Anomaly-Free Discrete Gauge Symmetries in Froggatt-Nielsen Models

Dissertation
zur
Erlangung des Doktorgrades (Dr. rer. nat.)
der
Mathematisch-Naturwissenschaftlichen Fakultät
der
Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von

Christoph Luhn

aus
Erfurt

Bonn 2006

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn.

Diese Dissertation ist auf dem Hochschulschriftenserver der ULB Bonn *http://hss.ulb.uni-bonn.de/diss_online* im Jahre 2006 elektronisch publiziert.

1. Gutachter:	Prof. Herbert K. Dreiner, Ph.D.
2. Gutachter:	Prof. Dr. Hans-Peter Nilles

Tag der Promotion: 23.06.2006

To my parents

Abstract

The supersymmetric extension of the Standard Model allows for baryon- and lepton-number violating operators in the Lagrangian. The experimentally observed longevity of the proton requires some of the corresponding interactions to be either immensely suppressed or absent. Imposing a discrete symmetry like *e.g.* R parity can forbid these dangerous operators. Due to the violation of global discrete symmetries by quantum gravity effects, the introduced discrete symmetry should be "gauged". Such a discrete *gauge* symmetry (DGS) is a discrete remnant of a spontaneously broken local gauge symmetry.

In this thesis we focus on Abelian DGSs, $\boldsymbol{Z}_N$, arising from a high-energy $U(1)_X$. We extend the work of Ibáñez and Ross with $N = 2, 3$ to arbitrary values of N by systematically investigating discrete $\boldsymbol{Z}_N$ symmetry extensions of G_{SM} in Part I. Demanding anomaly freedom of the high-energy gauge theory without invoking the existence of new light particles, we first determine all family-independent "anomaly-free $\boldsymbol{Z}_N$ symmetries" which are consistent with the trilinear MSSM superpotential terms. From the low-energy point of view, where heavy and possibly $\boldsymbol{Z}_N$ charged particles do not play a rôle, the infinite number of anomaly-free DGSs can be reduced to an equivalent finite set of DGSs, which we denote as fundamental. We are left with four $\boldsymbol{Z}_6$, nine $\boldsymbol{Z}_9$, and nine $\boldsymbol{Z}_{18}$ new symmetries, beyond the five $\boldsymbol{Z}_{2,3}$ symmetries of Ibáñez and Ross. Together these twenty-seven fundamental DGSs comprise a complete set.

Next, we investigate their effect on the baryon- and lepton-number violating operators. There is only one DGS which simultaneously allows the H_dH_u term and prohibits all dimension-three, dimension-four, and dimension-five baryon- and lepton-number violating operators, except for the dimension-five Majorana neutrino mass terms LH_uLH_u. We denote this outstanding $\boldsymbol{Z}_6$ symmetry as proton hexality, $\boldsymbol{P}_6$. *This we propose as the DGS of the MSSM, instead of R parity.*

In Part II, we combine the idea that a discrete symmetry should have a gauge origin with the scenario of Froggatt and Nielsen (FN). They introduce an extra $U(1)_X$ gauge symmetry for the sake of explaining the structure of the observed fermionic masses and mixings in terms of charges. In our work, we identify this flavor $U(1)_X$ with the high-energy gauge symmetry which breaks down to the phenomenologically required DGS. Guided by the principle of minimality and compactness, we introduce only two mass scales, the gravitational scale M_{grav} and the soft supersymmetry breaking scale m_{soft}. Therefore, the breaking of $U(1)_X$ must occur dynamically. Within a string-embedded framework, this is achieved by the Dine-Seiberg-Wen-Witten mechanism slightly below M_{grav}; the FN expansion parameter, ϵ, is then obtained naturally around the Wolfenstein parameter, λ_c.

Contrary to the situation assumed in Part I, this mechanism requires an anomalous $U(1)_X$. However, the effect of nonvanishing mixed anomaly coefficients can be compensated by the Green-Schwarz (GS) mechanism: Requiring the GS anomaly cancellation conditions, the theory is rendered mathematically consistent.

Within this framework, we construct concise $U(1)_X$ FN models in which the $\boldsymbol{Z}_3$ symmetry baryon triality, $\boldsymbol{B}_3$, arises from $U(1)_X$ breaking. We choose this specific DGS because it allows for R-parity violating interactions; thus neutrino masses and mixings can be explained without introducing right-handed neutrinos.

Demanding compatibility with the results of the atmospheric, solar, and reactor neutrino experiments, we find six phenomenologically viable $\boldsymbol{B}_3$-conserving FN models. Due to our ignorance about the details of the soft supersymmetry breaking parameters, we cannot distinguish between models with normal and inverse neutrino mass hierarchy. However, taking the smallness of the $(1,3)$-entry of the MNS matrix as a crucial criterion, we should prefer three models and an inverse hierarchy. In that sense, our investigation predicts inverse-hierarchical neutrino masses.

Acknowledgments

First and foremost, I am deeply indebted to my supervisor *Herbi Dreiner* for the support of my research and his encouragement whenever doubts lurked in my mind. His unaffected nature generates a very pleasant working atmosphere.
Furthermore, I would like to thank *Hans-Peter Nilles* for his willingness to be the second examiner for this thesis.
A special vote of thanks goes to *Marc Thormeier*. His idea of combining discrete gauge symmetries with Froggatt-Nielsen models paved the way for this work. Collaborating with him has always been a pleasure to me.
I am also grateful to the members of the theory department. In particular, I enjoyed helpful and stimulating discussions with *Manuel Drees*, *Michael Flohr*, *Rainald Flume*, *Mark Hillenbach*, *Jong-Soo Kim*, *Alexander Klitz*, *Hans-Peter Nilles*, *Patrick Vaudrevange*, and *Akın Wingerter*.
I also thank the – by now – numerous *members of the Dreiner group*, as well as my *room mates* for the inspiring and good atmosphere.
Without the staff, notably our secretaries *Dagmar Faßbender* and *Patricia Zündorf*, as well as our "operating manager" *Andreas Wißkirchen*, nothing would work in the department. Thanks for taking care of us and (most of) our problems.
I would like to express my deep gratitude to my parents *Eva* and *Heinrich*, as well as my sister *Roberta*, and my brother *Matthias* for supporting and encouraging me to follow my dreams.
I also thank the people and friends from the *Newmanhaus*. You have made my time in Bonn so unforgettable.
Last but not least, I thank the dearest of all, *Stephanie*, for loving me so much. Being with you opens my eyes to the true meaning of life.

Contents

Part I

Anomaly-Free Discrete Gauge Symmetries

Chapter 1

Need for Discrete Symmetries in Supersymmetry

The supersymmetric extension of the Standard Model allows for baryon- and lepton-number violating interactions, which, depending on their order of magnitude, are in conflict with the experimental observations. Imposing a discrete symmetry can forbid such dangerous operators in the Lagrangian.

1.1 The Supersymmetric Standard Model

The action of the Standard Model (SM) of particle physics [1, 2] is invariant under Poincaré transformations, as well as the gauge group $G_{\rm SM} = SU(3)_C \times SU(2)_W \times U(1)_Y$. Using path integral methods, 't Hooft [3, 4] could prove that gauge theories, to all orders of perturbation theory, lead to only a finite number of infinities. As all of these infinities can be absorbed into a redefinition of the parameters [5, 6], any gauge theory is renormalizable. In principle, one can therefore consider the SM the complete and final theory. Then, however, one must not add nonrenormalizable interactions, *i.e.* operators with mass dimension higher than four, to the Lagrangian of the theory. With only renormalizable interactions being allowed, baryon and lepton number are accidental global symmetries of the SM. When taking into account the sphaleron interactions [7], only $\frac{1}{3}B - L_i$, and $L_i - L_j$ are conserved in the SM. For the effect of sphaleron interactions in supersymmetry see for example Refs. [8, 9, 10].

Trying to grasp the full picture, one might step back and ask the question: Why should we formulate the final theory of nature in terms of Lagrangians, *i.e.* in the language of quantum fields? Indeed, many physicists today believe that some sort of string theory will eventually arise as the fundamental description of everything. However, provided that relativity, quantum mechanics and the cluster decomposition principle [11] are valid ingredients of the theory at sufficiently low energies, it is very likely that the low-energy approximation of such a theory will look like quantum field theory. In this case, we are not allowed to make any assumptions of simplicity about the low-energy *effective* Lagrangian. In particular the requirement of renormalizability is impermissible, see *e.g.* Ref. [12]. When considering the SM as a low-energy effective theory, G_{SM} allows for nonrenormalizable interactions which can violate baryon number. Such effective operators necessarily include at least four[1] Dirac spinors, *e.g.* three quarks and one lepton. For dimensional reason, the leading dimension-six operators are suppressed by two powers of an unknown mass scale M. These terms are unproblematic for proton decay (which gives the most stringent bound on baryon-number violation) if $M \gtrsim 10^{16}\,\mathrm{GeV}$.

Enlarging the Poincaré group, the action of the Supersymmetric SM (SSM) is invariant under supersymmetry, as well as G_{SM} [14, 15]. Within this class of theories, the situation concerning baryon- and lepton-number violation looks quite different. This is due to the fact that in supersymmetry there exist scalar superpartners to all fermions. Assigning identical baryon and lepton numbers to these pairs, we can construct baryon- and/or lepton-number violating interaction terms in the Lagrangian with mass dimension smaller than six. Explicitly we find that the *renormalizable* superpotential of the SSM is given by [16, 17, 18, 19]

$$\begin{aligned} W &= h^E_{ij}\, L_i H_d \bar{E}_j + h^D_{ij}\, Q_i H_d \bar{D}_j + h^U_{ij}\, Q_i H_u \bar{U}_j + \mu\, H_d H_u \\ &+ \frac{1}{2}\lambda_{ijk}\, L_i L_j \bar{E}_k + \lambda'_{ijk}\, L_i Q_j \bar{D}_k + \frac{1}{2}\lambda''_{ijk}\, \bar{U}_i \bar{D}_j \bar{D}_k + \mu_i\, L_i H_u\,. \end{aligned} \tag{1.1}$$

Here we employ the notation of Ref. [20], and $SU(3)_C$ and $SU(2)_W$ indices are suppressed. The fifth, sixth and eighth terms violate lepton number, and the seventh term violates baryon number.

Due to the unification of the G_{SM} gauge coupling constants in supersymmetry [21, 22, 23, 24], and also the automatic inclusion of gravity in local supersymmetry [25, 26], we expect the SSM to be a low-energy effective theory, embedded in a more complete theory formulated at the scale of Grand Unified Theories ($M_{\mathrm{GUT}} \sim 10^{16}\,\mathrm{GeV}$) [27], or above. Within the SSM, we must therefore take into account the possible nonrenormalizable operators which are consistent with G_{SM}. In particular, we are here interested in the dimension-five baryon- and/or

---

[1] Only an *even* number of Dirac spinors can combine to a Lorentz scalar. However, due to $SU(3)_C$ invariance, interactions with only two spinors do not lead to baryon-number violation. Thus the lowest-dimensional baryon-number violating operators comprise at least four spinors. Lepton number can be violated in the SM already by the effective dimension-five operator $\psi_L\phi\psi_L\phi$, where ψ_L denotes the lepton doublet and ϕ is the SM Higgs doublet, see *e.g.* Ref. [13].

lepton-number violating interactions. In Eq. (1.2), we list the complete set for the SSM [16, 17, 20, 28].

$$\begin{array}{lcllcl}
\mathcal{O}_1 &=& [QQQL]_F\,, & \mathcal{O}_2 &=& [\bar{U}\bar{U}\bar{D}\bar{E}]_F\,, \\
\mathcal{O}_3 &=& [QQQH_d]_F\,, & \mathcal{O}_4 &=& [Q\bar{U}\bar{E}H_d]_F\,, \\
\mathcal{O}_5 &=& [LH_uLH_u]_F\,, & \mathcal{O}_6 &=& [LH_uH_dH_u]_F\,, \\
\mathcal{O}_7 &=& [\bar{U}\bar{D}^*\bar{E}]_D\,, & \mathcal{O}_8 &=& [H_u{}^*H_d\bar{E}]_D\,, \\
\mathcal{O}_9 &=& [Q\bar{U}L^*]_D\,, & \mathcal{O}_{10} &=& [QQ\bar{D}^*]_D\,.
\end{array} \tag{1.2}$$

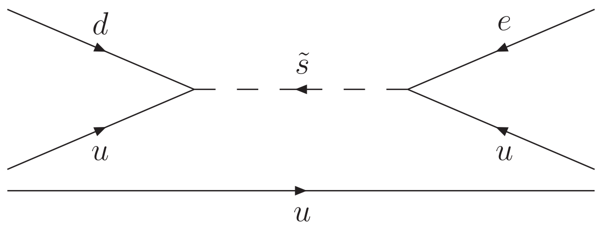
The subscripts F and D denote the F- and D-terms of the corresponding products of superfields. The F-term, *i.e.* the θ^2-term, of a product of left-chiral superfields is always invariant under supersymmetry; such a term derives from the (holomorphic) superpotential of a theory. The D-term, *i.e.* the $\theta^2\bar{\theta}^2$-term, of a product of left- and/or right-chiral superfields is also invariant under supersymmetry and derives from the (nonholomorphic) Kähler potential.

All operators of Eq. (1.2) are suppressed by only *one* power of an unknown mass scale. $\mathcal{O}_4$, $\mathcal{O}_5$, $\mathcal{O}_6$, $\mathcal{O}_7$, $\mathcal{O}_8$ and $\mathcal{O}_9$ violate only lepton number; Majorana-type neutrino masses of the order 10^{-1} eV can be explained if $\mathcal{O}_5$ is suppressed by a mass scale of about 10^{14} GeV. Baryon *and* lepton number are violated by $\mathcal{O}_1$ and $\mathcal{O}_2$; even if suppressed by the gravitational scale $M_{\rm grav} = 2.4 \times 10^{18}$ GeV, these two operators are potentially dangerous, depending on their flavor structure [16, 17, 29]. Finally, $\mathcal{O}_3$ and $\mathcal{O}_{10}$ violate only baryon number.

We see that, in the SSM, baryon- and lepton-number violating interactions occur abundantly already at mass dimensions smaller than six.

1.2 Getting Rid of Dangerous Operators

The most stringent bounds on baryon- and lepton-number violation derives from the longevity of the proton. Within the SSM, already the renormalizable operators $LQ\bar{D}$ and $\bar{U}\bar{D}\bar{D}$ together lead to rapid proton decay, *e.g.*



The lower experimental bound on the proton lifetime [30, 31] results in the very stringent bounds [32, 18, 33]

$$\lambda'_{i1j} \cdot \lambda''_{11j} < 2 \cdot 10^{-27} \left(\frac{M_{\tilde{d}_j}}{100\,\mathrm{GeV}} \right)^2 , \quad i = 1, 2\,,\ j \neq 1\,. \tag{1.3}$$

It seems rather unnatural to have nonvanishing coupling constants λ'_{i1j} and λ''_{11j} leading to this extremely tiny value. See Refs. [34, 35] for an extensive set of bounds on the products of similar coupling constants. Therefore the SSM must be considered incomplete. In order to obtain a natural and viable supersymmetric model, we must extend G_{SM} such that at least one of the operators $LQ\bar{D}$ or $\bar{U}\bar{D}\bar{D}$ is forbidden. This can be achieved by additionally imposing a suitable *discrete symmetry*.

Conventionally, the Minimal SSM (MSSM) is taken as the renormalizable SSM with the superpotential, Eq. (1.1), additionally constrained by the multiplicative discrete symmetry R parity, $\boldsymbol{R}_p \equiv (-\mathbf{1})^{2S+3B+L}$ [36], which acts on the components of the superfields. With S being spin, B baryon number and L lepton number, one easily finds that all SM particles, *i.e.* quarks, leptons, gauge and Higgs bosons, have R parity $+1$ while all superpartners, *i.e.* squarks, sleptons, gauginos and higgsinos, have R parity -1. Hence the superpotential of the renormalizable MSSM is given solely by the first line of Eq. (1.1), and baryon and lepton number are conserved.

Instead of R parity, it is sometimes more convenient to work with an equivalent $\boldsymbol{Z}_2$ symmetry, namely matter parity ($\boldsymbol{M}_p$) [37], which is defined by the *action on the superfields*. In its standard form, a matter-parity transformation leaves the Higgs and the gauge superfields invariant while multiplying the quark and lepton superfields by a factor of -1. In other words, the Higgs and gauge superfields are uncharged, the quark and the lepton superfields have $\boldsymbol{M}_p$ charge 1. A product of superfields is thus $\boldsymbol{M}_p$-invariant if its overall charge is an even number, *i.e.* 0 mod 2.

It can be shown easily that matter parity leads to exactly the same superpotential (and Kähler potential) as $\boldsymbol{R}_p$: Consider a general superpotential (or Kähler potential) operator with n_Q quark, n_L lepton, n_{H_d} down-type and n_{H_u} up-type Higgs doublets, as well as $n_{\bar{U}}$, $n_{\bar{D}}$, and $n_{\bar{E}}$ quark and lepton $SU(2)_W$-singlet superfields. Here, a right-chiral, *i.e.* charge conjugated left-chiral, superfield in the Kähler potential is counted negatively; for instance, the seventh term of Eq. (1.2), $\bar{U}\bar{D}^*\bar{E}$, has $n_{\bar{U}} = n_{\bar{E}} = 1$ and $n_{\bar{D}} = -1$. The total $\boldsymbol{M}_p$ charge of a general term is then given as

$$n_Q + n_{\bar{U}} + n_{\bar{D}} + n_L + n_{\bar{E}} \;\; = \;\; \iota_M \bmod 2, \tag{1.4}$$

with $\iota_M \in \{0, 1\}$. $\boldsymbol{M}_p$ is conserved or violated if $\iota_M = 0$ or 1, respectively. As all $n_{\ldots}$ are integers, we can subtract the even number $2 \cdot (n_{\bar{U}} + n_{\bar{D}} + n_{\bar{E}})$ from the left hand side without any changes on the right. Introducing baryon number $B \equiv \frac{1}{3} \cdot (n_Q - n_{\bar{U}} - n_{\bar{D}})$ and lepton number $L \equiv n_L - n_{\bar{E}}$ for terms, Eq. (1.4) can be rewritten as

$$3 \cdot B + L \;=\; \iota_M \bmod 2. \tag{1.5}$$

The interaction terms in the Lagrangian are obtained from these superpotential and Kähler potential terms by calculating the F- and the D-terms, respectively. It is at this stage that the spin enters the scene. The physical components of a chiral superfield are either spin zero or spin one-half; those of a gauge vector superfield are either spin one-half or spin one. Thus, the sum of the individual spins, S, in an interaction term of the Lagrangian is just 1/2 times the number of fermions plus the number of gauge bosons in this term. In order to build a Lorentz invariant Lagrangian, we always need an *even* number of fermions in one term. So, the sum of the individual spins is necessarily an integer for the interaction terms of the Lagrangian. Therefore, we can again add an even number, $2 \cdot S$, to the left hand side. Taking this as the exponent to the base (-1) yields the definition of $\boldsymbol{R}_p$ for interaction terms:

$$\boldsymbol{R}_p \equiv (-1)^{3 \cdot B + L + 2 \cdot S} \;=\; (-1)^{\iota_M}. \tag{1.6}$$

Hence, the conservation of R parity is equivalent to the conservation of matter parity as claimed above.

Matter parity (or R parity) forbids baryon- and lepton-number violation (thus proton decay) at the renormalizable level. However, as pointed out in Sect. 1.1, we should also include possible nonrenormalizable terms. Our working definition of the MSSM shall be the (nonrenormalizable) SSM constrained by $\boldsymbol{M}_p$. We return to this in Sect. 4.1. Imposing matter parity on the dimension-five operators of Eq. (1.2) leaves us with only three terms: $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_5$. In order to be compatible with the lower bound on the proton lifetime, there has to be an additional mechanism which suppresses the first two interactions, see *e.g.* Ref. [38]. Thus, even though $\boldsymbol{M}_p$ provides the SSM with an excellent candidate for cold dark matter, namely the lightest supersymmetric particle (LSP), it has a serious problem with baryon-number violation and proton decay.

As an alternative to matter parity, one can extend $G_{\rm SM}$ by a $\boldsymbol{Z}_3$ discrete symmetry which does the job of stabilizing the proton much better. This symmetry was originally introduced as baryon parity in [39, 28], however it is more appropriately called baryon triality ($\boldsymbol{B}_3$) [15, 40]. Baryon triality is defined on the superfields by the following generation-independent discrete charge assignment

	Q	$\bar{U}$	$\bar{D}$	L	$\bar{E}$	H_d	H_u
$\boldsymbol{B}_3$	0	2	1	2	2	2	1

.

Similar to the case of matter parity, an operator is invariant under $\boldsymbol{B}_3$ if the total discrete charge is 0 mod 3. Thus, baryon triality leads to the R-parity violating MSSM [20], which (up to dimension five) allows all but the baryon-number violating operators of Eqs. (1.1) and (1.2). That is, the renormalizable operators $\bar{U}\bar{D}\bar{D}$ and the nonrenormalizable operators $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_{10}$ are forbidden by $\boldsymbol{B}_3$; the rest is allowed.

Imposing baryon triality instead of matter parity is particularly attractive in the context of neutrino masses, as it is not necessary to introduce right-handed neutrinos: The nonvanishing $\boldsymbol{B}_3$ conserving, but $\boldsymbol{M}_p$ violating, bilinear terms LH_u mix neutrinos with neutralinos so that one neutrino becomes massive already at tree level. In addition to this, loop corrections to the neutrino mass matrix are possible if the trilinear $\boldsymbol{B}_3$ conserving terms $LQ\bar{D}$ and $LL\bar{E}$ are present. Then, the remaining two massless neutrinos can also acquire nonzero masses. We return to this in Part II of this thesis.

Besides matter parity and baryon triality there are many more discrete symmetries which can forbid dangerous operators of the SSM. In the following we study the extensions of the SSM by arbitrary discrete $\boldsymbol{Z}_N$ symmetries. When confronting the theory constrained by a certain discrete symmetry with its predictions concerning baryon- and lepton-number violation, we take into account the effects on the dimension-three, dimension-four and dimension-five operators of Eqs. (1.1) and (1.2). Throughout our study we take a bottom-up approach to discrete symmetries. At the LHC, we will hopefully discover supersymmetric fields and their interactions. Through the measured and thus allowed interactions we can infer the discrete symmetry. From this point of view, two discrete symmetries are equivalent, if they result in the same low-energy interaction terms.

Chapter 2

Discrete *Gauge* Symmetries – The Idea of Ibáñez and Ross

Discrete symmetries are typically violated by quantum gravity effects. However, if they arise as discrete remnants of a spontaneously broken high-energy *gauge* theory, they are good symmetries of the low-energy theory. In the case of $U(1)_X$ breaking down to a generation-independent $\boldsymbol{Z}_N$ symmetry, the discrete charges of the SSM particles can be parameterize by three integers.

2.1 Origin of Discrete Symmetries

We have seen in the previous chapter that the SSM needs to be augmented by some discrete symmetry in order to naturally explain the experimentally observed longevity of the proton. Naively, one would simply invoke the existence of *any* (global) discrete symmetry that forbids the dangerous baryon- and lepton-number violating operators. However, quantum gravity effects typically violate global discrete symmetries [41]. So, after all, *i.e.* after the inclusion of such exotic processes as black-hole evaporation or wormhole tunneling, the originally imposed discrete symmetry would not be a symmetry of the theory. On the other hand, we know that local gauge symmetries are unaffected by quantum gravity effects. This holds true for discrete subgroups of the continuous gauge group as well. Hence, we adopt the possibility of the discrete symmetry originating in a spontaneously broken high-energy gauge symmetry [41, 42]. A discrete symmetry with such an origin is denoted a discrete *gauge* symmetry (DGS).

Before continuing, we briefly comment on some related work in the literature. We do not consider discrete R symmetries. For an anomaly-free gauged $U(1)$ R symmetry in a local supersymmetric theory see Refs. [43, 44, 45]. This could be broken to a discrete R symmetry. Since R parity is inserted *ad hoc* in the SSM to give the MSSM, there is an extensive literature on "gauged" R parity, *i.e.* where R parity is the remnant of a broken gauge symmetry. Martin has considered R parity as embedded in a $U(1)_{B-L}$ gauge symmetry and classified the possible order parameters in extended gauge symmetries [$SO(10)$, $SU(5)$, $SU(5) \times U(1)$, E_6] which necessarily lead to R parity [46, 47]. Babu *et al.* [48] combine DGSs with an attempt to solve the μ problem. Chemtob *et al.* [49] deal with anomaly-free DGSs of the NMSSM.

In the following we focus on an *Abelian* DGS, *i.e.* a remnant of a spontaneously broken $U(1)$ gauge symmetry. For an explicit Lagrangian see, *e.g.*, Ref. [50]. This is appealing also from the string theory viewpoint, as one commonly encounters numerous $U(1)$ gauge symmetries and discrete subgroups of these in four-dimensional string models. Concerning the $U(1)$ charges of the chiral superfields, we additionally assume that they are quantized (*i.e.* the quotient of any two charges is rational), as usually happens in string theories. For later convenience we also normalize the charges to be integers.

Starting with an, in general, generation-dependent $U(1)_X$ extension of G_{SM}, one might ask the question: How does a discrete symmetry arise from a continuous symmetry? The general idea can be pictured as follows. Denoting the $U(1)_X$ charge (or simply X charge) of a chiral superfield ϕ_i by X_i, the field ϕ_i transforms under the high-energy gauge symmetry with a position-dependent phase factor. One has

$$\phi_i \quad \longrightarrow \quad e^{\mathrm{i}\alpha(x)X_i}\ \phi_i. \tag{2.1}$$

Note that the real gauge parameter $\alpha(x)$ is the same for all chiral superfields. Due to this and the assumed quantization of the X charges, the phase factors of the gauge transformation can only take discrete values for fixed $\alpha(x)$. This situation is pictured on the left side of Fig. 2.1. The dots indicate the allowed phases. The upper solid line represents the phase factor for the $U(1)_X$ breaking field, Φ, while the dashed line shows the smallest possible phase factor.[1] Now, $U(1)_X$ invariance requires the sum of the X charges to vanishes for allowed operators of the theory, that is $\sum_i X_i = 0$, with i running over the fields appearing in the operator. After the spontaneous breakdown of $U(1)_X$ by the scalar component of the superfield Φ, with $X_\Phi = N$, this condition is changed to a modulo N relation

$$\sum_i X_i \quad = \quad 0 \bmod N. \tag{2.2}$$

This relation implies that, from the low-energy point of view, the exact X charges are irrelevant; only the modulo N part counts. In the example of Fig. 2.1, where

[1] It is not necessary to have a superfield transforming with the smallest possible phase factor. Consider, for instance, the case with only two fields (Φ, ϕ) and X charges $X_\Phi = 3$ and $X_\phi = -2$; then the combined operator $\Phi \cdot \phi$ would transform with the smallest possible phase factor.

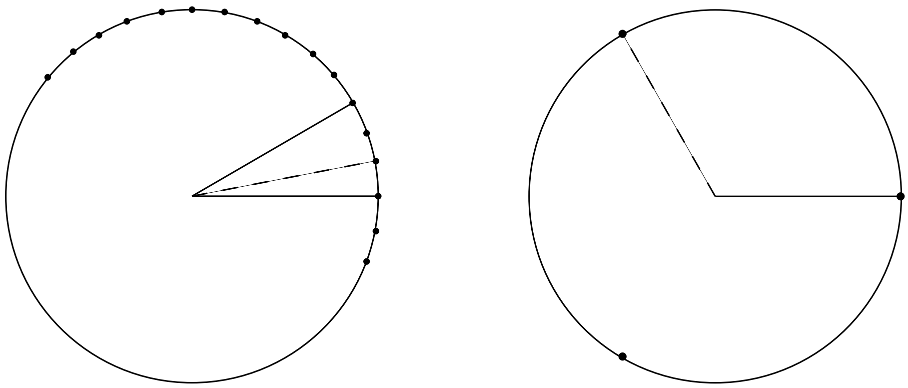


Figure 2.1: The origin of the discrete $\boldsymbol{Z}_N$ symmetry. Due to quantized $U(1)_X$ charges and the $U(1)_X$ breaking field acquiring a vacuum expectation value, the high-energy gauge symmetry (left) leads to a low-energy $\boldsymbol{Z}_N$ symmetry (right). The example shown here is for $N = 3$.

$N = 3$, the dots outside the slice in the left circle can be identified with dots inside the slice. One can therefore map the slice of the left circle to the full circle on the right. All low-energy effective operators derived from the high-energy gauge theory have to satisfy Eq. (2.2). This condition can be reformulated by the invariance of an operator under the newly defined discrete $\boldsymbol{Z}_N$ transformation:

$$\phi_i \quad \longrightarrow \quad e^{\frac{2\pi \mathrm{i}}{N}\cdot X_i}\ \phi_i. \tag{2.3}$$

Notice that, due to the integer X-charge normalization, the phase factor of the discrete transformation can take only N different values. Discrete symmetries arising by such a mechanism are what one means by the term discrete *gauge* symmetries, DGSs. Considering cases where $U(1)_X$ is spontaneously broken by more than one superfield, Φ_i, $i = 1, 2, ...$, the remnant discrete symmetry is a $\boldsymbol{Z}_N$, with N being the largest common factor of all X_{Φ_i}. *E.g.*, having two fields Φ_1 and Φ_2 with X charges $X_{\Phi_1} = 18$ and $X_{\Phi_2} = 30$, the resulting discrete symmetry is a $\boldsymbol{Z}_6$. See Appendix A for details.

2.2 Anomaly Conditions and Particle Content

Consistency of the high-energy $U(1)_X$ gauge theory demands vanishing anomalies. Following Fujikawa's interpretation [51], which adopts the path integral quantization of quantum field theory, an anomaly occurs if the path integral measure changes under a symmetry transformation of the classical action. Thus the quantum effective action, *i.e.* the one taking into account quantum fluctuations and loop effects, does not respect the symmetry of the classical (tree level) theory. For gauge theories, an anomaly would render the quantized theory mathematically inconsistent. Therefore, we have to make sure that certain anomaly conditions are satisfied. In the case of a chiral[2] gauge transformation, which maps the fermionic component ψ_{ϕ_i} of a left-chiral superfield ϕ_i to

$$\psi'_{\phi_i} \;=\; e^{2\,\mathrm{i}\,\alpha^a(x)\,T^a}\,\psi_{\phi_i}\,, \tag{2.4}$$

the path integral measure changes by the factor [52]

$$\exp\left[\frac{\mathrm{i}}{32\pi^2}\int d^4x\;\alpha^c(x)\;\epsilon^{\mu\nu\rho\sigma}\;g_a\,F^a_{\mu\nu}(x)\;g_b\,F^b_{\rho\sigma}(x)\;\mathrm{Trace}\left[\{T^a,T^b\}\cdot T^c\right]\right]. \tag{2.5}$$

$\alpha^a(x)$ are the parameters of the gauge transformations; $F^a_{\mu\nu}(x)$ denote the field strength tensors[3] of the gauge fields; T^a are the generators of the gauge groups, g_a the corresponding gauge coupling constants. $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor with $\epsilon^{0123}=1$, and the braces stand for the anticommutator. Defining the anomaly coefficient $\mathcal{A}_{abc}$ by the trace in the exponent of Eq. (2.5),

$$\mathcal{A}_{abc} \;\equiv\; \mathrm{Trace}\left[\{T^a,T^b\}\cdot T^c\right], \tag{2.6}$$

anomaly freedom can be expressed algebraically by the condition $\mathcal{A}_{abc}=0$ for all a,b,c. In the $U(1)_X$ extension of G_{SM} which we consider, there are $8+3+1+1$ gauge group generators, corresponding to $SU(3)_C$, $SU(2)_W$, $U(1)_Y$ and $U(1)_X$, respectively. So the indices a,b,c can run from 1 to 13. However, we need to consider only those combinations of generators for which the product of T^a, T^b and T^c is neutral under the gauge symmetry. For the $SU(3)_C$ [as well as the $SU(2)_W$] gauge group, invariants can be constructed out of zero, two or three generators. With regard to the $U(1)$ gauge groups, any number of generators is possible. In Chapter 3 we will investigate all anomaly coefficients in turn. Before, however, it is mandatory to specify the particle content of the class of models considered in this thesis, as it enters in the trace. It is easiest to first write all particles, taking into account the three differently colored quarks, in one "vector" and then expressing the gauge group generators in this basis of particles. That way the anomaly coefficients defined in Eq. (2.6) can be evaluated directly from the assumed particle content.

We have emphasized before, that the $U(1)_X$ gauge symmetry gets broken down to the discrete $\boldsymbol{Z}_N$ symmetry by the vacuum expectation value (VEV), υ, of the

[2]A chiral theory distinguishes between left- and right-handed components of particles, *e.g.* electrons. Here, we define the gauge transformation only on left-chiral superfields, that is we first take the complex conjugate of the right-handed particles, and then consider the action of the gauge transformation on the resulting left-chiral superfields.

[3]We adopt the standard definition of the field strength tensors where the gauge coupling constants are *not* included in the gauge fields A^a_μ. So we have $F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu-g_a f^{abc}A^b_\mu A^c_\nu$.

scalar component of a superfield Φ with $U(1)_X$ charge $X_\Phi \equiv N$. We assume here a single field Φ, or a vectorlike pair; *cf.* Sect. 4.3. Without loss of generality we can take $N > 0$. The mass scale of the broken symmetry is $M_X = \mathcal{O}(\upsilon) \gg M_W$.

In the low-energy theory, we definitely want to have all particles of the SSM. For these fields, the $\boldsymbol{Z}_N$ charges, q_i, are related to the integer $U(1)_X$ charges, X_i, via a modulo N shift

$$X_i = q_i + m_i N \, . \tag{2.7}$$

Here the index i labels the SSM particle species and q_i, m_i are integers. Just like the $U(1)_X$ charges, the m_i are in general generation *dependent*, whereas the q_i are assumed to be generation independent.[4] For convenience we do not require the discrete charges q_i to lie within the interval $[0, N-1]$, *cf.* Eq. (2.13).

In addition to the SSM superfields we *only* allow for particles which obtain their masses at the scale of $U(1)_X$ breaking, *i.e.* $\mathcal{O}(M_X)$. This means that the corresponding mass terms must be present immediately after the spontaneous breakdown of $U(1)_X$. As there are Dirac and Majorana fermions, we have to distinguish between these two cases. For a Dirac mass term, two chiral fields with $U(1)_X$ charges X^j_{D1} and X^j_{D2}, respectively, must pair-up, while for a Majorana mass term, one fields with charge $X^{j'}_{\mathrm{M}}$ suffices. The $\boldsymbol{Z}_N$ invariance of these mass terms yields the following constraints on the heavy particles' X charges:

$$X^j_{\mathrm{D1}} + X^j_{\mathrm{D2}} = p_j N, \qquad p_j \in \mathbb{Z}, \tag{2.8}$$

$$2 \cdot X^{j'}_{\mathrm{M}} = p'_{j'} N, \qquad p'_{j'} \in \mathbb{Z} \, . \tag{2.9}$$

The indices j and j' run over all heavy Dirac and Majorana particles, respectively. Notice that in the case of odd N, Eq. (2.9) requires $p'_{j'}$ to be even (as all X charges are normalized to be integers).

Having specified the particle content, we are now in the position to explicitly write down the anomaly conditions. Before, however, we want to make some simplifications. The discussion of the anomaly conditions is postponed to Chapter 3. Allowing for an arbitrary number of heavy Dirac and Majorana particles, anomaly freedom of the high-energy theory constrains the discrete charges of the low-energy SSM particles. We obtain additional constraints on the q_i by requiring a minimal set of interaction terms in the SSM superpotential which are responsible for the low-energy fermion masses, namely the first three terms in Eq. (1.1). In Sect. 4.1 we investigate the consequences of additionally imposing $H_d H_u$ invariance. The $\boldsymbol{Z}_N$-charge equations corresponding to the first three terms of Eq. (1.1)

[4] Note that due to the *three* nonvanishing mixing angles of the CKM matrix, one is forced to work with generation-independent discrete charges for the quarks. Concerning the leptons, generation dependence is only possible if one relies on radiatively generated neutrino masses. See, *e.g.*, Refs. [53, 54].

are

$$\begin{aligned} q_L + q_{H_d} + q_{\bar{E}} &= 0 \bmod N\,, && (2.10)\\ q_Q + q_{H_d} + q_{\bar{D}} &= 0 \bmod N\,, && (2.11)\\ q_Q + q_{H_u} + q_{\bar{U}} &= 0 \bmod N\,. && (2.12) \end{aligned}$$

These are three equations for seven unknowns. We can thus write the family-independent $\boldsymbol{Z}_N$ charges of the SSM chiral superfields in terms of four independent integers, which we choose as $m, n, p, r = 0, 1, ..., N-1$.

$$\begin{aligned} q_Q &= r, & q_{\bar{U}} &= -m-4r, & q_{\bar{D}} &= m-n+2r\,,\\ q_L &= -n-p-3r, & q_{\bar{E}} &= m+p+6r,\\ q_{H_d} &= -m+n-3r, & q_{H_u} &= m+3r\,. \end{aligned} \qquad (2.13)$$

The choice of integers m, n, p in Eq. (2.13) corresponds to the notation of IR, see Ref. [28]. The slightly unusual coefficients for the integer r correspond to the negative hypercharge given in the following (integer) normalization

$$Y(Q, \bar{U}, \bar{D}, L, \bar{E}, H_d, H_u) = (-1, 4, -2, 3, -6, 3, -3)\,. \qquad (2.14)$$

2.3 Hypercharge Shift and Termwise $\boldsymbol{Z_N}$

To simplify the up-coming calculations, we want to further reduce the number of parameters defining the low-energy discrete $\boldsymbol{Z}_N$ symmetry from four to three. This can be achieved by performing a shift of the integer $\boldsymbol{Z}_N$ charges by their integer hypercharges, such that the resulting charge ${q_Q}'$ is zero,

$$q_i \longrightarrow {q_i}' = q_i + Y_i \cdot r\,. \qquad (2.15)$$

Thus the parameter r in Eq. (2.13) drops out altogether and we are left with the three parameters m, n, and p, only. This choice of charges (where ${q_Q}' = 0$) is the basis in which IR work. They show that, in this case, any $\boldsymbol{Z}_N$ symmetry, g_N, can be expressed in terms of the product of powers of the three (mutually commuting) generators R_N, A_N and L_N [28]:

$$g_N = R_N^m \times A_N^n \times L_N^p, \quad \text{with the exponents} \quad m, n, p = 0, 1, ...N-1\,. \qquad (2.16)$$

The charges of the SSM chiral superfields under the three independent $\boldsymbol{Z}_N$ generators are given in Table 1 of Ref. [28]. L is connected to the (negative) lepton number, R is (the negative of) the third component of a right-handed weak isospin, and A corresponds to the Cartan subalgebra of the $SU(2)$ in $E_6 \supset SU(6) \times SU(2)$.

In terms of the powers m, n, p, the generation-independent $\boldsymbol{Z}_N$ charges of the SSM superfields are

$$\begin{aligned} q_Q{}' &= 0, & q_{\bar{U}}{}' &= -m, & q_{\bar{D}}{}' &= m-n \\ q_L{}' &= -n-p, & q_{\bar{E}}{}' &= m+p, & & \\ q_{H_d}{}' &= -m+n, & q_{H_u}{}' &= m\,. & & \end{aligned} \tag{2.17}$$

With this parameterization, the action of g_N on, *e.g.*, the chiral superfields $\bar{D}_i$ is given by $\bar{D}_i \to \exp\left[\frac{2\pi\mathrm{i}}{N}(m-n)\right]\bar{D}_i$. Note that the integers m, n, p here are the same as in Eq. (2.13). In the rest of this section we wish to discuss the effects of hypercharge shifts, finding that the set of possible anomaly-free DGSs is not altered by restricting to the discrete charges in Eq. (2.17), *i.e.* with $q_Q{}' = 0$.

Consider an anomaly-free $SU(3)_C \times SU(2)_W \times U(1)_Y \times U(1)_X$ gauge theory with integer Abelian charges for all particles. In general, the $U(1)_X$ charges of the quark doublets Q_i is *not* of the form $X_{Q_i} = q_Q + m_{Q_i}N$ with $q_Q = 0$.

When redefining the $U(1)_X$ charges of all fields by[5] $X_i{}' \equiv X_i + \alpha Y_i$, $\alpha \in \mathbb{R}$, it is obvious that any hypercharge-invariant superpotential or Kähler potential term does not change its total $U(1)_X$ charge; more formally $\sum_i X_i{}' = \sum_i X_i$, with i running over the particles of the term considered. Therefore one is tempted to conclude that such a hypercharge shift of X charges does not alter the theory at all. However, this is not true since the $U(1)_X$ charges also enter in the gauge couplings of the heavy $U(1)_X$ vector superfield to matter, which corresponds to a change in the underlying $U(1)_X$ gauge theory. This difference can lead to, in principle, observable effects, for example cross-sections which depend on X charges. We therefore consider such an modification of the theory not desirable.

Instead of applying the hypercharge shift to the X charges, we consider its action on the discrete $\boldsymbol{Z}_N$ charges. Similarly to the $U(1)_X$ case, a hypercharge shift of the discrete charges does not change the total $\boldsymbol{Z}_N$ charge of a hypercharge-invariant term: $\sum_i q_i{}' = \sum_i q_i$, with $q_i{}' \equiv q_i + \alpha Y_i$. Here α must be chosen so that αY_i is integer for all i. With this shift the theory remains exactly the same because the discrete $\boldsymbol{Z}_N$ transformation was defined only for the purpose of expressing the mod N condition of Eq. (2.2). As we demand hypercharge invariance for all renormalizable and nonrenormalizable superpotential or Kähler potential operators, the *definition* of the $\boldsymbol{Z}_N$ transformation is somewhat arbitrary; the one given in Eq. (2.3) is only one of N possible choices. Unlike before, the $U(1)_X$ charges, and thus the couplings of the gauge fields to matter, are not redefined under a hypercharge shift of the discrete charges.

[5]For the purpose of the rest of this section, we need not distinguish between light and heavy fields. Therefore the index i labels all particles; the separation of discrete and $U(1)_X$ charges, $X_i = q_i + m_i N$, is introduced also for the heavy particles.

However, the following question arises: How are the anomaly cancellation conditions affected if inserting $q_i{}'$ instead of q_i? This question is directly related to the behavior of the anomaly conditions under a hypercharge shift of the $U(1)_X$ charges: Instead of $X_i = q_i + m_i N$ we insert $q_i{}' + m_i N = X_i + \alpha Y_i \equiv X_i{}'$ into the anomaly cancellation equations. It is easy to see that the set of equations[6] $\mathcal{A}_{CCY} = \mathcal{A}_{CCX} = \mathcal{A}_{WWY} = \mathcal{A}_{WWX} = \mathcal{A}_{GGY} = \mathcal{A}_{GGX} = \mathcal{A}_{YYY} = \mathcal{A}_{YYX} = \mathcal{A}_{YXX} = \mathcal{A}_{XXX} = 0$ is equivalent to a similar set with X replaced by X'; for instance, $\mathcal{A}_{CCX'} = \mathcal{A}_{CCX} + \alpha \cdot \mathcal{A}_{CCY} = 0 + 0 = 0$. Therefore a redefinition of the X charges does not alter the anomaly constraints if the theory is free of anomalies.

With regard to anomalous theories which adopt the Green-Schwarz (GS) mechanism to cancel anomalies (see Sect. 6.3), the above hypercharge shift of the discrete charges is generally not allowed. GS requires the SM anomalies and $\mathcal{A}_{YXX}$ to vanish. The effects of the remaining (possibly nonzero) anomaly coefficients cancel by the transformation of the dilaton superfield, see *e.g.* Ref. [55]. Now, under a hypercharge shift, one finds $\mathcal{A}_{YX'X'} = \mathcal{A}_{YXX} + 2\alpha \cdot \mathcal{A}_{YYX} + \alpha^2 \cdot \mathcal{A}_{YYY} = 0 + 2\alpha \cdot \mathcal{A}_{YYX} + 0 = 2\alpha \cdot \mathcal{A}_{YYX}$, which is not zero in general. Hence, when considering discrete symmetries within GS scenarios, we should keep in mind that already the choice of $q_Q{}' = 0$ puts restrictions on the set of DGSs.

For anomaly-free DGSs, a hypercharge shift of the $\boldsymbol{Z}_N$ charges can be performed without changing the theory or the anomaly conditions. This freedom of shifting the q_i enables us to reduce the number N of the symmetry group $\boldsymbol{Z}_N$. We emphasize here that the X charge of the $U(1)_X$ breaking field Φ is not necessarily the same as the number N appearing in the final $\boldsymbol{Z}_N$ we obtain when restricting ourselves to the so-called "fundamental" DGSs. We discuss this in more detail in Sect. 3.5. Consider, *e.g.*, the following $\boldsymbol{Z}_{12}$ charge assignment with $n = 0$, $m = 10$, $p = 4$, $r = 1$ inserted into Eq. (2.13):

	Q	$\bar{U}$	$\bar{D}$	L	$\bar{E}$	H_d	H_u
q_i	1	−14	12	−7	20	−13	13
Y_i	−1	4	−2	3	−6	3	−3
$q_i{}' = q_i + Y_i$	0	−10	10	−4	14	−10	10

(2.18)

Starting with the original discrete $\boldsymbol{Z}_{12}$ charges q_i we can shift these by the hypercharges to obtain the last row of Eq. (2.18). As all $q_i{}'$ are even, we actually have a low-energy $\boldsymbol{Z}_6$ symmetry instead of a $\boldsymbol{Z}_{12}$. In the following, we always assume that the shift to the basis in Eq. (2.17) has been performed; therefore, we drop the prime on the discrete charges from now on.

[6] Here we adopt the notation of Ref. [38]. For example, the $SU(3)_C - SU(3)_C - U(1)_X$ anomaly is denoted as $\mathcal{A}_{CCX}$, and G stands for "gravity".

So far, our definition of $\boldsymbol{Z}_N$ symmetries has referred to the discrete charges of *individual fields*. One might be inclined to assume that the effective discrete symmetry reduces further if applied to $G_{\rm SM}$-invariant *terms*. Starting with the hypercharge-shifted discrete charges of Eq. (2.17), which, in addition, are maximally rescaled[7], the following discussion shows that this is, however, not the case. Consider an arbitrary term of the superpotential or Kähler potential, made up of exclusively SSM superfields with n_Q quark doublet superfields Q, n_L lepton doublet superfields L, *et cetera*. The $\boldsymbol{Z}_N$ charge of such a *term* is given by

$$q_{\rm total} \;=\; n_Q\, q_Q + n_{\bar U}\, q_{\bar U} + n_{\bar D}\, q_{\bar D} + n_L\, q_L + n_{\bar E}\, q_{\bar E} + n_{H_d}\, q_{H_d} + n_{H_u}\, q_{H_u}. \tag{2.19}$$

Inserting the discrete charges of Eq. (2.17), which take into account the requirement of $\boldsymbol{Z}_N$ invariance of the MSSM Yukawa terms $LH_d\bar{E}$, $QH_d\bar{D}$ and $QH_u\bar{U}$, we arrive at

$$\begin{aligned} q_{\rm total} \;=\;& m\cdot\left[-n_{\bar U} + n_{\bar D} + n_{\bar E} - n_{H_d} + n_{H_u}\right] \\ &+ n\cdot\left[-n_{\bar D} - n_L + n_{H_d}\right] + p\cdot\left[-n_L + n_{\bar E}\right]. \end{aligned} \tag{2.20}$$

Imposing the $G_{\rm SM}$ invariance conditions of Ref. [38], see also Eqs. (7.3), (7.4), and (7.5), we can express $n_{\bar U}$, $n_{\bar D}$, n_{H_d} and n_{H_u} in Eq. (2.20) in terms of n_Q, n_L, $n_{\bar E}$, $\mathcal{C}$, and $\mathcal{W}$; these five parameters can independently take arbitrary integer values. With this replacement we obtain

$$\begin{aligned} q_{\rm total} \;=\;& m\cdot\left[n_L - n_{\bar E} - \mathcal{C}\right] \\ &+ n\cdot\left[-n_Q - 2n_L + n_{\bar E} + 2\mathcal{C} + \mathcal{W}\right] + p\cdot\left[-n_L + n_{\bar E}\right]. \end{aligned} \tag{2.21}$$

Obviously, all three square brackets are independent integers. Hence, for all $\mathcal{J}_{m,n,p} \in \mathbb{Z}$ there exist $G_{\rm SM}$-invariant terms with $\boldsymbol{Z}_N$ charge

$$q_{\rm total} \;=\; m\cdot\mathcal{J}_m + n\cdot\mathcal{J}_n + p\cdot\mathcal{J}_p. \tag{2.22}$$

We now tackle the above question whether the $\boldsymbol{Z}_N$ symmetry reduces further if applied to terms instead of fields. Consider any $\boldsymbol{Z}_N$ symmetry defined by the maximally rescaled set of integers $(m, n, p; N)$, that is m, n, p, and N have no common factor F. However, in general it is possible that the pair (m, N) *has* a (largest) common factor F_m, such that $m \equiv F_m\overline{m}$ and $N \equiv F_m N_m$, with $\overline{m}, N_m \in \mathbb{Z}$. Analogously, we can have common factors for the pairs (n, N) and (p, N). The $\boldsymbol{Z}_N$ charge of terms with $\mathcal{J}_m = 1$ and $\mathcal{J}_n = \mathcal{J}_p = 0$ is $q_{\rm total} = m = F_m\overline{m}$; thus the N_mth power of such a term is $\boldsymbol{Z}_N$ allowed while smaller powers are forbidden. From this we conclude that the discrete symmetry of $G_{\rm SM}$-invariant terms is a

[7] In the example of Eq. (2.18) the maximally rescaled discrete charges of the last row are the $\boldsymbol{Z}_6$ charges with $n = 0$, $m = 5$, and $p = 2$.

$\boldsymbol{Z}_M$, with $M \geq N_m$. Similarly, there are terms whose N_nth or N_pth power is allowed in the Lagrangian, while smaller powers are forbidden. Combining these observations, it is clear that the discrete symmetry is a $\boldsymbol{Z}_M$, where M is greater than or equal to the least common multiple of N_m, N_n and N_p. We now show that this least common multiple is just N. Thus the $\boldsymbol{Z}_N$ symmetry does not reduce if applied to terms instead of individual fields; we have $M = N$. Obviously, N is a common multiple of N_m, N_n and N_p:

$$N = F_m N_m = F_n N_n = F_p N_p. \tag{2.23}$$

The question however is, whether there exists a smaller common multiple, $\frac{N}{f}$, such that

$$\frac{N}{f} = \frac{F_m}{f} N_m = \frac{F_n}{f} N_n = \frac{F_p}{f} N_p, \tag{2.24}$$

with integer $f \neq 1$ and $\frac{N}{f}, \frac{F_m}{f}, \frac{F_n}{f}, \frac{F_p}{f} \in \mathbb{Z}$. Assuming this to be true, we could rewrite the set $(m, n, p; N)$ as

$$(m, n, p; N) \;=\; (f \cdot \frac{F_m}{f} \cdot \overline{m}\ ,\ f \cdot \frac{F_n}{f} \cdot \overline{n}\ ,\ f \cdot \frac{F_p}{f} \cdot \overline{p}\ ;\ f \cdot \frac{N}{f}), \tag{2.25}$$

with only integer factors. The common factor of f however contradicts the initial assumption of maximally rescaled parameters $(m, n, p; N)$.

Chapter 3

The Set of Anomaly-Free DGSs

The linear and the cubic anomaly cancellation conditions significantly constrain the possible discrete gauge symmetries. Allowing for fractionally X-charged heavy fermions, the set of anomaly-free DGSs is further reduced to only 27 so-called "fundamental" DGSs. All of them require heavy particles.

3.1 The Linear Anomaly Constraints

Assuming the initial $U(1)_X$ is anomaly-free, Ibáñez and Ross determined the constraints on the remnant low-energy and family-independent DGSs [39, 28]. They classified all $\boldsymbol{Z}_N$ DGSs for $N = 2, 3$ according to their action on the baryon- and lepton-number violating operators and then determined which are discrete gauge anomaly-free. In this chapter we extend their work to arbitrary values of N. We first focus on constraints arising from the linear $U(1)_X$ anomalies $\mathcal{A}_{CCX}$, $\mathcal{A}_{WWX}$, and $\mathcal{A}_{GGX}$ and derive the resulting constraints on the $\boldsymbol{Z}_N$ charges q_i of Eq. (2.7). In Sections 3.3 and 3.4, we investigate the purely Abelian anomalies, *i.e.* $\mathcal{A}_{YYX}$, $\mathcal{A}_{YXX}$ and especially the cubic anomaly $\mathcal{A}_{XXX}$.

From the linear anomaly cancellation conditions $\mathcal{A}_{CCX} = \mathcal{A}_{WWX} = \mathcal{A}_{GGX} = 0$, we obtain

$$\sum_{i=\mathbf{3},\overline{\mathbf{3}}} q_i = -N \cdot \left[\sum_{i=\mathbf{3},\overline{\mathbf{3}}} m_i + \sum_{j=\mathbf{3},\overline{\mathbf{3}}} p_j \right], \tag{3.1}$$

$$\sum_{i=\mathbf{2}} q_i = -N \cdot \left[\sum_{i=\mathbf{2}} m_i + \sum_{j=\mathbf{2}} p_j \right], \tag{3.2}$$

$$\sum_{i} q_i = -N \cdot \left[\sum_{i} m_i + \sum_{j} p_j + \sum_{j'} \tfrac{1}{2} p'_{j'} \right]. \tag{3.3}$$

The sums in Eqs. (3.1) and (3.2) run over all color triplets and weak doublets, respectively, *i.e.* we restrict ourselves to only fundamental representations[1] of $SU(3)_C$ and $SU(2)_W$. As all particles couple gravitationally, we sum over the entire chiral superfield spectrum in Eq. (3.3). The heavy Majorana fermions of Eq. (2.9) only contribute to the gravitational anomaly coefficient $\mathcal{A}_{GGX}$.

Depending on the charge shifts, m_i, of the low-energy fields, as well as the heavy-fermion particle content, the square brackets in Eqs. (3.1)-(3.3) can take on arbitrary integer values. In the case of even N, any half-odd integer is allowed for the square bracket in Eq. (3.3). Hence, we can rewrite them symbolically as

$$\sum_{i=\mathbf{3},\overline{\mathbf{3}}} q_i = N \cdot \mathbb{Z}, \tag{3.4}$$

$$\sum_{i=\mathbf{2}} q_i = N \cdot \mathbb{Z}, \tag{3.5}$$

$$\sum_{i} q_i = N \cdot \mathbb{Z} + \eta \cdot \frac{N}{2} \cdot \mathbb{Z}, \tag{3.6}$$

with $\eta = 0, 1$ for $N = \text{odd}, \text{even}$, respectively. From the point of view of the low-energy theory, the various $\mathbb{Z}$s, including the two in Eq. (3.6), *each* represent an arbitrary and independent integer, which is fixed by the heavy-fermion content and the choice of m_i.

Inserting the charges q_i in terms of the parameters m, n, and p [*cf.* Eq. (2.17) with the primes dropped] into the left hand sides of Eqs. (3.4)-(3.6), and assuming the SSM light-fermion content (*i.e.* three generations of quarks and leptons as well as one pair of Higgs doublets) we arrive at the conditions

$$3n = N \cdot \mathbb{Z}\,, \tag{3.7}$$

$$3(n+p) - n = N \cdot \mathbb{Z}\,, \tag{3.8}$$

$$3(5\,n+p-m) - 2n = N \cdot \mathbb{Z} + \eta \cdot \tfrac{N}{2} \cdot \mathbb{Z}\,. \tag{3.9}$$

It should be pointed out that these three equations are r independent; *i.e.* they can be obtained by directly plugging Eq. (2.13) into Eqs. (3.4)-(3.6). However, when considering the Abelian anomalies in Sects. 3.3 and 3.4, the r dependence

[1] The contribution of a fermion to an $SU(M) - SU(M) - U(1)$ anomaly is proportional to the corresponding Dynkin index [56]. Particles constituting higher irreducible representations of $SU(M)$ have a Dynkin index, which is an integer multiple of that of the fundamental M-plet [48, 57]. Therefore heavy particles in higher irreducible representations need not be considered for our purposes, see Eqs. (3.4)-(3.6). Note that in Eqs. (3.1) and (3.2) we do not consider Majorana particles either, because all real representations of $SU(M)$ have a Dynkin index, which is an even multiple of that of the fundamental irreducible representation, see Refs. [48, 58]. Appendix B explicitly shows that, with only fundamental representations of $SU(2)$, it is impossible to write a bilinear term leading to Majorana masses.

does not cancel. Since all $\mathbb{Z}$s in Eqs.(3.7)-(3.9) stand for arbitrary and independent integers, we can combine these Diophantine equations to obtain a simpler set,

$$\begin{aligned} 3n &= N \cdot \mathbb{Z}\,, \quad &(3.10)\\ 3p - n &= N \cdot \mathbb{Z}\,, \quad &(3.11)\\ 3(m+p) &= N \cdot \mathbb{Z} + \eta \cdot \tfrac{N}{2} \cdot \mathbb{Z}\,. \quad &(3.12) \end{aligned}$$

This differs slightly from IR in notation, as we find it more convenient to retain the arbitrary integers $\mathbb{Z}$ on the right hand side. These three equations are the basis for our further study. DGSs satisfying all three equations will be called "anomaly-free DGSs", although these constraints are only necessary but not sufficient for complete anomaly freedom of the high-energy theory [58, 59].

3.2 Symmetries Allowed by Linear Constraints

Going beyond the work of IR, we now determine the solutions, $(n, p, m; N)$, to the Eqs. (3.10)-(3.12) for *general* values of N, not just $N = 2, 3$. We separately consider the two possibilities: either N *is not* or *is* a multiple of 3. We employ the notation:

$$\begin{aligned} (k|N) \;&:\Leftrightarrow\; N = 0 \bmod k\,,\\ \neg(k|N) \;&:\Leftrightarrow\; N \neq 0 \bmod k\,. \end{aligned}$$

$k \in \mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers including zero.

1. $\neg\,(\mathbf{3}\,|\boldsymbol{N})$**:** Since $n = 0, 1, ..., N-1$, Eq. (3.10) requires $n = 0$. Then Eq. (3.11) similarly gives $p = 0$. Finally, Eq. (3.12) then implies

   (a) $m = 0$ for odd N. This is the case of the trivial symmetry, the identity.

   (b) For even N there are two possibilities, either $m = 0$ (trivial) or $m = \frac{N}{2}$.

   We conclude that the only *nontrivial* anomaly-free DGSs here are

   $$g_N = R_N^{N/2}, \quad N = \text{even}\,. \qquad (3.13)$$

   The simplest case with $N = 2$ yields the discrete $\boldsymbol{Z}_2$ charges: $q_Q = q_L = 0$, $q_{\bar{D}} = q_{\bar{E}} = q_{H_u} = 1$, $q_{\bar{U}} = q_{H_d} = -1$. This charge assignment is, from the low-energy point of view, equivalent to standard matter parity [37]. A reversed hypercharge shift, Eq. (2.15), back to Eq. (2.13) with $r = 1$ yields: $q_Q = q_L = q_{\bar{D}} = q_{\bar{U}} = q_{\bar{E}} = 1 \bmod 2$, $q_{H_u} = q_{H_d} = 0$.

2. $(\mathbf{3}\,|\boldsymbol{N})$**:** Here we can define an $N' \in \mathbb{Z}$, such that $N \equiv 3N'$. From Eq. (3.10) we obtain $n = 0, N'$, or $2N'$:

(a) Focusing first on $n = 0$, we see that $p = \ell_p N'$, for $\ell_p = 0, 1, 2$. Concerning Eq. (3.12), it is again necessary to distinguish between odd and even N. Thus we find a set of anomaly-free DGSs

$$n = 0, \quad p = \ell_p N', \quad m = \begin{cases} \ell_m N', & N = \text{odd}, \\ s_m \frac{N'}{2}, & N = \text{even}, \end{cases} \tag{3.14}$$

with ℓ_p, $\ell_m = 0, 1, 2$ and $s_m = 0, 1, ..., 5$.

(b) Inserting $n = N'$ into Eq. (3.11), we obtain $p = \frac{N'}{3} + \ell_p N'$, again with $\ell_p = 0, 1, 2$. For $p \in \mathbb{Z}$, we need $(3\,|N')$ or equivalently $N' \equiv 3N''$, with $N'' \in \mathbb{Z}$. Taking into account Eq. (3.12), we now find

$$n = N', p = (1{+}3\,\ell_p)N''\,, \quad m = \begin{cases} (2 + 3\,\ell_m)\,N'', & N = \text{odd}, \\ (1 + 3\,s_m)\,\frac{N''}{2}, & N = \text{even}\,. \end{cases} \tag{3.15}$$

(c) Analogously, $n = 2N'$ gives

$$n = 2N', \quad p = (2 + 3\,\ell_p)N'', \quad m = \begin{cases} (1 + 3\,\ell_m)N'', & N = \text{odd}, \\ (2 + 3\,s_m)\frac{N''}{2}, & N = \text{even}. \end{cases} \tag{3.16}$$

The class of DGSs given in (c) need not be investigated any further for it is equivalent to the one in (b): A $\boldsymbol{Z}_N$ symmetry with charges q_i is indistinguishable from one with charges $-q_i$; therefore the sets (n, p, m) and $(N - n, N - p, N - m)$ yield equivalent DGSs. As an example, consider the integer p. For every p_2 in Eq. (3.16) require a p_1 in Eq. (3.15), such that $p_2 \stackrel{!}{=} N - p_1$. Inserting Eqs. (3.16) and (3.15), we obtain $(2 + 3\ell_{p_2})N'' \stackrel{!}{=} (9 - 1 - 3\ell_{p_1})N''$, which is solved for $\ell_{p_1} = 2 - \ell_{p_2} \in \{0, 1, 2\}$. Similarly, the integer m can be treated for even or odd N. Likewise, some DGSs of Eq. (3.14) are not independent of the others.

Table 3.1 summarizes the anomaly-free DGSs classified by N and the powers n, p and m. For example, the two rows with $(3\,|N)$ correspond to the DGSs of Eq. (3.14). The last column shows the number of independent nontrivial g_N. The 4 in the second row arises because there are three DGSs with $\ell_p = 1$ but only one with $\ell_p = 0$; with $p = 0$, the case $m = 0$ is trivial, whereas $m = N'$ and $m = 2N'$ lead to equivalent DGSs. Similarly, we get 9 DGSs instead of 12 for the third row.

3.3 The Purely Abelian Anomalies

So far, we have determined the constraints on DGSs arising from the three linear anomaly conditions of Eqs. (3.1)-(3.3). Next we consider the three purely Abelian anomalies $\mathcal{A}_{YYX}$, $\mathcal{A}_{YXX}$ and $\mathcal{A}_{XXX}$, respectively.

$\boldsymbol{Z}_N$ Category		n	p	m	# indep. g_N
$\neg(3\,\|N)$	N even	0	0	$\frac{N}{2}$	1
$(3\,\|N)$	N odd	0	$(0,1)\cdot N'$	$(0,1,2)\cdot N'$	4
	N even	0	$(0,1)\cdot N'$	$(0,1,2,3,4,5)\cdot\frac{N'}{2}$	9
$(9\,\|N)$	N odd	N'	$(1,4,7)\cdot N''$	$(2,5,8)\cdot N''$	9
	N even	N'	$(1,4,7)\cdot N''$	$(1,4,7,10,13,16)\cdot\frac{N''}{2}$	18

Table 3.1: The list of all DGSs satisfying the linear anomaly constraints of Ibáñez and Ross. N' and N'' are defined by $N = 3N' = 9N''$, where $N, N', N'' \in \mathbb{N}$. The $\ell_p = 2$ cases are not listed as they are equivalent to the sets of DGSs with $\ell_p = 1$. The last column gives the resulting number of independent nontrivial DGSs, g_N, for fixed N.

1. Analogously to Eqs. (3.1)-(3.3), we obtain from $\mathcal{A}_{YYX} = 0$ that

$$\sum_i Y_i^2\, q_i \;=\; -N\left[\sum_i Y_i^2\, m_i \;+\; \sum_j {Y_{\mathrm{D1}}^j}^2\, p_j\right]. \tag{3.17}$$

We have used $Y_{\mathrm{D2}}^j = -Y_{\mathrm{D1}}^j$ and $Y_{\mathrm{M}}^j = 0$, as well as Eq. (2.8). Note that each term, unlike those in Eqs. (3.1)-(3.3), contains a factor of ${Y_{\dots}}^2$, which is in general different for each field.[2] Recall, that we have chosen the hypercharges to be integer for all SSM particles, see Eq. (2.14). Thus the left hand side is integer. However, given this normalization, the hypercharges of the heavy fermions need not be integer and the quantity in square brackets need not be in $\mathbb{Z}$. Thus the right hand side can take on *any* value within $\mathbb{Z}$. Therefore Eq. (3.17) poses no constraint.

2. Now $\mathcal{A}_{YXX} = 0$. Analogously to Eq. (3.17), we get

$$\sum_i Y_i\, {q_i}^2 \;=\; -N\left[\sum_i Y_i\, m_i(m_i\, N + 2q_i) \;-\; \sum_j Y_{\mathrm{D1}}^j\, p_j(p_j N - 2X_{\mathrm{D1}}^j)\right]. \tag{3.18}$$

By considering only the Y_{D1}^j, we see that [...] is not necessarily an integer, just as in the previous case. Thus Eq. (3.18) is of no use from the low-energy point of view. Here we disagree with Refs. [39, 28] about the reason why

[2]In the case of the non-Abelian linear anomalies $\mathcal{A}_{CCX}$ and $\mathcal{A}_{WWX}$, one encounters a factor proportional to the Dynkin index instead. This is a common factor for all fields provided they are all in the fundamental representation of $SU(3)_C$ and $SU(2)_W$, respectively.

$\mathcal{A}_{YYX}$ and $\mathcal{A}_{YXX}$ do not impose useful constraints on $\boldsymbol{Z}_N$ symmetries. It is *not* the (overall) normalization of the Abelian charges, but the fact that these charges are in general different for each field.

3. Next, we consider the cubic anomaly $\mathcal{A}_{XXX}$. Here we do not have a mixture of known and unknown charges: We do not know any of the $U(1)_X$ charges. We obtain for the anomaly equation

$$\begin{aligned}\sum_i {q_i}^3 &= -\sum_i \left(3{q_i}^2 m_i N + 3q_i {m_i}^2 N^2 + {m_i}^3 N^3\right) \\ &\quad -\sum_j \left(3{X^j_{\rm D1}}^2 p_j N - 3X^j_{\rm D1} {p_j}^2 N^2 + {p_j}^3 N^3\right) \\ &\quad -\tfrac{1}{8}\sum_{j'} {p'_{j'}}^3 N^3 \,. \end{aligned} \tag{3.19}$$

If fractional $X^j_{\rm D1}$ were allowed, again no extraction of a meaningful constraint is feasible, since in this case the right hand side of Eq. (3.19) is not necessarily of the form $N\cdot\mathbb{Z}$. However, as outlined in Sects. 2.1 and 2.2, we only consider integer X charges here. We shall investigate the case of fractional X charges for the heavy fields in Sect. 3.5, since the difference can be meaningful in cosmology [60, 61, 62].

3.4 The Cubic Anomaly

In this section, we restrict ourselves to integer charges for *all* chiral superfields [39, 28] and investigate the resulting consequences of the cubic anomaly constraint on possible DGSs. The calculation is similar to the calculation in Sect. 3.2, *i.e.* it involves many case distinctions. Using Eq. (2.17), we can express the left hand side (LHS) of Eq. (3.19) in terms of n, p, and m

$$\begin{aligned}\text{LHS} &= -n\cdot\left(13n^2 + 18np - 21nm + 18p^2 + 21m^2\right) \\ &\quad + p\cdot\left(-3p^2 + 9pm + 9m^2\right) + 3m^3 \,, \end{aligned} \tag{3.20}$$

where we have made use of the fact that there are three generations of quarks and leptons in the SSM as well as one pair of Higgs doublets. Even when disregarding the restrictions on the heavy-particle content arising from the linear constraints, the right hand side (RHS) of Eq. (3.19) *cannot* take on arbitrary integer values. We shall denote it as $\text{RHS} \equiv \text{RHS}_1 + \text{RHS}_2 + \text{RHS}_3$, with a term for each line in Eq. (3.19). We now investigate these terms individually.

(a) $\underline{\text{RHS}_2}$: Factoring N, we see that the term RHS_2 contributes a multiple of N to the RHS. However, it might be that this term cannot take on every possible multiple of N, regardless of what the choice of heavy particles is. For $(3\,|N)$, we can again write $N = 3N'$ ($N' \in \mathbb{N}$), and rewrite the last term as $p_j^3 N^3 = 3p_j^3 N^2 N'$. We can thus factor $3N$ and therefore the term RHS_2 can take on *at most* values $\in 3N \cdot \mathbb{Z}$. By adding appropriate sets of heavy Dirac particles with simple charges, it is straightforward to show that *any* multiple of $3N$ can be obtained. For DGSs with $\neg\,(3\,|N)$, any element $\in N \cdot \mathbb{Z}$ can be obtained.

(b) $\underline{\mathrm{RHS}_3}$: For odd N, $p'_{j'}$ has to be even [see Eq. (2.9)], so that the term RHS_3 is an element of $N^3 \cdot \mathbb{Z}$. For even N, RHS_3 can take on all values $\in \left(\frac{N}{2}\right)^3 \cdot \mathbb{Z}$.

(c) $\underline{\mathrm{RHS}_1}$: The first two terms in RHS_1 are multiples of $3N$, which is included in (a), above. Similarly, the third term is a multiples of N^3 and therefore already included in (b).

Summarizing, the RHS of Eq. (3.19) can only take on values obeying

$$\begin{aligned} \mathrm{RHS} &= 3N \cdot \mathbb{Z} + \begin{cases} N^3 \cdot \mathbb{Z}, & N = \text{odd}, \\ \left(\frac{N}{2}\right)^3 \cdot \mathbb{Z}, & N = \text{even}, \end{cases} \qquad \text{for} \quad (3\,|N)\,, \\ &= \begin{cases} 9N' \cdot \mathbb{Z}, & (3\,|N), \qquad N = \text{odd}, \\ 9\frac{N'}{2} \cdot \mathbb{Z}, & \neg\,(12\,|N), \\ 9N' \cdot \mathbb{Z}, & (12\,|N), \end{cases} \Big\} \; N = \text{even}, \end{aligned} \tag{3.21}$$

where $N' = N/3$, as before. Furthermore

$$\mathrm{RHS} \;=\; N \cdot \mathbb{Z} + \left(\frac{N}{2}\right)^3 \cdot \mathbb{Z}, \qquad N = \text{even}, \qquad \text{for} \quad \neg\,(3\,|N). \tag{3.22}$$

Now consider the LHS, while taking the linear constraints of Sect. 3.1 into account. Again, we investigate the cases $\neg\,(3\,|N)$ and $(3\,|N)$ separately.

1. $\boldsymbol{\neg\,(3\,|N)}$**:** The DGSs of Eq. (3.13), satisfying the linear constraints, require $n = p = 0$ and $m = \frac{N}{2}$. Thus the LHS becomes $3 \cdot \left(\frac{N}{2}\right)^3$ [*cf.* Eq. (3.20)]. Comparing with Eq. (3.22), we see that the cubic anomaly cancellation condition can be satisfied for all anomaly-free DGSs of Table 3.1 with $\neg\,(3\,|N)$, *i.e.* the cubic anomaly results in no new constraint.

2. $\boldsymbol{(3\,|N)}$**:** We consider the remaining four categories of Table 3.1 in turn.

   (*i*) $\boldsymbol{(3\,|N),\ N = \mathrm{odd}}$**:** Eq. (3.21) shows that the RHS must be a multiple of $9N'$. Therefore the LHS must also be a multiple of $9N'$. From the corresponding row in Table 3.1, we see that in this case $n = 0$, $p = \ell_p N'$ and $m = \ell_m N'$. Inserting this into the LHS as given in Eq. (3.20) yields

$$\text{LHS} = \left(-3{\ell_p}^3 + 9{\ell_p}^2\ell_m + 9\ell_p{\ell_m}^2 + 3{\ell_m}^3\right)\cdot N'^3\,. \tag{3.23}$$

For the case where $\ell_p = \ell_m$, we can satisfy the condition $(9N'\,|\text{LHS})$ for all N which are subsumed in this category, *i.e.* any $N \in 6\cdot\mathbb{N}+3$. The remaining cases of Table 3.1, where $\ell_p \neq \ell_m$, require $(3\,|N'^2)$, and hence $N = 18\cdot\mathbb{N}+9$.

(*ii*) $\boldsymbol{(3\,|N),\ N = \textbf{even}}$**:** From Table 3.1 we have in this case: $n = 0$, $p = \ell_p N'$ and $m = s_m\frac{N'}{2}$. The LHS then becomes

$$\text{LHS} = \left(-24{\ell_p}^3 + 36{\ell_p}^2 s_m + 18\ell_p{s_m}^2 + 3{s_m}^3\right)\cdot\left(\frac{N'}{2}\right)^3. \tag{3.24}$$

Due to the form of the RHS for $\neg\,(12\,|N)$ [*cf.* Eq. (3.21)] , we need $(9\frac{N'}{2}\,|\text{LHS})$. This leads to three nontrivial possibilities for arbitrary N in this category ($N = 12\cdot\mathbb{N}+6$): $[\ell_p = 0 \wedge s_m = 3]$, $[\ell_p = 1 \wedge s_m = 2]$, and $[\ell_p = 1 \wedge s_m = 5]$. All DGSs can satisfy the cubic anomaly constraint if $(3\,|N'^2)$, hence if $N = 36\cdot\mathbb{N}+18$.

Considering $(12\,|N)$ yields exactly the same three sets (ℓ_p, s_m) for nontrivial possible DGSs with arbitrary $N \in 12\cdot\mathbb{N}$. All DGSs are allowed if $(3|N'^2)$, *i.e.* for $N = 36\cdot\mathbb{N}$.

Combining the results for $\neg\,(12\,|N)$ and $(12\,|N)$, we find that for each $N \in 6\cdot\mathbb{N}$ there are three allowed nontrivial DGSs. Taking $N \in 18\cdot\mathbb{N}$, *any* DGS satisfying the linear constraints is compatible with the cubic constraint.

(*iii*) $\boldsymbol{(9\,|N),\ N = \textbf{odd}}$**:** From Table 3.1 we obtain in this case $n = N'$, $p = (1+3\ell_p)N''$ and $m = (2+3\ell_m)N''$. Inserting this into Eq. (3.20) gives

$$\begin{aligned}\text{LHS} = \Big[&-27{\ell_p}^3 + 27{\ell_p}^2(-5+3\ell_m) + 9\ell_p(-23+18\ell_m+9{\ell_m}^2)\\ &+(-122+18\ell_m-108{\ell_m}^2+27{\ell_m}^3)\Big]\cdot N'\cdot N''^2.\end{aligned} \tag{3.25}$$

As 122 is not a multiple of 9, whereas the other coefficients in the square brackets are, $(9N'\,|\text{LHS})$ [which is necessary due to Eq. (3.21)] requires $(9\,|N''^2)$. Thus we need N to be an odd multiple of 27, *i.e.* $N = 54\cdot\mathbb{N}+27$. For such N, all linearly allowed DGSs are consistent with the cubic anomaly condition.

(*iv*) **$(9\,|N)$, $N =$ even:** From Table 3.1 we have in this case $n = N'$, $p = (1+3\ell_p)N''$ and $m = (1+3s_m)\frac{N''}{2}$. The LHS then becomes

$$\begin{aligned}\text{LHS} = \Big[&-216{\ell_p}^3 + 108{\ell_p}^2(-13+3s_m) + 18\ell_p(-119+18s_m+9{s_m}^2) \\ &+ (-1291+585s_m-297{s_m}^2+27{s_m}^3)\Big]\cdot\frac{N'}{2}\cdot\left(\frac{N''}{2}\right)^2. \end{aligned} \tag{3.26}$$

1291 is not a multiple of 9 (it is actually a prime), whereas the remaining coefficients in square brackets are multiples of 9. Therefore the LHS is not a multiple of $9\frac{N'}{2}$ in the case of $\neg\,(12\,|N)$, respectively $9N'$ in the case of $(12\,|N)$ [*cf.* Eq. (3.21)], unless $(9\,|N''^2)$. Thus the cubic anomaly constraint requires $N \in 54\cdot\mathbb{N}$ in this category. All linearly allowed DGSs are possible for these values of N.

In Table 3.2, we have summarized the results. We show those N, as well as the powers (n, p, m), in the case of only integer X charges, which satisfy both the linear anomaly constraints of Sect. 3.2 (*cf.* Table 3.1), as well as the cubic anomaly equation considered here. The main effect of the cubic anomaly constraint consists in reducing the (infinite) list of possible DGSs. Considering $N = 3$ for instance, there are four independent g_N symmetries allowed in Table 3.1. However, only one of these, namely the case where $(n, p, m) = (0, 1, 1)$, complies with Table 3.2. This corresponds to $\boldsymbol{B}_3$, *i.e.* baryon triality discussed by IR.

Another example is $N = 6$. Here we have nine linearly allowed DGSs, while only three are left after imposing the cubic anomaly constraint: R_6^3, $R_6^2L_6^2$ and $R_6^5L_6^2$. The first two are physically equivalent to $\boldsymbol{M}_p$ and $\boldsymbol{B}_3$ from the low-energy point of view. We shall denote $\boldsymbol{P}_6 \equiv R_6^5L_6^2$, as proton hexality. This is a special discrete symmetry, which we return to in Sect. 4.1. For $N = 9$ there are $4+9$ linearly allowed g_N, of which only four are also consistent with the cubic anomaly condition. $N = 27$ is the first case for $(3|N)$, where the cubic anomaly does not reduce the number of allowed DGSs.

3.5 Charge Rescaling

So far, we have assumed that hypercharge shifted discrete symmetries, as in Eq. (2.15), are equivalent and *all* chiral superfields have integer $U(1)_X$ charges. However, from the *low-energy* point of view, this latter assumption is too restrictive [58, 59]. To see this in our analysis, consider an example from Table 3.2, where $N = 18$. The powers of the elementary discrete gauge group generators, Eq. (2.16), are given by

$$n = 0, \qquad p = 6\cdot(0,1), \qquad m = 3\cdot s_m, \quad s_m = 0,1,\ldots,5\,, \tag{3.27}$$

$\boldsymbol{Z}_N$ Category		n	p	m	possible N
$\neg(3\,\vert N)$	N even	0	0	$\frac{N}{2}$	$2 \cdot \mathbb{N}$
$(3\,\vert N)$	N odd	0	$(0,1)\cdot N'$	$(0,1,2)\cdot N'$	$9\cdot(2{\cdot}\mathbb{N}+1)$
		0	N'	N'	$3\cdot(2{\cdot}\mathbb{N}+1)$
	N even	0	$(0,1)\cdot N'$	$(0,1,2,3,4,5)\cdot\frac{N'}{2}$	$18\cdot\mathbb{N}$
		0	0	$\frac{N}{2}$	$6\cdot\mathbb{N}$
		0	N'	N'	$6\cdot\mathbb{N}$
		0	N'	$5\cdot\frac{N'}{2}$	$6\cdot\mathbb{N}$
$(9\,\vert N)$	N odd	N'	$(1,4,7)\cdot N''$	$(2,5,8)\cdot N''$	$27\cdot(2{\cdot}\mathbb{N}+1)$
	N even	N'	$(1,4,7)\cdot N''$	$(1,4,7,10,13,16)\cdot\frac{N''}{2}$	$54\cdot\mathbb{N}$

Table 3.2: Compatibility of the linear and the cubic anomaly constraints in the case of integer $U(1)_X$ charges for *all* chiral superfields. For each $\boldsymbol{Z}_N$ category, the allowed values of N are given in the far right column. The DGSs are specified by the set (n,p,m), in accordance with Eq. (2.16). We employ the notation: $N' \equiv N/3$, $N'' \equiv N/9$, and N', $N'' \in \mathbb{N}$. For special values of N, all linearly allowed DGSs are compatible with the cubic anomaly condition. However, four classes of DGSs within the categories $(3\,|N)$ (rows 3, 5, 6, 7) are possible for less constrained N.

which are all multiples of the common factor $F = 3$. The X charges of the SSM fields, $q_i + m_i N$, are given [see Eq. (2.17)] as linear combinations of n, p, m, and N, and are therefore also all multiples of F, in our example. From the low-energy point of view, with the heavy fields integrated out, such a charge assignment is indistinguishable from a scaled one with charges $(q_i + m_i N)/F$. After the breakdown of $U(1)_X$, the residual DGS is then a $\boldsymbol{Z}_{N/F}$ instead of a $\boldsymbol{Z}_N$. However, the $\boldsymbol{Z}_{N/F}$ does not necessarily satisfy the cubic anomaly, with only integer X charges. In our example, we have $N/F = 6$, which, according to Table 3.2, satisfies the cubic anomaly only for very special values of (n,p,m).

This integer rescaling only applies to the X charges of the SSM chiral superfields. For the *heavy* fermions, it is typically not possible and leads to fractional charges. From a bottom-up approach, experiments would determine the rescaled DGS group $\boldsymbol{Z}_{N/F}$. When searching for the possible (low-energy) anomaly-free DGSs, we therefore relax our original assumption of integer X charges and instead allow fractional charges for the heavy sector, only. We then denote the DGS

N	n	p	m	DGSs
2	0	0	1	R_2
3	0	0	1	R_3
	0	1	$(0,1,2)$	$L_3,\ L_3R_3,\ L_3R_3^2$
6	0	0	1	R_6
	0	2	$(1,3,5)$	$L_6^2R_6,\ L_6^2R_6^3,\ L_6^2R_6^5$
9	3	1	$(2,5,8)$	$A_9^3L_9R_9^2,\ A_9^3L_9R_9^5,\ A_9^3L_9R_9^8$
	3	4	$(2,5,8)$	$A_9^3L_9^4R_9^2,\ A_9^3L_9^4R_9^5,\ A_9^3L_9^4R_9^8$
	3	7	$(2,5,8)$	$A_9^3L_9^7R_9^2,\ A_9^3L_9^7R_9^5,\ A_9^3L_9^7R_9^8$
18	6	2	$(1,7,13)$	$A_{18}^6L_{18}^2R_{18},\ A_{18}^6L_{18}^2R_{18}^7,\ A_{18}^6L_{18}^2R_{18}^{13}$
	6	8	$(1,7,13)$	$A_{18}^6L_{18}^8R_{18},\ A_{18}^6L_{18}^8R_{18}^7,\ A_{18}^6L_{18}^8R_{18}^{13}$
	6	14	$(1,7,13)$	$A_{18}^6L_{18}^{14}R_{18},\ A_{18}^6L_{18}^{14}R_{18}^7,\ A_{18}^6L_{18}^{14}R_{18}^{13}$

Table 3.3: All fundamental DGSs satisfying the linear and the cubic anomaly cancellation conditions. The heavy-fermion charges, X^j, are allowed to be fractional. The three DGSs highlighted by grey boxes can be realized with only integer heavy-fermion $U(1)_X$ charges.

$\boldsymbol{Z}_{N/F}$ with the *maximally* rescaled X charges as the *fundamental* DGS, *i.e.* F is the largest common factor of N and all $q_i + m_i N$. In Table 3.3, we present the complete list of fundamental DGSs, obtained from Table 3.2. We see that after rescaling, the infinite number of DGSs listed in Table 3.2 is reduced to a finite set of 27 fundamental $\boldsymbol{Z}_N$ symmetries: one with $N=2$, four with $N=3$, four with $N=6$, nine with $N=9$, and nine with $N=18$.

Refs. [58, 59] pointed out that the cubic anomaly constraint is in general too restrictive on *low-energy* anomaly-free DGSs due to possible rescalings. Comparing Table 3.2 with Table 3.3, presents a classification within the SSM of the solutions to this problem. As emphasized earlier, the cubic anomaly constraint is compatible with *all* five classes of linearly allowed DGSs presented in Table 3.1, however only for restricted values of N. Rescaling the charges and allowing for fractionally charged heavy fermions, eliminates the influence of the $\mathcal{A}_{XXX}$ condition on the fundamental DGSs completely. In other words, *all* linearly allowed *fundamental* DGSs are compatible with the cubic anomaly constraint. Therefore, Eq. (3.19) contains only information about whether or not the heavy-fermion $U(1)_X$ charges need to be fractional or not. Of the fundamental DGSs listed in Table 3.3, solely $\boldsymbol{M}_p \equiv R_2$, $\boldsymbol{B}_3 \equiv R_3 L_3$ and $\boldsymbol{P}_6 \equiv R_6^5 L_6^2$ are consistent with both the linear and the cubic anomaly conditions, without including fractionally charged heavy particles.

3.6 Isomorphisms

It is interesting to note that of the nine fundamental DGSs with $n = 0$, those with $N = 6$ are each equivalent to the requirement of imposing R_2 (*i.e.* matter parity) *along with* one of the four fundamental $\boldsymbol{Z}_3$ symmetries. Explicitly one has

$$R_2 \times R_3 L_3 \;\cong\; R_6^5 L_6^2\,, \qquad \Longleftrightarrow \qquad \boldsymbol{M}_p \times \boldsymbol{B}_3 \cong \boldsymbol{P}_6 \tag{3.28}$$

$$R_2 \times R_3 \;\cong\; R_6\,, \tag{3.29}$$

$$R_2 \times L_3 \;\cong\; R_6^3 L_6^2\,, \tag{3.30}$$

$$R_2 \times R_3^2 L_3 \;\cong\; R_6 L_6^2\,. \tag{3.31}$$

In the first line we have given the corresponding isomorphism in terms of matter parity, baryon triality and proton hexality. The reason for this is that the Cartesian product of the cyclic groups $\boldsymbol{Z}_2$ and $\boldsymbol{Z}_3$ is isomorphic to $\boldsymbol{Z}_6$, *i.e.* $\boldsymbol{Z}_2 \times \boldsymbol{Z}_3 \cong \boldsymbol{Z}_6$ [63]. This becomes evident by giving both possible isomorphisms $\boldsymbol{Z}_2 \times \boldsymbol{Z}_3 \to \boldsymbol{Z}_6$:

$$(0,0) \mapsto 0, \quad (0,1) \mapsto 2, \quad (0,2) \mapsto 4, \quad (1,0) \mapsto 3, \quad (1,1) \mapsto 5, \quad (1,2) \mapsto 1, \tag{3.32}$$

$$(0,0) \mapsto 0, \quad (0,1) \mapsto 4, \quad (0,2) \mapsto 2, \quad (1,0) \mapsto 3, \quad (1,1) \mapsto 1, \quad (1,2) \mapsto 5. \tag{3.33}$$

As an example, we calculate the discrete charges in the case of Eq. (3.28). Recalling the relations between q_i and the exponents m, n, and p given in Eq. (2.17), we find for the $\boldsymbol{Z}_2 \times \boldsymbol{Z}_3$ charges, where we compute modulo N [*e.g.* $q_{\bar{U}} = (-1,-1) = (1,2)$]:

$$q_Q = (0,0), \quad q_{\bar{U}} = (1,2), \quad q_{\bar{D}} = (1,1), \quad q_L = (0,2), \quad q_{\bar{E}} = (1,2), \quad q_{H_d} = (1,2), \quad q_{H_u} = (1,1), \tag{3.34}$$

and for the $\boldsymbol{Z}_6$ charges

$$q_Q = 0, \quad q_{\bar{U}} = 1, \quad q_{\bar{D}} = 5, \quad q_L = 4, \quad q_{\bar{E}} = 1, \quad q_{H_d} = 1, \quad q_{H_u} = 5. \tag{3.35}$$

Both charge assignments are related by the isomorphism of Eq. (3.32). Similarly, the $\boldsymbol{Z}_2 \times \boldsymbol{Z}_3$ and the $\boldsymbol{Z}_6$ charges in Eqs. (3.30) and (3.31) are related by this isomorphism. In the case of Eq. (3.29) we have to apply the isomorphism of Eq. (3.33).

3.7 Heavy-Fermion Sector

An interesting question to ask is: Given a DGS in Table 3.3, do we necessarily need heavy fermions in order to cancel the anomalies? In the case of matter parity, R_2, we can answer the question by considering Eq. (3.12). Here, the left hand side equals 3, while the right hand side is $2\cdot\mathbb{Z}+\eta\cdot\mathbb{Z}$. Recalling that the η term originates from heavy Majorana fermions [*cf.* Eq. (3.3)], we find that the symmetry R_2 is only possible if we include a heavy-fermion sector, *e.g.* one right-handed neutrino for each generation.

In the case of the other fundamental DGSs of Table 3.3, let us assume the absence of heavy fermions in what follows. Under this assumption, the anomaly cancellation conditions cannot be satisfied. Inserting the discrete charges of Eq. (2.17) into Eq. (3.3), we obtain

$$\begin{aligned} 13n+3p-3m &= N\cdot\Bigg[2m_{H_d}+2m_{H_u} \\ &\quad + \sum_k (6m_{Q_k}+3m_{\bar{U}_k}+3m_{\bar{D}_k}+2m_{L_k}+m_{\bar{E}_k})\Bigg], \end{aligned} \tag{3.36}$$

where k is a generation index. For even N, the right hand side in Eq. (3.36) is even. However, the left hand side is odd for the $\boldsymbol{Z}_2$, $\boldsymbol{Z}_6$ and $\boldsymbol{Z}_{18}$ DGSs. Therefore heavy fermions are necessary in these cases.

For the remaining 4+9 $\boldsymbol{Z}_3$ and $\boldsymbol{Z}_9$ symmetries, the right hand side of Eq. (3.36) can be both, even or odd. We thus employ the cubic anomaly constraint of Eq. (3.19). For the $\boldsymbol{Z}_9$ symmetries the RHS of Eq. (3.19) is always a multiple of 27. The left hand side of the cubic anomaly condition, given in Eq. (3.25), is $-122\cdot3+27\cdot\mathbb{Z}$, which is not a multiple of 27. Thus the fundamental $\boldsymbol{Z}_9$ symmetries also require heavy fermions.

For the four $\boldsymbol{Z}_3$ symmetries the RHS of Eq. (3.19) is always a multiple of 9. Eq. (3.23) shows that the LHS of Eq. (3.19) is a multiple of 9 only in the case of the R_3L_3 symmetry. Hence the other three fundamental $\boldsymbol{Z}_3$ symmetries require heavy fermions. But also R_3L_3 cannot satisfy the anomaly constraints without a heavy-fermion sector:[3] Although R_3L_3 is neither ruled out by $\mathcal{A}_{GGX}=0$ nor $\mathcal{A}_{XXX}=0$ alone, it is in conflict when combining both conditions; the LHS of Eq. (3.19) for R_3L_3 yields 18, [*cf.* Eq. (3.23)], whereas the RHS is a multiple of 27, as we now show. It is given by

$$-\sum_i \left(3{q_i}^2 m_i N + 3q_i {m_i}^2 N^2 + {m_i}^3 N^3\right), \tag{3.37}$$

[3]Here we disagree with Ibáñez's conclusion in Ref. [59]. See also Ref. [64].

where i runs over all chiral superfields. The last two terms within the parentheses are multiples of 27, which is not true for the first one. However, evaluating the sum and applying our knowledge of the q_i, we find

$$\sum_i 3{q_i}^2 m_i N = 3N \cdot \left[2 \cdot m_{H_d} + 2 \cdot m_{H_u} + \sum_k \Big(3 \cdot m_{\bar{U}_k} + 3 \cdot m_{\bar{D}_k} + 2 \cdot m_{L_k} + 4 \cdot m_{\bar{E}_k} \Big) \right], \tag{3.38}$$

where k denotes a generation index. The numerical coefficients inside the brackets are the product of the squared discrete charges and the multiplicity of the particle species. For example, we have 3 colors of quark fields $\bar{U}_k$ with $q_{\bar{U}_k} = -1$, thus $3 \cdot {q_{\bar{U}_k}}^2 = 3$. We can now adopt the gravity – gravity – $U(1)_X$ anomaly constraint of Eq. (3.36) to rewrite Eq. (3.38). Recalling that $N = 3$, $n = 0$, and $m = p = 1$ for $R_3 L_3$, we get

$$\sum_i 3{q_i}^2 m_i N \;\; = \;\; -9 \cdot \sum_k \Big(6 \cdot m_{Q_k} - 3 \cdot m_{\bar{E}_k} \Big) \,, \tag{3.39}$$

also a multiple of 27. This completes our proof.

In conclusion: The 27 fundamental DGSs we have found are *only* anomaly-free with a $U(1)_X$-charged heavy-fermion sector.

Chapter 4

Physics of Anomaly-Free DGSs

The fundamental anomaly-free discrete gauge symmetries forbid some operators in the superpotential and the Kähler potential while others are allowed. Experimentally motivated demands on the low-energy theory prefer two DGSs: baryon triality, $\boldsymbol{B}_3$, and proton hexality, $\boldsymbol{P}_6$; the latter we propose as the new discrete symmetry of the MSSM (instead of R parity). Both scenarios do not suffer the cosmological domain wall problem. Two explicit examples prove the existence of anomaly-free high-energy $U(1)_X$ gauge extensions of the SSM.

4.1 Physics of the Fundamental DGSs and the MSSM

In the previous chapter, we have derived a finite set of fundamental, anomaly-free low-energy DGSs. Now, we would like to confront the discrete symmetries with their phenomenology, *i.e.* we investigate the correspondingly allowed SSM operators. In particular, we study the effect of the 27 fundamental DGSs given in Table 3.3 on the crucial baryon- and/or lepton-number violating superpotential and Kähler potential operators [28, 20]; these are the four operators in the second line of Eq. (1.1) and all (dimension-five) operators of Eq. (1.2). Table 4.1 summarizes which operators are allowed for each fundamental anomaly-free DGS. The symbol ✓ indicates that an operator is allowed. Thus, for example, matter parity (R_2, *i.e.* $m = 1$, $n = p = 0$, $N = 2$) allows the operators $[H_d H_u]_F$ ($q_{H_d} + q_{H_u} = n = 0$), but also the dimension-five baryon-number violating operators $[QQQL]_F$ ($3q_Q + q_L = -n - p = 0$) and $[\bar{U}\bar{U}\bar{D}\bar{E}]_F$ ($2q_{\bar{U}} + q_{\bar{D}} + q_{\bar{E}} = -n + p = 0$), as well as the lepton-number violating operators $[LH_uLH_u]_F$ ($2q_L + 2q_{H_u} = 2m - 2n - 2p = 2 = 0 \bmod 2$). We have included the bilinear operators LH_u (unlike IR), since even under the most general complex field rotation [65], they cannot be eliminated, when taking into account the corresponding soft-breaking terms [66].

	R_2	R_3L_3	R_3	L_3	$R_3^2L_3$	$R_6^5L_6^2$	R_6	$R_6^3L_6^2$	$R_6L_6^2$	all $\boldsymbol{Z}_9$ & $\boldsymbol{Z}_{18}$
H_dH_u	✓	✓	✓	✓	✓	✓	✓	✓	✓	
LH_u		✓								
$LL\bar{E}$		✓								
$LQ\bar{D}$		✓								
$\bar{U}\bar{D}\bar{D}$				✓						
$QQQL$	✓		✓				✓			
$\bar{U}\bar{U}\bar{D}\bar{E}$	✓		✓				✓			
$QQQH_d$				✓						
$Q\bar{U}\bar{E}H_d$		✓								
LH_uLH_u	✓	✓				✓				
$LH_uH_dH_u$		✓								
$\bar{U}\bar{D}^*\bar{E}$		✓								
$H_u{}^*H_d\bar{E}$		✓								
$Q\bar{U}L^*$		✓								
$QQ\bar{D}^*$				✓						

Table 4.1: Physical consequences of the 27 fundamental DGSs. The Higgs Yukawa couplings $LH_d\bar{E}$, $QH_d\bar{D}$, and $QH_u\bar{U}$ are allowed for every DGS we consider by construction. The symbol ✓ denotes that the corresponding operator is possible for a given DGS. All anomaly-free fundamental $\boldsymbol{Z}_9$ and $\boldsymbol{Z}_{18}$ symmetries forbid the operators listed in the left column.

We now demand the existence or absence of certain operators on phenomenological grounds and thus further narrow down our choice of DGSs.

- We have not included the term $[\mu\, H_dH_u]_F$ in the original list leading to Eqs. (2.10)-(2.12), since in principle it can be generated dynamically [67, 68, 69, 70]. From a low-energy point of view we must have $\mu \neq 0$, and it must be of order the weak scale [71, 72]. There are attempts in the literature to combine a dynamical mechanism to generate $\mu \neq 0$ with an anomaly-free DGS, see for example Refs. [48, 49]. This is beyond the scope of this work.

If we explicitly require the $[\mu H_d H_u]_F$-operator in our theory, then as can be seen from Table 4.1, all fundamental $\boldsymbol{Z}_9$ and $\boldsymbol{Z}_{18}$ symmetries are excluded.

- Concerning proton decay, see *e.g.* Refs. [73, 74], if we wish to exclude up to dimension-five baryon-number violating operators, we are left with the DGSs: R_3L_3 ($\boldsymbol{B}_3$), $R_3^2L_3$, $R_6^5L_6^2$ ($\boldsymbol{P}_6$), $R_6^3L_6^2$, and $R_6L_6^2$. For R_2 ($\boldsymbol{M}_p$), R_3, or R_6, $QQQL$ and $\bar{U}\bar{U}\bar{D}\bar{E}$ must be suppressed by some mechanism due to the stringent bounds on proton decay, see *e.g.* Ref. [29, 75]. The DGS L_3 is significantly constrained by the bounds on $\bar{U}\bar{D}\bar{D}$ from heavy nucleon decay, see Ref. [32].

- Now, consider neutrino masses. Without right-handed neutrinos, we can generate masses at tree level through the terms LH_uLH_u and LH_u (via mixing with the neutralinos), or via loop diagrams involving $LL\bar{E}$ or $LQ\bar{D}$ [40, 76, 77, 78]. Hence, the DGSs R_2 ($\boldsymbol{M}_p$), R_3L_3 ($\boldsymbol{B}_3$), and $R_6^5L_6^2$ ($\boldsymbol{P}_6$) can incorporate neutrino masses without right-handed neutrinos.[1] However, right-handed neutrinos can easily be included as heavy Majorana fermions obeying Eq. (2.9). If the corresponding $U(1)_X$ charges allow Dirac neutrino mass terms, we obtain massive light neutrinos via the seesaw mechanism [79, 80, 81, 82]. But in this case, LH_uLH_u must be allowed by the $\boldsymbol{Z}_N$ symmetry as well: invariance of the Dirac mass terms for neutrinos as well as the Majorana mass terms implies a $\boldsymbol{Z}_N$-invariant LH_uLH_u term.

If we combine these phenomenological requirements, we are left with only two DGSs: baryon triality $\boldsymbol{B}_3$, and proton hexality $\boldsymbol{P}_6$. It is remarkable that these discrete symmetries also survived in Sect. 3.5, *i.e.* they can be discrete gauge anomaly-free with only *integer* heavy-fermion charges. However, we would like to go a step further. In Sect. 1.2, we defined the MSSM as the SSM restricted by $\boldsymbol{M}_p$. When considering the MSSM as a low-energy effective theory, the dangerous operators $QQQL$ and $\bar{U}\bar{U}\bar{D}\bar{E}$ are *allowed*. This is a highly unpleasant feature of the MSSM. IR already pointed this out as an advantage of the R-parity violating MSSM with $\boldsymbol{B}_3$, which does not suffer this problem. Here we propose a different solution: *We define the MSSM as the SSM which is restricted by proton hexality* $\boldsymbol{P}_6$. The only phenomenological difference to the conventional MSSM with $\boldsymbol{M}_p$ is with respect to baryon-number violation. However, given the stringent bounds on proton decay, we find this new definition of the MSSM significantly better motivated. Note that in the language of IR, $\boldsymbol{P}_6$ is a generalized matter parity (GMP).

[1] It is not possible to generate neutrino masses in the SSM in the case of R_3 or R_6. They allow for the lepton-number violating terms $QQQL$ and $\bar{U}\bar{U}\bar{D}\bar{E}$ but conserve $B - L$.

4.2 Domain Walls

Next, we consider domain walls which pose a severe cosmological problem if they occur [83]. It is commonly held that a spontaneously broken discrete symmetry leads to domain walls. In particular, this is expected to occur in the SSM with Higgs fields being charged under the $\boldsymbol{Z}_N$ symmetry. The reasoning goes as follows: If there existed different degenerate vacua which are related by a discrete $\boldsymbol{Z}_N$ transformation, the Universe would generally be composed of different domains. In the transition regions between such domains, energy would get stored. Regions in the Universe with different vacua would therefore be separated by so-called domain walls. A quantitative estimate shows that the amount of energy would overclose the Universe [84], hence contradicting the current standard model of cosmology, see *e.g.* Ref. [85] and references therein. In contrast, we do not expect domain walls if the Higgs' discrete charges are zero. However, by this reasoning the first set of charges below Eq. (3.13), ($q_{H_u} = 1$, $q_{H_d} = -1$) implies the existence of domain walls, whereas the second set, standard matter parity ($q_{H_u} = 0$, $q_{H_d} = 0$), does not. As stated in Sect. 3.2, these two symmetries are related by a simple hypercharge shift. They have the same low-energy superpotential and soft terms. Hence the resulting scalar potentials are identical. Therefore the two theories have the same vacuum structure, and either both have or both do not have domain walls.

If the SSM vacuum $\{\upsilon_{H_d}, \upsilon_{H_u}\}$ has zero $\boldsymbol{Z}_N$ charge, then it is unique. If it transforms nontrivially under $\boldsymbol{Z}_N$ then there are upto N distinct ground states $\{\upsilon_{H_d}, \upsilon_{H_u}\}$, $\{\upsilon_{H_d}{}', \upsilon_{H_u}{}'\}$, $\{\upsilon_{H_d}{}'', \upsilon_{H_u}{}''\}$, ... related by $\boldsymbol{Z}_N$ transformations. In the latter case, there are however no domain walls, if the $\boldsymbol{Z}_N$ transformation of the vacuum in a given domain can be compensated by a $U(1)_Y$ gauge transformation. Explicitly, we demand there exists a combined $\boldsymbol{Z}_N + \boldsymbol{Y}$-transformation $\boldsymbol{T}$, such that $\boldsymbol{T}(H_{d,u}) = H_{d,u}$, *i.e.*

$$\exists\, \alpha(x): \qquad \exp\left[\mathrm{i}\tfrac{2\pi}{N} \cdot q_{H_{d,u}} + \mathrm{i}\alpha(x) \cdot Y_{H_{d,u}}\right] H_{d,u} = H_{d,u}. \tag{4.1}$$

$\alpha(x) \in \mathbb{R}$ is the gauge parameter of $U(1)_Y$. This is equivalent to

$$\tfrac{2\pi}{N} \cdot q_{H_{d,u}} + \alpha(x) \cdot Y_{H_{d,u}} = 2\pi \cdot I_{d,u}\,, \qquad \text{with } \ I_{d,u} \in \mathbb{Z}\,. \tag{4.2}$$

These two equations can be combined to

$$I_u = \tfrac{1}{N \cdot Y_{H_d}} \cdot \left(q_{H_u} \cdot Y_{H_d} - q_{H_d} \cdot Y_{H_u} + N \cdot Y_{H_u} \cdot I_d\right), \tag{4.3}$$

$$\alpha(x) = \tfrac{2\pi}{N \cdot Y_{H_d}} \cdot (N \cdot I_d - q_{H_d}). \tag{4.4}$$

The second equation defines the required gauge transformation. We can simplify the first equation, using the hypercharge relation $Y_{H_u} = -Y_{H_d}$

$$N \cdot (I_u + I_d) = q_{H_d} + q_{H_u}. \tag{4.5}$$

This can only be fulfilled if the $\boldsymbol{Z}_N$ charges of the two Higgs, just like their hypercharges, are inverse to each other (in the sense of a mod N calculation). This is equivalent to the requirement that the μ term is allowed by $\boldsymbol{Z}_N$. This is *e.g.* the case for $\boldsymbol{M}_p$ as the Higgs fields are uncharged $(q_{H_d}, q_{H_u}) = (0,0)$, R_2 $(1,1)$, $\boldsymbol{B}_3$ $(2,1)$ and $\boldsymbol{P}_6$ $(1,5)$. We stress that this argument does not rely on $U(1)_X$ being nonanomalous.

Let us exemplarily illustrate the hypercharge shift to a DGS with uncharged Higgs doublets for the case of $\boldsymbol{P}_6$. In the basis of IR the discrete charges of the SSM chiral superfields are given by Eq. (2.17), with $n = 0$, $p = 2$ and $m = 5$. Adding $\frac{5}{3}$ times the hypercharge, leads to a discrete charge assignment with neutral Higgs fields, see Eq. (4.6).

	Q	$\bar{U}$	$\bar{D}$	L	$\bar{E}$	H_d	H_u
q_i	0	-5	5	-2	7	-5	5
Y_i	-1	4	-2	3	-6	3	-3
$q_i + \frac{5}{3} \cdot Y_i$	$-\frac{5}{3}$	$\frac{5}{3}$	$\frac{5}{3}$	3	-3	0	0

(4.6)

As the discrete charges of the quarks are now fractional, we should multiply them (and also $N = 6$) by three. This yields a *fieldwise* $\boldsymbol{Z}_{18}$ DGS which is explicitly free of domain walls. However, as has been shown in Sect. 2.3, the theory remains a *termwise* $\boldsymbol{P}_6$ model. Thus proton hexality does not suffer the domain wall problem.

4.3 Two Gauged $\boldsymbol{P}_6$ Models

In this section, we explicitly construct two $U(1)_X$ gauge models, which are spontaneously broken to proton hexality $\boldsymbol{P}_6$. We consider this a demonstration of existence, not necessarily optimized models. The first is only meant to be a toy model, which does not yield a phenomenologically viable fermionic mass spectrum. The second is much more elaborate, leading to a physically acceptable description of the quark and lepton masses and mixings in terms of X charges. Concerning the origin of the needed nonrenormalizable interaction terms, there are several sources imaginable, see *e.g.* [86]: Either the terms occur near the string scale or they are generated by integrating out heavy vectorlike pairs of $G_{\rm SM}$-charged states (the original Froggatt-Nielsen mechanism [87]). Here we adopt the first viewpoint and thus use a simple operator analysis. We assume the $U(1)_X$ breaking superfields to be suppressed by $M_{\rm grav}$.

4.3.1 A Toy Model

We first list the transformation properties of the SSM fields under a *generation-independent* $U(1)_X$ gauge symmetry, as well as under $U(1)_Y$. In our model, we wish to avoid all fractional charges. We have thus rescaled the hypercharges to integers [*cf.* Eq. (2.14)].

	Q	$\bar{U}$	$\bar{D}$	L	$\bar{E}$	H_d	H_u
$U(1)_X$	0	-5	5	-2	1	-5	11
$U(1)_Y$	-1	4	-2	3	-6	3	-3

(4.7)

As pointed out in Sect. 3.5, the cubic anomaly constraint of Eq. (3.19) can be satisfied, without including *fractionally* $U(1)_X$-charged heavy particles only in the cases where the fundamental DGS is either R_2, R_3L_3, or $R_6^5L_6^2 \equiv \boldsymbol{P}_6$, *i.e.* matter parity, baryon triality, or proton hexality.

Next, we introduce a set of heavy chiral superfields, $A_{k=1,\ldots,5}$, $B_{k=1,\ldots4}$, which are $SU(3)_C \times SU(2)_W$ singlets and have *integer* $U(1)_X \times U(1)_Y$ charges:

	$A_{\rm M}$	$A_{\rm D1}$	$A_{\rm D2}$	$A'_{\rm D1}$	$A'_{\rm D2}$	$B_{\rm D1}$	$B_{\rm D2}$	$B'_{\rm D1}$	$B'_{\rm D2}$
$U(1)_X$	-15	-2	8	-2	8	8	10	-8	-10
$U(1)_Y$	0	0	0	0	0	-7	7	5	-5

(4.8)

That is, the A_k are $G_{\rm SM}$ singlets, whereas the B_k have pairwise equal and opposite $U(1)_Y$ charges [see Sect. 3.3, below Eq. (3.17)]. In order to break the $U(1)_X$ symmetry to the DGS $\boldsymbol{P}_6$, we introduce two further chiral superfields, Φ_+, Φ_-, which are $G_{\rm SM}$ singlets and have the $U(1)_X$ charges $X_{\Phi_+} = 6$ and $X_{\Phi_-} = -6$. As a *pair*, $\Phi_\pm$ therefore do not contribute to any anomaly. The charges, 6, -6, are chosen to obtain a $\boldsymbol{Z}_6$ symmetry, after the scalar components of $\Phi_\pm$ acquire a vacuum expectation value, $\langle\Phi_\pm\rangle = v$.

Observe that the $U(1)_X$ charges of the light fields in Eq. (4.7) are those of the remnant DGS $\boldsymbol{P}_6 \equiv R_6^5L_6^2$: insert $m = 5$ and $p = 2$ into Eq. (2.17) and perform additional mod 6 shifts. Thus we do indeed obtain $\boldsymbol{P}_6$ after $U(1)_X$ breaking via the $\Phi_\pm$ VEVs. Furthermore, note that the $U(1)_X$ charges of $\Phi_\pm$ and of the fields in Eq. (4.8) satisfy the conditions for heavy fermions given in Eqs. (2.8) and (2.9). Explicitly, we can see that the A_k, B_k fields obtain a large mass at the scale of $U(1)_X$ breaking: the superpotential operators $M_{\rm grav} \cdot A_{\rm M}A_{\rm M}(\Phi_+/M_{\rm grav})^5$, $M_{\rm grav} \cdot A_{\rm D1}A_{\rm D2}(\Phi_-/M_{\rm grav})$, $M_{\rm grav} \cdot A'_{\rm D1}A'_{\rm D2}(\Phi_-/M_{\rm grav})$, $M_{\rm grav} \cdot B_{\rm D1}B_{\rm D2}(\Phi_-/M_{\rm grav})^3$ and $M_{\rm grav} \cdot B'_{\rm D1}B'_{\rm D2}(\Phi_+/M_{\rm grav})^3$ are $U(1)_X$ gauge invariant.

The particle content and the gauge charges, are chosen so as to cancel all anomalies of the high-energy theory. We shall consider in turn: $\mathcal{A}_{CCY}$, $\mathcal{A}_{WWY}$, $\mathcal{A}_{GGY}$, $\mathcal{A}_{YYY}$; $\mathcal{A}_{CCX}$, $\mathcal{A}_{WWX}$; $\mathcal{A}_{GGX}$; $\mathcal{A}_{XXX}$, $\mathcal{A}_{YYX}$, $\mathcal{A}_{YXX}$. The first four anomalies can only arise from particles with nonzero hypercharge, namely, the SSM particles and the B_k. However, the B_k do not contribute, as they have pairwise equal and opposite hypercharge. Thus the first four anomalies vanish as in the SSM.

Only SSM particles contribute to $\mathcal{A}_{CCX}$ and $\mathcal{A}_{WWX}$. Inserting the X charges of Eq. (4.7) one easily finds that $\mathcal{A}_{CCX}$ cancels in each generation, while $\mathcal{A}_{WWX}$ vanishes only for three generations of quarks and leptons and one pair of Higgs doublets.

When considering the $\mathcal{A}_{GGX}$ anomaly, we see that the B_k do not contribute; the fields with equal and opposite X charges cancel. Therefore we are left with

$$\begin{aligned}\mathcal{A}_{GGX} &= 3(6X_Q + 3X_{\bar{U}} + 3X_{\bar{D}} + 2X_L + X_{\bar{E}}) \\ &\quad +2X_{H_d} + 2X_{H_u} + X_{A_\mathrm{M}} + X_{A_\mathrm{D1}} + X_{A_\mathrm{D2}} + X_{A'_\mathrm{D1}} + X_{A'_\mathrm{D2}}, \qquad (4.9)\end{aligned}$$

which is zero for our choice of $U(1)_X$ charges.

The last three anomalies are the most difficult to cancel, since they involve squares and cubes of the Abelian charges. However, the B_k give no contribution to $\mathcal{A}_{XXX}$, while the hypercharge neutral A_k do not contribute to $\mathcal{A}_{YYX}$ and $\mathcal{A}_{YXX}$. It is a tedious but nevertheless straightforward calculation to show that they do indeed cancel with our choice of $U(1)_X$ charges.

Unfortunately, the fermionic mass spectrum of this toy model is completely unphysical. For instance, the mass terms for the up- and down-type quarks derive from the superpotential operators $QH_u\bar{U}(\Phi_-/M_\mathrm{grav})$ and $QH_d\bar{D}$, respectively; in this toy model, the mass of the bottom quark is therefore suppressed by a factor of $\upsilon/M_\mathrm{grav} < 1$ compared to the top quark. Besides, as the X charges are generation independent, we cannot explain the mass hierarchy between the the three families within this toy model.

4.3.2 A Physical Model

As a second example, we explicitly present a phenomenologically more convincing, *generation-dependent* $U(1)_X$ gauge model, constructed in collaboration with C. A. Savoy and S. Lavignac.

We first list in Table 4.2 the $U(1)_X$ charges of all the chiral superfields in this model. The G_SM singlets $\Phi_\pm$ constitute the vectorlike pair of $U(1)_X$ breaking superfields with equal VEVs υ. The $A_{\dots}$ are G_SM singlets as well but do not acquire VEVs, we introduce them solely for the sake to cancel $\mathcal{A}_{GGX}$ and $\mathcal{A}_{XXX}$. All the other (mixed) anomalies vanish within the particle content of the SSM. Contrary to the toy model of Subsection 4.3.1, all heavy particles are neutral under G_SM; however, some of the extra heavy fields are fractionally X-charged.

$X_{\Phi_+} = 6, \qquad X_{\Phi_-} = -6$

$X_{H_d} = 1, \qquad X_{H_u} = -49$

Generation i	X_{Q_i}	$X_{\bar{U}_i}$	$X_{\bar{D}_i}$	X_{L_i}	$X_{\bar{E}_i}$
1	−12	13	−25	40	−77
2	−12	37	−13	40	−17
3	0	49	−13	40	−53

$X_{A_{\mathrm{D1}}} = -\frac{27}{2}, \quad X_{A_{\mathrm{D2}}} = -\frac{45}{2}, \quad X_{A'_{\mathrm{D1}}} = \frac{1}{2}, \quad X_{A'_{\mathrm{D2}}} = \frac{71}{2}, \quad X_{A_{\mathrm{M}}} = 3$

Table 4.2: The $U(1)_X$ charges of all chiral superfields in our physical model. $\Phi_\pm$ break $U(1)_X$, the $A_{\dots}$ are G_{SM}-uncharged heavy particles.

The breaking of $U(1)_X$ generates the MSSM Yukawa coupling constants with textures that produce the observed fermionic mass spectrum as well as acceptable mixing matrices, see (a). Furthermore, $U(1)_X$ leaves a $\boldsymbol{Z}_{12}$ symmetry as a remnant which, after integrating out the $A_{\dots}$, yields $\boldsymbol{P}_6$, see (b).

(a) With

$$\epsilon \equiv \frac{\upsilon}{M_{\mathrm{grav}}} = 0.22, \tag{4.10}$$

we obtain an effective superpotential which contains the first line of Eq. (1.1) and the mass terms for the left-handed neutrinos ($h^\nu_{ij}/M_\nu \cdot L_i H_u L_j H_u$), where

$$\boldsymbol{h^U} \sim \begin{pmatrix} \epsilon^8 & \epsilon^4 & \epsilon^2 \\ \epsilon^8 & \epsilon^4 & \epsilon^2 \\ \epsilon^6 & \epsilon^2 & 1 \end{pmatrix}, \quad \boldsymbol{h^D} \sim \epsilon^2 \cdot \begin{pmatrix} \epsilon^4 & \epsilon^2 & \epsilon^2 \\ \epsilon^4 & \epsilon^2 & \epsilon^2 \\ \epsilon^2 & 1 & 1 \end{pmatrix}, \quad \boldsymbol{h^E} \sim \epsilon^2 \cdot \begin{pmatrix} \epsilon^4 & \epsilon^2 & 1 \\ \epsilon^4 & \epsilon^2 & 1 \\ \epsilon^4 & \epsilon^2 & 1 \end{pmatrix}, \tag{4.11}$$

$$\mu \sim \epsilon^8 \cdot M_\mu, \qquad \boldsymbol{h^\nu} \sim \epsilon^3 \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{4.12}$$

To get the μ term and the neutrino masses of the correct order of magnitude, we rely on the existence of intermediate mass scales: $M_\mu \sim 10^8$ GeV (which's necessity has been already anticipated by Refs. [88, 89] for anomaly-free Froggatt-Nielsen models without heavy $G_{\rm SM}$ charged matter) and $M_\nu \sim 10^{12}$ GeV. After diagonalization one gets for the masses of the electrically charged SM fermions $m_u : m_c : m_t \sim \epsilon^8 : \epsilon^4 : 1$, $m_d : m_s : m_b \sim \epsilon^4 : \epsilon^2 : 1$, $m_e : m_\mu : m_\tau \sim \epsilon^4 : \epsilon^2 : 1$, $m_\tau : m_b : m_t \sim \epsilon^2 : \epsilon^2 : 1$. For the mixing matrices we obtain an anarchical MNS matrix, which is compatible with experiment, see *e.g.* Refs. [90, 91, 92], as well as a CKM matrix which looks like

$$\boldsymbol{U}^{\mathbf{CKM}} \sim \begin{pmatrix} 1 & 1 & \epsilon^2 \\ 1 & 1 & \epsilon^2 \\ \epsilon^2 & \epsilon^2 & 1 \end{pmatrix}. \tag{4.13}$$

Thus we have to rely on some moderate fine-tuning among the unknown $\mathcal{O}(1)$ coefficients to be entirely satisfactory.
Furthermore, we get the following mass terms for the heavy fields:

$$\epsilon^6 \cdot M_{\rm grav}\ A_{\rm D1} A_{\rm D2}, \qquad \epsilon^6 \cdot M_{\rm grav}\ A'_{\rm D1} A'_{\rm D2}, \qquad \epsilon \cdot M_{\rm grav}\ A_{\rm M} A_{\rm M}. \tag{4.14}$$

(*b*) After $U(1)_X$ breaking we are left with an overall $\boldsymbol{Z}_{12}$ DGS, since $|X_{\Phi_\pm}| = 6$ and all SSM particles' X charges are integers, whereas the $A_{...}$'s X charges are half-odd integers. But as can be seen above, the $A_{...}$ are quite heavy, so that they all can be integrated out at around $\epsilon^6 M_{\rm grav} \sim 10^{14}$ GeV, leaving the *fundamental* (in the sense of Section 3.5) DGS $\boldsymbol{P}_6$.

With the above two models we have shown that it is indeed possible to construct anomaly-free models featuring a low-energy DGS (proton hexality) different from those found in Ref. [28]. In this respect our study complements that of Ibáñez and Ross.

Chapter 5

Summary and Outlook, Part I

In summary, we have systematically investigated discrete $\boldsymbol{Z}_N$ gauge symmetries for arbitrary values of N. We have classified the anomaly-free theories, depending on whether the necessary (see Sect. 3.7) heavy fermions must have fractional X charges or not. Through a rescaling of the X charges, we have, for a low-energy point of view, reduced this infinite set to a finite fundamental set: All theories related by rescaling lead to the same low-energy superpotential and Kähler potential. For this fundamental set we have investigated the phenomenological properties in detail. We have found two outstanding DGSs, the second of them being beyond IR: *(i)* baryon triality, $\boldsymbol{B}_3$, which allows for low-energy lepton-number violation, but no dimension-five or lower proton decay operators, and *(ii)* proton hexality, $\boldsymbol{P}_6$. The latter has a renormalizable superpotential which conserves lepton and baryon number and prohibits nonrenormalizable dimension-five proton decay operators. This is one of the main results of this paper and we propose $\boldsymbol{P}_6$ as *the* new discrete gauge symmetry of the MSSM, instead of matter parity. Both baryon triality and proton hexality are free of domain walls.

In our study of discrete gauge symmetries we did not consider (A) a possible generation dependence of the $\boldsymbol{Z}_N$ charges, (B) R symmetries and (C) the inclusion of the GS mechanism to cancel the effects of anomalies. We want to conclude with some remarks on these possible extensions of our investigation.

(A) We have already noted that generation dependence of the discrete charges is not allowed in the quark sector, *cf.* footnote 4 in Chapter 2. The discrete charges of L_i and $\bar{E}_i$ however might well be generation dependent. In this case one would encounter zero textures in the coupling constants h^E_{ij} of the superpotential terms $L_i H_d \bar{E}_j$. Due to the three nonvanishing charged lepton masses, $\boldsymbol{h^E}$ has to be a rank-three matrix. This can be achieved only if there is a generational index $i \in \{1, 2, 3\}$ for each $j \in \{1, 2, 3\}$ so that $q_{L_i} = -q_{\bar{E}_j} - q_{H_d} \bmod N$. For three *different* q_{L_i} there are only three nonzero entries in matrix $\boldsymbol{h^E}$, which however need not be diagonal. With regard to the neutrino masses, the matrix $\boldsymbol{h^\nu}$ [see above Eq. (4.12) in the example of Subsect. 4.3.2] is (at best) of rank one in the case of three different q_{L_i}. In order to obtain the experimentally required two or three nonzero neutrino masses, we need to rely on additional radiative contributions to the neutrino mass matrix in this case. See also Refs. [53, 54].

(B) In our generalization of the work on discrete symmetries by Ibáñez and Ross [28], we considered only $\boldsymbol{Z}_N$ symmetries commuting with supersymmetry. It is, in principle, straightforward to extend the study to $\boldsymbol{Z}_N$ R symmetries. Ibáñez and Ross have investigated the case of $N = 2, 3$. They introduce a generator P_N (*cf.* Sect. 4 of Ref. [28]), which only acts on the superspace coordinate θ and leaves invariant the chiral superfields of the theory.

(C) Gauge theories have to be either anomaly-free with vanishing anomaly coefficients $\mathcal{A}_{...}$ or anomaly-free due to the cancellation of possibly nonzero anomaly coefficients by the transformation of the dilaton superfield, see *e.g.* Ref. [55]. The latter is known as the Green-Schwarz (GS) mechanism and is explained in more detail in Sect. 6.3. Discretizing the GS anomaly conditions, one ends up with only two linear anomaly constraints, *i.e.* one less than in the non-GS case, *cf.* Ref. [59]. Thus the number of "GS anomaly-free" DGSs is enlarged compared to the number of anomaly-free (in the sense of Ibáñez and Ross) DGSs.

The number increases even further because, as pointed out in Sect. 2.3, hypercharge shifts are not allowed in a general GS scenario; so it is not possible to reduce the number of parameters defining the low-energy DGS: All values of r have to be considered in Eq. (2.13).

It would be interesting to see whether it is possible to systematically investigate these extensions. If feasible, one might find some other new discrete symmetries besides $\boldsymbol{P}_6$ which are attractive candidates for *the* discrete gauge symmetry of the MSSM.

Part II

Froggatt-Nielsen Models and DGSs

Chapter 6

The Framework of Froggatt and Nielsen in Supersymmetry

The mechanism of Froggatt and Nielsen explains the hierarchy of the fermionic mass spectrum in terms of generation-dependent $U(1)_X$ charges. The coupling constant of the bilinear MSSM superpotential term is naturally obtained at the phenomenologically required value by applying the Giudice-Masiero mechanism. In order to cancel the $U(1)_X$ gauge anomalies we invoke the Green-Schwarz mechanism. $U(1)_X$ is spontaneously broken by a nonvanishing Fayet-Iliopoulos term which originates in the Dine-Seiberg-Wen-Witten mechanism; thus the vacuum expectation value of the flavon field is determined slightly below the gravitational scale. Constraints on the $U(1)_X$ charges are derived from the quark and lepton masses and mixings as well as the requirement of the theory being (GS) anomaly-free.

6.1 Froggatt-Nielsen Mechanism

It is one of the most puzzling features of the Standard Model that the masses of quarks and leptons occupy such an enormous range of energies. The mass of the top quark is as large as 175 GeV while the electron weighs only 511 keV. Taking into account neutrinos as well, the mass ratio of quarks and leptons increases even further to a value of more than 10^{11}. Within the SM as well as the SSM, these wildly spread masses originate in the Yukawa coupling constants, which in turn are taken as free parameters of the theory and have to be put in by hand. This situation is rather unattractive and therefore regarded as a problem: the problem of flavor.

In 1979, Froggatt and Nielsen [87] put forward an idea to explain the masses of the fermions at least magnitudewise. Their approach requires the introduction of a conservation law that distinguishes between different flavors dynamically. The easiest possibility consists in enlarging $G_{\rm SM}$ by an additional *generation-dependent* $U(1)_X$ local gauge factor. In the supersymmetrized version of the SM, the chiral superfields thus carry different X charges in general. In the following, we discuss such an extension of the SSM and describe a model of flavor based on the Froggatt-Nielsen (FN) mechanism in its simplest form.

We introduce the above mentioned $U(1)_X$ gauge symmetry as well as an X-charged SM singlet left-chiral superfield, the so-called flavon superfield Φ. As we do not observe an additional gauge symmetry at low energies, we have to break the $U(1)_X$ at some higher energy. This is achieved by the scalar component of the flavon superfield acquiring a vacuum expectation value, $\langle\Phi\rangle \equiv \upsilon$, slightly below the gravitational scale $M_{\rm grav} = 2.4\times 10^{18}$ GeV. Note that this situation is very similar to the scenario considered in Part I of this thesis; there the $U(1)_X$ is spontaneously broken to a discrete gauge symmetry. The ratio $\epsilon \equiv \upsilon/M_{\rm grav}$ denotes the expansion parameter in which we formulate the hierarchy of the fermionic mass spectrum. Relying on the Dine-Seiberg-Wen-Witten mechanism, we show in Sect. 6.4 that the flavon VEV naturally takes a value such that $\epsilon \approx 0.2$.

The low-energy superpotential and Kähler potential terms are expected to originate from some sort of string theory. After compactification to four dimensions we therefore encounter renormalizable as well as nonrenormalizable $U(1)_X$-invariant operators. Contrary to our convention in Part I of this thesis, we normalize the X charges such that $X_\Phi = -1$; thus the X charges are fractional in general. Before the breakdown of $U(1)_X$, the "parent terms" of the MSSM superpotential operators $h^E_{ij} L_i H_d \bar{E}_j$ [*cf.* Eq. (1.1)] are given by

$$H^E_{ij} \left(\frac{\Phi}{M_{\rm grav}}\right)^{X_{L_i}+X_{H_d}+X_{\bar{E}_j}} L_i\, H_d\, \bar{E}_j \ , \tag{6.1}$$

where H^E_{ij} are unknown dimensionless complex coupling constants of order one, that is

$$\frac{1}{\sqrt{10}} \ \lesssim\ |H^E_{ij}|\ \lesssim\ \sqrt{10}. \tag{6.2}$$

At this point we emphasize that only operators with a nonnegative integer exponent of the flavon field are allowed in Eq. (6.1). Negative integers are forbidden due to holomorphy of the superpotential; fractional exponents are forbidden because the Lagrangian of any quantum field theory satisfying the principles of special relativity, quantum mechanics and cluster decomposition has to be a *polynomial* of quantum fields, see *e.g.* Ref. [93]. Given the case that the X charges yield negative or fractional exponents in Eq. (6.1) for some pairs of indices (i,j), we obtain zero textures in the corresponding Yukawa matrix. We will return to the blessings and the problems of such zero textures in Sect. 6.2 and in Subsect. 6.5.1. If not stated otherwise, we assume zero textures to be absent.

Neglecting the effects of the renormalization group equations, the low-energy Yukawa coupling constants h^E_{ij} are generated after the spontaneous breakdown of $U(1)_X$. They are related to the more fundamental $\mathcal{O}(1)$ parameters H^E_{ij} by

$$h^E_{ij} \;=\; H^E_{ij} \left(\frac{\langle\Phi\rangle}{M_{\text{grav}}}\right)^{X_{L_i}+X_{H_d}+X_{\bar{E}_j}} \;=\; H^E_{ij}\;\epsilon^{X_{L_i}+X_{H_d}+X_{\bar{E}_j}}. \tag{6.3}$$

It is straightforward to apply the above discussion also to the trilinear MSSM superpotential operators $h^D_{ij}Q_iH_d\bar{D}_j$ and $h^U_{ij}Q_iH_u\bar{U}_j$. We see that, in the framework of Froggatt and Nielsen, the hierarchical structure of the Yukawa coupling constants h^E_{ij}, h^D_{ij} and h^U_{ij} is related to the $U(1)_X$ charges of the chiral superfields. The unknown $\mathcal{O}(1)$ coefficients, *i.e.* H^E_{ij}, H^D_{ij} and H^U_{ij} cannot be predicted in by the FN mechanism and thus remain unspecified. Therefore it is only possible to calculate order-of-magnitude-wise. In our notation we thus neglect all $\mathcal{O}(1)$ factors and simply write:

$$h^E_{ij} \sim \epsilon^{X_{L_i}+X_{H_d}+X_{\bar{E}_j}}, \qquad h^D_{ij} \sim \epsilon^{X_{Q_i}+X_{H_d}+X_{\bar{D}_j}}, \qquad h^U_{ij} \sim \epsilon^{X_{Q_i}+X_{H_u}+X_{\bar{U}_j}}. \tag{6.4}$$

Given the numerical value of ϵ, *cf.* Sect. 6.4, it is possible to compare the above ϵ structure with the phenomenologically required Yukawa matrices. Thus the generation-dependent X charges are constrained significantly, see Sect. 6.5.

Besides the trilinear MSSM superpotential operators we obtain also other renormalizable and nonrenormalizable superpotential and Kähler potential terms via the FN mechanism. Similar to Eq. (6.1) we need nonnegative integer total X charges for all nonvanishing superpotential terms. With regard to the Kähler potential, negative integer total X charges are allowed as well. The reason being that the Kähler potential is in general not holomorphic. Thus we can add an appropriate power of *either* the flavon superfield Φ *or* its complex conjugate Φ^*, with $X_{\Phi^*} = 1$, to an effective low-energy operator in order to obtain $U(1)_X$ invariance above the scale of $U(1)_X$ breaking.

As an example of nontrilinear superpotential terms originating in the FN mechanism, let us consider the case of a $\boldsymbol{P}_6$-invariant scenario. From Table 4.1 we find that, up to dimension five, only the dimension-three operator H_dH_u and the dimension-five operators $L_iH_uL_jH_u$ are invariant under proton hexality. If we want these superpotential terms to be generated via the FN mechanism, the X charges have to satisfy

$$X_{H_d} + X_{H_u} \in \mathbb{N}, \qquad \text{and} \qquad X_{L_i} + X_{H_u} + X_{L_j} + X_{H_u} \in \mathbb{N}, \tag{6.5}$$

for all generational indices $i, j = 1, 2, 3$. Then the derived low-energy operators take the ϵ structure

$$\begin{aligned} \mu\ H_d H_u &\sim M_\mu\ \epsilon^{X_{H_d}+X_{H_u}}\ H_d H_u, \qquad (6.6)\\ \frac{h^\nu_{ij}}{M_\nu}\ L_i H_u L_j H_u &\sim \frac{1}{M_\nu}\ \epsilon^{X_{L_i}+X_{L_j}+2X_{H_u}}\ L_i H_u L_j H_u. \qquad (6.7) \end{aligned}$$

For dimensional reasons we need to introduce two mass scales, M_μ and M_ν, as in the example of Subsect. 4.3.2. It would be desirable to have only a few mass scales. However, identifying M_μ with the gravitational scale M_{grav} yields a μ parameter of the order $\epsilon^{X_{H_d}+X_{H_u}} \cdot 10^{18}$ GeV. Comparing with the phenomenologically required value of $\mathcal{O}(100 - 1000 \text{ GeV})$, the ϵ suppression would have to be very strong. For $\epsilon = 0.2$ the exponent in Eq. (6.6) must take a value of around 23. The unnaturalness of such a scenario constitutes the core of the μ problem. Giudice and Masiero [68] as well as Kim and Nilles [67] invented a method to generate the μ term dynamically at the correct order of magnitude more naturally. In Sect. 6.2 we explain this mechanism in detail.

So far we have given examples of superpotential operators only. As already mentioned above, the situation is a little different for Kähler potential terms. This is due to the fact that the Kähler potential is not holomorphic; so it is allowed to multiply the complex conjugate, Φ^*, of the left-chiral superfield Φ to the effective low-energy operators in order to achieve $U(1)_X$ invariance. Consider the baryon-triality conserving operators $\bar{U}_i \bar{D}^*_j \bar{E}_k$ (*cf.* Table 4.1) as an example. Assuming nonnegative total X charges for these terms, *i.e.* $X_{\text{total}} \geq 0$ with $X_{\text{total}} = X_{\bar{U}_i} - X_{\bar{D}_j} + X_{\bar{E}_k}$, we have the following Kähler potential terms before the breaking of $U(1)_X$

$$\sum_{n=0}^{\infty} \frac{C_{ijk}}{M_{\text{grav}}} \left(\frac{\Phi\,\Phi^*}{{M_{\text{grav}}}^2}\right)^n \left(\frac{\Phi}{M_{\text{grav}}}\right)^{X_{\bar{U}_i}-X_{\bar{D}_j}+X_{\bar{E}_k}} \bar{U}_i \bar{D}^*_j \bar{E}_k\ . \qquad (6.8)$$

C_{ijk} denote dimensionless $\mathcal{O}(1)$ coupling constants. Note that the X charges of the superfields $\bar{D}_j$ enter with a negative sign in the exponent. As $\Phi\Phi^*$ is a $G_{\text{SM}} \times U(1)_X$ singlet, it is possible to multiply the second factor without changing the gauge invariance of the terms. After the breakdown of $U(1)_X$, however, the terms with $n = 0$ dominate the infinite sum of Kähler potential operators in Eq. (6.8).
In the case of negative total X charges, *i.e.* $X_{\text{total}} < 0$, we similarly obtain the Kähler potential terms

$$\sum_{n=0}^{\infty} \frac{C_{ijk}}{M_{\text{grav}}} \left(\frac{\Phi\,\Phi^*}{{M_{\text{grav}}}^2}\right)^n \left(\frac{\Phi^*}{M_{\text{grav}}}\right)^{-(X_{\bar{U}_i}-X_{\bar{D}_j}+X_{\bar{E}_k})} \bar{U}_i \bar{D}^*_j \bar{E}_k\ , \qquad (6.9)$$

where the exponent is rendered positive by an overall sign compared to Eq. (6.8); $U(1)_X$ invariance is achieved by additionally exchanging the flavon superfield Φ

with its complex conjugate Φ^*. Again, the dominant terms after $U(1)_X$ breaking are the ones with $n = 0$.

Another type of Kähler potential operators is given by the terms leading to the kinetic part of the Lagrangian. Applying the FN mechanism also to these terms, we obtain for instance

$$\underbrace{K^Q_{ij}\ \epsilon^{|X_{Q_i}-X_{Q_j}|}}_{\equiv\, k^Q_{ij}}\ Q_i^{\dagger}\ e^{4\,T^a\,V^a}\ Q_j, \tag{6.10}$$

in the Kähler potential after $U(1)_X$ breaking, with unknown $\mathcal{O}(1)$ coefficients K^Q_{ij}. Q_i are the quark doublet superfields; with regard to the other chiral superfields, we get similar expressions as in Eq. (6.10). T^a denote the generators of the gauge groups; V^a are the corresponding gauge vector superfields.[1] Concerning the formulation of supersymmetric non-Abelian gauge theories in terms of superfields see, *e.g.*, Ref. [94].
Due to the hermiticity of the Lagrangian, the $\mathcal{O}(1)$ coefficients can be regarded as the entries of a hermitian matrix $\boldsymbol{K^Q}$. Hence, the matrix $\boldsymbol{k^Q}$ which includes the ϵ suppression is hermitian as well. In general, however, $\boldsymbol{k^Q}$ is not unity as assumed in the canonical form of the kinetic terms. Therefore, we must perform a basis transformation. We first diagonalize the hermitian matrix $\boldsymbol{k^Q}$ by a unitary transformation of the superfields Q_i. The new diagonal matrix is denoted as $\tilde{\boldsymbol{k}}^{\boldsymbol{Q}}$. In a second step we absorb $\tilde{\boldsymbol{k}}^{\boldsymbol{Q}}$ into the fields by multiplying the quark doublets by $1/\sqrt{\tilde{\boldsymbol{k}}^{\boldsymbol{Q}}}$. Note that this second transformation is not unitary.
This procedure of redefining the chiral superfields in order to obtain the kinetic terms of the Lagrangian in the canonical form is called the canonicalization of the Kähler potential, short CK.

In order to understand the ϵ structure of the CK basis transformation, let us first consider the 2×2 case as the easiest nontrivial example. The hermitian matrix

$$\boldsymbol{k^Q} \;=\; \begin{pmatrix} K^Q_{11} & K^Q_{12}\,\epsilon^{\Delta X} \\ K^{Q\,*}_{12}\,\epsilon^{\Delta X} & K^Q_{22} \end{pmatrix}, \tag{6.11}$$

with $K^Q_{ij} = \mathcal{O}(1)$ and $\Delta X = |X_{Q_1} - X_{Q_2}|$, can be diagonalized unitarily by

$$\begin{pmatrix} e^{\mathrm{i}\frac{\varphi}{2}} & 0 \\ 0 & e^{-\mathrm{i}\frac{\varphi}{2}} \end{pmatrix} \cdot \begin{pmatrix} \cos\kappa & -\sin\kappa \\ \sin\kappa & \cos\kappa \end{pmatrix}. \tag{6.12}$$

Here φ is the phase of the complex valued $(1,2)$-entry of $\boldsymbol{k^Q}$, *i.e.* $K^Q_{12} = |K^Q_{12}| \cdot e^{\mathrm{i}\varphi}$. The angle κ is given by $\tan 2\kappa = \frac{2\,|K^Q_{12}|}{K^Q_{11}-K^Q_{22}} \cdot \epsilon^{\Delta X}$. As we are only interested in the

[1] The gauge coupling constants are included in the definition of the gauge superfields. Compared to conventions in Ref. [94], our vector superfield includes a factor of $\frac{g_a}{2}$, *i.e.* $g_a V^a \longrightarrow 2V^a$.

ϵ structure of the CK transformation, we do neither worry about complex phases nor factors of $\mathcal{O}(1)$. Approximating the tangent by its Taylor expansion then yields $\sin\kappa \sim \epsilon^{\Delta X}$. Thus, the ϵ structure of the diagonalization matrix is given as

$$\begin{pmatrix} 1 & \epsilon^{\Delta X} \\ \epsilon^{\Delta X} & 1 \end{pmatrix}. \tag{6.13}$$

Having diagonalized the matrix $\boldsymbol{k^Q}$ of Eq. (6.11) we have to normalize it to unity in a second step. This nonunitary transformation does not change the ϵ structure of the CK transformation matrix because both eigenvalues of $\boldsymbol{k^Q}$ are of order one.

For a general hermitian $n \times n$ matrix $\boldsymbol{k^Q}$, the CK transformation matrix $\boldsymbol{C^Q}$ similarly takes the approximate form (see, *e.g.*, Ref. [38])

$$C^Q_{ij} \sim \epsilon^{|X_{Q_i} - X_{Q_j}|}. \tag{6.14}$$

Of course, the Kähler potential has to be canonicalized for *all* chiral superfields. The effects of these CK basis transformations on the FN-generated coupling constants are discussed in Subsect. 6.5.1.

6.2 Giudice-Masiero Mechanism

In the previous section we have seen that the μ term of the superpotential needs to be suppressed by large powers of ϵ if $M_\mu = M_{\text{grav}}$. To avoid such an unnatural scenario, Giudice and Masiero [68] as well as Kim and Nilles [67] proposed an alternative way to obtain the μ term. First we have to make sure that the operator $H_d H_u$ is forbidden in the superpotential due to negative integer total X charge. However, the μ term is finally obtained effectively from the Kähler potential. This dynamical generation of the μ term is known as the mechanism of Giudice and Masiero (GM) and discussed in the following.

Suppose $X_{H_d} + X_{H_u}$ is negative and integer. Then we can establish $U(1)_X$ invariance by multiplying appropriate powers of Φ^* to the product $H_d H_u$. This operator is not holomorphic and thus can only appear in the Kähler potential. The D-term of any Kähler potential operator is invariant under supersymmetry. However, our aim is to generate an effective $H_d H_u$ term in the superpotential from this Kähler potential term. In the GM mechanism a left-chiral $G_{\text{SM}} \times U(1)_X$ singlet superfield Z is introduced for this purpose. This hidden-sector superfield couples to the visible sector, *i.e.* the particles of the SSM, only via gravitation. We assume further that supersymmetry is broken in the hidden sector by the F-component of Z acquiring a VEV. This breaking of supersymmetry is mediated to the visible sector gravitationally, such that $\langle F_Z \rangle \sim m_{\text{soft}} M_{\text{grav}}$, with m_{soft} denoting the mass scale of the soft supersymmetry breaking masses.

The effective superpotential operator H_dH_u can now be traced back to the SM- and $U(1)_X$-invariant Kähler potential term

$$\left(\frac{Z^*}{M_{\text{grav}}}\right)\cdot\left(\frac{\Phi^*}{M_{\text{grav}}}\right)^{-(X_{H_d}+X_{H_u})} H_dH_u. \tag{6.15}$$

Here we omit the $\mathcal{O}(1)$ coefficient which appears in any FN-generated term. At low energies, both $U(1)_X$ and supersymmetry are broken spontaneously: the scalar component of Φ acquires a VEV and so does the F-term of the left-chiral superfield Z. The resulting D-term includes the low-energy operator

$$\int d^2\theta \int d^2\bar{\theta}\left(\frac{\langle F_Z\rangle^*\,\bar{\theta}^2}{M_{\text{grav}}}\right)\cdot\left(\frac{\langle\Phi\rangle^*}{M_{\text{grav}}}\right)^{-(X_{H_d}+X_{H_u})} H_dH_u, \tag{6.16}$$

with θ and $\bar{\theta}$ being the Grassmann coordinates of superspace. Observe that $\bar{\theta}$ does not appear in the product H_dH_u of left-chiral superfields. Performing the integration over $\bar{\theta}$ and inserting the vacuum expectation values $\langle F_Z\rangle \sim m_{\text{soft}}M_{\text{grav}}$ and $\langle\Phi\rangle$ into Eq. (6.16) yields

$$\int d^2\theta\; m_{\text{soft}}\;\epsilon^{-(X_{H_d}+X_{H_u})}\; H_dH_u. \tag{6.17}$$

This is the effective F-term of the bilinear MSSM superpotential operator. Comparing with Eq. (6.6) shows that the mass scale of the μ term is now obtained naturally at the phenomenologically required value; as $X_{H_d}+X_{H_u}$ is negative, the exponent of the ϵ suppression needs an additional minus sign. In the case of nonnegative and integer total X charge for the H_dH_u term, we could construct a $U(1)_X$-invariant Kähler potential operator similar to the one in Eq. (6.15) with Φ^* replaced by Φ and the sign of the exponent reversed. In addition to the FN-generated bilinear operator of Eq. (6.6) we obtain an effective contribution of the form

$$m_{\text{soft}}\;\epsilon^{(X_{H_d}+X_{H_u})}\; H_dH_u \tag{6.18}$$

in the superpotential. Assuming $M_\mu \sim M_{\text{grav}}$, we find that this GM contribution is negligible because $\frac{m_{\text{soft}}}{M_{\text{grav}}} \ll 1$. Concerning general bilinear superpotential operators, the GM-generated term has to be considered only in the case of negative total X charge, *i.e.* where the standard FN procedure is inapplicable due to the holomorphicity of the superpotential.

The above method to generate a superpotential operator dynamically from the Kähler potential is not restricted to bilinear terms. Provided integer total X charge, X_{total}, we can apply the GM mechanism to any superpotential operator. For dimensional reasons we need to multiply appropriate powers of some mass

Type of Operator	GM Mechanism	FN Mechanism
Linear	$m_{\text{soft}}\, M_{\text{grav}}\, \phi_1\, \epsilon^{\lvert X_{\phi_1}\rvert}$	${M_{\text{grav}}}^2\, \phi_1\, \epsilon^{X_{\phi_1}}$
Bilinear	$m_{\text{soft}}\, \phi_1\, \phi_2\, \epsilon^{\lvert X_{\phi_1}+X_{\phi_2}\rvert}$	$M_{\text{grav}}\, \phi_1\, \phi_2\, \epsilon^{X_{\phi_1}+X_{\phi_2}}$
Trilinear	$\frac{m_{\text{soft}}}{M_{\text{grav}}}\, \phi_1\, \phi_2\, \phi_3\, \epsilon^{\lvert X_{\phi_1}+X_{\phi_2}+X_{\phi_3}\rvert}$	$\phi_1\, \phi_2\, \phi_3\, \epsilon^{X_{\phi_1}+X_{\phi_2}+X_{\phi_3}}$

Table 6.1: The effective superpotential operators generated by the FN and GM mechanism, respectively. The general form of linear, bilinear and trilinear terms is given. X_{total} needs to be integer. The standard FN mechanism additionally requires $X_{\text{total}} \geq 0$; in this case the GM contribution can be neglected.

parameter to the corresponding Kähler potential operator. As we do not want to introduce too many mass scales, the most natural choice is M_{grav}. Table 6.2 lists the general form of linear, bilinear, and trilinear superpotential operators obtained via the FN mechanism as well as the GM mechanism. The left-chiral superfields are denoted as ϕ_i, with $i \in \mathbb{N}$. Integer total X charges are required in both cases; the standard FN mechanism however is only possible for $X_{\text{total}} \geq 0$. If allowed, the FN operator always dominates the GM contribution. So the mechanism of Giudice and Masiero is only relevant for negative X_{total}. Note that the coupling of trilinear GM-generated terms is suppressed by a factor of $\frac{m_{\text{soft}}}{M_{\text{grav}}}$ which is at least of $\mathcal{O}(10^{-15})$. Thus, although a FN-forbidden trilinear operator is generated via the GM mechanism, such a GM contribution is negligible and does not lead to any phenomenological consequences. However, it might well be that a FN-forbidden operator with integer total X charge can be obtained via the canonicalization of the Kähler potential (see Subsect. 6.5.1).

6.3 Green-Schwarz Mechanism

In Part I of this thesis we have considered $U(1)_X$ extensions of the SM gauge group G_{SM} which are anomaly-free in the sense that *all* anomaly coefficients vanish identically [*cf.* Eq. (2.6)]. There is, however, an alternative way to obtain an anomaly-free theory, first suggested by Green and Schwarz [95]. Their mechanism is formulated within the framework of superstring theory. From the phenomenological, *i.e.* low-energy point of view, we can restrict ourselves to the four-dimensional analog of the Green-Schwarz (GS) mechanism [96]. Here, nonvanishing anomaly coefficients can be compensated by the nonlinear transformation of the dilaton chiral superfield S under a $U(1)_X$ gauge transformation [97]. In the following, we sketch the four-dimensional GS mechanism and present a derivation of the GS anomaly cancellations conditions.

The compactification of a ten-dimensional superstring theory leads to an effective four-dimensional theory below the string cut-off scale. Generically, this theory is an $N=1$ supersymmetric gauge theory with a universal[2] gauge coupling constant, g_s, and the gauge group structure (see, *e.g.*, Ref. [98])

$$G_{\text{visible}} \times U(1)_X \times U(1) \times \cdots \times U(1) \times G_{\text{hidden}}. \tag{6.19}$$

G_{visible} denotes the gauge group of the visible sector and necessarily includes G_{SM}, *i.e.* $G_{\text{SM}} \subset G_{\text{visible}}$. The gauge group of the hidden sector is G_{hidden}. Both sectors interact only through the Abelian $U(1)$ gauge symmetries and gravitationally. Typically, some of the extra $U(1)$ factors are anomalous; it is, however, possible to rotate the total anomaly into a single $U(1)$, see *e.g.* Ref. [99]. In our notation, this anomalous $U(1)$ is denoted as $U(1)_X$.

This means that the path integral measure is not invariant under $U(1)_X$ but changes by the factor given in Eq. (2.5). In the language of superfields, the $U(1)_X$ gauge transformation of the left-chiral superfields ϕ_i and the $U(1)_X$ gauge vector superfield V_X is given by (see, *e.g.*, Ref. [94])

$$\phi_i \quad \longrightarrow \quad \phi_i' = e^{\mathrm{i}X_{\phi_i}\Lambda_X} \cdot \phi_i, \tag{6.20}$$

$$V_X \quad \longrightarrow \quad V_X' = V_X - \frac{\mathrm{i}}{2}\left(\Lambda_X - \Lambda_X^\dagger\right). \tag{6.21}$$

Here the left-chiral superfield Λ_X parameterizes the $U(1)_X$ gauge transformation; it is a *real function* of the superspace coordinate $y^\mu \equiv x^\mu - \mathrm{i}\theta\sigma^\mu\bar{\theta}$, with $\sigma^\mu = (\mathbb{1}, \vec{\sigma})$, see for example Ref. [100]. As in the case of the vector superfields our convention differs slightly from the one in Ref. [94]: We have defined the factor $-g_X$ into Λ_X. The generator T_X of the $U(1)_X$ gauge transformation is given by $\frac{X_{\phi_i}}{2}$, with X_{ϕ_i} denoting the X charge of the superfield ϕ_i.

The occurrence of nonvanishing anomaly coefficients is solely due to the transformation of the (chiral) fermions in the theory. In order to calculate the anomaly generated by the $U(1)_X$ transformation of Eq. (6.20), we first have to extract the corresponding change in the spin one-half components of the superfields ϕ_i. In the standard decomposition, any left-chiral superfield ϕ_i is written as[3]

$$\phi_i \;=\; \varphi_{\phi_i} \;+\; \sqrt{2}\,\psi_{\phi_i}\theta \;+\; F_{\phi_i}\theta^2, \tag{6.22}$$

with the scalar φ_{ϕ_i}, the fermion ψ_{ϕ_i} and the auxiliary field F_{ϕ_i}. Λ_X is decomposed analogously. We can now calculate the change of the fermionic components ψ_{ϕ_i}

[2] The universality of the gauge coupling constants in string theory is called string unification.

[3] The notation here is a little imprecise as the superfield and its component fields have different superspace arguments. See, for instance, Ref. [94].

resulting from the infinitesimal version of Eq. (6.20). Writing

$$\phi_i' - \phi_i \;=\; \mathrm{i}X_{\phi_i}\left(\varphi_{\Lambda_X} + \sqrt{2}\,\psi_{\Lambda_X}\theta + F_{\Lambda_X}\theta^2\right)\cdot\left(\varphi_{\phi_i} \;+\; \sqrt{2}\,\psi_{\phi_i}\theta \;+\; F_{\phi_i}\theta^2\right), \tag{6.23}$$

we obtain the following transformation of the spin one-half component

$$\psi_{\phi_i}' - \psi_{\phi_i} \;=\; \mathrm{i}X_{\phi_i}\varphi_{\Lambda_X}\psi_{\phi_i} + \mathrm{i}X_{\phi_i}\varphi_{\phi_i}\psi_{\Lambda_X}. \tag{6.24}$$

The second term vanishes because Λ_X is a *real function* of the superspace coordinate y^μ [100]; thus $\psi_{\phi_i} \equiv 0$. Comparing with Eq. (2.4), we find that the path integral measure of the fermionic fields, $\int \mathcal{D}\bar{\psi}_{\phi_i}\,\mathcal{D}\psi_{\phi_i}$, then changes by the factor given in Eq. (2.5) with $\alpha^c(x)\,T^c$ replaced by $\varphi_{\Lambda_X}(x)\,\frac{X_{\phi_i}}{2} = \varphi_{\Lambda_X}(x)\,T_X$. Therefore, the Lagrangian is not invariant under a $U(1)_X$ gauge transformation. We get

$$\begin{aligned}
\Delta\mathcal{L}_{\text{measure}} \;=\; & \frac{g_C^2}{32\pi^2}\,\varphi_{\Lambda_X}\,\epsilon^{\mu\nu\rho\sigma}\;F^a_{C\,\mu\nu}\;F^b_{C\,\rho\sigma}\;\mathrm{Trace}\left[\{T^a_C,T^b_C\}\cdot T_X\right] \\
&+ \frac{g_W^2}{32\pi^2}\,\varphi_{\Lambda_X}\,\epsilon^{\mu\nu\rho\sigma}\;F^a_{W\,\mu\nu}\;F^b_{W\,\rho\sigma}\;\mathrm{Trace}\left[\{T^a_W,T^b_W\}\cdot T_X\right] \\
&+ \frac{g_Y^2}{32\pi^2}\,\varphi_{\Lambda_X}\,\epsilon^{\mu\nu\rho\sigma}\;F_{Y\,\mu\nu}\;F_{Y\,\rho\sigma}\;\mathrm{Trace}\left[\{T_Y,T_Y\}\cdot T_X\right] \\
&+ \frac{g_X^2}{32\pi^2}\,\varphi_{\Lambda_X}\,\epsilon^{\mu\nu\rho\sigma}\;F_{X\,\mu\nu}\;F_{X\,\rho\sigma}\;\mathrm{Trace}\left[\{T_X,T_X\}\cdot T_X\right].
\end{aligned} \tag{6.25}$$

At first sight, the theory therefore looks anomalous if the anomaly coefficients defined in Eq. (2.6) do not vanish. However, the GS mechanism can compensate nonvanishing anomaly coefficients in the four-dimensional theory. It is a nontrivial task to boil down the string theoretical description of physics to the effective low-energy theory. One has to identify those components of the ten-dimensional fields which combine to the low-energy fields in four dimensions; see, *e.g.*, Ref. [101]. Among other fields, the dilaton S arises in such a way. It is a left-chiral superfield with the following nonlinear $U(1)_X$ gauge transformation property (see, *e.g.*, Refs. [97, 55])

$$S \;\longrightarrow\; S' \;=\; S \;-\; \frac{\mathrm{i}}{2}\,\delta_{\mathrm{GS}}\,\Lambda_X. \tag{6.26}$$

Here, δ_{GS} is a real parameter. The MSSM gauge kinetic terms of the low-energy Lagrangian originate in the interactions of the dilaton superfield with the gauge fields [97, 55]

$$\mathcal{L}_{\text{gauge}} \;=\; \frac{1}{4}\int d^2\theta\;k_a\,S\,W^{a\,\alpha}\,W^a{}_\alpha \;+\; \text{h.c.} \tag{6.27}$$

The index a labels the 8+3+1+1 gauge fields of $G_{\rm SM} \times U(1)_X$. The real coefficients k_a are the Kač-Moody levels of the gauge groups; they take positive integer values for non-Abelian symmetries. In the Wess-Zumino gauge, the supersymmetric field strength spinor superfields ${W^a}_\alpha$ are given as [94]

$$ {W^a}_\alpha \;=\; \frac{1}{2}\,\bar{D}^2\, D_\alpha\, V^a \;+\; {\rm i}\, f^{abc}\, \bar{D}^2\, (D_\alpha\, V^b)\, V^c\,, \tag{6.28} $$

with D_α and $\bar{D}^{\dot{\alpha}}$ denoting the supersymmetric covariant derivatives. f^{abc} are the structure constants of the corresponding gauge groups. Note that we have defined a factor of $\frac{g_a}{4}$ into the field strength superfields compared to Ref. [94], *i.e.* $g_a {W^a}_\alpha \longrightarrow 4{W^a}_\alpha$. Now, the scalar component φ_S of the dilaton left-chiral superfield S acquires a (real) vacuum expectation value $\langle\varphi_S\rangle$ which, then, leads to the generation of the MSSM gauge kinetic terms in the low-energy Lagrangian. Recalling our conventions used in Eq. (6.27), the gauge coupling constants are related to the dilaton VEV by $k_a \cdot \langle\varphi_S\rangle = \frac{1}{g_a^2}$. Assuming (see, *e.g.*, Refs. [65, 102])

$$ k_C \;=\; k_W \;=\; \frac{3}{5}\,k_Y\,, \tag{6.29} $$

we get the MSSM unification of the gauge coupling constants, $g_C = g_W = \sqrt{5/3}\,g_Y$. This relation only holds for the following normalization of the gauge group generators T_C^a, T_W^a, and T_Y (see, *e.g.*, Chapter 14 of Ref. [103])

$$ {\rm Trace}\,[\{T_C^a, T_C^b\}] = \delta_{ab}\,, \qquad {\rm Trace}\,[\{T_W^a, T_W^b\}] = \delta_{ab}\,, \qquad T_Y L = \frac{Y_L}{2} L = \frac{1}{2} L\,. \tag{6.30} $$

Thus we need $T_C^a = \frac{\lambda^a}{2}$ and $T_W^a = \frac{\sigma^a}{2}$, with λ^a and σ^a denoting the Gell-Mann and the Pauli matrices, respectively; the hypercharges are normalized so that $Y_L = 1$. Similarly, we can define the string coupling constant g_s by $2 \cdot \langle\varphi_S\rangle \equiv \frac{1}{g_s^2}$. The factor of 2 is discussed in Ref. [104] and becomes relevant in Sect. 6.4.

Having added the gauge kinetic Lagrangian of Eq. (6.27) to the total Lagrangian, we can calculate the variation of $\mathcal{L}_{\rm gauge}$ under a $U(1)_X$ gauge transformation. Requiring that this variation compensates $\Delta\mathcal{L}_{\rm measure}$ of Eq. (6.25), finally leads us to the GS anomaly cancellation conditions.

The $U(1)_X$ gauge transformation of the matter and gauge superfields as well as the dilaton is given by Eqs. (6.20), (6.21), and (6.26), respectively. Since the term ${W_X}^\alpha\, W_{X\alpha}$ is $U(1)_X$-invariant, we are left with the following change in the gauge kinetic Lagrangian

$$ \Delta\mathcal{L}_{\rm gauge} \;=\; -\frac{\delta_{\rm GS}}{8} \int d^2\theta\;k_a\,{\rm i}\,\Lambda_X\; W^{a\,\alpha}\, {W^a}_\alpha \;+\; {\rm h.c.} \tag{6.31} $$

Recalling that the only nonvanishing component field of the left-chiral superfield Λ_X is the *real* scalar function φ_{Λ_X}, we are lead to

$$\begin{aligned}\Delta\mathcal{L}_{\text{gauge}} &= -\frac{g_C^2}{16}\,\varphi_{\Lambda_X}\,\delta_{\text{GS}}\,k_C\,\epsilon^{\mu\nu\rho\sigma}\,F^a_{C\,\mu\nu}\,F^b_{C\,\rho\sigma}\,\delta_{ab}\\ &\quad-\frac{g_W^2}{16}\,\varphi_{\Lambda_X}\,\delta_{\text{GS}}\,k_W\,\epsilon^{\mu\nu\rho\sigma}\,F^a_{W\,\mu\nu}\,F^b_{W\,\rho\sigma}\,\delta_{ab}\\ &\quad-\frac{g_Y^2}{16}\,\varphi_{\Lambda_X}\,\delta_{\text{GS}}\,k_Y\,\epsilon^{\mu\nu\rho\sigma}\,F_{Y\,\mu\nu}\,F_{Y\,\rho\sigma}\\ &\quad-\frac{g_X^2}{16}\,\varphi_{\Lambda_X}\,\delta_{\text{GS}}\,k_X\,\epsilon^{\mu\nu\rho\sigma}\,F_{X\,\mu\nu}\,F_{X\,\rho\sigma}\,,\end{aligned}\qquad(6.32)$$

with the hermitian conjugate (h.c.) already included. Comparing this result with Eq. (6.25) and requiring that $\Delta\mathcal{L}_{\text{measure}}+\Delta\mathcal{L}_{\text{gauge}}\overset{!}{=}0$, we obtain the GS anomaly cancellation conditions. In order to state them in a compact form, recall that we can always find a basis in which the trace of two non-Abelian gauge group generators is given as [105, 103]

$$\text{Trace}\,[\{T^a,T^b\}] \;=\; \delta_{ab}. \qquad(6.33)$$

Thus we get rid of the Kronecker symbols in the first two lines of Eq. (6.32). Using the definition of the anomaly coefficients given in Eq. (2.6), the GS anomaly cancellation conditions finally read (*cf.* also Ref. [55])

$$\frac{\mathcal{A}_{CCX}}{k_C} \;=\; \frac{\mathcal{A}_{WWX}}{k_W} \;=\; \frac{\mathcal{A}_{YYX}}{k_Y} \;=\; \frac{\mathcal{A}_{XXX}}{k_X} \;=\; 2\,\pi^2\,\delta_{\text{GS}}. \qquad(6.34)$$

Similarly, the gravitational interaction generates a nonvanishing anomaly coefficient [105] which has to be canceled by the GS mechanism. With

$$\mathcal{A}_{GGX} \;\equiv\; \text{Trace}\,T_X \;=\; \frac{1}{2}\sum_i X_{\phi_i}\,, \qquad(6.35)$$

the corresponding GS anomaly condition is given by [98, 102, 38]

$$\frac{\mathcal{A}_{GGX}}{12} \;=\; 2\,\pi^2\,\delta_{\text{GS}}. \qquad(6.36)$$

Demanding that the relations of Eqs. (6.34) and (6.36) are satisfied, the effects of *some* nonvanishing anomaly coefficients are compensated by the nonlinear transformation of the dilaton. However, $\mathcal{A}_{YXX}$ as well as the SM anomalies $\mathcal{A}_{CCY}$, $\mathcal{A}_{WWY}$, $\mathcal{A}_{YYY}$, and $\mathcal{A}_{GGY}$ have to vanish identically, *i.e.*

$$\mathcal{A}_{YXX} \;=\; \mathcal{A}_{CCY} \;=\; \mathcal{A}_{WWY} \;=\; \mathcal{A}_{YYY} \;=\; \mathcal{A}_{GGY} \;=\; 0. \qquad(6.37)$$

6.4 The Flavon VEV

Collider experiments tell us that there is only *one* $U(1)$ gauge boson at low energies: the photon or, above the scale of the electroweak symmetry breaking, the B-boson. Therefore, we have to come up with a mechanism that spontaneously breaks the $U(1)_X$ gauge symmetry. Thanks to the anomalous nature of $U(1)_X$, this can be achieved by the Dine-Seiberg-Wen-Witten (DSWW) mechanism [106, 107, 108, 109]. Below the gravitational (or string) scale $M_{\mathrm{grav}} = M_{\mathrm{Planck}}/\sqrt{8\pi}$, an effective Fayet-Iliopoulos term is generated which in turn breaks $U(1)_X$ by the vacuum expectation value of the scalar component of the flavon superfield Φ.

In the four-dimensional language, the Kähler potential of the dilaton superfield S is given at the one-string loop level by[4] [110, 97, 55]

$$-\frac{1}{2}\ln\left(S + S^\dagger - \delta_{\mathrm{GS}}V_X\right). \tag{6.38}$$

The argument of the logarithm is invariant under the $U(1)_X$ transformation of Eqs. (6.26) and (6.21), as can be seen easily:

$$\begin{aligned} S' + S'^\dagger - \delta_{\mathrm{GS}}V'_X &= S - \frac{\mathrm{i}}{2}\delta_{\mathrm{GS}}\Lambda_X + S^\dagger + \frac{\mathrm{i}}{2}\delta_{\mathrm{GS}}\Lambda_X^\dagger - \delta_{\mathrm{GS}}\left[V_X - \frac{\mathrm{i}}{2}\left(\Lambda_X - \Lambda_X^\dagger\right)\right] \\ &= S + S^\dagger - \delta_{\mathrm{GS}}V_X\,. \end{aligned} \tag{6.39}$$

As the logarithm of superfields is defined by its Taylor series, we can expand the expression of Eq. (6.38) around the (real) VEV of the scalar component φ_S of the dilaton superfield. Using the expansion $\ln(1+x) = x - x^2/2 + x^3/3 - x^4/4 + \ldots$, we get

$$\begin{aligned} -\frac{1}{2}\ln\left(S + S^\dagger - \delta_{\mathrm{GS}}V_X\right) &= -\frac{1}{2}\ln 2\langle\varphi_S\rangle - \frac{1}{2}\ln\left(1 + \frac{\Delta S + \Delta S^\dagger - \delta_{\mathrm{GS}}V_X}{2\langle\varphi_S\rangle}\right) \\ &= -\frac{1}{2}\ln 2\langle\varphi_S\rangle - \frac{\Delta S + \Delta S^\dagger - \delta_{\mathrm{GS}}V_X}{4\langle\varphi_S\rangle} + \cdots \end{aligned} \tag{6.40}$$

to first order. ΔS is the dilaton superfield shifted by its VEV, *i.e.* $S = \langle\varphi_S\rangle + \Delta S$. Obviously, a Fayet-Iliopoulos term arises radiatively in the effective Lagrangian:

$$\int d^2\theta \int d^2\bar\theta \;\; 2\,\xi\cdot V_X\,, \qquad \text{with} \quad \xi = \frac{\delta_{\mathrm{GS}}}{8\langle\varphi_S\rangle}\,M^2_{\mathrm{grav}} = \frac{g_s^2\,\delta_{\mathrm{GS}}}{4}\,M^2_{\mathrm{grav}}\,. \tag{6.41}$$

Here we have restored the mass units, so that we can later relate the results of our calculations with experimental observations, as *e.g.* fermionic mass ratios.

[4] In order to be consistent with our conventions and normalizations, we include a factor of $\frac{1}{2}$.

Making use of the anomaly cancellation condition of Eq. (6.36), we can express δ_{GS} in terms of the gravitational anomaly coefficient. We find (*cf.* Ref. [104])

$$\xi \;=\; \frac{g_s^2\,\mathcal{A}_{GGX}}{96\,\pi^2}\,M_{\mathrm{grav}}^2 \;=\; \frac{g_s^2\,M_{\mathrm{grav}}^2}{192\,\pi^2}\sum_i X_{\phi_i}\,. \tag{6.42}$$

The existence of a nonvanishing Fayet-Iliopoulos term in the Lagrangian alters the D-term contribution to the scalar potential. Assuming the $U(1)_X$ gauge vector superfield V_X in the Wess-Zumino gauge, its component fields ${V_{V_X}}^{\mu}$, λ_{V_X}, $\bar{\lambda}_{V_X}$, and D_{V_X} are given by the decomposition (see, *e.g.*, Ref. [94])

$$V_X \;=\; \theta\sigma_\mu\bar{\theta}\,{V_{V_X}}^{\mu} \;+\; \mathrm{i}\theta^2\,\bar{\theta}\bar{\lambda}_{V_X} \;-\; \mathrm{i}\bar{\theta}^2\,\theta\lambda_{V_X} \;+\; \frac{1}{2}\,\theta^2\bar{\theta}^2\,D_{V_X}\,. \tag{6.43}$$

The auxiliary field D_{V_X} occurs in only a few terms of the Lagrangian. Taking into account the Fayet-Iliopoulos term of Eq. (6.41), we obtain

$$\mathcal{L}\,[D_{V_X}] \;=\; \frac{2}{g_X^2}\,D_{V_X}^2 \;+\; \sum_i \varphi_{\phi_i}^\dagger X_{\phi_i}\varphi_{\phi_i}\,D_{V_X} \;+\; \xi\,D_{V_X}. \tag{6.44}$$

The slightly unusual appearance is due to our conventions; compared to Ref. [94] we have $g_X V_X \longrightarrow 2V_X$ and therefore also $g_X D_{V_X} \longrightarrow 2D_{V_X}$ for the component field. Using the Euler-Lagrange equations, we can calculate D_{V_X} in terms of physical fields. This yields

$$D_{V_X} \;=\; -\,\frac{g_X^2}{4}\left(\sum_i \varphi_{\phi_i}^\dagger X_{\phi_i}\varphi_{\phi_i} \;+\; \xi\right). \tag{6.45}$$

The scalar potential of our anomalous $U(1)_X$ theory thus includes the following D-term contribution

$$-\,\mathcal{L}_{\mathrm{scalar}} \;\supset\; \frac{g_X^2}{8}\cdot\left(\sum_i \varphi_{\phi_i}^\dagger X_{\phi_i}\varphi_{\phi_i} \;+\; \xi\right)^2. \tag{6.46}$$

As the other contributions of the scalar potential are all nonnegative (they are squares of F- and D-terms), we get a *supersymmetric* theory only if the scalar component φ_Φ of the flavon superfield Φ acquires an appropriate vacuum expectation value. This VEV compensates the nonvanishing ξ of Eq. (6.46), so that the vacuum has zero potential and is therefore supersymmetric [94]. Normalizing the X charge of the flavon superfield Φ to $X_\Phi = -1$, we can calculate the flavon VEV

$$\upsilon \;=\; \langle\varphi_\Phi\rangle \;=\; \sqrt{\frac{\xi}{-X_\Phi}} \;=\; \sqrt{\frac{g_s^2\,\mathcal{A}_{GGX}}{96\,\pi^2}}\,M_{\mathrm{grav}}\,. \tag{6.47}$$

The expansion parameter ϵ of FN models is thus determined dynamically. Using the GS anomaly cancellation conditions of Eqs. (6.36) and (6.34) together with the relation $2g_s^2 = k_C g_C^2$, we get

$$\epsilon \;=\; \frac{\upsilon}{M_{\text{grav}}} \;=\; \frac{g_C}{4\,\pi}\,\sqrt{\mathcal{A}_{CCX}}\,. \tag{6.48}$$

Assuming that only the SM quarks are charged under $SU(3)_C$ and adopting the normalization of the non-Abelian gauge group generators given in Eq. (6.33), the anomaly coefficient $\mathcal{A}_{CCX}$ takes the form

$$\mathcal{A}_{CCX} \;=\; \frac{1}{2}\sum_k \left(2\,X_{Q_k} \,+\, X_{\bar{U}_k} \,+\, X_{\bar{D}_k}\right), \tag{6.49}$$

with k denoting the three generations. In Sect. 6.5 the phenomenological requirements on the X charges are included. Thus $\mathcal{A}_{CCX}$ can be constrained so that the expansion parameter ϵ is obtained around a value of 0.2 (*cf.* Sect. 8.1), which is close to the Wolfenstein parameter $\lambda_c \sim 0.22$, *i.e.* the sine of the Cabibbo angle.

6.5 Constraints on X Charges

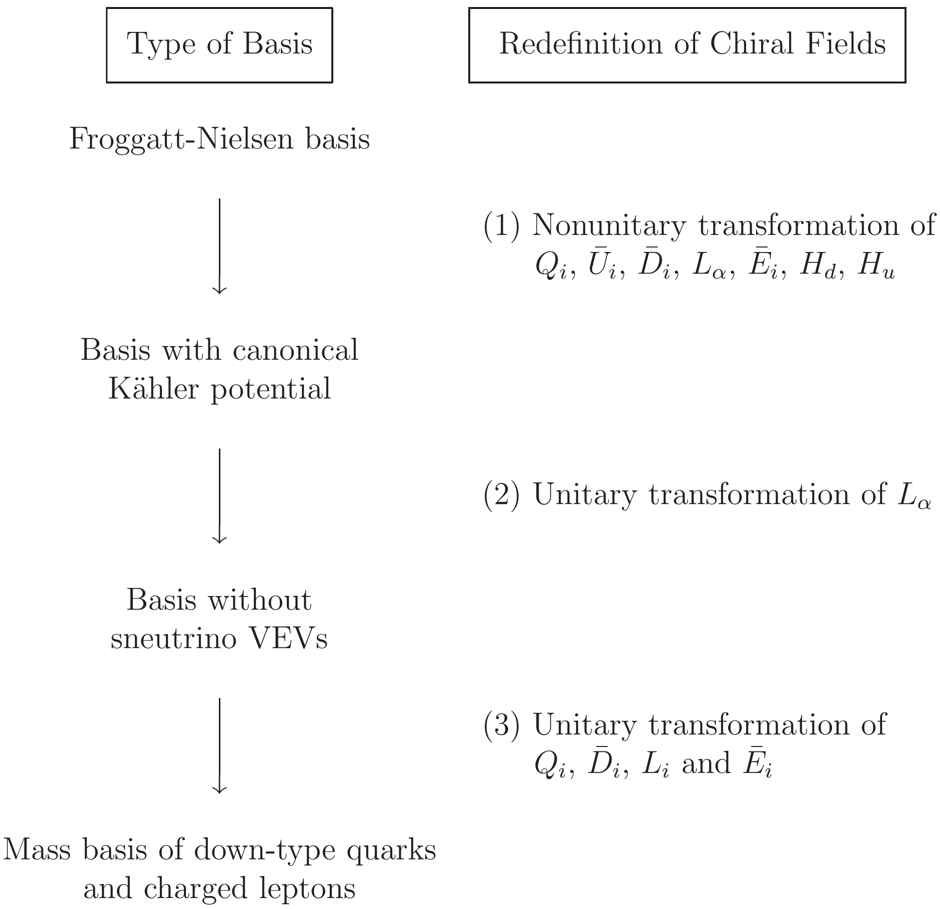
Before being able to extract constraints on the X charges from phenomenological requirements, it is necessary to relate the high-energy theory at the scale of $U(1)_X$ breaking to the effective theory at the electroweak and therefore experimentally accessible energy scale.

The Froggatt-Nielsen charges determine the structure of the theory just below the gravitational scale M_{grav}. The low-energy theory emerges after the successive breakdown of the $U(1)_X$ gauge theory, supersymmetry and finally the $SU(2)_W \times U(1)_Y$ gauge theory. The hierarchy of the fermion mass spectrum is given in terms of powers of the ratio $\epsilon \equiv \frac{\langle\varphi_\Phi\rangle}{M_{\text{grav}}}$ of the vacuum expectation value of the scalar component φ_Φ of the $U(1)_X$ flavon superfield field, Φ, and the gravitational scale. Within a string-embedded FN framework this expansion parameter originates in the Dine-Seiberg-Wen-Witten mechanism [106, 107, 108, 109], leading to a value of about $\epsilon \sim 0.2$ (see *e.g.* Ref. [38] as well as Sects. 6.4 and 8.1). Neglecting $\mathcal{O}(1)$ renormalization flow effects, we then obtain the effective Lagrangian at energy scales testable at colliders.

The kinetic terms obtained from the Kähler potential via the mechanism of Froggatt and Nielsen are noncanonical, *i.e.* they are not diagonal in generation space and not properly normalized. Furthermore, there is no reason why the sneutrino vacuum expectation values should be zero in general Supersymmetric Standard Models. It is usually more convenient to formulate $\not{M}_p$ theories where the neutrino masses can be induced radiatively in a basis with vanishing sneutrino VEVs and the down-type fields rotated to their mass bases [40].

6.5.1 Sequence of Basis Transformations

Therefore, we first apply the following sequence of basis transformations, depicted in the diagram below, to the chiral superfields ϕ_i and study its effects on the FN-generated coupling constants. Such redefinition of fields are only possible for particles with equal gauge quantum numbers. Evidently, this is the case for the three generations of quark and lepton chiral superfields. In addition, the down-type Higgs doublet can be combined with the lepton doublets as they have identical SM charges. Hence we can write $L_\alpha \equiv (H_d, L_1, L_2, L_3)_\alpha$, with $\alpha = 0, 1, 2, 3$. In the diagram, the numbers in brackets refer to the explanations of each step, below. Note that in the third step we only rotate the L_i, not the L_α. The transformations of the $\bar{U}_i$ do not affect any of the terms we are interested in, so that we do not further consider them. After the above transformations, we again find a FN structure for the coupling constants in the new basis. Working backwards, it is then possible to deduce phenomenologically viable X charge assignments from the experimentally observed masses and mixings of quarks and leptons.



1. *Canonicalization of the Kähler potential (CK):* The Kähler potential for n species of superfields[5] $\phi_i^{\rm FN}$ $(i = 1, ..., n)$ with equal gauge quantum numbers, *i.e.* which can mix, is canonicalized by the $n \times n$ *nonunitary* matrix $\boldsymbol{C^\phi}$, with the texture (see Sect. 6.1)

$$C^\phi{}_{ij} \quad \sim \quad \epsilon^{|X_{\phi_i} - X_{\phi_j}|} \,. \tag{6.50}$$

In terms of the canonicalized superfields $\phi_i \equiv C^\phi{}_{ij}\ \phi_j^{\rm FN}$, the kinetic operators are given in their standard diagonal and normalized form. The interaction coupling constants $c^{\rm FN}{}_i$ also change correspondingly through the basis transformation, *e.g.* for a trilinear interaction of superfields ϕ_i^1, ϕ_j^2 and ϕ_k^3

$$c^{\rm FN}{}_{ijk}\ \phi_i^{1\,\rm FN} \phi_j^{2\,\rm FN} \phi_k^{3\,\rm FN} \quad = \quad c_{ijk}\ \phi_i^1\ \phi_j^2\ \phi_k^3 \,, \tag{6.51}$$

with

$$c_{ijk} \quad \equiv \quad [\boldsymbol{C^{\phi^1}}^{-1}]_{i'i}\, [\boldsymbol{C^{\phi^2}}^{-1}]_{j'j}\, [\boldsymbol{C^{\phi^3}}^{-1}]_{k'k}\ c^{\rm FN}{}_{i'j'k'} \,. \tag{6.52}$$

Note that each index transforms separately. In the following, while discussing the general FN power structure, we focus on one index for notational simplicity, *i.e.* we suppress additional indices that might be attached to the coupling constants

$$c_i \quad \equiv \quad [\boldsymbol{C^{\phi}}^{-1}]_{ji}\ c^{\rm FN}{}_j \,. \tag{6.53}$$

The generalization to n indices is trivial. Considering superpotential couplings which are free of zero textures due to negative integer overall X charge, *i.e.* which are free of so-called supersymmetric zeros,[6] we have $c^{\rm FN}{}_i \propto \epsilon^{X_{\phi_i}}$. Under the above transformations, we obtain [111]

$$c_i \quad \propto \quad \epsilon^{|X_{\phi_j} - X_{\phi_i}|}\ \epsilon^{X_{\phi_j}} \quad \sim \quad \epsilon^{X_{\phi_i}} . \tag{6.54}$$

Coupling constants which are not generated by FN alone but involve a combination of the FN and the Giudice-Masiero mechanism (see *e.g.* Ref. [38]) are treated slightly differently:

- Later we are going to assume that, *e.g.*, the bilinear superpotential terms $\mu_\alpha L_\alpha H_u$ are *all* due to the GM mechanism. Coupling constants generated in this way have the X-charge dependence $c^{\rm FN}{}_i \propto \epsilon^{-X_{\phi_i}}$. In this case, the canonicalization of the Kähler potential yields

$$c_i \quad \propto \quad \epsilon^{|X_{\phi_j} - X_{\phi_i}|}\ \epsilon^{-X_{\phi_j}} \quad \sim \quad \epsilon^{-X_{\phi_i}} . \tag{6.55}$$

[5]The superscript FN stresses the fact that we start with the chiral superfields given in the Froggatt-Nielsen basis.

[6]The problems connected with having supersymmetric zeros in the Yukawa mass matrices are discussed in Appendix C.

- We also deal with the case where on the one hand the MSSM operators $L_i H_d \bar{E}_j$, $Q_i H_d \bar{D}_j$ are required to have overall positive integer X charges, whereas the corresponding $\not{\boldsymbol{M}}_p$ operators with $L_0 \equiv H_d \to L_i$ ($i = 1,2,3$) replaced, *i.e.* $L_i L_j \bar{E}_k$, $L_i Q_j \bar{D}_k$, have overall negative integer X charges. This assumption implies $X_{L_0} > X_{L_i}$. Due to the GM mechanism (*cf.* Sect. 6.2), so-called supersymmetric zeros of the coupling constants with negative overall X charge are actually not zero, however, for trilinear couplings the resulting terms are suppressed by a factor of $\mathcal{O}(\frac{m_{\text{soft}}}{M_{\text{grav}}})$ and therefore effectively absent. If *e.g.* $X_{Q_2} + X_{H_d} + X_{\bar{D}_2} = 2$ and $X_{L_0} - X_{L_1} = 5$, we obtain the FN operator
$$\epsilon^2 \cdot Q_2 H_d \bar{D}_2 \,; \tag{6.56}$$
for the corresponding $\boldsymbol{M}_p$ violating term $L_1 Q_2 \bar{D}_2$ we need to invoke the GM mechanism:
$$X_{L_1} + X_{Q_2} + X_{\bar{D}_2} = -3 \;\; \overset{\text{GM}}{\Longrightarrow} \;\; \frac{m_{\text{soft}}}{M_{\text{grav}}} \epsilon^3 \cdot L_1 Q_2 \bar{D}_2 \;\approx\; 0. \tag{6.57}$$
So for the coupling constants we thus have ${c^{\text{FN}}}_\alpha \propto (\epsilon^{X_{L_0}}, 0, 0, 0)_\alpha$. But thanks to the canonicalization of the kinetic terms, these "quasi supersymmetric zeros" are filled in so that
$$c_\alpha = [\boldsymbol{C}^{\boldsymbol{L}^{-1}}]_{0\alpha}\, {c^{\text{FN}}}_0 \;\;\propto\;\; \epsilon^{|X_{L_0} - X_{L_\alpha}|}\, \epsilon^{X_{L_0}} \;\;\sim\;\; \epsilon^{2X_{L_0} - X_{L_\alpha}}. \tag{6.58}$$
We can apply a similar consideration to superpotential terms containing $\epsilon_{ab} L^a_\alpha L^b_\beta$, where $a, b \in \{1, 2\}$ are $SU(2)$ doublet indices. As the symmetric part of the corresponding coupling constant ${c^{\text{FN}}}_{\alpha\beta}$ cancels automatically, it can be taken antisymmetric without loss of generality. Now, when constructing a viable model, we choose the X charges such that the terms $\epsilon_{ab} L^a_i L^b_j$, with $i, j = 1, 2, 3$, are forbidden by a negative integer total X charge, whereas $\epsilon_{ab} L^a_i L^b_0$ and $\epsilon_{ab} L^a_0 L^b_j$ are allowed. In this special case we find[7]
$$\begin{aligned} c_{\alpha\beta} \;&=\; [\boldsymbol{C}^{\boldsymbol{L}^{-1}}]_{0\alpha}\, [\boldsymbol{C}^{\boldsymbol{L}^{-1}}]_{j\beta}\, {c^{\text{FN}}}_{0j} \;-\; (\alpha \leftrightarrow \beta) \\ &\propto\; \epsilon^{2X_{L_0} - X_{L_\alpha} + X_{L_\beta}} \;-\; (\alpha \leftrightarrow \beta). \end{aligned} \tag{6.59}$$

[7] This can be seen as follows: $c_{\alpha\beta} = [\boldsymbol{C}^{\boldsymbol{L}^{-1}}]_{\alpha'\alpha} [\boldsymbol{C}^{\boldsymbol{L}^{-1}}]_{\beta'\beta}\, {c^{\text{FN}}}_{\alpha'\beta'}$. The assumption ${c^{\text{FN}}}_{ij} = 0$ together with the condition of antisymmetry, ${c^{\text{FN}}}_{\alpha 0} = -{c^{\text{FN}}}_{0\alpha}$, leads to
$$\begin{aligned} c_{\alpha\beta} \;&=\; [\boldsymbol{C}^{\boldsymbol{L}^{-1}}]_{0\alpha} [\boldsymbol{C}^{\boldsymbol{L}^{-1}}]_{j\beta} \cdot {c^{\text{FN}}}_{0j} + [\boldsymbol{C}^{\boldsymbol{L}^{-1}}]_{i\alpha} [\boldsymbol{C}^{\boldsymbol{L}^{-1}}]_{0\beta} \cdot {c^{\text{FN}}}_{i0}\,, \\ &=\; [\boldsymbol{C}^{\boldsymbol{L}^{-1}}]_{0\alpha} [\boldsymbol{C}^{\boldsymbol{L}^{-1}}]_{j\beta} \cdot {c^{\text{FN}}}_{0j} - (\alpha \leftrightarrow \beta). \end{aligned}$$
The ϵ structure is given by
$$\begin{aligned} c_{\alpha\beta} \;&\propto\; \epsilon^{|X_{L_0} - X_{L_\alpha}|}\, \epsilon^{|X_{L_j} - X_{L_\beta}|}\, \epsilon^{X_{L_0} + X_{L_j}} - (\alpha \leftrightarrow \beta) \\ &\propto\; \epsilon^{2X_{L_0} - X_{L_\alpha} + X_{L_\beta}} - (\alpha \leftrightarrow \beta)\,. \end{aligned}$$

After the canonicalization of the Kähler potential, all superpotential coupling constants of the fields ϕ_i will therefore include a factor of either $\epsilon^{X_{\phi_i}}$ or $\epsilon^{-X_{\phi_i}}$.

2. *Rotating away the sneutrino VEVs:* Next we perform a unitary transformation on the superfields L_α ($\alpha = 0, 1, 2, 3$) in order to get rid of the sneutrino VEVs. The four vacuum expectation values υ_α of the scalar component fields in L_α are determined by the minimization conditions for the neutral scalar potential. If we make the well-motivated [112] assumption of a FN structure in the soft supersymmetry breaking terms, we find (for details see Appendix D)

$$\upsilon_\alpha \;\; \propto \;\; \epsilon^{-X_{L_\alpha}}. \tag{6.60}$$

We eliminate the sneutrino VEVs υ_i ($i = 1, 2, 3$) by the unitary matrix which in Ref. [65] was used to rotate away the bilinear superpotential terms $L_i H_u$. In our case it has the texture[8]

$$\boldsymbol{U}^{\mathbf{VEVs}} \;\; \sim \;\; \begin{pmatrix} 1 & \epsilon^{X_{L_0}-X_{L_j}} \\ \epsilon^{X_{L_0}-X_{L_i}} & \delta_{ij} + \epsilon^{2X_{L_0}-X_{L_i}-X_{L_j}} \end{pmatrix}. \tag{6.61}$$

Accordingly, the coupling constants involving L_α have to be transformed. However, as $[\boldsymbol{U}^{\mathbf{VEVs}\dagger}]_{\beta\alpha}\ \epsilon^{\pm X_{L_\beta}} \;\; \sim \;\; \epsilon^{\pm X_{L_\alpha}}$, their ϵ structure remains unchanged.

3. *Rotation of the quarks and charged leptons into their mass bases:* In a third step, the down-type quark[9] and charged lepton mass matrices are diagonalized by the unitary transformations $\boldsymbol{U}^{\boldsymbol{Q}}$, $\boldsymbol{U}^{\bar{\boldsymbol{D}}}$, $\boldsymbol{U}^{\boldsymbol{L}}$ and $\boldsymbol{U}^{\bar{\boldsymbol{E}}}$ of the corresponding superfields. Their ϵ power structure is given by, see also Ref. [113]

$$\begin{aligned} {U^Q}_{ij} \;\; &\sim \;\; \epsilon^{|X_{Q_i}-X_{Q_j}|}, \qquad {U^{\bar{D}}}_{ij} \;\; \sim \;\; \epsilon^{|X_{\bar{D}_i}-X_{\bar{D}_j}|}, \\ {U^L}_{ij} \;\; &\sim \;\; \epsilon^{|X_{L_i}-X_{L_j}|}, \qquad {U^{\bar{E}}}_{ij} \;\; \sim \;\; \epsilon^{|X_{\bar{E}_i}-X_{\bar{E}_j}|}. \end{aligned} \tag{6.62}$$

---

[8] Replacing $\mu \to \upsilon_0$ and $K_i \to \upsilon_i$ in Eq. (4.10) of Ref. [65], we have $K = \sqrt{{\upsilon_i}^* \upsilon_i}$ and $\mathcal{M} = \sqrt{{\upsilon_\alpha}^* \upsilon_\alpha}$. For the matrix we then have

$$\begin{aligned} {U^{\text{VEVs}}}_{0j} \;\; &= \;\; \frac{|\upsilon_0|}{\mathcal{M}} \cdot \frac{{\upsilon_j}^*}{{\upsilon_0}^*} \sim \epsilon^{X_{L_0}-X_{L_j}}\ , \quad {U^{\text{VEVs}}}_{i0} = -\frac{|\upsilon_0|}{\mathcal{M}} \cdot \frac{\upsilon_i}{\upsilon_0} \sim \epsilon^{X_{L_0}-X_{L_i}}\ , \\ {U^{\text{VEVs}}}_{ij} \;\; &= \;\; \delta_{ij} + \frac{\upsilon_i {\upsilon_j}^*}{K^2} \cdot \left(\frac{|\upsilon_0|}{\mathcal{M}} - 1 \right) \approx \delta_{ij} - \frac{\upsilon_i {\upsilon_j}^*}{2|\upsilon_0|^2} \sim \delta_{ij} + \epsilon^{2X_{L_0}-X_{L_i}-X_{L_j}}\ . \end{aligned}$$

In the penultimate step we applied the approximation $K \ll \mathcal{M} \approx |\upsilon_0|$.

[9] As we apply the basis transformations equally on both components of the $SU(2)_W$ superfield doublets Q_i, we can diagonalize *either* the up- *or* the down-type quark mass matrix. The latter is more appropriate for our purpose because, in the context of radiatively generated neutrino masses, only down-type loops contribute to the neutrino mass matrix. After $SU(2)_W \times U(1)_Y$ breaking we rotate the left-and right-handed up-type quark superfields U_L and $\bar{U}$ into their mass basis.

Here we have to assume a decreasing X charge for increasing generation index.[10] The transformations of Eq. (6.62) diagonalize the down-type mass matrices. However, they do not alter the ϵ structure of the up-type Yukawa couplings and other renormalizable or nonrenormalizable coupling constants.

We are now in a position to compare the predictions of certain X charge assignments with the phenomenological requirements. As the neutrino sector is not known so well, we first consider the constraints coming from the masses of the quarks and the charged leptons as well as the CKM matrix. Additionally, we also demand that gauge theories should be free of anomalies.

6.5.2 Non-Neutrino Constraints

In the previous subsection, we translated our model from the scale of $U(1)_X$ breaking down to the electroweak scale. The FN charges of the MSSM superfields are now directly connected to the low-energy fermionic mass spectrum. For our model, we require the X charges to reproduce phenomenologically acceptable quark masses and mixings as well as charged lepton masses. Furthermore, we demand the GS anomaly cancellation conditions of Sect. 6.3 to be satisfied.

In terms of the Wolfenstein parameter $\lambda_c \sim 0.22$, the ratios of the fermionic masses are given at the GUT scale as [114, 115, 65, 38]

$$\begin{aligned} m_d : m_s : m_b &\sim \lambda_c^4 : \lambda_c^2 : 1 \,, \\ m_u : m_c : m_t &\sim \lambda_c^8 : \lambda_c^4 : 1 \,, \\ m_e : m_\mu : m_\tau &\sim \lambda_c^{4+z} : \lambda_c^2 : 1 \,, \\ m_b : m_t &\sim \lambda_c^x \cdot \cot\beta \,, \\ m_\tau : m_b &\sim 1 \,, \end{aligned} \tag{6.63}$$

with $z = 0, 1$ and $x = 0, 1, 2, 3$. $\tan\beta$ is the ratio of the up- and the down-type Higgs VEVs, *i.e.* $\tan\beta = \frac{\upsilon_u}{\upsilon_0}$. Recall that the sneutrino VEVs are rotated away, so $\sqrt{\upsilon_\alpha^* \upsilon_\alpha} \equiv \upsilon_d = \upsilon_0$. The absolute mass scale is determined by the top-quark mass

$$m_t \sim \upsilon_u \,. \tag{6.64}$$

Furthermore, the experimentally observed quark mixing is compatible with the

[10] The diagonalization matrices $\boldsymbol{U}^{\cdots}$ have the structure of Eq. (6.62) only if the X charges of the left- and the right-chiral superfields are ordered in the same way. Demanding further that the third generation is the heaviest and the first the lightest, we are restricted to decreasing X charge for increasing generation index.

following parameterization of the CKM matrix [38]

$$\boldsymbol{U^{\mathbf{CKM}}} \sim \begin{pmatrix} 1 & \lambda_c^{1+y} & \lambda_c^{3+y} \\ \lambda_c^{1+y} & 1 & \lambda_c^2 \\ \lambda_c^{3+y} & \lambda_c^2 & 1 \end{pmatrix}, \qquad (6.65)$$

with $y = -1, 0, 1$. As we are working in a basis of diagonal down-type fermions, we can easily extract constraints on the X charges from Eqs. (6.63), (6.64), and (6.65). Let us temporarily assume that the dynamically generated FN expansion parameter ϵ is equal to the Wolfenstein parameter λ_c. In Sect. 8.1, we will justify this assumption with hindsight.

Consider the CKM matrix first. It is obtained from the two diagonalization matrices $\boldsymbol{U^{U_L}}$ and $\boldsymbol{U^{D_L}}$ of the up- and the down-type quark mass matrices which act on the *left-handed* quarks. As the down-type quarks are already in their mass basis, we have $\boldsymbol{U^{D_L}} = \mathbb{1}$. Therefore the FN structure of the CKM matrix is given by

$$U^{\mathrm{CKM}}{}_{ij} \;=\; U^{U_L}{}_{ii'}\, U^{D_L}{}^{\dagger}{}_{i'j} \;=\; U^{U_L}{}_{ij} \;\sim\; \epsilon^{|X_{Q_i}-X_{Q_j}|}. \qquad (6.66)$$

The ϵ structure is completely analogous to the transformations of Eq. (6.62). Comparing with Eq. (6.65) and recalling that $X_{Q_3} \leq X_{Q_2} \leq X_{Q_1}$, we see that

$$X_{Q_2} \;=\; X_{Q_1} - 1 - y\,, \qquad X_{Q_3} \;=\; X_{Q_1} - 3 - y\,. \qquad (6.67)$$

The down-type quark mass ratios of FN models are related to the X charges by $m_d : m_s : m_b = \epsilon^{X_{Q1}+X_{H_d}+X_{\bar{D}_1}} : \epsilon^{X_{Q2}+X_{H_d}+X_{\bar{D}_2}} : \epsilon^{X_{Q3}+X_{H_d}+X_{\bar{D}_3}}$. Using Eq. (6.67) and the mass ratios of Eq. (6.63), we thus find

$$X_{\bar{D}_2} \;=\; X_{\bar{D}_1} - 1 + y\,, \qquad X_{\bar{D}_3} \;=\; X_{\bar{D}_1} - 1 + y\,. \qquad (6.68)$$

Concerning the up-type quarks, the corresponding Yukawa mass matrix must be diagonalized first. However, this transformation to the up-type quark mass basis does not change the ϵ structure of the diagonal entries in the Yukawa matrix. Hence, we similarly obtain

$$X_{\bar{U}_2} \;=\; X_{\bar{U}_1} - 3 + y\,, \qquad X_{\bar{U}_3} \;=\; X_{\bar{U}_1} - 5 + y\,. \qquad (6.69)$$

For the leptons we have

$$X_{\bar{E}_2} \;=\; X_{\bar{E}_1} - 2 - z - \Delta^L_{21}\,, \qquad X_{\bar{E}_3} \;=\; X_{\bar{E}_1} - 4 - z - \Delta^L_{31}\,, \qquad (6.70)$$

where we have defined the integers[11] $\Delta^L_{ij} \equiv X_{L_i} - X_{L_j}$. We are now able to express the X charges of the second and the third generations by those of the first, as well as four integer parameters: y, z, Δ^L_{21}, and Δ^L_{31}.

[11] Δ^L_{ij} has to be integer because, when exchanging L_i in any allowed superpotential or Kähler potential operator by L_j, we want the new term to be allowed as well.

With the last two relations of Eq. (6.63), and Eq. (6.64) we can write the X charges of the complex conjugated right-chiral superfields, *i.e.* $\bar{D}$, $\bar{E}$, and $\bar{U}$, in terms of the left-chiral superfields Q, L, H_u, and H_d. Starting with

$$\begin{aligned} X_{\bar{D}_3} &= X_{\bar{U}_3} + X_{H_u} - X_{H_d} + x\,, \\ X_{\bar{E}_3} &= X_{\bar{D}_3} + X_{Q_3} - X_{L_3}\,, \\ X_{\bar{U}_3} &= -X_{Q_3} - X_{H_u}\,, \end{aligned}$$

we obtain

$$X_{\bar{D}_1} = -X_{Q_1} - X_{H_d} + x + 4\,, \tag{6.71}$$
$$X_{\bar{E}_1} = -X_{L_1} - X_{H_d} + x + z + 4\,, \tag{6.72}$$
$$X_{\bar{U}_1} = -X_{Q_1} - X_{H_u} + 8\,. \tag{6.73}$$

All X charges are now determined by the parameters x, y, z, Δ^L_{21}, and Δ^L_{31}, and the four X charges X_{Q_1}, X_{L_1}, X_{H_u}, and X_{H_d}. With these constraints, the fermionic mass spectrum (except for the neutrinos masses) and the CKM quark mixing can be explained.

Further constraints are obtained from the anomaly cancellation conditions of Eq. (6.34) and Eq. (6.37). As there might be particles in the hidden sector, *i.e.* particles which are SM singlets, we only consider the anomaly coefficients which include some part of the SM gauge group. From

$$\frac{\mathcal{A}_{CCX}}{k_C} = \frac{\mathcal{A}_{WWX}}{k_W}\,,$$

together with the GUT relation $k_C = k_W$ of Eq. (6.29), we get

$$X_{Q_1} = \frac{1}{9}\left[-3X_{L_1} - 4\left(X_{H_u} + X_{H_d}\right) + 3x + 6y - \Delta^L_{21} - \Delta^L_{31} + 30\right]. \tag{6.74}$$

In order to make use of the GS anomaly cancellation condition

$$\frac{\mathcal{A}_{CCX}}{k_C} = \frac{\mathcal{A}_{YYX}}{k_Y}\,,$$

we have to specify the hypercharge normalization. This however is fixed in the GUT relation $g_C = \sqrt{5/3}\,g_Y$ to a normalization with $Y_L = 1$, see Eq. (6.30). Taking this and the relation $k_C = 3/5\,k_Y$ of Eq. (6.29) yields

$$X_{H_u} = -X_{H_d} - z\,. \tag{6.75}$$

The last GS anomaly condition arises from $\mathcal{A}_{YXX} = 0$ of Eq. (6.37). As it is quadratic in the unknown X charges, the resulting constraint is not very nice.

However, with this third anomaly condition, X_{H_d} can be written in terms of X_{L_1}, Δ^L_{21}, Δ^L_{31}, x, y, and z (see also Table 1 in Ref. [38]):

$$X_{H_d} = \frac{1}{54+9x+6z}\Big[-3X_{L_1}(12+2x+3z) - 2\Delta^L_{21}(6+x+z) - 2\Delta^L_{31}(3+x+z) + 6x(x+6) + 18y + z(2z+5x) - 18\Big]. \tag{6.76}$$

In conclusion, we have constrained the unknown X charges of the SSM superfields by the quark and charged lepton masses, the CKM mixing angles, and the mixed GS anomaly conditions. Thus we could determine all X charges except for X_{L_1} in terms of five integer parameters: Δ^L_{21}, Δ^L_{31}, x, y, and z.

Chapter 7

Implementing DGSs into FN Models

In Froggatt-Nielsen models, the SM gauge group is enlarged by a generation-dependent $U(1)_X$. This Abelian gauge symmetry is spontaneously broken by the flavon VEV slightly below the gravitational scale. Phenomenological demands determine the X charges significantly. Additionally requiring the conservation of a low-energy discrete gauge symmetry, yields further constraints on the possible X-charge assignment.

7.1 Anomalous $U(1)_X$ and DGSs

The most general Supersymmetric SM Lagrangian with one additional Higgs doublet leads to unobserved exotic processes, in particular rapid proton decay, inconsistent with the experimental bounds [18]. In the low-energy effective Lagrangian, this problem is resolved by introducing a global discrete multiplicative symmetry, which prohibits a subset of the superpotential and the Kähler potential interactions. Prominent examples are matter parity, $\boldsymbol{M}_p$ (or equivalently R parity), baryon triality, $\boldsymbol{B}_3$, and proton hexality, $\boldsymbol{P}_6$ [116]. In Part I of this thesis we have argued that such a low-energy discrete symmetry should be a "discrete *gauge* symmetry" (DGS), because gauge symmetries are not violated by quantum gravity effects. Hence, after the spontaneous breakdown of the gauge theory also the residual discrete symmetry remains intact [41, 50, 58]. In the case of Froggatt-Nielsen models, the Abelian $U(1)_X$ symmetry is broken near the gravitational scale.

In order to obtain a consistent quantum field theory, we demand that the underlying local $U(1)_X$ gauge theory is anomaly-free. In general, we include the possibility that the anomalies of the original gauge symmetry are canceled by the GS mechanism, see Sect. 6.3. Thus either

1) the low-energy DGS is a remnant of an anomaly-free local gauge symmetry, in which case the DGS is anomaly-free in the sense of Ibáñez and Ross [39], or

2) the DGS is a remnant of a local gauge symmetry whose anomalies are canceled by the GS mechanism. In this case the DGS can be either

  a) anomaly-free in the sense of Ibáñez and Ross or

  b) GS-anomalous, *i.e.* the DGS anomalies are canceled via a discrete version of the GS mechanism [59]. We emphasize that not every discrete symmetry is an *anomaly-free* DGS, *e.g.* baryon parity, $\boldsymbol{B}_p$, a $\boldsymbol{Z}_2$ symmetry[1] defined in Sect. 7.2.

The model we construct in this paper belongs to class *2a)*, *i.e.* the $U(1)_X$ gauge anomalies are canceled by the GS mechanism; however, the low-energy DGS satisfies the anomaly cancellation conditions of Ibáñez and Ross, without the GS mechanism.

The investigations in Part I of this thesis reveal that there are only three generation-independent anomaly-free DGSs which are compatible with the need for neutrino masses: matter parity [Eq. (7.11)], baryon triality [Eq. (7.12)] and proton hexality [Eq. (7.13)], which we discuss in more detail below.

In the following section, we wish to derive the necessary and sufficient conditions on the MSSM X charges for matter parity $\boldsymbol{M}_p$, baryon triality $\boldsymbol{B}_3$, and proton hexality $\boldsymbol{P}_6$ to arise as a DGS from a family-dependent local $U(1)_X$ gauge symmetry. To give an example of an *anomalous* DGS, we also present the case of baryon parity $\boldsymbol{B}_p$; additionally, we compare the physical implications of $\boldsymbol{B}_p$ and $\boldsymbol{B}_3$.

7.2 $\boldsymbol{M_p}$, $\boldsymbol{B_3}$, $\boldsymbol{P_6}$, and $\boldsymbol{B_p}$ Arising from $\boldsymbol{U(1)_X}$

Consider a general $SU(3)_C \times SU(2)_W \times U(1)_Y$ gauge invariant product of MSSM left-chiral superfields $\phi_i \in \{Q_k, \bar{U}_k, \bar{D}_k, L_k, \bar{E}_k, H_d, H_u\}$ and their charge conjugates $\bar{\phi}_i$,

$$R \equiv \prod_{i,j} (\phi_i)^{\alpha_i} \left(\bar{\phi}_j\right)^{\bar{\alpha}_j} \ . \tag{7.1}$$

In general, such an operator can appear in the Kähler potential or, if the $\bar{\alpha}_j$ vanish, in the superpotential. Imposing a discrete symmetry forbids some of these SM-invariant operators. We now wish to obtain a specific low-energy discrete

[1]As $\boldsymbol{B}_p$ is different from the only anomaly-free $\boldsymbol{Z}_2$ DGS, $\boldsymbol{M}_p$, it is evidently anomalous in the sense of Ibáñez and Ross.

symmetry by an appropriate $U(1)_X$ gauge charge assignment. As done in Sect. 6.1, we fix the X charge normalization such that the flavon superfield Φ has $U(1)_X$ charge $X_\Phi = -1$. It is then obvious that only those operators with an *integer* overall X charge, X_{total}, are allowed after the breaking of $U(1)_X$. We obtain further constraints on X_{total} by requiring $SU(3)_C \times SU(2)_W \times U(1)_Y$ gauge invariance of the given operator, as well as by demanding that the renormalizable MSSM superpotential operators are necessarily allowed. We thus have the conditions on X_{total} for an operator to be allowed or forbidden. We then make the connection with the corresponding discrete symmetry, originating from the MSSM X charges, stating the necessary and sufficient conditions thereof.

We shall denote the "combined" multiplicity of each superfield in a given operator by $n_{\phi_i} \equiv \alpha_i - \bar{\alpha}_i$. Thus, for example, the term $Q_1\bar{Q}_2\bar{U}_1\bar{D}_1\bar{D}_2$ has $n_{Q_1} = 1, n_{Q_2} = -1$, $n_{\bar{U}_1} = n_{\bar{D}_1} = n_{\bar{D}_2} = 1$. The total X charge of a general product, R, of superfields ϕ_i, $\bar{\phi}_j$ can then be expressed as

$$\begin{aligned} X_{\text{total}} &= n_{H_d}\, X_{H_d} + n_{H_u}\, X_{H_u} + \sum_i n_{Q_i}\, X_{Q_i} + \sum_i n_{\bar{D}_i}\, X_{\bar{D}_i} \\ &\quad + \sum_i n_{\bar{U}_i}\, X_{\bar{U}_i} + \sum_i n_{L_i}\, X_{L_i} + \sum_i n_{\bar{E}_i}\, X_{\bar{E}_i}. \end{aligned} \tag{7.2}$$

The coefficients $n_{...}$ and charges $X_{...}$ above are *not* all mutually independent:

- Since each product R should be $SU(3)_C \times SU(2)_W \times U(1)_Y$ gauge invariant, the $n_{...}$ are subject to the conditions (for the first equation see for example Chapter 10 in Ref. [117])

$$\sum_i n_{Q_i} - \sum_i n_{\bar{D}_i} - \sum_i n_{\bar{U}_i} = 3\mathcal{C}, \tag{7.3}$$

$$n_{H_d} + n_{H_u} + \sum_i n_{Q_i} + \sum_i n_{L_i} = 2\mathcal{W}, \tag{7.4}$$

$$\begin{aligned} & Y_{H_d}\, n_{H_d} + Y_{H_u}\, n_{H_u} + Y_Q \sum_i n_{Q_i} + Y_{\bar{D}} \sum_i n_{\bar{D}_i} \\ & + Y_{\bar{U}} \sum_i n_{\bar{U}_i} + Y_L \sum_i n_{L_i} + Y_{\bar{E}} \sum_i n_{\bar{E}_i} = 0. \end{aligned} \tag{7.5}$$

  Here, $\mathcal{C}$ is an integer, $\mathcal{W}$ is an integer which is nonnegative for terms in the superpotential. $Y_{...}$ denotes the hypercharge of the corresponding field. For the MSSM fields we have: $Y_{H_d} = -3Y_Q$, $Y_{H_u} = 3Y_Q$; $Y_L = -3Y_Q$, $Y_{\bar{E}} = 6Y_Q$; $Y_{\bar{U}} = -4Y_Q$, $Y_{\bar{D}} = 2Y_Q$. Solving Eqs. (7.3)-(7.5) for n_{Q_1}, $n_{\bar{D}_1}$, and $n_{\bar{E}_1}$ we obtain

$$
\begin{aligned}
n_{Q_1} &= 2\mathcal{W} - (n_{H_d} + n_{H_u}) - (n_{Q_2} + n_{Q_3}) - \sum_i n_{L_i}, &(7.6)\\
n_{\bar{D}_1} &= -3\mathcal{C} + 2\mathcal{W} - (n_{H_d} + n_{H_u}) - (n_{\bar{D}_2} + n_{\bar{D}_3}) - \sum_i n_{L_i} - \sum_i n_{\bar{U}_i}, &(7.7)\\
n_{\bar{E}_1} &= \mathcal{C} - \mathcal{W} + n_{H_d} - (n_{\bar{E}_2} + n_{\bar{E}_3}) + \sum_i n_{L_i} + \sum_i n_{\bar{U}_i}. &(7.8)
\end{aligned}
$$

- Since we assume that after the breaking of $U(1)_X$, all renormalizable MSSM superpotential operators are allowed, the corresponding gauge invariant products R must have nonfractional powers of the flavon superfield Φ, *i.e.* we require

  1. The renormalizable superpotential terms $Q_i H_d \bar{D}_j$, $Q_i H_u \bar{U}_j$, $L_i H_d \bar{E}_j$ and $H_d H_u$ have overall integer X charges.

  This corresponds to the conditions

$$
\begin{aligned}
X_{Q_1} + X_{H_d} + X_{\bar{D}_1} &= \text{integer},\\
X_{Q_1} + X_{H_u} + X_{\bar{U}_1} &= \text{integer},\\
X_{L_1} + X_{H_d} + X_{\bar{E}_1} &= \text{integer},\\
X_{Q_{2,3}} - X_{Q_1} &= \text{integer},\\
X_{L_{2,3}} - X_{L_1} &= \text{integer},\\
X_{\bar{D}_{2,3}} - X_{\bar{D}_1} &= \text{integer},\\
X_{\bar{U}_{2,3}} - X_{\bar{U}_1} &= \text{integer},\\
X_{\bar{E}_{2,3}} - X_{\bar{E}_1} &= \text{integer},\\
X_{H_d} + X_{H_u} &= \text{integer}. &(7.9)
\end{aligned}
$$

  We leave it open at the moment which other gauge invariant terms shall also have an overall integer X charge.

With the help of Eq. (7.9), we can express all X charges in terms of X_{L_1}, X_{Q_1}, X_{H_d}, and unknown integers. Inserting this and Eqs. (7.6)-(7.8) in Eq. (7.2) we get for the total X charge

$$
\begin{aligned}
X_{\text{total}} &= \mathcal{C} \cdot \left[3X_{Q_1} + X_{L_1} + 2(X_{H_d} - X_{L_1})\right]\\
&\quad + \left(n_{H_d} - \mathcal{W} + \sum_i n_{\bar{U}_i}\right) \cdot (X_{H_d} - X_{L_1}) + \text{integer}. &(7.10)
\end{aligned}
$$

If we now require *no* remnant DGS at low-energy whatsoever, *i.e.* if *all* renormalizable and nonrenormalizable terms which are $SU(3)_C \times SU(2)_W \times U(1)_Y$ gauge invariant have an overall integer X charge, then $X_{H_d} - X_{L_1}$ and $3X_{Q_1} + X_{L_1}$ must be integers. However, we wish to determine the constraints on the X charges, in order to obtain a remnant matter parity, baryon triality, proton hexality, or baryon parity DGS arising from the $U(1)_X$. Our treatment does not rely on the absence or cancellation of anomalies and is thus equally applicable to anomalous $\boldsymbol{Z}_N$ symmetries like, *e.g.*, baryon parity $\boldsymbol{B}_p$.

Under the respective DGSs, the MSSM left-chiral superfields transform as follows

- Matter parity[2] ($\boldsymbol{M}_p$)

$$\begin{aligned} \Big\{H_d, H_u\Big\} &\longrightarrow && \Big\{H_d, H_u\Big\}, \\ \Big\{Q_i, \bar{U}_i, \bar{D}_i, L_i, \bar{E}_i\Big\} &\longrightarrow && e^{2\pi \mathrm{i}/2} \Big\{Q_i, \bar{U}_i, \bar{D}_i, L_i, \bar{E}_i\Big\}, \end{aligned} \tag{7.11}$$

- Baryon triality ($\boldsymbol{B}_3$)

$$\begin{aligned} Q_i &\longrightarrow && Q_i, \\ \Big\{H_u, \bar{D}_i\Big\} &\longrightarrow && e^{2\pi \mathrm{i}/3} \Big\{H_u, \bar{D}_i\Big\}, \\ \Big\{H_d, \bar{U}_i, L_i, \bar{E}_i\Big\} &\longrightarrow && e^{4\pi \mathrm{i}/3} \Big\{H_d, \bar{U}_i, L_i, \bar{E}_i\Big\}, \end{aligned} \tag{7.12}$$

- Proton hexality ($\boldsymbol{P}_6$), (*cf.* Ref. [116])

$$\begin{aligned} Q_i &\longrightarrow && Q_i, \\ \Big\{H_d, \bar{U}_i, \bar{E}_i\Big\} &\longrightarrow && e^{2\pi \mathrm{i}/6} \Big\{H_d, \bar{U}_i, \bar{E}_i\Big\}, \\ L_i &\longrightarrow && e^{8\pi \mathrm{i}/6} \; L_i \;, \\ \Big\{H_u, \bar{D}_i\Big\} &\longrightarrow && e^{10\pi \mathrm{i}/6} \Big\{H_u, \bar{D}_i\Big\}, \end{aligned} \tag{7.13}$$

[2]In Part I we have given an alternative but physically equivalent definition of matter parity:

$$\begin{aligned} \Big\{Q_i, L_i\Big\} &\longrightarrow && \Big\{Q_i, L_i\Big\}, \\ \Big\{\bar{U}_i, \bar{D}_i, \bar{E}_i, H_d, H_u\Big\} &\longrightarrow && e^{2\pi \mathrm{i}/2} \Big\{\bar{U}_i, \bar{D}_i, \bar{E}_i, H_d, H_u\Big\}. \end{aligned}$$

Both are related by a hypercharge shift of the discrete charges. See Ref. [116] as well as Sect. 2.3 for details. With the definition of Part I it is easy to see that proton hexality is the direct product of matter parity and baryon triality.

- Baryon parity ($\boldsymbol{B}_p$)

$$\begin{aligned} \Big\{H_d, H_u, L_i, \bar{E}_i\Big\} &\longrightarrow & &\Big\{H_d, H_u, L_i, \bar{E}_i\Big\}, \\ \Big\{Q_i, \bar{U}_i, \bar{D}_i\Big\} &\longrightarrow & e^{2\pi \mathrm{i}/2} &\Big\{Q_i, \bar{U}_i, \bar{D}_i\Big\}. \end{aligned} \tag{7.14}$$

Note that none of these four symmetries has a domain wall problem, since the discrete charges of the two Higgs superfields are opposite to each other. For details see Sect. 4.2 and Ref. [116]. In other words, under $\boldsymbol{M}_p$, $\boldsymbol{B}_3$, $\boldsymbol{P}_6$, and $\boldsymbol{B}_p$ transformations a general product of MSSM superfields is multiplied by

- $$\Big(e^{2\pi \mathrm{i}/2}\Big)^{\sum_i n_{Q_i} \,+\, \sum_i n_{\bar{U}_i} \,+\, \sum_i n_{\bar{D}_i} \,+\, \sum_i n_{L_i} \,+\, \sum_i n_{\bar{E}_i}}, \tag{7.15}$$
- $$\Big(e^{2\pi \mathrm{i}/3}\Big)^{n_{H_u} \,+\, \sum_i n_{\bar{D}_i} \,+\, 2\, n_{H_d} \,+\, 2\sum_i n_{\bar{U}_i} \,+\, 2\sum_i n_{L_i} \,+\, 2\sum_i n_{\bar{E}_i}}, \tag{7.16}$$
- $$\Big(e^{2\pi \mathrm{i}/6}\Big)^{n_{H_d} \,+\, \sum_i n_{\bar{U}_i} \,+\, \sum_i n_{\bar{E}_i} \,+\, 4\sum_i n_{L_i} \,+\, 5\, n_{H_u} \,+\, 5\sum_i n_{\bar{D}_i}}, \tag{7.17}$$
- $$\Big(e^{2\pi \mathrm{i}/2}\Big)^{\sum_i n_{Q_i} \,+\, \sum_i n_{\bar{U}_i} \,+\, \sum_i n_{\bar{D}_i}}, \tag{7.18}$$

respectively. Thus we may write for $\boldsymbol{M}_p/\boldsymbol{B}_3/\boldsymbol{P}_6/\boldsymbol{B}_p$

$$\begin{aligned} \textstyle\sum_i n_{Q_i} \,+\, \sum_i n_{\bar{U}_i} \,+\, \sum_i n_{\bar{D}_i} \,+\, \sum_i n_{L_i} \,+\, \sum_i n_{\bar{E}_i} &= 2\mathcal{I}_M \,+\, \iota_M\,, \\ n_{H_u} \,+\, \textstyle\sum_i n_{\bar{D}_i} \,+\, 2\, n_{H_d} \,+\, 2\sum_i n_{\bar{U}_i} \,+\, 2\sum_i n_{L_i} \,+\, 2\sum_i n_{\bar{E}_i} &= 3\mathcal{I}_B \,+\, \iota_B\,, \\ n_{H_d} \,+\, \textstyle\sum_i n_{\bar{U}_i} \,+\, \sum_i n_{\bar{E}_i} \,+\, 4\,\sum_i n_{L_i} \,+\, 5\, n_{H_u} \,+\, 5\,\sum_i n_{\bar{D}_i} &= 6\mathcal{I}_P \,+\, \iota_P\,, \\ \textstyle\sum_i n_{Q_i} \,+\, \sum_i n_{\bar{U}_i} \,+\, \sum_i n_{\bar{D}_i} &= 2\mathcal{I}_M{}' \,+\, \iota_M{}'\,, \end{aligned} \tag{7.19}$$

respectively. $\mathcal{I}_M$, $\mathcal{I}_B$, $\mathcal{I}_P$, and $\mathcal{I}_M{}'$ are integers; $\iota_M/\iota_M{}'$ is 0 or 1 if matter parity / baryon parity is conserved or broken, ι_B is 0 or $1, 2$ if baryon triality is conserved or broken, ι_P is 0 or $1, 2, 3, 4, 5$ if proton hexality is conserved or broken. With

Eqs. (7.6)-(7.8) we get from Eq. (7.19) that

$$\boldsymbol{M}_p: \quad n_{H_d} = 3\mathcal{W} - \iota_M - 2(\mathcal{C} + \mathcal{I}_M + n_{H_u}) + \sum_i n_{\bar{U}_i}, \tag{7.20}$$

$$\boldsymbol{B}_3: \quad \mathcal{C} = 3\Big(- \mathcal{I}_B + n_{H_d} + \sum_i n_{L_i} + \sum_i n_{\bar{U}_i}\Big) - \iota_B, \tag{7.21}$$

$$\boldsymbol{P}_6: \quad n_{H_d} = 3\mathcal{W} - \sum_i n_{\bar{U}_i} - \frac{14\,\mathcal{C} + 6\mathcal{I}_P + \iota_P}{3}, \tag{7.22}$$

$$\boldsymbol{B}_p: \quad \mathcal{C} = 2\Big(- \mathcal{I}_M{}' - \mathcal{C} + 2\mathcal{W} - n_{H_d} - n_{H_u} - \sum_i n_{L_i}\Big) - \iota_M{}', \tag{7.23}$$

respectively. We now require

1′. All $SU(3)_C \times SU(2)_W \times U(1)_Y$ gauge invariant terms which conserve the discrete symmetry $\boldsymbol{M}_p/\boldsymbol{B}_3/\boldsymbol{P}_6/\boldsymbol{B}_p$ each have an overall integer X charge. This requirement is a generalization of Point 1. above Eq. (7.9).

2. All $SU(3)_C \times SU(2)_W \times U(1)_Y$ gauge invariant terms which do not conserve the discrete symmetry $\boldsymbol{M}_p/\boldsymbol{B}_3/\boldsymbol{P}_6/\boldsymbol{B}_p$ each have an overall fractional X charge. It follows that all superfield operators which violate $\boldsymbol{M}_p/\boldsymbol{B}_3/\boldsymbol{P}_6/\boldsymbol{B}_p$ are forbidden. $\boldsymbol{M}_p/\boldsymbol{B}_3/\boldsymbol{P}_6/\boldsymbol{B}_p$ is thus conserved exactly.

For any $SU(3)_C \times SU(2)_W \times U(1)_Y$ invariant operator $\phi_1\phi_2...\phi_n$ which violates $\boldsymbol{M}_p/\boldsymbol{B}_3/\boldsymbol{P}_6/\boldsymbol{B}_p$ one has that $(\phi_1\phi_2...\phi_n)^2/(\phi_1\phi_2...\phi_n)^3/(\phi_1\phi_2...\phi_n)^6/(\phi_1\phi_2...\phi_n)^2$ conserves $\boldsymbol{M}_p/\boldsymbol{B}_3/\boldsymbol{P}_6/\boldsymbol{B}_p$. From Point 1′. we find that the X charge of the latter operator, namely $2 \times \big(X_{\phi_1} + X_{\phi_2} + ... + X_{\phi_n}\big)/3 \times \big(X_{\phi_1} + X_{\phi_2} + ... + X_{\phi_n}\big)/6 \times \big(X_{\phi_1} + X_{\phi_2} + ... + X_{\phi_n}\big)/2 \times \big(X_{\phi_1} + X_{\phi_2} + ... + X_{\phi_n}\big)$ is integer. Point 2. demands that $X_{\phi_1} + X_{\phi_2} + ... + X_{\phi_n}$ is fractional. *It follows that all superfield operators which violate $\boldsymbol{M}_p/\boldsymbol{B}_3/\boldsymbol{P}_6/\boldsymbol{B}_p$ have an overall X charge of the form $\frac{1}{2}$+ integer/ $\frac{1 \text{ or } 2}{3}$ + integer/$\frac{1,2,3,4 \text{ or } 5}{6}$ + integer/$\frac{1}{2}$+ integer.* Bearing this in mind, we plug Eqs. (7.20) - (7.22) into Eq. (7.10) to eliminate $n_{H^D}/\mathcal{C}/n_{H^D}/\mathcal{C}$, respectively:

- We first treat $\boldsymbol{M}_p$. One has

$$\begin{aligned} X_{\text{total}} &= \mathcal{C} \cdot (3X_{Q_1} + X_{L_1}) \\ &\quad + \Big[2\Big(\mathcal{W} - \mathcal{I}_M - n_{H_u} + \sum_i n_{\bar{U}_i}\Big) - \iota_M\Big] \cdot (X_{H_d} - X_{L_1}) \;+\; \text{integer}. \end{aligned} \tag{7.24}$$

  Now consider a superpotential term forbidden by $\boldsymbol{M}_p$, *i.e.* with $\iota_M = 1$. X_{total} must then be $\frac{1}{2}$+integer. Choosing $\mathcal{C} = 0$, $\mathcal{W} = \mathcal{I}_M + n_{H_u} - \sum_i n_{\bar{U}_i}$, we obtain the condition

$$X_{H_d} \;-\; X_{L_1} \;\overset{!}{=}\; -\tfrac{1}{2} \;+\; \text{integer}. \tag{7.25}$$

We insert this into the expression for X_{total}:

$$X_{\text{total}} \;=\; \mathcal{C}\cdot(3X_{Q_1}+X_{L_1}) \;+\; \tfrac{\iota_M}{2} \;+\; \text{integer}\,. \tag{7.26}$$

For the terms which are allowed by $\boldsymbol{M}_p$, *i.e.* which have $\iota_M = 0$ (and thus X_{total} is integer), we get another condition on the X charges of the MSSM superfields when choosing $\mathcal{C} = 1$,

$$3X_{Q_1} \;+\; X_{L_1} \;\overset{!}{=}\; \text{integer}\,. \tag{7.27}$$

To check consistency, we plug Eqs. (7.25) and (7.27) into Eq. (7.24); we thus find that

$$X_{\text{total}} \;=\; \tfrac{\iota_M}{2} \;+\; \text{integer}\,. \tag{7.28}$$

In Refs. [38, 29, 75] the implications of Eqs. (7.9), (7.25), and (7.27) in combination with a viable phenomenology were studied in detail.

- Now $\boldsymbol{B}_3$. We get

$$\begin{aligned} X_{\text{total}} = & \Big[3\Big(n_{H_d} - \mathcal{I}_B + \sum_i n_{L_i} + \sum_i n_{\bar{U}_i}\Big) - \iota_B\Big]\cdot\Big[3X_{Q_1}+X_{L_1}+2(X_{H_d}-X_{L_1})\Big] \\ & + \Big(n_{H_d} \;-\; \mathcal{W} \;+\; \sum_i n_{\bar{U}_i}\Big)\cdot(X_{H_d}-X_{L_1}) \;+\; \text{integer}. \end{aligned} \tag{7.29}$$

With $\iota_B = 0$ (thus X_{total} is integer) and $\mathcal{I}_B = n_{H_d} + \sum_i n_{L_i} + \sum_i n_{\bar{U}_i}$ we arrive at

$$X_{\text{total}} \;=\; \Big(n_{H_d} - \mathcal{W} + \sum_i n_{\bar{U}_i}\Big)\cdot(X_{H_d}-X_{L_1}) \;+\; \text{integer}, \tag{7.30}$$

with the choice $\mathcal{W} = n_{H_d} + \sum_i n_{\bar{U}_i} + 1$ leading to the condition

$$X_{H_d} \;-\; X_{L_1} \;\overset{!}{=}\; \text{integer} \tag{7.31}$$

[to be compared with Eq. (7.25)]. We insert this into the expression for X_{total}, getting

$$X_{\text{total}} = \Big[3\big(n_{H_d} - \mathcal{I}_B + \sum_i n_{L_i} + \sum_i n_{\bar{U}_i}\big) - \iota_B\Big]\cdot(3X_{Q_1}+X_{L_1}) \;+\; \text{integer}. \tag{7.32}$$

Setting $\mathcal{I}_B = n_{H_d} + \sum_i n_{L_i} + \sum_i n_{\bar{U}_i}$, we arrive at

$$X_{\text{total}} \;=\; -\;\iota_B\cdot(3X_{Q_1}+X_{L_1}) \;+\; \text{integer}.$$

Setting $\iota_B = 1$ (thus $X_{\rm total}$ being $\frac{1 \text{ or } 2}{3}$+integer) we get

$$3X_{Q_1} + X_{L_1} \stackrel{!}{=} -\tfrac{b}{3} + \text{ integer}\,, \qquad (7.33)$$

with $b \in \{1,2\}$ [to be compared with Eq. (7.27)]. This is compatible with $\iota_B = 2$ also requiring $X_{\rm total}$ not to be an integer. To check consistency, we plug Eqs. (7.31) and (7.33) into Eq. (7.29); this gives

$$X_{\rm total} = \tfrac{b\cdot\iota_B}{3} + \text{ integer}\,. \qquad (7.34)$$

- For $\boldsymbol{P}_6$ we find that

$$\begin{aligned} X_{\rm total} &= \mathcal{C}\cdot(3X_{Q_1}+X_{L_1}) \qquad (7.35)\\ &+ \left[2\mathcal{W} - \frac{8\,\mathcal{C}+6\mathcal{I}_P+\iota_P}{3}\right]\cdot(X_{H_d}-X_{L_1}) + \text{ integer}\,. \end{aligned}$$

Consider $\iota_P = 3$ (already the square of such an operator is $\boldsymbol{P}_6$ invariant, therefore we have $X_{\rm total} = \frac{1}{2}$ + integer in this case) and $\mathcal{C} = \mathcal{W} = \mathcal{I}_P = 0$, giving that $X_{H_d} - X_{L_1}$ equals $\frac{1}{2}$ + integer. The question is: what kind of integer? To answer this, consider $\iota_P = 1$ and $\mathcal{C} = 0$. Here we obtain

$$\begin{aligned} X_{\rm total} &= \left[2\mathcal{W} - 2\mathcal{I}_P - \tfrac{1}{3}\right]\cdot(X_{H_d}-X_{L_1}) + \text{ integer},\\ &= -\tfrac{1}{3}\cdot(X_{H_d}-X_{L_1}) + \text{ integer}. \qquad (7.36) \end{aligned}$$

For $\iota_P = 1$ we need $X_{\rm total} = \frac{p}{6}$ + integer, with $p = 1, 5$: $p = 2, 3, 4$ are not allowed as these have common prime factors with 6; this would lead to a term in the Lagrangian whose square or cube is $\boldsymbol{P}_6$-invariant contrary to the assumption that ι_P is 1. This way we find

$$X_{H_d} - X_{L_1} \stackrel{!}{=} -\tfrac{p}{2} + 3\cdot\text{integer} \qquad (7.37)$$

[to be compared with Eq. (7.25)]. Inserting this into $X_{\rm total}$ of Eq. (7.35) we get

$$X_{\rm total} = \mathcal{C}\cdot(3X_{Q_1}+X_{L_1}) + p\cdot\frac{2\,\mathcal{C}+\iota_P}{6} + \text{ integer}\,. \qquad (7.38)$$

For $\iota_P = 0$ (thus $X_{\rm total}$ is integer) and $\mathcal{C} = 1$ the following condition is obtained

$$3X_{Q_1} + X_{L_1} \stackrel{!}{=} -\tfrac{p}{3} + \text{ integer}. \qquad (7.39)$$

Plugging Eqs. (7.37) and (7.39) into Eq. (7.35) we get

$$X_{\rm total} = \tfrac{p\cdot\iota_P}{6} + \text{ integer}\,. \qquad (7.40)$$

- Finally $\boldsymbol{B}_p$. In this case we have

$$\begin{aligned} X_{\text{total}} &= \left[2\Big(-\mathcal{I}_M{}' - \mathcal{C} + 2\mathcal{W} - n_{H_d} - n_{H_u} - \sum_i n_{L_i}\Big) - \iota_M{}'\right] \\ &\qquad \cdot \left[3X_{Q_1} + X_{L_1} + 2(X_{H_d} - X_{L_1})\right] \\ &\quad + \Big(n_{H_d} - \mathcal{W} + \sum_i n_{\bar{U}_i}\Big) \cdot (X_{H_d} - X_{L_1}) \ + \ \text{integer}. \end{aligned} \tag{7.41}$$

With $\iota_M{}' = 1$ (thus $X_{\text{total}} = \frac{1}{2} + \text{integer}$), $\mathcal{W} = n_{H_d} + \sum_i n_{\bar{U}_i} - 2$, and $\mathcal{C} = -\mathcal{I}_M{}' + n_{H_d} - n_{H_u} - \sum_i n_{L_i} + 2\sum_i n_{\bar{U}_i} - 4$ we obtain the condition

$$3X_{Q_1} + X_{L_1} \ \stackrel{!}{=} \ -\tfrac{1}{2} \ + \ \text{integer}. \tag{7.42}$$

Considering $\iota_M{}' = 0$ (thus X_{total} is integer), $\mathcal{W} = n_{H_d} + \sum_i n_{\bar{U}_i} - 1$, and $\mathcal{C} = -\mathcal{I}_M{}' + n_{H_d} - n_{H_u} - \sum_i n_{L_i} + 2\sum_i n_{\bar{U}_i} - 2$ then gives

$$X_{H_d} - X_{L_1} \ \stackrel{!}{=} \ \text{integer}. \tag{7.43}$$

Plugging the conditions of Eqs. (7.42) and (7.43) into Eq. (7.41) yields

$$X_{\text{total}} \ = \ \tfrac{\iota_M{}'}{2} \ + \ \text{integer}. \tag{7.44}$$

As a summary, in addition to Eq. (7.9), depending on the desired remnant low-energy discrete symmetry, we need to impose the following conditions on the X charges:

$$\begin{aligned} X_{H_d} - X_{L_1} &= \begin{cases} \text{integer} \\ \text{integer} - m/2 \\ \text{integer} \\ 3\cdot\text{integer} - p/2 \\ \text{integer} \end{cases}, \qquad 3X_{Q_1} + X_{L_1} = \begin{cases} \text{integer} \\ \text{integer} \\ \text{integer} - b/3 \\ \text{integer} - p/3 \\ \text{integer} - m'/2 \end{cases}, \\ \Longrightarrow \quad X_{\text{total}} &= \begin{cases} \text{integer} \\ \text{integer} + m\cdot\iota_M/2, \\ \text{integer} + b\cdot\iota_B/3 \\ \text{integer} + p\cdot\iota_P/6 \\ \text{integer} + m'\cdot\iota_M{}'/2 \end{cases}, \end{aligned} \tag{7.45}$$

(with $m, m' = 1$, $b \in \{1,2\}$, $p \in \{1,5\}$, $\iota_M, \iota_M{}' \in \{0,1\}$, $\iota_B \in \{0,1,2\}$, $\iota_P \in \{0,1,2,3,4,5\}$) to have *all* terms, only $\boldsymbol{M}_p$ terms, only $\boldsymbol{B}_3$ terms, only $\boldsymbol{P}_6$ terms, or only $\boldsymbol{B}_p$ terms allowed by virtue of the X charges, respectively. Note that in Ref. [38] it was shown that Eq. (7.9) together with the coefficients $\mathcal{A}_{CCX}$ and $\mathcal{A}_{WWX}$ of the $SU(3)_C$-$SU(3)_C$-$U(1)_X$ and $SU(2)_W$-$SU(2)_W$-$U(1)_X$ anomalies and the condition of Green-Schwarz anomaly cancellation requires

$$3X_{Q_1} + X_{L_1} = \frac{\text{integer}}{\mathcal{N}_g}, \tag{7.46}$$

where $\mathcal{N}_g$ symbolizes the number of generations. With $\mathcal{N}_g = 3$ all possibilities listed above except the anomalous $\boldsymbol{B}_p$ are compatible with Eq. (7.46).

We want to conclude this section by an investigation of the phenomenological differences between $\boldsymbol{B}_p$ and $\boldsymbol{B}_3$. One can quickly check that on the renormalizable level $\boldsymbol{B}_p$ and $\boldsymbol{B}_3$ allow and forbid exactly the same terms. This equality persists for terms made up of four superfields, so it is interesting to ask the question how to systematically find one of the lowest-dimensional terms which conserve $\boldsymbol{B}_p$ but violate $\boldsymbol{B}_3$. Since $\boldsymbol{B}_p$ is a $\boldsymbol{Z}_2$ symmetry, the last line of Eq. (7.19) can be recast as

$$\sum_i n_{Q_i} \; - \; \sum_i n_{\bar{U}_i} \; - \; \sum_i n_{\bar{D}_i} \; = \; 2\,\mathcal{I}_M{}' + \; \iota_M{}'. \tag{7.47}$$

This is to be compared with Eq. (7.3), leading to

$$3\mathcal{C} \; = \; 2\,\mathcal{I}_M{}' + \; \iota_M{}'. \tag{7.48}$$

Solving Eqs. (7.3)-(7.5) for $\sum_i n_{Q_i}$, $\sum_i n_{\bar{D}_i}$, $\sum_i n_{\bar{U}_i}$, and plugging the result into the second line of Eq. (7.19) we get

$$3\mathcal{W} \; + \; 3\sum_i n_{\bar{E}_i} \; - \; 4\mathcal{C} \; = \; 3\mathcal{I}_B \; + \; \iota_B. \tag{7.49}$$

Eqs. (7.48) and (7.49) imply that $\boldsymbol{B}_p$ is conserved if $\mathcal{C}$ is an integer multiple of two, whereas $\boldsymbol{B}_3$ is conserved if $4\mathcal{C}$ is an integer multiple of three.

We examine the cases with $|\mathcal{C}| \leq 6$. For $|\mathcal{C}| = 0, 6$ $\boldsymbol{B}_p$ and $\boldsymbol{B}_3$ are both conserved, for $|\mathcal{C}| = 1, 5$ $\boldsymbol{B}_p$ and $\boldsymbol{B}_3$ are both not conserved, for $|\mathcal{C}| = 2, 4$ $\boldsymbol{B}_p$ is conserved but $\boldsymbol{B}_3$ is not conserved, for $|\mathcal{C}| = 3$ $\boldsymbol{B}_p$ is not conserved but $\boldsymbol{B}_3$ is conserved. So $|\mathcal{C}| = 2$ is the lowest one with baryon parity and triality differing in their behavior. $|\mathcal{C}| = 2$ implies that the term contains at least six quark superfields [see Eq. (7.3)]. Hence the difference between baryon parity and and baryon triality arises only at the highly nonrenormalizable and thus highly suppressed level, so that the effective low-energy phenomenology is identical.

An easy example (a, b, c, e, f, g and i, j, k, l, m, n, p are color and generational indices, summation over repeated indices implied) of a term which conserves $\boldsymbol{B}_p$ but violates $\boldsymbol{B}_3$ is the "square" of the notorious ($\boldsymbol{B}_p$ and $\boldsymbol{B}_3$ violating) term $\bar{U}\bar{D}\bar{D}$, *i.e.* $\epsilon^{abc}\epsilon^{efg}\bar{U}_i^a\bar{D}_j^b\bar{D}_k^c\bar{U}_l^e\bar{D}_m^f\bar{D}_n^g$; another example which is not a "square" but so to speak "prime" is $\epsilon^{abc}\epsilon^{efg}\bar{U}_i^a\bar{U}_j^b\bar{U}_k^c\bar{D}_l^e\bar{D}_m^f\bar{D}_n^g\bar{E}_p$.

Chapter 8

B_3-Conserving FN Models

Baryon triality conserving Froggatt-Nielsen models are constructed. The DGS arises as a low-energy remnant of the $U(1)_X$ gauge symmetry by an appropriate choice of the X charges. The charge assignment is further constrained by the Green-Schwarz anomaly cancellation conditions as well as the requirement that the obtained orders of magnitude of the fermionic masses and mixings are phenomenologically acceptable. The constraints coming from the experimentally observed neutrino masses and mixing angles are derived for $\boldsymbol{B}_3$-conserving models.

8.1 X Charges for B_3-Conserving FN Models

In Sect. 7.2 we have derived the non-neutrino constraints on the X charges of generic FN models, *i.e.* FN models without a specific DGS. Requiring anomaly cancellation à la Green and Schwarz as well as a phenomenologically viable explanation of the observed quark and charged lepton masses and CKM mixing angles, we could reduce the number of independent parameters of the X-charge assignment significantly. All 17 X charges could be expressed in terms of only six real numbers [see Subsect. 6.5.2, Eqs. (6.67)-(6.76)]:

$$\begin{array}{lll} x = 0, 1, 2, 3 \ , & y = -1, 0, 1 \ , & z = 0, 1 \ , \\ \Delta^L_{31} \equiv X_{L_3} - X_{L_1} \ , & \Delta^L_{21} \equiv X_{L_2} - X_{L_1} \ , & X_{L_1} \ . \end{array} \tag{8.1}$$

Δ^L_{31} and Δ^L_{21} are necessarily integer, whereas X_{L_1} is arbitrary. For phenomenological reasons x, y, z can only take on the shown integer values. As we choose to generate the μ term via the GM mechanism (*cf.* Sect. 6.2), we need $X_{H_d}+X_{H_u} < 0$; taking into account Eq. (6.75) we are restricted to $-X_{H_d} - X_{H_u} = z \stackrel{!}{=} 1$ from now on. x is related to the ratio of the Higgs VEVs by $\epsilon^x \sim \frac{m_b}{m_t} \tan\beta$, with $\tan\beta = \left|\frac{\upsilon_u}{\upsilon_0}\right|$. Recall, the sneutrino VEVs are rotated away, so $|\upsilon_0| = |\upsilon_d| \equiv \sqrt{\upsilon_\alpha{}^* \upsilon_\alpha}$.

y parameterizes all phenomenologically viable CKM matrices. Our preferred choice is $y = 0$, resulting in ${U^{\rm CKM}}_{12} \sim \epsilon$, ${U^{\rm CKM}}_{13} \sim \epsilon^3$ and ${U^{\rm CKM}}_{23} \sim \epsilon^2$, see Eq. (6.65).

Assuming a string-embedded FN framework, the parameter ϵ originates solely in the Dine-Seiberg-Wen-Witten mechanism, see Sect. 6.4. Thus it is a derived quantity which via Eqs. (6.48) and (6.49) depends on the X charges of the quark superfields. Inserting the X charges of Eqs. (6.67),(6.68), (6.69), (6.71), (6.73), and (6.75) into the expression of the anomaly coefficient $\mathcal{A}_{CCX}$ given in Eq. (6.49) yields

$$\mathcal{A}_{CCX} \;=\; \frac{3}{2}\cdot(x+z+6)\,. \tag{8.2}$$

Thus the FN expansion parameter ϵ is determined by x and z. With Eq. (6.48) we get (*cf.* also Ref. [38])

$$\epsilon \;=\; \frac{g_C}{4\,\pi\,\sqrt{2}}\cdot\sqrt{3(x+z+6)}\,. \tag{8.3}$$

Taking $z = 1$, $x = 0, 1, 2, 3$, and the strong gauge coupling constant g_C around $M_{\rm GUT} \sim 10^{16}$ GeV, *i.e.* $g_C \approx 0.72$, we obtain ϵ within the interval $\epsilon \in [0.186, 0.222]$. This justifies the identification of ϵ with the Wolfenstein parameter λ_c made in Subsect. 6.5.2 with hindsight.[1]

Our goal is now to construct a conserved $\boldsymbol{B}_3$ model; as demonstrated in Sect. 7.2 this leads to additional constraints on the X charges. Rewriting Eqs. (7.31) and (7.33), the latter by making use of Eqs. (6.74) and (6.75), it is possible to define the *integers* Δ^H and ζ such that[2]

$$\Delta^H \equiv X_{L_1} - X_{L_0}\,, \qquad 3\zeta + b \equiv \Delta^L_{21} + \Delta^L_{31} - z, \tag{8.4}$$

where the parameter b is as introduced in Eq. (7.33) and only takes on the values 1 or 2. By demanding Δ^H and ζ to be integer, we guarantee $\boldsymbol{B}_3$ conservation. Demanding baryon triality by virtue of the X charges which are given in Eqs. (6.67)-(6.76) in terms of x, y, z, Δ^L_{31}, Δ^L_{21}, and X_{L_1}, we can replace X_{L_1} and Δ^L_{21} in favor of Δ^H, ζ and $b = 1, 2$. This has the advantage that the X charges of the SSM superfields are parameterized by integers only. We thus arrive at the constrained X charges of Table 8.1, which is the equivalent of Table 2 in Ref. [38] for the case of $\boldsymbol{B}_3$ instead of $\boldsymbol{M}_p$. Note that the parameters ζ and b appear in Table 8.1 only in the combination $3\zeta + b$.

---

[1] The parameterization of the mass ratios of the SM fermions in terms of ϵ is based on $\epsilon = \lambda_c = 0.22$, so that working with other values for ϵ is strictly speaking slightly inconsistent.

[2] The corresponding definitions for conserved $\boldsymbol{M}_p$ are

$$\Delta^H \equiv X_{L_1} - X_{L_0} - \tfrac{1}{2}\,, \qquad 3\zeta \equiv \Delta^L_{21} + \Delta^L_{31} - z.$$

$$
\begin{aligned}
X_{H_d} &= \tfrac{1}{5\ (6+x+z)} \Big(6y + x\ (2x + 12 + z - 2\Delta^H) \\
&\quad -z\ (4 + 3\Delta^H) - 2\ (3 + 6\Delta^H - \Delta^L_{31}) - \tfrac{2}{3}\ (6 + x + z)(3\zeta + b)\Big) \\
X_{H_u} &= -z - \ X_{H_d} \\
X_{Q_1} &= \tfrac{1}{3}\Big(10 - X_{H_d} + x + 2y + z - \Delta^H - \tfrac{1}{3}(3\zeta + b)\Big) \\
X_{Q_2} &= X_{Q_1} - 1 - y \\
X_{Q_3} &= X_{Q_1} - 3 - y \\
X_{\bar{U}_1} &= X_{H_d} - X_{Q_1} + 8 + z \\
X_{\bar{U}_2} &= X_{\bar{U}_1} - 3 + y \\
X_{\bar{U}_3} &= X_{\bar{U}_1} - 5 + y \\
X_{\bar{D}_1} &= -X_{H_d} - X_{Q_1} + 4 + x \\
X_{\bar{D}_2} &= X_{\bar{D}_1} - 1 + y \\
X_{\bar{D}_3} &= X_{\bar{D}_1} - 1 + y \\
X_{L_1} &= X_{H_d} + \Delta^H \\
X_{L_2} &= X_{L_1} - \Delta^L_{31} + z + (3\zeta + b) \\
X_{L_3} &= X_{L_1} + \Delta^L_{31} \\
X_{\bar{E}_1} &= -X_{H_d} + 4 - X_{L_1} + x + z \\
X_{\bar{E}_2} &= X_{\bar{E}_1} - 2 - 2z + \Delta^L_{31} - (3\zeta + b) \\
X_{\bar{E}_3} &= X_{\bar{E}_1} - 4 - \ z - \Delta^L_{31}
\end{aligned}
$$

Table 8.1: The constrained X charges with an acceptable low-energy phenomenology of quark and charged lepton masses and quark mixings. In addition, the GS anomaly cancellation conditions are satisfied and conservation of $\boldsymbol{B}_3$ is imposed. x, y, z and b are integers specified in Eqs. (7.33) and (8.1). Δ^H, Δ^L_{31} and ζ are integers as well but as yet unconstrained. $SU(5)$ invariance would require $y = 1$ and $z = \Delta^L_{21} = \Delta^L_{31} = 0$, but the latter is not compatible with the second condition in Eq. (8.4).

Table 8.1 summarizes the constraints on the X charges originating from the GS anomaly cancellation, the quark and charged lepton masses, the CKM matrix, and the conservation of baryon triality. The neutrino sector has not been included so far. Before discussing the restrictions on the X charges due to neutrino masses and mixings, we want to comment on (i) the possibility of baryon-number violation in $\boldsymbol{B}_3$-conserving FN models and (ii) the compatibility with grand unified theories.

(i) Phenomenologically, the conservation of $\boldsymbol{B}_3$ renders the proton stable. For the proton to decay we need a baryon-number violating operator. This in turn requires the parameter $\mathcal{C}$ in Eq. (7.3) to be non-zero. On the other hand, $\mathcal{C}$ must be an integer multiple of three in the case of $\boldsymbol{B}_3$ conservation [see Eq. (7.49), with $\iota_B = 0$]. Hence, only operators with $|\mathcal{C}| = 3, 6, 9, ...$ are $\boldsymbol{B}_3$-conserving and baryon-number violating. Comparing with Eq. (7.3) we see that at least nine quark (or antiquark) superfields are needed. Such a superpotential term, however, is suppressed by a factor of $\frac{1}{{M_{\text{grav}}}^6}$ and thus negligible.

(ii) Our baryon triality conserving models are not compatible with grand unified theories. Unlike in the $\boldsymbol{M}_p$-conserving model in Ref. [38], it is impossible to choose the parameters $x, y, z, \Delta^H, \zeta, b, \Delta^L_{31}$ such that the X charges of Table 8.1 are $SU(5)$-invariant. This should be obvious, since the trilinear GUT superpotential term $\bar{\mathbf{5}}\,\bar{\mathbf{5}}\,\mathbf{10}$, after symmetry breaking, produces $LL\bar{E}$, $LQ\bar{D}$ (both $\boldsymbol{B}_3$ conserving) and $\bar{U}\bar{D}\bar{D}$ ($\boldsymbol{B}_3$ violating). $SU(5)$ invariance requires $y = 1$ and $z = \Delta^L_{21} = \Delta^L_{31} = 0$. However, the latter is not compatible with the second condition in Eq. (8.4). For a review of models where horizontal symmetries are combined with unification see Ref. [118].

8.2 Neutrino Masses from $\boldsymbol{M_p}$ Violation

Neutrino oscillation experiments tell us that at least two neutrinos are massive, see, *e.g.*, Ref. [119] and references therein. Therefore, before going into the details of constructing a $\boldsymbol{B}_3$-conserving FN model, we need to discuss the possible origin of neutrino masses in a $\boldsymbol{B}_3$-conserving but $\boldsymbol{M}_p$-violating theory. In Eq. (7.9) we demand that the mass terms for the quarks and the charged leptons are present in the superpotential. However, there is no mention of a neutrino mass term.

Regarding the SSM as a low-energy effective theory, one can obtain neutrino masses simply from the nonrenormalizable terms $L_iH_uL_jH_u$ without questioning their origin. Doing so, one would expect some new physics to appear at an intermediate energy scale below M_{GUT}.

The introduction of heavy SM singlets, the right-handed neutrinos, $\bar{N}_i$, is the most popular idea of how this new physics might look. Then the Dirac mass terms

$L_iH_u\bar{N}_j$ together with the heavy Majorana mass terms $\bar{N}_i\bar{N}_j$ could in principle, *i.e.* if allowed by the discrete symmetry, lead to light Majorana-type neutrinos via the seesaw mechanism.

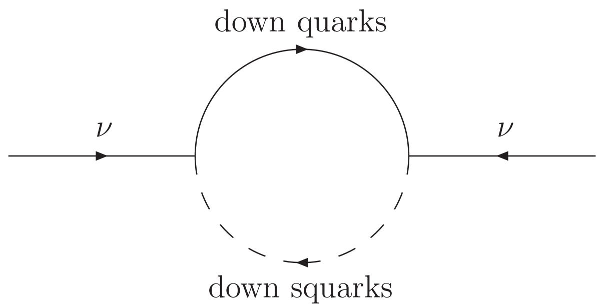
In the framework of an $\boldsymbol{M}_p$-violating theory, such as the $\boldsymbol{B}_3$-conserving SSM, there exists an alternative way of generating neutrino masses. Here, the existence of right-handed neutrinos need not be postulated. The renormalizable superpotential of the $\boldsymbol{B}_3$-conserving SSM is given by, see, *e.g.*, Ref. [40],

$$W \;=\; \frac{1}{2}\lambda_{\alpha\beta k}L_\alpha L_\beta \bar{E}_k + \lambda'_{\alpha jk}L_\alpha Q_j\bar{D}_k + h^U_{ij}Q_iH_u\bar{U}_j + \mu_\alpha L_\alpha H_u, \tag{8.5}$$

with Greek indices running from zero to three, while Latin indices only run from one to three. $\lambda_{\alpha\beta k}$ is taken antisymmetric in the first two indices. The MSSM Yukawa mass matrices $\boldsymbol{h^E}$ and $\boldsymbol{h^D}$ are included in the coupling constants $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda'}$, namely $h^E_{ij} = -\lambda_{0ij}$ and $h^D_{ij} = -\lambda'_{0ij}$, respectively.

Consider first the $\boldsymbol{M}_p$-violating but $\boldsymbol{B}_3$-conserving bilinear superpotential terms L_iH_u, which couple the neutrinos to the neutral higgsinos. The diagonalization of the 7×7 neutral fermion[3] mass matrix then leads to one massive Majorana-type neutrino, *cf.* Ref. [40] and references therein. Therefore, already at tree level, one neutrino can acquire a mass, while two remain massless.

Going beyond tree level, one encounters many contributions to the neutrino mass matrix, see *e.g.* Refs. [77, 120]. Especially, those arising from the quark-squark and the charged lepton-slepton loop are discussed extensively. Consider the $\boldsymbol{M}_p$-violating superpotential terms $L_iQ_j\bar{D}_k$ which evokes the interaction of a neutrino with a down-type quark and a down-type squark. Hence, the following loop diagram can be constructed.



Clearly, such a process contributes to the neutrino Majorana mass matrix. The derivation of the formula for this loop contribution can be found in Ref. [40, 121]. Analogously, the presence of the $\boldsymbol{M}_p$-violating superpotential terms $L_iL_j\bar{E}_k$ leads to a similar diagram with charged leptons and sleptons running in the loop. Due to possible mixing of the charged leptons with the charginos and the charged sleptons

[3]These are the four neutralinos and the three neutrinos.

with the charged Higgs bosons, respectively, it is actually the joint mass eigenstates that propagate in the loop.

The $\boldsymbol{B}_3$-conserving FN model which we are going to construct shall be realized without right-handed neutrinos. Thus, neutrinos acquire their mass via the above mentioned mechanisms, namely (a) mixing with the neutralinos, as well as (b) quark-squark and charged lepton-slepton loop corrections to the neutrino mass matrix.

8.3 Origin of Superpotential Couplings

We have already emphasized in Sects. 6.1 and 6.2 that the bilinear superpotential term $\mu H_d H_u$ should be generated by the GM mechanism; if this term had a pure FN origin, the μ parameter would naturally come out with an unacceptable high value. In this section, we want to discuss the origin of all superpotential interactions of Eq. (8.5). We first give an overview of our choices and state why they are necessary. Then we consider how the coupling constants are affected by the sequence of basis transformations listed in Subsect. 6.5.1.

- By choosing positive integer X charges for all trilinear MSSM interactions all Yukawa mass matrices are generated via the mechanism of Froggatt and Nielsen. Thus we avoid troubles in the fermionic mass spectrum associated with supersymmetric zeros (see Appendix C).

- The generalized μ problem ($\mu_\alpha \, L_\alpha H_u$) is solved by the mechanism of Giudice and Masiero. The reasoning is identical to the discussion of the μ problem in Sects. 6.1 and 6.2.

- As stated in Sect. 8.2, the bilinear terms $\mu_i L_i H_u$ can lead to one massive neutrino. In order to avoid that this mass is too heavy [20], we require $\frac{\mu_i}{\mu_0} \sim \epsilon^{X_{L_0} - X_{L_i}} < 1$, that is $X_{L_0} > X_{L_i}$.

- If the trilinear $\boldsymbol{M}_p$-violating interactions are only suppressed by powers of ϵ comparable to the trilinear MSSM terms, they would be in disagreement with the experimental bounds [38]. Therefore, we forbid the coupling constants λ_{ijk} and λ'_{ijk} by *negative integer* total X charge. On the other hand, $\boldsymbol{M}_p$ violation is necessary in order to generate neutrino masses in our models. As already discussed in Subsect. 6.5.1, the trilinear GM-generated effective couplings are negligibly small, so they cannot provide sufficient $\boldsymbol{M}_p$ violation. Fortunately, however, the trilinear $\boldsymbol{M}_p$-violating operators are also obtained by the canonicalization of the Kähler potential as described above Eq. (6.58).

This mechanism has also been employed in[4] Ref. [122]. The term $\bar{U}\bar{D}\bar{D}$ is forbidden altogether by $\boldsymbol{B}_3$.

Having specified the origin of all superpotential interactions of Eq. (8.5), we can now discuss the ϵ structure of the corresponding coupling constants after the sequence of basis transformations in Subsect. 6.5.1. The FN suppression of the coupling constants μ_α (GM origin) and ${h^U}_{ij}$ (FN origin) is not affected by these transformations, so the original ϵ structure is kept

$$\mu_\alpha \;\sim\; m_{\text{soft}}\,\epsilon^{-X_{L_\alpha}-X_{H_u}}\,, \qquad\qquad h^U_{ij} \;\sim\; \epsilon^{X_{Q_i}+X_{H_u}+X_{\bar{U}_j}}\,. \tag{8.6}$$

Concerning the down-type Yukawa mass matrices $\boldsymbol{h^D}$ and $\boldsymbol{h^E}$ (both of FN origin), notice that the ϵ structure is changed drastically in their off-diagonal entries by the transition to the mass basis.

Finally, consider the $\boldsymbol{M}_p$-violating couplings λ'_{ijk} and λ_{ijk}. As stated above, the corresponding interaction terms should be forbidden due to negative integer overall X charges. However, these supersymmetric zeros are filled in by the canonicalization of the Kähler potential. Obviously, the resulting $\boldsymbol{M}_p$-violating coupling constants are then proportional to the down-type mass matrices. Diagonalizing these by the transformations of Eq. (6.62) therefore also diagonalizes λ'_{ijk} and λ_{ijk} in the corresponding two indices. Due to the antisymmetry of λ_{ijk} in the first two indices, the situation is a little more involved for the $LL\bar{E}$ terms. To be more precise, we present the $\boldsymbol{M}_p$-violating coupling constants λ'_{ijk} and λ_{ijk} in terms of the corresponding $\boldsymbol{M}_p$-conserving Yukawa matrices $\lambda'_{0jk} = -h^D_{jk}$ and $\lambda_{0jk} = -h^E_{jk}$, which are nothing but the diagonalized down-type mass matrices after all required basis transformations. Explicitly, we find

$$\lambda'_{ijk} \;=\; \frac{[\boldsymbol{C^{L}}^{-1}\cdot\boldsymbol{U^{\mathbf{VEVs}}}^\dagger]_{0l}}{[\boldsymbol{C^{L}}^{-1}\cdot\boldsymbol{U^{\mathbf{VEVs}}}^\dagger]_{00}}\,[\boldsymbol{U^{L}}^\dagger]_{li}\,\lambda'_{0jk}\,, \tag{8.7}$$

with the coupling constants $\lambda'_{\alpha jk}$ now given in the basis of diagonal down-type mass matrices. Analogously, we obtain

$$\lambda_{ijk} \;=\; \frac{[\boldsymbol{C^{L}}^{-1}\cdot\boldsymbol{U^{\mathbf{VEVs}}}^\dagger]_{0l}}{[\boldsymbol{C^{L}}^{-1}\cdot\boldsymbol{U^{\mathbf{VEVs}}}^\dagger]_{00}}\,[\boldsymbol{U^{L}}^\dagger]_{li}\,\lambda_{0jk}\;-\;(i\leftrightarrow j). \tag{8.8}$$

Here we have neglected the second (antisymmetrizing) contribution of Eq. (6.59) when expressing λ_{0jk} in terms of ${\lambda^{\text{FN}}}_{0j'k'}$ because, compared to the first term, it is

[4]The authors of Ref. [122] construct their model such that $X_{\bar{U}_i\bar{D}_j\bar{D}_k} < 0$, so that it is GM-suppressed if $X_{\bar{U}_i\bar{D}_j\bar{D}_k}$ is integer. However, with $X_{L_i}+X_{H_u}$ required to be integer and working with $\Delta^L_{21}=\Delta^L_{31}=0$, $z=1$, their proposed X-charge assignment also accidentally generates $\boldsymbol{B}_3$, so that $\bar{U}_i\bar{D}_j\bar{D}_k$ is not only highly suppressed but absent altogether.

	$\frac{\mu_\alpha}{m_{\text{soft}}}$	$\lambda'_{\alpha jk}$	$\lambda_{\alpha\beta k}$
(0)	$\epsilon^{-X_{L_\alpha}-X_{H_u}}$	only $\lambda'_{0jk} \sim \epsilon^{X_{L_0}+X_{Q_j}+X_{\bar{D}_k}}$	only $\lambda_{0jk} \sim \epsilon^{X_{L_0}+X_{L_j}+X_{\bar{E}_k}}$
(1)	$\epsilon^{-X_{L_\alpha}-X_{H_u}}$	$\epsilon^{2X_{L_0}-X_{L_\alpha}+X_{Q_j}+X_{\bar{D}_k}}$	$\epsilon^{2X_{L_0}-X_{L_\alpha}+X_{L_\beta}+X_{\bar{E}_k}} - (\alpha \leftrightarrow \beta)$
(2)	$\epsilon^{-X_{L_\alpha}-X_{H_u}}$	$\lambda'_{0jk} \sim \delta_{jk}\, \epsilon^{X_{L_0}+X_{Q_k}+X_{\bar{D}_k}}$, $\lambda'_{ijk} \sim \epsilon^{X_{L_0}-X_{L_i}} \lambda'_{0jk}$,	$\lambda_{0jk} \sim \delta_{jk}\, \epsilon^{X_{L_0}+X_{L_k}+X_{\bar{E}_k}}$, $\lambda_{ijk} \sim \epsilon^{X_{L_0}-X_{L_i}} \lambda_{0jk} - (i \leftrightarrow j)$

Table 8.2: FN structure of superpotential couplings at various stages of the basis transformations: Before (0) and after (1) the canonicalization of the Kähler potential, and finally in the mass basis of the down-type quarks and the charged leptons (2).

suppressed by a factor of $\epsilon^{2(X_{L_0}-X_{L_j})}$. Hence, both types of trilinear $\boldsymbol{M}_p$-violating coupling constants are proportional to the corresponding Yukawa mass matrices,[5] which are diagonal in our basis.

Table 8.2 summarizes the FN structure of some important superpotential coupling constants at different steps in the sequence of basis transformations. We omitted the up-type quark Yukawa coupling constants in Table 8.2, as they have the standard FN structure, which does not change under the sequence of basis transformations.

In the following section we study the constraints on the X charges arising from the neutrino sector. We assume that the neutrino masses are generated through $\boldsymbol{M}_p$-violating interactions as described in Sect. 8.2.

8.4 Neutrino Sector

8.4.1 Experimental Results

In order to include the experimental constraints from the neutrino sector, in particular from the atmospheric, solar, and reactor experiments, we first need to bring the data into a form which can be compared directly to our FN models. Then we can further constrain the X charges.

In our $\boldsymbol{B}_3$-conserving model, there are no right-handed neutrinos. However, matter parity is broken. Hence we have only Majorana mass terms for the left-handed neutrinos with a symmetric mass matrix $\boldsymbol{M}^{\boldsymbol{\nu}}$. This is diagonalized by a

[5] In Ref. [123] quite generally models for radiatively generated neutrino masses are studied in which as it so happens *(1)* baryon triality is accidentally conserved and *(2)* the trilinear $\boldsymbol{M}_p$-violating coupling constants are proportional to the mass matrices of the down-type quarks and charged leptons. Our model belongs to this category, with both *(1)* and *(2)* arising by virtue of the X charges. The 5[th] charge assignment in Table 8.4 is presented in Ref. [123], as an example.

unitary matrix $\boldsymbol{U^\nu}$

$$\boldsymbol{M^\nu_{\rm diag}} \;=\; \boldsymbol{U^{\nu *}} \cdot \boldsymbol{M^\nu} \cdot \boldsymbol{U^{\nu\dagger}}, \tag{8.9}$$

with $\boldsymbol{M^\nu_{\rm diag}} = \mathbf{diag}(\boldsymbol{m_1}, \boldsymbol{m_2}, \boldsymbol{m_3})$. At this stage, we cannot make any statement on the relative size of the three mass eigenvalues. The connection between the structure of the original mass matrix $\boldsymbol{M^\nu}$ and the ordering of the masses exists only after pinning down the mixing matrix. Experimentally, we have access to the MNS matrix, which is a product of the left-handed charged lepton mixing matrix $\boldsymbol{U^{E_L}}$ and $\boldsymbol{U^\nu}$. As we are already in the basis with a diagonal charged lepton mass matrix, we have $\boldsymbol{U^{E_L}} = \mathbb{1}$ and thus

$$\boldsymbol{U^{\rm MNS}} \;\equiv\; \boldsymbol{U^{E_L}} \cdot \boldsymbol{U^{\nu\dagger}} \;=\; \boldsymbol{U^{\nu\dagger}}. \tag{8.10}$$

In the standard parameterization [124], $\boldsymbol{U^{\rm MNS}}$ is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \cdot \begin{pmatrix} c_{13} & 0 & s_{13}\, e^{\mathrm{i}\delta} \\ 0 & 1 & 0 \\ -s_{13}\, e^{-\mathrm{i}\delta} & 0 & c_{13} \end{pmatrix} \cdot \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} e^{-\mathrm{i}\alpha_1/2} & 0 & 0 \\ 0 & e^{-\mathrm{i}\alpha_2/2} & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

with $c_{ij} \equiv \cos\theta_{ij}$ and $s_{ij} \equiv \sin\theta_{ij}$. Here, $i, j = 1, 2, 3$ are the generation indices, and θ_{ij} denotes the mixing angles between the corresponding neutrino generations. δ and α_1, α_2 are the CP violating Dirac and Majorana phases, respectively. A global three-generation neutrino oscillation analysis yields the following 3σ CL allowed ranges [125]

$$\begin{aligned} \Delta m^2_{21} \;&\equiv\; {m_2}^2 - {m_1}^2 \;= 8.2^{+1.1}_{-0.9} \times 10^{-5}\,\mathrm{eV}^2, \qquad && \tan^2\theta_{12} = 0.39^{+0.21}_{-0.11}\,, \\ |\Delta m^2_{32}| &\equiv |{m_3}^2 - {m_2}^2| = 2.2^{+1.4}_{-0.6} \times 10^{-3}\,\mathrm{eV}^2, && \tan^2\theta_{23} = \;1.0\,^{+1.1}_{-0.5}\,, \\ & && \sin^2\theta_{13} \leq 0.041\,. \end{aligned} \tag{8.11}$$

Up to now, the CP phases have not been measured. Notice, that there are two possibilities for the mass eigenvalues [126]: Either $m_1 < m_2 < m_3$ or $m_3 < m_1 < m_2$. We discuss these two orderings and their individual implications in the context of our FN scenarios in Subsection 8.4.4.

The experimentally measured mixing angles together with their uncertainties can be translated [125] into allowed ranges for the entries of the MNS matrix in terms of the FN parameter ϵ. For the mixing angles θ_{12} and θ_{23} we assume Gaussian errors in their measured values.[6] Furthermore, assuming equal distributions for

[6]Disregarding systematic effects, measured quantities follow a Gaussian distribution in case of high enough statistics. Derived quantities such as the mixing angles θ might, however, show a distorted statistical spread. Taking the central values of $\tan^2\theta$ plus their 3σ CL limits and translating these into corresponding angles θ, we found approximately symmetrical distributions for the mixing angles. Thus we are led to our simplifying assumption of Gaussian errors.

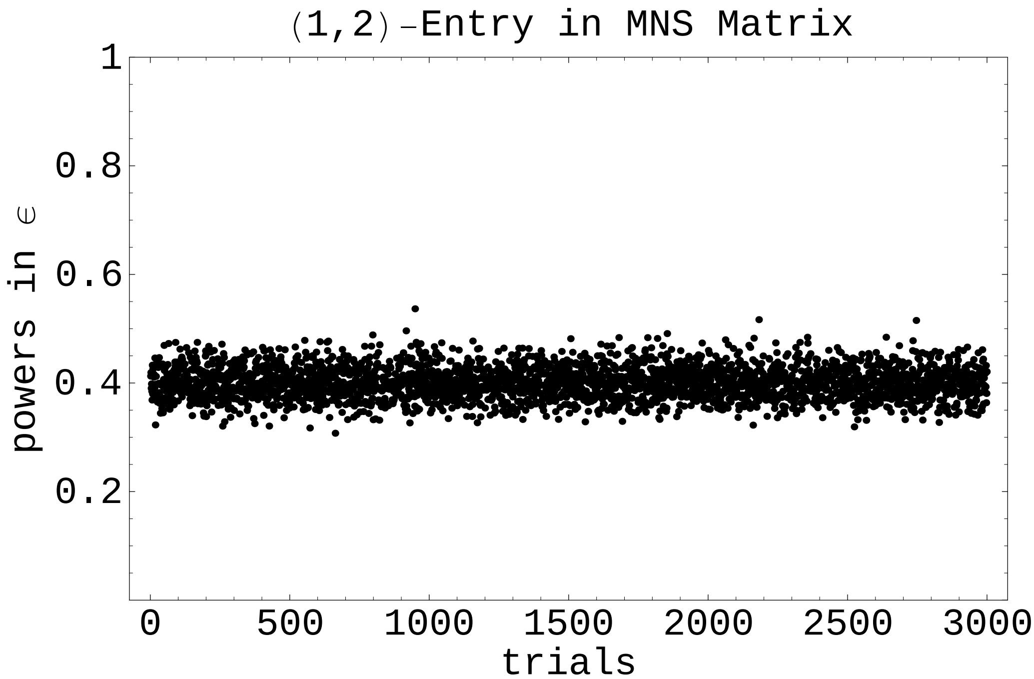


Figure 8.1: The powers in ϵ of the $(1,2)$-element for an ensemble of 3000 sets of mixing parameters obeying Gaussian statistics for θ_{12} and θ_{23}, whereas $\theta_{13} \in [0°, 11.7°]$ and $\delta \in [0, 2\pi]$ are taken from an equal distribution.

the unmeasured quantities $\theta_{13} \in [0°, 11.7°]$ and the Dirac phase $\delta \in [0, 2\pi]$, we can calculate the scatter of the absolute values of the MNS matrix elements. Figure 8.1 shows the powers in $\epsilon = 0.2$ of the $(1,2)$-element for an ensemble of 3000 sets of mixing parameters obeying the upper assumed statistics. From this we deduce an ϵ structure (by definition the exponents must be integer) of ϵ^0 or ϵ^1 for the (1,2)-element.

We employ a similar analysis of the other MNS matrix elements. Due to the unknown $\mathcal{O}(1)$ coefficients in FN models, we allow all (integer) powers in ϵ within about ± 1 of the center of the scattering region. We then obtain the experimentally acceptable ϵ structure for the MNS matrix

$$\boldsymbol{U}^{\mathbf{MNS}}_{\mathbf{exp}} \quad \sim \quad \begin{pmatrix} \epsilon^{0,1} & \epsilon^{0,1} & \epsilon^{0,1,2,3,4} \\ \epsilon^{0,1,2} & \epsilon^{0,1} & \epsilon^{0,1} \\ \epsilon^{0,1,2} & \epsilon^{0,1} & \epsilon^{0,1} \end{pmatrix}, \tag{8.12}$$

where multiple possibilities for the exponents are separated by commas. It should be mentioned again that this calculation is done for $\epsilon = 0.2$. However, varying ϵ within the interval $[0.18, 0.22]$ does not alter the allowed exponents in Eq. (8.12).

8.4.2 Neutrino Mass Matrix

In order to make use of the experimental information about the neutrino sector we need to specify the origin of the neutrino masses. It has already been pointed out that a $\boldsymbol{B}_3$ invariance allows for lepton-number violating $\not{\boldsymbol{M}}_p$ interactions. Due to the bilinear terms $\mu_i\, L_i\, H_u$ the neutrinos mix with the neutralinos, which leads to *one* massive neutrino at tree level.[7] The experimentally inferred mass squared differences Δm^2_{21} and Δm^2_{31} require at least *two* massive neutrinos. Therefore we must consider higher order contributions to the neutrino mass matrix. In the following, we concentrate on effects of quark-squark and charged lepton-slepton loop corrections [40, 127, 77, 121]. The resulting effective neutrino mass matrix in the flavor basis is given by

$$\boldsymbol{M^\nu} \;=\; \boldsymbol{M^\nu_{\text{tree}}} + \boldsymbol{M^\nu_{\lambda'\text{-loop}}} + \boldsymbol{M^\nu_{\lambda\text{-loop}}}, \tag{8.13}$$

with

$$M^\nu_{\text{tree}\,ij} \;=\; \frac{m_Z{}^2\; M_{\tilde\gamma}\; \mu_0\;\; \cos^2\beta}{m_Z{}^2\; M_{\tilde\gamma}\;\; \sin 2\beta \;\;-\;\; M_1\; M_2\; \mu_0} \cdot \frac{\mu_i\;\mu_j}{\mu_0{}^2}, \tag{8.14}$$

$$M^\nu_{\lambda'\text{-loop}\,ij} \;\simeq\; \frac{3}{32\pi^2} \sum_{l,n} (\,\lambda'_{iln}\;\lambda'_{jnl} \;+\; \lambda'_{jln}\;\lambda'_{inl}\,)\; m^d_l \sin 2\phi^{\tilde d}_n\;\; \ln\left[\left(\frac{m^{\tilde d}_{n_1}}{m^{\tilde d}_{n_2}}\right)^2\right], \tag{8.15}$$

$$M^\nu_{\lambda\text{-loop}\,ij} \;\simeq\; \frac{1}{32\pi^2} \sum_{l,n} (\,\lambda_{iln}\;\lambda_{jnl} \;+\; \lambda_{jln}\;\lambda_{inl}\,)\; m^e_l \sin 2\phi^{\tilde e}_n\;\; \ln\left[\left(\frac{m^{\tilde e}_{n_1}}{m^{\tilde e}_{n_2}}\right)^2\right]. \tag{8.16}$$

Here, m_Z is the Z-boson mass. $m_l^{d/e}$ denotes the masses of the down-type quarks/ charged leptons of generation $l = 1, 2, 3$. M_1 and M_2 are the gaugino mass parameters which, together with the weak mixing angle θ_W, define the photino mass parameter $M_{\tilde\gamma} = M_1 \cos^2\theta_W + M_2 \sin^2\theta_W$. In addition, we have the down-type squark/charged slepton masses $m^{\tilde d/\tilde e}_{n_i}$ of generation n; $i = 1, 2$ labels the two sfermion mass eigenstates in one generation. Finally, $\phi_n^{\tilde d/\tilde e}$ is the mixing angle in the sfermion sector, explicitly (no summation over repeated indices),

$$\tan 2\phi^{\tilde d}_n \;=\; \frac{2m^d_n \Big|[\boldsymbol{A_D}]_{0nn} - \mu_0{}^* \tan\beta\Big|}{[\boldsymbol{M^2_{\tilde Q}}]_{nn} - [\boldsymbol{M^2_{\tilde D}}]_{nn} - \frac{1}{24}(g_Y{}^2 - 3g_W{}^2)(\upsilon_u{}^2 - \upsilon_d{}^2)}, \tag{8.17}$$

$$\tan 2\phi^{\tilde e}_n \;=\; \frac{2m^e_n \Big|[\boldsymbol{A_E}]_{0nn} - \mu_0{}^* \tan\beta\Big|}{[\boldsymbol{M^2_{\tilde L}}]_{nn} - [\boldsymbol{M^2_{\tilde E}}]_{nn} - \frac{1}{8}(3g_Y{}^2 - g_W{}^2)(\upsilon_u{}^2 - \upsilon_d{}^2)}, \tag{8.18}$$

[7] Strictly speaking, the distinction between neutrino and neutralino mass eigenstates is no longer appropriate. However, due to stringent experimental constraints on the neutrino masses, $\mu_i/m_W \ll 1$. Thus the mixing is small so that the three light mass eigenstates, the so-called "neutrino mass eigenstates", comprise only small amounts of the neutralino states.

where the $\boldsymbol{A_{D,E}}$ are the coefficients of the soft supersymmetry breaking trilinear scalar interactions $[\boldsymbol{A_D}]_{\alpha jk}\ \lambda'_{\alpha jk}\ \widetilde{L}_\alpha \widetilde{Q}_j \widetilde{D}^*_k$ and $\frac{1}{2}[\boldsymbol{A_E}]_{\alpha\beta k}\ \lambda_{\alpha\beta k}\ \widetilde{L}_\alpha \widetilde{L}_\beta \widetilde{E}^*_k$. Here, $\widetilde{L}_\alpha$ refers to the scalar component of the chiral superfields L_α. The $[\boldsymbol{M^2_{\tilde{\dots}}}]_{ij}$ are the soft scalar masses squared. Assuming all soft supersymmetry breaking mass parameters are $\mathcal{O}(m_{\text{soft}}) > 100\,\text{GeV}$ and excluding accidental cancellations, the denominators of Eqs. (8.17) and (8.18) are of order m^2_{soft}. For the numerators we get $2\,m_n^{d/e} m_{\text{soft}}\,\mathcal{O}(1+\epsilon\tan\beta)$. Taking into account the lower limit for m_{soft} of about 300 GeV which originates from the combination of the experimental lower bound on $\mu_0 \geq 60\,\text{GeV}$ and its ϵ structure, $\mu_0 \sim m_{\text{soft}}\cdot\epsilon$, in our model (see also Appendix D), we conclude that even for large $\tan\beta \lesssim 50$ the left-right mixing in the down squark and charged slepton sectors is small. Thus, the sines in Eqs. (8.15) and (8.16) can be approximated by tangents. Furthermore, the logarithms become $\mathcal{O}(1)$ coefficients if the sfermion masses are nondegenerate but not too different either, *i.e.* $\mathcal{O}(1) \lesssim \left([m^{\tilde{d}/\tilde{e}}_{n_i}]^2 - [m^{\tilde{d}/\tilde{e}}_{n_j}]^2\right)\Big/[m^{\tilde{d}/\tilde{e}}_{n_j}]^2 \lesssim \mathcal{O}(10)$, where $[m^{\tilde{d}/\tilde{e}}_{n_i}] > [m^{\tilde{d}/\tilde{e}}_{n_j}]$. We consider these assumptions natural and apply the corresponding simplifications to Eqs. (8.15) and (8.16). Inserting the $\not{\boldsymbol{M}}_p$ parameters of the last line in Table 8.2, using the phenomenological constraints of Table 8.1 and keeping only leading terms, we obtain the FN structure of the tree and the loop contributions to the neutrino mass matrix

$$M^\nu_{\text{tree}\,ij} \sim \frac{{m_Z}^2\ M_{\tilde{\gamma}}\ \mu_0\ \cos^2\beta}{{m_Z}^2\ M_{\tilde{\gamma}}\ \sin 2\beta\ -\ M_1\ M_2\ \mu_0}\cdot\epsilon^{-2\Delta^H-\Delta^L_{i1}-\Delta^L_{j1}}, \tag{8.19}$$

$$M^\nu_{\lambda'\text{-loop}\,ij} \sim \frac{3}{8\pi^2}\frac{{m_b}^2\ \left|[\boldsymbol{A_D}]_{033} - {\mu_0}^*\tan\beta\right|\epsilon^{2x}}{[\boldsymbol{M^2_{\tilde{Q}}}]_{33} - [\boldsymbol{M^2_{\tilde{D}}}]_{33} - \frac{1}{24}({g_Y}^2 - 3{g_W}^2)({\upsilon_u}^2 - {\upsilon_d}^2)}\cdot\epsilon^{-2\Delta^H-\Delta^L_{i1}-\Delta^L_{j1}},$$

$$M^\nu_{\lambda\text{-loop}\,ij} \sim \frac{1}{8\pi^2}\frac{{m_\tau}^2\ \left|[\boldsymbol{A_E}]_{033} - {\mu_0}^*\tan\beta\right|\epsilon^{2x}}{[\boldsymbol{M^2_{\tilde{L}}}]_{33} - [\boldsymbol{M^2_{\tilde{E}}}]_{33} - \frac{1}{8}(3{g_Y}^2 - {g_W}^2)({\upsilon_u}^2 - {\upsilon_d}^2)}\cdot\epsilon^{-2\Delta^H-\Delta^L_{i1}-\Delta^L_{j1}}\cdot f_{ij}.$$

Here, we have replaced m_3^d by m_b and m_3^e by m_τ. The factors $f_{ij} = f_{ji}$ in the last term take care of the λ_{iln}'s direct dependence on the charged lepton mass matrix and its antisymmetry under interchange of the first two indices. Depending on i and j the tau-stau loop may be forbidden by symmetry and thus does not give the leading contribution. For $i,j = 1,2$ we find $f_{ij} \sim 1$, whereas $f_{23} \sim \epsilon^4$ and $f_{13} \sim f_{33} \sim \epsilon^8$. See Appendix E for details.

Some remarks are in order at this point. Compared to the quark-squark loop, the charged lepton-slepton loop does not contribute significantly to the neutrino mass matrix. Therefore we neglect it in our following discussion. There is a further source of neutrino masses: the nonrenormalizable but $\boldsymbol{B}_3$-conserving superpotential terms $L_i H_u L_j H_u$. In our model, these effective terms are generated via the GM mechanism and thus suppressed by a factor of $\frac{m_{\text{soft}}}{{M_{\text{grav}}}^2}\epsilon^{2z-2\Delta^H-\Delta^L_{i1}-\Delta^L_{j1}}$. Inserting the Higgs VEV υ_u for H_u we find that the resulting neutrino mass scale is negligibly small compared to the tree level contribution in Eq. (8.14). The ratio of the two is of the order $\frac{{m_{\text{soft}}}^2}{{M_{\text{grav}}}^2}(1+\tan^2\beta)$. So even for large $\tan\beta$ it can be safely discarded. Similarly, we find that the quark-squark loop contribution of Eq. (8.15) is huge compared to the mass scale of the nonrenormalizable operators $L_iH_uL_jH_u$.

8.4.3 Constraints from Neutrino Masses

In our model, we obtain one massive neutrino already at tree level. A second nonzero mass is supplied by the quark-squark loop. Notice that, except for an overall relative factor, the ϵ structure of the tree level and one-loop matrices is exactly the same. However, they are not aligned in the sense that one matrix is a (real or complex) multiple of the other; the μ_i and the λ'_{i33} have completely different origin, *i.e.* the $\mathcal{O}(1)$ coefficients are in general different. Adding the two terms we therefore expect not one but two nonzero mass eigenvalues. One neutrino remains massless since $\boldsymbol{M}^{\boldsymbol{\nu}}_{\textbf{tree}}$ and $\boldsymbol{M}^{\boldsymbol{\nu}}_{\boldsymbol{\lambda'}\textbf{-loop}}$ are both rank one matrices.[8] Hence, a degenerate neutrino scenario is excluded. Notice that this remains true even if we reconsider the possibility of the charged lepton-slepton loop contribution. The resulting third nonzero mass eigenvalue would be smaller by a factor of $\frac{{m_\tau}^2}{3{m_b}^2}\approx\frac{1}{15}$ compared to the quark-squark loop mass. This is inconsistent with degenerate neutrino masses which require $m_1\approx m_2\approx m_3\gg\sqrt{|{m_i}^2-{m_j}^2|}$ for all $i\neq j$.

To see whether our model is compatible with the hierarchical or the inverse-hierarchical neutrino scenario, we calculate the relative factor $\frac{m^{\text{tree}}}{m^{\text{loop}}}$ between the overall scales of the tree and the loop mass matrix [*cf.* Eq. (8.19)]. This factor must not come out larger than the ratio of the atmospheric and solar neutrino mass scales which is approximately 5. First, we do a rough estimate for $\tan\beta\lesssim 2$, that is $x=2,3$ (thus $\cos\beta\gtrsim 0.5$), where we assume all soft breaking parameters, even the gaugino masses M_1 and M_2, to be of the same order $\mathcal{O}(m_{\text{soft}})$. Neglecting the first term in the denominator of the tree level as well as the second term in the numerator of the loop level overall mass scale, we arrive at $\frac{m^{\text{tree}}}{m^{\text{loop}}}\sim\frac{8\pi^2}{3}\cos^2\!\beta\,\frac{{m_Z}^2}{{m_b}^2}\,\epsilon^{-2x}$. This is much too large for $x\geq 2$. So we are restricted to the cases with $x=0,1$, *i.e.* $\tan\beta\gtrsim 8$. We can then approximate $\cos\beta$ by $\cot\!\beta\sim\epsilon^{-x}\frac{m_b}{m_t}$. Neglecting again the first term in the denominator of the tree level overall mass scale, we get

$$\frac{m^{\text{tree}}}{m^{\text{loop}}}\;\sim\;\epsilon^{-4x}\;\frac{8\pi^2\;{m_Z}^2}{3{m_t}^2}\cdot\frac{M_{\tilde\gamma}}{M_1\;M_2}\cdot\frac{[\boldsymbol{M}^{\mathbf{2}}_{\tilde{\boldsymbol{Q}}}]_{33}-[\boldsymbol{M}^{\mathbf{2}}_{\tilde{\boldsymbol{D}}}]_{33}-\frac{1}{24}({g_Y}^2-3{g_W}^2)({\upsilon_u}^2-{\upsilon_d}^2)}{\left|[\boldsymbol{A_D}]_{033}-{\mu_0}^*\;\epsilon^x\;\frac{m_t}{m_b}\right|}. \tag{8.20}$$

[8]For the loop contribution this statement relies on the fact that the $\not{\!\boldsymbol{M}}_p$ coupling constants are generated via the canonicalization of the Kähler potential and are thus proportional to the down-type quark mass matrix, cf. Eq. (8.7).

Note that we have replaced $\tan\beta$ in the denominator of the last factor. The second factor numerically gives a value of about 7. Taking $x = 1$ requires the product of the last two factors to yield a tiny fraction of their natural value of about 1. Such a scenario where there is either fine tuning in the scalar masses or the gaugino mass parameters are about 1000 times larger than the scalar masses is very unnatural. We therefore reject this case and focus on $x = 0$. This together with $z = 1$ numerically determines the expansion parameter $\epsilon \equiv \frac{\langle A \rangle}{M_{\text{grav}}} = 0.186$, see Eq. (8.3). Notice that with $x = 0$ it seems reasonable to assume that the denominator of the last term in Eq. (8.20) is now dominated by the second term. Taking the gaugino masses at a common scale $M_{1/2}$, and the scalar mass parameters all of $\mathcal{O}(m_{\text{soft}})$, we can simplify Eq. (8.20) to

$$\frac{m^{\text{tree}}}{m^{\text{loop}}} \sim \frac{8\pi^2 {m_Z}^2 m_b}{3{m_t}^3} \cdot \epsilon^{-z} \cdot \frac{m_{\text{soft}}}{M_{1/2}} \sim \mathcal{O}(1)\, \frac{m_{\text{soft}}}{M_{1/2}}. \tag{8.21}$$

Often the scalar quark masses are taken bigger than the gaugino mass parameters by factors of about two to five [128, 129]. Hence, assuming $m_{\text{soft}} \approx 5\, M_{1/2}$ we predict a hierarchical neutrino scenario, with the tree level contribution providing for one relatively heavy neutrino while the other two neutrinos remain light. On the other hand, an inverse hierarchy is possible just as well. Then the tree and the quark-squark loop mass matrices must have the same order of magnitude, thus generating two relatively heavy neutrinos while the third neutrino remains light. Due to our ignorance of the soft breaking sector and the arbitrariness of all $\mathcal{O}(1)$ coefficients, our $\boldsymbol{B}_3$-conserving FN models allow both, the hierarchical and the inverse-hierarchical neutrino scenario.

In both cases however, the mass of the heaviest neutrino is given by the atmospheric neutrino mass scale $\sqrt{|\Delta m^2_{32}|}$. Thus the integer parameter Δ^H can be determined. Equating the eigenvalue of the tree level neutrino mass matrix, which is proportional to $\sum_i \frac{{\mu_i}^2}{{\mu_0}^2}$, with $\sqrt{|\Delta m^2_{32}|}$ and putting $M_{1/2} = \mathcal{O}(m_{\text{soft}})$ yields

$$-2\Delta^H \sim \frac{1}{\ln\epsilon} \cdot \ln \frac{{m_t}^2\, m_{\text{soft}} \sqrt{|\Delta m^2_{32}|}}{{m_b}^2\, {m_Z}^2}.$$

Here we made use of the ordering $X_{L_3} \leq X_{L_2} \leq X_{L_1}$, so that $\sum_i \epsilon^{-2\Delta^L_{i1}} \sim 1$. Inserting $\epsilon = 0.186$, $m_t = 175\,\text{GeV}$, $m_b = 4.2\,\text{GeV}$, $m_Z = 91.2\,\text{GeV}$, $\sqrt{|\Delta m^2_{32}|} = 0.047\,\text{eV}$ and $1000\,\text{GeV} \geq m_{\text{soft}} \geq 100\,\text{GeV}$ gives the following range

$$-2\Delta^H \in [\,11.0\,,\, 12.3\,]. \tag{8.22}$$

Here the lower bound corresponds to $m_{\text{soft}} = 1000\,\text{GeV}$ and the upper one to $m_{\text{soft}} = 100\,\text{GeV}$. Since Δ^H is integer, we end up with the single option

$$\Delta^H = -\,6. \tag{8.23}$$

At the end of Appendix D we argue that the sequence of basis transformations generates $\boldsymbol{M}_p$-violating coupling constants which are to some extent larger than expected. Taking this feature into account, the interval in Eq. (8.22) is shifted slightly to higher values. For $\mu_i \sim \epsilon^{-0.5} \cdot m_{\text{soft}}\, \epsilon^{-X_{L_i}-X_{H_u}}$, where the first factor quantifies such a systematic effect, this shift has magnitude one. So the constraint of Eq. (8.23) remains stable.

8.4.4 Constraints from Neutrino Mixing

We now turn to the conditions on the X charges imposed by the MNS matrix. The effective neutrino mass matrix of Eq. (8.13) is diagonalized by the unitary transformation $\mathfrak{U}^{\nu}{}_{ij} \sim \epsilon^{|X_{L_i}-X_{L_j}|}$ so that the eigenvalues $\mathfrak{m}_i$ are given in the order $\mathfrak{m}_3 \ll \mathfrak{m}_2 \lesssim \mathfrak{m}_1$ (for details see Appendix F). It is important to realize that $\boldsymbol{\mathfrak{U}^{\nu}}$ is different from $\boldsymbol{U^{\nu}} = \boldsymbol{U^{\mathbf{MNS}}}^{\dagger}$. Both matrices diagonalize $\boldsymbol{M^{\nu}}$, however, the ordering of the masses is not identical. The crucial question is then how to relate the eigenvalues $\mathfrak{m}_i$ with the masses m_i which are defined in Eq. (8.9). Here we have to distinguish between two different possibilities: hierarchy and inverse hierarchy in the neutrino masses.

Mass Ordering		Hierarchy	Inverse Hierarchy
Heaviest	$\mathfrak{m}_1$	m_3	m_2
Medium	$\mathfrak{m}_2$	m_2	m_1
Lightest	$\mathfrak{m}_3$	m_1	m_3

For the hierarchical scenario, m_1 must be the lightest and m_3 the heaviest neutrino mass. We can exchange $\mathfrak{m}_1$ and $\mathfrak{m}_3$ by multiplying the transposition

$$\boldsymbol{T}^{\mathbf{h.}} \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{8.24}$$

to $\boldsymbol{\mathfrak{U}^{\nu}}$. In that way we get the correct diagonalization matrix

$$\boldsymbol{U^{\nu}} \equiv \boldsymbol{T}^{\mathbf{h.}} \cdot \boldsymbol{\mathfrak{U}^{\nu}}, \tag{8.25}$$

for hierarchical neutrino masses. Combining Eqs. (8.10), (8.12) and (8.25) we find

$$\epsilon^{|X_{L_i}-X_{L_j}|} \sim \begin{pmatrix} \epsilon^{0,1,2,3,4} & \epsilon^{0,1} & \epsilon^{0,1} \\ \epsilon^{0,1} & \epsilon^{0,1} & \epsilon^{0,1} \\ \epsilon^{0,1} & \epsilon^{0,1,2} & \epsilon^{0,1,2} \end{pmatrix}_{ij}, \tag{8.26}$$

Δ^L_{31}	Δ^L_{21}	Hierarchy	Inverse Hierarchy	Conservation of $\boldsymbol{B}_3$
-1	-1	yes	no	no
-1	0	yes	yes	yes
0	0	yes	yes	yes

Table 8.3: All combinations of Δ^L_{i1} which are compatible with the experimental MNS matrix for the hierarchical and the inverse-hierarchical neutrino scenario. In addition, the condition of $\boldsymbol{B}_3$ conservation on the X charges as stated in Eq. (8.4) is checked.

which puts restrictions on Δ^L_{i1}, for $i = 2, 3$.[9] All acceptable combinations which in addition comply with the ordering $\Delta^L_{31} \leq \Delta^L_{21} \leq 0$ are listed in Table 8.3. As we impose conservation of $\boldsymbol{B}_3$, the second condition in Eq. (8.4) has to be satisfied, *i.e.* $\Delta^L_{31} + \Delta^L_{21} - z \neq 0 \bmod 3$. The last column of Table 8.3 shows which cases are compatible with $\boldsymbol{B}_3$ conservation for $z = 1$.

In the case of an inverse hierarchy, we need $m_3 \ll m_1 \lesssim m_2$. Similar to the hierarchical case it is necessary to introduce a transposition which, now, interchanges $\mathfrak{m}_1$ with $\mathfrak{m}_2$,

$$\boldsymbol{T}^{\mathbf{i.h.}} \equiv \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8.27}$$

Taking this into account leads to

$$\epsilon^{|X_{L_i} - X_{L_j}|} \sim \begin{pmatrix} \epsilon^{0,1} & \epsilon^{0,1} & \epsilon^{0,1} \\ \epsilon^{0,1} & \epsilon^{0,1,2} & \epsilon^{0,1,2} \\ \epsilon^{0,1,2,3,4} & \epsilon^{0,1} & \epsilon^{0,1} \end{pmatrix}_{ij}, \tag{8.28}$$

for inverse-hierarchical neutrinos. Again, the allowed Δ^L_{i1} are given in Table 8.3. In addition to the constraints arising from the experimental MNS matrix, we now

[9] In Ref. [38] with $\boldsymbol{M}_p$ conserved by virtue of the X charges, the MNS matrix was assumed with the structure

$$\boldsymbol{U}^{\mathbf{MNS}}_{\mathbf{exp}} \sim \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & \epsilon & \epsilon \\ \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \end{pmatrix}.$$

This resulted from a seesaw mass matrix, and thus no transposition $\boldsymbol{T}^{\cdots}$ was needed. With ${U^{\mathrm{MNS}}}_{ij} \sim \epsilon^{|X_{L_i} - X_{L_j}|}$ this led to $\Delta^L_{21} = \Delta^L_{31} \in \{-1, 0, 1\}$. Combining this with $z \in \{0, 1\}$ and the second equation in footnote 2 of this chapter, led to $-\Delta^L_{21} = -\Delta^L_{31} = \zeta = z = 0$ or 1. The authors then focused on the latter choice because it allows the nicer MNS matrix, the Giudice-Masiero mechanism takes care of the μ term and proton stability is greater.

have to ensure that the ratio $\frac{\mathfrak{m}_1}{\mathfrak{m}_2}$ of the two heavy masses is of order one. As shown in Appendix F, $\mathfrak{m}_2$ is not only determined by the scale of the second largest contribution to the neutrino mass matrix, but it is additionally suppressed by a factor of $\epsilon^{-2\Delta^L_{21}}$, *cf.* Eq. (F.11). For this reason $\Delta^L_{21} = -1$ is forbidden in the case of an inverse hierarchy.

8.4.5 Viable $\boldsymbol{X}$-Charge Assignments

In summary, we have fixed almost all parameters determining the FN charges by imposing conservation of $\boldsymbol{B}_3$, requiring GS anomaly cancellation and finally taking into account the phenomenological constraints of the low-energy fermionic mass spectrum, including the neutrinos. Starting with Table 8.1 we need $z = 1$ if the bilinear superpotential terms are to be generated via the Giudice-Masiero mechanism. Then, we have $x = 0$ due to the upper limit of $\frac{m^{\text{tree}}}{m^{\text{loop}}}$, which is given by the ratio of the atmospheric and the solar neutrino mass scale. Δ^H is fixed through the absolute neutrino mass scale. As degenerate neutrinos are excluded, this corresponds to the atmospheric mass scale. Hence we find $\Delta^H = -6$. Finally, the constraints coming from the MNS mixing matrix together with the requirement of $\boldsymbol{B}_3$ conservation yield $\Delta^L_{21} = 0$ and $\Delta^L_{31} = -1, 0$ (see Table 8.3). So, in the end we are left with only the choice of

$$y \;=\; -1, 0, 1, \quad \text{and} \quad 3\zeta + b \;\equiv\; \Delta^L_{31} + \Delta^L_{21} - z \;=\; \Delta^L_{31} - 1 \;=\; -2, -1. \qquad (8.29)$$

This leads to six sets of viable X-charge assignments displayed in Table 8.4. All sets are compatible with either a hierarchical or an inverse-hierarchical neutrino scenario, depending on the ratio $\frac{m^{\text{tree}}}{m^{\text{loop}}}$ [*cf.* Eq. (8.21)] and unknown $\mathcal{O}(1)$ coefficients in $\boldsymbol{M^\nu}$. Taking the smallness of the $(1,3)$-element of the MNS matrix in Eq. (8.12) [corresponding to the $(1,1)$-entry of Eq. (8.26) in the hierarchical, and the $(3,1)$-entry of Eq. (8.28) in the inverse-hierarchical case] as a crucial criterion, we prefer the inverse-hierarchical cases with $\Delta^L_{31} = -1$. It is only there, that the FN prediction for this entry is of $\mathcal{O}(\epsilon)$. In all other cases we have to assume an unattractively small "$\mathcal{O}(1)$ coefficient". Remarkably, there exists one set where all FN charges are multiples of one third. This salient charge assignment is obtained for $3\zeta + b = -2$ (or equivalently $\Delta^L_{31} = -1$) and $y = 1$. However, as $y \neq 0$, the CKM matrix does not come out too nice. All other sets contain highly fractional X charges (just like the sets in [38]) and are thus "esthetically disfavored". However, requiring that the FN scenario is in agreement with the very tight experimental bounds on exotic processes typically leads to highly fractional X-charge assignments [38, 65], so that the models presented in this subsection are so-to-speak in good company. In the manner of Ref. [65] we checked that the $\boldsymbol{M}_p$-violating coupling constants which are produced by the six sets of X charges

Input			**Output**							
Δ^L_{31}	$3\zeta+b$	y	X_{H_d}	X_{H_u}	i	X_{Q_i}	$X_{\bar{U}_i}$	$X_{\bar{D}_i}$	X_{L_i}	$X_{\bar{E}_i}$
-1	-2	-1	$\frac{244}{105}$	$-\frac{349}{105}$	1	$\frac{467}{105}$	$\frac{722}{105}$	$-\frac{97}{35}$	$-\frac{386}{105}$	$\frac{667}{105}$
					2	$\frac{467}{105}$	$\frac{302}{105}$	$-\frac{167}{35}$	$-\frac{386}{105}$	$\frac{352}{105}$
					3	$\frac{257}{105}$	$\frac{92}{105}$	$-\frac{167}{35}$	$-\frac{491}{105}$	$\frac{247}{105}$
-1	-2	0	$\frac{262}{105}$	$-\frac{367}{105}$	1	$\frac{177}{35}$	$\frac{676}{105}$	$-\frac{373}{105}$	$-\frac{368}{105}$	$\frac{631}{105}$
					2	$\frac{142}{35}$	$\frac{361}{105}$	$-\frac{478}{105}$	$-\frac{368}{105}$	$\frac{316}{105}$
					3	$\frac{72}{35}$	$\frac{151}{105}$	$-\frac{478}{105}$	$-\frac{473}{105}$	$\frac{211}{105}$
-1	-2	1	$\frac{8}{3}$	$-\frac{11}{3}$	1	$\frac{17}{3}$	6	$-\frac{13}{3}$	$-\frac{10}{3}$	$\frac{17}{3}$
					2	$\frac{11}{3}$	4	$-\frac{13}{3}$	$-\frac{10}{3}$	$\frac{8}{3}$
					3	$\frac{5}{3}$	2	$-\frac{13}{3}$	$-\frac{13}{3}$	$\frac{5}{3}$
0	-1	-1	$\frac{236}{105}$	$-\frac{341}{105}$	1	$\frac{458}{105}$	$\frac{241}{35}$	$-\frac{274}{105}$	$-\frac{394}{105}$	$\frac{683}{105}$
					2	$\frac{458}{105}$	$\frac{101}{35}$	$-\frac{484}{105}$	$-\frac{394}{105}$	$\frac{368}{105}$
					3	$\frac{248}{105}$	$\frac{31}{35}$	$-\frac{484}{105}$	$-\frac{394}{105}$	$\frac{158}{105}$
0	-1	0	$\frac{254}{105}$	$-\frac{359}{105}$	1	$\frac{174}{35}$	$\frac{677}{105}$	$-\frac{356}{105}$	$-\frac{376}{105}$	$\frac{647}{105}$
					2	$\frac{139}{35}$	$\frac{362}{105}$	$-\frac{461}{105}$	$-\frac{376}{105}$	$\frac{332}{105}$
					3	$\frac{69}{35}$	$\frac{152}{105}$	$-\frac{461}{105}$	$-\frac{376}{105}$	$\frac{122}{105}$
0	-1	1	$\frac{272}{105}$	$-\frac{377}{105}$	1	$\frac{586}{105}$	$\frac{631}{105}$	$-\frac{146}{35}$	$-\frac{358}{105}$	$\frac{611}{105}$
					2	$\frac{376}{105}$	$\frac{421}{105}$	$-\frac{146}{35}$	$-\frac{358}{105}$	$\frac{296}{105}$
					3	$\frac{166}{105}$	$\frac{211}{105}$	$-\frac{146}{35}$	$-\frac{358}{105}$	$\frac{86}{105}$

Table 8.4: All six sets of viable X-charge assignments, where $z = 1$, $x = 0$ (*i.e.* large $\tan\beta$) and $\Delta^H = -6$. The other input parameters of Table 8.1, namely Δ^L_{31}, $3\zeta + b$, and y differentiate between the various possible scenarios. All of them are compatible with hierarchical and inverse-hierarchical neutrino masses, depending on the ratio $\frac{m^{\text{tree}}}{m^{\text{loop}}}$ and unknown $\mathcal{O}(1)$ coefficients in $\boldsymbol{M^\nu}$. The former depends on the parameters of supersymmetry breaking. Here we assume gravity mediation so that all soft breaking mass parameters are of roughly $\mathcal{O}(m_{\text{soft}})$, with $m_{\text{soft}} \in [100\,\text{GeV}, 1000\,\text{GeV}]$. In order to determine the structure of the sneutrino VEVs, we have assumed a FN structure for b_α and $[\boldsymbol{M^2_{\tilde{L}}}]_{\alpha\beta}$, see Appendix D.

are all in agreement with the experimental bounds, unless there is an unnatural adding-up among the $\mathcal{O}(1)$ coefficients. In Appendix G we shall give an explicit example how high-energy physics, constrained by the third X-charge assignment in Table 8.4, boils down to a viable low-energy phenomenology.

Finally, one could raise the question: Is it possible to construct a scenario where no hidden sector fields are needed to cancel the effects of the cubic $\mathcal{A}_{XXX}$ and the gravitational $\mathcal{A}_{GGX}$ anomaly coefficients? In terms of the X charges, $\mathcal{A}_{GGX}$ [see Eq. (6.35)] and $\mathcal{A}_{CCX}$ [see Eq. (6.49)] are given as

$$\begin{aligned}\mathcal{A}_{GGX} &= X_{H_d} + X_{H_u} + \frac{1}{2}\sum_i \Big(6X_{Q_i} + 3X_{\bar{U}_i} + 3X_{\bar{D}_i} + 2X_{L_i} + X_{\bar{E}_i}\Big) \\ &\qquad + \frac{1}{2}X_\Phi + \mathcal{A}_{GGX}^{\text{hidden sector}}, \end{aligned} \tag{8.30}$$

$$\mathcal{A}_{CCX} = \frac{1}{2}\Big[\sum_i \Big(2\, X_{Q_i} + X_{\bar{U}_i} + X_{\bar{D}_i}\Big)\Big]. \tag{8.31}$$

Inserting the relations of Table 8.1 with $z = 1$ and $x = 0$ yields

$$\mathcal{A}_{GGX} = \frac{1}{2}\left[(3\zeta + b) + 3\Delta^H + 68\right] + \mathcal{A}_{GGX}^{\text{hidden sector}} \quad \text{and} \quad \mathcal{A}_{CCX} = \frac{21}{2}. \tag{8.32}$$

Requiring the Green-Schwarz anomaly cancellation conditions given in Eqs. (6.34) and (6.36) yields

$$\frac{\mathcal{A}_{CCX}}{k_C} = \frac{\mathcal{A}_{GGX}}{12}, \tag{8.33}$$

with k_C being the *positive integer* Kač-Moody level of $SU(3)_C$. Assuming that the hidden sector fields are uncharged under $U(1)_X$, *i.e.* $\mathcal{A}_{GGX}^{\text{hidden sector}} = 0$, we arrive at the condition

$$\frac{2\cdot 2\cdot 3\cdot 3\cdot 7}{k_C} = (3\zeta + b) + 3\Delta^H + 68. \tag{8.34}$$

As both sides of this equation have to be integer, k_C is restricted to some product of the primes in the numerator on the left. Thus the left hand side of Eq. (8.34) can only take the following values: $2, 3, 4, 6, 7, 9, 12, 14, 18, 21, 28, 36, 42, 63, 84, 126, 252$. However, with $3\zeta + b = -2, -1$ [*cf.* Eq. (8.29)] and $\Delta^H = -6$ [*cf.* Eq. (8.23)] the right hand side of Eq. (8.34) becomes 48 or 49, which certainly does not overlap with the needed values even if a variation of ± 1 in Δ^H was allowed. This shows that $U(1)_X$-charged hidden sector fields are necessary to cancel the gravity-gravity-$U(1)_X$ anomaly.

Chapter 9

Summary and Outlook, Part II

In Part II of this thesis we have combined the idea of a discrete symmetry emerging from a local gauge symmetry with the scenario of Froggatt and Nielsen. The Standard Model gauge group G_{SM} is augmented by a generation-dependent $U(1)_X$ symmetry. This additional gauge group is spontaneously broken by the vacuum expectation value of the scalar component of the flavon superfield Φ. Assuming the $U(1)_X$ to be anomalous, the breakdown occurs slightly below the gravitational scale via the Dine-Seiberg-Wen-Witten mechanism. Contrary to the situation investigated in Part I of this thesis, we require *nonvanishing* mixed anomaly coefficients. However, their overall effects are compensated by the nonlinear $U(1)_X$ transformation of the dilaton superfield, S, if the Green-Schwarz anomaly cancellation conditions are satisfied.

With regard to the low-energy theory, the purpose of the $U(1)_X$ local gauge symmetry is to

(i) provide an explanation of the fermionic mass spectrum and the observed mixings, as well as

(ii) generate a (phenomenologically necessary) discrete symmetry after its breakdown.

A discrete symmetry is needed to ensure a long-lived proton. Without enlarging the low-energy fermionic particle content of the MSSM and excluding a GS mechanism, there are only three "generalized matter parities" [39, 28, 116] which (a) can originate from an anomaly-free $U(1)_X$ gauge symmetry and thus do not experience violation by quantum gravity effects, and (b) are also phenomenologically acceptable because they allow for neutrino masses. These salient discrete symmetries are matter parity $\boldsymbol{M}_p$, baryon triality $\boldsymbol{B}_3$, and proton hexality $\boldsymbol{P}_6$ (see Sect. 4.1 of Part I).

The X-charge assignment of the MSSM superfields is now constrained by (i), the fermionic masses and mixings, including the neutrino sector, (ii), the specific choice of the DGS, and finally the GS anomaly cancellation conditions. Concerning the DGS, we have derived the conditions for $\boldsymbol{M}_p$, $\boldsymbol{B}_3$, $\boldsymbol{P}_6$, and baryon parity, $\boldsymbol{B}_p$, to arise as remnants of the continuous $U(1)_X$ symmetry by virtue of the X-charge assignment.

Then we have constructed a minimalist and compact $\boldsymbol{B}_3$-conserving $U(1)_X$ Froggatt-Nielsen scenario with the MSSM particle content plus one additional flavon superfield Φ. Furthermore, our model exhibits only two mass scales, M_{grav} and m_{soft}. The generalized μ problem is solved via the Giudice-Masiero mechanism. The DGS baryon triality has some attractive features: First, it phenomenologically stabilizes the proton. Second, it allows for bilinear and trilinear $\boldsymbol{M}_p$-violating coupling constants, so that neutrino masses are possible at the renormalizable level without the need to introduce right-handed neutrinos. Imposing further the restrictions of the measured fermionic masses and mixings as well as the GS anomaly cancellation conditions, we arrive at six phenomenologically viable sets of X charges presented in Table 8.4. All of them feature large $\tan\beta$ ($\gtrsim$ 40). Our ignorance about the details of the soft supersymmetry breaking parameters does not allow us to distinguish between models of normal and inverse neutrino mass hierarchy. However, taking the smallness of ${U^{\text{MNS}}}_{13}$ as a crucial criterion, we prefer the first three cases (*i.e.* those with $\Delta^L_{31} = -1$) of Table 8.4 and an inverse hierarchy. Doing so, our model then *predicts* inverse-hierarchical neutrino masses. Of all six possibilities, we find the third X-charge assignment of Table 8.4 the most pleasing: All X charges are integer multiples of one third in this case.

In our study of $\boldsymbol{B}_3$ conserving FN models we have restricted ourselves to cases where (A) the μ parameter is obtained through the mechanism of Giudice and Masiero, thus $z = 1$, and (B) the existence of supersymmetric zeros in the fermionic mass matrices is forbidden. It should be interesting to study the consequences of relaxing these restrictions. We conclude with some comments on the possibilities and the implications of such extensions. Furthermore, one can also combine (C) proton hexality $\boldsymbol{P}_6$, or (D) other discrete gauge symmetries with the idea of Froggatt and Nielsen.

(A) In constructing viable models of the fermionic mass spectrum we have been guided by the principle of minimality and compactness. With only the $U(1)_X$ gauge symmetry and two mass scales at hand, we had to exclude the choice $z = 0$ right from the beginning, as it is incompatible with a GM origin of the μ parameter. However, the quest for a dark matter candidate requires us to introduce at least one additional particle like, *e.g.*, the axion, which in turn would suggest the existence of a new global $U(1)$ symmetry [130]. Also superstring models often predict more than just one $U(1)$. So it is tempting to assume that the μ term is originally forbidden by such a symmetry. Effectively it may then be generated via some mechanism other than GM at the phenomenologically needed mass scale (see, *e.g.*, Refs. [131, 132, 133, 134]). In that case, the possibility of $z = 0$ should be reconsidered and investigated seriously.

(B) Supersymmetric zeros must not occur in the Yukawa quark mass matrices because, if present, they lead to an unacceptable FN structure of the CKM mixing matrix (see Appendix C and Refs. [135, 136]). However, allowing for supersymmetric zeros in the leptonic mass matrices is not excluded, as the origin and thus the X-charge dependence of the neutrino mass matrix is not clear. Similarly the MNS mixing matrix can be due to the charged lepton or the neutrino mass matrix. Therefore, investigating the possibility of supersymmetric zeros in the leptonic mass matrices could be another direction of further study. Maybe they are a blessing for the MNS matrix. It would be interesting to examine if a small MNS mixing angle θ_{13} can naturally arise from supersymmetric zeros appearing in the charged lepton and/or neutrino mass matrix (see, *e.g.*, Ref. [137]).

(C) In Part I we have introduced the anomaly-free DGS proton hexality $\boldsymbol{P}_6$, which we propose as *the* discrete symmetry of the MSSM, instead of $\boldsymbol{M}_p$. Consequently, when merging DGSs with the FN idea, one should also aim at constructing $\boldsymbol{P}_6$-conserving FN models. In such scenarios, the neutrino masses cannot be generated through $\not{\boldsymbol{M}}_p$ processes. We have to stick to the nonrenormalizable operators $L_i H_u L_j H_u$ which, assuming a FN origin, lead to a neutrino mass matrix with the ϵ structure

$$M^{\nu}{}_{ij} \quad \sim \quad \frac{v_u^2}{M_{\rm grav}} \cdot \epsilon^{2X_{H_u}+X_{L_i}+X_{L_j}} \,. \tag{9.1}$$

With $X_{L_3} \leq X_{L_2} \leq X_{L_1}$, the (3,3)-entry determines the absolute neutrino mass scale. Since this must be higher than the atmospheric neutrino mass scale, but lower than the WMAP constraint, we demand

$$M^{\nu}{}_{33} \quad \sim \quad [0.05\,\text{eV}, 0.5\,\text{eV}]. \tag{9.2}$$

Thus the corresponding exponent of Eq. (9.1), *i.e.* $2X_{H_u} + 2X_{L_3}$, can only take the integer values -5, -6, or -7. Here an exponent of -7 generates the higher absolute neutrino mass scale which in turn implies a degenerate neutrino scenario. Rewriting this condition on $2X_{H_u} + 2X_{L_3}$ with the help of Eq. (7.37), *i.e.* one requirement of $\boldsymbol{P}_6$ conservation, we get

$$2\Delta^L_{31} \quad = \quad 6 \cdot \text{integer} + 2z - p - \left\{ \begin{array}{c} 5 \\ 6 \\ 7 \end{array} \right. , \tag{9.3}$$

with $p = 1, 5$. Restricting to $\Delta^L_{31} \in \{0, -1, -2\}$ and[1] $z = 1, 2$, we obtain only eight possible sets of parameters (z, p, Δ^L_{31}) which satisfy Eq. (9.3):

z	1	1	1	1	2	2	2	2
p	1	1	5	5	1	1	5	5
Δ^L_{31}	0	-2	-1	-2	-1	-2	0	-1

Inserting Eq. (6.74) into Eq. (7.39), *i.e.* the second requirement of $\boldsymbol{P}_6$ conservation, gives

$$\Delta^L_{21} \quad = \quad 3 \cdot \text{integer} + z + p - \Delta^L_{31}. \tag{9.4}$$

As we need $\Delta^L_{31} \leq \Delta^L_{21} \leq 0$, only four of the eight sets survive, namely:

z	1	1	2	2
p	1	5	1	5
Δ^L_{31}	-2	-2	-2	-1
Δ^L_{21}	-2	-1	-1	-1

Comparing these four cases with Eq. (9.3), we find that all but the first yield an absolute neutrino mass scale which requires degenerate neutrinos. However, the second and the third sets yield hierarchical scenarios because $\Delta^L_{31} < \Delta^L_{21}$. Therefore, we are left with only two possibilities, either $z = 1$ and $\Delta^L_{31} = \Delta^L_{21} = -2$, or $z = 2$ and $\Delta^L_{31} = \Delta^L_{21} = -1$. Assuming $\boldsymbol{M^\nu}$ of the form given in Eq. (9.1), the above result relies solely on the $\boldsymbol{P}_6$ conditions and the experimentally allowed absolute neutrino mass scale. The constraints of the MNS matrix have not been taken into account so far.

A more systematic investigation necessarily includes the structure of the MNS matrix as well as the possibility to have a neutrino mass matrix of a form different from the one assumed in Eq. (9.1).

(D) Last but not least, we should not forget that our FN framework adopts the Dine-Seiberg-Wen-Witten mechanism to break the high-energy gauge symmetry. So we assume an *anomalous* $U(1)_X$. Hence, the remnant DGS need not be anomaly-free in the sense of Ibáñez and Ross. If we find phenomenologically acceptable DGSs which satisfy the discrete GS anomaly cancellation conditions, one should also construct FN models featuring these symmetries.

[1] We only consider cases where the μ term originates from the GM mechanism. To be more flexible, we also allow for $z = 2$.

Appendix A

More than one $U(1)_X$ Breaking Field

The following is a discussion of a scenario with two $U(1)_X$ breaking fields Φ_A and Φ_B carrying the X charges A' and B', respectively. For simplicity, we choose both A' and B' to be positive; the generalization to also negative X charges is straightforward. After the breakdown of $U(1)_X$, the effective operators in the Lagrangian can only have an overall X charge of the form

$$X_{\text{total}} = -a \cdot A' \ - \ b \cdot B', \tag{A.1}$$

where $a, b \in \mathbb{N}$ for superpotential terms and $a, b \in \mathbb{Z}$ for Kähler potential terms. Here, we assume a normalization in which all particles have integer $U(1)_X$ charges, *i.e.* $A', B' \in \mathbb{Z}$. If A' and B' have a largest common factor F, we can define new integers $A \equiv A'/F$ and $B \equiv B'/F$. With this, Eq. (A.1) can be rewritten as

$$X_{\text{total}} = -F \cdot [a \cdot A \ + \ b \cdot B] \,. \tag{A.2}$$

Obviously, X_{total} is a multiple of F. If the square bracket is *not* restricted to any subset of $\mathbb{Z}$, we will end up with a $\boldsymbol{Z}_F$ symmetry after $U(1)_X$ breaking. The question, however, remains whether or not the square bracket can take *any* integer value. In order to find an answer we first decompose A and B into their prime factors:

$$A = \prod_i \alpha_i, \qquad B = \prod_i \beta_i, \tag{A.3}$$

with α_i and β_i being primes. Since A and B do not have a common factor, one necessarily has that

$$\alpha_i \neq \beta_j, \qquad \text{for all } i, j. \tag{A.4}$$

Thus, the least common multiple of A and B is just their product $A \cdot B$. If one can obtain any integer within the interval $[0, A \cdot B[$ with an appropriate integer-valued

linear combination of A and B, then, the square bracket in Eq. (A.2) can take any integer value whatsoever.
Consider the two linear combinations

$$0 \leq a_1 \cdot A + b_1 \cdot B < A \cdot B, \tag{A.5}$$

$$0 \leq a_2 \cdot A + b_2 \cdot B < A \cdot B, \tag{A.6}$$

with $b_1, b_2 \in \{0, 1, ..., A-1\}$ and $a_1, a_2 \in \mathbb{Z}$ such that the linear combinations of A and B lie within the given interval. Assuming $b_1 \neq b_2$, we can show that the two linear combinations can never be matched within the interval $[0, A \cdot B[$ as

$$a_1 \cdot A + b_1 \cdot B = a_2 \cdot A + b_2 \cdot B, \tag{A.7}$$

can be rewritten as

$$\underbrace{(b_1 - b_2)}_{\equiv \Delta b} \cdot B = (a_2 - a_1) \cdot A. \tag{A.8}$$

For this to be true, Δb would have to be a multiple of A, which, however, is not the case for $b_1 \neq b_2$ and $b_1, b_2 \in \{0, 1, ..., A-1\}$. Therefore, two linear combinations of the form $0 \leq a \cdot A + b \cdot B < A \cdot B$ always yield different values for different b. Now there are A different b. For each b one finds B different possible values for a such that the linear combination lies within the interval $[0, A \cdot B[$. Thus we can obtain $A \cdot B$ *different* values, *i.e. all* integers, within the interval $[0, A \cdot B[$ by integer-valued linear combinations of A and B. This finally shows that the square bracket in Eq. (A.2) can take any integer value.

Likewise, this argumentation can be applied to cases with any number of $U(1)_X$ breaking fields, Φ_i $(i = 1, 2, ...)$. The remnant discrete symmetry is a $\boldsymbol{Z}_N$ with N being the largest common factor of all X_{Φ_i}.

Appendix B

Bilinear Terms of $SU(2)_W$ Doublets

Consider the left-chiral superfields U_i and D_i of an $SU(2)_W$ doublet $\begin{pmatrix} U_i \\ D_i \end{pmatrix}$. The index i enumerates the superfields in case there is more than one. The most general $SU(2)_W$-invariant bilinear terms are given by

$$\sum_{i,j} c_{ij} \left(U_i D_j - D_i U_j \right), \tag{B.1}$$

with c_{ij} being mass parameters. Only the antisymmetric part $c_{[i,j]} \equiv \frac{c_{ij} - c_{ji}}{2}$ of c_{ij} contributes to this sum. Thus we have

$$2 \cdot \sum_{i,j} c_{[i,j]} U_i D_j. \tag{B.2}$$

The fermionic mass terms are obtained by taking the F-term of Eq. (B.2) with the left-chiral Weyl spinors u_i of U_i and d_i of D_i. Due to the antisymmetry of $c_{[i,j]}$ we can simplify the sum as follows

$$-\frac{1}{2} \cdot 2 \cdot \sum_{i<j} c_{[i,j]} \left(u_i d_j - u_j d_i \right). \tag{B.3}$$

The factor of $-\frac{1}{2}$ is due to a Fierz reordering. Assuming real parameters $c_{[i,j]}$, we can add the hermitian conjugate to obtain the mass terms of the Lagrangian

$$-\sum_{i<j} c_{[i,j]} \left(u_i d_j + \bar{u}_i \bar{d}_j - u_j d_i - \bar{u}_j \bar{d}_i \right). \tag{B.4}$$

Fixing i and j, the term in the parentheses involves four Weyl degrees of freedom, namely u_i, u_j, d_i and d_j. Out of these four Weyl spinors, we now construct two

Dirac spinors Ψ_1^{ij} and Ψ_2^{ij}

$$\Psi_1^{ij} \equiv \begin{pmatrix} u_i \\ \bar{d}_j \end{pmatrix}, \qquad \Psi_2^{ij} \equiv \begin{pmatrix} d_i \\ -\bar{u}_j \end{pmatrix}. \tag{B.5}$$

In terms of these, Eq. (B.4) takes the form

$$-\sum_{i<j} c_{[i,j]} \left(\overline{\Psi_1^{ij}} \Psi_1^{ij} + \overline{\Psi_2^{ij}} \Psi_2^{ij} \right), \tag{B.6}$$

thus showing that the Dirac fields Ψ_1^{ij} and Ψ_2^{ij} both have the same mass $c_{[i,j]}$. We conclude that any $SU(2)_W$-invariant bilinear term formed out of $SU(2)_W$ doublets necessarily leads to Dirac mass terms.

Note, however, that any Dirac particle can be decomposed into two Majorana particles with the same mass. To see this, consider the Dirac spinor

$$\Psi \equiv \begin{pmatrix} \xi \\ \bar{\eta} \end{pmatrix}, \tag{B.7}$$

where ξ and η are left-chiral Weyl spinors. Defining the Majorana spinors

$$\Theta_1 \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \xi + \eta \\ \bar{\xi} + \bar{\eta} \end{pmatrix} \qquad \text{and} \qquad \Theta_2 \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \xi - \eta \\ -\bar{\xi} + \bar{\eta} \end{pmatrix}, \tag{B.8}$$

we easily find that

$$\overline{\Psi}\Psi = \frac{1}{2}\Big(\overline{\Theta}_1 \Theta_1 + \overline{\Theta}_2 \Theta_2 \Big). \tag{B.9}$$

These considerations show that we need not include heavy Majorana fermions in the $SU(2)_W - SU(2)_W - U(1)_X$ anomaly cancellation condition of Ibáñez and Ross as there are no genuine[1] heavy Majorana doublets.

[1]If we wish to decompose heavy Dirac particles into Majorana particles, we are always left with the *double* number of Majorana fermions.

Appendix C

Supersymmetric Zeros in Mass Matrices

The structure of the CKM matrix depends on the overall X charges of the terms in the quark mass matrices. Due to holomorphy of the superpotential, terms with negative X charge are forbidden. These supersymmetric zeros are filled in by the canonicalization of the Kähler potential. Naively, the resulting mass matrices might suggest a CKM matrix consistent with the experimentally measured quark mixings. However, if one allows for supersymmetric zeros, then, things are more involved since the matrix canonicalizing the kinetic terms of the quark doublet Q affects both, the up- *and* the down-type quark mass matrices. Diagonalizing these, we therefore encounter cancellations in the CKM matrix (which is a product of the two left-handed diagonalization matrices). These cancellations spoil the naively expected nice results. Espinosa and Ibarra [135, 136] have investigated the influence of supersymmetric zeros on the CKM matrix. In the following, we illustrate such a situation explicitly for *two* generations of quarks. Assume an X-charge assignment with

$$X_{Q_2} = X_{Q_1} + 1, \quad X_{\bar{U}_2} = X_{\bar{U}_1} - 5, \quad X_{\bar{D}_2} = X_{\bar{D}_1} - 3,$$
$$X_{H_u} = 4 - X_{Q_1} - X_{\bar{U}_1}, \quad X_{H_d} = 2 - X_{Q_1} - X_{\bar{D}_1}. \tag{C.1}$$

Then, the Yukawa matrices come out as

$$\boldsymbol{h}^{\boldsymbol{U}}_{\mathbf{FN}} \sim \begin{pmatrix} \epsilon^4 & 0 \\ \epsilon^5 & 1 \end{pmatrix}, \quad \boldsymbol{h}^{\boldsymbol{D}}_{\mathbf{FN}} \sim \begin{pmatrix} \epsilon^2 & 0 \\ \epsilon^3 & 1 \end{pmatrix}, \tag{C.2}$$

where the $(1,2)$-elements are supersymmetric zeros. The Kähler potential has to be canonicalized by matrices of the form

$$\boldsymbol{C}^{\boldsymbol{Q}^{-1}} \sim \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix}, \quad \boldsymbol{C}^{\bar{\boldsymbol{U}}^{-1}} \sim \begin{pmatrix} 1 & \epsilon^5 \\ \epsilon^5 & 1 \end{pmatrix}, \quad \boldsymbol{C}^{\bar{\boldsymbol{D}}^{-1}} \sim \begin{pmatrix} 1 & \epsilon^3 \\ \epsilon^3 & 1 \end{pmatrix}. \tag{C.3}$$

These transformations change the Yukawa matrices to

$$\boldsymbol{h^U} \sim \begin{pmatrix} \epsilon^4 & \epsilon \\ \epsilon^5 & 1 \end{pmatrix}, \qquad \boldsymbol{h^D} \sim \begin{pmatrix} \epsilon^2 & \epsilon \\ \epsilon^3 & 1 \end{pmatrix}, \tag{C.4}$$

which are diagonalized by unitary matrices with the textures[1]

$$\begin{aligned} \boldsymbol{U^{U_L}} &\sim \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix}, \qquad \boldsymbol{U^{\bar{U}}} \sim \begin{pmatrix} 1 & \epsilon^5 \\ \epsilon^5 & 1 \end{pmatrix}, \\ \boldsymbol{U^{D_L}} &\sim \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix}, \qquad \boldsymbol{U^{\bar{D}}} \sim \begin{pmatrix} 1 & \epsilon^3 \\ \epsilon^3 & 1 \end{pmatrix}. \end{aligned} \tag{C.5}$$

If we naively neglect possible cancellations between $\boldsymbol{U^{U_L}}$ and $\boldsymbol{U^{D_L}}$ we *wrongly* get

$$\boldsymbol{U^{\mathbf{CKM}}} \equiv \boldsymbol{U^{U_L}} \cdot \boldsymbol{U^{D_L}}^\dagger \sim \begin{pmatrix} 1 & \epsilon \\ \epsilon & 1 \end{pmatrix}. \tag{C.6}$$

In order to show that this argumentation is too simple, we numerically calculated the CKM matrix for an ensemble of 3000 *Mathematica*© -randomly generated sets of complex $\mathcal{O}(1)$ coefficients, which remain undetermined by any FN model and appear in both, the FN-generated Yukawa matrices and the kinetic (hermitian) Kähler potential terms. Figure C.1 shows the powers in $\epsilon = 0.2$ of the off-diagonal element in the CKM matrix. The quark mass ratios $\frac{m_u}{m_c}$ and $\frac{m_d}{m_s}$ are depicted in Figure C.2.

Obviously, allowing for supersymmetric zeros in the up- and down-type Yukawa mass matrices, the naive result of Eq. (C.6) is in gross disagreement with the numerically calculated CKM matrix. Cancellations between $\boldsymbol{U^{U_L}}$ and $\boldsymbol{U^{D_L}}$ render the off-diagonal entry of the CKM matrix from $\mathcal{O}(\epsilon)$ to $\mathcal{O}(\epsilon^5)$. Thus, the Cabibbo angle is actually obtained at a very tiny and therefore experimentally excluded magnitude, although the naive result suggests relatively large quark mixing. On the other hand, the quark mass ratios come out correct in the naive calculation, namely

$$\frac{m_u}{m_c} \sim \epsilon^4\,, \qquad \frac{m_d}{m_s} \sim \epsilon^2\,. \tag{C.7}$$

[1]Note that $\boldsymbol{U} = \begin{pmatrix} \xi & 0 \\ 0 & \tilde{\xi} \end{pmatrix} \cdot \begin{pmatrix} 1 & -\chi^* \epsilon^a \\ \chi \epsilon^a & 1 \end{pmatrix}$, with $|\xi|^{-1} = |\tilde{\xi}|^{-1} = \sqrt{1 + |\chi|^2\, \epsilon^{2a}}$ and $a \in \mathbb{N}$, is the most general unitary matrix with texture $\begin{pmatrix} 1 & \epsilon^a \\ \epsilon^a & 1 \end{pmatrix}$. Applying this form to calculate the off-diagonals of $\boldsymbol{U^{U_L}}^* \cdot \boldsymbol{h^U} \cdot \boldsymbol{U^{\bar{U}}}^\dagger$ we readily find that $\boldsymbol{h^U}$ is diagonalized if χ is of $\mathcal{O}(1)$. The same holds for $\boldsymbol{h^D}$. With our choice of X charges, the ratios of the quark masses are $\frac{m_u}{m_c} \sim \epsilon^4$ and $\frac{m_d}{m_s} \sim \epsilon^2$, respectively.

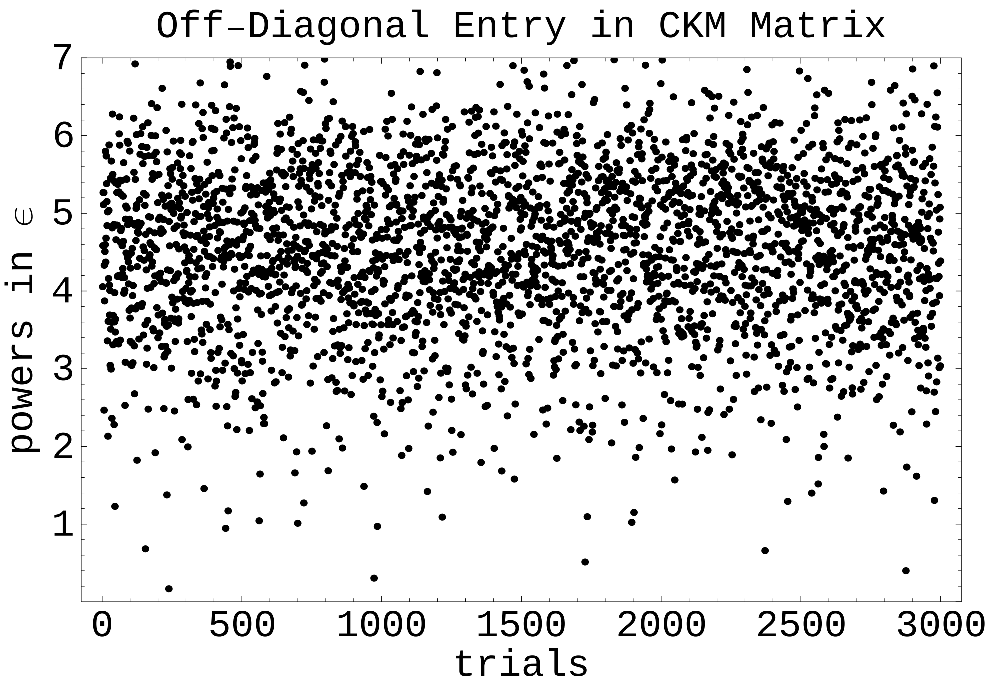


Figure C.1: The powers in ϵ of the off-diagonal entry in the two generation CKM matrix for 3000 sets of randomly generated complex $\mathcal{O}(1)$ coefficients in the FN-generated Yukawa matrices and the kinetic Kähler potential terms.

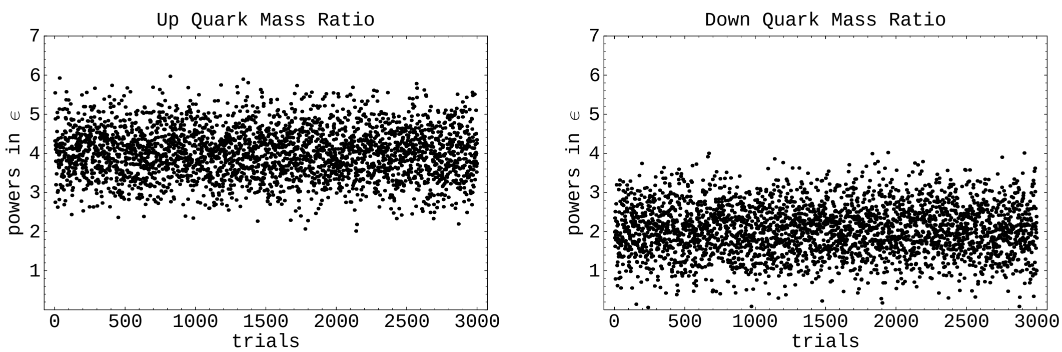


Figure C.2: The powers in ϵ of the ratios $\frac{m_u}{m_c}$ and $\frac{m_d}{m_s}$.

Appendix D

The Structure of Sneutrino VEVs

At tree level, the five minimization conditions in the $\boldsymbol{M}_p$-violating SSM are given as (see, *e.g.*, Ref. [40])

$$\Big({M_{H_u}}^2 + {\mu_\alpha}^* \; \mu_\alpha + \frac{{g_W}^2 + {g_Y}^2}{8} \left(|\upsilon_u|^2 - |\upsilon_d|^2 \right) \Big) \; \upsilon_u \; - \; {b_\alpha}^* \; {\upsilon_\alpha}^* \;\; \stackrel{!}{=} \;\; 0, \qquad \text{(D.1)}$$

$$\Big([\boldsymbol{M}^{\boldsymbol{2}}_{\widetilde{\boldsymbol{L}}}]_{\alpha\beta} + {\mu_\alpha}^* \; \mu_\beta + \frac{{g_W}^2 + {g_Y}^2}{8} \left(|\upsilon_d|^2 - |\upsilon_u|^2 \right) \delta_{\alpha\beta} \Big) \; \upsilon_\beta \; - {b_\alpha}^* \; {\upsilon_u}^* \;\; \stackrel{!}{=} \;\; 0, \qquad \text{(D.2)}$$

with $|\upsilon_d| \equiv \sqrt{{\upsilon_\alpha}^* \; \upsilon_\alpha}$. Unfortunately, not much is known about the soft supersymmetry breaking parameters. Assuming, however, minimal supergravity together with the separation of the hidden and the observable sector in the superpotential, we find for the bilinear terms $b_\alpha \widetilde{L_\alpha} H_u$ that [20]

$$b_\alpha \;\; \propto \;\; \mu_\alpha. \qquad \text{(D.3)}$$

Taking this as a motivation, we will from now on work with

$$b_\alpha \;\; \sim \;\; B \; m_{\text{soft}} \; \epsilon^{-X_{L_\alpha} - X_{H_u}}, \qquad \text{(D.4)}$$

where we do not demand a strict proportionality as in Eq. (D.3). B is a mass parameter of $\mathcal{O}(m_{\text{soft}})$. The other crucial ingredient is the structure of $[\boldsymbol{M}^{\boldsymbol{2}}_{\widetilde{\boldsymbol{L}}}]_{\boldsymbol{\alpha\beta}}$. For simplicity we take

$$[\boldsymbol{M}^{\boldsymbol{2}}_{\widetilde{\boldsymbol{L}}}]_{\alpha\beta} \;\; \sim \;\; {m_{\text{soft}}}^2 \; \epsilon^{|X_{L_\alpha} - X_{L_\beta}|}. \qquad \text{(D.5)}$$

This might originate either directly from a FN structure of the corresponding parent terms or via the CK transformation of a *diagonal* matrix $[\boldsymbol{M}^{\boldsymbol{2}}_{\widetilde{\boldsymbol{L}}}]$, whose eigenvalues are all of the same order but *not equal*.

Soft supersymmetry breaking parameters with the structure of Eqs. (D.4) and (D.5) have already been suggested in Ref. [112].

The FN structure is now passed on to the vacuum expectation values via the minimization conditions. To see this, we first simplify Eq. (D.2) by observing that $\frac{{g_W}^2+{g_Y}^2}{8}(|\upsilon_u|^2-|\upsilon_d|^2) < \frac{1}{10}\cdot(246\,\mathrm{GeV})^2$. On the other hand, the non-appearance of gauginos in electron-positron annihilation tells us that $\mu_0 \geq 60\,\mathrm{GeV}$ (see page 220 of Ref. [100]), which – in our case – gives a lower bound on m_{soft}: As we assume a GM-generated $\mu_0 \sim m_{\mathrm{soft}}\,\epsilon^{-X_{L_0}-X_{H_u}} \sim m_{\mathrm{soft}}\,\epsilon$, the lowest allowed value for m_{soft} is about 300 GeV. Putting both observations together we have

$$\frac{{g_W}^2+{g_Y}^2}{8}\,(|\upsilon_u|^2-|\upsilon_d|^2) \;\ll\; (246\,\mathrm{GeV})^2 \lesssim {m_{\mathrm{soft}}}^2 \;\sim\; [\boldsymbol{M^2_{\widetilde{L}}}]_{\alpha\alpha}\,, \qquad \text{(D.6)}$$

so that the cubic term of Eq. (D.2) is negligible. Applying this approximation, we are left with a set of linear equations in the VEVs, which is solved with the *ansatz*

$$\begin{pmatrix} \upsilon_u \\ \\ \upsilon_\alpha \end{pmatrix} = \begin{pmatrix} N_u\,\frac{m_{\mathrm{soft}}}{B} \\ \\ N_\alpha\,\epsilon^{-X_{L_\alpha}-X_{H_u}} \end{pmatrix}, \qquad \text{(D.7)}$$

if the coefficients N_u and N_α are all of the same order. This qualitative statement relies on the assumption commonly made in FN models that the sum of a few complex $\mathcal{O}(1)$ numbers is again of $\mathcal{O}(1)$. The overall scale of the VEVs is determined by the normalization requirement $\sqrt{|\upsilon_u|^2+|\upsilon_d|^2} = 246$ GeV. Eq. (D.1) does not constrain the sneutrino VEVs very much as the terms containing these are negligibly small, *i.e.* $\frac{b_i\ \upsilon_i}{b_0\ \upsilon_0} \sim \epsilon^{2\,(X_{L_0}-X_{L_i})}$.

Hence we finally end up with

$$\upsilon_\alpha \;\propto\; \epsilon^{-X_{L_\alpha}}. \qquad \text{(D.8)}$$

The apparent alignment of υ_α and μ_α is only magnitudewise and not exact. Both sets of $\mathcal{O}(1)$ coefficients differ from each other due to the VEVs' dependence on $[\boldsymbol{M^2_{\widetilde{L}}}]_{\alpha\beta}$. Therefore, excluding artificial exact alignment, υ_i and μ_i ($i = 1, 2, 3$) cannot be rotated away simultaneously.

Again, we checked the results numerically. Except for a slight tendency to have less ϵ suppression for the υ_i ($i = 1, 2, 3$) we found agreement with Eq. (D.8). The systematic effect of having bigger υ_i is caused by the "drunk man phenomenon", *i.e.*, on average, the distance covered in a two-dimensional random walk increases with every step.[1] Changing to a basis without sneutrino VEVs, this feature passes on to other coupling constants with the generation structure $\epsilon^{-X_{L_\alpha}}$.

[1]While $\mathcal{O}(1)\times\mathcal{O}(1) \sim \mathcal{O}(1)$, one has that $\mathcal{O}(1)+\mathcal{O}(1)$ is slightly bigger than $\mathcal{O}(1)$: $1\cdot 1 = 1$, but $1+1 = 2$. Less trivially, consider the sum of two phase factors. Assuming equal distribution of the phases, the absolute value of the sum with 2/3 probability is larger than 1, and only with 1/3 probability is smaller than 1; the average absolute value of the sum is calculated to be $\frac{1}{2\pi}\int_0^{2\pi}|1+e^{\mathrm{i}\phi}|\,d\phi = \frac{4}{\pi} > 1$.

We take account of this by preferring higher $\mathcal{O}(1)$ coefficients for those coupling constants which are proportional to $\epsilon^{-X_{L_i}}$, namely μ_i, λ'_{ijk}, and λ_{ijk}. Coupling constants which – after the canonicalization of the Kähler potential – have the structure $\epsilon^{+X_{L_\alpha}}$ are affected differently by the transformation $\boldsymbol{U}^{\mathbf{VEVs}}$: It is their zero component that gets somewhat enlarged. Thus the λ_{0jk} remain unchanged.

Appendix E

Symmetries of the $\boldsymbol{\lambda}$-Loop

The contribution of the charged lepton-slepton loop to the neutrino mass matrix is given in Eq. (8.16). The $\not{M}_p$ parameters λ_{iln} are generated from the charged lepton mass matrix via the canonicalization of the Kähler potential. This mechanism leads to Eq. (8.8), which can be written as

$$\lambda_{iln} \;=\; c_i\,\lambda_{0ln} - (i \leftrightarrow l), \tag{E.1}$$

with c_i being some coefficient. Recall that we are working in a basis with $\lambda_{0ln} = \delta_{ln}\,\lambda_{0ln}$. Using this structure of λ_{iln} we can calculate the first term of the mass matrix $\boldsymbol{M^\nu_{\lambda\text{-loop}}}$ in Eq. (8.16) symbolically

$$\begin{aligned}\sum_{l,n} \lambda_{iln}\,\lambda_{jnl}\,F_l^{(1)}F_n^{(2)} \;=\;& c_i\,c_j\,\Big(\lambda_{011}{}^2\,F_1^{(1)}F_1^{(2)} + \lambda_{022}{}^2\,F_2^{(1)}F_2^{(2)} \\ &+ \lambda_{033}{}^2\,F_3^{(1)}F_3^{(2)} + \lambda_{0ii}\,\lambda_{0jj}\,F_j^{(1)}F_i^{(2)} \\ &- \lambda_{0ii}{}^2\,F_i^{(1)}F_i^{(2)} - \lambda_{0jj}{}^2\,F_j^{(1)}F_j^{(2)}\Big).\end{aligned} \tag{E.2}$$

Here $F_l^{(1)}$ and $F_l^{(2)}$ are functions of the charged lepton masses as well as the charged slepton masses and mixing angles. For the purposes of this appendix it suffices to know the ratios $F_l^{(1)}/F_3^{(1)}$ and $F_l^{(2)}/F_3^{(2)}$. Eq. (8.16) together with the simplifications made below yields

$$\frac{F_l^{(1)}}{F_3^{(1)}} \;=\; \frac{m_l^e}{m_3^e} \;\sim\; \frac{F_l^{(2)}}{F_3^{(2)}}. \tag{E.3}$$

Depending on i and j, we encounter exact cancellations of seemingly dominating terms in Eq. (E.2). Applying the FN structure of the charged lepton masses, namely $m_1^e : m_2^e : m_3^e \sim \epsilon^{4+z} : \epsilon^2 : 1$, and keeping only the leading (nonzero) terms, we arrive at

$$\sum_{l,n} \lambda_{iln}\, \lambda_{jnl}\, F_l^{(1)} F_n^{(2)} \quad = \quad c_i\, c_j\, {\lambda_{033}}^2\, F_3^{(1)} F_3^{(2)}\, f_{ij} \ , \tag{E.4}$$

with $f_{ij} \sim 1$ for $i, j = 1, 2$, $f_{23} \sim f_{32} \sim \epsilon^4$ and $f_{13} \sim f_{31} \sim f_{33} \sim \epsilon^8$. Adding the second term of Eq. (8.16) symmetrizes the mass matrix $\boldsymbol{M^{\nu}_{\lambda\text{-loop}}}$. As our result in Eq. (E.4) is – concerning the magnitudes – already symmetric in i and j, we simply get a factor of two.

This shows that due to the λ_{iln}'s direct dependence on the diagonal charged lepton mass matrix and its antisymmetry under interchange of the first two indices, the $(i, 3)$- and the $(3, i)$-elements $(i = 1, 2, 3)$ of $\boldsymbol{M^{\nu}_{\lambda\text{-loop}}}$ are highly suppressed.

Appendix F

Diagonalizing the Neutrino Mass Matrix

The effective neutrino mass matrix $\boldsymbol{M^\nu}$ is diagonalized by the unitary matrix $\boldsymbol{U^\nu}$ defined in Eq. (8.9). In Subsection 8.4.2 we have seen that the mass matrix follows an ϵ structure

$$M^\nu{}_{ij} \quad \propto \quad \epsilon^{-X_{L_i}-X_{L_j}}. \tag{F.1}$$

Since $X_{L_1} \geq X_{L_2} \geq X_{L_3}$, we focus on finding the unitary matrix $\boldsymbol{\mathfrak{U}^\nu}$ which diagonalizes $\boldsymbol{M^\nu}$ such that the eigenvalues $\mathfrak{m}_i$ have the order $\mathfrak{m}_1 \gtrsim \mathfrak{m}_2 \gtrsim \mathfrak{m}_3$. Given (six) independent $\mathcal{O}(1)$ coefficients in the symmetric mass matrix, the structure of $\boldsymbol{\mathfrak{U}^\nu}$ is obtained to be [113]

$$\mathfrak{U}^\nu{}_{ij} \quad \sim \quad \epsilon^{|X_{L_i}-X_{L_j}|}, \tag{F.2}$$

as in the case of the down-type fermions, see Eq. (6.62). Unfortunately, our two main contributions are both of rank one, thus showing an additional symmetry, which obscures the validity of Eq. (F.2) in the model we consider. This appendix examines the case with only the tree level and the quark-squark loop contributing to the neutrino mass matrix. Hence, it can be written in the form

$$M^\nu{}_{ij} \quad = \quad A\; a_i\; a_j + B\; b_i\; b_j, \tag{F.3}$$

where the upper case letters define the overall (real-valued) scale of each term and the lower case letters give the generation structure $a_i \sim b_i \sim \epsilon^{X_{L_1}-X_{L_i}}$. Since $X_{L_1} \geq X_{L_2} \geq X_{L_3}$, we have $|a_1| \gtrsim |a_2| \gtrsim |a_3|$ and $|b_1| \gtrsim |b_2| \gtrsim |b_3|$. In addition, we take $A \gtrsim B$.[1]

[1] We do not specify which term is tree and which is loop level in order to stay general as long as possible. So our treatment in this appendix is valid also for $\frac{m^{\text{tree}}}{m^{\text{loop}}} \lesssim 1$.

Notice that there are six degrees of freedom in Eq. (F.3). So at first glance, the reasoning above Eq. (F.2) seems applicable. However, the mass matrix $\boldsymbol{M^\nu}$ exhibits an additional symmetry: It is of rank two, thus leaving one neutrino massless. It is therefore not obvious at all that Eq. (F.2) correctly describes the structure of the diagonalization matrix. We need to have a closer look at Eq. (F.3). In analogy to Eq. (6.61) we perform a unitary transformation which rotates away a_2 and a_3. Thus with

$$\boldsymbol{V}^* \quad \sim \quad \begin{pmatrix} 1 & \epsilon^{X_{L_1}-X_{L_2}} & \epsilon^{X_{L_1}-X_{L_3}} \\ \epsilon^{X_{L_1}-X_{L_2}} & 1 & \epsilon^{2X_{L_1}-X_{L_2}-X_{L_3}} \\ \epsilon^{X_{L_1}-X_{L_3}} & \epsilon^{2X_{L_1}-X_{L_2}-X_{L_3}} & 1 \end{pmatrix}, \tag{F.4}$$

and $V^*{}_{ij}\, a_j \equiv a'{}_i = \delta_{1i}\, a'{}_i$ and $V^*{}_{ij}\, b_j \equiv b'{}_i \sim \epsilon^{X_{L_1}-X_{L_i}}$ we have

$$\boldsymbol{M^{\nu\prime}} \equiv \boldsymbol{V}^* \cdot \boldsymbol{M^\nu} \cdot \boldsymbol{V}^\dagger = B \begin{pmatrix} \frac{A}{B}\, a'{}_1{}^2 + b'{}_1{}^2 & b'{}_1\, b'{}_2 & b'{}_1\, b'{}_3 \\ b'{}_2\, b'{}_1 & b'{}_2\, b'{}_2 & b'{}_2\, b'{}_3 \\ b'{}_3\, b'{}_1 & b'{}_3\, b'{}_2 & b'{}_3\, b'{}_3 \end{pmatrix}. \tag{F.5}$$

In the next step we want to find the unitary matrix $\boldsymbol{W}$ which, finally, diagonalizes $\boldsymbol{M^{\nu\prime}}$. We do so by investigation of

$$M^{\nu\prime}{}_{ij}\, W^\dagger{}_{jk} \quad = \quad W^T{}_{ik}\, \mathfrak{m}_k, \tag{F.6}$$

where $\mathfrak{m}_1 \geq \mathfrak{m}_2$ and $\mathfrak{m}_3 = 0$. For $k = 3$ we immediately find that

$$W^\dagger{}_{j3} \quad \sim \quad \frac{1}{b'_2} \begin{pmatrix} 0 \\ -b'_3 \\ b'_2 \end{pmatrix}_j \quad \sim \quad \begin{pmatrix} 0 \\ \epsilon^{X_{L_2}-X_{L_3}} \\ 1 \end{pmatrix}_j, \tag{F.7}$$

satisfies Eq. (F.6). Next, we turn to the conditions arising from $k = 1$. For $i = 1, 2, 3$ we get the following order of magnitude relations

$$\begin{aligned} \left(\frac{A}{B}\,\mathcal{O}(1) + \mathcal{O}(1)\right) W^\dagger{}_{11} + \mathcal{O}(\epsilon^{X_{L_1}-X_{L_2}}) W^\dagger{}_{21} + \mathcal{O}(\epsilon^{X_{L_1}-X_{L_3}}) W^\dagger{}_{31} &= \frac{\mathfrak{m}_1}{B} W^T{}_{11}, \\ \mathcal{O}(\epsilon^{X_{L_1}-X_{L_2}}) W^\dagger{}_{11} + \mathcal{O}(\epsilon^{2(X_{L_1}-X_{L_2})}) W^\dagger{}_{21} + \mathcal{O}(\epsilon^{2X_{L_1}-X_{L_2}-X_{L_3}}) W^\dagger{}_{31} &= \frac{\mathfrak{m}_1}{B} W^T{}_{21}, \\ \mathcal{O}(\epsilon^{X_{L_1}-X_{L_3}}) W^\dagger{}_{11} + \mathcal{O}(\epsilon^{2X_{L_1}-X_{L_2}-X_{L_3}}) W^\dagger{}_{21} + \mathcal{O}(\epsilon^{2(X_{L_1}-X_{L_3})}) W^\dagger{}_{31} &= \frac{\mathfrak{m}_1}{B} W^T{}_{31}. \end{aligned}$$

Assuming no accidental cancellations among $\mathcal{O}(1)$ coefficients and keeping only leading terms[2] we can determine the magnitudes of $W^\dagger{}_{21}$ and $W^\dagger{}_{31}$ from the last two equations:

$$W^\dagger{}_{21} \sim \frac{B}{\mathfrak{m}_1}\, \epsilon^{X_{L_1}-X_{L_2}}\, W^T{}_{11} \qquad \text{and} \qquad W^\dagger{}_{31} \sim \frac{B}{\mathfrak{m}_1}\, \epsilon^{X_{L_1}-X_{L_3}}\, W^T{}_{11}. \tag{F.8}$$

[2]Notice that $\frac{\mathfrak{m}_1}{B} \geq 1$. Hence, considering, for example, the last equation we can neglect the third term of the LHS compared to the RHS.

The first equation, then, does not contain any information on the ϵ structure of ${W^\dagger}_{j1}$; it simply states that $\mathfrak{m}_1$ is of $\mathcal{O}(A)$, the scale of the leading contribution to the neutrino mass matrix. By means of normalization arguments we conclude that

$$ {W^\dagger}_{j1} \;\sim\; \begin{pmatrix} 1 \\ \frac{B}{A}\,\epsilon^{X_{L_1}-X_{L_2}} \\ \frac{B}{A}\,\epsilon^{X_{L_1}-X_{L_3}} \end{pmatrix}_j . \tag{F.9} $$

The last vector of the unitary matrix $\boldsymbol{W}$ is obtained by figuring out a vector which is orthogonal to ${W^\dagger}_{j3}$ and ${W^\dagger}_{j1}$ and, in addition, normalized to one. Thus, we are led to

$$ {W^\dagger}_{j2} \;\sim\; \begin{pmatrix} \frac{B}{A}\,\epsilon^{X_{L_1}-X_{L_2}} \\ 1 \\ \epsilon^{X_{L_2}-X_{L_3}} \end{pmatrix}_j . \tag{F.10} $$

Inserting this result into Eq. (F.6) we find the magnitude of the second neutrino mass $\mathfrak{m}_2$

$$ \mathfrak{m}_2 \;\sim\; B \cdot \epsilon^{2(X_{L_1}-X_{L_2})}. \tag{F.11} $$

Notice that $\mathfrak{m}_2$ is not simply given by the scale B of the second largest contribution to the neutrino mass matrix, but it is additionally suppressed by a factor of $\epsilon^{2(X_{L_1}-X_{L_2})}$. This is of course only relevant if $X_{L_1} > X_{L_2}$.

We finally arrive at the diagonalization matrix $\boldsymbol{\mathfrak{U}}^{\boldsymbol{\nu}}$ which puts the eigenvalues into the order $\mathfrak{m}_1 \gtrsim \mathfrak{m}_2 > \mathfrak{m}_3 = 0$:

$$ {\mathfrak{U}^\nu}_{ij} \;\equiv\; W_{ik}\,V_{kj} \;\sim\; \epsilon^{|X_{L_i}-X_{L_j}|}. \tag{F.12} $$

The dependence on the factor $\frac{B}{A}$, which appears in $\boldsymbol{W}$, drops out in leading order.

Appendix G

A Top-Down Example

In this appendix we consider the X-charge assignment of Table 8.4 with $\Delta^L_{31} = -1$, $3\zeta + b = -2$ and $y = 1$. This is our preferred scenario since the resulting X charges are all multiples of one third, *i.e.* they are not highly fractional. With this choice we obtain the following Yukawa matrices for the superpotential terms $Q_i H_d \bar{D}_j$, $L_i H_d \bar{E}_j$, and $Q_i H_u \bar{U}_j$ without supersymmetric zeros:

$$\boldsymbol{h}^{\boldsymbol{D}}_{\mathbf{FN}} \sim \begin{pmatrix} \epsilon^4 & \epsilon^4 & \epsilon^4 \\ \epsilon^2 & \epsilon^2 & \epsilon^2 \\ 1 & 1 & 1 \end{pmatrix}, \quad \boldsymbol{h}^{\boldsymbol{E}}_{\mathbf{FN}} \sim \begin{pmatrix} \epsilon^5 & \epsilon^2 & \epsilon \\ \epsilon^5 & \epsilon^2 & \epsilon \\ \epsilon^4 & \epsilon & 1 \end{pmatrix}, \quad \boldsymbol{h}^{\boldsymbol{U}}_{\mathbf{FN}} \sim \begin{pmatrix} \epsilon^8 & \epsilon^6 & \epsilon^4 \\ \epsilon^6 & \epsilon^4 & \epsilon^2 \\ \epsilon^4 & \epsilon^2 & 1 \end{pmatrix}. \tag{G.1}$$

The trilinear $\not{\boldsymbol{M}}_p$ terms $L_i Q_j \bar{D}_k$ and $L_i L_j \bar{E}_k$ are disallowed due to negative integer overall X charges: $X_{L_1} + X_{Q_1} + X_{\bar{D}_1} = -2$ and $X_{L_1} + X_{L_2} + X_{\bar{E}_1} = -1$, respectively; for higher generational indices we obtain even smaller total X charges. Analogously, we have for the bilinear terms $L_\alpha H_u$: $X_{L_0} + X_{H_u} = -1$ (corresponding to the statement $z = 1$) and even smaller for the three lepton doublets L_i due to $X_{L_i} < X_{L_0}$. The $\boldsymbol{B}_3$-violating terms $\bar{U}_i \bar{D}_j \bar{D}_k$ are forbidden by noninteger overall X charge, $X_{\bar{U}_1} + X_{\bar{D}_1} + X_{\bar{D}_2} = -\frac{8}{3}$.

The Giudice-Masiero mechanism, however, reintroduces the terms disallowed by *negative integer* overall X charge in the effective superpotential. Thus we get

$$\frac{m_{\text{soft}}}{M_{\text{grav}}}\, \epsilon^{-(X_{L_i}+X_{Q_j}+X_{\bar{D}_k})}\; L_i Q_j \bar{D}_k, \qquad \frac{m_{\text{soft}}}{M_{\text{grav}}}\, \epsilon^{-(X_{L_i}+X_{L_j}+X_{\bar{E}_k})}\; L_i L_j \bar{E}_k, \tag{G.2}$$

and

$$\mu^{\text{FN}}_\alpha\; L_\alpha H_u \quad \text{with} \quad \mu^{\text{FN}}_\alpha \sim m_{\text{soft}}\; \epsilon^{-(X_{L_\alpha}+X_{H_u})}. \tag{G.3}$$

Since $\frac{m_{\text{soft}}}{M_{\text{grav}}} = \mathcal{O}(\frac{10^3\,\text{GeV}}{10^{18}\,\text{GeV}}) = \mathcal{O}(10^{-15})$ the GM-generated trilinear terms are negligibly small. In contrast, the bilinear terms, including the MSSM μ term, fall just in the right ballpark needed for susy phenomenology. Of course, care has to be taken for sufficient ϵ suppression of the $\not{M}_p$ bilinears. In the scenario considered here, we have

$$\mu_\alpha^{\rm FN} \;\sim\; m_{\rm soft} \begin{pmatrix} \epsilon \\ \epsilon^7 \\ \epsilon^7 \\ \epsilon^8 \end{pmatrix}_\alpha . \tag{G.4}$$

Next, we canonicalize the Kähler potential. The only CK transformation that changes the ϵ structure of the above coupling constants is the one connected to the superfields L_α [thus, *e.g.*, $\boldsymbol{h^{U,D}} \sim \boldsymbol{h^{U,D}_{\rm FN}}$ concerning the CK transformations of $Q_i, \bar{U}_i, \bar{D}_i, H_d, H_u$]. The corresponding transformation matrix takes the form [see Eq. (6.50)]

$$\boldsymbol{C^L} \;\sim\; \begin{pmatrix} 1 & \epsilon^6 & \epsilon^6 & \epsilon^7 \\ \epsilon^6 & 1 & 1 & \epsilon \\ \epsilon^6 & 1 & 1 & \epsilon \\ \epsilon^7 & \epsilon & \epsilon & 1 \end{pmatrix} . \tag{G.5}$$

The $\not{M}_p$ coupling constants λ'_{ijk} of the trilinear terms $L_i Q_j \bar{D}_k$ are now generated from $\lambda'^{\rm FN}_{0jk} \equiv -h^D_{{\rm FN}\,jk}$ as shown in Eq. (6.58):

$$\lambda'_{\alpha jk} \;=\; -\,[\boldsymbol{C^L}^{-1}]_{0\alpha}\; h^D_{{\rm FN}\,jk} \;\sim\; \begin{pmatrix} 1 \\ \epsilon^6 \\ \epsilon^6 \\ \epsilon^7 \end{pmatrix}_\alpha h^D_{{\rm FN}\,jk} . \tag{G.6}$$

Likewise, the $\not{M}_p$ coupling constants λ_{ijk} of the trilinear terms $L_i L_j \bar{E}_k$ are generated from $\boldsymbol{h^E_{\rm FN}}$. An additional antisymmetrizing term accounts for the antisymmetry of λ_{ijk} in the first two indices, see Eq. (6.59). The ϵ structure of the bilinear coupling constants μ_α is not affected by the CK transformation:

$$\mu_\alpha \;=\; [\boldsymbol{C^L}^{-1}]_{\beta\alpha}\, \mu_\beta^{\rm FN} \;\sim\; m_{\rm soft} \begin{pmatrix} 1 & \epsilon^6 & \epsilon^6 & \epsilon^7 \\ \epsilon^6 & 1 & 1 & \epsilon \\ \epsilon^6 & 1 & 1 & \epsilon \\ \epsilon^7 & \epsilon & \epsilon & 1 \end{pmatrix}_{\alpha\beta} \cdot \begin{pmatrix} \epsilon \\ \epsilon^7 \\ \epsilon^7 \\ \epsilon^8 \end{pmatrix}_\beta \;\sim\; m_{\rm soft} \begin{pmatrix} \epsilon \\ \epsilon^7 \\ \epsilon^7 \\ \epsilon^8 \end{pmatrix}_\alpha . \tag{G.7}$$

Neglecting renormalization flow effects, we now rotate away the sneutrino VEVs. To leading order in ϵ the necessary unitary transformation is given in Eq. (6.61). For our X-charge assignment it reads

$$\boldsymbol{U^{\rm VEVs}} \;\sim\; \begin{pmatrix} 1 & \epsilon^6 & \epsilon^6 & \epsilon^7 \\ \epsilon^6 & 1 & \epsilon^{12} & \epsilon^{13} \\ \epsilon^6 & \epsilon^{12} & 1 & \epsilon^{13} \\ \epsilon^7 & \epsilon^{13} & \epsilon^{13} & 1 \end{pmatrix} . \tag{G.8}$$

It is easy to see, that this transformation does not change the coupling constants $\lambda'_{\alpha jk}$, $\lambda_{\alpha\beta k}$, μ_α in their flavor structure.

Having generated the above $\not{M}_p$ couplings via the GM mechanism and the subsequent CK transformation, it is possible to have neutrino masses without introducing right-handed neutrinos. Assuming[1] $\frac{m^{\text{tree}}}{m^{\text{loop}}} \gtrsim 1$, we obtain an effective Majorana neutrino mass matrix of the structure [*cf.* Eq. (8.14) for $x = 0$ which corresponds to $\tan\beta \gtrsim 40$, thus $\cos^2\beta \approx \frac{1}{\tan^2\beta} \sim \frac{m_b{}^2}{m_t{}^2}$ and $\sin 2\beta = 2\,\sin\beta\cos\beta \approx \frac{2}{\tan\beta} \ll 1$]

$$\boldsymbol{M^\nu} \sim \frac{m_Z{}^2\; m_b{}^2}{m_t{}^2} \cdot \frac{M_{\tilde{\gamma}}}{M_1\; M_2} \cdot \epsilon^{12} \cdot \begin{pmatrix} 1 & 1 & \epsilon \\ 1 & 1 & \epsilon \\ \epsilon & \epsilon & \epsilon^2 \end{pmatrix} .$$

Differentiating between hierarchical and inverse-hierarchical neutrino scenarios (see Subsection 8.4.4), we arrive at an MNS mixing matrix with either

$$\boldsymbol{U^{\mathbf{MNS}}_{\mathbf{h.}}} \sim \begin{pmatrix} \epsilon & 1 & 1 \\ \epsilon & 1 & 1 \\ 1 & \epsilon & \epsilon \end{pmatrix} \quad \text{or} \quad \boldsymbol{U^{\mathbf{MNS}}_{\mathbf{i.h.}}} \sim \begin{pmatrix} 1 & 1 & \epsilon \\ 1 & 1 & \epsilon \\ \epsilon & \epsilon & 1 \end{pmatrix} . \tag{G.9}$$

Both scenarios are compatible with Eq. (8.12). However, due to the smallness of the $(1,3)$-element in the experimentally measured MNS matrix, leptonic mixing would suggest an inverse hierarchy. Then, consistency with the neutrino mass eigenvalues (see Sect. 8.4.3) would require two masses of similar magnitude, thus $\frac{m^{\text{tree}}}{m^{\text{loop}}} \sim \mathcal{O}(1)$.

As for the CKM matrix we refer to Eq. (6.65); with $y = 1$ we get

$$\boldsymbol{U^{\mathbf{CKM}}} \sim \begin{pmatrix} 1 & \epsilon^2 & \epsilon^4 \\ \epsilon^2 & 1 & \epsilon^2 \\ \epsilon^4 & \epsilon^2 & 1 \end{pmatrix} . \tag{G.10}$$

The price we have to pay for nice, *i.e.* not too fractional, X charges is a not-so-nice CKM matrix [*e.g.* the (1,2)-element is $\mathcal{O}(\epsilon^2)$ and not $\mathcal{O}(\epsilon)$].

---

[1]Equally, we could have taken $\frac{m^{\text{tree}}}{m^{\text{loop}}} \lesssim 1$ leading to the same flavor structure of $\boldsymbol{M^\nu}$. However, the prefactor of the neutrino mass matrix would then originate from Eq. (8.15).

Bibliography

[1] S. Weinberg. *Phys. Rev. Lett.*, 19:1264–1266, 1967.

[2] S. L. Glashow. *Nucl. Phys.*, 22:579–588, 1961.

[3] G. ‘t Hooft. *Nucl. Phys.*, B33:173–199, 1971.

[4] Gerard ‘t Hooft. *Nucl. Phys.*, B35:167–188, 1971.

[5] G. ‘t Hooft and M. J. G. Veltman. *Nucl. Phys.*, B44:189–213, 1972.

[6] G. ‘t Hooft and M. J. G. Veltman. *Nucl. Phys.*, B50:318–353, 1972.

[7] N. S. Manton. *Phys. Rev.*, D28:2019, 1983.

[8] J. A. Harvey and M. S. Turner. *Phys. Rev.*, D42:3344–3349, 1990.

[9] H. K. Dreiner and G. G. Ross. *Nucl. Phys.*, B410:188–216, 1993, hep-ph/9207221.

[10] M. Laine and M. E. Shaposhnikov. *Phys. Rev.*, D61:117302, 2000, hep-ph/9911473.

[11] E. Wichmann and J. Crichton. *Phys. Rev.*, 132:2788, 1963.

[12] S. Weinberg. hep-th/9702027.

[13] R. N. Mohapatra and P. B. Pal. Massive Neutrinos in Physics and Astrophysics. Second edition. *World Sci. Lect. Notes Phys.*, 60:1–397, 1998.

[14] H.-P. Nilles. *Phys. Rept.*, 110:1, 1984.

[15] S. P. Martin. 1997, hep-ph/9709356.

[16] N. Sakai and T. Yanagida. *Nucl. Phys.*, B197:533, 1982.

[17] S. Weinberg. *Phys. Rev.*, D26:287, 1982.

[18] H. K. Dreiner. hep-ph/9707435.

[19] H. K. Dreiner and G. G. Ross. *Nucl. Phys.*, B365:597–613, 1991.

[20] B. C. Allanach, A. Dedes, and H. K. Dreiner. *Phys. Rev.*, D69:115002, 2004, hep-ph/0309196.

[21] P. Langacker and M. Luo. *Phys. Rev.*, D44:817–822, 1991.

[22] J. R. Ellis, S. Kelley, and D. V. Nanopoulos. *Phys. Lett.*, B260:131–137, 1991.

[23] U. Amaldi, W. de Boer, and H. Fürstenau. *Phys. Lett.*, B260:447–455, 1991.

[24] B. C. Allanach, A. Dedes, and H. K. Dreiner. *Phys. Rev.*, D60:056002, 1999, hep-ph/9902251.

[25] D. Z. Freedman, P. van Nieuwenhuizen, and S. Ferrara. *Phys. Rev.*, D13:3214, 1976.

[26] S. Deser and B. Zumino. *Phys. Lett.*, B62:335, 1976.

[27] H. Georgi and S. L. Glashow. *Phys. Rev. Lett.*, 32:438–441, 1974.

[28] L. E. Ibáñez and G. G. Ross. *Nucl. Phys.*, B368:3–37, 1992.

[29] R. Harnik, D. T. Larson, H. Murayama, and M. Thormeier. *Nucl. Phys.*, B706:372–390, 2005, hep-ph/0404260.

[30] M. Shiozawa et al. *Phys. Rev. Lett.*, 81:3319–3323, 1998, hep-ex/9806014.

[31] Y. Hayato et al. *Phys. Rev. Lett.*, 83:1529–1533, 1999, hep-ex/9904020.

[32] J. L. Goity and M. Sher. *Phys. Lett.*, B346:69–74, 1995, hep-ph/9412208.

[33] P. F. Pérez. *J. Phys.*, G31:1025–1030, 2005, hep-ph/0412347.

[34] A. Y. Smirnov and F. Vissani. *Phys. Lett.*, B380:317–323, 1996, hep-ph/9601387.

[35] B. C. Allanach, A. Dedes, and H. K. Dreiner. *Phys. Rev.*, D60:075014, 1999, hep-ph/9906209.

[36] G. R. Farrar and P. Fayet. *Phys. Lett.*, B76:575–579, 1978.

[37] M. C. Bento, L. J. Hall, and G. G. Ross. *Nucl. Phys.*, B292:400, 1987.

[38] H. K. Dreiner, H. Murayama, and M. Thormeier. *Nucl. Phys.*, B729:278–316, 2005, hep-ph/0312012.

[39] L. E. Ibáñez and G. G. Ross. *Phys. Lett.*, B260:291–295, 1991.

[40] Y. Grossman and H. E. Haber. *Phys. Rev.*, D59:093008, 1999, hep-ph/9810536.

[41] L. M. Krauss and F. Wilczek. *Phys. Rev. Lett.*, 62:1221, 1989.

[42] J. Preskill and L. M. Krauss. *Nucl. Phys.*, B341:50–100, 1990.

[43] A. H. Chamseddine and H. K. Dreiner. *Nucl. Phys.*, B458:65–89, 1996, hep-ph/9504337.

[44] D. J. Castaño, D. Z. Freedman, and C. Manuel. *Nucl. Phys.*, B461:50–70, 1996, hep-ph/9507397.

[45] K. Kurosawa, N. Maru, and T. Yanagida. *Phys. Lett.*, B512:203–210, 2001, hep-ph/0105136.

[46] S. P. Martin. *Phys. Rev.*, D46:2769–2772, 1992, hep-ph/9207218.

[47] S. P. Martin. *Phys. Rev.*, D54:2340–2348, 1996, hep-ph/9602349.

[48] K. S. Babu, I. Gogoladze, and K. Wang. *Nucl. Phys.*, B660:322–342, 2003, hep-ph/0212245.

[49] M. Chemtob and P. N. Pandita. *Phys. Rev.*, D73:055012, 2006, hep-ph/0601159.

[50] T. Banks. *Nucl. Phys.*, B323:90, 1989.

[51] K. Fujikawa. *Phys. Rev. Lett.*, 42:1195, 1979.

[52] K. Fujikawa and H. Suzuki. Path Integrals and Quantum Anomalies. Oxford, UK: Clarendon (2004) 284 p.

[53] D. Kapetanakis, P. Mayr, and H.-P. Nilles. *Phys. Lett.*, B282:95–100, 1992.

[54] D. A. Demir, G. L. Kane, and T. T. Wang. *Phys. Rev.*, D72:015012, 2005, hep-ph/0503290.

[55] N. Maekawa. *Prog. Theor. Phys.*, 106:401–418, 2001, hep-ph/0104200.

[56] R. Slansky. *Phys. Rept.*, 79:1–128, 1981.

[57] K. Wang. 2004, hep-ph/0407234.

[58] T. Banks and M. Dine. *Phys. Rev.*, D45:1424–1427, 1992, hep-th/9109045.

[59] L. E. Ibáñez. *Nucl. Phys.*, B398:301–318, 1993, hep-ph/9210211.

[60] G. G. Athanasiu, J. J. Atick, M. Dine, and W. Fischler. *Phys. Lett.*, B214:55, 1988.

[61] J. R. Ellis, J. L. Lopez, and D. V. Nanopoulos. *Phys. Lett.*, B247:257, 1990.

[62] J. R. Ellis, V. E. Mayes, and D. V. Nanopoulos. *Phys. Rev.*, D70:075015, 2004, hep-ph/0403144.

[63] H. Kurzweil. Endliche Gruppen. Berlin, Germany: Springer-Verlag (1977) 187 p.

[64] I. Hinchliffe and T. Kaeding. *Phys. Rev.*, D47:279–284, 1993.

[65] H. K. Dreiner and M. Thormeier. *Phys. Rev.*, D69:053002, 2004, hep-ph/0305270.

[66] E. Nardi. *Phys. Rev.*, D55:5772–5779, 1997, hep-ph/9610540.

[67] J. E. Kim and H.-P. Nilles. *Mod. Phys. Lett.*, A9:3575–3584, 1994, hep-ph/9406296.

[68] G. F. Giudice and A. Masiero. *Phys. Lett.*, B206:480–484, 1988.

[69] A. H. Chamseddine and H. K. Dreiner. *Phys. Lett.*, B389:533–537, 1996, hep-ph/9607261.

[70] R. Harnik, G. D. Kribs, D. T. Larson, and H. Murayama. *Phys. Rev.*, D70:015002, 2004, hep-ph/0311349.

[71] M. Drees. hep-ph/9611409.

[72] LEPSUSYWG, ALEPH, DELPHI, L3 and OPAL experiments, note LEPSUSYWG/02-04.1.

[73] I. Dorsner and P. F. Pérez. *Phys. Lett.*, B625:88–95, 2005, hep-ph/0410198.

[74] P. Nath and P. F. Pérez. 2006, hep-ph/0601023.

[75] D. T. Larson. 2004, hep-ph/0410035.

[76] M. Hirsch, M. A. Diaz, W. Porod, J. C. Romao, and J. W. F. Valle. *Phys. Rev.*, D62:113008, 2000, hep-ph/0004115.

[77] S. Davidson and M. Losada. *JHEP*, 0005:021, 2000, hep-ph/0005080.

[78] A. Abada, G. Bhattacharyya, and M. Losada. *Phys. Rev.*, D66:071701, 2002, hep-ph/0208009.

[79] P. Minkowski. *Phys. Lett.*, B67:421, 1977.

[80] R. N. Mohapatra and G. Senjanovic. *Phys. Rev. Lett.*, 44:912, 1980.

[81] M. Gell-Mann, P. Ramond, and R. Slansky. in Sanibel Symposium, February, 1979: CALT 68–709, retroprint: hep-ph/9809459.

[82] Y. Yanagida. *Proc. of the Workshop on the Unified Theory*, 1979, edited by O. Sawada and A. Sugamoto: KEK Report N0. 79–18.

[83] T. W. B. Kibble. *J. Phys.*, A9:1387–1398, 1976.

[84] Y. B. Zel'dovich, I. Y. Kobzarev, and L. B. Okun. *Zh. Eksp. Teor. Fiz.*, 67:3–11, 1974.

[85] Douglas Scott. 2005, astro-ph/0510731.

[86] L. E. Ibáñez and G. G. Ross. *Phys. Lett.*, B332:100–110, 1994, hep-ph/9403338.

[87] C. D. Froggatt and H. B. Nielsen. *Nucl. Phys.*, B147:277, 1979.

[88] E. Dudas, S. Pokorski, and C. A. Savoy. *Phys. Lett.*, B356:45–55, 1995, hep-ph/9504292.

[89] I. Jack, D. R. T. Jones, and R. Wild. *Phys. Lett.*, B580:72–78, 2004, hep-ph/0309165.

[90] L. J. Hall, H. Murayama, and N. Weiner. *Phys. Rev. Lett.*, 84:2572–2575, 2000, hep-ph/9911341.

[91] N. Haba and H. Murayama. *Phys. Rev.*, D63:053010, 2001, hep-ph/0009174.

[92] A. de Gouvea and H. Murayama. *Phys. Lett.*, B573:94–100, 2003, hep-ph/0301050.

[93] S. Weinberg. The Quantum Theory of Fields. Vol. 1: Foundations. Cambridge, UK: Univ. Pr. (1995) 609 p.

[94] D. Bailin and A. Love. Supersymmetric Gauge Field Theory and String Theory. Bristol, UK: IOP (1994) 322 p.

[95] M. B. Green and J. H. Schwarz. *Phys. Lett.*, B149:117–122, 1984.

[96] R. Blumenhagen, G. Honecker, and T. Weigand. *JHEP*, 06:020, 2005, hep-th/0504232.

[97] T. Kobayashi and H. Nakano. *Nucl. Phys.*, B496:103–131, 1997, hep-th/9612066.

[98] P. Ramond. 1995, hep-ph/9604251.

[99] G. Cleaver, M. Cvetič, J. R. Espinosa, L. L. Everett, and P. Langacker. *Nucl. Phys.*, B525:3–26, 1998, hep-th/9711178.

[100] S. Weinberg. The Quantum Theory of Fields. Vol. 3: Supersymmetry. Cambridge, UK: Univ. Pr. (2000) 419 p.

[101] H.-P. Nilles. 2004, hep-th/0402022.

[102] K. S. Babu, T. Enkhbat, and I. Gogoladze. *Nucl. Phys.*, B678:233–257, 2004, hep-ph/0308093.

[103] T. P. Cheng and L. F. Li. Gauge Theory of Elementary Particle Physics. Oxford, Uk: Clarendon (1984) 536 p.

[104] M. Cvetič, L. L. Everett, and J. Wang. *Phys. Rev.*, D59:107901, 1999, hep-ph/9808321.

[105] S. Weinberg. The Quantum Theory of Fields. Vol. 2: Modern Applications. Cambridge, UK: Univ. Pr. (1996) 489 p.

[106] M. Dine, N. Seiberg, X. G. Wen, and E. Witten. *Nucl. Phys.*, B278:769, 1986.

[107] M. Dine, N. Seiberg, X. G. Wen, and E. Witten. *Nucl. Phys.*, B289:319, 1987.

[108] J. J. Atick, L. J. Dixon, and A. Sen. *Nucl. Phys.*, B292:109–149, 1987.

[109] M. Dine, I. Ichinose, and N. Seiberg. *Nucl. Phys.*, B293:253, 1987.

[110] M. Dine, N. Seiberg, and E. Witten. *Nucl. Phys.*, B289:589, 1987.

[111] P. Binétruy, S. Lavignac, and P. Ramond. *Nucl. Phys.*, B477:353–377, 1996, hep-ph/9601243.

[112] T. Banks, Y. Grossman, E. Nardi, and Y. Nir. *Phys. Rev.*, D52:5319–5325, 1995, hep-ph/9505248.

[113] L. J. Hall and A. Rašin. *Phys. Lett.*, B315:164–169, 1993, hep-ph/9303303.

[114] P. Ramond, R. G. Roberts, and G. G. Ross. *Nucl. Phys.*, B406:19–42, 1993, hep-ph/9303320.

[115] Y. Nir. *Phys. Lett.*, B354:107–110, 1995, hep-ph/9504312.

[116] H. K. Dreiner, C. Luhn, and M. Thormeier. *Phys. Rev.*, D73:075007, 2006, hep-ph/0512163.

[117] H. Georgi. *Front. Phys.*, 54:1–255, 1982.

[118] G. L. Kane, S. F. King, I. N. R. Peddie, and L. Velasco-Sevilla. 2005, hep-ph/0504038.

[119] G. L. Fogli, E. Lisi, A. Marrone, and A. Palazzo. 2005, hep-ph/0506083.

[120] S. K. Kang and O. C. W. Kong. *Phys. Rev.*, D69:013004, 2004, hep-ph/0206009.

[121] C. Luhn. diplom thesis, 2002: BONN–IB–2002–10.

[122] J. M. Mira, E. Nardi, D. A. Restrepo, and J. W. F. Valle. *Phys. Lett.*, B492:81–90, 2000, hep-ph/0007266.

[123] H. Dreiner, J. S. Kim, and M. Thormeier. preprint in preparation.

[124] S. Eidelman et al. *Phys. Lett.*, B592:1, 2004.

[125] M. C. Gonzalez-Garcia. 2004, hep-ph/0410030.

[126] C. Giunti. 2004, hep-ph/0412148.

[127] Y. Grossman and H. E. Haber. 1999, hep-ph/9906310.

[128] N. Ghodbane and H.-U. Martyn. 2002, hep-ph/0201233.

[129] B. C. Allanach et al. *Eur. Phys. J.*, C25:113–123, 2002, hep-ph/0202233.

[130] R. D. Peccei and H. R. Quinn. *Phys. Rev. Lett.*, 38:1440–1443, 1977.

[131] M. Cvetič, D. A. Demir, J. R. Espinosa, L. L. Everett, and P. Langacker. *Phys. Rev.*, D56:2861–2885, 1997, hep-ph/9703317.

[132] E. Keith and E. Ma. *Phys. Rev.*, D56:7155–7165, 1997, hep-ph/9704441.

[133] G. Cleaver, M. Cvetič, J. R. Espinosa, L. L. Everett, and P. Langacker. *Phys. Rev.*, D57:2701–2715, 1998, hep-ph/9705391.

[134] S. P. Martin. *Phys. Rev.*, D61:035004, 2000, hep-ph/9907550.

[135] J. R. Espinosa and A. Ibarra. *JHEP*, 0408:010, 2004, hep-ph/0405095.

[136] S. F. King, I. N. R. Peddie, G. G. Ross, L. Velasco-Sevilla, and O. Vives. 2004, hep-ph/0407012.

[137] A. Psallidas. 2005, hep-ph/0505093.