Aspects of Grand Unification in Higher Dimensions

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Chapter 1

Introduction

1.1 Motivation

Despite the fact that there is no convincing experimental evidence against the Standard Model [1–4], most physicists expect new physics beyond the scale of a few TeV. There are numerous reasons for this: The Standard Model has 26 free parameters (taking into account the recent developments indicating a non-vanishing neutrino mass), which seem to have arbitrary values. There is no explanation for the observed pattern of mass matrices, such as minimal and maximal mixing in the quark and lepton sectors, respectively. The cosmological constant which is now known to have a small, but non-vanishing value, cannot be consistently accommodated in the framework of the Standard Model. The repetition of families, where the respective particles in each family coincide in all quantum numbers and only differ in their masses, remains a puzzle. In addition to the above listed features which describe a certain lack of predictive power inherent to the Standard Model, the structure of the gauge symmetry, which is the direct product of three gauge groups (each with its own gauge coupling constant), is in many respects unattractive. The symmetries describing the strong and electro-weak interactions are not interrelated, thus spoiling a truly unified picture of the fundamental forces. By these considerations, one is naturally led to construct models with more symmetry.

Grand Unified Theories and Supersymmetry

The idea of unification is further supported by the running of the gauge coupling constants, which seem to meet in a single point at $\sim 3 \times 10^{16}$ GeV. By the time when the gauge coupling constants became to be known with higher precision, one realized that they do not exactly meet, but narrowly miss each other, forming the so-called GUT triangle. This, however, does not necessarily spoil our arguments, but may be in strong support of the idea of unification, as we will now explain.
The Standard Model is a renormalizable theory, and as such, infinities which may appear in the calculations can be consistently absorbed into a finite number of physical parameters. As soon as we give up the point of view that the Standard Model is the fundamental theory

which is valid up to arbitrarily high energies, we have to introduce a cut-off, where new physics takes over. The mass of the only fundamental spin-0 particle, the Higgs boson, receives corrections which are quadratic in the cut-off parameter, whereas the masses of the gauge bosons and of the fermions are protected by gauge invariance and chiral symmetries, respectively. These large corrections eventually drag the Higgs mass from the electroweak scale of ~ 200 GeV to the unification scale (*naturalness problem*). Supersymmetry [5] nicely resolves this problem by introducing fermionic partners for the bosons (and vice versa), thus protecting the Higgs mass against large corrections. Moreover, in the supersymmetric generalization of the Standard Model, the gauge couplings which previously missed each other, meet in one point, suggesting to consider supersymmetric Grand Unified Theories.

The smallest simple group which contains the Standard Model gauge symmetry is SU(5). In the (non-supersymmetric) Georgi-Glashow model [6], all particles of one generation (except the right handed neutrino) can be accommodated in two irreducible representations:

$$\begin{array}{ccccc} \mathrm{SO}(10) & \supset & \mathrm{SU}(5) & \supset & \mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \\ \mathbf{16} & \rightarrow & \mathbf{10}+\overline{\mathbf{5}}+\mathbf{1} & \rightarrow & \underset{Q_L}{(\mathbf{3},\mathbf{2})} + \underset{u_R}{(\overline{\mathbf{3}},\mathbf{1})} + \underset{e_R}{(\mathbf{1},\mathbf{1})} + \underset{d_R}{(\overline{\mathbf{3}},\mathbf{1})} + \underset{E_L}{(\mathbf{1},\mathbf{2})} + \underset{\nu_R}{(\mathbf{1},\mathbf{1})} \end{array}$$

We have also indicated how the SU(5) representations fit into the framework of the SO(10) gauge group, since this will shortly become relevant. The embedding of the hypercharge in the Grand Unified gauge group gives a prediction for the weak mixing angle $\sin^2\theta_w$, and the fact that there are only two Yukawa couplings corresponding to the two irreducible representations relates the b-quark and τ-lepton masses at the unification scale. In comparison to the Standard Model, this is a significant gain in predictive power. Unfortunately, the minimal SU(5) model (both in its supersymmetric and non-supersymmetric version) is ruled out by experiment.

Taking the idea of unification seriously, we notice that the SU(5) model has the rather disturbing feature that two irreducible representations are needed to account for one family of matter particles, and moreover, the right handed neutrino is conspicuously absent. The smallest gauge group, in which one family can be fit into one irreducible representation, is the Fritzsch-Minkowski model [7] with gauge group SO(10). The appearance of the singlet state in the decomposition of the **16** playing the role of the right handed neutrino and the absence of exotic particles can be considered to be a prediction of this model.

In spite of all the gain in predictive power and insight, Grand Unified Theories leave some questions open and introduce new problems. The Higgs particle is generically accompanied by color-triplet states, which must be absent in the low-energy theory (*doublet-triplet splitting problem*). There is still no explanation for the repetition of families, and we are as far as ever from unifying gravity with the other fundamental interactions. Particularly the quest for a unified picture including gravity led physicists to consider an even more daring idea, namely the existence of *extra dimensions*.

Strings and Extra Dimensions

The first attempt at unifying gravity with the electromagnetic interactions (which were at that time the only known fundamental forces apart from gravity) was made by Kaluza and Klein, who considered gravity in five dimensions and compactified the fifth dimension on a circle [8, 9], thus obtaining gravity in four dimensions and additionally one massless gauge boson and one scalar particle (plus an infinite series of heavy particles, now known as the *Kaluza-Klein tower*).
In higher dimensions, divergencies become more severe, thus spoiling renormalizability. Moreover, the interactions of quantum gravity are mediated by a massless spin-2 particle, the graviton, and the theory is not renormalizable even in four dimensions. At present, the most promising idea seems to be to consider *finite* theories in higher dimensions which include a massless spin-2 particle in their spectrum.

String theory is an excellent candidate: In the spectrum, there is indeed a massless spin-2 particle which is suitable to describe quantized gravity. Moreover, the theory is perfectly finite. Extra dimensions not only naturally arise from requiring mathematical consistency of the underlying theory, but their number is also predicted. Supersymmetry is realized (at the string scale), and the gauge symmetry follows from anomaly arguments[1] and is not arbitrary. In the following, we will describe how the Standard Model and Grand Unification fit into the picture suggested by string theory.
Notwithstanding all its mathematical beauty, it has still to be proved that string theory has anything to do with the real world. The final goal, of course, is to reproduce the Standard Model from string theory as the low-energy effective theory, preferably with a Grand Unified gauge group realized at some intermediate step. Any theory which is to describe the world we live in has to have only four observable space-time dimensions.

Compactification

Of the five consistent string theories, heterotic $\mathrm{E}_8\times\mathrm{E}_8'$ theory seems particularly promising.[2] The gauge group E_8 in ten dimensions is large enough to accommodate not only the Standard Model, but also all the other groups which have ever been used in successful Grand Unified Model building. In particular, we have the so-called series of GUTs:

$$\mathrm{E}_8 \;\supset\; \mathrm{E}_6 \;\supset\; \mathrm{SO}(10) \;\supset\; \mathrm{SU}(5) \;\supset\; \mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$$

Starting in ten dimensions, one can in principle confine the 6 extra dimensions to any compact manifold $\mathcal{M}$ such that the original space-time is given by a direct product

$$\mathbb{R}^4 \times \mathcal{M},$$

[1]This is true for the heterotic and type I theories. In the case of type II string theories, there is no gauge group in 10 dimensions.

[2]Type II constructions with intersecting D-branes are very successful in reproducing the particle content and some of the other aspects of the Standard Model. At the end of this section, we will briefly comment on their advantages and disadvantages as compared to the heterotic constructions.

where the four infinite dimensions correspond to our Minkowski space and the length scales associated with $\mathcal{M}$ correspond to very high energies which cannot be probed with present accelerators.
The simplest scheme of toroidal compactification [10] does not work, since the theory in four dimensions has then $\mathcal{N} = 4$ supersymmetry and is not chiral. Historically, the first serious attempt to derive the four dimensional phenomenology of strings was to compactify the heterotic $E_8 \times E_8'$ theory on so-called *Calabi-Yau manifolds* [11], which are special manifolds ensuring that the low-energy theory has exactly one supersymmetry. Unfortunately, Calabi-Yau manifolds are fairly complicated so that the computation of properties which are not of topological nature is very difficult and in many cases not possible.[3]

Orbifolds as Twisted Tori

Orbifold constructions [12, 13] which are the main objective of this thesis combine the desirable features of both toroidal compactification and of Calabi-Yau manifolds. Being defined as a six-torus[4], where points related by a discrete symmetry have been identified, the orbifold is Riemann flat with the exception of finitely many points. Therefore, the equations of motion remain exactly solvable and the parameters of the theory can easily be calculated: the gauge group, the matter content, the superpotential, the Yukawa couplings, the Kähler potential, the gauge kinetic function, the unification scale of the gauge coupling constants. Moreover, the discrete symmetry reduces the $\mathcal{N} = 4$ supersymmetry of the torus compactification scheme to $\mathcal{N} = 1$, thus allowing for chiral particles in the spectrum.

It is remarkable how unification is realized in the context of orbifolds. As we go to higher and higher energies, we probe not only the physics in four dimensions, but penetrate into the compactified six dimensions, which open up to display the full gauge symmetry $E_8 \times E_8'$. Conversely, starting in ten dimensions, in the process of compactification not all gauge bosons survive, but some are projected out, thus giving the gauge symmetry in four dimensions. It is this very same mechanism which solves one of the most prominent problems which plagues conventional GUTs in four dimensions, namely the doublet-triplet splitting problem. The Higgs, which is in the fundamental representation of the Grand Unified group (e.g. SU(5)), is accompanied by a color-triplet state,

$$\mathbf{5} \to (\mathbf{1}, \mathbf{2}) + \cancel{(\mathbf{3}, \mathbf{1})}.$$

which is projected out by the discrete symmetries defining the orbifold [14]. In orbifold constructions, the coexistence of these so-called *split multiplets* with the quark and lepton representations in *full multiplets* arises quite naturally.

[3]Recently, generalized Calabi-Yau manifolds have become popular after considering compactifications with fluxes. The presence of fluxes may stabilize quite a few of the moduli of the models both in heterotic and type II constructions.

[4]An orbifold is in general defined as a manifold divided by a discrete symmetry. In the context of string theory, this manifold is usually assumed to be a flat torus.

Another major step towards building realistic four dimensional models was taken when background fields, more specifically *Wilson lines* [15], were included, which subsequently led to the construction of three-generation models with Standard Model gauge group in the presence of extra U(1) factors [14]. In these models, the existence of three generations of quarks and leptons could often be related to the number of compactified dimensions [15] or to the number of twisted sectors [16], thus giving a geometric explanation for the repetition of families. This feature is characteristic of orbifold constructions: Most properties of the model can be related to the geometry of the orbifold.

Other Schemes of Compactification

Other schemes of compactification which have been proposed include (the already mentioned) Calabi-Yau manifolds [11], free fermion models [17], superconformal field theories [18] and brane world constructions [19]. We will now briefly comment on these schemes and indicate their relation to orbifolds.

In many respects, an orbifold can be viewed as a singular limit of a Calabi-Yau manifold, where the curvature has been concentrated at some finite number of points. For a number of cases, this relation has been explicitly illustrated by "blowing up" the orbifolds to Calabi-Yau manifolds, smoothing out the singularities. However, in some sense, orbifolds generalize compactification schemes on smooth manifolds, since they are not manifolds at all. The propagation of a string on an orbifold (in contrast to that of a point particle which could "run into" the singularities) is perfectly well-defined, so from the string point of view, there is no reason to prefer smooth manifolds. Furthermore, Calabi-Yau compactifications of the heterotic string (with 26 left moving and 10 right moving degrees of freedom) assume a ten dimensional intermediate theory, which is then compactified down to four dimensions. With so-called *asymmetric orbifolds* we can directly construct four dimensional strings, avoiding the detour over the ten dimensional picture.

Orbifolds are also closely linked to superconformal field theories. Strings are described by two-dimensional quantum field theories on the world-sheet. Consistency (in particular reparametrization invariance) requires these theories to be (super-)conformal, thus highly restricting their structure and allowing their classification. One class of theories describes bosons and fermions with periodic boundary conditions and considering their embedding in the target space, they can be shown to be equivalent to orbifold models.

In recent years, brane world constructions have played an important role in model building. Strictly speaking, brane world scenarios are not a compactification scheme. The fields of the Standard Model live on a four dimensional hypersurface embedded in ten dimensional space, whereas the mediators of gravity propagate in the full ten dimensions. The main new feature of brane worlds as compared to heterotic compactifications is that gauge and gravitational interactions propagate in different spaces, implying that the string scale

might be as low as a few TeV. The hierarchy problem (the large ratio of the Planck and electroweak scales) is reformulated, namely why this particular geometric setup is realized and whether it is stable.
Brane world models need quite a number of ad hoc assumptions. The placement of the fields on the branes is arbitrary, and contrary to the case of orbifold compactifications, the spectrum does not follow from the higher dimensional gauge group. Gauge coupling unification is generically absent, and the **16** of SO(10) (as explained above, our preferred matter multiplet) cannot be realized in this setup.

All these considerations make us believe that for constructing realistic string models, the heterotic $E_8 \times E_8'$ theory is the most promising starting point.

1.2 Organization of the Thesis

In the following, we will give an overview over the thesis and discuss the contents and the results of each section in some detail.

Chapter 2

We describe the generalities of orbifold constructions. The heterotic string is introduced, mainly to fix the notation and to set the stage. After discussing toroidal compactifications and their shortcomings, orbifolds are introduced, whereby we closely follow refs. [12,13,20]. Special emphasis is laid on the consistency conditions, which will become important when classifying orbifold models. The construction of the spectrum is discussed on an abstract level, deferring many details and examples to chapter 3. The $\mathbb{Z}_3$ orbifold is presented to illustrate the basic mechanism of using gauge background fields to control the number of families. The results in this chapter are standard.

Chapter 3

The early work on orbifolds concentrated on $\mathbb{Z}_3$, and other orbifolds were much less investigated. Orbifolds with point group symmetry $\mathbb{Z}_N$ (N not prime) and $\mathbb{Z}_N \times \mathbb{Z}_M$ (which we will also call *non-prime orbifolds*) have many features which distinguish them from the case of $\mathbb{Z}_N$, where N is a prime number. Maybe the most prominent feature is the appearance of the fixed tori. The states living there are then effectively six-dimensional (*brane states*), and the four dimensional spectrum is realized as the intersection of the higher dimensional spaces corresponding to these twisted sectors. The states on the branes feel the presence of a gauge symmetry which is generically larger than the gauge group surviving in four dimensions, thus allowing us the construction of GUTs in an intermediate step of our model. As explained before, the particle content of the Standard Model is reminiscent of a GUT structure with the appearance of full and split multiplets, which motivates the picture of the Grand Unified Group being realized in a higher dimension. In our publication [21], we focused on $\mathbb{Z}_2 \times \mathbb{Z}_2$ to present the general picture. Several three generation models were constructed, and claims in the literature [22] that there are no three generation models in

the context of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold could be disproved. Exploiting the brane world picture, we make contact to the recently much discussed orbifold GUTs [23, 24].
In this work, we chose to consider the $\mathbb{Z}_6$-II orbifold as the paradigm for general non-prime orbifolds, since this case is general enough to display all the complications which may arise in the course of the calculations. Subsequently, several three generation models are constructed. These results are as yet unpublished. Note that the first work done in this direction was by refs. [25, 26].
We have created computer programs for orbifold constructions. These programs are quite general in the sense that the point group, the Wilson lines and the type of the heterotic theory ($\mathrm{E}_8 \times \mathrm{E}_8'$ or SO(32)) may be specified at will and the program produces the spectrum in a *human readable form*. The programs have been developed in C++ and the construction of one model takes 3 seconds on a 2.7 GHz computer. So far, we have calculated orbifold models for $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_6$-II, $\mathbb{Z}_2 \times \mathbb{Z}_2$ in the context of the $\mathrm{E}_8 \times \mathrm{E}_8'$ theory, and for $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ in the context of the SO(32) theory. The results were compared to the literature, in case they were available.

Chapter 4

The simplest orbifold models generically give a large number of families which is incompatible with our experimental observations. As was previously pointed out, choosing an appropriate gauge field background is an effective method to arrive at phenomenologically viable three generation models. Unfortunately, the large number of parameters which one may choose at will makes a systematic search for interesting models rather difficult.
Since the gauge embedding of the orbifold twist is a Lie algebra automorphism, mathematical tools (in particular the Kač theorem) can be used to classify the models with and without Wilson lines. Scanning through the parameter space of all $\mathbb{Z}_6$-II orbifold models with one Wilson line, we have found some hundred three generation models with Standard Model, Pati-Salam and SO(10) gauge symmetry. Most remarkably, there is even one SO(10) three generation model with no Wilson lines at all.
The classification has been applied to the cases of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_6$-II for the $\mathrm{E}_8 \times \mathrm{E}_8'$ theory and to the case of $\mathbb{Z}_4$ for the SO(32) theory. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ classification has been published in ref. [21], but the other results are as yet unpublished.

Chapter 5

Contemporary orbifold constructions have one big disadvantage, which is a serious problem for them to make contact to established physics. Starting in ten dimensions with either $\mathrm{E}_8 \times \mathrm{E}_8'$ or SO(32), which are both of rank 16, the rank of the gauge group surviving in four dimensions can never be reduced. The Standard Model, however, has only 4 commuting operators, which corresponds to rank 4. The reason behind this can be traced back to the gauge embedding being realized as a shift. Ref. [27] broke new ground by realizing the twist as a rotation in the root lattice of the gauge symmetry, thereby successfully reducing the rank of the algebra in four dimensions. However, the gauge embedding was realized only in the root lattice and not on the algebra level, leaving many questions open. We have succeeded in lifting the rotation in the root lattice to an automorphism on the algebra and developed the calculational tools to unambiguously identify the unbroken gauge group corresponding to this automorphism. Moreover, the possible gauge embeddings could be linked to certain mathematical structures (the conjugacy classes of the Weyl group of the corresponding algebra), which allowed us to systematically search for interesting classes of models. In this chapter, the main focus was on the development of the necessary mathematical tools.

Chapter 6

The methods developed in chapter 5 were used to construct a semi-realistic orbifold GUT with E_6 gauge symmetry in the six-dimensional bulk, which was then reduced to Standard Model gauge group with one additional U(1) factor, thus reducing the rank by 1. Remarkably, the model can be viewed as a limit of a string model described in ref. [26], which has appealing phenomenological properties. The results obtained in this chapter have been published in ref. [28].

Chapter 7

The ultimate goal is to embed any lower dimensional orbifold GUT into the context of heterotic string theory. The methods of chapter 5 are completely general, also allowing the extension of the construction described in chapter 6 to the stringy case. We present the lifts of the 112 conjugacy classes of the E_8 and give the benchmarks for the SO(32) case. The actual construction of a string model will be presented in a forthcoming publication.

Publications

Some results in this thesis have been published in:

In journals

S. Forste, H. P. Nilles, P. K. S. Vaudrevange and A. Wingerter, “Heterotic brane world”, *Phys. Rev.* **D70** (2004) 106008, hep-th/0406208

S. Forste, H. P. Nilles and A. Wingerter, “Geometry of Rank Reduction”,
to appear in *Phys. Rev.* **D**, hep-th/0504117

In proceedings

S. Forste and A. Wingerter, “Bottom Up Meets Heterotic Strings”,
Fortsch. Phys. **53** (2005) 463-467, hep-th/0412066

Chapter 2

Orbifold Constructions

After a short discussion of the heterotic string, we proceed to explain orbifold constructions in some detail. Special emphasis is put on the consistency conditions of the construction, which run like a thread through this work. The discussion of the spectrum is quite abstract and should be read in parallel to the specific model constructions in chapter 3. We explain the role of background fields in the construction of three-generation models and illustrate the mechanism by means of the $\mathbb{Z}_3$ orbifold.

2.1 The Heterotic String

In the bosonic construction, the heterotic string [29, 30] is described by a 10 dimensional right moving superstring [31–34], and a 26 dimensional left moving bosonic string. We will denote the 8 right moving bosonic and fermionic coordinates of the superstring in the light-cone gauge by X_R^i and Ψ_R^i, $i = 1, \ldots, 8$, respectively. The left movers include 8 bosons X_L^i, $i = 1, \ldots, 8$, and another 16 bosons X_L^I, $I = 1, \ldots, 16$, which are compactified on the 16-torus T^{16}. From anomaly cancellation arguments in 10 dimensions it follows that T^{16} must correspond either to the root lattice of $\mathrm{E}_8 \times \mathrm{E}_8'$ or to that of $\mathrm{Spin}(32)/\mathbb{Z}_2$ [35]. While most of what we explain in this chapter also applies to the latter case, we will soon specialize to the case of $\mathrm{E}_8 \times \mathrm{E}_8'$, because this case is considered to be phenomenologically more promising.

To find the massless states is pretty straightforward. The mass equation for the right mover is given by

$$\frac{m_R^2}{4} = N_R - c_R \tag{2.1}$$

where N_R is the number operator, counting the excitations of the oscillators, and c_R is a constant which is 1/2 for Neveu-Schwarz states, and 0 for Ramond states. The only massless states are $b_{-1/2}^i|0\rangle_R$ in the Neveu-Schwarz sector, and $d_0^\alpha|0\rangle_R$ in the Ramond sector. The former state is an SO(8) vector, whereas the latter one is a spinor. To make this point explicit, we will represent these states by the weights of the respective representation in Cartan-Weyl labels:

$$b^i_{-1/2}|0\rangle_R \sim |\underline{\pm 1\,0\,0\,0}\rangle_R \sim \mathbf{8}_{\boldsymbol{v}}, \qquad d^\alpha_0|0\rangle_R \sim |\pm\tfrac{1}{2}\pm\tfrac{1}{2}\pm\tfrac{1}{2}\pm\tfrac{1}{2}\rangle_R \sim \mathbf{8}_{\boldsymbol{s}} \tag{2.2}$$

The underline denotes the permutation of the entries and the GSO projection [36] restricts number of plus signs in the spinor representation to be even. The mass formula is then rewritten to give

$$\frac{m_R^2}{4} = \frac{1}{2}q^2 - \frac{1}{2}, \tag{2.3}$$

where q is either $(\underline{\pm 1\,0\,0\,0})$ or $\left(\pm\frac{1}{2}\pm\frac{1}{2}\pm\frac{1}{2}\pm\frac{1}{2}\right)$.

The mass formula for the left movers is given by

$$\frac{m_L^2}{4} = N_L + \frac{(p_L^I)^2}{2} - 1. \tag{2.4}$$

The level matching condition $m_R^2 = m_L^2$ removes the tachyonic ground state ($N_L = p_L^I = 0$) of the bosonic string. The only massless states are

$$\begin{aligned} N_L = 1, \quad (p_L^I)^2 = 0: &\quad \tilde{\alpha}^i_{-1}|0\rangle_L \\ N_L = 1, \quad (p_L^I)^2 = 0: &\quad \tilde{\alpha}^I_{-1}|0\rangle_L \\ N_L = 0, \quad (p_L^I)^2 = 2: &\quad |p_L\rangle_L \end{aligned} \tag{2.5}$$

Putting everything together, we obtain the massless spectrum of the heterotic string in 10 dimensions:

$$\begin{aligned} b^i_{-1/2}|0\rangle_R \otimes \tilde{\alpha}^i_{-1}|0\rangle_L &\;:\; \text{Graviton, dilaton, B-field} \\ d^\alpha_0|0\rangle_R \otimes \tilde{\alpha}^i_{-1}|0\rangle_L &\;:\; \text{SUSY partners for graviton, dilaton, B-field} \\ b^i_{-1/2}|0\rangle_R \otimes \tilde{\alpha}^I_{-1}|0\rangle_L &\;:\; \text{16 uncharged gauge bosons} \\ d^\alpha_0|0\rangle_R \otimes \tilde{\alpha}^I_{-1}|0\rangle_L &\;:\; \text{16 uncharged gauginos} \\ b^i_{-1/2}|0\rangle_R \otimes |p_L\rangle_L &\;:\; \text{240+240 charged gauge bosons of } \mathrm{E}_8 \times \mathrm{E}_8' \\ d^\alpha_0|0\rangle_R \otimes |p_L\rangle_L &\;:\; \text{240+240 charged gauginos of } \mathrm{E}_8 \times \mathrm{E}_8' \end{aligned} \tag{2.6}$$

The states carrying a Lorentz index are identified with the corresponding particles using their transformation properties under the respective little group. More details can be found in ref. [37]. The Frenkel-Kač construction [38] establishes the gauge symmetry.

2.2 Toroidal Compactifications of the Heterotic String

Experience tells us that we live in 4 dimensions. Thus, for string theory to bear any relevance for low-energy particle physics phenomenology, the extra dimensions predicted have to be small as to have escaped detection so far. The first attempt at starting in a higher dimensional space-time and deriving the phenomenology in 4 dimensions was made by refs. [8, 9]: Compactifying 1 extra dimension on a circle, 5 dimensional gravity gives gravity in 4 dimensions, and additionally, one gauge boson and one massless scalar field. In our case, we have to compactify 6 extra dimensions. In principle, one can confine the 6 extra coordinates to any compact manifold $\mathcal{M}$ such that the original 10 dimensional space is given by a direct product

$$\mathbb{R}^4 \times \mathcal{M}, \tag{2.7}$$

where the 4 infinite dimensions correspond to our Minkowski space. The simplest idea is to compactify each extra coordinate on a circle, which is (topologically) equivalent to choosing $\mathcal{M}$ to be a 6-torus T^6 [10]. The virtue of this construction is not only its simplicity. The torus is Riemann flat, and as a consequence, the equations of motion remain exactly solvable even after compactification, so the spectrum and other properties of the model can easily be calculated.

However, torus compactification schemes do not lead to realistic models in 4 dimensions, the most prominent reason being the following. Consider the gravitino given in eq. (2.6):

$$|q\rangle_R \otimes \tilde{\alpha}^i_{-1}|0\rangle_L \tag{2.8}$$

The right mover is the 8 dimensional spinor representation of SO(8), which consists of all weight vectors

$$q = \left(\pm\tfrac{1}{2} \pm \tfrac{1}{2} \pm \tfrac{1}{2} \pm \tfrac{1}{2}\right), \quad \text{even number of + signs.} \tag{2.9}$$

In 10 dimensions, the state in eq. (2.8) corresponds to one gravitino, which establishes the fact that we have $\mathcal{N} = 1$ supersymmetry. After compactification, however, the last 3 entries of the weight vector in eq. (2.9) are internal indices, so we get a total of 4 gravitinos, thus giving $\mathcal{N} = 4$ supersymmetry in 4 dimensions.

Supersymmetric theories for $\mathcal{N} \geq 2$ have quite a few undesirable phenomenological properties. For such theories, the matter and vector representations are generally in the same supersymmetry multiplet. Since gauge and supersymmetry transformations commute, states in the same SUSY multiplet transform according to the same gauge group representation, and as the vector fields are required by local gauge invariance to transform in the adjoint representation, also matter representations are predicted to transform in the adjoint representation. This is in contradiction to the well-established fact that matter transforms in the fundamental representation of the gauge symmetry.

Although models with no supersymmetry are viable, supersymmetric models ($\mathcal{N} = 1$) can alleviate some of the problems encountered, when embedding the Standard Model into the context of a more fundamental theory [5]. Orbifold constructions, which we will introduce in the next section and which are the main objective of this thesis, combine the simplicity of torus compactifications with the virtues of $\mathcal{N} = 1$ supersymmetry in the low-energy theory. Since the orbifold is, with the exception of finitely many points, Riemann flat, one can comparatively easily calculate the gauge group, the matter content, the Yukawa couplings, the Kähler potential, the gauge kinetic function, and the unification scale of the coupling constants. The wealth of physical parameters which can be calculated in combination with the realistic features of the constructions are the major advantages of orbifolds over other compactification schemes.

2.3 Orbifolds

Our exposition closely follows ref. [20]. To construct a 4 dimensional string theory, 6 dimensions are compactified on a torus T^6. The resulting spectrum has $\mathcal{N}=4$ supersymmetry, and is thus non-chiral. To obtain a chiral theory with $\mathcal{N}=1$ supersymmetry, one compactifies on an orbifold [12, 13]:

$$\mathcal{O} = T^6 / P \otimes T_{\mathrm{E}_8\times\mathrm{E}_8'}/G \tag{2.10}$$

An orbifold is defined to be the quotient of a torus over a discrete set of isometries of the torus, called the *point group* P. Modular invariance requires the action of the point group to be embedded into the gauge degrees of freedom:

$$P \subset O(6) \quad \hookrightarrow \quad G \subset \mathrm{Aut}(\mathrm{E}_8 \times \mathrm{E}_8'), \qquad \theta \mapsto \sigma \tag{2.11}$$

G is in general a subgroup of the automorphisms of the $\mathrm{E}_8 \times \mathrm{E}_8'$ Lie algebra, and is called the *gauge twisting group*. In the absence of outer automorphisms, the Lie algebra automorphism on each E_8 factor can be realized as a shift $X_L \mapsto X_L + V$ in the root lattice, such that the automorphism is given by

$$\sigma_V(E_\alpha) = \exp\left(2\pi i\, \alpha \cdot V\right) E_\alpha. \tag{2.12}$$

The shift vector V is subject to conditions ensuring the modular invariance of the theory, as we will describe in section 2.4. The nature of the precise connection between shift vectors and Lie algebra automorphisms will be discussed in chapter 4.

An alternative description is to define an orbifold as

$$\mathcal{O} = \mathbb{R}^6 / S \otimes T_{\mathrm{E}_8\times\mathrm{E}_8'}/G, \tag{2.13}$$

where the lattice vectors e_α, $\alpha = 1, \ldots, 6$, defining the 6-torus T^6 have been added to the point group to form the *space group* $S = \left\{(\theta, n_\alpha e_\alpha) \,\middle|\, \theta \in P,\;\; n_\alpha \in \mathbb{Z}\right\}$. As before, modular invariance requires the action of the space group to be embedded into the gauge degrees of freedom,

$$S \hookrightarrow G, \qquad (\theta, n_\alpha e_\alpha) \mapsto \left(\sigma_V, n_\alpha \sigma_{A_\alpha}\right), \tag{2.14}$$

where the lattice vectors e_α are mapped to the automorphisms σ_{A_α} corresponding to the shifts A_α in the gauge lattice. Since the embedding is a homomorphism, the sum of lattice vectors, $n_\alpha e_\alpha$, is mapped to the sum of their images, $n_\alpha \sigma_{A_\alpha}$. The shifts A_α correspond to gauge transformations associated with the non-contractible loops given by e_α, and are thus interpreted as *Wilson lines.* The action of the orbifold group on all degrees of freedom is then given by

$$X^i \mapsto (\theta X)^i + n_\alpha e^i_\alpha, \qquad X^I_L \mapsto X^I_L + V^I + n_\alpha A^I_\alpha, \tag{2.15}$$

where $i = 3, \ldots, 8,\ \ I = 1, \ldots, 16$. Note that in this case, X^i are coordinates on the plane. Choosing complex coordinates on the torus,

$$Z^1 = X^3 + iX^4, \quad Z^2 = X^5 + iX^6, \quad Z^3 = X^7 + iX^8, \qquad X^i \sim X^i + n_\alpha e_\alpha, \tag{2.16}$$

the action of the point group on the space-time degrees of freedom can be neatly summarized as

$$Z^a \mapsto \exp\left(2\pi i v^a\right) Z^a, \quad a = 1, 2, 3, \tag{2.17}$$

where v is called the *twist vector.*

2.4 Consistency Conditions for Orbifold Constructions

Different 4 dimensional models can be constructed depending on the choice of the compactification torus T^6, the point group P, and the embedding into the gauge degrees of freedom $P \hookrightarrow G$. There are several constraints which must be fulfilled, which we will discuss now.

2.4.1 The Twist θ is well-defined

In eq. (2.10), an orbifold is defined by starting with a torus and moding out the action of a discrete group. The group action can only be consistently moded out, if it is a symmetry of the torus. Such groups are then said to act *crystallographically* on the lattice defining the torus. The fact that the list of lattices in dimensions $d \geq 2$ is finite, is a famous result due to Bieberbach [39, 40]. In 6 dimensions, there are 28,927,922 classes of space groups [41].

2.4.2 The Twist acts as Identity on Spinor Representation

From now on, we will restrict ourselves to abelian point groups. As such, $P \subset \mathrm{O}(6)$ is a subset of the Cartan algebra of $\mathrm{O}(6)$, and the twist $\theta \in P$ can be represented by

$$\theta = \exp\left(v^1 H_1 + v^2 H_2 + v^3 H_3\right). \tag{2.18}$$

Choosing the Cartan generators as rotations in the 3 orthogonal planes corresponding to X^{2a+1}, X^{2a+2}, $a = 1, 2, 3$, one arranges that the coefficients in eq. (2.18) are exactly the entries of the twist v. Denote the order of θ by N. Then, acting with $\theta^N = \mathbb{1}$ on the 8-dimensional spinor representation, we obtain

$$\theta^N \quad : \quad |\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle \quad \mapsto \quad e^{2\pi i\, N\left(\frac{1}{2}v^1 + \frac{1}{2}v^2 + \frac{1}{2}v^3\right)} |\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle, \tag{2.19}$$

concluding that

$$N\left(v^1+v^2+v^3\right)=0 \bmod 2. \tag{2.20}$$

The requirement of $\mathcal{N}=1$ supersymmetry in section 2.4.3 will give a tighter constraint, compare eq. (2.23). The above equation restricts the choices for the twist for orbifold models with no supersymmetry.

2.4.3 $\mathcal{N}=1$ Supersymmetry

Demanding $\mathcal{N}=1$ supersymmetry in 4 dimensions confines $\theta \in P$ to lie in a subset of the full holonomy group [11]:

$$P \subset \mathrm{SU}(3)$$

Remembering our restriction to abelian point groups, we find that P must either be $\mathbb{Z}_N$ with $N=3,4,6,7,8,12$, or $\mathbb{Z}_N \times \mathbb{Z}_M$ with $M,N=2,3,4,6$, and N a multiple of M [12,42]. For $N=6,8,12$, there are 2 different choices for the point group P. The lattices on which P acts as an isometry are the root lattices of semi-simple Lie algebras of rank 6. In some cases, there is more than one choice of lattice for a given set of symmetries P. A table of admissible lattices can be found in tab. 1 of ref. [20, 43].

To find the condition for $\mathcal{N}=1$ supersymmetry, consider the 4 gravitinos in 4 dimensions obtained after toroidal compactification (the first entry in the spinor corresponds to four dimensional chirality):

$$|\pm\tfrac{1}{2}\,|\,\pm\tfrac{1}{2}\ \pm\tfrac{1}{2}\ \pm\tfrac{1}{2}\rangle_R \otimes \tilde{\alpha}^{\mu}_{-1}|0\rangle_L \qquad \text{(even number of + signs)} \tag{2.21}$$

The left mover has only Lorentz indices in the uncompactified dimensions, and transforms trivially under the orbifold action. Thus, for the gravitino to be invariant, the right mover needs to transform trivially, too. Acting with $\theta \in \mathbb{Z}_N$ on a spinor representation of SO(8),

$$\theta \ :\ |\pm\tfrac{1}{2}\pm\tfrac{1}{2}\pm\tfrac{1}{2}\pm\tfrac{1}{2}\rangle \ \mapsto\ e^{2\pi i\left(\pm\frac{1}{2}v^1\pm\frac{1}{2}v^2\pm\frac{1}{2}v^3\right)}|\pm\tfrac{1}{2}\pm\tfrac{1}{2}\pm\tfrac{1}{2}\pm\tfrac{1}{2}\rangle, \tag{2.22}$$

we realize that demanding $\pm v^1 \pm v^2 \pm v^3 = 0$ for one combination of signs (all $v^i \neq 0$) guarantees that in eq. (2.21) exactly one gravitino will survive. In this case, one can always choose

$$v^1+v^2+v^3=0. \tag{2.23}$$

Knowing its order N, the form of the twist vector can easily be derived. Consider e.g. N=6. Since the sum $v^1+v^2+v^3$ is zero, not all v^i are positive. An overall sign is irrelevant, so we can assume that the first 2 entries are positive, and the last one is negative. As $v^1+v^2=|v^3|$, we are only free to choose v^1 and v^2, which we can take to be any positive integer from 1 to 5, divided by 6. Note that 0/6 is not an admissible choice, as it gives $\mathcal{N}=2$ supersymmetry, and 6/6 is equivalent to 0. Going through all possibilities and identifying twists which only differ in the permutation of their entries (which corresponds to interchanging the 2-tori), we find only 2 inequivalent twists, which are given in tab. 2.1.

Point Group	Twist v
$\mathbb{Z}_3$	(1/3, 1/3, -2/3)
$\mathbb{Z}_4$	(1/4, 1/4, -1/2)
$\mathbb{Z}_6$-I	(1/6, 1/6, -1/3)
$\mathbb{Z}_6$-II	(1/6, 1/3, -1/2)
$\mathbb{Z}_7$	(1/7, 2/7, -3/7)
$\mathbb{Z}_8$-I	(1/8, 1/4, -3/8)
$\mathbb{Z}_8$-II	(1/8, 3/8, -1/2)
$\mathbb{Z}_{12}$-I	(1/12, 1/3, -5/12)
$\mathbb{Z}_{12}$-II	(1/12, 5/12, -1/2)

Point Group	Twist v_1	Twist v_2
$\mathbb{Z}_2 \times \mathbb{Z}_2$	(1/2, 0, -1/2)	(0, 1/2, -1/2)
$\mathbb{Z}_3 \times \mathbb{Z}_3$	(1/3, 0, -1/3)	(0, 1/3, -1/3)
$\mathbb{Z}_2 \times \mathbb{Z}_4$	(1/2, 0, -1/2)	(0, 1/4, -1/4)
$\mathbb{Z}_4 \times \mathbb{Z}_4$	(1/4, 0, -1/4)	(0, 1/4, -1/4)
$\mathbb{Z}_2 \times \mathbb{Z}_6$-I	(1/2, 0, -1/2)	(0, 1/6, -1/6)
$\mathbb{Z}_2 \times \mathbb{Z}_6$-II	(1/2, 0, -1/2)	(1/6, 1/6, -1/3)
$\mathbb{Z}_3 \times \mathbb{Z}_6$	(1/3, 0, -1/3)	(0, 1/6, -1/6)
$\mathbb{Z}_6 \times \mathbb{Z}_6$	(1/6, 0, -1/6)	(0, 1/6, -1/6)

Table 2.1: All abelian point groups, which give $\mathcal{N} = 1$ SUSY.

Inspecting tab. 2.1, we see one striking feature of $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds, which will play an important role later. Moding out each twist leaves one 2-torus invariant, and the spectrum in 4 dimensions is obtained as an "intersection" of the spectra in 6 dimensions. The theory in the 6 dimensional intermediate will have $\mathcal{N} = 1$ supersymmetry, which is tantamount to $\mathcal{N} = 2$ supersymmetry in 4 dimensions. We will come back to this point in chapter 3.

2.4.4 The Embedding $P \hookrightarrow G$ is a Group Homomorphism

Let N denote the order of the twist. Since $\theta^N = \mathbb{1}$ and the embedding of the twist into the gauge degrees of freedom is a homomorphism, $N\,V$ must act as the identity on the root lattice. As such, $N\,V$ must lie in the dual root lattice, and from the self-duality of $\mathrm{E}_8 \times \mathrm{E}_8'$ it follows that $N\,V \in T_{\mathrm{E}_8\times\mathrm{E}_8'}$. The same reasoning applies to Wilson lines, and in summary, we have:

$$N\,V \in T_{\mathrm{E}_8\times\mathrm{E}_8'}, \quad N\,A_\alpha \in T_{\mathrm{E}_8\times\mathrm{E}_8'} \tag{2.24}$$

Since the number of independent cycles on a 6-torus is 6, one may be tempted to conclude that we can independently choose 6 Wilson lines, one for each cycle. However, if the lattice vectors defining the torus are related by the point group symmetry, the corresponding Wilson lines must be identified. An example is the case of the $\mathbb{Z}_3$ orbifold, which we will discuss in section 2.6. Consider e.g. the first torus in fig. 2.1. Under the action of θ, which is a 120° rotation, the first lattice vector is mapped onto the second one. Thus, we can only choose one Wilson line, say, along e_1. The second Wilson line along e_2 has to equal the first one.

The order of the Wilson line, which is by eq. (2.24) constrained to be N, also depends on the choice of the compactification lattice [43]. By exploiting the homomorphism property of the embedding, we will show in section 3.2, that for the $\mathbb{Z}_6$-II model with a specific choice of the 6-torus T^6, Wilson lines of order 6 are not allowed.

2.4.5 Modular Invariance

For the partition function of a $\mathbb{Z}_N$ orbifold to be modular invariant, the following conditions

$$N' \left[(mV + n_\alpha A_\alpha)^2 - m^2 v^2 \right] = 0 \text{ mod } 2, \quad m = 0, 1, \;\; n_\alpha \in \mathbb{Z}, \tag{2.25}$$

on the twist, gauge shift, and Wilson lines need to be fulfilled [20], where N' is the order of the local shift. Modular invariance automatically guarantees the anomaly freedom of orbifold models. These conditions can be recast into a more explicit form suitable for calculations:

$$\begin{gathered} N \left(V^2 - v^2 \right) = 0 \text{ mod } 2 \\ N\, V \cdot A_\alpha = 0 \text{ mod } 1 \\ N'\, A_\alpha \cdot A_\beta = 0 \text{ mod } 1, \quad \alpha \neq \beta \\ N''\, A_\alpha^2 = 0 \text{ mod } 2 \end{gathered} \tag{2.26}$$

The constants N', N'' can be derived from eq. (2.25). For $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds, the above conditions for modular invariance are generalized in a straightforward way. Let v_1, v_2 denote the twist vectors of $\mathbb{Z}_N \times \mathbb{Z}_M$, and V_1, V_2 the corresponding gauge shifts. Then we have

$$\begin{gathered} N' \left[(kV_1 + \ell V_2 + n_\alpha A_\alpha)^2 - (kv_1 + \ell v_2)^2 \right] = 0 \text{ mod } 2, \\ N' \text{ order of } kv_1 + \ell v_2, \\ k = 0, \ldots, N-1, \quad \ell = 0, \ldots, M-1, \quad n_\alpha \in \mathbb{Z}. \end{gathered} \tag{2.27}$$

For the Wilson lines, the conditions in eq. (2.26) are the same, except that they need to be fulfilled for both V_1, and V_2.

2.5 The Spectrum

On an orbifold, there are 2 types of strings, twisted and untwisted closed strings. An *untwisted string* is closed on the torus even before identifying points by the action of the twist:

$$X^i(\sigma + 2\pi) = X^i(\sigma) + n_\alpha e^i_\alpha, \quad i = 3, \ldots, 8 \tag{2.28}$$

A *twisted string* is closed on the torus only upon imposing the point group symmetry:

$$X^i(\sigma + 2\pi) = (\theta X(\sigma))^i + n_\alpha e^i_\alpha, \quad i = 3, \ldots, 8 \tag{2.29}$$

From the boundary conditions, it follows that the twisted strings are localized at the points which are left fixed under the action of some element $(\theta_i, n_\alpha e_\alpha)$ of the space group S. These

points are called the *fixed points* of the orbifold. We will call the element $g \equiv (\theta_i, n_\alpha e_\alpha)$ which corresponds to some given fixed point the *constructing element*, and denote the states which are localized at this fixed point by $\mathcal{H}_g$.
Since the strings live on the orbifold, we must project onto $S \otimes G$ invariant states. We will consider the twisted and untwisted sectors separately.

2.5.1 The Untwisted Sector

The states in the untwisted sector are those of the heterotic string compactified on a torus, where states which are not invariant under $S \otimes G$ have been projected out. The level matching condition for the massless states is given by

$$\frac{1}{2}q^2 - \frac{1}{2} = \frac{1}{4}m_R^2 = \frac{1}{4}m_L^2 = \frac{1}{2}p^2 + N_L - 1 = 0, \tag{2.30}$$

where q denotes the SO(8) weight vector of the right mover ground state, e.g. $|\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\rangle$ or $|1\ 0\ 0\ 0\rangle$. Under the action of the point group, the right and left mover states will transform as $\exp(-2\pi i q \cdot v)|q\rangle_R$, and $\exp(2\pi i p \cdot V)|p\rangle_L$, respectively[1]. Only states for which the product of these eigenvalues is 1 will survive the projection. The *gauge bosons* are formed by combining right movers which do not transform under the action of the point group with left movers satisfying

$$p \cdot V = 0 \bmod 1, \qquad p \cdot A_\alpha = 0 \bmod 1, \tag{2.31}$$

giving the *unbroken gauge group.* Right movers which transform non-trivially combine with left movers for which

$$p \cdot V = k/N \bmod 1, \quad k = 1 \ldots, N-1, \qquad p \cdot A_\alpha = 0 \bmod 1, \tag{2.32}$$

to give the *charged matter*. The states which include excitations for the left movers give uncharged gauge bosons (Cartan generators), the supergravity multiplet, and some number of singlets.

2.5.2 The Twisted Sectors

Without loss of generality, let us focus on the states corresponding to the constructing element $g \equiv (\theta_i, n_\alpha e_\alpha)$. The twist acts as a shift $p \mapsto p + V_i + n_\alpha A_\alpha$ on the momentum lattice, and as $q \mapsto q + v_i$ on the right mover ground state. In addition, the number operator N_L is moded. The zero point energy of the right and left movers is changed by [13]

$$\delta c = \frac{1}{2}\sum_k \eta^k(1-\eta^k), \tag{2.33}$$

[1] When we take the scalar product $q \cdot v$, we shall mean $q \cdot \tilde{v}$ with $\tilde{v} = (0, v)$.

where $\eta^k = v_i^k \bmod 1$ so that $0 \leq \eta^k < 1$. The level matching condition for the massless states then reads

$$\frac{1}{2}(q+v_i)^2 - \frac{1}{2} + \delta c = \frac{1}{4}m_R^2 = \frac{1}{4}m_L^2 = \frac{1}{2}(p+V_i+n_\alpha A_\alpha)^2 + N_L - 1 + \delta c = 0. \tag{2.34}$$

As compared to the untwisted sector, the projection conditions in the twisted sectors are slightly more complicated. Consider the subset Z_g of the space group S which commutes with the constructing element g. Acting with Z_g on the orbifold, the Hilbert space $\mathcal{H}_g$ is mapped into itself. Z_g should act as the identity on $\mathcal{H}_g$, thus all elements which are not invariant under $h \in Z_g$ are projected out.
If $h \in S$ does not commute with g, the action of h changes the boundary conditions of the states in $\mathcal{H}_g$, and states in $\mathcal{H}_g$ will be mapped to states in $\mathcal{H}_{hgh^{-1}}$. To form invariant states, one starts with some state in $\mathcal{H}_g$ and considers its image in $\mathcal{H}_{hgh^{-1}}$ for all $h \in S$. In each Hilbert space, we project onto its $Z_{hgh^{-1}}$ invariant subspace. The sum of these states is then invariant under the action of the space group S.

2.6 The $\mathbb{Z}_3$ Case as the Paradigm for Odd-N Orbifolds

We illustrate the discussion of the previous section by considering $\mathbb{Z}_N$ orbifolds with prime N, taking the $\mathbb{Z}_3$ orbifold as the paradigm. The lattice defining the 6-torus is the $\mathrm{SU}(3)^3$ root lattice as shown in figure 2.1. The point group $\mathbb{Z}_3$ is generated by θ which acts as a simultaneous rotation of 120° in the three 2-tori, and in the notation of eq. (2.17), this corresponds to the twist vector

$$v = \frac{1}{3}\,(1, \;\; 1, \;\; -2)\,. \tag{2.35}$$

The action of θ leaves 27 fixed points. The twisted sector corresponding to the action of θ^2 gives the anti-particles of the aforementioned sector, so we will not consider it separately.

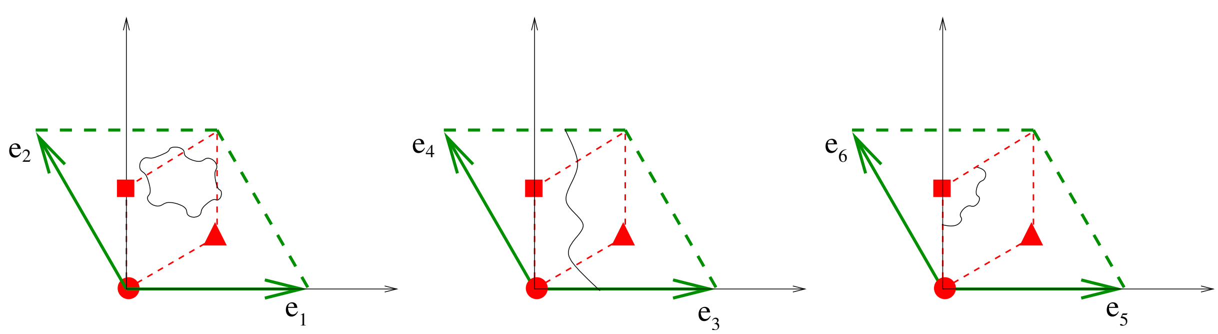


Figure 2.1: $\mathbb{Z}_3$ orbifold. The circle, triangle, and square denote the fixed points.

In figure 2.1, the strings in the first and the second torus are already closed on the torus (untwisted sector states), whereas in the third torus, the state only closes upon imposing the symmetry generated by the 120° rotations (twisted sector state).
Let us first consider the untwisted sector. The action of the orbifold twist is accompanied by an action in the gauge degrees of freedom realized as a shift. We choose the *standard embedding*

$$V = \frac{1}{3}\left(1,\ \ 1,\ \ -2,\ \ 0^5\right)\left(0^8\right), \tag{2.36}$$

where the first 3 components of the gauge shift[2] are equal to the components of the twist vector v. With this choice, the modular invariance condition eq. (2.26) is automatically satisfied, and the anomaly freedom of our model is guaranteed. From the 240 states in the first E_8, only $78 = 72 + 6$ survive the projection condition eq. (2.31), and yield the charged gauge bosons of $\mathrm{E}_6 \times \mathrm{SU}(3)$, whereas the second E_8' is left untouched.
The right mover ground state will decompose as $\mathbf{8} \to \mathbf{3} + \bar{\mathbf{3}} + \mathbf{1} + \mathbf{1}$ under $\mathrm{SU}(3) \subset \mathrm{SO}(8)$, i.e. there are 3 right mover states transforming as $|q\rangle_R \mapsto \exp\left(\text{-}2\pi i \cdot \frac{1}{3}\right)|q\rangle_R$ which will combine with left movers satisfying

$$p \cdot V = \frac{1}{3} \text{ mod } 1 \tag{2.37}$$

to give the charged matter representations $3 \times (\mathbf{27}, \mathbf{3})$. From the untwisted sector, we thus get 9 families of quarks and leptons.
Let us now discuss the twisted sector, and focus on the fixed point $(\bullet, \bullet, \bullet)$ in figure 2.1. The shift in the zero point energy as given by eq. (2.33) is $\delta c = 1/3$, and the level matching condition for the massless states reads

$$\frac{1}{2}(q+v) - \frac{1}{6} = \frac{1}{4}m_R^2 = \frac{1}{4}m_L^2 = \frac{1}{2}(p+V)^2 + N_L - \frac{2}{3} = 0. \tag{2.38}$$

The twisted right moving ground state $|q+v\rangle_R$ is a singlet under θ. (Note that $q+v$ must be shifted by a SO(8) root vector to fulfill the level matching condition). For $N_L = 0$, there are 27 elements $p+V$ satisfying $(p+V)^2 = 4/3$. These left movers transform as

$$|p+V\rangle_L \mapsto \exp\left(2\pi i (p+V)\cdot V\right)|p+V\rangle_L = \exp\left(2\pi i \cdot 1\right)|p+V\rangle_L, \tag{2.39}$$

and are invariant. They combine with the right mover to give the representation $(\mathbf{27}, \mathbf{1})$. For $N_L = 1/3$, there are 3 elements $p+V$ satisfying $(p+V)^2 = 2/3$. These left movers transform as

$$|p+V\rangle_L \mapsto \exp\left(2\pi i (p+V)\cdot V\right)|p+V\rangle_L = \exp\left(2\pi i \cdot \frac{2}{3}\right)|p+V\rangle_L, \tag{2.40}$$

whereas the oscillators (one for each complex dimension) transform as

$$\tilde{\alpha}^a \mapsto e^{2\pi i\, v^a}\tilde{\alpha}^a, \quad a = 1,2,3, \tag{2.41}$$

[2]Zero to the power of n is short for writing n zeros.

so that the states $|q\rangle_R \otimes \tilde{\alpha}^a |p\rangle_L$ are invariant, and give three copies of the representation $(\mathbf{1}, \bar{\mathbf{3}})$. Taking into account that there are 27 fixed points, the matter content of our orbifold model is

$$3 \times (\mathbf{27}, \mathbf{3}), \quad 27 \times (\mathbf{27}, \mathbf{1}), \quad 27 \times 3 \times (\mathbf{1}, \bar{\mathbf{3}}). \tag{2.42}$$

Thus, in the case of the standard embedding, we have 36 generations of quarks and leptons. All non-trivial embeddings of the point group into the gauge degrees of freedom have been classified [13]. For each model, we have listed the shift vector V, the resulting unbroken gauge group, and the number of generations in table 2.2.

Case	Shift V	Gauge Group	Gen.
1	$\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 0^5\right)(0^8)$	$\mathrm{E}_6 \times \mathrm{SU}(3) \times \mathrm{E}_8'$	36
2	$\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 0^5\right)\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 0^5\right)$	$\mathrm{E}_6 \times \mathrm{SU}(3) \times \mathrm{E}_6' \times \mathrm{SU}(3)'$	9
3	$\left(\frac{1}{3}, \frac{1}{3}, 0^6\right)\left(\frac{2}{3}, 0^7\right)$	$\mathrm{E}_7 \times \mathrm{U}(1) \times \mathrm{SO}(14)' \times \mathrm{U}(1)'$	0
4	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 0^3\right)\left(\frac{2}{3}, 0^7\right)$	$\mathrm{SU}(9) \times \mathrm{SO}(14)' \times \mathrm{U}(1)'$	9

Table 2.2: Inequivalent $\mathbb{Z}_3$ orbifold models without Wilson lines.

Note that the proliferation of the number of generations is due to the fact that the physics at each fixed point is the same. This dramatically changes in the presence of Wilson lines [15]. We will illustrate the lifting of the degeneracy at the fixed points using a specific example. Choose the standard embedding, and the Wilson lines

$$A_1 = A_2 = \left(0^6,\ \frac{1}{3},\ \frac{1}{3}\right)\left(\frac{2}{3},\ 0^7\right). \tag{2.43}$$

Applying the projection conditions eq. (2.31), we find that the surviving gauge symmetry is

$$\mathrm{SU}(6) \times \mathrm{SU}(3) \times \mathrm{U}(1) \times \mathrm{SO}(14)' \times \mathrm{U}(1)'. \tag{2.44}$$

From the untwisted sector, we obtain the charged matter representations $3 \times (\mathbf{15}, \mathbf{3})$. (We will only indicate the representations under the first 2 factors of the symmetry group.) Let us discuss the twisted sector in greater detail.

Consider the fixed points $(\bullet, \bullet, \bullet)$, $(\blacktriangle, \bullet, \bullet)$, and $(\blacksquare, \bullet, \bullet)$ as depicted in figure 2.2. The fixed point $(\bullet, \bullet, \bullet)$ is left invariant under the action of θ alone, i.e. the constructing element is $(\theta, 0)$. The level matching condition for massless states living at this fixed point will be the same as eq. (2.38). The states do not feel the presence of the Wilson lines. The fixed point $(\blacktriangle, \bullet, \bullet)$, however, is only invariant under the action of θ accompanied by the lattice

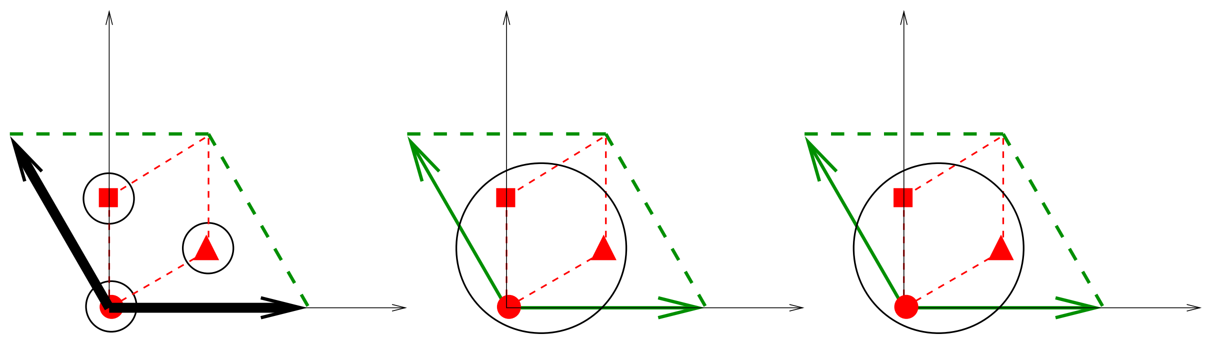


Figure 2.2: $\mathbb{Z}_3$ orbifold with non-vanishing Wilson lines A_1, A_2. The circles around the fixed points indicate that the degeneracy in the first torus is lifted.

shift e_1, and the constructing element is (θ, e_1). The immediate consequence is that the level matching condition for the massless states changes to

$$\frac{1}{2}(q+v) - \frac{1}{6} = \frac{1}{4}m_R^2 = \frac{1}{4}m_L^2 = \frac{1}{2}(p+V+A_1)^2 + N_L - \frac{2}{3} = 0. \tag{2.45}$$

Clearly, it is more difficult to fulfill the new relation, and the **27** of E_6 will preferentially decompose into small representations under the new gauge group. The level matching condition can only be satisfied for $N_L = 0$, and these states also survive the projection condition analogous to eq. (2.39) (where we have to substitute $V \to V + A_1$) to form the representations $(\mathbf{1}, \bar{\mathbf{3}}) + (\bar{\mathbf{6}}, \mathbf{1})$. As there are no Wilson lines in the second and third torus, the spectrum at the fixed point $(\blacktriangle, \bullet, \bullet)$ will still be 9-fold degenerate. All fixed points with $\blacktriangle$ as the first entry and an arbitrary one in the last two entries will have the same matter content. Analogous considerations also apply in the case of the fixed point $(\blacksquare, \bullet, \bullet)$. To summarize, the matter content of the model is (omitting the antiparticles)

Untwisted	$3 \times (\mathbf{15}, \mathbf{3})$,
$(\bullet, \cdot, \cdot)$	$9 \times (\mathbf{15}, \mathbf{1})$, $18 \times (\bar{\mathbf{6}}, \mathbf{1})$, $27 \times (\mathbf{1}, \bar{\mathbf{3}})$,
$(\blacktriangle, \cdot, \cdot)$	$9 \times (\mathbf{1}, \bar{\mathbf{3}})$, $9 \times (\bar{\mathbf{6}}, \mathbf{1})$,
$(\blacksquare, \cdot, \cdot)$	$9 \times (\mathbf{1}, \bar{\mathbf{3}})$, $9 \times (\bar{\mathbf{6}}, \mathbf{1})$.

From the untwisted sector, we obtain 9 families, which also have SU(3) quantum numbers, and from $(\bullet, \cdot, \cdot)$, we have another 9 families which are SU(3) singlets. The total number of 18 families is to be compared to the 36 families in the case of no Wilson lines. Using more Wilson lines, models with 3 families of quarks and leptons [15] and with standard model gauge group $SU(3) \times SU(2) \times U(1)^n$ can be constructed [14].

Chapter 3

Model Construction with the $\mathbb{Z}_6$-II Orbifold

The $\mathbb{Z}_3$ orbifold described in the previous chapter is the simplest supersymmetric abelian orbifold (cf. tab. 2.1), and most of the past research has concentrated on this case. In this chapter, we consider the $\mathbb{Z}_6$-II orbifold as a paradigm for the class of models with more general point groups $\mathbb{Z}_N$ or $\mathbb{Z}_N \times \mathbb{Z}_M$, which are much less investigated. The case of the $\mathbb{Z}_6$-II is complicated enough to illustrate all features which may arise in the construction of an orbifold with general point group. We describe the construction of one model in very great detail. Then we consider the inclusion of background fields and present several three-generation models with Standard Model and GUT gauge symmetry. These models are as yet unpublished. It should be noted that the study of $\mathbb{Z}_6$-II orbifold has been initiated by refs. [25, 26].

3.1 The Geometry

The twist vector for the $\mathbb{Z}_6$-II orbifold is given by

$$v = (\tfrac{1}{6}, \tfrac{1}{3}, -\tfrac{1}{2}), \tag{3.1}$$

which describes a rotation by 60° in the first torus, by 120° in the second one, and by 180° in the third one.

As explained in section 2.4.1, the twist acts on tori defined by the root lattices of semi-simple Lie algebras, and our choice of lattice is given by simple roots of

$$G_2 \times \mathrm{SU}(3) \times \mathrm{SO}(4).$$

Note, however, that there are several distinct lattices which are compatible with the above point group symmetry, as e.g. [43]

$$\mathrm{SU}(6) \times \mathrm{SU}(2), \quad \mathrm{SU}(3) \times \mathrm{SO}(8), \quad \mathrm{SU}(3) \times \mathrm{SO}(7) \times \mathrm{SU}(2).$$

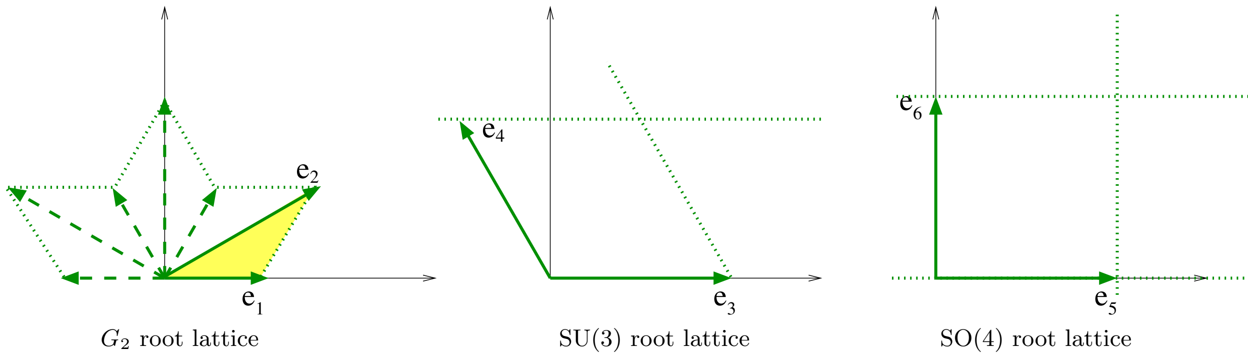


Figure 3.1: The geometry of the $\mathbb{Z}_6$-II orbifold.

The choice of the compactification lattice may influence some of the properties of the orbifold, such as the order of admissible Wilson lines. Whereas for our choice of the lattice, order 3 and 2 Wilson lines are allowed in the second and third torus, respectively, one can show that for SU(6) × SU(2) there are no Wilson lines in the SU(6) lattice at all, restricting the number of possible Wilson lines to 1.

3.2 Admissible Wilson Lines

The number of independent Wilson lines and their orders depend on the choice of the compactification lattice. We know from section 2.4.4 that if 2 lattice vectors e_{i_1}, e_{i_2} are related by the point group symmetry, i.e. $e_{i_2} = \theta^k e_{i_1}$ for some $k \in \{1, \dots, N\}$, then the associated Wilson lines must be identified: $A_{i_1} = A_{i_2}$.

Consider the G_2 lattice in fig. 3.1. We start with identities in the lattice and evaluate their gauge embeddings, where we denote the $\mathrm{E}_8 \times \mathrm{E}_8'$ root lattice by Λ:

$$\begin{aligned}
\theta^2 e_1 = \theta^2 e_2 + e_1 \quad &\rightsquigarrow \quad A_1 = A_2 + A_1 \bmod \Lambda \quad \rightsquigarrow \quad A_2 \in \Lambda \\
\theta e_1 = e_2 - e_1 \quad &\rightsquigarrow \quad A_1 = A_2 - A_1 \bmod \Lambda \quad \rightsquigarrow \quad A_1 \in \Lambda/2 \\
e_2 - \theta^2 e_2 = 3e_1 \quad &\rightsquigarrow \quad 0 = 3A_1 \bmod \Lambda \quad \rightsquigarrow \quad A_1 \in \Lambda/3
\end{aligned}$$

The first equation tells us that $A_2 = 0$, and since the last two equations cannot be fulfilled simultaneously, we conclude that no non-vanishing Wilson lines are allowed.

Next, consider the SU(3) torus. We proceed analogously:

$$\begin{aligned}
\theta e_3 = e_4 \quad &\rightsquigarrow \quad A_3 = A_4 \\
\theta^2 e_3 = -e_3 - e_4 \quad &\rightsquigarrow \quad A_3 = -A_3 - A_4 \bmod \Lambda \quad \rightsquigarrow \quad A_3 \in \Lambda/3
\end{aligned}$$

We find that there is 1 independent order 3 Wilson line.

Finally, consider the SO(4) lattice:

$$\begin{array}{lclcl} \theta e_5 = -e_5 & \rightsquigarrow & A_5 = -A_5 \bmod \Lambda & \rightsquigarrow & A_5 \in \Lambda/2 \\ \theta e_6 = -e_6 & \rightsquigarrow & A_6 = -A_6 \bmod \Lambda & \rightsquigarrow & A_6 \in \Lambda/2 \end{array}$$

We conclude that there are 2 independent Wilson lines of order 2.

3.3 The Standard Embedding

Modular invariance requires the orbifold twist

$$v = (\tfrac{1}{6},\, \tfrac{1}{3},\, -\tfrac{1}{2}) \tag{3.2}$$

to be embedded into the gauge degrees of freedom. In the absence of Wilson lines, the consistency conditions listed in eq. (2.26) in section 2.4.5 take on a very simple form:

$$6\left(V^2 - v^2\right) = 0 \bmod 2$$

Clearly, the easiest way to fulfill this equation is to choose a shift vector whose first three entries coincide with the twist:

$$V = (\tfrac{1}{6},\, \tfrac{1}{3},\, -\tfrac{1}{2},\, 0^5)\,(\,0^8\,) \tag{3.3}$$

This is the so-called *the standard embedding*.

3.3.1 The Untwisted Sector

The massless right movers are solutions to the equation

$$\frac{1}{4} m_R^2 = \frac{1}{2} q^2 - \frac{1}{2} = 0 \tag{3.4}$$

and are given by

$$|\underline{\pm 1,\, 0,\, 0,\, 0}\,\rangle_R \qquad \text{and} \qquad |\pm\tfrac{1}{2},\, \pm\tfrac{1}{2},\, \pm\tfrac{1}{2},\, \pm\tfrac{1}{2}\rangle_R. \tag{3.5}$$

As usual, the underline denotes the permutation of the entries, and in the second right mover, the number of + signs is even. Under the action of a space group element $(\theta^k, n_\alpha e_\alpha)$, the right mover transforms as

$$|q\rangle_R \;\mapsto\; e^{-2\pi i\, q\cdot kv}\, |q\rangle_R, \tag{3.6}$$

where for the purpose of taking scalar products, we extend the twist to a four dimensional vector $v = (0,\, v^1,\, v^2,\, v^3)$. We have summarized the transformation properties of the right movers in tab. 3.1.

	θ^1	θ^2	θ^3	θ^4	θ^5
$\lvert\pm 1, 0, 0, 0\rangle_R$	0	0	0	0	0
$\lvert 0, \pm 1, 0, 0\rangle_R$	$\pm\frac{1}{6}$	$\pm\frac{1}{3}$	$\pm\frac{1}{2}$	$\pm\frac{2}{3}$	$\pm\frac{5}{6}$
$\lvert 0, 0, \pm 1, 0\rangle_R$	$\pm\frac{1}{3}$	$\pm\frac{2}{3}$	0	$\pm\frac{1}{3}$	$\pm\frac{2}{3}$
$\lvert 0, 0, 0, \pm 1\rangle_R$	$\mp\frac{1}{2}$	0	$\mp\frac{1}{2}$	0	$\mp\frac{1}{2}$
$\lvert\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\rangle_R$	0	0	0	0	0
$\lvert\mp\frac{1}{2}, \mp\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\rangle_R$	$\mp\frac{1}{6}$	$\mp\frac{1}{3}$	$\mp\frac{1}{2}$	$\mp\frac{2}{3}$	$\mp\frac{5}{6}$
$\lvert\mp\frac{1}{2}, \pm\frac{1}{2}, \mp\frac{1}{2}, \pm\frac{1}{2}\rangle_R$	$\mp\frac{1}{3}$	$\mp\frac{2}{3}$	0	$\mp\frac{1}{3}$	$\mp\frac{2}{3}$
$\lvert\mp\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \mp\frac{1}{2}\rangle_R$	$\pm\frac{1}{2}$	0	$\pm\frac{1}{2}$	0	$\pm\frac{1}{2}$

Table 3.1: Transformation of the right movers. The k-th column lists $q \cdot kv$.

The massless left movers are solutions to the equation

$$\frac{1}{4}m_L^2 = \frac{1}{2}p^2 + N_L - 1 = 0. \tag{3.7}$$

For $N_L = 0$, the solution of the above equation, $p^2 = 2$, are the 480 root vectors of $\mathrm{E}_8 \times \mathrm{E}_8'$. Under the homomorphism, which embeds the space group into the gauge degrees of freedom, we have

$$(k \cdot v, n_\alpha e_\alpha) \mapsto k \cdot V, \tag{3.8}$$

so the left mover transforms as

$$|p\rangle_L \quad \mapsto \quad e^{2\pi i\, p\cdot kV}\, |p\rangle_L. \tag{3.9}$$

Note that we assume that the gauge background fields vanish, i.e. $A_\alpha = 0$. In tab. 3.2, we list the transformation properties of the left movers.

The states surviving the orbifold projection are the tensor products of right and left movers, which are invariant under the action of θ^k, $k = 1, \ldots, 5$ (see section 2.5.1).

Gauge Bosons

The gauge bosons have a Lorentz index only in the uncompactified dimensions, so the right mover does not transform at all. The right movers will combine with those left movers, which transform trivially. We find 240 charged gauge bosons

$$|\pm 1, 0, 0, 0\rangle_R \otimes |(0^8)\ (p^2 = 2)\rangle_L,$$

	#	θ^1	θ^2	θ^3	θ^4	θ^5
$\lvert(\pm1, \pm1, 0, 0, 0, 0, 0, 0)\ (0^8)\rangle_L$	2	$\pm\frac{1}{2}$	0	$\pm\frac{1}{2}$	0	$\pm\frac{1}{2}$
$\lvert(\pm1, \mp1, 0, 0, 0, 0, 0, 0)\ (0^8)\rangle_L$	2	$\mp\frac{1}{6}$	$\mp\frac{1}{3}$	$\mp\frac{1}{2}$	$\mp\frac{2}{3}$	$\mp\frac{5}{6}$
$\lvert(\pm1, 0, \pm1, 0, 0, 0, 0, 0)\ (0^8)\rangle_L$	2	$\mp\frac{1}{3}$	$\mp\frac{2}{3}$	0	$\mp\frac{1}{3}$	$\mp\frac{2}{3}$
$\lvert(\pm1, 0, \mp1, 0, 0, 0, 0, 0)\ (0^8)\rangle_L$	2	$\pm\frac{2}{3}$	$\pm\frac{1}{3}$	0	$\pm\frac{2}{3}$	$\pm\frac{1}{3}$
$\lvert(0, \pm1, \pm1, 0, 0, 0, 0, 0)\ (0^8)\rangle_L$	2	$\mp\frac{1}{6}$	$\mp\frac{1}{3}$	$\mp\frac{1}{2}$	$\mp\frac{2}{3}$	$\mp\frac{5}{6}$
$\lvert(0, \pm1, \mp1, 0, 0, 0, 0, 0)\ (0^8)\rangle_L$	2	$\pm\frac{5}{6}$	$\pm\frac{2}{3}$	$\pm\frac{1}{2}$	$\pm\frac{1}{3}$	$\pm\frac{1}{6}$
$\lvert(\pm1, 0, 0, \underline{\pm1, 0, 0, 0, 0})\ (0^8)\rangle_L$	10	$\pm\frac{1}{6}$	$\pm\frac{1}{3}$	$\pm\frac{1}{2}$	$\pm\frac{2}{3}$	$\pm\frac{5}{6}$
$\lvert(\pm1, 0, 0, \underline{\mp1, 0, 0, 0, 0})\ (0^8)\rangle_L$	10	$\pm\frac{1}{6}$	$\pm\frac{1}{3}$	$\pm\frac{1}{2}$	$\pm\frac{2}{3}$	$\pm\frac{5}{6}$
$\lvert(0, \pm1, 0, \underline{\pm1, 0, 0, 0, 0})\ (0^8)\rangle_L$	10	$\pm\frac{1}{3}$	$\pm\frac{2}{3}$	0	$\pm\frac{1}{3}$	$\pm\frac{2}{3}$
$\lvert(0, \pm1, 0, \underline{\mp1, 0, 0, 0, 0})\ (0^8)\rangle_L$	10	$\pm\frac{1}{3}$	$\pm\frac{2}{3}$	0	$\pm\frac{1}{3}$	$\pm\frac{2}{3}$
$\lvert(0, 0, \pm1, \underline{\pm1, 0, 0, 0, 0})\ (0^8)\rangle_L$	10	$\mp\frac{1}{2}$	0	$\mp\frac{1}{2}$	0	$\mp\frac{1}{2}$
$\lvert(0, 0, \pm1, \underline{\mp1, 0, 0, 0, 0})\ (0^8)\rangle_L$	10	$\mp\frac{1}{2}$	0	$\mp\frac{1}{2}$	0	$\mp\frac{1}{2}$
$\lvert(0, 0, 0, \underline{\pm1, \pm1, 0, 0, 0})\ (0^8)\rangle_L$	20	0	0	0	0	0
$\lvert(0, 0, 0, \underline{\pm1, \mp1, 0, 0, 0})\ (0^8)\rangle_L$	20	0	0	0	0	0
$\lvert(+\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})\ (0^8)\rangle_L$	16	0	0	0	0	0
$\lvert(-\frac{1}{2}, +\frac{1}{2}, +\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})\ (0^8)\rangle_L$	16	$-\frac{1}{6}$	$-\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{5}{6}$
$\lvert(+\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})\ (0^8)\rangle_L$	16	$-\frac{1}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	$-\frac{2}{3}$
$\lvert(+\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})\ (0^8)\rangle_L$	16	$+\frac{1}{2}$	0	$+\frac{1}{2}$	0	$+\frac{1}{2}$
$\lvert(-\frac{1}{2}, -\frac{1}{2}, +\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})\ (0^8)\rangle_L$	16	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$
$\lvert(-\frac{1}{2}, +\frac{1}{2}, -\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})\ (0^8)\rangle_L$	16	$+\frac{1}{3}$	$+\frac{2}{3}$	0	$+\frac{1}{3}$	$+\frac{2}{3}$
$\lvert(+\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})\ (0^8)\rangle_L$	16	$+\frac{1}{6}$	$+\frac{1}{3}$	$+\frac{1}{2}$	$+\frac{2}{3}$	$+\frac{5}{6}$
$\lvert(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})\ (0^8)\rangle_L$	16	0	0	0	0	0
$\lvert(0, 0, 0, 0, 0, 0, 0, 0)(p_L^2 = 2)\rangle_L$	240	0	0	0	0	0
	480					

Table 3.2: Transformation of the left movers. The second column counts the number of left movers (which sum up to 480, as they should), and of the last 5 columns, the k-th one lists $p \cdot kV$.

which correspond to the E_8' gauge symmetry (the second factor of the original gauge group). Concentrate now on those left movers which have gauge degrees of freedom in the first E_8. Picking those left movers from tab. 3.2 which transform trivially, we find 72 charged gauge bosons:

$$
\begin{aligned}
|\pm 1,\,0,\,0,\,0\rangle_R &\otimes |(0,\,0,\,0,\,\underline{\pm 1,\,\pm 1,\,0,\,0,\,0})\;(0^8)\rangle_L \\
|\pm 1,\,0,\,0,\,0\rangle_R &\otimes |(0,\,0,\,0,\,\underline{\pm 1,\,\mp 1,\,0,\,0,\,0})\;(0^8)\rangle_L \\
|\pm 1,\,0,\,0,\,0\rangle_R &\otimes |\big(+\tfrac{1}{2},+\tfrac{1}{2},+\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2}\big)\;(0^8)\rangle_L \\
|\pm 1,\,0,\,0,\,0\rangle_R &\otimes |\big(-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2},\pm\tfrac{1}{2}\big)\;(0^8)\rangle_L
\end{aligned}
$$

The 16 neutral gauge bosons

$$|\pm 1,\,0,\,0,\,0\rangle_R \otimes \tilde{\alpha}^I_{-1}|0\rangle_L$$

do not transform[1] at all, thus surviving all orbifold projections. Of these neutral gauge bosons, 8 correspond to the Cartan generators of E_8', and 8 to those of an algebra with 72 roots. It is easy to guess that this algebra is

$$E_6 \times U(1)^2.$$

Since identifying the algebra, and in particular, the irreducible representations, may not always be that easy, we explain how to prove the above statement in appendix B.1.

Charged Matter

The right movers which do not transform trivially can nevertheless form invariant states by combining with appropriate left movers. Without loss of generality, we will consider the right mover

$$|-\tfrac{1}{2},\,-\tfrac{1}{2},\,+\tfrac{1}{2},\,+\tfrac{1}{2}\rangle_R.$$

Under the twist θ, the above right mover transforms with $e^{-2\pi i\cdot(-1/6)}$. From tab. 3.2, we find 29 left movers transforming with $e^{2\pi i\cdot(-1/6)}$ or $e^{2\pi i\cdot(5/6)}$, which then combine with the right mover to give states invariant under θ^k for $k = 1, \ldots, 5$. Having already determined the gauge symmetry, we expect these states to transform in some representation of E_6. The lowest dimensional representations are $\mathbf{27}$, $\overline{\mathbf{27}}$ and $\mathbf{78}$, so the 29 states must necessarily transform as $\mathbf{27}+\mathbf{1}+\mathbf{1}$ or in the complex conjugate representation.

In appendix B.2 we outline an algorithm, which unambiguously determines these representations. We find that the 29 states correspond to $\overline{\mathbf{27}}_L + \mathbf{1}_L + \mathbf{1}_L$. The subscript denotes

[1] Here, we are explicitly confronted with the fact that shift embeddings can never project out neutral gauge bosons, thus preserving the rank of the original algebra. Needless to say, this is quite an obstacle for building realistic models, where the gauge groups typically of low rank. Beginning with chapter 5, we will present an alternative construction, which avoids this problem.

the four dimensional chirality, which is given by the first entry of the right mover.

The other cases can be analyzed analogously. Summarizing, the matter content of the untwisted sector is

$$3 \times \overline{\mathbf{27}}_L, \quad 1 \times \mathbf{27}_L, \quad 3 \times \mathbf{27}_R, \quad 1 \times \overline{\mathbf{27}}_R, \quad 3 \times \mathbf{1}_L, \quad 3 \times \mathbf{1}_R.$$

3.3.2 The T_1 Twisted Sector

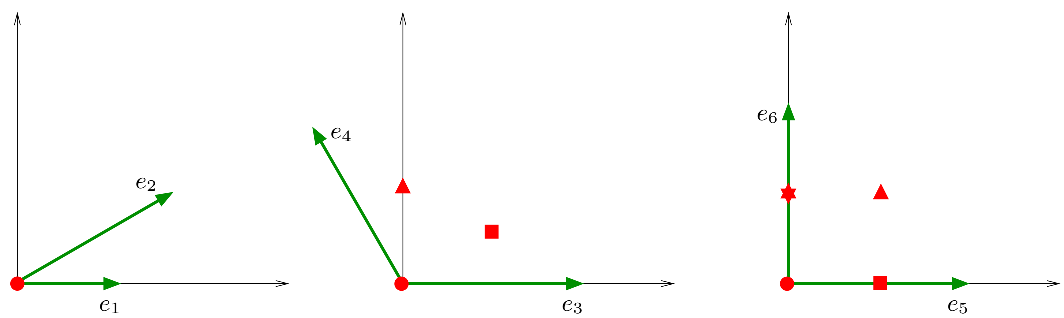


Figure 3.2: Geometry of the T_1 twisted sector.

Without loss of generality, we will determine the spectrum at the fixed point in the origin, corresponding to the constructing element $(\theta, 0)$. In the absence of Wilson lines, all fixed points of a given sector are degenerate, and the complete spectrum is obtained by multiplying the number of states by the number of fixed points, which is in this case 12.

The massless right movers of the first twisted sector are solutions to the equation

$$\frac{1}{4}m_R^2 = \frac{1}{2}(q+v)^2 - \frac{1}{2} + \delta c, \tag{3.10}$$

and are given by

$$|0, \tfrac{1}{6}, \tfrac{1}{3}, \tfrac{1}{2}\rangle_R, \qquad |\tfrac{1}{2}, -\tfrac{1}{3}, -\tfrac{1}{6}, 0\rangle. \tag{3.11}$$

Under the twist θ^k, the right movers transform as

$$|q+v\rangle_R \mapsto e^{-2\pi i\,(q+v)\cdot kv}|q+v\rangle_R, \tag{3.12}$$

which gives $e^{-2\pi i k\cdot(-1/9)}$ for both right movers.

The massless left movers are obtained by solving the equation

$$\frac{1}{4}m_L^2 = \frac{1}{2}(p+V)^2 + N_L - 1 + \delta c = 0 \tag{3.13}$$

on the root lattice of $\mathrm{E}_8 \times \mathrm{E}_8'$. We find 27 weight vectors, and they transform as

$$|p+V\rangle_L \mapsto e^{2\pi i\,(p+V)\cdot kV}|p+V\rangle_L. \tag{3.14}$$

All states transform with the same phase $e^{2\pi i k\cdot(-1/9)}$.

In section 2.5.2, we explained the projection conditions for the twisted sectors: We have to project onto those elements of the space group, which commute with the constructing element corresponding to the fixed point under consideration, and build a linear combination of those states which are mapped onto each other by elements which do not commute with the constructing element. In the T_1 twisted sector, only the former case plays a role, as we will now explain.

The Centralizer

The constructing element corresponding to the fixed point at the origin is given by $(\theta, 0)$. It is easy to see that all $(\theta^k, 0)$ commute with the constructing element, i.e.

$$Z_g = \left\{\,(\theta,0),\,(\theta^2,0),\,(\theta^3,0),\,(\theta^4,0),\,(\theta^5,0)\,\right\}.$$

This means that to build invariant states, we must project onto each θ^k. Now assume that we were determining the spectrum at another fixed point, e.g. (θ, e_3). This constructing element corresponds to the fixed point which is left invariant under the combined action of a rotation by 120°, and a translation by e_3. We calculate its commutator with $(\theta^2, 0)$:

$$\begin{aligned}(\theta,e_3)\star(\theta^2,0)\;x-(\theta^2,0)\star(\theta,e_3)\;x &= (\theta,e_3)\;\theta^2x-(\theta^2,0)\;[\theta x+e_3]\\ &= \theta^3x+e_3-[\theta^3x+\theta^2e_3]\\ &= e_3-\theta^2e_3\end{aligned}$$

Evidently, the two space group elements do not commute. This does not mean, however, that we do not project onto θ^2, since the centralizer of (θ, e_3) includes $(\theta^2, e_3 + e_4)$:

$$(\theta,e_3)\star(\theta^2,e_3+e_4)\;x-(\theta^2,e_3+e_4)\star(\theta,e_3)\;x=0$$

In the absence of Wilson lines, the space group elements $(\theta^2, 0)$ and $(\theta^2, e_3 + e_4)$ imply the same projection conditions,

$$|p+V\rangle_L \mapsto e^{2\pi i\,(p+V)\cdot 2V}|p+V\rangle_L, \qquad |p+V\rangle_L \mapsto e^{2\pi i\,(p+V)\cdot(2V+0+0)}|p+V\rangle_L, \tag{3.15}$$

respectively. Thus, the projection conditions are independent of the fixed point.

Having determined the projection conditions, we immediately verify that the combination of the right movers from eq. (3.11) and of the 27 left movers from eq. (3.13) survives the orbifold projections to give

$$1\times\mathbf{27}_R,$$

where we have only listed the fermionic state.

Oscillator States

In the twisted sectors, the zero point energy is sufficiently low so that we can have states arising from excited oscillators. Consider again the equation for massless left movers:

$$\frac{1}{4}m_L^2 = \frac{1}{2}(p+V)^2 + N_L - 1 + \delta c = 0 \tag{3.16}$$

In the first twisted sector, the oscillators are moded and N_L can have values which are multiples of $\frac{1}{6}$. As before, we will explain one case in detail. Assume that $N_L = \frac{1}{2}$, which corresponds to

$$\tilde{\alpha}^i_{\text{-}1/6}\tilde{\alpha}^j_{\text{-}1/6}\tilde{\alpha}^k_{\text{-}1/6}|p\rangle_L \quad \text{or} \quad \tilde{\alpha}^i_{\text{-}1/3}\tilde{\alpha}^j_{\text{-}1/6}|p\rangle_L \quad \text{or} \quad \tilde{\alpha}^i_{\text{-}1/2}|p\rangle_L. \tag{3.17}$$

For the left movers to be massless, we have to find p's such that $(p+V)^2 = \frac{7}{18}$. There is exactly one solution,

$$p + V = (\tfrac{1}{6},\, \tfrac{1}{3},\, -\tfrac{1}{2},\, 0,\, 0,\, 0,\, 0,\, 0)\,(0^8),$$

and according to eq. (3.14), this left mover transforms with $e^{2\pi i(7/18)}$. Since the right mover transforms with $e^{2\pi i(1/9)}$ (cf. eq. (3.12)), for the entire state to be invariant, we require the oscillator excitations to transform with $e^{2\pi i(1/2)}$.

The oscillators carry a Lorentz index $i = 1, \ldots, 8$, and as such, transforms as the coordinates do. Introducing complex coordinates also for the oscillators, we can neatly summarize their transformation properties, using the twist vector v:

$$\tilde{\alpha}^a \mapsto e^{2\pi i\, v^a}\ \tilde{\alpha}^a, \qquad \tilde{\alpha}^{\bar{a}} \mapsto e^{-2\pi i\, v^a}\ \tilde{\alpha}^{\bar{a}}, \qquad a, \bar{a} = 1, 2, 3$$

Referring back to eq. (3.17), we see that the oscillators

$$\tilde{\alpha}^{\bar{1}}_{\text{-}1/6}\tilde{\alpha}^{\bar{1}}_{\text{-}1/6}\tilde{\alpha}^{\bar{1}}_{\text{-}1/6}|p\rangle_L, \quad \tilde{\alpha}^{\bar{2}}_{\text{-}1/3}\tilde{\alpha}^{\bar{1}}_{\text{-}1/6}|p\rangle_L, \quad \tilde{\alpha}^{3}_{\text{-}1/2}|p\rangle_L, \quad \tilde{\alpha}^{\bar{3}}_{\text{-}1/2}|p\rangle_L \tag{3.18}$$

transform with $e^{2\pi i(1/2)}$, as required. To obtain the complete state, we have to form the tensor product with the right mover as given by eq. (3.11). Since we have only one left mover, these states transform as singlets under the gauge group. (A calculation reveals that all Dynkin labels are zero.)

In summary, in the T_1 twisted sector, we have

$$12 \times \mathbf{27}_R, \quad 12 \times 4 \times \mathbf{1}_R,$$

where the multiplicity of 12 comes from the number of fixed points.

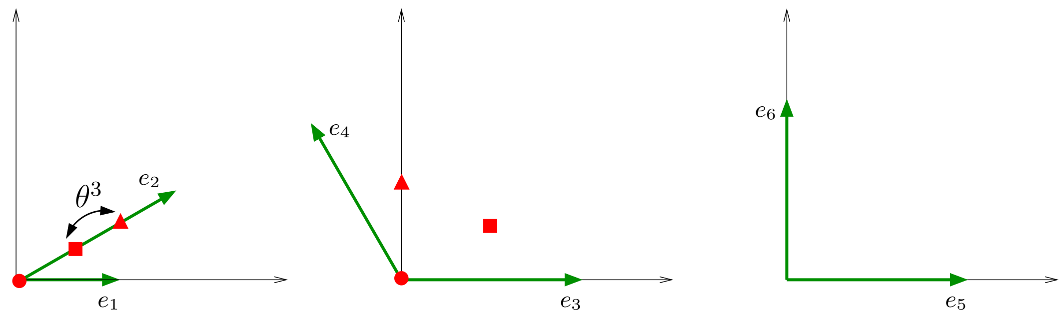


Figure 3.3: Geometry of the T_2 twisted sector.

3.3.3 The T_2 Twisted Sector

The second twisted sector is conceptionally more complicated than the first one. Consider the geometry of the orbifold, given in fig. 3.3.

The 9 fixed points of T_2 are left invariant under the action of the $\mathbb{Z}_3$ twist $2v$. In contrast to the first twisted sector T_1, not all 9 fixed points are invariant under the full point group symmetry. The $\mathbb{Z}_2$ twist $3v$ interchanges 2 of the fixed points in the first torus, as indicated in fig. 3.3. These 2 fixed points have to be identified, thus reducing the number of fixed points of the orbifold from 9 to 6. Referring back to section 2.5.2, we find that the prescription to build $S \otimes G$ invariant states instructs us to project onto $\mathbb{Z}_3$ invariant states and to sum over the states which are mapped onto each other by the $\mathbb{Z}_2$ symmetry. We will now explain these points in detail, using our concept of the centralizer. First, we determine the massless left and right movers.

The Massless Left and Right Movers

Solving the mass equation for the right and left movers, we find the states listed in the first part of tab. 3.3, where we have also indicated their transformation properties under the point group. There are 4 combinations, which we can build from the 2 right movers and the 2 left movers. These 4 combinations are given in the second part of tab. 3.3. Note that this is the spectrum *in each of the 9 fixed points before applying the projection conditions.*

Important note: Each **27** *is accompanied by a singlet representation. In order not to overload the following discussion with marginal details, we ignore these representations until the end of this section, where we list the full spectrum.*

	θ_1	θ_2	θ_3	θ_4	θ_5
Ramond, left-chiral	$\frac{7}{9}$	$\frac{5}{9}$	$\frac{2}{6}$	$\frac{1}{9}$	$\frac{8}{9}$
Ramond, right-chiral	$\frac{5}{18}$	$\frac{5}{9}$	$\frac{5}{6}$	$\frac{1}{9}$	$\frac{7}{18}$
Left mover **27**	$\frac{5}{18}$	$\frac{5}{9}$	$\frac{5}{6}$	$\frac{1}{9}$	$\frac{7}{18}$
Left mover $\overline{\mathbf{27}}$	$\frac{7}{9}$	$\frac{5}{9}$	$\frac{2}{6}$	$\frac{1}{9}$	$\frac{8}{9}$

	θ_1	θ_2	θ_3	θ_4	θ_5
$\overline{\mathbf{27}}_L$	0	0	0	0	0
$\overline{\mathbf{27}}_R$	$\frac{1}{2}$	0	-$\frac{1}{2}$	0	$\frac{1}{2}$
$\mathbf{27}_L$	-$\frac{1}{2}$	0	$\frac{1}{2}$	0	-$\frac{1}{2}$
$\mathbf{27}_R$	0	0	0	0	0

Table 3.3: Transformation properties of the right and left movers.

The $\mathbb{Z}_2$-invariant Fixed Points

Of the 9 $\mathbb{Z}_3$ fixed points in fig. 3.3,

$$(\bullet, \bullet, T^2), \quad (\bullet, \blacksquare, T^2), \quad (\bullet, \blacktriangle, T^2)$$

are also invariant under $\mathbb{Z}_2$, and thus under the full $\mathbb{Z}_6$ point group. For each fixed point, its centralizer is the full point group, so we must project onto all θ^k, $k = 1, \ldots, 5$. Consulting tab. 3.3, we find only 2 states which are invariant under the full point group, namely $\overline{\mathbf{27}}_L$ and $\mathbf{27}_R$, and since there are 3 fixed points, the spectrum is

$$3 \times \overline{\mathbf{27}}_L \quad \text{and} \quad 3 \times \mathbf{27}_R.$$

The Fixed Points not Invariant under $\mathbb{Z}_2$

The 6 remaining $\mathbb{Z}_3$ fixed points in fig. 3.3 form pairs,

$$(\blacksquare, \bullet, T^2) \leftrightarrow (\blacktriangle, \bullet, T^2), \quad (\blacksquare, \blacksquare, T^2) \leftrightarrow (\blacktriangle, \blacksquare, T^2), \quad (\blacksquare, \blacktriangle, T^2) \leftrightarrow (\blacktriangle, \blacktriangle, T^2)$$

where the fixed points in each pair are mapped onto each other. The geometry suggests to build the linear combination

$$|\cdot\rangle + \theta^3 |\cdot\rangle,$$

which is clearly $\mathbb{Z}_2$ invariant:

$$\theta^3 \ : \ |\cdot\rangle + \theta^3|\cdot\rangle \ \mapsto \ \theta^3|\cdot\rangle + \mathbb{1}|\cdot\rangle$$

This state will be also $\mathbb{Z}_3$ invariant, provided that $|\cdot\rangle$ is Z_3 invariant. Note that this is quite reminiscent of the construction described in section 2.5.2. We will now support the geometric picture with an algebraic prescription to build $S \otimes G$ invariant states.

Without loss of generality, consider the fixed point (not the pair) $(\blacksquare, \bullet, T^2)$. This fixed point is invariant under the combined action of the $\mathbb{Z}_3$ twist θ^2 and the translation by e_1, so the constructing element is (θ^2, e_1). The centralizer of the constructing element is

$$Z_g = \big\{(\mathbb{1}, 0), (\mathbb{1}, e_5), (\mathbb{1}, e_6), (\theta^2, e_1), (\theta^2, e_1 + e_5), (\theta^2, e_1 + e_6), (\theta^2, e_1 + e_5 + e_6)\big\}.$$

Space group elements of the form $(\theta^3, \cdot)$ and $(\theta^5, \cdot)$ are conspicuously absent. The centralizer instructs us to project onto θ^2, and since $\theta^3 \notin Z_g$, to sum over images of the states under θ^3.

From tab. 3.3 we see that all 4 states at $(\blacksquare, \bullet, \tau^2)$ are θ^2 invariant, and under the θ^3, they transform as

$$\overline{\mathbf{27}}_L \to +\overline{\mathbf{27}}_L, \quad \overline{\mathbf{27}}_R \to -\overline{\mathbf{27}}_R, \quad \mathbf{27}_L \to -\mathbf{27}_L, \quad \mathbf{27}_R \to +\mathbf{27}_R. \tag{3.19}$$

The invariant combinations are thus

$$|\overline{\mathbf{27}}_L\rangle_{\blacksquare} + |\overline{\mathbf{27}}_L\rangle_{\blacktriangle}, \quad |\overline{\mathbf{27}}_R\rangle_{\blacksquare} - |\overline{\mathbf{27}}_R\rangle_{\blacktriangle}, \quad |\mathbf{27}_L\rangle_{\blacksquare} - |\mathbf{27}_L\rangle_{\blacktriangle}, \quad |\mathbf{27}_R\rangle_{\blacksquare} + |\mathbf{27}_R\rangle_{\blacktriangle}.$$

Note that the linear combinations transform under the gauge group as their first summand, so we can simplify our notation. The other 2 pairs of fixed points give the same result. In summary, we obtain

$$3 \times \overline{\mathbf{27}}_L, \quad 3 \times \overline{\mathbf{27}}_R, \quad 3 \times \mathbf{27}_L, \quad 3 \times \mathbf{27}_R.$$

3.3.4 The T_3 Twisted Sector

The calculations in the third twisted sector are completely analogous to those in the previous section, so we restrict ourselves to giving only an outline.

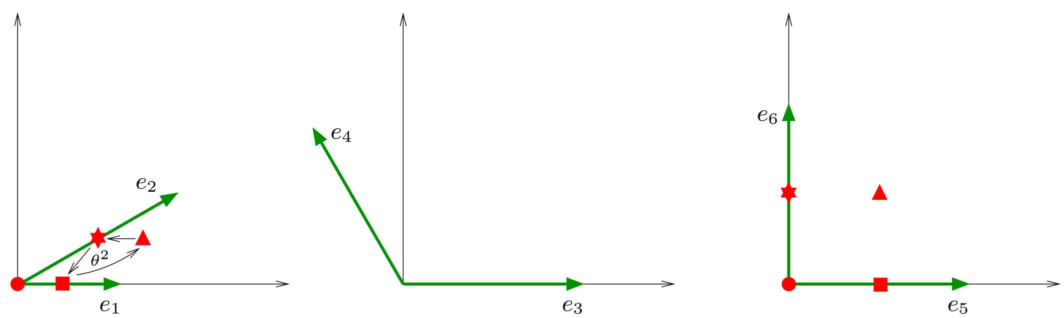


Figure 3.4: Geometry of the T_3 twisted sector.

Consider the geometry of the T_3 sector as given in fig. 3.4. There are 16 fixed points, which are invariant under the $\mathbb{Z}_2$ twist θ^3. However, not all fixed points are invariant under the full $\mathbb{Z}_6$ point group. In the first torus, the $\mathbb{Z}_3$ twist θ^2 interchanges 3 of the fixed points as indicated in fig. 3.4. Thus, these fixed points must be identified, and the number of $\mathbb{Z}_6$ invariant fixed points is reduced to $2 \times 4 = 8$.

The Massless Left and Right Movers

Solving the mass equation for the right and left movers, we find the states listed in the first part of tab. 3.4, where we have also indicated their transformation properties under the point group. There are 4 combinations, which we can build from the 2 right movers and the 2 left movers. These 4 combinations are given in the second part of tab. 3.4. As before, we do not count the singlets, but will list them in the final result.

	θ_1	θ_2	θ_3	θ_4	θ_5
Ramond, left-chiral	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$
Ramond, right-chiral	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$
Left-mover **27**	$\frac{5}{6}$	$\frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$
Left-mover $\overline{\mathbf{27}}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$

	θ_1	θ_2	θ_3	θ_4	θ_5
$\mathbf{27}_L$	$\frac{2}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$-\frac{2}{3}$
$\mathbf{27}_R$	0	0	0	0	0
$\overline{\mathbf{27}}_L$	0	0	0	0	0
$\overline{\mathbf{27}}_R$	$-\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{2}{3}$

Table 3.4: Transformation properties of the right and left movers.

The $\mathbb{Z}_3$ Invariant Fixed Points

Of the 16 fixed points in fig. 3.4,

$$(\bullet, T^2, \bullet), \quad (\bullet, T^2, \blacksquare), \quad (\bullet, T^2, \blacktriangle), \quad (\bullet, T^2, \star)$$

are also invariant under $\mathbb{Z}_3$, and thus under the full point group. From tab. 3.4, we find that there are 2 states which are invariant under all θ^k, $k = 1, \ldots, 5$, and since there are 4 fixed points, we have

$$4 \times \overline{\mathbf{27}}_L \quad \text{and} \quad 4 \times \mathbf{27}_R.$$

The Fixed Points not Invariant under $\mathbb{Z}_3$

The remaining 12 $\mathbb{Z}_2$ fixed points split into 4 groups of 3 fixed points,

$$(\blacksquare, T^2, \bullet) \leftrightarrow (\blacktriangle, T^2, \bullet) \leftrightarrow (\star, T^2, \bullet), \qquad (\blacksquare, T^2, \blacksquare) \leftrightarrow (\blacktriangle, T^2, \blacksquare) \leftrightarrow (\star, T^2, \blacksquare),$$

$$(\blacksquare, T^2, \blacktriangle) \leftrightarrow (\blacktriangle, T^2, \blacktriangle) \leftrightarrow (\star, T^2, \blacktriangle), \qquad (\blacksquare, T^2, \star) \leftrightarrow (\blacktriangle, T^2, \star) \leftrightarrow (\star, T^2, \star).$$

Again, concentrate on one group of fixed points, e.g. the first one. From tab. 3.4 we see that all states are θ^3 invariant, and under the θ^2, they transform as

$$\mathbf{27}_L \to e^{2\pi i\,(1/3)}\,\mathbf{27}_L, \qquad \mathbf{27}_R \to +\mathbf{27}_R, \qquad \overline{\mathbf{27}}_L \to +\overline{\mathbf{27}}_L, \qquad \overline{\mathbf{27}}_R \to e^{2\pi i\,(2/3)}\,\overline{\mathbf{27}}_R.$$

The invariant combinations are

$$|\mathbf{27}_L\rangle_\blacksquare + e^{2\pi i\,(1/3)}\,|\mathbf{27}_L\rangle_\blacktriangle + e^{2\pi i\,(2/3)}\,|\mathbf{27}_L\rangle_\star, \qquad |\mathbf{27}_R\rangle_\blacksquare + |\mathbf{27}_R\rangle_\blacktriangle + |\mathbf{27}_R\rangle_\star,$$

$$|\overline{\mathbf{27}}_R\rangle_\blacksquare + e^{2\pi i\,(2/3)}\,|\overline{\mathbf{27}}_R\rangle_\blacktriangle + e^{2\pi i\,(1/3)}\,|\overline{\mathbf{27}}_R\rangle_\star, \qquad |\overline{\mathbf{27}}_L\rangle_\blacksquare + |\overline{\mathbf{27}}_L\rangle_\blacktriangle + |\overline{\mathbf{27}}_L\rangle_\star.$$

Since there are 4 groups of fixed points, we have

$$4 \times \mathbf{27}_L, \quad 4 \times \overline{\mathbf{27}}_L, \quad 4 \times \mathbf{27}_R, \quad 4 \times \overline{\mathbf{27}}_R.$$

In summary, we have the following spectrum for the T_3 sector:

T_3	$8 \times \overline{\mathbf{27}}_L$, $4 \times \mathbf{27}_L$
	$8 \times \mathbf{27}_R$, $4 \times \overline{\mathbf{27}}_R$

3.3.5 The Twisted Sectors T_4 and T_5

As yet, we have covered all the features and calculational complications, which even order orbifolds in general and the $\mathbb{Z}_6$-II orbifold in particular have to offer. We will skip the calculations for the T_4 and T_5 twisted sectors, since they are completely analogous to those of T_2 and T_1, respectively. The result reads:

T_4	$6 \times \overline{\mathbf{27}}_L$, $6 \times \mathbf{27}_R$, $3 \times \mathbf{27}_L$, $3 \times \overline{\mathbf{27}}_R$,
T_5	$12 \times \overline{\mathbf{27}}_L$

Note that states in T_4 and T_5 are those of T_2 and T_1, respectively, with the chirality flipped and the representations complex conjugated, whereas T_3 closed w.r.t. these operations.

3.3.6 The Full Spectrum

We collect the results of the past sections in the following table (the gauge bosons are omitted):

U	$3 \times \mathbf{27}_R$, $3 \times \overline{\mathbf{27}}_L$, $1 \times \overline{\mathbf{27}}_R$, $1 \times \mathbf{27}_L$, $3 \times \mathbf{1}_R$, $3 \times \mathbf{1}_L$,
T_1	$12 \times \mathbf{27}_R$, $48 \times \mathbf{1}_R$
T_2	$6 \times \overline{\mathbf{27}}_L$, $6 \times \mathbf{27}_R$, $3 \times \mathbf{27}_L$, $3 \times \overline{\mathbf{27}}_R$, $9 \times \mathbf{1}_L$, $9 \times \mathbf{1}_R$
T_3	$8 \times \overline{\mathbf{27}}_L$, $4 \times \mathbf{27}_L$, $8 \times \mathbf{27}_R$, $4 \times \overline{\mathbf{27}}_R$, $52 \times \mathbf{1}_L$, $52 \times \mathbf{1}_R$
T_4	$6 \times \overline{\mathbf{27}}_L$, $6 \times \mathbf{27}_R$, $3 \times \mathbf{27}_L$, $3 \times \overline{\mathbf{27}}_R$, $9 \times \mathbf{1}_L$, $9 \times \mathbf{1}_R$
T_5	$12 \times \overline{\mathbf{27}}_L$, $48 \times \mathbf{1}_L$

For a consistent interpretation of these states as particles, we must construct the corresponding supersymmetry multiplets. Combining each representation of a given chirality with its complex conjugate representation of flipped chirality, we obtain exactly the degrees of freedom for a *massive chiral supermultiplet in four dimensions*.

U	$3\times\overline{\mathbf{27}},\quad 1\times\mathbf{27},\quad 3\times\mathbf{1},$
T_1+T_5	$12\times\overline{\mathbf{27}},\quad 48\times\mathbf{1}$
T_2+T_4	$12\times\overline{\mathbf{27}},\quad 6\times\mathbf{27},\quad 18\times\mathbf{1}$
T_3	$8\times\overline{\mathbf{27}},\quad 4\times\mathbf{27},\quad 52\times\mathbf{1}$

We have to combine states from different twisted sectors to build the supersymmetry multiplets. The reason can be traced back to the existence of a 6-dimensional intermediate space-time. Remember that the point group $\mathbb{Z}_6$-II is isomorphic to $\mathbb{Z}_2\times\mathbb{Z}_3$, where the $\mathbb{Z}_2$ twist is θ^3 and the $\mathbb{Z}_3$ twist is θ^2. Moding out the first twist, we obtain $\mathcal{N}=1$ hypermultiplets in 6 dimensions, where, in complete analogy to the $\mathbb{Z}_3$ case, the anti-particles of T_k are in T_{6-k}. Applying the second twist, projects out half of the states leaving us with half-hypermultiplets in 4 dimensions, which are then combined to give massive chiral multiplets.

Having determined the spectrum, we are interested to see how many generations of quarks and leptons we have obtained. The gauge symmetry being $\mathrm{E}_6\times\mathrm{U}(1)^2$, one $\mathbf{27}$ can accommodate one generation of quarks and leptons (including the right-handed neutrino), and the Higgs boson:

$$
\begin{array}{ccccccc}
\mathrm{E}_6 & \supset & \mathrm{SO}(10) & \supset & \mathrm{SU}(5) & \supset & \mathrm{SU}(3)\times\mathrm{SU}(2) \\[1em]
\mathbf{27} & \rightarrow & \mathbf{16}+\mathbf{10}+\mathbf{1} & \rightarrow & \underset{\sim\sim\sim\sim\sim\sim}{\mathbf{10}+\overline{\mathbf{5}}+\mathbf{1}}+\mathbf{5}+\overline{\mathbf{5}}+\mathbf{1} & \rightarrow & \begin{array}{l}(\mathbf{3},\mathbf{2})+(\overline{\mathbf{3}},\mathbf{1})+(\mathbf{1},\mathbf{1})+\\(\overline{\mathbf{3}},\mathbf{1})+(\mathbf{1},\mathbf{2})+(\mathbf{1},\mathbf{1})+\ldots\end{array}
\end{array}
$$

Since pairs of $\mathbf{27}$ and $\overline{\mathbf{27}}$ can be combined in the superpotential to obtain masses of the order of the string compactification scale, we are only interested in the so-called *net number of families*, which is the difference between the number of $\mathbf{27}$'s and the number of $\overline{\mathbf{27}}$'s. We have summarized these numbers in the following table:

	U	T_1	T_2	T_3	sum
$\mathbf{27}$	1	0	6	4	11
$\overline{\mathbf{27}}$	3	12	12	8	35

We obtain $24 \times \overline{\mathbf{27}}$ corresponding to 24 families of quarks and leptons. Clearly, this is in gross contradiction to experiment. In section 2.6, we explained by means of the $\mathbb{Z}_3$ orbifold how the mechanism of Wilson lines can reduce the number of families, thus allowing the construction of (semi-)realistic models [14, 15].

In the following chapters, we will present several 3 generation models with GUT and Standard Model gauge groups in 4 dimensions. The construction of these models proceed along the same lines as described in the present section. Therefore, we will skip the calculational details, confining ourselves to discussing the spectrum.

3.4 Non-Standard Embeddings

The modular invariance condition eq. (3.3) can also be satisfied by a number of shift vectors which do not correspond to the standard embedding discussed in section 3.3. In classifying all admissible shift vectors, we are confronted with the task to determine all order six Lie algebra automorphisms of $E_8 \times E_8'$, which fulfill the consistency requirements of the orbifold construction, in particular the condition on modular invariance. In chapter 4, we have explained the classification procedure in great detail. The results for the $\mathbb{Z}_6$-II orbifold are given in appendix E.

U	$1 \times (\overline{\mathbf{84}}, \mathbf{1}, \mathbf{1})_L$ $1 \times (\mathbf{1}, \mathbf{2}, \mathbf{16})_L$ $1 \times (\mathbf{1}, \mathbf{1}, \mathbf{1})_L$ $1 \times (\mathbf{1}, \mathbf{1}, \mathbf{10})_L$ $3 \times (\mathbf{1}, \mathbf{2}, \mathbf{1})_L$ $2 \times (\mathbf{1}, \mathbf{1}, \overline{\mathbf{16}})_L$ $1 \times (\mathbf{1}, \mathbf{1}, \mathbf{16})_L$ $1 \times (\mathbf{1}, \mathbf{2}, \mathbf{10})_L$
T_1	
T_2	$6 \times (\mathbf{9}, \mathbf{1}, \mathbf{1})_L$ $3 \times (\mathbf{9}, \mathbf{2}, \mathbf{1})_L$
T_3	$44 \times (\mathbf{1}, \mathbf{1}, \mathbf{1})_L$ $8 \times (\mathbf{1}, \mathbf{1}, \overline{\mathbf{16}})_L$ $12 \times (\mathbf{1}, \mathbf{2}, \mathbf{1})_L$ $4 \times (\mathbf{1}, \mathbf{2}, \mathbf{10})_L$ $4 \times (\mathbf{1}, \mathbf{1}, \mathbf{16})_L$
T_4	$6 \times (\overline{\mathbf{9}}, \mathbf{2}, \mathbf{1})_L$ $3 \times (\overline{\mathbf{9}}, \mathbf{1}, \mathbf{1})_L$
T_5	$12 \times (\mathbf{9}, \mathbf{1}, \mathbf{1})_L$

Table 3.5: The spectrum for the model corresponding to the 3rd shift in our classification. For the sake of clarity, we do not list the right chiral particles. It is understood that each left handed particle is accompanied by a right chiral one where the representation is complex conjugated.

Consider the 3rd shift from the classification:

$$V = \left(\tfrac{5}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}\right) \left(\text{-}\tfrac{5}{6}, \tfrac{5}{3}, \tfrac{5}{3}, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}, \tfrac{3}{2}, \text{-}\tfrac{3}{2}\right) \tag{3.20}$$

The unbroken gauge group in four dimensions is

$$\mathrm{SU}(9) \times \mathrm{SU}(2)' \times \mathrm{SO}(10)', \tag{3.21}$$

and we have summarized the spectrum of the model in tab. 3.5.

Remarkably, there are $10 \times \overline{\mathbf{16}}$ and $7 \times \mathbf{16}$ corresponding to a net number of 3 families. Whether the model has an interesting phenomenology has yet to be discussed. Without a classification, this and equally interesting models might have slipped through our fingers.

3.4.1 A Useful Consistency Check

The calculations leading to the spectrum of an orbifold model are long and tedious. The following criterion may be helpful for checking the correctness of the result.

Modular invariance guarantees the anomaly freedom of the orbifold spectrum. In four dimensions, only SU(N) groups for $N \geq 3$ can have anomalies. In appendix B.3, we list the anomalies for the irreducible representations. Specializing to $N = 9$, we find:

$$\mathbf{9} \quad \sim \quad [1,0,0,0,0,0,0,0]_{\mathrm{DL}} \quad \sim \quad \square \quad \rightsquigarrow \quad \text{anomaly is 1} \tag{3.22}$$

$$\mathbf{84} \quad \sim \quad [0,0,1,0,0,0,0,0]_{\mathrm{DL}} \quad \sim \quad \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} \quad \rightsquigarrow \quad \text{anomaly is 9} \tag{3.23}$$

For convenience, we reproduce the spectrum from tab. 3.5, but only list the representations which contribute an anomaly:

Sector	Irreps	Anomaly
U	$1 \times (\overline{\mathbf{84}}, \mathbf{1}, \mathbf{1})_L$	$1 \times (-9)$
T_1		0
T_2	$6 \times (\mathbf{9}, \mathbf{1}, \mathbf{1})_L \quad 3 \times (\mathbf{9}, \mathbf{2}, \mathbf{1})_L$	$6 \times 1 + 3 \times 2 \times 1$
T_3		0
T_4	$6 \times (\overline{\mathbf{9}}, \mathbf{2}, \mathbf{1})_L \quad 3 \times (\overline{\mathbf{9}}, \mathbf{1}, \mathbf{1})_L$	$6 \times 2 \times (-1) + 3 \times (-1)$
T_5	$12 \times (\mathbf{9}, \mathbf{1}, \mathbf{1})_L$	12×1
$\sum$		0

If the sum fails to be zero, it is a clear indication that something went wrong in the calculation.

3.5 Orbifold Models with Wilson Lines

3.5.1 The Construction

Using the formalism of the centralizer developed in section 3.3, constructing orbifold models with Wilson lines is not more difficult than the case without Wilson lines.

Modular invariance. In the presence of Wilson lines, the modular invariance conditions eq. (2.26) impose more restrictions on the shift vectors V and introduce new ones for the Wilson lines A_α.

Untwisted sector. In the untwisted sector, the Wilson lines act as additional projection conditions on the gauge and charged matter representations, cf. eqs. (2.31-2.32). Everything else in unchanged.

Twisted sectors. It is here that the Wilson lines have the most dramatic effects. First of all, the equation for massless states changes. Assume that we are at a fixed points which is invariant under the combined action of twist and lattice translations,

$$x_f = \theta^k x_f + n_\alpha e_\alpha . \tag{3.24}$$

Then the coefficients in the linear combination of Wilson lines entering the equation of the massless states

$$\frac{1}{2}(q+kv)^2 - \frac{1}{2} + \delta c = \frac{1}{4}m_R^2 = \frac{1}{4}m_L^2 = \frac{1}{2}(p + kV + n_\alpha A_\alpha)^2 + N_L - 1 + \delta c = 0 \tag{3.25}$$

are exactly those n_α's from eq. (3.24). The equation of massless states has not only changed, but this change is dependent on the fixed point.

The second important point is how the presence of Wilson lines changes the transformation properties of the left movers. Whereas in the case with no Wilson lines, the lattice vectors e_α were completely irrelevant, the gauge embedding now maps these e_α to Wilson lines A_α, so that left movers at different fixed points transform differently. Assume again that we are at the fixed point described by eq. (3.24). Then the transformation of the states living at this fixed point under an element $h = (\theta^\ell, m_\alpha e_\alpha)$ of the space group S will be given by:

$$|p + kV + n_\alpha A_\alpha\rangle_L \mapsto \exp\left(2\pi i (p + kV + n_\alpha A_\alpha)\cdot(\ell V + m_\alpha A_\alpha)\right)|p + kV + n_\alpha A_\alpha\rangle_L \tag{3.26}$$

Note that for the case of no Wilson lines, $k = 1$ (first twisted sector) and $h = (\theta^\ell, 0)$ this reduces to eq. (3.14).

Finally, the projection conditions have to be determined. The tensor product of left and right movers need not be invariant under the full space group S, but only under a subset, given by the *centralizer* of the space group element $g = (\theta^k, n_\alpha e_\alpha)$, which by virtue of eq. (3.24) defines the fixed point.

Putting these three pieces of information together allows us to determine the spectrum at the fixed point corresponding to the constructing element $g = (\theta^k, n_\alpha e_\alpha)$. First solve the equation for massless states eq. (3.25) corresponding to the fixed point described by g. The transformation of the left mover under an element $(\theta^\ell, m_\alpha e_\alpha)$ of the centralizer Z_g is given by eq. (3.26), whereas the right mover is not affected by the Wilson lines and transforms as

$$|q + kv\rangle \mapsto e^{-2\pi i (q+kv)\cdot \ell v} |q + kv\rangle. \tag{3.27}$$

The spectrum consists of those tensor products of left and right movers which are invariant under all $(\theta^\ell, m_\alpha e_\alpha) \in Z_g$.
If there are fixed points which are mapped onto each other, the centralizer also tells us which linear combinations of states to build in order to obtain invariant states. We will not go into any details here but refer back to section 3.3.3.

3.5.2 An SO(10) Orbifold Model With 3 Generations

Definition of the Model

We choose the shift

$$V = \left(-\tfrac{1}{3}, -\tfrac{2}{3}, \tfrac{1}{6}, \tfrac{1}{6}, 0, 0, 0, 0\right)\left(\tfrac{17}{6}, -\tfrac{5}{2}, -\tfrac{5}{2}, -\tfrac{5}{2}, -\tfrac{5}{2}, -\tfrac{5}{2}, -\tfrac{5}{2}, \tfrac{5}{2}\right) \tag{3.28}$$

and the Wilson lines

$$W_3 = W_4 = \left(\tfrac{1}{2}, -\tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, -\tfrac{1}{6}, -\tfrac{1}{6}, -\tfrac{1}{6}, -\tfrac{1}{6}\right)\left(0, 0, 0, 0, 0, 0, -\tfrac{4}{3}, -\tfrac{4}{3}\right) \tag{3.29}$$

in the 2nd torus. Notice that the geometry of the orbifold implies that the Wilson lines along e_3 and e_4 must be equal, cf. section 3.2.

Gauge Group and Spectrum

The gauge symmetry in four dimensions is given by

$$\mathrm{SU}(5) \times \mathrm{SU}(2) \times \mathrm{SO}(10)' \times \mathrm{SU}(2)'. \tag{3.30}$$

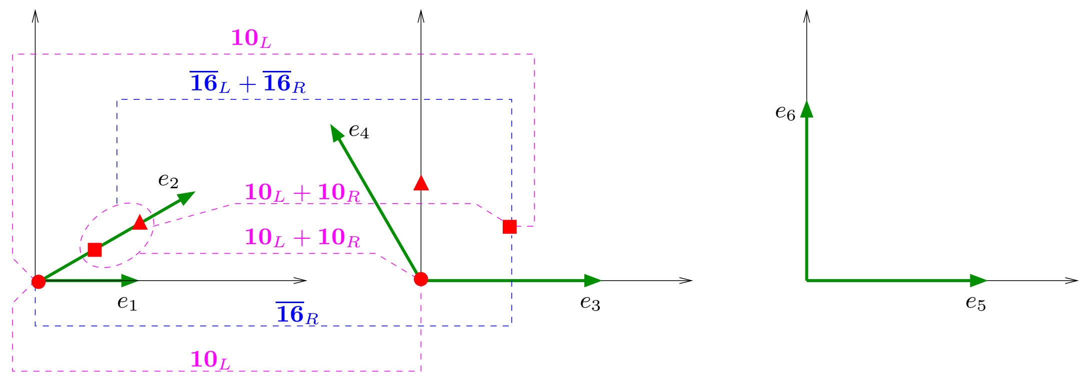
Keeping the remarks of section 3.5.1 in the back of our minds, the calculation of the spectrum proceeds along the same lines as described in great detail in section 3.3. We list the result in tab. 3.6.

In principle, we have again to combine the left and right handed particles from conjugate sectors to form chiral multiplets. However, we refer to a small trick, namely we only count left chiral particles and ignore the presence of right handed ones. This gives the right number of chiral multiplets in our model:

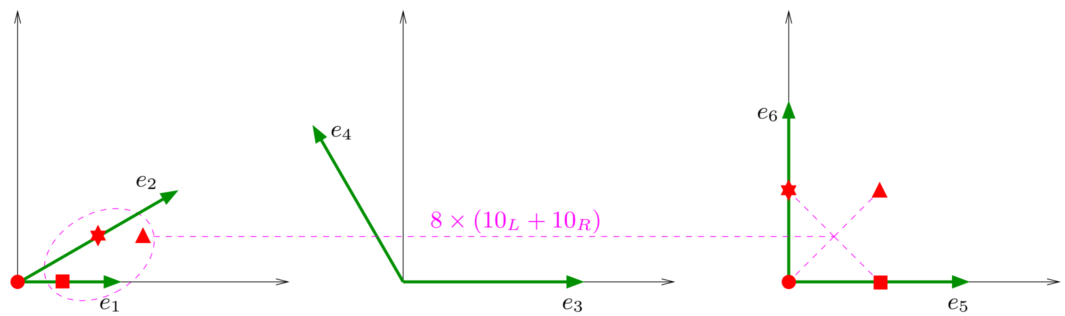
	U	T_1	T_2	T_3	T_4	T_5	sum
$\mathbf{16}_L$	2	0	0	0	2	0	4
$\overline{\mathbf{16}}_L$	0	0	1	0	0	0	1

Untwisted Sector	T_1 Sector	T_2 Sector	T_4 Sector
$1\times(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})_R$	$4\times(\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{1})_R$	$6\times(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})_R$	$4\times(\mathbf{1},\mathbf{1},\mathbf{10},\mathbf{1})_R$
$1\times(\mathbf{1},\mathbf{1},\mathbf{10},\mathbf{1})_R$	$8\times(\overline{\mathbf{5}},\mathbf{1},\mathbf{1},\mathbf{1})_R$	$2\times(\mathbf{1},\mathbf{1},\overline{\mathbf{16}},\mathbf{1})_R$	$10\times(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})_R$
$1\times(\mathbf{10},\mathbf{1},\mathbf{1},\mathbf{1})_R$	$24\times(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})_R$	$2\times(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})_R$	$2\times(\mathbf{5},\mathbf{1},\mathbf{1},\mathbf{1})_R$
$2\times(\overline{\mathbf{5}},\mathbf{2},\mathbf{1},\mathbf{1})_R$	$8\times(\mathbf{5},\mathbf{1},\mathbf{1},\mathbf{1})_R$	$2\times(\mathbf{5},\mathbf{1},\mathbf{1},\mathbf{1})_R$	$4\times(\overline{\mathbf{5}},\mathbf{1},\mathbf{1},\mathbf{1})_R$
$1\times(\mathbf{5},\mathbf{2},\mathbf{1},\mathbf{1})_R$	$4\times(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})_R$	$1\times(\overline{\mathbf{5}},\mathbf{1},\mathbf{1},\mathbf{1})_R$	$8\times(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})_R$
$1\times(\mathbf{1},\mathbf{1},\overline{\mathbf{16}},\mathbf{2})_R$	T(3,0) Sector	$2\times(\mathbf{1},\mathbf{1},\mathbf{10},\mathbf{1})_R$	$2\times(\overline{\mathbf{5}},\mathbf{1},\mathbf{1},\mathbf{2})_R$
$1\times(\overline{\mathbf{5}},\mathbf{2},\mathbf{1},\mathbf{1})_L$	$12\times(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})_R$	$5\times(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})_R$	$3\times(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})_R$
$2\times(\mathbf{5},\mathbf{2},\mathbf{1},\mathbf{1})_L$	$4\times(\mathbf{1},\mathbf{2},\mathbf{10},\mathbf{1})_R$	$4\times(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})_R$	$1\times(\mathbf{1},\mathbf{1},\mathbf{16},\mathbf{1})_R$
$1\times(\overline{\mathbf{10}},\mathbf{1},\mathbf{1},\mathbf{1})_L$	$12\times(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})_L$	$1\times(\mathbf{5},\mathbf{1},\mathbf{1},\mathbf{2})_R$	$1\times(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})_R$
$1\times(\mathbf{1},\mathbf{1},\mathbf{10},\mathbf{1})_L$	$4\times(\mathbf{1},\mathbf{2},\mathbf{10},\mathbf{1})_L$	$4\times(\mathbf{5},\mathbf{1},\mathbf{1},\mathbf{1})_L$	$6\times(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})_L$
$1\times(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})_L$	T(5,0) Sector	$2\times(\overline{\mathbf{5}},\mathbf{1},\mathbf{1},\mathbf{1})_L$	$2\times(\mathbf{1},\mathbf{1},\mathbf{16},\mathbf{1})_L$
$1\times(\mathbf{1},\mathbf{1},\mathbf{16},\mathbf{2})_L$	$8\times(\mathbf{5},\mathbf{1},\mathbf{1},\mathbf{1})_L$	$4\times(\mathbf{1},\mathbf{1},\mathbf{10},\mathbf{1})_L$	$2\times(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})_L$
	$4\times(\overline{\mathbf{10}},\mathbf{1},\mathbf{1},\mathbf{1})_L$	$10\times(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})_L$	$2\times(\mathbf{1},\mathbf{1},\mathbf{10},\mathbf{1})_L$
	$24\times(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})_L$	$8\times(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})_L$	$5\times(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})_L$
	$8\times(\overline{\mathbf{5}},\mathbf{1},\mathbf{1},\mathbf{1})_L$	$2\times(\mathbf{5},\mathbf{1},\mathbf{1},\mathbf{2})_L$	$1\times(\mathbf{5},\mathbf{1},\mathbf{1},\mathbf{1})_L$
	$4\times(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{2})_L$	$3\times(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{1})_L$	$2\times(\overline{\mathbf{5}},\mathbf{1},\mathbf{1},\mathbf{1})_L$
		$1\times(\mathbf{1},\mathbf{1},\overline{\mathbf{16}},\mathbf{1})_L$	$4\times(\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})_L$
		$1\times(\mathbf{1},\mathbf{2},\mathbf{1},\mathbf{2})_L$	$1\times(\overline{\mathbf{5}},\mathbf{1},\mathbf{1},\mathbf{2})_L$

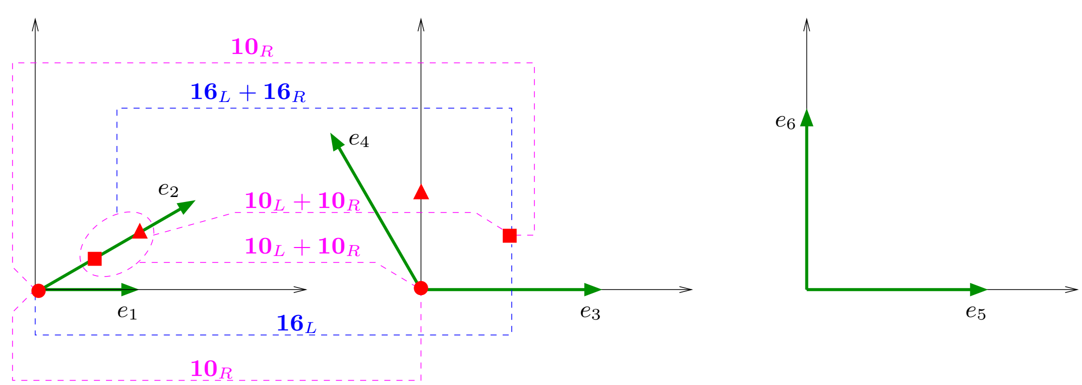
Table 3.6: Untwisted and twisted sectors for the three generation SO(10) model.



(a) 2nd Twisted Sector



(b) 3rd Twisted Sector



(c) 4th Twisted Sector

Figure 3.5: Twisted sectors for the SO(10) model. The matter representations are in blue and the Higgs representations in magenta. We have indicated fixed points which are mapped onto each other (and thus have to be considered as one fixed point) by encircling them with ellipses.

In fig. 3.5, we give a pictorial representation of the localization properties of the matter and Higgs fields.

Gauge Symmetry in the Six-Dimensional Intermediate Picture

In even order orbifolds in general and in our case with point group $\mathbb{Z}_6 \simeq \mathbb{Z}_2 \times \mathbb{Z}_3$ in particular we can mod out the symmetry in 2 steps:

$$T^6 \quad \longrightarrow \quad T^6 \big/ \left\{\theta^3\right\} \ \simeq \ \mathcal{O}' \times T^2 \quad \longrightarrow \quad \mathcal{O}' \times T^2 \big/ \left\{\theta^2\right\} \ \simeq \ \mathcal{O} \tag{3.31}$$

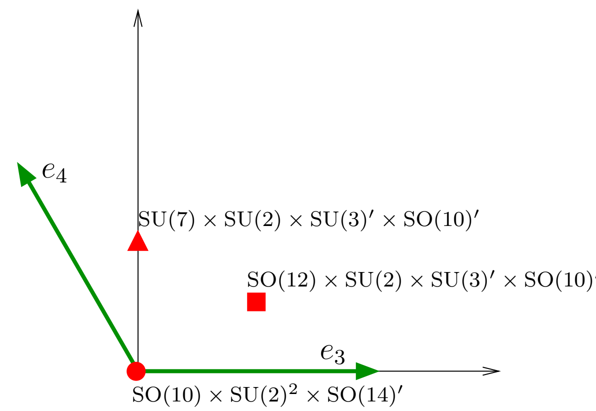


Figure 3.6: Geometry of the gauge symmetry breakdown.

After moding out the θ^3 symmetry, we arrive at a situation where the states in the twisted sector are fixed in four dimensions and free to move on the 2-torus T^2 and the uncompactified dimensions $\mathbb{R}^4$. The gauge group surviving in these six dimensions is obtained from $\mathrm{E}_8 \times \mathrm{E}_8'$ by requiring invariance under the gauge embedding θ^3:

$$\mathrm{E}_8 \times \mathrm{E}_8' \quad \stackrel{3V}{\longrightarrow} \quad \mathrm{E}_7 \times \mathrm{SU}(2) \times \mathrm{SO}(16)' \tag{3.32}$$

Forgetting about the ambient space and taking the six dimensions as our starting point, the above gauge group is the *bulk symmetry*. The spectrum in four dimensions is obtained by moding out the θ^2 symmetry from this six dimensional theory. In particular, the gauge symmetry at the fixed points is reduced, since we have the additional projection conditions from the Wilson lines. Thus, at each fixed point, a different gauge group will appear, as presented in fig. 3.6. The unbroken gauge group in four dimensions is the intersection of these three symmetries.

The particles at the fixed points which survive in four dimensions transform in full or split multiplets of the larger gauge symmetries. This is how the gauge groups in higher dimensions become relevant for the low-energy phenomenology in four dimensions. Looking at the *gauge group geography* may give us some ideas which route to take for constructing realistic models.

3.5.3 An SU(5) Orbifold Model With 3 Generations

Definition of the Model

We choose the shift

$$V = \left(-\tfrac{1}{3},\, -\tfrac{2}{3},\, \tfrac{1}{6},\, \tfrac{1}{6},\, 0,\, 0,\, 0,\, 0\right)\,\left(\tfrac{5}{3},\, -1,\, -1,\, -1,\, -1,\, -1,\, -1,\, 1\right) \tag{3.33}$$

and the Wilson lines

$$W_3 = W_4 = \left(\tfrac{1}{3},\, 0,\, \tfrac{1}{3},\, -\tfrac{1}{3},\, -\tfrac{1}{3},\, 0,\, 0,\, 0\right)\,\left(\tfrac{1}{6},\, -\tfrac{1}{2},\, -\tfrac{1}{2},\, \tfrac{1}{2},\, \tfrac{1}{6},\, \tfrac{1}{6},\, \tfrac{7}{6},\, -\tfrac{7}{6}\right). \tag{3.34}$$

The model has been found by the classification described in chapter 4.

U	$1\times(\overline{\mathbf{5}},\mathbf{1},\mathbf{1})_L$ $4\times(\mathbf{5},\mathbf{1},\mathbf{1})_L$ $1\times(\overline{\mathbf{10}},\mathbf{1},\mathbf{1})_L$ $1\times(\mathbf{1},\overline{\mathbf{4}},\overline{\mathbf{4}})_L$ $1\times(\mathbf{1},\mathbf{4},\mathbf{1})_L$ $1\times(\mathbf{1},\mathbf{1},\mathbf{4})_L$ $2\times(\mathbf{1},\mathbf{1},\mathbf{1})_L$
T_1	
T_2	$4\times(\overline{\mathbf{5}},\mathbf{1},\mathbf{1})_L$ $3\times(\mathbf{5},\mathbf{1},\mathbf{1})_L$ $2\times(\mathbf{1},\mathbf{6},\mathbf{1})_L$ $6\times(\mathbf{1},\mathbf{1},\mathbf{4})_L$ $2\times(\mathbf{1},\mathbf{1},\overline{\mathbf{4}})_L$ $5\times(\mathbf{1},\mathbf{4},\mathbf{1})_L$ $14\times(\mathbf{1},\mathbf{1},\mathbf{1})_L$ $1\times(\mathbf{1},\mathbf{1},\mathbf{6})_L$ $1\times(\mathbf{1},\overline{\mathbf{4}},\mathbf{1})_L$
T_3	$8\times(\mathbf{10},\mathbf{1},\mathbf{1})_L$ $12\times(\mathbf{1},\mathbf{1},\mathbf{1})_L$ $4\times(\mathbf{5},\mathbf{1},\mathbf{1})_L$ $4\times(\overline{\mathbf{10}},\mathbf{1},\mathbf{1})_L$ $4\times(\overline{\mathbf{5}},\mathbf{1},\mathbf{1})_L$
T_4	$19\times(\mathbf{1},\mathbf{1},\mathbf{1})_L$ $7\times(\mathbf{1},\overline{\mathbf{4}},\mathbf{1})_L$ $6\times(\mathbf{1},\mathbf{1},\overline{\mathbf{4}})_L$ $3\times(\overline{\mathbf{5}},\mathbf{1},\mathbf{1})_L$ $2\times(\mathbf{1},\mathbf{4},\mathbf{1})_L$ $2\times(\mathbf{1},\mathbf{1},\mathbf{6})_L$ $1\times(\mathbf{1},\mathbf{6},\mathbf{1})_L$ $1\times(\mathbf{1},\mathbf{1},\mathbf{4})_L$ $2\times(\mathbf{5},\mathbf{1},\mathbf{1})_L$
T_5	$20\times(\mathbf{1},\mathbf{1},\mathbf{1})_L$ $4\times(\mathbf{1},\mathbf{1},\mathbf{4})_L$ $4\times(\mathbf{1},\mathbf{4},\mathbf{1})_L$ $4\times(\overline{\mathbf{5}},\mathbf{1},\mathbf{1})_L$

Table 3.7: The spectrum for our SU(5) model. We only list the left chiral particles.

Gauge Group and Spectrum

The gauge symmetry in four dimensions is given by

$$\mathrm{SU}(5)\times\mathrm{SU}(4)'\times\mathrm{SU}(4)'. \tag{3.35}$$

The spectrum is given in tab. 3.7.

3.5.4 A Standard Model like Orbifold Model With 3 Generations

Definition of the Model

We choose the shift

$$V = \left(\tfrac{1}{4},\, \tfrac{7}{12},\, \text{-}\tfrac{1}{12},\, \text{-}\tfrac{1}{12},\, \text{-}\tfrac{1}{4},\, \text{-}\tfrac{1}{4},\, \text{-}\tfrac{1}{4},\, \text{-}\tfrac{1}{4}\right) \left(\tfrac{1}{3},\, \tfrac{1}{3},\, \text{-}\tfrac{1}{6},\, \text{-}\tfrac{1}{6},\, \text{-}\tfrac{1}{6},\, \text{-}\tfrac{1}{6},\, \tfrac{5}{6},\, \tfrac{5}{6}\right) \tag{3.36}$$

and the Wilson lines

$$W_3 = W_4 = \left(\tfrac{1}{3},\, \text{-}\tfrac{1}{3},\, 0,\, \tfrac{1}{3},\, \tfrac{1}{3},\, 0,\, 0,\, 0\right) \left(\tfrac{5}{6},\, \text{-}\tfrac{5}{6},\, \text{-}\tfrac{5}{6},\, \text{-}\tfrac{5}{6},\, \text{-}\tfrac{5}{6},\, \text{-}\tfrac{5}{6},\, \tfrac{1}{6},\, \tfrac{11}{6}\right) \tag{3.37}$$

The model has been found by the classification described in chapter 4.

Gauge Group and Spectrum

The gauge symmetry in four dimensions is given by

$$\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)^5 \times \mathrm{SO}(10)' \times \mathrm{U}(1)'^3. \tag{3.38}$$

The spectrum is given in tab. 3.8.

U	$6 \times (\mathbf{1},\mathbf{1},\mathbf{1})_L$ $1 \times (\mathbf{3},\mathbf{2},\mathbf{1})_L$ $2 \times (\overline{\mathbf{3}},\mathbf{2},\mathbf{1})_L$ $1 \times (\mathbf{1},\mathbf{1},\mathbf{16})_L$ $2 \times (\mathbf{1},\mathbf{1},\overline{\mathbf{16}})_L$ $4 \times (\mathbf{1},\mathbf{2},\mathbf{1})_L$ $3 \times (\overline{\mathbf{3}},\mathbf{1},\mathbf{1})_L$ $1 \times (\mathbf{1},\mathbf{1},\mathbf{10})_L$ $2 \times (\mathbf{3},\mathbf{1},\mathbf{1})_L$
T_1	
T_2	$15 \times (\mathbf{1},\mathbf{2},\mathbf{1})_L$ $10 \times (\overline{\mathbf{3}},\mathbf{1},\mathbf{1})_L$ $36 \times (\mathbf{1},\mathbf{1},\mathbf{1})_L$ $6 \times (\mathbf{3},\mathbf{1},\mathbf{1})_L$ $2 \times (\mathbf{3},\mathbf{2},\mathbf{1})_L$
T_3	$12 \times (\mathbf{3},\mathbf{1},\mathbf{1})_L$ $24 \times (\mathbf{1},\mathbf{1},\mathbf{1})_L$ $8 \times (\overline{\mathbf{3}},\mathbf{1},\mathbf{1})_L$
T_4	$36 \times (\mathbf{1},\mathbf{1},\mathbf{1})_L$ $4 \times (\overline{\mathbf{3}},\mathbf{2},\mathbf{1})_L$ $12 \times (\mathbf{1},\mathbf{2},\mathbf{1})_L$ $8 \times (\mathbf{3},\mathbf{1},\mathbf{1})_L$ $9 \times (\overline{\mathbf{3}},\mathbf{1},\mathbf{1})_L$
T_5	$40 \times (\mathbf{1},\mathbf{1},\mathbf{1})_L$ $8 \times (\mathbf{3},\mathbf{1},\mathbf{1})_L$ $12 \times (\mathbf{1},\mathbf{2},\mathbf{1})_L$

Table 3.8: The spectrum for our $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)^5$ model. We only list the left chiral particles.

We count the representations $(\mathbf{3},\mathbf{2})$ as families. Counting again only one chirality, we obtain a net number of three families:

	U	T_1	T_2	T_3	T_4	T_5	sum
$(\mathbf{3},\mathbf{2})_L$	1	0	2	0	0	0	3
$(\overline{\mathbf{3}},\mathbf{2})_L$	2	0	0	0	4	0	6

Gauge Group Geography

It is interesting to trace back the origin of the Standard Model gauge symmetry in four dimensions. To that end, we proceed along the same lines as in section 3.5.2. The bulk gauge group in six dimensions is $\mathrm{SO}(16) \times \mathrm{SU}(2)' \times \mathrm{E}_7'$.

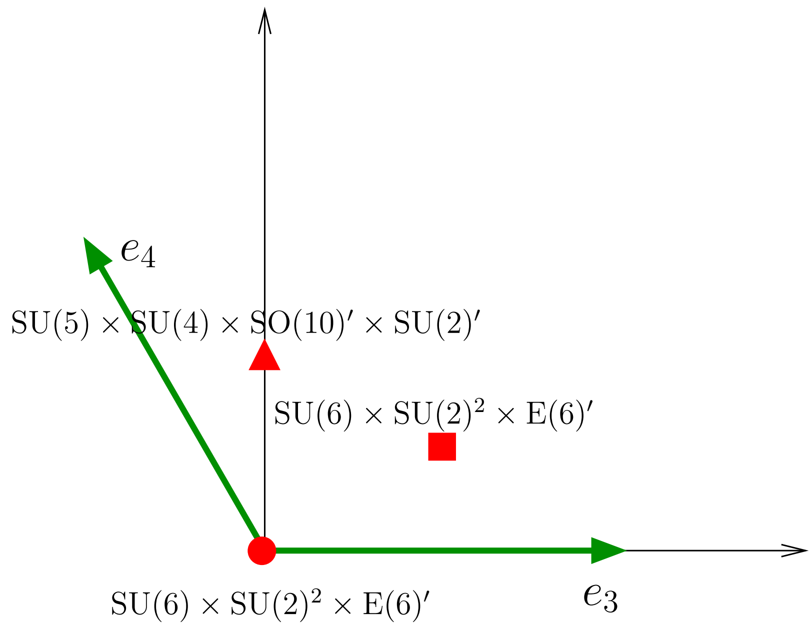


Figure 3.7: Geometry of the gauge symmetry breakdown.

The intersection of the bulk gauge groups with the symmetries at the fixed points yields the gauge group in four dimensions, $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{SO}(10)'$.

Chapter 4

Classification of Orbifold Models

In the last chapter, we considered the case of the $\mathbb{Z}_6$-II orbifold in some detail. The large number of parameters (i.e. the choice of shift vectors and Wilson lines) makes the construction of three-generation models a challenging task and raises the question of a classification of the inequivalent models. Shift vectors and Wilson lines correspond to automorphisms in the gauge symmetry. We will first give an overview of the necessary mathematical background, and then apply these tools to classify $\mathbb{Z}_6$-II orbifold models with one Wilson line.

4.1 Automorphisms of Simple Lie Algebras

Let $\mathfrak{g}$ denote a simple Lie algebra of rank ℓ. To describe its automorphism group $\mathrm{Aut}(\mathfrak{g})$, we first have to introduce some more terminology.

Diagram Automorphisms

The Cartan matrix of $\mathfrak{g}$ is defined by

$$A_{ij} = 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}, \tag{4.1}$$

where α_i with $i = 1, \ldots, \ell$ are the simple roots. Let Γ denote the associated Dynkin diagram. Define the automorphism group of the Dynkin diagram defined to be

$$\mathrm{Aut}(\Gamma) = \left\{ \pi \in S_\ell \;\middle|\; A_{\pi(i)\,\pi(j)} = A_{ij} \right\}, \tag{4.2}$$

i.e. the set of all permutations of the ℓ simple roots such that their scalar product is unchanged. Note that we can visualize $\mathrm{Aut}(\Gamma)$ as the symmetry group of the Dynkin diagram. Since the nodes correspond to the simple roots and the lines joining them to the scalar products of the respective simple roots, every automorphism which leaves the Cartan matrix in eq. (4.1) invariant must also leave the corresponding Dynkin diagram invariant.

Aut(Γ) can naturally be interpreted as a subgroup of Aut($\mathfrak{g}$). Let $\pi \in$ Aut(Γ) be a Dynkin diagram automorphism. Then define the corresponding element in $\sigma_\pi \in$ Aut($\mathfrak{g}$) by specifying its action on a set of step operators $E_{\pm\alpha_i}$ corresponding to the simple roots α_i, $i = 1, \ldots, \ell$:

$$\sigma_\pi(E_{\alpha_i}) = E_{\pi(\alpha_i)}, \qquad \sigma_\pi(E_{-\alpha_i}) = E_{-\pi(\alpha_i)}. \tag{4.3}$$

Inner Automorphisms

An automorphism $\sigma \in$ Aut($\mathfrak{g}$) is called an *inner automorphism*, if it is a product of automorphisms of the form

$$\mu_x \ : \ y \mapsto e^{\mathrm{ad}\, x}\, y, \qquad x \in \mathfrak{g}, \tag{4.4}$$

where the right hand side is defined by its power series expansion and the so-called *adjoint action of x on y* is given by

$$(\mathrm{ad}\ x)(y) \equiv [x, y]\,. \tag{4.5}$$

We will denote the set of all inner automorphisms by Int($\mathfrak{g}$). An element of Aut($\mathfrak{g}$) which is not an inner automorphism is called an *outer automorphism.* Note that the inner automorphisms Int($\mathfrak{g}$) form a normal subgroup of Aut($\mathfrak{g}$).

Automorphisms

The group of automorphisms Aut($\mathfrak{g}$) has the structure of the *semidirect product* of the diagram and inner automorphisms [44]:

$$\mathrm{Aut}(\mathfrak{g}) \simeq \mathrm{Aut}(\Gamma) \ltimes \mathrm{Int}(\mathfrak{g}) \tag{4.6}$$

Thus, the study of Aut($\mathfrak{g}$) can be reduced to the study of the smaller groups Aut(Γ) and Int($\mathfrak{g}$). The reader who is not familiar with the notion of a semidirect product may wish to refer to ref. [45].

Automorphisms of Finite Order

Since we know the diagram automorphisms and the inner automorphisms, eq. (4.6) allows us to give a nice characterization for elements in Aut($\mathfrak{g}$). The following theorem which can be found in ref. [46].

Theorem 4.1.1 *Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra, $\mathfrak{h}$ its Cartan subalgebra and $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ its simple roots. Then every automorphism $\sigma \in Aut(\mathfrak{g})$ which is of order N is conjugate to an automorphism of the form*

$$\sigma = \mu \exp\left(\tfrac{2\pi i}{N}\ ad\ H_\sigma\right) \tag{4.7}$$

such that

$$\mu \in Aut(\Gamma), \qquad H_\sigma \in \mathfrak{h}, \qquad \mu(H_\sigma) = H_\sigma, \qquad \langle \alpha_i, H_\sigma \rangle \in \mathbb{Z}. \tag{4.8}$$

σ is an inner automorphism if and only if $\mu = \mathbb{1}$.

4.2 Theorem of Kač

Theorem 4.1.1 describes all automorphisms of a simple Lie algebra up to conjugation. Unfortunately, the theorem gives no information on how to find H_σ, so it is not suitable for an explicit construction of all automorphisms of a given Lie algebra $\mathfrak{g}$. The following theorem 4.2.1 which is due to V. Kač [46, 47] provides us with the necessary tools.

Before we can formulate the theorem, some notions and definitions have to be introduced.

Kač Labels and Dynkin Diagrams of Affine Lie Algebras

The expansion coefficients of the highest root θ in terms of the simple roots,

$$\theta = a_1\alpha_1 + \ldots + a_\ell\alpha_\ell, \tag{4.9}$$

are called *Kač labels* and will be denoted by a_i, $i = 1, \ldots, \ell$. The Kač labels are tabulated and can be found in the literature [48]. It should be noted that for convenience, the Kač label of the most negative root α_0, which is introduced below, is set to $a_0 = 1$.

The notion of an *extended Dynkin diagram* is important for a couple of reasons, e.g. for deriving the regular subalgebras of a given Lie algebra $\mathfrak{g}$. The extended Dynkin diagram follows from the Dynkin diagram by adjoining the so-called *most negative root* $\alpha_0 \equiv -\theta$, which is the negative of the highest root defined above. The most negative root α_0 is connected to those roots α_i of the Dynkin diagram which have a non-vanishing scalar product $\langle\alpha_0, \alpha_i\rangle \neq 0$.

The extended Dynkin diagrams are sometimes also called *affine Dynkin diagrams* and will be denoted by $\Gamma^{(1)}$. They are listed in the literature [48]. Since we are interested in inner automorphisms, this is all we need. For constructing outer automorphisms, the affine diagrams $\Gamma^{(k)}$ of higher level are needed and these can be found in ref. [46].

We can now formulate the theorem.

Theorem 4.2.1 *Let $\mathfrak{g}$ be a simple Lie algebra, and $s = (s_0, \ldots, s_\ell)$ a sequence of non-negative, relatively prime integers. Put*

$$N = k\sum_{i=0}^{\ell} a_i s_i, \tag{4.10}$$

where a_i are the aforementioned Kač labels. Then the following statements hold:

(i) The relations

$$\sigma_{s,k}(E_{\alpha_j}) = \exp\left(2\pi i s_j/N\right) E_{\alpha_j}, \quad j = 0, \ldots, \ell, \tag{4.11}$$

define uniquely an automorphism $\sigma_{s,k}$ of $\mathfrak{g}$ of order N.

(ii) Up to conjugation by an automorphism of $\mathfrak{g}$, the automorphisms $\sigma_{s,k}$ exhaust all N-th order automorphisms of $\mathfrak{g}$.

(iii) The elements $\sigma_{s,k}$ and $\sigma_{s',k'}$ are conjugate by an automorphism of $\mathfrak{g}$ if and only if $k = k'$ and the sequence s can be transformed into the sequence s' by an automorphism of the diagram $\Gamma^{(k)}$.

Some remarks are in order.

Remarks

(i) The sequence $s = (s_0, \ldots, s_\ell)$ is called relatively prime if there is no integer except 1 which divides all s_i, $i = 0, \ldots, \ell$.

(ii) The automorphisms of a simple Lie algebra can be inner or outer. k is the least positive integer for which $(\sigma_{s,k})^k$ is an inner automorphism. The outer automorphisms of a simple Lie algebra are given by the automorphisms of its diagram Γ. The Dynkin diagrams for E_8 and E_7 have no symmetries, so $k = 1$. The diagrams of A_ℓ, D_ℓ, and E_6 have a 2-fold mirror symmetry, so k can be either 1 or 2. The diagram of D_4 has the symmetry group S_3, so k can be either 1, 2, or 3.

(iii) The automorphism $\sigma_{s,k}$ is of order N, i.e. N is the least positive integer for which $(\sigma_{s,k})^N = \mathbb{1}$. In orbifold constructions, we are interested in all inner automorphisms of order m, where $1 \leq m \leq N$, and m divides N. To this end, we can either determine all m-th order automorphisms separately, or alternatively, we determine all N-th order automorphisms, but drop the condition that the sequence s be relatively prime.

(iv) Two elements $\sigma, \sigma' \in \mathrm{Aut}(\mathfrak{g})$ are called *conjugate*, if there exists an automorphism $\tau \in \mathrm{Aut}(\mathfrak{g})$ such that

$$\sigma' = \tau\,\sigma\,\tau^{-1}. \tag{4.12}$$

In order to find out whether two inner automorphisms are conjugate to each other, we have to consider the diagram automorphisms of $\Gamma^{(1)}$. It is not clear whether the corresponding τ in this case is inner or outer.

(v) The theorem *does not assert* that by the construction we get all automorphisms, where the conjugate ones are given by reordering the s_i according to the symmetries of the affine Dynkin diagram. We obtain a subset of all automorphisms such that we have *at least* one representative from each conjugacy class in this subset. In this subset, there may still be automorphisms, which are conjugate to each other. These are then characterized by theorem 4.2.1.(iii).

4.3 The Gauge Embedding of the Orbifold Twist

Each automorphism $\sigma \in \mathrm{Aut}(\mathfrak{g})$ naturally defines a subalgebra of $\mathfrak{g}$ by

$$\mathfrak{g}_0^\sigma \equiv \{X \in \mathfrak{g} \,|\, \sigma(X) = X\}\,, \tag{4.13}$$

which is sometimes called *fixed-point algebra* of σ. In orbifold constructions, modular invariance requires us to associate the twist in the space-time with an automorphism of the gauge symmetry. The unbroken gauge group after the orbifold compactification is then given by the elements of the gauge symmetry which are invariant under the action of the automorphism, i.e. if $\mathfrak{g}$ is the gauge symmetry, then $\mathfrak{g}_0^\sigma$ is the unbroken gauge group in four dimensions.

In chapters 2 and 3, we described the gauge embedding of the twist by the shift vector V. We will now establish the connection between the automorphism as given in Theorem 4.2.1 and the definition of the shift vector.

It should be noted that in orbifold constructions, the automorphism is usually restricted to be inner. From now on we consider only inner automorphisms. Where appropriate, we point out where this specialization is relevant.

4.3.1 The Shift Vector

Theorem 4.2.1 specifies the action of the automorphism σ on the operators E_α for the simple roots α_i, $i = 1, \ldots, \ell$, and the extended root α_0:

$$E_{\alpha_i} \to \exp\left(2\pi i s_i/N\right) E_{\alpha_i} \tag{4.14}$$

On the other hand, in the orbifold literature, the automorphism is described in terms of the shift vector V by the following equation:

$$E_\alpha \to \exp\left(2\pi i \alpha \cdot V\right) E_\alpha \tag{4.15}$$

It is immediately clear that this implies

$$\alpha_i \cdot V = \frac{s_i}{N}, \quad i = 1, \ldots, \ell \tag{4.16}$$

for the ℓ linearly independent roots α_i. We want to determine V. To that end, we write V in terms of the fundamental weights[1]:

$$V = c_1 \alpha_1^* + \ldots + c_\ell \alpha_\ell^* \tag{4.17}$$

Substituting eq. (4.17) into eq. (4.16), and using $\alpha_i^* \cdot \alpha_j = \delta_{ij}$, we find

$$V = \frac{s_1}{N}\alpha_1^* + \ldots + \frac{s_\ell}{N}\alpha_\ell^*, \tag{4.18}$$

[1] For ADE algebras, the fundamental weights are equal to the dual simple roots. Also note that by definition, the expansion coefficients of a weight in terms of the fundamental weights are the Dynkin labels.

i.e. the integers s_i divided by the order N are the Dynkin labels of V.

To conclude the proof of equivalence between the two descriptions of the automorphism as given by eq. (4.14) and eq. (4.15), we must also check their action on the step operator corresponding to the extended root α_0. Take the former equation and calculate the transformation of E_{α_0}:

$$E_{\alpha_0} \to \exp(2\pi i s_0/N)\, E_{\alpha_0} \tag{4.19}$$

We use eq. (4.10) to express s_0 in terms of the other s_i, and noting that $a_0 = 1$ by definition and $k = 1$ for inner automorphisms, we have

$$1 \cdot s_0 = N - (a_1 s_1 + \ldots a_\ell s_\ell). \tag{4.20}$$

Now consider eq. (4.15) which gives the transformation of E_{α_0} in terms of the shift V:

$$E_{\alpha_0} \to \exp(2\pi i \alpha_0 \cdot V)\, E_{\alpha_0} \tag{4.21}$$

Using eq. (4.9) and eq. (4.18), we can easily evaluate the scalar product in the exponent:

$$\alpha_0 \cdot V = -(a_1\alpha_1 + \ldots + a_\ell\alpha_\ell) \cdot \frac{1}{N}(s_1\alpha_1^* + \ldots + s_\ell\alpha_\ell^*) = -\frac{1}{N}(a_1 s_1 + \ldots + a_\ell s_\ell) \tag{4.22}$$

Taking into account the definition of s_0 as given by eq. (4.16), we find

$$s_0 = -(a_1 s_1 + \ldots + a_\ell s_\ell). \tag{4.23}$$

Comparing eq. (4.20) to eq. (4.23), we conclude that they describe the same transformation, since numbers modulo N in the exponent are irrelevant.

Obtaining V in the Cartan-Weyl Basis

For practical calculations, we need V in the Cartan-Weyl basis. In principle, we could calculate the expressions for the dual simple roots α_i^* in the Cartan-Weyl basis, and then eq. (4.18) would give the desired expression for V.
Technically, it is more convenient to work in the basis of the simple roots. The simple roots are known, both in the Dynkin and the Cartan-Weyl basis. We decompose the expression of V in the Dynkin basis in terms of the simple roots in the Dynkin basis. The coefficients we obtain will be the same, if we now change to the Cartan-Weyl basis, i.e. in the expansion, we now insert the expressions for the simple roots in the Cartan-Weyl basis, and we obtain the expression for the shift vector V in the Cartan-Weyl basis.

4.3.2 Determining the Unbroken Gauge Group

The shift vector is a handy tool for determining the unbroken gauge group. For a given Lie algebra $\mathfrak{g}$, the transformation of the step operators is given by

$$\sigma\ :\ E_\alpha \mapsto \exp(2\pi i \alpha \cdot V)\, E_\alpha, \tag{4.24}$$

so only those step operators E_α for which

$$\alpha \cdot V = 0 \bmod 1 \tag{4.25}$$

will be invariant under the automorphism. To see how the Cartan generators transform, we consider theorem 4.1.1. The Cartan generators will transform according to eq. (4.7), and since we consider only inner automorphisms, we have $\mu = \mathbb{1}$:

$$\sigma \,:\, H_i \mapsto \exp\left(\tfrac{2\pi i}{N}\ \mathrm{ad}\ H_\sigma\right)\ H_i \tag{4.26}$$

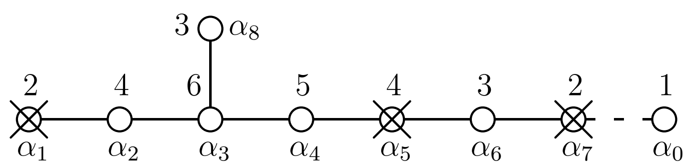
Since Cartan generators mutually commute, the action of $\mathrm{ad}\, H_\sigma$ on H_i will be trivial, and all Cartan generators will survive.

This explicitly proves that *inner automorphisms cannot reduce the rank.* This statement is independent of the gauge embedding being realized as a shift (chapters 2-4) or as a rotation in the root lattice (chapters 5-7). In the latter case, we will show how the interplay of the rotation in the root lattice with the concept of background fields nevertheless leads to rank reduction.

A Diagrammatic Approach to Symmetry Breaking

In fact, determining the unbroken gauge group is even easier than described above. Given an n-tuple $(s_0, s_1, \ldots, s_\ell)$, the fixed-point algebra under the corresponding automorphism $\sigma \in \mathrm{Int}(\mathfrak{g})$ is obtained from the Dynkin diagram of $\mathfrak{g}$ by deleting the nodes corresponding to $s_i \neq 0$. The order of the automorphism will be given by eq. (4.10).

An example is in order. Assume e.g. $\mathfrak{g} = \mathrm{E}_8$ and $s = (0, 1, 0, 0, 0, 3, 0, 2, 0)$:



The unbroken gauge group is

$$\mathrm{SO}(8) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)^2. \tag{4.27}$$

The numbers next to each node are the Kač labels. The order of the automorphism is then

$$N = \sum_{i=0}^{\ell} s_i a_i = 1 \cdot 2 + 3 \cdot 4 + 2 \cdot 2 = 18. \tag{4.28}$$

Since the embedding of the twist into the gauge degrees of freedom is a homomorphism and there is no point group with order greater than 12 (cf. tab. 2.1), the constructed automorphism cannot be realized in an orbifold construction.

4.4 Inequivalent $\mathbb{Z}_6$-II Orbifold Models

We will now apply the techniques which we described in the previous sections to specific examples. Since we studied in chapter 3 the case of the $\mathbb{Z}_6$-II orbifold in great detail, we start with classifying its inequivalent gauge shifts. Since the gauge symmetry of the orbifold is the direct product

$$\mathrm{E}_8 \times \mathrm{E}_8' \tag{4.29}$$

of two simple Lie algebras, we break up the study of its automorphisms to the study of the automorphisms of each factor.

4.4.1 Automorphisms of E_8

Theorem 4.2.1 allows us to construct the automorphisms of E_8 of a given order N. First note that we only consider inner automorphisms, thus $k = 1$. Second, since the gauge embedding is a homomorphism, the order of the automorphism must divide the order of the twist, so we only need to consider the cases $N = 1, 2, 3, 6$. Third, we note that the extended Dynkin diagram of E_8 has no symmetries, so that all automorphisms which we construct will be inequivalent.

The Construction

Without loss of generality, assume $N = 6$. The Kač labels of E_8 are

$$(a_0, a_1, \ldots, a_8) = (1, 2, 4, 6, 5, 4, 3, 2, 3). \tag{4.30}$$

Eq. (4.10) instructs us to find all $s = (s_0, s_1, \ldots, s_8)$ such that the s_i are non-negative integers, relatively prime and

$$1 \cdot s_0 + 2 \cdot s_1 + 4 \cdot s_2 + 6 \cdot s_3 + 5 \cdot s_4 + 4 \cdot s_5 + 3 \cdot s_6 + 2 \cdot s_7 + 3 \cdot s_8 = 6. \tag{4.31}$$

Each such 9-tuple s corresponds to a shift vector

$$V = \frac{1}{6}(s_1\,\alpha_1^* \,+\, \ldots \,+\, s_8\,\alpha_8^*). \tag{4.32}$$

To find the fundamental weights α_i^* in the Cartan-Weyl basis, collect the simple roots α_i as given in tab. E.1 in the rows of a matrix M. Then the rows of the matrix $(M^{-1})^T$ are the dual simple roots, which are for algebras of type ADE equal to the fundamental weights.

The consistency of the construction (cf. section 2.4.4) requires $N\,V$ to lie in the $\mathrm{E}_8 \times \mathrm{E}_8'$ root lattice. Since E_8 is *self-dual*, the dual simple roots α_i^* are in the root lattice, so it follows from eq. (4.32) that $6\,V$ is also in the lattice.

No.	Shift Vector								Algebra
1	1/6	5/6	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	A_8
2	1/4	1/4	-1/4	-1/4	-1/4	-1/4	-1/4	-1/4	$E_7 + A_1$
3	1/3	2/3	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	$A_5 + A_2$
4	1/2	1/2	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	$E_6 + A_2$
5	1/3	1/2	-1/6	-1/6	-1/6	-1/6	-1/6	-1/3	$D_5 + A_1 + A_1$
6	1/4	3/4	-1/12	-1/12	-1/12	-1/4	-1/4	-1/4	$A_5 + A_2 + A_1$
7	1/4	7/12	-1/12	-1/12	-1/4	-1/4	-1/4	-1/4	$A_5 + A_1 + A_1$
8	1/4	5/12	-1/12	-1/4	-1/4	-1/4	-1/4	-1/4	$D_6 + A_1$
9	1/3	2/3	0/1	-1/6	-1/6	-1/6	-1/6	-1/3	$D_4 + A_3$
10	1/4	3/4	1/12	-1/12	-1/4	-1/4	-1/4	-1/4	A_7
11	1/4	7/12	1/12	-1/4	-1/4	-1/4	-1/4	-1/4	$D_6 + A_1$
12	1/4	3/4	1/4	-1/4	-1/4	-1/4	-1/4	-1/4	D_8
13	1/6	1/2	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	A_6
14	1/3	1/3	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	E_6
15	1/4	7/12	-1/12	-1/12	-1/12	-1/12	-1/4	-1/4	$A_4 + A_3$
16	1/6	2/3	0/1	-1/6	-1/6	-1/6	-1/6	-1/6	A_6
17	1/3	1/2	0/1	-1/6	-1/6	-1/6	-1/6	-1/6	$D_5 + A_1$
18	1/6	1/6	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	E_7
19	1/4	5/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/4	$D_5 + A_2$
20	1/6	1/2	0/1	0/1	-1/6	-1/6	-1/6	-1/6	$A_6 + A_1$
21	1/6	1/3	0/1	-1/6	-1/6	-1/6	-1/6	-1/6	D_6
22	1/6	1/2	1/6	-1/6	-1/6	-1/6	-1/6	-1/6	D_7
23	1/12	5/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	A_7
24	1/4	1/4	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	$E_6 + A_1$
25	1/12	1/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	E_7
26	1/12	1/4	1/12	-1/12	-1/12	-1/12	-1/12	-1/12	D_7
27	0/1	0/1	0/1	0/1	0/1	0/1	0/1	0/1	E_8

Table 4.1: Inequivalent $\mathbb{Z}_6$ gauge shifts.

The Trick

We could repeat the above procedure separately with $N = 1$, $N = 2$ and $N = 3$, but there is a more efficient way to do it. The condition that the s_i be relatively prime assures that we obtain automorphisms which are exactly of order 6 and not lower. Assume that we drop this condition and that the greatest common divisor of the s_i is 2. Then the corresponding shift vector

$$V = \frac{1}{6}(s_1\,\alpha_1^* + \ldots + s_8\,\alpha_8^*) = \frac{1}{3}(s_1'\,\alpha_1^* + \ldots + s_8'\,\alpha_8^*) \tag{4.33}$$

can be written in terms of the $s_i' = s_i/2$ which are now relatively prime, and theorem 4.2.1 tells us that the shift corresponds to an automorphism of order 3. In the same way, we also obtain the automorphisms of order 2 and 1. (The automorphism of order 1 is, of course, the identity.)

Carrying out this construction, we find the 27 shift vectors listed in tab. 4.1.

4.4.2 Automorphisms of $\mathbf{E}_8 \times \mathbf{E}_8'$

So far, we have a characterization of $\mathrm{Aut}(\mathrm{E}_8)$, but for the orbifold constructions, we need to know $\mathrm{Aut}(\mathrm{E}_8 \times \mathrm{E}_8')$.

Let $\mathfrak{a}$, $\mathfrak{b}$ be two groups and let $\mathfrak{a}\times\mathfrak{b}$ denote their direct product. Remember that the group operation on $\mathfrak{a}\times\mathfrak{b}$ is given by

$$(a_1, b_1) \star (a_2, b_2) = (a_1 \circ a_2, b_1 \circ b_2), \tag{4.34}$$

where $\circ$ denotes the group operation in $\mathfrak{a}$ and $\mathfrak{b}$. In general, the automorphism group of the direct product will not be equal to the direct product of the automorphism groups, but rather

$$\mathrm{Aut}(\mathfrak{a}) \times \mathrm{Aut}(\mathfrak{b}) \subset \mathrm{Aut}(\mathfrak{a}\times\mathfrak{b}), \tag{4.35}$$

where the right hand side may or may not be a proper subset of the left hand side. However, for inner automorphisms, we can go one step further.

By definition, an inner *group* automorphism acts on $\mathfrak{a}$ by conjugation with an arbitrary, but fixed element,

$$\sigma_x \,:\, \mathfrak{a} \to \mathfrak{a}, \qquad a \mapsto x \circ a \circ x^{-1}, \tag{4.36}$$

and an analogous statement holds for the group $\mathfrak{b}$:

$$\sigma_y \,:\, \mathfrak{b} \to \mathfrak{b}, \qquad b \mapsto y \circ b \circ y^{-1}, \tag{4.37}$$

Now consider an inner automorphism on $\mathfrak{a}\times\mathfrak{b}$:

$$\sigma \,:\, \mathfrak{a}\times\mathfrak{b} \to \mathfrak{a}\times\mathfrak{b}, \qquad (a,b) \mapsto (x,y)\star(a,b)\star(x,y)^{-1} \tag{4.38}$$

Using the group operation on $\mathfrak{a}\times\mathfrak{b}$, we can evaluate the right hand side and obtain

$$(x,y)\star(a,b)\star(x,y)^{-1} = (x\circ a\circ x^{-1}, y\circ b\circ y^{-1}), \tag{4.39}$$

which shows that every inner automorphism on $\mathfrak{a} \times \mathfrak{b}$ is given by a pair of inner automorphisms, one on $\mathfrak{a}$, and the other on $\mathfrak{b}$. This proves

$$\mathrm{Int}(\mathfrak{a}) \times \mathrm{Int}(\mathfrak{b}) \supset \mathrm{Int}(\mathfrak{a} \times \mathfrak{b}), \tag{4.40}$$

and taking eq. (4.35) into account, we can conclude that

$$\mathrm{Int}(\mathfrak{a}) \times \mathrm{Int}(\mathfrak{b}) \simeq \mathrm{Int}(\mathfrak{a} \times \mathfrak{b}). \tag{4.41}$$

Interpreting this result for our case, we find that every inner automorphism on $\mathrm{E}_8 \times \mathrm{E}_8'$ is such that its restriction on E_8 and E_8' is an inner automorphism on the respective factor and can thus be described by a 8+8 dimensional shift vector which is conveniently denoted by

$$V = (V_1)\,(V_2), \tag{4.42}$$

where each V_i, $i = 1, 2$, corresponds to an inner automorphism in the sense of section 4.4.1.

4.4.3 Modular Invariance

However, not all of the $27 \times 27 = 729$ combinations correspond to admissible choices. For the $\mathbb{Z}_6$-II orbifold with

$$N = 6, \qquad v = (\tfrac{1}{6}, \tfrac{1}{3}, -\tfrac{1}{2}), \tag{4.43}$$

the modular invariance conditions eq. (2.26) read

$$N\left(V^2 - v^2\right) = 0 \bmod 2 \qquad \rightsquigarrow \qquad 6\left(V^2 - \tfrac{7}{18}\right) = 0 \bmod 2, \tag{4.44}$$

giving only 61 consistent gauge embeddings. We list the corresponding shift vectors and the unbroken gauge groups in appendix E.

The $\mathbb{Z}_6$-I vs. the $\mathbb{Z}_6$-II Point Group. Note that it is the modular invariance condition which distinguishes between the $\mathbb{Z}_6$-I and $\mathbb{Z}_6$-II orbifolds. For both orbifolds, the automorphisms are of order 1,2,3 or 6, but the value of v^2 is different.

The choice of V is not unique. The Weyl group[2] of E_8 is the symmetry group of the associated root lattice. As a consequence, two shifts V and V' which are related by the action of a Weyl group element will yield not only the same symmetry breakdown but also the same spectra. Since the Weyl group is generated by reflections, the length of V will be equal to the length of V', and either both shifts fulfill the modular invariance condition eq. (4.44), or none of them does. Thus, in the case of no Wilson lines, this freedom of choice is not relevant.

[2]We will have much more to say about the Weyl group in chapters 5-7. For a definition, the reader may want to refer to section 5.2.

4.5 Classification of Orbifold Models in the Presence of Wilson Lines

From the viewpoint of group theory, Wilson lines are additional shifts which cause a further symmetry breakdown of the gauge group. Thus, they can be classified using the very same techniques developed in the previous sections.

Unfortunately, the consistency conditions of the orbifold construction spoil this simple picture. First, we list the relevant conditions and indicate the problems. Then, we describe the construction of the automorphisms and the corresponding Wilson lines by means of examples rather than giving an abstract overview of the algorithms.

4.5.1 The Consistency Conditions

The Gauge Embedding is a Group Homomorphism

As explained in section 2.4.4, for a twist of order N we require

$$N\,V \in T_{\mathrm{E}_8\times\mathrm{E}_8'} \quad \text{and} \quad N\,A_\alpha \in T_{\mathrm{E}_8\times\mathrm{E}_8'}. \tag{4.45}$$

Whereas $N\,V$ is in the root lattice by construction (cf. section 4.4.1), this will not necessarily be true for NA_α. However, by adding fundamental weights which correspond to U(1) directions, one can restore NA_α to lie in the lattice.

Modular Invariance

Specializing the modular invariance conditions given in section 2.4.5 to the case of the $\mathbb{Z}_6$-II orbifold, we get

$$6\left(V^2 - v^2\right) = 0 \bmod 2, \tag{4.46}$$
$$6\,V\cdot A_\alpha = 0 \bmod 1, \tag{4.47}$$
$$N'\,A_\alpha\cdot A_\beta = 0 \bmod 1, \quad \alpha\neq\beta, \tag{4.48}$$
$$N''\,A_\alpha^2 = 0 \bmod 2, \tag{4.49}$$

where N' is the lowest common multiple of the orders of A_α and A_β, and N'' is the order of A_α.

Remember the discussion at the end of section 4.4.3, where we pointed out that two shift vectors V and V' which are related by the action of a Weyl group element describe the same orbifold model. Such two shift vectors correspond to automorphisms which are related by conjugation. Since theorem 4.2.1 gives us not all automorphisms, but only representatives of each conjugacy class,[3] it can happen that we construct the shift vector, say, V, but miss V'. In the case of no Wilson lines, this is not relevant, since elements of the Weyl group are

[3]Note that the theorem may give more than one representative of each conjugacy class.

generated by reflections and these do not change the length of vectors, so that eq. (4.46) is either fulfilled by both shift vectors or by none of them.

In the presence of Wilson lines, the situation is different. Assume that we have constructed a Wilson line A_α. Then, $V' \cdot A_\alpha$ may fulfill eq. (4.47) whereas the other combination $V \cdot A_\alpha$ may not. Moreover, the Wilson line itself is determined only up to an element of the Weyl group of the *unbroken gauge group*. As a consequence, we have to check the modular invariance conditions not only for V and A_α, but for all combinations of V' and A'_α. Note that elements of the Weyl group do not change lengths, so eq. (4.49) is unaffected.

There is yet another complication, namely the two Wilson lines A_α and $A_\alpha + \lambda V$, $\lambda \in \mathbb{Z}$, induce the same gauge symmetry breakdown, since A_α acts on those roots of the algebra which have survived the projection by V. This operation does not only change the direction of A_α in root space, but also affects its length, so for one choice of $\lambda \in \mathbb{Z}$, eq. (4.49) may be fulfilled, whereas for another choice it may not.
Still, this is not the most general term one may add to A_α without changing the gauge symmetry. Consider the orthogonal complement of the roots surviving the projection by V. Adding an element in this complement to A_α does not affect the gauge symmetry. Note that the orthogonal complement corresponds to U(1) directions in the unbroken gauge group, and choosing different linear combinations of U(1) directions corresponds to different embeddings of the unbroken gauge group in E_8.

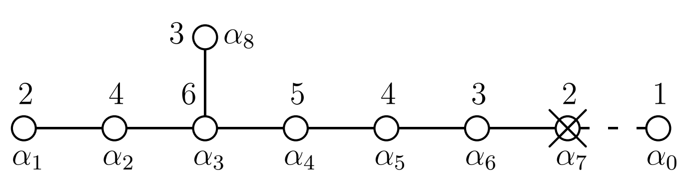
4.5.2 Classification of $\mathbb{Z}_6$-II Orbifold Models with 1 Wilson Line

We break up the problem of finding the shifts and Wilson lines for $\mathrm{E}_8 \times \mathrm{E}'_8$ to the study of a single E_8 factor. We already know that there are 27 inequivalent $\mathbb{Z}_6$ shifts which are given in tab. 4.1. To find all admissible Wilson lines, we have to determine the unbroken gauge group $\mathfrak{g}$ corresponding to each shift, and then consider the group of inner automorphisms of $\mathfrak{g}$.

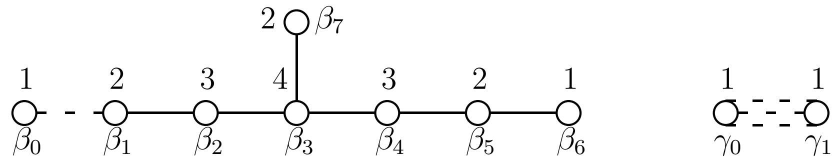
Without loss of generality, assume that we consider the second shift in tab. 4.1:

$$V = \left(\tfrac{1}{4}, \tfrac{1}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}, -\tfrac{1}{4}\right) = \tfrac{1}{6}[0, 0, 0, 0, 0, 0, 3, 0]_{\mathrm{DL}} \qquad (4.50)$$

We present the shift both in the Cartan-Weyl and in the Dynkin basis. The unbroken gauge group is given by the Dynkin diagram



and corresponds to $\mathfrak{g} = \mathrm{E}_7 \times \mathrm{SU}(2)$. The inner automorphisms of $\mathfrak{g}$ will be given by the direct product of the inner automorphism groups of E_7 and of SU(2). We draw the corresponding extended Dynkin diagrams with the associated Kač labels:



We determine all n-tuples

$$s = (s_0, \dots, s_7) \quad \text{and} \quad r = (r_0, r_1) \tag{4.51}$$

with non-negative integers s_i, r_i such that

$$1 \cdot s_0 + 2 \cdot s_1 + 3 \cdot s_2 + \dots + 2 \cdot s_7 = 6 \quad \text{and} \quad 1 \cdot r_0 + 1 \cdot r_1 = 6 \tag{4.52}$$

and construct the associated shift vectors (which are in this case actually the Wilson lines)

$$A_1 = \frac{1}{6}(s_1 \beta_1^* + \dots + s_7 \beta_7^*) \quad \text{and} \quad A_2 = \frac{1}{6}(r_1 \gamma_1^* + \dots + r_1 \gamma_1^*). \tag{4.53}$$

Note that $A_1 \perp A_2$, because every dual simple root can be expressed as a linear combination of the simple roots[4] and the two diagrams are disconnected, thus implying that each simple root in one diagram is orthogonal to every simple root in the other diagram. For the same reason,

$$A_1 \perp \gamma_i \quad \text{for all} \quad i = 1, \dots, 2 \qquad \text{and} \qquad A_2 \perp \beta_j \quad \text{for all} \quad j = 1, \dots, 7, \tag{4.54}$$

or in other words, A_1 acts as the identity on the second diagram and A_2 acts as the identity on the first diagram. This allows us to write the Wilson line for the direct product as

$$A = A_1 + A_2. \tag{4.55}$$

This gives the Wilson lines for one E_8 factor. To find the Wilson lines for $E_8 \times E_8'$, we first split up the shift $V = (V_1)\,(V_2)$ into its components acting on the first and the second E_8, respectively. Then we determine separately all Wilson lines for V_1 and for V_2. Next, we combine all Wilson lines from the first set with all Wilson lines from the second set to form 16-vectors acting on $E_8 \times E_8'$. If the modular invariance conditions are not fulfilled, we may perform the following operations on the components of the shifts and Wilson lines in each E_8 factor (cf. section 4.5.1):

1. Change the shift by the action of an element $w \in \mathcal{W}(E_8)$ in the Weyl group of E_8.
2. Change the Wilson line by the action of an element $w \in \mathcal{W}(\mathfrak{g})$ in the Weyl group of $\mathfrak{g}$, which is the unbroken gauge group surviving the projection by V.
3. Change the Wilson line by adding a linear combination of U(1) directions.

We illustrate this procedure by means of specific examples.

[4] The rows of the inverse Cartan matrix are the expansion coefficients of the fundamental weights in terms of the simple roots.

Rotating the Shift Saves Modular Invariance

Consider the following combination of shift vector and Wilson line:

$$V = \left(\tfrac{5}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}\right) \left(\tfrac{1}{2}, \tfrac{1}{2}, 0, 0, 0, 0, 0, 0\right) \tag{4.56}$$

$$A = \left(\tfrac{1}{3}, \text{-}\tfrac{1}{3}, \text{-}\tfrac{1}{3}, \text{-}\tfrac{1}{3}, \text{-}\tfrac{1}{3}, \text{-}\tfrac{1}{3}, \text{-}\tfrac{1}{3}, \text{-}\tfrac{1}{3}\right) \left(0, \text{-}1, \tfrac{1}{3}, 0, 0, 0, 0, 0\right) \tag{4.57}$$

The shift vector corresponds to model 1 in appendix E. Checking the modular invariance conditions given by eqs. (4.46-4.49), we find:

$$6\left(V^2 - v^2\right) = 0 \bmod 2 \quad \rightsquigarrow \quad 6\left(\tfrac{25}{18} - \tfrac{7}{18}\right) = 6 \ ✔ \tag{4.58}$$

$$6\, V \cdot A = 0 \bmod 1 \quad \rightsquigarrow \quad 6\, V \cdot A = -\tfrac{11}{3} \ ✗ \tag{4.59}$$

$$3\, A^2 = 0 \bmod 2 \quad \rightsquigarrow \quad 3\, A^2 = 2 \ ✔ \tag{4.60}$$

Rotating V, we find an equivalent shift vector

$$V' = \left(0, 0, \text{-}\tfrac{1}{3}, \text{-}\tfrac{1}{3}, \text{-}\tfrac{1}{3}, \text{-}\tfrac{1}{3}, \text{-}\tfrac{2}{3}, 0, \tfrac{1}{4}\right) \left(\tfrac{1}{4}, \text{-}\tfrac{1}{4}, \text{-}\tfrac{1}{4}, \text{-}\tfrac{1}{4}, \text{-}\tfrac{1}{4}, \text{-}\tfrac{1}{4}, \text{-}\tfrac{1}{4}\right) \tag{4.61}$$

which fulfills all modular invariance conditions:

$$6\left(V^2 - v^2\right) = 0 \bmod 2 \quad \rightsquigarrow \quad 6\left(\tfrac{25}{18} - \tfrac{7}{18}\right) = 6 \ ✔ \tag{4.62}$$

$$6\, V \cdot A = 0 \bmod 1 \quad \rightsquigarrow \quad 6\, V \cdot A = 2 \ ✔ \tag{4.63}$$

$$3\, A^2 = 0 \bmod 2 \quad \rightsquigarrow \quad 3\, A^2 = 2 \ ✔ \tag{4.64}$$

Adding U(1)'s Saves Modular Invariance

Consider the shift vector and the Wilson line

$$V = \left(\tfrac{5}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}\right) \left(\tfrac{2}{3}, \tfrac{1}{6}, \tfrac{1}{6}, 0, 0, 0, 0, 0\right), \tag{4.65}$$

$$A = \left(\tfrac{1}{3}, \text{-}\tfrac{1}{3}, \text{-}\tfrac{1}{3}, \text{-}\tfrac{1}{3}, \text{-}\tfrac{1}{3}, \text{-}\tfrac{1}{3}, \text{-}\tfrac{1}{3}, \text{-}\tfrac{1}{3}\right) \left(\tfrac{19}{6}, \tfrac{7}{6}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right). \tag{4.66}$$

The shift vector corresponds to model 3 in appendix E. Checking the modular invariance conditions, we see that two of them are not satisfied:

$$6\left(V^2 - v^2\right) = 0 \bmod 2 \quad \rightsquigarrow \quad 6\left(\tfrac{25}{18} - \tfrac{7}{18}\right) = 6 \ ✔ \tag{4.67}$$

$$6\, V \cdot A = 0 \bmod 1 \quad \rightsquigarrow \quad 6\, V \cdot A = \tfrac{41}{3} \ ✗ \tag{4.68}$$

$$3\, A^2 = 0 \bmod 2 \quad \rightsquigarrow \quad 3\, A^2 = \tfrac{124}{3} \ ✗ \tag{4.69}$$

Fortunately, we have two U(1) directions to play with, since the shift V breaks

$$\mathrm{E}_8 \times \mathrm{E}_8' \to \mathrm{SU}(9) \times \mathrm{SO}(10)' \times \mathrm{SU}(2)' \times \mathrm{U}(1)'^2. \tag{4.70}$$

Adding the U(1) direction

$$U = \tfrac{1}{3}\,(0,\,0,\,0,\,0,\,0,\,0,\,0,\,0,\,0)\;(4,\,0,\,0,\,0,\,0,\,0,\,0,\,0) \tag{4.71}$$

to the Wilson line, $A \to A' = A + U$, changes both its length and its direction, and the new Wilson line A' together with the shift V which is unchanged now fulfill the modular invariance conditions:

$$6\left(V^2 - v^2\right) = 0 \bmod 2 \qquad \rightsquigarrow \qquad 6\left(\tfrac{25}{18} - \tfrac{7}{18}\right) = 6 \;\checkmark \tag{4.72}$$

$$6\,V \cdot A = 0 \bmod 1 \qquad \rightsquigarrow \qquad 6\,V \cdot A = 19 \;\checkmark \tag{4.73}$$

$$3\,A^2 = 0 \bmod 2 \qquad \rightsquigarrow \qquad 3\,A^2 = 72 \;\checkmark \tag{4.74}$$

Chapter 5

Continuous Wilson Lines and Rank Reduction

The constructions considered so far have one obvious drawback, namely the rank of the gauge group can never be lowered, thus making it impossible to obtain realistic gauge groups in the low-energy limit. This shortcoming can be traced back to our embedding the space-time twist of the orbifold as a shift in the gauge degrees of freedom. With this chapter, we start pursuing another line of thought, namely realizing the twist in the space-time degrees of freedom as a rotation in the root lattice of the respective algebra. It will turn out that in combination with Wilson lines, this will lead to rank reduction. This chapter concentrates on the main ideas and technical details, deferring the construction of explicit models to the next chapter.

5.1 Embedding the Twist as a Rotation

In orbifold constructions of the heterotic string, modular invariance requires the twist in the space-time to be accompanied by an action on the gauge degrees of freedom, more specifically on the generators of the gauge symmetry, in our case $\mathrm{E}_8 \times \mathrm{E}_8'$. In other words, the twist is accompanied by an element of the automorphism group of the gauge symmetry, and in the literature, there is usually the additional assumption that the automorphism is inner:

$$P \subset \mathrm{O}(6) \;\;\hookrightarrow\;\; G \subset \mathrm{Int}(\mathrm{E}_8 \times \mathrm{E}_8') \tag{5.1}$$

In chapter 4, we explained in great detail how any inner automorphism can be represented by a shift on the $\mathrm{E}_8 \times \mathrm{E}_8'$ root lattice, and exploited this fact to classify orbifold models.
It may seem more natural to realize the twist in the space-time directly as a twist in the root lattice of the gauge group. In the following chapters, we will pursue this line of thought and explore its consequences. The root lattice of $\mathrm{E}_8 \times \mathrm{E}_8'$ defines a 16-torus in a way completely analogous to the space-time torus entering the definition of an orbifold. For

the construction of modding out the twist to be consistent, we demand that it maps the 16-torus onto itself. This requirement is tantamount to the twist being a lattice automorphism, i.e. its preserving the scalar products between root vectors spanning the torus. Such automorphisms of a root lattice constitute its Weyl group, which we denote by $\mathcal{W}$. In a next step, this element of the Weyl group which has been assigned to the space-time twist has to be lifted to an automorphism of the algebra:

$$P \subset \mathrm{O}(6) \quad \hookrightarrow \quad \mathcal{W} \quad \overset{\text{lift}}{\hookrightarrow} \quad G \subset \mathrm{Int}(\mathrm{E}_8 \times \mathrm{E}_8') \tag{5.2}$$

Since the gauge symmetry $\mathrm{E}_8 \times \mathrm{E}_8'$ is the direct product of two groups, the corresponding root lattices are orthogonal to each other. Consequently, the elements of the Weyl group will preserve this structure, because they are lattice automorphisms. Thus, they either map each root lattice onto itself or exchange the two lattices as a whole. The latter case corresponds to exchanging the visible and hidden sectors, and is a matter of choice. From this reasoning we learn that we can concentrate on one E_8 at a time. The automorphisms of $\mathrm{E}_8 \times \mathrm{E}_8'$ will then be given by all possible combinations of the automorphisms of the factor groups, where, of course, further consistency conditions arising from the orbifold construction will have to be taken into account.

The discussion in this chapter is fairly general, and applies to all simply laced, semi-simple Lie algebras. To present the developed methods in full detail, we will specialize from section 5.5 on to the case of SO(10).

5.2 Lattice and Algebra Automorphisms

Given a Lie algebra $\mathfrak{g}$, the corresponding Weyl group $\mathcal{W}$ is by definition generated by the reflections r_α, where α is a root. (Actually, one can constrain oneself to the simple roots.) The scalar product preserving symmetry group of the root lattice (sometimes called *lattice automorphisms*), $S(\Delta)$, is related to the Weyl group $\mathcal{W}$ by the symmetries of the Dynkin diagram, $D(\mathfrak{g})$ [48]:

$$D(\mathfrak{g}) = S(\Delta)/\mathcal{W} \tag{5.3}$$

Thus, if the Dynkin diagram of an algebra has no symmetries, the lattice automorphisms are given by the Weyl group $\mathcal{W}$. The embedding of the orbifold twist into the gauge degrees of freedom as a rotation is an automorphism of the lattice Λ, and can thus be described by an element of $\mathcal{W}$.

The automorphism of the lattice can be lifted to an automorphism of the algebra. To this end, we observe that the Weyl group $\mathcal{W}$ is generated by the so called *simple Weyl reflections*

$$r_{\alpha_i} : \beta \mapsto \beta - 2\,\frac{\langle \beta, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\,\alpha_i, \tag{5.4}$$

where α_i are the simple roots of the algebra $\mathfrak{g}$. The action of any Weyl reflection on the Lie algebra lattice can be lifted to an action on the Lie algebra itself,

$$r_{\alpha_i} : \beta \mapsto r_{\alpha_i}\beta \quad \rightsquigarrow \quad E_\beta \mapsto \tilde{r}_{\alpha_i} E_\beta \tilde{r}_{\alpha_i}^{-1}, \tag{5.5}$$

where the lift $\tilde{r}_{\alpha_i}$ is defined as

$$\tilde{r}_{\alpha_i} = \exp\left(i\frac{\pi}{2}\left(E_{\alpha_i} + E_{-\alpha_i}\right)\right). \tag{5.6}$$

To calculate the lift of an arbitrary element of $w \in \mathcal{W}$, write w in terms of the Weyl reflections. The lift of w will then be given by k consecutive lifts of Weyl reflections:

$$w = r_1 r_2 \dots r_k \quad \rightsquigarrow \quad \tilde{r}_1\tilde{r}_2 \dots \tilde{r}_k E_\beta \tilde{r}_k^{-1} \dots \tilde{r}_2^{-1}\tilde{r}_1^{-1} \tag{5.7}$$

5.3 The Transformation of the Generators Under the Lift

In the following, we will show that the generators of the Lie algebra transform as

$$\begin{aligned} &\tilde{r}_\beta \left(\lambda \cdot H\right) \tilde{r}_\beta^{-1} = \left(r_\beta(\lambda)\right) \cdot H, \\ &\tilde{r}_\beta E_\alpha \tilde{r}_\beta^{-1} = c_\beta(\alpha)\, E_{r_\beta(\alpha)}, \end{aligned} \tag{5.8}$$

under the lift of a single Weyl reflection r_β. The complex phases $c_\beta(\alpha)$ will be explicitly evaluated. For our calculations, we will make frequent use of the Baker-Campbell-Hausdorff formula

$$e^A B\, e^{-A} = \sum_{m=0}^{\infty} \frac{1}{m!} \left[A, B\right]_m, \quad \left[A, B\right]_m \equiv \left[A, \left[A, \dots, \left[A, B\right]\right]\right]. \tag{5.9}$$

Also, we want to remind the reader of the commutation relations in the Cartan-Weyl basis:

$$\begin{aligned} [H_i, H_j] &= 0 \\ [H_i, E_\alpha] &= \alpha^i E_\alpha \\ [E_\alpha, E_\beta] &= N_{\alpha,\beta} E_{\alpha+\beta} \quad \text{if } \alpha+\beta \in \Delta \\ &= \frac{2}{|\alpha|^2}\, \alpha \cdot H \quad \text{if } \alpha = -\beta \\ &= 0 \qquad\qquad \text{otherwise} \end{aligned} \tag{5.10}$$

In this section, we will assume that all roots are of length-squared 2.

5.3.1 The Transformation of the Step Operators E_α

To evaluate the expression

$$\tilde{r}_\beta E_\alpha \tilde{r}_\beta^{-1} = \exp\left(i\frac{\pi}{2}\left(E_\beta + E_{-\beta}\right)\right) E_\alpha \exp\left(-i\frac{\pi}{2}\left(E_\beta + E_{-\beta}\right)\right), \tag{5.11}$$

we set

$$A = \frac{i\pi}{2}\left(E_\beta + E_{-\beta}\right), \quad B = E_\alpha, \tag{5.12}$$

in eq. (5.9) and calculate its right hand side:

$$\exp\left(i\frac{\pi}{2}\left(E_\beta + E_{-\beta}\right)\right) E_\alpha \exp\left(-i\frac{\pi}{2}\left(E_\beta + E_{-\beta}\right)\right) = \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{i\pi}{2}\right)^m \left[E_\beta + E_{-\beta}, E_\alpha\right]_m \tag{5.13}$$

For the following calculations, it will be convenient to extend the definition of the structure constants in the sense that $N_{\alpha,\beta} = 0$, if $\alpha + \beta \notin \Delta$. Strictly speaking, the structure constants are not defined in this case.

Assume $\alpha \neq \pm\beta$.

<u>$m = 0$</u>:

$$\left[E_\beta + E_{-\beta}, E_\alpha\right]_0 = E_\alpha$$

<u>$m = 1$</u>:

$$\left[E_\beta + E_{-\beta}, E_\alpha\right]_1 = \left[E_\beta + E_{-\beta}, E_\alpha\right] = N_{\beta,\alpha}E_{\alpha+\beta} + N_{-\beta,\alpha}E_{\alpha-\beta}$$

Assume that $N_{\beta,\alpha} \neq 0$. Then $\alpha + \beta \in \Delta$, i.e. $|\alpha+\beta|^2 = 2$, from which we can infer that $\langle\alpha,\beta\rangle = -1$. Now we can calculate $|\alpha-\beta|^2 = |\alpha|^2 + |\beta|^2 - 2\cdot\langle\alpha,\beta\rangle = 2 + 2 - 2\cdot(-1) = 6$, so $\alpha - \beta \notin \Delta$, and thus $N_{-\beta,\alpha} = 0$:

$$\left[E_\beta + E_{-\beta}, E_\alpha\right]_1 = N_{\beta,\alpha}E_{\alpha+\beta}.$$

Assume $N_{\beta,\alpha} \neq 0$, $N_{-\beta,\alpha} = 0$ for the rest of the calculations. We will generalize the final result to arbitrary values of $N_{\beta,\alpha}$.

<u>$m = 2$</u>:

$$\begin{aligned}
\left[E_\beta + E_{-\beta}, E_\alpha\right]_2 &= \left[E_\beta + E_{-\beta}, \left[E_\beta + E_{-\beta}, E_\alpha\right]\right] = \left[E_\beta + E_{-\beta}, N_{\beta,\alpha}E_{\alpha+\beta}\right] \\
&= N_{\beta,\alpha}\left[E_\beta, E_{\alpha+\beta}\right] + N_{\beta,\alpha}\left[E_{-\beta}, E_{\alpha+\beta}\right] \\
&= N_{\beta,\alpha}N_{\beta,\alpha+\beta}E_{\alpha+2\beta} + N_{\beta,\alpha}N_{-\beta,\alpha+\beta}E_\alpha
\end{aligned}$$

We show that $\alpha+2\beta$ is not a root, and thus $N_{\beta,\alpha+\beta}=0$:
$|\alpha+2\beta|^2=|\alpha+\beta|^2+|\beta|^2+2\langle\alpha,\beta\rangle+2\langle\beta,\beta\rangle=2+2+2\cdot(-1)+2\cdot 2\neq 2$

By using eqs. (5.25-5.26), we show that $N_{-\beta,\alpha+\beta}=N_{\beta,\alpha}$:
$(-\beta)+(\alpha+\beta)+(-\alpha)=0 \quad\rightarrow\quad N_{-\beta,\alpha+\beta}=N_{\alpha+\beta,-\alpha}=N_{-\alpha,-\beta}=-N_{\alpha,\beta}=N_{\beta,\alpha}$

The expression for the commutator can then be simplified:

$$[E_\beta,E_{-\beta},E_\alpha]_2=N_{\beta,\alpha}^2E_\alpha$$

<u>$m=3$</u>:

$$\begin{aligned}[E_\beta+E_{-\beta},E_\alpha]_3 &= \left[E_\beta+E_{-\beta},[E_\beta+E_{-\beta},E_\alpha]_2\right] = \left[E_\beta+E_{-\beta},N_{\beta,\alpha}^2E_\alpha\right]\\ &= N_{\beta,\alpha}^2\,[E_\beta,E_\alpha]+N_{\beta,\alpha}^2\,[E_{-\beta},E_\alpha]\\ &= N_{\beta,\alpha}^3E_{\alpha+\beta}\end{aligned}$$

<u>$m=4$</u>:

$$\begin{aligned}[E_\beta+E_{-\beta},E_\alpha]_4 &= \left[E_\beta+E_{-\beta},[E_\beta+E_{-\beta},E_\alpha]_3\right] = \left[E_\beta+E_{-\beta},N_{\beta,\alpha}^3E_{\alpha+\beta}\right]\\ &= N_{\beta,\alpha}^3\,[E_\beta,E_{\alpha+\beta}]+N_{\beta,\alpha}^3\,[E_{-\beta},E_{\alpha+\beta}]\\ &= N_{\beta,\alpha}^4E_\alpha\end{aligned}$$

We substitute the results which we obtained for $m=0,\ldots,4$ into eq. (5.13):

$$\begin{aligned}&\exp\left(\frac{i\pi}{2}\left(E_\beta+E_{-\beta}\right)\right)E_\alpha\exp\left(-\frac{i\pi}{2}\left(E_\beta+E_{-\beta}\right)\right)\\ &= \sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{i\pi}{2}\right)^m[E_\beta+E_{-\beta},E_\alpha]_m\\ &= E_\alpha+i\frac{\pi}{2}N_{\beta,\alpha}E_{\alpha+\beta}-\frac{1}{2!}\left(\frac{\pi}{2}\right)^2N_{\beta,\alpha}^2E_\alpha-\frac{1}{3!}i\left(\frac{\pi}{2}\right)^3N_{\beta,\alpha}^3E_{\alpha+\beta}+\frac{1}{4!}\left(\frac{\pi}{2}\right)^4N_{\beta,\alpha}^4E_\alpha+\ldots\\ &= \left(1-\frac{1}{2!}\left(\frac{\pi}{2}N_{\beta,\alpha}\right)^2+\frac{1}{4!}\left(\frac{\pi}{2}N_{\beta,\alpha}\right)^4-\ldots\right)E_\alpha+i\left(\frac{\pi}{2}N_{\beta,\alpha}-\frac{1}{3!}\left(\frac{\pi}{2}N_{\beta,\alpha}\right)^3+\ldots\right)E_{\alpha+\beta}\\ &= \cos\left(\frac{\pi}{2}N_{\beta,\alpha}\right)E_\alpha+i\sin\left(\frac{\pi}{2}N_{\beta,\alpha}\right)E_{\alpha+\beta}\end{aligned}\tag{5.14}$$

For algebras of type ADE, the non-vanishing structure constants are ±1. In this case, eq. (5.14) can be simplified to

$$\tilde{r}_\beta E_\alpha\tilde{r}_\beta^{-1}=iN_{\beta,\alpha}E_{\alpha+\beta},\quad \alpha\neq\pm\beta,\quad N_{\beta,\alpha}\neq 0.\tag{5.15}$$

For the case $N_{-\beta,\alpha}\neq 0$, we substitute $\beta\rightarrow-\beta$ in this formula. The left hand side of the eq. (5.15) is invariant under this substitution, because the definition of the lift $\tilde{r}_\beta$ is symmetric in $\pm\beta$. Thus, we obtain

$$\tilde{r}_\beta E_\alpha \tilde{r}_\beta^{-1} = iN_{-\beta,\alpha} E_{\alpha-\beta}, \quad \alpha \neq \pm\beta, \quad N_{-\beta,\alpha} \neq 0. \tag{5.16}$$

For the case $\alpha \neq \pm\beta$, and $N_{\beta,\alpha} = N_{-\beta,\alpha} = 0$, all commutators vanish, and going through the previous calculations once again shows that the action of the lift is the identity:

$$\tilde{r}_\beta E_\alpha \tilde{r}_\beta^{-1} = E_\alpha, \quad \alpha \neq \pm\beta, \quad N_{\beta,\alpha} = N_{-\beta,\alpha} = 0 \tag{5.17}$$

Now we discuss the case $\alpha = \beta$. We repeat the calculations following eq. (5.13):

$$[E_\alpha + E_{-\alpha}, E_\alpha]_0 = E_\alpha$$

$$[E_\alpha + E_{-\alpha}, E_\alpha]_1 = [E_\alpha + E_{-\alpha}, E_\alpha] = [E_\alpha, E_\alpha] - [E_\alpha, E_{-\alpha}] = -[E_\alpha, E_{-\alpha}] = -\alpha \cdot H$$

$$\begin{aligned} [E_\alpha + E_{-\alpha}, E_\alpha]_2 &= [E_\alpha + E_{-\alpha}, [E_\alpha + E_{-\alpha}, E_\alpha]] \;=\; [E_\alpha + E_{-\alpha}, (-\alpha \cdot H)] \\ &= \alpha^i [H_i, E_\alpha] + \alpha^i [H_i, E_{-\alpha}] \;=\; |\alpha|^2 E_\alpha - |\alpha|^2 E_{-\alpha} \\ &= |\alpha|^2 (E_\alpha - E_{-\alpha}) \end{aligned}$$

$$\begin{aligned} [E_\alpha + E_{-\alpha}, E_\alpha]_3 &= [E_\alpha + E_{-\alpha}, [E_\alpha + E_{-\alpha}, E_\alpha]_2] \\ &= |\alpha|^2 [E_\alpha + E_{-\alpha}, E_\alpha - E_{-\alpha}] = |\alpha|^2 (-[E_\alpha, E_{-\alpha}] + [E_{-\alpha}, E_\alpha]) \\ &= -|\alpha|^2 \cdot 2 \cdot [E_\alpha, E_{-\alpha}] = -2|\alpha|^2 \, \alpha \cdot H \end{aligned}$$

$$\begin{aligned} [E_\alpha + E_{-\alpha}, E_\alpha]_4 &= [E_\alpha + E_{-\alpha}, [E_\alpha + E_{-\alpha}, E_\alpha]_3] \\ &= [E_\alpha + E_{-\alpha}, (-2|\alpha|^2 \, \alpha \cdot H)] = 2|\alpha|^2 \, \alpha^i \; [H_i, E_\alpha + E_{-\alpha}] \\ &= 2|\alpha|^2 \alpha^i (\alpha^i E_\alpha - \alpha^i E_{-\alpha}) \\ &= 2|\alpha|^4 (E_\alpha - E_{-\alpha}) \end{aligned}$$

$$\begin{aligned} [E_\alpha + E_{-\alpha}, E_\alpha]_5 &= [E_\alpha + E_{-\alpha}, [E_\alpha + E_{-\alpha}, E_\alpha]_4] = 2|\alpha|^4 [E_\alpha + E_{-\alpha}, E_\alpha - E_{-\alpha}] \\ &= 2|\alpha|^4 (-[E_\alpha, E_{-\alpha}] + [E_{-\alpha}, E_\alpha]) \\ &= -4|\alpha|^4 [E_\alpha, E_{-\alpha}] = -4|\alpha|^4 \, \alpha \cdot H \end{aligned}$$

We substitute the results which we obtained for $m = 0, \dots, 5$ into eq. (5.13), and set $\alpha = \beta$:

$$\begin{aligned}
&\exp\left(\frac{i\pi}{2}\left(E_\alpha + E_{-\alpha}\right)\right) E_\alpha \exp\left(-\frac{i\pi}{2}\left(E_\alpha + E_{-\alpha}\right)\right) \\
&= \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{i\pi}{2}\right)^m \left[E_\alpha + E_{-\alpha}, E_\alpha\right]_m \\
&= E_\alpha + \frac{i\pi}{2}\left(-\alpha \cdot H\right) + \frac{1}{2!}\left(\frac{i\pi}{2}\right)^2 |\alpha|^2 \left(E_\alpha - E_{-\alpha}\right) + \frac{1}{3!}\left(\frac{i\pi}{2}\right)^3 \left(-2|\alpha|^2 \alpha \cdot H\right) \\
&\quad + \frac{1}{4!}\left(\frac{i\pi}{2}\right)^4 \left(2|\alpha|^4 \left(E_\alpha - E_{-\alpha}\right)\right) + \frac{1}{5!}\left(\frac{i\pi}{2}\right)^5 \left(-4|\alpha|^4 \alpha \cdot H\right) + \dots \\
&= E_\alpha - \frac{1}{2}\left(E_\alpha - E_{-\alpha}\right) + \frac{1}{2}\left(E_\alpha - E_{-\alpha}\right)\left[1 - \frac{1}{2!}\pi^2 + \frac{1}{4!}\pi^4 - \dots\right] \\
&\quad + \frac{i}{2}\alpha \cdot H \left[-\pi + \frac{1}{3!}\pi^3 - \frac{1}{5!}\pi^5 + \dots\right] \\
&= E_\alpha - \frac{1}{2}\left(E_\alpha - E_{-\alpha}\right) + \frac{1}{2}\left(E_\alpha - E_{-\alpha}\right)\cos\pi + \frac{i}{2}\,\alpha \cdot H\left(\sin\pi\right) \\
&= E_{-\alpha}
\end{aligned}$$

It is clear that the same result holds for $\alpha = -\beta$, because eq. (5.11) is symmetric in $\pm\beta$.

We summarize the transformation properties of the step operators below for further reference:

$$\begin{array}{llcl}
\alpha + \beta \in \Delta, & \alpha \neq \pm\beta & \to & \tilde{r}_\beta E_\alpha \tilde{r}_\beta^{-1} = i N_{\beta,\alpha} E_{\alpha+\beta} \\
\alpha - \beta \in \Delta, & \alpha \neq \pm\beta & \to & \tilde{r}_\beta E_\alpha \tilde{r}_\beta^{-1} = i N_{-\beta,\alpha} E_{\alpha-\beta} \\
\alpha \pm \beta \notin \Delta, & \alpha \neq \pm\beta & \to & \tilde{r}_\beta E_\alpha \tilde{r}_\beta^{-1} = E_\alpha \\
 & \alpha = \pm\beta & \to & \tilde{r}_\beta E_\alpha \tilde{r}_\beta^{-1} = E_{-\alpha}
\end{array} \tag{5.18}$$

These transformation properties can easily be cast into the form given in eq. (5.8). Consider the first relation in eq. (5.18). As before, $\alpha + \beta \in \Delta$ implies $\langle\alpha, \beta\rangle = -1$. Then,

$$r_\beta(\alpha) = \alpha - 2\,\frac{\langle\alpha, \beta\rangle}{\langle\beta, \beta\rangle}\,\beta = \alpha + \beta, \tag{5.19}$$

so that we can rewrite the first relation in eq. (5.18) as

$$\tilde{r}_\beta E_\alpha \tilde{r}_\beta^{-1} = c_\beta(\alpha)\, E_{r_\beta(\alpha)}, \tag{5.20}$$

where $c_\beta(\alpha) = i N_{\beta,\alpha}$. The other relations are equally easy to show.

5.3.2 The Transformation of the Cartan Generators H_i

To evaluate the expression

$$\tilde{r}_\beta H_i\, \tilde{r}_\beta^{-1} = \exp\left(i\frac{\pi}{2}\left(E_\beta + E_{-\beta}\right)\right) H_i \exp\left(-i\frac{\pi}{2}\left(E_\beta + E_{-\beta}\right)\right), \tag{5.21}$$

we set

$$A = \frac{i\pi}{2}\left(E_\beta + E_{-\beta}\right), \quad B = H_i, \tag{5.22}$$

in eq. (5.9) and calculate its right hand side:

$$\exp\left(i\frac{\pi}{2}\left(E_\beta + E_{-\beta}\right)\right) H_i \exp\left(-i\frac{\pi}{2}\left(E_\beta + E_{-\beta}\right)\right) = \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{i\pi}{2}\right)^m \left[E_\beta + E_{-\beta}, H_i\right]_m \tag{5.23}$$

As before, we calculate the first 5 terms of the series, and then generalize the result to arbitrary $m \in \mathbb{N}$.

$\underline{m=0}$:

$$\left[E_\beta + E_{-\beta}, H_i\right]_0 = H_i$$

$\underline{m=1}$:

$$\left[E_\beta + E_{-\beta}, H_i\right]_1 = \left[E_\beta + E_{-\beta}, H_i\right] = -\beta_i E_\beta + \beta_i E_{-\beta} = -\beta_i\left(E_\beta - E_{-\beta}\right)$$

$\underline{m=2}$:

$$\begin{aligned}
\left[E_\beta + E_{-\beta}, H_i\right]_2 &= \left[E_\beta + E_{-\beta}, \left[E_\beta + E_{-\beta}, H_i\right]\right] \quad = \quad -\beta_i\left[E_\beta + E_{-\beta}, E_\beta - E_{-\beta}\right] \\
&= -\beta_i\Big(\left[E_\beta, E_\beta\right] - \left[E_\beta, E_{-\beta}\right] + \left[E_{-\beta}, E_\beta\right] - \left[E_{-\beta}, E_{-\beta}\right]\Big) \\
&= -\beta_i\Big(-2\beta\cdot H\Big) \quad = \quad 2\beta_i\ \beta\cdot H
\end{aligned}$$

$\underline{m=3}$:

$$\begin{aligned}
\left[E_\beta + E_{-\beta}, H_i\right]_3 &= \Big[E_\beta + E_{-\beta}, \left[E_\beta + E_{-\beta}, H_i\right]_2\Big] \quad = \quad \left[E_\beta + E_{-\beta}, 2\beta_i\ \beta\cdot H\right] \\
&= -2\beta_i\beta_j\left[H_j, E_\beta + E_{-\beta}\right] \quad = \quad -2\beta_i\beta_j\Big(\beta_j E_\beta - \beta_j E_{-\beta}\Big) \\
&= -2\beta_i\,\beta^2\left(E_\beta - E_{-\beta}\right)
\end{aligned}$$

$\underline{m=4}$:

$$\begin{aligned}
\left[E_\beta + E_{-\beta}, H_i\right]_4 &= \Big[E_\beta + E_{-\beta}, \left[E_\beta + E_{-\beta}, H_i\right]_3\Big] \quad = \quad -2\beta_i\,\beta^2\left[E_\beta + E_{-\beta}, E_\beta - E_{-\beta}\right] \\
&= -2\beta_i\,\beta^2\Big(-\left[E_\beta, E_{-\beta}\right] + \left[E_{-\beta}, E_\beta\right]\Big) \\
&= 4\beta_i\,\beta^2\ \beta\cdot H
\end{aligned}$$

$\underline{m=5}$:

$$
\begin{aligned}
[E_\beta + E_{-\beta}, H_i]_5 &= \big[E_\beta + E_{-\beta}, [E_\beta + E_{-\beta}, H_i]_4\big] \;=\; [E_\beta + E_{-\beta}, 4\beta_i\,\beta^2\;\beta\cdot H] \\
&= -4\beta_i\,\beta^2\,\beta_j\,[H_j, E_\beta + E_{-\beta}] \;=\; -4\beta_i\,\beta^2\,\beta_j\Big(\beta_j E_\beta - \beta_j E_{-\beta}\Big) \\
&= -4\beta_i\,\beta^4\,(E_\beta - E_{-\beta})
\end{aligned}
$$

We substitute the results which we obtained for $m = 0, \ldots, 5$ into eq. (5.21). Note that we use the normalization $\beta^2 = 2$ of the roots, and by e_i we denote the ith basis vector of the standard basis of Euclidean space.

$$
\begin{aligned}
&\exp\left(\frac{i\pi}{2}\,(E_\beta + E_{-\beta})\right) H_i \exp\left(-\frac{i\pi}{2}\,(E_\beta + E_{-\beta})\right) \\
&= \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{i\pi}{2}\right)^m [E_\beta + E_{-\beta}, H_i]_m \\
&= H_i + \frac{1}{1!}\left(\frac{i\pi}{2}\right)(-\beta_i)\,(E_\beta - E_{-\beta}) + \frac{1}{2!}\left(\frac{i\pi}{2}\right)^2 2\beta_i\;\beta\cdot H + \frac{1}{3!}\left(\frac{i\pi}{2}\right)^3 (-2\beta_i\beta^2)\,(E_\beta - E_{-\beta}) \\
&\quad + \frac{1}{4!}\left(\frac{i\pi}{2}\right)^4 4\beta_i\beta^2\;\beta\cdot H + \frac{1}{5!}\left(\frac{i\pi}{2}\right)^5 (-4\beta_i\beta^4)\,(E_\beta - E_{-\beta}) + \ldots \\
&= H_i - \frac{1}{2}\beta_i\;\beta\cdot H \\
&\quad + \left(\frac{1}{2} + \frac{1}{2!}\left(\frac{i\pi}{2}\right)^2 2 + \frac{1}{4!}\left(\frac{i\pi}{2}\right)^4 4\beta^2 + \ldots\right)\beta_i\;\beta\cdot H \\
&\quad + \left(\frac{1}{1!}\left(\frac{i\pi}{2}\right)(-1) + \frac{1}{3!}\left(\frac{i\pi}{2}\right)^3 (-2\beta^2) + \frac{1}{5!}\left(\frac{i\pi}{2}\right)^5 (-4\beta^4)\right)\beta_i\;(E_\beta - E_{-\beta}) \\
&= H_i - \frac{1}{2}\beta_i\;\beta\cdot H \\
&\quad + \frac{1}{2}\left(1 - \frac{\pi}{2!} + \frac{\pi}{4!} + \ldots\right)\beta_i\;\beta\cdot H - \frac{i}{2}\left(\frac{\pi}{1!} - \frac{\pi}{3!} + \frac{\pi}{5!}\right)\beta_i\;(E_\beta - E_{-\beta}) \\
&= H_i - \frac{1}{2}\beta_i\;\beta\cdot H + \frac{1}{2}\cos\pi\;\beta_i\;\beta\cdot H - \frac{i}{2}\sin\pi\;\beta_i\;(E_\beta - E_{-\beta}) \\
&= H_i - \beta_i\;\beta\cdot H \\
&= e_i\cdot H - \frac{2}{\beta^2}\,e_i\cdot\beta\;\beta\cdot H \\
&= r_\beta(e_i)\cdot H
\end{aligned}
$$

As we have proved this vector equation for any basis vector e_i, it is clear that the relation immediately generalizes to

$$\tilde{r}_\beta\,(\lambda\cdot H)\,\tilde{r}_\beta^{-1} = (r_\beta(\lambda))\cdot H. \tag{5.24}$$

Knowing the transformation properties of the step operators and Cartan generators, the structure constants are the final piece of information which is missing.

5.4 The Structure Constants

In this section, we are closely following ref. [49], where the determination of the structure constants has been described. The structure constants of a Lie algebra are not uniquely determined. We will show that for certain ordered pairs, to be called *extra special pairs*, the sign of the corresponding structure constants may be chosen arbitrarily, thus determining the values of all the others. The following theorem will be crucial for our considerations.

Theorem: Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, and let Δ denote the set of its roots. The structure constants of $\mathfrak{g}$ satisfy the following relations:

(i) For $\alpha, \beta \in \Delta$:
$$N_{\alpha,\beta} = -N_{\beta,\alpha} \tag{5.25}$$

(ii) If $\alpha, \beta, \gamma \in \Delta$ satisfy $\alpha + \beta + \gamma = 0$, then:
$$\frac{N_{\alpha,\beta}}{\langle \gamma, \gamma \rangle} = \frac{N_{\beta,\gamma}}{\langle \alpha, \alpha \rangle} = \frac{N_{\gamma,\alpha}}{\langle \beta, \beta \rangle} \tag{5.26}$$

(iii) For $\alpha, \beta \in \Delta$,
$$N_{\alpha,\beta}\, N_{-\alpha,-\beta} = -(p+1)^2, \tag{5.27}$$
where p is the integer such that $-p\alpha + \beta \in \Delta$, and $-(p+1)\alpha + \beta \notin \Delta$.

(iv) If $\alpha, \beta, \gamma, \delta \in \Delta$ satisfy $\alpha + \beta + \gamma + \delta = 0$, and no pair are opposite, then:
$$\frac{N_{\alpha,\beta} N_{\gamma,\delta}}{\langle \alpha+\beta, \alpha+\beta \rangle} + \frac{N_{\beta,\gamma} N_{\alpha,\delta}}{\langle \beta+\gamma, \beta+\gamma \rangle} + \frac{N_{\gamma,\alpha} N_{\beta,\delta}}{\langle \gamma+\alpha, \gamma+\alpha \rangle} = 0. \tag{5.28}$$

An ordered pair (r, s) is called *special*, if $r+s \in \Delta$ and $0 < r < s$. An ordered pair is called *extra special*, if (r, s) is a special pair and if for all special pairs (r_1, s_1) with $r+s = r_1+s_1$ we have $r \leq r_1$. Thus, every positive root which is the sum of two positive roots can be written uniquely as the sum of an extra special pair. We now show how the values of the structure constants $N_{r,s}$ can be derived from the structure constants on the extra special pairs using the theorem above.

Consider the structure constant $N_{r,s}$. Either $r+s \in \Delta$ or $r+s \notin \Delta$. In the latter case, $N_{r,s} = 0$ by definition. Assume the former case, i.e. $r+s \in \Delta$. It then follows that the 12 pairs,

$$\begin{array}{cccccc} (r,s), & (s,-r-s), & (-r-s,r), & (-r,-s), & (-s,r+s), & (r+s,-r), \\ (s,r), & (-r-s,s), & (r,-r-s), & (-s,-r), & (r+s,-s), & (-r,r+s), \end{array} \tag{5.29}$$

are such that the sum of the 2 roots in each pair is again a root. Since $r+s+(-r-s) = 0$, not all roots, r, s, $-r-s$, can be positive (neither can all be negative), so either 1 is positive and 2 negative, or 2 are positive and 1 is negative. Therefore, of the above 12 pairs, exactly one is a special pair.

Assume that e.g. $(r, -r-s)$ is the special pair and that we know how to determine the structure constants of special pairs. (We will show in a moment how the structure constants of special pairs follow from those of extra special pairs.) Choose $\alpha = r$, $\beta = s$, $\gamma = -r-s$, and use eq. (5.26). It follows that

$$N_{r,s} = \frac{\langle r+s, r+s\rangle}{\langle s,s\rangle} N_{-r-s,r} = -\frac{\langle r+s, r+s\rangle}{\langle s,s\rangle} N_{r,-r-s}, \tag{5.30}$$

i.e. the structure constant corresponding to the pair (r,s) is determined by that of the special pair $(r,-r-s)$.

Now we will concentrate our attention on determining the structure constants of the special pairs from those of the extra special pairs. Assume that (r,s) is special, but not extra special. Let r_1, s_1 denote the associated extra special pair. Then by definition, $r_1 \leq r$, and by assumption that (r,s) is not extra special, $r_1 < r$. Because $r+s = r_1+s_1$, it follows that $s_1 > s$. Furthermore, as (r,s) is special, $r < s$. Putting all this together, we have

$$0 < r_1 < r < s < s_1. \tag{5.31}$$

Since $r+s+(-r_1)+(-s_1) = 0$, we can use eq. (5.28),

$$\frac{N_{r,s}N_{-r_1,-s_1}}{\langle r+s, r+s\rangle} + \frac{N_{s,-r_1}N_{r,-s_1}}{\langle s-r_1, s-r_1\rangle} + \frac{N_{-r_1,r}N_{s,-s_1}}{\langle -r_1+r, -r_1+r\rangle} = 0, \tag{5.32}$$

which expresses $N_{r,s}$ in terms of the structure constants associated with the pairs

$$(-r_1,-s_1), \quad (s,-r_1), \quad (r,-s_1), \quad (-r_1,r), \quad (s,-s_1). \tag{5.33}$$

For each such pair, we repeat the steps following eq. (5.29). This gives us a special pair, and this special pair determines the structure constant of the original pair according to eq. (5.30). The special pairs corresponding to eq. (5.33) are

$$(r_1,s_1), \quad (r_1, s-r_1), \quad (s_1-r, r), \quad (r-r_1, r_1), \quad (s_1-s, s) \tag{5.34}$$

or transpositions thereof. For the first special pair, $r_1+s_1 = r+s$ and $r_1 < r$. However, for the remaining four pairs (r', s'), we have $r'+s' < r+s$. This means that we can express the structure constant of (r,s) either

1. by the structure constants of special pairs with the same sum $r+s$, but the first entry less than and not equal to r, or

2. by the structure constants of special pairs (r', s') for which $r'+s' < r+s$.

This algorithm will only terminate, when we have no further possibility to lower the first component of the pair, i.e. when we have arrived at an extra special pair. For ADE algebras, the structure constants of the extra special pairs can be chosen to be ± 1.

The structure constants in appendix D have been computed using this recursive algorithm, implemented in a C++ program.

5.5 The Lift of a Single Weyl Reflection

To illustrate the ideas of the previous sections, we consider the Lie algebra SO(10). Using the transformation properties for the Lie algebra generators as given in eqs. (5.18, 5.24), we will calculate the lift for a single Weyl reflection. To be specific, we concentrate on r_{α_1}.

We start with calculating the action of $\tilde{r}_{\alpha_1}$ on the Cartan generators, as given by eq. (5.24). To this end, we need the matrix representation of r_{α_1}.

Remember the general rule that the i-th column of the matrix representing the linear map is the coordinate vector of the image of the i-th basis vector. In the case of the standard basis of $\mathbb{R}^5$, the i-th column of the matrix is simply the image of the i-th basis vector. The images of the basis vectors e_i are

$$r_{\alpha_1} e_i = e_i - \langle e_i, \alpha_1 \rangle \, \alpha_1, \tag{5.35}$$

where the simple root α_1 is listed in tab. (D.1). The matrix representation of r_{α_1} in the standard basis is then given by

$$r_{\alpha_1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.36}$$

The matrix representations for the other simple Weyl reflections can be found in appendix D.

The transformation properties of the basis $H_1, \ldots, H_5$ of the Cartan subalgebra are then given by eq. (5.24), substituting $\lambda = e_i$, and using r_{α_1} as given in eq. (5.36):

$$\begin{aligned} \tilde{r}_{\alpha_1} H_1 \tilde{r}_{\alpha_1}^{-1} &= r_{\alpha_1}(e_1) \cdot H = H_2 \\ \tilde{r}_{\alpha_1} H_2 \tilde{r}_{\alpha_1}^{-1} &= r_{\alpha_1}(e_2) \cdot H = H_1 \\ \tilde{r}_{\alpha_1} H_3 \tilde{r}_{\alpha_1}^{-1} &= r_{\alpha_1}(e_3) \cdot H = H_3 \\ \tilde{r}_{\alpha_1} H_4 \tilde{r}_{\alpha_1}^{-1} &= r_{\alpha_1}(e_4) \cdot H = H_4 \\ \tilde{r}_{\alpha_1} H_5 \tilde{r}_{\alpha_1}^{-1} &= r_{\alpha_1}(e_5) \cdot H = H_5 \end{aligned} \tag{5.37}$$

The transformation law for the 40 root operators E_{β_i} is given by eq. (5.18). Once we know the structure constants $N_{\beta,\alpha}$, the action of the lift is trivial to calculate. Calculating the $N_{\beta,\alpha}$'s, however, is not at all a trivial undertaking. The structure constants of SO(10) are listed in appendix D. The roots, and the simple roots are given in appendix D.

Table 5.1: Action of the lift $\tilde{r}_{\alpha_1}$ on the root generators.

E_{β_i}	$\tilde{r}_{\alpha_1} E_{\beta_i} \tilde{r}_{\alpha_1}^{-1}$	$\tilde{r}_{\alpha_1}^2 E_{\beta_i} \tilde{r}_{\alpha_1}^{-2}$	E_{β_i}	$\tilde{r}_{\alpha_1} E_{\beta_i} \tilde{r}_{\alpha_1}^{-1}$	$\tilde{r}_{\alpha_1}^2 E_{\beta_i} \tilde{r}_{\alpha_1}^{-2}$
E_{β_1}	E_{β_1}	E_{β_1}	$E_{\beta_{21}}$	iE_{β_9}	$-E_{\beta_{21}}$
E_{β_2}	E_{β_3}	E_{β_2}	$E_{\beta_{22}}$	$-iE_{\beta_{10}}$	$-E_{\beta_{22}}$
E_{β_3}	E_{β_2}	E_{β_3}	$E_{\beta_{23}}$	$iE_{\beta_{11}}$	$-E_{\beta_{23}}$
E_{β_4}	E_{β_4}	E_{β_4}	$E_{\beta_{24}}$	$-iE_{\beta_{12}}$	$-E_{\beta_{24}}$
E_{β_5}	$-iE_{\beta_{17}}$	$-E_{\beta_5}$	$E_{\beta_{25}}$	$-iE_{\beta_{13}}$	$-E_{\beta_{25}}$
E_{β_6}	$-iE_{\beta_{18}}$	$-E_{\beta_6}$	$E_{\beta_{26}}$	$-iE_{\beta_{14}}$	$-E_{\beta_{26}}$
E_{β_7}	$iE_{\beta_{19}}$	$-E_{\beta_7}$	$E_{\beta_{27}}$	$iE_{\beta_{15}}$	$-E_{\beta_{27}}$
E_{β_8}	$iE_{\beta_{20}}$	$-E_{\beta_8}$	$E_{\beta_{28}}$	$iE_{\beta_{16}}$	$-E_{\beta_{28}}$
E_{β_9}	$iE_{\beta_{21}}$	$-E_{\beta_9}$	$E_{\beta_{29}}$	$E_{\beta_{29}}$	$E_{\beta_{29}}$
$E_{\beta_{10}}$	$-iE_{\beta_{22}}$	$-E_{\beta_{10}}$	$E_{\beta_{30}}$	$E_{\beta_{30}}$	$E_{\beta_{30}}$
$E_{\beta_{11}}$	$iE_{\beta_{23}}$	$-E_{\beta_{11}}$	$E_{\beta_{31}}$	$E_{\beta_{31}}$	$E_{\beta_{31}}$
$E_{\beta_{12}}$	$-iE_{\beta_{24}}$	$-E_{\beta_{12}}$	$E_{\beta_{32}}$	$E_{\beta_{32}}$	$E_{\beta_{32}}$
$E_{\beta_{13}}$	$-iE_{\beta_{25}}$	$-E_{\beta_{13}}$	$E_{\beta_{33}}$	$E_{\beta_{33}}$	$E_{\beta_{33}}$
$E_{\beta_{14}}$	$-iE_{\beta_{26}}$	$-E_{\beta_{14}}$	$E_{\beta_{34}}$	$E_{\beta_{34}}$	$E_{\beta_{34}}$
$E_{\beta_{15}}$	$iE_{\beta_{27}}$	$-E_{\beta_{15}}$	$E_{\beta_{35}}$	$E_{\beta_{35}}$	$E_{\beta_{35}}$
$E_{\beta_{16}}$	$iE_{\beta_{28}}$	$-E_{\beta_{16}}$	$E_{\beta_{36}}$	$E_{\beta_{36}}$	$E_{\beta_{36}}$
$E_{\beta_{17}}$	$-iE_{\beta_5}$	$-E_{\beta_{17}}$	$E_{\beta_{37}}$	$E_{\beta_{37}}$	$E_{\beta_{37}}$
$E_{\beta_{18}}$	$-iE_{\beta_6}$	$-E_{\beta_{18}}$	$E_{\beta_{38}}$	$E_{\beta_{38}}$	$E_{\beta_{38}}$
$E_{\beta_{19}}$	iE_{β_7}	$-E_{\beta_{19}}$	$E_{\beta_{39}}$	$E_{\beta_{39}}$	$E_{\beta_{39}}$
$E_{\beta_{20}}$	iE_{β_8}	$-E_{\beta_{20}}$	$E_{\beta_{40}}$	$E_{\beta_{40}}$	$E_{\beta_{40}}$

In tab. (5.1), we list the action of $\tilde{r}_{\alpha_1}$ and its square on the 40 step operators of SO(10). Although the Weyl reflection r_{α_1} is a $\mathbb{Z}_2$ map on the root lattice, its lift $\tilde{r}_{\alpha_1}$ clearly does not square to identity.

5.6 The Lift of the Coxeter Element

For an arbitrary element w of the Weyl group, $w^2 = \mathbb{1}$ only implies $\tilde{w}^4 = \mathbb{1}$ [50]. Only if w is *non-degenerate*, w and $\tilde{w}$ are of the same order, and $w^2 = \mathbb{1}$ implies $\tilde{w}^2 = \mathbb{1}$. An element w of the Weyl group is called *non-degenerate*, if all its eigenvalues are different from 1, i.e. if w does not leave any direction fixed.

Consider the Coxeter element $r \equiv r_{\alpha_1} r_{\alpha_2} r_{\alpha_3} r_{\alpha_4} r_{\alpha_5}$. By multiplying the matrices in eq. (D.2), and determining the eigenvalues of the resulting matrix, one immediately verifies that r has no fixed directions, and is of order 8, i.e. $r^8 = \mathbb{1}$. Thus, we expect that its lift $\tilde{r}$ has the same order as r.

We are interested in breaking SO(10) using a $\mathbb{Z}_2$ automorphism, so set $s \equiv r^4$. The transformation law of the 40 step operators under $\tilde{s}$ can be immediately calculated as we know the lifts for the constituting Weyl reflections. The result is given in tab. (5.2). It is clear that $\tilde{s}$ is of order 2.

Table 5.2: The action of $\tilde{s}$ on the root operators of SO(10), where s is the Coxeter element to the fourth.

E_{β_i}	$\tilde{s}E_{\beta_i}\tilde{s}^{-1}$			E_{β_i}	$\tilde{s}E_{\beta_i}\tilde{s}^{-1}$		
E_{β_1}	E_{β_4}	$=$	$E_{-\beta_1}$	$E_{\beta_{21}}$	$-E_{\beta_{24}}$	$=$	$-E_{-\beta_{21}}$
E_{β_2}	E_{β_3}	$=$	$E_{-\beta_2}$	$E_{\beta_{22}}$	$-E_{\beta_{23}}$	$=$	$-E_{-\beta_{22}}$
E_{β_3}	E_{β_2}	$=$	$E_{-\beta_3}$	$E_{\beta_{23}}$	$-E_{\beta_{22}}$	$=$	$-E_{-\beta_{23}}$
E_{β_4}	E_{β_1}	$=$	$E_{-\beta_4}$	$E_{\beta_{24}}$	$-E_{\beta_{21}}$	$=$	$-E_{-\beta_{24}}$
E_{β_5}	$-E_{\beta_8}$	$=$	$-E_{-\beta_5}$	$E_{\beta_{25}}$	$E_{\beta_{26}}$	$=$	$E_{-\beta_{27}}$
E_{β_6}	$-E_{\beta_7}$	$=$	$-E_{-\beta_6}$	$E_{\beta_{26}}$	$E_{\beta_{25}}$	$=$	$E_{-\beta_{28}}$
E_{β_7}	$-E_{\beta_6}$	$=$	$-E_{-\beta_7}$	$E_{\beta_{27}}$	$E_{\beta_{28}}$	$=$	$E_{-\beta_{25}}$
E_{β_8}	$-E_{\beta_5}$	$=$	$-E_{-\beta_8}$	$E_{\beta_{28}}$	$E_{\beta_{27}}$	$=$	$E_{-\beta_{26}}$
E_{β_9}	$E_{\beta_{12}}$	$=$	$E_{-\beta_9}$	$E_{\beta_{29}}$	$E_{\beta_{32}}$	$=$	$E_{-\beta_{29}}$
$E_{\beta_{10}}$	$E_{\beta_{11}}$	$=$	$E_{-\beta_{10}}$	$E_{\beta_{30}}$	$E_{\beta_{31}}$	$=$	$E_{-\beta_{30}}$
$E_{\beta_{11}}$	$E_{\beta_{10}}$	$=$	$E_{-\beta_{11}}$	$E_{\beta_{31}}$	$E_{\beta_{30}}$	$=$	$E_{-\beta_{31}}$
$E_{\beta_{12}}$	E_{β_9}	$=$	$E_{-\beta_{12}}$	$E_{\beta_{32}}$	$E_{\beta_{29}}$	$=$	$E_{-\beta_{32}}$

continued ...

continuation ...								
$E_{\beta_{13}}$	$E_{\beta_{14}}$	$=$	$E_{-\beta_{15}}$		$E_{\beta_{33}}$	$-E_{\beta_{34}}$	$=$	$-E_{-\beta_{35}}$
$E_{\beta_{14}}$	$E_{\beta_{13}}$	$=$	$E_{-\beta_{16}}$		$E_{\beta_{34}}$	$-E_{\beta_{33}}$	$=$	$-E_{-\beta_{36}}$
$E_{\beta_{15}}$	$E_{\beta_{16}}$	$=$	$E_{-\beta_{13}}$		$E_{\beta_{35}}$	$-E_{\beta_{36}}$	$=$	$-E_{-\beta_{33}}$
$E_{\beta_{16}}$	$E_{\beta_{15}}$	$=$	$E_{-\beta_{14}}$		$E_{\beta_{36}}$	$-E_{\beta_{35}}$	$=$	$-E_{-\beta_{34}}$
$E_{\beta_{17}}$	$E_{\beta_{20}}$	$=$	$E_{-\beta_{17}}$		$E_{\beta_{37}}$	$E_{\beta_{38}}$	$=$	$E_{-\beta_{39}}$
$E_{\beta_{18}}$	$E_{\beta_{19}}$	$=$	$E_{-\beta_{18}}$		$E_{\beta_{38}}$	$E_{\beta_{37}}$	$=$	$E_{-\beta_{40}}$
$E_{\beta_{19}}$	$E_{\beta_{18}}$	$=$	$E_{-\beta_{19}}$		$E_{\beta_{39}}$	$E_{\beta_{40}}$	$=$	$E_{-\beta_{37}}$
$E_{\beta_{20}}$	$E_{\beta_{17}}$	$=$	$E_{-\beta_{20}}$		$E_{\beta_{40}}$	$E_{\beta_{39}}$	$=$	$E_{-\beta_{38}}$

Considering the transformation law as given in tab. (5.2), we see that the step operators are not invariant under the action of $\tilde{s}$, but there are linear combinations which are invariant, and which survive the symmetry breakdown induced by the action of the automorphism. The 20 invariant combinations are listed in tab. (5.3).

We now consider the transformation of the 5 Cartan generators $H_1, \dots, H_5$. The matrix representation of $s = (r_{\alpha_1} r_{\alpha_2} r_{\alpha_3} r_{\alpha_4} r_{\alpha_5})^4$ is

$$s = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.38}$$

the from eq. (5.24), it follows that

$$H_1 \to -H_1, \quad H_2 \to -H_2, \quad H_3 \to -H_3, \quad H_4 \to -H_4, \quad H_5 \to +H_5, \tag{5.39}$$

i.e. the Cartan generator H_5 survives, the rest is projected out.

To summarize, 21 generators survive the symmetry breakdown. Comparing the number of generators to those of the subgroups of SO(10), we may state:

Conjecture : The Lie algebra automorphism $\tilde{s}$ breaks SO(10) to SO(6) $\times$ SO(4).

In the following sections, we will prove this conjecture.

$16 \times E_\alpha \to E_{-\alpha}$	$12 \times E_\alpha \to E_\beta,\ \ E_\beta \to E_\alpha$
$E_1 + E_4$	$E_{13} + E_{14}$
$E_2 + E_3$	$E_{15} + E_{16}$
$E_9 + E_{12}$	$E_{25} + E_{26}$
$E_{10} + E_{11}$	$E_{27} + E_{28}$
$E_{17} + E_{20}$	$E_{37} + E_{38}$
$E_{18} + E_{19}$	$E_{39} + E_{40}$
$E_{29} + E_{32}$	
$E_{30} + E_{31}$	
$8 \times E_\alpha \to -E_{-\alpha}$	$4 \times E_\alpha \to -E_\beta,\ \ E_\beta \to -E_\alpha$
$E_5 - E_8$	$E_{33} - E_{34}$
$E_6 - E_7$	$E_{35} - E_{36}$
$E_{21} - E_{24}$	
$E_{22} - E_{23}$	

Table 5.3: The 20 invariant combinations of SO(10) step operators.

5.7 The Symmetry Breakdown Induced by the Lift of the Coxeter Element

In order to find out which algebra is described by the 21 operators, we follow the by now standard procedure described in any textbook on the theory of Lie algebras, e.g. refs. [48, 49].

Choose a maximal commuting set which will play the role of the new Cartan generators:

$$\tilde{H}_1 = E_1 + E_4, \quad \tilde{H}_2 = E_2 + E_3, \quad \tilde{H}_3 = E_{29} + E_{32}, \quad \tilde{H}_4 = E_{30} + E_{31}, \quad \tilde{H}_5 = H_5 \tag{5.40}$$

That these operators mutually commute can be immediately verified by direct calculation.

Now calculate the root vectors. The i-th entry of a root α is the eigenvalue of the corresponding step operator E_α under the adjoint action of the i-th Cartan generator:

$$\mathrm{ad}(H_i)E_\alpha = \alpha^i\, E_\alpha \tag{5.41}$$

The adjoint action of the Cartan generators given in eq. (5.40) will in general not be diagonal from the start, but since they mutually commute, their action on the step operators can be simultaneously diagonalized. To this end, we derive the matrix representation for the adjoint action, which we will then diagonalize. The eigenvalues are then given by the diagonal elements.

5.7.1 Matrix Representation of the Adjoint Action

The adjoint action of an operator on its algebra,

$$\mathrm{ad}(\mathcal{O}) : \mathcal{X} \mapsto [\mathcal{O}, \mathcal{X}]\,, \tag{5.42}$$

is a linear map, and can thus be represented by a matrix. Remember once again the general rule already stated earlier that the i-th column of the matrix representing the linear map is the coordinate vector of the image of the i-th basis vector.

The algebra is spanned by the 20 invariant combinations of root operators given in

$\tilde{E}_1$	$=$	$E_1 + E_4$	$\tilde{E}_{12}$	$=$	$E_{22} - E_{23}$
$\tilde{E}_2$	$=$	$E_2 + E_3$	$\tilde{E}_{13}$	$=$	$E_{25} + E_{26}$
$\tilde{E}_3$	$=$	$E_5 - E_8$	$\tilde{E}_{14}$	$=$	$E_{27} + E_{28}$
$\tilde{E}_4$	$=$	$E_6 - E_7$	$\tilde{E}_{15}$	$=$	$E_{29} + E_{32}$
$\tilde{E}_5$	$=$	$E_9 + E_{12}$	$\tilde{E}_{16}$	$=$	$E_{30} + E_{31}$
$\tilde{E}_6$	$=$	$E_{10} + E_{11}$	$\tilde{E}_{17}$	$=$	$E_{33} - E_{34}$
$\tilde{E}_7$	$=$	$E_{13} + E_{14}$	$\tilde{E}_{18}$	$=$	$E_{35} - E_{36}$
$\tilde{E}_8$	$=$	$E_{15} + E_{16}$	$\tilde{E}_{19}$	$=$	$E_{37} + E_{38}$
$\tilde{E}_9$	$=$	$E_{17} + E_{20}$	$\tilde{E}_{20}$	$=$	$E_{39} + E_{40}$
$\tilde{E}_{10}$	$=$	$E_{18} + E_{19}$	$\tilde{E}_{21}$	$=$	H_5
$\tilde{E}_{11}$	$=$	$E_{21} - E_{24}$			

Table 5.4: The 21 basis vectors of the surviving gauge group.

tab. (5.3) and the one invariant Cartan generator given in eq. (5.39). For the matrix representation, we have to fix an enumeration for these basis vectors, as we do in tab. (5.4). Note that

$$\tilde{E}_1 = \tilde{H}_1, \quad \tilde{E}_2 = \tilde{H}_2, \quad \tilde{E}_{15} = \tilde{H}_3, \quad \tilde{E}_{16} = \tilde{H}_4, \quad \tilde{E}_{21} = \tilde{H}_5 \tag{5.43}$$

are the Cartan generators of the new algebra.

To be specific, let us determine the matrix representation of $\mathrm{ad}(\tilde{E}_1)$. We calculate the commutator of $\tilde{E}_1$ with all the other generators of the algebra, and then express the result as a linear combination of these 21 basis vectors. The coefficients of this expansion form the corresponding column of the matrix representation of $\mathrm{ad}(\tilde{E}_1)$. We give the first lines of the calculation:

$$
\begin{aligned}
[\tilde{E}_1, \tilde{E}_1] &= [E_{\alpha_1} + E_{\alpha_4}, E_{\alpha_1} + E_{\alpha_4}] = 0 \\
[\tilde{E}_1, \tilde{E}_2] &= [E_{\alpha_1} + E_{\alpha_4}, E_{\alpha_2} + E_{\alpha_3}] \\
&= N_{\alpha_1,\alpha_4} E_{\alpha_1+\alpha_4} + N_{\alpha_1,\alpha_2} E_{\alpha_1+\alpha_2} + N_{\alpha_4,\alpha_2} E_{\alpha_4+\alpha_2} + N_{\alpha_4,\alpha_3} E_{\alpha_4+\alpha_3} \\
&= 0 \\
[\tilde{E}_1, \tilde{E}_3] &= [E_{\alpha_1} + E_{\alpha_4}, E_{\alpha_5} - E_{\alpha_8}] \\
&= N_{\alpha_1,\alpha_5} E_{\alpha_1+\alpha_5} - N_{\alpha_1,\alpha_8} E_{\alpha_1+\alpha_8} + N_{\alpha_4,\alpha_5} E_{\alpha_4+\alpha_5} - N_{\alpha_4,\alpha_8} E_{\alpha_4+\alpha_8} \\
&= 0 \cdot E_{\alpha_1+\alpha_5} - 1 \cdot E_{\alpha_1+\alpha_8} + (-1) \cdot E_{\alpha_4+\alpha_5} - 0 \cdot E_{\alpha_4+\alpha_8} \\
&= -E_{\alpha_1+\alpha_8} - E_{\alpha_4+\alpha_5} \\
&= -E_{\alpha_{18}} - E_{\alpha_{19}} \\
&= -\tilde{E}_{\alpha_{10}}
\end{aligned}
$$

Thus, the first and second columns of the matrix are $(0, \ldots, 0)$, and the third column is $(0, \ldots, -1, \ldots, 0)$, where the -1 is at position 10.

Note that we made extensive use of the structure constants which are listed in tab. (D.3).

5.7.2 Calculating the Roots of the New Algebra

The matrix representation of the Cartan generators is given in appendix D. These matrices are not diagonal, but can be simultaneously diagonalized. The simultaneously diagonalized matrices are given in appendix D.

The root vectors are now trivial to find. Remember the general definition of the root α:

$$
\mathrm{ad}(H_i) E_\alpha = \alpha^i \, E_\alpha \tag{5.44}
$$

Thus, the i-th eigenvalues of the five diagonalized matrices $(\mathrm{ad}\ \tilde{H}_1), \ldots, (\mathrm{ad}\ \tilde{H}_5)$ give the five entries of the root $\alpha_i = (\alpha_i^1, \ldots, \alpha_i^5)$. The new weight system is given in tab. (5.5). Diagonalizing the adjoint action of the Cartan generators corresponds to a change of basis, which, for the sake of completeness, is given in appendix D.

Table 5.5: Roots and Cartan generators of the unbroken gauge group.

	Cartan-Weyl labels					Roots in basis γ_i					Pos.?
ω_1	1	1	1	1	0	0	0	1	0	0	> 0
ω_2	1	1	-1	-1	0	0	0	0	1	0	> 0
ω_3	1	-1	1	-1	0	0	0	0	0	1	> 0
ω_4	1	-1	-1	1	0	-1	1	0	0	1	< 0
ω_5	-1	1	1	-1	0	1	-1	0	0	-1	> 0
ω_6	-1	1	-1	1	0	0	0	0	0	-1	< 0
ω_7	-1	-1	1	1	0	0	0	0	-1	0	< 0
ω_8	-1	-1	-1	-1	0	0	0	-1	0	0	< 0
ω_9	0	0	1	-1	1	1	0	0	0	0	> 0
ω_{10}	0	0	-1	1	1	0	1	0	0	0	> 0
ω_{11}	0	0	1	-1	-1	0	-1	0	0	0	< 0
ω_{12}	0	0	-1	1	-1	-1	0	0	0	0	< 0
ω_{13}	-1	1	0	0	1	1	0	0	0	-1	> 0
ω_{14}	1	-1	0	0	1	0	1	0	0	1	> 0
ω_{15}	-1	1	0	0	-1	0	-1	0	0	-1	< 0
ω_{16}	1	-1	0	0	-1	-1	0	0	0	1	< 0
ω_{17}	0	0	0	0	0	0	0	0	0	0	$= 0$
ω_{18}	0	0	0	0	0	0	0	0	0	0	$= 0$
ω_{19}	0	0	0	0	0	0	0	0	0	0	$= 0$
ω_{20}	0	0	0	0	0	0	0	0	0	0	$= 0$
ω_{21}	0	0	0	0	0	0	0	0	0	0	$= 0$

To identify which algebra is described by this set of roots, we have to calculate the Cartan matrix. First, choose a basis

$$\gamma_1 \equiv \omega_9 = (0,\,0,\,1,\,-1,\,1), \quad \gamma_2 \equiv \omega_{10} = (0,\,0,\,-1,\,1,\,1), \quad \gamma_3 \equiv \omega_1 = (1,\,1,\,1,\,1,\,0),$$
$$\gamma_4 \equiv \omega_2 = (1,\,1,\,-1,\,-1,\,0), \quad \gamma_5 \equiv \omega_3 = (1,\,-1,\,1,\,-1,\,0), \tag{5.45}$$

and introduce a semi-ordering by expanding every ω_i in terms of $\gamma_1, \ldots, \gamma_5$. Define a root to be positive, if its first non-vanishing coefficient in the expansion is positive. The expansion coefficients are given in tab. (5.5), where we have also indicated whether the root is positive, negative, or zero. We will say that ω_i is greater than ω_j, if $\omega_i - \omega_j$ is positive.

Next, list the positive roots from tab. (5.5) separately, and reorder them in ascending order:

$$
\begin{array}{lclclcl}
\omega_3 & = & (1 \;\; -1 \;\; 1 \;\; -1 \;\; 0) & = & [0 \;\; 0 \;\; 0 \;\; 0 \;\; 1]_{\gamma_i} & = & \alpha_1 \text{ simple} \\
\omega_2 & = & (1 \;\; 1 \;\; -1 \;\; -1 \;\; 0) & = & [0 \;\; 0 \;\; 0 \;\; 1 \;\; 0]_{\gamma_i} & = & \alpha_4 \text{ simple} \\
\omega_1 & = & (1 \;\; 1 \;\; 1 \;\; 1 \;\; 0) & = & [0 \;\; 0 \;\; 1 \;\; 0 \;\; 0]_{\gamma_i} & = & \alpha_2 \text{ simple} \\
\omega_{10} & = & (0 \;\; 0 \;\; -1 \;\; 1 \;\; 1) & = & [0 \;\; 1 \;\; 0 \;\; 0 \;\; 0]_{\gamma_i} & = & \alpha_3 \text{ simple} \\
\omega_{14} & = & (1 \;\; -1 \;\; 0 \;\; 0 \;\; 1) & = & [0 \;\; 1 \;\; 0 \;\; 0 \;\; 1]_{\gamma_i} & = & \omega_3 + \omega_{10} \\
\omega_5 & = & (-1 \;\; 1 \;\; 1 \;\; -1 \;\; 0) & = & [1 \;\; -1 \;\; 0 \;\; 0 \;\; -1]_{\gamma_i} & = & \alpha_5 \text{ simple} \\
\omega_{13} & = & (-1 \;\; 1 \;\; 0 \;\; 0 \;\; 1) & = & [1 \;\; 0 \;\; 0 \;\; 0 \;\; -1]_{\gamma_i} & = & \omega_5 + \omega_{10} \\
\omega_9 & = & (0 \;\; 0 \;\; 1 \;\; -1 \;\; 1) & = & [1 \;\; 0 \;\; 0 \;\; 0 \;\; 0]_{\gamma_i} & = & \omega_5 + \omega_{14}
\end{array}
\tag{5.46}
$$

A *simple root* is by definition a root which cannot be written as the sum of two positive roots. From eq. (5.46), we see immediately that the roots

$$
\alpha_1 \equiv \omega_3, \quad \alpha_2 \equiv \omega_1, \quad \alpha_3 \equiv \omega_{10}, \quad \alpha_4 \equiv \omega_2, \quad \alpha_5 \equiv \omega_5 \tag{5.47}
$$

are simple. Now we calculate the scalar products of the simple roots in the Cartan-Weyl basis to obtain the Cartan matrix:

$$
A_{ij} = 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \begin{pmatrix} 2 & 0 & -4/3 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -4/3 & 0 & 2 \end{pmatrix} \tag{5.48}
$$

The Cartan matrix does not look familiar. Reordering the simple roots, so that those with non-vanishing scalar products are adjacent, we obtain

$$
A_{ij} = \left(\begin{array}{ccc|cc} 2 & -4/3 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -4/3 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right). \tag{5.49}
$$

This Cartan matrix already looks very similar to the Cartan matrix of the algebra $SU(4) \times SU(2) \times SU(2) \simeq SO(6) \times SO(4)$. One might think that all we have to do is to rescale one or more simple roots to get a -1 instead of a $-4/3$ in the Cartan matrix. Unfortunately, this does not work. We conclude that there is something wrong with the scalar product itself. In the next section, we will derive the correct expression for the scalar product after the symmetry breaking.

5.7.3 The Scalar Product of the Unbroken Gauge Group

For two elements $a, b \in \mathfrak{g}$, the *Killing form* is defined as

$$(a,b)_K \equiv \mathrm{Tr}\left((\mathrm{ad}\ a)(\mathrm{ad}\ b)\right). \tag{5.50}$$

The trace on the right hand side is most easily evaluated using the matrix representation of the operators ad a and ad b. For a detailed description on how to calculate the matrix representations, see sec. 5.7.1.

The Killing form cannot be used to define a scalar product on the Lie algebra $\mathfrak{g}$, because it is not positive definite. However, the restriction of the Killing form to the Cartan subalgebra $\mathfrak{h}$ is positive definite, and defines a scalar product on this space.

The Riesz representation theorem establishes a one-to-one correspondence between the Cartan subalgebra $\mathfrak{h}$ and its dual $\mathfrak{h}^*$. For each $\alpha \in \mathfrak{h}^*$, there exists a unique element $H_\alpha \in \mathfrak{h}$ such that

$$\forall t \in \mathfrak{h}: \quad \alpha(t) = (H_\alpha, t)_K\,. \tag{5.51}$$

This one-to-one correspondence allows us to define a scalar product on the root space $\mathfrak{h}^*$ as

$$\langle \alpha, \beta \rangle \equiv (H_\alpha, H_\beta)_K\,, \tag{5.52}$$

where H_α and H_β are the elements of the Cartan subalgebra $\mathfrak{h}$ which correspond to the roots α, β as described by eq. (5.51).

We now return to the task of calculating the scalar products of the simple roots $\alpha_1, \ldots, \alpha_5$. It is clear how to proceed. We first determine the elements of the Cartan subalgebra $h_{\alpha_1}, \ldots, h_{\alpha_5}$ which correspond to the simple roots. Then, the scalar products of the roots are given by eq. (5.52).

In the following, let α denote any one of the five simple roots. Choose a basis $H_1, \ldots, H_5$ of the Cartan subalgebra $\mathfrak{h}$, and expand h_α in terms of this basis:

$$h_\alpha = c_1 H_1 + \ldots + c_5 H_5 \tag{5.53}$$

Evaluating eq. (5.51) for each basis element H_i, we obtain five linear equations which will yield the values for the unknown coefficients $c_1, \ldots, c_5$:

$$\alpha(H_i) = (h_\alpha, H_i)_K\,, \quad i = 1, \ldots 5, \tag{5.54}$$

The right hand side of eq. (5.54) is easily calculated:

$$\begin{aligned}
(h_\alpha, H_1) &= c_1 (H_1, H_1)_K + \ldots + c_5 (H_5, H_1)_K = 12c_1 - 4c_2 \\
(h_\alpha, H_2) &= -4c_1 + 12c_2 \\
(h_\alpha, H_3) &= 12c_3 - 4c_4 \\
(h_\alpha, H_4) &= -4c_3 + 12c_4 \\
(h_\alpha, H_5) &= 8c_5
\end{aligned} \tag{5.55}$$

The expressions $(H_i, H_j)_K$ are evaluated using the definition of the Killing form in eq. (5.50), and the explicit expressions for ad H_i as given in eqs. (D.2-D.11). The results are listed in tab. (5.6).

	H_1	H_2	H_3	H_4	H_5
H_1	12	-4	0	0	0
H_2	-4	12	0	0	0
H_3	0	0	12	-4	0
H_4	0	0	-4	12	0
H_5	0	0	0	0	8

Table 5.6: The Killing form $(H_i, H_j)_K$ evaluated for the basis elements.

To derive the left hand side of eq. (5.54), remember that α^i (i-th entry of the root α) is by definition the eigenvalue of the operator E_α under the adjoint action of the Cartan generator H_i:

$$(\text{ad } H_i)\, E_\alpha = [H_i, E_\alpha] = \alpha(H_i)\, E_\alpha = \alpha^i\, E_\alpha. \tag{5.56}$$

The left hand side of eq. (5.54) is thus given by

$$\alpha(H_i) = \alpha^i \tag{5.57}$$

Now specialize to $\alpha = \alpha_5$ to calculate the constants $c_1, \ldots, c_5$ for the first root:

$$\left.\begin{array}{lcrclcl}
\alpha_5(H_1) &=& -1 &=& (h_{\alpha_5}, H_1) &=& 12c_1 - 4c_2 \\
\alpha_5(H_2) &=& 1 &=& (h_{\alpha_5}, H_2) &=& -4c_1 + 12c_2 \\
\alpha_5(H_3) &=& 1 &=& (h_{\alpha_5}, H_3) &=& 12c_3 - 4c_4 \\
\alpha_5(H_4) &=& -1 &=& (h_{\alpha_5}, H_4) &=& -4c_3 + 12c_4 \\
\alpha_5(H_5) &=& 0 &=& (h_{\alpha_5}, H_5) &=& 8c_5
\end{array}\right\} \begin{array}{l} c_1 = c_4 = -\frac{1}{16}, \\ c_2 = c_3 = \frac{1}{16}, \quad c_5 = 0 \end{array} \tag{5.58}$$

The calculations for $\alpha_1, \ldots, \alpha_4$ are completely analogous:

$$\begin{array}{ll}
\alpha = \alpha_1: & c_1 = c_3 = \frac{1}{16}, \quad c_2 = c_4 = -\frac{1}{16}, \quad c_5 = 0, \\
\alpha = \alpha_2: & c_1 = c_2 = c_3 = c_4 = \frac{1}{8}, \quad c_5 = 0, \\
\alpha = \alpha_3: & c_1 = c_2 = 0, \quad c_3 = -\frac{1}{16}, \quad c_4 = \frac{1}{16}, \quad c_5 = \frac{1}{8}, \\
\alpha = \alpha_4: & c_1 = c_2 = \frac{1}{8}, \quad c_3 = c_4 = -\frac{1}{8}, \quad c_5 = 0.
\end{array} \tag{5.59}$$

Summarizing, the Cartan generators corresponding to the simple roots are:

$$\begin{aligned}
h_{\alpha_1} &= \tfrac{1}{16}H_1 - \tfrac{1}{16}H_2 + \tfrac{1}{16}H_3 - \tfrac{1}{16}H_4 \\
h_{\alpha_2} &= \tfrac{1}{8}H_1 + \tfrac{1}{8}H_2 + \tfrac{1}{8}H_3 + \tfrac{1}{8}H_4 \\
h_{\alpha_3} &= -\tfrac{1}{16}H_3 + \tfrac{1}{16}H_4 + \tfrac{1}{8}H_5 \\
h_{\alpha_4} &= \tfrac{1}{8}H_1 + \tfrac{1}{8}H_2 - \tfrac{1}{8}H_3 - \tfrac{1}{8}H_4 \\
h_{\alpha_5} &= -\tfrac{1}{16}H_1 + \tfrac{1}{16}H_2 + \tfrac{1}{16}H_3 - \tfrac{1}{16}H_4
\end{aligned} \tag{5.60}$$

We can now calculate the scalar products of the simple roots using eq. (5.52), e.g. for $\alpha = \beta = \alpha_5$:

$$\begin{aligned}
\langle\alpha_5, \alpha_5\rangle &= (h_{\alpha_5}, h_{\alpha_5})_K \\
&= \left(-\tfrac{1}{16}H_1 + \tfrac{1}{16}H_2 + \tfrac{1}{16}H_3 - \tfrac{1}{16}H_4, -\tfrac{1}{16}H_1 + \tfrac{1}{16}H_2 + \tfrac{1}{16}H_3 - \tfrac{1}{16}H_4\right)_K \\
&= \tfrac{1}{16^2}\cdot 12 + \tfrac{1}{16^2}\cdot 4 + \tfrac{1}{16^2}\cdot 4 + \tfrac{1}{16^2}\cdot 12 + \tfrac{1}{16^2}\cdot 12 + \tfrac{1}{16^2}\cdot 4 + \tfrac{1}{16^2}\cdot 4 + \tfrac{1}{16^2}\cdot 12 \\
&= \tfrac{1}{4}
\end{aligned}$$

We have calculated all scalar products between the simple roots and listed them in tab. (5.7).

	α_1	α_2	α_3	α_4	α_5
α_1	1/4	0	-1/8	0	0
α_2	0	1/2	0	0	0
α_3	-1/8	0	1/4	0	-1/8
α_4	0	0	0	1/2	0
α_5	0	0	-1/8	0	1/4

Table 5.7: The scalar products of simple roots.

Using the scalar products given in tab. (5.7), the Cartan matrix is immediately calculated:

$$A_{ij} = 2\frac{\langle\alpha_i, \alpha_j\rangle}{\langle\alpha_j, \alpha_j\rangle} = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}. \tag{5.61}$$

After reordering the rows and columns, which corresponds to a renumbering of the simple roots, we obtain

$$A_{ij} = \left(\begin{array}{ccc|cc} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right), \tag{5.62}$$

and this proves the conjecture that the unbroken gauge group is SU(4) × SU(2) × SU(2).

5.8 Reducing the Rank of the New Algebra

If a direction in the Cartan subalgebra acquires a vacuum expectation value, the gauge symmetry will further break down. For definiteness, we consider $\langle H_2 \rangle \neq 0$, where H_2 denotes the Cartan generator of the original algebra SO(10).

Only those operators which commute with H_2 will survive. Consider tab. (5.4), where we have listed the 21 operators of the unbroken gauge group SU(4) × SU(2) × SU(2). The commutators

$$[H_2, E_\alpha \pm E_\beta] = \alpha^{(2)} E_\alpha \pm \beta^{(2)} E_\beta \tag{5.63}$$

are easily calculated, and it is clear that these commutators vanish if and only if $\alpha^{(2)} = \beta^{(2)} = 0$. The roots of SO(10) are given in tab. (D.2). From the 21 generators of the Pati-Salam group, 13 generators commute with H_2, and these are given in tab. (5.8).

$\tilde{E}_3$	$=$	$E_5 - E_8$	$\tilde{E}_{16}$	$=$	$E_{30} + E_{31}$
$\tilde{E}_4$	$=$	$E_6 - E_7$	$\tilde{E}_{17}$	$=$	$E_{33} - E_{34}$
$\tilde{E}_5$	$=$	$E_9 + E_{12}$	$\tilde{E}_{18}$	$=$	$E_{35} - E_{36}$
$\tilde{E}_6$	$=$	$E_{10} + E_{11}$	$\tilde{E}_{19}$	$=$	$E_{37} + E_{38}$
$\tilde{E}_7$	$=$	$E_{13} + E_{14}$	$\tilde{E}_{20}$	$=$	$E_{39} + E_{40}$
$\tilde{E}_8$	$=$	$E_{15} + E_{16}$	$\tilde{E}_{21}$	$=$	H_5
$\tilde{E}_{15}$	$=$	$E_{29} + E_{32}$			

Table 5.8: The 13 generators which survive for $\langle H_2 \rangle \neq 0$.

To see which algebra is described by these 13 generators, we proceed as before. A maximal commuting set of operators is given by

$$U_1 \equiv E_{29} + E_{32}, \quad U_2 \equiv E_{30} + E_{31}, \quad U_3 \equiv H_5. \tag{5.64}$$

Simultaneously diagonalizing the matrix representations of (ad U_1), (ad U_2), and (ad U_3), we find the root vectors of the algebra, which we list in tab. (5.9).

No.	Root			No.	Root		
ω_1	1	1	0	ω_8	1	-1	-1
ω_2	-1	1	0	ω_9	0	0	1
ω_3	1	-1	0	ω_{10}	0	0	-1
ω_4	-1	-1	0	ω_{11}	0	0	0
ω_5	-1	1	1	ω_{12}	0	0	0
ω_6	1	-1	1	ω_{13}	0	0	0
ω_7	-1	1	-1				

Table 5.9: The roots and Cartans of the surviving gauge group.

Fix the vectors

$$v_1 = (1,\, 1,\, 0), \quad v_2 = (-1,\, 1,\, 0), \quad v_3 = (-1,\, 1,\, 1) \tag{5.65}$$

as the basis to introduce the semi-ordering. Then, the simple roots are

$$\alpha_1 = (1,\, 1,\, 0), \quad \alpha_2 = (0,\, 0,\, -1), \quad \alpha_3 = (-1,\, 1,\, 1). \tag{5.66}$$

The Killing form is easily calculated and is given in tab. (5.10).

	U_1	U_2	U_3
U_1	8	-4	0
U_2	-4	8	0
U_3	0	0	6

Table 5.10: The Killing form.

The Cartan generators corresponding to the simple roots are

$$\begin{aligned} h_{\alpha_1} &= \tfrac{1}{4}U_1 + \tfrac{1}{4}U_2, \\ h_{\alpha_2} &= -\tfrac{1}{6}U_3, \\ h_{\alpha_3} &= -\tfrac{1}{12}U_1 + \tfrac{1}{12}U_2 + \tfrac{1}{6}U_3. \end{aligned} \tag{5.67}$$

The scalar products of the simple roots are then easily calculated. We list them in tab. (5.11).

	α_1	α_2	α_3
α_1	1/2	0	0
α_2	0	1/6	-1/6
α_3	0	-1/6	1/3

Table 5.11: The scalar products.

The Cartan matrix is then given by

$$A_{ij} = 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \left(\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & 2 & -1 \\ 0 & -2 & 2 \end{array} \right), \tag{5.68}$$

which corresponds to the algebra SU(2) $\times$ Sp(2) $\simeq$ SU(2) $\times$ SO(5).

5.9 Possible Rank Reducing Symmetry Breakings Starting from Pati-Salam

The directions in the Cartan subalgebra corresponding to $H_1, \ldots, H_4$ are fully rotated by the $\mathbb{Z}_2$ symmetry, so the Wilson lines in these directions will be continuous (see section 6.1.2). To find all possible symmetry breaking patterns, we consider the cases

$$\langle n_1 H_1 + n_2 H_2 + n_3 H_3 + n_4 H_4 \rangle \neq 0, \quad n_i \in \mathbb{Z}, \quad n_i \text{ relatively prime.} \tag{5.69}$$

Note that each Cartan generator acts on the root operators of the Pati-Salam group as

$$[H_i, E_\alpha \pm E_\beta] = \alpha^{(i)} E_\alpha \pm \beta^{(i)} E_\beta, \tag{5.70}$$

and clearly these root operators only survive for $\alpha^{(i)} = \beta^{(i)} = 0$. If we consider a linear combination of Cartan generators $H = n_1 H_1 + n_2 H_2 + n_3 H_3 + n_4 H_4$, its action on the root operators is

$$[H, E_\alpha \pm E_\beta] = (n_1\, \alpha^{(1)} + \ldots + n_4\, \alpha^{(4)}) E_\alpha \pm (n_1\, \beta^{(1)} + \ldots + n_4\, \beta^{(4)}) E_\beta, \tag{5.71}$$

and we may satisfy the relations

$$n_1\, \alpha^{(1)} + \ldots + n_4\, \alpha^{(4)} = n_1\, \beta^{(1)} + \ldots + n_4\, \beta^{(4)} = 0 \tag{5.72}$$

# generators	Algebra	Wilson line
1	U(1)	$\langle 2H_1 + H_2 + 3H_3 + 4H_4 \rangle$
2	$\mathrm{U}(1)^2$	$\langle 2H_1 + 3H_2 + H_3 + H_4 \rangle$
3	$\mathrm{U}(1)^3$	$\langle 2H_1 + 2H_2 + H_3 + H_4 \rangle$
4	SU(2) × U(1)	$\langle 2H_1 + H_2 + 2H_3 \rangle$
6	SU(2) × SU(2)	$\langle 2H_1 + H_2 + H_3 \rangle$
7	SU(2) × SU(2) × U(1)	$\langle 2H_1 + H_2 + H_3 + H_4 \rangle$
8	SU(2) × SU(2) × $\mathrm{U}(1)^2$	$\langle 2H_1 + H_2 \rangle$
13	SO(5) × SU(2)	$\langle H_2 \rangle$
21	SO(6) × SO(4)	

Table 5.12: All symmetry breakings of the type $\mathrm{SO}(10) \stackrel{\tilde{s}}{\rightarrow}$ Pati-Salam $\stackrel{\mathrm{WL}}{\rightarrow} \mathfrak{g}$.

for $\alpha^{(i)}, \beta^{(i)} \neq 0$, i.e. it may occur that although neither H_i nor H_j commutes with a root operator, that the linear combination does.

The above linear combinations exhaust all possible symmetry breaking patterns for continuous Wilson lines. The result is presented in tab. (5.12).

5.10 From SO(10) to Other Subalgebras

In sections 5.7 - 5.9 we explored the symmetry breakdown of SO(10) to Pati-Salam by s, which is a rotation in the root lattice, and investigated all possible subalgebras, which can be obtained from the Pati-Salam group via continuous Wilson lines, thereby reducing the rank of the original algebra.
We are now interested in obtaining all symmetry breakdowns of SO(10), induced by rotations in root space. Considering the extended Dynkin diagram of SO(10), we find all maximal, regular subalgebras:

$$\begin{aligned} \mathrm{SO}(10) &\rightarrow \mathrm{SU}(5) \times \mathrm{U}(1) \\ \mathrm{SO}(10) &\rightarrow \mathrm{SO}(8) \times \mathrm{U}(1) \\ \mathrm{SO}(10) &\rightarrow \mathrm{SO}(6) \times \mathrm{SO}(4) \end{aligned} \tag{5.73}$$

The last line, $\mathrm{SO}(10) \rightarrow \mathrm{SO}(6) \times \mathrm{SO}(4) \simeq \mathrm{SU}(4) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$, corresponds to $\tilde{s}$. To find the other symmetry breakdowns necessitates a systematic study.

The Weyl group of SO(10) has 1920 elements. Noting that elements in the same conjugacy class give rise to the same symmetry breaking pattern, we can concentrate on the 18 conjugacy classes. The problem has now been reduced to finding representatives of the 18 conjugacy classes of SO(10). Then we can repeat the analysis of sections 5.7 - 5.9 to find the gauge symmetry breakdown for each class.

5.11 Conjugacy Classes of the Weyl Group and Carter Diagrams

The Weyl groups and their conjugacy classes have been determined separately for each type of Lie algebra in the first half of the last century, but a uniform approach to finding the conjugacy classes has first been developed in ref. [51]. In the following, we give a short account of the basic ideas. Special emphasis is placed on the calculational methods, and all proofs of the theorems we use have been omitted. Unless otherwise indicated, all results in this section are due to ref. [51].

5.11.1 Carter Diagrams

Every element $w \in \mathcal{W}$ can be written as a product of 2 involutions, i.e. $w = w_1 w_2$, where $w_1^2 = w_2^2 = \mathbb{1}$. Furthermore, every involution in $\mathcal{W}$ can be expressed as a product of reflections corresponding to mutually orthogonal roots. Putting this together, we have the following decomposition for an arbitrary element of the Weyl group:

$$w = w_1 w_2 = \left(r_{\alpha_1} r_{\alpha_2} \ldots r_{\alpha_k}\right)\left(r_{\alpha_{k+1}} r_{\alpha_{k+2}} \ldots r_{\alpha_{k+h}}\right) \tag{5.74}$$

It can be proved that the roots $\alpha_1, \ldots, \alpha_{k+h}$ are linearly independent. For each such decomposition, we define a graph Γ as follows. For every α_i, $i = 1, \ldots, k+h$, we introduce a node, and the nodes corresponding to α_k and α_ℓ are joined by $n_{k\ell} \cdot n_{\ell k}$ lines, where

$$n_{k\ell} = 2\frac{\alpha_k \cdot \alpha_\ell}{\alpha_k \cdot \alpha_k}, \qquad n_{\ell k} = 2\frac{\alpha_\ell \cdot \alpha_k}{\alpha_\ell \cdot \alpha_\ell}. \tag{5.75}$$

The alert reader will have noticed the striking similarity to the definition of Dynkin diagrams. In fact, it will turn out that the graphs we have just defined are the Dynkin diagrams, plus an additional set of diagrams containing loops.
It is easy to see that if $w \in \mathcal{W}$ corresponds to the graph Γ, then any conjugate element w' also has a decomposition with the same graph Γ:
Let $w = r_{\alpha_1} r_{\alpha_2} \ldots r_{\alpha_k} \cdot r_{\alpha_{k+1}} r_{\alpha_{k+2}} \ldots r_{\alpha_{k+h}}$. Then $w' = \sigma w \sigma^{-1}$ corresponds to the decomposition $w' = r_{\sigma(\alpha_1)} r_{\sigma(\alpha_2)} \ldots r_{\sigma(\alpha_k)} \cdot r_{\sigma(\alpha_{k+1})} r_{\sigma(\alpha_{k+2})} \ldots r_{\sigma(\alpha_{k+h})}$, because conjugation is in this case just a change of basis. As σ is generated by reflections and thus orthogonal, it preserves the scalar products, and the graph of w' is the same as the graph of w.
It is important to point out that what we have proved does not mean that there is only one graph corresponding to w. One element of the Weyl group may correspond to more

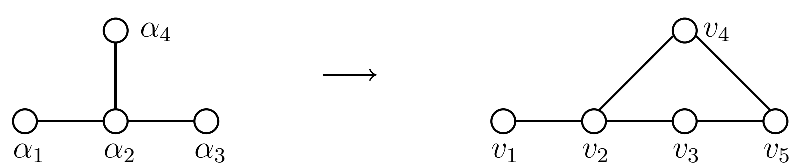
than one graph. Any conjugate element will then have a decomposition corresponding to the same graph. This is not really a problem, because in the worst case, we obtain too many diagrams, which will in any case cover all the conjugacy classes.
To classify the conjugacy classes, we need the converse correspondence, i.e. which conjugacy classes arise from a given diagram, and a method to determine all admissible diagrams. Unfortunately, also in this case the correspondence is not unique, i.e. to one diagram there may correspond more than one conjugacy class. This shortcoming is more serious. However, the exceptional cases, in which one diagram corresponds to more than one conjugacy class, are listed in ref. [51].
For the diagram associated to a conjugacy class of the Weyl group of a simple Lie algebra $\mathfrak{g}$, one of the following two statements is correct:

- Γ is a Dynkin diagram of $\mathfrak{g}$ or of one of its subgroups $\mathfrak{h} \subset \mathfrak{g}$.
- Γ can be obtained from a Dynkin diagram Γ' of $\mathfrak{g}$ or of one of its subgroups $\mathfrak{h} \subset \mathfrak{g}$ by replacing one or more connected sub-diagrams of Γ' by diagrams with cycles. Each sub-diagram of Γ' can only be replaced by an exhaustive list of diagrams with cycles, corresponding to the Lie algebra which the sub-diagram represents.

The diagrams with cycles are listed tab. 2 of ref. [51]. Obtaining all Carter diagrams for a given algebra $\mathfrak{g}$ is now straightforward. Start from the extended Dynkin diagram of $\mathfrak{g}$, and remove one or more nodes in all possible ways. Then recursively repeat this process with the diagrams just obtained. (So far, this is the standard Dynkin algorithm to obtain all subgroups of a given algebra.) In each diagram obtained by this procedure, replace an arbitrary number (including zero) of connected sub-diagrams by diagrams with cycles corresponding to the Lie algebra which the sub-diagram represents.
Before considering the case of SO(10) in some detail, we show how to obtain the conjugacy class representative corresponding to a given Carter diagram.

5.11.2 An Example: The Carter Diagram $D_5(a_2)$

We will illustrate the procedure of relating Carter diagrams to representatives of conjugacy classes by means of an explicit example. We want to realize the Carter diagram $D_5(a_2)$, and first note that it may be obtained from the Dynkin diagram of SO(8) by the addition of one further root, which is connected to α_3 and α_4:



The four simple roots of SO(8) are given by

$$\alpha_1 = (1\text{-}1\,0\,0), \quad \alpha_2 = (0\,1\text{-}1\,0), \quad \alpha_3 = (0\,0\,1\text{-}1), \quad \alpha_4 = (0\,0\,1\,1)\,.$$

We can now guess the v_i for the above Carter diagram. Take as four of the v_i's the simple roots of SO(8) (with the juxtaposition of an additional zero in the last column of each root, as we are looking for the Carter diagram of a rank 5 algebra), and then look for a 5th root v_5 which fulfills $v_5 \cdot v_3 = v_5 \cdot v_4 = \pm 1$, and $v_5 \cdot v_i = 0$ for $i \neq 3, 4$. Note that v_5 need not be simple, it is just required to be linearly independent from the other roots:

$$v_1 = (1\,\text{-}1\,0\,0\,0), \quad v_2 = (0\,1\,\text{-}1\,0\,0), \quad v_3 = (0\,0\,1\,\text{-}1\,0), \quad v_4 = (0\,0\,\text{-}1\,\text{-}1\,0), \quad v_5 = (0\,0\,0\,1\,1)$$

Actually, for convenience we have set $v_4 = -\alpha_4$, but this is irrelevant for the calculation.

Now we decompose $\{v_1 \ldots, v_5\}$ into 2 disjoint sets $\{v_1, v_3, v_4\}$ and $\{v_2, v_5\}$, where in each set, the vectors are mutually orthogonal. This is the representation of the Weyl group element in terms of 2 involutions:

$$w = (r_{v_1} r_{v_3} r_{v_4})\,(r_{v_2} r_{v_5})$$

This element of the Weyl group induces the symmetry breakdown SO(10) $\rightarrow$ SO(8)$\times$U(1). In section 5.11.3 we will see that w corresponds to the conjugacy class 8. Here, we have an example for the aforementioned fact that one conjugacy class may correspond to more than one diagram, in this case $D_5(a_2)$ and D_2.

5.11.3 The Classification

To find the conjugacy classes of SO(10), we could in principle proceed as follows. First, we apply the algorithm described at the end of section 5.11.1 to find all Carter diagrams, and next, we determine the Weyl group element corresponding to each Carter diagram along the lines described in section 5.11.2.

In practice, however, this procedure turns out to be too tedious. The conjugacy classes of Weyl groups have meanwhile been incorporated into several computer algebra programs. We used the Maple package Coxeter-Weyl [52] to find representatives of each conjugacy class in terms of simple Weyl reflections. The determination of the gauge symmetry breakdown associated to the conjugacy class is then easily calculated using the techniques developed in section 5.7. Our results are presented in tab. 5.13.

The symmetry breakings associated with Weyl group conjugacy classes have been determined before [53, 54]. Inspired by heterotic string constructions, ref. [53] concentrates on the case of E_8. Using techniques of conformal field theory, the conjugacy classes are related to shift vectors, and the gauge symmetry breakdown can then be easily calculated. This procedure is to be contrasted with our approach, where we take no detour over the equivalent shift and need not immerse ourselves in conformal field theory.

Ref. [54] presents the Weyl group conjugacy classes for all classical and exceptional Lie algebras of rank less than or equal to eight. Comparing tab. XI in ref. [54] to our results in tab. 5.13, we find an apparent contradiction. For the conjugacy class 12, we have the gauge group SU(3) $\times$ U(1)^3, whereas in ref. [54] we find SU(3) $\times$ SU(2). Note that both

No.	Carter Diagram	Weyl Group Element	ord	Gauge group
1	$\emptyset$	$\mathbb{1}$	1	SO(10)
2	A_1	r_5	2*	SU(4) × SU(2) × U(1)
3	$A_1 + A_1$	$r_5\, r_2$	2*	SU(4) × U(1)^2
4	A_2	$r_5\, r_3$	3	SU(4) × SU(2) × U(1)
5	$A_2 + A_1$	$r_5\, r_3\, r_1$	6*	SU(2)^2 × U(1)^3
6	A_3	$r_5\, r_3\, r_2$	4*	SU(2)^2 × U(1)^3
7	A_4	$r_5\, r_3\, r_2\, r_1$	5	SU(2)^2 × U(1)^3
8	D_2	$r_5\, r_4$	2	SO(8) × U(1)
9	$A_1 + D_2$	$r_5\, r_4\, r_2$	2*	SU(2)^3 × U(1)^2
10	$A_2 + D_2$	$r_5\, r_4\, r_2\, r_1$	6	SU(2)^3 × U(1)^2
11	D_3	$r_5\, r_4\, r_3$	4	SU(4) × U(1)^2
12	$A_1 + D_3$	$r_5\, r_4\, r_3\, r_1$	4	SU(3) × U(1)^3
13	$D_2 + D_2$	$r_5\, r_4\, r_3\, r_5\, r_4\, r_3\, r_2\, r_3\, r_5\, r_4\, r_3\, r_2$	2	SU(4) × SU(2)^2
14	$D_4(a_1)$	$r_5\, r_4\, r_3\, r_5\, r_2\, r_3$	4	SU(2)^3 × U(1)^2
15	D_4	$r_5\, r_4\, r_3\, r_2$	6	SU(2)^2 × U(1)^3
16	$D_3 + D_2$	$r_5\, r_4\, r_3\, r_5\, r_4\, r_3\, r_2\, r_3\, r_5\, r_4\, r_3\, r_2\, r_1$	4	SU(2)^4 × U(1)
17	$D_5(a_1)$	$r_5\, r_4\, r_3\, r_5\, r_2\, r_3\, r_1$	12	U(1)^5
18	D_5	$r_5\, r_4\, r_3\, r_2\, r_1$	8	U(1)^5

Table 5.13: Weyl group conjugacy classes of SO(10) and the associated symmetry breakings. The 4th column lists the order of the Weyl group element on the root lattice, whereby a star indicates, when the order of its lift on the algebra is doubled.

gauge groups have dimension 11, which is in accordance with the number of invariants we found. As only our gauge group is of rank 5, we are confident that our result is correct.

5.12 Discussion of the Results

The established orbifold constructions (both in the context of heterotic string theory and field theory) associate the orbifold twist in space-time with a shift in the gauge degrees of freedom. Comparing the shift embedding (chapters 2-4) to the rotational embedding described in this chapter, we find that, as far as the calculations are concerned, the shift embedding is simpler. Furthermore, the Kač theorem provides us with the necessary tools for the classification of our models (chapter 4). Despite of all these features, the failure of this construction to reduce the rank of the original gauge group represents a serious obstacle in obtaining (semi-)realistic models of particle physics.

In the present chapter, we have systematically developed the ideas and calculational tools for realizing the gauge embedding as a rotation in the root lattice of the gauge group. The starting point for our investigations was ref. [27], where the idea of embedding the twist as a rotation was considered for the first time. There, the action of the rotation in the lattice was not lifted to the algebra level. In particular, the phase in the transformation of the step operators (cf. eq. (5.8)) was not determined, which may be problematic in some cases.

Anticipating the results of chapter 6, consider the gauge group E_6 and the conjugacy class 4 of its Weyl group (cf. tab. 6.1). Disregarding the phases, naive counting arguments suggest 46 invariants and the unbroken gauge group to be $SO(10) \times U(1)$. Calculating the phases, we find that some of them are actually -1 so that some of the invariant combinations vanish, thus giving only 30 invariants corresponding to the unbroken gauge group $SO(8) \times U(1)^2$.

Even if the dimension and the rank of the gauge group are known beyond doubt, this information may not be enough to identify the gauge group. In section 7.2 we discuss several examples, where the dimension and the rank of gauge groups coincide, although the gauge groups are distinct. There are other pieces of information which one may add, such as the order of the automorphism or the number of U(1) factors, but there are cases, where this approach is not successful. When all indirect methods fail, one must resort to the methods developed in this chapter. Needless to say, analogous statements apply to the case of identifying the representations transforming under the unbroken gauge group.

The gauge symmetries associated to the conjugacy classes of Weyl groups have been determined before, namely in ref. [53] for E_8 and in ref. [54] for all simply laced algebras[1] of rank less than or equal to 8. Both publications have in common that they first derive the shift which is equivalent to the automorphism corresponding to the Weyl group conjugacy class, and then calculate the associated symmetry breakdown. Using techniques of conformal field theory, the equivalent shift is found by comparing its properties to those of the automorphism, and in the end, is based on going through all possibilities. (This procedure represents an alternative to the methods described in section 5.7, but not for the following sections.)

In contrast, we directly calculate the Lie algebra automorphism corresponding to a Weyl group element. The calculation of the lift only involves elementary algebra, and the methods applied are conceptionally easier to grasp. The disadvantage might be that we have to keep track of the structure constants and handle large matrix representations, which necessitates the use of computers.

[1]For the conjugacy class 12 in tab. 5.13, we find a contradiction to the result in ref. [54]. Details are given in section 5.11.3.

Chapter 6

An E_6 Orbifold GUT

We will use the methods developed in the previous chapter to construct a six dimensional orbifold model with an E_6 bulk gauge group, which breaks down to the Standard Model with one extra U(1) factor in four dimensions. The rank of the bulk gauge group is reduced by the continuous Wilson line. This symmetry breakdown is smooth and corresponds to a standard field theory Higgs mechanism. We present an appealing geometric picture of the symmetry breakdown, and discuss the embedding of our model into heterotic string theory.

6.1 Orbifold Constructions in Six Dimensions

We will briefly review orbifold constructions [12, 13]. Starting with six dimensions, two dimensions are compactified on an orbifold

$$\mathcal{O} = T^2 \big/ P. \tag{6.1}$$

An orbifold is defined to be the quotient of a torus over a discrete set of isometries, called the *point group* P. Alternatively, one can start with the complex plane, and first identify points which differ by translations (lattice shifts) L in order to arrive at the torus, and then mod out the action of P:

$$T^2 = \mathbb{C}\big/L \quad \rightsquigarrow \quad \mathcal{O} = T^2\big/P = \mathbb{C}\big/S. \tag{6.2}$$

S is called the *space group*, and is the semidirect product of the point group P and the translation group L defining the torus. For the action of the point group to be well-defined, elements of P must be automorphisms of the lattice defining the torus.
The original theory in six dimensions is taken to be a grand unified gauge theory. The action of the point group on the space-time degrees of freedom is generically accompanied by an action on the gauge degrees of freedom, $P \hookrightarrow G$, where the embedding is a homomorphism. G is a subgroup of the automorphisms of the Lie algebra $\mathfrak{g}$ describing the gauge symmetry, and is called the *gauge twisting group*.

6.1.1 Embedding the Twist in the Gauge Degrees of Freedom

Any inner Lie algebra automorphism σ of finite order N can be realized as a shift $X \mapsto X + V$, in the root lattice Λ of $\mathfrak{g}$ such that its action on the step operators corresponding to the simple roots α_k and the extended root α_0 is given by [47]

$$\sigma\left(E_{\alpha_k}\right) = \exp\left(2\pi i \alpha_k \cdot V\right) E_{\alpha_k}, \quad k = 0, \ldots, \text{rank } \mathfrak{g}, \tag{6.3}$$

with $\alpha_0 = -\sum_{j=1}^{\text{rank } \mathfrak{g}} \alpha_j k_j$, where k_j are the Kac labels. On the Cartan generators H_i, the action of σ is trivial. Thus none of the Cartan generators is projected out. It is therefore clear that by this construction the rank of the algebra cannot be reduced.
To be able to reduce the rank of the gauge group we need an alternative approach to symmetry breaking in which the action of the twist in the gauge algebra transforms some of the Cartan generators non-trivially. Such a mechanism can be realized as follows. As the elements of P are automorphisms of the lattice defining T^2, it is natural to associate with them automorphisms of the root lattice Λ, i.e. to realize the twist in the space-time as a twist in the gauge degrees of freedom [27]. The automorphisms of the root lattice, the Weyl group $\mathcal{W}$ of $\mathfrak{g}$, is generated by the reflections

$$r_{\alpha_k} : \xi \mapsto \xi - 2\frac{\langle \alpha_k, \xi\rangle}{\langle \alpha_k, \alpha_k\rangle}\alpha_k, \tag{6.4}$$

where the α_k are the simple roots of $\mathfrak{g}$. There is a natural lift of r_{α_k} to the Lie algebra $\mathfrak{g}$ given by [50, 55, 56]

$$\tilde{r}_{\alpha_k} = \exp\left(\frac{i\pi}{2}\left(E_{\alpha_k} + E_{-\alpha_k}\right)\right), \tag{6.5}$$

so that the lift $\tilde{w}$ of an arbitrary element of $w \in \mathcal{W}$ is given by the product of lifts of simple Weyl reflections. Under the lift of a single Weyl reflection r_α, the generators of the Lie algebra transform as

$$\begin{aligned} \tilde{r}_\alpha\left(\lambda \cdot H\right)\tilde{r}_\alpha^{-1} &= \left(r_\alpha(\lambda)\right) \cdot H, \\ \tilde{r}_\alpha E_\beta \tilde{r}_\alpha^{-1} &= c_\alpha(\beta)\, E_{r_\alpha(\beta)}. \end{aligned} \tag{6.6}$$

For an explicit calculation, the complex phases $c_\alpha(\beta)$ must be determined. In section 5.3, we express these phases in terms of the structure constants of the algebra.
As the order of the Weyl group is finite, and the automorphism group of a Lie algebra is itself a Lie algebra, it is clear that the relation between the lift described above and the automorphisms realized by shifts cannot be one-to-one. Refs. [50,53,54] are concerned with determining the shift vectors corresponding to the conjugacy classes of the Weyl group.
The definition of the lift as given by eq. (6.5) is not unique. One can also first shift the lattice by an arbitrary vector v before twisting it with $\tilde{r}_{\alpha_k}$ [50, 56] :

$$\tilde{r}'_{\alpha_k} = \tilde{r}_{\alpha_k} \exp\left(2\pi v \cdot H\right) = \exp\left(\frac{i\pi}{2}\left(E_{\alpha_k} + E_{-\alpha_k}\right)\right) \exp\left(2\pi v \cdot H\right). \tag{6.7}$$

In the following, we shall see that this generalization will lead to many new possibilities for model building.
A nontrivial consistency condition is that the order of the algebra automorphism should be a divisor of the order of the point group. Note that the Cartan generators transform non-trivially, and step operators need not be eigenstates under the orbifold action. Starting from a set of generators in the Cartan-Weyl basis yields a set of invariant generators which are typically not in the Cartan-Weyl basis of the unbroken algebra. The invariant generators consist of the sum of a generator and its images. In order to find the Cartan-Weyl basis of the unbroken algebra one proceeds as follows. First, we identify a Cartan subalgebra by taking the (linear combinations of) Cartan generators of the original algebra which are invariant under the orbifold action. These we supplement by linear combinations of step operators which are even under the orbifold group and are not charged under the invariant Cartan generators. If the number of invariant Cartan generators differs from the rank of the original gauge symmetry by more than one we need to supplement by more than one linear combination of step operators. These should mutually commute.
After a Cartan subalgebra is chosen one has to simultaneously diagonalize its adjoint action on the remaining invariant combinations of step operators. Eventually, this procedure leads to the unbroken gauge symmetry written in the Cartan-Weyl basis allowing for an identification of the group from the Cartan matrix or the Dynkin diagram.
Once we have clarified this, we shall see that the rank of the gauge group still remains the same. Some of the Cartan operators in the original gauge group have been projected out, but some new ones (linear combinations of the step operators) appear and will replace the former ones. This is a result of the fact that, if one just considers the point group and not the full space group of the orbifold, any rotation in the gauge group can be represented by a shift [20]. Rank reduction needs more than just this. It also needs a representation of the full space group in the gauge group, thus additional Wilson lines.

6.1.2 Wilson Lines

So far we have embedded the twist in space-time as a twist in the gauge degrees of freedom. Analogously, each shift in the space-time defining the torus can be associated with a shift in the co-root lattice[1] of $\mathfrak{g}$, and this corresponds to a Wilson line W. Around non-contractible loops, the operators will then transform with a phase,

$$E_\alpha \to \exp\left(2\pi i \alpha \cdot W\right) E_\alpha. \tag{6.8}$$

A Wilson line might thus remove some of the step operators E_α. If it projects out step operators that play the role of Cartan operators in the "twisted" gauge group the rank of the gauge group can thus be reduced.
When considering Wilson lines in the presence of the space-time twist realized as a rotation, there are 3 cases to be distinguished:

(i) W is left invariant by the twist s,

[1] For algebras of type ADE, the co-root lattice is equal to the root lattice.

(ii) W is completely rotated by the twist s,

(iii) Some of the components of W are rotated by the twist s.

In the following, we will concentrate on the first 2 cases. Consider the case, when the Wilson line is invariant, and for the sake of briefness, assume that the twist is of order 2. Applying twice the same gauge transformation must act as the identity. Denoting the gauge degrees of freedom by X, we have

$$X \stackrel{s}{\to} s(X)+W \stackrel{s}{\to} s^2(X)+s(W)+W = X+2W, \tag{6.9}$$

where we used the fact that s is of order 2, and leaves W invariant. We conclude that $2W$ must be in the co-root lattice, and hence is discrete.
Let us now consider the case, when the Wilson line is completely rotated by the twist s, and repeat the previous arguments:

$$X \stackrel{s}{\to} s(X)+W \stackrel{s}{\to} s^2(X)+s(W)+W = X-W+W = X. \tag{6.10}$$

In contrast to the case, where the Wilson line was invariant, W is now not restricted to lie in a lattice, and hence can be rescaled by an arbitrary real parameter λ.
The step operators are still subject to the transformation law given in eq. (6.8) around non-contractible loops, but this time the continuous parameter λ in

$$E_\alpha \to \exp\left(2\pi i\alpha\cdot\lambda W\right) E_\alpha \tag{6.11}$$

will have the effect that there will always be a non-trivial phase, if $\alpha\cdot W \neq 0$. Hence, the corresponding operators will be projected out and this leads to rank reduction. In section 6.2.2 we shall discuss this in detail in an explicit example.

6.2 The Breaking of E$_6$

Let us now consider E$_6$ as an explicit example. It appears at an intermediate stage in many of the phenomenologically interesting models derived from the heterotic E$_8$ × E$_8$ string theory. The Weyl group of E$_6$ has 51,840 elements, each of which is in one of 25 conjugacy classes [51]. In tab. 6.1, we list for each conjugacy class one representative in terms of simple Weyl reflections [52], its order on the root lattice, the order of the corresponding lift to the algebra, and the associated symmetry breaking.
Naively one might think that with this, all the possibilities of breakdown of E$_6$ would be classified. But this is actually not the case. As we have explained in the last section, we can generalize the lift (cf. eq. (6.7)) by first shifting the lattice by an arbitrary vector v before twisting it by an element of one of the conjugacy classes. This opens up many more possibilities which, unfortunately, are difficult to classify in full generality.
In fact we shall here consider an example with a generalized lift. Following the argumentation of ref. [21, 57] we are particularly interested in an SO(10) gauge symmetry at some

No.	Weyl Group Element	ord	Gauge group
1	$\mathbb{1}$	1	E_6
2	r_1	2^*	$SU(6) \times U(1)$
3	$r_1 r_2$	3	$SU(6) \times U(1)$
4	$r_1 r_6$	2^*	$SO(8) \times U(1)^2$
5	$r_1 r_2 r_3$	4^*	$SU(4) \times U(1)^3$
6	$r_6 r_2 r_3 r_6 r_4 r_3$	4	$SU(3)^2 \times SU(2) \times U(1)$
7	$r_1 r_6 r_2$	6^*	$SU(4) \times U(1)^3$
8	$r_6 r_2 r_3 r_4$	6	$SU(3)^2 \times U(1)^2$
9	$r_1 r_6 r_2 r_3$	5	$SU(4) \times U(1)^3$
10	$r_1 r_6 r_2 r_3 r_6 r_4 r_3$	12^*	$U(1)^6$
11	$r_1 r_6 r_2 r_1 r_3 r_6 r_2 r_3 r_4 r_3 r_5 r_4$	6	$SU(2)^3 \times U(1)^3$
12	$r_1 r_6 r_4$	2^*	$SU(3)^2 \times SU(2) \times U(1)$
13	$r_1 r_6 r_3 r_4$	4^*	$SU(3) \times U(1)^4$
14	$r_1 r_6 r_2 r_3 r_4$	8	$SU(2)^2 \times U(1)^4$
15	$r_1 r_2 r_4 r_5$	3	$SO(8) \times U(1)^2$
16	$r_1 r_2 r_3 r_4 r_5$	6^*	$SU(2) \times U(1)^5$
17	$r_1 r_6 r_2 r_3 r_6 r_4 r_3 r_5$	9	$SU(2) \times U(1)^5$
18	$r_1 r_6 r_2 r_4$	6^*	$SU(2)^3 \times U(1)^3$
19	$r_1 r_6 r_2 r_3 r_4 r_5$	12	$U(1)^6$
20	$r_1 r_6 r_2 r_3 r_5$	10^*	$SU(2) \times U(1)^5$
21	$r_6 r_2 r_3 r_6 r_2 r_3 r_4 r_3 r_6 r_2 r_3 r_4$	2	$SU(6) \times SU(2)$
22	$r_1 r_6 r_2 r_3 r_6 r_2 r_3 r_4 r_3 r_6 r_2 r_3 r_4$	4^*	$SU(2)^4 \times U(1)^2$
23	$r_1 r_6 r_2 r_4 r_5$	6^*	$SU(2)^3 \times U(1)^3$
24	$r_1 r_6 r_2 r_3 r_6 r_2 r_3 r_4 r_3 r_6 r_2 r_3 r_4 r_5$	6	$SU(2)^4 \times U(1)^2$
25	$r_1 r_6 r_2 r_1 r_3 r_6 r_2 r_1 r_3 r_4 r_3 r_6 r_2 r_1 r_3 r_4 r_5 r_4 r_3$ $r_6 r_2 r_3 r_4 r_5$	3	$SU(3)^3$

Table 6.1: Weyl group conjugacy classes of E_6 and the associated symmetry breakings. The entry 'ord' denotes the order of the root lattice automorphism, and a star indicates, when the order of the associated Lie algebra automorphism does not coincide, but is twice as large. The Carter diagrams are listed in the second column for convenience.

intermediate stage, and SO(10) is not included in the list of possible symmetry breakings of table 1. Introducing an appropriate shift, the fourth conjugacy class in tab. 6.1 will correspond to a breakdown of the gauge symmetry $E_6 \to SO(10) \times U(1)$, and moreover, the order of the twist in the root lattice and of its lift will coincide, which is required for the consistency of the construction.

6.2.1 Breaking E_6 to $SO(10) \times U(1)$

For our purposes, E_6 is best described in terms of its embedding in E_8, cf. appendix C. We consider the fourth conjugacy class $s \equiv r_1 r_6$ which is of order 2. The lift of s will be given by

$$\tilde{s} \equiv \tilde{r}_1 \tilde{r}_6 \exp(2\pi v \cdot H), \quad v = (0,\, 1/4,\, 0,\, 0,\, 0,\, 1/4,\, 0,\, 0), \tag{6.12}$$

where the choice of v is motivated by the considerations discussed above. Using eq. (6.6), it is straightforward to determine the images of the 6 Cartan and 72 step operators of E_6 under the transformation. From the step operators 12 are invariant, and the rest pair up to give 30 invariant combinations. Of the 6 Cartan generators, 2 are invariant, and 4 pair up to give 2 invariant combinations. Thus, 2 of the original Cartan generators are projected out. Looking for a maximal commuting subalgebra, we find that there are 2 invariant combinations of step operators, namely $E_2 + E_3$, and $E_{37} + E_{40}$, which commute with the 4 original Cartan generators, forming the Cartan subalgebra of the unbroken gauge group. The 46 invariant combinations are summarized in tab. 6.3 in section 6.5.
Knowing the dimension and rank of the unbroken gauge group, it is not difficult to conclude that it corresponds to the subgroup $SO(10) \times U(1)$ of E_6. It will prove useful to verify this conclusion by an explicit calculation, using Dynkin's approach to group theory. Even though the dimension and the rank may uniquely specify the gauge group in this particular case, as soon as the dimension becomes small, ambiguities arise, necessitating a more thorough investigation. We present the details of the calculation in section 6.5.
The 32 linear combinations of operators, which are not invariant under the gauge twist, but transform with a minus sign (cf. tab. 6.4 in section 6.5) correspond to the irreducible representations $\mathbf{16} + \overline{\mathbf{16}}$ of SO(10). Again, the details are given in section 6.5.

6.2.2 Reducing the Rank of the Gauge Group

We are now ready to consider possible Wilson lines that lead to rank reduction. The matrix representation of the gauge twist in the the standard basis of $\mathbb{R}^8$ is

$$s = r_1\, r_6 = \mathrm{diag}(1,1,1,\sigma,1,-\sigma), \qquad \text{with} \qquad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.13}$$

Diagonalizing the matrix representation we find that there are 2 directions which are completely rotated, namely

$$\lambda\,(0,\, 0,\, 0,\, 1,\, -1,\, 0,\, 0,\, 0), \quad \lambda'\,(0,\, 0,\, 0,\, 0,\, 0,\, 0,\, 1,\, 1), \quad \lambda, \lambda' \in \mathbb{R}, \tag{6.14}$$

which can be switched on as continuous Wilson lines. As we can read off immediately, these Wilson lines are in the direction of the broken Cartan generators $H_4 - H_5$ and $H_7 + H_8$ of the original E_6 algebra. Switching on these Wilson lines leads to a symmetry breaking pattern as summarized in tab. 6.2. The reduction of the rank of the gauge group is clearly demonstrated. We are now ready to use this result in a framework of a more realistic model.

First Wilson Line	Second Wilson Line	Unbroken Gauge Group
$\lambda\langle H_4 - H_5\rangle$		$\mathrm{SU}(5) \times \mathrm{U}(1)$
$\lambda\langle H_7 + H_8\rangle$		$\mathrm{SU}(5) \times \mathrm{U}(1)$
$\lambda\langle H_4 - H_5\rangle + \lambda\langle H_7 + H_8\rangle$		$\mathrm{SO}(7) \times \mathrm{U}(1)$
$\lambda\langle H_4 - H_5\rangle$	$\lambda'\langle H_7 + H_8\rangle$	$\mathrm{SU}(4) \times \mathrm{U}(1)$
$\lambda\langle H_4 - H_5\rangle$	$\lambda'\langle H_4 - H_5\rangle + \lambda'\langle H_7 + H_8\rangle$	$\mathrm{SU}(4) \times \mathrm{U}(1)$
$\lambda\langle H_7 + H_8\rangle$	$\lambda'\langle H_4 - H_5\rangle + \lambda'\langle H_7 + H_8\rangle$	$\mathrm{SU}(4) \times \mathrm{U}(1)$

Table 6.2: All symmetry breakings of the type $E_6 \xrightarrow{s} \mathrm{SO}(10) \times \mathrm{U}(1) \xrightarrow{W} \mathfrak{g}$ with continuous Wilson lines W.

6.3 A $\mathbb{Z}_2$ Orbifold in 6 Dimensions

We shall now try to see how the technology developed so far can be implemented in the framework of (semi) realistic model building. We envisage a situation where we start with gauge symmetry E_6 in the bulk, broken by continuous and discrete Wilson lines to the standard model gauge group. We shall consider the symmetry breakings discussed above in the context of a 6-dimensional orbifold model. Because the embedding of the point group in the gauge degrees of freedom $P \hookrightarrow G$ is a homomorphism, the order of the space-time twist is required to be a multiple the order of the root lattice automorphism, which happens to be 2 in our example. In the following, we shall therefore consider the simplest case, the $\mathbb{Z}_2$ orbifold.

The action of the translation group L, and of the point group P are illustrated in fig. 6.1. The first picture shows how the torus is defined by identifying points which differ by lattice shifts: $x \sim x + ne_5 + me_6$, $n, m \in \mathbb{Z}$. In the second picture, the action of the point group P identifies points on the torus, $x \sim -x$, and we obtain the *fundamental domain* of the orbifold. The 4 special points on the torus which are mapped onto themselves by the action of P are called *fixed points*.

With the twist in the space-time we associate $s = r_1 r_6$ as the twist in the gauge degrees of freedom, as discussed in the last section. This would lead us to the gauge group $\mathrm{SO}(10) \times$

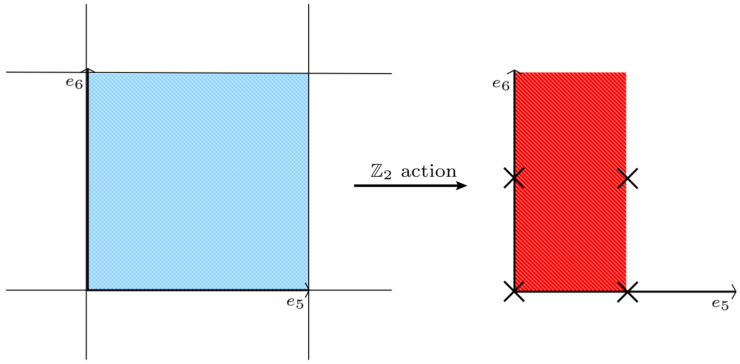


Figure 6.1: The geometry of the $\mathbb{Z}_2$ orbifold.

U(1). The lattice vectors e_5 and e_6 defining the torus can be embedded non-trivially as Wilson lines and break the gauge symmetry further. Our previous analysis makes it clear that the breakdown to the standard model can not be achieved by just one continuous Wilson line. Thus we need a discrete Wilson line as well. We choose

$$W_5 = (1/2,\, 1/2,\, -1,\, 0,\, 0,\, 0,\, -1/2,\, 1/2), \quad W_6 = \lambda\,(0,\, 0,\, 0,\, 1,\, -1,\, 0,\, 0,\, 0), \tag{6.15}$$

where W_5 represents the discrete Wilson line, while W_6 corresponds to a continuous Wilson line in the $H_4 - H_5$ direction as discussed previously. The choice of W_5 is motivated by the desire to obtain a realistic gauge group. A more detailed discussion will be given in section 6.4. In the following we shall now exhibit the "gauge group geography" [21] in the 2-dimensional orbifold. The result is displayed in fig.2. In the bulk we have the gauge group E_6. At the fixed point $(0, 0)$ the gauge symmetry is only affected by the twist and not by the Wilson lines, thus SO(10) $\times$ U(1).

6.3.1 The Symmetry at the Nontrivial Fixed Points

Before discussing our concrete E_6 model it turns out to be useful to have a look at the projection conditions in a general situation. A non trivial fixed point is associated to a space group element (θ, l) which means that a 180° rotation (θ) is followed by a shift with the lattice vector l. The non trivial fixed points on our torus correspond to space group elements with $l = e_5, e_6$, or $e_5 + e_6$ (see fig. 6.1). We want to discuss the embedding of the space group element separately for Cartan directions of the bulk group, step operators belonging to invariant root vectors and step operators belonging to non invariant roots.
The transformation rule for Cartan generators is not sensitive to Wilson lines. The projections are the same at all the fixed points.
If the embedding of the root lattice automorphism into the algebra is not degenerate, step operators belonging to invariant roots are invariant under the rotation [50]. Those operators are sensitive only to the Wilson line. They transform as

$$E_\alpha \to e^{2\pi i \alpha \cdot W_l} E_\alpha, \tag{6.16}$$

where W_l is the Wilson line on the non contractible cycle spanned by the lattice vector l. If that happens to be a continuous Wilson line, even the remaining phase is trivial because the scalar product of an invariant root and a continuous Wilson line vanishes. The orbifold acts on the root lattice as a rotation and hence

$$\alpha \cdot W_l = s\alpha \cdot sW_l = -\alpha \cdot W_l,$$

where in the last step we have used that α is invariant and a continuous Wilson line points into an odd Cartan direction. The situation is different for discrete Wilson lines. The scalar product of a discrete Wilson line with an invariant root can be half integer and the corresponding step operator is projected out.
In addition we need the transformation properties of step operators belonging to non invariant roots:

$$E_\alpha \to e^{2\pi i s\alpha \cdot W_l} \tilde{s} E_\alpha \tilde{s}^{-1} \tag{6.17}$$

(recall that we first rotate and then shift). Since for non invariant roots the orbifold image is a different step operator, the sum of algebra element and orbifold image never vanishes for those roots. The number of invariant sums is not altered by W_l. What is changed is the way these invariant sums are embedded into the bulk gauge algebra. In particular with a continuous Wilson line one can *continuously rotate* the embedding.
These observations give an appealing geometric picture for the symmetry breakdown by a continuous Wilson line. If the degeneracy of two fixed points is lifted by a continuous Wilson line our above arguments imply that the unbroken gauge groups are still the same at these fixed points. The lifted degeneracy shows up in a misaligned embedding of these unbroken gauge groups into the bulk group. This results in a smaller overlap of the gauge groups at the two fixed points and hence yields a reduced gauge symmetry in four dimensions. The overlap does not contain invariant combinations coming from step operators with non invariant roots and a non trivial phase under eq. (6.17).
Our geometric understanding of the symmetry breakdown with a continuous Wilson line yields also a consistent picture in the limit of vanishing Wilson line. Geometrically, this is the limit where the alignment in the embedding of the gauge groups at two different fixed points is restored.
Let us now give the detailed picture for our particular E_6 model. The gauge symmetry at the fixed point $(1/2, 0)$ is sensitive to the Wilson line

$$W_5 = (1/2,\ 1/2,\ -1,\ 0,\ 0,\ 0,\ -1/2,\ 1/2), \tag{6.18}$$

which is invariant under s, and hence discrete. According to eq. (6.16) eight step operators to the s-invariant spinorial roots $\beta_{57\text{–}60}$, $\beta_{66\text{–}69}$ (see table C.1 in the appendix) are projected out. The bulk symmetry is broken to SU(6)×SU(2) at the fixed point $(1/2, 0)$:

$$\mathrm{E}_6 \to \mathrm{SU}(6) \times \mathrm{SU}(2), \qquad \mathbf{78} \to (\mathbf{35}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{20}, \mathbf{2}). \tag{6.19}$$

The Wilson line along e_6

$$W_6 = \lambda\,(0,\ 0,\ 0,\ 1,\ -1,\ 0,\ 0,\ 0), \quad \lambda \in \mathbb{R}, \tag{6.20}$$

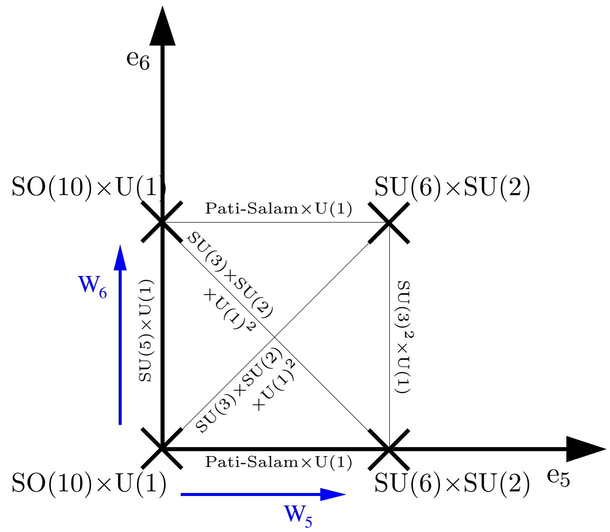


Figure 6.2: The unbroken gauge groups at the 4 fixed points. The gauge group in the bulk is E_6. A detailed explanation is given in the text.

is completely rotated, and hence continuous. According to our general discussion in the beginning of the section, the unbroken gauge group for fixed points differing only in that direction is the same. The situation is summarized in figure 6.2 where so far we have discussed the gauge symmetry breaking at the fixed points.
In order to visualize the fact that the groups unbroken at the fixed points are embedded differently into the bulk gauge group we have displayed the overlapping gauge group at lines connecting different fixed points. Technically, these groups are obtained by imposing two projection conditions, one for each fixed point involved. For example at the line connecting the origin with the fixed point $(0, 1/2)$ only those operators which are invariant under the action of $\tilde{s}$, and W_6, will survive:

$$E_\alpha \ \stackrel{\tilde{s}}{\rightarrow} \ E'_\beta \equiv \tilde{s} E_\alpha \tilde{s}^{-1}, \qquad E'_\beta \stackrel{W_6}{\rightarrow} \ \exp\left(2\pi i \beta \cdot W_6\right) E'_\beta. \tag{6.21}$$

The continuous Wilson line W_6 has the effect of projecting out all operators whose commutator with $H_4 - H_5$ is non-zero. The 25 surviving operators correspond to the unbroken gauge group $\mathrm{SU}(5) \times \mathrm{U}(1)$, as explained in the last section. The groups written at the other lines in fig. 6.2 are computed in an analogous way.

6.3.2 The Spectrum in Four Dimensions

The unbroken gauge group in four dimensions consists of those operators which are invariant under all the symmetry transformations:

$$E_\alpha \ \stackrel{\tilde{s}}{\rightarrow} \ E'_\beta \equiv \tilde{s} E_\alpha \tilde{s}^{-1}, \quad E'_\beta \stackrel{W_5}{\rightarrow} \ \exp\left(2\pi i \beta \cdot W_5\right) \ E'_\beta, \quad E''_\gamma \stackrel{W_6}{\rightarrow} \ \exp\left(2\pi i \gamma \cdot W_6\right) \ E''_\gamma. \tag{6.22}$$

Alternatively, we can say that the symmetry in four dimensions is the intersection of the gauge groups at the fixed points.
In fact we would like to discuss the symmetry breaking pattern in two steps. The Wilson line W_5 is discrete and its value will therefore be of the order of the string scale. W_6 is continuous and will be assumed to have a smaller vacuum expectation value that breaks the remaining gauge group via a Higgs mechanism. In the first step we then have a resulting gauge symmetry from the overlap of the gauge symmetries at the fixed points $(0,0)$ and $(1/2,0)$. Thus the first step amounts to

$$\mathrm{SO}(10)\times\mathrm{U}(1)\overset{W_5}{\rightarrow}\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1). \tag{6.23}$$

At the intermediate scale we thus obtain the Pati-Salam gauge group (with an additional U(1) factor). The details of the calculation can be found in section 6.7.
In the next step we then have the breakdown of $\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{SU}(2)$ due to the action of the continuous Wilson line W_6:

$$\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\overset{W_6}{\rightarrow}\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1). \tag{6.24}$$

The continuous Wilson line thus provides a smooth breakdown of the Pati-Salam gauge group to the standard model. From the low energy effective field theory point of view this corresponds to a Higgs mechanism.
The full chain of symmetry breakdown is

$$\mathrm{E}_6\overset{\tilde{s}}{\rightarrow}\mathrm{SO}(10)\times\mathrm{U}(1)\overset{W_5}{\rightarrow}\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)\overset{W_6}{\rightarrow}\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)^2. \tag{6.25}$$

The details of the calculation are given in section 6.7.

6.4 Possible Relations to String Theory Constructions

So far, we have concentrated on a field theoretic orbifold model in $d=6$. Of course, this is just a first step to understand the mechanism of rank reduction. The final aim would be to implement the scheme in the framework of a consistent string orbifold construction. This would assure the quantum consistency of the theory and it would also give us a hint about the incorporation and location of matter fields. The example discussed in the last section should be used as a tool to implement the mechanism in $d=10$ string orbifolds. To find such applications, let us look at some string constructions of models with Pati-Salam gauge symmetry, as e.g. given in ref. [26]. The model A1 of this paper seems to be particularly suited to our discussion. It is the result of a $\mathbb{Z}_6$ orbifold of the $\mathrm{E}_8\times\mathrm{E}_8$ heterotic string with 3 families of quarks and leptons and a Pati-Salam (PS) gauge group. We would like to see whether our mechanism can be applied to such a model by providing a smooth breakdown of the PS gauge group.
Let us start our discussion using the notation of ref. [26] where the authors describe their model in a certain approximation as a 5d orbifold GUT with an E_6 gauge symmetry and

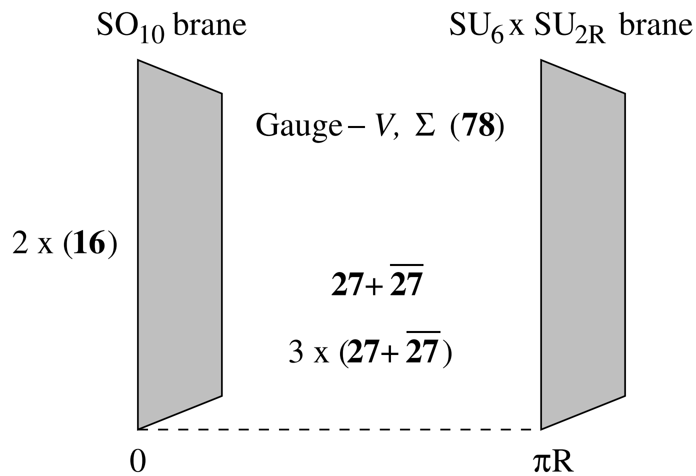


Figure 6.3: Setup of the 5d orbifold GUT with an E_6 gauge symmetry of ref. [26].

point group $\mathbb{Z}_2$. In the bulk, there are the gauge fields in the adjoint representation **78**, and $4 \times (\mathbf{27} + \overline{\mathbf{27}})$ hypermultiplets.

The 2 orbifold parities break the bulk gauge group E_6 to SO(10) at the $y = 0$ brane, and to SU(6) $\times$ SU(2) at the $y = \pi R$ brane, where y denotes the coordinate of the extra dimension. The setup of the model is summarized in fig. 6.3. The gauge group in 4 dimensions is realized as the intersection of the symmetries at the 2 branes, and yields SU(4) $\times$ SU(2) $\times$ SU(2). This Pati-Salam symmetry should be spontaneously broken to the Standard Model gauge group via a Higgs mechanism.

As the model of [26] has a string theory origin in $d = 10$ it naturally allows its interpretation as a six dimensional orbifold GUT model. The $d = 6$ model is obtained from ten dimensional heterotic string theory by compactifying on $T^4/\mathbb{Z}_3$ with a discrete (third order) Wilson line. The orbifold group in the string model is $\mathbb{Z}_6 = \mathbb{Z}_3 \times \mathbb{Z}_2$. The $\mathbb{Z}_2$ factor acts on the remaining T^2 exactly in the same way as in our orbifold GUT. For details see [26]. The string model yields also E_6 gauge symmetry in the bulk of $T^2/\mathbb{Z}_2$ but there are additional U(1) and hidden sector gauge group factors. Moreover, there is bulk matter coming from twisted and untwisted sectors in the $T^4/\mathbb{Z}_3$ compactified heterotic model. Finally, the $\mathbb{Z}_2$ twisted sector gives information on the localization of matter at the $T^2/\mathbb{Z}_2$ fixed points.

Our discrete Wilson line W_5 is equivalent to the second order Wilson line of ref. [26] which lifts the degeneracy of the left and right plane in figure 6.3. As long as we do not switch on a continuous Wilson line, the gauge group geography in our model and the one derived from the string model are identical when we focus on E_6.

The continuous Wilson line breaks the Pati-Salam group to the standard model gauge group. In the low-energy effective field theory this corresponds to a nontrivial vacuum expectation value of a bulk field. Our analysis demonstrates that such a smooth breakdown can be achieved within the string model considered here. This actually proves that there exists a bulk field in the theory that has all the properties of a modulus, i.e. a flat direction in the full scalar potential. Without the argument given above one would have needed to

compute the full low-energy effective potential and prove that it has a flat direction.
This is in fact a nontrivial statement. To see this let us consider the possibility to realize the breakdown of the the standard model gauge group SU(3)×SU(2)×U(1) to SU(3)×U(1): the standard Higgs mechanism of electroweak symmetry breakdown. We still have the option of switching on the second continuous Wilson line of section 3 along the $H_7 + H_8$ direction. Unfortunately, it leads to a breakdown of SU(3)×SU(2)×U(1) to SU(2)×SU(2)×U(1). This shows that in the model under consideration, the Higgs field of the standard model cannot correspond to a bulk field. If the standard model Higgs mechanism could be achieved within the present framework, the Higgs boson would have to be localized on one of the branes. Our analysis thus clarifies properties of the model which otherwise would have been difficult to explain.

6.5 Calculational Details of Section 6.2.1

In the following, we use standard techniques of group theory to identify the gauge symmetry and the irreducible representations [55, 58].
We start with the 46 invariant combinations of operators listed in tab. 6.3. As one can easily verify, the operators

$$\begin{aligned} \tilde{H}_1 &= H_1 + H_2 + H_3, \quad \tilde{H}_2 = H_4 + H_5, \quad \tilde{H}_3 = H_6, \\ \tilde{H}_4 &= H_7 - H_8, \quad \tilde{H}_5 = E_2 + E_3, \quad \tilde{H}_6 = E_{37} + E_{40}, \end{aligned} \tag{6.26}$$

form the Cartan subalgebra of the unbroken gauge group. As a first step, we identify the U(1) generator, which is given by a linear combination of the Cartans, commuting with all operators in the algebra. In practical terms, evaluating the Killing form on the Cartan generators, $\tilde{K}_{ij} = \mathrm{Tr}\ \mathrm{ad}\ \tilde{H}_i\ \mathrm{ad}\ \tilde{H}_j$, and calculating the kernel of $\tilde{K}_{ij}$ gives the U(1) generator

$$U = \frac{1}{16}\left(\tilde{H}_1 + 3\,\tilde{H}_3 + 3\,\tilde{H}_5 + 3\,\tilde{H}_6\right). \tag{6.27}$$

E_1	$E_{10}+E_{28}$	$E_{30}+E_{36}$	$E_{43}+E_{70}$	$E_{51}-E_{71}$	E_{68}
E_2+E_3	$E_{11}+E_{25}$	$E_{31}+E_{33}$	$E_{44}+E_{65}$	$E_{52}-E_{64}$	E_{69}
E_4	$E_{12}+E_{26}$	$E_{32}+E_{34}$	$E_{45}-E_{72}$	E_{57}	$H_1+H_2+H_3$
E_5-E_{17}	$E_{13}+E_{23}$	$E_{37}+E_{40}$	$E_{46}-E_{63}$	E_{58}	H_4+H_5
E_6+E_{18}	$E_{14}+E_{24}$	E_{38}	$E_{47}+E_{53}$	E_{59}	H_6
E_7+E_{19}	$E_{15}+E_{21}$	E_{39}	$E_{48}+E_{54}$	E_{60}	H_7-H_8
E_8-E_{20}	$E_{16}+E_{22}$	$E_{41}+E_{61}$	$E_{49}+E_{55}$	E_{66}	
E_9+E_{27}	$E_{29}+E_{35}$	$E_{42}+E_{62}$	$E_{50}+E_{56}$	E_{67}	

Table 6.3: The 46 invariant combinations corresponding to SO(10) × U(1).

Next, we perform the Levi decomposition [59], i.e. we separate the U(1) factor from the semisimple part of the algebra [60, 61]. The Cartan generators of the semisimple part of the unbroken gauge group are then given by

$$\begin{aligned} \tilde{H}'_1 &= \tilde{H}_2, \quad \tilde{H}'_2 = \tilde{H}_3 - U, \quad \tilde{H}'_3 = \tilde{H}_4, \\ \tilde{H}'_4 &= \tilde{H}_5 - 2U, \quad \tilde{H}'_5 = \tilde{H}_6 - 2U, \end{aligned} \tag{6.28}$$

whereas the other 40 operators are unaffected. The Killing form of the semisimple part is

$$\tilde{K}'_{ij} = \begin{pmatrix} 32 & 0 & 0 & 0 & 0 \\ 0 & 13 & 0 & -6 & -6 \\ 0 & 0 & 32 & 0 & 0 \\ 0 & -6 & 0 & 20 & -12 \\ 0 & -6 & 0 & -12 & 20 \end{pmatrix}. \tag{6.29}$$

To find the roots, we calculate the adjoint action ad $\tilde{H}'_i$ of the Cartan generators on the algebra. These 45×45 matrices will in general not be diagonal, but since the Cartan generators commute, they can be simultaneously diagonalized. The kth eigenvalue of the ith matrix is then the ith entry of the kth root, $\tilde{\alpha}_k^{(i)}$.

We introduce a semi-ordering by fixing the basis

$$\begin{gathered} (2,\ 0,\ 0,\ 0,\ 0), \quad (0,\ 0,\ -2,\ 0,\ 0) \\ (-1,\ 0,\ 1,\ 1,\ -1), \quad (1,\ 1,\ 0,\ -1,\ 0), \quad (0,\ -1/2,\ 0,\ 1,\ -1). \end{gathered} \tag{6.30}$$

Among the positive roots, we identify the simple roots as those which cannot be written as the sum of 2 positive ones:

$$\begin{gathered} \alpha_1 = (0,\ -1/2,\ 0,\ 1,\ -1), \quad \alpha_2 = (1,\ 1,\ 0,\ -1,\ 0), \\ \alpha_3 = (0,\ 0,\ -2,\ 0,\ 0), \quad \alpha_4 = (1,\ -1/2,\ 0,\ 0,\ 1), \quad \alpha_5 = (-1,\ 1/2,\ 1,\ 0,\ 0) \end{gathered} \tag{6.31}$$

The canonical isomorphism between the Cartan subalgebra $\mathfrak{h}$ and its dual $\mathfrak{h}^*$ maps each simple root to an element of the Cartan subalgebra:

$$\begin{gathered} h_{\alpha_1} = -\frac{1}{8}\tilde{H}'_2 - \frac{1}{16}\tilde{H}'_4 - \frac{1}{8}\tilde{H}'_5, \quad h_{\alpha_2} = \frac{1}{32}\tilde{H}'_1 + \frac{1}{16}\tilde{H}'_2 - \frac{1}{32}\tilde{H}'_4, \quad h_{\alpha_3} = -\frac{1}{16}\tilde{H}'_3 \\ h_{\alpha_4} = \frac{1}{32}\tilde{H}'_1 + \frac{1}{16}\tilde{H}'_2 + \frac{3}{32}\tilde{H}'_4 + \frac{1}{8}\tilde{H}'_5, \quad h_{\alpha_5} = -\frac{1}{32}\tilde{H}'_1 + \frac{1}{8}\tilde{H}'_2 + \frac{1}{32}\tilde{H}'_3 + \frac{3}{32}\tilde{H}'_4 + \frac{3}{32}\tilde{H}'_5 \end{gathered} \tag{6.32}$$

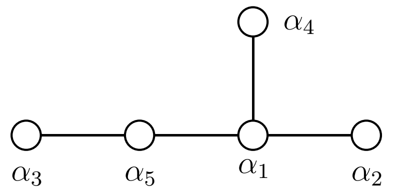
The scalar product for the root vectors $\alpha, \beta \in \mathfrak{h}^*$ is then defined by

$$\langle \alpha, \beta \rangle \equiv \mathrm{Tr}\ \mathrm{ad}\ h_\alpha\ \mathrm{ad}\ h_\beta. \tag{6.33}$$

Using the Killing form given in eq. (6.29), we can evaluate the right-hand-side, and calculate the Cartan matrix

$$A_{ij} \equiv 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \begin{pmatrix} 2 & -1 & 0 & -1 & -1 \\ -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 \\ -1 & 0 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 & 2 \end{pmatrix}. \tag{6.34}$$

From the corresponding Dynkin diagram



we see that the semisimple part of the gauge group is SO(10).

$E_2 - E_3$	$E_{12} - E_{26}$	$E_{32} - E_{34}$	$E_{47} - E_{53}$
$E_5 + E_{17}$	$E_{13} - E_{23}$	$E_{37} - E_{40}$	$E_{48} - E_{54}$
$E_6 - E_{18}$	$E_{14} - E_{24}$	$E_{41} - E_{61}$	$E_{49} - E_{55}$
$E_7 - E_{19}$	$E_{15} - E_{21}$	$E_{42} - E_{62}$	$E_{50} - E_{56}$
$E_8 + E_{20}$	$E_{16} - E_{22}$	$E_{43} - E_{70}$	$E_{51} + E_{71}$
$E_9 - E_{27}$	$E_{29} - E_{35}$	$E_{44} - E_{65}$	$E_{52} + E_{64}$
$E_{10} - E_{28}$	$E_{30} - E_{36}$	$E_{45} + E_{72}$	$H_4 - H_5$
$E_{11} - E_{25}$	$E_{31} - E_{33}$	$E_{46} + E_{63}$	$H_7 + H_8$

Table 6.4: The 32 combinations which transform with a minus.

Now consider the 32 operators in tab. 6.4 transforming with a minus sign. To calculate their weight vectors is completely analogous to the case of the adjoint representation. First, we determine the adjoint action of the 5 Cartan generators on these operators, which is given by 32×32 matrices. Second, these matrices are simultaneously diagonalized, and the 32 eigenvalues in the 5 diagonal matrices constitute the 32 weight vectors. Third, we take the scalar products of the weights with the simple roots to calculate the Dynkin labels. Looking for the highest weights, we find

$$(0\,0\,0\,1\,0) \text{ and } (0\,0\,0\,0\,1), \tag{6.35}$$

corresponding to the representations **16** and $\overline{\mathbf{16}}$.

6.6 Calculational Details of Section 6.3.1

In tab. 6.7, we list the transformation of the E_6 operators,

$$E_\alpha \stackrel{\tilde{s}}{\longrightarrow} \tilde{s}E_\alpha\tilde{s}^{-1} \stackrel{W_5}{\longrightarrow} \exp\left(2\pi i\alpha \cdot W_5\right)\, \tilde{s}E_\alpha\tilde{s}^{-1}, \tag{6.36}$$

and tab. 6.5 summarizes the 38 invariant combinations which survive the gauge symmetry breakdown.
Repeating the analysis of section 6.5 in the present context, we find that the 38 operators correspond to the unbroken gauge group $SU(6) \times SU(2)$.
The 40 combinations which transform with a minus sign are given in tab. 6.6. Calculating their Dynkin labels, we identify the highest weight $(00100|1)$, which corresponds to the representation $(\mathbf{20}, \mathbf{2})$.

E_1	$E_{10} - E_{28}$	$E_{30} - E_{36}$	$E_{43} + E_{70}$	$E_{51} - E_{71}$
$E_2 + E_3$	$E_{11} - E_{25}$	$E_{31} - E_{33}$	$E_{44} + E_{65}$	$E_{52} - E_{64}$
E_4	$E_{12} - E_{26}$	$E_{32} - E_{34}$	$E_{45} - E_{72}$	$H_1 + H_2 + H_3$
$E_5 - E_{17}$	$E_{13} - E_{23}$	$E_{37} + E_{40}$	$E_{46} - E_{63}$	$H_4 + H_5$
$E_6 + E_{18}$	$E_{14} - E_{24}$	E_{38}	$E_{47} - E_{53}$	H_6
$E_7 + E_{19}$	$E_{15} - E_{21}$	E_{39}	$E_{48} - E_{54}$	$H_7 - H_8$
$E_8 - E_{20}$	$E_{16} - E_{22}$	$E_{41} + E_{61}$	$E_{49} - E_{55}$	
$E_9 - E_{27}$	$E_{29} - E_{35}$	$E_{42} + E_{62}$	$E_{50} - E_{56}$	

Table 6.5: The 38 invariant combinations corresponding to SU(6) $\times$ SU(2).

$E_2 - E_3$	$E_{12} + E_{26}$	$E_{32} + E_{34}$	$E_{47} + E_{53}$	E_{59}
$E_5 + E_{17}$	$E_{13} + E_{23}$	$E_{37} - E_{40}$	$E_{48} + E_{54}$	E_{60}
$E_6 - E_{18}$	$E_{14} + E_{24}$	$E_{41} - E_{61}$	$E_{49} + E_{55}$	E_{66}
$E_7 - E_{19}$	$E_{15} + E_{21}$	$E_{42} - E_{62}$	$E_{50} + E_{56}$	E_{67}
$E_8 + E_{20}$	$E_{16} + E_{22}$	$E_{43} - E_{70}$	$E_{51} + E_{71}$	E_{68}
$E_9 + E_{27}$	$E_{29} + E_{35}$	$E_{44} - E_{65}$	$E_{52} + E_{64}$	E_{69}
$E_{10} + E_{28}$	$E_{30} + E_{36}$	$E_{45} + E_{72}$	E_{57}	$H_4 - H_5$
$E_{11} + E_{25}$	$E_{31} + E_{33}$	$E_{46} + E_{63}$	E_{58}	$H_7 + H_8$

Table 6.6: The 40 combinations which transform with a minus sign.

6.7 Calculational Details of Section 6.3.2

We first give a short description of how E_6 breaks to Pati-Salam $\times$ U(1). Consider

$$E_\alpha \ \stackrel{\tilde{s}}{\longrightarrow}\ E'_\beta \equiv \tilde{s} E_\alpha \tilde{s}^{-1}, \qquad E'_\beta \stackrel{W_5}{\longrightarrow}\ \exp\left(2\pi i \beta \cdot W_5\right)\, E'_\beta. \tag{6.37}$$

The first transformation breaks E_6 to SO(10) $\times$ U(1), see section 6.5. In tab. 6.8, we list the invariant combinations of the SO(10) $\times$ U(1) operators under the Wilson line W_5.

The 24 combinations of operators which transform with a minus sign are given in tab. 6.9. Calculating the Dynkin labels, we identify the highest weight $(010|1|1)$ corresponding to the representation $(\mathbf{6}, \mathbf{2}, \mathbf{2})$.

Next, we give a detailed description of how the Pati-Salam gauge group breaks down to the Standard Model gauge symmetry.
The continuous Wilson line W_6 will project out all those step operators in tab. 6.8, which have a non-vanishing scalar product with $H_4 - H_5$. Equivalently we can say that a step

$E_1 \rightarrow E_1$	$E_{27} \rightarrow -E_9$	$E_{53} \rightarrow -E_{47}$
$E_2 \rightarrow E_3$	$E_{28} \rightarrow -E_{10}$	$E_{54} \rightarrow -E_{48}$
$E_3 \rightarrow E_2$	$E_{29} \rightarrow -E_{35}$	$E_{55} \rightarrow -E_{49}$
$E_4 \rightarrow E_4$	$E_{30} \rightarrow -E_{36}$	$E_{56} \rightarrow -E_{50}$
$E_5 \rightarrow -E_{17}$	$E_{31} \rightarrow -E_{33}$	$E_{57} \rightarrow -E_{57}$
$E_6 \rightarrow E_{18}$	$E_{32} \rightarrow -E_{34}$	$E_{58} \rightarrow -E_{58}$
$E_7 \rightarrow E_{19}$	$E_{33} \rightarrow -E_{31}$	$E_{59} \rightarrow -E_{59}$
$E_8 \rightarrow -E_{20}$	$E_{34} \rightarrow -E_{32}$	$E_{60} \rightarrow -E_{60}$
$E_9 \rightarrow -E_{27}$	$E_{35} \rightarrow -E_{29}$	$E_{61} \rightarrow E_{41}$
$E_{10} \rightarrow -E_{28}$	$E_{36} \rightarrow -E_{30}$	$E_{62} \rightarrow E_{42}$
$E_{11} \rightarrow -E_{25}$	$E_{37} \rightarrow E_{40}$	$E_{63} \rightarrow -E_{46}$
$E_{12} \rightarrow -E_{26}$	$E_{38} \rightarrow E_{38}$	$E_{64} \rightarrow -E_{52}$
$E_{13} \rightarrow -E_{23}$	$E_{39} \rightarrow E_{39}$	$E_{65} \rightarrow E_{44}$
$E_{14} \rightarrow -E_{24}$	$E_{40} \rightarrow E_{37}$	$E_{66} \rightarrow -E_{66}$
$E_{15} \rightarrow -E_{21}$	$E_{41} \rightarrow E_{61}$	$E_{67} \rightarrow -E_{67}$
$E_{16} \rightarrow -E_{22}$	$E_{42} \rightarrow E_{62}$	$E_{68} \rightarrow -E_{68}$
$E_{17} \rightarrow -E_5$	$E_{43} \rightarrow E_{70}$	$E_{69} \rightarrow -E_{69}$
$E_{18} \rightarrow E_6$	$E_{44} \rightarrow E_{65}$	$E_{70} \rightarrow E_{43}$
$E_{19} \rightarrow E_7$	$E_{45} \rightarrow -E_{72}$	$E_{71} \rightarrow -E_{51}$
$E_{20} \rightarrow -E_8$	$E_{46} \rightarrow -E_{63}$	$E_{72} \rightarrow -E_{45}$
$E_{21} \rightarrow -E_{15}$	$E_{47} \rightarrow -E_{53}$	$H_1 + H_2 + H_3 \rightarrow H_1 + H_2 + H_3$
$E_{22} \rightarrow -E_{16}$	$E_{48} \rightarrow -E_{54}$	$H_4 \rightarrow H_5$
$E_{23} \rightarrow -E_{13}$	$E_{49} \rightarrow -E_{55}$	$H_5 \rightarrow H_4$
$E_{24} \rightarrow -E_{14}$	$E_{50} \rightarrow -E_{56}$	$H_6 \rightarrow H_6$
$E_{25} \rightarrow -E_{11}$	$E_{51} \rightarrow -E_{71}$	$H_7 \rightarrow -H_8$
$E_{26} \rightarrow -E_{12}$	$E_{52} \rightarrow -E_{64}$	$H_8 \rightarrow -H_7$

Table 6.7: Transformation of the E_6 operators, when the twist $\tilde{s}$ and the Wilson line W_5 are applied simultaneously.

operator is projected out, if the corresponding root has a non-vanishing scalar product

E_1	$E_8 - E_{20}$	$E_{43} + E_{70}$	$H_1 + H_2 + H_3$
$E_2 + E_3$	$E_{37} + E_{40}$	$E_{44} + E_{65}$	$H_4 + H_5$
E_4	E_{38}	$E_{45} - E_{72}$	H_6
$E_5 - E_{17}$	E_{39}	$E_{46} - E_{63}$	$H_7 - H_8$
$E_6 + E_{18}$	$E_{41} + E_{61}$	$E_{51} - E_{71}$	
$E_7 + E_{19}$	$E_{42} + E_{62}$	$E_{52} - E_{64}$	

Table 6.8: The 22 invariant combinations corresponding to $\mathrm{SU}(4) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$.

$E_9 + E_{27}$	$E_{15} + E_{21}$	$E_{47} + E_{53}$	E_{59}
$E_{10} + E_{28}$	$E_{16} + E_{22}$	$E_{48} + E_{54}$	E_{60}
$E_{11} + E_{25}$	$E_{29} + E_{35}$	$E_{49} + E_{55}$	E_{66}
$E_{12} + E_{26}$	$E_{30} + E_{36}$	$E_{50} + E_{56}$	E_{67}
$E_{13} + E_{23}$	$E_{31} + E_{33}$	E_{57}	E_{68}
$E_{14} + E_{24}$	$E_{32} + E_{34}$	E_{58}	E_{69}

Table 6.9: The 24 combinations corresponding transforming with a minus sign.

E_1	E_{38}	$E_{42} + E_{62}$	$H_1 + H_2 + H_3$	$H_7 - H_8$
E_4	E_{39}	$E_{43} + E_{70}$	$H_4 + H_5$	
$E_{37} + E_{40}$	$E_{41} + E_{61}$	$E_{44} + E_{65}$	H_6	

Table 6.10: The 13 invariant combinations corresponding to $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)$.

with W_6. The surviving operators are listed in tab. 6.10.
The Cartan subalgebra is given by

$$\begin{gathered} \bar{H}_1 = E_{37} + E_{40}, \quad \bar{H}_2 = H_1 + H_2 + H_3, \\ \bar{H}_3 = H_4 + H_5, \quad \bar{H}_4 = H_6, \quad \bar{H}_5 = H_7 - H_8. \end{gathered} \tag{6.38}$$

Calculating the kernel of the Killing form

$$\bar{K}_{ij} \equiv \mathrm{Tr}\ \mathrm{ad}\ \bar{H}_i\ \mathrm{ad}\ \bar{H}_j = \begin{pmatrix} 4 & -6 & 0 & -2 & 0 \\ -6 & 9 & 0 & 3 & 0 \\ 0 & 0 & 12 & 0 & 0 \\ -2 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{pmatrix}, \tag{6.39}$$

we find the U(1) generators,

$$\bar{U}_1 = \bar{H}_2 - 3\bar{H}_4, \quad \bar{U}_2 = \bar{H}_1 + 2\bar{H}_4, \tag{6.40}$$

and after the Levi decomposition, the Cartan generators of the semisimple part of the algebra are

$$\bar{H}'_1 = \bar{H}_3, \quad \bar{H}'_2 = \bar{H}_4 + \frac{1}{6}\bar{U}_1 - \frac{1}{6}\bar{U}_2, \quad \bar{H}'_3 = \bar{H}_5. \tag{6.41}$$

The Killing form of the semisimple part is then

$$\bar{K}'_{ij} \equiv \mathrm{Tr}\ \mathrm{ad}\ \bar{H}'_i\ \mathrm{ad}\ \bar{H}'_j = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{pmatrix}. \tag{6.42}$$

The adjoint action ad $\bar{H}'_i$ of the Cartan generators on the 11 operators of the semisimple part of the algebra,

$$\begin{array}{llllll} E_1, & E_{38}, & E_{41}+E_{61}, & E_{43}+E_{70}, & \bar{H}'_1, & \bar{H}'_3, \\ E_4, & E_{39}, & E_{42}+E_{62}, & E_{44}+E_{65}, & \bar{H}'_2, & \end{array}$$

is given by 11×11 matrices. Since the Cartan generators mutually commute, these matrices can be simultaneously diagonalized. The kth eigenvalue of the ith matrix is then the ith entry of the kth root, $\bar{\alpha}_k^{(i)}$:

$$\begin{array}{llllll} (2,0\ ,0), & (0,0,2), & (\text{-}1,\ 1/2,0), & (0,0,\text{-}2), & (0,0,0), & (0,0,0), \\ (\text{-}2,0,0), & (1,1/2,0), & (1,\text{-}1/2,0), & (\text{-}1,\text{-}1/2,0) & (0,0,0) & \end{array}$$

We introduce a semi-ordering by fixing the basis

$$(2,\ 0,\ 0), \quad (0,\ 0,\ -2), \quad (1,\ 1/2,\ 0). \tag{6.43}$$

Among the positive roots,

$$(2,\ 0,\ 0), \quad (0,\ 0,\ -2), \quad (1,\ 1/2,\ 0), \quad (1,\ -1/2,\ 0), \tag{6.44}$$

we select those, which cannot be written as the sum of two positive ones (*simple roots*):

$$\alpha_1 = (0,\ 0,\ -2), \quad \alpha_2 = (1,\ 1/2,\ 0), \quad \alpha_3 = (1,\ -1/2,\ 0). \tag{6.45}$$

The canonical isomorphism between the Cartan subalgebra $\mathfrak{h}$, and it dual $\mathfrak{h}^*$ assigns to each simple root α_i an element of the Cartan subalgebra h_{α_i},

$$h_{\alpha_1} = -\frac{1}{4}\bar{H}'_3, \quad h_{\alpha_2} = \frac{1}{12}\bar{H}'_1 + \frac{1}{2}\bar{H}'_2, \quad h_{\alpha_3} = \frac{1}{12}\bar{H}'_1 - \frac{1}{2}\bar{H}'_2. \tag{6.46}$$

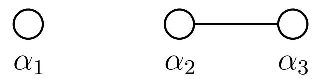
The scalar product in root space is then given by

$$\langle \alpha_i, \alpha_j \rangle \equiv \mathrm{Tr}\ \mathrm{ad}\ h_{\alpha_i}\ \mathrm{ad}\ h_{\alpha_j}. \tag{6.47}$$

Using the Killing form (cf. eq. (6.42)), the right-hand-side can easily be evaluated. From the Cartan matrix

$$A_{ij} \equiv 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \tag{6.48}$$

and its corresponding Dynkin diagram



we see that the semisimple part of the gauge group is SU(3) × SU(2).
Now consider the operators transforming under the SU(4) × SU(2) × SU(2) symmetry in the $(\mathbf{6}, \mathbf{2}, \mathbf{2})$ representation. After switching on the continuous Wilson line W_6, only 12 of them will survive:

$$\begin{gathered} E_{29} + E_{35}, \quad E_{30} + E_{36}, \quad E_{31} + E_{33}, \quad E_{32} + E_{34}, \quad E_{57}, \\ E_{58}, \quad E_{59}, \quad E_{60}, \quad E_{66}, \quad E_{67}, \quad E_{68}, \quad E_{69} \end{gathered} \tag{6.49}$$

Calculating the adjoint action of the Cartan generators and diagonalizing the 12 × 12 matrices, we find the weight vectors:

$$\begin{gathered} (0,\ 1/3,\ 1), \quad (0,\ \text{-}1/3\ 1), \quad (0,\ 1/3,\ \text{-}1), \quad (0,\ \text{-}1/3,\ \text{-}1), \\ (1,\ 1/6,\ \text{-}1), \quad (\text{-}1,\ \text{-}1/6,\ 1), \quad (1,\ 1/6,\ 1), \quad (\text{-}1,\ \text{-}1/6,\ \text{-}1), \\ (1,\ \text{-}1/6,\ \text{-}1), \quad (1,\ \text{-}1/6,\ 1), \quad (\text{-}1,\ 1/6,\ \text{-}1), \quad (\text{-}1,\ 1/6,\ 1) \end{gathered} \tag{6.50}$$

Using the metric in root space, we calculate the Dynkin labels by taking scalar products of the weights with the simple roots. We find the representations

$$(1|01) \quad \text{and} \quad (1|10) \tag{6.51}$$

corresponding to the representations $(\overline{\mathbf{3}}, \mathbf{2})$ and $(\mathbf{3}, \mathbf{2})$ of SU(3) × SU(2).

Chapter 7

Lattice Automorphisms of E_8

So far, we considered the gauge groups SO(10) and E_6 in some detail. These gauge groups appear in Grand Unified Theories in four or more dimensions. The final aim, of course, is to apply our construction to heterotic orbifold models in ten dimensions. In the following, we present all possible symmetry breaking patterns, induced by the shiftless lifts of the 112 conjugacy classes of the E_8 Weyl group.

7.1 Automorphisms of the E_8 Root Lattice

The Weyl group of E_8 has 696,729,600 elements. Investigating the unbroken gauge groups for such a large number of models is not feasible, even with today's resources, i.e. computer based calculations.
The task is greatly simplified by noting that all elements in the same conjugacy class lead to the same unbroken gauge group. Although methods to determine the conjugacy classes of the Weyl groups for each series of the classical algebras and for the exceptional algebras were known, a standardized approach applicable to the conjugacy classes of all Weyl groups was only introduced in ref. [51]. The Carter diagrams, which we discussed in section 5.11, are a powerful, and yet easily accessible means to classify and find a representative for each conjugacy class.
The 112 conjugacy classes of the E_8 Weyl group, and the associated symmetry breakings are listed in tab. 7.1. The first column enumerates the conjugacy classes. In the second column, a representative of each conjugacy class in terms of simple Weyl reflections is listed. The third column gives the order of the conjugacy class on the root lattice, whereby a star indicates, if its order on the Lie algebra is doubled. Finally, in the last column we present the symmetry breakdown associated with each conjugacy class.

Table 7.1: The 112 breakings of E_8.

No.	Weyl Group Element	ord	Gauge Symmetry
1	$\mathbb{1}$	1	E_8
2	r_1	2^*	$E_7 \times U(1)$
3	$r_1 r_2$	3	$E_7 \times U(1)$
4	$r_1 r_8$	2^*	$SO(14) \times U(1)$
5	$r_1 r_2 r_3$	4^*	$SO(12) \times U(1)^2$
6	$r_8 r_2 r_3 r_8 r_4 r_3$	4	$E_6 \times SU(2) \times U(1)$
7	$r_1 r_8 r_2 r_1 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8 r_2 r_1 r_3$ $r_4 r_5 r_4 r_3 r_8 r_2 r_1 r_3 r_4 r_5 r_6 r_5 r_4 r_3 r_8 r_2$ $r_3 r_4 r_5 r_6 r_7 r_6 r_5 r_4$	6	$SU(5) \times SU(4) \times U(1)$
8	$r_1 r_8 r_2$	6^*	$SO(12) \times U(1)^2$
9	$r_8 r_2 r_3 r_4$	6	$E_6 \times U(1)^2$
10	$r_1 r_8 r_2 r_3$	5	$SO(12) \times U(1)^2$
11	$r_1 r_8 r_2 r_3 r_8 r_4 r_3$	12^*	$SO(8) \times U(1)^4$
12	$r_1 r_8 r_2 r_1 r_3 r_8 r_2 r_3 r_4 r_3 r_5 r_4$	6	$SO(10) \times SU(2) \times U(1)^2$
13	$r_1 r_8 r_2 r_1 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8 r_2 r_3 r_5$ $r_4 r_3 r_6 r_5 r_4$	6^*	$SU(3)^2 \times SU(2)^2 \times U(1)^2$
14	$r_1 r_8 r_4$	2^*	$E_6 \times SU(2) \times U(1)$
15	$r_1 r_8 r_3 r_4$	4^*	$SO(8) \times SU(3) \times U(1)^2$
16	$r_1 r_8 r_2 r_3 r_4$	8	$SO(10) \times U(1)^3$
17	$r_1 r_2 r_4 r_5$	3	$SO(14) \times U(1)$
18	$r_1 r_2 r_3 r_4 r_5$	6^*	$SO(8) \times SU(2) \times U(1)^3$
19	$r_1 r_8 r_2 r_3 r_8 r_4 r_3 r_5$	9	$SO(8) \times SU(2) \times U(1)^3$
20	$r_8 r_2 r_3 r_8 r_4 r_3 r_5 r_4 r_3 r_6$	6^*	$SU(4) \times SU(2)^2 \times U(1)^3$
21	$r_8 r_2 r_3 r_8 r_4 r_3 r_6$	4	$SU(8) \times U(1)$
22	$r_8 r_2 r_3 r_8 r_4 r_3 r_5 r_6$	8	$SU(6) \times U(1)^3$
			continued ...

continuation ...			
23	$r_1\, r_8\, r_2\, r_1\, r_3\, r_8\, r_2\, r_3\, r_4\, r_3\, r_5\, r_4\, r_6$	30*	$SU(2) \times U(1)^7$
24	$r_1\, r_8\, r_2\, r_1\, r_3\, r_8\, r_2\, r_1\, r_3\, r_2\, r_4\, r_3\, r_8\, r_2\, r_3\, r_5\, r_4\, r_3\, r_6\, r_5\, r_4\, r_7$	12	$SU(3)^2 \times SU(2) \times U(1)^3$
25	$r_1\, r_8\, r_2\, r_1\, r_3\, r_8\, r_2\, r_1\, r_3\, r_4\, r_3\, r_8\, r_2\, r_3\, r_4\, r_5\, r_4\, r_3\, r_6\, r_5\, r_4\, r_7\, r_6\, r_5$	10	$SU(3)^2 \times SU(2)^2 \times U(1)^2$
26	$r_1\, r_8\, r_2\, r_4$	6*	$SO(10) \times SU(2) \times U(1)^2$
27	$r_8\, r_2\, r_3\, r_4\, r_6$	6*	$SU(6) \times U(1)^3$
28	$r_1\, r_8\, r_2\, r_3\, r_4\, r_5$	12	$SO(8) \times U(1)^4$
29	$r_1\, r_8\, r_2\, r_3\, r_5$	10*	$SO(8) \times SU(2) \times U(1)^3$
30	$r_8\, r_2\, r_3\, r_4\, r_5\, r_6$	10*	$SU(4) \times U(1)^5$
31	$r_1\, r_8\, r_2\, r_3\, r_8\, r_4\, r_3\, r_5\, r_4\, r_3\, r_6$	12	$SU(4) \times SU(2) \times U(1)^4$
32	$r_1\, r_2\, r_3\, r_5\, r_6$	12*	$SO(8) \times SU(2) \times U(1)^3$
33	$r_1\, r_2\, r_3\, r_4\, r_5\, r_6$	7	$SO(8) \times SU(3) \times U(1)^2$
34	$r_1\, r_8\, r_2\, r_3\, r_8\, r_4\, r_3\, r_5\, r_6$	14*	$SU(2)^2 \times U(1)^6$
35	$r_1\, r_8\, r_2\, r_3\, r_8\, r_4\, r_3\, r_6$	12*	$SU(3) \times SU(2)^2 \times U(1)^4$
36	$r_1\, r_8\, r_2\, r_1\, r_3\, r_8\, r_2\, r_3\, r_4\, r_3\, r_5\, r_4\, r_6\, r_5\, r_4\, r_7$	15	$SU(2)^4 \times U(1)^4$
37	$r_8\, r_2\, r_3\, r_8\, r_4\, r_3\, r_5\, r_4\, r_3\, r_6\, r_7$	24	$SU(3)^2 \times U(1)^4$
38	$r_8\, r_2\, r_3\, r_8\, r_4\, r_3\, r_5\, r_6\, r_7$	20*	$SU(2)^2 \times U(1)^6$
39	$r_1\, r_8\, r_2\, r_1\, r_3\, r_8\, r_2\, r_3\, r_4\, r_3\, r_5\, r_4\, r_6\, r_7$	18	$SU(2)^3 \times U(1)^5$
40	$r_1\, r_8\, r_2\, r_1\, r_3\, r_8\, r_2\, r_3\, r_4\, r_3\, r_5\, r_4\, r_7$	6*	$SU(4) \times SU(2)^2 \times U(1)^3$
41	$r_1\, r_8\, r_2\, r_1\, r_3\, r_8\, r_2\, r_1\, r_3\, r_2\, r_4\, r_3\, r_8\, r_2\, r_1\, r_3\, r_2\, r_4\, r_3\, r_8\, r_5\, r_4\, r_3\, r_8\, r_2\, r_1\, r_3\, r_4\, r_5\, r_6\, r_5\, r_4\, r_3\, r_8\, r_2\, r_3\, r_4\, r_5\, r_6\, r_7\, r_6\, r_5$	6	$SU(6) \times SU(2)^2 \times U(1)$
42	$r_8\, r_2\, r_3\, r_8\, r_4\, r_3\, r_6\, r_7$	12	$SU(6) \times SU(2) \times U(1)^2$
43	$r_1\, r_8\, r_4\, r_6$	2*	$SU(8) \times U(1)$
44	$r_8\, r_2\, r_3\, r_8\, r_2\, r_3\, r_4\, r_3\, r_8\, r_2\, r_3\, r_4$	2	$E_7 \times SU(2)$
45	$r_1\, r_8\, r_3\, r_4\, r_6$	4*	$SU(5) \times SU(2)^2 \times U(1)^2$
			continued ...

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46	$r_1\,r_8\,r_2\,r_3\,r_8\,r_2\,r_3\,r_4\,r_3\,r_8\,r_2\,r_3\,r_4$	4^*	$SO(10) \times SU(2)^2 \times U(1)$
47	$r_1\,r_8\,r_2\,r_3\,r_4\,r_6$	8	$SU(5) \times SU(3) \times U(1)^2$
48	$r_1\,r_8\,r_2\,r_1\,r_3\,r_8\,r_2\,r_3\,r_4\,r_3\,r_8\,r_2\,r_5\,r_4\,r_3\,r_6\,r_5\,r_4\,r_3\,r_7$	12	$SU(3) \times SU(2)^3 \times U(1)^3$
49	$r_1\,r_8\,r_2\,r_4\,r_5$	6^*	$SO(10) \times SU(2) \times U(1)^2$
50	$r_1\,r_8\,r_2\,r_3\,r_4\,r_5\,r_6$	18^*	$SU(2) \times U(1)^7$
51	$r_1\,r_8\,r_3\,r_4\,r_5\,r_6$	6^*	$SU(4) \times SU(2)^2 \times U(1)^3$
52	$r_8\,r_2\,r_3\,r_4\,r_6\,r_7$	6	$SU(7) \times U(1)^2$
53	$r_1\,r_8\,r_2\,r_3\,r_8\,r_2\,r_3\,r_4\,r_3\,r_8\,r_2\,r_3\,r_4\,r_5$	6	$SO(10) \times SU(2)^2 \times U(1)$
54	$r_8\,r_2\,r_3\,r_4\,r_5\,r_6\,r_7$	12^*	$SU(2)^2 \times U(1)^6$
55	$r_1\,r_8\,r_2\,r_3\,r_8\,r_4\,r_3\,r_5\,r_4\,r_3\,r_6\,r_7$	20	$SU(2)^2 \times U(1)^6$
56	$r_1\,r_8\,r_2\,r_3\,r_8\,r_4\,r_3\,r_5\,r_7$	18^*	$SU(2)^4 \times U(1)^4$
57	$r_1\,r_8\,r_2\,r_1\,r_3\,r_8\,r_2\,r_1\,r_3\,r_2\,r_4\,r_3\,r_8\,r_2\,r_1\,r_3\,r_2\,r_4\,r_3\,r_5\,r_4\,r_6\,r_5$	6^*	$SU(4) \times SU(2)^3 \times U(1)^2$
58	$r_1\,r_8\,r_2\,r_3\,r_5\,r_6$	15	$SO(8) \times SU(2) \times U(1)^3$
59	$r_1\,r_2\,r_3\,r_5\,r_6\,r_7$	4^*	$SU(4)^2 \times U(1)^4$
60	$r_8\,r_2\,r_3\,r_8\,r_2\,r_3\,r_4\,r_3\,r_8\,r_2\,r_3\,r_4\,r_5\,r_4\,r_6\,r_5$	4	$SO(12) \times SU(2) \times U(1)$
61	$r_1\,r_8\,r_2\,r_3\,r_8\,r_4\,r_3\,r_5\,r_6\,r_7$	24	$SU(2) \times U(1)^7$
62	$r_1\,r_2\,r_3\,r_4\,r_5\,r_6\,r_7$	8^*	$SU(2)^4 \times U(1)^4$
63	$r_1\,r_8\,r_2\,r_3\,r_8\,r_2\,r_3\,r_4\,r_3\,r_8\,r_2\,r_3\,r_4\,r_5\,r_4\,r_6\,r_5$	8	$SO(8) \times SU(2)^2 \times U(1)^2$
64	$r_1\,r_8\,r_2\,r_1\,r_3\,r_8\,r_2\,r_1\,r_3\,r_2\,r_4\,r_3\,r_8\,r_2\,r_1\,r_3\,r_5\,r_4\,r_3\,r_8\,r_2\,r_3\,r_6\,r_5\,r_4\,r_3\,r_7\,r_6\,r_5\,r_4$	8	$SU(4) \times SU(3) \times SU(2)^2 \times U(1)$
65	$r_1\,r_8\,r_2\,r_1\,r_3\,r_8\,r_2\,r_1\,r_3\,r_4\,r_3\,r_8\,r_2\,r_1\,r_3\,r_2\,r_4\,r_5\,r_4\,r_3\,r_6\,r_5\,r_4\,r_7\,r_6\,r_5$	30	$SU(2)^4 \times U(1)^4$
66	$r_1\,r_8\,r_2\,r_3\,r_8\,r_4\,r_3\,r_6\,r_7$	12^*	$SU(3) \times SU(2)^2 \times U(1)^4$
67	$r_1\,r_8\,r_2\,r_3\,r_8\,r_2\,r_3\,r_4\,r_3\,r_8\,r_2\,r_3\,r_4\,r_5\,r_4\,r_3\,r_6\,r_5\,r_4\,r_7\,r_6\,r_5$	12	$SU(3) \times SU(2)^4 \times U(1)^2$
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68	$r_8\,r_2\,r_3\,r_8\,r_2\,r_3\,r_4\,r_3\,r_8\,r_2\,r_3\,r_4\,r_5\,r_4\,r_3\,r_6\,r_5\,r_4\,r_7\,r_6\,r_5$	4^*	$\mathrm{SU}(4)\times\mathrm{SU}(3)\times\mathrm{SU}(2)^2\times\mathrm{U}(1)$
69	$r_1\,r_8\,r_2\,r_1\,r_3\,r_8\,r_2\,r_1\,r_3\,r_2\,r_4\,r_3\,r_8\,r_2\,r_1\,r_3\,r_2\,r_4\,r_3\,r_5\,r_4\,r_3\,r_8\,r_2\,r_3\,r_4\,r_5\,r_6\,r_5\,r_4\,r_3\,r_2\,r_1\,r_7\,r_6\,r_5\,r_4\,r_3\,r_8\,r_2\,r_3\,r_4\,r_5\,r_6$	6	$\mathrm{SU}(6)\times\mathrm{SU}(3)\times\mathrm{U}(1)$
70	$r_1\,r_8\,r_2\,r_1\,r_3\,r_8\,r_2\,r_1\,r_3\,r_2\,r_4\,r_3\,r_8\,r_2\,r_1\,r_3\,r_2\,r_4\,r_3\,r_8\,r_5\,r_4\,r_3\,r_8\,r_2\,r_3\,r_4\,r_6\,r_5\,r_4\,r_3\,r_8\,r_2\,r_1\,r_3\,r_2\,r_4\,r_3\,r_7\,r_6\,r_5\,r_4\,r_3\,r_8\,r_2\,r_1\,r_3\,r_2\,r_4\,r_3\,r_8\,r_5\,r_4\,r_3\,r_6\,r_5\,r_4\,r_7\,r_6\,r_5$	4	$\mathrm{SO}(10)\times\mathrm{SU}(4)$
71	$r_1\,r_8\,r_2\,r_4\,r_6$	6^*	$\mathrm{SU}(6)\times\mathrm{SU}(2)\times\mathrm{U}(1)^2$
72	$r_8\,r_2\,r_3\,r_8\,r_2\,r_3\,r_4\,r_3\,r_8\,r_2\,r_3\,r_4\,r_5\,r_6$	6^*	$\mathrm{SO}(8)\times\mathrm{SU}(2)^2\times\mathrm{U}(1)^2$
73	$r_1\,r_8\,r_2\,r_3\,r_4\,r_5\,r_7$	12	$\mathrm{SU}(3)^2\times\mathrm{SU}(2)\times\mathrm{U}(1)^3$
74	$r_1\,r_8\,r_2\,r_3\,r_5\,r_7$	10^*	$\mathrm{SU}(4)\times\mathrm{SU}(2)^2\times\mathrm{U}(1)^3$
75	$r_1\,r_8\,r_2\,r_3\,r_8\,r_2\,r_3\,r_4\,r_3\,r_8\,r_2\,r_3\,r_4\,r_5\,r_6$	10^*	$\mathrm{SU}(2)^5\times\mathrm{U}(1)^3$
76	$r_1\,r_8\,r_2\,r_1\,r_3\,r_8\,r_2\,r_1\,r_3\,r_2\,r_4\,r_3\,r_8\,r_2\,r_1\,r_3\,r_2\,r_4\,r_3\,r_5\,r_4\,r_6\,r_5\,r_7$	12	$\mathrm{SU}(4)\times\mathrm{SU}(2)^2\times\mathrm{U}(1)^3$
77	$r_1\,r_8\,r_2\,r_4\,r_5\,r_6$	12^*	$\mathrm{SU}(3)\times\mathrm{SU}(2)^3\times\mathrm{U}(1)^3$
78	$r_1\,r_8\,r_2\,r_3\,r_4\,r_5\,r_6\,r_7$	30	$\mathrm{U}(1)^8$
79	$r_1\,r_8\,r_2\,r_3\,r_4\,r_6\,r_7$	24	$\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{U}(1)^4$
80	$r_1\,r_8\,r_3\,r_4\,r_5\,r_6\,r_7$	14^*	$\mathrm{SU}(3)\times\mathrm{SU}(2)^3\times\mathrm{U}(1)^3$
81	$r_1\,r_8\,r_2\,r_3\,r_8\,r_2\,r_3\,r_4\,r_3\,r_8\,r_2\,r_3\,r_4\,r_5\,r_4\,r_6\,r_5\,r_7$	14	$\mathrm{SU}(2)^5\times\mathrm{U}(1)^3$
82	$r_8\,r_2\,r_3\,r_8\,r_2\,r_3\,r_4\,r_3\,r_8\,r_2\,r_3\,r_4\,r_5\,r_4\,r_6\,r_5\,r_7$	12^*	$\mathrm{SU}(4)\times\mathrm{SU}(2)^2\times\mathrm{U}(1)^3$
83	$r_1\,r_8\,r_2\,r_1\,r_3\,r_8\,r_2\,r_1\,r_3\,r_4\,r_3\,r_8\,r_2\,r_1\,r_3\,r_4\,r_5\,r_4\,r_3\,r_8\,r_2\,r_3\,r_4\,r_5$	3	$\mathrm{E}_6\times\mathrm{SU}(3)\times\mathrm{U}(1)$
84	$r_1\,r_8\,r_2\,r_1\,r_3\,r_8\,r_2\,r_1\,r_3\,r_4\,r_3\,r_8\,r_2\,r_1\,r_3\,r_4\,r_5\,r_4\,r_3\,r_8\,r_2\,r_3\,r_4\,r_5\,r_6$	6^*	$\mathrm{SU}(3)^3\times\mathrm{SU}(2)\times\mathrm{U}(1)$
85	$r_1\,r_8\,r_2\,r_3\,r_5\,r_6\,r_7$	20^*	$\mathrm{SU}(2)^4\times\mathrm{U}(1)^4$
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86	$r_1 r_8 r_2 r_1 r_3 r_8 r_2 r_1 r_3 r_4 r_3 r_8 r_2 r_1 r_3 r_4 r_5 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_6 r_5 r_7 r_6$	9	$SU(3)^3 \times SU(2) \times U(1)$
87	$r_1 r_8 r_2 r_1 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8 r_2 r_1 r_3 r_4 r_5 r_4 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_5 r_6 r_5 r_4 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_5 r_4 r_6 r_5 r_7 r_6 r_5 r_4$	12	$SU(4)^2 \times SU(2) \times U(1)$
88	$r_8 r_2 r_3 r_8 r_2 r_3 r_4 r_3 r_8 r_2 r_3 r_4 r_6$	2^*	$SO(12) \times SU(2) \times U(1)$
89	$r_1 r_8 r_2 r_3 r_8 r_2 r_3 r_4 r_3 r_8 r_2 r_3 r_4 r_6$	4^*	$SU(4) \times SU(3) \times SU(2)^2 \times U(1)$
90	$r_8 r_2 r_3 r_8 r_2 r_3 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_6 r_7$	8	$SU(6) \times SU(2) \times U(1)^2$
91	$r_1 r_8 r_2 r_4 r_5 r_7$	6^*	$SU(3)^2 \times U(1)^2$
92	$r_1 r_8 r_2 r_3 r_8 r_2 r_3 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_6 r_7$	18	$SU(2)^4 \times U(1)^4$
93	$r_1 r_8 r_2 r_3 r_8 r_2 r_3 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_7$	6^*	$SU(4) \times SU(2)^3 \times U(1)^2$
94	$r_1 r_8 r_2 r_1 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_4 r_3 r_8 r_2 r_1 r_3 r_4 r_5 r_6 r_5 r_4 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_5 r_4 r_6 r_5 r_7 r_6 r_5 r_4$	6	$SO(8) \times SU(4) \times U(1)$
95	$r_1 r_8 r_2 r_1 r_3 r_8 r_2 r_1 r_3 r_4 r_3 r_8 r_2 r_1 r_3 r_4 r_5 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_6 r_7$	12	$SU(3)^3 \times U(1)^2$
96	$r_1 r_8 r_2 r_4 r_5 r_6 r_7$	30^*	$SU(2)^4 \times U(1)^4$
97	$r_1 r_8 r_2 r_1 r_3 r_8 r_2 r_3 r_4 r_3 r_8 r_2 r_1 r_3 r_4 r_5 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_6 r_5 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_6$	4	$SO(10) \times SU(4)$
98	$r_1 r_8 r_2 r_1 r_3 r_8 r_2 r_3 r_4 r_3 r_8 r_2 r_1 r_3 r_4 r_5 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_6 r_5 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_6 r_7$	8	$SU(4)^2 \times SU(2) \times U(1)$
99	$r_1 r_8 r_2 r_1 r_3 r_8 r_2 r_1 r_3 r_4 r_3 r_8 r_2 r_1 r_3 r_4 r_5 r_4 r_3 r_8 r_2 r_1 r_3 r_4 r_5 r_6 r_5 r_4 r_3 r_8 r_2 r_1 r_3 r_4 r_5 r_6 r_7 r_6 r_5 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_6 r_7$	5	$SU(5)^2$
100	$r_8 r_2 r_3 r_8 r_2 r_3 r_4 r_3 r_8 r_2 r_3 r_4 r_6 r_7$	6	$SO(12) \times SU(2) \times U(1)$
101	$r_1 r_8 r_2 r_3 r_8 r_2 r_3 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_6 r_5 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_6$	6^*	$SU(4)^2 \times SU(2) \times U(1)$

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102	$r_1 r_8 r_2 r_3 r_8 r_2 r_3 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_4 r_3$ $r_8 r_2 r_3 r_4 r_5 r_6 r_5 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_6 r_7$	10	$SU(3)^2 \times U(1)^4$
103	$r_1 r_8 r_2 r_3 r_8 r_2 r_3 r_4 r_3 r_8 r_2 r_3 r_4 r_6 r_7$	12^*	$SU(4) \times SU(2)^3 \times U(1)^2$
104	$r_1 r_8 r_2 r_1 r_3 r_8 r_2 r_1 r_3 r_4 r_3 r_8 r_2 r_1 r_3 r_4$ $r_5 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_7$	6^*	$SU(6) \times SU(3) \times U(1)$
105	$r_1 r_8 r_2 r_1 r_3 r_8 r_2 r_1 r_3 r_4 r_3 r_8 r_2 r_1 r_3 r_4$ $r_5 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_6 r_5 r_4 r_3 r_8 r_2 r_3 r_4$ $r_5 r_6 r_7 r_6 r_5 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_6 r_7$	6	$SU(6) \times SU(3) \times SU(2)$
106	$r_8 r_2 r_3 r_8 r_2 r_3 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_4 r_3 r_8$ $r_2 r_3 r_4 r_5 r_6 r_5 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_6$	2^*	$SO(10) \times SU(4)$
107	$r_8 r_2 r_3 r_8 r_2 r_3 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_4 r_3 r_8$ $r_2 r_3 r_4 r_5 r_6 r_5 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_6 r_7$	4^*	$SO(8) \times SU(4) \times U(1)$
108	$r_1 r_8 r_2 r_1 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8 r_2 r_1 r_3$ $r_2 r_4 r_3 r_8 r_5 r_4 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8 r_5$ $r_4 r_3 r_2 r_1 r_6 r_5 r_4 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8$ $r_5 r_4 r_3 r_2 r_1 r_6 r_7 r_6 r_5 r_4 r_3 r_8 r_2 r_3 r_4 r_5$ $r_6 r_7$	4	$SU(8) \times SU(2)$
109	$r_1 r_8 r_2 r_1 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8 r_2 r_1 r_3$ $r_2 r_4 r_5 r_4 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8 r_5 r_6 r_5$ $r_4 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8 r_5 r_4 r_3 r_2 r_1 r_6$ $r_7 r_6 r_5 r_4 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8 r_5 r_4 r_3$ $r_2 r_1 r_6 r_5 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_6 r_7 r_6 r_5 r_4$	3	$SU(9)$
110	$r_1 r_8 r_2 r_1 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8 r_2 r_1 r_3$ $r_2 r_4 r_3 r_8 r_5 r_4 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8 r_5$ $r_4 r_3 r_2 r_1 r_6 r_5 r_4 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8$ $r_5 r_4 r_3 r_2 r_1 r_6 r_5 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_6 r_7$	6	$SU(8) \times U(1)$
111	$r_1 r_8 r_2 r_1 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8 r_2 r_1 r_3$ $r_2 r_4 r_3 r_8 r_5 r_4 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8 r_5$ $r_4 r_3 r_2 r_1 r_6 r_5 r_4 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8$ $r_5 r_4 r_3 r_2 r_1 r_6 r_5 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_6$	2^*	$SU(8) \times SU(2)$
			continued ...

continuation ...			
112	$r_1 r_8 r_2 r_1 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8 r_2 r_1 r_3$ $r_2 r_4 r_3 r_8 r_5 r_4 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8 r_5$ $r_4 r_3 r_2 r_1 r_6 r_5 r_4 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8$ $r_5 r_4 r_3 r_2 r_1 r_6 r_5 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_6 r_7$ $r_6 r_5 r_4 r_3 r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8 r_5 r_4 r_3 r_2$ $r_1 r_6 r_5 r_4 r_3 r_8 r_2 r_3 r_4 r_5 r_6 r_7 r_6 r_5 r_4 r_3$ $r_8 r_2 r_1 r_3 r_2 r_4 r_3 r_8 r_5 r_4 r_3 r_2 r_1 r_6 r_5 r_4$ $r_3 r_8 r_2 r_3 r_4 r_5 r_6 r_7$	2	SO(16)

The correspondence between the conjugacy classes and the Carter diagrams was established in ref. [51]. To calculate the symmetry breakdown associated with each conjugacy class, there are two approaches. Refs. [53, 54] calculate for each conjugacy class the corresponding equivalent shift vector, and then determine the unbroken gauge group using methods described in chapter 2. Our approach is not to take the detour over the equivalent shift, but to work from the very beginning with the conjugacy class representative along the lines of the methods described in great detail in chapter 5.

To determine all admissible Carter diagrams for E_8, and then to calculate the corresponding conjugacy class representative in terms of Weyl reflections is doable, but quite cumbersome. Instead, we use the Maple package Coxeter-Weyl [52], which gives all 112 representatives of the conjugacy classes in terms of simple Weyl reflections. For each conjugacy class, we calculate the gauge symmetry breakdown, as listed in the last column of tab. 7.1.

7.2 The 112 Breakings of E_8

The calculation of the gauge symmetry breakdown for the 112 conjugacy classes proceeds along the same lines as described in chapters 5 and 6, and will therefore not be described here in detail.

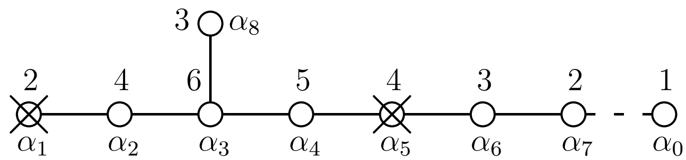
Some remarks, however, are in order here. Knowing the structure constants (section 5.4), it is not difficult to calculate the transformation properties of the operators (section 5.3), and the invariant combinations (section 5.6) which survive the gauge symmetry breakdown.

Knowing the rank and the dimension of the unbroken gauge group may in many cases identify the symmetry uniquely. In those cases, in which ambiguities arise, there are still ways to avoid the lengthy calculation of the gauge symmetry, which was carried out in section 5.7.

As soon as we have found a Cartan subalgebra, we can calculate the Killing form, and the dimension of its kernel gives the number of U(1) factors of the unbroken gauge group. Thus, we can distinguish between two groups, if their dimensions and ranks coincide, but they contain a different number of ideals. An explicit example for this situation is the following. The conjugacy classes 18 and 101 correspond to $\mathrm{SO}(8) \times \mathrm{SU}(2) \times \mathrm{U}(1)^3$, and

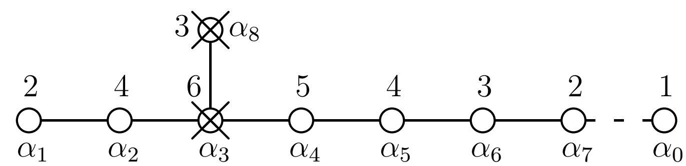
$SU(4)^2 \times SU(2) \times U(1)$, respectively. As is readily verified, both gauge groups are of rank 8, and contain 34 elements.

Another piece of information which we can exploit is the order of the automorphism on the Lie algebra. Consider the conjugacy classes 69, 104, 94, 107. The first two conjugacy classes correspond to the gauge group $SU(6) \times SU(3)$, and the last two correspond to $SO(8) \times SU(4)$. All four gauge groups have rank 8, dimension 44, and contain one $U(1)$ factor. As we have one $U(1)$ factor, we know that we have to delete exactly two roots from the extended Dynkin diagram of E_8 to arrive at $SO(8) \times SU(4)$. It turns out that we only have one choice, namely to remove the roots α_1, and α_5:

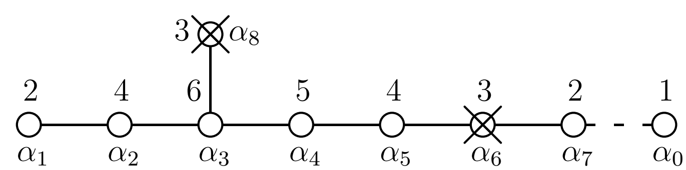


Thus, the order of the automorphism realizing this gauge symmetry breakdown can be $n \times 2 + m \times 4$ for n, m any positive integer. In particular, the order can be 6, 8, or 12, and as far as we can tell from this argument, all four conjugacy classes may realize $SO(8) \times SU(4)$.

Next, consider the gauge group $SU(6) \times SU(3)$. Here, we have two choices, namely we can either remove α_3, and α_8, leading to an automorphism of order $n \times 6 + m \times 3$,



or we can remove α_6, and α_8, leading to an automorphism of order $n \times 3 + m \times 3$:



In neither case we can construct an automorphism of order 8, thereby proving that the conjugacy class 107 cannot correspond to the gauge symmetry $SU(6) \times SU(3)$, whereas we can make no statements about the other three cases. Considering the effort it took to arrive at this conclusion, we think that this is a rather poor result.

We want to emphasize that for the conjugacy classes 69 and 94, the rank, dimension, number of $U(1)$ factors, and the order of the automorphism both on the lattice and on the Lie algebra coincide, although they correspond to distinct gauge groups. In these cases, there is no other way than to determine the gauge symmetry by direct calculation.

Chapter 8

Conclusions

In this thesis, we considered various aspects of string phenomenology in the context of heterotic orbifold constructions, where special emphasis was laid on the connection between GUT models in extra dimensions and their relation to string theory.

We investigated orbifold models with more general structure than the $\mathbb{Z}_3$ orbifold, on which most of the past research had focused. The picture of the heterotic brane world which naturally emerged allowed us to make contact to field theoretic orbifold constructions in five and six dimensions, which have recently attracted much attention.

For a systematic study of the large number of models which can be constructed, we developed computer programs which are quite general and powerful in the sense that we may specify all the parameters of the theory and we obtain the gauge symmetry and the spectrum within a few seconds. In this thesis, we restricted ourselves to the case of the $\mathbb{Z}_6$-II orbifold, whose structure is rich enough to display all the features encountered in constructions of models with any other point group.

It should be noted, however, that the tools we developed are completely general and have already been applied to a number of other orbifold models. One example is the $\mathbb{Z}_2 \times \mathbb{Z}_2$ case, where we have constructed a number of three-generation models, thereby disproving claims in the literature that there are no three-generation models for this type of orbifolds.

Another focus of this thesis has been a classification of orbifold models, i.e. the construction of shift vectors and Wilson lines leading to inequivalent gauge groups and spectra. In connection with our tools which can determine the spectrum of a model within a few seconds, this progress in methodology allowed the construction of hundreds of models with interesting characteristics. As a matter of fact, all of the models we presented have been found by classifying $\mathbb{Z}_6$-II orbifold models with zero or one Wilson line.

Based on the work done by ref. [27], we developed the mathematical background for a *stringy Higgs mechanism* which allows us to lower the rank of the gauge group in the higher dimensions, which cannot be achieved by contemporary orbifold constructions. We provide all the calculational methods needed to unambiguously identify the gauge symmetry and to construct the matter representations.

For specific model constructions, we focused on two promising gauge groups, namely on SO(10) and E_6. In the latter case, we could derive a GUT model in six dimensions which has a Standard Model like gauge symmetry $SU(3) \times SU(2) \times U(1) \times U(1)'$ in four dimensions, and discussed its embedding into string theory.

Some results in this thesis have been published in:

In journals

S. Forste, H. P. Nilles, P. K. S. Vaudrevange and A. Wingerter, "Heterotic brane world", *Phys. Rev.* **D70** (2004) 106008, hep-th/0406208

S. Forste, H. P. Nilles and A. Wingerter, "Geometry of Rank Reduction", to appear in *Phys. Rev.* **D**, hep-th/0504117

In proceedings

S. Forste and A. Wingerter, "Bottom Up Meets Heterotic Strings", *Fortsch. Phys.* **53** (2005) 463-467, hep-th/0412066

Appendix A

Fixed Points

A.1 The Theorem

Calculating the number of fixed points may sometimes be subtle, especially when we run out of geometric imagination. The following theorem is useful:

Theorem A.1.1 *Let Λ denote an arbitrary lattice, and θ a symmetry of Λ. Define*

$$I = \left\{ w \in \Lambda \,|\, (1-\theta^T)w = 0 \right\}, \quad N = I^\perp = \{ w \in \Lambda \,|\, \forall v \in I : \langle v, w\rangle = 0 \}. \tag{A.1}$$

Then the number of fixed points of θ is given by

$$\#FP = \left| N \Big/ (1-\theta)\Lambda \right|. \tag{A.2}$$

Proof:

Let x_f denote a fixed point. Then, x_f is invariant up to a lattice vector:

$$\theta\, x_f = x_f + \ell, \qquad \ell \in \Lambda. \tag{A.3}$$

We recast this equation into the following form:

$$(\mathbb{1} - \theta)\, x_f = \ell, \qquad \ell \in \Lambda \tag{A.4}$$

Next, we prove that ℓ is actually an element of the sub-lattice N. To that end, let $w \in I$ be an arbitrary element in I:

$$\langle w, \ell\rangle \overset{A.4}{=} \langle w, (1-\theta)\, x_f\rangle = \langle (1-\theta^T)\, w, x_f\rangle \overset{w\in I}{=} \langle 0, x_f\rangle = 0$$

This proves that $\ell \in N$. For each $\ell \in N$, we obtain a fixed point x_f, which is given by eq. (A.4). Many of the fixed points we obtain will be equivalent, i.e. they will correspond to the same point on the torus. To get the true number of fixed points, we must identify the ones which differ by lattice shifts:

$$x_f \sim x_f + \lambda, \qquad \lambda \in \Lambda \tag{A.5}$$

We know that x_f is the fixed point which corresponds to ℓ. In order to find out to what ℓ' the fixed point $x_f + \lambda$ corresponds, we substitute it into eq. (A.4):

$$(\mathbb{1} - \theta)\,(x_f + \lambda) = \ell, \quad \ell \in N, \lambda \in \Lambda, \qquad \leftrightarrow \qquad (\mathbb{1} - \theta)\,x_f = \ell - (\mathbb{1} - \theta)\lambda \tag{A.6}$$

The calculation shows that $x_f + \lambda$ corresponds to $\ell' = \ell - (\mathbb{1} - \theta)\lambda$ for an arbitrary $\lambda \in \Lambda$. If we now identify all x_f with $x_f + \lambda$ to get the true number of fixed points, we must identify all $\ell \in N$ with ℓ':

$$\#\mathrm{FP} = \left| N \big/ (\mathbb{1} - \theta)\Lambda \right| . \tag{A.7}$$

This proves the theorem.

Interpreting the Result

Using this theorem is not as straightforward as it may seem at first glance. Eq. (A.7) requires the the calculation of the kernel and the image of $\mathbb{1}-\theta$ *on the lattice*, and not on some $\mathbb{R}^n$. To illustrate this point, consider the $\mathbb{Z}_3$ point group on T^2, and assume in the following, that all mappings act on $\mathbb{R}^2$. The mapping $\mathbb{1} - \theta$ has an inverse, so $I = \ker(\mathbb{1} - \theta) = \{0\}$, and $N = I^\perp = \mathbb{R}^2$. Furthermore, $\mathbb{1} - \theta$ has rank 2, so $\mathrm{Im}\,(\mathbb{1} - \theta) = \mathbb{R}^2$. According to the theorem, we have 1 fixed point, whereas we know that there are in fact 3.

The reason why we obtain the wrong result is that we have assumed the mappings to act on $\mathbb{R}^2$, whereas as a matter of fact, they act on the lattice Λ. In eq. (A.7) both N and $(1 - \theta)\Lambda$ are lattices, and to determine the number of fixed points, we must count the equivalence classes of N, where 2 vectors are considered equivalent, if they differ by a vector in $(1 - \theta)\Lambda$. This can be quite cumbersome. However, there is a trick.

The Fundamental Volume of a Lattice

The fundamental volume of a lattice Λ is the volume of its fundamental region, and is given by

$$\mathrm{Vol}(\Lambda) = \sqrt{\det(A\,A^T)}, \tag{A.8}$$

where A is a matrix, whose rows (or columns) are the basis vectors of Λ. The fundamental volume is independent of the choice of basis for the lattice, since any 2 bases for the same lattice are related by an integral, unimodular matrix, which does not contribute to the determinant in eq. (A.8).

If Λ is a lattice and $\Lambda' \subset \Lambda$ is a sub-lattice, one can prove the interesting relation

$$\left| \Lambda \big/ \Lambda' \right| = \frac{\mathrm{Vol}(\Lambda')}{\mathrm{Vol}(\Lambda)}, \tag{A.9}$$

which is exactly what we need to calculate the fixed points. This relation is intuitively clear: The sub-lattice Λ' is coarser than Λ, and in 1 fundamental region of Λ', there are many fundamental regions of Λ, the number being the number of cosets.

A.2 The $\mathbb{Z}_3$ Symmetry on the Root Lattice of SU(3)

The root lattice of $\Lambda \equiv \mathrm{SU}(3)$ is spanned by the vectors

$$e_1 = (1,\, 0), \quad e_2 = \left(-\tfrac{1}{2},\, \tfrac{\sqrt{3}}{2}\right).$$

The matrix representations of θ and $\mathbb{1} - \theta$ are given by

$$\theta = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}, \qquad \mathbb{1} - \theta = \begin{pmatrix} 3/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 3/2 \end{pmatrix}. \tag{A.10}$$

Since $\det(\mathbb{1} - \theta) \neq 0$, there is no vector $v \neq 0$ which is mapped to 0 under $\mathbb{1} - \theta$, thus $I = \{0\}$ and $N = \Lambda$.

The lattice $\Lambda' \equiv (\mathbb{1} - \theta)\,\Lambda$ is spanned by

$$(\mathbb{1} - \theta)\, e_1 = e_1 - e_2 = (3/2, -\sqrt{3}/2), \qquad (\mathbb{1} - \theta)\, e_2 = e_1 + 2e_2 = (0, \sqrt{3}).$$

Define the matrices comprising the basis vectors of Λ and Λ', respectively:

$$A_\Lambda = \begin{pmatrix} 1 & 0 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}, \qquad A_{\Lambda'} = \begin{pmatrix} 3/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3} \end{pmatrix} \tag{A.11}$$

The fundamental volumes of Λ and Λ' are given by

$$\mathrm{Vol}(\Lambda) = \sqrt{\det(A_\Lambda\, A_\Lambda^T)} = \sqrt{\frac{3}{4}}, \qquad \mathrm{Vol}(\Lambda') = \sqrt{\det(A_{\Lambda'}\, A_{\Lambda'}^T)} = \sqrt{\frac{27}{4}} \tag{A.12}$$

and from eq. (A.9) it follows that the number of fixed points is 3.

A.3 The $\mathbb{Z}_2$ Symmetry on the Root Lattice of $\mathrm{SO}(4)^3$

The root lattice of $\Lambda \equiv \mathrm{SO}(4)^3$ is spanned by the vectors

$$\begin{aligned} e_1 &= (1,\, 0,\, 0,\, 0,\, 0,\, 0), & e_4 &= (0,\, 0,\, 0,\, 1,\, 0,\, 0), \\ e_2 &= (0,\, 1,\, 0,\, 0,\, 0,\, 0), & e_5 &= (0,\, 0,\, 0,\, 0,\, 1,\, 0), \\ e_3 &= (0,\, 0,\, 1,\, 0,\, 0,\, 0), & e_6 &= (0,\, 0,\, 0,\, 0,\, 0,\, 1). \end{aligned}$$

We define the $\mathbb{Z}_2$ action θ as

$$\theta \;:\; e_i \mapsto \begin{cases} \text{-}e_i & \text{for } i = 1, \ldots, 4, \\ e_i & \text{for } i = 5,\, 6. \end{cases} \tag{A.13}$$

The matrix representation of θ in the standard basis of $\mathbb{R}^6$ is given by

$$J \equiv \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A.14}$$

We will need in the following the matrix representation of $\mathbb{1} - \theta$, which is then given by

$$\mathbb{1} - J \equiv \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.15}$$

Using the above results, we can calculate I, N, and Im $(\mathbb{1} - \theta)$. The SO(4)^3 root lattice is given by

$$\Lambda = \{(n_1,\, n_2,\, n_3,\, n_4,\, n_5,\, n_6) \;|\; n_i \in \mathbb{Z}\}\,.$$

To vector space I is by definition the set of all $v \in \Lambda$ such that $(\mathbb{1} - J)\, v = 0$, so we conclude

$$I = \{(0,\, 0,\, 0,\, 0,\, n_5,\, n_6) \;|\; n_i \in \mathbb{Z}\}\,.$$

N is by definition the set of lattice vectors in Λ which are orthogonal to I, and we get

$$N = \{(n_1,\, n_2,\, n_3,\, n_4,\, 0,\, 0) \;|\; n_i \in \mathbb{Z}\}\,.$$

The final ingredient needed to apply eq. (A.7) is $(\mathbb{1} - J)\,\Lambda$. The image of an arbitrary element $(n_1, n_2, n_3, n_4, n_5, n_6) \in \Lambda$ under the mapping $(\mathbb{1} - J)$ is

$$\Lambda' \equiv (\mathbb{1} - J)\,\Lambda = \{(2n_1, 2n_2, 2n_3, 2n_4, 0, 0) \;|\; n_i \in \mathbb{Z}\}\,.$$

Define matrices which comprise the basis vectors of the lattices N and Λ', respectively:

$$A_N = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \qquad A_{\Lambda'} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{pmatrix}$$

The fundamental volumes of N and Λ' are given by

$$\mathrm{Vol}(N) = \sqrt{\det(A_N\, A_N^T)} = \sqrt{1}, \qquad \mathrm{Vol}(\Lambda') = \sqrt{\det(A_{\Lambda'}\, A_{\Lambda'}^T)} = \sqrt{256} \tag{A.16}$$

and from eq. (A.9) it follows that the number of fixed points is 16.

A.4 The $\mathbb{Z}_2$ Symmetry on the Root Lattice of SO(12)

The root lattice of $\Lambda \equiv \mathrm{SO}(12)$ is spanned by the vectors

$$
\begin{aligned}
e_1 &= (1,\text{-}1,0,0,0,0), & e_4 &= (0,0,0,1,\text{-}1,0),\\
e_2 &= (0,1,\text{-}1,0,0,0), & e_5 &= (0,0,0,0,1,\text{-}1),\\
e_3 &= (0,0,1,\text{-}1,0,0), & e_6 &= (0,0,0,0,1,1).
\end{aligned}
$$

We define the $\mathbb{Z}_2$ action θ as

$$
\theta \;:\; e_i \mapsto \begin{cases} \text{-}e_i & \text{for } i = 1,\ldots,4,\\ e_i & \text{for } i = 5,\,6. \end{cases} \tag{A.17}
$$

The matrix representation of θ in the standard basis of $\mathbb{R}^6$ is given by

$$
\underbrace{\begin{pmatrix} 1&0&0&0&0&0\\ -1&1&0&0&0&0\\ 0&-1&1&0&0&0\\ 0&0&-1&1&0&0\\ 0&0&0&-1&1&1\\ 0&0&0&0&-1&1 \end{pmatrix}}_{\text{Columns are old basis vectors } e_i}
\underbrace{\begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{pmatrix}}_{\text{Matrix representation of } \theta \text{ in basis } \{e_i\}}
\underbrace{\begin{pmatrix} 1&0&0&0&0&0\\ -1&1&0&0&0&0\\ 0&-1&1&0&0&0\\ 0&0&-1&1&0&0\\ 0&0&0&-1&1&1\\ 0&0&0&0&-1&1 \end{pmatrix}^{-1}}_{\text{Inverse of the first matrix}},
$$

which evaluates to

$$
J \equiv \begin{pmatrix} -1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&-1&0&0\\ 2&2&2&2&1&0\\ 0&0&0&0&0&1 \end{pmatrix}. \tag{A.18}
$$

We will need in the following the matrix representation of $\mathbb{1} - \theta$, which is then given by

$$
\mathbb{1} - J \equiv \begin{pmatrix} 2&0&0&0&0&0\\ 0&2&0&0&0&0\\ 0&0&2&0&0&0\\ 0&0&0&2&0&0\\ -2&-2&-2&-2&0&0\\ 0&0&0&0&0&0 \end{pmatrix}. \tag{A.19}
$$

Using the above results, we can calculate I, N, and Im $(\mathbb{1} - \theta)$. The SO(12) root lattice is given by

$$
\Lambda = \left\{ (n_1,\, n_2,\, n_3,\, n_4,\, n_5,\, n_6) \;\middle|\; n_i \in \mathbb{Z},\; \sum n_i = \text{even} \right\}.
$$

To determine I, we have to solve the matrix equation

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ -2 & -2 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which gives

$$n_1 = n_2 = n_3 = n_4 = 0, \quad n_5, n_6 \in \mathbb{Z}, \quad n_5 + n_6 \text{ even.}$$

N is by definition the set of lattice vectors in Λ which are orthogonal to I, i.e. we have to determine the $m_i \in \mathbb{Z}$, $\sum m_i =$ even, such that

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ n_5 \\ n_6 \end{pmatrix} \cdot \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \end{pmatrix} = 0$$

for all n_5, $n_6 \in \mathbb{Z}$ and $n_5 + n_6$ even. Clearly,

$$N = \left\{ (m_1, m_2, m_3, m_4, 0, 0) \;\middle|\; m_i \in \mathbb{Z}, \; \sum m_i \text{ even} \right\}.$$

The final ingredient needed to apply eq. (A.7) is $(\mathbb{1} - J)\,\Lambda$. The image of an arbitrary element $(n_1, n_2, n_3, n_4, n_5, n_6) \in \Lambda$ under the mapping $(\mathbb{1} - J)$ is

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ -2 & -2 & -2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \end{pmatrix} = \begin{pmatrix} 2n_1 \\ 2n_2 \\ 2n_3 \\ 2n_4 \\ -2n_1 - 2n_2 - 2n_3 - 2n_4 \\ 0 \end{pmatrix},$$

so we conclude

$$(\mathbb{1} - J)\,\Lambda = \left\{ (2n_1, 2n_2, 2n_3, 2n_4, M, 0) \;\middle|\; n_i \in \mathbb{Z}, \; \sum n_i = -M \right\}.$$

In forming the cosets

$$N \Big/ (\mathbb{1} - J)\Lambda, \tag{A.20}$$

we identify elements in N, which differ by lattice vectors in $(\mathbb{1} - J)\Lambda$. To that end, we can set $M = 0$, since the 5th entry of any element in N is always 0. Define

$$\Lambda' = \{ (2n_1, 2n_2, 2n_3, 2n_4, 0, 0) \;|\; n_i \in \mathbb{Z} \}.$$

Λ' is a sub-lattice of Λ and

$$N\Big/(\mathbb{1}-J)\Lambda \;=\; N\Big/\Lambda'. \tag{A.21}$$

Note that for the purpose of determining the equivalence classes, it is all the same whether we take the left or the right hand side of the above equation. We have to resort to this little trick of using a sub-lattice of Λ, since in the end, we do not calculate these equivalence classes, but look at how often the volume of a fundamental cell in Λ fills out the volume of a fundamental cell in Λ'. This only works if $\Lambda' \subset \Lambda$.

Define matrices which comprise the basis vectors of the lattices N and Λ', respectively:

$$A_N = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}, \qquad A_{\Lambda'} = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \end{pmatrix}$$

The lattice N is basically the SO(8) root lattice, which explains the choice of the basis vectors, and in the definition of Λ', we do not have the condition that the sum be even, so it is clear that the above choice is indeed a basis.

The fundamental volumes of N and Λ' are given by

$$\mathrm{Vol}(N) = \sqrt{\det(A_N\, A_N^T)} = \sqrt{4}, \qquad \mathrm{Vol}(\Lambda') = \sqrt{\det(A_{\Lambda'}\, A_{\Lambda'}^T)} = \sqrt{256} \tag{A.22}$$

and from eq. (A.9) it follows that the number of fixed points is 8.

Appendix B

Details of the $\mathbb{Z}_6$-II Model Construction

B.1 Deriving the Gauge Symmetry

(i) Collect all root vectors corresponding to the invariant left movers.

(ii) Fix a basis and expand all root vectors in terms of this basis.

(iii) Introduce a semi-ordering in root space: A root vector is positive (negative), if the first non-zero coefficient in the above expansion is positive (negative), and it is zero, if all coefficients are zero.

(iv) Find the positive roots. Of all root vectors, 8 are zero, and of the rest, one half is positive and the other half is negative.

(v) Define a simple root to be a positive root which cannot be written as the sum of 2 positive ones.

(vi) Draw the Dynkin diagram corresponding to the simple roots. For each simple root, there is a node, and two nodes are connected, if their roots have a non-vanishing scalar product.

(vii) The Dynkin diagram unambiguously identifies the algebra. Since we already know that we have 8 Cartan generators, the number of U(1) factors is 8 minus the number of simple roots.

The relevant methods of group theory are described in great detail in ref. [58]. As reference works, refs. [48, 62] are recommended.

B.2 Identifying the Irreducible Representations

(i) Find the Dynkin labels of each weight. The i-th Dynkin label of a weight is equal to its scalar product with the i-th simple root. From now on, we will work with the weight vectors in the Dynkin basis.

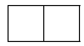
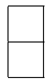
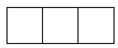
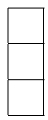
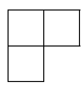
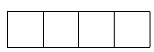
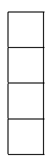
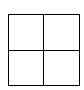
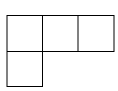
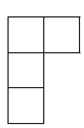
(ii) Identify the highest weights, i.e. the weights, all of whose entries are positive or zero.

(iii) To each highest weight, there corresponds an irreducible representation, whose dimensionality can be found by the standard Dynkin algorithm [48]. The representations can also be directly identified, using the extensive tables of ref. [62].

(iv) In most cases, this is the end of the analysis: The dimensions of the irreducible representations add up to the number of states.

(v) In some cases, one irreducible representation might be a subset of another irreducible representation. (In this case, the highest weight vector of the former representation is included in the list of weights of the latter one.) It is clear that only the larger representation is counted.

B.3 Anomalies for SU(N)

Highest weight	Young tableaux	Cubic Index
$[1\,0\,0\,0\ldots0]_{\mathrm{DL}}$		1
$[2\,0\,0\,0\ldots0]_{\mathrm{DL}}$		$N+4$
$[0\,1\,0\,0\ldots0]_{\mathrm{DL}}$		$N-4$
$[3\,0\,0\,0\ldots0]_{\mathrm{DL}}$		$\frac{1}{2}(N+3)(N+6)$
$[0\,0\,1\,0\ldots0]_{\mathrm{DL}}$		$\frac{1}{2}(N-3)(N-6)$
$[1\,1\,0\,0\ldots0]_{\mathrm{DL}}$		N^2-9
$[4\,0\,0\,0\ldots0]_{\mathrm{DL}}$		$\frac{1}{6}(N+3)(N+4)(N+8)$
$[0\,0\,0\,1\ldots0]_{\mathrm{DL}}$		$\frac{1}{6}(N-3)(N-4)(N-8)$
$[0\,2\,0\,0\ldots0]_{\mathrm{DL}}$		$\frac{1}{3}N(N^2-16)$
$[2\,1\,0\,0\ldots0]_{\mathrm{DL}}$		$\frac{1}{2}(N+4)(N^2+N-8)$
$[1\,0\,1\,0\ldots0]_{\mathrm{DL}}$		$\frac{1}{2}(N-4)(N^2-N-8)$

Appendix C

Details of the E_6 Calculation

We describe E_6 in terms of its embedding in E_8. The roots of E_6 are those roots of E_8, whose first 3 components are equal. For convenience and reference, we list the roots in tab. C.1. Using the standard metric on root space, we find the simple roots as listed in tab. C.2.

β_1	0	0	0	1	1	0	0	0
β_2	0	0	0	-1	1	0	0	0
β_3	0	0	0	1	-1	0	0	0
β_4	0	0	0	-1	-1	0	0	0
β_5	0	0	0	1	0	1	0	0
β_6	0	0	0	-1	0	1	0	0
β_7	0	0	0	1	0	-1	0	0
β_8	0	0	0	-1	0	-1	0	0
β_9	0	0	0	1	0	0	1	0
β_{10}	0	0	0	-1	0	0	1	0
β_{11}	0	0	0	1	0	0	-1	0
β_{12}	0	0	0	-1	0	0	-1	0
β_{13}	0	0	0	1	0	0	0	1
β_{14}	0	0	0	-1	0	0	0	1
β_{15}	0	0	0	1	0	0	0	-1
β_{16}	0	0	0	-1	0	0	0	-1
β_{17}	0	0	0	0	1	1	0	0
β_{18}	0	0	0	0	-1	1	0	0
β_{19}	0	0	0	0	1	-1	0	0
β_{20}	0	0	0	0	-1	-1	0	0
β_{21}	0	0	0	0	1	0	1	0
β_{22}	0	0	0	0	-1	0	1	0
β_{23}	0	0	0	0	1	0	-1	0
β_{24}	0	0	0	0	-1	0	-1	0
β_{25}	0	0	0	0	1	0	0	1
β_{26}	0	0	0	0	-1	0	0	1
β_{27}	0	0	0	0	1	0	0	-1
β_{28}	0	0	0	0	-1	0	0	-1
β_{29}	0	0	0	0	0	1	1	0
β_{30}	0	0	0	0	0	-1	1	0
β_{31}	0	0	0	0	0	1	-1	0
β_{32}	0	0	0	0	0	-1	-1	0
β_{33}	0	0	0	0	0	1	0	1
β_{34}	0	0	0	0	0	-1	0	1
β_{35}	0	0	0	0	0	1	0	-1
β_{36}	0	0	0	0	0	-1	0	-1

β_{37}	0	0	0	0	0	0	1	1
β_{38}	0	0	0	0	0	0	-1	1
β_{39}	0	0	0	0	0	0	1	-1
β_{40}	0	0	0	0	0	0	-1	-1
β_{41}	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2
β_{42}	-1/2	-1/2	-1/2	-1/2	-1/2	-1/2	-1/2	-1/2
β_{43}	1/2	1/2	1/2	-1/2	-1/2	1/2	1/2	1/2
β_{44}	-1/2	-1/2	-1/2	1/2	1/2	-1/2	-1/2	-1/2
β_{45}	1/2	1/2	1/2	-1/2	1/2	-1/2	1/2	1/2
β_{46}	-1/2	-1/2	-1/2	1/2	-1/2	1/2	-1/2	-1/2
β_{47}	1/2	1/2	1/2	-1/2	1/2	1/2	-1/2	1/2
β_{48}	-1/2	-1/2	-1/2	1/2	-1/2	-1/2	1/2	-1/2
β_{49}	1/2	1/2	1/2	-1/2	1/2	1/2	1/2	-1/2
β_{50}	-1/2	-1/2	-1/2	1/2	-1/2	-1/2	-1/2	1/2
β_{51}	1/2	1/2	1/2	1/2	-1/2	-1/2	1/2	1/2
β_{52}	-1/2	-1/2	-1/2	-1/2	1/2	1/2	-1/2	-1/2
β_{53}	1/2	1/2	1/2	1/2	-1/2	1/2	-1/2	1/2
β_{54}	-1/2	-1/2	-1/2	-1/2	1/2	-1/2	1/2	-1/2
β_{55}	1/2	1/2	1/2	1/2	-1/2	1/2	1/2	-1/2
β_{56}	-1/2	-1/2	-1/2	-1/2	1/2	-1/2	-1/2	1/2
β_{57}	1/2	1/2	1/2	1/2	1/2	-1/2	-1/2	1/2
β_{58}	-1/2	-1/2	-1/2	-1/2	-1/2	1/2	1/2	-1/2
β_{59}	1/2	1/2	1/2	1/2	1/2	-1/2	1/2	-1/2
β_{60}	-1/2	-1/2	-1/2	-1/2	-1/2	1/2	-1/2	1/2
β_{61}	1/2	1/2	1/2	1/2	1/2	1/2	-1/2	-1/2
β_{62}	-1/2	-1/2	-1/2	-1/2	-1/2	-1/2	1/2	1/2
β_{63}	-1/2	-1/2	-1/2	-1/2	1/2	1/2	1/2	1/2
β_{64}	-1/2	-1/2	-1/2	1/2	-1/2	1/2	1/2	1/2
β_{65}	-1/2	-1/2	-1/2	1/2	1/2	-1/2	1/2	1/2
β_{66}	-1/2	-1/2	-1/2	1/2	1/2	1/2	-1/2	1/2
β_{67}	-1/2	-1/2	-1/2	1/2	1/2	1/2	1/2	-1/2
β_{68}	1/2	1/2	1/2	-1/2	-1/2	-1/2	-1/2	1/2
β_{69}	1/2	1/2	1/2	-1/2	-1/2	-1/2	1/2	-1/2
β_{70}	1/2	1/2	1/2	-1/2	-1/2	1/2	-1/2	-1/2
β_{71}	1/2	1/2	1/2	-1/2	1/2	-1/2	-1/2	-1/2
β_{72}	1/2	1/2	1/2	1/2	-1/2	-1/2	-1/2	-1/2

Table C.1: Roots of E_6.

α_1	0	0	0	1	-1	0	0	0
α_2	0	0	0	0	1	-1	0	0
α_3	0	0	0	0	0	1	-1	0
α_4	0	0	0	0	0	0	1	-1
α_5	1/2	1/2	1/2	-1/2	-1/2	-1/2	-1/2	1/2
α_6	0	0	0	0	0	0	1	1

Table C.2: Simple roots of E_6.

Appendix D

Details of the SO(10) Calculation

Roots and Simple Roots

We list the simple roots of SO(10) in tab. D.1 below. Note that the simple roots depend on the choice of semi-ordering in the root lattice.

Table D.1: Simple roots of SO(10).

No.	Simple Root				
α_1	1	-1	0	0	0
α_2	0	1	-1	0	0
α_3	0	0	1	-1	0
α_4	0	0	0	1	-1
α_5	0	0	0	1	1

Since the structure constants depend on the choice of roots and their enumeration, we give the 40 roots of SO(10) for reference.

Table D.2: Roots of SO(10).

No.	Root					No.	Root				
β_1	1	1	0	0	0	β_{21}	0	1	0	1	0
β_2	-1	1	0	0	0	β_{22}	0	-1	0	1	0
β_3	1	-1	0	0	0	β_{23}	0	1	0	-1	0
β_4	-1	-1	0	0	0	β_{24}	0	-1	0	-1	0
β_5	1	0	1	0	0	β_{25}	0	1	0	0	1
											continued ...

continuation ...											
β_6	-1	0	1	0	0	β_{26}	0	-1	0	0	1
β_7	1	0	-1	0	0	β_{27}	0	1	0	0	-1
β_8	-1	0	-1	0	0	β_{28}	0	-1	0	0	-1
β_9	1	0	0	1	0	β_{29}	0	0	1	1	0
β_{10}	-1	0	0	1	0	β_{30}	0	0	-1	1	0
β_{11}	1	0	0	-1	0	β_{31}	0	0	1	-1	0
β_{12}	-1	0	0	-1	0	β_{32}	0	0	-1	-1	0
β_{13}	1	0	0	0	1	β_{33}	0	0	1	0	1
β_{14}	-1	0	0	0	1	β_{34}	0	0	-1	0	1
β_{15}	1	0	0	0	-1	β_{35}	0	0	1	0	-1
β_{16}	-1	0	0	0	-1	β_{36}	0	0	-1	0	-1
β_{17}	0	1	1	0	0	β_{37}	0	0	0	1	1
β_{18}	0	-1	1	0	0	β_{38}	0	0	0	-1	1
β_{19}	0	1	-1	0	0	β_{39}	0	0	0	1	-1
β_{20}	0	-1	-1	0	0	β_{40}	0	0	0	-1	-1

Structure Constants

The structure constants depend not only on the algebra, but also on the choice of the basis for the roots. Below we list the structure constants of SO(10) for the roots as given in tab. (D.2).

Table D.3: Structure constants of SO(10).

i	j	$N_{i,j}$	i	j	$N_{i,j}$	i	j	$N_{i,j}$	i	j	$N_{i,j}$	i	j	$N_{i,j}$
1	6	1	1	8	-1	1	10	-1	1	12	-1	1	14	1
1	16	-1	1	18	1	1	20	-1	1	22	1	1	24	1
1	26	1	1	28	-1	2	5	-1	2	7	1	2	9	1
2	11	1	2	13	-1	2	15	1	2	18	-1	2	20	1
2	22	-1	2	24	-1	2	26	-1	2	28	1	3	6	-1
3	8	1	3	10	-1	3	12	-1	3	14	-1	3	16	1
3	17	-1	3	19	1	3	21	1	3	23	1	3	25	-1
3	27	1	4	5	1	4	7	-1	4	9	1	4	11	1
4	13	1	4	15	-1	4	17	1	4	19	-1	4	21	-1
4	23	-1	4	25	1	4	27	-1	5	2	1	5	4	-1
5	10	1	5	12	-1	5	14	1	5	16	-1	5	19	1
5	20	-1	5	30	1	5	32	-1	5	34	1	5	36	-1
														continued ...

continuation ...														
6	1	-1	6	3	1	6	9	1	6	11	-1	6	13	1
6	15	-1	6	19	-1	6	20	1	6	30	1	6	32	-1
6	34	1	6	36	-1	7	2	-1	7	4	1	7	10	1
7	12	-1	7	14	1	7	16	-1	7	17	-1	7	18	1
7	29	1	7	31	-1	7	33	1	7	35	-1	8	1	1
8	3	-1	8	9	1	8	11	-1	8	13	1	8	15	-1
8	17	1	8	18	-1	8	29	1	8	31	-1	8	33	1
8	35	-1	9	2	-1	9	4	-1	9	6	-1	9	8	-1
9	14	1	9	16	-1	9	23	1	9	24	1	9	31	1
9	32	1	9	38	1	9	40	-1	10	1	1	10	3	1
10	5	-1	10	7	-1	10	13	1	10	15	-1	10	23	-1
10	24	-1	10	31	1	10	32	1	10	38	1	10	40	-1
11	2	-1	11	4	-1	11	6	1	11	8	1	11	14	1
11	16	-1	11	21	1	11	22	1	11	29	-1	11	30	-1
11	37	1	11	39	-1	12	1	1	12	3	1	12	5	1
12	7	1	12	13	1	12	15	-1	12	21	-1	12	22	-1
12	29	-1	12	30	-1	12	37	1	12	39	-1	13	2	1
13	4	-1	13	6	-1	13	8	-1	13	10	-1	13	12	-1
13	27	1	13	28	-1	13	35	1	13	36	1	13	39	1
13	40	1	14	1	-1	14	3	1	14	5	-1	14	7	-1
14	9	-1	14	11	-1	14	27	-1	14	28	1	14	35	1
14	36	1	14	39	1	14	40	1	15	2	-1	15	4	1
15	6	1	15	8	1	15	10	1	15	12	1	15	25	-1
15	26	1	15	33	-1	15	34	-1	15	37	-1	15	38	-1
16	1	1	16	3	-1	16	5	1	16	7	1	16	9	1
16	11	1	16	25	1	16	26	-1	16	33	-1	16	34	-1
16	37	-1	16	38	-1	17	3	1	17	4	-1	17	7	1
17	8	-1	17	22	-1	17	24	1	17	26	-1	17	28	-1
17	30	-1	17	32	1	17	34	1	17	36	1	18	1	-1
18	2	1	18	7	-1	18	8	1	18	21	1	18	23	-1
18	25	-1	18	27	-1	18	30	1	18	32	-1	18	34	1
18	36	1	19	3	-1	19	4	1	19	5	-1	19	6	1
19	22	1	19	24	-1	19	26	1	19	28	1	19	29	1
19	31	-1	19	33	-1	19	35	-1	20	1	1	20	2	-1
20	5	1	20	6	-1	20	21	-1	20	23	1	20	25	1
														continued ...

continuation …

20	27	1	20	29	-1	20	31	1	20	33	-1	20	35	-1
21	3	-1	21	4	1	21	11	-1	21	12	1	21	18	-1
21	20	1	21	26	1	21	28	1	21	31	-1	21	32	1
21	38	-1	21	40	-1	22	1	-1	22	2	1	22	11	-1
22	12	1	22	17	1	22	19	-1	22	25	-1	22	27	-1
22	31	1	22	32	-1	22	38	1	22	40	1	23	3	-1
23	4	1	23	9	-1	23	10	1	23	18	1	23	20	-1
23	26	1	23	28	1	23	29	1	23	30	-1	23	37	-1
23	39	-1	24	1	-1	24	2	1	24	9	-1	24	10	1
24	17	-1	24	19	1	24	25	-1	24	27	-1	24	29	-1
24	30	1	24	37	1	24	39	1	25	3	1	25	4	-1
25	15	1	25	16	-1	25	18	1	25	20	-1	25	22	1
25	24	1	25	35	1	25	36	-1	25	39	-1	25	40	-1
26	1	-1	26	2	1	26	15	-1	26	16	1	26	17	1
26	19	-1	26	21	-1	26	23	-1	26	35	1	26	36	-1
26	39	1	26	40	1	27	3	-1	27	4	1	27	13	-1
27	14	1	27	18	1	27	20	-1	27	22	1	27	24	1
27	33	1	27	34	-1	27	37	-1	27	38	-1	28	1	1
28	2	-1	28	13	1	28	14	-1	28	17	1	28	19	-1
28	21	-1	28	23	-1	28	33	1	28	34	-1	28	37	1
28	38	1	29	7	-1	29	8	-1	29	11	1	29	12	1
29	19	-1	29	20	1	29	23	-1	29	24	1	29	34	-1
29	36	-1	29	38	1	29	40	1	30	5	-1	30	6	-1
30	11	1	30	12	1	30	17	1	30	18	-1	30	23	1
30	24	-1	30	33	-1	30	35	-1	30	38	1	30	40	1
31	7	1	31	8	1	31	9	-1	31	10	-1	31	19	1
31	20	-1	31	21	1	31	22	-1	31	34	1	31	36	1
31	37	-1	31	39	-1	32	5	1	32	6	1	32	9	-1
32	10	-1	32	17	-1	32	18	1	32	21	-1	32	22	1
32	33	1	32	35	1	32	37	-1	32	39	-1	33	7	-1
33	8	-1	33	15	1	33	16	1	33	19	1	33	20	1
33	27	-1	33	28	-1	33	30	1	33	32	-1	33	39	1
33	40	-1	34	5	-1	34	6	-1	34	15	1	34	16	1
34	17	-1	34	18	-1	34	27	1	34	28	1	34	29	1
34	31	-1	34	39	1	34	40	-1	35	7	1	35	8	1

continued …

continuation ...														
35	13	-1	35	14	-1	35	19	1	35	20	1	35	25	-1
35	26	-1	35	30	1	35	32	-1	35	37	1	35	38	-1
36	5	1	36	6	1	36	13	-1	36	14	-1	36	17	-1
36	18	-1	36	25	1	36	26	1	36	29	1	36	31	-1
36	37	1	36	38	-1	37	11	-1	37	12	-1	37	15	1
37	16	1	37	23	1	37	24	-1	37	27	1	37	28	-1
37	31	1	37	32	1	37	35	-1	37	36	-1	38	9	-1
38	10	-1	38	15	1	38	16	1	38	21	1	38	22	-1
38	27	1	38	28	-1	38	29	-1	38	30	-1	38	35	1
38	36	1	39	11	1	39	12	1	39	13	-1	39	14	-1
39	23	1	39	24	-1	39	25	1	39	26	-1	39	31	1
39	32	1	39	33	-1	39	34	-1	40	9	1	40	10	1
40	13	-1	40	14	-1	40	21	1	40	22	-1	40	25	1
40	26	-1	40	29	-1	40	30	-1	40	33	1	40	34	1

Weyl Reflections

Below, we list the matrix representation of the simple Weyl reflections in the standard basis of $\mathbb{R}^5$.

$$
\begin{aligned}
r_{\alpha_1} &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
r_{\alpha_2} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad r_{\alpha_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\
r_{\alpha_4} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad r_{\alpha_5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}
\end{aligned}
\tag{D.1}
$$

Adjoint Action of the Cartan Generators

Matrix Representation of the Adjoint Action of the Cartan Generators Corresponding to Pati-Salam Group

$$\mathrm{ad}(\tilde{H}_1) = \left(\begin{array}{ccccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) \tag{D.2}$$

$$\mathrm{ad}(\tilde{H}_2) = \left(\begin{array}{ccccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) \tag{D.3}$$

$$\mathrm{ad}(\tilde{H}_3) = \left(\begin{array}{ccccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) \tag{D.4}$$

$$
\mathrm{ad}(\tilde{H}_4) = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix} \tag{D.5}
$$

$$
\mathrm{ad}(\tilde{H}_5) = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix} \tag{D.6}
$$

Matrix Representation of the Adjoint Action of the Cartan Generators Corresponding to Pati-Salam Group Simultaneously Diagonalized

Since the matrices given in section D mutually commute, they can be simultaneously diagonalized:

$$
A\ \mathrm{ad}(\tilde{H}_1)\, A^{-1} = \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix} \tag{D.7}
$$

$$A\ \mathrm{ad}(\tilde{H}_2)\,A^{-1} = \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix} \tag{D.8}$$

$$A\ \mathrm{ad}(\tilde{H}_3)\,A^{-1} = \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix} \tag{D.9}$$

$$A\ \mathrm{ad}(\tilde{H}_4)\,A^{-1} = \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix} \tag{D.10}$$

$$
A\ \mathrm{ad}(\tilde{H}_5)\, A^{-1} = \begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix} \tag{D.11}
$$

Although not relevant for the calculation, it is interesting to see the structure of the Pati-Salam generators in the basis, where the adjoint action of the corresponding Cartan generators is diagonal. These results are listed below. Note that, expressed in the old basis of SO(10), the expressions are quite complicated, and without an explicit calculation, there is scarcely hope to find them.

$\tilde{E}'_1$	$E_5 - E_8 - E_6 + E_7 + E_9 + E_{12} - E_{10} - E_{11} + E_{17} + E_{20} - E_{18} - E_{19} - E_{21} + E_{24} - E_{22} + E_{23}$
$\tilde{E}'_2$	$E_5 - E_8 - E_6 + E_7 - E_9 - E_{12} + E_{10} + E_{11} + E_{17} + E_{20} - E_{18} - E_{19} + E_{21} - E_{24} + E_{22} - E_{23}$
$\tilde{E}'_3$	$E_5 - E_8 + E_6 - E_7 - E_9 - E_{12} - E_{10} - E_{11} - E_{17} - E_{20} - E_{18} - E_{19} - E_{21} + E_{24} + E_{22} - E_{23}$
$\tilde{E}'_4$	$E_5 - E_8 + E_6 - E_7 + E_9 + E_{12} + E_{10} + E_{11} - E_{17} - E_{20} - E_{18} - E_{19} + E_{21} - E_{24} - E_{22} + E_{23}$
$\tilde{E}'_5$	$E_5 - E_8 + E_6 - E_7 - E_9 - E_{12} - E_{10} - E_{11} + E_{17} + E_{20} + E_{18} + E_{19} + E_{21} - E_{24} - E_{22} + E_{23}$
$\tilde{E}'_6$	$E_5 - E_8 + E_6 - E_7 + E_9 + E_{12} + E_{10} + E_{11} + E_{17} + E_{20} + E_{18} + E_{19} - E_{21} + E_{24} + E_{22} - E_{23}$
$\tilde{E}'_7$	$E_5 - E_8 - E_6 + E_7 + E_9 + E_{12} - E_{10} - E_{11} - E_{17} - E_{20} + E_{18} + E_{19} + E_{21} - E_{24} + E_{22} - E_{23}$
$\tilde{E}'_8$	$E_5 - E_8 - E_6 + E_7 - E_9 - E_{12} + E_{10} + E_{11} - E_{17} - E_{20} + E_{18} + E_{19} - E_{21} + E_{24} - E_{22} + E_{23}$
$\tilde{E}'_9$	$E_{33} - E_{34} - E_{37} - E_{38}$
$\tilde{E}'_{10}$	$E_{33} - E_{34} + E_{37} + E_{38}$
$\tilde{E}'_{11}$	$E_{35} - E_{36} - E_{39} - E_{40}$
$\tilde{E}'_{12}$	$E_{35} - E_{36} + E_{39} + E_{40}$
$\tilde{E}'_{13}$	$E_{13} + E_{14} + E_{25} + E_{26}$
$\tilde{E}'_{14}$	$E_{13} + E_{14} - E_{25} - E_{26}$
$\tilde{E}'_{15}$	$E_{15} + E_{16} - E_{27} - E_{28}$
$\tilde{E}'_{16}$	$E_{15} + E_{16} + E_{27} + E_{28}$
$\tilde{E}'_{17}$	$E_1 + E_4$
$\tilde{E}'_{18}$	$E_2 + E_3$
$\tilde{E}'_{19}$	$E_{29} + E_{32}$
$\tilde{E}'_{20}$	$E_{30} + E_{31}$
$\tilde{E}'_{21}$	H_5

Appendix E

Inequivalent $\mathbb{Z}_6$-II Gauge Shifts

E.1 Simple Roots of E_8

In tab. E.1 we present the simple roots of E_8. Note that the choice of simple roots is not unique, but depends on the semi-ordering introduced in root space.

No.	Simple Root							
α_1	1/2	-1/2	-1/2	-1/2	-1/2	-1/2	-1/2	1/2
α_2	0	0	0	0	0	0	1	-1
α_3	0	0	0	0	0	1	-1	0
α_4	0	0	0	0	1	-1	0	0
α_5	0	0	0	1	-1	0	0	0
α_6	0	0	1	-1	0	0	0	0
α_7	0	1	-1	0	0	0	0	0
α_8	0	0	0	0	0	0	1	1
α_0	-1	-1	0	0	0	0	0	0

Table E.1: Simple roots of E_8 in some standard basis.

For constructing the gauge shift according to eq. (4.32) we need the dual simple roots, which are the rows of the matrix $(M^{-1})^T$, where M is the matrix whose rows are the simple roots.

E.2 The 61 $\mathbb{Z}_6$-II Models

1	1/6	5/6	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	1/4	1/4	-1/4	-1/4	-1/4	-1/4	-1/4	-1/4	$A_8 + E_7' + A_1'$
2	1/6	5/6	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	1/4	3/4	-1/12	-1/12	-1/12	-1/4	-1/4	-1/4	$A_8 + A_5' + A_2' + A_1'$
3	1/6	5/6	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	1/3	1/2	0/1	-1/6	-1/6	-1/6	-1/6	-1/6	$A_8 + D_5' + A_1'$
4	1/6	5/6	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	1/4	1/4	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	$A_8 + E_6' + A_1'$
5	1/4	1/4	-1/4	-1/4	-1/4	-1/4	-1/4	-1/4	1/4	5/12	-1/12	-1/4	-1/4	-1/4	-1/4	-1/4	$E_7 + A_1 + D_6' + A_1'$
6	1/4	1/4	-1/4	-1/4	-1/4	-1/4	-1/4	-1/4	1/4	3/4	1/12	-1/12	-1/4	-1/4	-1/4	-1/4	$E_7 + A_1 + A_7'$
7	1/4	1/4	-1/4	-1/4	-1/4	-1/4	-1/4	-1/4	1/4	7/12	-1/12	-1/12	-1/12	-1/12	-1/4	-1/4	$E_7 + A_1 + A_4' + A_3'$
8	1/4	1/4	-1/4	-1/4	-1/4	-1/4	-1/4	-1/4	1/6	1/6	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	$E_7 + A_1 + E_7'$
9	1/4	1/4	-1/4	-1/4	-1/4	-1/4	-1/4	-1/4	1/12	5/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	$E_7 + A_1 + A_7'$
10	1/3	2/3	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	1/2	1/2	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	$A_5 + A_2 + E_6' + A_2'$
11	1/3	2/3	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	1/4	7/12	-1/12	-1/12	-1/4	-1/4	-1/4	-1/4	$A_5 + A_2 + A_5' + A_1' + A_1'$
12	1/3	2/3	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	1/4	3/4	1/4	-1/4	-1/4	-1/4	-1/4	-1/4	$A_5 + A_2 + D_8'$
13	1/3	2/3	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	1/4	5/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/4	$A_5 + A_2 + D_5' + A_2'$
14	1/3	2/3	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	0/1	0/1	0/1	0/1	0/1	0/1	0/1	0/1	$A_5 + A_2 + E_8'$
15	1/2	1/2	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	1/4	7/12	1/12	-1/4	-1/4	-1/4	-1/4	-1/4	$E_6 + A_2 + D_6' + A_1'$
16	1/2	1/2	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	1/3	1/3	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	$E_6 + A_2 + E_6'$
17	1/2	1/2	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	1/6	1/2	0/1	0/1	-1/6	-1/6	-1/6	-1/6	$E_6 + A_2 + A_6' + A_1'$
18	1/2	1/2	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	1/12	1/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	$E_6 + A_2 + E_7'$
19	1/3	1/2	-1/6	-1/6	-1/6	-1/6	-1/6	-1/3	1/3	2/3	0/1	-1/6	-1/6	-1/6	-1/6	-1/3	$D_5 + A_1 + A_1 + D_4' + A_3'$
20	1/3	1/2	-1/6	-1/6	-1/6	-1/6	-1/6	-1/3	1/6	1/2	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	$D_5 + A_1 + A_1 + A_6'$

21	1/3	1/2	-1/6	-1/6	-1/6	-1/6	-1/6	-1/3	1/6	1/2	1/6	-1/6	-1/6	-1/6	-1/6	-1/6	$D_5 + A_1 + A_1 + D_7'$
22	1/3	1/2	-1/6	-1/6	-1/6	-1/6	-1/6	-1/3	1/12	1/4	1/12	-1/12	-1/12	-1/12	-1/12	-1/12	$D_5 + A_1 + A_1 + D_7'$
23	1/4	3/4	-1/12	-1/12	-1/12	-1/4	-1/4	-1/4	1/4	5/12	-1/12	-1/4	-1/4	-1/4	-1/4	-1/4	$A_5 + A_2 + A_1 + D_6' + A_1'$
24	1/4	3/4	-1/12	-1/12	-1/12	-1/4	-1/4	-1/4	1/4	3/4	1/12	-1/12	-1/4	-1/4	-1/4	-1/4	$A_5 + A_2 + A_1 + A_7'$
25	1/4	3/4	-1/12	-1/12	-1/12	-1/4	-1/4	-1/4	1/4	7/12	-1/12	-1/12	-1/12	-1/12	-1/4	-1/4	$A_5 + A_2 + A_1 + A_4' + A_3'$
26	1/4	3/4	-1/12	-1/12	-1/12	-1/4	-1/4	-1/4	1/6	1/6	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	$A_5 + A_2 + A_1 + E_7'$
27	1/4	3/4	-1/12	-1/12	-1/12	-1/4	-1/4	-1/4	1/12	5/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	$A_5 + A_2 + A_1 + A_7'$
28	1/4	7/12	-1/12	-1/12	-1/4	-1/4	-1/4	-1/4	1/4	7/12	1/12	-1/4	-1/4	-1/4	-1/4	-1/4	$A_5 + A_1 + A_1 + D_6' + A_1'$
29	1/4	7/12	-1/12	-1/12	-1/4	-1/4	-1/4	-1/4	1/3	1/3	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	$A_5 + A_1 + A_1 + E_6'$
30	1/4	7/12	-1/12	-1/12	-1/4	-1/4	-1/4	-1/4	1/6	1/2	0/1	0/1	-1/6	-1/6	-1/6	-1/6	$A_5 + A_1 + A_1 + A_6' + A_1'$
31	1/4	7/12	-1/12	-1/12	-1/4	-1/4	-1/4	-1/4	1/12	1/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	$A_5 + A_1 + A_1 + E_7'$
32	1/4	5/12	-1/12	-1/4	-1/4	-1/4	-1/4	-1/4	1/3	1/2	0/1	-1/6	-1/6	-1/6	-1/6	-1/6	$D_6 + A_1 + D_5' + A_1'$
33	1/4	5/12	-1/12	-1/4	-1/4	-1/4	-1/4	-1/4	1/4	1/4	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	$D_6 + A_1 + E_6' + A_1'$
34	1/3	2/3	0/1	-1/6	-1/6	-1/6	-1/6	-1/3	1/6	2/3	0/1	-1/6	-1/6	-1/6	-1/6	-1/6	$D_4 + A_3 + A_6'$
35	1/3	2/3	0/1	-1/6	-1/6	-1/6	-1/6	-1/3	1/6	1/3	0/1	-1/6	-1/6	-1/6	-1/6	-1/6	$D_4 + A_3 + D_6'$
36	1/4	3/4	1/12	-1/12	-1/4	-1/4	-1/4	-1/4	1/3	1/2	0/1	-1/6	-1/6	-1/6	-1/6	-1/6	$A_7 + D_5' + A_1'$
37	1/4	3/4	1/12	-1/12	-1/4	-1/4	-1/4	-1/4	1/4	1/4	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	$A_7 + E_6' + A_1'$
38	1/4	7/12	1/12	-1/4	-1/4	-1/4	-1/4	-1/4	1/4	3/4	1/4	-1/4	-1/4	-1/4	-1/4	-1/4	$D_6 + A_1 + D_8'$
39	1/4	7/12	1/12	-1/4	-1/4	-1/4	-1/4	-1/4	1/4	5/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/4	$D_6 + A_1 + D_5' + A_2'$
40	1/4	7/12	1/12	-1/4	-1/4	-1/4	-1/4	-1/4	0/1	0/1	0/1	0/1	0/1	0/1	0/1	0/1	$D_6 + A_1 + E_8'$

41	1/4	3/4	1/4	-1/4	-1/4	-1/4	-1/4	-1/4	1/3	1/3	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	$D_8 + E_6'$
42	1/4	3/4	1/4	-1/4	-1/4	-1/4	-1/4	-1/4	1/6	1/2	0/1	0/1	-1/6	-1/6	-1/6	-1/6	$D_8 + A_6' + A_1'$
43	1/4	3/4	1/4	-1/4	-1/4	-1/4	-1/4	-1/4	1/12	1/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	$D_8 + E_7'$
44	1/6	1/2	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	1/6	2/3	0/1	-1/6	-1/6	-1/6	-1/6	-1/6	$A_6 + A_6'$
45	1/6	1/2	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	1/6	1/3	0/1	-1/6	-1/6	-1/6	-1/6	-1/6	$A_6 + D_6'$
46	1/3	1/3	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	1/4	5/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/4	$E_6 + D_5' + A_2'$
47	1/3	1/3	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	0/1	0/1	0/1	0/1	0/1	0/1	0/1	0/1	$E_6 + E_8'$
48	1/4	7/12	-1/12	-1/12	-1/12	-1/12	-1/4	-1/4	1/3	1/2	0/1	-1/6	-1/6	-1/6	-1/6	-1/6	$A_4 + A_3 + D_5' + A_1'$
49	1/4	7/12	-1/12	-1/12	-1/12	-1/12	-1/4	-1/4	1/4	1/4	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	$A_4 + A_3 + E_6' + A_1'$
50	1/6	2/3	0/1	-1/6	-1/6	-1/6	-1/6	-1/6	1/6	1/2	1/6	-1/6	-1/6	-1/6	-1/6	-1/6	$A_6 + D_7'$
51	1/6	2/3	0/1	-1/6	-1/6	-1/6	-1/6	-1/6	1/12	1/4	1/12	-1/12	-1/12	-1/12	-1/12	-1/12	$A_6 + D_7'$
52	1/3	1/2	0/1	-1/6	-1/6	-1/6	-1/6	-1/6	1/6	1/6	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	$D_5 + A_1 + E_7'$
53	1/3	1/2	0/1	-1/6	-1/6	-1/6	-1/6	-1/6	1/12	5/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	$D_5 + A_1 + A_7'$
54	1/6	1/6	-1/6	-1/6	-1/6	-1/6	-1/6	-1/6	1/4	1/4	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	$E_7 + E_6' + A_1'$
55	1/4	5/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/4	1/6	1/2	0/1	0/1	-1/6	-1/6	-1/6	-1/6	$D_5 + A_2 + A_6' + A_1'$
56	1/4	5/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/4	1/12	1/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	$D_5 + A_2 + E_7'$
57	1/6	1/2	0/1	0/1	-1/6	-1/6	-1/6	-1/6	0/1	0/1	0/1	0/1	0/1	0/1	0/1	0/1	$A_6 + A_1 + E_8'$
58	1/6	1/3	0/1	-1/6	-1/6	-1/6	-1/6	-1/6	1/6	1/2	1/6	-1/6	-1/6	-1/6	-1/6	-1/6	$D_6 + D_7'$
59	1/6	1/3	0/1	-1/6	-1/6	-1/6	-1/6	-1/6	1/12	1/4	1/12	-1/12	-1/12	-1/12	-1/12	-1/12	$D_6 + D_7'$
60	1/12	5/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	1/4	1/4	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	$A_7 + E_6' + A_1'$
61	1/12	1/12	-1/12	-1/12	-1/12	-1/12	-1/12	-1/12	0/1	0/1	0/1	0/1	0/1	0/1	0/1	0/1	$E_7 + E_8'$

Glossary

a_i Kač labels

A_{ij} Cartan matrix of semisimple Lie algebra

α_0 Most negative root of simple Lie algebra

$\mathbf{Aut}(\mathfrak{g})$ Automorphism group of Lie algebra

$D(\mathfrak{g})$ Symmetry group of the Dynkin diagram

Δ Set of roots

Δ^+ Set of positive roots

Δ^- Set of negative roots

G Subset of the automorphism group of a Lie algebra

$\mathfrak{g}$ Semi-simple Lie algebra

Γ Dynkin diagram

$\Gamma^{(k)}$ Affine Dynkin diagram

$\mathfrak{h}$ Cartan subalgebra

$\mathbf{Int}(\mathfrak{g})$ Group of inner automorphisms of Lie algebra

Λ Root lattice of semisimple Lie algebra

$\mathcal{M}$ Compact manifold

μ Dynkin diagram automorphism

$\mathcal{O}$ Orbifold

P Point group

Π Set of simple roots

S Space group

$S(\Delta)$ Lattice automorphisms of Λ

σ Lie algebra automorphism

S_ℓ Symmetric group of ℓ elements

$T_{\mathbf{E}_8\times\mathbf{E}_8'}$ Root lattice of $\mathrm{E}_8 \times \mathrm{E}_8'$

θ Highest root of Lie algebra

$\mathbf{Vol}(\Lambda)$ Fundamental volume of lattice Λ

$\mathcal{W}$ Weyl group

$|\cdot\rangle_L$ Left mover of string state

$|\cdot\rangle_R$ Right mover of string state

$[\quad]_{DL}$ Dynkin labels of weight vector

$(\quad)_K$ Killing form

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