \\ (a_i, a_j) & \mapsto & \xi(a_i, a_j), \end{array}$$

where $\xi(a_i, a_j)$ is the number of edges $a \in \mathcal{A}, a \neq a_i$ and $a \neq a_j$ with $a \subset \gamma_{x,y}$ for some $x \in a_i$ and for some $y \in a_j$.

Definition 3.4 *For each edge $a_i, i \in \mathbb{N}$, we define the projection v_i of a measurable map $v : \tilde{M} \to N$ by*

$$v_i(x) := \begin{cases} 0, & \text{if } x \in v^{-1}(L_i) \\ d(e_i^-, e_i^+), & \text{if } x \in v^{-1}(R_i) \\ d(e_i^-, v(x)), & \text{if } x \in a_i \end{cases}$$

with

$$\begin{array}{rcl} L_i & := & \{a \in \mathcal{A} : a \neq a_i \text{ and } dist(e_i^-, a) < dist(e_i^+, a)\} \\ R_i & := & \{a \in \mathcal{A} : a \neq a_i \text{ and } dist(e_i^-, a) > dist(e_i^+, a)\}. \end{array}$$

In the sequel, we denote by G_N the group of covering transformations of N and we fix a group homomorphism $\rho : G_M \to G_N$. We will write $\rho(\gamma)x$ for $\rho(\gamma)(x)$.

A map $v : \tilde{M} \to N$ is said to be ρ-equivariant if

$$v(\gamma x) = \rho(\gamma) v(x)$$

for all $x \in \tilde{M}$ and $\gamma \in G_M$.

Definition 3.5 *Let $\rho : G_M \to G_N$ be a group homomorphism. For all $\gamma \in G_M$ we define functions $\rho_\gamma : \mathbb{N} \to \mathbb{N}$ such that $\rho(\gamma)x \in a_{\rho_\gamma(i)}$ for any $x \in a_i$.*

Lemma 3.6 *Let $v : \tilde{M} \to N$ be a ρ-equivariant measurable map. For all $i \in \mathbb{N}$ and for all $\gamma \in G_M$ one has*

$$v_i(x) = v_{\rho_\gamma(i)}(\gamma x) \qquad \forall x \in \tilde{M}.$$

This lemma leads to the following definition.

Definition 3.7 *Let $v : \tilde{M} \to N$ be a ρ-equivariant measurable map. We say that two projections v_i and v_j of v are related ($v_i \sim v_j$) if there is $\gamma \in G_M$ such that $v_i(x) = v_j(\gamma x)$ for all $x \in \tilde{M}$.*

It is easy to see that for each ρ-equivariant measurable map $v : \tilde{M} \to N$ the relation $\sim$ is an equivalence relation on the set of all projections.

Definition 3.8 *Let us denote the set of all projections $v_i, i \in \mathbb{N}$, of a ρ-equivariant measurable map v by $\mathbb{F}(v)$. Then we define*

$$\Pi(v) := \mathbb{F}(v)/_{\sim}.$$

Since N is the universal cover of a finite graph and $\rho : G_M \to G_N$ is a group homomorphism, it holds $\#\Pi(v) < \infty$.

In the next lemma, we will analyze the energy of equivalent projections.

Lemma 3.9 *Let $v : \tilde{M} \to N$ be a ρ-equivariant measurable map. Assume there is a projection v_i with $v_i \in \mathcal{D}_{loc}(\mathcal{E})$. It holds for all projections v_j equivalent w.r.t the relation $\sim$ to v_i*

$$v_j \in \mathcal{D}_{loc}(\mathcal{E})$$

and

$$\mu_{<v_j>} = \gamma(\mu_{<v_i>})$$

with $\gamma \in G_M$ such that $v_i(x) = v_j(\gamma x)$.

Proof: In the sequel, let us denote by $\mathcal{C}_c(\tilde{M})$ the set of all continuous functions on $\tilde{M}$ with compact support. Given $\phi \in \mathcal{C}_c(\tilde{M})$ one has

$$\begin{aligned}
& \int_{\tilde{M}}\int_{\tilde{M}} \phi(x)|v_j(x) - v_j(y)|^2 p_t(x,dy)\tilde{m}(dx) \\
\overset{(3.1)}{=} & \int_{\tilde{M}}\int_{\tilde{M}} \phi(x)|v_j(x) - v_j(y)|^2 \gamma(p_t \times \tilde{m})(dx,dy) \\
= & \int_{\tilde{M}}\int_{\tilde{M}} \phi(\gamma x)|v_j(\gamma x) - v_j(\gamma y)|^2 p_t(x,dy)\tilde{m}(dx) \\
= & \int_{\tilde{M}}\int_{\tilde{M}} \phi(\gamma x)|v_i(x) - v_i(y)|^2 p_t(x,dy)\tilde{m}(dx).
\end{aligned}$$

Hence, (cf. [BM95])

$$\begin{aligned}
\int_{\tilde{M}} \phi(x)\mu_{<v_j>}(dx) &= \int_{\tilde{M}} \phi(\gamma x)\mu_{<v_i>}(dx) \\
&= \int_{\tilde{M}} \phi(x)\gamma(\mu_{<v_i>})(dx) \qquad \forall \phi \in \mathcal{C}_c(\tilde{M})
\end{aligned}$$

which leads to

$$\mu_{<v_j>} = \gamma(\mu_{<v_i>}).$$

□

Motivated by the analysis of the nonlinear energy for maps with values in finite trees, we present the following definition of a nonlinear energy function for ρ-equivariant measurable maps:

Definition 3.10 *Let $v : \tilde{M} \to N$ be a ρ-equivariant measurable map. If all projections v_i are in $\mathcal{D}^b_{loc}(\mathcal{E})$ (for the definition of $\mathcal{D}^b_{loc}(\mathcal{E})$ see Chapter 2) and if*

$$\lim_{t\to 0} \frac{1}{t} \int_{v^{-1}(a_i)} \int_{v^{-1}(a_j)} \phi(x) p_t(x, dy) \tilde{m}(dx) = 0 \tag{3.2}$$

for all $a_i, a_j \in \mathcal{A}$ with $\xi(a_i, a_j) > 0$ and for all $\phi \in \mathcal{C}_c(\tilde{M})$ holds we define the nonlinear energy function $\mathcal{E}^\rho_N$ by

$$\mathcal{E}^\rho_N(v) := \sum_{v_i \in \Pi(v)} \mu_{<v_i>}(\tilde{M}) < \infty.$$

For all other ρ-equivariant measurable maps v we set

$$\mathcal{E}^\rho_N(v) := \infty.$$

Thus, we define $\mathcal{D}(\mathcal{E}^\rho_N) := \{v : M \to N \ \rho\text{-equivariant, measurable} : v_i \in \mathcal{D}^b_{loc},\ (3.2) \text{ holds}\}$.

The nonlinear energy defined above has the following property:

Theorem 3.11 *Let $v \in \mathcal{D}(\mathcal{E}^\rho_N)$ be given and let M_0 be any fundamental domain for M, in $\tilde{M}$, such that $\bar{M}_0$ is compact and ∂M_0 has measure zero. Then one has*

$$\mathcal{E}^\rho_N(v) = \sum_{i\in\mathbb{N}} \mu_{<v_i>}(M_0). \tag{3.3}$$

On the other hand, if v is a ρ-equivariant measurable map, such that $v_i \in \mathcal{D}_{loc}(\mathcal{E}), \forall i \in \mathbb{N}$, condition (3.2) and

$$\sum_{i\in\mathbb{N}} \mu_{<v_i>}(M_0) < \infty$$

hold, one can deduce $v \in \mathcal{D}(\mathcal{E}^\rho_N)$.

Proof: For each equivalence class in $\Pi(v)$ let us choose a representative $v_i, i \in I$, with $I = \{1, \ldots, \#\Pi(v)\}$. Defining $D_i := v_i^{-1}((0,1))$ one has (cf. [BH91])

$$\mu_{<v_i>}(\tilde{M}) = \mu_{<v_i>}(D_i).$$

For $i \in I$ and $\gamma \in G_M$ we define the sets $G^i_\gamma \subset M_0$ by

$$G^i_\gamma := \gamma^{-1}(\gamma M_0 \cap D_i).$$

This yields a family of disjoint subset of M_0. It holds for fixed $i \in I$

$$\begin{aligned}
\mu_{<v_i>}(D_i) &= \mu_{<v_i>}(\dot{\bigcup}_{\gamma \in G_M} \gamma M_0 \cap D_i) \\
&= \sum_{\gamma \in G_M} \mu_{<v_i>}(\gamma M_0 \cap D_i) \\
&= \sum_{\gamma \in G_M} \gamma^{-1} \mu_{<v_{\rho_\gamma(i)}>}(\gamma M_0 \cap D_i) \\
&= \sum_{\gamma \in G_M} \mu_{<v_{\rho_\gamma(i)}>}(G^i_\gamma)
\end{aligned}$$

Furthermore, one has

$$\begin{aligned}
\mu_{<v_{\rho_\gamma(i)}>}(G^i_\gamma) &= \gamma^{-1} \mu_{<v_{\rho_\gamma(i)}>}(\gamma M_0 \cap D_i) \\
&= \mu_{<v_i>}(\gamma M_0 \cap D_i) \\
&= \mu_{<v_i>}(\gamma M_0) \\
&= \gamma \mu_{<v_i>}(M_0) \\
&= \mu_{<v_{\rho_\gamma(i)}>}(M_0)
\end{aligned}$$

Hence,

$$\begin{aligned}
\mu_{<v_i>}(\tilde{M}) &= \mu_{<v_i>}(D_i) \\
&= \sum_{\gamma \in G_M} \mu_{<v_{\rho_\gamma(i)}>}(G^i_\gamma) \\
&= \sum_{\gamma \in G_M} \mu_{<v_{\rho_\gamma(i)}>}(M_0)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}^\rho_N(v) &= \sum_{i \in I} \mu_{<v_i>}(\tilde{M}) \\
&= \sum_{i \in I} \sum_{\gamma \in G_M} \mu_{<v_{\rho_\gamma(i)}>}(M_0).
\end{aligned}$$

We still have to prove

$$\sum_{i \in I} \sum_{\gamma \in G_M} \mu_{<v_{\rho_\gamma(i)}>}(M_0) = \sum_{j \in \mathbb{N}} \mu_{<v_j>}(M_0). \tag{3.4}$$

Let v_j be a projection with $\mu_{<v_j>}(M_0) > 0$ and let v_i be the representative chosen above with $v_i \sim v_j$, i.e. there is $\gamma \in G_M$ with $v_i(x) = v_j(\gamma x)$. One has $\gamma M_0 \cap D_i \neq \emptyset$ and

$$\begin{aligned}
\mu_{<v_j>}(M_0) &= \gamma\mu_{<v_i>}(M_0) \\
&= \mu_{<v_i>}(\gamma M_0) \\
&= \mu_{<v_i>}(\gamma M_0 \cap D_i) \\
&= \gamma^{-1}\mu_{<v_{\rho_\gamma(i)}>}(\gamma M_0 \cap D_i) \\
&= \mu_{<v_{\rho_\gamma(i)}>}(G^i_\gamma) \\
&= \mu_{<v_{\rho_\gamma(i)}>}(M_0)
\end{aligned}$$

which yields equation (3.4).

On the other hand, from $v_i \in \mathcal{D}_{loc}(\mathcal{E}), \forall i \in \mathbb{N}$, (3.2) holds and

$$\sum_{i\in\mathbb{N}} \mu_{<v_i>}(M_0) < \infty$$

one can deduce in the same way $v_i \in \mathcal{D}^b_{loc}(\mathcal{E})$ and

$$\sum_{v_i\in\Pi(v)} \mu_{<v_i>}(\tilde{M}) = \sum_{j\in\mathbb{N}} \mu_{<v_j>}(M_0).$$

□

Remark 3.12 *Given $v \in \mathcal{D}(\mathcal{E}^\rho_N)$, let M_0 and M_1 be two different fundamental domains such that $\bar{M}_0, \bar{M}_1$ are compact and $\partial M_0, \partial M_1$ has measure zero. Then it holds*

$$\sum_{i\in\mathbb{N}} \mu_{<v_i>}(M_0) = \sum_{i\in\mathbb{N}} \mu_{<v_i>}(M_1).$$

3.1.1 Nonlinear Dirichlet Problem for Equivariant Maps

In this subsection, we define the nonlinear Dirichlet problem for equivariant maps. In the sequel, let M_0 be any fundamental domain for M, in $\tilde{M}$, with $\bar{M}_0$ compact and ∂M_0 has measure zero.

Definition 3.13 (Nonlinear Dirichlet problem) *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a regular Dirichlet form on $L^2(\tilde{M}, \tilde{m})$ which fulfills the conditions (A1) - (A4). Given a map $g \in \mathcal{D}(\mathcal{E}^\rho_N)$ and a set D with $\bar{D} \subsetneq M_0$, let us define the class of maps*

$$V^\rho_N(g) := \{v \in \mathcal{D}(\mathcal{E}^\rho_N) : \tilde{v} = \tilde{g} \text{ quasi everywhere on } M_0\backslash D\}$$

where $\tilde{v}, \tilde{g}$ denotes quasi-continuous versions of v and g, resp. A map $u \in V^\rho_N(g)$ is called solution to the nonlinear Dirichlet problem for g and D whenever

$$\mathcal{E}^\rho_N(u) = \min_{v\in V^\rho_N(g)} \mathcal{E}^\rho_N(v).$$

The next result states a sufficient condition for the existence and uniqueness of a solution to the nonlinear Dirichlet problem.

Theorem 3.14 *Let* $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ *be a regular Dirichlet form on* $L^2(\tilde{M}, \tilde{m})$ *which fulfills the conditions (A1) - (A4) with diffusion* X_t*. Given a relatively compact open subset* D *with* $dist(\bar{D}, \partial M_0) > 0$ *such that* X_t *quits* D *during its lifetime and a* ρ*-equivariant continuous map* $g \in \mathcal{D}(\mathcal{E}_N^\rho)$ *there exists a unique (up to modifications) map* $u \in \mathcal{D}(\mathcal{E}_N^\rho)$ *which solves the nonlinear Dirichlet problem for* g *and* D*.*

Proof: Since g is continuous and $\bar{M}_0$ is compact it holds $g(\bar{M}_0) \subset N_b$ where N_b is a finite subtree of N. Given a map $v \in V_N^\rho(g)$ with $N_b \subset v(M_0)$ let v_b the projection of $v|_{M_0}$ to N_b. The map v_b can be continued to a ρ-equivariant map v_{bc} (because one changes only values on D). Obviously, it holds $v_{bc} \in \mathcal{D}(\mathcal{E}_N^\rho)$ and

$$\mathcal{E}_N^\rho(v_{bc}) \leq \mathcal{E}_N^\rho(v)$$

such that to solve the nonlinear Dirichlet problem one can restrict oneself to the set

$$V_N^{\rho,b}(g) := \{v \in \mathcal{D}(\mathcal{E}_N^\rho) : \tilde{v} = \tilde{g} \text{ quasi everywhere on } M_0 \backslash D,\ v(\bar{M}_0) \subset N_b\}.$$

Now, let g_b be the projection of g on N_b. The map g_b is defined on $\tilde{M}$ and has values in the finite tree N_b but this map is not ρ-equivariant anymore. In addition, let $\mathcal{E}_{N_b}$ be the nonlinear energy for maps defined on $\tilde{M}$ with values in the finite tree N_b (see Section 2.1). One has for a map $w \in \mathcal{D}(\mathcal{E}_{N_b})$

$$\mathcal{E}_{N_b}(w) = \sum_{i=1}^{l} \mu_{<w_i>}(\tilde{M})$$

with l being the number of edges of N_b. From Definition 3.10 it follows

$$g_b \in \mathcal{D}(\mathcal{E}_{N_b}).$$

We know that there is a solution to the nonlinear Dirichlet form for g_b and D (see Section 2.2). Let us denote this solution by u_b. It holds

$$\mathcal{E}_{N_b}(u_b) = \sum_{i=1}^{l} \int_D \Gamma((u_b)_i) d\tilde{m} + \sum_{i=1}^{l} \int_{M_0 \backslash D} \Gamma((g_b)_i) d\tilde{m} + \sum_{i=1}^{l} \int_{\tilde{M} \backslash M_0} \Gamma((g_b)_i) d\tilde{m}. \tag{3.5}$$

One can continue $u_b|_{M_0}$ to a ρ-equivariant map u which is in $V_N^{\rho,b}(g)$. One has (cf. Theorem 3.11)

$$\mathcal{E}_N^\rho(u) = \sum_{i=1}^{l} \int_D \Gamma(u_i) d\tilde{m} + \sum_{i=1}^{l} \int_{M_0 \backslash D} \Gamma(g_i) d\tilde{m} \tag{3.6}$$

This map u solves the nonlinear Dirichlet problem for g and D.
Assume there is a map w in $V_N^{\rho,b}(g)$ with $\mathcal{E}_N^\rho(w) < \mathcal{E}_N^\rho(u)$. Then we define a map w_b by

$$w_b := \begin{cases} w, & \text{if } x \in M_0 \\ g_b, & \text{otherwise.} \end{cases}$$

This map would be in $V_{N_b}(g_b)$ and it would hold (cf. (3.5), (3.6))

$$\mathcal{E}_{N_b}(w_b) < \mathcal{E}_{N_b}(u_b)$$

which is a contradiction. □

3.2 Nonlinear Energy for Maps with Values in Finite Graphs

In this section, we define the nonlinear energy for maps with values in a finite graph using the results of the previous section.

Let $(M, \mathcal{M}, m)$ and (N, d) be as described in the introduction of this chapter. Furthermore, we assume that there is a regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(\tilde{M}, \tilde{m})$ which fulfills the conditions (A1) - (A4).

Note that each continuous map

$$v : M \to N$$

induces a homomorphism

$$v_* : \pi_1(M, z) \to \pi_1(N, g(z))$$

between fundamental groups by $v_*(\gamma)$ being the loop $t \mapsto v(\gamma_t)$ in N with base point $v(z)$ whenever $\gamma : t \mapsto \gamma_t$ is a loop in M with base point z. Equivalently, it can be regarded as homomorphism $v_* : G_M \to G_N \subset isom(N)$.

Lemma 3.15

(i) For every continuous map $v : M \to N$ and every $\tilde{n} \in \varphi_N^{-1}(v(z))$ there exists a unique continuous map $\tilde{v} : \tilde{M} \to \tilde{N}$ with

$$\varphi_N \circ \tilde{v} = v \circ \varphi_M$$

and $\tilde{v}(\tilde{z}) = \tilde{n}$ ("$\tilde{v}$ is the lifting of v"). The map $\tilde{v}$ is v_*-equivariant.

(ii) Let $\rho : G_M \to G_N$ be a group homomorphism. Then every ρ-equivariant map $\tilde{v} : \tilde{M} \to \tilde{N}$ defines a unique map $v : M \to N$ ("projection of $\tilde{v}$") with $v(G_M x) = G_N \tilde{v}(x)$ for all $x \in \tilde{M}$.

Proof: See [Stu02].

Definition 3.16 *Let Ξ be the set of all group homomorphism between G_M and G_N. Then we define for each $\rho \in \Xi$*

$$\mathcal{P}_\rho(M,N) := \{v : M \to N : v \text{ is the projection of a } \rho\text{-equivariant measurable map } \tilde{v}\}$$

and

$$\mathcal{P}(M,N) := \bigcup_{\rho\in\Xi} \mathcal{P}_\rho(M,N).$$

Let $v \in \mathcal{P}(M,N)$ be given. Since G_N acts freely on $\tilde{N}$, there exists a unique $\rho \in \Xi$ and a ρ-equivariant measurable map $\tilde{v} : \tilde{M} \to \tilde{N}$ such that v is the projection of $\tilde{v}$. We define the nonlinear energy $\mathcal{E}_N$ of v by

$$\mathcal{E}_N(v) := \mathcal{E}^\rho_{\tilde{N}}(\tilde{v}) \tag{3.7}$$

where $\mathcal{E}^\rho_{\tilde{N}}$ is the nonlinear energy function defined for ρ-equivariant measurable maps with values in $\tilde{N}$. In addition, we define $\mathcal{D}(\mathcal{E}_N) := \{v \in \mathcal{P}(M,N) : \mathcal{E}_N(v) < \infty\}$.

3.2.1 Nonlinear Dirichlet Problem

Now, we define the nonlinear Dirichlet problem for maps with values in finite graphs.

Let $g \in \mathcal{D}(\mathcal{E}_N)$ be a continuous map. Then the lifting $\tilde{g} : \tilde{M} \to \tilde{N}$ is a g_*-equivariant map. Furthermore, let D be a domain in M. We define the nonlinear Dirichlet problem for g and D as follows:

Definition 3.17 *Given g and D as described above, let us define the class of maps*

$$V_N(g) := \{v \in \mathcal{D}(\mathcal{E}_N) \cap \mathcal{P}_{g_*}(M,N) : \hat{v} = \hat{g} \text{ quasi everywhere on } M\backslash D\}$$

where $\hat{v}, \hat{g}$ denote the projection of a quasi-continuous version of $\tilde{v}$ and $\tilde{g}$, resp. A map $u \in V_N(g)$ is called solution to the nonlinear Dirichlet problem for g and D whenever

$$\mathcal{E}_N(u) = \min_{v\in V_N(g)} \mathcal{E}_N(v).$$

The next result states a sufficient condition for the existence and uniqueness of a solution to the nonlinear Dirichlet problem.

Theorem 3.18 *Let $g \in \mathcal{D}(\mathcal{E}_N)$ be continuous and let D be a subset of M which fulfills the following conditions*

1. *There is fundamental domain* M_0 *of* M*, in* $\tilde{M}$*, with* $\bar{M}_0$ *compact and* ∂M_0 *has measure zero such that* $D_0 := \varphi_M^{-1}(D) \cap M_0$ *is relatively compact and open.*

2. *It holds* $dist(\bar{D}_0, \partial M_0) > 0$.

3. *The process* X_t *corresponding to* $\mathcal{E}$ *quits* D_0 *during its lifetime.*

Then there exists a unique map $u \in \mathcal{D}(\mathcal{E}_N) \cap \mathcal{P}_{g_*}(M, N)$ *which solves the nonlinear Dirichlet problem to* g *and* D*.*
In particular, the three conditions are fulfilled, if D *is a simply connected relatively compact open subset of* M *such that* $\varphi_M(X_t)$ *quits* D *during its lifetime.*

Proof: The claim is a conclusion of Theorem 3.14, Lemma 3.15 (ii) and Equation (3.7). □

3.2.2 Homotopy Problems

Denoting the homotopy class of a continuous map $g : M \to N$ by $Hom(g)$ we call a map $u \in Hom(g)$ harmonic if it is a minimizer of the nonlinear energy in this homotopy class. For particular domain spaces M we will show the existence of harmonic maps.

We start with the following lemma.

Lemma 3.19

(i) *Let* $u : M \to N$ *be continuous. Given a map* $v : M \to N$ *homotop to* u *one has that the lifting* $\tilde{v}$ *is* u_**-equivariant.*

(ii) *If* $\tilde{u}$ *and* $\tilde{v}$ *are continuous* ρ*-equivariant maps (*$\rho \in \Xi$*) then the corresponding continuous projections* u *and* $v : M \to N$ *are homotop.*

Proof: See [Stu02], [KS93].

In the sequel, we will denote the homotopy class of a continuous map $g : M \to N$ by $Hom(g)$.

Definition 3.20 *We call a map* $u \in Hom(g)$ *harmonic if*

$$\mathcal{E}_N(u) = \min_{v \in \mathcal{D}(\mathcal{E}_N) \cap Hom(g)} \mathcal{E}_N(v).$$

Theorem 3.21 *Let* M *be a connected compact Riemannian manifold with* $\partial M = \emptyset$ *and let* $\mathcal{E}$ *be the classical Dirichlet form on* $\tilde{M}$ *given by the Laplace-Beltrami operator.*
Given a continuous map $g : M \to N$ *there exists a map* $u \in Hom(g)$ *which is harmonic and Lipschitz continuous.*

Before we start with the proof of the theorem let us introduce some notations and definitions from the work [KS93] of Korevaar/Schoen.

We will say that Ω is a Riemannian domain if it is a connected, open subset of a k-dimensional Riemannian manifold M having the property that its metric completion $\bar{\Omega}$ is a compact subset of M. Furthermore, let μ be the Riemannian volume measure on M and let (X, d) be an NPC-space.

Definition 3.22 *We define the space $L^2(\Omega, X)$ as the set of Borel-measurable maps $v : \Omega \to X$ for which*

$$\int_\Omega d^2(u(x), q) d\mu(x) < \infty$$

for some $q \in X$.

The space $L^2(\Omega, X)$ is a complete metric space, with distance function D defined by

$$D^2(u, v) = \int_\Omega d^2(u(x), v(x)) d\mu(x).$$

Definition 3.23 (Nonlinear Energy of Korevaar/Schoen) *Let $v \in L^2(\Omega, X)$ be given and let $\mathcal{C}_c(\Omega)$ be the set of all continuous maps on Ω with compact support. Then for $\epsilon > 0$ and $f \in \mathcal{C}_c(\Omega)$ define*

$$E_{\epsilon,f}(v) = c_k \cdot \int_{\Omega_\epsilon} f(x) \underbrace{\int_{S(x,\epsilon)} \epsilon^{-k-1} d^2(v(x), v(y))\, d\sigma_{x,\epsilon}(y)}_{e_\epsilon(x)}\, d\mu(x)$$

with $\Omega_\epsilon = \{x \in \Omega : dist(x, \partial\Omega) > \epsilon\}$, $S(x,\epsilon) = \{y \in \Omega : |y - x| = \epsilon\}$, $d\sigma_{x,\epsilon}(y)$ the $(k-1)$-dimensional surface measure on $S(x,\epsilon)$ and

$$c_k = \int_{S^{k-1}} |x^1|^2 d\sigma(x)$$

(where $x = (x^1, \dots, x^k) \in \mathbb{R}^k$ and $S^{k-1} = \{|x| = 1\}$).
The map v has finite (nonlinear) energy (and one writes $v \in W^{1,2}(\Omega, X)$), if

$$\sup_{\substack{f \in \mathcal{C}_c(\Omega)\\ 0 \le f \le 1}} \left(\limsup_{\epsilon \to 0} E_{\epsilon,f}(v) \right) \equiv E(v) < \infty.$$

Now, let $\tilde{v} : \tilde{M} \to X$ be a ρ-equivariant map and let M_0 be any connected compact fundamental domain for M, in $\tilde{M}$, which boundary has measure zero. In the definition of Korevaar/Schoen the map $\tilde{v}$ has finite energy if

$$\tilde{v} \in W^{1,2}(\mathring{M}_0, X)$$

Thus, the nonlinear energy of a ρ-equivariant map $\tilde{v} : \tilde{M} \to X$ is given by

$$E^\rho(\tilde{v}) := E(\tilde{v}).$$

For the proof of Theorem 3.21, we will show the following three steps:

1. Let (X, d) be a tree with l edges each with length equal to one. Then for any map $v \in W^{1,2}(\overset{\circ}{M}_0, X)$ it holds

$$E(v) = \sum_{i=1}^{l} \int_{M_0} |\nabla v_i(x)|^2 d\tilde{\mu}(x) \tag{3.8}$$

   with $v_i, 1 \leq i \leq l$, being the projections of v and $\tilde{\mu}$ the Riemannian volume measure on $\tilde{M}$.

2. For the lifting $\tilde{v}$ of any map $v \in \mathcal{D}(\mathcal{E}_N) \cap Hom(g)$ it holds

$$E^{g_*}(\tilde{v}) = \mathcal{E}^{g_*}_{\tilde{N}}(\tilde{v}) = \mathcal{E}_N(v) < \infty.$$

3. For any g_*-equivariant continuous map $\tilde{v} : \tilde{M} \to \tilde{N}$ with $E^{g_*}(\tilde{v}) < \infty$ one has

$$\mathcal{E}_N(v) = \mathcal{E}^{g_*}_{\tilde{N}}(\tilde{v}) = E^{g_*}(\tilde{v}),$$

   i.e. the projection v of the map $\tilde{v}$ is an element of $\mathcal{D}(\mathcal{E}_N) \cap Hom(g)$.

Then the claim follows from Theorem 2.7.1 in [KS93].

Step 1:
Equation (3.8) can be proven with the same arguments used in Subsection 2.1.2 taking into account the following remark.

Remark 3.24 *Let $\mathcal{E}$ be the classical Dirichlet form on $L^2(\tilde{M}, \tilde{\mu})$. Then the energy measure $d\tilde{\mu}_{<u>}$ of a function $u \in \mathcal{D}_{loc}(\mathcal{E})$ is given by $d\tilde{\mu}_{<u>} = |\nabla u|^2 d\tilde{\mu}$.*

Step 2:
Let $v \in \mathcal{D}(\mathcal{E}_N) \cap Hom(g)$ be given. The lifting $\tilde{v}$ is continuous and bounded on any compact fundamental domain (i.e. on M_0 the lifting $\tilde{v}$ has values in a finite tree with l edges). Thus, it follows from Theorem 3.11, Step 1, and the fact that ∂M_0 has measure zero

$$E^{g_*}(\tilde{v}) = \sum_{i=1}^{l} \int_{M_0} |\nabla \tilde{v}_i|^2 d\tilde{\mu} = \mathcal{E}^{g_*}_{\tilde{N}}(\tilde{v}) < \infty.$$

Step 3:
Given a g_*-equivariant continuous map $\tilde{v}$ with $E^{g_*}(\tilde{v}) < \infty$ one can deduce with similar

arguments as used in Lemma 2.18 and Proposition 2.17 that $\nabla\tilde{v}_i$ exists for all projections $\tilde{v}_i, i \in \mathbb{N}$. Hence Step 1 and Theorem 3.11 yield $\tilde{v} \in \mathcal{D}(\mathcal{E}_{\tilde{N}}^{g_*})$ and

$$\mathcal{E}_{\tilde{N}}^{g_*}(\tilde{v}) = \sum_{i=1}^{l} \int_{M_0} |\nabla\tilde{v}_i|^2 d\tilde{\mu} = E^{g_*}(\tilde{v}) < \infty.$$

□

Corollary 3.25 *Let M be a compact admissible Riemannian polyhedron with $\partial M = \emptyset$ and let $\mathcal{E}$ be the classical Dirichlet form on $\tilde{M}$ given by the Laplace-Beltrami operator.*
For any continuous map $g : M \to N$ there exists a map $u \in Hom(g)$ which is harmonic and Hölder continuous.

Proof: The proof that our energy coincides with the nonlinear energy for maps between Riemannian polyhedra introduced in [EF01] works out in the same way as in Theorem 3.21. For the existence and the Hölder continuity of the energy minimizer we refer to Theorem 11.1 in [EF01]. □

Final Remarks: Our intention was to study harmonic maps from a measure space (M, m) equipped with a local regular conservative Dirichlet form $\mathcal{E}$ into a tree N. For this we defined a extension $\mathcal{E}_N$ of the energy functional for maps with values in trees. This definition of the nonlinear energy was motivated by the approach of Jost presented in [Jos94]. He introduced for a map f defined on a locally compact metric space M equipped with an abstract Dirichlet form $\mathcal{E}$ with values in an NPC space the following definition of the nonlinear energy

$$\tilde{\mathcal{E}}_N(f) := \Gamma - \lim_{t \to 0} \frac{1}{2t} \int_M \int_M d^2(f(x), f(y)) p_t(x, dy) m(dx)$$

where $(p_t)_t$ denotes the semigroup of Markov kernels corresponding to the Dirichlet form $\mathcal{E}$. The main difference to our definition is that we replace the $\Gamma - \lim$ by $\limsup$. Our approach is more restrictive than Jost's approach because of the restrictions on the target space. However, for our setting we proved a decomposition of the nonlinear energy (cf. Theorem 2.7) which yields an energy measure for our nonlinear energy. Obviously, this decomposition depends on the structure of the target space and will not hold for more general NPC spaces. For the special case that M is a Riemannian manifold equipped with the classical Dirichlet form we showed that our definition of nonlinear energy coincides with energy given by Korevaar/Schoen in [KS93]. In addition, we proved that our nonlinear energy is identical to the energy for maps with values in trees defined by Picard in [Pic04].

Another issue of this work was to present conditions for the existence and uniqueness of a solution to the nonlinear Dirichlet problem for tree-valued maps and to provide a rigorously numerical approach. We constructed a numerical algorithm to solve the nonlinear Dirichlet problem for maps from a two dimensional Euclidean domain into trees and we proved the convergence of our numerical method. For the proof of convergence we used regularity results of the solution to the nonlinear Dirichlet problem given in [KS93] and [Ser94].

In addition, we implemented the algorithm and visualized solutions to the nonlinear Dirichlet problem. Because of the consistence of our nonlinear energy with the energy given by Korevaar/Schoen and Picard one can use our algorithm to construct and visualize solutions to the nonlinear Dirichlet problem given by their nonlinear energy.

A further intention of this work was to study harmonic maps from a compact measure space equipped with a Dirichlet form into graphs. Our approach to analyze this problem was motivated by the works [Jos94], [Jos96] of Jost and [KS93] of Korevaar/Schoen.
They investigated a nonlinear energy for equivariant maps and analyzed the properties of energy minimizers. Korevaar/Schoen defined a nonlinear energy for equivariant maps from the universal cover of a Riemannian manifold into an NPC space and showed the existence of a Lipschitz continuous energy minimizing map. Jost studied a nonlinear energy for equivariant maps from a locally compact metric space equipped with a Dirichlet form $\mathcal{E}$ into an NPC space. He presented conditions on the domain and the target space for the existence of a Hölder continuous energy minimizer.
In general, Jost's conditions on the target space are not fulfilled if the NPC space is the universal cover of a graph. Therefore, we presented another nonlinear energy for equivariant maps from a measure space into in the universal cover of a graph. Our approach is motivated by our analysis of the nonlinear energy for tree-valued maps and the fact that the universal cover of a graph is a tree (with an infinite number of edges).
Finding conditions for the existence of Hölder or Lipschitz continuous equivariant energy minimizing maps is still an open question. But, we proved that our energy for equivariant maps with values in the universal cover of a graph coincides with the energy given by Korevaar/Schoen resp. Eells/Fuglede if the domain space is the universal cover of a Riemannian manifold resp. Riemannian polyhedron. Hence, for our nonlinear energy we could show the existence of a Lipschitz resp. Hölder continuous harmonic map between a Riemannian manifold resp. Riemannian polyhedron and a graph in a given homotopy class.
A further open question is the development of a numerical algorithm to construct a harmonic map in a given homotopy class with values in a graph.

Appendix

A.1 Locality for Regular Dirichlet Forms

In this part of the work, we will show that a regular Dirichlet form whose corresponding semigroup $(T_t)_{t\geq 0}$ is given by a semigroup of Markov kernels $p_t(x,dy)$ and which is local in the sense of Fukushima is also strongly local.
Furthermore, we discuss for regular Dirichlet forms the equivalence of the strong locality property in the sense of Fukushima and the locality property in the sense of Bouleau/Hirsch.

Let us start with the definitions of locality and strong locality in the sense of Fukushima.

Definition A.1 *A Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is said to be local in the sense of Fukushima (cf. [FOT94]) if*

> *$u,v\in\mathcal{D}(\mathcal{E})$, supp$[u]$ and supp$[v]$ are compact, $v\equiv 0$ on a neighbourhood of supp$[u]$ $\Rightarrow\quad \mathcal{E}(u,v)=0$.*

Definition A.2 *A Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is said to be strongly local in the sense of Fukushima (cf. [FOT94]) if*

(ST) $u,v\in\mathcal{D}(\mathcal{E})$, supp$[u]$ and supp$[v]$ are compact, v is constant on a neighbourhood of supp$[u]$ $\quad\Rightarrow\quad \mathcal{E}(u,v)=0$.

For a regular Dirichlet form whose corresponding semigroup $(T_t)_{t\geq 0}$ is given by a semigroup of Markov kernels $p_t(x,dy)$ one has the following result.

Theorem A.3 *Let $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ be a regular local (in the sense of Fukushima) Dirichlet form on $L^2(X,m)$ whose corresponding semigroup $(T_t)_{t\geq 0}$ is given by a semigroup of Markov kernels $p_t(x,dy)$. Then $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ is also strongly local.*

Proof: Let $u,v\in\mathcal{D}(\mathcal{E})$ be given with supp$[u]$ and supp$[v]$ compact and $v\equiv c$ on a neighbourhood of supp$[u]$. Defining $u_c := u-c$ one has

$$\begin{aligned}
\mathcal{E}(u,v) &= \lim_{t\to 0}\frac{1}{t}\int_{\{v\neq 0\}}\left[u(x)-p_tu(x)\right]v(x)m(dx)\\
&= \lim_{t\to 0}\frac{1}{t}\int_{\{v\neq 0\}}\left[u_c(x)-p_tu_c(x)\right]v(x)m(dx)\\
&= -\lim_{t\to 0}\frac{1}{t}\int_{\{v\neq 0\}}p_tu_c(x)\cdot v(x)m(dx)\\
&= 0
\end{aligned}$$

because of $dist(\text{supp}[u_c], \text{supp}[v]) > 0$ and because of the locality of p_t. □

Lemma A.4 *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be regular Dirichlet form on $L^2(X, m)$. Given a compact set $K \subset X$ and a relatively compact set $V \subset X$ with $K \subset V$. Then there exists a function $v \in \mathcal{D}(\mathcal{E}) \cap C_0(X)$ with $v \equiv 1$ on K and supp$[v] \subset V$.*

Proof: There exists a function $u \in C_0(X)$ with $u \equiv 1$ on K and $\text{supp}[u] \subset V$. Since $\mathcal{E}$ is regular there exists a sequence of functions $u_n \in \mathcal{D}(\mathcal{E}) \cap C_0(X)$ with $\text{supp}[u_n] \subset \{x \in X : u(x) \neq 0\} \subset V$ which converges to u uniformly (cf. Lemma 1.4.2 in [FOT94]). Thus, there exists $n_0 \in \mathbb{N}$ such that $|u_{n_0}(x) - 1| < \frac{1}{2} \quad \forall x \in K$. Defining $v := 2u_{n_0} \wedge 1$ yields the claim. □

With Lemma A.4 we obtain the following theorem.

Theorem A.5 *Let $(\mathcal{E}, (\mathcal{D}(\mathcal{E}))$ by a regular strongly local Dirichlet form. Given two functions $u, v \in \mathcal{D}(\mathcal{E})$ with v being constant on a neighbourhood of supp$[u]$ then one has $\mathcal{E}(u, v) = 0$.*

Proof:
Step I: Let $u, v \in \mathcal{D}(\mathcal{E})$ with support of v compact and $u \equiv c$ on a neighbourhood N of $\text{supp}[v]$ be given. (The function u doesn't need to have compact support.) Then there exists a relatively compact set U with $\text{supp}[u] \subset U \subset N$. Since the Dirichlet form $\mathcal{E}$ is regular there is a function $w \in \mathcal{D}(\mathcal{E})$ with compact support, with $\text{supp}[w] \subset \text{supp}[u]$ and with $w \equiv c$ on U (cf. Lemma A.4). It follows

$$\mathcal{E}(u, v) = \mathcal{E}(u - w, v) + \mathcal{E}(w, v).$$

Furthermore, one has by the definition of strongly local that $\mathcal{E}(w, v) = 0$, because of the compact supports of w and v. In addition, it follows from Proposition 1.2 in [MR92] that $\mathcal{E}(u - w, v) = 0$, because of $\text{supp}[u - w] \cap \text{supp}[v] = \emptyset$. Thus $\mathcal{E}(u, v) = 0$.

Step II: Now let $u, v \in \mathcal{D}(\mathcal{E})$ with $u \equiv c$ on a neighbourhood N of $\text{supp}[v]$ be given. It follows from the proof of Proposition 1.2 in [MR92] that there exists a sequence $v_n \in \mathcal{D}(\mathcal{E})$ with $v_n \to v$ in $\mathcal{D}(\mathcal{E})$, $\text{supp}[v_n]$ compact and $\text{supp}[v_n] \subset \text{supp}[v]$. Hence, for all n one has $u \equiv 1$ on a neighbourhood of $\text{supp}[v_n]$ and it follows by *Step I*

$$\mathcal{E}(u, v_n) = 0 \qquad \forall n.$$

Thus, it holds $\mathcal{E}(u, v) = 0$, because of $v_n \to v$ in $\mathcal{D}(\mathcal{E})$. □

Remark: Theorem A.5 yields that for regular Dirichlet forms one can replace in definitions A.1 and A.2 compact support just by support.

Definition A.6 *A Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is said to be local in the sense of Bouleau/Hirsch (cf. [BH91]) if it satisfies:*

(L0) $\forall u \in \mathcal{D}(\mathcal{E}) \quad \forall F, G \in C_0^\infty(\mathbb{R})$

$$\text{supp}[F] \cap \text{supp}[G] = \emptyset \quad \Longrightarrow \quad \mathcal{E}(F_0(u), G_0(u)) = 0$$

with $F_0(x) := F(x) - F(0)$.

Theorem A.7 *Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a regular Dirichlet form. Then the following properties are equivalent:*

(ST) $\mathcal{E}$ is strongly local in the sense of Fukushima.

(L0) $\mathcal{E}$ is local in the sense of Bouleau/Hirsch.

(L1) $\forall u, v \in \mathcal{D}(\mathcal{E}) \qquad \forall a \in \mathbb{R}$

$$(v + a)u = 0 \quad \Longrightarrow \quad \mathcal{E}(u, v) = 0.$$

For the proof of Theorem A.7 we need the following Proposition.

Proposition A.8 *Let F be a normal contraction from $\mathbb{R}$ into $\mathbb{R}$. Then the map*

$$u \in \mathcal{D}(\mathcal{E}) \longrightarrow F \circ u \in \mathcal{D}(\mathcal{E}) \tag{A.1}$$

is continuous (for the Hilbert structure of $\mathcal{D}(\mathcal{E})$).

For the proof see [Anc76].

In the sequel, we will denote the set of normal contractions of $\mathbb{R}$ into $\mathbb{R}$ by $\mathcal{T}_1^0$.

Proof of Theorem A.7:

$(ST) \Rightarrow (L0)$:
Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a regular strongly local Dirichlet form and let $F, G \in C_0^\infty(\mathbb{R})$ with $\text{supp}[F] \cap \text{supp}[G] = \emptyset$ and $u \in \mathcal{D}(\mathcal{E}) \cap C_0(X)$ be given. We may assume that $G(0) = 0$. One has that $f := F_0(u)$ is constant on $\text{supp}(g)$ with $g := G_0(u)$. Defining for $\epsilon > 0$ the function $g_\epsilon := g - ((-\epsilon) \vee g \wedge \epsilon)$ it holds $g_\epsilon \in \mathcal{D}(\mathcal{E})$ and f is constant on a neighbourhood of $\text{supp}(g_\epsilon)$, because of the continuity of g. Applying Theorem A.5 one obtains $\mathcal{E}(f, g_\epsilon) = 0$ and $\epsilon \to 0$ yields $\mathcal{E}(f, g) = 0$.
Now, let $u \in \mathcal{D}(\mathcal{E})$ be given. Since $\mathcal{E}$ is regular, there exists a sequence of functions $u_n \in \mathcal{D}(\mathcal{E}) \cap C_0(X)$ with $u_n \to u$ in $\mathcal{D}(\mathcal{E})$. From Proposition A.8 it follows $F_0(u_n) \to F_0(u)$ and $G_0(u_n) \to G_0(u)$ in $\mathcal{D}(\mathcal{E})$, because the set of Lipschitz functions from $\mathbb{R}$ into $\mathbb{R}$ is homothetic to $\mathcal{T}_1^0$ and $C_0^\infty(\mathbb{R})$ is a subset of the set of Lipschitz functions from $\mathbb{R}$ into $\mathbb{R}$. Defining $f_n := F_0(u_n), f := F_0(u), g_n := G_0(u_n)$ and $g := G_0(u)$ one has

$$\mathcal{E}(f, g) = \mathcal{E}(f, g - g_n) + \mathcal{E}(f - f_n, g_n) + \mathcal{E}(f_n, g_n).$$

Since $\mathcal{E}_1(g_n)$ is bounded and $f_n \to f, g_n \to g$ in $\mathcal{D}(\mathcal{E})$ it holds $\lim_{n\to\infty} \mathcal{E}(f, g - g_n) = 0$ and $\lim_{n\to\infty} \mathcal{E}(g_n, f - f_n) = 0$. In addition, one has $\mathcal{E}(f_n, g_n) = 0 \; \forall n$. Hence, we obtain $\mathcal{E}(f, g) = 0$.
$(L0) \Rightarrow (L1)$:
This follows from Proposition 5.1.3 in [BH91].
$(L1) \Rightarrow (ST)$:
Let $u, v \in \mathcal{D}(\mathcal{E})$ with $\text{supp}[u]$ and $\text{supp}[v]$ compact and $v \equiv a \in \mathbb{R}$ on a neighbourhood of $\text{supp}[u]$ be given. It holds $(v - a)u = 0$ and thus $\mathcal{E}(u, v) = 0$. □

Bibliography

[Anc76] ANCONA, A. Continuite des contractions dans les espaces de Dirichlet. *Semin. Theor. Potent., Paris, No.2, Lect. Notes Math.* **563** (1976), 1–26.

[Bra92] BRAESS, D. *Theorie, schnelle Löser und Anwendungen in der Elastizitätstheorie*. Springer Verlag, Berlin, 1992.

[BM95] BIROLI, M., MOSCO, U. A Saint-Venant type principle for Dirichlet forms on discontinuous media. *Ann. Mat. Pura Appl., IV. Ser.* **169** (1995), 125–181.

[BH91] BOULEAU, N., HIRSCH, F. *Dirichlet Forms and Analysis on Wiener Spaces*. Walter de Gruyter, Berlin, 1991.

[Cia78] CIARLET, P. G. *The Finite Element Method for Elliptic Problems*. North-Holland, Amsterdam, 1978.

[Cle75] CLEMENT, PH. Approximation by finite element functions using local regularization. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge Anal. Numér.* **9** (R-2) (1975), 77–84.

[EF01] EELLS, J., FUGLEDE, B. *Harmonic maps between Riemannian polyhedra*. 142 *Cambridge Tracts in Mathematics*. Cambridge University Press, 2001.

[EL78] EELLS, J., LEMAIRE, L. A report on harmonic maps. *Bull. Lond. Math. Soc.* **10** (1978), 1–68.

[EL88] EELLS, J., LEMAIRE, L. Another report on harmonic maps. *Bull. Lond. Math. Soc.* **20** (5) (1988), 385–524.

[Fug01] FUGLEDE, B. The Dirichlet problem for harmonic maps from Riemannian polyhedra to geodesic spaces of upper bounded curvature. Preprint, 2001.

[Fug03a] FUGLEDE, B. Finite energy maps from Riemannian polyhedra to metric spaces. *Ann. Acad. Sci. Fenn. Math.* **28** (2) (2003), 433–458.

[Fug03b] FUGLEDE, B. Hölder continuity of harmonic maps from Riemannian polyhedra to spaces of upper bounded curvature, *Calc. Var. Partial Differential Equations* **16** (4) (2003), 375–403.

[FOT94] FUKUSHIMA, M., OSHIMA, Y., TAKEDA, M. *Dirichlet Forms and Symmetrical Markov Processes*. Walter de Gruyter, Berlin, 1994.

[GS92] GROMOV, M., SCHOEN, R. M. Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one. *Inst. Hautes Études Sci. Publ. Math.* (76) (1992), 165–246.

[JY93] JOST, J., YAU, S. T. Harmonic maps and superrigidity, differential geometry: partial differential equations on manifolds, Proc. Symp. Pure Math. **54**, Part I, 1993, 245–280.

[Jos94] JOST, J. Equilibrium maps between metric spaces. *Calc. Var. Partial Differential Equations* **2** (2) (1994), 173–204.

[Jos95] JOST, J. *Riemannian Geometry and Geometric Analysis*. Springer Verlag, Berlin, 1995.

[Jos96] JOST, J. Generalized harmonic maps between metric spaces. In: Jost, J. (ed.), Geometric analysis and the calculus of variations. Dedicated to Stefan Hildebrandt on the occasion of his 60th birthday. International Press (1996), 143–174.

[Jos97a] JOST, J. *Nonpositive curvature: geometric and analytical aspects*. Lectures in Mathematics, ETH Zürich. Birkhäuser Verlag, Basel, 1997.

[Jos97b] JOST, J. Generalized Dirichlet forms and harmonic maps. *Calc. Var. Partial Differential Equations* **5** (1) (1997), 1–19.

[KS93] KOREVAAR, N. J., SCHOEN, R. M. Sobolev spaces and harmonic maps for metric space targets. *Comm. Anal. Geom.* **1** (3–4) (1993), 561–659.

[KS97] KOREVAAR, N. J., SCHOEN, R. M. Global existence theorems for harmonic maps to non-locally compact spaces. *Comm. Anal. Geom.* **5** (2) (1997), 333–387.

[MR92] MA, Z.-M., RÖCKNER, M. *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms*. Springer Verlag, Berlin, 1992.

[Pic04] PICARD, J. Stochastic calculus and martingales on trees, Preprint, 2003.

[Ser94] SERBINOWSKI, T. Boundary regularity of harmonic maps to nonpositively curved metric spaces. *Comm. Anal. Geom.* **2** (1) (1994), 139–153.

[SZ88] STÖCKER, R., ZIESCHANG, H. *Algebraische Topologie. Eine Einführung.* Teubner Verlag, Stuttgart, 1988.

[Stu01] STURM, K.-T. Nonlinear Markov operators associated with symmetric Markov kernels and energy minimizing maps between singular spaces. *Calc. Var. Partial Differential Equations* **12** (4) (2001), 317–357.

[Stu02] STURM, K.-T. Nonlinear Markov operators, discrete heat flow, and harmonic maps between singular spaces. *Potential Anal.* **16** (4) (2002), 305–340.

[Tho97] THOMÉE, V. *Galerkin Finite Element Methods for Parabolic Problems.* Springer Verlag, Berlin, 1997, Springer Series in Computational Mathematics, 25.

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