Risk Management of Life Insurance Contracts with Interest Rate and Return Guarantees and an Analysis of Chapter 11 Bankruptcy Procedure

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Chapter 1

Introduction: problems, definitions and solution concepts

Equity–linked life insurance contracts are an example of the interplay between insurance and finance. By considering some specific equity–linked life insurance contracts, this thesis mainly studies risk management methods, i.e., the insurance company hedges its exposure to risk by using certain conventional hedging criteria for an incomplete market, like risk–minimizing, quantile and efficient hedging. In addition to the untradable insurance risk, different sources of incompleteness are analyzed, such as the incompleteness from trading restrictions or from model misspecification. Furthermore, this thesis provides an insight to the net loss of the insurer, given that the insurer trades in the financial market according to risk–minimizing hedging criterion. However, under no circumstances, the untradable insurance risk can be hedged completely, i.e., there always exists a positive probability that the considered insurance company defaults. In this context, the chapter before last is designed to consider the insurance company as an aggregate and to analyze the market value of this company if default risk and different bankruptcy procedures are taken into consideration. In this analysis, the mortality risk is neglected and no specific contracts are studied.

1.1 Equity–linked life insurance

Equity–linked life insurance combines life insurance and investment strategies. Therefore, first of all, it is an insurance contract which contains insurance risk. This risk can have effect on both the inflow and outflow side of an insurance company, i.e., the premium payments and the promised payout to its clients. In contrast to an upfront premium, periodic premium payments make the analysis more complicated.[1] Because if the issued contracts contain periodic premiums, mortality risk indicates that the periodic premiums can only be paid if the insured is still alive. The exercise time of the promised payout is

[1]For instance, Hipp (1996), Ekern and Persson (1996) and Aase and Persson (1997) construct insurance contracts with a single premium. In contrast, Aase and Persson (1994), Boyle and Schwartz (1977), Nielsen and Sandmann (1995) consider periodic premiums.

conditional on the random death/survival of the insured. Furthermore, sometimes, the insurance risk plays a role in the size of the payoff of the contract.

It is well known that the individual insurance risk is non–tradable in the sense of a secondary market with required liquidity. The death and survival distribution the insurer uses for the pricing and hedging purposes are usually estimated by using historical data. In the literature on life insurance mathematics, frequently, several assumptions concerning the insurance risk are made. First, for simplicity reasons, the death distribution is assumed stochastically independent of the financial risks. It allows a separate analysis of both uncertainties.[2] Second, the insurer is assumed risk neutral with respect to mortality. Third, it is assumed that the insurer can perfectly diversify the death uncertainty within each group. Diversification can be realized over subpopulation (law of large numbers)[3], over time or over insurers (reinsurance). It is a usual and acceptable assumption for standard endowment life insurance contracts. Although diversification and hedging are both used to eliminate the risks, they are two basically different concepts. Diversification should not be confused with hedging. Diversification makes uses of the uncorrelated characteristic of the portfolio or subpopulation. On some level, diversification is "free lunch" of finance, because the investor can reduce market risk simply by investing in many uncorrelated instruments, without affecting the expected return. While trading/hedging is taking of offsetting risks. These assumptions are made in order to allow the use of standard financial valuation and hedging techniques for complete markets. However, the number of equity–linked life insurance contracts is much smaller than those of the standard case, so it is not so obvious that the law of large numbers can be applied, i.e., the additional insurance risk introduces incompleteness to the model. Further incompleteness can be caused by mortality misspecification which is either caused by a false estimation or an intentional abuse of the insurer.[4] The main purpose of this thesis is to deal with the non–tradable insurance risk, either by taking account of this in hedging analysis or consider this implicitly.

Most of equity–linked life insurance contracts are provided with guarantees, i.e., a guaranteed amount is ensured to the customer. The guarantee can be offered at the maturity date, known as "long or maturity guarantee" [5], or periodically, known as "periodic guarantee". [6] The guaranteed amount provides a floor of the future payoff to the customer.

[2]In reality, insurance risk and financial risk are not always uncorrelated. For example, the health condition of a president can affect the financial market trend.

[3]Following Brennan and Schwartz (1979), most authors (Bacinello and Ortu (1993); Bacinello and Persson (2002); Miltersen and Persson (2003)) replace the uncertainty of the insured individuals' death/survival by the expected values according to the law of large numbers. So, the actual insurance claims including mortality risk as well as financial risk are replaced by modified claims, which only contain financial uncertainty.

[4]How mortality misspecification influences the insurers' hedging decisions is investigated explicitly in Chapter 4.

[5]See for example Brennan and Schwartz (1976), Briys and de Varenne (1994a), Boyle and Hardy (1997).

[6]See for example Grosen and Jørgensen (2002), Hansen and Miltersen (2002) and Miltersen and

In addition to the guaranteed amount, possibly the insurance company promises to offer a fraction of its surpluses to its customer. Due to this promise, the equity–linked life insurance products contain usually option components. The entire benefit of an equity–linked insurance is the guaranteed amount plus a participation in the surpluses of an underlying index or fund or combination of funds. Similar to the guaranteed payment, the surplus can be provided as a participation in maturity surplus or periodic surpluses. In addition, this participation can be offered by diverse option forms,[7] e.g. a (sequence of) European option(s) or even some exotic options. Nielsen and Sandmann (2002) introduce a surplus whose feature corresponds to average options, so called Asia–type options. Furthermore, since the equity–linked life insurance contracts are long–term contracts, [8] it is necessary to take account of the fluctuation in the term structure of the interest rate. These together lead to the consequence that equity–linked life insurance contracts contain at least two financial risks: the financial risk related to the underlying index and the interest rate risk.

Neglecting some detailed subtleties, many life insurance products provided outside Europe have similar features to equity–linked life insurance, although they own completely different names. Therefore, risk management and modelling issues introduced in this thesis can be applied to these contracts too. Some examples for these contracts are: segregated fund contracts in Canada which has become a popular alternative to mutual fund investment, unit–linked insurance products in United Kingdom and variable annuities in United States which are similar to segregated funds. A detailed description of the history of equity–linked life insurance and diverse contract forms can be found e.g. in Hardy (2003).

1.2 Some empirical observations

Since the 1980s a long list of defaulted life insurance companies in Europe, Japan and USA has been reported. A few examples are illustrated in Table 1.1.[9] It is worth mentioning that First Executive Life Insurance Co. constituted the 12th largest bankruptcy in the United States in the period 1980-2005 and Conceso Inc. the 3rd largest bankruptcy in this time period.

There are a variety of causes which lead to the bankruptcy of the insurance companies.[10] Some insolvencies were caused by natural catastrophes and some resulted from the losses

Persson (2003).

[7]Sometimes, an option to surrender the contract is provided to the customer. This surrender option feature is ignored in this thesis. But a relevant discussion can be found for instance in Grosen and Jørgensen (2000).

[8]For instance, in Germany, returns of life insurance contracts are free of tax only when the period of the contracts is longer than 12 years. Therefore, the average period of life insurance contracts is 25–26 years.

[9]The life insurance insolvency cases up to 1994 are taken from the 1999 Special Comment "Life After Death" by Moody's Investors Service, Global Credit Research, available at http://www.moodys.com. The other cases are taken from http://www.bankruptcydata.com.

[10]See Jørgensen (2004).

Country	Company	Default year
Australia	HIH Insurance	2001
France	Garantie Mutuelle des Fonctionnaires	1993
Germany	Mannheimer Leben	2003
Japan	Nissan Mutual Life	1997
	Chiyoda Mutual Life Insurance Co.	2000
	Kyoei Life Insurance Co.	2000
	Tokyo Mutual Life Insurance	2001
U.K.	Equitable Life	2000
USA	First Farwest Corp.	1989
	Integrated Resource Life Insurance Co.	1989
	Pacific Standard Life Insurance Co.	1989
	Mutual Security Life Insurance Co.	1990
	First Executive Life Insurance Co.	1991
	First Stratford Life Insurance Co.	1991
	Executive Life Insurance Company of New York	1991
	Fidelity Bankers Life Insurance Co.	1991
	First Capital Life Insurance Co.	1991
	Mutual Benefit Life Insurance Co.	1991
	Guarantee Security Life Insurance Co.	1991
	Fidelity Mutual Life Insurance Co.	1992
	Summit National Life Insurance Co.	1994
	Monarch Life Insurance Co.	1994
	Confederation Life Insurance Co.	1994
	ARM Financial Group	1999
	Penn Corp. Financial Group	2000
	Conseco Inc.	2002
	Metropolitan Mortgage & Securities	2004

Table 1.1: Some examples of defaulted life insurance companies in Australia, Europe, Japan and USA.

on assets. Furthermore, there have existed some internal administration problems within the insurance company, like mismanagement of the interest rate guarantee, mismanagement of the credit risk and application of poor or inappropriate accounting principles. Since insurance contracts are usually long–term contracts, many insurance companies cannot afford the guarantees which they issued before 2000. In addition, mispricing of insurance liabilities is another important reason to cause insolvency.[11] A concrete insolvency is usually caused not by a single reason but by a combination of the reasons mentioned above.

Whatever causes the insolvency, one thing holds for sure. This dramatic rise in the number of insolvent insurers has set off an alarm among the insurance companies. They notice their inadequate risk management and become more and more aware of the importance of managing the risk caused by issuing life insurance liabilities. In particular, how to eliminate financial risks resulting from the issued contracts have become a real challenge for the insurance companies. Therefore, it is high time for the insurance company to handle the hedging perspective of life insurance contracts with interest rate and return guarantees in addition to eliminating the insurance risk by diversification. Besides, to consider the insolvency risk in a contingent claim model would provide the insurance company some insights to its risk management. Therefore, the main purpose of this dissertation is to highlight the importance of managing the financial risks.

1.3 Some solution concepts

A straightforward effort which was made to solve the guarantee problem is to decrease the level of the minimum interest rate guarantee. In fact, it is observed that at the beginning of 1990s, there is a dramatic decline of the interest rate through Europe, Japan and USA. As a consequence of the decreasing interest rate, a dramatic reduction in the guaranteed interest rate occurred. E.g., in Japan, the maximum interest rate guarantee is reduced from previous 4.5% to 2.5% since 2000, and in Germany, the minimum interest rate guarantee has experienced a reduction from 4% to 3.25% in 2000 and recently even to 2.75%. This adjustment of the interest rate guarantee is management in the liability side. Another reaction to the low interest rate is an increasing investment in the fonds or other indexes, hence, leads to an increasing volume of equity-linked life insurance contracts. In this thesis, we are mainly interested in the management of the asset side. Since risk management of an insurance company is very complex and relies on a variety of techniques in its risk management system, the analysis made here can just provide a very limited

[11] In this context, so–called market–consistent valuation methods have attracted a great deal of attention recently and are highly recommended for the valuation of life insurance liabilities. A valuation is called "market–consistent" if it replicates the market prices of the calibration assets to within an acceptable tolerance. Market–consistent valuation can take many different forms and two different valuation methods can be market–consistent at the same time. This topic will not be studied in this dissertation and a detailed discussion to this topic can be found e.g. in Sheldon and Smith (2004).

insight into the risk management of insurance companies.

This dissertation presents a risk management approach where the insurance company actively manages its risk exposures by appropriately hedging the risks of the issued contracts, i.e., the effort is made to hedge the combined actuarial and financial risk. It is well–known that no perfect hedge is possible in an incomplete market, i.e., it is impossible to find a strategy whose final value corresponds to the payoff of the contingent claim and which is at the same time self–financing. In order to deal with this problem, different "optimality" criteria which lead to different hedging methods are used to tackle the combined financial and insurance risk. Different hedging criteria lead to different hedging strategies. For instance, risk–minimizing hedging is a dynamic hedging approach which looks for an admissible strategy which basically amounts to minimizing the variance of the hedger's future costs. However, this approach has the undesirable property that minimization of the variance (or the expected value of the square of the future costs) implies that relative losses and relative gains are treated equally, c.f. Föllmer and Sondermann (1986) and Föllmer and Schweizer (1988). Quantile and efficient hedging are two concepts introduced in Föllmer and Leukert (1999, 2000), where the former concept can be expressed as a special case of the latter. The investor has usually two motives by using these hedging methods: they are either unwilling to spend too much capital which is needed for perfect hedges (complete market) or super hedges (incomplete market) or ready to take some risks given a certain shortfall probability. Based on a specific equity–linked life insurance contract, all these hedging criteria are applied to the insurance contract and quantile, efficient and risk–minimizing hedging strategy are calculated respectively for this contract in **Chapter 2**. Furthermore, in case of quantile and efficient hedging, an "optimal" survival probability is derived. This analysis allows a transfer between the financial and insurance risk.

Chapter 3 is designed to answer the question "what happens to the hedger's net loss given that the hedger trades according to the hedging strategies introduced in Chapter 2". In other words, the chapter is drawn up to learn to what extent the hedger benefits from using the introduced hedging strategy. Net loss of the hedger or more specifically the ruin probability is taken as the criterion of goodness. As an example, risk–minimizing hedging strategies are applied by the hedger. In particular, due to high transaction costs and trading restrictions, discrete–time risk–minimizing hedging strategies are investigated. A discrete–time hedging strategy can be obtained either by time–discretizing a continuous strategy, i.e., a trading restriction is imposed on a continuous–time risk–minimizing strategy or by assuming a discrete–time model (e.g. binomial model) for the underlying asset process. In the analysis of the former discrete–time strategy, the considered model is incomplete where the incompleteness results not only from the mortality risk but also from the trading restrictions. The asset price dynamics are assumed to be in the framework of Black–Scholes (1973), but the hedging of the contingent claims occurs at discrete times instead of continuously.

It is observed from the simulations that the use of the time–discretized risk–minimizing

strategy yields a substantial reduction in the ruin probability. This argument is based on the comparison with the scenario where the premiums are invested in a risk free asset with a rate of return corresponding to the market interest rate. However, the extent of the reduction becomes less apparent and the advantage of using this strategy almost disappears when the trading frequency is increased. This is due to the fact that extra duplication errors are caused when the original mean–self–financing risk–minimizing hedging strategy is discretized with respect to time and that these errors increase with the frequency, i.e., a higher frequency leads to more hedging errors which constitute a vital part of the hedger's net loss. In order to improve the numerical results, the second type of discrete–time risk–minimizing strategy is taken into consideration. When comparing the simulation results with the scenario where the strategy is discretized, we observe considerably smaller ruin probabilities, in particular, when the frequency is increased.

Chapter 4 investigates another source of the market incompleteness, i.e., the incompleteness caused by the model risk, or more precisely parameter misspecifications. In pricing and hedging the issued contracts, the hedger chooses a certain model (and the corresponding model parameters) to describe the term structure of the interest rate and the death distribution. In contrast, the fairness of the price and the effectiveness of the hedging depend on the true dynamics. The fairness is defined in the sense that the present value of the contract payment to the contract–holder corresponds to the present value of his contributions. The effectiveness analysis of the trading strategies is based on the variance comparison of the hedging errors associated with the strategies. Evidently, the assumed model or model parameters can deviate from the real ones. This is so–called model risk or model misspecification. By considering life insurance products which give a minimum return guarantee on a periodic premium together with an endowment protection, we analyze effective risk management strategies under interest rate and mortality misspecification. More specifically, we consider risk management strategies combining diversification and hedging effect, when it is both possible to misspecify the interest rate dynamic which is used for hedging the contracts and the mortality distributions which are used for the diversification effects. We study the distribution of the total hedging errors which depends on interaction of the true and assumed interest rate and mortality distributions. In particular, we look for a combination of diversification and hedging effects which is robust in the sense of partially independence against model misspecification.

It is shown that independent of the choice of the hedging instruments, the insurer who issues specific contracts introduced in Chapter 4 stays on the safe side on average, i.e. a superhedge is achieved in the mean, when he overestimates the death probability.[12] In fact, it is mortality risk that decides whether a superhedge can be achieved. If there exists no mortality misspecification, the considered strategies are mean–self–financing, independent of the evolution of the term structure of the interest rate. If there does exist parameter misspecification associated with the mortality risk, the model risk related to the interest rate plays a role in deciding the size of the expected value too. However, the

[12]This result is contrary to the case of a pure endowment insurance contract, where a superhedge is achieved when the survival probability is overestimated.

effect of model misspecification related to the interest rate is highly enforced when the variance of the hedging errors is taken into consideration. No model risk related to the interest rate implies that different hedging strategies considered lead to the same variance level of the total cost from both the asset and liability side. This argument holds independent of the mortality misspecification and the choice of the hedging instruments. However, if there does exist model risk related to the term structure, the importance of the choice of the hedging instruments is highlighted. Even when there exists no mortality misspecification, a variance markup always results when only a subset of hedging instruments are traded and when interest rate misspecification is present. I.e., the restriction on the hedging instruments results in a variance markup under model risk concerning the interest rate. Taking account of the combined effect of these two sources of model risk, we observe that if the set of hedging instruments is restricted, an overestimation of the death probability combined with a huge misspecification associated with the interest rate leads to a very high variance markup, and consequently it could lead to an increase in the shortfall probability.

Through the analysis in this chapter, it is shown that neither the model risk which is related to the death distribution nor the one associated with the financial market model is negligible for a meaningful risk management. There is an interaction between these two sources of model misspecification. In the analysis of the expected value of the hedging errors, the effect of the interest rate misspecification depends on the mortality misspecification. On the contrary, the effect of mortality risk depends on the model risk related to the interest rate when it comes to the analysis of the variance.

Due to the fact that the relevant financial market is incomplete (the incompleteness can result from the untradable insurance risk alone, from trading restrictions or from model misspecification), it is impossible to eliminate/hedge the insurance risk completely. This leads to the fact that default risk occurs in a life insurance with a positive possibility. Therefore, **Chapter 5** is constituted to investigate the market value of the life insurer if default risk and different bankruptcy procedures are taken into account, the mortality risk is ignored and the firm values of the insurance company are considered as an aggregate.

In Grosen and Jørgensen (2002), bankruptcy and liquidation are described by standard knock–out barrier options[13]. The firm defaults and is liquidated if up to the maturity time the value of the total assets has not been sufficiently high to cover the barrier. The barrier level is set by the regulator as an intervention rule in order to control the strictness of intervention. If the US Bankruptcy procedure is used as an example, then the model of Grosen and Jørgensen (2002) corresponds to Chapter 7 Bankruptcy Code, where default and liquidation are treated as equivalent events. At the moment when the firm defaults, it is liquidated immediately. Obviously, this bankruptcy procedure is unrealistic, therefore, Chapter 5 can be considered as an extension of Grosen and Jørgensen (2002). More general bankruptcy procedures such as Chapter 11 Bankruptcy Code are investi-

[13]Bernard, Le Courtois and Quittard-Pinon (2005) extended the model of Grosen and Jørgensen (2002) by incorporating the stochastic interest rate.

gated. According to Chapter 11 Bankruptcy Code, default and liquidation are modelled as distinguishable events. A certain period is given for the defaulted company for reorganization and renegotiation. Mathematically, this is realized by so–called standard and cumulative Parisian barrier option frameworks. In case of a standard Parisian option, the option loses its value when the underlying asset stays consecutively under the barrier longer than a certain length of excursion before the maturity date. In case of a cumulative Parisian option, the knock–out condition is that the asset stays under the barrier in total a certain length of excursion before the maturity date. It is shown that these options have appealing interpretations in terms of the bankruptcy mechanism and indeed describe two extreme cases. Any realistic bankruptcy mechanism stays between these two extreme scenarios. In this chapter, mortality risk is ignored for a while, and the question is mainly investigated of how the value of the equity and of the liability of a life insurance company are affected by the default risk and the choice of the relevant bankruptcy procedure.

1.4 Main sources of uncertainty

This section introduces the underlying index, interest rate dynamics, and the underlying death/survival distribution in a mathematical manner. As stated before, in an equity–linked life insurance contract, the exercising time of the contract is determined by the insurance risk; and the size of the payoff can be determined by the financial risk only or both the mortality risk and the asset risk. The long–term characteristic of the equity–linked life insurance contracts makes the analysis of the interest rate risk not negligible. Therefore, pricing and hedging of equity–linked life insurance contracts contains analysis of at least three sources of risks: the financial risk related to the asset and the interest rate risk and the untradable insurance risk. Before we come to each single risk, several common assumptions concerning the financial market are made for the valuation and hedging purpose:

- It is assumed there exists a continuous–time frictionless economy with a perfect financial market, no tax effects, no transaction costs and no other imperfection. Trading restrictions and volatility misspecification are permitted.

- Diverse risky assets are traded in the financial market, in particular, the underlying assets of the considered contracts. This assumption is of high relevance, because synthesizing of the underlying assets becomes necessary if they are not traded in the financial market. Synthesization makes the hedging analysis more complicated and could lead to the consequence that some robust results become invalid.

- A bank account and zero coupon bonds with different maturities are traded at any given time.

Hence, we can rely on martingale techniques introduced by Harrison and Kreps (1979) for the valuation of the contingent claim. I.e., if the considered financial market is complete, the arbitrage free condition implies the existence of a unique equivalent martingale measure, under which the present values of all the contingent claims correspond to the

expected discounted future payments. If the considered financial market is incomplete, a sequence of equivalent martingale measures exist, under which no arbitrage profit can be made. A discussion on the arbitrage free argument and the existence of martingale measures can be found e.g. Delbaen and Schachermeyer (1994).

1.4.1 Index dynamics

As mentioned, often a return guarantee (bonus payment) based on a reference index or portfolio is offered to the customer in equity–linked life insurance. It is assumed that the price process of the reference portfolio is driven by a n–dimensional standard Brownian motion in the filtered probability space $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}, P^*)$

$$dS(t) \;=\; r(t)\, S(t)\, dt + \sigma(t)\, S(t)\, dW^*(t) \tag{1.1}$$

where $r(t) > 0$ is the risk free continuously compounded interest rate and the volatility of the reference portfolio $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}^n_{>0}$ is a n–dimensional bounded, deterministic function. [14]. $\{W^*(t)\}_t$ is a n–dimensional standard Brownian motion under the martingale measure P^*. Solving the differential Equation (1.1), we obtain

$$S(t) \;=\; S(t_0) \exp\left\{ \int_{t_0}^{t} (r(u) - \frac{1}{2}\|\sigma(u)\|^2)\, du + \int_0^t \sigma(u)\, dW^*(u) \right\}. \tag{1.2}$$

where $S(t_0)$ gives the initial value of the index.

1.4.2 Interest rate risk

When the stochastic interest rate comes into consideration, it is necessary to assume a stochastic model which drives the interest rate dynamic. We consider it by studying the dynamics of zero coupon bonds. As a benchmark case, we assume that the dynamic of a zero coupon bond $D(., \bar{t})$ paying one monetary unit at maturity $\bar{t} \in [0, T]$ is a lognormal stochastic process. More specifically, it evolves as follows:

$$\begin{aligned} dD(t, \bar{t}) \;&=\; D(t, \bar{t})\, (r(t)\, dt + \sigma_{\bar{t}}(t)\, dW^*(t))\,, \\ D(\bar{t}, \bar{t}) \;&=\; 1,\; P^* - \text{a.s.}\;\; \forall\;\; \bar{t} \in [0, T], \end{aligned} \tag{1.3}$$

where W^* denotes a n–dimensional Brownian motion with respect to P^*. $D(\bar{t}, \bar{t}) = 1$ gives the terminal value condition of a zero coupon bond. It says that the price of a zero coupon bond corresponds to its face value at the maturity date if no credit risk is available. $\sigma_{\bar{t}}(t)$ describes the corresponding volatility of the zero coupon bond with maturity $\bar{t}$. Due to

[14] The assumption of a deterministic volatility function is made in order to apply Black and Scholes (1973) economy. In fact, Gaussian hedge is still robust even when the volatility is stochastic but has an upper bound. Please refer to e.g. Avellaneda, Levy and Parás (1995) for a detailed discussion on this topic.

the terminal value condition, the volatility function has to be a time–dependent function, and at maturity date the volatility should be zero, i.e.:

$$\sigma_{\bar{t}}(\cdot) \ : \ [0,\bar{t}] \rightarrow \mathbb{R}^n_{\geq 0} \ \text{ with } \ \sigma_{\bar{t}}(\bar{t}) = 0, \quad \forall\, \bar{t}.$$

In addition, if for every $t \in [0,\bar{t}]$, it holds that $\sigma_{\bar{t}}(t)$ is differentiable with respect to $\bar{t}$ and $\sigma_{\bar{t}}(t)$ is square integrable, i.e.,

$$\int_0^{\bar{t}} ||\sigma_{\bar{t}}(t)||^2 \, dt < \infty.$$

The above term structure of the interest rate is a Gaussian term structure. This means, the corresponding conform spot rate is normally distributed. Due to two reasons, the Gaussian term structure is widespread. First, these models are analytically tractable and can generate a bunch of other different term structures. Although there exists a positive possibility that negative spot rates are generated, it can be considerably avoided through proper parameter choices. Second, Gaussian term structures are in accord with Black and Schole's (1973) model. The solution to Equation (1.3) has a form of

$$D(t,\bar{t}) = D(t_0,\bar{t}) \exp\left\{ \int_{t_0}^{t} (r(u) - \frac{1}{2}\|\sigma_{\bar{t}}(u)\|^2)\, du + \sigma_{\bar{t}}(u)\, dW^*(u) \right\}.$$

A prominent example for Gaussian term structure is Vasiček–model, [15] one of the earliest and most famous term structure model, and it has received a wide application. The conform short rate follows a diffusion process, i.e. an Ornstein–Uhlenbeck process,

$$dr = (b - \kappa\, r(t))dt + \bar{\sigma} dW(t),$$

where κ, b and $\bar{\sigma}$ are non–negative parameters of Vasiček model. $\bar{\sigma}$ is the volatility of the short rate and $\frac{b}{\kappa}$ the long–run mean. This model incorporates mean reversion. The short rate is pulled to a level $\frac{b}{\kappa}$ at a speed rate κ. Superimposed upon this "pull" is a normally distributed stochastic term $\sigma dW(t)$. This implies that the volatility of the zero coupon bond $\sigma_{\bar{t}}(t)$ has a form of

$$\sigma_{\bar{t}}(t) = \frac{\bar{\sigma}}{\kappa}(1 - \exp\{-\kappa(\bar{t} - t)\}).$$

Both conditions $\sigma_{\bar{t}}(\bar{t}) = 0$ and a decreasing function of $\sigma_{\bar{t}}(.)$ in the remaining time are satisfied here. [16] A drawback of Vasiček model is that it does not automatically fit today's term structure. In order to overcome this drawback, Hull and White (1990, 1993) made the parameters b and κ time-dependent and developed the Hull–White model and the extended Vasiček model.[17] The extended Vasiček model is still very popular in the market today with practitioners.

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[15] C.f. Vasiček (1977). Vasičeck developed a one–factor Gaussian interest rate model.

[16] This model converges to deterministic term structure model when θ goes to infinity. $\lim_{\theta\rightarrow\infty} \sigma_{\bar{t}}(t) = 0$. And it converges to the Ho-Lee Model, $\lim_{\theta\rightarrow 0} \sigma_{\bar{t}}(t) = \lim_{\kappa\rightarrow 0} \bar{\sigma}(\bar{t} - t)\exp\{-\kappa(\bar{t} - t)\} = \bar{\sigma}(\bar{t} - t)$, when κ goes to zero.

[17] In the model of Hull and White (1990), only b is time-dependent, and in the model of Hull and White (1993) (extended Vasiček model), both b and κ are time-dependent.

1.4.3 Non–traded insurance risk

The non–tradable insurance risk usually determines when the payout happens. According to this, three different life insurance contracts are distinguished: pure endowment life insurance, term life insurance and endowment life insurance. In a pure endowment life insurance, the contract will be paid out at a fixed maturity date only when the customer of this contract survives that time point. In a term life insurance, the contract becomes due when the customers dies before a fixed date. Endowment insurance combines the above two concepts and pays out at either an early death date or a fixed date, whichever comes first. Now let us have a look at the death distribution. For the pricing (and hedging), the insurer assumes a certain distribution of the death time of his customers, which must not necessarily coincide with the real one. Usually, historical data are used by the insurer for hedging purpose to obtain the death distribution. The implications of the mortality misspecification will be considered in Chapter 4. That means, except in Chapter 4, it is assumed that the assumed and real death distribution coincide. Here, we adopt the simple version of the notations in the life insurance mathematics[18]:

$$ {}_tp_x := P\left(\tau^x > t\right); \qquad {}_tq_x := P\left(\tau^x \leq t\right); \quad {}_{u|t-u}q_x := P\left(u < \tau^x \leq t\right); t > u $$

where τ^x is the remaining life of an x-aged life. ${}_tp_x$ denotes the probability of an x–aged life surviving time t, ${}_tq_x$ the probability of an x–aged life dying before time t and ${}_{u|t-u}q_x$ the probability that he dies between u and t. In addition, we use

$$ {}_{t-v}p_{x+v} := P\left(\tau^x > t|\tau^x > v\right); \qquad {}_{u|t}q_{x+v} := P\left(u < \tau^x \leq t|\tau^x > v\right) $$

to denote the corresponding conditional survival/death probabilities, i.e., given that he has survived time v.[19] For example, the insurer might use the death distribution according to Makeham where

$$ \begin{aligned} {}_tp_x &= \exp\left\{-\int_0^t \mu_{x+s}\, ds\right\}, \\ \mu_{x+t} &:= H + Q\, c^{x+t}. \end{aligned} \tag{1.4} $$

As a benchmark case, we use a parameter constellation along the lines of Delbaen (1990), i.e., $H = 0.0005075787$, $Q = 0.000039342435$, and $c = 1.10291509$.

All of these three sources of uncertainty are of vital importance in the risk management of an insurance company. It is particularly interesting to look at the interaction of these risks and how the insurance company tackles all these risks simultaneously. However, for simplification reasons, these three risk sources are discussed later step by step. In Chapter 2, we begin with the non–tradable insurance risk. It is likely to incorporate a

[18] C.f. Gerber (1997).

[19] Conventionally, ${}_{u|t}q_{x+v}$ is used to denote the probability that an x–aged life dies between]x+v+u, x+v+u+t] given that he has survived time v. In order to simplify the notation, this dissertation misuses this notation to denote the probability that an x–aged life dies between $]u,\, t]$ given that he has survived time v.

stochastic interest rate throughout all the chapters. However, by doing this, the importance of the other perspectives, such as diverse hedging strategies in Chapters 2 and 3, and bankruptcy procedures in Chapter 5 cannot be demonstrated. Therefore, except in Chapters 4, a deterministic interest rate is assumed in order to simplify the analysis and emphasize other effects. In Chapters 4, a stochastic term structure of the interest rate is used because the emphasis in this chapter is laid on the effects of model risk, i.e., model misspecification associated with the interest rate risk and death distribution. In addition, in Chapter 5 where the effect of default risk and bankruptcy procedures is emphasized, both mortality and interest rate risk is neglected. Unless mentioned additionally in the chapters, the above models and parameters are used as a benchmark scenario.

This thesis proceeds as follows: Chapter 2 calculates different hedging strategies according to different hedging criteria for a specific life insurance contract, namely, risk–minimizing, quantile and efficient hedging strategies are derived. Chapter 3 places an emphasis on the discrete–time risk–minimizing hedging strategies and investigates and compares the insurer's net loss after applying these strategies. Another source of incompleteness, i.e., the incompleteness resulting from the model misspecification caused by the term structure of the interest rate and the mortality risk, is analyzed in Chapter 4. There, a very popular insurance contract in Germany, mixed life insurance contract is investigated. In Chapter 5, mortality risk is neglected and we mainly look at how the valuation of the life insurance liabilities are affected by default risk and bankruptcy procedures. A conclusion concerning the future research is made in Chapter 6.

Chapter 2

Continuous-time hedging and non–tradable insurance risk

The essential non–tradable characteristic of the individual insurance risk[1] makes the considered financial market incomplete. This leads to the fact that no perfect hedging is possible, i.e., no self–financing hedging strategies which duplicate the final payment of the contingent claim at the same time can be found to eliminate the combined financial and insurance risk. In order to hedge its exposure to these risks, the insurance company has to choose certain "optimality" criterion to tackle this problem. The emphasis of this chapter is placed on introducing and explaining several "optimality" criteria, such as risk–minimizing, quantile and efficient hedging. That means that time–discretization or model misspecification is completely ignored in this chapter. In order to strengthen the understanding of these diverse criteria, concrete specific contracts are investigated and corresponding hedging strategies are derived. As examples, mostly, we consider pure endowment contracts, in which a payment (defined at the beginning of the contract) is provided to the customer if he survives the maturity date. I.e., the payoff of the contract is contingent on the survival of the customer. However, it should be noticed that the entire analysis is very general because the specific contracts are just considered as an example. The study could be extended to the equity-linked structured products sold by banks or the general equity-indexed annuities marketed by most insurance companies.

2.1 Strategy, value process, cost process...

Before we come to different hedging concepts, this section defines some concepts such as strategy, cost processes etc. which play an important role in the hedging analysis.

All the stochastic processes we consider are defined on an underlying stochastic basis $(\Omega, \mathcal{G}, \mathcal{G} = (\mathcal{G}_t)_{t\in[0,T^*]}, P)$, which satisfies the "usual hypotheses".[2]

[1] However, aggregated insurance risk is possibly tradable. A series of mortality derivative have developed recently, see e.g. Milevsky and Promislow (2001) and Blake, Cairns and Dowd (2004) for a discussion on these products.

[2] In this thesis, both t_0 and 0 are used to denote the initial (contract–issuing) time.

- $\mathcal{G}$ is P–complete. We call a probability space P–complete if for each $B \subset A \in \mathcal{G}$ such that $P(A) = 0$, we have that $B \in \mathcal{G}$.
- $\mathcal{G}_0$ contains all P–null set of Ω. Intuitively, it means that we know which events are possible and which are not.
- $\mathcal{G}$ is right–continuous, i.e. $\mathcal{G}_t = \cap_{s>t} \mathcal{G}_s$.

Trading terminates at time $T^* > 0$. We assume that the price processes of underlying assets are described by strictly positive, continuous semimartingales. By a contingent claim X_T with maturity $T \in [0, T^*]$, we simply mean a random payoff received at time T, which is a random variable adapted to the filtration $(\mathcal{G}_t)_{t\in[0,T]}$.

Definition 2.1.1 (Trading strategy, value process, duplication)**.** *Let $S^{(1)}, \ldots, S^{(N)}$ denote the price processes of underlying assets. A trading strategy ϕ in these assets is given by a $\mathbb{R}^N$–valued, predictable process which is integrable with respect to S. The value process $V(\phi)$ associated with ϕ is defined by*

$$V_t(\phi) = \sum_{i=1}^{N} \phi_t^{(i)} S_t^{(i)}.$$

If X_T is a contingent claim with maturity T, then ϕ duplicates X_T iff

$$V_T(\phi) = X_T, \quad P\text{–a.s.}.$$

The deviation of the terminal value of the strategy from the payoff is called duplication cost C^{dup}, i.e.,

$$C_T^{dup} := X_T - V_T(\phi).$$

Definition 2.1.2 (Cost process, gain process)**.** *If ϕ is a trading strategy in the assets $S^{(1)}, \ldots, S^{(N)}$, the (rebalancing) cost process $C(\phi)$ associated with ϕ is defined as follows:*[3]

$$\begin{aligned} C_t(\phi) &:= V_t(\phi) - V_0(\phi) - I_t(\phi) \\ I_t(\phi) &:= \sum_{i=1}^{N} \int_0^t \phi_u^{(i)}\, dS_u^{(i)}. \end{aligned}$$

where $I_t(\phi)$ is the accumulated gain until time t from using strategy ϕ.

Lemma 2.1.3. *The cost process can be reformulated to*

$$C_t(\phi) := \sum_{i=1}^{N} \int_0^t S_u^{(i)}\, d\phi_u^{(i)} + \sum_{i=1}^{N} \int_0^t d\langle \phi.^{(i)}, S.^{(i)} \rangle_u.$$

[3]With loss of generality, an implicit assumption is made in the following definitions, i.e., $r = 0$.

PROOF:

$$\begin{aligned} C_t(\phi) &:= V_t(\phi) - V_0(\phi) - I_t(\phi) \\ &= \sum_{i=1}^{N} \phi_t^{(i)} S_t^{(i)} - \sum_{i=1}^{N} \phi_0^{(i)} S_0^{(i)} - \sum_{i=1}^{N} \int_0^t \phi_u^{(i)} dS_u^{(i)} \\ &\overset{\text{Itô}}{=} \sum_{i=1}^{N} \left(\int_0^t \phi_u^{(i)} dS_u^{(i)} + \int_0^t S_u^{(i)} d\phi_u^{(i)} + \int_0^t d\langle S_\cdot^{(i)}, \phi_\cdot^{(i)} \rangle_u - \int_0^t \phi_u^{(i)} dS_u^{(i)} \right) \\ &= \sum_{i=1}^{N} \int_0^t S_u^{(i)}\, d\phi_u^{(i)} + \sum_{i=1}^{N} \int_0^t d\langle \phi_\cdot^{(i)}, S_\cdot^{(i)} \rangle_u. \end{aligned}$$

□

Definition 2.1.4 (Self–financing, mean–self–financing)**.** *The trading strategy ϕ is called self–financing if between the beginning and ending dates it neither receives nor pays out anything, i.e., it changes a portfolio value only through price changes:*

$$V_t(\phi) = V_0(\phi) + \sum_{i=1}^{N} \int_0^t \phi_u^{(i)} dS_u^{(i)}.$$

A trading strategy ϕ is called mean–self–financing if the cost process $C(\phi) = (C_t(\phi))_{0 \le t \le T}$ is a martingale.

In a self–financing hedging strategy, except an initial investment no further inflows and outflows are necessary. I.e., after time t_0, any fluctuations in the underlying assets can be neutralized by rebalancing ϕ in such a way no further gains or losses result. Furthermore, self–financing can be considered as a special case of mean–self–financing, i.e., when the cost is a constant, $C_t(\phi) = C_0(\phi) = 0$.

Definition 2.1.5 (Attainable, completeness)**.** *A contingent claim X_T with maturity T is called attainable if there exists a self–financing portfolio ϕ such that $V_T(\phi) = X_T \; P-a.s.$ If all contingent claims are attainable, then the market is said to be complete, otherwise incomplete.*

Notice that in the above definition of the cost process, the rebalancing costs at two different trading dates are equally weighted when the costs are due. In order to take account of this, a numeraire is used, i.e., all the rebalancing costs occurring at different dates are measured in terms of one reference date.

Definition 2.1.6 (Bank (money) account)**.** *A bank account B_t corresponds to an accumulation factor. An investment in a bank account is equivalent to investing in a self–financing rolling strategy which consists of just maturing bonds, i.e., bonds due at time $t + dt$.*

$$B_t = e^{\int_0^t r_u du}, \qquad dB_t = r_t\, B_t\, dt.$$

where r_t is instantaneous interest rate at time t.

Unless mentioned otherwise, we use the money account as numeraire and denote the discounted versions of X, S, V, C and C^{dup} with a superscript *, i.e.,

$$\begin{aligned}
X_T^* &= e^{-\int_0^T r_u\,du}\,X_T = B_T^{-1}X_T \\
S_t^{(i)*} &= e^{-\int_0^t r_u\,du}\,S_t^{(i)} = B_t^{-1}S_t^{(i)} \\
V_t^*(\phi) &= e^{-\int_0^t r_u\,du}\,V_t(\phi) = B_t^{-1}V_t(\phi) \\
C_t^*(\phi) &= V_t^*(\phi) - V_0^*(\phi) - \sum_{i=1}^N \int_0^t \phi_u^{(i)}\,dS_u^{(i)*} \\
C_T^{\mathrm{dup},*} &= e^{-\int_0^T r_u\,du}\,(X_T - V_T(\phi)) = X_T^* - V_T^*(\phi).
\end{aligned}$$

Lemma 2.1.7. *C and C^* are related as follows*

$$\begin{aligned}
C_t^* &= \int_0^t e^{-\int_0^u r_s\,ds}\;dC_u + \int_0^t d\left\langle e^{-\int_0 r_s\,ds}, C\right\rangle_u, \\
C_t &= \int_0^t e^{\int_0^u r_s\,ds}\;dC_u^* + \int_0^t d\left\langle e^{\int_0 r_s\,ds}, C^*\right\rangle_u.
\end{aligned}$$

Proof: According to the definitions of (discounted) cost processes and an application of Itô's product chain rule, we obtain

$$\begin{aligned}
dC_t^*(\phi) &= d\left(V_t^*(\phi) - V_0^*(\phi) - \sum_{i=1}^N \int_0^t \phi_u^{(i)}\,dS_u^{(i)*}\right) \\
&= dV_t^*(\phi) - \sum_{i=1}^N \phi_t^{(i)} dS_t^{(i)^*} \\
&= B_t^{-1}dV_t(\phi) + V_t(\phi)\,d(B_t^{-1}) + d\langle V.(\phi), B.^{-1}\rangle_t \\
&\quad - \sum_{i=1}^N \phi_t^{(i)}\left(B_t^{-1}dS_t^{(i)} + S_t^{(i)}d(B_t^{-1}) + d\langle S.^{(i)}, B.^{-1}\rangle_t\right) \\
&= B_t^{-1}dC_t(\phi) + d\langle C.(\phi), B.^{-1}\rangle_t.
\end{aligned}$$

$$\begin{aligned}
dC_t &= d\left(V_t(\phi) - V_0(\phi) - \sum_{i=1}^N \int_0^t \phi_u^{(i)}\,dS_u^{(i)}\right) \\
&= dV_t(\phi) - \sum_{i=1}^N \phi_t^{(i)} dS_t^{(i)} \\
&= B_t\,dV_t^*(\phi) + V_t^*(\phi)\,dB_t + d\langle V^*.(\phi), B.\rangle_t \\
&\quad - \sum_{i=1}^N \phi_t^{(i)}\left(B_t dS_t^{(i)*} + S_t^{(i)*}dB_t + d\langle S.^{(i)*}, B.\rangle_t\right) \\
&= B_t\,dC_t^*(\phi) + d\langle C^*.(\phi), B.\rangle_t.
\end{aligned}$$

□

It is observed that if the paths of C are locally of bounded variation, i.e., $d\langle C., B.^{-1}\rangle_t = 0$, then the paths of C^* are also locally of bounded variation.

After all of these general definitions which are needed for the hedging analysis are set up, in the following we go through several hedging criteria one after another. Throughout the chapter, the interest rate is assumed to be deterministic, i.e., $r_u = r$.

2.2 Risk–minimizing hedging

Risk–minimizing hedging is a dynamic hedging approach which relies on the condition that contingent claims can be duplicated by the final value of the hedging portfolio and basically amounts to minimizing the variance of the hedger's future costs. However, this approach has the undesirable property that minimization of the variance (or the expected value of the square of the future costs) implies that relative losses and relative gains are treated equally. Now let us have a look at how the concept of risk–minimizing hedging has been developed.

2.2.1 A short review

The concept of risk–minimizing was introduced by Föllmer and Sondermann (1986), who extended the theory from hedging in a complete market to the case of an incomplete market. They obtained strategies in the sense of minimization of a certain squared error process and proved the resulting risk–minimizing hedging strategy is mean–self–financing. Assume, we consider a financial market consisting of a risky asset and a risk free bond only. Therefore, the trading strategy or portfolio strategy is an adapted process $\phi = (\xi_t, \eta_t)_{t\in[0,T]}$, where ξ_t is the number of the stock S and η_t the number of the bond B held in the portfolio at time t. The value process associated with ϕ is given by

$$V_t(\phi) = \xi_t S_t + \eta_t B_t.$$

This strategy is self–financing if $V_t(\phi) = V_0(\phi) + \int_0^t \xi_u dS_u + \int_0^t \eta_u dB_u$ for all $0 \le t \le T$, i.e., after time t_0, no further inflows or outflows are needed. Therefore, in this case, a self–financing strategy is completely described by the initial investment and ξ since the self–financing constraint determines $(V_t)_{0<t\le T}$ and hence also η.

In the following, the deflated value process is considered and it is given by:

$$V_t^*(\phi) = V_t(\phi)B_t^{-1} = \xi_t S_t^* + \eta_t,$$

where S^* the discounted asset price and $B^* = 1$. The discounted cost process $C^*(\phi)$ associated with the strategy ϕ is defined by

$$C_t^*(\phi) = V_t^*(\phi) - \int_0^t \xi_u dS_u^*.$$

The cost $C^*(\phi)$ is the value of the portfolio less the accumulated income from the asset S. The total costs $C_t^*(\phi)$ incurred in $[0,t]$ can be decomposed into the costs incurred during $(0,t]$ and an initial cost $C_0^*(\phi) = V_0^*(\phi)$, which is typically greater than 0.

Definition 2.2.1 (Risk process)**.** *The risk process* $(R_t(\phi))_{\{t\in[0,T]\}}$ *of a trading strategy* ϕ *is defined by*

$$R_t(\phi) = E^*[(C_T^*(\phi) - C_t^*(\phi))^2|\mathcal{G}_t].$$

It corresponds to the conditional expected squared value of future costs under the minimal martingale measure.[4]

This usage differs from the traditional actuarial one, where "risk process" denotes the cash flow of premiums and benefits.

According to the requirement for risk–minimizing strategies that the final portfolio value should duplicate the contingent claim, only admissible strategies come into consideration. First, an admissible strategy which minimizes the mean squared error $R_0(\phi)$ can be determined. The cost process associated with an admissible strategy is given as follows:

$$C_T^*(\phi) \;=\; V_T^*(\phi) - \int_0^T \xi_u dS_u^* = X_T^* - \int_0^T \xi_u dS_u^*. \tag{2.1}$$

Consequently, the risk at the initial time can be determined by

$$R_0(\phi) \;=\; E^*\left[(C_T^*(\phi) - C_0^*(\phi))^2\right] = E^*\left[\left(X_T^* - \int_0^T \xi_u dS_u^* - C_0^*(\phi)\right)^2\right]. \tag{2.2}$$

In order to minimize the $R_0(\phi)$, first order condition of R_0 with respect to $C_T^*(\phi)$ is looked at, i.e.

$$\frac{\partial E^*\left[(C_T^*(\phi) - C_0^*(\phi))^2\right]}{\partial C_T^*(\phi)} = E^*\left[\frac{\partial (C_T^*(\phi) - C_0^*(\phi))^2}{\partial C_T^*(\phi)}\right] = E^*\left[2(C_T^*(\phi) - C_0^*(\phi))\right] \stackrel{!}{=} 0.$$

This requires $C_0^*(\phi) = E^*[C_T^*(\phi)]$. Due to the expression in Equation (2.1) and the fact that $(S_t^*)_{t\in[0,T]}$ is a P^*–martingale, $E^*[C_T^*(\phi)] = E^*[X_T^*]$ results. Hence, $R_0(\phi)$ is minimized iff $C_0^*(\phi) = E^*[X_T^*] = E^*[C_T^*(\phi)]$. We should choose ξ so as to minimize the variance

$$R_0(\phi) \;=\; E^*[(C_T^*(\phi) - E^*[C_T^*(\phi)])^2]. \tag{2.3}$$

This criterion does **not** yield a unique strategy, but it characterizes an entire class of strategies all minimizing the mean squared error. The non–uniqueness of the optimal admissible strategy is a natural consequence of the simple criterion of minimizing Equation (2.3), which only involves the value of the cost process $C^*(\phi)$ at time T. Furthermore, note that $X_T^* = \xi_T S_T^* + \eta_T$, which does not depend on $(\eta_t)_{0\le t<T}$. Thus, we should not expect the

[4] A detailed study on the minimal martingale measure can be found e.g. in Schweizer (1991, 1995).

minimization criterion associated with the squared error to impose any constraints on the number of bonds held in the time interval $(0, T)$. An application of the Galtchouk–Kunita–Watanabe decomposition enables the construction of the strategy, i.e., determination of the number of the risky asset and that of the risk free bonds. Defining the intrinsic value process $(V_t^*)_{\{t\in[0,T]\}}$ by

$$V_t^* = E^*[X_T^*|\mathcal{G}_t]$$

and noting that $(V^*)_t$ is an $(\mathcal{G}, P^*)$–martingale, the Galtchouk–Kunita–Watanabe decomposition theorem allows us to write V_t^* **uniquely** in the form

$$V_t^* = E^*[X^*] + \int_0^t \xi_u^H dS_u^* + L_t^H,$$

where $(L_t^H)_{0\le t\le T}$ is a zero–mean $(\mathcal{G}, P^*)$–martingale, L^H and S^* are orthogonal martingales, and ξ^H is a predictable process. By applying the orthogonality of the martingales L^H and S^* and using $V_T^* = X_T^*$, the following proposition results.

Proposition 2.2.2. *An admissible strategy $\phi = (\xi, \eta)$ has a minimal variance*

$$E^*[(C_T^*(\phi) - E^*[C_T^*(\phi)])^2] = E^*[(L_T^H)^2],$$

if and only if $\xi = \xi^H$.

PROOF: Proof can be found e.g. in Föllmer and Sondermann (1986). □

A more precise result is obtained by looking for admissible strategies which satisfy $V_T(\phi) = X$ and minimize the remaining risk, defined by $R_t(\phi)$ at any time $t \in [0, T]$. Such strategies are said to be risk–minimizing.

Proposition 2.2.3. *There exists a unique admissible risk–minimizing strategy $\phi = (\xi, \eta)$ given by*

$$(\xi_t, \eta_t) = (\xi_t^H, V_t^* - \xi_t^H S_t^*), \qquad 0 \le t \le T,$$

the associated risk process is given by $R_t(\phi) = E^*[(L_T^H - L_t^H)^2|\mathcal{G}_t]$. The risk process associated with the risk–minimizing strategy is also called the intrinsic risk process.

PROOF: Proof can be found e.g. in Föllmer and Sondermann (1986). □

2.2.2 Application in equity–linked life insurance

Møller (1998) applied risk–minimizing concept to the context of equity–linked life insurance and derived risk–minimizing hedging strategies for different equity–linked life insurance contracts. Since the entire analysis in Chapter 3 and partial analysis in Chapter 4 are based on Møller's hedging strategy, we give a brief review of the derivation of this strategy.

We begin with the insurance risk. An insurance group with n insured with age x at time t_0 is considered. τ_i^x is used to denote the remaining life time of insured i, $i = 1, 2, \cdots, n$

at age x and assumed to be identically and independently distributed. In the following, we use

$$ {}_tp_x = \text{Prob}(\tau_i^x > t) $$

denoting the probability that the remaining time of an x–aged life larger than t. The relation between the survival probability and the hazard rate of mortality μ_x

$$ {}_tp_x = \exp\left\{ -\int_0^t \mu_{x+s} ds \right\} $$

leads to

$$ \frac{d\,{}_tp_x}{dt} \;=\; -\mu_{x+t} \cdot {}_tp_x. \tag{2.4} $$

This equality is very often used in the derivation of the risk–minimizing hedging strategy. Another attribute to describe all the remaining times of the insured is the count process $(N_t^x)_{t\in[0,T]}$ defined by

$$ N_t^x = \sum_{i=1}^{n} 1_{\{\tau_i^x \le t\}}. $$

N_t^x counts the number of the insured with age x who do not survive time t. The natural filtration generated by this count process is the following σ–field

$$ \mathcal{H}_t = \sigma\{N_u^x \mid u \le t\}. $$

Obviously, by studying (N_t^x) it is possible to catch the change of the original insurance group with n insured at the beginning over the time. From today's perspective, the expected number of the insured who do not survive time t is determined by

$$ E[N_t^x \,|\, \mathcal{H}_0] = E\left[\sum_{i=1}^{n} 1_{\{\tau_i^x \le t\}}\right] = \sum_{i=1}^{n} E[1_{\{\tau_i^x \le t\}}] = n \cdot (1 - {}_tp_x). $$

In addition, the expected number of the insured who die between a time interval $]t, t+\Delta t]$ at the viewpoint of time t is determined by

$$ \begin{aligned} E[N_{t+\Delta t}^x - N_t^x \,|\, \mathcal{H}_t] &= E\left[\sum_{i=1}^{n} 1_{\{t<\tau_i^x \le t+\Delta t\}} \Big| \mathcal{H}_t\right] \\ &= \sum_{i=1}^{n-N_t^x} E[1_{\{t<\tau_i^x \le t+\Delta t\}} \,|\, \tau_i^x > t] \\ &= (n - N_t^x)\, P(t < \tau^x \le t + \Delta t \,|\, \tau^x > t) \\ &= (n - N_t^x) \left(\frac{{}_tp_x - {}_{t+\Delta t}p_x}{{}_tp_x} \right) \\ &= (n - N_t^x)\,(1 - {}_{\Delta t}p_{x+t}). \end{aligned} \tag{2.5} $$

The second equality results because at time t the insurer has observed how many insured survived time t and only those who survive that time point (namely $n - N_t^x$ insured) have effect on this expectation. As Δt goes to 0, we obtain

$$\begin{aligned}
\lim_{\Delta t\to 0} \frac{1}{\Delta t} E[N_{t+\Delta t}^x - N_t^x \,|\, \mathcal{H}_t] &= \lim_{\Delta t\to 0} (n - N_t^x) \left(\frac{{}_tp_x - {}_{t+\Delta t}p_x}{{}_tp_x\, \Delta t} \right) \\
&= \frac{n - N_t^x}{{}_tp_x} \lim_{\Delta t\to 0} \left(-\frac{{}_{t+\Delta t}p_x - {}_tp_x}{\Delta t} \right) \\
&= \frac{n - N_t^x}{{}_tp_x}\, \mu_{x+t}\, {}_tp_x \\
&= (n - N_t^x)\, \mu_{x+t}.
\end{aligned}$$

As a consequence, it holds

$$E[dN_t^x|\mathcal{H}_t] = (n - N_t^x)\mu_{x+t}dt := \lambda_t\, dt.$$

The new parameter (λ_t) gives the intensity of the process (N_t^x), i.e., describes how many insured die within an infinitesimal small time interval. Furthermore, for $u > t$ it holds

$$\begin{aligned}
\lim_{\Delta t\to 0} \frac{1}{\Delta t} E[N_{u+\Delta t}^x - N_u^x \,|\, \mathcal{H}_t] &= \lim_{\Delta t\to 0} \frac{1}{\Delta t} \left(E[N_{u+\Delta t}^x - N_t^x \,|\, \mathcal{H}_t] - E[N_u^x - N_t^x \,|\, \mathcal{H}_t] \right) \\
&= \lim_{\Delta t\to 0} (n - N_t^x) \left(\frac{{}_tp_x - {}_{u+\Delta t}p_x}{{}_tp_x\, \Delta t} - \frac{{}_tp_x - {}_up_x}{{}_tp_x\, \Delta t} \right) \\
&= \frac{n - N_t^x}{{}_tp_x} \lim_{\Delta t\to 0} \left(-\frac{{}_{u+\Delta t}p_x - {}_up_x}{\Delta t} \right) \\
&= \frac{n - N_t^x}{{}_tp_x}\, \mu_{x+u} \cdot {}_up_x \\
&= (n - N_t^x)\, \mu_{x+u} \cdot {}_{u-t}p_{x+t}. \qquad (2.6)
\end{aligned}$$

This limes–value is the expected number of deaths in a future infinitesimal time interval $]u, u + dt]$ given that the insurer is situated at time point t. Since μ or λ is usually not equal to zero, the death count process (N_t^x) is no $\mathcal{H}$–martingale. Consequently, the compensated counting process $(M_t)_{t\in[0,T]}$ is defined:

$$M_t = N_t^x - \int_0^t \lambda_u\, du.$$

It is noticed that (M_t) gives an $\mathcal{H}$–martingale. [5] It is shown in Møller (1998) that the change of measure from P to P^* does not affect the distribution of N and that M stays

[5] Apparently, M_t is $\mathcal{H}_t$–measurable, and it holds

$$\begin{aligned}
E[M_t|\mathcal{H}_s] &= E\left[N_t^x - \int_0^t \lambda_u du|\mathcal{H}_s \right] = N_s^x - \int_0^s \lambda_u du + E\left[N_t^x - N_s^x - \int_s^t \lambda_u du|\mathcal{H}_s \right] \\
&= N_s^x - \int_0^s \lambda_u du + E\left[N_t^x - E[N_t^x|\mathcal{F}_s] \,|\, \mathcal{H}_s \right] = M_s,\ s \le t.
\end{aligned}$$

an $(\mathcal{H}, P^*)$–martingale.

Concerning the financial market model, we stay in the Black–Scholes Economy introduced in Section 1.4.1. In Møller (1998), it is mentioned that the unique equivalent martingale measure derived in a complete financial market described by the Black–Scholes Economy coincides with the minimal martingale measure which is needed for risk–minimization analysis. In order to proceed with the analysis, it is necessary to introduce the probability space $(\Omega, \mathcal{F}, \mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}, P^*)$ for the combined market. I.e., the combined filtration $\mathcal{F}_t$ is the union of $\mathcal{G}_t$ and $\mathcal{H}_t$:

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t = \sigma\{(S_u, B_u, N_u^x) \,|\, u \leq t\} = \sigma\{(S_u^*, N_u^x) \,|\, u \leq t\}.$$

After the model is set up, the concept of risk–minimizing trading strategies is studied for diverse equity–linked life insurance contracts. Overall, discounted processes are considered. The first contract category we study is pure endowments which pay off only when the insured survives the maturity date. At time maturity date T, the relevant contingent claim owns the form of

$$X_T^* \;=\; B_T^{-1}\,(n - N_T^x)\,g(T, S_T), \tag{2.7}$$

where $g(T, S_T)$ is the payoff of each contract. It can be a function of the final asset value or a function of the entire evolution of the asset. Due to the stochastic independence between N^x and (B, S) under P^*, the deflated intrinsic value process $V^* = (V_t^*)_{0\leq t\leq T}$ for this specific claim is given by

$$\begin{aligned}
V_t^* &= E^*[X_T^*|\mathcal{F}_t] \\
&= E^*[(n - N_T^x)|\mathcal{H}_t]B_t^{-1}\,\underbrace{E^*[g(T, S_T)\,B_t\,B_T^{-1}|\mathcal{F}_t]}_{:=F^g(t,T,S_t)} \\
&= (E^*[(n - N_t^x)|\mathcal{H}_t] - E^*[(N_T^x - N_t^x)|\mathcal{H}_t])\,B_t^{-1}F^g(t, T, S_t) \\
&= (n - N_t^x)\,\cdot\,{}_{T-t}p_{x+t}\,\cdot\,B_t^{-1}\,\cdot\,F^g(t, T, S_t).
\end{aligned}$$

In the above derivation, the independence assumption between the financial market and insurance risk, and Equation (2.5) are used. Furthermore, it is worth mentioning that in the function $F^g(t, T, S_t)$, t gives the valuation time, T the payout time and S the underlying asset. The process V_t^* can be interpreted as the market value process associated with the entire portfolio of pure endowment contracts, using the pricing rule P^*. It can be decomposed into two parts: the first part is $(n-N_t^x)\cdot{}_{T-t}p_{x+t}$, i.e., the expected number of insured who survive the maturity date given that they have survived time t, and the rest is the expected discounted value of the financial part of the contingent claim under the equivalent martingale measure. In particular, the initial value $V_{t_0}^* = n\,.\,{}_Tp_x\,.\,F^g(t_0, T, S_{t_0})$ is a natural candidate for the single premium for the entire portfolio. In order to derive the risk–minimizing hedging strategy, Itô's Lemma for diffusions with jumps is applied.

Lemma 2.2.4 (Itô's formula for semimartingales)**.** *Let $(Y_t)_{t\geq 0}$ be a semimartingale. For*

any $C^{1,2}$ function $f : \mathbb{R} \times [0,T] \mapsto \mathbb{R}$,

$$\begin{aligned} f(Y_t,t) &= f(Y_0,0) + \int_0^t \frac{\partial f}{\partial Y}(Y_{s-},s)dY_s + \int_0^t \frac{\partial f}{\partial s}(Y_{s-},s)ds + \int_0^t \frac{1}{2}\frac{\partial^2 f}{\partial Y^2}(Y_{s-},s)d\langle Y\rangle_s \\ &\quad + \sum_{0\le s\le t,\, \Delta Y_s\neq 0} \left[f(Y_s,s) - f(Y_{s-},s) - \Delta Y_s \frac{\partial f}{\partial Y}(Y_{s-},s) \right] \end{aligned}$$

PROOF: A proof for this lemma can be found e.g. in Chapter 8 of Cont and Tankov (2004). □

Applying this lemma to the value process $V_t^* = (n - N_t^x)\, {}_{T-t}p_{x+t}\, B_t^{-1}\, F^g(t,T,S_t)$, and using a simplified notation $Q_t := B_t^{-1}\, F^g(t,T,S_t)$ we obtain

$$\begin{aligned} V_t^* &= V_0^* + \int_0^t (n - N_{u-}^x)\, Q_{u-} \frac{\partial\, {}_{T-u}p_{x+u}}{\partial u}\, d\,u - \int_0^t {}_{T-u}p_{x+u}\, Q_{u-}\, dN_u^x \\ &\quad + \int_0^t (n - N_{u-}^x)\, {}_{T-u}p_{x+u}\, dQ_u + \sum_{0\le s\le t} [V_s^* - V_{s-}^* - \Delta N_s^x(-{}_{T-s}p_{x+s}\; Q_{s-})] \\ &= V_0^* + \int_0^t (n - N_{u-}^x) Q_u\, {}_{T-u}p_{x+u}\; \mu_{x+u}\, d\,u + \int_0^t (n - N_{u-}^x)\, {}_{T-u}p_{x+u}\; d\,Q_u \\ &\quad + \sum_{0\le u\le t} [V_u^* - V_{u-}^*] - \int_0^t {}_{T-u}p_{x+u}\; Q_{u-}\, dN_u^x + \sum_{0\le u\le t} [\Delta N_u^x\, ({}_{T-u}p_{x+u}\, Q_{u-})]. \end{aligned}$$

For the above derivation, Equation (2.4) is applied. Since (N_t^x) is a pure jump process with e.g. jump times $T_1 < T_2 \cdots$, $Q_s\; {}_{T-s}p_{x+s}$ is constant between two jumps. This implies

$$\begin{aligned} \int_0^t {}_{T-u}p_{x+u}\; Q_{u-}\, dN_u^x &= \sum_{0\le u\le t} [\Delta N_u^x\, ({}_{T-u}p_{x+u}\; Q_{u-})] \\ \sum_{0\le u\le t} [V_u^* - V_{u-}^*] &= -\int_0^t {}_{T-u}p_{x+u} Q_{u-}\, dN_u^x. \end{aligned}$$

Furthermore, Itô's Lemma leads to

$$dQ_t = d(B_t^{-1}F^g(t,T,S_t)) = F_s^g(t,T,S_t)dS_t^*,$$

with $F_s^g(t,T,S_t) = \frac{\partial F^g(t,T,S_t)}{\partial S_t}$ denoting the derivative of $F(t,T,S_t)$ with respect to S. Finally we obtain:

$$\begin{aligned} V_t^* &= V_0^* + \int_0^t (n - N_{u-}^x)\; {}_{T-u}p_{x+u}\; F_s^g(u,T,S_u)\, dS_u^* \\ &\quad + \int_0^t (-B_u^{-1}F^g(u,T,S_u){}_{T-u}p_{x+u})\, (\, dN_u^x - (n - N_{u-}^x)\, \mu_{x+u}\, d\,u) \\ &= V_0^* + \int_0^t (n - N_{u-}^x)\; {}_{T-u}p_{x+u}\; F_s^g(u,T,S_u)\, dS_u^* + \int_0^t (-B_u^{-1}\, F^g(u,T,S_u)\, {}_{T-u}p_{x+u}) dM_u. \end{aligned}$$

The following two propositions are direct consequences of the above derivation.

Proposition 2.2.5. *For the contingent claim X_T^* given in Equation (2.7), the process V^* defined by $V_t^* = E^*[X_T^*|\mathcal{F}_t]$ has the decomposition*

$$V_t^* = V_0^* + \int_0^t \xi_u^H dS_u^* + \int_0^t v_u^H dM_u,$$

where (ξ^H, v^H) are given by

$$\begin{aligned} \xi_t^H &= (n - N_{t-}^x)\, {}_{T-t}p_{x+t}\, F_s^g(t,T,S_t) \\ v_t^H &= -B_t^{-1}\, F^g(t,T,S_t)\, {}_{T-t}p_{x+t}. \end{aligned}$$

Proposition 2.2.6. *For a pure endowment insurance contract X_T^* given in Equation (2.7), an admissible strategy $(\phi_t^*) = (\xi_t^*, \eta_t^*)$, $0 \le t \le T$, minimizing the variance is determined by*

$$\begin{aligned} \xi_t^* &= \xi_t^H = (n - N_{t-}^x)\, {}_{T-t}p_{x+t}\, F_s^g(t,T,S_t), \\ \eta_t^* &= V_t^* - \xi_t^* S_t^*. \end{aligned}$$

Consequently, the variance of this variance–minimizing strategy can be calculated as follows:

$$\begin{aligned} & E^*[(C_T^*(\phi) - E^*[C_T^*(\phi)])^2] \\ = \;& E^*\left[\left(\int_0^T v_u^H\, d\, M_u\right)^2\right] = E^*\left[\int_0^T \left(v_u^H\right)^2 d\langle M\rangle_u\right] \\ = \;& E^*\left[\int_0^T (B_u^{-1}\, F^g(u,T,S_u)\, {}_{T-u}p_{x+u})^2\, \lambda_u\, d\, u\right] \\ = \;& \int_0^T E^*\left[\left(B_u^{-1} F^g(u,T,S_u)\right)^2\right]\, {}_{T-u}p_{x+u}^2\, E^*[(n - N_u^x)\, \mu_{x+u}]\, d\, u \\ = \;& \int_0^T E^*\left[\left(B_u^{-1}\, F^g(u,T,S_u)\,\right)^2\right]\, ({}_{T-u}p_{x+u})^2\; n\; {}_up_x\; \mu_{x+u}]\, d\, u \\ = \;& n\, {}_Tp_x \int_0^T E^*\left[\left(B_u^{-1}\, F^g(u,T,S_u)\right)^2\right]\, {}_{T-u}p_{x+u}\, \mu_{x+u}\, d\, u. \end{aligned}$$

Second, we consider the term insurance contracts whose payoffs are conditioned on the death of the insured. The discounted final payment of such contracts are described by

$$X_T^* = \int_0^T g(u,S_u) B_u^{-1} dN_u^x, \tag{2.8}$$

where $g(u,S_u)$ is a positive $C^{1,2}$ function of time and the stock price. The intrinsic value process of X_T^* is calculated by

$$\begin{aligned} V_t^* &= E^*[X_T^*|\mathcal{F}_t] = \int_0^t g(u,S_u) B_u^{-1} dN_u^x + E^*\left[\int_t^T g(u,S_u)\, B_u^{-1}\, dN_u^x|\mathcal{F}_t\right] \\ &= \int_0^t g(u,S_u)\, B_u^{-1}\, dN_u^x + \int_t^T B_t^{-1}\, F^g(t,u,S_t)\, (n - N_t^x)\; {}_{u-t}p_{x+t}\; \mu_{x+u}\, du, \end{aligned} \tag{2.9}$$

where $F^g(t,u,S_t) = E^*[e^{-\int_t^u r_\tau d\tau} g(u,S_u)|\mathcal{F}_t]$ is the time t $(t<u)$ value of a contingent claim whose payoff is due at time u. Equation (2.6) is needed in the above derivation. Using the general Itô's formula and the Fubini Theorem for Itô processes, see Ikeda and Watanabe (1981), the value process $(V_t^*)_{t\in[0,T]}$ can be formulated as follows:

$$V_t^* = V_0^* + \int_0^t \frac{\partial V_\tau^*}{\partial \tau}\, d\tau + \int_0^t \frac{\partial V_\tau^*}{\partial N_{\tau-}^x}\, dN_\tau^x + \int_0^t \frac{\partial V_\tau^*}{\partial Q_{\tau-}}\, dQ_\tau + \sum_{0\le u\le t}\left[V_u^* - V_{u-}^* - \Delta N_u^x \frac{\partial V_u^*}{\partial N_{u-}^x}\right].$$

Again, $Q_t := B_t^{-1}F^g(t,u,S_t)$ and it holds here $dQ_t = F_s^g(t,u,S_t)dS_t^*$. Furthermore, in the pure jump process, the last expression equals 0. A straightforward use of this generalized Itô's Lemma in the intrinsic value process V_t^* in Equation (2.9) leads to the following rephrasing of V_t^*:

$$\begin{aligned} V_t^* &= V_0^* + \int_0^t (-B_\tau^{-1}\, F^g(\tau,\tau,S_\tau)\, \mu_{x+\tau}\, (n - N_\tau^x))d\tau \\ &\quad + \int_0^t \underbrace{\left(g(\tau,S_\tau)B_\tau^{-1} - \int_\tau^T B_\tau^{-1}F^g(\tau,u,S_\tau)\ {}_{u-\tau}p_{x+\tau}\ \mu_{x+\tau}\ du\right)}_{:=v_t^H} dN_\tau^x \\ &\quad + \int_0^t \left(\int_\tau^T B_\tau^{-1}F^g(\tau,u,S_\tau)\, {}_{u-\tau}p_{x+\tau}\, du\right)\, (n - N_{\tau-}^x)\, \mu_{x+\tau}\, d\,\tau \\ &\quad + \int_0^t \underbrace{(n - N_{\tau-}^x)\int_\tau^T F_s^g(\tau,u,S_\tau)\, {}_{u-\tau}p_{x+\tau}\, \mu_{x+u}\, d\,u)}_{:=\xi_t^H} dS_\tau^*. \end{aligned} \tag{2.10}$$

The result for the term insurance contracts is summarized in the following proposition.

Proposition 2.2.7. *The expression of V_t^* leads to the unique admissible risk–minimizing strategy for a term insurance given in Equation (2.8)*

$$\begin{aligned} \xi_t^* &= \xi_t^H = (n - N_{t-}^x)\int_t^T F_s^g(t,u,S_t)\, {}_{u-t}p_{x+t}\, \mu_{x+u} du \\ \eta_t^* &= \int_0^t g(u,S_u)B_u^{-1}dN_u^x + (n - N_t^x)\int_t^T B_t^{-1}F^g(t,u,S_t)\, {}_{u-t}p_{x+t}\, \mu_{x+u} du \\ &\quad -\xi_t^* S_t^*, \qquad 0\le t\le T. \end{aligned}$$

Besides, the intrinsic risk process

$$R_t(\phi) = (n - N_t^x)\int_t^T E^*[(v_u^H)^2|\mathcal{F}_t]\, {}_{u-t}p_{x+t}\, \mu_{x+u} d\,u,$$

where v_t^H is given in Equation (2.10).

And the endowment insurance contract is the combination of the pure endowment and term insurance contract.

Because equity–linked products with an asset value guarantee have become very popular both as pure investment contracts and in the context of life insurance policies, a specific guaranteed equity–linked insurance contract is considered as an illustrative example. Our goal is not only to price the issued contract, but to derive the continuous risk–minimizing strategy. First, we consider a specific guaranteed equity–linked pure endowment life insurance contract, which provides the buyer of such a contract the payoff at time $T = t_M$

$$f(t_M, S) \;=\; G + \alpha \sum_{i=0}^{M-1} (i+1) K \left[\frac{S(t_{i+1})}{S(t_i)} - e^{g(t_{i+1}-t_i)} \right]^+, \tag{2.11}$$

if he survives the maturity of the contract T. This payoff first of all ensures the insured a guaranteed amount G, and allows him a possibility to participate in the surpluses of the insurance company with a participation rate α. The surpluses are determined periodically and linked to the reference portfolio. I.e., the periodic surpluses are described by a sequence of European call options with the strike $e^{g(t_{i+1}-t_i)}$, $i = 0, \cdots, M-1$. If the distance between two time points is assumed to be equidistant, i.e. $\Delta t = t_{i+1} - t_i$, then all European options have the same strike. The parameter K can be interpreted as a periodic premium. The periodic surplus at time t_{i+1} is based on all the premiums the insurer has obtained, i.e., $(i+1)\,K$.

In order to simplify matters, the asset price is assumed to follow a one–dimensional geometric Brownian motion, in addition, the interest rate is assumed to be deterministic. Therefore, under the equivalent martingale measure, we have

$$dS_t = S_t(r\,dt + \sigma\,dW_t^*),$$

where r is risk–free interest rate and $(W_t^*)_{t\in[0,T]}$ a Brownian motion under the equivalent martingale measure P^*. In order to derive the risk–minimizing hedging strategy for $f(t_M, S)$, we need to calculate $F(t, T, S)$ and $F_s(t, T, S)$ for this specific equity–linked life insurance contract.

Proposition 2.2.8. *At time t, the fair price $F(t, T, S)$ of the contingent claim $f(t_M, S)$ given in Equation (2.11) is determined by*

$$\begin{aligned} F(t,T,S) \;=\; & e^{-r(T-t)}G + \alpha K \sum_{i=0}^{M-1} (i+1) \Bigg\{ 1_{\{t>t_{i+1}\}} e^{-r(T-t)} \left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t} \right]^+ \\ & + 1_{\{t_i<t\le t_{i+1}\}} e^{-r(T-t_{i+1})} \left(\frac{S(t)}{S(t_i)} N(d_1^{(t,t_i)}) - e^{g\,\Delta t} e^{-r(t_{i+1}-t)} N(d_2^{(t,t_i)}) \right) \\ & + 1_{\{t\le t_i\}} e^{-r(t_{M-1}-t)} \left(N(d_1) - e^{(g-r)\,\Delta t} N(d_2) \right) \Bigg\}, \end{aligned}$$

with

$$\begin{aligned} d_{1/2}^{(t,t_i)} &= \frac{\ln S(t)/S(t_i) - g\,\Delta t + (r \pm \frac{1}{2}\sigma^2)(t_{i+1}-t)}{\sigma\sqrt{t_{i+1}-t}} \\ d_{1/2} &= \frac{(r - g \pm \frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}, \end{aligned}$$

where $N(t) = \int_0^t \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ *is the cumulative standard normal distribution function.*

PROOF: It is well–known that the price of a T–contingent claim at time t equals the expected discounted value of the terminal payoff conditional on the information structure till time t, $t \in [0,T]$, under the equivalent martingale measure, i.e.,

$$\begin{aligned} F(t,T,S) &= E^*[e^{-r(T-t)} f(t_M, S)|\mathcal{F}_t] \\ &= E^*\left[e^{-r(T-t)}\left(G + \alpha\sum_{i=0}^{M-1}(i+1)K\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\,\Delta t}\right]^+\right)\Big|\mathcal{F}_t\right] \\ &= e^{-r(T-t)}G + \alpha K\sum_{i=0}^{M-1}(i+1)E^*\left[e^{-r(T-t)}\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+\Big|\mathcal{F}_t\right] \\ &= e^{-r(T-t)}G + \alpha K\sum_{i=0}^{M-1}(i+1)\Bigg(E^*\left[e^{-r(T-t)}\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\,\Delta t}\right]^+ 1_{\{t>t_{i+1}\}}\Big|\mathcal{F}_t\right] \\ &\quad + E^*\left[e^{-r(T-t)}\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\,\Delta t}\right]^+ 1_{\{t_i<t\le t_{i+1}\}}\Big|\mathcal{F}_t\right] \\ &\quad + E^*\left[e^{-r(T-t)}\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\,\Delta t}\right]^+ 1_{\{t\le t_i\}}\Big|\mathcal{F}_t\right]\Bigg) \\ &= e^{-r(T-t)}G + \alpha K\sum_{i=0}^{M-1}(i+1)\Bigg(e^{-r(T-t)}\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\,\Delta t}\right]^+ 1_{\{t>t_{i+1}\}} \\ &\quad + e^{-r(T-t)}e^{r(t_{i+1}-t)}E^*\left[e^{-r(t_{i+1}-t)}\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\,\Delta t}\right]^+ 1_{\{t_i<t\le t_{i+1}\}}\Big|\mathcal{F}_t\right] \\ &\quad + e^{-r(T-t)}e^{r\,\Delta t}E^*\left[E^*\left[e^{-r\,\Delta t}\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\,\Delta t}\right]^+ 1_{\{t\le t_i\}}\Big|\mathcal{F}_{t_i}\right]\Big|\mathcal{F}_t\right]\Bigg) \\ &= e^{-r(T-t)}G + \alpha K\sum_{i=0}^{M-1}(i+1)\Bigg(e^{-r(T-t)}\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\,\Delta t}\right]^+ 1_{\{t>t_{i+1}\}} \\ &\quad + e^{-r(T-t_{i+1})}\left(\frac{S(t)}{S(t_i)}N(d_1^{(t,t_i)}) - e^{g\,\Delta t}e^{-r(t_{i+1}-t)}N(d_2^{(t,t_i)})\right)1_{\{t_i<t\le t_{i+1}\}} \\ &\quad + e^{-r(T-t)}e^{r\,\Delta t}\left(N(d_1) - e^{(g-r)\,\Delta t}N(d_2)\right)1_{\{t\le t_i\}}\Bigg). \end{aligned}$$

□

Several remarks concerning Proposition 2.2.8 are necessary. The considered payoff structure contains in total M European options. At time t, the realizations of those options whose payoff are conditioned on the price processes $\frac{S(t_{i+1})}{S(t_i)}$, $t > t_{i+1}$, are already observable in the financial market. Hence, it is not necessary to build an expectation on these options. On the contrary, for those options who payoff are conditioned on the price processes $\frac{S(t_{i+1})}{S(t_i)}$, $t < t_i$, i.e., both $S(t_i)$ and $S(t_{i+1})$ are still not observable on the market, therefore, expectations should be taken for the fraction $\frac{S(t_{i+1})}{S(t_i)}$:

$$E^*\left[e^{-r(T-t)}\frac{S(t_{i+1})}{S(t_i)}\Big|\mathcal{F}_t\right] = E^*\left[e^{-r(T-t)}e^{r\Delta t}E^*\left[e^{-r\Delta t}\frac{S(t_{i+1})}{S(t_i)}\Big|\mathcal{F}_{t_i}\right]\Big|\mathcal{F}_t\right] = e^{-r(T-t)}e^{r\Delta t}.$$

The time t–value of all of these options do not depend on the asset price S_t. Exclusively for the call option with $t_i < t \leq t_{i+1}$, the time t–value for this option depends on the asset price t. This is due to the fact that $S(t_i)$ is already observable at time t, while $S(t_{i+1})$ not. The validity of the expectation $E^*[e^{-r(t_{i+1}-t)}S(t_{i+1})|\mathcal{F}_t] = S(t)$ leads to the above result. In particular, for $t = 0$, all of the M options are not realized, and we obtain the following simple expression for the initial price of this pure endowment product:

$$F(t_0,T,S) = {}_Tp_x\left[e^{-r(T-t_0)}G + \alpha K\sum_{i=0}^{M-1}(i+1)e^{-r(t_{M-1}-t_0)}\left(N(d_1) - e^{(g-r)\,\Delta t}N(d_2)\right)\right] \quad (2.12)$$

Before we come to the derivation of the risk–minimizing hedging strategy, let us have a short look at how to determine the fair premium for this contract.

Definition 2.2.9 (Fair premium principle)**.** *A premium is called fair if the accumulated expected discounted premium is equal to the accumulated expected discounted payments of the contract under the equivalent martingale measure under consideration of insurance risk.* [6]

As mentioned in Chapter 1, premiums can be provided either as a single premium which is usually paid at the initial time of the contract or periodically. In case of a single premium, the initial value of the final payoff given in Equation (2.12) is the only candidate for the initial fair premium. In case of a periodic premium, it can be determined either explicitly or by an implicit relation between the guaranteed amount G and the participation rate α.

Proposition 2.2.10 (Fair participation rate $\alpha^*(G)$)**.** *In case of a periodic premium, a fair combination of α and G for the considered contract payoff given in Equation (2.11) results from the fair premium principle as follows:*

$$\alpha^*(G) = \frac{K\sum_{i=0}^{M-1} e^{-rt_i}\,{}_{t_i}p_x - {}_Tp_x\,e^{-r(T-t_0)}\,G}{{}_Tp_x\,K\sum_{i=0}^{M-1}(i+1)e^{-r(t_{M-1}-t_0)}\left(N(d_1) - e^{(g-r)\Delta t}N(d_2)\right)}.$$

[6] Due to the deterministic interest rate, the expectations are taken under the equivalent martingale measure. If a stochastic term structure of the interest rate is taken into account, forward–risk–adjusted measure is applied.

Fair parameter combinations (G^*, α^*)

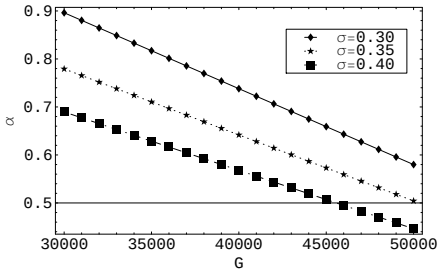

Figure 2.1: Fair $\alpha - G$ combinations with parameters: $r = 0.05$, $g = 0.0275$, $x = 35$, $M = 30$, $K = 1000$ and the other parameters are given in P12.

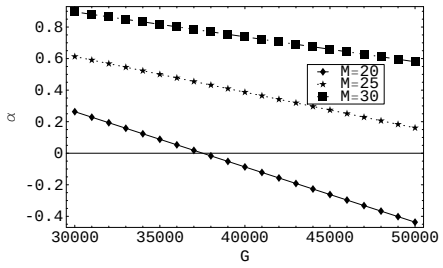

Figure 2.2: Fair $\alpha - G$ combinations with parameters: $r = 0.05$, $g = 0.0275$, $x = 35$, $\sigma = 0.3$, $K = 1000$ and the other parameters are given in P12.

PROOF: On the one hand, assume a periodic premium K are provided by the insured at a set of equidistant time points $\{0 = t_0 < t_1 < \cdots < t_{M-1}\}$, as long as he is still alive at that point. I.e., due to the independence assumption between the financial market and insurance risk, the expected discounted accumulated contributions of the insured is determined by

$$E^* \left[\sum_{i=0}^{M-1} e^{-r\,t_i} K 1_{\{\tau^x > t_i\}} \right] = K \sum_{i=0}^{M-1} e^{-rt_i}\ {}_{t_i}p_x.$$

On the other hand, the expected discounted payout of the contract is given by the value in Equation (2.12). Equating these two parts and solving the equation with respect to α, we reach the desired result. □

Apparently, there exists a negative relation between the participation rate α and the guaranteed amount G. Since the contract value goes up with an increase in the guaranteed amount or a higher participation in the periodic surpluses, in the viewpoint of a fair contract, a tradeoff between these two parameters results. Since the European call option goes up with the σ value, which then leads to a higher contract value, there exists a negative relation between the participation rate and the volatility of the asset. These two effects are observed in Figure 2.1. Furthermore, the duration of the contract M has diverse effects. As M goes up, on the one side, more premiums are accumulated, but on the other side, more participation in the surpluses becomes possible. For the given parameters, the longer the contract, the higher the resulting fair participation rate. This relation is demonstrated graphically in Figure 2.2.

Now we proceed with the derivation of the risk–minimizing hedging strategy. From the derived price of the contingent claim in Proposition 2.2.8, we take the derivative with

respect to S_t and obtain for $t \in]t_i, t_{i+1}]$:

$$F_s(t,T,S) = \frac{\partial F(t,T,S)}{\partial S_t} = \alpha(i+1)\, K\, e^{-r(T-t_{i+1})} \frac{1}{S_{t_i}} N(d_1^{(t,t_i)}). \tag{2.13}$$

Therefore, in this pure endowment life insurance, the risk–minimizing hedging strategy at $t \in]t_i, t_{i+1}]$ is determined as follows:

$$\begin{aligned} \xi_t^* &= (n - N_{t-}^x)\, {}_{T-t}p_{x+t}\, \alpha\, (i+1)\, K e^{-r(T-t_{i+1})} \frac{1}{S_{t_i}} N(d_1^{(t,t_i)}) \\ \eta_t^* &= V_t^* - \xi_t^* S_t^* \\ &= (n - N_t^x)\, {}_{T-t}p_{x+t}\, F_s(t,T,S) - \xi_t^* S_t^*. \end{aligned}$$

Furthermore, the intrinsic risk is given as follows:

$$\begin{aligned} R_t(\phi) &= (n - N_t^x) \int_t^T E^*[(v_u^H)^2 | \mathcal{F}_t]\, {}_{u-t}p_{x+t}\, \mu_{x+u}\, d\, u \\ &= (n - N_t^x) \int_t^T E^*[(-B_u^{-1} F(u,T,S)\, {}_{T-u}p_{x+u})^2 | \mathcal{F}_t]\, {}_{u-t}p_{x+t}\, \mu_{x+u}\, d\, u \\ &= (n - N_t^x)\, {}_{T-t}p_{x+t} \int_t^T E^*[(-B_u^{-1} F(u,T,S))^2 | \mathcal{F}_t]\, {}_{T-u}p_{x+u}\, \mu_{x+u}\, d\, u. \end{aligned}$$

In particular, the initial intrinsic risk has a form of

$$R_{t_0}(\phi) = n\ {}_Tp_x \int_{t_0}^T E^*[(-B_u^{-1} F(u,T,S))^2 | \mathcal{F}_{t_0}]\ {}_{T-u}p_{x+u}\, \mu_{x+u}\ d\, u.$$

In order to give you an idea how to derive the risk–minimizing hedging strategy for a term insurance contract, a very simple example is illustrated. In a term insurance contract, the payout time of the contract is conditional on the death of the insured. Assume, At a premature death time $u < T$, the contract pays out $g(u, S_u)$ which is given by

$$\max\left\{ K \frac{S(u)}{S(t_0)}, K e^{g\, u} \right\} = K e^{g\, u} + K \left[\frac{S(u)}{S(t_0)} - e^{g\, u}) \right]^+.$$

In such a contract specification, K can be understood as a single premium determined at the contract–issuing time. This premium is connected with a reference portfolio S, i.e., eventually the insurer can buy $\frac{K}{S(t_0)}$ portfolio S. The insured obtains either the premium accumulated with a guaranteed interest rate g or the value of portfolio at time u, whichever is larger. In the Black–Scholes Economy, the time t–value ($t < u$) of this payoff is determined by

$$\begin{aligned} F^g(t,u,S_t) &= E^*[e^{-r(u-t)} g(u,S_u) | \mathcal{F}_t] = K e^{gu} e^{-r(u-t)} N(-d_2^{(u,t)}) + K \frac{S(t)}{S(t_0)} N(d_1^{(u,t)}) \\ d_{1/2}^{(u,t)} &= \frac{\ln\left(\frac{S(t)}{S(t_0)}\right) - g\, u + (r \pm \frac{1}{2}\sigma^2)(u-t)}{\sigma\sqrt{u-t}}. \end{aligned}$$

Applying the result of Proposition 2.2.7 leads to the following risk–minimizing hedging strategy for this illustrative term insurance payoff:

$$\begin{aligned}
\xi_t^* &= (n - N_{t-}^x) \int_t^T \frac{K}{S(t_0)} N(d_1^{(u,t)})\, {}_{u-t}p_{x+t}\, \mu_{x+u}\, d\, u \\
\eta_t^* &= \int_0^t g(u, S_u)\, B_u^{-1}\, dN_u^x + (n - N_t^x) \int_t^T B_t^{-1}\, F^g(t, u, S_t)\, {}_{u-t}p_{x+t}\, \mu_{x+u}\, d\, u - \xi_t^* S_t^*,
\end{aligned}$$

for $0 \leq t \leq T$. In addition, the intrinsic risk process is determined by

$$\begin{aligned}
R_t(\phi) &= (n - N_t^x) \int_t^T E^* \left[\left(g(u, S_u) B_u^{-1} - \int_u^T F^g(t, \tau, S_t) B_t^{-1}\, {}_{\tau-t}p_{x+t}\, \mu_{x+\tau}\, d\, \tau \right)^2 \Big| \mathcal{F}_t \right] \\
&\quad {}_{u-t}p_{x+t}\ \mu_{x+u}\, d\, u.
\end{aligned}$$

2.3 Quantile and efficient hedging

In a pure endowment contract, the risk–minimizing hedging strategy is described by the product of the expected number of customers who survive the maturity date and the hedge ratio of the pure financial risk. In a term insurance contract, the risk–minimizing hedging strategy is based on the expected number of deaths. To sum up, by using a risk–minimizing hedging, no transfer between the financial and insurance risk is made. In this place, the question is asked whether there exists a hedging method which constrains the financial exposure to some extent and reduces the insurance risk, or which allows a transfer between the financial and insurance risk. This section is designed to answer this question. For this purpose, quantile and efficient hedging come into consideration. Furthermore, the entire study in this section focuses on the pure endowment insurance. And in contrast to the last section, we begin the analysis by considering a specific pure endowment equity–linked insurance contract. In particular, efforts are made to derive the implicit survival probability of this specific pure endowment contract by applying the quantile and efficient hedging methods. In other words, here, we do not take the mortality risk explicitly as given or follow a certain mortality function but consider it implicitly and try to answer the question whether it is possible to make some transfer between the financial and insurance risk.

Quantile and efficient hedging are two concepts introduced in Föllmer and Leukert (1999, 2000), where the former concept can be expressed as a special case of the second one. This argument will be proven for our specific contract later, too. The investors have usually two incentives to use these hedging methods: they are either unwilling to spend so much capital which is needed for perfect hedges (complete market) or super hedges (incomplete market) or ready to take some risks given a certain shortfall probability. The purpose of our analysis here belongs to the latter category. In this place, two recent works by Melnikov (2004a, b) should be mentioned. To our knowledge, he is the first one that applies the concepts of quantile and efficient hedging in equity–linked life insurance. He investigates efficient hedging methodology for a contract with a flexible guarantee in the

framework of Black and Scholes (1973). The difference between Melnikov (2004a, b) and the analysis here lies in the considered contingent claims. The contract considered by Melnikov is a simple European call option whose final payoff depends on the final asset price S_T only, while here we attempt to quantile/efficient hedge a sequence of European call options which is based on the entire development of the asset price.

Again, we consider the equity–linked life insurance contract with a guarantee given in Equation (2.11), which owns the form of

$$g(t_M, S) \;=\; G + \alpha \sum_{i=0}^{M-1} (i+1) K \left[\frac{S(t_{i+1})}{S(t_i)} - e^{g(t_{i+1}-t_i)} \right]^+ ,$$

if the customer survives the maturity of the contract. In order to simplify matters, several assumptions are made.

- Equidistant time periods are assumed and Δt is used the time different between two periods.
- The interest rate is assumed to be zero.
- The asset price is driven by a one–dimensional Brownian motion.
- It is assumed that the mortality and financial risk are independent.

The goal of quantile or efficient hedging is either to control the shortfall probability or to constrain the expected loss. In order to consider these risk measures, it only makes sense to use the real world market measure instead of the equivalent martingale measure. Under the real world measure, the asset price is assumed to follow a geometric Brownian motion:

$$dS_t = S_t(\mu\, dt + \sigma\, dW_t),$$

where μ is the rate of return and $(W_t)_{t\in[0,T]}$ is a Brownian motion under the market measure P. Under the market measure, $\frac{S(t_{i+1})}{S(t_i)}$ can be expressed as

$$\frac{S(t_{i+1})}{S(t_i)} \;=\; \exp\left\{ \left(\mu - \frac{1}{2}\sigma^2 \right) \Delta t + \sigma (W_{t_{i+1}} - W_{t_i}) \right\}.$$

I.e., $\frac{S(t_{i+1})}{S(t_i)}$ depends on $W_{t_{i+1}} - W_{t_i}$ exclusively. Due to the fact that the increments of the Brownian motion $W_{t_{i+1}} - W_{t_i}$ for $i = 0, \cdots, M-1$ are independent, we can quantile/efficient hedge the sequence of the European call options and consequently the considered payoff by using a two–step procedure. Namely, we quantile/efficient hedge each European call option $\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\,\Delta t} \right]^+$, $i = 0, \cdots, M-1$, separately and then take the sum of all the values caused by quantile/efficient hedging.

Through the Radon–Nikodym density

$$\frac{dP^*}{dP}\Big|_{\mathcal{F}_T} = \exp\left\{-\frac{\mu}{\sigma}W_T - \frac{1}{2}\frac{\mu^2}{\sigma^2}T\right\}, \tag{2.14}$$

the equivalent martingale measure P^* is defined, namely, $W_T^* = W_T + \frac{\mu}{\sigma}T$. Under the equivalent martingale measure, the price process $\left(\frac{S(t_{i+1})}{S(t_i)}\right)$ can be expressed as

$$\frac{S(t_{i+1})}{S(t_i)} = \exp\left\{-\frac{1}{2}\sigma^2\,\Delta t + \sigma(W^*_{t_{i+1}} - W^*_{t_i})\right\}. \tag{2.15}$$

According to Proposition 2.2.8, the arbitrage price of this contingent claim at time zero is obtained as follows:

$$\begin{aligned}
{}_Tp_x \cdot \Pi_0 &= {}_Tp_x\, G + {}_Tp_x\,\alpha\, K \sum_{i=0}^{M-1}(i+1)\; E^*\left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\,\Delta t}\right]^+ \Big| \mathcal{F}_{t_0}\right] \\
&= {}_Tp_x\, G + {}_Tp_x\,\alpha\, K \sum_{i=0}^{M-1}(i+1)\; E^*\left[E^*\left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\,\Delta t}\right]^+ \Big| \mathcal{F}_{t_i}\right] \Big| \mathcal{F}_{t_0}\right] \\
&= {}_Tp_x\, G + {}_Tp_x\,\alpha\, K \sum_{i=0}^{M-1}(i+1)\; E^*\left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\,\Delta t}\right]^+ \Big| \mathcal{F}_{t_i}\right] \\
&= {}_Tp_x\, G + {}_Tp_x\,\alpha\, K \sum_{i=0}^{M-1}(i+1)\,(N(d_1) - e^{g\Delta t}N(d_2)) \\
&= {}_Tp_x\, G + {}_Tp_x\,\alpha\, K\,\frac{M(M+1)}{2}\,(N(d_1) - e^{g\Delta t}N(d_2)).
\end{aligned} \tag{2.16}$$

In the above derivation, the third equality holds because a zero-interest-rate economy is assumed.

2.3.1 Quantile hedging

In a quantile hedging methodology, the hedging is implemented by two steps: first, find a modified contingent claim; second, super/perfect–hedge this modified claim. The problem of a quantile hedging is formulated as follows: to construct an admissible strategy ϕ^* such that $V_T(\phi^*)$ is close enough to $H = f(T,S)1_{\{\tau^x > T\}}$, i.e.,

$$P(V_T(\phi^*) \geq H) = \max_{\phi} P(V_T(\phi) \geq H)$$

under the initial–capital–constraint

$$V_0(\phi) \leq {}_Tp_x \cdot \Pi_0 \leq \Pi_0.$$

How to transform this idea to our specific contract is the main concern in the rest of this subsection. Since $S_{t_{i+1}}/S_{t_i}$, $i = 0, \cdots M-1$ are independently and identically distributed,

it is sufficient to investigate how to quantile hedge one single European call option. I.e., every single European call option can be considered as a one–period option and it begins at time t_i and matures at t_{i+1}. From Equation (2.16), we obtain

$$\begin{aligned} {}_Tp_x \frac{2(\Pi_0 - G)}{\alpha K M(M+1)} &= {}_Tp_x \; E^* \left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t} \right]^+ \Big| \mathcal{F}_{t_i} \right] \\ &= {}_Tp_x \; E^* \left[\left[\frac{S(t_1)}{S(t_0)} - e^{g\Delta t} \right]^+ \Big| \mathcal{F}_{t_0} \right]. \end{aligned} \tag{2.17}$$

Therefore, the value in the left–hand side of Equation (2.17) is the bound of the initial available capital for the each relevant call option. In a quantile hedging, a maximal success set which satisfies the above requirements, i.e., it makes the value of the admissible strategy be close enough to the considered contingent claim and the initial capital requirement stay below the constraint. As mentioned, the goal of this section is to analyze the mortality risk implicitly. In other words, we are looking for the survival probability induced by the maximal success set:

$${}_Tp_x^* \; E^* \left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t} \right]^+ \Big| \mathcal{F}_{t_i} \right] = E^* \left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t} \right]^+ \Big| \mathcal{F}_{t_i} \right] \mathbf{1}_{A^*},$$

where A^* is the maximal success set. Before we discuss this set in detail, we come to the following reformulation at first:

$$\begin{aligned} \frac{\frac{dP}{dP^*}|_{\mathcal{F}_{t_{i+1}}}}{\frac{dP}{dP^*}|_{\mathcal{F}_{t_i}}} &= \exp\left\{ \frac{\mu}{\sigma}(W^*_{t_{i+1}} - W^*_{t_i}) - \frac{1}{2}\frac{\mu^2}{\sigma^2}\,\Delta t \right\} \\ &= \exp\left\{ \frac{\mu}{\sigma} \cdot \frac{1}{\sigma} \left(\ln\left(\frac{S(t_{i+1})}{S(t_i)} \right) + \frac{1}{2}\sigma^2 \Delta t \right) - \frac{1}{2}\frac{\mu^2}{\sigma^2}\Delta t \right\} \\ &= \exp\left\{ \frac{1}{2}\mu\,\Delta t - \frac{1}{2}\frac{\mu^2}{\sigma^2}\Delta t \right\} \left(\frac{S(t_{i+1})}{S(t_i)} \right)^{\frac{\mu}{\sigma^2}} \\ &:= \text{constant} \cdot \left(\frac{S(t_{i+1})}{S(t_i)} \right)^{\frac{\mu}{\sigma^2}} \end{aligned} \tag{2.18}$$

The second step above is obtained by solving Equation (2.15) with respect to $W^*_{t_{i+1}} - W^*_{t_i}$. According to Föllmer and Leukert (1999), the **maximal** success set is given by

$$\left\{ \frac{\frac{dP}{dP^*}|_{\mathcal{F}_{t_{i+1}}}}{\frac{dP}{dP^*}|_{\mathcal{F}_{t_i}}} \geq \text{constant} \cdot \left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t} \right]^+ \right\}. \tag{2.19}$$

Due to the above reformulation of $\frac{\frac{dP}{dP^*}|_{\mathcal{F}_{t_{i+1}}}}{\frac{dP}{dP^*}|_{\mathcal{F}_{t_i}}}$, the maximal success set is transformed into

$$\left\{ \left(\frac{S(t_{i+1})}{S(t_i)} \right)^{\frac{\mu}{\sigma^2}} \geq \text{constant} \cdot \left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t} \right]^+ \right\}. \tag{2.20}$$

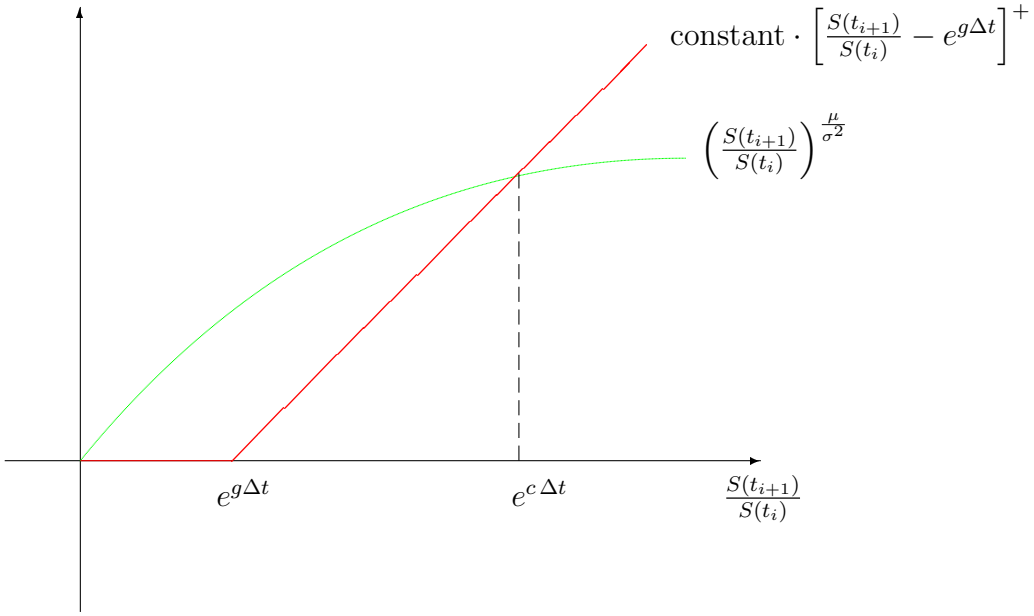

Figure 2.3: $\left(\frac{S(t_{i+1})}{S(t_i)}\right)^{\frac{\mu}{\sigma^2}}$ and constant · $\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+$ for $\mu \leq \sigma^2$

Of course these two constants in (2.19) and (2.20) are not the same. The latter one is the former one times $\exp\left\{-\frac{1}{2}\,\mu\,\Delta t + \frac{1}{2}\,\frac{\mu^2}{\sigma^2}\,\Delta t\right\}$. For the case of $\underline{\mu \leq \sigma^2}$, which implies that $\left(\frac{S(t_{i+1})}{S(t_i)}\right)^{\frac{\mu}{\sigma^2}}$ is a concave function of $\left(\frac{S(t_{i+1})}{S(t_i)}\right)$, the equation

$$\left(\frac{S(t_{i+1})}{S(t_i)}\right)^{\frac{\mu}{\sigma^2}} \quad = \quad \text{constant} \cdot \left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+ \tag{2.21}$$

has only one solution (here we consider $\frac{S(t_{i+1})}{S(t_i)}$ as a variable). Assume $e^{c\,\Delta t}$ is the solution of the above equation, [7] then the **maximal** success set in (2.20) is equivalent to

$$\left\{\frac{S(t_{i+1})}{S(t_i)} \leq e^{c\,\Delta t}\right\},$$

c.f. Figure 2.3.1. The green curve is above the red one for the area of $\left\{\frac{S(t_{i+1})}{S(t_i)} \leq e^{c\,\Delta t}\right\}$.

[7] Due to the lognormally distributed asset price, it makes sense to choose this expression.

Hence,

$$\begin{aligned} & E^*\left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+ \mathbf{1}_{A^*}\middle|\mathcal{F}_{t_i}\right] \\ = \; & E^*\left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+ \mathbf{1}_{\left\{\frac{S(t_{i+1})}{S(t_i)} \le e^{c\,\Delta t}\right\}}\middle|\mathcal{F}_{t_i}\right] \\ = \; & N\left(\frac{(c-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right) - N\left(\frac{(g-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right) \\ & -e^{g\Delta t}N\left(\frac{(c+\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right) + e^{g\Delta t}N\left(\frac{(g+\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right). \end{aligned} \tag{2.22}$$

In the quantile hedging, the financial risk is described by the shortfall probability, i.e., the hedger constrains the shortfall probability (under the market measure) to a certain level and strives for a goal by regulating the insurance risk. Now we let ϵ denote this constrained shortfall probability. Usually c is determined by the level of ϵ, i.e.,

$$P(A^*) = P\left(\frac{S(t_{i+1})}{S(t_i)} \le e^{c\Delta t}\right) = 1-\epsilon.$$

In addition, it holds

$$\begin{aligned} P\left(\frac{S(t_{i+1})}{S(t_i)} \le e^{c\,\Delta t}\right) &= P\left(\exp\left\{\left(\mu-\frac{1}{2}\sigma^2\right)\Delta t + \sigma(W_{t_{i+1}} - W_{t_i})\right\} \le e^{c\,\Delta t}\right) \\ &= N\left(\frac{(c-\mu+\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right). \end{aligned}$$

Consequently, we obtain

$$c = \frac{1}{\Delta t}\left(N^{-1}(1-\epsilon)\sigma\sqrt{\Delta t} + (\mu - \frac{1}{2}\sigma^2)\Delta t\right) \tag{2.23}$$

with $N^{-1}(1-\epsilon) = \{x \in \mathbb{R} | N(x) = 1-\epsilon\}$. That is, the concrete value of $E^*\left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+ \mathbf{1}_{A^*}\middle|\mathcal{F}_{t_i}\right]$ is derived for a given ϵ, which accordingly results in the following proposition.

Proposition 2.3.1 (Implied survival probability)**.** *Given that the financial risk is characterized by the constrained shortfall probability ϵ the insurer wishes, i.e., $P\left(\frac{S(t_{i+1})}{S(t_i)} \le e^{c\,\Delta t}\right) = 1-\epsilon$, for the case of $\mu < \sigma^2$, the implied survival probability resulting from quantile hedging for the contract given in Equation (2.11) has a form of*

$${}_Tp_x^* = \frac{E^*\left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+ \mathbf{1}_{A^*}\middle|\mathcal{F}_{t_i}\right]}{E^*\left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+ \middle|\mathcal{F}_{t_i}\right]}.$$

μ	${}_Tp_x^*$	σ	${}_Tp_x^*$	g	${}_Tp_x^*$	ϵ	${}_Tp_x^*$
$\mu = 0.03$	0.702308	$\sigma = 0.3$	0.746807	$g = 0.02$	0.746807	$\epsilon = 0.01$	0.93566
$\mu = 0.05$	0.732469	$\sigma = 0.4$	0.704046	$g = 0.03$	0.739995	$\epsilon = 0.02$	0.882329
$\mu = 0.07$	0.760644	$\sigma = 0.5$	0.665453	$g = 0.04$	0.732897	$\epsilon = 0.03$	0.833927
$\mu = 0.09$	0.786803	$\sigma = 0.6$	0.628399	$g = 0.05$	0.725502	$\epsilon = 0.04$	0.788996

Table 2.1: Implied survival probability with parameters: $\epsilon = 0.05$, $T = t_M = 12$, $g = 0.02$, $\mu = 0.06$, $\sigma = 0.3$

with $E^*\left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+ \mathbf{1}_{A^*}\middle|\mathcal{F}_{t_i}\right]$ *determined in Equation (2.22) and* c *given in Equation (2.23).*

It is noticed that by choosing the size of ϵ, the insurer chooses how much financial risk he is willing to bear. At the same time, the ϵ determines the c–value, which consequently affects the size of the implied survival probability. Thus, by setting the ϵ–value, the insurer aims at controlling the value of the implied survival probability, namely, a transfer between the financial and insurance risk becomes possible. On some level, we always obtain an optimal survival probability for a given financial risk. The lower the resulting optimal survival probability is, the more old and safe customers should the insurer acquire. On the contrary, the higher the resulting optimal survival probability is, the more younger customers are allowed to be taken. Furthermore, if an increase in ϵ leads to a decrease in ${}_Tp_x^*$, it implies there exists some transfer between the financial and insurance risk. In other words, when the insurer takes more financial risk, as a compensation, more safe old customers are preferred by the insurer.

In the following some numerical results for the implied survival probability are calculated. Table 2.1 demonstrates how the implied survival probability changes with μ, σ and g. There are some obvious effects, e.g., the positive influence of μ on the considered probability and the negative effect of σ and g on the probability. As μ goes up, in the expectation, the insurer's paying ability goes up, and it allows the insurer to accept more younger customers. While σ and g has reversed effects on the implied survival probability. Furthermore, it is observed that the implied survival probability decreases with the significance level ϵ for a given T. As ϵ goes up, i.e, the hedger (the insurance company) is ready to take more risk and hedge with a smaller success probability $(1 - \epsilon)$, as a compensation, a smaller implied survival probability results. That means, some safer old customers should be chosen because with a higher probability they are not going to survive the maturity date, which is good for the insurer who just issues pure endowment contracts. Thereby you see that the insurance company transfers part of its financial risks to insurance risks.

Definition 2.3.2 (Reduction level)**.** *Let* ${}_Tp_x$ *denote the empirical survival probability the*

	$\epsilon = 0.01$	$\epsilon = 0.03$	$\epsilon = 0.05$
$T = 12$	4.24894	14.6598	23.5753
$T = 18$	2.04975	12.6997	21.8200
$T = 24$	0.00000	9.2789	18.7565

Table 2.2: Reduction level (%) with parameters: $g = 0.02$, $\mu = 0.06$, $\sigma = 0.3$, $x = 30$.

insurer uses for pricing and hedging. By using the quantile hedging method, there is a reduction in the premium and this reduction level is given by

$$\textit{reduction level} = 1 - \frac{E^*\left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+ \mathbf{1}_{A^*}\middle|\mathcal{F}_{t_i}\right]}{{}_Tp_x\ E^*\left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+ \middle|\mathcal{F}_{t_i}\right]} = 1 - \frac{{}_Tp_x^*}{{}_Tp_x}.$$

Apparently, the resulting premium from using quantile hedging method owns a smaller value than that determined by arbitrage pricing because a certain shortfall probability is permitted in the quantile hedging. The reduction level indicates how many percents this premium is reduced by applying quantile hedging method. Only this empirical survival probability coincides with the above calculated implied one, no essential reductions occur. In the following we use the benchmark death distribution as an empirical mortality introduced in Subsection 1.4.3 of Chapter 1 and some numbers are given in Table 2.2. Obviously the premium is reduced to a much bigger extent as the insurance company is permitted to take more risk, namely the risk is increased that the company will fail to hedge successfully. While for given ϵ–values, the reduction becomes less apparent as the maturity of the contract is lengthened. Combined with small ϵ's, say $\epsilon = 0.01$, almost no reduction of the premium is possible.

If $\underline{\mu > \sigma^2}$, then $\left(\frac{S(t_{i+1})}{S(t_i)}\right)^{\frac{\mu}{\sigma^2}}$ is a convex function of $\frac{S(t_{i+1})}{S(t_i)}$. In this case, the Equation (2.21) might have one, two or no solutions. If there is no solution, i.e., the function $\left(\frac{S(t_{i+1})}{S(t_i)}\right)^{\frac{\mu}{\sigma^2}}$ lies always above the constant $\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+$, i.e.,

$$E^*\left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+ \mathbf{1}_{A^*}\middle|\mathcal{F}_{t_i}\right] \quad = \quad E^*\left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+ \middle|\mathcal{F}_{t_i}\right].$$

Obviously, it contradicts the initial investment constraints. If there is only one solution, the analysis is in analogy to the case of $\mu \leq \sigma^2$. If there are two solutions, say $e^{c_1\,\Delta t}$ and $e^{c_2\,\Delta t}$ and we assume $c_1 < c_2$, then the maximal success set can be described

$$\left\{\frac{S(t_{i+1})}{S(t_i)} \leq e^{c_1\Delta t}\right\} \qquad \text{or} \qquad \left\{\frac{S(t_{i+1})}{S(t_i)} \geq e^{c_2\Delta t}\right\}.$$

In the case, the price of the call option under the quantile hedging is given by

$$\begin{aligned} & E^*\left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+ \mathbf{1}_{A^*}\middle|\mathcal{F}_{t_i}\right] \\ = \; & N\left(\frac{-(g-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right) - e^{g\Delta t}N\left(\frac{-(g+\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right) - N\left(\frac{(c_2-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right) \\ & +N\left(\frac{(c_1-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right) + e^{g\Delta t}N\left(\frac{(c_2+\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right) - e^{g\Delta t}N\left(\frac{(c_1+\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right). \end{aligned}$$

As a consequence, the implied survival probability can be determined by the quotient of the above expression and the price of the call option. And again c_1 and c_2 can be determined by the shortfall probability the insurer chooses.

2.3.2 Efficient hedging with a power loss function

In the following we consider a more general hedging method, namely efficient hedging with a power loss function:

$$l(y) = y^p, \quad y \geq 0,\, p > 0.$$

In this case the optimal strategy ϕ^* for a given contingent claim H is defined

$$E[l([H - V_T(\phi^*)]^+)] = \min_{\phi} E[l([H - V_T(\phi)]^+)].$$

Again ϕ are all self–financing strategies with nonnegative values satisfying the budget restriction

$$V_0(\phi) \leq {}_Tp_x \cdot E^*[H].$$

Efficient hedging proceeds similarly to quantile hedging, i.e., after a modified claim is found, super/perfect this modified claim. Föllmer and Leukert (2000) show that the solution for the efficient hedge exists and coincides with a perfect hedge for a modified contingent claim H_P with the following structure:

$$\begin{aligned} H_P &= H - \alpha_p\left(\frac{dP^*}{dP}\right)^{\frac{1}{p-1}} \wedge H, && p > 1, \\ H_P &= H1_{\{\frac{dP}{dP^*} > \alpha_p H^{1-p}\}}, && 0 < p < 1, \\ H_P &= H1_{\{\frac{dP}{dP^*} > \alpha_p\}}, && p = 1. \end{aligned}$$

Overall α_p is determined by the equation $E^*[H_P] = {}_Tp_xE^*[H]$. It is observed that three different cases are distinguished according to the p–values. The implicit survival probability has the form of

$${}_Tp_x = \frac{E^*\left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+ \middle|\mathcal{F}_{t_i}\right]_P}{E^*\left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+ \middle|\mathcal{F}_{t_i}\right]}.$$

The notation P in the nominator is taken to denote the modified contingent claim as in H_P. For the case of $p = 1$, at the first sight, the resulting expression for the modified contingent claim is different from the one obtained in the quantile hedging, but we prove these two expressions are the same later. I.e., we are back to the quantile hedging.

We start with the case of $p > 1$. For this purpose, the expression of dP^*/dP is needed. According to Equation (2.14) and the analogous derivation in Equation (2.18), the Random–Nikodym density can be reformulated into

$$\frac{\frac{dP^*}{dP}|_{\mathcal{F}_{t_{i+1}}}}{\frac{dP^*}{dP}|_{\mathcal{F}_{t_i}}} = \exp\left\{\frac{1}{2}\frac{\mu^2}{\sigma^2}\Delta t - \frac{1}{2}\mu\Delta t\right\}\left(\frac{S(t_{i+1})}{S(t_i)}\right)^{-\frac{\mu}{\sigma^2}} := R\cdot\left(\frac{S(t_{i+1})}{S(t_i)}\right)^{-\frac{\mu}{\sigma^2}}$$

Substituting this density in the modified contingent claim, we obtain

$$\begin{aligned} H_P &= H - \alpha_p\cdot\left[R\left(\frac{S(t_{i+1})}{S(t_i)}\right)^{-\frac{\mu}{\sigma^2}}\right]^{\frac{1}{p-1}} \wedge H \\ &= \left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+ - \alpha_p\, R^{\frac{1}{p-1}}\left(\frac{S(t_{i+1})}{S(t_i)}\right)^{-\frac{\mu}{\sigma^2(p-1)}} \wedge \left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+ \end{aligned} \quad (2.24)$$

Lemma 2.3.3. *We claim that*

(a) The value of α_p given in Equation (2.24) is larger than 0.

(b) If $e^{c\,\Delta t}$ is the solution of $\frac{S(t_{i+1})}{S(t_i)}$ of the equation $\alpha_p\, R^{\frac{1}{p-1}}\left(\frac{S(t_{i+1})}{S(t_i)}\right)^{-\frac{\mu}{\sigma^2(p-1)}} = \left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+$, then it holds $\alpha_p = (e^{c\,\Delta t} - e^{g\Delta t})^+ R^{\frac{-1}{p-1}}\cdot\exp\left\{\frac{c\,\mu\,\Delta t}{\sigma^2(p-1)}\right\}$.

Proof:

(a) Assume that α_p is smaller than 0, then it follows

$$\alpha_p\, R^{\frac{1}{p-1}}\left(\frac{S(t_{i+1})}{S(t_i)}\right)^{-\frac{\mu}{\sigma^2(p-1)}} \wedge \left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+ = \alpha_p\, R^{\frac{1}{p-1}}\left(\frac{S(t_{i+1})}{S(t_i)}\right)^{-\frac{\mu}{\sigma^2(p-1)}}.$$

Consequently, in this case

$$\begin{aligned} E^*[H_P] &= E^*\left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+\right] - \alpha_p\, R^{\frac{1}{p-1}}\, E^*\left[\left(\frac{S(t_{i+1})}{S(t_i)}\right)^{-\frac{\mu}{\sigma^2(p-1)}}\right] \\ &> E^*\left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+\right] \\ &\geq {}_Tp_x\; E^*\left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+\right]. \end{aligned}$$

This contradicts the budget condition and implies that the equality that $E^*[H_P] = {}_Tp_x E^*[H]$ cannot hold. Therefore, α_p must be larger than 0.

(b) For $\alpha_p > 0$, a unique solution is found for the equation

$$\alpha_p \, R^{\frac{1}{p-1}} \left(\frac{S(t_{i+1})}{S(t_i)} \right)^{-\frac{\mu}{\sigma^2(p-1)}} = \left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t} \right]^+ ,$$

because $\alpha_p \, R^{\frac{1}{p-1}} \left(\frac{S(t_{i+1})}{S(t_i)} \right)^{-\frac{\mu}{\sigma^2(p-1)}}$ is a convex and decreasing function of $\frac{S(t_{i+1})}{S(t_i)}$. We denote $e^{c\Delta t}$ this solution. Obviously $e^{c\Delta t}$ have to be larger than $e^{g\Delta t}$. Consequently, α_p equals

$$(e^{c\,\Delta t} - e^{g\Delta t})^+ R^{\frac{-1}{p-1}} \cdot \exp\left\{ \frac{c\,\mu\,\Delta t}{\sigma^2(p-1)} \right\}.$$

□

With this cognition, the modified contingent claim H_P is transformed into

$$\begin{aligned} H_P &= \left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t} \right]^+ - \left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t} \right]^+ 1_{\left\{ \frac{S(t_{i+1})}{S(t_i)} \le e^{c\Delta t} \right\}} \\ &\quad -\alpha_p R^{\frac{1}{p-1}} \left(\frac{S(t_{i+1})}{S(t_i)} \right)^{-\frac{\mu}{\sigma^2(p-1)}} 1_{\left\{ \frac{S(t_{i+1})}{S(t_i)} > e^{c\Delta t} \right\}} \\ &= \left(\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t} \right]^+ - (e^{c\Delta t} - e^{g\Delta t})^+ \, e^{\frac{c\,\mu\,\Delta t}{\sigma^2(p-1)}} \left(\frac{S(t_{i+1})}{S(t_i)} \right)^{-\frac{\mu}{\sigma^2(p-1)}} \right) 1_{\left\{ \frac{S(t_{i+1})}{S(t_i)} > e^{c\Delta t} \right\}} \end{aligned}$$

In order to value H_P for the case $p > 1$, we have to calculate

$$\begin{aligned} & E^* \left[\left(\frac{S(t_{i+1})}{S(t_i)} \right)^{-\frac{\mu}{\sigma^2(p-1)}} 1_{\left\{ \frac{S(t_{i+1})}{S(t_i)} > e^{c\Delta t} \right\}} \right] \\ &= E^* \left[\exp\left\{ -\frac{\mu}{\sigma(p-1)} (W^*_{t_{i+1}} - W^*_{t_i}) + \frac{1}{2} \frac{\mu \Delta t}{p-1} \right\} \mathbf{1}_{\left\{ W^*_{t_{i+1}} - W^*_{t_i} > \frac{(c+\frac{1}{2}\sigma^2)\Delta t}{\sigma} \right\}} \right] \\ &= \exp\left\{ \frac{\mu^2 \Delta t + \mu\sigma^2(p-1)\Delta t}{2\sigma^2(p-1)^2} \right\} N\left(-\frac{(c+\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}} - \frac{\mu\sqrt{\Delta t}}{\sigma(p-1)} \right). \end{aligned}$$

Consequently, the price for the modified contingent claim for the case $p > 1$ and $\alpha_p > 0$ is

$$\begin{aligned} E^*[H_P] &= N\left(\frac{-(g-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}} \right) - e^{g\Delta t} N\left(\frac{-(g+\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}} \right) - N\left(\frac{(c-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}} \right) \\ &\quad + N\left(\frac{(g-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}} \right) + e^{g\Delta t} N\left(\frac{(c+\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}} \right) - e^{g\Delta t} N\left(\frac{(g+\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}} \right) \\ &\quad -(c^{g\Delta t} - e^{g\Delta t})^+ \, e^{\frac{c\,\mu\,\Delta t}{\sigma^2(p-1)}} \exp\left\{ \frac{\mu^2 \Delta t + \mu\sigma^2(p-1)\Delta t}{2\sigma^2(p-1)^2} \right\} \\ &\quad N\left(-\frac{(c+\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}} - \frac{\mu\sqrt{\Delta t}}{\sigma(p-1)} \right), \end{aligned}$$

μ	${}_Tp_x^*$	σ	${}_Tp_x^*$	g	${}_Tp_x^*$
$\mu = 0.03$	0.0729241	$\sigma = 0.2$	0.0592240	$g = 0.02$	0.0592240
$\mu = 0.05$	0.0637018	$\sigma = 0.3$	0.0780432	$g = 0.03$	0.0625627
$\mu = 0.07$	0.0548723	$\sigma = 0.4$	0.0968713	$g = 0.04$	0.0661808
$\mu = 0.09$	0.0466340	$\sigma = 0.5$	0.1172620	$g = 0.05$	0.0701061

Table 2.3: Implied survival probability with parameters: $\epsilon = 0.05$, $T = t_M = 12$, $g = 0.02$, $\mu = 0.06$, $\sigma = 0.2, p = 2$.

p	${}_Tp_x^*$	ϵ	${}_Tp_x^*$
$p = 2$	0.0592240	$\epsilon = 0.01$	0.0100341
$p = 3$	0.0515828	$\epsilon = 0.02$	0.0214264
$p = 4$	0.0488637	$\epsilon = 0.03$	0.0335235
$p = 5$	0.0474696	$\epsilon = 0.04$	0.0461523

Table 2.4: Implied survival probability with parameters: $T = t_M = 12$, $g = 0.02$, $\mu = 0.06$, $\sigma = 0.2$, $\epsilon = 0.05$, $p = 2$.

where the critical value $e^{c\Delta t}$ is determined by the equation: ${}_Tp_x = \frac{E^*[H_P]}{E^*[H]}$. Of course this is only possible if the survival probability is already known. Still we can go one step further to determine the efficient hedging strategy by taking the derivative of $E^*[H_P]$ with respect to the stock price. However, here we follow the idea of Melnikov (2004b), i.e., as in the quantile hedging, we mainly analyze the implied survival probability after determining the critical value by fixing a constrained shortfall probability

$$P\left(\frac{S(t_{i+1})}{S(t_i)} \leq e^{c\Delta t}\right) = 1 - \epsilon,$$

i.e.,

$$c = \frac{1}{\Delta t}\left(N^{-1}(1-\epsilon)\sigma\sqrt{\Delta t} + (\mu - \frac{1}{2}\sigma^2)\Delta t\right). \tag{2.25}$$

In Tables 2.3 and 2.4, it is observed that all the implied survival probabilities are quite small and close if the insurance company bears its risk with a power loss function ($p > 1$). This indicates that the company cannot accept many transfers between the financial risk to the insurance risk. Table 2.3 demonstrates how the implied survival probability depends on the market return of the asset, its volatility and the strike parameter g. Compared to the quantile case, it is observed that all the effects of these parameters are reversed. In

the quantile hedging, the survival probability is given by the ratio

$$\frac{E^*\left[\left(\frac{S(t_{i+1})}{S(t_i)}-e^{g\Delta t}\right)^+ 1_{\left\{\frac{S(t_{i+1})}{S(t_i)}<e^{c\Delta t}\right\}}\right]}{E^*\left[\left(\frac{S(t_{i+1})}{S(t_i)}-e^{g\Delta t}\right)^+\right]},$$

which decreases with g. On the contrary, in the efficient hedging ($p > 1$), it is given by the sum of a similar ratio which is conditional on a counter–event and a term which increases with g. Therefore, a rise in g leads to a rise in the survival probability. Furthermore, a higher p value leads to a smaller implied survival probability. As p goes up, the degree of risk aversion increases, as a compensation, the insurance company would rather choose some safer older customers than young customers. The effects of ϵ on the considered survival probability are listed in Table 2.4. As ϵ goes up, more financial risk is borne, unexpectedly, the insurance company will sign contracts with young clients. Hence, no transfer from the financial risk to the mortality risk is possible.

We proceed with the second case $\underline{0 < p < 1}$, where the modified contingent claim for our case is of the form of

$$H_P = H 1_{\left\{\frac{dP}{dP^*}|_{\mathcal{F}_{t_{i+1}}}/\frac{dP}{dP^*}|_{\mathcal{F}_{t_i}}>\alpha_p H^{1-p}\right\}}$$

with

$$\frac{\frac{dP}{dP^*}|_{\mathcal{F}_{t_{i+1}}}}{\frac{dP}{dP^*}|_{\mathcal{F}_{t_i}}} = \exp\left\{\frac{1}{2}\mu\Delta t - \frac{1}{2}\frac{\mu^2}{\sigma^2}\Delta t\right\}\left(\frac{S(t_{i+1})}{S(t_i)}\right)^{\frac{\mu}{\sigma^2}} := R_1 \cdot \left(\frac{S(t_{i+1})}{S(t_i)}\right)^{\frac{\mu}{\sigma^2}}$$

The event under the indicator function is equivalent to

$$\left\{\frac{\frac{dP}{dP^*}|_{\mathcal{F}_{t_{i+1}}}}{\frac{dP}{dP^*}|_{\mathcal{F}_{t_i}}} > \alpha_p H^{1-p}\right\} = \left\{(R_1\cdot\alpha_p)^{-\frac{1}{1-p}}\left(\frac{S(t_{i+1})}{S(t_i)}\right)^{\frac{\mu}{\sigma^2(1-p)}} > \left[\frac{S(t_{i+1})}{S(t_i)}-e^{g\Delta t}\right]^+\right\}.$$

Apparently for $\alpha_p < 0$ or for $\alpha_p > 0$ and $\mu > \sigma^2(1-p)$, this above event occurs with a probability of 1, consequently it leads to an equivalence between the modified contingent claim with the original claim. This contradicts the initial budget constraint and accordingly the idea of efficient hedging. Therefore, the only interesting case here is that α_p is larger than 0 and $\mu < \sigma^2(1-p)$. In this case the equation owns a unique solution. Assume now $e^{c\Delta t}$ is the solution of $(R_1\cdot\alpha_p)^{-\frac{1}{1-p}}\left(\frac{S(t_{i+1})}{S(t_i)}\right)^{\frac{\mu}{\sigma^2(1-p)}} = \left[\frac{S(t_{i+1})}{S(t_i)}-e^{g\Delta t}\right]^+$, then

$$\alpha_p = \frac{1}{R_1}\left(\frac{(e^{c\Delta t}-e^{g\Delta t})^+}{e^{\frac{c\,\mu\,\Delta t}{\sigma^2(1-p)}}}\right)^{1-p} > 0.$$

μ	${}_Tp_x^*$	σ	${}_Tp_x^*$	g	${}_Tp_x^*$	ϵ	${}_Tp_x^*$
$\mu = 0.03$	0.702308	$\sigma = 0.3$	0.746807	$g = 0.02$	0.746807	$\epsilon = 0.01$	0.93566
$\mu = 0.04$	0.717634	$\sigma = 0.4$	0.704046	$g = 0.03$	0.739995	$\epsilon = 0.01$	0.882329
$\mu = 0.05$	0.732469	$\sigma = 0.5$	0.665453	$g = 0.04$	0.732897	$\epsilon = 0.01$	0.833927
$\mu = 0.06$	0.746807	$\sigma = 0.6$	0.628399	$g = 0.05$	0.725502	$\epsilon = 0.01$	0.788996

Table 2.5: Implied survival probability with parameters: $\epsilon = 0.05$, $T = t_M = 12$, $g = 0.02$, $\mu = 0.06$, $\sigma = 0.3$, $0 < p < 1$

Following this we obtain the price of the modified contingent claim:

$$\begin{aligned} E^*[H_P] &= E^*\left[H 1_{\left\{\frac{S(t_{i+1})}{S(t_i)} < e^{c\Delta t}\right\}}\right] \\ &= N\left(\frac{(c-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right) - N\left(\frac{(g-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right) - e^{g\Delta t}N\left(\frac{(c+\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right) \\ &\quad + e^{g\Delta t}N\left(\frac{(g+\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right). \end{aligned}$$

Again here we can determine the value of c and that of α_p through the relation ${}_Tp_x = \frac{E^*[H_P]}{E^*[H]}$, given that the survival probability is known. As before, we are more interested in the implied survival probability. Hence, as in the quantile hedging case c can be derived as a function of the given significance level:

$$P\left(\frac{S(t_{i+1})}{S(t_i)} \leq e^{c\Delta t}\right) = 1 - \epsilon,$$

i.e.,

$$c = \frac{1}{\Delta t}\left(N^{-1}(1-\epsilon)\sigma\sqrt{\Delta t} + (\mu - \frac{1}{2}\sigma^2)\Delta t\right).$$

Plugging this value in the expression for the modified contingent claim, we obtain the price we look for and the resulting survival probability for different hedge values.

Table 2.5 is displayed for the scenario $\mu < \sigma^2(1-p)$, where a unique solution for c is found. Above all, $p < 1$ indicates that the hedger is a risk–taking insurance company. If you look at the expression for the survival probability carefully, you will observe that it does not depend on p at all, i.e., same survival probabilities result for all p's which satisfy the condition $\mu < \sigma^2(1-p)$. In this scenario, the transfer between the financial and insurance risk becomes possible again because you observe quite big survival probabilities overall. When ϵ is increased (more financial risks), as a consequence, the survival probabilities are decreased (more older safer customers are preferred). All of the other effects are the same as in the quantile hedging case.

Finally, the case of $\underline{p=1}$ can be constructed as follows:

$$\begin{aligned}
& E^*[H_P] = E^*\left[H1_{\left\{\frac{dP}{dP^*}|_{\mathcal{F}_{t_{i+1}}}/\frac{dP}{dP^*}|_{\mathcal{F}_{t_i}}>\alpha_p\right\}}\right] \\
= \; & E^*\left[\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t}\right]^+ 1_{\left\{\frac{S(t_{i+1})}{S(t_i)}>(R_1\alpha_p)^{-\frac{\mu}{\sigma^2}}\right\}}\right] \\
= \; & N\left(\frac{-(g-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right) - e^{g\Delta t}N\left(\frac{-(g+\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right) - N\left(\frac{-\frac{\mu}{\sigma^2}\ln(R_1\alpha_p) - \frac{1}{2}\sigma^2\Delta t}{\sigma\sqrt{\Delta t}}\right) \\
& + N\left(\frac{-(g+\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right) + e^{g\Delta t}N\left(\frac{-\frac{\mu}{\sigma^2}\ln(R_1\alpha_p) + \frac{1}{2}\sigma^2\Delta t}{\sigma\sqrt{\Delta t}}\right) - e^{g\Delta t}N\left(\frac{-(g-\frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right).
\end{aligned} \tag{2.26}$$

Here α_p is determined by

$$\begin{aligned}
P(A^*) &= P\left(\frac{\frac{dP}{dP^*}|_{\mathcal{F}_{t_{i+1}}}}{\frac{dP}{dP^*}|_{\mathcal{F}_{t_i}}} > \alpha_p\right) \\
&= P\left(\frac{S(t_{i+1})}{S(t_i)} > (R_1\alpha_p)^{-\frac{\mu}{\sigma^2}}\right) \\
&= 1 - N\left(\frac{-\frac{\mu}{\sigma^2}(\ln R_1 + \ln\alpha_p) - (\mu - \frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}}\right) := 1-\epsilon.
\end{aligned}$$

Consequently,

$$\alpha_p = \exp\left\{-\frac{\sigma^2}{\mu}\left(N^{-1}(\epsilon)\sigma\sqrt{\Delta t} + (\mu - \frac{1}{2}\sigma^2)\Delta t\right) - \ln R_1\right\}.$$

In fact, $p=1$ reflects exactly the quantile hedging. Although the quantile price looks different from that in Equation (2.26), this gives precisely the same value obtained in Subsection 2.3.1. On the one hand, in the quantile hedging, the maximal success set is given by

$$\left\{\frac{S(t_{i+1})}{S(t_i)} < \exp\left\{N^{-1}(1-\epsilon)\sigma\sqrt{\Delta t} + (\mu - \frac{1}{2}\sigma^2)\Delta t\right\}\right\} = \{X < N^{-1}(1-\epsilon)\},$$

where X is a standard normal distributed random variable. On the other hand, the maximal success set in the efficient hedging for the case $p=1$ is expressed by

$$\left\{\frac{S(t_{i+1})}{S(t_i)} > (R_1\alpha_p)^{-\frac{\mu}{\sigma^2}}\right\} = \{X > N^{-1}(\epsilon)\}.$$

The above equation holds because it is valid that

$$(R_1\alpha_p)^{-\frac{\mu}{\sigma^2}} = \exp\left\{N^{-1}(\epsilon)\sigma\sqrt{\Delta t} + (\mu - \frac{1}{2}\sigma^2)\Delta t\right\}.$$

Obviously, these two events $\{X < N^{-1}(1-\epsilon)\}$ and $\{X > N^{-1}(\epsilon)\}$ occur with an identical probability.

2.4 Summary

The chapter investigates several hedging methods in order to deal with the incompleteness caused by the untradable insurance risk. Different hedging goals of the insurer lead to different hedging strategies. We consider several popular hedging/optimality criteria in a continuous setup, such as risk–minimizing, quantile and efficient hedging. Risk–minimizing is a criterion which strives to achieve a minimum variance of the hedger's future cost. While in a quantile or efficient hedging, the insurer constrain his shortfall probability to a certain level. By looking at a specific guaranteed equity–linked life insurance contract, we investigate how these corresponding strategies are derived. In particular, in Section 2.3, we start from another point of view, i.e., to derive mortality risk explicitly when quantile and efficient hedging come into consideration. Besides, we prove that quantile hedging is indeed a special case of efficient hedging with a power loss function.

In a pure endowment contract, the risk–minimizing criterion suggests the insurer to trade with a hedge ratio which corresponds to the hedge ratio if there is financial risk only weighted with the expected number of customer who survive the maturity date. While quantile and efficient hedging give completely different suggestions to the insurer. Given that how many financial risks they would like to take, quantile or efficient hedging give them a hint what kinds of customers to choose in order to reach this goal. In other words, this allows a transfer between financial and insurance risk. However, it seems not very realistic to choose customers rather than attract more customers. In fact, in reality, for their own benefits, many brokers of the insurance company try to acquire as many customers as possible even when sometimes they have to misinterpret the customers' life style or physical conditions. This is a main cause which leads to the phenomenon of mortality misspecification. A detailed discussion on this topic will be followed in Chapter 4.

Assume that an insurer really applies the introduced hedging strategy, what effects does this have on the insurer's loss? The following chapter takes risk–minimizing hedging strategy as an example and is designed to answer this question.

Chapter 3

Loss analysis under discrete–time risk–minimization[1]

In Chapter 2, the concept of "risk–minimizing" is reviewed and the risk–minimizing hedging strategies are derived for different insurance contracts. In the present chapter, we assume that the considered life insurance company sets risk–minimizing (variance–minimizing) as its striving aim, and applies this optimality criterion for its hedging purpose. The goal of this chapter is to investigate the net loss of the insurance company and to figure out whether the insurer can benefit from using this hedging method. More concretely, we compare simulated ruin probabilities resulting from diverse trading strategies the hedger uses. Here, the ruin probability is defined as the relative frequency that the net loss of the insurer at the maturity of the contract is larger than 0.[2] For simplification reasons, it is assumed that the insurance company issues pure endowment contracts only and that a deterministic interest rate r is used. Mainly two discrete–time risk–minimizing hedging strategies are investigated. One is obtained by discretizing a continuous risk–minimizing hedging strategy and the other is obtained by straight discretizing the relevant underlying asset process, i.e., a binomial hedging model instead of Black–Scholes model comes into consideration. Since trading restriction is a source of market incompleteness, in the analysis of the time–discretized risk–minimizing strategy, there exist two sources of market incompleteness: the untradable insurance risk and the trading restrictions. The asset price dynamics are assumed to be in the framework of Black and Scholes (1973), but the hedging of the contingent claims occurs at discrete times instead of continuously. In order to make the analysis even more interesting, the case when the insurer invests the premiums in a risk free asset is taken as a basis scenario.

Through an illustrative example, it is observed numerically that a substantial reduction in the ruin probability is achieved by using the time–discretized risk–minimizing strategy, in comparison with the scenario, where the insurer invests the premiums in a risk free asset with a rate of return corresponding to the market interest rate. However, the

[1] This chapter is based on Chen (2005).

[2] Usually ruin is defined as the first passage time, but due to our contract specification, namely pure endowment insurance contract, ruin probability can be simplified to the definition in the text.

extent of the reduction becomes less apparent and the advantage of using this strategy almost disappears when the trading frequency is increased. This is due to the fact that extra duplication errors are caused when the original mean–self–financing risk–minimizing hedging strategy is discretized with respect to time and that these errors increase with the frequency. This numerical result motivates to consider another type of discrete–time risk–minimizing hedging strategy. I.e., it is obtained by discretizing the hedging model instead of the hedging strategy. Since the binomial model of Cox, Ross and Rubinstein (1979) (CRR) converges in the limit to the model of Black and Scholes (1973), it is self–evident to consider a discrete–time risk–minimizing hedging strategy obtained e.g. in a binomial model. For this purpose, the binomial risk–minimizing strategy derived in Møller (2001) is adopted. When comparing the simulation results with the scenario where the strategy is discretized, we observe considerably smaller ruin probabilities, in particular, when the frequency is increased.

This chapter proceeds as follows: First, the net loss of an insurance company is defined and for two simple scenarios the loss is computed. Second, we focus on the net loss caused by using the time–discretized originally continuous risk–minimizing hedging strategies. Third, the risk–minimizing hedging strategy is derived for a specific insurance contract. Fourth, some simulation results related to the time–discretized strategy are demonstrated. Consequently, the binomial risk–minimizing hedging strategy is regarded as a comparison. Finally, this chapter concludes with a short summary.

3.1 Net loss, two extreme scenarios

This section aims at defining the net loss of a life insurance company and at exhibiting two extreme cases. Suppose that at the beginning n identical customers of age x engage in the same pure endowment contract with the insurance company, which promises each of them a payment of $f(T,S)$ at the maturity date if they survive until this point in time. In other words, a population of n x–aged customers are considered. Again, the function $f(T,S)$ describes the dependence of the final payment on the evolution of the stock price. It can be a function of the terminal stock price S_T only or of the whole path of the stock and possibly it contains embedded options[3]. In return, each customer pays a premium of K periodically, i.e., the premium payments occur at

$$\{0 = t_0 < t_1 < \cdots < t_{M-1}\}$$

where $T = t_M$ denotes the maturity date of the contract. The periodic premium is determined at the beginning of the contract and will be kept constant throughout the duration of the contract. In the last chapter, N_t^x is used to denote the number of the customers who dies before time t. Since most of time we work with the number of survived customers in this chapter, the new random variable Y_t^x with

$$Y_t^x := n - N_t^x = \sum_{i=1}^{n} 1_{\{\tau_i^x > t\}},$$

[3]In Section 3.3, a specific payment $f(T,S)$ is illustrated.

is introduced. Y_t^x gives the number of customers who survive time t, in particular, it holds $Y_0^x = n$. Again the assumption that τ_i^x are i.i.d leads to a binomial distribution of Y_t^x with parameters $(n, {}_tp_x)$. By this definition of the contract, we observe that both the payment of the insurance company and that of the customers depend on the mortality uncertainty, while the size of the payment of the insurer also hinges on the performance of the stock. Consequently, the net loss of the insurance company at the maturity date of the contract is defined as the difference of its accumulated outflows and its accumulated inflows by that point in time.

$$\begin{aligned}
&\text{(Discounted) net loss of the insurer at time } T \\
&= \text{(Discounted) payment of the insurer at time } T \\
&\quad -\text{(Discounted) accumulated premium incomes till time } T \\
&\quad -\text{(Discounted) trading gains (losses) from investment strategies} \\
&= e^{-rT} Y_T^x f(T,S) - \sum_{i=0}^{M-1} \left((Y_{t_i}^x - Y_{t_{i+1}}^x) \sum_{j=0}^{i} K e^{-rt_j} \right) - Y_T^x \sum_{j=0}^{M-1} K e^{-rt_j} \\
&\quad -\text{(Discounted) trading gains (losses) from investment strategies,}
\end{aligned}$$

where r denotes the deterministic discount factor. This definition is partially motivated by Møller (2001). Those, who die during $]t_i, t_{i+1}]$, $i \leq M-2$, only pay the premiums till t_i and those who survive the time point t_{M-1} pay all of the premiums. Naturally, the trading gains (losses) depend on the hedging/investment strategies the insurer chooses.

3.1.1 Investment in a risk free asset

As a starting point, we consider the net loss of the company when the insurance company invests the premiums in a risk free asset with a rate of return r. Hence, the discounted net loss of the insurer at time T is simplified to:

$$L_T^{rf} = e^{-rT} Y_T^x f(T,S) - \sum_{i=0}^{M-1} (Y_{t_i}^x - Y_{t_{i+1}}^x) \sum_{j=0}^{i} K e^{-rt_j} - Y_T^x \sum_{j=0}^{M-1} K e^{-rt_j}. \tag{3.1}$$

The expected loss can be derived as follows:

$$\begin{aligned}
\mathrm{E}[L_T^{rf}] &= \mathrm{E}\left[Y_T^x f(S) e^{-rT} - \sum_{i=0}^{M-1} (Y_{t_i}^x - Y_{t_{i+1}}^x) \sum_{j=0}^{i} K e^{-rt_j} - Y_T^x \sum_{j=0}^{M-1} K e^{-rt_j} \right] \\
&= e^{-rT} \mathrm{E}[f(S)] \mathrm{E}[Y_T^x] - \sum_{i=0}^{M-1} \sum_{j=0}^{i} K e^{-rt_j} E[(Y_{t_i}^x - Y_{t_{i+1}}^x)] - E[Y_T^x] \sum_{j=0}^{M-1} K e^{-rt_j} \\
&= n \, {}_Tp_x \, e^{-rT} \mathrm{E}[f(S)] - n \sum_{i=0}^{M-1} \sum_{j=0}^{i} K e^{-rt_j} ({}_{t_i}p_x - {}_{t_{i+1}}p_x) - n \, {}_Tp_x \sum_{j=0}^{M-1} K e^{-rt_j}
\end{aligned} \tag{3.2}$$

The independence assumption between financial and mortality risk and the equality $\mathrm{E}[Y_{t_i}^x] = n \cdot {}_{t_i}p_x$ are needed for the above derivation. It is observed that the expected loss is equal to 0, if and only if

$$K^* = \frac{e^{-rT}\,{}_Tp_x\,\mathrm{E}[f(T,S)]}{\sum_{i=0}^{M-1}\sum_{j=0}^{i} e^{-rt_j}({}_{t_i}p_x - {}_{t_{i+1}}p_x) + {}_Tp_x\sum_{j=0}^{M-1} e^{-rt_j}}. \tag{3.3}$$

It is observed that the optimal K^* does not depend on the number of the contracts n the insurer issues. Only with this premium, $E[L_T^{rf}]/n = 0$ holds, i.e., the expected loss per contract is equal to zero. If the charged premium is larger than K^*, then $E[L_T^{rf}]/n < 0$, i.e., $\lim\limits_{n\to\infty} E[L_T^{rf}] = -\infty$. This means that in the expectation, the company makes an infinitely large profit as the number of the contract holders is increased to infinity. On the contrary, if the charged premium is smaller than K^*, this will result in an infinitely large expected loss for the company as the number of the contract–holders goes to infinity.

Due to the impact of the mortality risk on both the payment of the insurer and the payment of the insured, the variance of the loss is much more complicated than in the single premium case. In order to calculate the variance of the discounted net loss L_T^{rf}, the relation between variance and conditional variance is applied, i.e.,

$$\mathrm{Var}[X] = \mathrm{Var}[E[X|Z]] + \mathrm{E}[\mathrm{Var}[X|Z]], \tag{3.4}$$

where X, Z are two arbitrary random variables. Sophisticated choices of Z can simplify the calculation of $\mathrm{Var}[X]$ to a big extent. In our context, if the stock price S is chosen as the random variance which L_T^{rf} is conditioned on, the independence between the financial market risk and insurance risk can be exploited. Hence,

$$\begin{aligned}
\mathrm{Var}[L_T^{rf}] &= \mathrm{Var}[E[L_T^{rf}|S]] + E[\mathrm{Var}[L_T^{rf}|S]] \\
&= \mathrm{Var}\left[\mathrm{E}\left[Y_T^x f(T,S)e^{-rT} - \sum_{i=0}^{M-1}(Y_{t_i}^x - Y_{t_{i+1}}^x)\sum_{j=0}^{i} Ke^{-rt_j} - Y_T^x\sum_{j=0}^{M-1} Ke^{-rt_j}\,\middle|\,S\right]\right] \\
&\quad +\mathrm{E}\left[\mathrm{Var}\left[Y_T^x f(T,S)e^{-rT} - \sum_{i=0}^{M-1}(Y_{t_i}^x - Y_{t_{i+1}}^x)\sum_{j=0}^{i} Ke^{-rt_j} - Y_T^x\sum_{j=0}^{M-1} Ke^{-rt_j}\,\middle|\,S\right]\right] \\
&= \mathrm{Var}\left[f(T,S)e^{-rT}\mathrm{E}[Y_T^x] - \sum_{i=0}^{M-1}\sum_{j=0}^{i} Ke^{-rt_j}\mathrm{E}[Y_{t_i}^x - Y_{t_{i+1}}^x] - \mathrm{E}[Y_T^x]\sum_{j=0}^{M-1} Ke^{-rt_j}\right] \\
&\quad +\mathrm{E}\left[\left(f(T,S)e^{-rT} - \sum_{j=0}^{M-1} Ke^{-rt_j}\right)^2 \mathrm{Var}[Y_T^x] + \mathrm{Var}\left[\sum_{i=0}^{M-1}(Y_{t_i}^x - Y_{t_{i+1}}^x)\sum_{j=0}^{i} Ke^{-rt_j}\right]\right. \\
&\quad \left. -2\left(f(T,S)e^{-rT} - \sum_{j=0}^{M-1} Ke^{-rt_j}\right)\mathrm{Cov}\left[Y_T^x, \sum_{i=0}^{M-1}(Y_{t_i}^x - Y_{t_{i+1}}^x)\sum_{j=0}^{i} Ke^{-rt_j}\right]\right],
\end{aligned} \tag{3.5}$$

where $\mathrm{Var}[Y_T^x] = n\,{}_Tp_x\,(1 - {}_Tp_x)$ because of the binomial distribution of Y_T^x. In order to calculate the variance further, the following lemma is needed.

Lemma 3.1.1. *$Cov[Y_T^x, Y_{t_i}^x] = n\,{}_Tp_x\,(1 - {}_{t_i}p_x)$.*

PROOF: Analogous relation as in Equation (3.4) holds for the covariance too, i.e.,

$$\mathrm{Cov}[X, Y] = \mathrm{E}[\mathrm{Cov}[X, Y|Z]] + \mathrm{Cov}[\mathrm{E}[X|Z], \mathrm{E}[Y|Z]],$$

where X, Y, Z are three arbitrary random variables. If $Y_{t_i}^x$ is chosen as the random variable which is conditional on, we obtain

$$\begin{aligned}
\mathrm{Cov}[Y_T^x, Y_{t_i}^x] &= \mathrm{E}\left[\mathrm{Cov}[Y_T^x, Y_{t_i}^x|Y_{t_i}^x]\right] + \mathrm{Cov}\left[\mathrm{E}[Y_T^x|Y_{t_i}^x], \mathrm{E}[Y_{t_i}^x|Y_{t_i}^x]\right] \\
&= 0 + \mathrm{Cov}\left[Y_{t_i}^x\,{}_{T-t_i}p_{x+t_i}, Y_{t_i}^x\right] \\
&= {}_{T-t_i}p_{x+t_i}\,\mathrm{Var}[Y_{t_i}^x] \\
&= {}_{T-t_i}p_{x+t_i} \cdot n \cdot {}_{t_i}p_x \cdot (1 - {}_{t_i}p_x) \\
&= n \cdot {}_Tp_x \cdot (1 - {}_{t_i}p_x).
\end{aligned}$$

$\mathrm{Cov}[Y_T^x, Y_{t_i}^x|Y_{t_i}^x]$ equals zero because $Y_{t_i}^x$ can be considered as a constant if it is conditional on itself. Furthermore, the equality ${}_Tp_x = {}_{t_i}p_x \cdot {}_{T-t_i}p_{x+t_i}$ is used. □

In the following, every separate component in Equation (3.5) is calculated. By using Lemma 3.1.1 and the fact that $Y_{t_i}^x - Y_{t_{i+1}}^x$, $i = 0, \cdots, M-1$ are independent, we obtain

$$\begin{aligned}
&\mathrm{Var}\left[\sum_{i=0}^{M-1}(Y_{t_i}^x - Y_{t_{i+1}}^x)\sum_{j=0}^{i} Ke^{-rt_j}\right] \\
&= \sum_{i=0}^{M-1}\left(\sum_{j=0}^{i} Ke^{-rt_j}\right)^2 \mathrm{Var}[Y_{t_i}^x - Y_{t_{i+1}}^x] \\
&= \sum_{i=0}^{M-1}\left(\sum_{j=0}^{i} Ke^{-r\,t_j}\right)^2 \left(\mathrm{Var}[Y_{t_i}^x] + \mathrm{Var}[Y_{t_{i+1}}^x] - 2\mathrm{Cov}[Y_{t_i}^x, Y_{t_{i+1}}^x]\right) \\
&\overset{Lemma 3.1.1}{=} \sum_{i=0}^{M-1}\left(\sum_{j=0}^{i} Ke^{-rj}\right)^2 \left(n\,{}_{t_i}p_x\,(1 - {}_{t_i}p_x) + n\,{}_{t_{i+1}}p_x\,(1 - {}_{t_{i+1}}p_x) - 2\,n\,{}_{t_{i+1}}p_x\,(1 - {}_{t_i}p_x)\right) \\
&= \sum_{i=0}^{M-1}\left(\sum_{j=0}^{i} Ke^{-rj}\right)^2 \left(n\,{}_{t_i}p_x\,(1 - {}_{t_i}p_x) - n\,({}_{t_{i+1}}p_x)^2 - n\,{}_{t_{i+1}}p_x + 2\,n\,{}_{t_{i+1}}p_x\,{}_{t_i}p_x\right) \\
&= \sum_{i=0}^{M-1}\left(\sum_{j=0}^{i} Ke^{-rj}\right)^2 \left(n\,{}_{t_i}p_x\,(1 - {}_{t_i}p_x) + n\,{}_{t_{i+1}}p_x({}_{t_i}p_x - {}_{t_{i+1}}p_x) + n\,{}_{t_{i+1}}p_x({}_{t_i}p_x - 1)\right) \\
&= \sum_{i=0}^{M-1}\left(\sum_{j=0}^{i} Ke^{-rj}\right)^2 \left(n\,{}_{t_i}p_x\,(1 - {}_{t_i}p_x) + n\,{}_{t_{i+1}}p_x({}_{t_i}p_x - {}_{t_{i+1}}p_x) + n\,{}_{t_i}p_x\,{}_{\Delta t}p_{x+t_i}\,({}_{t_i}p_x - 1)\right) \\
&= \sum_{i=0}^{M-1}\left(\sum_{j=0}^{i} Ke^{-r\,t_j}\right)^2 \left(n\,{}_{t_i}p_x\,(1 - {}_{t_i}p_x)(1 - {}_{\Delta t}p_{x+t_i}) + n\,{}_{t_{i+1}}p_x({}_{t_i}p_x - {}_{t_{i+1}}p_x)\right).
\end{aligned}$$

Due to a repeated use of the independence of the increments in the survived customers, we obtain

$$\begin{aligned}
&\mathrm{Cov}\left[Y_T^x, \sum_{i=0}^{M-1}(Y_{t_i}^x - Y_{t_{i+1}}^x)\sum_{j=0}^{i} K e^{-r\,t_j}\right] \\
&\quad = \sum_{i=0}^{M-1}\left(\sum_{j=0}^{i} K e^{-r\,t_j}\right)\Big(\mathrm{Cov}[Y_T^x, Y_{t_i}^x] - \mathrm{Cov}[Y_T^x, Y_{t_{i+1}}^x]\Big) \\
&\quad \overset{Lemma 3.1.1}{=} \sum_{i=0}^{M-1}\left(\sum_{j=0}^{i} K e^{-rt_j}\right) n\,{}_Tp_x\,({}_{t_{i+1}}p_x - {}_{t_i}p_x).
\end{aligned}$$

Finally, we obtain the variance of the net loss when the insurer invests the premium incomes in a risk free asset by substituting the above two results to the variance expression in Equation (3.5):

$$\begin{aligned}
\mathrm{Var}[L_T^{rf}] &= \mathrm{Var}\left[f(T,S)e^{-rT}\,n\,{}_Tp_x - \sum_{i=0}^{M-1}\sum_{j=0}^{i} K e^{-rt_j} n\,({}_{t_i}p_x - {}_{t_{i+1}}p_x) - n\,{}_Tp_x \sum_{j=0}^{M-1} K e^{-rt_j}\right] \\
&\quad + \mathrm{E}\Bigg[\left(f(T,S)e^{-rT} - \sum_{j=0}^{M-1} K e^{-rt_j}\right)^2 n\,{}_Tp_x(1 - {}_Tp_x) \\
&\quad + \sum_{i=0}^{M-1}\left(\sum_{j=0}^{i} K e^{-r\,t_j}\right)^2 \big(n\,{}_{t_i}p_x\,(1 - {}_{t_i}p_x)(1 - {}_{\Delta t}p_{x+t_i}) + n\,{}_{t_{i+1}}p_x({}_{t_i}p_x - {}_{t_{i+1}}p_x)\big) \\
&\quad - 2\left(f(T,S)e^{-rT} - \sum_{j=0}^{M-1} K e^{-rt_j}\right)\sum_{i=0}^{M-1}\left(\sum_{j=0}^{i} K e^{-rt_j}\right) n\,{}_Tp_x\,({}_{t_{i+1}}p_x - {}_{t_i}p_x)\Bigg] \\
&= (n\,{}_Tp_x\,e^{-rT})^2\,\mathrm{Var}[f(T,S)] + n\,{}_Tp_x\,(1 - {}_Tp_x)e^{-2\,r\,T}\,\mathrm{E}[(f(T,S))^2] \\
&\quad - \Bigg\{2\,n\,e^{-rT}\,{}_Tp_x\,(1 - {}_Tp_x)\left(\sum_{j=0}^{M-1} K e^{-r\,t_j}\right) + 2\,e^{-rT}\sum_{i=0}^{M-1}\left(\sum_{j=0}^{i} K e^{-r\,t_j}\right) \\
&\quad n\,{}_Tp_x\,({}_{t_{i+1}}p_x - {}_{t_i}p_x)\Bigg\}\mathrm{E}[f(T,S)] + n\,{}_Tp_x\,(1 - {}_Tp_x)\left(\sum_{j=0}^{M-1} K e^{-rt_j}\right)^2 \\
&\quad + \sum_{i=0}^{M-1}\left(\sum_{j=0}^{i} K e^{-r\,t_j}\right)^2 \big(n\,{}_{t_i}p_x\,(1 - {}_{t_i}p_x)(1 - {}_{\Delta t}p_{x+t_i}) + n\,{}_{t_{i+1}}p_x({}_{t_i}p_x - {}_{t_{i+1}}p_x)\big) \\
&\quad + 2\left(\sum_{j=0}^{M-1} K e^{-rt_j}\right)\left[\sum_{i=0}^{M-1}\left(\sum_{j=0}^{i} K e^{-r\,t_j}\right) n\,{}_Tp_x\,({}_{t_{i+1}}p_x - {}_{t_i}p_x)\right]. \qquad (3.6)
\end{aligned}$$

It is observed that the variance of the discounted net loss in the case of periodical premium is much more complicated than that of the single premium case, where the variance corresponds to the first two terms of Equation (3.6). This is due to the fact that the

mortality risk not only decides the occurrence of the claim payment but also influences the payment of the periodic premiums. Asymptotically, that is, as $n \to \infty$, it is noticed that

$$\mathrm{Var}\left[\frac{1}{n}L_T^{rf}\right] \;\to\; {}_Tp_x^2\, e^{-2\,r\,T}\mathrm{Var}[f(T,S)].$$

As expected, by increasing the number of the insured, the insurer can eliminate all the mortality risk, i.e., the risk related to the uncertainty concerning the number of the policyholders that will survive to time T and the uncertainty concerning the number of the policyholders that will die between $(t_i, t_{i+1}]$, $\forall\, i \leq M-1$. This is so–called diversification effect over sub–population. While the financial uncertainty concerning the future development of the stock remains with the insurer, since all contracts are linked to the same stock. Since the insurer does not really hedge against the risk in this case, the resulting variance gives an upper bound of the risk that the insurer can reach.

Since there is an upper bound for the variance, naturally, the question will be asked whether there exists a lower bound. If there are some static hedging strategies which duplicate the final payment $f(T,S)$ perfectly, the insured can eliminate all the financial risk by certain "buy–and–hold" strategies.

3.1.2 Net loss in the case of a static hedge

In contrast to the above extreme scenario, we now assume that there are some static ("buy–and–hold") hedging strategies which completely duplicate the final payment $f(T,S)$, so that the insurer can eliminate the entire risk. Assume that the company applies the static strategy, i.e., it purchases $n \cdot {}_Tp_x$ financial contracts at the beginning of the insurance contract and holds them until the maturity date of the insurance contract. Each of these financial contracts pays the amount $f(T,S)$ at time T. Let V_0 be today's price of such a financial contract. Hence, the net loss for this case is described as the difference of the loss in the first case and the profit/loss from trading:

$$\begin{aligned} L_T^s &= L_T^{rf} - \text{(discounted) profit/loss from trading} \\ &= L_T^{rf} - \left(e^{-rT}\, n\,{}_Tp_x\, f(T,S) - n\,{}_Tp_x\, V_0\right) \\ &= n\,{}_Tp_x\, V_0 + e^{-rT}\, f(T,S)\,(Y_T^x - n\,{}_Tp_x) - \sum_{i=0}^{M-1}(Y_{t_i}^x - Y_{t_{i+1}}^x)\sum_{j=0}^{i} K e^{-rt_j} - Y_T^x \sum_{j=0}^{M-1} K e^{-rt_j}. \end{aligned} \tag{3.7}$$

The expected loss is described by

$$\begin{aligned} E[L_T^s] &= E[L_T^{rf}] - n\,{}_Tp_x\, e^{-rT}\, E[f(T,S)] + n\,{}_Tp_x\, V_0 \\ &= n\,{}_Tp_x\, V_0 - n\sum_{i=0}^{M-1}\sum_{j=0}^{i} K\, e^{-rt_j}\,({}_{t_i}p_x - {}_{t_{i+1}}p_x) - n\,{}_Tp_x \sum_{j=0}^{M-1} K e^{-rt_j}. \end{aligned}$$

In this situation the expected loss is equal to 0 if and only if

$$\tilde{K} = \frac{{}_T p_x V_0}{\sum_{i=0}^{M-1} \sum_{j=0}^{i} e^{-r t_j} ({}_{t_i} p_x - {}_{t_{i+1}} p_x) + {}_T p_x \sum_{j=0}^{M-1} e^{-r\, t_j}}.$$

Similarly, just with this premium $\tilde{K}$, $\lim_{n\to\infty} E[L_T^s]/n = 0$ holds. If the charged premium is larger than $\tilde{K}$, then $\lim_{n\to\infty} E[L_T^s]/n < 0$, i.e., $E[L_T^s] = -\infty$ as $n \to \infty$. That means, it leads to an infinitely large expected profits for the company as the size of the contract–holders is increased to ∞. On the contrary, if the charged premium is smaller than $\tilde{K}$, it will lead to an infinitely large expected loss for the company as the size of the contract–holders goes up to ∞. Besides, $\tilde{K}$ corresponds to K^* if $V_0 = e^{-rT} E[f(T,S)]$[4]. The variance of the net loss for this static hedge is calculated as follows:

$$\begin{aligned}
\mathrm{Var}[L_T^s] &= \mathrm{Var}[\mathrm{E}[L_T^s|S]] + \mathrm{E}[\mathrm{Var}[L_T^s|S]] \\
&= \mathrm{Var}\left[n\, {}_T p_x\, V_0 - \sum_{i=0}^{M-1} \left(\sum_{j=0}^{i} K e^{-r t_j} \right) E[Y_{t_i}^x - Y_{t_{i+1}}^x] - E[Y_T^x] \sum_{j=0}^{M-1} K e^{-r t_j} \right] \\
&\quad + \mathrm{E}\Big[\mathrm{Var}[L_T^{rf} - n\, {}_T p_x\, e^{-rT} f(T,S) + n\, {}_T p_x\, V_0 | S] \Big] \\
&= 0 + \mathrm{E}\Big[\mathrm{Var}[L_T^{rf}|S] + \mathrm{Var}[n\, {}_T p_x\, f(T,S) e^{-rT}|S] + 2\mathrm{Cov}[L_T^{rf}, n\, {}_T p_x f(T,S)\, e^{-rT}|S] \Big] \\
&= \mathrm{E}\Big[\mathrm{Var}[L_T^{rf}|S] \Big] \\
&= n\, {}_T p_x\, (1 - {}_T p_x) e^{-2\, r\, T}\, \mathrm{E}[(f(T,S))^2] \\
&\quad - \Bigg\{ 2\, n\, e^{-rT}\, {}_T p_x\, (1 - {}_T p_x) \left(\sum_{j=0}^{M-1} K e^{-r\, t_j} \right) + 2\, e^{-rT} \sum_{i=0}^{M-1} \left(\sum_{j=0}^{i} K e^{-r\, t_j} \right) \\
&\quad n\, {}_T p_x\, ({}_{t_{i+1}} p_x - {}_{t_i} p_x) \Bigg\} \mathrm{E}[f(T,S)] + n\, {}_T p_x\, (1 - {}_T p_x) \left(\sum_{j=0}^{M-1} K e^{-r t_j} \right)^2 \\
&\quad + \sum_{i=0}^{M-1} \left(\sum_{j=0}^{i} K e^{-r\, t_j} \right)^2 \left(n\, {}_{t_i} p_x\, (1 - {}_{t_i} p_x)(1 - {}_{\Delta t} p_{x+t_i}) + n\, {}_{t_{i+1}} p_x ({}_{t_i} p_x - {}_{t_{i+1}} p_x) \right) \\
&\quad + 2 \left(\sum_{j=0}^{M-1} K e^{-r t_j} \right) \left[\sum_{i=0}^{M-1} \left(\sum_{j=0}^{i} K e^{-r\, t_j} \right) n\, {}_T p_x\, ({}_{t_{i+1}} p_x - {}_{t_i} p_x) \right].
\end{aligned}$$

Opposite to the first case, it is noticed that

$$\mathrm{Var}\left[\frac{L_T^s}{n} \right] \to 0$$

as $n \to \infty$, i.e., in this case, the total risk (mortality risk + financial risk) can be eliminated by increasing the number of policies in the portfolio and buying the options on the stock.

[4] E.g., the equality holds if we take the expected value under the equivalent martingale.

However, is this simple strategy suggestive and applicable in reality? Are there any other better strategies for the insurance company who sells equity–linked life insurance contracts to eliminate their risk? The answer to the first question is "no", because the usual term of these insurance contracts is quite long, e.g. it takes 12 to 30 years in Germany, while standard options are typically short–term transactions, say, less than one year. Due to this unrealistic restriction, this second scenario will not be considered later. In order to answer the second question, the first thought is to take into consideration time–discretized risk–minimizing hedging strategies and to derive the corresponding net loss function. Before coming to the risk–minimizing hedging strategy introduced in Møller (1998), we review some fundamentals about cost processes and duplication errors caused by using time–discretized originally continuous hedging strategies.

3.2 Time–discretized risk–minimizing strategy

Due to two reasons, namely, high transaction costs and the fact that security markets do not operate but are closed at nights, at weekends and on holidays, it is impossible for a hedger to trade continuously or make continuous adjustments to his hedging portfolio. In this context, discrete–time strategies receive a wide application. There are two methods to derive the discrete–time trading strategies:

- It is generated from discretizing a continuous–time hedging strategy with respect to time. That is, the relevant price dynamic is assumed to be a continuous–time stochastic process so that a continuous–time hedging strategy is received at first.[5]
- It is generated directly from assuming that the relevant price dynamic is driven by a binomial model.

Furthermore, it is assumed that transactions are carried out at a given trading set τ^Q. This set of trading dates is characterized by a sequence of equidistant refinements of the interval $[0, T]$, namely,

$$\tau^Q := \{t_0 = 0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_Q = T\}$$

with $|\tau_{k+1} - \tau_k| \to 0$ for $Q \to \infty$. In the following, we denote $\Delta\tau = \frac{T}{Q}$, hence $\tau_j = j \cdot \Delta\tau$. For simplification reasons, Q is assumed to be a multiple of M. Therefore, τ^Q can be considered as a refinement of the premium payment dates. For example, Q is assumed to be equal to $2M$ in Figure 3.1, i.e., trading occurs twice as frequently as premium payments, i.e. $\Delta\tau = \frac{1}{2}\Delta t$.

In the following, let $\phi_\tau^Q = (\xi_\tau^Q, \eta_\tau^Q)$ describe a discrete–time trading strategy in the stock and bond account under the set of trading dates τ^Q. It is assumed that the trading strategy

[5]For instance, the relevant stochastic process is assumed to be lognormal, i.e., the derived continuous strategy depends on the usual assumptions of the Black and Scholes (1973) model.

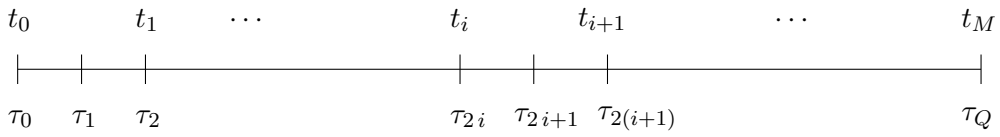

Figure 3.1: Partition of T according to premium payment dates ($\{t_0, t_1, \cdots, t_{M-1}\}$) and trading dates ($\{\tau_0, \tau_1, \cdots, \tau_Q\}$). Trading occurs e.g. twice as frequently as premium payments.

ϕ^Q_τ satisfies the usual integrability condition, but it is not necessarily self–financing. In addition, discounted processes are considered. Bank account $(B_t)_{t\in[0,T]}$ with

$$B_t = \exp\{rt\}$$

is used as a numeraire and a star is put in the superscript to denote the discounted value. In other words, by discounting we transform the financial market with two assets (B, S) to a market with $(1, S^*)$ where the interest rate is zero. I.e., $S^*_t = S_t\, e^{-rt}$. More specifically, the discrete–time strategy is defined by

$$\begin{aligned} \phi^Q_{t_0} &:= \phi^Q_{\tau_0}, \\ \phi^Q_t &:= \phi^Q_{\tau_k}, \quad t \in]\tau_k, \tau_{k+1}], \quad k = 0, \cdots, Q-1. \end{aligned}$$

That is, transactions are carried out immediately after the prices are announced at a certain discrete point in time and are kept constant throughout the time period until the next trading decision takes place.

The latter case, i.e., a risk–minimizing hedging strategy resulting from a binomial asset evolution is analyzed in Section 3.4. In the following, we consider the former case at first, i.e., a time–discretized originally continuous hedging strategy is analyzed. In the risk–minimizing hedging, only admissible hedging strategies are investigated, i.e., the contingent claim is always duplicated by the final payment of the considered strategies. However, by discretization, duplication becomes impossible. I.e., some discretization and duplication errors are generated. Therefore, before the loss analysis is taken into consideration, the corresponding cost process and the duplication error resulting from the use of a time–discretized hedging strategy are studied.

3.2.1 Cost process and duplication error

For the sake of clarity, we introduce $(\phi^c_t)_{t\in[0,T]} = (\xi^c_t, \eta^c_t)$ to denote a continuous–time trading strategy derived in a continuous–time model. The time–discretized version of this continuous strategy is given by

$$\phi^Q_{\tau_k} = \phi^c_{\tau_{k-1}}, \quad k = 1, \cdots, Q.$$

A straightforward consequence of time–discretizing a continuous–time hedging strategy is that some extra rebalancing or/and duplication costs might result. I.e., if the relevant continuous–time strategy is self–financing, after time–discretization, it might lose its self–financing property. Due to the definition of the discounted net loss in Section 3.1, trading gains/losses are an component of the net loss. More concretely, according to the relation between gain and cost processes, the discounted net loss can be rephrased as follows:

$$\begin{aligned}\text{Net loss} &= e^{-rT} Y_T^x f(T,S) - \text{Trading gains/losses} - \text{Premium incomes} \\ &= e^{-rT} Y_T^x f(T,S) - [V_T^*(\phi^Q) - V_0(\phi^Q) - C_T^*(\phi^Q)] - \text{Premium incomes},\end{aligned}$$

where $V_T^*(\phi^Q) = e^{-rT} V_T(\phi^Q)$ and $C_T^*(\phi^Q) = e^{-rT} C_T(\phi^Q)$, corresponding to the discounted value and accumulated rebalancing cost until the maturity resulting from the strategy ϕ^Q. Furthermore, if we can assume an equality between $Y_T^x f(T,S)$ and $V_T(\phi^c)$, the expression can be further simplified. This assumption holds e.g. for the continuous risk–minimizing hedging given in Chapter 2. As a consequence, the net loss has a form of

$$\text{Net loss} = V_0(\phi^Q) + C_T^*(\phi^Q) + [V_T^*(\phi^c) - V_T^*(\phi^Q)] - \text{Premium incomes}. \quad (3.8)$$

It is noticed that $(V_T^*(\phi^c) - V_T^*(\phi^Q))$ describes the discounted duplication error caused by using the time–discretized hedging strategy. It is well–known that the time–discretized version of Gaussian hedging strategies could lead to an extra duplication bias, even when there are no model or parameter misspecifications, see e.g. Mahayni (2003). It is observed that the net loss can be decomposed into four parts: the initial investment, the accumulated rebalancing costs associated with ϕ^Q, the duplication costs, and minus the premium incomes. Since the premium incomes are known and since the initial value of ϕ^Q equals the initial value of ϕ^c, the net loss can be readily obtained as soon as the rebalancing cost and the duplication cost with respect to ϕ^Q are calculated.

Due to the definition of cost process stated in Section 2.1, the corresponding discounted rebalancing cost process $C^*(\phi^Q)$ associated with ϕ^Q is given as follows:

$$\begin{aligned}C_t^*(\phi^Q) &= V_t^*(\phi^Q) - V_0(\phi^Q) - \sum_{j=0}^{k-1} \left[\xi_{\tau_j}^c (S_{\tau_{j+1}}^* - S_{\tau_j}^*) + \xi_{\tau_k}^c (S_t^* - S_{\tau_k}^*) \right] \\ &= \xi_{\tau_k}^c S_t^* + \eta_{\tau_k}^c - V_0(\phi^c) - \sum_{j=0}^{k-1} \left[\xi_{\tau_j}^c (S_{\tau_{j+1}}^* - S_{\tau_j}^*) + \xi_{\tau_k}^c (S_t^* - S_{\tau_k}^*) \right] \\ &= \xi_{\tau_k}^c S_{\tau_k}^* + \eta_{\tau_k}^c - \left(V_0(\phi^c) + \sum_{j=0}^{k-1} \xi_{\tau_j}^c (S_{\tau_{j+1}}^* - S_{\tau_j}^*) \right), \ t \in]\tau_k, \tau_{k+1}]. \quad (3.9)\end{aligned}$$

The discounted bond value disappears in the above equation because discounted assets are considered and hence the value of B_t^*, $t \leq T$ is identical to 1. It is noted that ϕ^Q is

not necessarily self–financing or mean–self–financing even if this holds for ϕ^c. Assume ϕ^c is continuous–time self–financing strategy, i.e.,

$$\xi^c_{\tau_k} S^*_{\tau_k} = V_0(\phi^c) + \int_0^{\tau_k} \xi^c_u \, dS^*_u,$$

then for $t \in]\tau_k, \tau_{k+1}]$,

$$\begin{aligned} C^*_t(\phi^Q) &= V_0(\phi^c) + \int_0^{\tau_k} \xi^c_u dS^*_u + \eta^c_{\tau_k} - V_0(\phi^c) - \sum_{j=0}^{k-1} \xi^c_{\tau_j}(S^*_{\tau_{j+1}} - S^*_{\tau_j}) \\ &= \sum_{j=0}^{k-1} \int_{\tau_j}^{\tau_{j+1}} (\xi^c_u - \xi^c_{\tau_j}) \, dS^*_u + \eta^c_{\tau_k}. \end{aligned}$$

From this, without extra conditions, even $E[C^*_t(\phi^Q)]$ is not equal to zero, i.e., ϕ^Q is not mean–self–financing.

Moreover, the value of the time–discretized version ϕ^Q differs from that of the continuous strategy ϕ^c by an amount, which is given by:

$$V^*_t(\phi^c) - V^*_t(\phi^Q) = (\xi^c_t - \xi^c_{\tau_k}) S^*_t + (\eta^c_t - \eta^c_{\tau_k}), \; t \in]\tau_k, \tau_{k+1}]. \tag{3.10}$$

That means, if the contingent claim is duplicated by the value of ϕ^c_T, in general it cannot be duplicated by the value of the time–discretized strategy for maturity date T simultaneously, because it takes the value of $\phi^c_{\tau_{Q-1}}$. In the following we denote by $C^{*,tot}_T(\phi^Q)$ the total hedging error of the insurer by trading according to ϕ^Q, which is defined as the sum of the discounted rebalancing cost until time T and the generated duplication error. Putting together the rebalancing cost in Equation (3.9) and the duplication cost for $T \in]\tau_{Q-1}, \tau_Q]$ in Equation (3.10), we obtain

$$\begin{aligned} C^{*,tot}_T(\phi^Q) &:= C^*_T(\phi^Q) + V^*_T(\phi^c) - V^*_T(\phi^Q) \\ &= \xi^c_{\tau_{Q-1}} S^*_{\tau_{Q-1}} + \eta^c_{\tau_{Q-1}} - \left(V_0(\phi^c) + \sum_{j=0}^{Q-2} \xi^c_{\tau_j}(S^*_{\tau_{j+1}} - S^*_{\tau_j}) \right) + (\xi^c_{\tau_Q} - \xi^c_{\tau_{Q-1}}) S^*_{\tau_Q} \\ &\quad + (\eta^c_{\tau_Q} - \eta^c_{\tau_{Q-1}}) \\ &= \xi^c_{\tau_{Q-1}} S^*_{\tau_{Q-1}} + \eta^c_{\tau_{Q-1}} - \left(\xi^c_{\tau_0} S_{\tau_0} + \eta^c_{\tau_0} + \sum_{j=0}^{Q-2} \xi^c_{\tau_j}(S^*_{\tau_{j+1}} - S^*_{\tau_j}) \right) \\ &\quad + (\xi^c_{\tau_Q} - \xi^c_{\tau_{Q-1}}) S^*_{\tau_Q} + (\eta^c_{\tau_Q} - \eta^c_{\tau_{Q-1}}) \\ &= \sum_{j=1}^{Q} (\xi^c_{\tau_j} - \xi^c_{\tau_{j-1}}) S^*_{\tau_j} + (\eta^c_{\tau_Q} - \eta^c_{\tau_{Q-1}}). \end{aligned} \tag{3.11}$$

So far we have investigated the impact of the time–discretization of a continuous–time hedging strategy on the cost processes, in particular, on the rebalancing cost and duplication costs. Accordingly, below we will specify this continuous strategy, and have a look

at Møller's (1998) risk–minimizing hedging strategy.

According to the analysis of Chapter 2, Møller's dynamic risk–minimizing hedging strategy is derived in the Black and Scholes' (1973) economy. And in particular, in the pure endowment insurance, it is of the form:

$$\begin{aligned} \xi_t^Q &:= \xi_{\tau_k}^c = Y_{\tau_k}^x \, {}_{T-\tau_k}p_{x+\tau_k} \, F_s(\tau_k, T, S) && t \in]\tau_k, \tau_{k+1}], \\ \eta_t^Q &:= \eta_{\tau_k}^c = V_{\tau_k}^* - \xi_{\tau_k}^c \, S_{\tau_k}^* && t \in]\tau_k, \tau_{k+1}], \end{aligned} \tag{3.12}$$

where $F(t, T, S)$ represents the no–arbitrage value of the contingent claim at time t and $F_s(t, T, S)$ the corresponding derivative of $F(t, T, S)$ with respect to the stock price S_t. The hedge ratio (the number of stocks the insurer should hold) at time $t \in]\tau_k, \tau_{k+1}]$ is described as the product of the hedge ratio in the case of financial risk only and the average number of customers who survive the contract's maturity time T given that they have survived time τ_k. The number of bonds is determined as the difference between the discounted value of the portfolio and the amount invested in the stock. Substituting Equation (3.12) in Equation (3.11), we obtain the discounted total hedging error with respect to the above risk–minimizing hedging strategy as follows:

$$\begin{aligned} C_T^{*,tot}(\phi^Q) &= \sum_{j=1}^{Q} (\xi_{\tau_j}^c - \xi_{\tau_{j-1}}^c) S_{\tau_j}^* + \eta_{\tau_Q}^c - \eta_{\tau_0}^c \\ &= \sum_{j=1}^{Q} \left(Y_{\tau_j}^x \, {}_{T-\tau_j}p_{x+\tau_j} \, F_s(\tau_j, T, S) - Y_{\tau_{j-1}}^x \, {}_{T-\tau_{j-1}}p_{x+\tau_{j-1}} F_s(\tau_{j-1}, T, S) \right) S_{\tau_j}^* \\ &\quad + Y_{\tau_Q}^x f(T, S) \, e^{-rT} - Y_{\tau_Q}^x F_s(\tau_Q, T, S) S_{\tau_Q}^* - V_{\tau_0}(\phi^c) + n \, {}_T p_x \, F_s(\tau_0, T, S) S_{\tau_0}^* \end{aligned} \tag{3.13}$$

It is known that the hedger could eliminate all the financial risk in the case of a continuous strategy, i.e., the hedging errors left to the hedger completely result from the mortality risk. However, this argument loses its validity if the continuous risk–minimizing strategy is applied discretely. This implies that the time–discretized version of a continuous risk–minimizing hedging strategy cannot be variance–minimizing. It is observed from Equation (3.13) that the accumulated hedging error hinges not only on the mortality risk, but also on the financial risk. In other words, by applying the time–discretized risk–minimizing hedging strategy, not all the financial risks can be eliminated.

3.2.2 Net loss

According to the rephrased net loss in Equation (3.8) and the discounted total hedge costs in Equation (3.13), the net loss of the insurer using the discretized risk–minimizing hedging strategy consists of the initial investment plus the discounted total hedging error

with respect to ϕ^Q less the premium inflows of the hedger:

$$\begin{aligned} L_T^{rm} &= V_0(\phi^Q) + C_T^{*,tot}(\phi^Q) - \sum_{i=0}^{M-1}(Y_{t_i}^x - Y_{t_{i+1}}^x)\sum_{j=0}^{i} Ke^{-rt_j} - Y_T^x \sum_{j=0}^{M-1} Ke^{-rt_j} \\ &= \sum_{j=1}^{Q}\left(Y_{\tau_j}^x \, {}_{T-\tau_j}p_{x+\tau_j} \, F_s(\tau_j, T, S) - Y_{\tau_{j-1}}^x \, {}_{T-\tau_{j-1}}p_{x+\tau_{j-1}} F_s(\tau_{j-1}, T, S)\right) S_{\tau_j}^* \\ &\quad + Y_{\tau_Q}^x f(T,S)\, e^{-rT} - Y_{\tau_Q}^x F_s(\tau_Q, T, S) S_{\tau_Q}^* + n \, {}_Tp_x \, F_s(\tau_0, T, S) S_{\tau_0}^* \\ &\quad - \sum_{i=0}^{M-1}(Y_{t_i}^x - Y_{t_{i+1}}^x)\sum_{j=0}^{i} Ke^{-rt_j} - Y_T^x \sum_{j=0}^{M-1} Ke^{-rt_j}. \end{aligned} \tag{3.14}$$

It is noticed that in this expression, two partitions of T are existent, i.e., $\{t_0, t_1, \cdots, t_M\}$ are used to denote premium payment dates and $\{\tau_0, \tau_1, \cdots, \tau_Q\}$ the trading dates. Later, Equation (3.14) is used in order to simulate the ruin probability of the hedger in this case.

3.3 An illustrative example

Again, because of its popularity, an equity–linked life insurance contract with guarantees is applied as an illustrative example. Our goal is not only to price the issued contract, but to derive the discretized originally continuous risk–minimizing strategy, to study the cost process, and further to investigate the hedger's net loss.

We consider a specific guaranteed equity–linked pure endowment life insurance contract, which provides the buyer of such a contract the payoff

$$f(t_M, S) = \sum_{i=0}^{M-1} Ke^{gt_{i+1}} + \alpha \sum_{i=0}^{M-1}(i+1)K\left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\,\Delta t}\right]^+, \tag{3.15}$$

if he survives the maturity of the contract.[6] In comparison with the illustrative example given in Equation (2.11) of Chapter 2, a periodic instead of a constant guarantee is provided to the insured. In this specific case, the final payment is dependent on the minimum guaranteed interest rate g, the participation rate in the surpluses α, the duration of the contract M and more importantly the whole stock prices. Specified at the beginning of

[6] At the sight, $\sum_{i=0}^{M-1} Ke^{g(T-t_i)}$ might be more intuitive in comparison with $\sum_{i=0}^{M-1} Ke^{gt_{i+1}}$. But in fact, these two sums are equal.

$$\begin{aligned} \sum_{i=0}^{M-1} Ke^{gt_{i+1}} &= \sum_{i=0}^{M-1} Ke^{g\,(i+1)\Delta t} = \frac{Ke^{g\,\Delta t}(1 - e^{g\,M\,\Delta t})}{1 - e^{g\,\Delta t}} = \frac{Ke^{g\,\Delta t}(1 - e^{gT})}{1 - e^{g\,\Delta t}} \\ \sum_{i=0}^{M-1} Ke^{g(T-t_i)} &= Ke^{gT}\sum_{i=0}^{M-1} e^{-gt_i} = \frac{Ke^{gT}(1 - e^{-g\,M\,\Delta t})}{(1 - e^{-g\Delta t})} = \sum_{i=0}^{M-1} Ke^{gt_{i+1}}. \end{aligned}$$

the contract, the premium K is paid periodically by the insured till the maturity of the contract or the death of the insured, whichever comes first. If the insured survives the maturity of the contract, he obtains the guaranteed amount and the accumulated boni (participation in the surplus of the company), which are represented by a sequence of European call options with strike $e^{g\,\Delta t}$.

After substituting the $f(t_M, S)$–value into Equation (3.1), we easily obtain the loss of the company for the first situation, where the insurer invest all the premiums in the risk free asset with a rate of return r.

$$\begin{aligned} L_T^{rf} &= e^{-rT} Y_T^x \left(\sum_{i=0}^{M-1} K e^{g t_{i+1}} + \alpha \sum_{i=0}^{M-1} (i+1) K \left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t} \right]^+ \right) \\ &\quad - \sum_{i=0}^{M-1} (Y_{t_i}^x - Y_{t_{i+1}}^x) \sum_{j=0}^{i} K e^{-r t_j} - Y_T^x \sum_{j=0}^{M-1} K e^{-r t_j}. \end{aligned} \tag{3.16}$$

Due to the unrealistic constraint of the second extreme case, we skip this case and jump to the third case, where the insurer hedges his risk by using the risk–minimizing strategy. Above all, the discretized risk–minimizing strategy for this specific contract is to be derived in order to be able to computer the loss of the insurer.

3.3.1 Time–discretized risk–minimizing strategy

Following Equation (3.12), we need to calculate $F_s(t, T, S)$, $t \in \tau^Q$, for this specific equity–linked life insurance contract in order to obtain the discrete–time version of the continuous risk-minimizing strategy. It is well–known that the price of a contingent claim at time t equals the expected discounted value of the terminal payoff conditional on the information structure till time t, $t \in [0, T]$, under the equivalent martingale measure. According to the same calculation in Proposition 2.2.8, we obtain for $t \in \tau^Q$:

$$\begin{aligned} F(t,T,S) &= E^*[e^{-r(T-t)} f(t_M, S) | \mathcal{F}_t] \\ &= e^{-r(T-t)} \sum_{i=0}^{M-1} K e^{g t_{i+1}} + \alpha K \sum_{i=0}^{M-1} (i+1) \Bigg\{ 1_{\{t > t_{i+1}\}} e^{-r(T-t)} \left[\frac{S(t_{i+1})}{S(t_i)} - e^{g\Delta t} \right]^+ \\ &\quad + 1_{\{t_i < t \le t_{i+1}\}} e^{-r(T-t_{i+1})} \left(\frac{S(t)}{S(t_i)} N(d_1^{(t,t_i)}) - e^{g\,\Delta t} e^{-r(t_{i+1}-t)} N(d_2^{(t,t_i)}) \right) \\ &\quad + 1_{\{t \le t_i\}} e^{-r(t_{M-1}-t)} \left(N(d_1) - e^{(g-r)\,\Delta t} N(d_2) \right) \Bigg\}, \end{aligned}$$

with

$$\begin{aligned} d_{1/2}^{(t,t_i)} &= \frac{\ln S(t)/S(t_i) - g\,\Delta t + (r \pm \frac{1}{2}\sigma^2)(t_{i+1} - t)}{\sigma \sqrt{t_{i+1} - t}} \\ d_{1/2} &= \frac{(r - g \pm \frac{1}{2}\sigma^2)\Delta t}{\sigma \sqrt{\Delta t}}. \end{aligned}$$

where $N(\cdot)$ is the cumulative standard normal distribution function. From the derived price of the contingent claim we take the derivative with respect to S_t and obtain for $t \in]t_i, t_{i+1}]$

$$F_s(t,T,S) \;=\; \alpha K\,(i+1)e^{-r(T-t_{i+1})}\frac{1}{S_{t_i}}N(d_1^{(t,t_i)}). \tag{3.17}$$

In our context, only those derivative at $t \in \tau^Q$ are of importance. At different trading dates, $F_s(t,T,S)$ could take very similar values as long as all of these trading dates lie in a same time interval resulting from the partition of T according to the premium payments. For instance, in Figure 3.1, both τ_1 and τ_2 lie in the interval $]t_0,t_1]$, the derivatives F_s distinguish from each other only by $N(d_1^{(\tau_1,t_0)})$ and $N(d_1^{(\tau_2,t_0)})$. Plugging Equation (3.17) in Equation (3.14), we obtain the net loss of the insurer for this specific contract straightforwardly. This resulting net loss together with the net loss for the case where the premiums are invested in the risk–free asset given in Equation (3.16) is simulated in the next section. Monte Carlo simulation method is applied to obtain some numerical results and to figure out which strategy is more beneficial to the insurance company by comparing the simulated technical ruin probabilities. Usually, ruin is defined as a "first passage" event, but due to the contract specification (pure endowment), ruin is defined as the event that the discounted net loss of the insurance company is larger than zero. Hence, the ruin probability is given as the frequency of the net loss of the insurer is larger than zero. Hence, an insurance company aims at reaching a ruin probability which is as small as possible.

3.3.2 Numerical results

This section targets at simulating the insurer's losses for different cases:

1) The insurer invests the premiums in the risk free asset at a fixed rate of interest r (Equation (3.16)).

2) The hedger uses a time–discretized risk–minimizing hedging strategy and the hedging frequency is the same as the premium payments ($Q = M \Rightarrow \Delta t = \Delta\tau$). If it is assumed that premiums are paid yearly, then the adjustment of the hedging strategies occurs yearly as well.

3) The hedger uses a time–discretized risk–minimizing hedging strategy and the hedger adjusts his trading strategy 12 times as frequently as premium payments ($Q = 12M \Rightarrow \Delta t = 12\Delta\tau$). That means the insurer adjusts the trading portfolio monthly.

The distinguishing between the last two scenarios is done in order to find out whether the hedger is able to reach a smaller ruin probability by increasing the trading frequency.

Due to the independence assumption between the mortality risk and the financial risk, in principle, the simulation of the losses reduces to simulating: a) the survival process $\{Y_t^x\}_{t\in\tau^Q}$ and b) the payoff of the pure endowment insurance contract $f(t_M,S)$ (or the

corresponding derivative of $f(t_M, S)$ with respect to the stock) respectively. In order to simulate the survival process, we just need to know the survival probability $\{{}_tp_x\}_{t\in[0,T]}$, which can be calculated by a hazard rate function. In this place, the Gompertz–Makeham hazard rate function from Møller (1998) is adopted, i.e:

$$\mu_{x+t} = 0.0005 + 0.000075858 \cdot 1.09144^{x+t} \qquad t \geq 0.$$

This function was used in the Danish 1982 technical basis for men. Consequently, the survival probability of an x–aged life is given by

$${}_tp_x = \exp\left\{-\int_0^t (0.0005 + 0.000075858 \cdot 1.09144^{x+u})du\right\}.$$

Another parameter which should be considered before starting a simulation is the fair premium K^*. According to the analyses in Section 3.1, non–optimal K–values could cause infinite losses or profits to the hedger asymptotically. According to Definition 2.2.9, a premium is called fair, if the expected discounted accumulated premium income equals the expected discounted accumulated payoff of the contract under the equivalent martingale measure. In addition, a fair periodic premium can be calculated explicitly or implicitly by fair combinations of two parameters. In comparison with Example given in Equation (2.11), the fair premium cannot be determined explicitly, because the final payment of the contract depends on the periodic premiums. Substituting this final payment in the expression of the optimal premium equation (Equation (3.3)), the K–terms would be left out in the calculation. Hence, the optimal K^* can only be determined implicitly through the fair relationship between the participation rate α and the minimum guaranteed interest rate g. That is, for a given g, we obtain a corresponding participation α^*, which makes the contract fair. A straightforward application of Equation (3.3) for the equivalent martingale measure leads to α as a function of g:

$$\alpha^*(g) = \frac{\sum_{i=0}^{M-1}\sum_{j=0}^{i} e^{-rt_j}({}_{t_i}p_x - {}_{t_{i+1}}p_x) + {}_Tp_x \sum_{j=0}^{M-1} e^{-rt_j} - {}_Tp_x e^{-rT}\sum_{i=0}^{M-1} e^{gt_{i+1}}}{{}_Tp_x \sum_{i=0}^{M-1}(i+1)e^{-rt_{M-1}}(N(d_1) - e^{(g-r)\Delta t}N(d_2))}. \quad (3.18)$$

Those scrupulous readers might have noticed the finer difference between the α–expressions in Equation (3.18) and in Proposition 2.2.10. However, when the difference in the guarantee part is ignored, these two expressions are exactly the same due to the following lemma.

Lemma 3.3.1. *It holds*

$$\sum_{i=0}^{M-1} e^{-rt_i}\,{}_{t_i}p_x = \sum_{i=0}^{M-1}\sum_{j=0}^{i} e^{-rt_j}({}_{t_i}p_x - {}_{t_{i+1}}p_x) + {}_Tp_x \sum_{j=0}^{M-1} e^{-rt_j}.$$

PROOF:

$$\begin{aligned}
&\sum_{i=0}^{M-1}\sum_{j=0}^{i} e^{-rt_j}({}_{t_i}p_x - {}_{t_{i+1}}p_x) + {}_{T}p_x \sum_{j=0}^{M-1} e^{-rt_j} \\
= &\sum_{j=0}^{M-1}\sum_{i=j}^{M-1} e^{-rt_j}({}_{t_i}p_x - {}_{t_{i+1}}p_x) + {}_{T}p_x \sum_{j=0}^{M-1} e^{-rt_j} \\
= &\sum_{j=0}^{M-1} e^{-rt_j}\left(\sum_{i=j}^{M-1}({}_{t_i}p_x - {}_{t_{i+1}}p_x)\right) + {}_{T}p_x \sum_{j=0}^{M-1} e^{-rt_j} \\
= &\sum_{j=0}^{M-1} e^{-rt_j}\left({}_{t_j}p_x - {}_{t_M}p_x\right) + {}_{T}p_x \sum_{j=0}^{M-1} e^{-rt_j} \\
= &\sum_{j=0}^{M-1} e^{-rt_j}\,{}_{t_j}p_x
\end{aligned}$$

For the first equality, Fubini's theorem for summations (switching the order of summation in multiple sums) is applied. The last equality results from the fact that ${}_{t_M}p_x = {}_{T}p_x$. □

In Table 3.1, some exemplary fair values of α are listed. Obviously, there exists a negative relationship between fair α's and g's. Furthermore, the fair α^* rises substantially as the duration of the contract increases. This is due to the fact that the periodic bonuses in the issued contract are held by the insurer till the maturity date, without giving any compensations to the customer. A long duration of the contract implies that the insurer keeps more bonuses of his customers for a longer time, which hampers the insured to reinvest the periodic bonuses to a large extent. According to the fair premium principle, a larger α–value becomes necessary to make the contract fair. These values for the fair participation rate α^* combined with the corresponding g's and M's are used in simulating the ruin probabilities. Of course the fair participation rate also depends on some other parameters like σ and the survival probabilities. However, these dependencies are not of interest in the analysis of this chapter.

Simulating the net loss in the first case, where the company invests the premium incomes in a risk free asset, is relatively simple. Simulate the price processes $\frac{S(t_{i+1})}{S(t_i)}$, $i = 0, \cdots M-1$ under the market measure and substitute them into the $f(t_M, S)$ expression, then one sample of the claim $f(t_M, S)$ is obtained. Combined with the simulated $Y^x_{\tau_1}, \cdots, Y^x_{\tau_Q}$, one path of the loss is generated. In the Monte Carlo simulation, if m paths are generated, the ruin probability of the insurance company is approximated as the ratio:

$$\frac{\text{the number of the paths where the simulated loss is above 0}}{m}.$$

By additionally taking account of the derivatives $F_s(t, T, S)$, $t \in \tau^{\mathcal{Q}}$, the ruin probabilities for the risk–minimizing strategies are achieved similarly. Following the procedure we introduced above, the ruin probabilities for Cases 1), 2) and 3) are obtained after simulating

Duration M	Minimum Guarantee g	Fair Participation Rate α^*
$M = 12$	$g = 0.0275$	0.37587
$M = 12$	$g = 0.0325$	0.31939
$M = 12$	$g = 0.0375$	0.25634
$M = 20$	$g = 0.0275$	0.49067
$M = 30$	$g = 0.0275$	0.70779

Table 3.1: Fair participation rates α's with following parameters: $r = 0.05$, $x = 35$, $\sigma = 0.2$.

the losses 100000 times.

Table 3.2 exhibits how the ruin probability depends on the market performance of the stock, which is described by the rate of return μ. Three different μ values, $\mu < r$, $\mu = r$, and $\mu > r$ are used. The percentage numbers in the last column of the table give the ratio of the ruin probability in the case of $Q = M$ and $Q = 12M$ to the ruin probability in Case 1 respectively. First of all, it is observed that the ruin probability in the case of discretized risk–minimizing hedging is considerably smaller than in the first case. In the situation $Q = M$, the ruin probabilities are reduced by 69.28%, 62.47% and 77.95% respectively for $\mu = 0.04$, $\mu = 0.05$ and $\mu = 0.06$. The same phenomenon is observed for the situation of $Q = 12M$ with the percentage numbers 76.02%, 74.45% and 77.74%. Second, a common observation for the first case and the case $Q = 12M$ is that the ruin probability increases with the value of μ. This is due to the fact that a better performance of the stock leads to a higher liability of the insurer. However, this relationship between μ and the ruin probability in the discretized risk–minimizing hedge ($Q = 12M$) is not so noticeable as in Case 1. And in case $Q = M$ this relationship ceases to be valid, i.e. the relationship between the ruin probability and μ is quite ambiguous (see also Tables 3.3–3.5). Theoretically, it is valid that the more frequently the insurer updates his risk–minimizing hedging strategies, the more the financial risks are reduced. Furthermore, the insurer can eliminate all the financial risks if he could hedge continuously. However, the accumulated hedging error caused by discretizing the continuous risk–minimizing hedging strategy destroyed this argument. This is why it is observed that not all the ruin probabilities in the case $Q = 12M$ are smaller than in the case $M = Q$.

The relation between the ruin probability and the duration of the contract is illustrated in Tables 3.3, 3.4 and 3.5 for different μ–values. Above all, M plays a very important role in determining the fair participation rate α (c.f. Table 3.1). For different g's and M's different fair α's are obtained. Also in these cases the ruin probabilities are reduced substantially, with the use of discretized risk–minimizing strategies. Almost overall a positive relationship between the ruin probability and M is observed. In the first case, obviously the effect of M on the insurer's liability dominates that of M on his accumulated premium incomes. Ruin appears more likely as M increases. In the second case,

Case 1	
μ	Ruin Prob.
0.04	0.45291
0.05	0.47996
0.06	0.53353

Case 2: $Q = M$		
μ	Ruin Prob.	%
0.04	0.13914	30.72%
0.05	0.13213	27.53%
0.06	0.11762	22.05%

Case 2: $Q = 12M$		
μ	Ruin Prob.	%
0.04	0.10861	23.98%
0.05	0.12262	25.55%
0.06	0.12412	23.26%

Table 3.2: Ruin probabilities for different μ's with parameters: $n = 100$, $\alpha = 0.37587$, $g = 0.0275$, $M = 12$, $r = 0.05$, $x = 35$, $\sigma = 0.2$.

Case 1	
M	Ruin Prob.
12	0.45291
20	0.47796
30	0.55110

Case 2: $Q = M$		
M	Ruin Prob.	%
12	0.13914	30.72%
20	0.12112	25.34%
30	0.19770	35.87%

Case 2: $Q = 12M$		
M	Ruin Prob.	%
12	0.10861	23.98%
20	0.15732	32.91%
30	0.27158	49.28%

Table 3.3: Ruin probabilities for different M with parameters: $n = 100$, $\alpha = 0.37587\,(M = 12)$, $\alpha = 0.49067\,(M = 20)$, $\alpha = 0.70779\,(M = 30)$, $g = 0.0275$, $\mu = 0.04$, $r = 0.05$, $x = 35$, $\sigma = 0.2$.

on the one hand, it is known that some discretization and duplication errors exist when the discretized risk–minimizing hedging strategy is used and that they are an essential part of the hedger's loss. As time goes by, the hedge errors accumulate (negative effect). On the other hand, a longer duration of the contract leads to higher premium inflows. Consequently, in the long run this reduces the insurer's loss to a certain extent (positive effect). Here the negative effect dominates the positive effect overall. This negative impact is so distinct that quite big ruin probabilities have resulted for $M = 30$ for the case of $Q = 12M$. In this subcategory, the insurer adjusts his portfolio much more frequently than the premium payment dates occur. The more often the hedger updates his strategy, the more duplication and discretization errors arise. Consequently, relatively high ruin probabilities are caused as the duration of the contract increases.

Case 1	
M	Ruin Prob.
12	0.47996
20	0.51102
30	0.57715

Case 2: $Q = M$		
M	Ruin Prob.	%
12	0.13213	27.53%
20	0.14114	27.62%
30	0.20320	35.21%

Case 2: $Q = 12M$		
M	Ruin Prob.	%
12	0.12262	25.55%
20	0.19740	38.63%
30	0.31127	53.93%

Table 3.4: Ruin probabilities for different M with parameters: $n = 100$, $\alpha = 0.37587\,(M = 12)$, $\alpha = 0.49067\,(M = 20)$, $\alpha = 0.70779\,(M = 30)$, $g = 0.0275$, $\mu = 0.05$, $r = 0.05$, $x = 35$, $\sigma = 0.2$.

Case 1	
M	Ruin Prob.
12	0.53353
20	0.58912
30	0.62525

Case 2: $Q = M$		
M	Ruin Prob.	%
12	0.11762	22.05%
20	0.14314	24.30%
30	0.23073	36.90%

Case 2: $Q = 12M$		
M	Ruin Prob.	%
12	0.12412	23.26%
20	0.20641	35.03%
30	0.38382	61.39%

Table 3.5: Ruin probabilities for different M with parameters: $n = 100$, $\alpha = 0.37587\,(M = 12)$, $\alpha = 0.49067\,(M = 20)$, $\alpha = 0.70779\,(M = 30)$, $g = 0.0275$, $\mu = 0.06$, $r = 0.05$, $x = 35$, $\sigma = 0.2$.

g, α	Case 1	$Q = M$	%	$Q = 12M$	%
$g = 0.0275$, $\alpha = 0.37587$	0.53353	0.11762	22.05%	0.13527	25.35%
$g = 0.0325$, $\alpha = 0.31939$	0.53607	0.12112	22.59%	0.14214	26.52%
$g = 0.0375$, $\alpha = 0.25634$	0.54609	0.13563	24.84%	0.15231	27.89%

Table 3.6: Ruin probabilities for different combinations of α and g with parameters: $n = 100$, $\mu = 0.06$, $M = 12$, $r = 0.05$, $x = 35$, $\sigma = 0.2$.

Table 3.6 demonstrates how the ruin probability changes with the fair combination of α and g. Overall, the effect of the minimum guarantee g dominates that of α. This is due to the fact that the resulting α's are relatively small, and consequently the bonuses part of the payment does not play a role as important as the minimum guarantee parameter g. Hence, a higher minimum interest rate guarantee leads to a higher ruin probability. Conversely, it is expected that the effect of the α's will dominate that of the g's for relatively small minimum interest rate guarantees g, say near 0, and relatively high participation rates.

3.4 Loss analysis in a binomial risk–minimizing hedge

Some of the numerical results obtained in the last section are not very satisfactory. The reduction in the ruin probabilities is relatively small when a high rebalancing frequency is combined with a long duration. Naturally, the question will be asked whether discretizing the hedging model instead of discretizing the strategy would improve the results. [7] It is well known the binomial model converges to the Black–Scholes model when the number of time periods increases to infinity and the length of each time period is infinitesimally

[7] According to Mahayni (2003), discretizing the hedging model (CRR–based hedging model) yields a more favorable result for the hedger than discretizing the continuous hedging strategy, in the sense that the binomial hedge with a suitably adjusted drift component is mean–self–financing, while the discretized Gaussian hedge sub–replicates the convex payoff for both a positive or a negative drift component.

short. This argument is proven e.g. by Cox et al. (1979) and Hsia (1983).[8] Due to this convergence reason, a binomial model is used as the relevant discrete–time model setup. We consider Møller (2001) risk–minimizing strategy for equity–linked life insurance contracts derived in the CRR model. There, for all $t \in \tau^Q = \{\tau_1, \cdots, \tau_Q\}$, the trading strategy has the form of

$$\begin{aligned} \xi_t^B &= Y^x_{t-\Delta\tau}\ {}_{T-(t-\Delta\tau)}p_{x+t-\Delta\tau}\ \alpha_t^f, \\ \eta_t^B &= Y^x_t\ {}_{T-t}p_{x+t}\, F(t,T,S) - Y^{(x)}_{t-\Delta\tau}\ {}_{T-(t-\Delta\tau)}p_{x+t-\Delta\tau}\ \alpha_t^f S_t^*, \end{aligned} \tag{3.19}$$

where $F(t,T,S)$ gives the value of the contingent claim at time t and α_t^f stands for the hedging strategy calculated in the binomial model without mortality risk. In addition, the binomial model contains Q periods. The discounted accumulated hedging error from using the risk–minimizing strategy at time $\tau_Q = T$ has the form of

$$C_T^{*,tot}(\phi^B) = \sum_{j=1}^{Q} e^{-r\tau_j}\, F(\tau_j,T,S)\, {}_{T-\tau_j}p_{x+\tau_j}\, (Y^x_{\tau_j} - Y^x_{\tau_{j-1}\ \Delta\tau}p_{x+\tau_{j-1}}). \tag{3.20}$$

The last term of the above equation $(Y^x_{\tau_j} - Y^x_{\tau_{j-1}\ \Delta\tau}p_{x+\tau_{j-1}})$ indicates that this unhedgeable risk results exactly from the difference between the actual number of survivors at time τ_j and the expected number of survivors at time τ_j conditional on time τ_{j-1}. In this case all the hedge errors are caused by mortality risk and the expected hedge errors are zero under both the subjective and the martingale measure, i.e., the binomial risk–minimizing strategy is mean–self–financing.

Similarly, the net loss of the insurance company is decomposed into three parts: the initial investment plus the hedging errors and minus the premium incomes.

$$\begin{aligned} L_T^{rb} &= V_0(\phi^B) + C_T^{*,tot}(\phi^B) - \sum_{i=0}^{M-1}(Y^x_{t_i} - Y^x_{t_{i+1}}) \sum_{j=0}^{i} K e^{-rt_j} - Y^x_T \sum_{j=0}^{M-1} K e^{-rt_j} \\ &= V_0(\phi^B) \sum_{j=1}^{Q} e^{-r\tau_j}\, F(\tau_j,T,S)\, {}_{T-\tau_j}p_{x+\tau_j}\, (Y^x_{\tau_j} - Y^x_{\tau_{j-1}\ \Delta\tau}p_{x+\tau_{j-1}}) \\ &\quad - \sum_{i=0}^{M-1}(Y^x_{t_i} - Y^x_{t_{i+1}}) \sum_{j=0}^{i} K e^{-rt_j} - Y^x_T \sum_{j=0}^{M-1} K e^{-rt_j}. \end{aligned} \tag{3.21}$$

In accordance with the net loss expression when the hedging model is discretized (Equation (3.21)), only the values of the contingent claims at certain discrete trading times

[8] The proof of Cox et al. (1979) is elegant but long and specific. It relies on Central Limit Theorem. Furthermore, they choose specific up and down factors so that the distribution of the stock return to have the same parameters as the desired lognormal distribution in the limit. While no restrictions are imposed on up and down parameters in Hsia (1983). His proof is shorter and requires few cases of taking limits.

$F(\tau_j, T, S)$ are relevant for the examination of the net loss of the hedger, where $F(\tau_j, T, S)$ denotes the time τ_j–value of the contract's payoff in the binomial model. In the following, again the specific contract construction introduced in Section 3.3 is used to obtain some numerical results in the binomial model.

Here, we follow the original parameter constellation as in Cox et al. (1979), i.e., specific up and down factors are chosen so that the distribution of the stock return to have the same parameters as the desired lognormal distribution in the limit. First, the market rate of return μ in the binomial model can be expressed as a function of the weighted sum of up and down values as follows:

$$\mu M = \mathrm{E}\left[\ln\left(\frac{S(T)}{S(t_0)}\right)\Big|\mathcal{F}_{t_0}\right] = Q(w \ln \mathrm{up} + (1-w) \ln \mathrm{down}), \tag{3.22}$$

where w gives the probability that the stock moves upwards under the market measure and E denotes the corresponding expected value under this measure. As in Cox et al. (1979), the up, the down movement and the interest rate per period are set as follows:

$$\mathrm{up} = \exp\left\{\sigma\sqrt{\frac{M}{Q}}\right\},\ \mathrm{down} = \exp\left\{-\sigma\sqrt{\frac{M}{Q}}\right\},\ r(Q) = \exp\left\{r\frac{M}{Q}\right\} - 1. \tag{3.23}$$

Plugging Equation (3.23) in (3.22), the market performance can also be characterized consequently by w:

$$w = \frac{1}{2} + \frac{\mu}{2\sigma}\sqrt{\frac{M}{Q}}.$$

Although μ/w is irrelevant in determining the hedging strategy in the binomial model, it does decide how the market performs and with which probability that the underlying asset reaches a certain knot under the market measure. Table 3.7 demonstrates several values of up, down and w, which are used later for the calculation of the ruin probability. In order to determine the loss of the insurer (Equation (3.21)), only the values of the contingent claims at $\tau_i, i = 0, 1, \cdots, Q$ together with the survival probabilities and processes matter. Since in the binomial model the calculations of these values and of the risk–minimizing strategy are quite simple, we immediately come to the results, which are demonstrated in Tables 3.8 and 3.9.

Table 3.8 illustrates how the ruin probability depends on the market performance of the stock for two subcases $Q = M$ and $Q = 12M$. First, an increase in the ruin probability is observed as μ goes up for $M = Q$, but this effect is not so pronounced as in the first case. Furthermore, it ceases to be valid as the trading frequency increases to $Q = 12M$. Second, with a more frequent rebalancing of the portfolio ($Q = M \to Q = 12M$) the ruin probability becomes very small. Almost all the financial risks are eliminated when the trading occurs 12 times as often as the premium payment. Accordingly, quite small ruin probabilities result in the scenario $Q = 12M$ in the binomial hedge. This advantage obtained from the binomial hedge can be explained by the following theory to some extent.

μ	up		down		w	
	$M = Q$	$Q = 12M$	$M = Q$	$Q = 12M$	$M = Q$	$Q = 12M$
0.04	1.2214	1.05943	0.818731	0.9439	0.600	0.529
0.05	1.2214	1.05943	0.818731	0.9439	0.625	0.536
0.06	1.2214	1.05943	0.818731	0.9439	0.650	0.543

Table 3.7: up, down and w-values with $\sigma = 0.2$.

Binomial Hedge: $Q = M$		Binomial Hedge: $Q = 12M$	
μ	Ruin Prob.	μ	Ruin Prob.
0.04	0.33283	0.04	0.04372
0.05	0.34284	0.05	0.06553
0.06	0.34689	0.06	0.03924

Table 3.8: Ruin probabilities with a binomial hedge with parameters: $n = 100$, $x = 35$, $\sigma = 0.2, M = 12$, $r = 0.05$, $g = 0.0275$, $\alpha = 0.203596$.

Binomial Hedge: $Q = M$		Binomial Hedge: $Q = M$	
M	Ruin Prob.	g, α	Ruin Prob.
12	0.34689	$g = 0.0275, \alpha = 0.37587$	0.34689
20	0.27327	$g = 0.0325, \alpha = 0.31939$	0.35986
30	0.14014	$g = 0.0375, \alpha = 0.25654$	0.42693

Table 3.9: Ruin probabilities with a binomial hedge with parameters: $n = 100$, $x = 35$, $\sigma = 0.2, \mu = 0.06$ Left: $g = 0.0275$; Right: $M = 12$.

In the binomial hedge, there exists no duplication errors, and all the hedging errors are caused by mortality risk. While in the use of the time–discretized risk–minimizing hedging strategy, duplication errors are encountered with each adjustment of the portfolio. As the adjustment frequency rises, the advantages from this rise can be largely destroyed by these duplication errors and consequently higher ruin probabilities are caused (c.f. Tables 3.2–3.5).

Table 3.9 is generated for the case $M = Q$ and shows the dependence of the ruin probabilities on the duration of the contract M (left table) and on the different α–g–combinations (right table). In contrast to Case 1 and the case of the originally continuous risk–minimizing strategy, the ruin probability does not go up with the duration of the contract M. It is known that only some intrinsic hedging errors will result from the use of this binomial hedging strategy, which are completely caused by the mortality risk. The size of these intrinsic hedging errors is small in comparison with the premium inflows of the insurer. Therefore, a quite small ruin probability is observed, e.g. 0.14014 for $M = 30$. It could easily be shown that almost no ruin probability will result if a long duration of the contract is combined with a high adjustment frequency. Hence, a binomial hedge improves the stability of those insurers, who mainly deal with long-term contracts or/and adjust their trading portfolio very frequently. The effect of the combination of α and g on the ruin probability remains unchanged (the effect of g dominates α). Rather, larger values of the ruin probability are observed compared to the originally continuous risk–minimizing strategy. This is due to the fact that both the duration of the contract ($M = 12$) and the frequency of adjusting the trading portfolio are chosen quite low ($Q = 12$). Consequently, the advantages from the binomial hedge are not so pronounced.

3.5 Summary

This chapter represents a simulation study to examines the goodness of applying risk–minimizing hedging strategies by investigating the net loss of a life insurance company issuing identical pure endowment contracts to n identical customers. More specifically, ruin probability is used as the criterion. It is observed that a considerable decrease in the ruin probability is achieved when the hedger uses a time–discretized risk–minimizing strategy. Nevertheless, the magnitude of the reduction becomes quite small and the advantage of using this time–discretized strategy almost disappears as the hedging frequency is increased. This is due to the fact that by discretization the originally mean–self–financing continuous risk–minimizing hedging strategy is not mean–self–financing any more. Furthermore, it causes some extra duplication errors, which may increase the insurer's net loss considerably. It is shown that the simulation results are greatly improved when the hedging model instead of the hedging strategy is discretized. The effect is particularly distinct when long–term contracts are taken into consideration or when the hedging strategy is adjusted quite frequently.

In this chapter, the simulation errors are not taken into consideration. However, since

the results for these two discrete–time hedging strategies differ much from each other, analogous results could be expected after the simulation errors are taken into account. Furthermore, the result in this paper is contract– and model–dependent, i.e., another specification of the contract or another dynamics of the underlying asset could lead to different results.

The contract considered in the present chapter is a pure endowment contract. It will be a natural extension to analyze an endowment contract, in which the insured will get paid both on an early death and on survival of the maturity date. Furthermore, all customers and all the issued insurance contracts are assumed to be identical in this paper. It would be interesting to study the net loss and the corresponding ruin probability when different customers, e.g., customers with different entering or/and exiting times are considered.

Chapter 4

Hedging interest rate guarantees under mortality risk[1]

In life insurance mathematics, the insurer uses a certain death/survival distribution for pricing and hedging. Previous analyses indeed contain an implicit assumption that the certain future trend of a life expectancy the insurer assumes for pricing and hedging purposes coincides with the true trend of this life. In other words, the discrepancy between the true and assumed death/survival distribution is ignored so far. However, in reality, the phenomenon of model misspecification concerning the mortality and longevity risk is unavoidable and has attracted more and more attention.[2] It can be caused either by a false estimation or by an intentional abuse of the insurer. As to the former case, for instance, a medical breakthrough or a catastrophe could increase or decrease life spans to a big extent. This changes the insurer's expectation substantially. Furthermore, mortality can change a lot according to e.g. AIDS, new treatments, global warming, floods, etc. In reality, it is not uncommon that an annuity provider deliberately underestimates the survival probability of a potential customer, by which the insurer assumes the period of the annuity payment will become shorter and consequently, he can offer a higher annuity payment so that more customers are acquired. This phenomenon demonstrates a good example for the second cause of mortality misspecification.[3] Therefore, it's high time to analyze the effect of model misspecification associated with mortality and longevity risk on the pricing and hedging decisions of the insurance companies.

In addition to mortality misspecification, another source of model risk is investigated, i.e., model risk associated with the dynamic of the interest rate. In a standard approach in mathematical finance, a certain stochastic model is assumed to describe the underlying

[1] This chapter is based on a joint work with Antje Mahayni, c.f. Chen and Mahayni (2006).

[2] For example, Wilmott (2006) mentioned in his book a factor with which the death probabilities change over time, i.e., it is normally assumed that there is a trend which reduces the death probabilities with respect to each age class. In general, there exist different possible aspects which can change the death distribution in both ways. Therefore, it is in fact realistic to assume that these distributions change in a random way.

[3] This phenomenon appears in reality customarily because the brokers of the insurance company might have incentives which do not have to coincide with the insurer's.

asset or the term structure of the interest rate. However, the fairness of pricing and the effectiveness of hedging depend on the true dynamics. The fairness is defined in the sense that the present value of the contract payment to the contract–holder corresponds to the present value of his contributions. The effectiveness analysis of the trading strategies is based on the variance comparison of the hedging errors associated with the strategies. What if a wrong model is assumed? How does this wrong choice affect the insurer's pricing and hedging decisions? Most of insurers have a severe problem with the data inputs which are necessary to generate useful outputs. They have either collected inadequate data or encountered difficulties e.g in organizing their lapse, surrender and claims data in a manner which allows them to accurately model the interest rate. Therefore, the term structure models assumed by the insurer to describe the dynamic of the interest rate really only partially reflect the true evolution. It is high time to have a look at the model misspecification related to the interest rate risk.

In addition, due to the fact that life insurance contracts are most of time long–term contracts, it is not a good idea to assume an exogenous curve of the discount factor as in the option pricing. I.e., an endogenous model for the interest rate should be assumed. There should exist not only uncertainty about the further discount factor but also uncertainty about the initial discount factor. In other words, by varying the parameters of the assumed interest rate model, the variance and the expected value of the future discount factor are changed. Furthermore, the initial discount factor is changed, too.

Therefore, this chapter is designed to answer the question how the model misspecification concerning both the dynamics of the interest rate and the mortality risk affects the insurer's pricing and hedging decisions. How does the market incompleteness resulting from the uncertainty of the true stochastic processes which drive the interest rate dynamic and the uncertainty of the true (stochastic) death distribution play a role in the insurer's decisions? Misspecification of the interest rate dynamic may lead to a hedging error associated with each strategy concerning the payout at one particular maturity date. Misspecification of the death distribution can be interpreted in the sense that the hedger assumes a wrong number of bonds concerning one particular maturity date. Since both kinds of model misspecification are of high interest and importance, a combination of both is investigated in detail. The main contribution of this chapter is to analyze the distribution of the hedging errors resulting from the combination of these two sources of model misspecification.

For this purpose, life insurance products with guarantees are taken as examples. They combine endowment life insurances and an investment strategy with a minimum guarantee. In this chapter, we consider a very simple sort of guaranteed life insurance products[4], in which the customer pays periodic premiums. The benefit of the contract is determined by a guaranteed amount together with an endowment protection, i.e., the maximum of a

[4]Although the contract introduced in the following is quite simple, it is a very common and popular contract form in Germany. About 75% of the life insurance products sold in Germany belong to this category. This is so called mixed life insurance.

fixed amount and the guaranteed one is paid to the customer. [5] The maturity date of the contract is conditioned on the death time of the customer. It is either given by a fixed terminal date or the nearest future reference date after an early death.[6] As a compensation, the customer pays a periodic premium which is contingent on his death evolution, too. Obviously, periodic premiums make the insurer exposed to more risk, because he has no idea whether future periodic premium payments will be forthcoming. Hence, the contracts contain both mortality and interest rate uncertainty.

Usually, the financial market and mortality risk are assumed to be independent, which allows a separate analysis of both uncertainties, in particular if the market is complete. The mortality risk can be diversified by a continuum of contract policies. This is justified by the law of large numbers which states that the random maturity times can be replaced by deterministic numbers, i.e., the number of contracts which mature at each reference date is known with probability one. In addition, the financial market risk can be hedged perfectly by self–financing and duplicating trading strategies which are adjusted to the numbers of contracts which mature at each date.

In an incomplete market model, a separate analysis of financial market and mortality risk is no longer possible. Caused either by the financial market model and/or by a death distribution which changes over time stochastically, the market incompleteness makes it impossible to achieve a risk management strategy which exactly matches the liabilities. Therefore, it results in a non–zero hedging error with positive probabilities. It is intuitively clear that the distribution of the hedging error is influenced by the true death distribution and interest rate dynamic.

In the literature on model risk, there is an extensive analysis of financial market risk. Without postulating completeness, we refer to the papers of Avellaneda et al. (1995), Lyons (1995), Bergman, Grundy and Wiener (1996), El Karoui, Jeanblanc-Picqué and Shreve (1998), Hobsen (1998), Dudenhausen, Schlögl and Schlögl (1998) and Mahayni (2003). Certainly, there are also papers dealing with different scenarios of mortality risk and/or stochastic death distributions, for instance, Milevsky and Promislow (2001), Ballotta and Haberman (2006), Blake et al. (2004), and Gründl, Post and Schulze (2006). However, to our knowledge, there are no papers which analyze the distribution of the hedging errors resulting from the combination of both. Therefore, the purpose of this chapter is to analyze the effectiveness of risk management strategies stemming from the combination of diversification and hedging effects. In particular, it is interesting to look for a combination of diversification and hedging effects which is robust against model misspecification.

[5] It is not uncommon that additional option features are offered to the customer. One might think of an additional participation in the excess return of a benchmark index, c.f. Mahayni and Sandmann (2005). One can also or additionally think of an option to surrender the contract, c.f. for example Grosen and Jørgensen (2000).

[6] An early death means that the customer dies before t_{N-1} if t_N is the contract maturity implied by survival.

Neglecting model risk, the strategies which are considered are risk–minimizing. Intuitively, these strategies can be explained as follows. Without the uncertainty about the random times of death, the cash flow of the benefits and contributions is deterministic. In particular, the benefits can be hedged perfectly by long–positions in bonds with matching maturities. Therefore, the most natural hedging instruments are given by the corresponding set of zero coupon bonds. Apparently, a strategy containing the entire term structure is an ideal case. Because of liquidity constraints in general or transaction costs in particular[7], it is not possible or convenient for the hedger to trade in all the bonds. Hence, we consider hedging strategies containing a subset of the above zero bonds. Independent of the optimality criterion which is used to construct the hedging strategy, the effectiveness of the optimal strategy can be improved if there are additional hedging instruments available. In the case of restricting the set of hedging instruments, the bonds which are unavailable must be synthesized by the traded ones. Obviously, in contrast to the strategy in all bonds, the resulting strategy depends on the assumed interest model. In particular, we study the impact of model risk on the variance of the total duplicating costs.

In order to initialize the above strategies, the insurer needs an amount corresponding to the initial contract value, while he only obtains the first periodic premium at the beginning. Therefore, a credit corresponding to the (assumed) expected discounted value of the delayed periodic premiums should be taken by the insurer, because the initial contract value equals the (assumed) present value of the entire periodic premiums. The insurance company trades with a simple selling strategy to pay back this loan. Apparently, the effectiveness of this strategy in the liability side depends on the model risk too.

It is shown that, independent of the choice of the hedging instruments, the insurer stays on the safe side on average, i.e., a superhedge is achieved in the mean, when he overestimates the death probability. Thus, dominating the true death probabilities can also be explained by the use of conservative hedging strategies. In fact, it is the mortality risk which decides the sign of the expected discounted hedging costs and consequently determines whether a superhedge in the mean can be achieved. It is worth mentioning that the effect of model risk related to the interest rate depends on the mortality misspecification. In particular, if there is no mortality misspecification, all the considered strategies from the asset side are mean–self–financing, i.e., model risk related to the interest rate has no effect on the expected discounted hedging errors. If there does exist mortality misspecification, the model risk associated with the interest rate influences the size of the expected hedging costs.

However, when it comes to the variance analysis, the effect of the model associated with the interest rate is highlighted and so is the effect of restricting the hedging instruments. No model risk related to the interest rate implies that different hedging strategies considered in this paper lead to the same variance level of the total cost from both the asset

[7]For instance, there are trading constraints in the sense that not all zero coupons (maturities of zero coupon bonds) are traded at the financial market.

and liability side. This argument holds independent of the mortality misspecification and the choice of the hedging instruments. However, if there does exist model risk related to the term structure, the choice of the hedging instruments plays a very important role. Even when there exists no mortality misspecification, a variance markup always results when only a subset of hedging instruments are traded. I.e., the restriction on the hedging instruments results in a variance markup under model risk concerning the interest rate. Taking account of the combined effect of these two sources of model risk, we observe that if the set of hedging instruments is restricted, an overestimation of the death probability combined with a huge misspecification associated with the interest rate leads to a very high variance markup, and consequently it could lead to an increase in the shortfall probability. Therefore, we can conclude that the model misspecification resulting from both the interest rate and mortality risk has a pronounced effect on the risk management of the insurer.

The rest of this chapter is structured as follows. Section 4.1 specifies the product and states the basic model assumptions. In particular, we discuss the problem of fair contract specification. In Section 4.2, the hedging problem which is associated with model risk is introduced and some definitions which are needed for the analysis are given. Furthermore, this section analyzes hedging strategies consisting of a subset of zero coupon bonds and their cost processes under model risk. Mainly, we discuss the distribution of the hedging errors. Section 4.3 illustrates and discusses the cost distributions under different scenarios of model misspecification. Section 4.4 summarizes this chapter.

4.1 Product and model description

This section specifies the contract and introduces some terminology. In addition, the model risk is neglected for a while and fair parameter combinations are analyzed.

The contract considered in this chapter is a guaranteed endowment insurance contract with a periodic premium payment. I.e., the customer's death determines when the customer stops paying the premiums and obtains the benefits from the insurer and eventually the size of the benefits as well. For simplicity, we assume that the customer pays, as long as he lives, a constant periodic premium K until the last reference date before the contract maturity date t_N. Let $\mathcal{S} = \{t_0, \ldots, t_N\}$ denote a set of equidistant reference dates and $\Delta t = t_{i+1} - t_i$ the distance between two reference dates. In addition, if τ^x specifies the death time of a life aged x, then the set of premium dates $\mathcal{S}^P$ is given by

$$\mathcal{S}^P = \left\{t_0, \ldots, \min\left\{t_{N-1}, t_{n^*(\tau^x)}\right\}\right\},$$

where $n^*(t) := \max\{j \in \mathbb{N}_0 | t_j < t\}$. The investor receives his payoff at time $T = \min\left\{t_N, t_{n^*(\tau^x)+1}\right\}$, i.e., the earlier date of the fixed contract maturity t_N and the nearest future reference date after the death time τ^x. In particular, the payoff to the customer is given by

$$\bar{G}_T = \max\{h, G_T\}, \qquad \text{for} \qquad h > 0.$$

Thus, independent of the actual time of death, the insured (or his heirs) receive at least the amount h, i.e., h can be interpreted as the endowment part of the contract. In addition, we also consider a nominal capital guarantee, i.e., the insured gets back his paid premiums accrued with an interest guarantee. In general, the guaranteed part G resembles an insurance account where for each premium K a minimum interest rate g ($g \geq 0$) is granted. We use the following definitions

$$\begin{aligned} \tilde{K}_{t_i} &:= \sum_{j=0}^{i} K e^{g(t_i - t_j)},\ i = 0, 1, \ldots, N-1 \\ \text{and } G_{t_i} &:= \tilde{K}_{t_{i-1}} e^{g(t_i - t_{i-1})},\ i = 1, \ldots, N. \end{aligned}$$

Therefore, the contract payoff at the random maturity time T is given by

$$\begin{aligned} \bar{G}_T &= \max\{G_T, h\} = \max\{\tilde{K}_{t_s} e^{g\Delta t}, h\} \\ \text{where} \quad s &= \min\left\{N-1, n^*\left(\tau^x\right)\right\}. \end{aligned} \tag{4.1}$$

To sum up the contract specification, there are two basic death scenarios. One scenario is given by an early death, i.e., the insured dies before t_{N-1}, i.e., $\tau^x \in]t_{i-1}, t_i]$ ($i = 1, \ldots, N-1$). Here, the insured pays his periodic premiums until t_{i-1} and his heirs obtain the payoff $\bar{G}_{t_i}$ at t_i. In contrast, the other scenario is given when the insured survives the last premium date t_{N-1}, then he receives the payoff $\bar{G}_{t_N}$ at t_N. Thus, a death which occurs in the interval $]t_{N-1}, t_N]$ is not an early death in the sense of the insurance contract.

Since the chapter mainly discusses the effect of the discrepancy between the assumed and true death/survival probability, it is necessary to distinguish them by notation. It is assumed that the conventional notations introduced in Section 1.4.3 illustrate the real trend of life expectancies, i.e., the real death/survival probabilities. In the following, we use a tilde to denote the assumed death/survival probabilities which are used by the insurer for pricing and hedging. For instance, ${}_t\tilde{p}_x$ gives the assumed probability that the insured survives time t. The assumed probabilities do not have to coincide with the real ones, i.e. ${}_t\tilde{p}_x \neq {}_tp_x$. The rest analysis of this section focuses at determining the initial value of the contract and making analysis of fair contracts. Hence, only the assumed death/survival distribution is needed in this section.

Proposition 4.1.1 (Initial contract value)**.** *Let $D(t_0, t_i)$ $(i = 1, \ldots, N)$ denote the current (observable) market price of a zero coupon bond with maturity t_i. In a complete arbitrage free market, the present value of the benefit (under the assumed death and survival probability) is given by*

$$X_{t_0} = \sum_{i=0}^{N-1} \max\{G_{t_{i+1}}, h\}\ D(t_0, t_{i+1})\ {}_{t_i|t_{i+1}}\tilde{q}_x + \max\{G_{t_N}, h\}\ D(t_0, t_N)\ {}_{t_N}\tilde{p}_x.$$

PROOF: In order to determine the present value of the benefit $\bar{G}_T$, it is convenient to notice that

$$\bar{G}_T = \sum_{i=0}^{N-1} \bar{G}_{t_{i+1}} \, 1_{\{t_i < \tau^x \leq t_{i+1}\}} + \bar{G}_{t_N} \, 1_{\{\tau^x > t_N\}}.$$

Thus, applying the independence assumption between the financial market and mortality risk, the present value in the sense of the expected discounted value of $\bar{G}_T$ is given by

$$\begin{aligned} & E^* \left[e^{-\int_0^T r_u du} \bar{G}_T \right] \\ = & \sum_{i=0}^{N-1} E^* \left[\bar{G}_{t_{i+1}} e^{-\int_0^{t_{i+1}} r_u du} \right] E^* \left[1_{\{t_i < \tau^x \leq t_{i+1}\}} \right] + E^* \left[\bar{G}_{t_N} e^{-\int_0^{t_N} r_u du} \right] E^* \left[1_{\{\tau^x > t_N\}} \right] \\ = & \sum_{i=0}^{N-1} \bar{G}_{t_{i+1}} \, D(t_0, t_{i+1}) \, {}_{t_i|t_{i+1}}\tilde{q}_x + \bar{G}_{t_N} \, D(t_0, t_N) \, {}_{t_N}\tilde{p}_x \end{aligned}$$

□

Notice that the above expectation coincides with the initial investment in a risk management strategy which gives a perfect hedge under full diversification. The underlying strategy is discussed in detail in Section 4.2.

Now, we are able to answer the question how to specify a *fair* contract. The so–called equivalence principle states that a contract is fair if the present value of the contributions is equal to the present value of the benefits. The present value of the contributions of the customer under the assumed death and survival probabilities is given by the discounted expected value, i.e.,

$$P_{t_0} := E^* \left[\sum_{i=0}^{N-1} K e^{-\int_0^{t_i} r_s ds} \, 1_{\{\tau^x > t_i\}} \right] = K \sum_{i=0}^{N-1} D(t_0, t_i) \, {}_{t_i}\tilde{p}_x,$$

where the independence assumption between the financial and mortality risk is needed again. Therefore, a fair contract results from the following equality:

$$X_{t_0} \;=\; P_{t_0} \,. \tag{4.2}$$

It implies that, for a given premium payment K, any contract with a parameter combination (h, g) leading to $X_{t_0} - P_{t_0} > 0$ is an unfair contract in favor of the insured. On the contrary, any contract combinations of (h, g) resulting in $X_{t_0} - P_{t_0} < 0$ are in the benefit of the insurer. Since $X_{t_0} - P_{t_0}$ can be written as

$$\sum_{i=0}^{N-1} \left(\bar{G}_{t_{i+1}} \, D(t_0, t_{i+1}) \, {}_{t_i|t_{i+1}}\tilde{q}_x + \frac{1}{N} \bar{G}_{t_N} \, D(t_0, t_N) \, {}_{t_N}\tilde{p}_x - K D(t_0, t_i) \, {}_{t_i}\tilde{p}_x \right),$$

a sufficient technical condition for $X_{t_0} - P_{t_0} > 0$ is e.g. that for every $i = 0, \cdots, N-1$, it holds:

$$\bar{G}_{t_{i+1}}\ D(t_0, t_{i+1})\ {}_{t_i|t_{i+1}}\tilde{q}_x + \frac{1}{N}\,\bar{G}_{t_N}\ D(t_0, t_N)\ {}_{t_N}\tilde{p}_x - K D(t_0, t_i)\ {}_{t_i}\tilde{p}_x > 0.$$

Furthermore, since the insurer will not offer a negative minimum interest rate g to the insured, in the case of $g = 0$, i.e., $G_{t_{i+1}} = (i+1)K$, the resulting fair h as a function of K provides an upper bound for h. Intuitively, it is clear that a very high h–value should be offered to the customer if no interest rate is provided to the insured's contributions. This indicates that **probably** an h–value smaller than G_{t_N} would not give a fair contract. If this is the case, i.e., if the fair h shall be larger than G_{t_N}, the relation between h and K can be calculated from Equation (4.2) explicitly, i.e.,

$$h^*(K) \;=\; \frac{K \sum_{i=0}^{N-1} D(t_0, t_i)\ {}_{t_i}\tilde{p}_x}{\sum_{i=0}^{N-1} D(t_0, t_{i+1})\ {}_{t_i|t_{i+1}}\tilde{q}_x + \ D(t_0, t_N)\ {}_{t_N}\tilde{p}_x}.$$

Otherwise, if the fair h shall be smaller than G_{t_N}, say $k \cdot K = G_k \leq h < G_{t_{k+1}} = (k+1) \cdot K$, $k = 1, \cdots, N-1$, then in this case, the upper bounder for the fair h as a function of K results from Equation (4.2) straightforwardly:

$$h^*(K) \;=\; \frac{K\left(\sum_{i=0}^{N-1} D(t_0, t_i)\ {}_{t_i}\tilde{p}_x - \sum_{i=k}^{N-1}(i+1)\ D(t_0, t_{i+1})\ {}_{t_i|t_{i+1}}\tilde{q}_x - N\, D(t_0, t_N)\ {}_{t_N}\tilde{p}_x\right)}{\sum_{i=0}^{k-1} D(t_0, t_{i+1})\ {}_{t_i|t_{i+1}}\tilde{q}_x}.$$

In the following, we mainly focus on the contracts with a positive minimum interest rate guarantee. Besides, the case $h \geq G_{t_N}$ implies $\max\{h, G_T\} = h$ such that the asymmetry which is introduced by the maximum operator vanishes. This case is neither economically nor technically interesting. Of course, the condition $h < G_{t_N}$ would restrict the set of fair parameter constellations, because under our contract specification, small guarantee values could lead to some h^*–values which are much higher than G_T. However, this problem is unlikely to appear in reality because most of realistic insurance products incorporate additional options, c.f. footnote 5. These additional options reduce the value of the resulting fair parameter h^* to a big extent if K stays the same as in the case of no additional options. Thus, we study how to specify the fair contract parameters h^* and g^* for a given periodic premium K for $g > 0$ and $h < G_{t_N}$.

Corollary 4.1.2. *For the case $h < G_{t_N}$, i.e., let h be a constant such that there exists a $k \in \{1, \ldots, N-1\}$ with $G_{t_k} < h \leq G_{t_{k+1}}$, the fair contract is specified by a fair combination*

of h^* and g^* as follows:

$$h^*(g) = \frac{K \sum_{i=0}^{N-1} D(t_0,t_i)\ {}_{t_i}\tilde{p}_x - \sum_{i=k}^{N-1} G_{t_{i+1}}\ D(t_0,t_{i+1})\ {}_{t_i|t_{i+1}}\tilde{q}_x - G_{t_N}\ D(t_0,t_N)\ {}_{t_N}\tilde{p}_x}{\sum_{i=0}^{k-1} D(t_0,t_{i+1})\ {}_{t_i|t_{i+1}}\tilde{q}_x}.$$

PROOF: This corollary is a straightforward consequence of using Proposition 4.1.1, Equation (4.2) and the fact that $G_{t_k} < h \leq G_{t_{k+1}}$. □

Notice that h is a decreasing function of g in view of fair contract analysis. As g goes up, G_T increases and so does $\bar{G}_T$. A rise in h leads to an increase in $\bar{G}_T$ as well. I.e., the customer of such a contract benefits from both a higher h and a higher g.

4.1.1 Example

Recall that it is not necessary to specify a term structure model if one assumes that the relevant bond prices are given by market data. However, to avoid the summary of all prices with respect to the long contract maturities, the following examples are given according to a term structure which fits to a Vasiček–model with a parameter constellation summarized in Table 4.1.[8] As an example for the death distribution, the insurer might use the death distribution according to Makeham where

$$\begin{aligned} {}_t\tilde{p}_x &= \exp\left\{-\int_0^t \mu_{x+s}\,ds\right\}, \\ \mu_{x+t} &:= H + Kc^{x+t}. \end{aligned} \tag{4.3}$$

As a benchmark case, we use a parameter constellation along the lines of Delbaen (1990) which is given in Table 4.1. Based on the specific death distribution and the assumed term structure model of the interest rate given in Subsection 4.1.1, a product example is given in Table 4.2.

Using the illustrative parameters, fair values $h^*(g)$ are demonstrated in Figures 4.1 and 4.2 for two different spot rate volatilities $\bar{\sigma}$. For comparison reasons, the curve describing the evolution of G_{t_N} as a function of g is given in the figures additionally. First of all, an upper bound for the fair h^* results for $g = 0$. In case of $\bar{\sigma} = 0.02$, the upper bounds for h^* are given by 112926, 52297.3, 25628.4 for $x = 30, 40, 50$ respectively, and in case of $\bar{\sigma} = 0.03$, they are 102396, 47937.4, 24045.4 respectively. As we estimated, for the small values of g, the fair values of h lie mostly above the curve describing the evolution of G_{t_N} in g. However, these contracts are not of much interest in reality. Therefore, we mainly

[8] The Vasiček–model implies that the volatility $\sigma_{\bar{t}}(t)$ of a zero coupon bond with maturity $\bar{t}$ is $\sigma_{\bar{t}}(t) = \frac{\bar{\sigma}}{\kappa}(1 - \exp\{-\kappa(\bar{t} - t)\})$ where κ and $\bar{\sigma}$ are non–negative parameters. $\bar{\sigma}$ is the volatility of the short rate and κ the speed factor of mean reversion.

Benchmark parameter

contract parameter	interest rate parameter (Vasiček model)	mortality parameter (Makeham)
$g = 0.05$	initial spot rate = 0.05	
$h = 20673.6$ ($G_{t_N} = 35694.6$)	spot rate volatility = 0.03	$H = 0.0005075787$
$t_N = 30$ (years)	speed of mean reversion = 0.18	$Q = 0.000039342435$
$x = 40$, $K = 500$	long run mean = 0.07	$c = 1.10291509$

Table 4.1: Basic (assumed) model parameter.

Product Example

i	G_{t_i}	h	$\bar{G}_{t_i}$	$P(\tau^x \in]t_{i-1}, t_i])$	$D(t_0, t_{i+1})$
1	525.6	20 673.6	20 673.6	0.00178031	0.949742
2	1078.2	20 673.6	20 673.6	0.00190781	0.899889
...					
...					
22	20 546.9	20 673.6	20 673.6	0.00947623	0.289887
23	22 126.0	20 673.6	22 126.0	0.01029050	0.274033
24	23 786.0	20 673.6	23 786.0	0.01116730	0.259051
...					
...					
≥ 30	35 694.6	20 673.6	35 694.6	0.789179	0.184932

Table 4.2: Insurance account G and death dependent payoff $\bar{G}$ for an insurance contract with maturity in $t_N = 30$ years, guaranteed rate $g = 0.05$ and $h = 20673.6$ and a life aged $x = 40$. In particular, the parameter constellation is summarized in Table 4.1.

consider contracts which offer a minimum interest rate guarantee (slightly) smaller than (or equal to) the instantaneous risk free rate of interest at the contract–issuing date, but as a compensation, that a minimum amount of money (h) will be guaranteed to the customer if an early death occurs. Finally, notice that an increase in the spot rate volatility leads to a rise in the price of zero coupon bonds. Consequently, this results in a lower fair value for h, i.e. a little more intersection areas between G_{t_N} and fair–h–curves are observed in the case of $\sigma = 0.03$ illustrated in Figure 4.2 than in the case of $\sigma = 0.02$, i.e. Figure 4.1.

Fair parameter combinations (g^*, h^*)

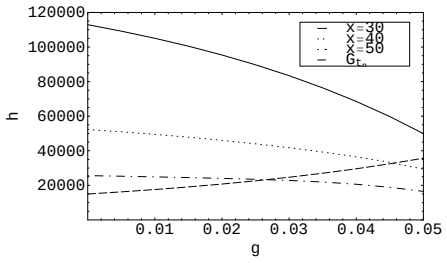

Figure 4.1: Fair parameter combinations for a contract as given in Table 4.1. In particular, the spot rate volatility is 0.02.

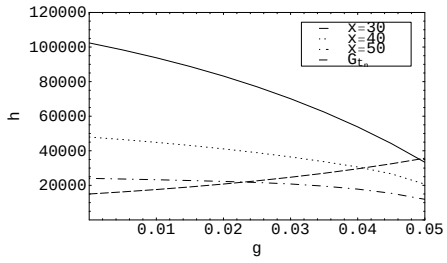

Figure 4.2: Fair parameter combinations for a contract as given in Table 4.1 and a spot rate volatility of 0.03 instead of 0.02.

4.2 Hedging

In reality, there are sources of market incompleteness which impede the concept of perfect hedging[9]. I.e., there are reasons why the strategies under consideration are not self–financing and duplicating with probability one. First, the insurance risk is a non–tradable risk. It cannot be hedged away by trading on the financial market and can only be reduced by diversification. Hence, the relevant hedging strategy cannot be perfect, in the sense of self–financing and duplicating perfectly. Second, it can be caused by model risk/misspecification. Model misspecification includes the possibility of a wrong choice of the stochastic processes which describe the dynamic of the zero coupon bonds as well as the possibility that the hedger assumes a death distribution which deviates from the true one. The deviation from the self–financing property is described by a continuous–time rebalancing cost process. Besides, the random death time can be reinterpreted as the real maturity of the insurance contract. This implies that even a hedge which is a perfect hedge under full diversification, i.e., when the random time of death can be replaced by deterministic numbers, gives a deviation between the value of the hedging strategy and the payoff of the insurance contract at the maturity. In the following, such deviations are called duplication costs. We adopt a more general definition of trading strategies which does not include the self–financing requirement. In particular, the strategies under consideration are only self–financing with full diversification and no model misspecification. The main purpose of the next subsection is to introduce some general and with respect to our contract specific definitions which are needed for the hedging analysis.

[9]By a perfect hedge, we mean the considered strategies duplicate the final payment of the contracts in addition to their self–financing characteristics. A perfect hedge can only be realized under full diversification, because this condition implies that the real death/survival numbers of the customer exactly correspond to the (assumed) expected death/survial number by the insurer as the number of the customers goes to infinity.

4.2.1 General Definitions

All the stochastic processes we consider are defined on an underlying stochastic basis $(\Omega, \mathcal{F}, \mathcal{F} = (\mathcal{F}_t)_{t\in[0,T^*]}, P^*)$, which satisfies the usual conditions stated in Section 2.1. Trading terminates at time $T^* > 0$. We assume that the price processes of underlying assets are described by strictly positive, continuous semimartingales. By a contingent claim X with maturity $T \in [0, T^*]$, we simply mean a random payoff received at time T, which is described by the $\mathcal{F}_T$–measurable random variable X.

Definition 4.2.1 (Trading strategy, value process, duplication)**.** *Let $D(., t_1), \ldots, D(., t_N)$ denote the price processes of underlying zero coupon bonds with maturities $t_1, \cdots, t_N$. A trading strategy ϕ in these assets is given by a $\mathbb{R}^N$–valued, predictable process which is integrable with respect to D. The value process $V(\phi)$ associated with ϕ is defined by*

$$V_t(\phi) = \sum_{i=1}^{N} \phi_t^{(i)} D(t, t_i).$$

If X is a contingent claim with maturity T, then ϕ duplicates X iff

$$V_T(\phi) = X, \quad P\text{–a.s.}.$$

The deviation of the terminal value of the strategy from the payoff is called duplication cost C^{Dup}, i.e.,

$$C_T^{dup} := X - V_T(\phi).$$

Definition 4.2.2 (Rebalancing cost process)**.** *If ϕ is a trading strategy in the assets $D(., t_1), \ldots, D(., t_N)$, the rebalancing cost process $(C_t^{reb}(\phi))_{t\in[0,T]}$ associated with ϕ is defined as follows:*

$$C_t^{reb}(\phi) := V_t(\phi) - V_0(\phi) - \sum_{i=1}^{N} \int_0^t \phi_u^{(i)} dD(u, t_i).$$

In particular, Itô's Lemma implies

$$C_t^{reb}(\phi) := \sum_{i=1}^{N} \int_0^t D(u, t_i)\, d\phi_u^{(i)} + \sum_{i=1}^{N} \int_0^t d\langle \phi^{(i)}, D(., t_i)\rangle_u.$$

By this definition, the rebalancing costs at two different trading dates are equally weighted when the costs are due. Recall that D^*, V^*, $C^{reb,*}$ and $C^{dup,*}$ denote the discounted asset, value, rebalancing and duplication cost respectively, e.g. $D^*(u, t_i) = \exp\{-\int_0^u r_s\, d\, s\} D(u, t_i)$.

Lemma 4.2.3. *C^{reb} and $C^{reb,*}$ are related as follows*

$$\begin{aligned} C_t^{reb,*} &= \int_0^t e^{-\int_0^u r_s\, ds}\; dC_u^{reb} + \int_0^t d\left\langle e^{-\int_0 r_s\, ds}, C^{reb}\right\rangle_u, \\ C_t^{reb} &= \int_0^t e^{\int_0^u r_s\, ds}\; dC_u^{reb,*} + \int_0^t d\left\langle e^{\int_0 r_s\, ds}, C^{reb,*}\right\rangle_u. \end{aligned}$$

PROOF: To simplify the notation, let $Y_u^{(0)} := e^{\int_0^u r_s\,ds}$. The first part of the proof follows from

$$dC^{reb,*}(\phi) = dV_t^*(\phi) - \sum_{i=1}^{N} \phi_t^{(i)}\, dD^*(t,t_i)$$

and an application of Itô's product chain rule in $V_t^* = \frac{V_t}{Y_t^{(0)}}$ and $D_t^* = \frac{D_t}{Y_t^{(0)}}$. The second part follows with similar reasonings. For those who are interested in the detailed derivation of this lemma, please refer to Section 2.1. □

Furthermore, for the purpose of a later use, the definition of (discounted) gain process is introduced.

Definition 4.2.4 ((Discounted) gain process)**.** *If ϕ is a trading strategy in the assets $D(.,t_1),\ldots,D(.,t_N)$, the discounted gain $(I_t^*(\phi))_{t\in[0,T]}$ and gain process $(I_t(\phi))_{t\in[0,T]}$ associated with ϕ are defined as follows:*

$$I_t(\phi) \;=\; \sum_{i=1}^{N} \int_0^t \phi_u^{(i)} dD(u,t_i); \qquad I_t^*(\phi) = \sum_{i=1}^{N} \int_0^t \phi_u^{(i)} dD^*(u,t_i),$$

with $D^*(u,t_i) = e^{-\int_0^u r_s ds} D(u,t_i)$, the discounted bond price.

Definition 4.2.5 ((Discounted) Total Cost)**.** *The (discounted) total cost is described as the sum of (discounted) rebalancing and duplication cost:*

$$C_t^{tot} \;=\; C_t^{reb} + C_t^{dup}, \qquad C_t^{tot,*} = C_t^{reb,*} + C_t^{dup,*}.$$

Definition 4.2.6 (Super– and Subhedge)**.** *A hedging strategy ϕ for the claim X is called superhedge (subhedge) iff $C_t^{tot}(\phi) \le 0$ ($C_t^{tot}(\phi) \ge 0$) for all $t \in [0,T]$. In particular, a strategy which is a superhedge and a subhedge at the same time is called perfect hedge.*

It is noticed that super– and subhedge in the mean can be defined similarly, when the expectation of the total cost is considered. A strategy which is super– and subhedge at the same time in the mean is called mean–self–financing.

Lemma 4.2.7. *The (discounted) total hedging cost C_T^{tot} and $C_T^{*,tot}$ are given by*

$$C_T^{tot}(\phi) = X_T - (V_0(\phi) + I_T(\phi)), \qquad C_T^{tot,*}(\phi) = X_T^* - (V_0^*(\phi) + I_T^*(\phi)).$$

PROOF: According to the above definitions, we have

$$C_T^{tot} = C_T^{reb} + C_t^{dup} = V_T - (V_0 + I_T) + X_T - V_T = X_T - (V_0 + I_T).$$

□

4.2.2 Contract–specific definitions

In this subsection, several definitions related to the insurance risk and our specific contract are explained.

Definition 4.2.8 (Diversification within a subpopulation, full diversification)**.**

(a) Diversification within a subpopulation: The insurance risk can be (completely) reduced by diversification within a subpopulation, who are exposed to the same danger. We can understand diversification within a subpopulation as the effect, that the relative total loss of an insurance company in a certain risk class, which can be regarded as a probability distribution, is more stable than the individual loss . This technique makes the use of the law of large numbers. The larger the number of the identically, but not independently distributed, i.e. correlated incidents (casualties) is, the less the actual loss in a risk class deviates from the expected loss. That is, the risk decreases as the number of the insured increases because the loss for the entire group can be looked at predictable.

(b) Full diversification: An insurance risk is completely diversifiable (full diversification), if the law of large numbers shows an asymptotical convergence of expected value.

Theoretically the mortality risk is completely diversifiable, if catastrophe could be excluded. In our contract specification, full diversification implies that the random time of death can be replaced by deterministic numbers, i.e., the insurer can predict how many contracts become due at t_i, $i = 1, \cdots, N$. It's a usual and acceptable assumption in life insurance.

Definition 4.2.9 (Model risk, model risk related to the interest rate and mortality risk)**.**

(a) Model risk: It is also called model misspecification. We say institutions are exposed to model risk when they rely heavily on models for pricing and hedging financial transactions or monitoring risks. This is the risk that models are applied to tasks for which they are inappropriate or are otherwise implemented incorrectly.

(b) Model risk related to the interest rate: We use this to refer to the deviation of the assumed parameters used in the term structure model from the true ones. This results in that the assumed bond price processes deviate from the true ones.

(c) Model risk related to the mortality risk: It gives the deviation of the true death/survival distribution from the real trend of life expectancies.

Throughout this chapter, we put a tilde above the true parameters to denote the assumed ones, not only for mortality parameters but also parameters concerning the interest risk.

In the context of the insurance contract under consideration, the maturity T is stochastic. In particular, the payoff of the claim at T is $\bar{G}_T$. With respect to the terminal costs C_T^{reb} and C_T^{dup}, the following lemma is useful:

Lemma 4.2.10. *Let* $T = \min\left\{t_N, t_{n^*(\tau^x)+1}\right\}$, $n^*(t) := \max\{j \in I\!N_0 | t_j < t\}$ *and* $X_T = \bar{G}_T$. *For* $i =$ *reb, dup, tot, it holds*

$$C_T^i(\phi) \;=\; C_{t_N}^i(\phi) 1_{\{\tau^x > t_N\}} + \sum_{i=0}^{N-1} C_{t_{i+1}}^i(\phi) 1_{\{\tau^x \in]t_i, t_{i+1}]\}}.$$

In particular, we have

$$C_T^{dup}(\phi) \;=\; \left(\bar{G}_{t_N} - V_{t_N}(\phi)\right) 1_{\{\tau^x > t_N\}} + \sum_{i=0}^{N-1} \left(\bar{G}_{t_{i+1}} - V_{t_{i+1}}(\phi)\right) 1_{\{\tau^x \in]t_i, t_{i+1}]\}}.$$

PROOF: The above lemma is a straightforward consequence of the definition of T and of duplication cost in Definition 4.2.1. □

After the needed definitions are established, in the following we would have a look at the hedging perspective.

4.2.3 Hedging with subsets of bonds

The hedging possibility and the hedging effectiveness of a claim depend on the set of available hedging instruments. Hedging is easy if the hedging instrument coincides with the claim to be hedged, i.e. its payoff is given by a random variable which is indistinguishable from the one which represents the claim. However, this is not the case in our context. With respect to the insurance contract under consideration, the most natural hedging instruments are given by the set of zero coupon bonds with maturities $t_1, \ldots, t_N$, i.e., by the set $\{D(., t_1), \ldots, D(., t_N)\}$.[10] Thus, we consider the set Φ of hedging strategies which consist of these bonds, i.e.,

$$\Phi = \left\{ \phi = (\phi^{(1)}, \ldots, \phi^{(N)}) \,\middle|\, \phi \text{ is trading strategy with } V(\phi) = \sum_{j=1}^{N} \phi^{(i)} D(., t_i) \right\}.$$

However, due to liquidity constraints in general or transaction costs in particular, it is not possible or convenient to use all bonds for the hedging purpose. This is modelled in the following by restricting the class of strategies Φ. The relevant subset is denoted by $\Psi \subset \Phi$. Obviously, independent of the optimality criterion which is used to construct the hedging strategy, the effectiveness of the optimal strategy $\psi^* \in \Psi$ can be improved if there

[10] This is motivated by the contract value given in Proposition 4.1.1.

are additional hedging instruments available. To simplify the exposition, we propose that the assumed interest rate dynamic is given by a one–factor term structure model and set

$$\Psi = \left\{ \psi \in \Phi \,\middle|\, \psi = (0, \ldots, 0, \psi^{(N-1)}, \psi^{(N)}) \right\}.$$

Two comments are necessary. First, the assumption of a one–factor term structure model implies that two bonds are enough to synthesize any bond with maturity $\{t_1, \ldots, t_N\}$. However, the following discussion can easily be extended to a multi–factor term structure model. Second, as the bonds cease to exist as time goes by, it is simply convenient to use the two bonds with the longest time to maturity[11]. For a discussion on an optimal choice of bonds c.f. Dudenhausen and Schlögl (2002).

Apparently, the insurer has to decide what hedging strategies to use and what kind of aims he is striving after. I.e., certain hedging criteria should be imposed on the hedging strategies. The first criterion we come up with is that the considered trading strategies should be mean–self–financing if no model risk exists. However, we argue that the mean–self–financing feature is not enough to give a meaningful strategy. This is reasoned by the following proposition:

Proposition 4.2.11. *For $\phi \in \Phi$ and a claim with payoff $X_T = \bar{G}_T$ at random time $T = \min\left\{t_N, t_{n^*(\tau^x)+1}\right\}$, it holds*

$$E^*\left[C_T^{tot,*}(\phi)\right] = X_{t_0}^* - V_{t_0}^*(\phi)$$

where $X_{t_0}^* = X_{t_0}$ is given as in Proposition 4.1.1.

Proof: According to Lemma 4.2.7, and the fact that X^* and I^* are P^*–martingales,[12] we obtain

$$E^*\left[C_T^{tot,*}(\phi)\right] = E^*[X_T^*] - (V_0^*(\phi) + E^*[I_T^*(\phi)]) = X_{t_0}^* - V_{t_0}^*(\phi).$$

□

The above proposition states that any strategy where the initial investment coincides with the price of the claim to be hedged is self–financing in the mean. Therefore, it is necessary to use an additional hedging criterion. In the following, we consider a conventional hedging criterion used in the incomplete market, i.e., the considered hedging strategies are risk–minimizing if model risk is neglected. First of all, if a strategy is risk–minimizing, it is mean–self–financing. Therefore, risk–minimizing feature contains mean–self–financing feature. In the analysis of risk–minimizing hedging, we look for an admissible strategy which minimizes the the remaining risk at any time $t \in [0, T]$.[13] Along the lines of Møller

[11] It is more realistic to use two bonds where their maturities are not very close, e.g t_1– and t_N–bond. However, by using these two bonds, an extra problem appears because t_1–bond ceases to exist in the market after time t_1.

[12] In the above context, the martingale measure coincides with the real world measure P.

[13] This remaining risk is so–called intrinsic risk process $(R_t(\phi))_{\{t\in[0,T]\}}$ and it is defined by

$$R_t(\phi) = E^*[(C_T^{tot,*}(\phi) - C_t^{tot,*}(\phi))^2 | \mathcal{F}_t].$$

It corresponds to the conditional expected squared value of future costs under the equivalent martingale measure, c.f. Chapter 2.

(1998), we derive the risk–minimizing hedging strategy for both cases: when the entire term structure or when only the last two zero bonds are used. They are simply denoted by ϕ and ψ respectively. The motivation and derivation of the hedging strategies is based on the value process of the claim to be hedged.

Proposition 4.2.12 (Value Process)**.** *In our arbitrage–free model setup, the contract value at time $t \in [0, \tau^x]$ is given by*

$$X_t = \left[\sum_{j=n^*(t)+1}^{N-1} \bar{G}_{t_j} \, D(t,t_j) \, {}_{t_{j-1}|t_j}\tilde{q}_{x+t} + \bar{G}_{t_N} \, D(t,t_N) \, \underbrace{\left({}_{t_{N-1}|t_N}\tilde{q}_{x+t} \, + {}_{t_N-t}\tilde{p}_{x+t}\right)}_{={}_{t_{N-1}-t}\tilde{p}_{x+t}} \right].$$

Proof: Obviously, (X_t) is a stopped process. Using standard theory of pricing by no arbitrage implies that the contract value at t $(0 \leq t < T)$ is given by the expected discounted payoff under the martingale measure P^*,i.e.,

$$\begin{aligned} X_t &= E^*[e^{-\int_t^T r(u)du}\bar{G}_T|\mathcal{F}_t] \\ &= E^*[e^{-\int_t^T r(u)du}\bar{G}_T|\mathcal{F}_t] \, 1_{\{t\leq\tau^x\}} + E^*[e^{-\int_t^T r(u)du}\bar{G}_T|\mathcal{F}_t] \, 1_{\{t>\tau^x\}} \\ &= E^*[e^{-\int_t^T r(u)du}\bar{G}_T|\mathcal{F}_t] \, 1_{\{t\leq\tau^x\}} + D(t,t_{n^*(t)+1})G_{n^*(t)+1}1_{\{t>\tau^x\}}. \end{aligned}$$

Notice that on the set $\{t > \tau^x\}$ (in addition it holds $t < T$), the maturity of the contract is known, i.e., $T = t_{n^*(t)+1} = t_{n^*(\tau^x)+1}$. On the set $\{t \leq \tau^x\}$, the calculation is proceeded as follows:

$$\begin{aligned} E^*[e^{-\int_t^T r(u)du}\bar{G}_T|\mathcal{F}_t] \, 1_{\{t\leq\tau^x\}} &= \sum_{j=n^*(t)+1}^{N} E^*[e^{-\int_t^{t_j} r(u)du} \, \bar{G}_{t_j} \, 1_{\{T=t_j\}}|\mathcal{F}_t] \, 1_{\{t\leq\tau^x\}} \\ &= \Bigg[\sum_{j=n^*(t)+1}^{N} \bar{G}_{t_j} \, E^*[e^{-\int_t^{t_j} r(u)du}|\mathcal{F}_t] \, E^*[1_{\{\tau^x\in]t_{j-1},t_j]\}}|\mathcal{F}_t] \\ &\quad + \bar{G}_{t_N} \, E^*[e^{-\int_t^{t_j} r(u)du}|\mathcal{F}_t] \, E^*[1_{\{\tau^x>t_N\}}|\mathcal{F}_t] \Bigg] \, 1_{\{t\leq\tau^x\}}. \end{aligned}$$

Finally, we achieve

$$\begin{aligned} E^*[e^{-\int_t^T r(u)du}\bar{G}_T|\mathcal{F}_t] \, 1_{\{t\leq\tau^x\}} &= \Bigg[\sum_{j=n^*(t)+1}^{N-1} \bar{G}_{t_j} \, D(t,t_j) \, {}_{t_{j-1}|t_j}\tilde{q}_{x+t} \\ &\quad + \bar{G}_{t_N} \, D(t,t_N) \, \underbrace{\left({}_{t_{N-1}|t_N}\tilde{q}_{x+t} \, + {}_{t_N-t}\tilde{p}_{x+t}\right)}_{={}_{t_{N-1}-t}\tilde{p}_{x+t}} \Bigg] \, 1_{\{t\leq\tau^x\}}. \end{aligned}$$

□

The above proposition immediately motivates a duplication strategy on the set $\{t \leq \tau^x\}$. Prior to the death time τ^x, the contract value (at time t) can be synthesized by a trading strategy which consists of bonds with maturities t_i $(i = n^*(t)+1, \ldots, N)$. Assuming that the insurance company will not learn the death of the customer until no further premiums are paid by the insured implies that the strategy proceeds on the set $t \in]\tau^x, T]$ in a same way as on the set $t \in [0, \tau^x]$. Notice that the number of available instruments, i.e. the number of bonds, decreases as time goes by. At time t, only bonds with maturities later than $n^*(t)$ are traded, i.e., the hedger buys $\bar{G}_{t_i \cdot \, t_{i-1}|t_i} \tilde{q}_x$ units of $D(t,t_i)$ and $\bar{G}_{t_N \cdot \, t_{N-1}-t} \tilde{p}_{x+t}$ units of $D(t,t_N)$. The advantage of using this strategy is that the strategy itself is not dependent of the model assumptions of the interest rate, which is indeed a consequence of the contract specification (no options).

Proposition 4.2.13. *Let $\phi \in \Phi$ denote a risk– (variance–) minimizing trading strategy with respect to the set of trading strategies Φ. Assume that the insurance company notices the death of the customer only when no further premium is paid by the insured. If one additionally restricts the set of admissible strategies to the ones which are independent of the term structure, then it holds: ϕ is uniquely determined and for $t \in [0,T]$*

$$\begin{aligned} \phi_t^{(i)} &= 1_{\{t \leq t_i\}} \bar{G}_{t_i \; t_{i-1}|t_i} \tilde{q}_{x+t} \qquad i = 1, \cdots, N-1 \\ \phi_t^{(N)} &= \bar{G}_{t_N \; t_{N-1}-t} \tilde{p}_{x+t} \end{aligned}$$

Proof: Without the introduction of model risk it is easily seen that V_{t_0} and the contract value X_{t_0} according to Proposition 4.2.12 coincide. Thus, with Proposition 4.2.12 it follows that ϕ is self–financing in the mean. The rest is an immediate consequence of the combination of Theorem 4.2 and Theorem 4.9 of Møller (1998) because endowment insurance is a mixture of pure endowment and term insurance. □

Notice that in a complete one–factor model, the variance–minimizing strategy is not uniquely defined since any bond can be synthesized by two bonds. However, any variance –minimizing strategy which is not of the form of the above proposition depends on the model assumptions of the interest rate when only a subset of bonds are used.

Proposition 4.2.14. *Let ψ denote the risk– (variance–) minimizing trading strategy with respect to the set of trading strategies $\Psi \subset \Phi$. Assuming that the insurance company notices the death of the customer only when no further premiums are paid by the insured*

implies that for $t \in [0,T]$

$$\begin{aligned}
\psi_t^{(N-1)} &= 1_{\{\tau^x \geq t\}} \Bigg(1_{\{t \leq t_{N-2}\}} \sum_{i=n^*(t)+1}^{N-2} \bar{G}_{t_i\ t_{i-1}|t_i} \tilde{q}_{x+t} \frac{D(t,t_i)}{D(t,t_{N-1})} \lambda_1^{(i)}(t) \\
&\quad + 1_{\{t \leq t_{N-1}\}} \bar{G}_{t_{N-1}\ t_{N-2}|t_{N-1}} \tilde{q}_{x+t} \Bigg) \\
\psi_t^{(N)} &= 1_{\{\tau^x \geq t\}} \Bigg(1_{\{t \leq t_{N-2}\}} \sum_{i=n^*(t)+1}^{N-2} \bar{G}_{t_i\ t_{i-1}|t_i} \tilde{q}_{x+t} \frac{D(t,t_i)}{D(t,t_N)} \lambda_2^{(i)}(t) \\
&\quad + \bar{G}_{t_N} \left({}_{t_{N-1}|t_N} \tilde{q}_{x+t} + {}_{t_N - t}\tilde{p}_{x+t} \right) \Bigg) \\
\textit{where } \lambda_1^{(i)}(t) &:= \frac{\tilde{\sigma}_{t_i}(t) - \tilde{\sigma}_{t_N}(t)}{\tilde{\sigma}_{t_{N-1}}(t) - \tilde{\sigma}_{t_N}(t)} \textit{ and } \lambda_2^{(i)}(t) = \frac{\tilde{\sigma}_{t_{N-1}}(t) - \tilde{\sigma}_{t_i}(t)}{\tilde{\sigma}_{t_{N-1}}(t) - \tilde{\sigma}_{t_N}(t)}.
\end{aligned}$$

PROOF: With respect to one–factor model, it holds that any bond can be hedged perfectly, i.e. there is a self–financing strategy $\tilde{\phi}^{(i)} = \left(\alpha^{(i)}, \beta^{(i)}\right)$ with value process $V_t\left(\tilde{\phi}^{(i)}\right) = \alpha_t^{(i)} D(t,t_{N-1}) + \beta^{(i)} D(t,t_N) = D(t,t_i)$ for $i = 1,\ldots,N$. With Proposition 7.1.1 of Appendix 7.1, one immediately can write down the strategy for $D(.,t_i)$, i.e.

$$\alpha_t^{(i)} = \frac{D(t,t_i)}{D(t,t_{N-1})} \lambda_1^{(i)}(t), \quad \beta_t^{(i)} = \frac{D(t,t_i)}{D(t,t_N)} \lambda_2^{(i)}(t)$$

where $\lambda_1^{(i)}(t)$ and $\lambda_2^{(i)}(t)$ are given as above. Notice that $V_t\left(\tilde{\phi}^{(i)}\right) = D(t,t_i)$ P^*–almost surely implies $Var^*[C_T^*(\psi)] = Var^*[C_T^*(\phi)]$ (alternatively, this can be deducted from Proposition 4.2.17). This together with $\Psi \subset \Phi$ ends the proof. □

The above proposition states that ψ corresponds to the strategy which is defined along the lines of Proposition 4.2.13 where the hedging instruments $D(.,t_1),\ldots,D(.,t_{N-2})$ are synthesized by the traded zero bonds $D(.,t_{N-1})$ and $D(.,t_N)$. Obviously, the strategy depends on the term structure model. Basically, by using a one–factor interest model, the risk–minimizing strategy for the insurance contract can be implemented in any subset of bonds with at least two elements. A generalization is straightforward if a hedging instrument is added for every dimension. In addition, it is important to notice that the periodic premium contributions of the insurance taker implies that the insurance must borrow the initial investment in order to implement the hedging strategy. This is especially important if one considers model risk.

Just because of the existence of model risk, an extra cost from the liability side is not negligible in addition to the total cost (under model misspecification) from the asset

side. It is noticed that the implementation of the above strategies is based on taking a credit at t_0. Since the initial value of the hedging strategies is given by the expected value of the (delayed) premium inflows, the insurer must in fact sell the amount $\sum_{i=1}^{N-1} K\,{}_{t_i}\tilde{p}_x\, D(t_0,t_i)$. The underpinning strategy for this is to borrow $K\,{}_{t_i}\tilde{p}_x$ bonds with maturity t_i $(i=1,\ldots,t_{N-1})$. Under model risk, it is not necessarily the case that the insurer achieves exactly the number of periodic premiums which are necessary to pay back the credit. These discrepancies lead to extra costs. In particular, these costs can be understood as a sequence of cash flows, i.e., the insurer has to pay back $K\,{}_{t_i}\tilde{p}_x$ at each time t_i $(i=1,\ldots,t_{N-1})$, i.e. independent of whether the insured survives. Therefore, the additional discounted costs $C_T^{add,*}$ associated with the above borrowing strategy are given by

$$C_T^{add,*} = \sum_{i=0}^{N-1} e^{-\int_0^{t_i} r_u\,du} K\,({}_{t_i}\tilde{p}_x - Y)$$

where $Y := 1_{\{\tau^x > t_i\}}$ is a random variable which takes 1 if the x–aged life survives time t_i and zero otherwise. It is straightforward that due to the independence assumption between the mortality and financial market risk, under consideration of mortality misspecification, we obtain

$$E^*[C_T^{add,*}] = \sum_{i=0}^{N-1} D(t_0,t_i) K\,({}_{t_i}\tilde{p}_x - {}_{t_i}p_x)\,.$$

In the following, we have a look at the distribution of the total hedging errors under consideration of model misspecification.

Proposition 4.2.15 (Expected total discounted hedging costs)**.** *Under the consideration of the model risk, the total discounted hedging costs associated with ϕ and ψ (from the asset side) are given by*

$$E^*[C_T^{tot,*}] = D(t_0,t_N)\bar{G}_{t_N}({}_{t_N}p_x - {}_{t_N}\tilde{p}_x) + \sum_{j=1}^{N-1}({}_{t_{j-1}|t_j}q_x - {}_{t_{j-1}|t_j}\tilde{q}_x) D(t_0,t_j)\bar{G}_{t_j}.$$

Proof: The proof is an immediate consequence of Proposition 4.2.11 and Proposition 4.2.12. □

The expected discounted total hedging costs depend on the deviation of the assumed death/survival probabilities from the true ones. This deviation of the expected discounted total hedging costs from 0 caused by mortality misspecification can be explained intuitively. If n identical contracts are issued, in the expectation, at time t_j, $j=1,\cdots,N-1$, the hedger **can** ensure $n\cdot {}_{t_{j-1}|t_j}\tilde{q}_x$ customers to obtain the defined payoff and at time t_N, he **can** ensure $n\cdot {}_{t_N}\tilde{p}_x$ customers to obtain the contract payoff. However, the number of the customers the hedger **should** ensure in the expectation is $n\cdot {}_{t_{j-1}|t_j}q_x$ at time t_j and $n\cdot {}_{t_N}p_x$ at time t_N. Therefore, the expected (discounted) cost results from the difference

between these "should" and "can" magnitudes, i.e., from the deviation of the assumed death distribution from the true one as described in Proposition 4.2.16. Some graphics are exhibited in a later section in order to illustrate the effect of mortality misspecification on this expectation.

In the following, we denote C_T^* the discounted total costs from both asset and liability side. I.e.,

$$C_T^* = C_T^{tot,*} + C_T^{add,*}.$$

Proposition 4.2.16 (Expected discounted total costs from both asset and liability side)**.** *Under the consideration of the model risk, the expected discounted total costs from both asset and liability side is given by*

$$\begin{aligned} E^*[C_T^*] &= D(t_0,t_N)\bar{G}_{t_N}({}_{t_N}p_x - {}_{t_N}\tilde{p}_x) + \sum_{j=1}^{N-1}({}_{t_{j-1}|t_j}q_x - {}_{t_{j-1}|t_j}\tilde{q}_x)D(t_0,t_j)\bar{G}_{t_j} \\ &\quad + \sum_{i=1}^{N-1} D(t_0,t_i)K\left({}_{t_i}p_x - {}_{t_i}\tilde{p}_x\right). \end{aligned}$$

Proof: This is a straightforward result of the relation $E^*[C_T^*] = E^*[C_T^{tot,*}] + E^*[C_T^{add,*}]$. □

In particular, the above proposition states that, independent of the set of bonds which are available for hedging, the expected costs are the same. Furthermore, independent of the model risk related to the interest rate, it is mortality misspecification that determines the sign of the expected value, i.e., that decides when a superhedge in the mean can be achieved. When no mortality misspecification is available, the model risk related to the interest rate has no impact on the expected value. When there exists mortality misspecification, the model risk related to the interest rate will influence the size of the expected value. Therefore, the effect of model risk associated with the interest rate depends on the mortality misspecification. However, when it comes to the analysis of the variance, model risk associated with the interest rate has a more pronounced effect than mortality misspecification.

Proposition 4.2.17 (Additional variance)**.** *It holds*

$$\begin{aligned} (i) \quad Var^*[C_T^{tot,*}(\psi)] &= Var^*[C_T^{tot,*}(\phi)] + AV_T \\ (ii) \quad Var^*[C_T^*(\psi)] &= Var^*[C_T^*(\phi)] + AV_T \end{aligned}$$

with $AV_T = 0$ when there exists no model risk related to the interest rate, otherwise

$$AV_T = {}_{t_N}p_x\, E^*\left[(I_{t_N}^*(\psi) - I_{t_N}^*(\phi))^2\right] + \sum_{j=0}^{N-1} {}_{t_j|t_{j+1}}q_x\, E^*\left[(I_{t_{j+1}}^*(\psi) - I_{t_{j+1}}^*(\phi))^2\right].$$

PROOF: A detailed proof can be found in Appendix 7.2. □

In this place, it should be emphasized that the effect of mortality misspecification depends on the model risk related to the interest rate when it comes to the analysis of the variance. If there exists no interest rate misspecification, mortality misspecification plays no role in the additional variance. However, if there exists model risk related to the interest rate, an additional variance part results always when only a subset of zero coupon bonds are used as hedging instruments.

As stated in the introduction, mortality misspecification can be caused by a deliberate use of the insurance company for certain purposes, e.g. safety reasons. I.e., a deviation of the assumed mortality from the true one is generated by a shift in the parameter x. For this purpose, we let ${}_tp_{\tilde{x}}$ and ${}_tq_{\tilde{x}}$ denote the assumed probabilities ${}_t\tilde{p}_x$ and ${}_t\tilde{q}_x$.[14]

Proposition 4.2.18. *For any realistic death/survival probability which satisfies*

$$\frac{\partial\, {}_tp_x}{\partial x} < 0 \text{ and } \frac{\partial {}_{u|t}q_{x+v}}{\partial x} > 0,\ v \leq u < t,$$

we obtain that

(i) $\frac{\partial E^*[C_T^*]}{\partial\, \tilde{x}} < 0$. *Furthermore, an overestimation of the death probability (an underestimation of the survival probability) leads to a superhedge in the mean, i.e.,* $E^*[C_T^*] \leq 0$.
(ii) The additional variance given in Proposition 4.2.17 is increasing in $\tilde{x}$.

PROOF: The proof is given in Appendix 7.3. □

Independent of the choice of the hedging instruments, an overestimation of the death probability ($\tilde{x} > x$) makes the insurance company achieve a superhedge in the mean. However, as the assumed $\tilde{x}$ goes up, the additional variance increases. I.e., a traditional tradeoff between the expected hedging costs and the additional variance is observed here. Furthermore, the impact of restricting the set of hedging instruments is highlighted only when the variance is taken into consideration and when the model risk related to the interest rate is available.

4.3 Illustration of results

To illustrate the results of the last sections, we use a one–factor Vasiček–type model framework to describe the financial market risk and a death distribution according to

[14]Since we want to obtain some general results, we make the sensitivity analysis with respect to $\tilde{x}$. If a specific death/survival distribution is used, similar sensitivity analyses can be made. For instance, concerning the illustrative death/survival distribution according to Makeham, naturally a sensitivity analysis can be made with respect to the parameter $\tilde{c}$. However, it should be emphasized that the same consequence will result, because only the effect of these parameters on the death/survival probabilities is of importance.

Death and Survival Probabilities for Varying x Values

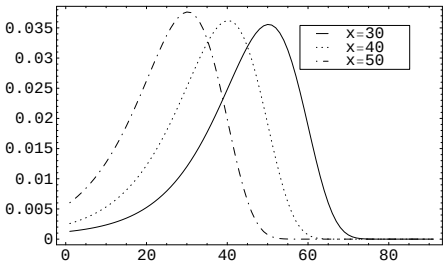

Figure 4.3: ${}_{t_{j-1}|t_j}q_x$ for $x = 30, 40, 50$. The other parameters are given in Table 4.1

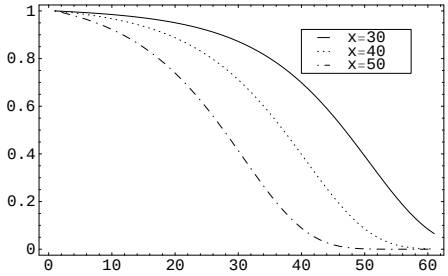

Figure 4.4: ${}_tp_x$ for $x = 30, 40, 50$. The other parameters are given in Table 4.1

Makeham. The benchmark parameter constellation is given in Table 4.1.

Figures 4.3 and 4.4 demonstrate how the death and survival probability, i.e., ${}_{t_{j-1}|t_j}q_x$ and ${}_tp_x$ change with the age x. With the change of x, the death and survival probability demonstrate a parallel shift. If the true age of the customer is 40, then an assumed age of 50 leads to an overestimation of the death probability and an assumed age of 30 results in an underestimation of the death probability. Of course the survival probability has exactly a reversed trend.

4.3.1 Fair parameter h?

Since an endogenous term structure of the interest rate is assumed in this chapter, model misspecification concerning the interest rate (change of the relevant parameters in the assumed model) leads to a change in the initial discount factor $D(t_0, t_i)$, $i = 1, \cdots, N$. This results in a deviation of the "fair" h–value. Analogously, the deviation of the assumed death/survival distribution from the real one leads to a shift in the fair h–value too. Hence, this subsection is designed to answer the question how far the "fair" h–values obtained by using the assumed model parameters are from the one obtained under the real parameter constellations. By using the "fair" h–value obtained from the assumed model, which party of this contract benefits from the misspecification?

Assuming that the short rate is driven by a one–factor Vasiček model, model risk associated with the interest rate can be characterized either by the mismatch of the volatility ($\bar{\sigma}$) or the speed factor (κ). For compatibility reasons (as in the forthcoming subsection 4.3.3), only mismatch of the parameter κ is considered. First, we neglect the mortality misspecification, i.e., we look at the case of $\tilde{x} = x$. The increase in κ has a consequence that the initial value of a zero coupon bond becomes smaller, which leads to a decline in the denominator of the right–hand side of the equation in Corollary 4.1.2. How the

	h		
$\tilde{\kappa}$	$\tilde{x} = 35$	$\tilde{x} = 40$	$\tilde{x} = 45$
0.150	13233.2	10629.4	8426.28
0.155	16650.9	13225.3	10346.2
0.160	19318.0	15255.8	11842.9
0.165	21512.8	16923.7	13084.5
0.170	23388.5	18345.0	14142.1
0.175	25016.0	19579.6	15065.5
0.180	26451.1	20673.6	15866.3
0.185	27738.2	21635.2	16585.0
0.190	28892.5	22506.1	17228.5
0.195	29935.8	23291.6	17814.0
0.200	30892.4	24011.8	18337.5
0.205	31765.8	24660.3	18826.3
0.210	32565.9	25267.5	19269.6

Table 4.3: Fair h–values for varying $\tilde{\kappa}$ with $x = 40$ and $\kappa = 0.18$ and the other parameters are given in Table 4.1.

corresponding nominator changes with κ is not that clear, but in total, as κ goes up, the fair value of h rises. In other words, if the insurer overestimates the speed factor ($\tilde{\kappa} > \kappa$) and uses the resulting h (e.g. $h = 25267.5$ for $\tilde{\kappa} = 0.21$) instead of the true fair value ($h^* = 20673.6$), the resulting h–value is larger than the true fair h value, i.e., the insured benefits from this overestimation. On the contrary, an underestimation of κ is in the benefit of the insurer. Second, assume, there is no model misspecification associated with the interest rate $\tilde{\kappa} = \kappa = 0.18$, an overestimation of the death probability ($\tilde{x} > x$) leads to a smaller fair value of h than the one obtained under the real parameter constellations. Therefore, the insurer is better off by overestimating the death probability but worse off by underestimating the death probability. Third, if both model misspecification are taken into account, the resulting h–value is extremely in the benefit of the insured when a very high mean–reverting speed factor is combined with an extreme underestimation of the death probability.

4.3.2 Expected total costs

How the expected discounted total costs from both asset and liability side change with the assumed age $\tilde{x}$ is depicted by Figures 4.5 and 4.6. It is noticed that, for the given parameters, the expected discounted total cost exhibits a negative relation in $\tilde{x}$. It is a monotonically decreasing concave function of $\tilde{x}$. Especially, for a given t_N value in Figure 4.6, the higher $\tilde{x}$, the lower the expected total costs. From both figures, it is observed that, independent of the set of hedging instruments (bonds), the hedger achieves profits in

Expected Discounted Cost for Varying $\tilde{x}$

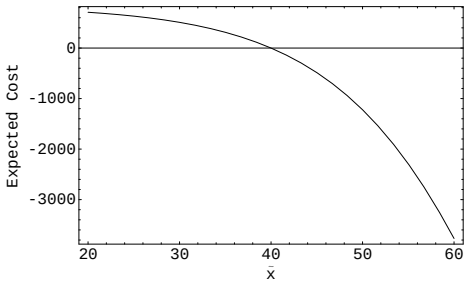

Figure 4.5: Expected cost as a function of $\tilde{x}$ with $x = 40$. The other parameters are given in Table 4.1

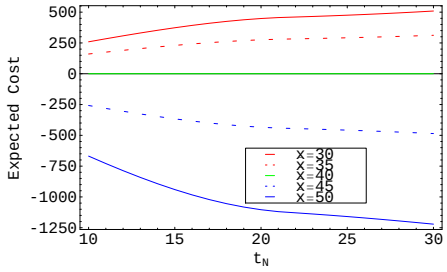

Figure 4.6: Expected cost for $\tilde{x} = 30, 35, 40, 4550$ with the real $x = 40$. The other parameters are given in Table 4.1

mean (negative expected discounted cost) if he overestimates the death probabilities. [15] Hence, negative expected discounted costs result when true x is smaller than the assumed one. Converse effects are observed when the insurer underestimates the death probability. Here, a real age of 40 is taken and it is observed that for $\tilde{x} = 45, 50$, the expected costs have negative values (blue curves), and for $\tilde{x} = 30, 35$, the expected costs exhibit positive values. When the true age coincides with the assumed one, the considered strategy is mean–self–financing because the expected discounted cost equals zero. These observations coincide with the result stated in Proposition 4.2.18.

4.3.3 Variance of total costs/ distribution of total costs

In contrast to the expected total costs, the distribution of the costs depends on the set of hedging instruments. This subsection attempts to illustrate how the variance difference depends on the model risk, i.e., some illustrations are exhibited to support Proposition 4.2.18. The model risk associated with the interest rate influences the variance difference through the functions $|g^{(i)}|$, $i = 1, \cdots, N-2$, which is given by

$$|g_u^{(i)}| = \left| \frac{\tilde{\sigma}_{t_i}(u) - \tilde{\sigma}_{t_N}(u)}{\tilde{\sigma}_{t_{N-1}}(u) - \tilde{\sigma}_{t_N}(u)} \sigma_{t_{N-1}}(u) + \frac{\tilde{\sigma}_{t_{N-1}}(u) - \tilde{\sigma}_{t_i}(u)}{\tilde{\sigma}_{t_{N-1}}(u) - \tilde{\sigma}_{t_N}(u)} \sigma_{t_N}(u) - \sigma_{t_i}(u) \right|.$$

Only if it holds that

$$\sigma_{t_i}(u) = \frac{\tilde{\sigma}_{t_i}(u) - \tilde{\sigma}_{t_N}(u)}{\tilde{\sigma}_{t_{N-1}}(u) - \tilde{\sigma}_{t_N}(u)} \sigma_{t_{N-1}}(u) + \frac{\tilde{\sigma}_{t_{N-1}}(u) - \tilde{\sigma}_{t_i}(u)}{\tilde{\sigma}_{t_{N-1}}(u) - \tilde{\sigma}_{t_N}(u)} \sigma_{t_N}(u), \tag{4.4}$$

i.e., only if it is possible to write the volatility of the t_i–bond as a linear combination of the hedge instruments' volatilities, it is possible to find a self–financing replicating strategy

[15] This result is opposite to the result in pure endowment insurance contracts, where a negative expected discounted cost is achieved when an overestimation of the survival probability exists.

for the bond with maturity t_i, and consequently, it is possible that no variance difference results, independent of mortality misspecification. This indicates, if there is no model misspecification associated with the interest rate, the choice of the hedging instruments has no impact on the variance of the total cost. However, condition (4.4) is a very demanding condition, i.e., there always exists model misspecification related to the interest rate.

Assuming that the short rate is driven by a one–factor Vasiček model, model risk associated with the interest rate can be characterized either by the mismatch of the volatility ($\bar{\sigma}$) or the speed factor (κ), which are determining factors in the volatility function of the zero coupon bonds. Due to the Vasiček modelling, the misspecification of $\bar{\sigma}$ has no impact on $g_u^{(i)}$ functions, hence, no impact on the variance difference. Therefore, in the following, we concentrate on the interest rate misspecification characterized by the deviation of the assumed $\tilde{\kappa}$ from the true κ.

The volatility of the zero coupon bond (with any maturities) is a decreasing function of κ. I.e., a $\tilde{\kappa} < \kappa$ leads to an overestimation of the bond volatility. Under this condition, $|g^{(i)}|$ is a decreasing function of $\tilde{\kappa}$. On the contrary, in the case of $\tilde{\kappa} > \kappa$ (underestimation of the bond volatility), $|g^{(i)}|$ is a increasing function of $\tilde{\kappa}$. Therefore, we obtain some values for the variance difference as exhibited in Table 4.4. Firstly, there exists a deviation of $\tilde{\kappa}$ from κ, the variances of these two strategies differ, even when there is no mortality misspecification. Secondly, mortality misspecification does not have impact on the variance difference, if there are no interest rate misspecification available. I.e., these two strategies make no difference to the variance of the total cost if no model risk associated with the interest rate appears. Therefore, for $\tilde{\kappa} = \kappa = 0.18$, overall the variance difference exhibits a value of 0. These two observations validate the argument that the model misspecification resulting from the term structure of the interest rate has a substantial effect when the variance is taken into account. The effect of mortality risk is partly contingent on the model risk associated with the interest rate. Thirdly, only the absolute distance of $\tilde{\kappa}$ from κ counts. The bigger this absolute distance is, the higher variance differences these two strategies result in. Therefore, overall parabolic curves for the variance difference are observed. In addition, the variance difference increases in $\tilde{x}$, as stated in Proposition 4.2.18. This positive effect can be observed in Figures 4.7 and 4.8.

To sum up, if the hedger substantially overestimates ($\tilde{\kappa} << \kappa$) or underestimates ($\tilde{\kappa} >> \kappa$) the bond volatilities, and if at the same time he highly overestimates the death probability ($\tilde{x} >> x$), the diverse choice of the hedging instruments leads to a huge difference in the variance. On the contrary, a $\tilde{\kappa}$ value close to κ combined with a big overestimation of the survival probability ($\tilde{x} << x$) almost leads to very small variance difference. I.e., very close variances result. The choice of the hedging instrument does not have a significant effect under this circumstance. These result leads to a very interesting phenomenon, with an overestimation of the death probability ($\tilde{x} > x$), the insurance company is always on the safe side in mean, i.e., it achieves a superhedge in the mean. However, if the set of hedging instruments is restricted, an overestimation of the death probability does not nec-

	Expected total cost			Variance Difference			The Ratio		
$\tilde{\kappa}$	$\tilde{x}=35$	$\tilde{x}=40$	$\tilde{x}=45$	$\tilde{x}=35$	$\tilde{x}=40$	$\tilde{x}=45$	$\tilde{x}=35$	$\tilde{x}=40$	$\tilde{x}=45$
0.150	287.795	0	-449.842	774.983	1876.71	4660.65	0.0967	-	-0.1518
0.155	293.229	0	-457.831	568.654	1375.73	3414.78	0.0813	-	-0.1276
0.160	297.998	0	-464.843	385.079	930.677	2308.87	0.0659	-	-0.1034
0.165	302.192	0	-471.014	229.521	554.135	1373.97	0.0501	-	-0.0787
0.170	305.891	0	-476.458	108.254	261.073	646.952	0.0340	-	-0.0534
0.175	309.159	0	-481.270	28.7651	69.2933	171.608	0.0173	-	-0.0272
0.180	312.054	0	-485.532	0	0	0	0	-	0
0.185	314.621	0	-489.315	32.6569	78.4791	194.106	0.0182	-	-0.0285
0.190	316.903	0	-492.677	139.546	334.922	827.806	0.0373	-	-0.0584
0.195	318.934	0	-495.670	336.029	805.425	1989.29	0.0575	-	-0.0890
0.200	320.745	0	-498.338	640.547	1533.21	3783.96	0.0789	-	-0.1234
0.205	322.361	0	-500.719	1075.28	2570.10	6338.04	0.1017	-	-0.1590
0.210	323.805	0	-502.847	1666.92	3978.35	9802.85	0.1261	-	-0.1969

Table 4.4: Expected total cost, variance differences and the ratio of the standard deviation of the variance difference and the expected total cost for varying $\tilde{\kappa}$ with $x = 40$ and the other parameters are given in Table 4.1.

essarily decrease the shortfall probability under a huge misspecification associated with the interest rate (characterized by a big deviation of $\tilde{\kappa}$ from κ). This is due to the observation that a quite high variance difference is reached under this parameter constellation.

In addition, due to the tradeoff between the expected value and the variance difference[16], it is interesting to have a look at the relative size, like the ratio of the standard deviation of the variance difference and the expected value of the total cost from both asset and liability side. First of all, this ratio is not defined when the assumed and real age coincide. Second of all, here for the given parameters, an overestimation of the death probability ($\tilde{x} = 45$) has a higher effect than an underestimation ($\tilde{x} = 35$), i.e. the absolute value of this ratio is larger for the case of $\tilde{x} = 45$. Finally, this ratio can give a hint to the safety loading factor. Assume, the insurer uses standard–deviation premium principle. The ratio given in Table 4.4 suggests him how much safety loading to take when he uses the last two bonds instead of the entire term structure.

4.4 Summary

Based on a simple endowment life insurance contract specification, a contract paying out a guarantee together with an endowment, this paper analyzes how the model misspeci-

[16] An overestimation of the death probability leads to a superhedge in the mean but at the same time a higher variance difference.

Variance Difference

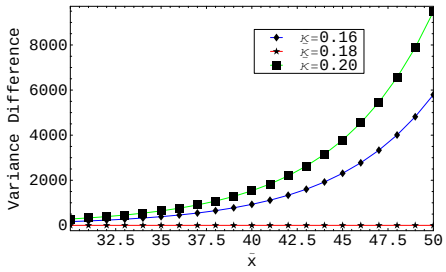

Figure 4.7: Variance difference as function of $\tilde{x}$ with the real $x = 40$ for $\tilde{\kappa} = 0.16$, $\tilde{\kappa} = 0.18$ and $\tilde{\kappa} = 0.20$. The other parameters are given in Table 4.1.

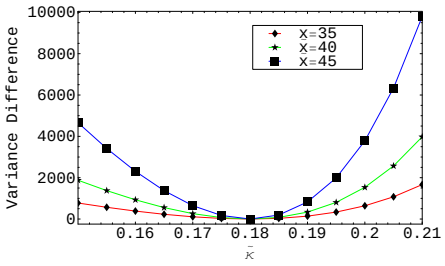

Figure 4.8: Variance difference as function of $\tilde{\kappa}$ with the real $\kappa = 0.18$ for $\tilde{x} = 35$, $\tilde{x} = 40$ and $\tilde{x} = 45$. The other parameters are given in Table 4.1.

fication related to the death distribution of the customer and the term structure of the interest rate plays a role in the insurer's hedging decision. The insured pays a periodic premium and receives at the contract maturity the maximum of an insurance account and an endowment amount. In the case of an early death, the contract maturity is given by the nearest future reference date immediately after the death. Otherwise, the contract maturity is equal to a date which is fixed at the initialization of the contract.

For this purpose, we investigate trading strategies consisting of a subset of most natural hedging instruments which are given by a set of zero coupon bonds whose maturities are given by a discrete set of reference dates. The non–available hedging instruments have to be synthesized by the traded bonds. Therefore, the construction of the hedging strategy is not only based on the assumed death distribution but also on the assumed interest rate dynamic. It is shown that model misspecification in the interest rate model has a more pronounced effect when the set of hedging instruments is restricted.

Besides the asset side, model risk influences the liability side too, because a periodic instead of an upfront premium is paid by the insured, i.e., a credit must be taken by the insurer in order to implement the considered hedging strategies in the asset side. Under mortality risk, it is not necessarily the case that the insurer achieves exactly the number of periodic premiums which are necessary to pay back the credit. These discrepancies lead to extra costs. Hence, the total cost of the insurer under model misspecification has to be taken account of from both the asset and liability side.

Not like in the option pricing, an endogenous model of the interest rate implies that there exists uncertainty concerning the initial bond value, which influences both the initial contract value and the expected discounted contribution of the insured. Similarly, there exists an uncertainty concerning the death/survival distribution. Therefore, first and fore-

most, the existent model parameter mismatches related to both the interest rate and the mortality risk lead to a deviation of the fair endowment value h from the one obtained under the true parameter constellations. When issuing the contracts, since the h–values provided to the insured results from the assumed (mismatched) parameter/model, they are not fair under the true model parameter scenarios. By illustrating some numerical results, we observe that an extreme overestimation of the mean–reverting speed factor ($\rightarrow$ an extreme small initial bond value) together with an extreme underestimation of the death probability yields an h–value much larger than the real optimal one. Hence, in this parameter scenarios, the insured is much better off while the insurer much worse off.

Independent of the choice of the hedging instruments, an overestimation of the death probability leads to some profits for the insurer in the mean. This is exactly opposite to a pure endowment contract. However, when it comes to the analysis of the variance of the hedging errors and the total cost, the effect of mortality misspecification depends on the model risk concerning the interest rate. No model risk associated with the interest rate implies no variance difference by using different subsets of the hedging instruments, even when mortality misspecification is present. On the contrary, even when there exists no mortality misspecification, a variance markup always results when an interest rate misspecification is existent. This variance markup goes up with the magnitude of the death probability overestimation. In particular, we observe that if the set of hedging instruments is restricted, an overestimation of the death probability combined with a huge misspecification associated with the interest rate leads to a very high variance markup, and consequently it could lead to an increase in the shortfall probability. To sum up, neither the model risk which is related to the death distribution nor the one associated with the financial market model is negligible for a meaningful risk management.

In reality, it is not uncommon that an additional bonus payment is offered to the customer in such kinds of contracts. Usually the bonus payment is constructed as a call option (or a sequence of call options) on the asset. Hence, these contracts maintain the financial risk related to the stock. Therefore, a natural further research interest lies in how model risk related not only to the mortality, interest rate but also to the stock affects insurers' hedging decisions.

Chapter 5

Default risk and Chapter 11 bankruptcy procedure [1]

Diverse perspectives related to hedging equity–linked life insurance contracts have been gone through in Chapters 2–4. It varies from deriving different hedging strategies, considering the net loss when certain hedging strategies are used by the hedger, to analyzing the effect of model misspecification associated with interest rate and mortality risk on the hedging decisions. It implies that so far different sources of incompleteness have been investigated, i.e., the incompleteness results from the untradable insurance risk alone, from trading restrictions or from model misspecifications. However, in all of these analyses, it is observed that it is impossible to eliminate the insurance risk completely. In other words, default occurs in a life insurance company with a positive probability. The numerous defaulted life insurers reported in Europe, Japan and the United States listed in Chapter 1 provide best proof for this argument. Therefore, this chapter is designed to answer the question how default risk and liquidation ($\rightarrow$ relevant bankruptcy procedures) influence the insurance company, i.e., how the market value of the life insurer is affected by default risk. For this purpose, the life insurer is indeed considered as an aggregate and the firm's value is assumed to follow a geometric Brownian motion. $(A_t)_{t\in[0,T]}$ is used in this chapter to denote the firm's value. Throughout this chapter, we neglect the mortality risk and we use a deterministic interest rate rather than a stochastic one.

Different approaches have been developed to describe default and liquidation. However, in life insurance mathematics, so far only standard knock–out barrier option is used to construct default and liquidation event. According to it, when a certain barrier level which is set ex ante is hit, the firm defaults and is liquidated immediately. I.e., in a knock–out barrier option framework, default and liquidation are formulated as equivalent events. Evidently, this is not very realistic, because default and liquidation cannot be considered as equivalent events.

Hence, it is worth having a close look at the bankruptcy procedures. We take the United

[1] This chapter is based on a joint work with Michael Suchanecki, c.f. Chen and Suchanecki (2006) which is forthcoming in Insurance: Mathematics and Economics.

American defaulted companies	Year	Bankruptcy code	Days spent in default
Executive Life Insurance Co.	1991	Ch. 11	462
First Capital Life Insurance Co.	1991	Ch. 11	1669
Monarch Life Insurance Co.	1994	Ch. 11	392
ARM Financial Group	1999	Ch. 11	245
Penn Corp. Financial Group	2000	Ch. 11	119
Conseco Inc.	2002	Ch. 11	266
Metropolitan Mortgage & Securities	2004	Ch. 11	n/a

Table 5.1: Some defaulted insurance companies in the United States.

States' Bankruptcy Code as an example. Similar bankruptcy laws are also applied in Japan and in France. In the U.S. Bankruptcy Code, there are two possible procedures: Chapter 7 and Chapter 11 bankruptcy code. It is generally assumed that a firm is in financial distress when the value of its assets is lower than the default threshold. As mentioned, with Chapter 7 bankruptcy, the firm is liquidated immediately after default, i.e., no renegotiations or reorganizations are possible. With Chapter 11 bankruptcy, first the reality of the financial distress is checked before the firm is definitively liquidated, i.e., the defaulted firm is granted some "grace" period during which a renegotiation process between equity and debt holders may take place and the firm is given the chance to reorganize. If, during this period, the firm is unable to recover then it is liquidated. Hence, the firm's asset value can cross the default threshold without causing an immediate liquidation. Thus, the default event is only signalled.

Table 5.1 provides some realistic detailed information on the bankruptcy procedure and the number of days spent in default for some exemplary bankruptcies of life insurance companies in the United States.[2] It is observed that all the filed life insurers adopt Chapter 11 bankruptcy code. For the above mentioned cases from the United States for which data were available, the "grace" period lasted from 119 days up to 1669 days. In France, a legal 3–month observation period before a possible liquidation is systematically granted to firms in financial distress by the courts. This period can be renewed once and can be exceptionally prolonged in the limit of six months. As these examples show, it is important to consider bankruptcy procedures that are explicitly based on the time spent in financial distress and to include such a "grace" period into the model if one wants to capture the effects of an insurance company's default risk on the value of its liabilities and on the value of the insurance contracts more realistically.

In the present chapter, we construct a contingent claim model along the lines of Briys and de Varenne (1994b, 1997) and Grosen and Jørgensen (2002) for the valuation of the equity and the liability of a life insurance company where the liability consists only of

[2]These data are taken from Lynn M. LoPucki's Bankruptcy Research Database, http://lopucki.law.ucla.edu/index.htm.

the policy holder's payments. Their main contribution is to explicitly consider default risk in a contingent claim model to value the equity and the liability of a life insurance company. In Briys and de Varenne (1994b, 1997) , default can only occur at the maturity date, whereas in Grosen and Jørgensen (2002) default can occur at any time before the maturity date, i.e., they introduce the risk of a premature default to the valuation of a life insurance contract[3]. In order to model the default event, they build into the model a regulatory mechanism in the form of an intervention rule, i.e., they add a simple knock–out barrier option feature to the different components of the insurance contract. The default event is defined so that the value of the total assets of the life insurance company must always be sufficient to cover the life insurance policy holder's initial deposit compounded with the guaranteed rate of return. Otherwise the firm defaults and is immediately liquidated. Absolute priority is assumed, i.e., the holder of the life insurance contract (= liability holder) has the first claim on the firm's assets. This corresponds to a Chapter 7 bankruptcy procedure, where default and liquidation times coincide.

However, as we have explained above, the approach to modelling the insolvency risk in Grosen and Jørgensen (2002) does not reflect the reality well. Default and liquidation cannot be considered as equivalent events. We therefore extend their model in order to be able to capture the effects of the Chapter 11 (or of the other countries' codes corresponding to Chapter 11) bankruptcy procedure and to study the impact of a delayed liquidation on the valuation of the insurance company's liabilities and on the ex–ante pricing of the life insurance contracts. We do this by using so–called Parisian barrier option frameworks. Here we distinguish between two kinds of Parisian barrier options: standard Parisian barrier options and cumulative Parisian barrier options.

Assume, we are interested in the modelling of a Parisian down–and–out option. With standard Parisian barrier options, the option contract is knocked out if the underlying asset value stays consecutively below the barrier for a time longer than some predetermined time d before the maturity date. With cumulative Parisian barrier options, the option contract is terminated if the underlying asset value spends until maturity in total at least d units of time below the barrier. In a corporate bankruptcy framework these two Parisian barrier options have appealing interpretations. Think of the idea that a regulatory authority takes its bankruptcy filing actions according to a hypothetical default clock. In the case of standard Parisian barrier options, this default clock starts ticking when the asset price process breaches the default barrier and the clock is reset to zero if the firm recovers from the default. Thus, successive defaults are possible until one of these defaults lasts d units of time. One may say that in this case the default clock is memoryless, i.e., earlier defaults which may last a very long time but not longer than d do not have any consequences for eventual subsequent defaults. In the case of cumulative Parisian barrier options, the default clock is not reset to zero when a firm emerges from default, but it is only halted and restarted when the firm defaults again. Here d denotes the maximum authorized total time in default until the maturity of the debt. This

[3]Bernard et al. (2005) recently extended this model by taking into account stochastic interest rates.

corresponds to a full memory default clock, since every single moment spent in default is remembered and affects further defaults by shortening the maximum allowed length of time that the company can spend in default without being liquidated.[4] Thus, in the limiting case when d is set equal to zero (or is going to zero), we are back in the model of in Grosen and Jørgensen (2002). Our model therefore encompasses that of Grosen and Jørgensen (2002) and also those of Briys and de Varenne (1994b, 1997). Both kinds of Parisian options are of course not new in the literature on exotic options. They have been introduced by Chesney, M. and Yor (1997) and subsequently developed further in Hugonnier (1999), Moraux (2002), Anderluh and van der Weide (2004) and Bernard et. al. (2005b).

There are two related papers in the credit risk literature analyzing the effects of bankruptcy procedures: Moraux (2003) extends the model of Black and Cox (1976) and models the value of debt and equity of a company in a structural model of credit risk when the default barrier is not an absorbing one. He is mainly concerned with valuing various forms of debt and analyzes the obtained credit spreads. François and Morellec (2004) perform a similar analysis in a time–independent framework extending Leland (1994) model. However, these authors are more interested in credit spreads, debt subordination or agency conflicts. Bernard et al. (2005c) consider a model of bank deposit insurance with Parisian options.

The remainder of this chapter is structured as follows. In Section 5.1, we briefly review the model of Grosen and Jørgensen (2002) because we will place our model in almost the same basic setup. Moreover, we already introduce the standard Parisian barrier feature along the lines of Chesney et al. (1997). In the numerical analysis, in order to invert the Laplace transforms involved, we use the procedure introduced by Bernard et. al. (2005b) . Hence we are able to obtain approximate solutions for the components of the life insurance company's balance sheet and for the issued equity–linked life insurance contract. In the case of the cumulative Parisian barrier feature, we deduce quasi–closed–form solutions for the different components of the life insurance company's liabilities and the life insurance contract following and extending Hugonnier (1999) and Moraux (2002). In Section 5.2, we perform a number of representative numerical analyses and comparative statics for both cases in order to investigate the effects of different parameter changes on the value of the insurance company's equity and liability, and hence on the life insurance contract. In particular, we study the impact of the new regulation parameter d and compare it with the old regulation parameter η which determines the barrier level. In Section 5.3, we calculate the shortfall probabilities for both standard and cumulative Parisian options in order to analyze the incentives for the customers to engage in a life insurance contract in this model framework. Section 5.4 gives a short summary to this chapter.

[4]The real life bankruptcy procedures lie somewhere in between these two extreme cases.

5.1 Model

This section mainly consists of two parts. The first part reviews the basic model of Grosen and Jørgensen (2002) succinctly, and more importantly, the Parisian barrier option features are introduced to describe the different default and liquidation events. Accordingly, the rebate payment used by the above mentioned authors has to be altered because it does not make sense in our framework. The remaining part of this section focuses on the valuation of the life insurance company's equity and liability and of the issued life insurance contract.

5.1.1 Contract specification

As in the original work of Grosen and Jørgensen (2002), which is an extension of the early models merging default risk and life insurance contracts of Briys and de Varenne (1994b, 1997) , we assume that at time $t = 0$ the insurance company owns a capital structure as illustrated in the following balance sheet:

Assets	Liabilities
A_0	$E_0 \equiv (1-\alpha)A_0$
	$L_0 \equiv \alpha A_0$
A_0	A_0

That is, for simplicity, we suppose that the representative policy holder (also liability holder) whose premium payment at the beginning of the contract constitutes the liability of the insurance company, denoted by $L_0 = \alpha A_0$, $\alpha \in [0, 1]$, and the representative equity holder whose equity is accordingly denoted by $E_0 = (1-\alpha)A_0$ form a mutual company, the life insurance company. Through their initial investments in the company, both acquire a claim on the firm's assets for a payoff at maturity (or before maturity).

The following notations are used for the specification of the insurance contract:

T	:=	the maturity date
$L_T = L_0 e^{gT}$	:=	the guaranteed payment to the policy holder at maturity, where g is the minimum guaranteed interest rate
A_t	:=	the value of the firm's assets at time $t \in [0, T]$
δ	:=	the participation rate, i.e., to which extent the policy holder participates in the firm's surpluses at maturity.

Since an interest rate guarantee and the contribution principle which entitles the policy holder to a participation in the insurer's investment surpluses are common features of today's life insurance contracts, we consider the following simplified version of a participating life insurance contract incorporating all these features. The total payoff to the holder of such an insurance contract at maturity, $\psi_L(A_T)$, is given by:

$$\psi_L(A_T) = \delta[\alpha A_T - L_T]^+ + L_T - [L_T - A_T]^+.$$

This payment consists of three parts: a bonus (call option) paying to the policy holder a fraction δ of the positive difference of the actual performance of his share in the insurance company's assets and the guaranteed amount at maturity, a guaranteed fixed payment which is the initial premium payment compounded by the interest rate guarantee and a short put option resulting from the fact that the equity holder has a limited liability. In Grosen and Jørgensen (2002), a rebate payment,

$$\Theta_L(\tau) = \min\{L_0 e^{g\tau}, B_\tau\} = \min\{1, \eta\} L_0 e^{g\tau},$$

is offered to the liability holder in the case of a premature closure of the firm, where τ denotes the liquidation date. Analogously, the total payoff to the equity holder at maturity, $\psi_E(A_T)$, is given by:

$$\psi_E(A_T) = [A_T - L_T]^+ - \delta[\alpha A_T - L_T]^+.$$

This payment consists of two call options: a long call option on the assets with strike equal to the promised payment at maturity, called the residual call, and a short call option offsetting exactly the bonus call option of the liability holder. For the equity holder a rebate is offered, too, in the case of a premature liquidation of the firm:

$$\Theta_E(\tau) = \max\{0, (\eta - 1)\} L_0 e^{g\tau}.$$

Grosen and Jørgensen (2002) model their regulatory intervention rule in the form of a boundary, i.e., an *exponential* barrier $B_t = \eta L_0 e^{gt}$ is imposed on the underlying asset value process, where η is a regulation parameter. When the asset price reaches this boundary, namely, $A_\tau = B_\tau$ with $\tau \in [0, T]$, the company defaults and is liquidated immediately, i.e., default and liquidation coincide. If the regulatory authority chooses $\eta \geq 1$, in the case of liquidation, the liability holder obtains his initial deposit plus the accrued guaranteed interest up to the liquidation date. If an $\eta < 1$ is chosen, no such payment can be made to the full extent. Obviously, the specified contract contains standard down–and–out barrier options. Therefore, the requirement $A_0 > B_0 = \eta L_0$ must be satisfied initially. It should be noted that in the case of liquidation, any recovered funds will be distributed to the company's stake holders according to the usual procedure. The liability holder enjoys absolute priority, i.e., he has the first claim on the company's assets.

The bankruptcy procedure described above where default and liquidation occur at the same time corresponds to Chapter 7 of the U.S. Bankruptcy Code. As mentioned in the introduction, we generalize the model of Grosen and Jørgensen (2002) in order to allow for Chapter 11 bankruptcy. This can be realized by adding a Parisian barrier option feature instead of the standard knock–out barrier option feature to the model. Before we come

to this point, we have to make a small change on the rebate term of the issued contract. Both Parisian barrier option features could lead to the result that at the liquidation time the asset price falls far below the barrier value, which makes it impossible for the insurer to offer the above mentioned rebate. Hence, a new rebate for the liability holder is introduced to the model and it has the form of

$$\Theta_L(\tau) = \min\{L_\tau, A_\tau\},$$

where τ is the liquidation time. The rebate term implicitly depends on the regulation parameter η. Because of the following inequality

$$A_\tau \leq B_\tau = \eta L_\tau,$$

it is observed that for $\eta < 1$, the rebate corresponds to the asset value A_τ.

Correspondingly, the new rebate for the equity holder can be expressed as follows:

$$\Theta_E(\tau) = A_\tau - \min\{L_\tau, A_\tau\} = \max\{A_\tau - L_\tau, 0\},$$

i.e., the equity holder obtains the remaining asset value if there is any. Clearly, in the case of $\eta < 1$, all the asset value goes to the liability holder.

In this chapter, we differentiate between two categories of Parisian barrier features:

- Standard Parisian barrier feature: This corresponds to a procedure where the liquidation of the firm is declared when the financial distress has lasted successively at least a period of length d.
- Cumulative Parisian barrier feature: This corresponds to a procedure where the liquidation is declared when the financial distress has lasted in total at least a period of length d during the life of the contract.

It is noted that the original model by Grosen and Jørgensen (2002) is a special case in both scenarios described above, namely when the time window d is set to 0. Observe that with $\eta \downarrow 0$, we are back in the model of Briys and de Varenne (1994b) because in that situation premature default and liquidation are impossible.

5.1.2 Valuation

This subsection aims at valuing the liabilities of the life insurance company and of the issued life insurance contract. In the literature, different methods have been applied to value standard and cumulative Parisian options. The inverse Laplace transform method originally introduced by Chesney et al. (1997) is adopted to price the standard Parisian claims. The results of Hugonnier (1999) and Moraux (2002) and some newly derived extensions are used to value the cumulative Parisian claims.

In general, for the valuation framework, we assume a continuous–time frictionless economy with a perfect financial market, no tax effects, no transaction costs and no other imperfections. Hence, we can rely on martingale techniques for the valuation of the contingent claims.

Under the equivalent martingale measure, the price process of the insurance company's assets $\{A_t\}_{t\in[0,T]}$ is assumed to follow a geometric Brownian motion

$$dA_t = A_t(rdt + \sigma dW_t^*),$$

where r denotes the deterministic interest rate, σ the deterministic volatility of the asset price process $\{A_t\}_{t\in[0,T]}$ and $\{W_t^*\}_{t\in[0,T]}$ the equivalent P^*–martingale. Solving this differential equation, we obtain

$$A_t = A_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma W_t^*\right\}.$$

Standard Parisian barrier framework

Before we come to the general valuation of standard Parisian barrier options, some special cases are considered:

- $A_t > B_t$ and $d \geq T - t$: In this case, it is impossible to have an excursion below B_t between t and T of length at least equal to d. Therefore, the value of a Parisian down–and–out call just corresponds to the Black–Scholes (Black and Scholes (1973)) price of a standard European call option.
- $d \geq T$: In this case, the Parisian option actually becomes a standard call option.
- $A_t > B_t$ and $d = 0$: As already mentioned, this corresponds to the scenario which Grosen and Jørgensen (2002) introduced.

Apart from these special cases, the standard Parisian option is priced as follows. In the standard Parisian down–and–out option framework, the final payoff $\psi_L(A_T)$ is only paid if the following technical condition is satisfied:

$$T_B^- \quad = \quad \inf\left\{t > 0 \middle| \left(t - g_{B,t}^A\right)\mathbf{1}_{\{A_t<B_t\}} > d\right\} > T \tag{5.1}$$

with

$$g_{B,t}^A = \sup\{s \leq t | A_s = B_s\},$$

where $g_{B,t}^A$ denotes the last time before t at which the value of the assets A hits the barrier B. T_B^- gives the first time at which an excursion below B lasts more than d units of time. In fact, T_B^- is the liquidation date of the company if $T_B^- < T$. It is noted that the condition in Inequality 5.1 is equivalent to

$$T_b^- := \inf\left\{t > 0 \middle| \left(t - g_{b,t}\right)\mathbf{1}_{\{Z_t<b\}} > d\right\} > T$$

where

$$g_{b,t} := \sup\{s \leq t | Z_s = b\}; \qquad b = \frac{1}{\sigma} \ln\left(\frac{\eta L_0}{A_0}\right),$$

and $\{Z_t\}_{0 \leq t \leq T}$ is a martingale under a new probability measure P which is defined by the Radon–Nikodym density

$$\frac{dP^*}{dP}\Big|_{\mathcal{F}_T} = \exp\left\{mZ_T - \frac{m^2}{2}T\right\}, \qquad m = \frac{1}{\sigma}\left(r - g - \frac{1}{2}\sigma^2\right),$$

i.e., $Z_t = W_t^* + m\,t$. The following derivation enlightens this equivalence argument:

$$\begin{aligned} g_{B,t}^A &= \sup\{s \leq t | A_s = B_s\} \\ &= \sup\left\{s \leq t \Big| A_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)s + \sigma W_s\right\} = \eta L_0 e^{gs}\right\} \\ &= \sup\{s \leq t | Z_s = b\} = g_{b,t}. \end{aligned}$$

Thereby, we transform the event "the excursion of the value of the assets below the exponential barrier $B_t = \eta L_0 e^{gt}$" to the event "the excursion of the Brownian motion Z_t below a constant barrier $b = \frac{1}{\sigma}\ln\left(\frac{\eta L_0}{A_0}\right)$". This simplifies the entire valuation procedure. Under the new probability measure P the value of the assets A_t can be expressed as

$$A_t = A_0 \exp\left\{\sigma\, Z_t\right\} \exp\{g\, t\}.$$

It is well known that in a complete financial market, the price of a T–contingent claim with the payoff $\phi(A_T)$ corresponds to the expected discounted payoff under the equivalent martingale measure P^*, i.e.,

$$E^*\left[e^{-rT}\phi(A_T)\mathbf{1}_{\left\{T_B^- > T\right\}}\right].$$

This can be rephrased as follows:

$$e^{-\left(r+\frac{1}{2}m^2\right)T} E_P\left[\mathbf{1}_{\left\{T_b^- > T\right\}} \phi\left(A_0 \exp\{\sigma Z_T\} \exp\{gT\}\right) \exp\{mZ_T\}\right].$$

Therefore, the value of the liability of the life insurance company, i.e., the price of the issued life insurance contract is determined by:

$$\begin{aligned}
V_L(A_0,0) &= E^*\left[e^{-rT}\left(\delta[\alpha A_T - L_T]^+ - [L_T - A_T]^+ + L_T\right)\mathbf{1}_{\{T_B^- > T\}}\right] \\
&\quad + E^*\left[e^{-rT_B^-}\min\left\{L_{T_B^-}, A_{T_B^-}\right\}\mathbf{1}_{\{T_B^- \le T\}}\right] \\
&= \delta\,\alpha\, e^{-\left(r-g+\frac{1}{2}m^2\right)T} E_P\left[\left[A_0 e^{\sigma Z_T} - \frac{L_0}{\alpha}\right]^+ e^{mZ_T}\mathbf{1}_{\{T_b^- > T\}}\right] \\
&\quad - e^{-\left(r-g+\frac{1}{2}m^2\right)T} E_P\left[\left[L_0 - A_0 e^{\sigma Z_T}\right]^+ e^{mZ_T}\mathbf{1}_{\{T_b^- > T\}}\right] + E^*\left[e^{-rT}L_T\mathbf{1}_{\{T_b^- > T\}}\right] \\
&\quad + E_P\left[e^{-\left(r+\frac{1}{2}m^2\right)T_b^-}\exp\left\{mZ_{T_b^-}\right\}\min\left\{L_{T_b^-}, A_{T_b^-}\right\}\mathbf{1}_{\{T_b^- \le T\}}\right] \\
&:= \delta\,\alpha\, PDOC\left[A_0, B_0, \frac{L_0}{\alpha}, r, g\right] - PDOP[A_0, B_0, L_0, r, g] + E^*\left[e^{-rT}L_T\mathbf{1}_{\{T_b^- > T\}}\right] \\
&\quad + E_P\left[e^{-\left(r+\frac{1}{2}m^2\right)T_b^-}\exp\left\{mZ_{T_b^-}\right\}\min\left\{L_{T_b^-}, A_{T_b^-}\right\}\mathbf{1}_{\{T_b^- \le T\}}\right].
\end{aligned}$$

It is observed that the price of this contingent claim consists of four parts: A Parisian down–and–out call option with strike $\frac{L_T}{\alpha}$ (multiplied by $\delta\,\alpha$), i.e., the bonus part, a Parisian down–and–out put option with strike L_T, a deterministic guaranteed part L_T which is paid at maturity when the value of the assets has not stayed below the barrier for a time longer than d and a rebate paid immediately when the liquidation occurs.

Various approaches are applied to valuing standard Parisian products, such as Monte Carlo algorithms (Andersen and Brotherton-Ratcliffe (1996)), binomial or trinomial trees (Avellaneda and Wu (1999), Costabile (2002)), PDEs (Haber, Schönbucher and Wilmott (1999)), finite–element methods (Stokes and Zhu (1999)) or the implied barrier concept (Anderluh and van der Weide (2004)). In this article, we adopt the original Laplace transform approach initiated by Chesney et al. (1997). Later, in the numerical analysis, for inverting the Laplace transforms, we rely on the recently introduced and more easily implementable procedure by Bernard, Le Courtois and Quittard-Pinon (2005b). They approximate the Laplace transforms needed to value standard Parisian barrier contingent claims by a linear combination of a number of fractional power functions in the Laplace parameter. The inverse Laplace transforms of these functions are well–known analytical functions. Therefore, due to the linearity, the needed inverse Laplace transforms are obtained by summing up the inverse Laplace transforms of the approximate fractional power functions. In the following, we apply this technique to each component of the liabilities and of the issued contract.

It is well known that the price of a Parisian down–and–out call option can be described as the difference of the price of a plain–vanilla call option and the price of a Parisian

down–and–in call option with the same strike and maturity date, i.e.,

$$\begin{aligned} & PDOC\left[A_0, B_0, \frac{L_0}{\alpha}, r, g\right] \\ &= e^{-\left(r-g+\frac{1}{2}m^2\right)T} E_P\left[\left[A_0 e^{\sigma Z_T} - \frac{L_0}{\alpha}\right]^+ \exp\left\{mZ_T\right\} \mathbf{1}_{\left\{T_b^- > T\right\}}\right] \\ &= BSC\left[A_0, \frac{L_0}{\alpha}, r, g\right] - \underbrace{e^{-\left(r-g+\frac{1}{2}m^2\right)T} E_P\left[\left[A_0 e^{\sigma Z_T} - \frac{L_0}{\alpha}\right]^+ \exp\left\{mZ_T\right\} \mathbf{1}_{\left\{T_b^- \leq T\right\}}\right]}_{:=PDIC\left[A_0, B_0, \frac{L_0}{\alpha}, r, g\right]}. \end{aligned}$$

The price of the plain–vanilla call option is obtained by the Black–Scholes formula as follows:

$$\begin{aligned} BSC\left[A_0, \frac{L_0}{\alpha}, r, g\right] &= E^*\left[e^{-rT}\left[A_T - \frac{L_T}{\alpha}\right]^+\right] = A_0 N(d_1) - \frac{L_0}{\alpha} e^{-(r-g)T} N(d_2) \\ d_{1/2} &= \frac{\ln\left(\frac{\alpha A_0}{L_0}\right) + \left(r - g \pm \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} = \frac{\left(r - g \pm \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}. \end{aligned}$$

Here $N(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ gives the cumulative distribution function of the standard normal distribution. Since it is valid that $\frac{L_0}{\alpha} \geq B_0$, $PDIC\left[A_0, B_0, \frac{L_0}{\alpha}, r, g\right]$ can be calculated as follows:

$$PDIC\left[A_0, B_0, \frac{L_0}{\alpha}, r, g\right] = e^{-\left(r-g+\frac{1}{2}m^2\right)T} A_0 \int_k^\infty e^{my}\left(e^{\sigma y} - \frac{L_0}{\alpha}\right) h_1(T, y) dy$$

with $k = \frac{1}{\sigma}\ln\left(\frac{L_0}{\alpha\, A_0}\right) = 0$. The density $h_1(T, y)$ is uniquely determined by inverting the corresponding Laplace transform which is given by

$$\begin{aligned} \hat{h}_1(\lambda, y) &= \frac{e^{(2b-y)\sqrt{2\lambda}}\psi\left(-\sqrt{2\lambda d}\right)}{\sqrt{2\lambda}\psi\left(\sqrt{2\lambda d}\right)} \\ \text{with} \qquad \psi(z) &= \int_0^\infty x \exp\left\{-\frac{x^2}{2} + zx\right\} dx = 1 + z\sqrt{2\pi} e^{\frac{z^2}{2}} N(z), \end{aligned}$$

and λ the parameter of Laplace transform.

The Parisian down–and–out put option can be derived by the following in–out–parity:

$$PDOP[A_0, B_0, L_0, r, g] := BSP[A_0, L_0, r, g] - PDIP[A_0, B_0, L_0, r, g].$$

Here $BSP[A_0, L_0, r, g]$ gives the price of the plain–vanilla put option and $PDIP[A_0, B_0, L_0, r, g]$ the price of the Parisian down–and–in put option. $BSP[A_0, L_0, r, g]$ is derived by the Black–Scholes formula:

$$\begin{aligned} BSP[A_0, L_0, r, g] &= E^* \left[e^{-rT} \left[L_T - A_T \right]^+ \right] = L_0 e^{-(r-g)T} N(-d_2) - A_0 N(-d_1) \\ d_{1/2} &= \frac{\ln\left(\frac{A_0}{L_0}\right) + \left(r - g \pm \frac{1}{2}\sigma^2\right) T}{\sigma\sqrt{T}}. \end{aligned}$$

Due to the different possible choices of the η–value, different pricing formulas are obtained for the Parisian down–and–in put option. An $\eta < 1$, which leads to the fact that the strike is larger than the barrier, results in

$$\begin{aligned} PDIP[A_0, B_0, L_0, r, g] &= e^{-\left(r-g+\frac{1}{2}m^2\right)T} \left(\int_{-\infty}^{b} e^{my} \left(L_0 - A_0 e^{\sigma y} \right) h_2(T, y) dy \right. \\ &\quad \left. + \int_{b}^{k_1} e^{my} \left(L_0 - A_0 e^{\sigma y} \right) h_1(T, y) dy \right) \end{aligned}$$

with $k_1 = \frac{1}{\sigma} \ln\left(\frac{L_0}{A_0}\right)$. As before, $h_1(T, y)$ and $h_2(T, y)$ are calculated by inverting the corresponding Laplace transforms. $\hat{h}_1(T, y)$ has the same value as before and the Laplace transform of $h_2(T, y)$ is given by

$$\begin{aligned} \hat{h}_2(\lambda, y) &= \frac{e^{y\sqrt{2\lambda}}}{\sqrt{2\lambda}\psi\left(\sqrt{2\lambda d}\right)} + \frac{\sqrt{2\lambda d} e^{\lambda d}}{\psi\left(\sqrt{2\lambda d}\right)} \left(e^{y\sqrt{2\lambda}} \left(N\left(-\sqrt{2\lambda d} - \frac{y-b}{\sqrt{d}} \right) - N\left(-\sqrt{2\lambda d} \right) \right) \right. \\ &\quad \left. - e^{(2b-y)\sqrt{2\lambda}} N\left(-\sqrt{2\lambda d} + \frac{y-b}{\sqrt{d}} \right) \right). \end{aligned}$$

Analogously, for the case of $\eta \geq 1$, the Parisian down–and–in put option has the form of

$$PDIP[A_0, B_0, L_0, r, g] = e^{-\left(r-g+\frac{1}{2}m^2\right)T} \int_{-\infty}^{k_1} e^{my} \left(L_0 - A_0 e^{\sigma y} \right) h_2(T, y) dy.$$

The third term in the payoff function can be calculated as follows:

$$\begin{aligned} E^* \left[e^{-rT} L_T \mathbf{1}_{\{T_b^- > T\}} \right] &= e^{-rT} L_T - E^* \left[e^{-rT} L_T \mathbf{1}_{\{T_b^- \leq T\}} \right] \\ &= e^{-rT} L_T \left[1 - e^{-\frac{1}{2}m^2 T} \left(\int_{-\infty}^{b} h_2(T, y) e^{my} dy + \int_{b}^{\infty} h_1(T, y) e^{my} dy \right) \right]. \end{aligned}$$

As mentioned before, in the numerical analysis, we adopt the technique developed by Bernard et al. (2005b) to invert $\hat{h}_1$ and $\hat{h}_2$.

In the calculation of the expected rebate, distinction of cases becomes necessary again. For the case of $\eta < 1$, the liability holder will get $A_{T_b^-}$ if an early liquidation occurs. Therefore, the expected rebate can be calculated as follows:

$$\begin{aligned} & E_P\left[e^{-\left(r+\frac{1}{2}m^2\right)T_b^-}\exp\left\{mZ_{T_b^-}\right\}\min\left\{L_{T_b^-},A_{T_b^-}\right\}\mathbf{1}_{\left\{T_b^-\leq T\right\}}\right] \\ = \; & A_0E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)T_b^-}\exp\left\{(m+\sigma)Z_{T_b^-}\right\}\mathbf{1}_{\left\{T_b^-\leq T\right\}}\right] \\ = \; & A_0E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)T_b^-}\mathbf{1}_{\left\{T_b^-\leq T\right\}}\right]E_P\left[\exp\left\{(m+\sigma)Z_{T_b^-}\right\}\right]. \end{aligned}$$

The last equality follows from the fact that T_b^- and $Z_{T_b^-}$ are independent, which is shown in the appendix of Chesney et al. (1997). Furthermore, the corresponding laws for these two random variables are given in Chesney et al. (1997), too. As a consequence, we obtain

$$E_P\left[\exp\left\{(m+\sigma)Z_{T_b^-}\right\}\right]=\int_{-\infty}^{b}e^{(m+\sigma)x}\,\frac{b-x}{d}\,\exp\left\{-\frac{(x-b)^2}{2d}\right\}dx$$

and

$$E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)T_b^-}\mathbf{1}_{\left\{T_b^-<T\right\}}\right] \;=\; \int_d^T e^{-\left(r-g+\frac{1}{2}m^2\right)t}\,h_3(t)\,dt,$$

where $h_3(t)$ denotes the density of the stopping time T_b^-. This density can be calculated by inverting the following Laplace transform

$$\hat{h}_3(\lambda)=\frac{\exp\left\{\sqrt{2\lambda}b\right\}}{\psi\left(\sqrt{2\lambda d}\right)}.$$

For the case of $\eta \geq 1$, we obtain

$$\begin{aligned} & E_P\left[e^{-\left(r+\frac{1}{2}m^2\right)T_b^-}\exp\left\{mZ_{T_b^-}\right\}\min\left\{L_{T_b^-},A_{T_b^-}\right\}\mathbf{1}_{\left\{T_b^-\leq T\right\}}\right] \\ = \; & A_0E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)T_b^-}\exp\left\{(m+\sigma)Z_{T_b^-}\right\}\mathbf{1}_{\left\{T_b^-\leq T\right\}}\mathbf{1}_{\left\{Z_{T_b^-}\leq k_1\right\}}\right] \\ & +L_0E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)T_b^-}\exp\left\{mZ_{T_b^-}\right\}\mathbf{1}_{\left\{T_b^-\leq T\right\}}\mathbf{1}_{\left\{k_1<Z_{T_b^-}<b\right\}}\right] \\ = \; & A_0E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)T_b^-}\mathbf{1}_{\left\{T_b^-\leq T\right\}}\right]E_P\left[\exp\left\{(m+\sigma)Z_{T_b^-}\right\}\mathbf{1}_{\left\{Z_{T_b^-}\leq k_1\right\}}\right] \\ & +L_0E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)T_b^-}\mathbf{1}_{\left\{T_b^-\leq T\right\}}\right]E_P\left[\exp\left\{mZ_{T_b^-}\right\}\mathbf{1}_{\left\{k_1<Z_{T_b^-}<b\right\}}\right]. \end{aligned}$$

This expression can be calculated further similarly as in the case of $\eta < 1$. Inspired by Bernard et al. (2005b), we invert $\hat{h}_3$ numerically in the same way.

For the equity holder, we have the following value for his contingent claim

$$\begin{aligned} V_E(A_0,0) &= E^*\left[e^{-rT}[A_T-L_T]^+\mathbf{1}_{\{T_B^-<T\}}\right]-E^*\left[e^{-rT}\delta\left[\alpha A_T-L_T\right]^+\mathbf{1}_{\{T_B^-<T\}}\right] \\ &\quad +E^*\left[e^{-rT_B^-}\max\left\{A_{T_B^-}-L_{T_B^-},0\right\}\mathbf{1}_{\{T_B^-\le T\}}\right] \\ &= PDOC\left[A_0,B_0,L_0,r,g\right]-\delta\,\alpha\,PDOC\left[A_0,B_0,\frac{L_0}{\alpha},r,g\right] \\ &\quad +E_P\left[e^{-\left(r+\frac{1}{2}m^2\right)T_b^-}\exp\left\{mZ_{T_b^-}\right\}\max\left\{A_{T_B^-}-L_{T_B^-},0\right\}\mathbf{1}_{\{T_b^-\le T\}}\right]. \end{aligned}$$

It is composed of three parts: A Parisian down–and–out call option with strike L_T, called the residual claim, a short Parisian down–and–out call option with strike $\frac{L_T}{\alpha}$ (multiplied by $\delta\,\alpha$), i.e., the negative value of the liability holder's bonus option and a rebate paid immediately when the liquidation occurs. It is noted that the second component has already been calculated above. The first component is given by the price difference of the corresponding plain–vanilla and the Parisian down–and–in option. The price of the plain–vanilla option is described by

$$\begin{aligned} BSC[A_0,L_0,r,g] &= E^*\left[e^{-rT}\left[A_T-L_T\right]^+\right]=A_0N(d_1)-L_0e^{-(r-g)T}N(d_2) \\ d_{1/2} &= \frac{\ln\left(\frac{A_0}{L_0}\right)+\left(r-g\pm\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}. \end{aligned}$$

In order to calculate the relevant Parisian down–and–in option, again, two cases are distinguished. For $\eta<1$,

$$PDIC[A_0,B_0,L_0,r,g] = e^{-\left(r-g+\frac{1}{2}m^2\right)T}\int_{k_1}^{\infty}e^{my}\left(A_0e^{\sigma y}-L_0\right)h_1(T,y)dy$$

and for $\eta\ge 1$,

$$\begin{aligned} &PDIC[A_0,B_0,L_0,r,g] \\ &= e^{-\left(r-g+\frac{1}{2}m^2\right)T}\left(\int_{k_1}^{b}e^{my}\left(A_0e^{\sigma y}-L_0\right)h_2(T,y)\,d\,y+\int_{b}^{\infty}e^{my}\left(A_0e^{\sigma y}-L_0\right)h_1(T,y)\,d\,y\right). \end{aligned}$$

Finally, we come to the value of the equity holder's rebate. Only in the case of $\eta\ge 1$, he would possibly obtain a rebate payment.

$$\begin{aligned} &E_P\left[e^{-\left(r+\frac{1}{2}m^2\right)T_b^-}\exp\left\{mZ_{T_b^-}\right\}\max\left\{A_{T_B^-}-L_{T_B^-},0\right\}\mathbf{1}_{\{T_b^-\le T\}}\right] \\ &= A_0E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)T_b^-}\exp\left\{(m+\sigma)Z_{T_b^-}\right\}\mathbf{1}_{\{T_b^-\le T\}}\mathbf{1}_{\left\{k_1<Z_{T_b^-}<b\right\}}\right] \\ &\quad -L_0E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)T_b^-}\exp\left\{mZ_{T_b^-}\right\}\mathbf{1}_{\{T_b^-\le T\}}\mathbf{1}_{\left\{k_1<Z_{T_b^-}<b\right\}}\right]. \end{aligned}$$

Further calculations can be done analogously to the derivation of the expected rebate for the liability holder.

Cumulative Parisian barrier framework

In this case, the options are lost by their owners when the underlying asset has stayed below the barrier for at least d units of time during the entire duration of the contract. Therefore, the options do not lose their values when the following condition holds:

$$\Gamma_T^{-,B} = \int_0^T \mathbf{1}_{\{A_t \leq B_t\}} dt < d,$$

where $\Gamma_T^{-,B}$ denotes the occupation time of the process describing the value of the assets $\{A_t\}_{t\in[0,T]}$ below the barrier B during $[0,T]$. The condition is equivalent to

$$\Gamma_T^{-,b} := \int_0^T \mathbf{1}_{\{Z_t \leq b\}} dt < d,$$

where b and Z_t are the same value or process, respectively, as in the standard Parisian option case. Since τ is defined as the premature liquidation date, it implies:

$$\Gamma_\tau^{-,b} := \int_0^\tau \mathbf{1}_{\{\tau \leq T\}} \mathbf{1}_{\{Z_t \leq b\}} dt = d.$$

Consequently, we obtain the present value of the liability or of the contract issued to the policy holder in the cumulative Parisian framework:

$$\begin{aligned}
V_L^C(A_0,0) &= E^*\left[e^{-rT}\left(\delta\left[\alpha A_T - L_T\right]^+ + L_T - \left[L_T - A_T\right]^+\right)\mathbf{1}_{\left\{\Gamma_T^{-,b}<d\right\}}\right] + E^*\left[e^{-r\tau}\Theta_L(\tau)\right] \\
&= E^*\left[e^{-rT}\left(\delta\left[\alpha A_T - L_T\right]^+ - \left[L_T - A_T\right]^+\right)\mathbf{1}_{\left\{\Gamma_T^{+,b}\geq T-d\right\}}\right] + E^*\left[e^{-rT}L_T\mathbf{1}_{\left\{\Gamma_T^{-,b}<d\right\}}\right] \\
&\quad + E^*\left[e^{-r\tau}\Theta_L\left(\tau\right)\right] \\
&= e^{-\left(r-g+\frac{1}{2}m^2\right)T}\left(E_P\left[\delta\alpha\left[A_0e^{\sigma Z_T} - \frac{L_0}{\alpha}\right]^+ e^{mZ_T}\mathbf{1}_{\left\{\Gamma_T^{+,b}\geq T-d\right\}}\right]\right. \\
&\quad \left. - E_P\left[\left[L_0 - A_0e^{\sigma Z_T}\right]^+ e^{mZ_T}\mathbf{1}_{\left\{\Gamma_T^{+,b}\geq T-d\right\}}\right]\right) + E^*\left[e^{-(r-g)T}L_0\mathbf{1}_{\left\{\Gamma_T^{-,b}<d\right\}}\right] \\
&\quad + E^*\left[e^{-r\tau}\min\left\{A_\tau, L_\tau\right\}\right] \\
&:= \delta\alpha\, C^+\left[0, A_0, \frac{L_0}{\alpha}, B_0, T-d, r-g\right] - P^+[0, A_0, L_0, B_0, T-d, r-g] \\
&\quad + E^*\left[e^{-(r-g)T}L_0\mathbf{1}_{\left\{\Gamma_T^{-,b}<d\right\}}\right] + E^*\left[e^{-r\tau}\min\left\{A_\tau, L_\tau\right\}\right].
\end{aligned}$$

Here, the first equality results from the equivalence of two events, i.e., the event that the occupation time of the asset process below the barrier is shorter than d during $[0,T]$ and the event that the occupation time of the asset price process above the barrier is longer than $T-d$, i.e.,

$$\left\{\Gamma_T^{+,b} := \int_0^T \mathbf{1}_{\{Z_t > b\}} dt \geq T-d\right\} = \left\{\int_0^T \mathbf{1}_{\{Z_t < b\}} dt := \Gamma_T^{-,b} < d\right\}.$$

First, let us consider the cumulative Parisian down–and–out call option. According to Hugonnier (1999) and the correction in Moraux (2002), the $(r-g,m)$ discounted price at time 0 of a cumulative Parisian call option with maturity T, strike $\frac{L_0}{\alpha}$, excursion level B_0, and window d is given by

$$C^+\left[0,A_0,\frac{L_0}{\alpha},B_0,T-d,r-g\right] = e^{-\left(r-g+\frac{1}{2}m^2\right)T}\left(A_0\Psi^+_{m+\sigma}(T,k,b,T-d)-\frac{L_0}{\alpha}\Psi^+_m(T,k,b,T-d)\right)$$

with $k=\frac{1}{\sigma}\ln\left(\frac{L_0/\alpha}{A_0}\right)$ and $b=\frac{1}{\sigma}\ln\left(\frac{\eta L_0}{A_0}\right)$. $\Psi^+_\mu(T,k,b,T-d)$ takes different values for different cases. The only interesting case for us is $b<0$, i.e., $B_0<A_0$, and in this case $\Psi^+_\mu(T,k,b,T-d)$ assumes the following value:

$$\begin{aligned}&\Psi^+_\mu(T,k,b,T-d)\\ &= e^{\frac{1}{2}\mu^2T}\left[N\left(d^{\Xi(\mu)}\left(A_0,B_0\vee\frac{L_0}{\alpha},T\right)\right)-\left(\frac{B_0}{A_0}\right)^{2\mu/\sigma}N\left(d^{\Xi(\mu)}\left(\frac{B_0^2}{A_0},B_0\vee\frac{L_0}{\alpha},T\right)\right)\right]\\ &\quad+\int_{T-d}^{T}ds\left\{\int_{k\wedge b}^{b}e^{\mu x}\gamma(b-x,-b,s,T-s)dx+\int_{k\vee b}^{\infty}e^{\mu x}\gamma(0,x-2b,s,T-s)dx\right\},\end{aligned} \tag{5.2}$$

where

$$\Xi(\mu)=\begin{cases}+ & \text{if}\quad \mu=m+\sigma\\ - & \text{if}\quad \mu=m\end{cases}$$

$$d^\pm(x,K,t) = \frac{\ln\left(\frac{x}{K}\right)+(r-g\pm\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}$$

$$\begin{aligned}\gamma(a,b,u,v) &= \int_0^\infty\frac{(z+a)(z+b)}{\pi(uv)^{3/2}}\exp\left\{-\frac{(z+a)^2}{2v}\right\}\exp\left\{-\frac{(z+b)^2}{2u}\right\}dz\\ &= \frac{1}{\pi}\left\{\frac{av+bu}{(u+v)^2(uv)^{1/2}}\right\}\exp\left\{-\frac{a^2}{2v}-\frac{b^2}{2u}\right\}+\sqrt{\frac{2}{\pi}}\left(\frac{1}{u+v}\right)^{3/2}\\ &\quad\cdot\left(1-\frac{(b-a)^2}{u+v}\right)\exp\left\{-\frac{(b-a)^2}{2(u+v)}\right\}N\left(\frac{-au-bv}{(uv(u+v))^{1/2}}\right).\end{aligned}$$

Second, let us consider the embedded cumulative Parisian down–and–out put option:

$$\begin{aligned}&P^+[0,A_0,L_0,B_0,T-d,r-g]\\ &= A_0L_0\left(BSC\left[0,\frac{1}{A_0},\frac{1}{L_0},g-r\right]-C^+\left[0,\frac{1}{A_0},\frac{1}{L_0},\frac{1}{B_0},d,g-r\right]\right),\end{aligned}$$

where the put–call–symmetry is used. Furthermore, BSC is the Black–Scholes value of the corresponding call, i.e.,

$$\begin{aligned} BSC\left[0, \frac{1}{A_0}, \frac{1}{L_0}, g-r\right] &= e^{(g-r)T}\frac{1}{A_0}N(d_1) - \frac{1}{L_0}N(d_2) \\ d_{1/2} &= \frac{\ln\left(\frac{L_0}{A_0}\right) + \left(g - r \pm \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}. \end{aligned}$$

And analogous to the call,

$$C^+\left[0, \frac{1}{A_0}, \frac{1}{L_0}, \frac{1}{B_0}, d, g-r\right] = e^{-\frac{1}{2}m_2^2 T}\left(\frac{1}{A_0}\Psi^+_{m_2+\sigma}(T, k_2, b_2, d) - \frac{1}{L_0}\Psi^+_{m_2}(T, k_2, b_2, d)\right)$$

with now $k_2 = -k_1 = \frac{1}{\sigma}\ln\left(\frac{A_0}{L_0}\right)$, $b_2 = -b = \frac{1}{\sigma}\ln\left(\frac{A_0}{\eta L_0}\right) > 0$ and $m_2 = \frac{1}{\sigma}\left(g - r - \frac{1}{2}\sigma^2\right)$.
Hence, Ψ^+_μ owns a different value, namely,

$$\Psi^+_\mu(T, k_2, b_2, d) = \int_d^T ds\left\{\int_{k_2\wedge b_2}^{b_2} e^{\mu x}\gamma(2b_2 - x, 0, s, T-s)dx + \int_{k_2\vee b_2}^{\infty} e^{\mu x}\gamma(b_2, x - b_2, s, T-s)dx\right\}.$$

Third, we come to the valuation of the fixed payment. With a close look, the discounted expected fixed payment under the martingale measure Q is nothing but the product of $e^{-(r-g)T}L_0$ and the price of a cumulative binary option paying 1 at maturity if the occupation time below the barrier is shorter than d. Hence, we can use the representation for the cumulative binary option derived in Hugonnier (1999) to obtain:

$$\begin{aligned} & E^*\left[e^{-(r-g)T}L_0\mathbf{1}_{\left\{\Gamma_T^{-,b} < d\right\}}\right] \\ &= e^{-\left(r-g+\frac{1}{2}m^2\right)T}L_0\Psi^+_m(T, -\infty, b, T-d), \end{aligned}$$

where $\Psi^+_m(T, -\infty, b, T-d)$ takes its value according to Equation (5.2).
Finally, we come to the derivation of the expected rebate payment:

$$E^*\left[e^{-r\tau}\min\left\{A_\tau, L_\tau\right\}\right].$$

Above all, it is noted that τ can be described as the inverse of the occupation time d, namely,

$$\tau = \Gamma_-^{-1}(d) = \inf\left\{t \geq 0 \;\middle|\; \Gamma_t^{-,b} = d\right\}, \quad t \leq T.$$

Here, two cases are distinguished: $\eta < 1$ and $\eta \geq 1$. First, let us look at the case of $\eta < 1$. In this case, the expected rebate is simplified to

$$E^*\left[e^{-r\tau}A_\tau\right].$$

It can be further calculated as follows:

$$\begin{aligned}
E^*\left[e^{-r\tau}A_\tau\right] &= A_0 E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)\tau}e^{(\sigma+m)Z_\tau}\right] \\
&= A_0 e^{(\sigma+m)b} E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)\tau}e^{(\sigma+m)(Z_\tau-b)}\right] \\
&= A_0 e^{(\sigma+m)b} E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)\tau}e^{(\sigma+m)(Z^*_\tau)}\right] \\
&= A_0 e^{(\sigma+m)b}\int_{-\infty}^{0}\int_{d}^{T} e^{-\left(r-g+\frac{1}{2}m^2\right)s}e^{(\sigma+m)x} \\
&\quad \cdot \int_0^\infty \frac{|l-b||-x+l|}{\pi(s-d)^{3/2}d^{3/2}}\exp\left\{-\frac{(l-b)^2}{2(s-d)}-\frac{(-x+l)^2}{2d}\right\} dl\, ds\, dx,
\end{aligned}$$

where $Z^*_\tau = Z_\tau - b$ is a $P-$Brownian motion with initial value $-b$. The first equality results from Girsanov's theorem and the second and third step are done by using the argument that the law of a Brownian motion with initial value 0 staying below a negative barrier b should be equivalent to the law of a Brownian motion with initial value $-b$ staying below the barrier value of 0. The expression in the last integral gives the joint law of a Brownian motion with initial value $-b > 0$, the inverse of the occupation time of length d below 0 and the local time of this Brownian motion at the level 0 which is e.g. given as formula 1.1.5.8 in Borodin and Salminen (1996). In addition, we applied the results given in Chapter 6.3, Section C of Karatzas and Shreve (1991). By solving the integral with respect to the local time, we obtain the law of the Brownian motion and the inverse of the occupation time. Similarly, we can calculate the expected rebate payment for the case of $\eta \geq 1$:

$$\begin{aligned}
&E^*\left[e^{-r\tau}\min\{A_\tau, L_\tau\}\right] \\
&= E^*\left[e^{-r\tau}A_\tau \mathbf{1}_{\{Z_\tau<k_1\}}\right] + E^*\left[e^{-r\tau}L_\tau \mathbf{1}_{\{k_1<Z_\tau<b\}}\right] \\
&= A_0 E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)\tau}e^{(m+\sigma)Z_\tau}\mathbf{1}_{\{Z_\tau<k_1\}}\right] + L_0 E_P\left[e^{-\left(r-g+\frac{1}{2}m^2\right)\tau}e^{mZ_\tau}\mathbf{1}_{\{k_1<Z_\tau<b\}}\right] \\
&= A_0 e^{(\sigma+m)b}\int_{-\infty}^{\frac{1}{\sigma}\ln\left(\frac{1}{\eta}\right)}\int_{d}^{T} e^{-(r-g+\frac{1}{2}m^2)s}e^{(\sigma+m)x} \\
&\quad \cdot \int_0^\infty \frac{|l-b||-x+l|}{\pi(s-d)^{3/2}d^{3/2}}\exp\left\{-\frac{(l-b)^2}{2(s-d)}-\frac{(-x+l)^2}{2d}\right\} dl\, ds\, dx \\
&\quad + L_0 e^{mb}\int_{\frac{1}{\sigma}\ln\left(\frac{1}{\eta}\right)}^{0}\int_{d}^{T} e^{-(r-g+\frac{1}{2}m^2)s}e^{mx}\int_0^\infty \frac{|l-b||-x+l|}{\pi(s-d)^{3/2}d^{3/2}}\exp\left\{-\frac{(l-b)^2}{2(s-d)}-\frac{(-x+l)^2}{2d}\right\} dl\, ds\, dx,
\end{aligned}$$

where $k_1 = \frac{1}{\sigma}\ln\left(\frac{L_0}{A_0}\right)$ and $b = \frac{1}{\sigma}\ln\left(\frac{\eta L_0}{A_0}\right)$ as before. For the equity holder, we have the following value for his contingent claim

$$\begin{aligned}
V_E(A_0,0)) &= E^*\left[e^{-rT}\left[A_T-L_T\right]^+\mathbf{1}_{\left\{\Gamma_T^{-,b}<d\right\}}\right] - E^*\left[e^{-rT}\delta\left[\alpha A_T - L_T\right]^+\mathbf{1}_{\left\{\Gamma_T^{-,b}<d\right\}}\right] \\
&\quad + E^*\left[e^{-r\tau}\max\{A_\tau - L_\tau, 0\}\right].
\end{aligned}$$

The value of the residual call is given by:

$$\begin{aligned} & E^*\left[e^{-rT}\left[A_T - L_T\right]^+ \mathbf{1}_{\left\{\Gamma_T^{-,b}<d\right\}}\right] \\ &= C^+\left[0, A_0, L_0, B_0, T-d, r-g\right] \\ &= e^{-\left(r-g+\frac{1}{2}m^2\right)T}\left(A_0\Psi^+_{m+\sigma}(T, k_1, b, T-d) - L_0\Psi^+_m(T, k_1, b, T-d)\right). \end{aligned}$$

Ψ^+_μ is given in 5.2. Again, the value of the short bonus option can be taken from the computations for the liability holder. Obviously, for the case of $\eta < 1$ the equity holder does not obtain any rebate payment. Consequently, we just look at the value of the equity holder's rebate when $\eta \geq 1$. Since the derivation is analogous to that for the policy holder, we jump to the result:

$$\begin{aligned} & E^*\left[e^{-r\tau}\max\left\{A_\tau - L_\tau, 0\right\}\right] \\ &= E^*\left[e^{-r\tau}\left(A_\tau - L_\tau, 0\right)\mathbf{1}_{\{L_\tau<A_\tau<\eta L_\tau\}}\right] \\ &= A_0 e^{(\sigma+m)b}\int_{\frac{1}{\sigma}\ln\left(\frac{1}{\eta}\right)}^0\int_d^T e^{-\left(r-g+\frac{1}{2}m^2\right)s}e^{(\sigma+m)x} \\ &\quad \cdot\int_0^\infty \frac{|l-b||-x+l|}{\pi(s-d)^{3/2}d^{3/2}}\exp\left\{-\frac{(l-b)^2}{2(s-d)} - \frac{(-x+l)^2}{2d}\right\} dl\, ds\, dx \\ &\quad -L_0 e^{mb}\int_{\frac{1}{\sigma}\ln\left(\frac{1}{\eta}\right)}^0\int_d^T e^{-\left(r-g+\frac{1}{2}m^2\right)s}e^{mx} \\ &\quad \cdot\int_0^\infty \frac{|l-b||-x+l|}{\pi(s-d)^{3/2}d^{3/2}}\exp\left\{-\frac{(l-b)^2}{2(s-d)} - \frac{(-x+l)^2}{2d}\right\} dl\, ds\, dx. \end{aligned}$$

In the next section, we calculate the contract for these two kinds of Parisian barrier frameworks numerically.

5.1.3 Fair contract principle

A contract is called fair if the accumulated expected discounted premium is equal to the accumulated expected discounted payments of the contract under consideration. This principle requires the equality between the initial investment of the policy holder and his expected benefit from the contract, namely the value of the contract equals the initial liability,

$$V_L(A_0, 0) = \alpha A_0 = L_0.$$

Alternatively, we could also take the equity holder's point of view, since $A_0 = V_L(A_0, 0) + V_E(A_0, 0)$. Then,

$$V_E(A_0, 0) = (1-\alpha)A_0 = E_0.$$

Certainly, these equations hold for both standard and cumulative Parisian barrier claims.

Relation between the participation rate and the minimum guaranteed interest rate for a fair contract

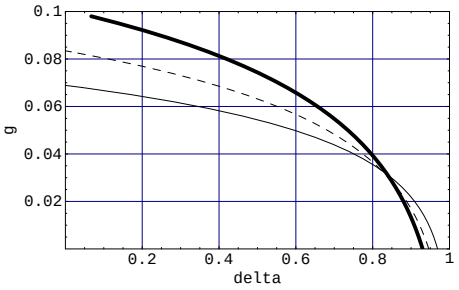

Figure 5.1: Relation between δ and g for different σ with parameters (case (a)): $A_0 = 100$; $L_0 = T$ with parameters (case (a)): $A_0 = 100$; $L_0 = 80$; $\alpha = 0.8$; $r = 0.05$; $\eta = 0.8$; $T = 12$; $d = 1$; $\sigma = 80$; $\alpha = 0.8$; $r = 0.05$; $\eta = 0.8$; $\sigma = 0.2$; $d = 1$; $T = 0.15$ (solid); $\sigma = 0.20$ (dashed); $\sigma = 0.25$ (thick).

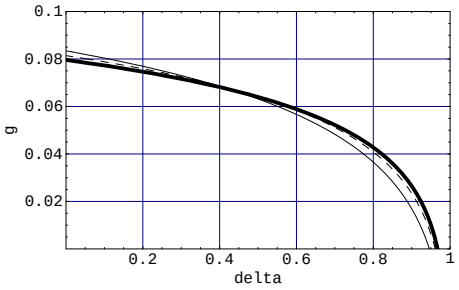

Figure 5.2: Relation between δ and g for different σ with parameters (case (a)): $A_0 = 100$; $L_0 = 80$; $\alpha = 0.8$; $r = 0.05$; $\eta = 0.8$; $\sigma = 0.2$; $d = 1$; $T = 12$ (solid); $T = 18$ (dashed); $T = 24$ (thick).

5.2 Numerical analysis

5.2.1 Fair combination analysis

According to the fair premium principle introduced in Subsection 5.1.3, we can determine the fair premium implicitly through a fair combination of the parameters. In this subsection, we mainly look at the fair combination of δ and g given various parameter constellations. As before, we consider two cases: standard and cumulative Parisian options.

Standard Parisian Barrier Framework

Again, two subcases are distinguished because different relations between the strike and the barrier require different valuation formulas.

$$\text{(a)}\ \ \eta \in [0,1] \quad \Longleftrightarrow \quad \frac{L_0}{\alpha} \geq L_0 \geq B_0$$

$$\text{(b)}\ \ \eta \in \left[1, \frac{1}{\alpha}\right] \quad \Longleftrightarrow \quad \frac{L_0}{\alpha} \geq B_0 \geq L_0.$$

We start our analysis with four graphics for the first subcase. The relation between the participation rate δ and the minimum guarantee g for different volatilities is demonstrated in Figure 5.1. First, it is quite obvious to observe a negative relation between the participation rate and the minimum guarantee (decreasing concave curves) which results from the fair contract principle. Similarly to Grosen and Jørgensen (2002), for smaller values of δ ($\delta < 0.83$), either higher values of g or of δ are required for a higher volatility in order

Relation between the participation rate and the minimum guaranteed interest rate for a fair contract

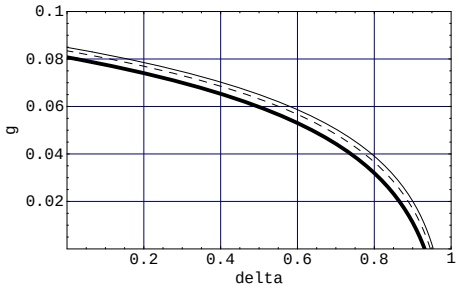

Figure 5.3: Relation between δ and g for different η with parameters (case (a)): $A_0 = 100$; $L_0 = 80$; $\alpha = 0.8$; $r = 0.05$; $\sigma = 0.2$; $T = 12$; $d = 1$; $\eta = 0.7$ (solid); $\eta = 0.8$ (dashed); $\eta = 0.9$ (thick).

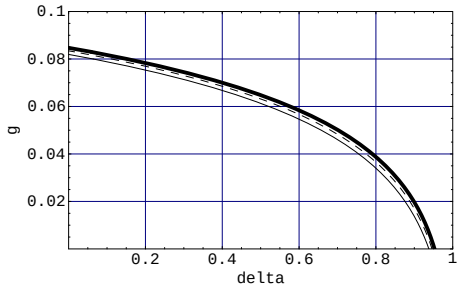

Figure 5.4: Relation between δ and g for different d with parameters (case (a)): $A_0 = 100$; $L_0 = 80$; $\alpha = 0.8$; $r = 0.05$; $\eta = 0.8$; $\sigma = 0.2$; $T = 12$; $d = 0.5$ (solid); $d = 1$ (dashed); $d = 2$ (thick).

to make the contract fair. For higher values of δ ($\delta > 0.83$), this effect is reversed. As the volatility goes up, the value of Parisian down–and–out call increases, while the value of the Parisian down–and–out put increases with the volatility at first and then decreases (hump–shaped). The value of the fixed payment goes down and the rebate term behaves similarly to the Parisian down–and–out put, i.e., goes up at first then goes down after a certain level of volatility is reached. For low values of δ, the fixed payment dominates. Therefore, a positive relation between δ and σ (also g and σ) is generated. In contrast, the reversed effect is observed for high values of δ. Therefore, a volatility–neutral fair combination of $(\delta^*, g^*) \approx (0.83, 0.033)$ is observed.

Figure 5.2 gives the relation between δ and g for different maturity dates T. The value of the Parisian down–and–out call rises with the time to maturity (positive effect), while the value of the Parisian down–and–out put increases with the time to maturity for a while then decreases (hump–shaped). For the chosen parameter values, the put value begins to go down when the maturity time is chosen larger than three years. Hence, this value decreases with T locally (positive effect)[5]. The expected value of the fixed payment declines when the issued contracts have a longer duration (negative effect), while the expected rebate payment increases (positive effect). Before a certain δ is reached, namely $\delta < 0.47$, the positive effect dominates the negative one. The reversed effect is observed for $\delta > 0.47$. Hence, a T–neutral fair combination is also observed here. It is worth mentioning that the magnitude of the effect of T is quite small because the three curves almost overlap.

How δ (or g) changes with η is illustrated in Figure 5.3. First of all, it is noted that different η–values lead to different values of the barrier ($B_0 = \eta L_0$). In Grosen and Jørgensen

[5]Because the three T–values applied in Figure 5.2 are $T = 12$, 18 and 24 years, all of them are larger than $T = 3$ years.

(2002), the liability holder benefits much from a higher regulation parameter η because higher values of η provide the liability holder a better protection against losses. The same effect can also be found here. As the barrier is set higher, the values of the Parisian down–and–out call and put decrease, so does the value of the fixed payment. In contrast, the expected value of the rebate increases with the barrier. In all, the contract value rises when the barrier is set higher. This is why the solid curve ($\eta = 0.7$) lies above the thick one ($\eta = 0.9$). However, the effect is not as large as in the case of a standard knock–out barrier option (the distances among these three curves are not that big) because the introduction of the Parisian barrier feature diminishes the knock–out probability (the factor d, i.e., the length of the excursion reduces the effect caused by the magnitude of the barrier). This positive effect of η (barrier) on the contract value becomes more obvious when the length of excursion d is smaller. Apparently, the adjustment of the parameter d has a considerable impact on the effect of η. Therefore, the regulator controls the strictness of the regulation by adjusting these two parameters. Later, Tables 5.2– 5.4 will show a more intuitive effect of these two parameters.

The last figure for the first case exhibits how the contract value changes with the length of excursion d. Since it is the main concern of this paper to capture the effect of d, three tables are listed (Tables 5.2–5.4) for this purpose. Table 5.2 helps to understand the following argument. Obviously, a positive relation exists between the Parisian down–and–out call and the length of excursion (positive effect). The longer the allowed excursion is, the larger the value of the option. In fact, the value of the call does not change much with the length of excursion when a certain level of d is reached, i.e., the value of the Parisian down–and–out call is a concave increasing function of d. The put option changes with the length of excursion in a similar way. It increases with d but the extent to which it increases becomes smaller after a certain level of d is reached. The fixed payment arises only when the asset price process does not stay below the barrier for a time longer than d. Hence, as the size of d goes up, the probability increases that the fixed payment will become due. Consequently, the expected value of the fixed payment rises. Its magnitude is bounded from above by the payment $L_T e^{-rT}$. In contrast, the rebate payment appears only when the considered insurance company is liquidated, i.e., when the asset price process stays below the barrier for a time period which is longer than d. Therefore, the longer the length of excursion is, the smaller the expected rebate payment. To sum up, the entire contract value diminishes with the length of excursion, i.e., the contract can only remain fair when a high d is combined with a high participation rate or a high minimum interest rate guarantee.

The same figures are provided for case (b) where the barrier value is larger than L_0. Since most of the graphics are similar to those of case (a), we do not want to repeat all the details. However, some further differences are discovered when the effect of d on the contract value is considered. In comparison with case (a), the length of excursion d shows a bigger effect here (the curves are more distant here). In case (b), the Parisian down–and–out call option exhibits considerably smaller values for very small values of d. This fact becomes especially evident for d near zero, since the barrier level is much

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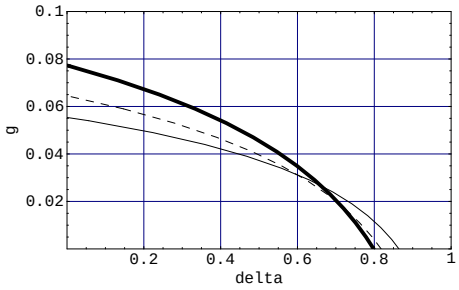

Figure 5.5: Relation between δ and g for different σ with parameters (case (b)): $A_0 = 100$; $L_0 = T$ with parameters (case (b)): $A_0 = 100$; $L_0 = 80; \alpha = 0.8$; $r = 0.05$; $\eta = 1.2$; $T = 12$; $d = 1$; $\sigma = 0.15$ (solid); $\sigma = 0.20$ (dashed); $\sigma = 0.25$ (thick).

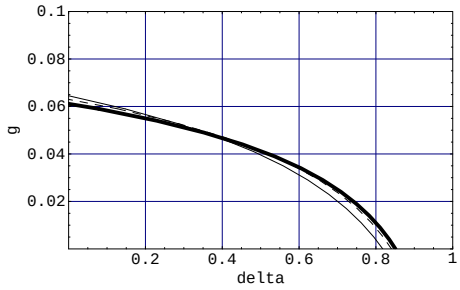

Figure 5.6: Relation between δ and g for different σ with parameters (case (b)): $A_0 = 100$; $L_0 = 80; \alpha = 0.8$; $r = 0.05$; $\eta = 1.2$; $\sigma = 0.2$; $d = 1$; $T = 12$ (solid); $T = 18$ (dashed); $T = 24$ (thick).

higher in the present case (barrier $\geq L_0$) than in case (a) (barrier $< L_0$). It is well known that higher barriers lead to lower prices for down–and–out options (negative effect). If smaller values of d are used, this negative effect of the barrier cannot be reduced or even offset by the positive effect of d. Second, an extraordinarily small value of the expected fixed payment and on the contrary an extraordinary big value of the expected rebate are observed for d close to zero. Altogether, very small values of d, say close to zero, combined with high barrier levels cause small contract values. This is the reason why a relatively more pronounced effect of d results for the case of $\eta \geq 1$ (c.f. Tables 5.2–5.4).

Cumulative Parisian barrier framework

As for the standard Parisian barrier options discussed above, in the cumulative Parisian option framework, a negative relation between the participation rate and the minimum interest rate guarantee is observed. Due to the fact that different η–values require the use of different valuation formulas, again two subcategories can be distinguished: (a) $\eta \in [0, 1]$ $\left(\Leftrightarrow \frac{L_0}{\alpha} \geq L_0 \geq B_0\right)$ and (b) $\eta \in \left[1, \frac{1}{\alpha}\right]$ $\left(\Leftrightarrow \frac{L_0}{\alpha} \geq B_0 \geq L_0\right)$. For each of these subcategories, four figures are plotted. We illustrate how the participation rate and the minimum interest rate guarantee (δ and g) change with the volatility (σ), the maturity date (T), the regulation parameter (η) and the length of excursion (d). Since most of the results are similar to the standard Parisian option case, we only discuss the points where we observe differences. In the following, we first consider category (a).

Overall, it is observed that in this case the resulting values for the fair participation rate are slightly smaller than those in the standard Parisian option case. Although this difference can hardly be seen in the graphics, it is observable in Tables 5.2–5.4. It is justified as follows. The cumulative Parisian down–and–out call, the down–and–out put

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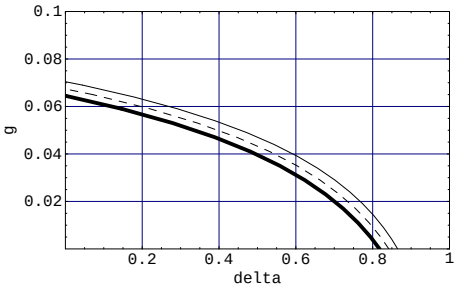

Figure 5.7: Relation between δ and g for different η with parameters (case (b)): $A_0 = 100$; $L_0 = 80$; $\alpha = 0.8$; $r = 0.05$; $\sigma = 0.2$; $T = 12$; $d = 1$; $\eta = 1.1$ (solid); $\eta = 1.15$ (dashed); $\eta = 1.2$ (thick).

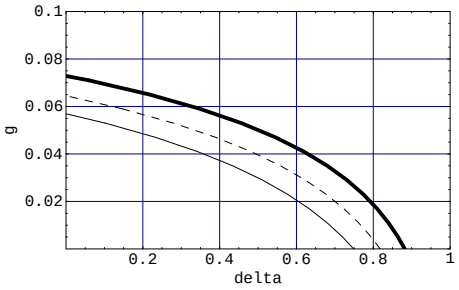

Figure 5.8: Relation between δ and g for different d with parameters (case (b)): $A_0 = 100$; $L_0 = 80$; $\alpha = 0.8$; $r = 0.05$; $\eta = 1.2$; $\sigma = 0.2$; $T = 12$; $d = 0.5$ (solid); $d = 1$ (dashed); $d = 2$ (thick).

and the fixed payment assume smaller values than the corresponding standard Parisian contingent claims. This is due to the fact that the knock–out probability becomes higher in the cumulative case, given the same parameters. This is quite obvious because the knock–out condition for standard Parisian barrier options is that the underlying asset stays *consecutively* below barrier for a time longer than d before the maturity date, while the knock–out condition for cumulative Parisian barrier options is that the underlying asset value spends until the maturity *in total* d units of time below the barrier. In contrast, the expected cumulative rebate part of the payment assumes larger values because it is contingent on the reversed condition compared to the other three parts of the payment. Moreover, (usually) the total effect of these other parts together dominates that of the rebate.

Figure 5.9 depicts how the participation rate δ (or the minimum guarantee g) varies with the volatility. The figure is very similar to Figure 5.1. The fair combinations of g and δ for different maturity dates T are plotted in Figure 5.10 which resembles Figure 5.2.

How the regulation parameter η influences the fair combination of δ and g is demonstrated in Figure 5.11. In contrast to the standard Parisian case (Figure 5.3), η has a bigger impact on the fair parameter combination: the differences of the three curves are more pronounced. Intuitively, it is clear that the value of cumulative Parisian barrier options depends more on the magnitude of the barrier than the value of standard Parisian barrier options does.

Figure 5.12 illustrates the effect of the length of excursion d on the fair combination of δ and g. As in the standard Parisian case (c.f. Figure 5.4), the parameter d does not show a big influence (but bigger than in the standard Parisian case) on the fair combination of δ and g. All four parts of the payment change with d similarly to the standard Parisian

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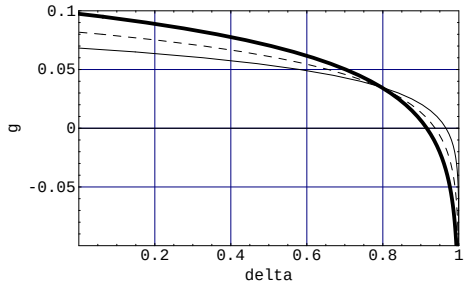

Figure 5.9: Relation between δ and g for different σ with parameters (case (a)): $A_0 = 100$; $L_0 = 80$; $\alpha = 0.8$; $r = 0.05$; $\eta = 0.8$; $T = 12$; $d = 1$; $\sigma = 0.15$ (solid); $\sigma = 0.20$ (dashed); $\sigma = 0.25$ (thick).

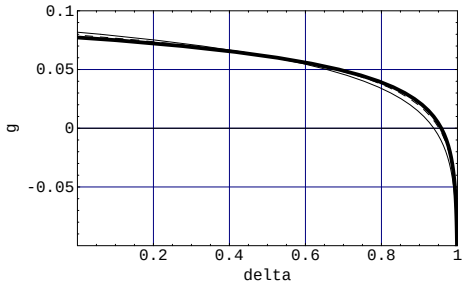

Figure 5.10: Relation between δ and g for different T with parameters (case (a)): $A_0 = 100$; $L_0 = 80$; $\alpha = 0.8$; $r = 0.05$; $\eta = 0.8$; $\sigma = 0.2$; $d = 1$; $T = 12$ (solid); $T = 18$ (dashed); $T = 24$ (thick).

case, namely the cumulative Parisian down–and–out call, the cumulative Parisian down–and–out put and the expected fixed payment go up when d is increased (positive effect). The opposite is true for the rebate part (negative effect). However, the magnitude of the changes in the values is bigger.

Figures 5.13–5.16 are plotted for the case where $\eta \in \left[1, \frac{1}{\alpha}\right]$. This parameter choice leads to a considerably higher barrier level which reduces the values of the cumulative Parisian down–and–out call, the cumulative Parisian down–and–out put and of the expected payment to a big extent and increases the expected rebate part (c.f. Tables 5.2–5.4). Since Figures 5.13–5.16 are quite similar to Figures 5.5–5.8, we do not discuss them in detail.

5.2.2 Value decomposition for fair contracts

In the above numerical analysis, it could be noticed that the choice of the η–parameter influences the effect of d. In the following, the separate effect of d and η is analyzed through some tables. In Tables 5.2–5.4, it is investigated how the fair participation rate and the different components of the liability holder's and the equity holder's payoff change with the length of excursion d for different η–values. Since we do not want to repeat the results of the last subsection, we just mention several important aspects and concentrate on the liability holder's claims. First, assume that the regulation parameter is set to be zero which results in a barrier level of zero. It then follows that the length of excursion d has no effect on the components of the liability holder's payoff because the asset price can never hit the barrier in this situation due to the log–normal assumption of the asset dynamics. That means, the insurance company never defaults and hence is never liquidated. Then we are back in the standard call and put case. Therefore, we obtain the same values for the standard and cumulative Parisian option, and also for the case in Grosen and Jørgensen

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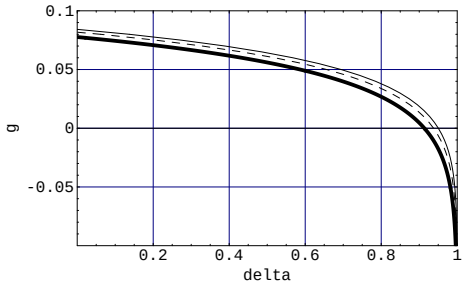

Figure 5.11: Relation between δ and g for different η with parameters (case (a)): $A_0 = 100$; $L_0 = 80$; $\alpha = 0.8$; $r = 0.05$; $\sigma = 0.2$; $T = 12$; $d = 1$; $\eta = 0.7$ (solid); $\eta = 0.8$ (dashed); $\eta = 0.9$ (thick).

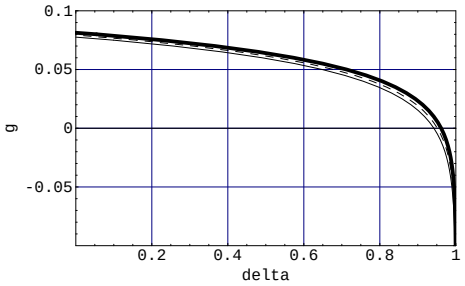

Figure 5.12: Relation between δ and g for different d with parameters (case (a)): $A_0 = 100$; $L_0 = 80$; $\alpha = 0.8$; $r = 0.05$; $\eta = 0.8$; $\sigma = 0.2$; $T = 12$; $d = 0.5$ (solid); $d = 1$ (dashed); $d = 2$ (thick).

(2002). Second, except in this extreme case, smaller participation rates result from the cumulative Parisian option framework than from the standard Parisian modelling given the same parameters. Obviously, for the same parameters, the cumulative down–and–out contingent claims exhibit smaller values than the standard Parisian ones. Third, we emphasize here that the effect of η is twofold. On the one hand, an increase in η leads to a rise of barrier level which accelerates the default of the company, especially when d is set to a small value. On the other hand, a larger expected rebate results from a higher η. Finally, we summarize how different combinations of d and η affect the different components of the liability holder's payoff. If small η–values ($\eta = 0.8$ or 0.9) are combined with long d–values (e.g. $d = 5$), the probability that the firm defaults before the maturity date is small. Hence, very high bonus values, very high expected fixed payments and very small rebate values are observed. As the barrier level rises gradually, the default probability climbs up, and so does the expected rebate. However, in the other extreme case, where high barrier levels (e.g. for the cases $\eta = 1.1$ and $\eta = 1.2$) are combined with a very short length of excursion (say $d = 0.25$ in a 20–year contract), relatively small bonus values, small fixed payments and relatively large expected rebate payments result.

5.3 Shortfall probability

Until now we have not raised the question of how attractive the issued contract is to the liability holder. The liability holder might be interested in getting to know with exactly what probability he will get the rebate payment at the liquidation time instead of the contract value at the maturity date. Therefore, in this section, we would like to have a look at the shortfall probability, i.e., the probability of an early liquidation (liquidation occurs before the maturity date).

$\eta = 0 \Rightarrow$ Barrier $= 0$											
	d	δ	BO	SP	CFP	RL	V_L	RC	SBO	RE	V_E
GJ		0.951	41.49	-5.39	43.90	0.00	80.00	61.49	-41.49	0.00	20.00
PA	0-T	0.951	41.49	-5.39	43.90	0.00	80.00	61.49	-41.49	0.00	20.00
CP	0-T	0.951	41.49	-5.39	43.90	0.00	80.00	61.49	-41.49	0.00	20.00
$\eta = 0.8 \Rightarrow$ Barrier $= 64$											
GJ		0.836	30.91	-0.03	19.84	29.28	80.00	50.91	-30.91	0.00	20.00
PA	0	0.836	30.91	-0.03	19.84	29.28	80.00	50.91	-30.91	0.00	20.00
	0.25	0.888	35.60	-0.15	24.13	20.42	80.00	55.60	-35.60	0.00	20.00
	0.50	0.902	36.94	-0.23	25.83	17.46	80.00	56.94	-36.94	0.00	20.00
	1.00	0.917	38.38	-0.40	28.29	13.73	80.00	58.38	-38.38	0.00	20.00
	1.50	0.926	39.20	-0.57	30.23	11.14	80.00	59.20	-39.20	0.00	20.00
	2.00	0.932	39.75	-0.73	31.41	9.57	80.00	59.75	-39.75	0.00	20.00
	5.00	0.945	41.02	-1.94	36.46	4.46	80.00	61.02	-41.02	0.00	20.00
CP	0	0.836	30.91	-0.03	19.84	29.28	80.00	50.91	-30.91	0.00	20.00
	0.25	0.874	34.26	-0.09	22.75	23.08	80.00	54.26	-34.26	0.00	20.00
	0.50	0.886	35.41	-0.13	23.93	20.79	80.00	55.41	-35.41	0.00	20.00
	1.00	0.901	36.81	-0.22	25.59	17.82	80.00	56.81	-36.81	0.00	20.00
	1.50	0.910	37.72	-0.30	26.85	15.73	80.00	57.72	-37.72	0.00	20.00
	2.00	0.917	38.39	-0.40	27.90	14.10	80.00	58.39	-38.39	0.00	20.00
	5.00	0.938	40.39	-1.03	32.43	8.21	80.00	60.39	-40.39	0.00	20.00
$\eta = 0.9 \Rightarrow$ Barrier $= 72$											
GJ		0.743	23.87	0.00	15.23	40.90	80.00	43.87	-23.87	0.00	20.00
PA	0	0.743	23.87	0.00	15.23	40.90	80.00	43.87	-23.87	0.00	20.00
	0.25	0.840	31.27	-0.04	20.14	28.63	80.00	51.27	-31.27	0.00	20.00
	0.50	0.865	33.44	-0.09	21.91	24.74	80.00	53.44	-33.44	0.00	20.00
	1.00	0.891	35.83	-0.20	24.23	20.24	80.00	55.83	-35.83	0.00	20.00
	1.50	0.905	37.23	-0.32	26.79	16.30	80.00	57.23	-37.23	0.00	20.00
	2.00	0.915	38.17	-0.44	29.10	13.17	80.00	58.17	-38.17	0.00	20.00
	5.00	0.940	40.52	-1.42	33.63	7.27	80.00	60.52	-40.52	0.00	20.00
CP	0	0.743	23.87	0.00	15.23	10.90	80.00	43.87	-23.87	0.00	20.00
	0.25	0.814	29.11	-0.02	18.47	32.44	80.00	49.11	-29.11	0.00	20.00
	0.50	0.836	30.93	-0.04	19.78	29.32	80.00	50.93	-30.93	0.00	20.00
	1.00	0.862	33.19	-0.08	21.63	25.26	80.00	53.19	-33.19	0.00	20.00
	1.50	0.878	34.68	-0.13	23.03	22.42	80.00	54.68	-34.68	0.00	20.00
	2.00	0.890	35.80	-0.18	24.21	20.18	80.00	55.80	-35.80	0.00	20.00
	5.00	0.925	39.24	-0.63	29.30	12.09	80.00	59.24	-39.24	0.00	20.00

Table 5.2: **Decomposition of fair contracts with** $A_0 = 100$, $r = 0.05$, $g = 0.02$, $\alpha = 0.8$, $\sigma = 0.2$, $T = 20$.

$\eta = 1 \Rightarrow$ Barrier $= 80$											
	d	δ	BO	SP	CFP	RL	V_L	RC	SBO	RE	V_E
GJ		0.569	14.50	0.00	10.71	54.79	80.00	34.50	-14.50	0.00	20.00
PA	0	0.569	14.50	0.00	10.71	54.79	80.00	34.50	-14.50	0.00	20.00
	0.25	0.764	25.29	-0.01	16.19	38.53	80.00	45.29	-25.29	0.00	20.00
	0.50	0.807	28.54	-0.02	18.40	33.08	80.00	48.54	-28.54	0.00	20.00
	1.00	0.851	32.19	-0.09	20.26	27.64	80.00	52.19	-32.19	0.00	20.00
	1.50	0.875	34.36	-0.18	21.95	23.87	80.00	54.36	-34.36	0.00	20.00
	2.00	0.891	35.83	-0.28	25.99	18.46	80.00	55.83	-35.83	0.00	20.00
	5.00	0.931	39.70	-0.93	29.68	11.55	80.00	59.70	-39.70	0.00	20.00
CP	0	0.569	14.50	0.00	10.71	54.79	80.00	34.50	-14.50	0.00	20.00
	0.25	0.715	22.10	0.00	14.22	43.69	80.00	42.10	-22.10	0.00	20.00
	0.50	0.756	24.77	-0.01	15.65	39.59	80.00	44.77	-24.77	0.00	20.00
	1.00	0.801	28.12	-0.03	17.65	34.26	80.00	48.12	-28.12	0.00	20.00
	1.50	0.829	30.36	-0.05	19.18	30.51	80.00	50.36	-30.36	0.00	20.00
	2.00	0.848	32.05	-0.08	20.46	27.57	80.00	52.05	-32.05	0.00	20.00
	5.00	0.906	37.44	-0.36	26.02	16.91	80.00	57.44	-37.44	0.00	20.00
$\eta = 1.1 \Rightarrow$ Barrier $= 88$											
GJ		0.540	9.10	0.00	6.31	64.58	80.00	22.64	-9.10	6.46	20.00
PA	0	0.540	9.10	0.00	6.31	64.58	80.00	22.64	-9.10	6.46	20.00
	0.25	0.659	18.23	0.00	12.05	49.72	80.00	37.53	-18.23	0.70	20.00
	0.50	0.726	22.43	0.00	14.28	43.29	80.00	42.09	-22.43	0.34	20.00
	1.00	0.795	27.41	-0.02	17.32	35.27	80.00	47.26	-27.41	0.15	20.00
	1.50	0.833	30.52	-0.08	19.70	29.28	80.00	50.42	-30.52	0.10	20.00
	2.00	0.858	32.70	-0.18	21.73	25.57	80.00	52.64	-32.70	0.06	20.00
	5.00	0.918	38.47	-0.48	28.13	13.40	80.00	58.45	-38.47	0.02	20.00
CP	0	0.540	9.10	0.00	6.31	64.58	80.00	22.64	-9.10	6.46	20.00
	0.25	0.612	14.97	0.00	10.06	54.97	80.00	33.06	-14.97	1.91	20.00
	0.50	0.666	18.06	0.00	11.58	50.36	80.00	36.77	-18.06	1.29	20.00
	1.00	0.731	22.26	-0.01	13.72	44.03	80.00	41.44	-22.26	0.82	20.00
	1.50	0.771	25.21	-0.02	15.34	39.47	80.00	44.60	-25.21	0.61	20.00
	2.00	0.799	27.48	-0.03	16.71	35.84	80.00	46.99	-27.48	0.49	20.00
	5.00	0.882	35.04	-0.20	22.66	22.50	80.00	54.84	-35.04	0.20	20.00

Table 5.3: **Decomposition of fair contracts with** $A_0 = 100$, $r = 0.05$, $g = 0.02$, $\alpha = 0.8$, $\sigma = 0.2$, $T = 20$.

Relation between the participation rate and the minimum guaranteed interest rate for a fair contract

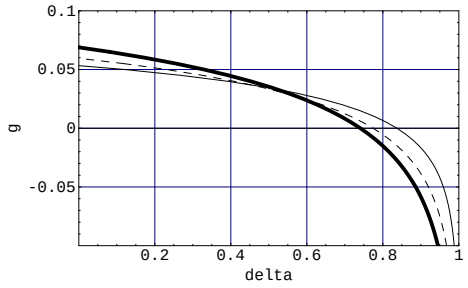

Figure 5.13: Relation between δ and g for different σ with parameters (case (b)): $A_0 = 100$; $L_0 = 80$; $\alpha = 0.8$; $r = 0.05$; $\eta = 1.2$; $T = 12$; $d = 1$; $\sigma = 0.15$ (solid); $\sigma = 0.20$ (dashed); $\sigma = 0.25$ (thick).

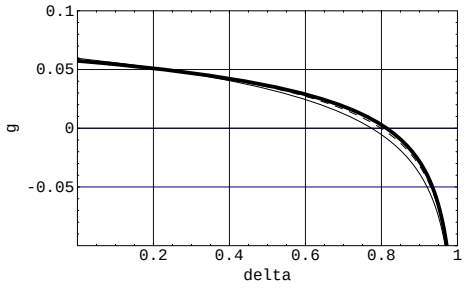

Figure 5.14: Relation between δ and g for different T with parameters (case (b)): $A_0 = 100$; $L_0 = 80$; $\alpha = 0.8$; $r = 0.05$; $\eta = 1.2$; $\sigma = 0.2$; $d = 1$; $T = 12$ (solid); $T = 18$ (dashed); $T = 24$ (thick).

Obviously, it only makes sense to consider the shortfall probability under the subjective probability measure, under which the assets are assumed to evolve as:

$$dA_t = A_t \left(\mu \, dt + \sigma d \widetilde{W}_t \right),$$

where $\mu > 0$ is the instantaneous expected return of the asset and $\widetilde{W}_t$ is a martingale under the subjective measure. In the case of the standard Parisian framework, the shortfall probability is given by

$$\begin{aligned} \widetilde{P}^{SF} &= \widetilde{P}\left(T_B^- = \inf\left\{t > 0 \middle| \left(t - g_{B,t}^A\right) \mathbf{1}_{\{A_t < B_t\}} > d\right\} \leq T\right) \\ &= e^{-\frac{1}{2}\tilde{m}^2 T} \left(\int_{-\infty}^{b} h_2(T,y) e^{\tilde{m}y} \, d\, y + \int_{b}^{\infty} h_1(T,y) e^{\tilde{m}y} \, d\, y \right) \end{aligned}$$

with $\tilde{m} = \frac{1}{\sigma}\left(\mu - g - \frac{1}{2}\sigma^2\right)$.

In case of the cumulative Parisian framework, the shortfall probability is determined by

$$\begin{aligned} \widetilde{P}^{SF} &= \widetilde{P}(\tau \leq T) = \widetilde{P}\left(\frac{1}{T} \int_0^T \mathbf{1}_{\left\{\widetilde{W}_u + \tilde{m}\, u \leq b\right\}} du \geq \frac{d}{T} \right) \\ &= \widetilde{P}\left(\frac{1}{T} \int_0^T \mathbf{1}_{\left\{\widetilde{W}_u - \tilde{m}\, u \leq -b\right\}} du \leq 1 - \frac{d}{T} \right) \\ &= 2 \int_0^{1-\frac{d}{T}} \left\{ \left[\frac{n\left(-\tilde{m}\sqrt{T}\sqrt{1-u}\right)}{\sqrt{1-u}} + \left(-\tilde{m}\sqrt{T}\right) N\left(-\tilde{m}\sqrt{T}\sqrt{1-u}\right) \right] \right. \\ &\quad \left. \cdot \left[\frac{1}{\sqrt{u}} n\left(\frac{(-b)/\sqrt{T} + \tilde{m}\sqrt{T}u}{\sqrt{u}} \right) + \tilde{m}\sqrt{T} e^{2\tilde{m}b} N\left(\frac{b/\sqrt{T} + \tilde{m}\sqrt{T}u}{\sqrt{u}} \right) \right] \right\} du, \end{aligned}$$

Relation between the participation rate and the minimum guaranteed interest rate for a fair contract

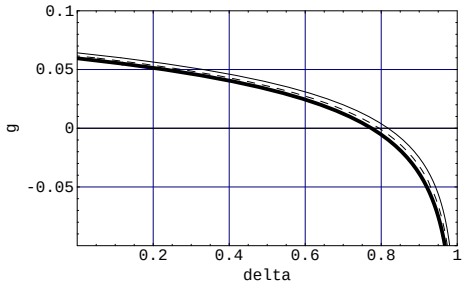

Figure 5.15: Relation between δ and g for different η with parameters (case (b)): $A_0 = 100$; $L_0 = 80$; $\alpha = 0.8$; $r = 0.05$; $\sigma = 0.2$; $T = 12$; $d = 1$; $\eta = 1.1$ (solid); $\eta = 1.15$ (dashed); $\eta = 1.2$ (thick).

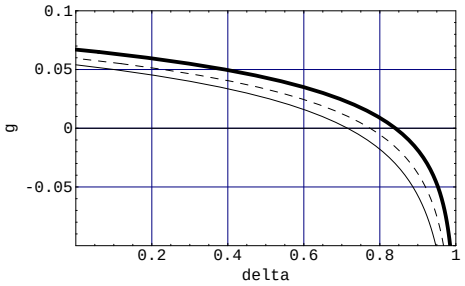

Figure 5.16: Relation between δ and g for different d with parameters (case (b)): $A_0 = 100$; $L_0 = 80$; $\alpha = 0.8$; $r = 0.05$; $\eta = 1.2$; $\sigma = 0.2$; $T = 12$; $d = 0.5$ (solid); $d = 1$ (dashed); $d = 2$ (thick).

where $n(\cdot)$ is the density function of the standard normal distribution.

In the above derivation, Equation (12) of Takács (1996) is applied. In Table 5.5, several shortfall probabilities are calculated for both standard and cumulative Parisian frameworks. First, apparently, shortfall occurs with a higher probability in the case of cumulative than in that of standard Parisian options. This is due to the fact that the knock–out condition is less demanding for the cumulative Parisian option. All the other effects, e.g. that the shortfall probability increases in σ and η and decreases in μ and d, are quite straightforward. Therefore, the insurance company can offer customers with different risk aversions (willingness to accept a certain shortfall probability) different insurance contracts according to varying parameter choices.

5.4 Summary

In the present chapter, we extend the model of Grosen and Jørgensen (2002) and investigate the question of how to value an equity–linked life insurance contract when considering the default risk (and the liquidation risk) under different bankruptcy procedures. In order to take into account the realistic bankruptcy procedure Chapter 11, these risks are modelled in both standard and cumulative Parisian frameworks. In the numerical analysis part, we perform several sensitivity analyses to see how the fair combinations of the participation rate and the minimum interest rate guarantee depend on the volatility of the company's assets, the maturity dates of the contract, the regulation parameter and the length of excursion. In addition, due to their importance, a number of tables are given which help to catch and to compare the effects of the two regulation parameters d and η. Furthermore, we consider how likely it is that the liability holder will obtain the

$\eta = 1.2 \Rightarrow$ Barrier = 96											
	d	δ	BO	SP	CFP	RL	V_L	RC	SBO	RE	V_E
GJ		0.514	3.16	0.00	2.07	74.77	80.00	8.21	-3.16	14.95	20.00
PA	0	0.514	3.16	0.00	2.07	74.77	80.00	8.21	-3.16	14.95	20.00
	0.25	0.580	12.00	0.00	8.14	59.86	80.00	27.87	-12.00	4.13	20.00
	0.50	0.645	16.20	0.00	10.53	53.27	80.00	33.94	-16.20	2.26	20.00
	1.00	0.737	22.18	0.00	13.65	44.17	80.00	41.09	-22.18	1.09	20.00
	1.50	0.782	25.91	-0.02	16.08	38.01	80.00	45.20	-25.91	0.71	20.00
	2.00	0.818	28.83	-0.09	18.42	32.75	80.00	48.32	-28.83	0.51	20.00
	5.00	0.901	36.76	-0.10	24.71	18.53	80.00	56.61	-36.76	0.15	20.00
CP	0	0.514	3.16	0.00	2.07	74.77	80.00	8.21	-3.16	14.95	20.00
	0.25	0.561	9.16	0.00	6.00	64.84	80.00	21.91	-9.16	7.25	20.00
	0.50	0.607	12.09	0.00	7.60	60.30	80.00	26.82	-12.09	5.28	20.00
	1.00	0.677	16.57	0.00	9.85	53.58	80.00	33.04	-16.57	3.53	20.00
	1.50	0.725	19.96	0.00	11.56	48.48	80.00	37.28	-19.96	2.68	20.00
	2.00	0.759	22.67	-0.01	13.00	44.33	80.00	40.52	-22.68	2.16	20.00
	5.00	0.862	32.24	-0.10	19.27	28.59	80.00	51.33	-32.24	0.91	20.00

Table 5.4: **Decomposition of fair contracts with** $A_0 = 100$, $r = 0.05$, $g = 0.02$, $\alpha = 0.8$, $\sigma = 0.2$, $T = 20$.

	Shortfall Probability PA						Shortfall Probability CP					
σ	μ		η		d		μ		η		d	
	0.06	0.08	0.9	1.1	0.5	2	0.06	0.08	0.9	1.1	0.5	2
0.10	0.013	0.000	0.000	0.000	0.000	0.000	0.018	0.003	0.010	0.093	0.004	0.002
0.15	0.125	0.052	0.092	0.240	0.068	0.035	0.159	0.070	0.124	0.318	0.085	0.053
0.20	0.289	0.180	0.251	0.403	0.222	0.132	0.347	0.227	0.308	0.507	0.259	0.184

Table 5.5: Shortfall probabilities for standard and cumulative Parisian frameworks with parameters: $A_0 = 100, L_0 = 80,\ g = 0.02, T = 20$, $\mu = 0.08$, $\sigma = 0.2$, $\eta = 0.8$, $d = 1$.

rebate payment whose size is uncertain at the point in time when the contract is signed. Based on the analysis in Section 5.3, the insurance company can offer different contracts to customers with different willingness to accept certain shortfall probabilities.

The incentives for the customers to buy this kind of contracts introduced in this chapter are not very high due to two reasons: first, the guaranteed interest rate is usually smaller than the market interest rate; and second, probably the customers can even not obtain the guaranteed amount which is against the nature of an insurance contract, although the probability for this event can be controlled quite low. Therefore, it seems not very interesting to price this kind of contracts, but to analyze the risk management of these contracts. Moreover, the role of the regulator should be somehow strengthened. Usually, they would take some measures in order not to let the insurer go bust immediately, in addition to setting the bankruptcy and liquidation rules for the insurance companies. Concretely, one possibility is for instance to formulate the regulation parameter η endogenously.

Chapter 6

Conclusion

This thesis studies several risk management methods of a life insurance company issuing equity–linked life insurance contracts. Due to the characteristics of these contracts, the question is indeed investigated of how to hedge the combined insurance and financial risk in the present dissertation. I.e., the emphasis is placed on dealing with the non–tradable insurance risk, which alone already makes the considered financial market incomplete. Therefore, as a starting point, several popular hedging/optimality criteria in a continuous–time setup, such as risk–minimizing, quantile and efficient hedging, are analyzed. Risk–minimizing is a criterion which suggests the hedger to trade in such a strategy that achieves a minimum variance of the hedger's future cost. While in a quantile or efficient hedging, the hedger is ready to bear a certain shortfall probability and trades in a self–financing strategy whose initial price is constrained. Another point of view is taken in the analysis of quantile and efficient hedging, i.e., by considering the mortality risk explicitly, a transfer between the insurance and financial risk is allowed. In other words, knowing that how much shortfall risk he is ready to bear, the hedger can learn what kinds of customers to take (e.g. how large the resulting survival probability is in a pure endowment contract).

In addition to the insurance risk, some other sources of incompleteness are analyzed, e.g., incompleteness resulting from trading restrictions. A chapter is designed to tackle this problem and additionally attempts to answer the question "what happens to the hedger's net loss given that the hedger trades in discrete–time risk–minimizing hedging strategies". In other words, the chapter is drawn up to learn whether the hedger benefits from using the introduced hedging strategy. Net loss of the hedger or more specifically the ruin probability is taken as the criterion of goodness. A further chapter is drawn up to consider the incompleteness caused by the model risk, i.e., the incompleteness results when the assumed model or model parameters deviate from the real ones. There, we look at effective risk management strategies which combine diversification and hedging effect, when it is both possible to misspecify the interest rate dynamic which is used for hedging the contracts and the mortality distributions which are used for the diversification effects. We study the distribution of the total hedging errors which depends on interaction of the true and assumed interest rate and mortality distributions. In particular, we look for a

combination of diversification and hedging effects which is robust against model misspecification.

The fact that the relevant financial market is incomplete results in a positive probability that default occurs in a life insurance. Therefore, a chapter is constituted to consider the default risk and the relevant bankruptcy procedures, i.e., to build these in a contingent claim model and the firm values of the insurance company are considered as an aggregate. In comparison with Grosen and Jørgensen (2002) whose model corresponds to Chapter 7 bankruptcy procedure, a more realistic and general bankruptcy procedure Chapter 11 is taken into account. Mathematically, it is realized by using both standard and cumulative Parisian options because they have appealing interpretations in terms of the bankruptcy mechanism.

Most of the analyses in the present thesis are based on a contract–specific level, in particular, specific pure endowment insurance. Further research activities in this direction can be undertaken by analyzing other forms of insurance contracts or more general contracts. It would be interesting to figure out whether some arguments can be made or some suggestions can be given to a life insurance company beyond the contract specification. Furthermore, customers are always assumed to be identical when a cohort of customers are included in the model. Hence, to take into consideration different customers, e.g customers with different age, entering or/and exiting times, might bring some interesting insights.

In Chapter 4, by considering a simple guaranteed endowment insurance, model misspecification associated with mortality and the interest rate risk is analyzed. The risk linked to the asset is neglected. However, in reality, it is not uncommon that an additional bonus payment is offered to the customer in such kinds of contracts. Usually the bonus payment is constructed as a call option (or a sequence of call options) on the asset. Hence, these contracts maintain the financial risk related to the asset. Therefore, a natural further interest lies in how model risk related not only to the mortality, interest rate but also to the asset affects insurers' hedging decisions.

Through the analysis of Chapter 5, we learn that the shortfall probability can be regulated to a quite small one by setting different regulation parameters. However, there is a positive probability with which the customer obtains an amount smaller than the guaranteed amount, which is against the nature of an insurance contract with a guaranteed payment. Therefore, it will be interesting to ask whether the role of the regulator can be reinforced. Or more specifically, the regulator not only sets the default or liquidation rules for the insurance company, but more importantly, they would take some measures to help the insurance company get out of default when it is in a financial distress. This then ensures the customer to obtain the guaranteed amount.

Chapter 7

Appendix

7.1 Synthesising the pseudo asset X

We study the case where the hedge instrument X is not liquidly traded in the market and a potential hedger must use other assets $Y^1, ..., Y^n$ to synthesize X. We place ourselves in a diffusion setting, i.e. the prices X, $Y^1, \ldots, Y^n$ are given by Itô processes which are driven by a d-dimensional Brownian motion W defined on $\left(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, P\right)$:

$$dX_t = X_t\{\mu_t^X dt + \sigma_t^X dW_t\}$$
$$dY_t^i = Y_t^i\{\mu_t^i dt + \sigma_t^i dW_t\}$$

where μ^X, σ^X and μ^i, σ^i are suitably integrable stochastic processes. We assume the prices X, $Y^1, \ldots, Y^n$ are arbitrage-free. This implies that there is a "market price of risk" process φ such that for any $i \in \{1, \ldots, n\}$:

$$\mu^X - \sigma^X\varphi = \mu^i - \sigma^i\varphi.$$

Synthesizing X out of $Y^1, \ldots, Y^n$ involves finding a self-financing strategy ϕ with a position of ϕ^i in asset Y^i for each $i \in \{1, \ldots, n\}$ such that $X = \sum_{i=1}^n \phi^i Y^i$. The following proposition characterizes these strategies ϕ.

Proposition 7.1.1. *Suppose that $\lambda^1, \ldots, \lambda^n$ are predictable processes satisfying the following two conditions:*

$$(1)\ \ \sum_{i=1}^n \lambda_t^i = 1 \text{ and } (2)\ \ \sum_{i=1}^n \lambda_t^i\sigma_t^i = \sigma_t^X.$$

For each $i \in \{1, \ldots, n\}$, we set $\phi^i := \frac{X}{Y^i}\lambda^i$. Then ϕ is a self-financing strategy which identically duplicates X. In particular, any such strategy is of the form above.

Proof: Suppose that weights $\lambda^1, \ldots, \lambda^n$ are given and satisfy conditions (1) and (2) and that ϕ is the corresponding strategy. By condition (1), it is clear that $\sum_{i=1}^n \phi^i Y^i = X$. By the no-arbitrage condition and because of (2) we have

$$\sum_{i=1}^n \lambda^i\mu^i = \sum_{i=1}^n \lambda^i\{\mu^X + \varphi(\sigma^i - \sigma^X)\} = \mu^X.$$

From this we see that ϕ is also self-financing because

$$\sum_{i=1}^{n} \phi_t^i dY_t^i = X_t \sum_{i=1}^{n} \lambda_t^i \{\mu_t^i dt + \sigma_t^i dW_t\} = X_t \{\mu^X dt + \sigma_t^X dW_t\} = dX_t.$$

Conversely, if ϕ is a self-financing strategy which identically duplicates X, then the weights $\lambda^1, \ldots, \lambda^n$ determined by $\lambda^i := \frac{Y^i}{X}\phi^i$ will satisfy the two conditions. □

The weights $\lambda^1, \ldots, \lambda^n$ are to be interpreted as portfolio weights, i.e. λ^i is the proportion of total capital to be invested in the asset Y^i.

7.2 Proof of Proposition 4.2.17

(i)
Since the variance of the discounted total hedging costs is given by

$$\mathrm{Var}^*[C_T^{tot,*}(\phi)] \;=\; \mathrm{Var}^*[X_T^* - I_T^*(\phi)].$$

it holds that

$$\mathrm{Var}^*[C_T^{tot,*}(\phi)] - \mathrm{Var}^*[C_T^{tot,*}(\psi)] \;=\; \mathrm{Var}^*[I_T^*(\phi)] - \mathrm{Var}^*[I_T^*(\psi)] + 2\mathrm{Cov}^*[X_T^*, I_T^*(\psi) - I_T^*(\phi)].$$

Since ϕ and ψ own the following relation:

$$\begin{aligned} \psi^{(N-1)} &= \sum_{i=1}^{N-2} \alpha^i \phi^{(i)} + \phi^{(N-1)} \\ \psi^{(N)} &= \sum_{i=1}^{N-2} \beta^i \phi^{(i)} + \phi^{(N)}, \end{aligned}$$

where α_i is function of $\frac{D(t,t_i)}{D(t,t_{N-1})}$ and β_i function of $\frac{D(t,t_i)}{D(t,t_N)}$ given as in Proposition 4.2.14. First, the gain process of ψ is given as follows

$$\begin{aligned} I_t^*(\psi) &= \int_0^t \psi_u^{(N-1)} dD^*(u, t_{N-1}) + \int_0^t \psi_u^{(N)} dD^*(u, t_N) \\ &= \int_0^t \left(\sum_{i=1}^{N-2} \alpha_u^i \phi_u^{(i)} + \phi_u^{(N-1)} \right) dD^*(u, t_{N-1}) + \int_0^t \left(\sum_{i=1}^{N-2} \beta_u^i \phi_u^{(i)} + \phi_u^{(N)} \right) dD^*(u, t_N) \\ &= \sum_{i=1}^{N-2} \int_0^t \alpha_u^i \phi_u^{(i)} dD^*(u, t_{N-1}) + \int_0^t \phi_u^{(N-1)} dD^*(u, t_{N-1}) \\ &\quad + \sum_{i=1}^{N-2} \int_0^t \beta_u^i \phi_u^{(i)} dD^*(u, t_N) + \int_0^t \phi_u^{(N)} dD^*(u, t_N), \end{aligned}$$

while the gain process of ϕ owns the following simple form

$$I_t^*(\phi) = \sum_{i=1}^{N} \int_0^t \phi_u^{(i)} dD^*(u, t_i).$$

This leads to

$$\begin{aligned}
I_t^*(\psi) - I_t^*(\phi) &= \sum_{i=1}^{N-2} \int_0^t \alpha_u^i \phi_u^{(i)} dD^*(u, t_{N-1}) + \sum_{i=1}^{N-2} \int_0^t \beta_u^i \phi_u^{(i)} dD^*(u, t_N) - \sum_{i=1}^{N-2} \int_0^t \phi_u^{(i)} dD^*(u, t_i) \\
&= \sum_{i=1}^{N-2} \int_0^t \left(\alpha_u^i \phi_u^{(i)} dD^*(u, t_{N-1}) + \beta_u^i \phi_u^{(i)} dD^*(u, t_N) - \phi_u^{(i)} dD^*(u, t_i) \right) \\
&= \sum_{i=1}^{N-2} \int_0^t \phi_u^{(i)} \left(\lambda_1^{(i)}(u) \frac{D(u, t_i)}{D(u, t_{N-1})} dD^*(u, t_{N-1}) + \lambda_2^{(i)}(u) \frac{D(u, t_i)}{D(u, t_N)} dD^*(u, t_N) \right. \\
&\quad \left. - dD^*(u, t_i) \right).
\end{aligned}$$

The following equalities

$$\begin{aligned}
D^*(u, t_i) = e^{-\int_0^u r_s ds} D(u, t_i) &= D(t_0, t_i) \exp\left\{ -\frac{1}{2} \int_0^u (\sigma_{t_i}(s))^2 ds + \int_0^u \sigma_{t_i}(s) dW_s^* \right\} \\
dD^*(u, t_i) &= D^*(u, t_i) \sigma_{t_i}(u) dW_u^*,
\end{aligned}$$

result in

$$\begin{aligned}
I_t^*(\psi) - I_t^*(\phi) &= \sum_{i=1}^{N-2} \int_0^t \phi_u^{(i)} \left(\lambda_1^{(i)}(u) D^*(u, t_i) \sigma_{t_{N-1}}(u) dW_u^* + \lambda_2^{(i)} D^*(u, t_i) \sigma_{t_N}(u) dW_u^* \right. \\
&\quad \left. - D^*(u, t_i) \sigma_{t_i}(u) dW_u^* \right) \\
&= \sum_{i=1}^{N-2} \int_0^t \phi_u^{(i)} D^*(u, t_i) \left(\lambda_1^{(i)}(u) \sigma_{t_{N-1}}(u) + \lambda_2^{(i)}(u) \sigma_{t_N}(u) - \sigma_{t_i}(u) \right) dW_u^* \\
&:= \sum_{i=1}^{N-2} \int_0^t \phi_u^{(i)} D^*(u, t_i)\, g_u^{(i)}\, dW_u^*.
\end{aligned}$$

If there exists **no** model risk concerning the interest rate, i.e., $\sigma_{t_i}(u) = \tilde{\sigma}_{t_i}(u)$, $u \leq T$, the gain process of ϕ coincides with that of ψ. Therefore, under this circumstance, it holds

$$\mathrm{Var}^*[I_t^*(\psi)] = \mathrm{Var}^*[I_t^*(\phi)].$$

Consequently, it leads to

$$\mathrm{Var}^*[C_T^{tot,*}(\phi)] = \mathrm{Var}^*[C_T^{tot,*}(\psi)].$$

The following transformation enlightens this argument.

$$\begin{aligned}
& \mathrm{Var}^*[C_T^{tot,*}(\phi)] - \mathrm{Var}^*[C_T^{tot,*}(\psi)] \\
= \; & \mathrm{Var}^*[I_T^*(\phi)] - \mathrm{Var}^*[I_T^*(\psi)] + 2\mathrm{Cov}^*[X_T^*, I_T^*(\psi) - I_T^*(\phi)] \\
= \; & \mathrm{Var}^*[I_T^*(\phi)] - \mathrm{Var}^*[I_T^*(\phi) + I_T^*(\psi) - I_T^*(\phi)] + 2\mathrm{Cov}^*[X_T^*, I_T^*(\psi) - I_T^*(\phi)] \\
= \; & \mathrm{Var}^*[I_T^*(\phi)] - \mathrm{Var}^*[I_T^*(\phi)] - \mathrm{Var}^*[I_T^*(\psi) - I_T^*(\phi)] - 2\mathrm{Cov}^*[I_T^*(\phi), I_T^*(\psi) - I_T^*(\phi)] \\
& +2\mathrm{Cov}^*[X_T^*, I_T^*(\psi) - I_T^*(\phi)] \\
= \; & -\mathrm{Var}^*[I_T^*(\psi) - I_T^*(\phi)] + 2\mathrm{Cov}^*[C_T^* - I_T^*(\phi), I_T^*(\psi) - I_T^*(\phi)].
\end{aligned}$$

Now let us have a look at the variance difference if there does exist model misspecification related to the interest rate. If T is a deterministic time point,

$$\begin{aligned}
\mathrm{Var}^*[I_T^*(\psi) - I_T^*(\phi)] &= \mathrm{Var}^* \left[\sum_{i=1}^{N-2} \int_0^T \phi_u^{(i)} D^*(u,t_i)\, g_u^{(i)}\, dW_u^* \right] \\
&= \sum_{i=1}^{N-2} \mathrm{Var}^* \left[\int_0^T \phi_u^{(i)} D^*(u,t_i)\, g_u^{(i)}\, dW_u^* \right] \\
&= \sum_{i=1}^{N-2} E^* \left[\left(\int_0^T \phi_u^{(i)} D^*(u,t_i)\, g_u^{(i)}\, dW_u^* \right)^2 \right] \\
&= \sum_{i=1}^{N-2} E^* \left[\left(\int_0^T \phi_u^{(i)} D^*(u,t_i)\, g_u^{(i)} \right)^2 du \right].
\end{aligned}$$

Since

$$\begin{aligned}
\phi_u^{(i)} &= 1_{u \le t_i} \bar{G}_{t_i t_{i-1}|t_i} \tilde{q}_{x+u} \\
E^*[(D^*(u,t_i))^2] &= (D(t_0,t_i))^2 \exp\left\{ \int_0^u (\sigma_{t_i}(s))^2 ds \right\},
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}^*[I_T^*(\psi) - I_T^*(\phi)] &= \sum_{i=1}^{N-2} \int_0^{t_i} \left(\bar{G}_{t_i t_{i-1}|t_i} \tilde{q}_{x+u} \right)^2 \left(g_u^{(i)} \right)^2 E^*[(D^*(u,t_i))^2] du \\
&= \sum_{i=1}^{N-2} (D(t_0,t_i))^2 \int_0^{t_i} \left(\bar{G}_{t_i t_{i-1}|t_i} \tilde{q}_{x+u} \right)^2 \left(g_u^{(i)} \right)^2 \exp\left\{ \int_0^u (\sigma_{t_i}(s))^2 ds \right\} du.
\end{aligned}$$

And if T is a stopping time as specified in our contract, we obtain

$$\begin{aligned}
& \mathrm{Var}^*[I_T^*(\psi) - I_T^*(\phi)] \\
= \; & E^*\left[(I_T^*(\psi) - I_T^*(\phi))^2 1_{\{\tau^x > t_N\}}\right] + \sum_{i=0}^{N-1} E^*\left[(I_T^*(\psi) - I_T^*(\phi))^2 1_{\{t_j < \tau^x \le t_{j+1}\}}\right] \\
= \; & E^*\left[(I_{t_N}^*(\psi) - I_{t_N}^*(\phi))^2 1_{\{\tau^x > t_N\}}\right] + \sum_{j=0}^{N-1} E^*\left[(I_{t_{j+1}}^*(\psi) - I_{t_{j+1}}^*(\phi))^2 1_{\{t_j < \tau^x \le t_{j+1}\}}\right] \\
= \; & {}_{t_N}p_x \, E^*\left[(I_{t_N}^*(\psi) - I_{t_N}^*(\phi))^2\right] + \sum_{j=0}^{N-1} {}_{t_j|t_{j+1}}q_x \, E^*\left[(I_{t_{j+1}}^*(\psi) - I_{t_{j+1}}^*(\phi))^2\right],
\end{aligned}$$

where

$$\begin{aligned}
E^*[(I_{t_j}^*(\psi) - I_{t_j}^*(\phi))^2] \; = \; & \sum_{i=1}^{\min\{j, N-2\}} (D(t_0, t_i))^2 \int_0^{t_i} \left(\bar{G}_{t_i t_{i-1}|t_i} \tilde{q}_{x+u}\right)^2 \left(g_u^{(i)}\right)^2 \\
& \exp\left\{\int_0^u (\sigma_{t_i}(s))^2 ds\right\} du, \qquad j = 1, \cdots, N.
\end{aligned}$$

In addition,

$$\begin{aligned}
& \mathrm{Cov}^*[X_T^* - I_T^*(\phi), I_T^*(\psi) - I_T^*(\phi)] \\
= \; & \mathrm{Cov}^*[C_T^{tot,*}(\phi), I_T^*(\psi) - I_T^*(\phi)] \\
= \; & {}_{t_N}p_x \, \mathrm{Cov}^*[C_{t_N}^{tot,*}(\phi), I_{t_N}^*(\psi) - I_{t_N}^*(\phi)] + \sum_{j=0}^{N-1} {}_{t_j|t_{j+1}}q_x \, \mathrm{Cov}^*[C_{t_j+1}^{tot,*}(\phi), I_{t_{j+1}}^*(\psi) - I_{t_{j+1}}^*(\phi)]
\end{aligned}$$

Due to the fact that $I_t^*(\psi) - I_t^*(\phi)$ is not of bounded variation, but $C_T^{tot,*}(\phi)$ is, the above covariance equals zero. To sum up, after taking account of the mortality risk, the variance difference is given by

$$\begin{aligned}
& \mathrm{Var}^*[C_T^{tot,*}(\phi)] - \mathrm{Var}^*[C_T^{tot,*}(\psi)] \\
= \; & -\mathrm{Var}^*[I_T^*(\psi) - I_T^*(\phi)] + 2\mathrm{Cov}^*[X_T^* - I_T^*(\phi), I_T^*(\psi) - I_T^*(\phi)] \\
= \; & -\left({}_{t_N}p_x \, E^*\left[(I_{t_N}^*(\psi) - I_{t_N}^*(\phi))^2\right] + \sum_{j=0}^{N-1} {}_{t_j|t_{j+1}}q_x \, E^*\left[(I_{t_{j+1}}^*(\psi) - I_{t_{j+1}}^*(\phi))^2\right]\right) \\
< \; & 0.
\end{aligned}$$

(ii) Now we come to the second part of proof:

$$\begin{aligned}
\mathrm{Var}^*[C_T^*(\phi)] \; &= \; \mathrm{Var}^*[C_T^{tot,*}(\phi) + C_T^{add,*}] \\
&\Rightarrow \\
\mathrm{Var}^*[C_T^*(\psi)] - \mathrm{Var}^*[C_T^*(\phi)] \; &= \; \mathrm{Var}^*[C_T^{tot,*}(\psi)] - \mathrm{Var}^*[C_T^{tot,*}(\phi)] + 2\mathrm{Cov}^*[C_T^{tot,*}(\psi) - C_T^{tot,*}(\phi), C_T^{add,*}
\end{aligned}$$

Since it holds that

$$\begin{aligned} C_T^{tot,*}(\psi) - C_T^{tot,*}(\phi) &= I_T^*(\phi) - I_T^*(\psi) \\ &= -\sum_{i=1}^{N-2} \int_0^t 1_{\{u \le t_i\}} \bar{G}_{t_i\ t_{i-1}|t_i} \tilde{q}_{x+u} D^*(u,t_i)\, g_u^{(i)}\, dW_u^* \\ C_T^{add,*} &= \sum_{i=0}^{N-1} e^{-\int_0^{t_i} r_u\, du} K \left({}_{t_i}\tilde{p}_x - 1_{\{\tau^x > t_i\}} \right), \end{aligned}$$

the covariance part is given by

$$\mathrm{Cov}^* \left[\sum_{i=1}^{N-2} \int_0^t 1_{\{u \le t_i\}} \bar{G}_{t_i\ t_{i-1}|t_i} \tilde{q}_{x+u} D^*(u,t_i)\, g_u^{(i)}\, dW_u^*, \sum_{i=0}^{N-1} e^{-\int_0^{t_i} r_u\, du} K \cdot 1_{\{\tau^x > t_i\}} \right].$$

Now we claim it equals zero because of the independence assumption between the financial and mortality risk. It is observed that the first part depends only on the financial risk, while the second only on the mortality risk.

7.3 Proof of Proposition 4.2.18

First, we want to show that the two assumptions about the death/survival probabilities are not quite realistic and not very demanding. Recall that

$$\begin{aligned} {}_tp_x &= e^{-\int_0^t \mu_{x+s} ds} \\ {}_{u|t}q_x &= {}_up_x - {}_tp_x = e^{-\int_0^t \mu_{x+s} ds} - e^{-\int_0^u \mu_{x+s} ds}, \quad t > u \end{aligned}$$

μ is the so called hazard rate of mortality. Furthermore, concerning the death/survival probabilities, we make the following assumptions:

$$
\begin{array}{lrcl}
(a) & \dfrac{\partial_t p_x}{\partial x} & = & {}_tp_x\left(-\displaystyle\int_0^t \frac{\partial \mu_{x+s}}{\partial x} ds\right) < 0 \\
\Leftrightarrow & \dfrac{\partial \mu_{x+s}}{\partial x} & > & 0 \\
(b) & \dfrac{\partial_t p_x}{\partial t} & = & {}_tp_x\left(-\displaystyle\int_0^t \frac{\partial \mu_{x+s}}{\partial t} ds\right) \\
 & & = & -{}_tp_x\,\mu_{x+t} < 0 \\
(c) & \dfrac{\partial\, {}_{u|t}q_x}{\partial x} & = & \dfrac{\partial_u p_x}{\partial x} - \dfrac{\partial_t p_x}{\partial x} > 0 \\
\Leftrightarrow & \dfrac{\frac{\partial_s p_x}{\partial x}}{\partial s} & < & 0 \quad \Leftrightarrow \quad {}_sp_x\left(\mu_{x+s}\displaystyle\int_0^s \frac{\partial \mu_{x+v}}{\partial x}\, dv - \frac{\partial \mu_{x+s}}{\partial x}\right) < 0 \\
(d) & \dfrac{\partial\, {}_{t_{i-1}|t_i}q_{x+u}}{\partial x} & = & \dfrac{\partial\left({}_{t_{i-1}-u}p_{x+u} - {}_{t_i-u}p_{x+u}\right)}{\partial x} = \dfrac{\partial\left(\frac{{}_{t_{i-1}}p_x}{{}_up_x} - \frac{{}_{t_i}p_x}{{}_up_x}\right)}{\partial x} > 0 \\
\Leftrightarrow & \dfrac{\partial\left(\partial \frac{{}_sp_x}{{}_up_x}/\partial x\right)}{\partial s} & < & 0 \quad \Leftrightarrow \quad \dfrac{{}_sp_x}{{}_up_x}\left(\mu_{x+s}\displaystyle\int_u^s \frac{\partial \mu_{x+v}}{\partial x}\, dv - \frac{\partial \mu_{x+s}}{\partial x}\right) < 0,\ s > u
\end{array}
$$

These assumptions are indeed quite realistic. Assumption (a) says that the survival probability decreases in the age. Assumptions (c) and (d) tell that the (conditional) death probability increases in the age. Condition (b) holds always. Technically, it should hold

$$
\frac{\partial \mu_{x+s}}{\partial x} > 0, \qquad \mu_{x+s}\int_u^s \frac{\partial \mu_{x+v}}{\partial x}\, dv - \frac{\partial \mu_{x+s}}{\partial x} < 0,\ u < s.
$$

E.g these conditions hold e.g. for De Moivre hazard rate, where $\mu_{x+t} = \frac{1}{w-x-t}$ with w the highest attainable age, and Makeham hazard rate, where $\mu_{x+t} = H + Q\, e^{x+t}$ etc. Since we use the Makeham hazard rate, it is proven shortly that all the four conditions hold for

this death distribution.

$$
\begin{aligned}
(i) \qquad \frac{\partial_t p_x}{\partial x} &= {}_tp_x \left(- \int_0^t \frac{\partial \mu_{x+s}}{\partial x} ds \right) \\
&= -Q \, {}_tp_x \, e^x \, (e^t - 1) < 0; \\
(ii) \qquad \frac{\partial_t p_x}{\partial t} &= {}_tp_x \left(- \int_0^t \frac{\partial \mu_{x+s}}{\partial t} ds \right) \\
&= -{}_tp_x \, \mu_{x+t} < 0 \\
(iii) \qquad \frac{\partial \, {}_{u|t}q_x}{\partial x} &= \frac{\partial_u p_x}{\partial x} - \frac{\partial_t p_x}{\partial x} \\
&= -Q \, {}_up_x \, e^x \, (e^u - 1) + Q \, {}_tp_x \, e^x \, (e^t - 1) \\
&= -Q \, e^x (\, {}_up_x \, (e^u - 1) - \, {}_tp_x \, (e^t - 1)) > 0 \\
(iv) \qquad \frac{\partial \, {}_{t_{i-1}|t_i}q_{x+u}}{\partial x} &= \frac{\partial \left({}_{t_{i-1}-u}p_{x+u} - {}_{t_i-u}p_{x+u} \right)}{\partial x} \\
&= \frac{\partial \left(\frac{{}_{t_{i-1}}p_x}{{}_up_x} - \frac{{}_{t_i}p_x}{{}_up_x} \right)}{\partial x} \\
&= \frac{\partial \left(\exp \left\{ - \int_u^{t_{i-1}} \mu_{x+s} ds \right\} - \exp \left\{ - \int_u^{t_i} \mu_{x+s} ds \right\} \right)}{\partial x} \\
&= {}_{t_{i-1}-u}p_{x+u} \left(-Qe^x (e^{t_{i-1}} - e^u) \right) - {}_{t_i-u}p_{x+u} \left(-Qe^x (e^{t_i} - e^u) \right) \\
&= -Qe^x \left(e^u ({}_{t_i-u}p_{x+u} - {}_{t_{i-1}-u}p_{x+u}) + ({}_{t_{i-1}-u}p_{x+u} e^{t_{i-1}} - {}_{t_i-u}p_{x+u} e^{t_i}) \right) \\
&= -Qe^x \left(-e^u \, {}_{t_{i-1}|t_i}q_{x+u} + ({}_{t_{i-1}-u}p_{x+u} e^{t_{i-1}} - {}_{t_i-u}p_{x+u} e^{t_i}) \right) \\
&< 0
\end{aligned}
$$

The first two arguments are obvious, and the last inequality in the last derivative is a straightforward result of the following lemma.

Lemma 7.3.1. *For $0 < u < t$, it holds that*

$$ {}_up_x \, (e^u - 1) - \, {}_tp_x \, (e^t - 1) < 0 $$

PROOF:

$$
\begin{aligned}
\lim_{u \to 0} \, {}_up_x \, (e^u - 1) - \, {}_tp_x \, (e^t - 1) &= 1 \cdot (1 - 1) - \, {}_tp_x \, (e^t - 1) < 0 \\
\lim_{u \to t} \, {}_up_x \, (e^u - 1) - \, {}_tp_x \, (e^t - 1) &= \, {}_tp_x \, (e^t - 1) - \, {}_tp_x \, (e^t - 1) = 0 \\
\frac{\partial \, ({}_up_x \, (e^u - 1) - \, {}_tp_x \, (e^t - 1))}{\partial u} &= {}_up_x \, e^u + (e^u - 1)(-{}_up_x \, \mu_{x+u}) \\
&= {}_up_x e^u (1 - \mu_{x+u}) + {}_up_x \, \mu_{x+u} > 0
\end{aligned}
$$

Due to the fact that the hazard rate is a positive function which is smaller than 1, we obtain the last inequality. We have shown that the function ${}_up_x \, (e^u - 1) - \, {}_tp_x \, (e^t - 1)$ is a monotonically increasing function, and its lower and upper bound are $- \, {}_tp_x \, (e^t - 1)$ and

0 respectively, therefore we come to our argument.

(i) It holds

$$\begin{aligned}\frac{\partial E^*[C_T^*]}{\partial \tilde{x}} &= \frac{\partial E^*[C_T^{tot,*}]}{\partial \tilde{x}} + \frac{\partial E^*[C_T^{add,*}]}{\partial \tilde{x}};\\ \frac{\partial E^*[C_T^{add,*}]}{\partial \tilde{x}} &= \sum_{i=0}^{N-1} D(t_0,t_i) K \frac{\partial\, {}_{t_i}p_{\tilde{x}}}{\partial \tilde{x}} < 0.\end{aligned}$$

In addition, it is known that the expected discounted total hedging cost is the difference between the initial price of the contract conditional on the true death distribution and that conditional on the true one.

$$\begin{aligned}E^*[C_T^{tot,*}(\phi)] &= D(t_0,t_N)\bar{G}_{t_N}({}_{t_N}p_x - {}_{t_N}\tilde{p}_x) + \sum_{j=1}^{N-1}({}_{t_{j-1}|t_j}q_x - {}_{t_{j-1}|t_j}\tilde{q}_x) D(t_0,t_j)\bar{G}_{t_j}\\ &= f(x) - f(\tilde{x})\end{aligned}$$

Since the true x is always considered given, we are interested in how exactly this expected cost depends on the assumed age $\tilde{x}$, i.e.,

$$\frac{\partial E^*[C_T^{tot,*}(\phi)]}{\partial \tilde{x}} = -\frac{\partial f(\tilde{x})}{\partial \tilde{x}}$$

Since the initial value can be reformulated as follows:

$$\begin{aligned}f(\tilde{x}) &= \bar{G}_{t_N}\, D(t_0,t_N)\, {}_{t_N}p_{\tilde{x}} + \sum_{i=0}^{N-1} \bar{G}_{t_{i+1}}\, D(t_0,t_{i+1})\, {}_{t_i|t_{i+1}}q_{\tilde{x}}\\ &= \bar{G}_{t_N}\, D(t_0,t_N)(1 - {}_{t_N}q_{\tilde{x}}) + \sum_{i=0}^{N-1} \bar{G}_{t_{i+1}}\, D(t_0,t_{i+1})\, {}_{t_i|t_{i+1}}q_{\tilde{x}}\\ &= \bar{G}_{t_N}\, D(t_0,t_N) - \bar{G}_{t_N}\, D(t_0,t_N) \sum_{i=0}^{N-1} {}_{t_i|t_{i+1}}q_{\tilde{x}} + \sum_{i=0}^{N-1} \bar{G}_{t_{i+1}}\, D(t_0,t_{i+1})\, {}_{t_i|t_{i+1}}q_{\tilde{x}}\\ &= \bar{G}_{t_N}\, D(t_0,t_N) + \sum_{i=0}^{N-1} \left(\bar{G}_{t_{i+1}}\, D(t_0,t_{i+1}) - \bar{G}_{t_N}\, D(t_0,t_N)\right) {}_{t_i|t_{i+1}}q_{\tilde{x}}\end{aligned}$$

And

$$\frac{\partial E^*[C_T^{tot,*}(\phi)]}{\partial \tilde{x}} = -\frac{\partial f(\tilde{x})}{\partial \tilde{x}} = -\sum_{i=0}^{N-1} \underbrace{\left(\bar{G}_{t_{i+1}}\, D(t_0,t_{i+1}) - \bar{G}_{t_N}\, D(t_0,t_N)\right)}_{>0} \underbrace{\frac{\partial_{t_i|t_{i+1}}q_{\tilde{x}}}{\partial \tilde{x}}}_{>0} < 0$$

Since under this condition $E^*[C_T^{*,tot}]$ is a decreasing monotonic function of $\tilde{x}$ and $E^*[C_T^{*,tot}]|_{x=\tilde{x}} = 0$, for the region $\{\tilde{x} > x\}$ (overestimation of the death probability), a superhedge in the

mean results.

(ii) The derivative of the variance difference with respect to $\tilde{x}$.

$$\begin{aligned}
& \frac{\partial \left(\mathrm{Var}^*[C_T^{tot,*}(\psi)] - \mathrm{Var}^*[C_T^{tot,*}(\phi)] \right)}{\partial \tilde{x}} \\
= \quad & {}_{t_N}p_x \frac{\partial \left(E_{P^*}\left[(I^*_{t_N}(\psi) - I^*_{t_N}(\phi))^2 \right] \right)}{\partial \tilde{x}} + \sum_{j=0}^{N-1} {}_{t_j|t_{j+1}}q_x \, \frac{\partial \left(E^*\left[(I^*_{t_{j+1}}(\psi) - I^*_{t_{j+1}}(\phi))^2 \right] \right)}{\partial \tilde{x}} \\
> 0 &
\end{aligned}$$

because

$$\begin{aligned}
& \frac{\partial E^*\left[(I^*_{t_N}(\psi) - I^*_{t_N}(\phi))^2 \right]}{\partial x} \\
= \quad & \sum_{i=1}^{N-2} (D(t_0,t_i))^2 \int_0^{t_i} (\bar{G}_{t_i})^2 2_{t_{i-1}|t_i}\tilde{q}_{x+u} \underbrace{\frac{\partial_{t_{i-1}|t_i}\tilde{q}_{x+u}}{\partial x}}_{>0} \left(g_u^{(i)}\right)^2 \exp\left\{ \int_0^u (\sigma_{t_i}(s))^2 ds \right\} du \\
> \quad & 0.
\end{aligned}$$

□

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