On affine Deligne-Lusztig varieties for GL_n

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Eva Mierendorff

aus

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1. Referent: Prof. Dr. M. Rapoport
2. Referent: Prof. Dr. G. Faltings

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Introduction

Let $F = \mathrm{Quot}(W(\mathbb{F}_q))$ or $F = \mathbb{F}_q((t))$ where $q = p^r$ for some prime p. Let L be the completion of the maximal unramified extension of F. Let $\mathcal{O}_F$ and $\mathcal{O}_L$ be the valuation rings. We denote by $\sigma : x \mapsto x^q$ the Frobenius of $\overline{\mathbb{F}}_q$ over $\mathbb{F}_q$ and also of L over F.

Let $G = GL_n$ over F and let A be the diagonal torus. Let B be the Borel subgroup of lower triangular matrices. For $\mu, \mu' \in X_*(A)_{\mathbb{Q}}$ we say that $\mu \geq \mu'$ if $\mu - \mu'$ is a non-negative linear combination of positive coroots. An element $\mu = (\mu_1, \dots, \mu_n) \in X_*(A) \cong \mathbb{Z}^n$ is dominant if $\mu_1 \leq \cdots \leq \mu_n$. We write μ_{dom} for the dominant element in the orbit of $\mu \in X_*(A)$ under the Weyl group of A in G. For $\alpha \in X_*(A)$ we denote by $t^\alpha \in A(F)$ the image of the uniformiser of $\mathcal{O}_F$ under the homomorphism $\alpha : \mathbb{G}_m \to A$.

We recall the definitions of affine Deligne-Lusztig sets and closed affine Deligne-Lusztig sets from [Ra1], [GHKR]. Let $K = G(\mathcal{O}_L)$ and let $X = G(L)/K$. For $b \in G(L)$ and a dominant coweight $\mu \in X_*(A)$, the affine Deligne-Lusztig set $X_\mu(b)$ is the subset of X defined by

$$X_\mu(b) = \{g \in G(L)/K \mid g^{-1} b \sigma(g) \in K t^\mu K\}. \tag{0.0.1}$$

The closed affine Deligne-Lusztig set is the subset of X defined by

$$X_{\leq\mu}(b) = \bigcup_{\mu' \leq \mu} X_{\mu'}(b).$$

Let $\nu \in \mathbb{Q}^n$ be the Newton vector associated to b. In [KR] Kottwitz and Rapoport prove that $X_\mu(b)$ is nonempty if and only if $\nu \leq \mu$. From now on we assume this.

If F is a function field, then $X_\mu(b)$ and $X_{\leq\mu}(b)$ have the structure of reduced subschemes of the affine Grassmannian X. Both are locally of finite type. Besides, $X_\mu(b)$ is a locally closed subscheme and $X_{\leq\mu}(b)$ is a closed subscheme of X. We call this the function field case or the equal characteristic case.

If $F = \mathrm{Quot}(W(\mathbb{F}_q))$, a scheme structure on the affine Deligne-Lusztig sets is in general not known. However, if $q = p$ and if $\mu = (0, \dots, 0, 1, \dots, 1)$ is minuscule, $X_\mu(b)$ has an interpretation as the set of geometric points of a moduli space of quasi-isogenies of p-divisible groups.

We now describe these moduli spaces. Let k be a perfect field of characteristic p and $W = W(k)$ its ring of Witt vectors. Let $\sigma : x \mapsto x^p$ be the Frobenius automorphism on k as well as on W. By Nilp_W we denote the category of schemes S over $\mathrm{Spec}(W)$ such that p is locally nilpotent on S. Let $\overline{S}$ be the closed subscheme of S that is defined by the ideal sheaf $p\mathcal{O}_S$. Let $\mathbb{X}$ be a decent p-divisible group over k.

We consider the functor

$$\mathcal{M} = \mathcal{M}(\mathbb{X}) : \mathrm{Nilp}_W \to \mathrm{Sets},$$

which assigns to $S \in \mathrm{Nilp}_W$ the set of isomorphism classes of pairs (X, ρ), where X is a p-divisible group over S and $\rho : \mathbb{X}_{\overline{S}} = \mathbb{X} \times_{\mathrm{Spec}(k)} \overline{S} \to X \times_S \overline{S}$ is a quasi-isogeny. Two pairs (X_1, ρ_1) and (X_2, ρ_2) are isomorphic if $\rho_1 \circ \rho_2^{-1}$ lifts to an isomorphism $X_2 \to X_1$. Rapoport and Zink prove that this functor is representable by a formal scheme $\mathcal{M} = \mathcal{M}(\mathbb{X})$, which is locally formally of finite type over $\mathrm{Spf}(W)$ (see [RZ], Thm. 2.16). Let $\mathcal{M}_{\mathrm{red}}$ be its reduced subscheme. The irreducible components of $\mathcal{M}_{\mathrm{red}}$ are projective varieties ([RZ], Prop. 2.32).

Let (N, F) be the rational Dieudonné module of $\mathbb{X}$ and M_0 its Dieudonné module. Assume $k = \mathbb{F}_p$ and let $G = GL(N)$. We write $F = b\sigma$ with $b \in G$. Let $\mu = \mathrm{inv}(M_0, F(M_0))$ be the relative position of M_0 and $F(M_0)$. Then

$$\begin{array}{rcl} X_\mu(b) & \to & \mathcal{M}_{\mathrm{red}}(\overline{\mathbb{F}}_p) \\ g & \mapsto & gM_0 \end{array}$$

is a bijection. As μ is minuscule, we have $X_\mu(b) = X_{\leq\mu}(b)$. This case is called the unequal characteristic case.

Both in this and in the equal characteristic case, $X_\mu(b)$ and $X_{\leq\mu}(b)$ are called affine Deligne-Lusztig varieties. From now on we only consider these two cases where $X_\mu(b)$ has the structure of a reduced scheme.

Left multiplication by $g \in G(L)$ induces an isomorphism between $X_\mu(b)$ and $X_\mu(gb\sigma(g)^{-1})$. Thus the isomorphism class of the affine Deligne-Lusztig variety only depends on the σ-conjugacy class of b.

We write $\pi_1(G)$ for the quotient of $X_*(A)$ by the coroot lattice of G. In [K2] Kottwitz defines a homomorphism

$$\kappa_G : G(L) \to \pi_1(G)$$

which induces a locally constant map $\kappa_G : X \to \pi_1(G)$. For $G = GL_n$ we have $\pi_1(G) \cong \mathbb{Z}$ and $\kappa_G(g)$ is the valuation of $\det(g)$.

Let $\boldsymbol{P}$ be a standard parabolic subgroup of G. Then $\boldsymbol{P} = \boldsymbol{M}\boldsymbol{N}$, where $\boldsymbol{N}$ is the unipotent radical of $\boldsymbol{P}$ and where $\boldsymbol{M}$ is the unique Levi subgroup of $\boldsymbol{P}$ containing A. Applying the construction of κ to $\boldsymbol{M}$ rather than G we obtain a homomorphism $\kappa_{\boldsymbol{M}} : \boldsymbol{M}(L) \to \pi_1(\boldsymbol{M})$. The inclusion $\boldsymbol{M}(L)/\boldsymbol{M}(\mathcal{O}_L) \hookrightarrow G(L)/G(\mathcal{O}_L)$ induces for each μ and each $b \in \boldsymbol{M}(L)$ an inclusion $X_\mu^{\boldsymbol{M}}(b) \hookrightarrow X_\mu^G(b)$. Here $X_\mu^{\boldsymbol{M}}(b)$ denotes the affine Deligne-Lusztig variety for $\boldsymbol{M}$.

Let $A_{\boldsymbol{P}}$ denote the identity component of the center of $\boldsymbol{M}$. Let

$$\mathfrak{a}_{\boldsymbol{P}}^+ = \{x \in X_*(A_{\boldsymbol{P}}) \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \alpha, x \rangle > 0 \text{ for every root } \alpha \text{ of } A_{\boldsymbol{P}} \text{ in } \boldsymbol{N}\}.$$

In [K2] it is shown that there is a unique standard parabolic $\boldsymbol{P}_b = \boldsymbol{M}_b\boldsymbol{N}_b$ of G such that the σ-conjugacy class of b contains an element b' with the following properties: b' is basic in $\boldsymbol{M}_b$ and $\kappa_{\boldsymbol{M}_b}(b')$, considered as an element of $\mathfrak{a}_{\boldsymbol{P}_b}$, lies in $\mathfrak{a}_{\boldsymbol{P}_b}^+$. We assume that $b = b'$. The proof of the Hodge-Newton decomposition by Kottwitz (see [K3]) yields: Let $\boldsymbol{P} = \boldsymbol{MN} \subseteq G$ be a standard parabolic subgroup with $\boldsymbol{P}_b \subseteq \boldsymbol{P}$. If $\kappa_{\boldsymbol{M}}(b) = \mu$, then the morphism $X_\mu^{\boldsymbol{M}}(b) \hookrightarrow X_\mu^G(b)$ is an isomorphism. We call a pair (μ, b) indecomposable with respect to the Hodge-Newton decomposition if for all standard parabolic subgroups $\boldsymbol{P}$ with $\boldsymbol{P}_b \subseteq \boldsymbol{P} = \boldsymbol{MN} \subsetneq G$ we have $\kappa_{\boldsymbol{M}}(b) \neq \mu$. Given G, μ, and b we may always pass to a Levi subgroup $\boldsymbol{M}$ of G in which (μ, b) is indecomposable. For a description of the affine Deligne-Lusztig varieties it is therefore sufficient to consider pairs (μ, b) which are indecomposable with respect to the Hodge-Newton decomposition.

Let

$$J = \{g \in GL_n(L) \mid g \circ b\sigma = b\sigma \circ g\}.$$

Then there is a canonical J-action on $X_\mu(b)$.

In this thesis, we study the global structure of affine Deligne-Lusztig varieties. More precisely, we address the following questions.

Question 1: What are the sets of connected components of $X_\mu(b)$ and $X_{\leq\mu}(b)$?

For the closed affine Deligne-Lusztig varieties we prove both in the equal and in the unequal characteristic case that J acts transitively on the set of connected components. We obtain the following description of $\pi_0(X_{\leq\mu}(b))$.

Theorem. *Let (μ, b) be as above and indecomposable with respect to the Hodge-Newton decomposition.*

(i) Either $\kappa_{\boldsymbol{M}}(b) \neq \mu$ for all proper standard parabolic subgroups $\boldsymbol{P}$ of G with $b \in \boldsymbol{M}$ or the σ-conjugacy class $[b]$ is central and equal to $[t^\mu]$.

(ii) In the first case, κ_G induces a bijection between $\pi_0(X_{\leq\mu}(b))$ and $\pi_1(GL_n) \cong \mathbb{Z}$.

(iii) In the second case, $X_\mu(b) = X_{\leq\mu}(b) \cong J/(J \cap K) \cong GL_n(F)/GL_n(\mathcal{O}_F)$.

For the moduli spaces $\mathcal{M}(\mathbb{X})$ this result leads to an explicit description of the set of connected components without assuming that (μ, b) is indecomposable: Let $\mathbb{X} = \mathbb{X}_{\mathrm{m}} \times \mathbb{X}_{\mathrm{bi}} \times \mathbb{X}_{\mathrm{et}}$ be the decomposition of $\mathbb{X}$ into its multiplicative, bi-infinitesimal and étale part. Then $\mathcal{M}(\mathbb{X}_{\mathrm{m}})$ and $\mathcal{M}(\mathbb{X}_{\mathrm{et}})$ are discrete and as sets isomorphic to $GL_{\mathrm{ht}(\mathbb{X}_{\mathrm{m}})}(F)/GL_{\mathrm{ht}(\mathbb{X}_{\mathrm{m}})}(\mathcal{O}_F)$ and $GL_{\mathrm{ht}(\mathbb{X}_{\mathrm{et}})}(F)/GL_{\mathrm{ht}(\mathbb{X}_{\mathrm{et}})}(\mathcal{O}_F)$.

Theorem.

$$\pi_0(\mathcal{M}(\mathbb{X})) \cong \begin{cases} \mathcal{M}(\mathbb{X}_{\mathrm{m}}) \times \mathcal{M}(\mathbb{X}_{\mathrm{et}}) & \text{if } \mathbb{X} \text{ is ordinary} \\ \mathcal{M}(\mathbb{X}_{\mathrm{m}}) \times \mathcal{M}(\mathbb{X}_{\mathrm{et}}) \times \mathbb{Z} & \text{else.} \end{cases}$$

The factors on the right hand side correspond to the indecomposable factors of (μ, b) after applying the Hodge-Newton decomposition.

For the non-closed varieties our calculations seem to support the following conjecture.

Conjecture. *The action of J on $\pi_0(X_\mu(b))$ is transitive.*

We do not have a precise conjecture for $\pi_0(X_\mu(b))$. The theorem implies that the map $\pi_0(X_\mu(b)) \to \pi_0(X_{\leq\mu}(b))$ induced by the inclusion is surjective. We give an example to show that in general it is not injective.

Question 2: What is the set of irreducible components of $X_\mu(b)$?

For the moduli spaces $\mathcal{M}$ we show

Theorem. *There is a bijection between the set of irreducible components of $\mathcal{M}$ and $J/(K \cap J)$.*

Guided by this result we arrive at the following conjecture about the set of irreducible components of $X_\mu(b)$.

Conjecture. *The action of J on the set of irreducible components of $X_\mu(b)$ has only finitely many orbits.*

However, we give an example to show that in general the action of J on the set of irreducible components is not transitive for non-minuscule μ.

Question 3: What is the dimension of $X_\mu(b)$?

Affine Deligne-Lusztig varieties $X_\mu(b)$ can also be defined as in (0.0.1) when GL_n is replaced by an unramified connected reductive group G. There is a conjectural formula for the dimension of $X_\mu(b)$ by Rapoport (see [Ra2], Conj. 5.10). For split groups G it takes the form

Conjecture. (Rapoport)

$$\dim X_\mu(b) = \langle 2\rho, \mu - \nu \rangle + \sum_i \left[\langle \omega_i, \nu - \mu \rangle \right]$$

Here ρ is the half-sum of the positive roots and ω_i are the fundamental weights of G_{ad}. By $[x]$ we denote the greatest integer which is less or equal to x.

In [GHKR], Görtz, Haines, Kottwitz, and Reuman reduce the proof of the dimension formula for Deligne-Lusztig varieties in the function field case to the case that $G = GL_n$ and that the σ-conjugacy class of b is superbasic. Here superbasic means that no σ-conjugate element is contained in a proper Levi subgroup of G. They prove the conjecture for $b \in A(L)$. For moduli spaces of p-divisible groups whose isocrystal is simple, the conjecture is shown by de Jong and Oort in [JO]. We prove the dimension conjecture for moduli spaces of p-divisible groups without this restriction. For the relation to results of Chai and Oort see the introduction to Section 1.

Theorem. *Let ν be the Newton vector of $\mathbb{X}$ and $\mu = (0, \dots, 0, 1, \dots, 1)$ minuscule. Then*

$$\dim \mathcal{M}_{\mathrm{red}} = \langle 2\rho, \mu - \nu \rangle + \sum_i \left[\langle \omega_i, \nu - \mu \rangle \right].$$

The conjecture leads to the following conjectural description of the set of pairs (μ, b) with $\dim X_\mu(b) = 0$. It is a modification of a conjecture by Rapoport.

Conjecture. (Rapoport) *Let G be split. Assume that $X_\mu(b)$ is nonempty and that (μ, b) is indecomposable with respect to the Hodge-Newton decomposition. Then $\dim X_\mu(b) > 0$ unless either $[b]$ is μ-ordinary or the adjoint group G_{ad} is equal to PGL_n and $\mu = (0, \dots, 0, 1)$ or $\mu = (0, 1, \dots, 1)$.*

Our methods seem to give lower bounds on $\dim X_\mu(b)$. One instance is given by the following result on affine Deligne-Lusztig varieties in the function field case, which proves the preceding conjecture for $G = GL_n$.

Theorem. *Let $G = GL_n$ and let (μ, b) be indecomposable with respect to the Hodge-Newton decomposition. Then $\dim X_\mu(b) = 0$ if and only if μ is of one of the following forms: $\mu = (a, \dots, a)$, $\mu = (a, \dots, a, a+1)$ or $\mu = (a-1, a, \dots, a)$ for some $a \in \mathbb{Z}$.*

Question 4: Which of the moduli spaces $\mathcal{M}_{\mathrm{red}}$ are smooth?

Our results on the sets of connected components and of irreducible components show that the connected components of a moduli space $\mathcal{M}(\mathbb{X})$ are not irreducible and thus not smooth unless the isocrystal of the bi-infinitesimal part of $\mathbb{X}$ is simple. In this case, the connected components of the moduli space are irreducible and projective. By $\mathcal{M}^0_{\mathrm{red}}$ we denote the connected component of the identity in the reduced subscheme of the moduli space $\mathcal{M}$.

Theorem. *Let $\mathbb{X}$ be bi-infinitesimal and let its isocrystal be simple. Let $l \neq p$ be prime. Then for all j*

$$\begin{aligned} H^{2j+1}(\mathcal{M}^0_{\mathrm{red}}, \mathbb{Q}_l) &= 0 \\ H^{2j}(\mathcal{M}^0_{\mathrm{red}}, \mathbb{Q}_l) &= \mathbb{Q}_l(-j)^{d(j)}, \end{aligned}$$

for some $d(j) \in \mathbb{Z}$.

A combinatorical description of the dimensions $d(j)$ in terms of the slope of the isocrystal is given. The proof of this theorem uses a paving of $\mathcal{M}^0_{\mathrm{red}}$ by affine spaces which resembles the description of the geometric points in [JO], 5. As an application we show

Theorem. *Let $\mathbb{X}$ be a p-divisible group over k. Then $\mathcal{M}^0_{\mathrm{red}}$ is smooth if and only if one of the following holds: $\dim \mathcal{M}^0_{\mathrm{red}} = 0$ or the isocrystal N of $\mathbb{X}_{\mathrm{bi}}$ is simple of slope $\frac{2}{5}$ or $\frac{3}{5}$. In these last cases, $\mathcal{M}^0_{\mathrm{red}} \cong \mathbb{P}^1$.*

In this situation our description of the set of zero-dimensional affine Deligne-Lusztig varieties takes the following form: The condition $\dim \mathcal{M}^0_{\mathrm{red}} = 0$ holds if and only if $\mathbb{X}$ is ordinary or the isocrystal of $\mathbb{X}_{\mathrm{bi}}$ is simple of slope $\frac{m}{m+n}$ with $\min\{m,n\} = 1$.

We obtain our results by explicit calculations for the elements of $X_\mu(b)$. For these calculations it is essential to find bases for the isocrystal $(N, b\sigma)$ and the lattices in which the different problems raised above take a particularly simple form. To obtain a basis for the isocrystal we choose a decomposition into simple summands. For each simple summand of N we take the basis defined in [JO],5. Then $b\sigma$ only permutes the basis elements and multiplies them by powers of the uniformising element. The bases for the lattices are adjusted to the different questions. As an example we sketch how to choose the basis for the description of the irreducible components of $\mathcal{M}$: Results of Oort [O1] and Oort and Zink [OZ] show that the Dieudonné lattices that are generated by a single element form a dense subset. Thus it is enough to consider these lattices. We define a normal form of the generator of such a lattice. Then a suitable basis for the lattice is given by the images of the normalised generator under powers of F and V.

This paper consists of two parts: In the first section we answer the above questions for moduli spaces of p-divisible groups. In the second section we consider generalisations of these results to affine Deligne-Lusztig varieties for the function field case. Each section has its own introduction where one can find more precise versions of the theorems stated above.

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Contents

1 Moduli spaces of p-divisible groups

In this section we study the global structure of moduli spaces of quasi-isogenies of p-divisible groups introduced by Rapoport and Zink. We determine their dimensions and their sets of connected components and of irreducible components. If the isocrystals of the p-divisible groups are simple, we compute the cohomology of the moduli space. As an application we determine which moduli spaces are smooth.

1.1 Introduction

Let k be a perfect field of characteristic p and $W = W(k)$ its ring of Witt vectors. Let σ be the Frobenius automorphism on k as well as on W. By Nilp_W we denote the category of schemes S over $\mathrm{Spec}(W)$ such that p is locally nilpotent on S. Let $\overline{S}$ be the closed subscheme of S that is defined by the ideal sheaf $p\mathcal{O}_S$. Let $\mathbb{X}$ be a decent p-divisible group over k.

We consider the functor

$$\mathcal{M} : \mathrm{Nilp}_W \to \mathrm{Sets},$$

which assigns to $S \in \mathrm{Nilp}_W$ the set of isomorphism classes of pairs (X, ρ), where X is a p-divisible group over S and $\rho : \mathbb{X}_{\overline{S}} = \mathbb{X} \times_{\mathrm{Spec}(k)} \overline{S} \to X \times_S \overline{S}$ is a quasi-isogeny. Two pairs (X_1, ρ_1) and (X_2, ρ_2) are isomorphic if $\rho_1 \circ \rho_2^{-1}$ lifts to an isomorphism $X_2 \to X_1$. This functor is representable by a formal scheme $\mathcal{M}$, which is locally formally of finite type over $\mathrm{Spf}(W)$ (see [RZ], Thm. 2.16). Let $\mathcal{M}_{\mathrm{red}}$ be its reduced subscheme. The irreducible components of $\mathcal{M}_{\mathrm{red}}$ are projective varieties ([RZ], Prop. 2.32).

These moduli spaces and their generalisations for moduli problems of type (EL) or (PEL) serve to analyse the local structure of Shimura varieties which have an interpretation as moduli spaces of abelian varieties. In [RZ] they are used to prove a uniformization theorem for Shimura varieties along Newton strata. Mantovan (see [Ma]) computes the cohomology of certain (PEL) type Shimura varieties in terms of the cohomology of Igusa varieties and of the corresponding (PEL) type Rapoport-Zink spaces. In [F], Fargues shows that the cohomology of basic unramified Rapoport-Zink spaces realises local Langlands correspondences.

For p-divisible groups whose rational Dieudonné module is simple, the moduli spaces have been studied by de Jong and Oort in [JO]. They show that the connected components are irreducible and determine their dimension. In the general case very little is known besides the existence theorem. This section is directed towards a better understanding of the global structure of $\mathcal{M}_{\mathrm{red}}$.

We now state our main results.

Let $\mathbb{X} = \mathbb{X}_{\mathrm{m}} \times \mathbb{X}_{\mathrm{bi}} \times \mathbb{X}_{\mathrm{et}}$ be the decomposition of $\mathbb{X}$ into its multiplicative, bi-infinitesimal, and étale part. To formulate the result about the set of connected components we exclude the trivial case $\mathbb{X}_{\mathrm{bi}} = 0$.

Theorem A. *Let $\mathbb{X}$ be non-ordinary. Then*

$$\pi_0(\mathcal{M}_{\mathrm{red}}) \cong GL_{\mathrm{ht}(\mathbb{X}_{\mathrm{m}})}(\mathbb{Q}_p)/GL_{\mathrm{ht}(\mathbb{X}_{\mathrm{m}})}(\mathbb{Z}_p) \times GL_{\mathrm{ht}(\mathbb{X}_{\mathrm{et}})}(\mathbb{Q}_p)/GL_{\mathrm{ht}(\mathbb{X}_{\mathrm{et}})}(\mathbb{Z}_p) \times \mathbb{Z}.$$

Next we consider the set of irreducible components of $\mathcal{M}_{\mathrm{red}}$. From now on we assume that k is algebraically closed. Let (N, F) be the rational Dieudonné module of $\mathbb{X}$ and

$$J_N = \{g \in GL(N) \mid g \circ F = F \circ g\}.$$

We choose a decomposition $N = \bigoplus_{j=1}^{l} N_j$ with N_j simple of slope $\lambda_j = \frac{m_j}{m_j+n_j}$ with $(m_j, n_j) = 1$ and $\lambda_j \leq \lambda_{j'}$ for $j < j'$. Let $M_0 \subset N$ be the lattice associated to a minimal p-divisible group (see [O2] or Definition 1.4.2).

Theorem B. *(i) There is a bijection between the set of irreducible components of $\mathcal{M}_{\mathrm{red}}$ and $J_N/(J_N \cap \mathrm{Stab}(M_0))$.*

(ii) $\mathcal{M}_{\mathrm{red}}$ is equidimensional with

$$\dim \mathcal{M}_{\mathrm{red}} = \sum_j \frac{(m_j-1)(n_j-1)}{2} + \sum_{j<j'} m_j n_{j'}. \tag{1.1.1}$$

Let $G = GL(N)$ and let $\nu = (\lambda_1, \dots, \lambda_l)$ be the Newton vector associated to N. Here each λ_j occurs $m_j + n_j$ times. Let $\mu = (1, \dots, 1, 0 \dots, 0)$ be the corresponding minuscule Hodge vector. Let ρ be the half-sum of the positive roots and let ω_i be the fundamental weights of G_{ad}. Then one can reformulate (1.1.1) as

$$\dim \mathcal{M}_{\mathrm{red}} = \langle 2\rho, \mu - \nu \rangle + \sum_i \left[\langle \omega_i, \nu - \mu \rangle\right]. \tag{1.1.2}$$

Here $[x]$ denotes the greatest integer which is less or equal to x.

In [O3], Oort defines an almost product structure (that is, up to a finite morphism) on Newton strata of moduli spaces of abelian varieties. It is given by the corresponding Rapoport-Zink space and a central leaf for the p-divisible group. He announces a joint paper with Chai, in which they prove a dimension formula for central leaves (compare [O3], Remark 2.8). The dimension of the Newton polygon stratum itself is known from [O1]. Then the dimension of $\mathcal{M}_{\mathrm{red}}$ can also be computed as the difference of the dimensions of the Newton polygon stratum and the central leaf.

Let G be an unramified connected reductive group over a finite extension F of $\mathbb{Q}_p$ and K a parahoric subgroup. Let L be the completion of the maximal unramified extension of F. Let μ be a conjugacy class of one-parameter subgroups of G and $b \in B(G, \mu)$ (compare [Ra2], 5). Let

$$X_\mu(b)_K = \{g \in G(L)/K \mid g^{-1} b \sigma(g) \in K p^\mu K\} \tag{1.1.3}$$

be the generalised affine Deligne-Lusztig set associated to μ and b. In general it is not known whether $X_\mu(b)_K$ is the set of $\mathbb{F}$-valued points of a scheme, where $\mathbb{F}$ is the residue field of $\mathcal{O}_L$. In our case choose $G = GL(N)$ and $K = \mathrm{Stab}(M_0)$. Let μ be as above. We write $F = b\sigma$ with $b \in G$. Then

$$\begin{array}{rcl} X_\mu(b)_K & \to & \mathcal{M}_{\mathrm{red}}(k) \\ g & \mapsto & gM_0 \end{array}$$

is a bijection. If K is a hyperspecial maximal parahoric, there is a conjecture of Rapoport ([Ra2], Conj. 5.10) for the dimension of generalised affine Deligne-Lusztig varieties. In our case this conjecture is (1.1.2).

Reuman considers Deligne-Lusztig sets for the cases $b = 1$, $G = SL_2$, SL_3, or Sp_4 and various parahoric subgroups K (compare [Re1] and [Re2]). For hyperspecial K, his explicit calculations support the conjecture for the dimension of $X_\mu(b)_K$. In [GHKR] the proof of the dimension formula for Deligne-Lusztig varieties in the function field case is reduced to the case that $G = GL_n$ and that the σ-conjugacy class of b is superbasic. In our situation, this is the case considered in [JO]. Hence this is another approach to proving (1.1.2).

If the isocrystal of $\mathbb{X}_{\mathrm{bi}}$ is not simple, Theorems A and B imply that the connected components of the moduli space are not irreducible and thus not smooth. Now assume that the isocrystal of $\mathbb{X}_{\mathrm{bi}}$ is simple of slope $\frac{m}{m+n}$. Then the connected components are irreducible and projective. By $\mathcal{M}^0_{\mathrm{red}}$ we denote the connected component of the identity in the moduli space.

Let $m, n \in \mathbb{N}$ with $(m, n) = 1$ be as above. A normalised cycle is a $m+n$-tuple of integers $B = (b_0, \ldots, b_{m+n-1})$ with $b_0 > b_i$, $b_{m+n-1} + m = b_0$, $\sum_i b_i = \sum_i i$ and $b_{i+1} \in \{b_i + m, b_i - n\}$ for all i (compare [JO], 6). There are only finitely many such cycles. Let $B^+ = \{b_i \in B \mid b_i + m \in B\}$ and $B^- = \{b_i \mid b_i - n \in B\}$. Then $B = B^+ \sqcup B^-$. For $j \in \mathbb{N}$ let $d(j)$ be the number of cycles B such that

$$\mathcal{V}(B) = \{(d, i) \mid b_d \in B^+, b_i \in B^-, b_i < b_d\}$$

has j elements.

Theorem C. *Let $\mathbb{X}$ be bi-infinitesimal and let its isocrystal be simple. Let m, n and $d(j)$ be as above. Let $l \neq p$ be prime. Then*

$$\begin{array}{rcll} H^{2j+1}(\mathcal{M}^0_{\mathrm{red}}, \mathbb{Q}_l) & = & 0 & (1.1.4) \\ H^{2j}(\mathcal{M}^0_{\mathrm{red}}, \mathbb{Q}_l) & = & \mathbb{Q}_l(-j)^{d(j)}, & (1.1.5) \end{array}$$

for all j.

This description uses a paving of $\mathcal{M}^0_{\mathrm{red}}$ by affine spaces which resembles the description of the geometric points in [JO], 5. As an application we show

Theorem D. *Let $\mathbb{X}$ be a p-divisible group over k. Then $\mathcal{M}^0_{\mathrm{red}}$ is smooth if and only if one of the following holds:* $\dim \mathcal{M}^0_{\mathrm{red}} = 0$ *or the isocrystal N of $\mathbb{X}_{\mathrm{bi}}$ is simple of slope $\frac{2}{5}$ or $\frac{3}{5}$. In this case, $\mathcal{M}^0_{\mathrm{red}} \cong \mathbb{P}^1$.*

The condition $\dim \mathcal{M}^0_{\mathrm{red}} = 0$ holds if and only if $\mathbb{X}$ is ordinary or the isocrystal of $\mathbb{X}_{\mathrm{bi}}$ is simple of slope $\frac{m}{m+n}$ with $\min\{m, n\} = 1$.

1.2 Review of methods

Let R be a commutative ring of characteristic $p > 0$.

1.2.1 Witt vectors

Let $W(R)$ be the ring of Witt vectors of R. The Frobenius operator $\sigma : R \to R$ with $\alpha \mapsto \alpha^p$ induces an operator $W(R) \to W(R)$ which we also denote by σ. We will also write a^σ instead of $\sigma(a)$.

Let $a \in R$. By $[a] = (a, 0, \dots) \in W(R)$ we denote the Teichmüller representative of a. This defines a multiplicative embedding $R \to W(R)$.

Remark 1.2.1. Let $a = (a_0, a_1, \dots), b = (0, \dots, 0, b_n, b_{n+1}, \dots) \in W(R)$ and $\lambda \in R$. Then

$$a + b = (a_0, \dots, a_{n-1}, a_n + b_n, c_{n+1}, \dots) \quad (1.2.1)$$
$$[\lambda]b = (0, \dots, 0, \lambda^{p^n} b_n, d_{n+1}, \dots) \quad (1.2.2)$$

with $c_i, d_i \in R$ for $i \geq n + 1$. Assume that $b_n \in R^\times$ and $-a_n b_n^{-1} = \lambda^{p^n}$ for some $\lambda \in R$. Then from (1.2.1) and (1.2.2) we get that

$$a + [\lambda]b = (a_0, \dots, a_{n-1}, 0, c_{n+1}, \dots)$$

with $c_i \in R$.

1.2.2 Dieudonné modules

Let

$$\mathcal{D}(R) = W(R)[F, V]/(FV = VF = p, F\alpha = \alpha^\sigma F, \alpha V = V\alpha^\sigma) \quad (1.2.3)$$

be the Dieudonné ring of R. Then each element $A \in \mathcal{D}(R)$ has a normal form

$$A = \sum_{i,j \geq 0} V^i a_{ij} F^j$$

with $a_{ij} \in W(R)$. If $R = K$ is a perfect field, A can also be written as

$$A = \sum_{i,j \geq 0} [a_{ij}] V^i F^j \quad (1.2.4)$$

with $a_{ij} \in K$ and $| i - j |$ bounded.

Let M be a Dieudonné module over a field k of characteristic p and N its rational Dieudonné module. For a k-algebra R denote

$$\begin{aligned} M_R &= M \otimes_{W(k)} W(R) \\ N_R &= N \otimes_{\mathrm{Quot}(W(k))} W(R)[1/p]. \end{aligned}$$

A lattice in N which is also a Dieudonné module is called a Dieudonné lattice.

1.2.3 Displays

To fix notation we give a summary of some definitions and results of [Z] on displays of p-divisible groups.

Let R be an excellent p-adic ring and let p be nilpotent in R.

Definition 1.2.2. A *display over* R is a quadrupel $\mathcal{P} = (P, Q, F, V^{-1})$, where P is a finitely generated projective $W(R)$-module, $Q \subseteq P$ is a submodule and F and V^{-1} are σ-linear maps, $F : P \to P$ and $V^{-1} : Q \to P$, such that the following properties are satisfied:

(i) Let I_R be the ideal in $W(R)$ defined by the condition that the first Witt polynomial w_0 vanishes. Then $I_R P \subseteq Q \subseteq P$ and there exists a decomposition $P = L \oplus T$ into a direct sum of $W(R)$-modules such that $Q = L \oplus I_R T$. It is called a normal decomposition.

(ii) $V^{-1} : Q \to P$ is a σ-linear epimorphism.

(iii) For $x \in P$ and $w \in W(R)$ we have $V^{-1}({}^V wx) = wFx$ where ${}^V \cdot : W(R) \to W(R)$ is the Verschiebung.

Besides, a nilpotence condition for V is required, see [Z], Def. 11.

Example 1.2.3. If M is the Dieudonné module of a formal p-divisible group $\mathbb{X}$ over a perfect field k, then (M, VM, F, V^{-1}) is a display over k. We refer to it as the display associated to M. In this case a normal decomposition is easily obtained: We choose representatives $w_1, \ldots, w_m$ in VM of a basis of the k-vector space VM/pM and set $L = \langle w_1, \ldots, w_m \rangle_{W(k)}$. Similarly, we choose representatives $v_1, \ldots, v_n$ of a basis of M/VM and set $T = \langle v_1, \ldots, v_n \rangle_{W(k)}$.

If R is an excellent local ring or if R/pR is of finite type over a field, there is an equivalence of categories between the category of displays over R and the category of p-divisible formal groups over $\mathrm{Spec}(R)$. ([Z], Thm. 103)

To the base change of p-divisible groups corresponds a base change for displays. More precisely, let S be another excellent ring and $\varphi : R \to S$ a morphism. Then for any display $\mathcal{P} = (P, Q, F, V^{-1})$ over R there is an associated display

$$\mathcal{P}_S = (P_S, Q_S, F_S, V_S^{-1})$$

over S with $P_S = W(S) \otimes_{W(R)} P$, called the base change of $\mathcal{P}$ with respect to φ. We call the second component of the base change Q_S, although in general, we only have $Q_S \supseteq W(S) \otimes_{W(R)} Q$. For a definition of the base change see [Z], Def. 20.

Definition 1.2.4. An isodisplay over R is a pair $(\mathcal{I}, F)$ where $\mathcal{I}$ is a finitely generated projective $W(R) \otimes \mathbb{Q}$-module and $F : \mathcal{I} \to \mathcal{I}$ is a σ-linear isomorphism.

Let $\mathcal{P} = (P, Q, F, V^{-1})$ be a display over R. Then the pair $(P \otimes \mathbb{Q}, F)$, where F is the extension to $P \otimes \mathbb{Q}$, is an isodisplay over R.

Let $\mathbb{X}$ be a p-divisible group over k and N its rational Dieudonné module. Let R be a k-algebra of finite type and let $\mathcal{P} = (P, Q, F, V^{-1})$ be a display over R with $P \otimes \mathbb{Q} \cong N_R$. Then by [Z], Prop. 66, this isomorphism induces a quasi-isogeny between $\mathbb{X}_R$ and the p-divisible group corresponding to $\mathcal{P}$.

1.3 Connected Components

In this section we determine the set of connected components of $\mathcal{M}_{\mathrm{red}}$. By $\mathcal{M}^0_{\mathrm{red}}$ we denote the connected component of $(\mathbb{X}, \mathrm{id})$ in $\mathcal{M}_{\mathrm{red}}$.

Let $\mathbb{X} = \mathbb{X}_{\mathrm{m}} \times \mathbb{X}_{\mathrm{bi}} \times \mathbb{X}_{\mathrm{et}}$ be the decomposition of $\mathbb{X}$ into its multiplicative, bi-infinitesimal and étale part. The moduli spaces $\mathcal{M}(\mathbb{X}_{\mathrm{m}})$ and $\mathcal{M}(\mathbb{X}_{\mathrm{et}})$ corresponding to $\mathbb{X}_{\mathrm{m}}$ and $\mathbb{X}_{\mathrm{et}}$ are discrete. As sets,

$$\mathcal{M}(\mathbb{X}_{\mathrm{m}}) \cong GL_{\mathrm{ht}(\mathbb{X}_{\mathrm{m}})}(\mathbb{Q}_p)/GL_{\mathrm{ht}(\mathbb{X}_{\mathrm{m}})}(\mathbb{Z}_p)$$

and

$$\mathcal{M}(\mathbb{X}_{\mathrm{et}}) \cong GL_{\mathrm{ht}(\mathbb{X}_{\mathrm{et}})}(\mathbb{Q}_p)/GL_{\mathrm{ht}(\mathbb{X}_{\mathrm{et}})}(\mathbb{Z}_p).$$

We define

$$\Delta = \begin{cases} \mathcal{M}(\mathbb{X}_{\mathrm{m}}) \times \mathcal{M}(\mathbb{X}_{\mathrm{et}}) \times \mathbb{Z} & \text{if } \mathbb{X}_{\mathrm{bi}} \text{ is nontrivial} \\ \mathcal{M}(\mathbb{X}_{\mathrm{m}}) \times \mathcal{M}(\mathbb{X}_{\mathrm{et}}) & \text{else.} \end{cases} \tag{1.3.1}$$

Let $S \in \mathrm{Nilp}_W$ and let $\rho : \mathbb{X}_{\overline{S}} \to X_{\overline{S}}$ be a quasi-isogeny where X is a p-divisible group over S. From [Me], Lemma II.4.8 we get a factorisation $X \twoheadrightarrow X_{\mathrm{et}} \to S$ such that $X \to X_{\mathrm{et}}$ is infinitesimal and $X_{\mathrm{et}} \to S$ is étale, as well as a quasi-isogeny $\rho_{\mathrm{et}} : \mathbb{X}_{\mathrm{et},\overline{S}} \to X_{\mathrm{et},\overline{S}}$, functorially in ρ. This defines a morphism

$$\kappa_{\mathrm{et}} : \mathcal{M}_{\mathrm{red}} \to \mathcal{M}(\mathbb{X}_{\mathrm{et}}).$$

By duality one also gets a morphism

$$\kappa_{\mathrm{m}} : \mathcal{M}_{\mathrm{red}} \to \mathcal{M}(\mathbb{X}_{\mathrm{m}}).$$

Finally, the morphism $\mathrm{ht} : \mathcal{M}_{\mathrm{red}} \to \mathbb{Z}$ maps a quasi-isogeny to its height. Let

$$\begin{aligned} \kappa &: \mathcal{M}_{\mathrm{red}} \to \Delta \\ \kappa &= \begin{cases} (\kappa_{\mathrm{m}}, \kappa_{\mathrm{et}}, \mathrm{ht}) & \text{if } \mathbb{X}_{\mathrm{bi}} \text{ is nontrivial} \\ (\kappa_{\mathrm{m}}, \kappa_{\mathrm{et}}) & \text{else.} \end{cases} \end{aligned} \tag{1.3.2}$$

Theorem 1.3.1. *κ identifies Δ with the set of connected components of $\mathcal{M}_{\mathrm{red}}$.*

This follows from Lemma 1.3.2 and Proposition 1.3.3.

Lemma 1.3.2. *κ is surjective.*

Proof. It is enough to show that for nontrivial $\mathbb{X}_{\mathrm{bi}}$ there is a quasi-isogeny $\mathbb{X}_{\mathrm{bi},K} \to X$ of height 1 over some algebraically closed field K. There is a quasi-isogeny ρ from $\mathbb{X}_{\mathrm{bi},K}$ to a product of groups that are up to isogeny simple. If we restrict F and V to such a factor, the greatest common divisor of their heights is 1. Thus there are integers a and b such that V^aF^b is a quasi-isogeny of this factor of height 1. By extending this map by the identity on the other factors we obtain a quasi-isogeny $\mathbb{X}_{\mathrm{bi},K} \to X$ of height 1 for some p-divisible group X over K. □

Proposition 1.3.3. *Let K over k be a perfect field and $(X,\rho),(X',\rho')$ two K-valued points of $\mathcal{M}_{\mathrm{red}}$ with $\kappa((X,\rho)) = \kappa((X',\rho'))$. Then the two points are in the same connected component of $\mathcal{M}_{\mathrm{red}}$.*

For the proof we need the following lemma.

Lemma 1.3.4. *Let $M \subset N_K$ be a Dieudonné lattice and*

$$v_1, v_2 \in (F^{-1}M \cap V^{-1}M) \setminus M.$$

Then $\langle M, v_1\rangle$ and $\langle M, v_2\rangle$ are Dieudonné lattices and the corresponding points of $\mathcal{M}_{\mathrm{red}}$ lie in the same connected component.

Here $\langle M, v\rangle$ denotes the $W(K)$-module generated by M and v.

Proof. We may assume that $\langle M, v_1\rangle \neq \langle M, v_2\rangle$. We define a quasi-isogeny of p-divisible groups over $\mathrm{Spec}(K[t])$ such that $\langle M, v_1\rangle$ and $\langle M, v_2\rangle$ are the lattices corresponding to the specialisations at $t = 1$ and $t = 0$, respectively. To do this we describe the corresponding subdisplay (P', Q', F, V^{-1}) of the isodisplay $N_{K[t]}$ of $\mathbb{X}_{K[t]}$. We use the notation of 1.2.3. Let

$$T = \langle Vv_1, Vv_2, w_1, \ldots, w_{n-2}\rangle \tag{1.3.3}$$
$$L = \langle pv_1, pv_2, x_1, \ldots, x_{m-2}\rangle \tag{1.3.4}$$

be a normal decomposition of the display associated to M. As the classes of Vv_1 and Vv_2 in M/VM are linearly independent over K, we can choose such w_i, x_i that the elements on the right hand sides of (1.3.3) and (1.3.4) are representatives of bases of the K-vector spaces T/pT and L/pL. We now set

$$T' = \langle [t]^\sigma \otimes v_1 + [1-t]^\sigma \otimes v_2, 1 \otimes (Vv_1 - Vv_2), 1 \otimes w_1, \ldots, 1 \otimes w_{n-2}\rangle_{W(K[t])}$$
$$L' = \langle [t] \otimes Vv_1 + [1-t] \otimes Vv_2, p \otimes (v_1 - v_2), 1 \otimes x_1, \ldots, 1 \otimes x_{m-2}\rangle_{W(K[t])}$$

and further $P' = L' + T'$ and $Q' = L' + I_{K[t]}T'$. Here $\langle\cdot\rangle_{W(K[t])}$ denotes the $W(K[t])$-submodule of $N_{K[t]}$ generated by the elements in the brackets. In every closed point of $\mathrm{Spec}(K[t])$, the two modules L' and T' of rank m and n generate a lattice, which is equivalent to the fact that their intersection is trivial. To show that P' and Q' define a subdisplay we have only to verify that V^{-1} is a σ-linear epimorphism from Q' to P'. This follows from

$$V^{-1}([t]\otimes Vv_1 + [1-t]\otimes Vv_2) = [t]^\sigma \otimes v_1 + [1-t]^\sigma \otimes v_2.$$

The specialisations of this display for $t=0$ and $t=1$ are as desired. □

Proof of Proposition 1.3.3. As K is a perfect field, we can decompose X and ρ into

$$\rho = (\rho_{\mathrm{m}}, \rho_{\mathrm{bi}}, \rho_{\mathrm{et}}) : \mathbb{X}_{\mathrm{m},K} \times \mathbb{X}_{\mathrm{bi},K} \times \mathbb{X}_{\mathrm{et},K} \longrightarrow X_{\mathrm{m}} \times X_{\mathrm{bi}} \times X_{\mathrm{et}},$$

and similarly for ρ'. The morphism κ maps ρ to $(\rho_{\mathrm{m}}, \rho_{\mathrm{et}}, \mathrm{ht}(\rho))$. The assumption implies $\rho_{\mathrm{m}} = \rho'_{\mathrm{m}}$, $\rho_{\mathrm{et}} = \rho'_{\mathrm{et}}$ and $\mathrm{ht}(\rho) = \mathrm{ht}(\rho')$. Assume that the proposition is proved for $\mathbb{X}_{\mathrm{bi}}$. Then we can construct a quasi-isogeny over a connected base $S = \overline{S}$ with fibres ρ and ρ' by extending a quasi-isogeny with fibres ρ_{bi} and ρ'_{bi} on $\mathbb{X}_{\mathrm{bi},S}$ by the constant isogeny $(\rho_{\mathrm{m}} \times \rho_{\mathrm{et}})_S = (\rho'_{\mathrm{m}} \times \rho'_{\mathrm{et}})_S$ on $(\mathbb{X}_{\mathrm{m}} \times \mathbb{X}_{\mathrm{et}})_S$. Thus for the rest of the proof we may assume that $\mathbb{X} = \mathbb{X}_{\mathrm{bi}}$.

From the two quasi-isogenies we get Dieudonné lattices $M, M' \subset N_K$ with $\mathrm{vol}(M) = \mathrm{vol}(M')$. We prove the proposition by induction on the length of $M'/M \cap M'$. If the length is 0, the lattices are equal and the statement is trivial. Let now $M' \neq M$. As $\mathbb{X}$ is bi-infinitesimal, both F and V are topologically nilpotent on M. As $M \cap M' \subsetneq M$, there is an element

$$v_1 \in M \setminus (FM + VM + M'). \tag{1.3.5}$$

Let further $v' \in M' \setminus M$. Let i' be maximal with $F^{i'}v' \notin M$ and j' maximal with $V^{j'}F^{i'}v' = v_2 \notin M$. Then

$$v_2 \in M' \cap F^{-1}(M' \cap M) \cap V^{-1}(M' \cap M) \setminus M. \tag{1.3.6}$$

Let $\{v_1, x_1, \dots x_l\}$ be a basis of the K-vector space $M/(FM + VM + (M' \cap M))$. We choose representatives of the x_i in M, which we also denote by x_i. Let $\widetilde{M}$ be the smallest $\mathcal{D}(K)$-module containing $FM, VM, M' \cap M$, and all x_i. By the choice of the x_i we have $v_1 \notin \widetilde{M}$. As $\widetilde{M} \subset M$ we have $v_2 \notin \widetilde{M}$. We also get $Fv_2, Vv_2 \in M' \cap M \subseteq \widetilde{M}$. Thus the tuple $(\widetilde{M}, v_1, v_2)$ satisfies the assumption of Lemma 1.3.4. Hence $\langle \widetilde{M}, v_1\rangle = M$ and $\langle \widetilde{M}, v_2\rangle$ correspond to points in the same connected component of the moduli space. As $M' \cap M \subseteq \widetilde{M}$ and $v_2 \in M' \setminus \widetilde{M}$, the length of $M'/(\langle \widetilde{M}, v_2\rangle \cap M')$ is smaller than that of $M'/(M' \cap M)$. Thus the assertion follows from the induction hypothesis. □

1.4 Irreducible Components

From now on we assume that k is algebraically closed.

1.4.1 Statement of the Theorem

To formulate the main theorem of this section we need some notation. We introduce a system of generators for the rational Dieudonné module N of $\mathbb{X}$. Let

$$N = \bigoplus_{j=1}^{j_0} N_{\lambda_j} \tag{1.4.1}$$

be the isotypic decomposition of N with $\lambda_j < \lambda_{j'}$ for $j < j'$. There are coprime integers $0 \leq m_j \leq h_j$ with $h_j > 0$ and $\lambda_j = \frac{m_j}{h_j}$. Let $n_j = h_j - m_j$. For each j we choose $a_j, b_j \in \mathbb{Z}$ with

$$a_j h_j + b_j m_j = 1. \tag{1.4.2}$$

We define additive maps $\pi_j : N \to N$ by

$$\pi_j \mid_{N_{\lambda_{j'}}} = \begin{cases} p^{a_j} F^{b_j} & \text{if } j' = j \\ \mathrm{id}_{N_{\lambda_{j'}}} & \text{else} \end{cases} \tag{1.4.3}$$

and $\sigma_j : N \to N$ by

$$\sigma_j \mid_{N_{\lambda_{j'}}} = \begin{cases} V^{-m_j} F^{n_j} & \text{if } j' = j \\ \mathrm{id}_{N_{\lambda_{j'}}} & \text{else.} \end{cases} \tag{1.4.4}$$

There is an algebraic group $J = J_N$ over $\mathbb{Q}_p$ associated to the moduli problem and the isocrystal N, see [RZ], 1.12. For each $\mathbb{Q}_p$-algebra R its R-valued points are defined as

$$J_N(R) = \{g \in GL(N \otimes_{\mathbb{Q}_p} R) \mid g \circ F = F \circ g\}.$$

In the following we will write J_N instead of $J_N(\mathbb{Q}_p)$ to simplify the notation.

Remark 1.4.1. Let $g \in GL(N)$. Then $g \in J_N$ if and only if g commutes with all π_j and σ_j. Indeed, $g \in J_N$ if and only if $g = \bigoplus_j g|_{N_{\lambda_j}}$ and $g|_{N_{\lambda_j}} \in J_{N_{\lambda_j}}$ for all j. On N_{λ_j} we have $\pi_j = p^{a_j} F^{b_j}$ and $\sigma_j = p^{-m_j} F^{m_j + n_j}$, and for the other direction $F = \pi_j^{m_j} \sigma_j^{a_j}$.

Let

$$N_{\lambda_j} = \bigoplus_{i=1}^{l_j} N_{j,i} \tag{1.4.5}$$

be a decomposition into simple isocrystals. Let $e_{ji0} \in N_{j,i} \setminus \{0\}$ with

$$F^{h_j} e_{ji0} = p^{m_j} e_{ji0}. \tag{1.4.6}$$

For $l \in \mathbb{Z}$ let

$$e_{jil} = \pi_j^l e_{ji0}. \tag{1.4.7}$$

By (1.4.6) the e_{jil} are independent of the choice of a_j and b_j in (1.4.2). Besides

$$\begin{aligned} e_{j,i,l+h_j} &= \pi_j^{l+h_j}(e_{ji0}) \\ &= \pi_j^l p^{h_j a_j} F^{h_j b_j} e_{ji0} \\ &= \pi_j^l p^{1-m_j b_j} F^{h_j b_j} e_{ji0} \\ &= p e_{jil} \end{aligned} \tag{1.4.8}$$

and analogously

$$F(e_{jil}) = e_{j,i,l+m_j}, \tag{1.4.9}$$

$$V(e_{jil}) = e_{j,i,l+n_j}, \tag{1.4.10}$$

$$\sigma_{j'}(e_{jil}) = e_{jil} \tag{1.4.11}$$

for $1 \leq j' \leq j_0$. The e_{jil} with $0 \leq l < h_j$ form a basis of $N_{j,i}$ over $\mathrm{Quot}(W(k))$.

Let $K \mid k$ be a perfect field. For $a \in K$ let $[a] \in W(K)$ be the Teichmüller representative as in 1.2.1. By (1.4.8) each $v \in N_K$ can be written as

$$v = \sum_{j=1}^{j_0} \sum_{i=1}^{l_j} \sum_{l \in \mathbb{Z}} [a_{jil}] e_{jil} \tag{1.4.12}$$

with $a_{jil} \in K$ and $a_{jil} = 0$ for l small enough.

Definition 1.4.2. (i) Let $M_0 \subset N$ be the lattice generated by the e_{jil} with $l \geq 0$.

(ii) For a lattice M in some sub-isocrystal $\widetilde{N} \subseteq N$ let

$$\mathrm{vol}_{\widetilde{N}}(M) = \lg((M_0 \cap \widetilde{N})/(M_0 \cap M)) - \lg(M/(M_0 \cap M)). \tag{1.4.13}$$

If $\widetilde{N} = N$ we write vol instead of vol_N.

Then $\pi_j(M_0) \subseteq M_0$ and $\sigma_j(M_0) \subseteq M_0$ for all j, and $\mathrm{vol}(M_0) = 0$.

Using this notation we can formulate the main result of this section.

Theorem 1.4.3. *(i) There is a bijection between the set of irreducible components of $\mathcal{M}_{\mathrm{red}}$ and $J_N/(J_N \cap \mathrm{Stab}(M_0))$.*

(ii) $\mathcal{M}_{\mathrm{red}}$ is equidimensional with

$$\dim \mathcal{M}_{\mathrm{red}} = \sum_j l_j \frac{(m_j - 1)(n_j - 1)}{2} + \sum_{(j,i)<(j',i')} m_j n_{j'},$$

where the pairs (j, i) are ordered lexicographically.

To prove this theorem, we define an open and dense subscheme $\mathcal{S}_1$ of $\mathcal{M}_{\mathrm{red}}$ and show the corresponding results for this subscheme. We assume until Section 1.4.5 that $\mathbb{X}$ is bi-infinitesimal. The general case is discussed in Section 1.4.5.

1.4.2 Definition of the stratum $\mathcal{S}_1$

Definition 1.4.4. Let $\mathcal{S}_1 \subseteq \mathcal{M}_{\text{red}}$ be the open subscheme defined by the following condition: Every point on $\mathcal{S}_1$ has an open affine neighbourhood $\operatorname{Spec}(R)$, such that the display (P, Q, F, V^{-1}) over R of the corresponding p-divisible group has the property that there is an element in P generating $P/(Q + F(P))$ as $W(R)$-module. Here, $(Q+F(P))$ denotes the $W(R)$-submodule of P generated by $F(P)$ and Q.

Recall that the a-invariant $a(M)$ of a Dieudonné lattice $M \subseteq N_K$ over a perfect field K is defined as the dimension of the K-vector space $M/(FM + VM)$. As we assumed $\mathbb{X}$ to be bi-infinitesimal, the a-invariant is always positive.

Lemma 1.4.5. *Let $K \mid k$ be a perfect field and let $M \subset N_K$ be the lattice associated to a K-valued point x of $\mathcal{M}_{\text{red}}$. The following statements are equivalent.*

(i) $x \in \mathcal{S}_1$

(ii) $a(M) = 1$

(iii) There is a $v \in M$ such that M is the $\mathcal{D}(K)$-submodule of N_K generated by v.

Proof. As F and V are topologically nilpotent on M, (ii) and (iii) are equivalent. It remains to show that the second assertion implies the first. Let $\operatorname{Spec}(R)$ be an open affine neighbourhood of x in $\mathcal{M}_{\text{red}}$ and (P, Q, F, V^{-1}) the corresponding display. Then $F(Q) \subseteq pP \subseteq Q$, and $F : P/Q \to P/Q$ is a σ-linear morphism of R-modules of constant rank. As the rank of its cokernel in x is $a(M) = 1$, it is also 1 in an open neighbourhood of x, implying the first assertion. □

Lemma 1.4.6. *The open subscheme $\mathcal{S}_1 \subseteq \mathcal{M}_{\text{red}}$ is dense.*

Proof. Let X_0 be the p-divisible group of a K-valued point in $\mathcal{M}_{\text{red}} \setminus \mathcal{S}_1$. By Proposition 2.8 of [O1], there exists a deformation of X_0 with constant Newton polygon such that the a-invariant at the generic fibre is 1. By [OZ], Cor. 3.2 we get a deformation of the quasi-isogeny after a suitable base change preserving the generic fibre. □

1.4.3 K-valued points of $\mathcal{S}_1$

Let $K \mid k$ be a perfect field. In this section we classify the K-valued points of $\mathcal{S}_1$ by introducing a normal form for the corresponding lattices in N_K. We will write $N_{j,i}$ instead of $(N_{j,i})_K$.

Lemma 1.4.7. *Let $M \subset N_K$ be the lattice associated to a K-valued point of $\mathcal{S}_1$ and v a generator of M as $\mathcal{D}(K)$-submodule of N_K as in Lemma 1.4.5 (iii). Let $g \in J_{N_K}$ with $M \subseteq gM_0$ and maximal $v_p(\det(g))$. Then*

$$v = \sum_{j=1}^{j_0} \sum_{i=1}^{l_j} \sum_{l \geq 0} [a_{jil}] g(e_{jil}) \tag{1.4.14}$$

with $a_{jil} \in K$ and

$$\text{for each } j, \text{ the } a_{ji0} \text{ for } 1 \leq i \leq l_j \text{ are linearly independent over } \mathbb{F}_{p^{h_j}}. \tag{1.4.15}$$

Proof. We may assume that N is isoclinic: Otherwise we can write v as a sum of elements of the $(N_{\lambda_j})_K$ and show the claim for each summand separately. Assume that there is a nontrivial relation

$$\sum_{i=1}^{l_1} \alpha_i a_{1i0} = 0 \tag{1.4.16}$$

with $\alpha_i \in \mathbb{F}_{p^{h_1}}$. After permuting the simple summands of N_K we may assume that α_1 is nonzero. Then we may also assume that $\alpha_1 = -1$. We define $\delta \in GL(N_K)$ by

$$\delta(e_{1il}) = \begin{cases} e_{1,1,l+1} & \text{if } i = 1 \\ e_{1il} + [\alpha_i]^{\sigma^{lb_1}} e_{1,1,l} & \text{if } i \geq 2 \end{cases} \tag{1.4.17}$$

for $l \in \mathbb{Z}$. This map is well defined as

$$\begin{aligned} \delta(pe_{1il}) &= \delta(e_{1,i,l+h_1}) \\ &= pe_{1il} + [\alpha_i]^{\sigma^{(l+h_1)b_1}} e_{1,1,l+h_1} \\ &= pe_{1il} + [\alpha_i]^{\sigma^{lb_1}} e_{1,1,l+h_1} \\ &= p\delta(e_{1il}) \end{aligned}$$

for $i > 1$. We also have

$$\begin{aligned} \delta \circ F(e_{1il}) &= \delta(e_{1,i,l+m_1}) \\ &= e_{1,i,l+m_1} + [\alpha_i]^{\sigma^{(l+m_1)b_1}} e_{1,1,l+m_1} \\ &= e_{1,i,l+m_1} + [\alpha_i]^{\sigma^{lb_1+1-a_1h_1}} e_{1,1,l+m_1} \\ &= F(e_{1il} + [\alpha_i]^{\sigma^{lb_1}} e_{1,1,l}) \\ &= F \circ \delta(e_{1il}), \end{aligned}$$

for $i > 1$ and $\delta \circ F(e_{11l}) = F \circ \delta(e_{11l}) = e_{1,1,l+m_1+1}$, so $\delta \in J_{N_K}$. Besides, $v_p(\det(\delta)) = 1$. (1.4.14) and (1.4.16) imply that $v \in g \circ \delta(M_0)$. As v generates M, we have $M \subseteq g \circ \delta(M_0)$ in contradiction to the maximality of $v_p(\det(g))$. $\square$

Let $M \subset N_K$ be a Dieudonné lattice. Let $P(M)$ be the smallest $\mathcal{D}(K)$-submodule of N_K containing M with

$$\sigma_j(P(M)) \subseteq P(M)$$

and

$$\pi_j(P(M)) \subseteq P(M)$$

for all j. There exists a $c \in \mathbb{Z}$ with $M \subseteq p^c M_0$. As $\pi_j(M_0) \subseteq M_0$ and $\sigma_j(M_0) \subseteq M_0$ for all j, we get $M \subseteq P(M) \subseteq p^c M_0$. Hence $P(M)$ is also a lattice in N_K. As all π_j and σ_j commute with J_{N_K}, we have $P(gM) = gP(M)$ for all $g \in J_{N_K}$.

Lemma 1.4.8. *Let M be the Dieudonné lattice corresponding to a K-valued point of $\mathcal{S}_1$ and let $v \in M$ be a generator. Let g be as in Lemma 1.4.7. Then*

$$P(M) = gM_0. \tag{1.4.18}$$

Especially, the class of g in $J_{N_K}/(\mathrm{Stab}(M_0) \cap J_{N_K})$ is uniquely determined by M.

Proof. The inclusion $P(M) \subseteq gM_0$ follows from $v \in gM_0$ and $P(gM_0) = gM_0$. As π_j commutes with g, the other inclusion follows as soon as we know $g(e_{ji0}) \in P(M)$ for all i and j. We show this by induction on $\sum_{j=1}^{j_0} l_j$, the number of simple summands of N. As all π_j and σ_j commute with J we may assume $g = \mathrm{id}$.

For $j_0 = l_1 = 1$ we have $v = \sum_{l \geq 0}[a_{11l}]e_{11l}$ with $a_{11l} \in K$ for all l and $a_{110} \neq 0$. This implies

$$\pi_1^k v = \sum_{l \geq 0}[a_{11l}^{\sigma^{b_1 k}}]e_{1,1,l+k}.$$

By Remark 1.2.1, e_{110} is a linear combination of these elements, hence in $P(M)$.

Let now $\sum_{j=1}^{j_0} l_j > 1$. On N_{λ_j}, the map π_j is elementwise topologically nilpotent, while it is the identity on each $N_{\lambda_{j'}}$ with $j' \neq j$. Thus $v \in P(M)$ implies that its image $v_j = \sum_{i,l}[a_{jil}]e_{jil}$ under the projection to $(N_{\lambda_j})_K$ is in $P(M)$ for each j. If N is not isoclinic, we may apply the induction hypothesis to each $v_j \in N_{\lambda_j}$. Let M_j be the $\mathcal{D}(K)$-submodule of $(N_{\lambda_j})_K$ generated by v_j. By induction $e_{ji0} \in P(M_j) \subset P(M)$ for all i, which shows the claim in this case. Let now N_K be isoclinic. By multiplying v with $[a_{110}^{-1}]$ and subtracting multiples of the $\pi_1^k(v)$ for $k > 0$ as in the case $j_0 = l_1 = 1$, we may assume that

$$v = e_{110} + \sum_{i>1}\sum_{l \geq 0}[a_{1il}]e_{1il}. \tag{1.4.19}$$

Here 1 and the a_{1i0} with $2 \leq i \leq l_1$ are again linearly independent over $\mathbb{F}_{p^{h_1}}$. Then

$$\sigma_1(v) - v = \sum_{i=2}^{l_1}\sum_{l \geq 0}([a_{1il}^{\sigma^{h_1}}] - [a_{1il}])e_{1il} \in N' = \bigoplus_{i=2}^{l_1} N_{1,i}. \tag{1.4.20}$$

By Remark 1.2.1 the new coefficient of e_{1i0} is $[a_{1i0}^{\sigma^{h_1}} - a_{1i0}]$. We have to check that the $a_{1i0}^{\sigma^{h_1}} - a_{1i0}$ for $i \geq 2$ are linearly independent over $\mathbb{F}_{p^{h_1}}$. Otherwise we would have a nontrivial relation

$$\sum_{i>1} \alpha_i (a_{1i0}^{\sigma^{h_1}} - a_{1i0}) = 0. \tag{1.4.21}$$

As $\alpha_i \in \mathbb{F}_{p^{h_1}}$ this yields

$$\sum_{i>1} \alpha_i a_{1i0} \in \mathbb{F}_{p^{h_1}}.$$

This is a contradiction as we assumed $a_{110} = 1$ and as the a_{1i0} for $1 \leq i \leq l_1$ were linearly independent over $\mathbb{F}_{p^{h_1}}$. Let $M(\sigma_1(v) - v)$ be the $\mathcal{D}(K)$-submodule of $N' \cap P(M)$ generated by $\sigma_1(v) - v$. The induction hypothesis shows that all e_{1i0} for $i > 1$ are in $P(M(\sigma_1(v) - v)) \subset P(M)$. Thus $\sum_{i>1} \sum_{l \geq 0} [a_{1il}] e_{1il} \in P(M)$, and (1.4.19) implies $e_{110} \in P(M)$. $\square$

Theorem 1.4.9. *Let*

$$v = \sum_{j=1}^{j_0} \sum_{i=1}^{l_j} \sum_{l \geq 0} [a_{jil}] g(e_{jil}) \in N_K \tag{1.4.22}$$

with $a_{jil} \in K$ and $g \in J_{N_K}$ such that (1.4.15) is satisfied. Let M be the smallest $\mathcal{D}(K)$-submodule of N_K containing v. Then

(i) M is a lattice in N_K.

(ii)

$$\mathrm{vol}(M) = v_p(\det(g)) + c \tag{1.4.23}$$

with

$$c = \sum_j l_j \frac{(m_j - 1)(n_j - 1)}{2} + \sum_{(j,i)<(j',i')} m_j n_{j'}, \tag{1.4.24}$$

where the pairs (j, i) are ordered lexicographically.

(iii) Let $I = I(N) \subset \coprod_{j,i} \mathbb{N}$ with

$$(\coprod_{j,i} \mathbb{N}) \setminus I = \{(j, i, l) \mid l = am_j + bn_j + \sum_{(j',i')<(j,i)} m_{j'} n_j \text{ for some } a, b \geq 0\}. \tag{1.4.25}$$

Then $\mid I \mid = c$ and $(1, 1, 0) \notin I$, but $(j, i, 0) \in I$ for $(j, i) \neq (1, 1)$. There is an element $w \in M$ such that

$$w = \sum_{(j,i)} \sum_{l \geq 0} [b_{jil}] g(e_{jil}) \tag{1.4.26}$$

with (1.4.15), $b_{110} = 1$, and $b_{jil} = 0$ if $(1,1,0) \neq (j,i,l) \notin I$. This element w is a generator of M as $\mathcal{D}(K)$-module, and is called a normalised generator. It only depends on the choice of the representative $g \in [g] \in J/(J \cap \mathrm{Stab}(M_0))$.

(iv) Let

$$x = \sum_{j,i,l} [b_{jil}] g(e_{jil}) \in M. \tag{1.4.27}$$

We order the set $\coprod_{j,i} \mathbb{N}$ lexicographically. Then the index of the first nonzero summand of x is not in I.

(v) For all $(j,i,l) \notin I$ there is an $x \in M$ as in (1.4.27) such that $[b_{jil}]$ is its first nonzero coefficient.

For the proof of the theorem we need two technical lemmas.

Lemma 1.4.10. *Let $a, b, m, n \in \mathbb{N}$ with $an + bm > mn$ and $\lambda = \frac{m'}{m'+n'} \in (0,1)$. Then*

$$an' + bm' > \min\{nm', mn'\}.$$

Proof. We may assume $0 < a \leq m$ and $b > 0$, because otherwise the implication is evident. If the claim were false, there would be some λ with $\lambda(n + a - b) \geq a$ and $\lambda(m + b - a) \leq m - a$. Especially, this implies $n + a - b > 0$. From our assumptions we get $m + b - a > 0$. Thus

$$a(m + b - a) \leq \lambda(m + b - a)(n + a - b) \leq (m - a)(n + a - b),$$

in contradiction to $an + bm > mn$. $\square$

Lemma 1.4.11. *Let v and M be as in the theorem and assume $g = \mathrm{id}$ and $a_{110} = 1$. Then there is an $A \in \mathcal{D}(K)$ of the form*

$$A = F^{n_1} - V^{m_1} + \sum_{k > m_1 n_1} [\alpha_k] V^{a(k)} F^{b(k)}$$

with $\alpha_k \in K$ and the following properties:

(i) For $k > m_1 n_1$, the exponents $a(k), b(k)$ are the unique positive integers with $-n_1 < a(k) - b(k) \leq m_1$ and $a(k)n_1 + b(k)m_1 = k$.

(ii) $Av \in N' = \bigoplus_{(j,i) \neq (1,1)} N_{j,i}$.

(iii) Av generates $M \cap N'$ as a Dieudonné lattice in N'.

(iv) Let $g' \in J_{N'}(K)$ with

$$g'(e_{jil}) = e_{j,i,l+m_1n_j}.$$

Then

$$Av = \sum_{(j,i)\neq(1,1)} \sum_{l\geq 0} [a'_{jil}] g'(e_{jil}),$$

with $a'_{jil} \in K$ such that for each j all a'_{ji0} (with $1 \leq i \leq l_j$ if $j \neq 1$ and $2 \leq i \leq l_1$ if $j = 1$) are linearly independent over $\mathbb{F}_{p^{h_j}}$.

Proof. As $a_{110} = 1$ and $F^{n_1}(e_{110}) = V^{m_1}(e_{110}) = e_{1,1,m_1n_1}$, we have

$$F^{n_1}v - V^{m_1}v = \sum_{l>m_1n_1} [c_{11l}]e_{11l} + \sum_{(j,i)\neq(1,1)} \sum_{l\geq 0} [c_{jil}]e_{jil} \tag{1.4.28}$$

with $c_{jil} \in K$ for all j, i, l. For each $k > m_1n_1$ choose $a(k)$ and $b(k)$ as in (i). Then

$$\begin{aligned} V^{a(k)}F^{b(k)}v &= \sum_{j,i} \sum_{l\geq 0} [a_{jil}^{\sigma^{b(k)-a(k)}}] e_{j,i,l+a(k)n_j+b(k)m_j} \\ &= e_{11k} + \sum_{l>k} [d_{11l}]e_{11l} + \sum_{(j,i)\neq(1,1)} \sum_{l\geq 0} [d_{jil}]e_{jil} \end{aligned}$$

with $d_{jil} \in K$. By Remark 1.2.1 we can inductively define $\alpha_k \in K$ for $k > m_1n_1$ such that

$$(F^{n_1} - V^{m_1} + \sum_{k=m_1n_1+1}^{k_0} [\alpha_k]V^{a(k)}F^{b(k)})(v)$$

is a linear combination of the e_{11l} with $l > k_0$ and of an element of N'. Thus the element

$$A = F^{n_1} - V^{m_1} + \sum_{k>m_1n_1} [\alpha_k]V^{a(k)}F^{b(k)} \tag{1.4.29}$$

satisfies $Av \in N'$. As $a(k) - b(k)$ is bounded, we have $A \in \mathcal{D}(K)$.

Let $B = \sum_{a,b\geq 0} [\beta_{ab}]V^aF^b \in \mathcal{D}(K)$ with $\beta_{ab} \in K$ and $Bv \in M \cap N'$. We want to show that $B = CA$ for some $C \in \mathcal{D}(K)$. We assume $B \neq 0$. For each $k \in \mathbb{N}$ such that there exists a $\beta_{ab} \neq 0$ with $an_1 + bm_1 = k$ let

$$d_k(B) = \min\{a - b \mid an_1 + bm_1 = k, \beta_{ab} \neq 0\}$$

and

$$d^k(B) = \max\{a - b \mid an_1 + bm_1 = k, \beta_{ab} \neq 0\}.$$

Furthermore let

$$\begin{aligned} d(B) &= \min\{d_k(B)\} && (1.4.30) \\ \tilde{d}(B) &= \max\{d^k(B)\} && (1.4.31) \end{aligned}$$

The existence of this minimum and maximum is equivalent to $B \in \mathcal{D}(K)$. Inductively we construct $C_k \in \mathcal{D}(K)$ with the following properties:

(a) The coefficient of V^cF^d in the representation of $B - \sum_{k' \leq k} C_{k'}A$ as in (1.2.4) vanishes for all c, d with $cn_1 + dm_1 \leq k$.

(b) If there exists a $\beta_{ab} \neq 0$ with $an_1 + bm_1 = k$, then $d(C_k) \geq d(B)$ and $\tilde{d}(C_k) \leq \tilde{d}(B)$. Otherwise, $C_k = 0$.

(c) If $B - \sum_{k' \leq k} C_{k'}A \neq 0$ then

$$\begin{aligned} d(B) &\leq d(B - \sum_{k' \leq k} C_{k'}A) \\ \tilde{d}(B) &\geq \tilde{d}(B - \sum_{k' \leq k} C_{k'}A). \end{aligned}$$

If $C = \sum_{k \geq 0} C_k$ exists in $\mathcal{D}(K)$, then this implies $B = CA$. By replacing B by $B - \sum_{k' < k} C_{k'}A$ we may assume that k is the least integer such that there exist a, b with $an_1 + bm_1 = k$ and $\beta_{ab} \neq 0$. We want to show that $d_k(B) \neq d^k(B)$. Assume that $d_k(B) = d^k(B)$. Then there is only one $\beta_{a_0b_0} \neq 0$ with $a_0n_1 + b_0m_1 = k$. Denote by p_1 the projection to $N_{1,1}$. We have

$$\begin{aligned} 0 &= B(p_1(v)) \\ &= [\beta_{a_0b_0}]V^{a_0}F^{b_0}(p_1(v)) + \sum_{\{(a,b)|an_1+bm_1>k\}} [\beta_{ab}]V^aF^b(p_1(v)) \\ &= \sum_{l \geq 0} [\beta_{a_0b_0} a_{11l}^{\sigma^{b_0-a_0}}] e_{1,1,l+k} + \sum_{\{(a,b)|an_1+bm_1>k\}} \sum_{l \geq 0} [\beta_{ab} a_{11l}^{\sigma^{b-a}}] e_{1,1,l+an_1+bm_1}. \end{aligned}$$

Hence the coefficient of $e_{1,1,k}$ in the expression above is $[\beta_{a_0b_0} a_{110}^{\sigma^{b_0-a_0}}] = [\beta_{a_0b_0}]$. This implies $\beta_{a_0b_0} = 0$, a contradiction. Thus $d^k(B) > d_k(B)$. Note that $m_1 + n_1$ divides $d^k(B) - d_k(B)$. Let a, b with $a - b = d^k(B)$ be the pair of indices realising the maximum. Let $C_{k,1} = [\beta_{ab}]V^{a-m_1}F^b$. From $d^k(B) \geq d_k(B) + m_1 + n_1$ we see that $d_k(B) < a - b - m_1 < d^k(B)$ and that $d^k(B) > d^k(B - C_{k,1}A)$ and $d_k(B) \leq d_k(B - C_{k,1}A)$. Hence $d^k(B) - d_k(B) > d^k(B - C_{k,1}A) - d_k(B - C_{k,1}A)$. Using a second induction on this difference, we can construct C_k as a finite sum of such expressions $C_{k,1}$. The fact that each pair (a, b) occurs at most once in the construction of some C_k together with (b) implies that the sum $C = \sum_{k \geq 0} C_k$ exists in $\mathcal{D}(K)$. This proves (iii).

Now we want to show (iv). We have

$$\begin{aligned}
Av &= F^{n_1}v - V^{m_1}v + \sum_{k>m_1n_1}[\alpha_k]V^{a(k)}F^{b(k)}v \\
&= \sum_{(j,i)\neq(1,1)}\sum_{l\geq 0}\left(F^{n_1}[a_{jil}]e_{jil} - V^{m_1}[a_{jil}]e_{jil} + \sum_{k>m_1n_1}[\alpha_k]V^{a(k)}F^{b(k)}[a_{jil}]e_{jil}\right) \\
&= \sum_{(j,i)\neq(1,1)}\sum_{l\geq 0}\Big([a_{jil}^{\sigma^{n_1}}]e_{j,i,l+n_1m_j} - [a_{jil}^{\sigma^{-m_1}}]e_{j,i,l+m_1n_j} \\
&\qquad\qquad + \sum_{k>m_1n_1}[\alpha_k a_{jil}^{\sigma^{b(k)-a(k)}}]e_{j,i,l+a(k)n_j+b(k)m_j}\Big)
\end{aligned}$$

For each j and i we determine the first nonvanishing coefficient of some e_{jil}. First we consider summands with $j = 1$ and $i > 1$. In this case $V^{a(k)}F^{b(k)}e_{1i0} = e_{1ik}$ with $k > m_1n_1$. Thus a candidate for the first coefficient is that of e_{1,i,m_1n_1}, namely $[a_{1i0}^{\sigma^{n_1}} - a_{1i0}^{\sigma^{-m_1}}]$. (Here we used Remark 1.2.1 to determine $w_0([a_{1i0}^{\sigma^{n_1}}] - [a_{1i0}^{\sigma^{-m_1}}])$.) As in the proof of Lemma 1.4.8 one sees that these coefficients are again linearly independent over $\mathbb{F}_{p^{h_1}}$. Now we consider summands with $j > 1$. From Lemma 1.4.10 and the ordering of the λ_j we get

$$a(k)n_j + b(k)m_j > \min\{n_1m_j, m_1n_j\} = m_1n_j.$$

Thus the first nonzero coefficient is that of e_{j,i,m_1n_j}, namely $a_{ji0}^{\sigma^{-m_1}}$. For fixed j the a_{ji0} were linearly independent over $\mathbb{F}_{p^{h_j}}$, hence the new first nonzero coefficients are again linearly independent. This proves (iv). $\square$

Corollary 1.4.12. *Let N be bi-infinitesimal and simple and $v \in N_K \setminus \{0\}$. Then* $\mathrm{Ann}(v) \subset \mathcal{D}(K)$ *is a principal left ideal.*

Proof of Theorem 1.4.9. Both F and V commute with g. Thus $M = gM'$ where M' is generated by

$$g^{-1}v = \sum_{j,i}\sum_{l\geq 0}[a_{jil}]e_{jil}.$$

Hence we may assume that $g = \mathrm{id}$.

Assertion (iii) is implied by (iv) and (v). We show that (i) and (ii) also follow from (iv) and (v): We consider the $\mathcal{D}(K)$-modules

$$M^{j_0i_0l_0} = \langle M, \{e_{jil} \mid (j,i,l) \geq (j_0,i_0,l_0), l \geq 0\}\rangle_{W(K)}. \tag{1.4.32}$$

Then

$$M^{110} = M_0.$$

Using (v) one sees

$$M^{j_0,l_{j_0},d} = M$$

where $d = \sum_{(j,i)\neq(j_0,l_{j_0})} m_j n_{j_0} + (m_{j_0}-1)(n_{j_0}-1)$. For $(j,i,l) < (j',i',l')$ we have $M^{jil} \supseteq M^{j'i'l'}$ with equality if and only if

$$I \cap \{(j_1,i_1,l_1) \mid (j,i,l) \leq (j_1,i_1,l_1) < (j',i',l')\} = \emptyset.$$

Indeed, by (iv) and (v) this is equivalent to the condition that for each (j_1,i_1,l_1) with $(j,i,l) \leq (j_1,i_1,l_1) < (j',i',l')$, there is already an element of M whose first nonzero coefficient has index (j_1,i_1,l_1). As the colength of $M^{j,i,l+1}$ in M^{jil} is at most 1, this implies $\mathrm{vol}(M) =\mid I \mid= c$.

We now prove (iv) and (v) using induction on $\sum_{j=1}^{j_0} l_j$, the number of simple summands of N_K.

By multiplying v with $[a_{110}^{-1}] \in W(K)^\times$ we may assume that $a_{110} = 1$. First we consider the case that N is simple. We have

$$V^a F^b v = e_{1,1,an_1+bm_1} + \sum_{l>0} [a_{11l}^{\sigma^{b-a}}] e_{1,1,l+an_1+bm_1}.$$

Thus for all l that can be written as $l = an_1 + bm_1$ with $a, b \geq 0$ there is an element in M of the form $e_{11l} + \sum_{l'>l} [b_{l'}] e_{11l'}$, proving (v). Let now $x \in M \setminus \{0\}$ and assume that

$$x = \sum_{l \geq 0} [b_{11l}] e_{11l}$$

such that (iv) is not satisfied for x. Let $[b_{11l_0}]$ with $(1,1,l_0) \in I$ be the first nonvanishing coefficient. We also have a representation $x = \sum_{a,b\geq 0} [c_{a,b}] V^a F^b(v)$. Let (a_0, b_0) be a pair with $c_{a_0,b_0} \neq 0$ and minimal $a_0 n_1 + b_0 m_1$. Then $a_0 n_1 + b_0 m_1 \leq l_0$. As no $l > (m_1-1)(n_1-1)$ is in I, we get $a_0 n_1 + b_0 m_1 \leq (m_1-1)(n_1-1)$. Especially, (a_0,b_0) is the unique pair of nonnegative integers (a,b) with $an_1 + bm_1 = a_0 n_1 + b_0 m_1$. Hence the coefficient of $e_{1,1,a_0 n_1 + b_0 m_1}$ is the first nonzero coefficient of x, proving (iv).

Let now N be the sum of more than one simple summand. Let $p_1 : N_K \to N_{1,1}$ be the projection and

$$N' = \bigoplus_{(j,i)\neq(1,1)} N_{j,i}.$$

Note that $p_1(M)$ is the lattice in $N_{1,1}$ generated by $p_1(v)$. Thus the theorem applied to the simple isocrystal $N_{1,1}$ yields that each $x \in M \setminus (M \cap N')$ satisfies (iv), and that for each $(1,1,l) \notin I$ there is an element $x \in M$ as in (v). We now consider elements of $M \cap N'$. Let $I(N')$ be the index set corresponding to N' as in (1.4.25), viewed as a subset of $\coprod_{(j,i)\neq(1,1)} \mathbb{N}$. Then one easily checks that

$$I \cap \coprod_{(j,i)\neq(1,1)} \mathbb{N} = \{(j,i,l) \mid (j,i,l - m_1 n_j) \in I(N')\}. \tag{1.4.33}$$

In Lemma 1.4.11 we proved that $M \cap N'$ is generated by Av for some $A \in \mathcal{D}(K)$ and determined the corresponding $g' \in J_{N'}$, which only shifts the last indices of the basis by $m_1 n_j$. Thus the induction hypothesis implies that there is an $x \in M \cap N'$ with $x = \sum_{(j,i)\neq(1,1)}[c_{jil}]e_{jil}$ and first nonzero coefficient $[c_{jil}]$ if and only if $(j,i,l-m_1n_j) \notin I(N')$. Together with (1.4.33), this implies the theorem. $\square$

1.4.4 Irreducible subvarieties of $\mathcal{S}_1$

Let I be the index set defined in Theorem 1.4.9(iii). Denote the coordinates of a point in $\mathbb{A}_k^I$ by a_{jil} with $(j,i,l) \in I$. Let $U = U(N) \subseteq \mathbb{A}_k^I$ be the affine open subvariety defined by (1.4.15). Let $a_{110} = 1$. Then U is defined by the condition that for each j, the a_{ji0} for $1 \leq i \leq l_j$ have to be linearly independent over $\mathbb{F}_{p^{h_j}}$. We write $U = \mathrm{Spec}(R)$.

For each $g \in J$ we want to define a morphism

$$\varphi_g : U \to \mathcal{S}_1.$$

For $g = \mathrm{id}$ we describe the corresponding quasi-isogeny of p-divisible groups over U via the display of the p-divisible group. As J acts on $\mathcal{M}_{\mathrm{red}}$ we can define φ_g for general g as the composition of φ_{id} and the action of g.

Let (P, Q, F, V^{-1}) be the base change of the display of $\mathbb{X}$ from k to R. Let

$$v = e_{110} + \sum_{(j,i,l)\in I} [\sigma^{\sum_{j,i} m_j}(a_{jil})]e_{jil} \in P \otimes \mathbb{Q} = N_R$$

and

$$\begin{aligned} \tilde{T} &= \langle v, Fv, \dots, F^{\sum_{i,j} n_j - 1}v\rangle_{W(R)} && (1.4.34)\\ \tilde{L} &= \langle Vv, \dots, V^{\sum_{i,j} m_j}v\rangle_{W(R)} && (1.4.35)\end{aligned}$$

as $W(R)$-submodules of $P \otimes \mathbb{Q}$. Let $\tilde{P} = \tilde{L} + \tilde{T}$ and $\tilde{Q} = I_R\tilde{T} + \tilde{L}$. We have to show that $(\tilde{P}, \tilde{Q}, F, V^{-1})$ is a display where F and V^{-1} are the restrictions of F and V^{-1} on $P \otimes \mathbb{Q}$. By construction we have $I_R\tilde{P} \subset \tilde{Q} \subset \tilde{P}$ and $\tilde{P}$ and $\tilde{Q}$ are finitely generated $W(R)$-modules. The results of the preceding section show that the reduction of $\tilde{P} \otimes \mathbb{Q}$ in a K-valued point of U is N_K. Thus $\tilde{P} \otimes \mathbb{Q} \cong P \otimes \mathbb{Q}$, and for dimension reasons, $\tilde{P}$ has to be free and $\tilde{L} \cap \tilde{T} = (0)$. Hence $\tilde{L}$ and $\tilde{T}$ form a normal decomposition. The third condition for a display and the nilpotence condition on V are satisfied because they were satisfied on P. We now determine the matrix associated to $F|_{\tilde{T}}$ and $V^{-1}|_{\tilde{L}}$ as in [Z],(9) to show that the image of F is again in $\tilde{P}$ and that $V^{-1} : \tilde{Q} \to \tilde{P}$ is a σ-linear epimorphism. The matrix is of the following form:

$$\begin{pmatrix} 0 & \cdots & 0 & * & 1 & & & \\ 1 & \ddots & \vdots & \vdots & & & & \\ & \ddots & 0 & \vdots & & \text{\Large 0} & & \\ & & 1 & * & & & & \\ & & & * & 0 & 1 & & \\ & \text{\Large 0} & & \vdots & & \ddots & \ddots & \\ & & & \vdots & \text{\Large 0} & & \ddots & 1 \\ & & & * & & & & 0 \end{pmatrix}$$

All columns except the one corresponding to $F^{\sum_{j,i} n_j - 1} v$ have exactly one nonzero entry, which is 1. We now have to show that the remaining column has entries in $W(R)$. We use induction on $\sum l_j$, the number of simple summands of N, to show the following property: Let $v = \sum_{j,i,l \geq 0} [b_{jil}] e_{jil} \in N_R$ for some R such that for each j all non-trivial linear combinations of its coefficients b_{ji0} with coefficients in $\mathbb{F}_{p^{h_j}}$ are in $R^\times$ and that $b_{110} = 1$. Assume furthermore that the coefficients of v are in $\sigma^{\sum_{j,i} m_j}(R)$. Then

$$F^{\sum_{j,i} n_j} v = \sum_{0 < k < \sum_{j,i} n_j} \gamma_{-k} F^k v + \sum_{0 \leq k \leq \sum_{j,i} m_j} \gamma_k V^k v$$

with $\gamma_k \in W(R)$.

Let $A \in \operatorname{Ann}(p_1(v)) \subseteq \mathcal{D}(R)$ of the same form as in Lemma 1.4.11. The construction for this over R is the same as over K. As we chose $a(k) - b(k) \leq m_1$, all coefficients of $(F^{n_1} - V^{m_1})(v)$ and $V^{a(k)} F^{b(k)}(v)$ are in $\sigma^{\sum_{(j,i) \neq (1,1)} m_j}(R)$. Thus the coefficients of A are also in $\sigma^{\sum_{(j,i) \neq (1,1)} m_j}(R)$. We can write A in the form

$$A = \sum_{0 < k \leq n_1} \alpha_{-k} F^k + \sum_{0 \leq k \leq m_1} \alpha_k V^k \tag{1.4.36}$$

with $\alpha_k \in W(\sigma^{\sum_{(j,i) \neq (1,1)} m_j}(R))$ and $\alpha_{-n_1} = 1$. Hence if N is simple, the equation $Av = 0$ gives the desired relation for $F^{n_1} v$. We now consider the case that N is not simple. As in (1.4.21), the linear independence condition on the coefficients of v implies a similar condition for Av: For each j, all non-trivial linear combinations of the first coefficients of the projections of Av on all $(N_{j,i})_R \subseteq N'_R$ with coefficients in $\mathbb{F}_{p^{h_j}}$ are invertible in R. Especially the projection of Av on the second simple summand of N_R is nonzero, and its first nonzero coefficient $[\beta]$ is invertible. This implies that $[\beta^{-1}] Av$ is (up to an index shift as in Lemma 1.4.11(iv)) an element of N'_R satisfying the conditions needed to apply the induction hypothesis. Thus

$$F^{\sum_{(j,i)\neq(1,1)} n_j}([\beta^{-1}]Av) \\ = \sum_{0<k<\sum_{(j,i)\neq(1,1)} n_j} \gamma_{-k} F^k([\beta^{-1}]Av) + \sum_{0\leq k\leq\sum_{(j,i)\neq(1,1)} m_j} \gamma_k V^k([\beta^{-1}]Av)$$

with $\gamma_k \in R$. Together with (1.4.36) this leads to the desired relation for $F^{\sum_{j,i} n_j} v$. Hence $F^{\sum_{j,i} n_j} v \in \tilde{P}$ and $(\tilde{P}, \tilde{Q}, F, V^{-1})$ is a display.

This display, together with the identity as isomorphism of isodisplays, induces a quasi-isogeny of p-divisible groups over U, that is a morphism $\varphi_{\mathrm{id}} : U \to \mathcal{M}_{\mathrm{red}}$. For all $g \in J$ let

$$\mathcal{S}(g) = \varphi_g(U) = g \circ \varphi_{\mathrm{id}}(U).$$

As can be seen on K-valued points, the subvarieties $\mathcal{S}(g)$ and $\mathcal{S}(g')$ for $g, g' \in J$ are equal if and only if $[g] = [g']$ in $J/(J \cap \mathrm{Stab}(M_0))$.

Lemma 1.4.13. *For all $g \in J/(J\cap\mathrm{Stab}(M_0))$, the subscheme $\mathcal{S}(g)$ is a connected component of $\mathcal{S}_1$.*

Remark 1.4.14. Let $M \subset N$ be a lattice and let $\mathcal{P} = (P, Q, F, V^{-1})$ be the display associated to an S-valued point of $\mathcal{M}_{\mathrm{red}}$. We consider P as a submodule of N_S. Then the condition that P is contained in M_S is a closed condition on S.

Proof of Lemma 1.4.13. Let B be a set of representatives of $J/(J \cap \mathrm{Stab}(M_0))$. Then the $\mathcal{S}(g)$ for $g \in B$ are disjoint and cover $\mathcal{S}_1$. This holds as it is true for their sets of K-valued points for every algebraically closed K, compare Lemma 1.4.8. The height of the quasi-isogeny is constant on each connected component of $\mathcal{M}_{\mathrm{red}}$ and thus of $\mathcal{S}_1$. Let M be a lattice associated to a K-valued point of $\mathcal{S}_1$. Then $\mathrm{vol}(M) - \mathrm{vol}(P(M)) = c$ is a constant only depending on N. If $M \subseteq gM_0$ for some $g \in J$ with $v_p(\det(g)) = \mathrm{vol}(P(M))$, then $P(M) = gM_0$. Thus $\mathcal{S}(g)(K)$ consists of the lattices M with $\mathrm{vol}(M) = v_p(\det(g)) + c$ and $M \subseteq gM_0$. Hence $\mathcal{S}(g)$ is closed. The fact that $\mathcal{M}$ is locally formally of finite type implies that the disjoint union is locally finite. Thus the $\mathcal{S}(g)$ are also open. $\square$

1.4.5 The general case

Now we consider the case of general $\mathbb{X}$ over k, that is we do not assume that $\mathbb{X}$ is bi-infinitesimal. The results obtained for the set of irreducible components and the dimension in the bi-infinitesimal case also hold in this more general context. To see this we again consider the set of K-valued points for an algebraically closed field K. Over K each quasi-isogeny $\rho : \mathbb{X}_K \to X$ splits into a product of quasi-isogenies between the étale, multiplicative, and bi-infinitesimal parts of $\mathbb{X}_K$ and X. The results of Section 1.3 show that the connected component of the point $x \in \mathcal{M}_{\mathrm{red}}(K)$ corresponding to ρ is given by fixing the étale and multiplicative part of the quasi-isogeny and its height. Thus all points of one connected component may be classified by considering the bi-infinitesimal parts of the quasi-isogenies. Quasi-isogenies corresponding to the irreducible subvarieties of $\mathcal{M}_{\mathrm{red}}$ of Section 1.4.4 can be defined in this context as a product of a constant quasi-isogeny of the étale and multiplicative parts of $\mathbb{X}_K$ and the quasi-isogeny of Section 1.4.4 for the bi-infinitesimal part.

1.5 Cohomology

1.5.1 A paving of $\mathcal{M}_{\mathrm{red}}$ for N simple

Let $\mathbb{X}$ be a bi-infinitesimal p-divisible group over an algebraically closed field k of characteristic p whose rational Dieudonné module N is simple. In the following we use e_l instead of $e_{1,1,l}$ for the basis of N and $n = n_1$, $m = m_1$. Let $\pi \in GL(N)$ with $\pi(e_l) = e_{l+1}$ for all l. The description of J in Remark 1.4.1 shows that an element of J is determined by the image of e_0. As $F^{h_1}e_0 = p^{m_1}e_0$, the image has to be invariant under σ^{h_1}. Thus $J \cong \mathrm{Quot}(W(\mathbb{F}_{p^{h_1}})[\pi])$ and $\mathrm{Stab}(M_0) \cap J \cong W(\mathbb{F}_{p^{h_1}})[\pi]^{\times}$. Therefore $J/(\mathrm{Stab}(M_0) \cap J) \cong \mathbb{Z}$. Theorem 1.3.1 and Theorem 1.4.3 then show that $\mathcal{M}^0_{\mathrm{red}}$ is irreducible. This implies that $\mathcal{M}^0_{\mathrm{red}}$ is projective (compare [RZ], Prop. 2.32). We now pave $\mathcal{M}^0_{\mathrm{red}}$ with affine spaces to compute its cohomology. This is inspired by a description of the geometric points of $\mathcal{M}_{\mathrm{red}}$ by de Jong and Oort, see [JO].

Let K be a perfect field over k. We recall a combinatorical invariant for K-valued points of $\mathcal{M}_{\mathrm{red}}$ from [JO], 5. A subset $A \subseteq \mathbb{Z}$ is called a semimodule if it is bounded below and satisfies $m + A \subseteq A$ and $n + A \subseteq A$. It is called normalised if $\mid \mathbb{N} \setminus A(M) \mid = \mid A(M) \setminus \mathbb{N} \mid$. One easily sees that there are only finitely many normalised semimodules. In fact, their number is $\binom{m+n}{m}/(m+n)$, see [JO], 6.3. For every semimodule A, there is a unique integer l such that $l + A$ is normalised. We call $l + A$ the normalisation of A. Each element of N_K can be uniquely written as $\sum_l [a_l]e_l$ with $a_l \in K$ and $a_l = 0$ for l small enough. We call the least $l \in \mathbb{Z}$ with $a_l \neq 0$ the first index of the element. Let $M \subset N_K$ be the lattice associated to $x \in \mathcal{M}_{\mathrm{red}}(K)$. As M is a Dieudonné lattice,

$$A = A(M) = \{l \in \mathbb{Z} \mid l \text{ first index of some } v \in M\} \tag{1.5.1}$$

is a semimodule called the semimodule of x or M. From the definition of the volume we get

$$\mathrm{vol}(M) = \mid \mathbb{N} \setminus A(M) \mid - \mid A(M) \setminus \mathbb{N} \mid . \tag{1.5.2}$$

We may assume that $\mathrm{id}_{\mathbb{X}}$ corresponds to a lattice of volume 0. Then the semimodules of K-valued points of $\mathcal{M}^0_{\mathrm{red}}$ are normalised.

Proposition 1.5.1. *For each normalised semimodule A there is a constructible subscheme $\mathcal{M}_A \subseteq \mathcal{M}^0_{\mathrm{red}}$ which is defined by the property that for each perfect* *field K, the set $\mathcal{M}_A(K)$ consists of the points with semimodule A. The $\mathcal{M}_A$ are disjoint and cover $\mathcal{M}^0_{\mathrm{red}}$.*

Proof. It is enough to show that for every normalised semimodule A there is an open subscheme $\mathcal{M}_{\leq A}$ of $\mathcal{M}^0_{\mathrm{red}}$, such that for every perfect field K the set $\mathcal{M}_{\leq A}(K)$ consists of all points whose semimodules A' satisfy the following condition: There is a bijection $f : A' \to A$ with $f(a) \geq a$ for all a. A Dieudonné lattice M of volume 0 corresponds to an element of $\mathcal{M}_{\leq A}(K)$ if and only if for all $a \in A$, the length of $M/\langle e_{a+1}, e_{a+2}, \dots \rangle_{W(K)}$ is at least $\mid A \cap \mathbb{Z}_{\leq a} \mid$. Every normalised semimodule A' contains an element $a_0 \leq 0$. All $a > mn-m-n \geq a_0+mn-m-n$ can be written as $a = a_0 + \alpha m + \beta n$ with $\alpha, \beta \geq 0$. Thus $a \in A'$ for all $a > mn - m - n$ and all normalised semimodules A'. Hence the condition above is an intersection of finitely many open conditions on $\mathcal{M}^0_{\mathrm{red}}$. $\square$

We want to identify each $\mathcal{M}_A$ with an affine space. To do this we need further combinatorical invariants from [JO]. Let A be a semimodule. We arrange the $m + n$ elements of $A \setminus (m + n + A)$ in the following way: Let b_0 be the largest element. For $i = 1, \dots, m+n-1$ we choose inductively $b_i \in A \setminus (m+n+A)$ to be $b_{i-1} - n$ or $b_{i-1} + m$, depending on which of the elements lies in $A \setminus (m + n + A)$. Then $b_0 = b_{m+n-1} + m$. The tuple

$$B = B(A) = (b_0, \dots, b_{m+n-1}) \tag{1.5.3}$$

is called the cycle of A. One can recover A as $A = \{b_i + l(m+n) \mid b_i \in B, l \geq 0\}$. This defines a bijection between the set of semimodules and the set of cycles, that is of $m + n$-tuples of integers b_i satisfying $b_0 > b_i$, $b_{m+n-1} + m = b_0$ and $b_i \in \{b_{i-1}-n, b_{i-1}+m\}$ for all $i \neq 0$. The normalisation condition for semimodules is equivalent to the condition

$$\sum_i b_i = \frac{(m+n-1)(m+n)}{2}. \tag{1.5.4}$$

We split each cycle B in two parts:

$$B^+ = \{b_i \mid b_i + m \in B\} \tag{1.5.5}$$
$$B^- = \{b_i \mid b_i - n \in B\}. \tag{1.5.6}$$

Let

$$\mathcal{V}(B) = \{(d, i) \mid b_d \in B^+, b_i \in B^-, b_i < b_d\} \tag{1.5.7}$$

and

$$R = k[a_{d,i} \mid (d, i) \in \mathcal{V}(B)], \tag{1.5.8}$$

$$S = \mathrm{Spec}(R) = \mathbb{A}_k^{\mathcal{V}(B)}. \tag{1.5.9}$$

We define a quasi-isogeny $\mathbb{X}_S \to X$ by describing the display of X as a subdisplay of the isodisplay N_R of $\mathbb{X}_S$. For each $b_i \in B$ we want to define an element $v_i \in N_R$ which has first index b_i and first coefficient 1 in all closed points of S. We want the v_i to satisfy the following relations:

$$v_0 = e_{b_0} \tag{1.5.10}$$

and

$$v_{i+1} = \begin{cases} Fv_i & \text{if } b_i, b_{i+1} \in B^+ \\ Fv_i + \sum_{(d,i+1)\in\mathcal{V}(B)} [a_{d,i+1}]v_d & \text{if } b_i \in B^+, b_{i+1} \in B^- \\ V^{-1}v_i & \text{if } b_i \in B^-, b_{i+1} \in B^+ \\ V^{-1}v_i + \sum_{(d,i+1)\in\mathcal{V}(B)} [a_{d,i+1}]v_d & \text{if } b_i, b_{i+1} \in B^-. \end{cases} \tag{1.5.11}$$

We set

$$v_i = \sum_{j=0}^{m+n-1} c_{i,j} e_{b_i+j}$$

with $c_{i,j} \in W(R)$ and write $c_{i,j} = (c_{i,j,l})_{l\in\mathbb{N}}$. Let

$$\varphi(j,l) = j + l(m+n). \tag{1.5.12}$$

As φ is a bijection between $\{0,\dots,m+n-1\} \times \mathbb{N}$ and $\mathbb{N}$, we may write $\tilde{c}_{i,\varphi(j,l)}$ instead of $c_{i,j,l}$. Let $\tilde{c}_{i,0} = 1$ for all i. Then in every point of S, the first index of v_i is b_i, and its first coefficient is 1. We define the $\tilde{c}_{i,\varphi}$ by induction on $\varphi(j,l)$, and for fixed φ by induction on i: For $\varphi > 0$ let $\tilde{c}_{0,\varphi} = 0$ to satisfy (1.5.10). If $b_{i+1} \in B^+$, (1.5.11) implies that $\tilde{c}_{i+1,\varphi} = \tilde{c}^{\sigma}_{i,\varphi}$. If $b_{i+1} \in B^-$, then $\tilde{c}_{i+1,\varphi}$ is equal to $\tilde{c}^{\sigma}_{i,\varphi}$ plus a polynomial in the $\tilde{c}_{i',\varphi'}$ with $\varphi' < \varphi$ and coefficients in R. Hence we can inductively define $\tilde{c}_{i,\varphi(j,l)} = c_{i,j,l} \in R$, and obtain $c_{i,j} = (c_{i,j,l}) \in W(R)$.

With these v_i let

$$\begin{aligned} L &= \langle v_i \mid b_i \in B^- \rangle_{W(R)} \\ T &= \langle v_i \mid b_i \in B^+ \rangle_{W(R)} \\ P &= L \oplus T \\ Q &= L \oplus I_R T. \end{aligned}$$

The first indices b_i of the v_i are pairwise non-congruent modulo $m+n$, hence the v_i are linearly independent over $W(R)[1/p]$ and $P \otimes \mathbb{Q} = N_R$. To show that this defines a display over S, we have to verify that $F(P) \subseteq P$ and that $V^{-1}(Q)$ generates P. The only assertions that do not immediately follow from (1.5.11) are $Fv_{m+n-1} \in P$ and $v_0 \in \langle V^{-1}(Q) \rangle_{W(R)}$. As $l \in A$ for all $l \geq b_0$, all those e_l are in P. As $b_{m+n-1} + m = b_0$, we have $Fv_{m+n-1} = \sum_{j=0}^{m+n-1} c^{\sigma}_{m+n-1,j} e_{b_0+j}$ with $w_0(c_{m+n-1,0}) = 1$. Therefore, $Fv_{m+n-1} \in P$, and $Fv_{m+n-1} = \sum_{i\in B} \delta_i v_i$

with $\delta_i \in W(R)$ and $w_0(\delta_0) = 1$. All v_i with $i > 0$ are in $\langle V^{-1}(Q)\rangle_{W(R)}$ and $Fv_{m+n-1} = V^{-1}(pv_{m+n-1})$. Thus also $v_0 \in \langle V^{-1}(Q)\rangle_{W(R)}$.

Let B be the cycle corresponding to a normalised semimodule A. Then this display induces a quasi-isogeny and thus a morphism

$$f_A : \mathbb{A}^{\mathcal{V}(B)} \to \mathcal{M}_A. \tag{1.5.13}$$

Lemma 1.5.2. *Let R be an excellent local ring and a k-algebra. Let $x \in \mathcal{M}_A(R)$ and let $\mathcal{P} = (P, Q, F, V^{-1})$ be the corresponding display over R. Assume that for every $b_i \in B^+$ there is a $w_i^0 \in P$ and for every $b_i \in B^-$ a $w_i^0 \in Q$ with $w_i^0 = \sum_{j=0}^{m+n-1} c_{i,j}^0 e_{b_i+j}$ with $c_{i,j}^0 \in W(R)$ and the following properties: $w_0(c_{i,0}^0) = 1$ for all i and the w_i^0 generate P. Then there is a unique $\tilde{x} \in \mathbb{A}^{\mathcal{V}(B)}(R)$ with $f_A(\tilde{x}) = x$.*

Proof. We want to show that there exist unique $a_{d,i} \in R$ such that for the corresponding display $\mathcal{P}' = (P', Q', F, V^{-1})$ we have that $P' \subseteq P$ and $Q' \subseteq Q$. As x and $f_A((a_{d,i}))$ are in the same connected component of $\mathcal{M}$, the displays then have to be equal.

We show by induction on h that for each $b_i \in B^-$ there is a $w_i^h \in Q$ and for each $b_i \in B^+$ a $w_i^h \in P$ of the form $w_i^h = \sum_{j=0}^{m+n-1} c_{i,j}^h e_{b_i+j}$ with $c_{i,j}^h = (c_{i,j,l}^h) \in W(R)$ and the following property: The coefficients $c_{i,j,l}^h$ for $\varphi(j,l) \leq h$ are equal to those of the basis v_i of the display of a point of $\mathbb{A}^{\mathcal{V}(B)}$ which only depends on $\mathcal{P}$ and not on the chosen basis w_i^0. The coordinate $a_{d,i}$ of the point of $\mathbb{A}^{\mathcal{V}(B)}$ will be determined in the step where $h = b_d - b_i$. Especially, the point is fixed after finitely many steps.

For $h = 0$ the claim follows from the assumptions of the lemma. As $b_0 + \mathbb{N} \subseteq n + A$, we can choose $w_0^h = e_{b_0} = v_0 \in Q$ for all h. Now suppose that the w_i^h are defined for some fixed h. We use a second induction on i to define

$$\tilde{w}_{i+1}^{h+1} = \begin{cases} Fw_i^{h+1} & \text{if } b_i \in B^+ \\ V^{-1}w_i^{h+1} & \text{if } b_i \in B^-. \end{cases}$$

From $w_i^{h+1} \in Q$ for $b_i \in B^-$ we obtain that $\tilde{w}_{i+1}^{h+1} \in P$ also for those i. If $b_{i+1} \in B^+$ let

$$w_{i+1}^{h+1} = \tilde{w}_{i+1}^{h+1}.$$

If $b_{i+1} \in B^-$, we modify $\tilde{w}_{i+1}^{h+1} \in P$ to obtain an element of Q: A basis of the free R-module $P/Q \cong T/I_R T$ is given by the w_d^h with $b_d \in B^+$. As the first indices of the elements of Q are in $n + A$, there are unique $\alpha_{d,i+1}^{h+1} \in R$ with

$$w_{i+1}^{h+1} = \tilde{w}_{i+1}^{h+1} + \sum_{b_d > b_{i+1}, b_d \in B^+} [\alpha_{d,i+1}^{h+1}] w_d^h \in Q. \tag{1.5.14}$$

By the induction hypothesis, the coefficients $c_{i,j,l}^{h+1}$ of w_i^{h+1} with $\varphi(j,l) \leq h+1$ and the coefficients of all w_d^h with $\varphi(j,l) \leq h$ are uniquely defined by $\mathcal{P}$ and independent of the chosen w_i^0. This implies $c_{i,j,l}^{h+1} = c_{i,j,l}^h$ for all j,l with $\varphi(j,l) \leq h$. Especially, $\alpha_{d,i+1}^{h+1} = a_{d,i+1}$ for all $b_d \leq b_{i+1} + h$. If $(d, i+1) \in \mathcal{V}(B)$ with $b_d - b_{i+1} = h+1$, then let $a_{d,i+1} = \alpha_{d,i+1}^{h+1}$. Then $a_{d,i+1}$ also only depends on $\mathcal{P}$. This defines w_i^{h+1} for all i and unique $a_{d,i}$ satisfying the condition above for $h+1$.

Each coefficient $c_{i,j,l}^h$ remains fixed after $\varphi(j,l)$ steps. Hence the sequences w_i^h converge in P, and their limits w_i are as desired. $\square$

Theorem 1.5.3. *Let A be a normalised semimodule. Then $f_A : \mathbb{A}^{\mathcal{V}(B)} \to \mathcal{M}_A$ is an isomorphism.*

Proof. If $K \mid k$ is a perfect field, each lattice corresponding to an element of $\mathcal{M}_A(K)$ has a basis satisfying the assumptions of Lemma 1.5.2. Hence $f_A(K) : \mathbb{A}^{\mathcal{V}(B)}(K) \to \mathcal{M}_A(K)$ is a bijection.

We want to show that f_A is proper by verifying the valuation criterion. Let $x \in \mathcal{M}_A(k[[t]])$ and let x_η and x_0 be its generic and special point. Let $\tilde{x}_\eta \in \mathbb{A}^{\mathcal{V}(B)}(k((t)))$ be a point mapping to x_η. Let $\mathcal{P} = (P, Q, F, V^{-1})$ be the display of x. The $W(k[[t]])$-module P is a submodule of $P_\eta = P \otimes_{W(k[[t]])} W(k((t)))$, the first component of the display $\mathcal{P}_\eta$ of x_η. As $\tilde{x}_\eta$ maps to x_η, we can describe $\mathcal{P}_\eta$ as generated by elements v_i as above. By Lemma 1.5.2, $x_0 \in \mathcal{M}_A(k)$ is also in the image of f_A. Hence we can choose generators v_i' of $\mathcal{P}$ which for $b_i \in B^-$ are in Q, and which modulo (t) reduce to the standard generators of the display of the inverse image of x_0 under f_A. Let y be the minimal element of B. As $v_i' \in N_{k[[t]]}$, there is an $a \in \mathbb{N}$ such that $v_i' \in \langle e_y, \ldots, e_{y+m+n-1} \rangle_{W(k[[s]])}$ where $s^{p^a} = t$. In the following we consider x as a $k[[s]]$-valued point of $\mathcal{M}_A$. The reduction of v_i' modulo (s) has first index b_i. As $x_\eta \in \mathcal{M}_A$, the index of the first nonzero coefficient of each v_i' is in A. Thus we can modify each v_i' by a linear combination of the v_j' with $b_j < b_i$ and coefficients in $W(k[[s]])$ which reduce to 0 modulo (s) such that the new first nonzero coefficient is that of e_{b_i}. Besides, we only have to modify the elements $v_i \in Q$ by other elements of Q. Therefore we may in addition assume that the first nonzero coefficient of v_i' has index b_i, and is 1. After passing to a larger a we may assume that $v_i' \in \langle e_{b_i}, \ldots, e_{b_i+m+n-1} \rangle_{W(k[[s]])}$. By Lemma 1.5.2 for $w_i^0 = v_i'$ we obtain a unique point $\tilde{x}' = (b_{d,i}) \in \mathbb{A}^{\mathcal{V}(B)}(k[[s]])$ mapping to x. But as its generic point $\tilde{x}'_\eta$ maps to x_η, the uniqueness in Lemma 1.5.2 implies that $\tilde{x}'_\eta = \tilde{x}_\eta \in \mathbb{A}^{\mathcal{V}(B)}(k((t)))$. Hence $b_{d,i} \in k[[s]] \cap k((t)) = k[[t]]$, and x is in the image of $f_A(k[[t]])$.

Lemma 1.5.2 further implies that the tangent morphism of f_A is injective at every closed point. The theorem now follows from [V], Lemma 5.13. $\square$

Using the paving of $\mathcal{M}_{\mathrm{red}}^0$ by affine spaces and that $\mathcal{M}_{\mathrm{red}}^0$ is projective, one obtains the following result about its cohomology.

Theorem 1.5.4. *Let $l \neq p$ be prime. Then*

$$H^{2i+1}(\mathcal{M}^0_{\mathrm{red}}, \mathbb{Q}_l) = 0 \tag{1.5.15}$$
$$H^{2i}(\mathcal{M}^0_{\mathrm{red}}, \mathbb{Q}_l) = \mathbb{Q}_l(-i)^{d(i)}, \tag{1.5.16}$$

for all i, where $d(i)$ is the number of normalised cycles B with $\mid \mathcal{V}(B) \mid= i$.

Proposition 1.5.5. *(i) $d(0) = d(\frac{(m-1)(n-1)}{2}) = 1$ for all m and n. If $m, n > 1$, also $d(1) = 1$.*

(ii) Let $\min\{m, n\} = 2$. Then $d(i) = 1$ for $0 \leq i \leq \dim \mathcal{M}^0_{\mathrm{red}}$.

(iii) Let $\min\{m, n\} > 2$. Then $d(\frac{(m-1)(n-1)}{2} - 1) > 1$.

Proof. The equation $d(\frac{(m-1)(n-1)}{2}) = 1$ is shown in [JO], 6. They also show that for a semimodule A, the dimension $\mid \mathcal{V}(B(A)) \mid$ is bounded below by the number of positive integers s such that there exists an $a \in A$ with $a + s \notin A$ (see [JO], 6.12). Let A be a normalised semimodule with $\mid \mathcal{V}(B(A)) \mid= 0$. Then $a \in A$ implies $a' \in A$ for all $a' > a$. Thus $A = \mathbb{N}$. One easily sees that this semimodule indeed leads to a zero-dimensional subscheme. Let now A be normalised with $\mid \mathcal{V}(B(A)) \mid= 1$. Then $a \in A$ implies that there is at most one element of $\mathbb{Z} \setminus A$ that is larger than a. Analogously, for $a \notin A$ there is at most one element of A smaller than a. This leaves only $A = \{-1, 1, 2, 3, \dots\}$ as a candidate for a contribution to $d(1)$. It is a semimodule if and only if $m, n > 1$. Again one can see (using the combinatorics explained in [JO], 6) that $\mid \mathcal{V}(B(A)) \mid= 1$.

To show (ii), we may assume that $m = 2$ and $n = 2l + 1$ for some l. Each normalised semimodule is of the form $A = A_i = (2\mathbb{N} - i) \cup (\mathbb{N} + i + 1)$ for some $i \in \{0, \dots, l\}$. The cycle $B_i = B(A_i)$ is

$$(2l + 2 + i, i + 1, i + 3, \dots, 2l - i - 1, 2l - i + 1, -i, -i + 2, \dots, 2l + i)$$

with $B^- = \{2l+2+i, 2l-i+1\}$. The element $2l+2+i$ is the largest element of B_i, so it does not contribute to $\mathcal{V}(B_i)$. The other element of B^- is smaller than the i elements $2l - i + 2, \dots, 2l + i$ of B^+. Hence $\mid \mathcal{V}(B_i) \mid= i$.

For (iii) we have to construct two normalised semimodules leading to subschemes of codimension 1. Assume that $m < n$, the other case is completely analogous. Let

$$A_1 = \{am + bn \mid a, b \geq 0\} \cup \{mn - m - n\}. \tag{1.5.17}$$

There are $\frac{(m-1)(n-1)}{2}$ natural numbers which cannot be written as $am + bn$ with $a, b \geq 0$, and $mn - m - n$ is the largest. Thus the lower bound on $\mid \mathcal{V}(A_1) \mid$ used in the proof of (i) shows that the codimension of the subscheme corresponding to the normalisation of A_1 is at most 1. But there is only one semimodule leading to a subscheme of codimension 0, and this is obtained by normalising $A_0 = \{am+bn \mid a, b \geq 0\}$ (see [JO], 6). Thus the normalisation of the semimodule A_1 leads to a subscheme of codimension 1. The cycle corresponding to A_0 is given by $B^+ = \{0, m, \ldots, (n-1)m\}$ and $B^- = \{n, 2n, \ldots, mn\}$. Let B_2^+ be the index set obtained from B^+ by replacing $(n-1)m$ by $(n-1)m - 2n$, and let B_2^- be obtained from B^- by replacing mn by $mn - m$ and $mn - n$ by $mn - m - n$. One can easily check that this defines a cycle. A pair of elements $(i, j) \in B^+ \times B^-$ with $i > j$ is then replaced by a pair in $B_2^+ \times B_2^-$. The pair $((n-1)m, mn - n)$ which is replaced by $(mn - m - 2n, mn - m - n)$ is the only pair with larger first entry which is replaced by a pair such that the first entry is smaller than the second. After normalising the cycle we get again a subscheme of codimension 1. The smallest element of A_1 and A_2 is 0. As $mn - m - 2n \in A_2 \setminus A_1$, the normalisations of the two semimodules are different. $\square$

1.5.2 Application to smoothness

In this section we show the following

Theorem 1.5.6. *Let $\mathbb{X}$ be an arbitrary p-divisible group over an algebraically closed field of characteristic p. Then $\mathcal{M}^0_{\mathrm{red}}$ is smooth if and only if one of the following holds: $\dim \mathcal{M}^0_{\mathrm{red}} = 0$ or the isocrystal N of $\mathbb{X}_{\mathrm{bi}}$ is simple of slope $\frac{2}{5}$ or $\frac{3}{5}$.*

Remark 1.5.7. The condition $\dim \mathcal{M}^0_{\mathrm{red}} = 0$ is equivalent to the condition that $\mathbb{X}$ is ordinary or that the isocrystal of $\mathbb{X}_{\mathrm{bi}}$ is simple of slope $\frac{m}{m+n}$ with $\min\{m, n\} = 1$.

Once we have shown the theorem for bi-infinitesimal $\mathbb{X}$, we can treat the general case as in Section 1.4.5. We may thus assume that $\mathbb{X}$ is bi-infinitesimal. The results of Sections 1.3 and 1.4 imply that the connected components of $\mathcal{M}_{\mathrm{red}}$ are irreducible if and only if N is simple. From now on we assume this. Let $\frac{m}{m+n}$ be its slope with $(m, n) = 1$. We also assume that $\mathrm{id}_{\mathbb{X}}$ corresponds to a lattice of volume 0. We consider the following cases:

Case 1: $\min\{m, n\} = 1$.

If m or n is 1, the dimension of $\mathcal{M}^0_{\mathrm{red}}$ is 0 and the scheme is smooth.

Case 2: $\{m, n\} = \{2, 3\}$.

We assume that $m = 2$ and $n = 3$. The case $n = 2$ and $m = 3$ is similar and thus omitted.

Theorem 1.5.8. *Let $\mathbb{X}$ be bi-infinitesimal and let its rational Dieudonné module N be simple of slope $\frac{2}{5}$. Then $\mathcal{M}^0_{\mathrm{red}} \cong \mathbb{P}^1$.*

Proof. Let $\mathbb{P}^1 = U_0 \cup U_1$ be the standard open covering. We denote the points of $\mathbb{P}^1$ by $[a_{-1} : a_0]$. Over $U_0 \cong \mathrm{Spec}(k[a_0])$ let

$$\begin{aligned} L_0 &= \langle e_2 + [a_0]e_3, e_5\rangle_{W(k[a_0])} \\ T_0 &= \langle e_{-1} + [a_0]^\sigma e_0, e_1, e_3\rangle_{W(k[a_0])} \end{aligned}$$

and

$$\begin{aligned} P_0 &= L_0 \oplus T_0 = \langle e_{-1} + [a_0]^\sigma e_0, e_1, e_2, e_3, e_5\rangle_{W(k[a_0])} \\ Q_0 &= L_0 \oplus I_{k[a_0]}T_0. \end{aligned}$$

On the other hand let P and Q be the display from the definition of f_A for the semimodule $A = \{-1, 1, 2 \ldots\}$. Then $\{e_1, e_2, \ldots\} \subset P$ and $\{e_4, e_5, \ldots\} \subset Q$. Using this and the first steps of the recursion for the generators of P, one can see that $P = P_0$ and $Q = Q_0$. Thus P_0 and Q_0 define a display. As A is the minimal semimodule, the corresponding morphism $\mathbb{A}^1 \to \mathcal{M}^0_{\mathrm{red}}$ is an open immersion. Over $U_1 \cong \mathrm{Spec}(k[a_{-1}])$ let

$$\begin{aligned} L_1 &= \langle [a_{-1}]e_2 + e_3, e_4\rangle_{W(k[a_{-1}])} \\ T_1 &= \langle [a_{-1}]^\sigma e_{-1} + e_0, e_1, e_2\rangle_{W(k[a_{-1}])} \end{aligned}$$

and choose

$$\begin{aligned} P_1 &= L_1 \oplus T_1 = \langle [a_{-1}]^\sigma e_{-1} + e_0, e_1, e_2, e_3, e_4\rangle_{W(k[a_{-1}])} \\ Q_1 &= L_1 \oplus I_{k[a_{-1}]}T_1. \end{aligned}$$

One easily checks that this defines a display. It is obvious that the corresponding morphism $\varphi_1 : \mathbb{A}^1 \to \mathcal{M}^0_{\mathrm{red}}$ is injective on R-valued points. As for f_A one can show that φ_1 is an immersion. The complement of its image consists of the image of the origin in U_0. We can glue the morphisms corresponding to the displays over U_0 and U_1 to obtain an isomorphism $\mathbb{P}^1 \to \mathcal{M}^0_{\mathrm{red}}$. $\square$

Case 3: $\min\{m, n\} = 2$ **and** $\max\{m, n\} > 3$.

We consider the case $m = 2$ and $n = 2l + 1$ with $l > 1$. The case $n = 2$ and $m = 2l + 1$ is similar and thus omitted. We have $\dim \mathcal{M}^0_{\mathrm{red}} = l$. Let $M \subset N$ be the Dieudonné lattice generated by e_{-l+2} and e_{l-1}. Then this lattice corresponds to the k-valued point $x = f_{A_{l-2}}(0) \in \mathcal{M}_{A_{l-2}}(k)$ where A_{l-2} is as in the proof of Proposition 1.5.5.

Proposition 1.5.9. *The dimension of the tangent space of $\mathcal{M}^0_{\mathrm{red}}$ in x is at least $l + 1$.*

Proof. For $(a_0, \dots, a_{l-1}, b_0, b_1) \in k^{l+2}$ consider the following submodules of $N_{k[\varepsilon]}$ where $k[\varepsilon] \cong k[t]/(t^2)$. Let

$$v_1 = e_{l+3} + [\varepsilon]([a_0]e_{l+1} + \sum_{i=1}^{l-1}[a_i]e_{l+2i}) \tag{1.5.18}$$

and

$$v_2 = e_{3l} + [\varepsilon b_0]e_{l+1} + [\varepsilon b_1]e_{l+2}. \tag{1.5.19}$$

Let

$$\begin{aligned} L &= \langle v_1, v_2 \rangle_{W(k[\varepsilon])} \\ T &= \langle e_{-l+2}, e_{-l+4}, \dots, e_{3l-2}, e_{l-1}, e_{l+1} \rangle_{W(k[\varepsilon])}, \end{aligned}$$

then

$$P = L \oplus T = M \otimes_{W(k)} W(k[\varepsilon]) \tag{1.5.20}$$

$$Q = L \oplus I_{k[\varepsilon]}T. \tag{1.5.21}$$

As $\sigma(\varepsilon) = 0$, this defines a display. For $i_0 \in \{l+1, l+2, l+4, \dots, 3l-2\}$ there is no element of Q of the form $\sum_{i \geq i_0}[\delta_i]e_i$ with $\delta_i \in k[\varepsilon]$ and $\delta_{i_0} \neq 0$. This implies that the display leads to an $l+2$-dimensional subspace of the tangent space of $\mathcal{M}$ at x. We now have to construct an $l+1$-dimensional subspace that lies in the tangent space of $\mathcal{M}_{\mathrm{red}}$. For $a_0, a_1 \neq 0$ let

$$v_1 = [ta_0]e_{l+1} + [ta_1]e_{l+2} + e_{l+3} + \sum_{i=2}^{l-1}[ta_i]e_{l+2i} \in N_{k((t))}. \tag{1.5.22}$$

Let further

$$\begin{aligned} L_1 &= \langle e_{3l+2}, v_1 \rangle_{W(k((t)))} \\ T_1 &= \langle V^{-1}v_1, FV^{-1}v_1, \dots, F^{2l}V^{-1}v_1 \rangle_{W(k((t)))} \\ P_1 &= L_1 \oplus T_1 \\ Q_1 &= L_1 \oplus I_{k((t))}T_1. \end{aligned}$$

As $a_0 \neq 0$, there is an element of P_1 with first index i for all $i \geq l$ and of Q_1 for all $i \geq 3l+1$. Using this one can easily see that $F(P_1) \subseteq P_1$ and that $V^{-1}(Q_1)$ generates P_1. Thus P_1 and Q_1 define a display over $k((t))$, and a $k((t))$-valued point of $\mathcal{M}^0_{\mathrm{red}}$. As $\mathcal{M}^0_{\mathrm{red}}$ is projective this point is induced by a $k[[t]]$-valued point. Its display is $(P_1 \cap N_{k[[t]]}, Q_1 \cap N_{k[[t]]}, F, V^{-1})$. We want to show that the special point corresponds to x. The element $V^{-1}v_1$ of $P_1 \cap N_{k[[t]]}$ reduces to $V^{-1}(e_{l+3}) = e_{-l+2}$ modulo (t). To show that M is contained in the reduction of $P_1 \cap N_{k[[t]]}$ modulo (t), it remains to see that e_{l-1} is contained in this reduction.

For all $i \geq 3l+1$ the vector e_i is in Q_1. We consider the following element of Q_1 modulo the lattice generated by these elements e_i:

$$\begin{aligned}(F^{l-1} - [(ta_0)^{\sigma^{l-1}}]F^{l-2})v_1 \equiv [t^{\sigma^{l-1}}] \Big(& -[a_0^{\sigma^{l-1}}(ta_0)^{\sigma^{l-2}}]e_{3l-3} - [a_0^{\sigma^{l-1}}(ta_1)^{\sigma^{l-2}}]e_{3l-2} \\ & +([a_1^{\sigma^{l-1}}] - [a_0^{\sigma^{l-1}}(ta_2)^{\sigma^{l-2}}])e_{3l}\Big) \\ = [t^{\sigma^{l-1}}]v &\end{aligned}$$

for some $v \in Q_1 \cap N_{k[[t]]}$. The reduction of v modulo (t) is $[a_1^{\sigma^{l-1}}]e_{3l}$. Thus $e_{l-1} = V^{-1}(e_{3l})$ is contained in the lattice at the special point. Hence the special point of this $k[[t]]$-valued point is x. If $l > 2$, the reduction of v modulo (t^2) is $[a_1^{\sigma^{l-1}}]e_{3l}$. Hence e_{3l} is in the projection of $Q_1 \cap N_{k[[t]]}$ to $N_{k[\varepsilon]}$. If $l = 2$, the reduction of $([a_1^{\sigma^{l-1}}] - [a_0^{\sigma^{l-1}}(ta_2)^{\sigma^{l-2}}])^{-1}v$ modulo (t^2) is equal to v_2 as in (1.5.19) with $b_0 = -a_0^\sigma a_0/a_1^\sigma$ and $b_1 = -a_0^\sigma a_1/a_1^\sigma$. Comparing the image of $Q_1 \cap N_{k[[t]]}$ under the projection to $N_{k[\varepsilon]}$ to the definition of Q in (1.5.21) we see that the tangent vector of this $k[[t]]$-valued point at x corresponds to the tangent vector $(a_0, \ldots, a_{l-1}, 0, 0) \in k^{l+2}$ if $l > 2$ and to

$$(a_0, a_1, \frac{-a_0^\sigma a_0}{a_1^\sigma}, \frac{-a_0^\sigma a_1}{a_1^\sigma})$$

if $l = 2$.

For $b_0 \neq 0$ let

$$v_2 = [tb_0]e_{l+1} + e_{3l}$$

and

$$\begin{aligned} L_2 &= \langle e_{3l+2}, v_2 \rangle_{W(k((t)))} \\ T_2 &= \langle V^{-1}v_2, FV^{-1}v_2, \ldots, F^{2l}V^{-1}v_2 \rangle_{W(k((t)))} \\ P_2 &= L_2 \oplus T_2 \\ Q_2 &= L_2 \oplus I_{k((t))}T_2. \end{aligned}$$

The same reasoning as above shows that this defines a display. To show that it leads to a $k[[t]]$-valued point of $\mathcal{M}_{\mathrm{red}}$ with special point x, we have to check that e_{l-1} and e_{-l+2} are in the lattice at the special point. The reduction of $V^{-1}v_2 = [tb_0]^\sigma e_{-l} + e_{l-1}$ modulo (t) is e_{l-1}. Besides, $Fv_2 - e_{3l+2} = [t^\sigma b_0^\sigma]e_{l+3} \in Q_2$, hence $e_{l+3} \in Q_2 \cap N_{k[[t]]}$. As $e_{-l+2} = V^{-1}e_{l+3}$, the lattice M is contained in the reduction of P_2 modulo (t). The fact that $e_{l+3} \in Q_2 \cap N_{k[[t]]}$ also shows that the tangent vector of this $k[[t]]$-valued point in x corresponds to $(0, \ldots, 0, b_0, 0) \in k^{l+2}$. Thus we constructed elements of the tangent space of $\mathcal{M}_{\mathrm{red}}$ in x which generate an $l+1$-dimensional subspace. $\square$

Case 4: $\min\{m,n\} > 2$

In this case Theorem 1.5.4 and Proposition 1.5.5 show that

$$\dim H^2(\mathcal{M}^0_{\mathrm{red}}, \mathbb{Q}_l) \neq \dim H^{2\dim \mathcal{M}^0_{\mathrm{red}}-2}(\mathcal{M}^0_{\mathrm{red}}, \mathbb{Q}_l).$$

Hence $\mathcal{M}^0_{\mathrm{red}}$ does not satisfy Poincaré duality and it cannot be smooth. $\square$

2 The function field case

2.1 Introduction

Let k be a finite field with $q = p^r$ elements and let $\overline{k}$ be an algebraic closure. Let $F = k((t))$ and let $L = \overline{k}((t))$. Let $\mathcal{O}_F$ and $\mathcal{O}_L$ be the valuation rings. We denote by $\sigma : x \mapsto x^q$ the Frobenius of $\overline{k}$ over k and also of L over F.

Let $G = GL_n$ over F and let A be the diagonal torus. Let B be the Borel subgroup of lower triangular matrices. For $\mu, \mu' \in X_*(A)_{\mathbb{Q}}$ we say that $\mu \geq \mu'$ if $\mu - \mu'$ is a non-negative linear combination of positive coroots. An element $\mu = (\mu_1, \dots, \mu_n) \in X_*(A) \cong \mathbb{Z}^n$ is dominant if $\mu_1 \leq \cdots \leq \mu_n$. We write μ_{dom} for the dominant element in the orbit of $\mu \in X_*(A)$ under the Weyl group of A in G. For $\alpha \in X_*(A)$ we denote by $t^\alpha \in A(F)$ the image of t under the homomorphism $\alpha : \mathbb{G}_m \to A$.

We recall the definitions of affine Deligne-Lusztig varieties and closed affine Deligne-Lusztig varieties from [Ra1], [GHKR]. Let $K = G(\mathcal{O}_L)$ and let $X = G(L)/K$ be the affine Grassmannian. For $b \in G(L)$ and a dominant coweight $\mu \in X_*(A)$ the affine Deligne-Lusztig variety $X_\mu(b)$ is the locally closed reduced subscheme of X defined by

$$X_\mu(b) = \{g \in G(L)/K \mid g^{-1} b \sigma(g) \in K t^\mu K\}. \tag{2.1.1}$$

The closed affine Deligne-Lusztig variety is the closed reduced subscheme of X defined by

$$X_{\leq\mu}(b) = \bigcup_{\mu' \leq \mu} X_{\mu'}(b). \tag{2.1.2}$$

Both $X_\mu(b)$ and $X_{\leq\mu}(b)$ are locally of finite type. Let $\nu \in \mathbb{Q}^n$ be the Newton vector associated to b. In [KR] Kottwitz and Rapoport prove that $X_\mu(b)$ is nonempty if and only if $\nu \leq \mu$. From now on we only consider this case.

Left multiplication by $g \in G(L)$ induces an isomorphism between $X_\mu(b)$ and $X_\mu(gb\sigma(g)^{-1})$. Thus the isomorphism class of the affine Deligne-Lusztig variety only depends on the σ-conjugacy class of b. For each central $\alpha \in X_*(A)$ there is the trivial isomorphism

$$X_\mu(b) \to X_{\mu+\alpha}(t^\alpha b). \tag{2.1.3}$$

We write $\pi_1(G)$ for the quotient of $X_*(A)$ by the coroot lattice of G. In [K2] Kottwitz defines a homomorphism

$$\kappa_G : G(L) \to \pi_1(G) \tag{2.1.4}$$

which induces a locally constant map $\kappa_G : X \to \pi_1(G)$. For $G = GL_n$ we have $\pi_1(G) \cong \mathbb{Z}$ and $\kappa_G(g) = v_t(\det g)$.

Let $\boldsymbol{P}$ be a standard parabolic subgroup of G. Then $\boldsymbol{P} = \boldsymbol{M}\boldsymbol{N}$, where $\boldsymbol{N}$ is the unipotent radical of $\boldsymbol{P}$ and where $\boldsymbol{M}$ is the unique Levi subgroup of $\boldsymbol{P}$

containing A. Applying the construction of κ to $\boldsymbol{M}$ rather than G we obtain a homomorphism $\kappa_{\boldsymbol{M}} : \boldsymbol{M}(L) \to \pi_1(\boldsymbol{M})$. The inclusion $\boldsymbol{M}(L)/\boldsymbol{M}(\mathcal{O}_L) \hookrightarrow G(L)/G(\mathcal{O}_L)$ induces for each μ and each $b \in \boldsymbol{M}(L)$ an inclusion $X_\mu^{\boldsymbol{M}}(b) \hookrightarrow X_\mu^G(b)$. Here $X_\mu^{\boldsymbol{M}}(b)$ denotes the affine Deligne-Lusztig variety for $\boldsymbol{M}$.

Let $A_{\boldsymbol{P}}$ denote the identity component of the center of $\boldsymbol{M}$. Let

$$\mathfrak{a}_{\boldsymbol{P}}^+ = \{x \in X_*(A_{\boldsymbol{P}}) \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle \alpha, x\rangle > 0 \text{ for every root } \alpha \text{ of } A_{\boldsymbol{P}} \text{ in } \boldsymbol{N}\}. \qquad (2.1.5)$$

In [K2] it is shown that there is a unique standard parabolic $\boldsymbol{P}_b = \boldsymbol{M}_b\boldsymbol{N}_b$ of G such that the σ-conjugacy class of b contains an element b' with the following properties: b' is basic in $\boldsymbol{M}_b$ and $\kappa_{\boldsymbol{M}_b}(b')$, considered as an element of $\mathfrak{a}_{\boldsymbol{P}_b}$, lies in $\mathfrak{a}_{\boldsymbol{P}_b}^+$. We assume that $b = b'$. The proof of the Hodge-Newton decomposition by Kottwitz (see [K3]) yields: Let $\boldsymbol{P} = \boldsymbol{M}\boldsymbol{N} \subseteq G$ be a standard parabolic subgroup with $\boldsymbol{P}_b \subseteq \boldsymbol{P}$. If $\kappa_{\boldsymbol{M}}(b) = \mu$, then the morphism $X_\mu^{\boldsymbol{M}}(b) \hookrightarrow X_\mu^G(b)$ is an isomorphism. We call a pair (μ, b) indecomposable with respect to the Hodge-Newton decomposition if for all standard parabolic subgroups $\boldsymbol{P}$ with $\boldsymbol{P}_b \subseteq \boldsymbol{P} = \boldsymbol{M}\boldsymbol{N} \subsetneq G$ we have $\kappa_{\boldsymbol{M}}(b) \neq \mu$. Given G, μ, and b we may always pass to a Levi subgroup $\boldsymbol{M}$ of G in which (μ, b) is indecomposable. For a description of the affine Deligne-Lusztig varieties it is therefore sufficient to consider pairs (μ, b) which are indecomposable with respect to the Hodge-Newton decomposition.

Let

$$J = \{g \in GL_n(L) \mid g \circ b\sigma = b\sigma \circ g\}.$$

Then there is a canonical J-action on $X_\mu(b)$.

In this section we address the following three questions:

Question 1: What are the sets of connected components of $X_\mu(b)$ and $X_{\leq\mu}(b)$?

For the closed affine Deligne-Lusztig varieties we prove that J acts transitively on the set of connected components. Using this we obtain the following description of $\pi_0(X_{\leq\mu}(b))$.

Theorem E. *Let (μ, b) be as above and indecomposable with respect to the Hodge-Newton decomposition.*

(i) Either $\kappa_{\boldsymbol{M}}(b) \neq \mu$ for all proper standard parabolic subgroups $\boldsymbol{P}$ of G with $b \in \boldsymbol{M}$ or the σ-conjugacy class $[b]$ is central and equal to $[t^\mu]$.

(ii) In the first case, κ_G induces a bijection between $\pi_0(X_{\leq\mu}(b))$ and $\pi_1(GL_n) \cong \mathbb{Z}$.

(iii) In the second case, $X_\mu(b) = X_{\leq\mu}(b) \cong J/(J \cap K) \cong GL_n(F)/GL_n(\mathcal{O}_F)$.

For the non-closed varieties our calculations seem to support the following conjecture.

Conjecture 2.1.1. *The action of J on $\pi_0(X_\mu(b))$ is transitive.*

We do not have a precise conjecture for $\pi_0(X_\mu(b))$. Theorem E implies that the map $\pi_0(X_\mu(b)) \to \pi_0(X_{\leq\mu}(b))$ induced by the inclusion is surjective. We give an example to show that in general it is not injective.

Question 2: What is the set of irreducible components of $X_\mu(b)$?

In Section 1 we showed that J acts transitively on the set of irreducible components of the moduli spaces $\mathcal{M}$. Guided by this result we arrive at the following conjecture about the set of irreducible components of $X_\mu(b)$.

Conjecture 2.1.2. *The action of J on the set of irreducible components of $X_\mu(b)$ has only finitely many orbits.*

However, we give an example to show that in general the action of J on the set of irreducible components is not transitive for non-minuscule μ.

Question 3: What is the dimension of $X_\mu(b)$?

Affine Deligne-Lusztig varieties $X_\mu(b)$ can also be defined as in (2.1.1) when GL_n is replaced by an unramified connected reductive group G. There is a conjectural formula for the dimension of $X_\mu(b)$ by Rapoport (see [Ra2], Conj. 5.10). For split groups G it takes the form

Conjecture 2.1.3. (Rapoport)

$$\dim X_\mu(b) = \langle 2\rho, \mu - \nu \rangle + \sum_i \left[\langle \omega_i, \nu - \mu \rangle\right]$$

Here ρ is the half-sum of the positive roots and ω_i are the fundamental weights of G_{ad}. By $[x]$ we denote the greatest integer which is less or equal to x. In [GHKR], Görtz, Haines, Kottwitz, and Reuman reduce the proof of the dimension formula to the case that $G = GL_n$ and that the σ-conjugacy class of b is superbasic. Here superbasic means that no σ-conjugate element is contained in a proper Levi subgroup of G. They prove the conjecture for $b \in A(L)$.

Conjecture 2.1.3 leads to the following conjectural description of the set of pairs (μ, b) with $\dim X_\mu(b) = 0$. It is a modification of a conjecture by Rapoport.

Conjecture 2.1.4. (Rapoport) *Let G be split. Assume that $X_\mu(b)$ is nonempty and that (μ, b) is indecomposable with respect to the Hodge-Newton decomposition. Then* $\dim X_\mu(b) > 0$ *unless either $[b]$ is μ-ordinary or the adjoint group G_{ad} is equal to PGL_n and $\mu = (0, \ldots, 0, 1)$ or $\mu = (0, 1, \ldots, 1)$.*

Our methods seem to give lower bounds on $\dim X_\mu(b)$. One instance is given by the second main result of this section, which proves Conjecture 2.1.4 for $G = GL_n$.

Theorem F. *Let $G = GL_n$ and let (μ, b) be indecomposable with respect to the Hodge-Newton decomposition. Then* $\dim X_\mu(b) = 0$ *if and only if μ is of one of the following forms: $\mu = (a, \ldots, a)$, $\mu = (a, \ldots, a, a+1)$ or $\mu = (a-1, a, \ldots, a)$ for some $a \in \mathbb{Z}$.*

Theorems E and F together imply that J acts transitively on $X_\mu(b)$ if $G = GL_n$ and $\dim X_\mu(b) = 0$.

Notation and conventions

Our notation in this section will differ slightly from that of the previous section: Let $N = L^n$ and consider $b\sigma$ as an automorphism of N. Let $N = \bigoplus_{i=1}^{l} N_i$ be a decomposition of $(N, b\sigma)$ into simple summands. Let $p_i : N \to N_i$ be the projection. Let $h_i = \dim_L N_i$ and let m_i/h_i be the slope of $(N_i, (b\sigma)|_{N_i})$. As in Section 1.4.1 we choose a basis $e_{i,0}, \ldots, e_{i,h_i-1}$ of N_i with the following property: For $j \in \mathbb{Z}$ let $e_{i,j} = te_{i,j-h_i}$. Then

$$b\sigma(e_{i,j}) = e_{i,j+m_i}. \tag{2.1.6}$$

Let $M_0 = \langle e_{i,j} \mid j \geq 0 \rangle_{\mathcal{O}_L}$. Then $K = \mathrm{Stab}(M_0)$. Mapping $g \in X_\mu(b)$ to gM_0 defines a bijection between the $\overline{k}$-rational points of $X_\mu(b)$ and the set of lattices $M \subset N$ with $\mathrm{inv}(M, b\sigma(M)) = \mu$. Here $\mathrm{inv}(M, b\sigma(M)) = \mu = (\mu_1, \ldots, \mu_n)$ for some dominant $\mu \in X_*(A)$ if and only if there is an $\mathcal{O}_L$-basis v_i for M such that $t^{\mu_i} v_i$ is an $\mathcal{O}_L$-basis for $b\sigma(M)$.

A vector $v \in N$ has a unique representation

$$v = \sum_{i=1}^{l} \sum_{j \in \mathbb{Z}} a_{i,j} e_{i,j}$$

with $a_{i,j} \in \overline{k}$ and $a_{i,j} = 0$ for j small enough. Let $I(v) \in \{1, \ldots, l\} \times \mathbb{Z}$ be the smallest pair (i, j) (in the lexicographic order) with $a_{i,j} \neq 0$. It is called the first index of v. If $a \in \mathbb{Z}$ we write $I(v) + a$ instead of $I(v) + (0, a)$.

Remark 2.1.5. In [KR], 4 it is shown that there is a unique minimal dominant $\mu_{\min}$ with $\mu_{\min} \geq \nu$. We can inductively define $\mu_{\min}$ as follows:

$$\mu_i = \max\{j \mid \sum_{i' < i} \mu_{i'} + j(l - i) \leq \sum_{l' \leq l} \nu_{l'} \text{ for all } l \geq i + 1\}. \tag{2.1.7}$$

The minimality is then easily verified. If $\nu_i \in \mathbb{Z}$ then $\mu_i = \nu_i$. Let $\nu_{i_0}, \ldots, \nu_{i_1}$ be all entries of ν in the interval $(a, a+1)$ for some integer a. Then $\mu_i \in \{a, a+1\}$ for $i_0 \leq i \leq i_1$, with suitable multiplicities of a and $a+1$ to ensure that $\sum_{i=i_0}^{i_1} \nu_i = \sum_{i=i_0}^{i_1} \mu_i$. With a formula analogous to (2.1.7) one can for each $i_0 \in \{1, \ldots, n-1\}$ find the unique minimal $\mu^{(i_0)}$ with $\mu^{(i_0)} \geq \nu$ and $\sum_{i=1}^{i_0} \mu_i^{(i_0)} < \sum_{i=1}^{i_0} \nu_i$.

We have $M_0 = \bigoplus(M_0 \cap N_i)$. If N_i has slope in $[a, a+1]$, equation (2.1.6) implies that $p^{a+1}(M_0 \cap N_i) \subseteq b\sigma(M_0 \cap N_i) \subseteq p^a(M_0 \cap N_i)$. Hence $\mathrm{inv}(M_0, b\sigma(M_0)) = \mu_{\min}$. As J commutes with $b\sigma$, its image $J/(K \cap J)$ in X is also contained in $X_{\mu_{\min}}(b)$.

The volume of a lattice M is defined as

$$\mathrm{vol}(M) = \lg(M_0/(M \cap M_0)) - \lg(M/(M_0 \cap M)).$$

Let $M \in X_\mu(b)$. As in the preceding chapters we denote by $P(M) \in X_{\mu_{\min}}(b)$ the lattice with maximal volume containing M and of the form $P(M) = gM_0$ for some $g \in J$.

2.2 Connected components of closed affine Deligne-Lusztig varieties

In this section we determine the set of connected components of $X_{\leq\mu}(b)$ for $G = GL_n$.

Theorem 2.2.1. *Let μ and b be as above and indecomposable with respect to the Hodge-Newton decomposition.*

(i) Either $\kappa_{\boldsymbol{M}}(b) \neq \mu$ for all proper standard parabolic subgroups $\boldsymbol{P}$ of G with $b \in \boldsymbol{M}$ or the σ-conjugacy class $[b]$ is central and equal to $[t^\mu]$.

(ii) In the first case, κ_G induces a bijection between $\pi_0(X_{\leq\mu}(b))$ and $\pi_1(GL_n) \cong \mathbb{Z}$.

(iii) In the second case, $X_\mu(b) = X_{\leq\mu}(b) \cong J/(J \cap K) \cong GL_n(F)/GL_n(\mathcal{O}_F)$.

Proof of Theorem 2.2.1 (i) and (iii). To show (i) we assume that $\kappa_{\boldsymbol{M}}(b) = \mu$ for some $\boldsymbol{M}$ and $\boldsymbol{P}$ as above. After possibly enlarging $\boldsymbol{P}$ we may assume that $\boldsymbol{M} = GL_{i_0} \times GL_{n-i_0}$ for some $0 < i_0 < n$. Then $\nu_1 + \cdots + \nu_{i_0} = \mu_1 + \cdots + \mu_{i_0}$. The assumption that (μ, b) is indecomposable implies that $\boldsymbol{M}_b \not\subseteq \boldsymbol{M}$, hence $\nu_{i_0} = \nu_{i_0+1}$. As μ is dominant and $\mu \geq \nu$, we obtain

$$\mu_{i_0+1} \geq \mu_{i_0} \geq \nu_{i_0} = \nu_{i_0+1} \geq \mu_{i_0+1}. \tag{2.2.1}$$

Thus $\nu_{i_0} = \nu_{i_0+1} = \mu_{i_0} = \mu_{i_0+1}$. Repeating this argument with i_0 replaced by $i_0 - 1$ and $i_0 + 1$ we inductively obtain

$$\nu_1 = \cdots = \nu_n = \mu_1 = \cdots = \mu_n.$$

Assertion (iii) is easy. $\square$

The strategy of the proof of the second part of the theorem is as follows: We first show in Proposition 2.2.5 that each connected component contains an element of J and afterwards connect elements of $J \cap \ker(\kappa)$ by one-dimensional subvarieties in $X_{\leq\mu}(b)$. To show Proposition 2.2.5 we start with an arbitrary element M of the affine Deligne-Lusztig variety and define a one-dimensional connected subvariety which connects M to an element which is in a certain sense closer to J. As a preparation we define a basis for lattices $M \in X_\mu(b)$ and consider a special case.

Lemma 2.2.2. *Let $M \subset N$ be a lattice with $M \in X_\mu(b)$. Then there is a basis $v_1, \dots, v_n$ of M and $\tilde{\mu} = (\tilde{\mu}_1, \dots, \tilde{\mu}_n) \in X_*(A)$ with $\tilde{\mu}_{\mathrm{dom}} = \mu$ and the following properties:*

(i) $\langle \{t^{-\tilde{\mu}_i} b\sigma(v_i)\}_{i=1,\dots,n} \rangle_{\mathcal{O}_L} = M$

(ii) $I(v_1) = (1, c)$ *for some* $c \in \mathbb{Z}$

(iii) Let p_1 be the projection to N_1. Then $I(v_i) < I(v_j)$ for all $i < j$ with $p_1(v_i) \neq 0$.

(iv) If $p_1(v_i), p_1(v_j) \neq 0$ for some $i < j$ and $I(v_j) = I(v_i) + h_1 a$ with $a \in \mathbb{N}$, then $\tilde{\mu}_i + a < \tilde{\mu}_j$.

Proof. As $M \in X_\mu(b)$ we may choose a basis v_i that satisfies (i) for $\tilde{\mu} = \mu$. We renumber the v_i such that $I(v_i) \leq I(v_j)$ if $i < j$. If $I(v_i) = I(v_j)$ we may further achieve that for the permuted element $\tilde{\mu}$ we have $\tilde{\mu}_i \geq \tilde{\mu}_j$. If now (iv) is not satisfied for some pair i, j, we modify v_j by adding $ct^a v_i$ for some $c \in \overline{k}$ such that $I(v_j + ct^a v_i) > I(v_j)$. Then the new basis also satisfies (i) and (ii). We permute the elements again to achieve the weak inequality in (iii) and that $\tilde{\mu}_i \geq \tilde{\mu}_j$ for each $i < j$ with $I(v_i) = I(v_j)$ and $p_1(v_i) \neq 0$. As long as (iv) is not satisfied we can continue enlarging the first indices of basis elements in this way. This process has to stop as the v_i generate the lattice M. Thus after finitely many steps we obtain a basis satisfying (i), (ii), (iv), the weak inequality in (iii), and $\tilde{\mu}_i \geq \tilde{\mu}_j$ for each $i < j$ with $I(v_i) = I(v_j)$ and $p_1(v_i) \neq 0$. But then (iii) holds as well. ☐

Lemma 2.2.3. *Let M, v_i and $\tilde{\mu}$ be as in the preceding lemma. Then $\nu_1 \geq \tilde{\mu}_1$. If $\nu_1 = \tilde{\mu}_1$ then $p_1(v_i) = 0$ for all $i > 1$.*

Proof. Let $p_1(v_i) \neq 0$ for some $i \geq 1$. Then

$$I(t^{-\tilde{\mu}_i} b\sigma(v_i)) = I(v_i) + m_1 - \tilde{\mu}_i h_1. \tag{2.2.2}$$

As $I(v) \geq I(v_1)$ for all $v \in M$, this equation for $i = 1$ implies $m_1 - \tilde{\mu}_1 h_1 \geq 0$, hence $\nu_1 \geq \tilde{\mu}_1$. Assume that $\nu_1 = \tilde{\mu}_1$ and that $p_1(v_i) \neq 0$ for some $i > 1$. As

$\nu_1 \in \mathbb{Z}$, we have $h_1 = 1$ and $m_1 = \nu_1$. Then $I(v_i) = I(v_1) + ah_1$ for some $a > 0$. Equation (2.2.2) together with Lemma 2.2.2 (iv) yields

$$\begin{aligned} I(t^{-\tilde{\mu}_i} b\sigma(v_i)) &= I(v_i) + \nu_1 - \tilde{\mu}_i \\ &= I(v_1) + a + \tilde{\mu}_1 - \tilde{\mu}_i \\ &< I(v_1). \end{aligned}$$

This is a contradiction to $t^{-\tilde{\mu}_i} b\sigma(v_i) \in M$. $\square$

Proposition 2.2.4. *Let $\boldsymbol{P} \subset GL_n$ be the parabolic subgroup consisting of the $(g_{i,j}) \in GL_n$ with $g_{1,j} = 0$ for $j > 1$. Let $\boldsymbol{M}$ be the associated Levi subgroup containing A. Assume that the first slope ν_1 of b is an integer. Let $g \in \boldsymbol{P}$ and assume that $g^{-1}b\sigma(g) = k_0 t^{\tilde{\mu}}$ for some $k_0 \in K \cap \boldsymbol{P}$ and $\tilde{\mu} \in X_*(A)$ not necessarily dominant. Let $\mu = \tilde{\mu}_{\mathrm{dom}}$.*

(i) The connected component of g in $X_\mu(b)$ contains some g_1 with $g_1^{-1}b\sigma(g_1) = k_1 t^{\tilde{\mu}}$ and $k_1 \in K \cap \boldsymbol{M}$.

(ii) Let g_1 be as in (i). Then $g_1 = jg'$ with $j \in J$ and $g' \in \boldsymbol{M}$.

Proof. We write all matrices as $(1, n-1) \times (1, n-1)$ block matrices. From $\nu_1 \in \mathbb{Z}$ and (2.1.6) we see that $b\sigma$ is of the form

$$b\sigma(e_{i,j}) = \begin{pmatrix} b_1 & 0 \\ 0 & b' \end{pmatrix} e_{i,j}$$

with $b_1 = t^{\nu_1}$ and $b' \in GL_{n-1}(L)$. Multiplying $g \in \boldsymbol{P}$ by a scalar we may assume that

$$g = \begin{pmatrix} 1 & 0 \\ v_0 & g' \end{pmatrix}$$

with $v_0 \in L^{n-1}$ and $g' \in GL_{n-1}(L)$. We write

$$k_0 = \begin{pmatrix} 1 & 0 \\ y & k_0' \end{pmatrix}$$

with $y \in (t^c\mathcal{O}_L)^{n-1} \setminus (t^{c+1}\mathcal{O}_L)^{n-1}$ for some $c \geq 0$. The assumption implies that $\tilde{\mu} = (\tilde{\mu}_1, \dots, \tilde{\mu}_n)$ with $\tilde{\mu}_1 = \nu_1$.

It is enough to show that there is an integer a only depending on b and g' and an element $g^{(1)}$ in the connected component of g as follows: $g^{(1)}$ is obtained from g by replacing v_0 by $v_0 + x$ where $x \in (t^{a+c}\mathcal{O}_L)^{n-1}$. Besides, $k_1 = (g^{(1)})^{-1}b\sigma(g^{(1)})t^{-\tilde{\mu}} \in K$ satisfies $(k_1)_{i,1} \in t^{c+1}\mathcal{O}_L$ for all $i > 1$. Indeed, assume that such a $g^{(1)}$ is constructed. Then we may inductively construct $g^{(i+1)} = (g^{(i)})^{(1)}$. The existence of a ensures that the sequence $g^{(i)}$ converges in the t-adic topology. Let $g_1 \in GL_n(L)$ be the limit. Then $g_1^{-1}b\sigma(g_1)$ is the limit of the $(g^{(i)})^{-1}b\sigma(g^{(i)})$, hence of the desired form. Especially, $g_1 \in X_\mu(b)$.

Let R be an L-algebra and let $g_w \in GL_n(R)$ be obtained from g by replacing v_0 by $v_0 + w \in R^{n-1}$ for some $w \in R^{n-1}$. Then

$$g_w^{-1} b\sigma(g_w) t^{-\tilde{\mu}} = g^{-1} b\sigma(g) t^{-\tilde{\mu}} + \begin{pmatrix} 0 & 0 \\ (g')^{-1}(b'\sigma(w) - wb_1)t^{-\tilde{\mu}_1} & 0 \end{pmatrix}. \tag{2.2.3}$$

We write

$$g'y = \sum_{i=2}^{l} \sum_{j \geq j_0} a_{i,j} e_{i,j}$$

with $j_0 \in \mathbb{Z}$ and choose $j_1 > j_0$ with

$$(g')^{-1}(\sum_i \sum_{j \geq j_1} a_{i,j} e_{i,j}) \in (t^{c+1}\mathcal{O}_L)^{n-1}. \tag{2.2.4}$$

Let $\beta = (g')^{-1}(\sum_i \sum_{j=j_0}^{j_1-1} a_{i,j} e_{i,j})$. Then $y \in (t^c \mathcal{O}_L)^{n-1}$ and (2.2.4) imply that $\beta \in (t^c \mathcal{O}_L)^{n-1}$.

We want to construct a finite connected covering V of $\mathbb{A}^1_{\overline{k}} = \operatorname{Spec}(\overline{k}[s])$ and a morphism $V \to X_\mu(b)$ such that the points of V correspond to modifications of g by some $w \in (\mathcal{O}_V((t)))^{n-1}$ as described above. More precisely we want that a point over $s \in \mathbb{A}^1(\overline{k}) \cong \overline{k}$ corresponds to a modification of g by some w with

$$(g')^{-1}(b'\sigma(w) - wb_1)t^{-\tilde{\mu}_1} = s\beta \tag{2.2.5}$$

which is equivalent to

$$b'\sigma(w) - wb_1 = s \sum_{i \geq 2} \sum_{j=j_0}^{j_1-1} a_{i,j} e_{i,j} t^{\tilde{\mu}_1}. \tag{2.2.6}$$

Besides we require that g itself corresponds to a point of V over $0 \in \mathbb{A}^1$. Then this implies that g lies in the same connected component as a point of V over $-1 \in \mathbb{A}^1$. By (2.2.3) and (2.2.5) we see that this point corresponds to an element $g^{(1)}$ with

$$(g^{(1)})^{-1} b\sigma(g^{(1)}) = \left(\begin{pmatrix} 1 & 0 \\ y & k_0' \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} \right) t^{\tilde{\mu}}.$$

Thus y is replaced by $y - \beta$. By (2.2.4) this is an element of $(t^{c+1}\mathcal{O}_L)^{n-1}$.

We now have to construct V and w. Let N_i be a simple summand of N with $i > 1$ and $b_i = b|_{N_i}$. If the slope of b_i is $\nu_1 = \tilde{\mu}_1$, then $\dim N_i = 1$. In this case let $R = \overline{k}[s, x_{j_0}, \ldots, x_{j_1-1}]/(x_j^\sigma - x_j - sa_{i,j})$ and V_i the connected component of 0 in $\operatorname{Spec}(R)$. As the canonical projection $\operatorname{Spec}(R) \to \operatorname{Spec}(\overline{k}[s])$ is an étale covering, this induces an étale covering $V_i \to \operatorname{Spec}(\overline{k}[s])$. Over V_i let

$w_i = \sum_{j=j_0}^{j_1-1} x_j e_{i,j} \in N_{i,V_i} \subseteq N_{V_i}$. Then

$$\begin{aligned} b'\sigma(w_i) - w_i b_1 &= (\sigma(w_i) - w_i)t^{\tilde{\mu}_1} \\ &= s \sum_{j=j_0}^{j_1-1} a_{i,j} e_{i,j} t^{\tilde{\mu}_1}. \end{aligned}$$

If the slope of b_i is greater than ν_1, we choose $V_i = \mathbb{A}^1 = \operatorname{Spec}(\overline{k}[s])$. We set $w = \sum_j \delta_j e_{i,j}$ with $\delta_j \in s\overline{k}[s]$ to be determined. We have $b'\sigma(e_{i,j}) = e_{i,j+m_i}$ with $m_i > \nu_1 h_i$ and $b_1 e_{i,j} = t^{\nu_1} e_{i,j} = e_{i,j+\nu_1 h_i}$. The projection of (2.2.6) to N_i reads in this basis

$$\sum_j \left(\delta_j^\sigma e_{i,j+m_i} - \delta_j e_{i,j+\nu_1 h_i}\right) = s \sum_{j=j_0}^{j_1-1} a_{i,j} e_{i,j+\nu_1 h_i}. \tag{2.2.7}$$

This is equivalent to

$$s a_{i,j} = \delta^\sigma_{j-m_i+\nu_1 h_i} - \delta_j \tag{2.2.8}$$

for every j. We can solve these equations by induction on j: If $j < j_0$, we choose $\delta_j = 0$. In the equation for $a_{i,j}$ we assume that we already know $\delta_{j-m_i+\nu_1 h_i} \in s\overline{k}[s]$, the coefficient with the smaller index. Then we can choose $\delta_j \in s\overline{k}[s]$ such that (2.2.8) is satisfied.

Let V be the connected component of 0 in the fibre product over $\operatorname{Spec}(\overline{k}[s])$ of the V_i associated to the simple summands. This V and $w = \sum_i w_i$ are as desired.

For (ii) we may choose $j(e_{1,0}) = g(e_{1,0})$ and $j|_{N_i} = \mathrm{id}$ for $i > 1$. $\square$

Proposition 2.2.5. *Each connected component of $X_{\leq\mu}(b)$ contains an element of J.*

Proof. Recall that $J/(K \cap J) \subseteq X_{\mu_{\min}}(b)$. Using induction on μ and n we may assume that the proposition is shown for all components containing an element of $X_{\mu'}(b)$ for some $\mu' < \mu$ or a lattice M with $M = p_1(M) \oplus (M \cap \bigoplus_{i\neq 1} N_i)$ and $p_1(M) = g(M_0 \cap N_1)$ for some $g \in J_{N_1}$.

Let $M \in X_\mu(b)$. Let $v_1, \ldots, v_n$ and $\tilde{\mu}$ be as in Lemma 2.2.2. If $\nu_1 = \tilde{\mu}_1$, we showed in Proposition 2.2.4 that the connected component of M contains an element of the form jg' with $j \in J$ and g' and b in the Levi subgroup $\boldsymbol{M} \cong GL(N_1) \times GL(\bigoplus_{i>1} N_i)$. Thus in this case the proposition follows from the induction hypothesis. From now on assume $\nu_1 > \tilde{\mu}_1$.

Let $c \in \mathbb{Z}$ with $I(v_1) = (1, c)$. Let d be maximal with $e_{1,d} \notin M$. Then $d - c \geq -1$. If $d - c = -1$ then $p_1(M) = \langle e_{1,j} \mid j \geq c\rangle_{\mathcal{O}_L}$ is a direct summand of M, a case that is covered by the induction hypothesis. Hence we may assume that $d - c \geq 0$.

For $s = [s_0 : s_1] \in \mathbb{P}^1_{\overline{k}}$ let $v_{1,s} = s_0 v_1 + s_1 e_{1,d}$. Let

$$M' = \langle \{v_i \mid i > 1\} \cup \{tv_1\}\rangle_{\mathcal{O}_L} \tag{2.2.9}$$

and

$$M_s = \langle M', v_{1,s} \rangle_{\mathcal{O}_L}. \qquad (2.2.10)$$

Note that the minimality of $I(v_1)$ implies that an element $x \in M$ with $I(x) > I(v_1)$ is contained in M'. As $v_{1,s} \in t^{-1}M' \setminus M'$ for all s, this defines a morphism $\mathbb{P}^1_{\overline{k}} \to X$. We want to show that its image lies in $X_{\leq\mu}(b)$. To do this, we show that for almost all $\overline{k}$-valued points s of $\mathbb{P}^1$, there are generators $v_{i,s} \in M_s$ with

$$\langle t^{-\tilde{\mu}_i} b\sigma(v_{i,s}) \mid i = 1, \ldots, n \rangle_{\mathcal{O}_L} = M_s. \qquad (2.2.11)$$

As $\sum \tilde{\mu}_i = v_t(\det(b))$, it is enough to show that the left hand side is contained in M_s. We have

$$t^{-\tilde{\mu}_1} b\sigma(v_{1,s}) = s_0^\sigma t^{-\tilde{\mu}_1} b\sigma(v_1) + s_1^\sigma t^{-\tilde{\mu}_1} b\sigma(e_{1,d}) \qquad (2.2.12)$$

and $t^{-\tilde{\mu}_1} b\sigma(v_1) \in M$. As $\nu_1 > \tilde{\mu}_1$, the first index of $t^{-\tilde{\mu}_1} b\sigma(v_1)$ is greater than that of v_1. Thus $t^{-\tilde{\mu}_1} b\sigma(v_1) \in M'$. On the other hand we have $t^{-\tilde{\mu}_1} b\sigma(e_{1,d}) = e_{1,d+(\nu_1-\tilde{\mu}_1)h_1}$. The maximality of d implies that this element lies in M. As $(1,d) \geq I(v_1)$, this also shows that $t^{-\tilde{\mu}_1} b\sigma(e_{1,d}) \in M'$. Thus $t^{-\tilde{\mu}_1} b\sigma(v_{1,s}) \in M_s$. Let now $i > 1$. For all v_i with $\tilde{\mu}_i \leq \nu_1$ and $p_1(v_i) \neq 0$, the first index $I(t^{-\tilde{\mu}_i} b\sigma(v_i)) = I(v_i) + h_1(\nu_1 - \tilde{\mu}_i)$ is not smaller than $I(v_i)$. Hence for all $i > 1$ with $\tilde{\mu}_i \leq \nu_1$ or $p_1(v_i) = 0$ we have $I(t^{-\tilde{\mu}_i} b\sigma(v_i)) > I(v_1)$ which implies $t^{-\tilde{\mu}_i} b\sigma(v_i) \in M' \subset M_s$. In these cases we may choose $v_{i,s} = v_i$. Now let $\tilde{\mu}_i > \nu_1$. We write $t^{-\tilde{\mu}_i} b\sigma(v_i) = \varepsilon_i + \lambda_i v_1$ with $\varepsilon_i \in M'$ and $\lambda_i \in \overline{k}$. We assume $s_0 \neq 0$ and choose

$$v_{i,s} = v_i + \left(\frac{\lambda_i s_1}{s_0}\right)^{\sigma^{-1}} (b\sigma)^{-1} t^{\tilde{\mu}_i}(e_{1,d}). \qquad (2.2.13)$$

We have $(b\sigma)^{-1} t^{\tilde{\mu}_i}(e_{1,d}) = e_{1,d+(\tilde{\mu}_i-\nu_1)h_1}$. As $d + (\tilde{\mu}_i - \nu_1)h_1 > d$, this is an element of M', thus $v_{i,s} \in M_s$. Besides,

$$\begin{aligned} t^{-\tilde{\mu}_i} b\sigma(v_{i,s}) &= t^{-\tilde{\mu}_i} b\sigma \left(v_i + \left(\frac{\lambda_i s_1}{s_0}\right)^{\sigma^{-1}} (b\sigma)^{-1} t^{\tilde{\mu}_i}(e_{1,d}) \right) \\ &= \varepsilon_i + \frac{\lambda_i}{s_0}(s_0 v_1 + s_1 e_{1,d}) \in M_s. \end{aligned}$$

For $s = [1:0]$, the $v_{i,s} = v_i$ generate the lattice $M_s = M$. Thus for s in an open dense subvariety of $\mathbb{P}^1$, the $v_{i,s} \in M_s$ generate M_s. This implies $\mathrm{inv}(M_s, b\sigma(M_s)) = \mu$ for those s. As $X_{\leq\mu}(b)$ is closed in X, the family of lattices induces a morphism $\mathbb{P}^1 \to X_{\leq\mu}(b)$. For $s = [1:0]$, the $v_{i,s}$ generate $M_s = M$. Let $\widetilde{M} = M_{[0:1]}$. Then M and $\widetilde{M}$ are in the same connected component of $X_{\leq\mu}(b)$.

For the new lattice $\widetilde{M}$ we can similarly define $(1, c(\widetilde{M}))$ to be the minimal first index of an element of $\widetilde{M}$, and $d(\widetilde{M})$ to be maximal with $e_{1,d(\widetilde{M})} \notin \widetilde{M}$. Then $d - c > d(\widetilde{M}) - c(\widetilde{M})$. Thus after finitely many such replacements of the lattice within its connected component, we arrive at a lattice with $d - c < 0$. As seen above, the proposition then follows from the induction hypothesis. $\square$

Proof of Theorem 2.2.1 (ii). By Proposition 2.2.5, the inclusion induces a surjective morphism $J \to \pi_0(X_{\leq\mu}(b))$. We have to show that $\ker(\kappa_G) \cap J$ is in the kernel of this surjection. For $i_0 \in \{1, \ldots, l\}$ let $x_{i_0} \in J$ with

$$x_{i_0}(e_{i,j}) = \begin{cases} e_{i,j+1} & \text{if } i = i_0 \\ e_{i,j} & \text{else.} \end{cases} \tag{2.2.14}$$

Then $\ker(\kappa_G) \cap J$ is generated by $J \cap K$ and the $x_{i_0+1} \circ x_{i_0}^{-1}$ for $1 \leq i_0 < l$. As the surjection factors through $J/(J \cap K)$, it remains to show that $x_{i_0+1} \circ x_{i_0}^{-1}$ is also mapped to the connected component of the identity. To simplify the notation we assume that $l = 2$. We consider the following family over $\mathbb{P}^1_{\bar{k}}$: For $s = [s_0 : s_1]$ let

$$M_s = \langle M_0, s_0 e_{1,-1} + s_1 e_{2,-1} \rangle_{\mathcal{O}_L}. \tag{2.2.15}$$

Then $M_{[1:0]}$ is in the connected component of x_1^{-1} and $M_{[0:1]}$ is in the connected component of x_2^{-1}. It is enough to show that $M_s \in X_{\leq\mu}(b)$ for all $s \notin \mathbb{P}^1(\mathbb{F}_q)$. A basis v_i for M_0 as in Lemma 2.2.2 is given by $e_{1,0}, \ldots, e_{1,h_1-1}, e_{2,0}, \ldots, e_{2,h_2-1}$. Let $\tilde{m}_i = h_i(\nu_i - \lfloor \nu_i \rfloor)$. Then the corresponding $\tilde{\mu}$ consists of $h_1 - \tilde{m}_1$ entries $\lfloor \nu_1 \rfloor$ and $\tilde{m}_1$ entries $\lfloor \nu_1 \rfloor + 1$ followed by $h_2 - \tilde{m}_2$ entries $\lfloor \nu_2 \rfloor$ and $\tilde{m}_2$ entries $\lfloor \nu_2 \rfloor + 1$. We want to construct a basis $v_{i,s}$ of M_s with

$$\langle \{ t^{-\tilde{\mu}_{s,i}} b\sigma(v_{i,s}) \} \rangle_{\mathcal{O}_L} = M_s. \tag{2.2.16}$$

We replace e_{1,h_1-1} by $s_0 e_{1,-1} + s_1 e_{2,-1}$. Before the replacement, the corresponding entry of $\tilde{\mu}$ was ν_1, if ν_1 is an integer and $\lfloor \nu_1 \rfloor + 1$ otherwise. It is replaced by $\tilde{\mu}_{s,h_1} = \tilde{\mu}_{h_1} - 1 = \lceil \nu_1 \rceil - 1$. Then

$$\begin{aligned} t^{-\tilde{\mu}_{s,h_1}} b\sigma(s_0 e_{1,-1} + s_1 e_{2,-1}) &= s_0^{\sigma} e_{1,m_1-(\lceil \nu_1 \rceil -1)h_1-1} + s_1^{\sigma} e_{2,m_2-(\lceil \nu_1 \rceil -1)h_2-1} \\ &\in M_0 \setminus tM_s. \end{aligned}$$

Let $\iota = (\lfloor \nu_2 \rfloor + 1)h_2 - m_2 = h_2 - \tilde{m}_2$. Then we also replace $e_{2,\iota-1}$ by

$$v = s_0^{\sigma^{-1}} e_{1,(\lfloor \nu_2 \rfloor+1)h_1-m_1-1} + s_1^{\sigma^{-1}} e_{2,\iota-1}$$

and let $\tilde{\mu}_{s,h_1+\iota} = \tilde{\mu}_{h_1+\iota} + 1 = \lfloor \nu_2 \rfloor + 1$. Then

$$b\sigma(v) = t^{\tilde{\mu}_{s,h_1+\iota}}(s_0 e_{1,-1} + s_1 e_{2,-1}).$$

For all other i we have $v_i \in M_s \setminus tM_s$ and $t^{-\tilde{\mu}_i} b\sigma(v_i) \in M_s \setminus tM_s$. We thus defined a basis for M_s as in (2.2.16). The corresponding $\tilde{\mu}_s$ is obtained from $\tilde{\mu}$ by replacing an element $\lceil \nu_1 \rceil$ by $\lceil \nu_1 \rceil - 1$ and an element $\lfloor \nu_2 \rfloor$ by $\lfloor \nu_2 \rfloor + 1$. Thus $\tilde{\mu}_{s,\mathrm{dom}}$ is the minimal element of $X_*(A)$ with $\tilde{\mu}_{s,\mathrm{dom}} \geq \nu$ and $\sum_{i=1}^{h_1} (\tilde{\mu}_{s,\mathrm{dom}})_i < \sum_{i=1}^{h_1} \nu_i$, compare Remark 2.1.5. This shows $M_s \in X_{\leq \tilde{\mu}_s}(b) \subseteq X_{\leq\mu}(b)$.

If $l > 2$, the assertion follows similarly by using a sum of the lattices defined above and a constant family of lattices in the other summands of N. $\square$

2.3 Examples

In this section we discuss two explicit examples. In the first example we describe a non-closed affine Deligne-Lusztig variety and show that in that case $\pi_0(X_\mu(b)) \neq \pi_0(X_{\leq\mu}(b))$. In the second example we show that the action of J on the set of irreducible components is not transitive.

2.3.1. Let $N = L^5$ with basis $e_{1,0}$, $e_{1,1}$, $e_{2,0}$, $e_{2,1}$, and $e_{2,2}$. For $i \in \mathbb{Z}$ let $e_{1,i} = te_{1,i-2}$ and $e_{2,i} = te_{2,i-3}$. Let $b \in G(L)$ with $b\sigma(e_{i,j}) = e_{i,j+1}$. Let further $\mu = (0,0,0,0,2)$. Then Theorem 2.2.1 implies $\pi_0(X_{\leq\mu}(b)) \cong \mathbb{Z}$. We prove in this section that

$$\pi_0(X_\mu(b)) \cong \mathbb{Z}^2 \cong J/(K \cap J). \tag{2.3.1}$$

More precisely, we define a morphism $\coprod_{\mathbb{Z}^2} \mathbb{A}^2 \to X_\mu(b)$ which is a bijection on $\overline{k}$-valued points.

Let $M \subset N$ be the lattice corresponding to a point of $X_\mu(b)$ and assume that $P(M) = M_0$. As there is only one $\mu_i \neq 0$, we have $M/(b\sigma(M) + tM) \cong \overline{k}$. Each $v \in M \setminus (b\sigma(M) + tM)$ generates M as a $b\sigma$-invariant $\mathcal{O}_L$-module. As $P(M) = M_0$, we have that $v = \sum_{i\in\{1,2\}, j\geq 0} \alpha_{i,j} e_{i,j}$ with $\alpha_{i,j} \in \overline{k}$ and $\alpha_{i,0} \neq 0$. By multiplying with $\alpha_{1,0}^{-1}$ we may assume that $\alpha_{1,0} = 1$. We have $(b\sigma)^l v = e_{1,l} + \sum_{j>0} \alpha_{1,j}^{\sigma^l} e_{1,j+l} + \sum_{j\geq 0} \alpha_{2,j}^{\sigma^l} e_{2,j+l}$, hence we can modify v by a linear combination of these elements for $l > 0$ to obtain an element of $M \setminus (b\sigma(M) + tM)$ of the form $e_{1,0} + \sum_{j\geq 0} \alpha_{2,j} e_{2,j}$. We assume that v is already of this form. Then

$$\begin{aligned} w &= (b\sigma)^2(v) - tv \\ &= \alpha_{2,0}^{\sigma^2} e_{2,2} + (\alpha_{2,1}^{\sigma^2} - \alpha_{2,0}) e_{2,3} + \cdots \in b\sigma(M) + tM. \end{aligned}$$

By dividing w by $\alpha_{2,0}^{\sigma^2}$ and after subtracting a suitable linear combination of the $(b\sigma)^l w$, we obtain that $e_{2,2} \in b\sigma(M) + tM$. Hence also $e_{2,j} \in b\sigma(M) + tM$ for all $j \geq 2$. We can now modify v by a suitable linear combination of these vectors to obtain an element of the form

$$v' = e_{1,0} + a_0 e_{2,0} + a_1 e_{2,1} \in M \setminus (b\sigma(M) + tM)$$

for some $a_0, a_1 \in \overline{k}$. This implies

$$(b\sigma)^2(v') - a_0^{\sigma^2} e_{2,2} - a_1^{\sigma^2} e_{2,3} = e_{1,2} \in M.$$

Finally we obtain

$$M = \langle e_{1,0} + a_0 e_{2,0} + a_1 e_{2,1}, e_{1,1} + a_0^\sigma e_{2,1}, e_{i,j} \mid j \geq 2 \rangle_{\mathcal{O}_L} \tag{2.3.2}$$

for some $a_0, a_1 \in \overline{k}$. We can now define a morphism $\mathbb{A}^2 \to X_\mu(b)$ by mapping (a_0, a_1) to the lattice in (2.3.2). We choose $\{x_1^a \circ x_2^{a'} \mid (a, a') \in \mathbb{Z}^2\}$ as a system of representatives of $J/(K \cap J) \cong \mathbb{Z}^2$ in J (see (2.2.14) for the definition of the x_i). As $X_\mu(b)$ is invariant under J, this yields a morphism

$$\coprod_{\mathbb{Z}^2} \mathbb{A}^2 \to X_\mu(b).$$

The lattices in the image of the summand (a, a') satisfy

$$P(M) = \langle e_{1,i}, e_{2,j} \mid i \geq a, j \geq a' \rangle_{\mathcal{O}_L}.$$

The morphism defines a bijection of geometric points which implies

$$\pi_0(X_\mu(b)) \cong J/(K \cap J) \cong \mathbb{Z}^2.$$

2.3.2. Let $N = L^3$ with basis e_1, e_2, e_3. The lattice generated by these basis elements is again denoted by M_0. Let $b = t \cdot \mathrm{id}$ and $\mu = (0, 1, 2)$. From the results of [GHKR] it follows that $X_\mu(b)$ is equidimensional with $\dim X_\mu(b) = 2$. We now construct two subvarieties of $X_\mu(b)$ of dimension 2 whose J-orbits are disjoint.

Let $U \subseteq \mathrm{Spec}(\overline{k}[a_2, a_3])$ be the open subvariety where $1, a_2$, and a_3 are linearly independent over $\mathbb{F}_q$. We define a morphism $\varphi_1 : U \to X_\mu(b)$ by describing the corresponding family of lattices in $N \otimes_{\overline{k}} \overline{k}[a_2, a_3]$. Let

$$v = e_1 + a_2^\sigma e_2 + a_3^\sigma e_3$$

and let M_1 be generated by $t^{-1}v, e_2$, and e_3. A different basis of M_1 is given by $v_1 = t^{-1}v$, $v_2 = e_1$ and $v_3 = \sigma^{-1}(v)$. One easily sees that $t^{-\mu_i} b\sigma(v_i) \in M_1$. As $v_t(\det(b)) = \sum \mu_i$, this implies that the $t^{-\mu_i} b\sigma(v_i)$ generate M_1. Hence $\mathrm{inv}(M_1, b\sigma(M_1)) = \mu$. In every point x of U we have $P(M_{1,x}) = t^{-1}M_0$ and $\mathrm{vol}(M_{1,x}) = -1$. For every $g \in J$ one has $gP(M_{1,x}) = P(gM_{1,x})$. Thus

$$\mathrm{vol}(gM_{1,x})) - \mathrm{vol}(P(gM_{1,x}))) = 2.$$

We now define a second morphism $\varphi_2 : U \to X_\mu(b)$: Let

$$w = e_1 + a_2 e_2 + a_3 e_3 \in N \otimes_{\overline{k}} \overline{k}[a_2, a_3]$$

and let M_2 be generated by $t^{-1}\sigma(w) = w_1, t^{-1}w = w_2$, and $\sigma^{-1}(w) = w_3$. Then $M_0 \subseteq M_2$. One easily sees that $t^{-\mu_i} b\sigma(w_i) \in M_2$. Hence $\mathrm{inv}(M_2, b\sigma(M_2)) = \mu$. In every point x of U we have $P(M_{2,x}) = t^{-1}M_0$. Thus

$$\mathrm{vol}(gM_{2,x})) - \mathrm{vol}(P(gM_{2,x}))) = 1$$

for every $g \in J$. Especially, the J-orbits of the images of φ_1 and φ_2 are disjoint. As both morphisms are injective on geometric points, their images are two-dimensional. As their dimensions are equal to the dimension of $X_\mu(b)$, this shows that the action of J on the set of irreducible components is not transitive.

2.4 Zero-dimensional affine Deligne-Lusztig varieties

Theorem 2.4.1. *If $X_\mu(b)$ has an isolated point, then its dimension is* 0. *This holds if and only if the following conditions are satisfied:*

(i) $\mu = \mu_{\min}$

(ii) Each slope of b is of the form $a + \frac{m}{m+n}$ with $a \in \mathbb{Z}$, $n > 0$, and $\min\{m, n\} \in \{0, 1\}$.

(iii) For each $a \in \mathbb{Z}$, there exists at most one simple summand of $(N, b\sigma)$ with slope in $(a, a+1)$.

Proof. Let $\mu \neq \mu_{\min}$ and $x \in X_\mu(b)$. Hence $x \notin J/(K \cap J)$. In the proof of Proposition 2.2.5 we constructed a one-dimensional connected subvariety of $X_{\leq\mu}(b)$ containing x and an element of $J/(K \cap J) \subseteq X_{\mu_{\min}}(b)$. As $X_\mu(b)$ is open in $X_{\leq\mu}(b)$, we obtain a one-dimensional subvariety of $X_\mu(b)$ containing x. This implies that $X_\mu(b)$ has no isolated points.

From now on let $\mu = \mu_{\min}$. In Proposition 2.2.5 we saw that the action of J is transitive on the set of connected components of $X_\mu(b) = X_{\leq\mu}(b)$. Thus the existence of an isolated point is equivalent to $\dim X_\mu(b) = 0$. The minimality of μ (compare Remark 2.1.5) implies that the Hodge polygon contains all lattice points (i_0, y_{i_0}) on the Newton polygon where $\nu_{i_0} \leq a$ and $\nu_{i_0+1} \geq a$ for some integer a. We may assume that (μ, b) is indecomposable with respect to the Hodge-Newton decomposition. As $\mu = \mu_{\min}$, this implies that all slopes of b are either contained in an interval $(a, a+1)$ or are equal to a. By (2.1.3), we may further assume that $a = 0$. If all slopes are equal to 0, then b is σ-conjugate to 1, $J = G(F)$, and the condition on $g \in X_\mu(b)$ reads $g^{-1}\sigma(g) \in K$. Hence $X_\mu(b) \cong J/(K \cap J)$ is zero-dimensional. Otherwise $\mu = (0, \ldots, 0, 1, \ldots, 1)$ and a lattice M is in $X_\mu(b)$ if and only if $pM \subseteq b\sigma(M) \subseteq M$.

Assume that N has two simple summands with slope in $(0, 1)$. Then the one-dimensional family of (2.2.15) is contained in $X_\mu(b)$. (In this case $\lceil \nu_1 \rceil - 1 = \lfloor \nu_2 \rfloor$, thus μ is not changed.)

Now assume that $(N, b\sigma)$ is simple of slope $\frac{m}{m+n}$ with $(m, n) = 1$ and $m, n \geq 2$. We consider the family of lattices M_s with $s \in \mathbb{A}^1$ where M_s is generated by $e_{1,0} + se_{1,1}, e_{1,2}, e_{1,3}, \ldots$. As $b\sigma(e_{1,i}) = e_{1,i+m}$ and $p(b\sigma)^{-1}(e_{1,i}) = e_{1,i+n}$ with $m, n \geq 2$, one easily sees that $M_s \in X_\mu(b)$. Thus the dimension of $X_\mu(b)$ is at least 1.

Finally, we consider the case of a single slope $1/h$. The case of slope $(h-1)/h$ is similar and thus omitted. Let M be the lattice corresponding to a point of $X_\mu(b)$. By multiplying with a suitable power of $x_1 \in J$ (compare (2.2.14)), we may assume that $M \subseteq M_0$ and that there is an element $v \in M$ of the form $v = e_{1,0} + \sum_{j>0} a_j e_{1,j}$ with $a_j \in \overline{k}$. Then $(b\sigma)^l(v) = e_{1,l} + \sum_{j>0} a_j^{\sigma^l} e_{1,j+l} \in M$ for each $l \geq 0$. As $e_{1,0}$ is a linear combination of these elements, we have $e_{1,0} \in M$. Thus

$e_{1,l} = (b\sigma)^l(e_{1,0}) \in M$ and $M = M_0$. Hence in this case $X_\mu(b) \cong J/(K \cap J) \cong \mathbb{Z}$ is zero-dimensional. $\square$

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