Dynamic Matching and Bargaining Games: Towards a General Perspective

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vorgelegt von
Stephan Lauermann
aus Seefeld, Stadland.

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Dekan: Professor Dr. Gerhard Wagner
Erstreferent: Professor Dr. Georg Nöldeke
Zweitreferen: Professor Dr. Benny Moldovanu

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Introduction

"*Nature* makes no jumps," according to a famous saying - but what about *economies*? Is economic welfare continuous in the size of the frictions of trading and do the welfare theorems hold approximately when frictions are small? We will look at a specific institution - a decentralized market - and at a specific problem in decentralized markets that might be induced by frictions - market power. We will ask: When frictions in the decentralized market are small, will the trading outcome be approximately efficient?

Suppose you are looking for a job that pays a good wage. Most likely, you will have to exert effort to find some vacancies - these are the *frictions* of trade. Once you have found some vacancies, you will have to bargain with each single employer separately - this is the decentralized nature of the market: Despite the fact that there are potentially hundreds of other employers and workers in the market, within each negotiation, you will find yourself in a bilateral, one-on-one relationship. In this relationship, each of you might enjoy some market power and the outside option of searching for a new partner is of limited help since search is costly. In addition, both of you might not know how important a deal is to the other, and both of you are probably low-balling: The employer offers only low wages (since, unfortunately, he can hardly afford a new employee), while you ask for a high wage (since, actually, you do not really need the job right now). Is there reason to believe that a labor market can nevertheless be well approximated by a general equilibrium model, which assumes that wages will be market clearing and which predicts that the trading outcome is efficient?

Situations as described before can be modeled as dynamic matching and bargaining games, which have been introduced by Gale (1987).[1] He considers bilateral trade between one buyer and one seller and embeds it into a larger dynamic market game as follows: There is a continuum of buyers and sellers who are matched into pairs at the beginning of each period. Within each pair, they bargain over the terms of trade. The pairs are connected by allowing an unsuccessful trader to be matched with another partner in a new pair in the next period. However, there is a friction that makes waiting for the next period costly, so the integration of the market is not perfect. Here, this friction is an exogeneous probability $\delta \in (0,1)$ that a trader cannot enter the next period and exits (dies).

Formally, we will look at the limit of the equilibrium outcomes of such setups when δ converges to zero. The first chapter of this thesis illustrates how market power in the bilateral bargaining situation can make the overall trading outcome inefficient. We show that if sellers can observe the valuation of the buyers, i.e., if information is symmetric, and if sellers make price offers, then the overall surplus is *increasing* in the size of frictions δ. In particular, the outcome does not become efficient when δ converges to zero. The second chapter shows that the outcome does become efficient if sellers can *not* observe the valuation of the buyers, i.e., contrary to intuition, asymmetric information makes trading more efficient. We relate the positive convergence result in this second chapter to the informal reasoning that prices must be market clearing (implying an efficient trading outcome) since otherwise sellers would be *rationed*, giving them an incentive to offer lower prices. The third and final chapter of this thesis explores the general structure

[1] This description is taken from the introduction of Chapter 1.

of the first two results. We observe that each specification of a dynamic matching and bargaining game together with a decreasing sequence of frictions defines a sequence of trading outcomes. We will discuss structural properties of such sequences, which on the one hand ensure that its limit is efficient, while relating on the other hand to economic properties of the underlying games. Thereby, we highlight a common cause behind existing positive convergence results, e.g., those by Gale (1987) and Satterthwaite and Shneyerov (2007). We also illustrate the structural properties of some other specifications for which trading outcomes fail to converge to efficiency, e.g., Serrano (2002) and DeFraja and Sakovics (2001).

Although we will concentrate on the characterization of trading outcomes in the limit, it should not go unnoticed that the first two chapters provide characterizations of trading outcomes for every level of frictions. The first main result is that asymmetric information, i.e., *consumer privacy*, can be good for efficiency in a market: In contrast to bilateral interaction, in a market the distribution of rents between the trading partner matters, and this distribution of rents is influenced by the degree of information. The second main result is the provision of a *generalized Lerner formula* for dynamic markets in which buyers can time their purchases. We show that the markup of prices over costs is proportional to the *dynamic elasticity of demand.* Nevo and Hendel (2006) empirically analyse the market for laundry detergents and show that the possibility to store these goods makes demand more elastic. Both results of these chapters are of direct relevance for economic policy evaluation.

1 When Less Information is Good for Efficiency: Private Information in Bilateral Trade and in Markets

We consider a simple bilateral trading game between a seller and a buyer who have private valuations for an indivisible good. The seller makes a price offer which the buyer can either accept or reject. If the seller can observe the valuation of the buyer (if information is symmetric), the trading outcome is trivially efficient. With asymmetric information, the outcome must be inefficient, as is known from the Myerson-Satterthwaite Theorem. We embed this bilateral trading game into a matching market and show that this relation between information and efficiency is reversed. In particular, if information is symmetric, trading in the market is inefficient.

1.1 Introduction

Asymmetric information makes bilateral trade inefficient, as is known from the Myerson-Satterthwaite impossibility theorem. With symmetric information however, bilateral trade is efficient. Embedding a bilateral trading game into a larger market, we show that this connection between efficiency and information is reversed: With symmetric information, the market outcome is bounded away from the efficient one, even when trading frictions are small. In the same model, trading with asymmetric information becomes efficient once frictions vanish, as shown in chapter two.

To model a decentralized market, we use a steady state, dynamic matching and bargaining game with an exogeneous inflow, building on Gale (1987). He considers bilateral trade between one buyer and one seller and embeds it into a larger dynamic market game as follows: There is a continuum of buyers and sellers who are matched into pairs at the beginning of each period. Within each pair, they bargain over the terms of trade. The pairs are connected by allowing an unsuccessful trader to be matched with another partner in a new pair in the next period. However, there is a friction that makes waiting for the next period costly, so the integration of the market is not perfect. Here, this friction is an exogeneous probability $\delta \in (0, 1)$ that a trader cannot enter the next period and will exit (die).

The equilibrium outcome of versions of the dynamic matching and bargaining game have been shown to be efficient if frictions are small and any of the following three *market clearing forces* is present: if bargaining power is symmetric between buyers and sellers (e.g., Gale (1987)); if there is a chance that one buyer receives prices from several competing sellers (Satterthwaite and Shneyerov (2005));[2] or if information is asymmetric (chapter two). Formally, with δ converging to one, equilibrium outcomes of these models become efficient. Here we show the opposite: If sellers have all the bargaining power, if bargaining takes place only in pairs, and if information is symmetric, then the outcome can never become efficient, not even for small frictions when δ converges to zero.

The intuition for the negative result is the following: If sellers can observe the buyers' willingness to pay and if sellers have all the bargaining power, then sellers will be able to

[2] Satterthwaite and Shneyerov (2005) assume that there is an exogeneous exit rate, as we do here. In their later 2007 paper, they assume that there is no exogeneous exit, see the discussion in Section 1.3.1.

perfectly price discriminate among buyers. This allows sellers to make strictly positive profits - even in the limit for small δ. Thus, sellers are not willing to trade with those buyers whose valuations are just above the cost of the sellers. Therefore, marginal buyers who should trade in the efficient allocation do not find a trading partner, and the allocation is inefficient.

The first step of the intuition is well-known from the Diamond paradox (Diamond (1971), see the discussion in the next section). But note that price discrimination by itself does not create inefficiencies. Inefficiencies arise only because the bilateral bargaining problem is part of a market in which sellers can make profits on other buyers in the market in the future.[3]

On an applied level, our result suggests that, in a market with a very skewed distribution of market power, the ability of the stronger side to price discriminate is harmful. Therefore, our results suggest the economic importance of private information, i.e., "consumer privacy" (see Varian (1996)). This paper is also related to the literature on embedding problems of "contract design" into (matching) markets (see e.g., Inderst (2001, 2004) or Felli and Roberts (2002)). As in their models, a property of exchange between a small set of agents in isolation is fundamentally altered when considered as part of the equilibrium of a market.

To further analyze the role of information, we change the model slightly and assume that sellers do not observe the type of the buyer directly but only a noisy signal of it. Thereby, buyers of different types can mimic each other. With the possibility of mimicry in place, the reasoning put forth in chapter two implies that prices converge to their competitive level of zero. We sketch out this model and the proof of the convergence result in the appendix.

1.2 Model and Analysis

1.2.1 The Model

The model is taken from chapter two, adding the assumption that sellers can observe buyers' types:[4] There is a continuum of buyers and sellers. Sellers are endowed with one unit of an indivisible good, and their costs of trading are $c = 0$. Buyers want to buy one unit of the good, and their valuation for the good is $v \in [0, 1]$. Buyers and sellers interact in a repeated market over infinitely many periods, with time running from minus to plus infinity. At the beginning of each period, there is some pool of buyers and sellers. All traders from this pool are matched into pairs consisting of one seller and one buyer. Within each pair, the seller observes the type of the buyer and then announces a price p. The buyer announces whether he accepts or rejects the offer. If he accepts, the seller receives a payoff p, while the buyer receives $v - p$; If he rejects, they receive nothing. Then, all those agents who have traded leave the market together with a share δ of those who have not. After that, new players enter the market. The inflow of buyers and the inflow of sellers has mass one each. The distribution of valuations among entering buyers

[3] In Diamond's orginal paper, it is the assumption of linear prices, coupled with elastic demand, which causes the inefficiency of the equilibrium outcome.

[4] In addition to the different bargaining protocol, another main difference from Gale (1987) is the existence of an exit rate $\delta \in (0, 1)$. If traders can leave the market only through trading, as in Gale, our results do not hold, see Section 1.3.1.

is given by the c.d.f. $G(\cdot)$. With the inflow of new traders, the period ends, and the next period starts according to the same rules. Finally, we describe the pool by M, and $\Phi^B(\cdot)$: M denotes the mass of buyers in the pool at the beginning of each period. This mass is equal to the mass of sellers. The c.d.f. $\Phi^B(\cdot)$ describes the distribution of buyers' types v in the pool.

The mass of entering buyers with valuations above some given v, $(1 - G(v)) \in [0, 1]$ can be interpreted as "demand," and we assume that it is strictly decreasing, i.e., its density $g(\cdot)$ is strictly positive. "Supply," i.e., the mass of entering sellers, is equal to one. Let p^w be the Walrasian price at which demand is equal to supply, i.e.,

$$1 - G(p^w) = 1,$$

and this price is clearly $p^w = 0$. In a more complicated model, sellers could be heterogeneous as well, and in this case p^w would be interior (see Section 1.3.3).

We restrict attention to equilibria in which sellers use symmetric and stationary pricing strategies $P(\cdot, \cdot)$, where $P(p', v)$ is the probability to offer a price $p \leq p'$ to a buyer of type v, i.e., $P(\cdot, v)$ is a c.d.f.. The payoff to a seller in a steady state (see below) who uses a pricing strategy $P(\cdot, \cdot)$ can be derived as follows. Denote by $q^S(P(\cdot, \cdot))$ the probability that this seller can trade some time during his entire lifetime. Let $r(v)$ be the highest price accepted by a buyer with valuation v (see below). With $D(P(\cdot, \cdot)) = \int_0^1 P(r(\tau), \tau)\, d\Phi^B(\tau)$ being the probability to trade in any given period, we can derive $q^S(P(\cdot, \cdot))$ recursively from

$$q^S(P(\cdot, \cdot)) = D(P(\cdot, \cdot)) + (1 - D(P(\cdot, \cdot)))(1 - \delta)\, q^S(P(\cdot, \cdot)),$$

as

$$q^S(P(\cdot, \cdot)) \equiv \frac{D(P(\cdot, \cdot))}{1 - (1 - \delta) D(P(\cdot, \cdot))}.$$

Denote by $E[p|P(\cdot, \cdot)]$ the expected price conditional on being able to trade.[5] Then expected profits $\Pi(\cdot)$ are

$$\Pi(P(\cdot, \cdot)) = q^S(P(\cdot, \cdot))\, E[p|P(\cdot, \cdot)] \ .$$

To derive the optimal search strategy of a buyer, observe that he is essentially sampling without recall from a known and constant distribution of prices. For this problem, it is well known that the optimal solution can be described by a threshold, a reservation price r, such that a price p is accepted if and only if $p \leq r$ (see McMillan and Rothschild (1994)). The payoff to a buyer of type v with a reservation price r depends on the expected price offer, $E[p|p \leq r, v]$,[6] and the probability to trade some time during his lifetime, i.e., to receive an acceptable offer $p \leq r$ in any period, denoted by $q^B(r, v)$:

$$U^B(r, v) = q^B(r, v)(v - E[p|p \leq r, v]) \ .$$

Let $V(v) = \max_r U^B(r, v)$ be the maximized expected lifetime payoff. At the reservation price $r(v)$, buyers must be indifferent between acceptance and rejection, so $v - r(v) =$

[5] Let $E[p|P(\cdot, \cdot)] = 0$ if $q^S(P(\cdot, \cdot)) = 0$.

[6] Let $E[p|p \leq r, v] = r$ if $q^B(r, v) = 0$.

$(1-\delta) V(v)$. Rewriting yields

$$r(v) = v - (1-\delta) V(v). \tag{1.1}$$

We will require the price offer $P(\cdot, v)$ to be optimal for every possible type v. For this, let $U^S(p, v|P(\cdot,\cdot))$ be the profit of a seller who offers a price p to a buyer of type v and continues according to the strategy $P(\cdot,\cdot)$, i.e.,

$$U^S(p, v|P(\cdot,\cdot)) = \begin{cases} p & \text{if } p \leq r(v) \\ (1-\delta)\Pi(P(\cdot,\cdot)) & \text{if } p > r(v). \end{cases}$$

Formally, we require that every price in the support of $P(\cdot, v)$ be optimal:

$$p \in \arg\max U^S(p, v|P(\cdot,\cdot)) \text{ for all } v \text{ and } p \in \text{supp}P(\cdot,\cdot). \tag{1.2}$$

The market is in a steady state if the inflow is equal to the outflow. The inflow of buyers with valuations below v is $G(v)$, while the outflow consists of all buyers who trade plus those buyers who die. Equality of in- and outflows holds if

$$G(v) = M \int_0^v [P(r(\tau), \tau) + \delta(1 - P(r(\tau), \tau))] \, d\Phi^B(\tau), \tag{1.3}$$

and similarly for sellers:

$$1 = M[D(P(\cdot,\cdot)) + \delta(1 - D(P(\cdot,\cdot)))]. \tag{1.4}$$

We define a steady state equilibrium as a vector consisting of a pair of two strategies, $P(\cdot,\cdot)$ and $r(\cdot)$, the steady state distribution $\Phi^B(\cdot)$, and the mass M of traders, such that $P(\cdot,\cdot) \in \arg\max \Pi(\cdot)$, (1.2) holds, reservation prices satisfy (3.5), and the steady state conditions (1.3) and (1.4) hold.

1.2.2 Results

We want to characterize the set of equilibria with $\delta \to 0$. For this, we will look at a strictly decreasing sequence of exit rates $\{\delta_k\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} \delta_k = 0$. We will see that for each k at least one equilibrium exists, and we fix one equilibrium for each k, yielding a sequence $\left\{\left(P_k(\cdot,\cdot), r_k(\cdot), \Phi_k^B(\cdot), M_k\right)\right\}_{k=0}^{\infty}$. Let $l_k(v)$ denote the lowest price offered to a type v, defined as $l_k(v) \equiv \inf\{p : P_k(p, v) > 0\}$.

Our main result is this: No buyer receives a price offer that is strictly below his valuation, i.e., $l_k(v) \geq v$ for all v. And for every k, the pricing strategy is characterized by a unique cutoff $\bar{v}_k$: The price offer is unacceptable for all buyers with a valuation below $\bar{v}_k$, i.e., the probability to offer a price at or below the valuation v, $P(v, v)$ is zero for all $v < \bar{v}_k$. For all other buyers the price offer is just acceptable, i.e., $P(v, v) = 1$ for all $v > \bar{v}_k$. This cutoff type is decreasing in δ_k. Therefore, more buyers can trade when frictions are large and the equilibrium outcome is more efficient. Finally, $\bar{v}_k$ does not converge to the efficient level (zero) for vanishing frictions:

Proposition 1 *For every* δ_k *and in every equilibrium,* $l_k(v) \geq v$ *and* $r_k(v) = v$*. In addition, there is a unique cutoff* $\bar{v}_k \in (0,1)$ *such that*

$$P_k(v,v) = \begin{cases} 0 & \text{if } v < \bar{v}_k \\ 1 & \text{if } v > \bar{v}_k. \end{cases}$$

The cutoff $\bar{v}_k$ *is decreasing in* δ_k*, and* $\lim_{k\to\infty} \bar{v}_k \equiv \bar{v}_* > 0$*.*

We prove the proposition in the remainder of this section. The first statement, $l_k(v) \geq v$, follows from reasoning familiar from the Diamond paradox: Suppose there is some equilibrium with a pricing strategy $P(\cdot,\cdot)$ such that $l_k(v) < v$ for some v and δ_k. Because of the probability to die while waiting, a buyer of type v is willing to pay a price above $l_k(v)$: $r_k = v - (1-\delta_k) V_k(v)$ and $V_k(v) \leq (v - l_k(v))$ together imply that $r_k(v) > l_k(v)$. So by definition of $l_k(v)$, there is some $p' \in suppP(\cdot,\cdot)$ such that $p' < r_k(v)$. However, offering a price p equal to the reservation price would strictly increase profits, i.e.,

$$U^S(p', v|P(\cdot,\cdot)) = p' < r_k(v) = U^S(r_k(v), v|P(\cdot,\cdot)),$$

and so $P(\cdot,\cdot)$ fails the optimality condition (1.2). Thus, $l_k(v) \geq v$. Noting that buyers reject all prices $p > v$, $l_k(v) \geq v$ implies that in every equilibrium sellers either offer an acceptable price $p = v$ or some unacceptable price. In both cases payoffs to the buyer are zero. Hence, $r_k(v) = v$.

The intuition for the remainder of the results is this: If sellers are able to price discriminate among buyers, they can offer a price equal to their valuation, $p(v) = v$. Given $p(v) = v$ for all v, the expected profit of sellers is strictly positive in every equilibrium, because the expected valuation of a buyer is strictly positive, $E[v] > 0$. For the outcome to become efficient, however, sellers must be willing to trade with *all* types $v \in (0,1]$. But if their expected continuation profits are strictly positive, no seller is willing to trade with a buyer with a valuation close to zero. Thus, the equilibrium outcome does not become efficient.

Next we go into the details. Suppose there is some equilibrium given δ_k (we will prove existence of equilibrium below) and let Π_k^* be the associated profit of sellers. Given these profits, suppose a seller is matched with a buyer with a valuation v': If this valuation is strictly above the continuation value of the seller, i.e., if $v' > (1-\delta_k)\Pi_k^*$, then the optimal offer is clearly a price $p = v'$; in this case, the seller makes more revenue from trading than from waiting further. If the valuation is below the continuation value, then any unacceptable price $p > v'$ is optimal (recall $v' = r_k(v')$). Let $\bar{v}_k^* \equiv (1-\delta_k)\Pi_k^*$ be the cut-off type, then every strategy in this equilibrium is characterized by $l_k(v) \geq v$ (from before) and

$$P_k^*(v,v) = \begin{cases} 0 & \text{if } v < \bar{v}_k^* \\ 1 & \text{if } v > \bar{v}_k^*. \end{cases} \tag{1.5}$$

Note that the exact unacceptable offer is not specified uniquely, and the optimal offer to the type $\bar{v}$ might or might not be acceptable.

Take any pricing strategy that has this structure, i.e., take any $P(\cdot,\cdot)$ such that $l_k(v) \geq v$ and such that $P(v,v) = 1$ if $v > \bar{v}$ and $P(v,v) = 0$ if $v < \bar{v}$, as in (1.5). Denote this strategy by a subscript $\bar{v}$, $P_{\bar{v}}(\cdot,\cdot)$. If sellers use this strategy, we can calculate expected profits and denote them by $\Pi_k(P_{\bar{v}}(\cdot,\cdot))$. A strategy $P_{\bar{v}}(\cdot,\cdot)$ is part of an equilibrium if and only if

$$\bar{v} = (1-\delta_k)\,\Pi_k(P_{\bar{v}}(\cdot,\cdot)), \tag{1.6}$$

where the "only if" part was shown before; the "if" part follows from the one time deviation principle. To derive $\Pi_k(P_{\bar{v}}(\cdot,\cdot))$, we need the trading probability of a seller and we need the expected price he receives. For the former, note that all entering buyers with valuations above $\bar{v}$ will trade immediately; no other buyer can not trade. So the total mass of buyers who enter the market and who trade is $(1-G(\bar{v}))$. In a steady state, this mass of trading buyers must be exactly equal to the mass of trading sellers. Therefore, a mass $(1-G(\bar{v}))$ of all sellers in the inflow will trade, and hence[7]

$$q_k^S(P_{\bar{v}}(\cdot,\cdot)) = 1 - G(\bar{v}).$$

The expected price a seller receives conditional on trading, $E[p|P_{\bar{v}}(\cdot,\cdot)]$, is simply the expected valuation conditional on $v \geq \bar{v}$. Note that buyers with such a valuation remain in the market only for one period; consequently, the distribution of their types is given by $G(\cdot)$. Thus:

$$E[p|P_{\bar{v}}(\cdot,\cdot)] = \frac{1}{1-G(\bar{v})}\int_{\bar{v}}^{1} vg(v)\,dv,$$

and profits are

$$\Pi_k(P_{\bar{v}}(\cdot,\cdot)) = q_k^S(P_{\bar{v}}(\cdot,\cdot))\,E[p|P_{\bar{v}}(\cdot,\cdot)] = \int_{\bar{v}}^{1} vg(v)\,dv. \tag{1.7}$$

Using this observation and condition (1.6), $\bar{v} = (1-\delta_k)\,\Pi_k(P_{\bar{v}}(\cdot,\cdot))$, we find that $P_{\bar{v}}(\cdot,\cdot)$ is an equilibrium pricing strategy if and only if $\bar{v}$ satisfies

$$\bar{v} = (1-\delta_k)\int_{\bar{v}}^{1} vg(v)\,dv. \tag{1.8}$$

A solution to this equation exists by the intermediate value theorem: Both sides are continuous; at $\bar{v} = 0$ the right hand side is strictly above zero, and at $\bar{v} = 1$ the right hand side is zero. Therefore, an equilibrium exists. Furthermore, the right hand side is strictly decreasing in $\bar{v}$, so the solution is unique. Denoting this value by $\bar{v}_k$ for given δ_k, an inspection of (1.8) reveals that $\bar{v}_k$ must be decreasing in δ_k.

Finally, let $\bar{v}_*$ be the limit of $\bar{v}_k$ as δ_k becomes zero, $\bar{v}_* = \lim_{k\to\infty}\bar{v}_k$ with $\delta_k \to 0$. This limit exists by $\bar{v}_k$ being decreasing in δ_k. In the limit, (1.8) becomes

$$\bar{v}_* = \int_{\bar{v}_*}^{1} vg(v)\,dv,$$

and clearly $\bar{v}_* = 0$ is not a solution by $g(v) > 0$ for all v (from the assumption that "demand" $(1-G(\cdot))$ is strictly decreasing). Because the right hand side of (1.8) is continuous in δ_k and in $\bar{v}_k$, this implies $\bar{v}_* > 0$. This completes the proof.

[7] Although intuitive, the identity of the masses of trading buyers and sellers must be proven. This is done in chapter two. The formula for q^S, however, can also be derived directly by rewriting the steady state conditions (1.3) and (1.4) (see again chapter two). The same is true later for the expected price $E[p|P_v(\cdot,\cdot)]$.

1.3 Remarks and Conclusion

1.3.1 Infinitely Lived Agents

In some set-ups of dynamic matching and bargaining games, traders never die (i.e., $\delta = 0$); so they can exit the market only through trade (e.g., Gale (1987) or Satterthwaite, Shneyerov (2007)). In these models, the frictions (search costs) stemming from the exit rate are replaced by frictions stemming from discounting. On the individual level, this replacement does not change the incentives of the traders. But the absence of an exit rate does change the composition of the pool. In particular, in a steady state of such a model, all traders who choose to enter must be able to trade at some point, for otherwise they would accumulate over time. To ensure the existence of a steady state, one therefore has to introduce an entry stage at which traders can decide whether they want to enter the market. Without going into the very subtle details of such models, we can already see that our line of reasoning does not apply here: Suppose we would like to support some inefficient equilibrium in which only buyers with valuations above some threshold $\bar{v} > 0$ trade. So, at most, buyers with such a valuation can enter the market, and the total mass of buyers who enter in every period is strictly below one, since by assumption $(1 - G(\bar{v})) < 1$ for all $\bar{v} > 0$. In a steady state, the mass of entering buyers must be equal to the mass of entering sellers. Hence, only a mass $(1 - G(\bar{v}))$ of sellers becomes active. This, however, requires that some sellers stay out - which they do only if their expected profits are zero. Of course, if sellers make zero expected profits, then they are willing to trade with all buyers, including those with a valuation close to zero. So the proposed threshold $\bar{v} > 0$ is upset by the "free entry condition," which is implicitly included in the assumption of infinitely lived agents (see also the discussion in Chapter 2).

1.3.2 Asymmetric Information and Bargaining Power

As said in the introduction, the equilibrium becomes efficient with $\delta \to 0$ if information is asymmetric, i.e., if sellers do not observe the valuation of buyers before they make an offer. Recall that the source of the inefficiency in our model is the possibility of sellers to make strictly positive profits. This is true even if sellers are willing to trade with all buyers since the expected profit from price discrimination is $E[v|v \geq 0]$, which is strictly positive (see Equation 1.7). This is different with asymmetric information: If the sellers are willing to trade with all buyers, they must be offering a uniform price $p = 0$ *to all buyers.* This implies that their profits are zero, and they would indeed be willing to trade with low valuation buyers. This is the reason why asymmetric information is important for efficiency: With asymmetric information, (perfect) price discrimination becomes impossible; so profits when trading with all types above some threshold $\bar{v}$ are at most $\bar{v}$ itself. With symmetric information, profits when trading with all buyers with a valuation above some threshold $\bar{v}$ might be strictly larger than $\bar{v}$.

Similarly, perfect price discrimination becomes impossible if the bargaining power of sellers is reduced. This is the case if buyers themselves can make counter offers (as in Gale (1987)) or if several sellers are directly competing for a single buyer, as in Satterthwaite and Shneyerov (2005, 2007)). This explains why in Gale (1987) the outcome becomes efficient with vanishing frictions. It is an open question whether the equilibrium outcome converges to efficiency in the setup by Satterthwaite and Shneyerov (2005) if we assume that information is symmetric. However, in the much simpler setup by Burdett and Judd

(1983), who also analyse competing offers, it is possible to show that prices do become competitive, even though the valuation of the buyer is known.

1.3.3 Heterogeneous Sellers and One-Time Entry

The inefficiency result hinges on the fact that sellers can make profits in the future on newly arriving buyers. What will happen if traders enter the pool only in the first period and if there is no inflow in the subsequent periods? While one will have to adjust several details of the model, the result will be that trading becomes efficient. Essentially, there will be a sequence of cutoffs for each period, $\left\{\bar{v}^t\right\}_{t=1}^{\infty}$, such that in period t sellers trade with all buyers with $v \geq \bar{v}^t$ and with $t \to \infty$, $\bar{v}^t \to 0$. To simplify the analysis, however, we assume in our model that sellers are homogeneous with costs $c = 0$.[8] If sellers are heterogeneous, with costs distributed according to some smooth c.d.f. $G^S(\cdot)$, then trading is not efficient in the limit. First, there will be a "market-clearing" price p^w such that $G^S(p^w) = 1 - G(p^w)$, and in the efficient allocation all buyers with $v \geq p^w$ must trade with sellers with $c \leq p^w$ - and no one else. Second, by the same reasoning as in our model, buyers will make zero payoffs and accept all prices below their valuation. Thus, sellers with costs above p^w can trade (although they should not) with all those buyers who have a valuation above their own costs, i.e., with all $v \geq c > p^w$. Basically, price discrimination allows unproductive sellers to trade, which makes the outcome inefficient. This is yet another reason for inefficiencies arising from symmetric information, which is distinct from the one analyzed in our main model.

[8] If we allow for heterogeneous costs in the main model, then we can no longer provide simple closed form solutions for the equilibrium outcome. Furthermore, we would intermingle two separate sources of inefficiency; see the end of this section.

2 A Dynamic Matching and Bargaining Game with Asymmetric Information and Price Offers

We study a dynamic matching and bargaining game in which traders are matched into pairs and sellers make price offers. Traders exit the market with a constant rate δ. We show that for every δ, an equilibrium exists. With vanishing δ, the market converges to the competitive outcome. Additional assumptions that can be found in the literature and that are favorable to the competitive outcome are not needed.

2.1 Introduction

It is a common claim that decentralized markets clear and become efficient as *frictions* vanish. Decentralized markets include the markets for housing, used cars, and labor. Economists refer to the following informal story: Suppose prices in a market are constantly too high. Then some sellers must be *rationed* and trade less than they desire. This gives them an incentive to decrease their price in order to increase the trading volume by making the offer acceptable to more buyers. This incentive upsets any equilibrium candidate in which prices are too high.

The story relies on two ingredients: the rationing of sellers, and the existence of additional buyers at lower prices. Here, we show that one can model the story formally, i.e., the two ingredients are indeed sufficient for a decentralized market to become efficient once frictions vanish. Additional assumptions which are favorable to the competitive outcome that are not part of the story but that are made in the existing literature on decentralized trading are not needed for the convergence result. The results of this paper suggest that convergence to efficiency is a robust property of decentralized trading that is largely independent of the exact trading rules.

We use the following dynamic matching and bargaining game, similar to the model used by Gale (1987): There are infinitely many periods, and in each period there is a large pool of traders who want to trade an indivisible good. The pool consists of a continuum of buyers and sellers: Sellers have costs $c \in [0, 1]$ and buyers have valuations $v \in [0, 1]$. These *types* are private information. At the beginning of every period, all sellers and all buyers from the pool are matched into pairs. In each pair, the seller makes a price offer to the buyer. If the buyer accepts the price, the pair exits the market. If he declines, the match is broken up, and both traders return to the pool and wait to be rematched with new partners in the next period. While waiting, traders exit with a constant hazard rate δ. The hazard rate introduces costs of waiting for better offers, and we say that δ is the *friction* in the market. At the end of every period, an equal mass of new buyers and new sellers enters the market.

Let p^w be defined as the price at which the mass of entering sellers with costs below p^w is equal to the mass of entering buyers with valuations above p^w. So p^w is the *competitive* or *market clearing* price relative to the inflow. The trading outcome is *Walrasian* if all buyers with valuations above p^w and all sellers with costs below p^w can trade. Our main result characterizes the trading outcome with small δ: With $\delta \to 0$, all trade happens at the price p^w (Proposition 3), and the trading outcome becomes the Walrasian outcome (Corollary 1).

To illustrate the result, it is helpful to consider the case where sellers are homogeneous and their costs are $c = 0$. Buyers are assumed to be heterogeneous. The unique market clearing price is zero, $p^w = 0$: At any price p^N above zero, the mass of the sellers is strictly larger than the mass of buyers in the inflow with valuations above p^N. We show that if sellers set a price $p^N > 0$, then some of them will be *rationed*, where rationing means that the probability to trade some time during their life must be strictly smaller than one. However, setting any price $p' < p^N$, would in addition allow them to trade with those buyers who have valuations $v \in \left[p', p^N\right]$. Because these buyers never trade, they make up a strictly positive share of the pool. When δ becomes small, sellers are matched with more and more buyers; in the limit, a seller will become certain to be matched with a buyer of type $v \in \left[p', p^N\right]$, who accepts his price p' below p^N. Because p' is arbitrarily close to p^N, for δ small enough, this implies that already an infinitesimal decrease of the price ensures a trading probability close to one. At p^N, however, the trading probability is bounded away from one because of rationing. This incentive to decrease prices upsets every equilibrium candidate with $p^N > 0$.

The convergence result is not immediate: Diamond (1971) shows that even with small trading frictions sellers can have considerable market power: Given any common price p^N set by sellers and any arbitrarily small friction δ, buyers with valuations $v > p^N$ are willing to pay an additional premium of $\delta \left(vp^N\right)$ to save on waiting costs. This allows all sellers to mark up the price p^N and provides incentives to increase their prices. With homogeneous buyers, this implies that sellers offer monopolistic prices in the unique equilibrium. This is known as the *Diamond paradox*. Here, prices are not monopolistic because buyers are heterogeneous, and sellers have a countervailing incentive to decrease their price to reach additional buyers with valuations below p^N. This becomes apparent when we construct an equilibrium in the example in Section 2.3: We prove that for each δ there exists a common price $p^* (\delta)$ set by all sellers, at which the incentives for sellers to mark up the price by the waiting costs are just balanced by the incentives to decrease the price to reach additional buyers. With decreasing δ, the potential premium $\delta (v - p^* (\delta))$ decreases, while the incentives to reach additional buyers remain and $\lim_{\delta \to 0} p^* (\delta) = 0$.

When sellers have heterogeneous costs, however, we have an additional complication: sellers with different costs might set different prices, i.e., we need to account for *price dispersion*. With dispersed prices, it might be the case that sellers set prices just in the right way to give incentives to buyers to accept high prices (by setting high prices most of the time), while balancing the distribution of buyers to avoid accumulation of low valuation buyers (by setting low prices some of the time). The main part of the proof with heterogeneous sellers consists in showing that price dispersion does not occur with vanishing δ.

The fact that we give sellers all the bargaining power is our crucial departure from the existing literature; our model is standard in most other respects: The basic framework of the steady state model with heterogeneous agents, pairwise matching, and an exogeneous inflow of agents was introduced by Gale (1987). Recent models like those of Inderst (2001) and Satterthwaite and Shneyerov (2005, 2007) have extended this framework to asymmetric information.[9] Following McAfee (1993) and Satterthwaite and Shneyerov (2005), we introduce an exogeneous exit rate. Given the exit rate, we drop time discounting as an additional friction.

We start with a section introducing the model. Then we illustrate the model by assuming homogeneous sellers, and we characterize the unique equilibrium in pure strategies. Then we go on to the heterogeneous case. First, we show that an equilibrium exists for all levels of δ. Second, we prove our main result, showing convergence to the competitive outcome. We go on to an extensive discussion of our modelling choices and extensions. In particular, we show which additional assumptions the existing literature makes to ensure convergence, and how these assumptions translate into forces promoting market clearance. We also discuss extensions to the analysis of non-steady states, the relation to the non-convergence result in chapter two, the role of the assumption that buyers are heterogeneous, and the role of the exit rate.

2.2 Model

There is a continuum of buyers and sellers who interact in a repeated market over infinitely many periods. Sellers have one unit of an indivisible good and their costs of trading are $c \in [0,1]$. Buyers want to buy one unit of the good and their valuation for the good is $v \in [0,1]$. At the beginning of each period there is some pool of buyers and sellers. The traders from this pool are matched into pairs consisting of one seller and one buyer. Within each pair the seller announces a price offer $p \in [0,1]$ and the buyer announces whether he rejects or accepts the offer. If he accepts, the seller receives $p - c$ while the buyer receives $v - p$. Next, all buyers and sellers who have traded exit the pool. Likewise, a share δ of all those traders who failed to trade exits. Finally, new players enter the market and the period ends. The next period starts according to the same rules.

The inflow of buyers and the inflow of sellers has mass one each. The distribution of valuations among buyers in the inflow is exogeneously given by some c.d.f. $G^B(\cdot)$ and similarly, the distribution of costs is given by some distribution $G^S(\cdot)$. We assume that $G^B(\cdot)$ has a continuous and strictly positive density $g^B(\cdot)$.[10] The function $G^S(\cdot)$ can be interpreted as *supply,* and $1 - G^B(p^w)$ can be interpreted as *demand.* Let p^w be the Walrasian price at which demand is equal to supply, i.e.,

$$G^S(p^w) = 1 - G^B(p^w). \tag{2.1}$$

Since the former is weakly increasing and the latter is strictly decreasing by our assumptions, the solution to (3.1) is the unique.

The market constellation is characterized by a vector $\sigma = \left[p(\cdot), r(\cdot), \Phi^S(\cdot), \Phi^B(\cdot), M\right]$ where $p(c) \in [0,1]$ is the price offered by a seller of type c, $r(v) \in [0,1]$ is the highest price accepted by a buyer of type v, $\Phi^S(\cdot)$ is the cumulative distribution function of costs in the pool of sellers, $\Phi^B(\cdot)$ is the corresponding distribution function for buyers, and M is the total mass of buyers in the pool which is equal to the total mass of sellers in

[9] Moreno and Wooders (2001) also analyse convergence with asymmetric information but in a non-stationary market with one-time inflow and only two types.

[10] We do not assume that $G^S(\cdot)$ is strictly increasing, since we want to give an example with homogeneous sellers where $c \equiv 0$.

a steady state. For the analysis, we assume that all functions under consideration are measurable. With Σ_M being the set of measurable functions $f : [0,1] \to [0,1]$, σ is an element of $\Sigma \equiv \Sigma_M^4 \times [0,1]$.

We say that a vector σ constitutes an *equilibrium* if strategies are mutually optimal given the distribution of types and if the distribution of types in the pool is consistent with the trading strategies and the exogeneous inflow. These conditions are now spelled out in detail.

First we turn to the sellers. Let us denote by $D(p|\sigma)$ the probability that the buyer in any given pair accepts an offer p. Buyers accept a price p if $p \leq r(v)$ (see below) so $D(p|\sigma)$ is

$$D(p|\sigma) \equiv \int_{\{v|p \leq r(v)\}} d\Phi^B(v) \quad .^{11} \tag{2.2}$$

Let $q^S(p|\sigma)$ be the probability that a seller can trade some time during his lifetime

$$q^S(p|\sigma) \equiv \frac{D(p|\sigma)}{1-(1-D(p|\sigma))(1-\delta)}, \tag{2.3}$$

which is the solution to the recursive formula

$$q^S(p|\sigma) = D(p|\sigma) + (1-D(p|\sigma))(1-\delta)\, q^S(p|\sigma).$$

The expected payoff to a seller when offering a price p is defined as

$$U^S(p,c|\sigma) \equiv q^S(p|\sigma)(p-c),$$

and we require that $p(c) \in \arg\max U^S(\cdot, c|\sigma)$ for all c in equilibrium.

To derive the optimal search strategy of a buyer, note that he is essentially sampling without recall from a known and constant distribution of prices. For this problem, it is well known that the optimal solution can be described by a threshold, a reservation price r, such that a price p is accepted if and only if $p \leq r$ (see McMillan and Rothschild (1994)). The payoff to a buyer of type v with a reservation price r depends on the expected price offer, $E[p|p \leq r, \sigma]$ and the probability to trade some time during his lifetime, i.e., to receive an acceptable offer $p \leq r$ in any period, denoted by $q^B(r|\sigma)$ and derived just as $q^S(p|\sigma)$ as

$$q^B(r|\sigma) \equiv \frac{S(r|\sigma)}{1-(1-S(r|\sigma))(1-\delta)}.$$

Payoffs are given by

$$U^B(r,v|\sigma) \equiv q^B(r|\sigma)(v - E[p|p \leq r, \sigma]). \tag{2.4}$$

Let $V^B(v|\sigma) \equiv \max_r U^B(r,v|\sigma)$ be the maximized expected lifetime payoff. At the reservation price $r(v)$ buyers must be indifferent between acceptance and rejection, so $v - r(v) = (1-\delta)V^B(v|\sigma)$. Rewriting yields

$$r(v) = v - (1-\delta)V^B(v|\sigma). \tag{2.5}$$

[11] If $r(\cdot)$ is monotone, $D(p|\sigma)$ simplifies to $\left(1-\Phi^B\left(r^{-1}(p)\right)\right)$.

As said in the introduction, we restrict attention to equilibria in which the pool does not change over time. If the distribution at the beginning of a period is given by $\Phi_t^S(\cdot)$ and the trading strategies are $r(\cdot)$ and $p(\cdot)$, then the distribution of sellers at the end of the period is sum of the entering sellers and the initial sellers who did neither trade nor die:

$$\Phi_{t+1}^S(c|\sigma) = G^S(c) + (1-\delta)\int_0^c (1 - D(p(\tau)))\, d\Phi_t^S(\tau).$$

The pool is in a steady-state distribution of sellers if and only if the distribution does not change over time that is if $\Phi_{t+1}^S(c|\sigma) = \Phi_t^S(c) = \Phi^S(c)$ for all c. This condition can be written as[12]

$$\Phi^S(c) = \int_0^c \frac{dG^S(\tau)}{M\left(D(p(\tau)|\sigma) + \delta(1 - D(p(\tau)|\sigma))\right)} \quad \text{for all } c. \tag{2.6}$$

A similar condition can be obtained for buyers:[13]

$$\Phi^B(v) = \int_0^v \frac{dG^B(\tau)}{M\left(S(r(\tau)|\sigma) + \delta(1 - S(r(\tau)|\sigma))\right)} \quad \text{for all } v. \tag{2.7}$$

Summing up, we say σ is an equilibrium if it satisfies the above conditions:

Definition 1 *A steady-state equilibrium vector $\sigma^* \in \Sigma$ consist of an optimal pair of strategies and a corresponding steady-state pool, i.e., $\sigma^* = \left[p(\cdot), r(\cdot), \Phi^S(\cdot), \Phi^B(\cdot), M\right]$ such that*

- $p(c) \in \arg\max U^S(p, c|\sigma^*)$ *for all c,*
- $r(v) = v - (1-\delta) V^B(v|\sigma^*)$ *for all v,*
- $\Phi^S(\cdot)$*,* $\Phi^B(\cdot)$*, and M satisfy the steady-state conditions (3.6), (3.7).*

We will show that an equilibrium exists in Section 2.3. Since the proof is quite technical and non-constructive, we will use the next section to introduce a simpler case to prove existence and also to provide a characterization of equilibrium.

2.3 Homogeneous Sellers: Existence and Characterization

In this section we analyze an example in which all sellers in the inflow have costs of zero, i.e., $G^S(c) = 1$ for all $c \in [0,1]$. We keep the assumption that buyers' valuations are smoothly distributed according to some continuously differentiable, strictly increasing c.d.f. $G^B(\cdot)$. We assume in addition that *demand* $\left(1 - G^B(\cdot)\right)$ is concave which allows us to utilize the sufficiency of the first order condition. In this example, the market clearing price p^w is zero.

Let $\sigma(p^*)$ be an equilibrium in which sellers offer the price p^*, $p(0) = p^*$. A necessary condition that must be satisfied by $\sigma(p^*)$ is that p^* is a best response to itself, i.e., $p^* \in \arg\max U^S(\cdot, 0|\sigma(p^*))$. We will see that $U^S(\cdot, 0|\sigma(p^*))$ is continuously differentiable

[12] Rewriting $\Phi^S(c) = \Phi_{t+1}^S(c)$ as
$\int_0^c d\Phi^S(\tau) = \int_0^c dG^S(\tau) + \int_0^c (1-\delta)(1 - D(p(\tau)))\, d\Phi^S(c)$ we get $\int_0^c d\Phi^S(\tau) - \frac{dG^S(\tau)}{1-(1-\delta)(1-D(p(\tau)))} = 0$.

[13] Implicitly, these conditions imply mass consistency, see lemma 2 and they imply $\Phi^B(1) = \Phi^S(1) = 1$ by reasoning similar to A.5.

and strictly concave in p. Hence, a necessary and sufficient condition for p^* to be a best response to $\sigma(p^*)$ is that the first order condition $\frac{\partial}{\partial p} U^S(\cdot, 0|\sigma(p^*))_{p=p^*} = 0$ holds. This first order condition can be written similar to the familiar Lerner pricing formula (see equation (2.15) in the proof below):

$$\frac{p^* - \tilde{c}(p^*)}{p^*} = \frac{1}{\tilde{\varepsilon}(p^*|p^*)}, \tag{2.8}$$

where $\tilde{\varepsilon}(\cdot|p^*)$ is equal to $-\frac{\partial}{\partial p} D(\cdot|\sigma(p^*))\, pD(\cdot|\sigma(p^*))^{-1}$, the *dynamic elasticity* of demand that accounts for the possibility of buyers to substitute intertemporally. The function $\tilde{c}(p^*)$ is equal to $(1-\delta)\, q^S(p^*|\sigma(p^*))\, p^*$, with $q^S(p^*|\sigma(p^*))$ being the lifetime trading probability when setting p^*. It can be interpreted as dynamic opportunity costs of selling the good: By not selling today and offering the good at a price p^* from tomorrow onwards, a seller would get $\tilde{c}(p^*)$. We will prove that there exists indeed an equilibrium $\sigma(p^*)$. The proof essentially uses the intermediate value theorem to show that (2.8) has a solution. Furthermore, we show that concavity of *demand* $\left(1 - G^B(\cdot)\right)$ implies that this solution is unique and decreasing in δ.

Now, suppose there is some sequence of exit rates $\{\delta_k\}_{k=1}^{\infty}$ such that $\delta_k \to 0$. Let p_k^* be the unique price offered by the sellers for given δ_k. We prove that the price p_k^* converges to zero, i.e., the price converges to the market clearing price $p^w = 0$ with vanishing frictions δ_k. The proof of $p_k^* \to 0$ is by contradiction: If it does not converge to zero, then there is some (sub-)sequence such that p_k^* converges to $p^N \in (0, 1]$ by the Bolzano-Weierstrass theorem. For δ_k close enough to zero, we show that this implies that (2.8) is violated and p_k^* is not be a best response to itself, i.e., sellers have an incentive to deviate. To see why, let us rewrite (2.8) by cancelling p_k^* on the left hand side:

$$1 - (1-\delta)\, q^S(p_k^*) = \frac{1}{\tilde{\varepsilon}\left(p_k^*|p_k^*\right)}. \tag{2.9}$$

The intuition for this equality not to hold if p_k^* converges to some $p^N > 0$ is nothing more than the formalization of the intuition for convergence given in the introduction. We said that if sellers offer a price $p_k^* > 0$, then some of them must be *rationed.* Formally, their lifetime trading probability $q^S(p_k^*)$ must be below one. In the proof we show that $q^S(p_k^*)$ is simply the mass of entering buyers with a valuation above p_k^*, independent of δ (see equation (2.11) and the subsequent remark):

$$q^S(p_k^*) = 1 - G^B(p_k^*).$$

By assumption, the mass of buyers with valuations above any $p_k^* > 0$ is smaller than one and hence $q^S(p_k^*) = \left(1 - G^B(p_k^*)\right) < 1$. Then we went on in the introduction and said that sellers would have an incentive to decrease their price marginally, because this would increase their trading probability strictly. Formally, the elasticity $\tilde{\varepsilon}(p_k^*|p_k^*)$ of demand becomes *infinite* and hence the right hand side becomes zero. Therefore, whenever p_k^* converges to some $p^N > 0$ we have

$$\lim_{k\to\infty} 1 - (1-\delta)\, q^S(p_k^*) = G^B\left(p^N\right) > 0 = \lim_{k\to\infty} \frac{1}{\tilde{\varepsilon}\left(p_k^*|p_k^*\right)}.$$

Hence, for δ_k small enough, the condition (2.8) is violated. But this condition is necessary for $p^* \in \arg\max U^S(\cdot, 0|\sigma(p^*))$ and therefore sellers deviate from the proposed equilibrium. In the remainder of this section we provide the proof of the following proposition which summarizes our findings:

Proposition 2 *If $G^S(c) = 1$ for all $c \in [0,1]$, then there exists a steady-state equilibrium for any $\delta \in (0,1)$. For each $\delta \in (0,1)$ there is a unique $p^*_\delta \in [0,1]$ such that $p(0) = p^*_\delta$ in every equilibrium. The price p^*_δ is strictly decreasing in δ. For every sequence $\{\delta_k\}_{k=1}^\infty$ with $\delta_k \to 0$, $\lim_{k\to\infty} p^*_{\delta_k} = 0$.*

For the proof, we first reduce the problem of finding an equilibrium in the function space Σ to the simpler problem of finding an equilibrium price p^0 in the unit interval. For this we define the function $\sigma(\cdot) : [0,1] \to \Sigma$ with

$$\sigma\left(p^0\right) \equiv \left(p\left(\cdot|p^0\right), r\left(\cdot|p^0\right), \Phi^S\left(\cdot|p^0\right), \Phi^B\left(\cdot|p^0\right), M\left(p^0\right)\right),$$

derived from the equilibrium implications of $p(0) = p^0$ as follows: For reservation prices, note that $V^B\left(v|p^0\right) = 0$ for all $v < p^0$ and $V^B\left(v|p^0\right) = v - p^0$ otherwise. Therefore the unique function satisfying the equilibrium condition (3.5) for reservation prices is

$$r\left(v|p^0\right) \equiv \min\left\{v, p^0 + \delta\left(v - p^0\right)\right\}.$$

By $G^S(c) = 1$ for all c, (3.6) implies

$$\Phi^S\left(c|p^0\right) = 1 \quad \forall c.$$

The trading probability of buyers is $S(r(\cdot)) = 1_{v \geq p^0}$. So (3.7) implies

$$\Phi^B\left(v|p^0\right) = \begin{cases} G^B(v) / (\delta M) & \text{if } v \leq p^0 \\ G^B\left(p^0\right) / (\delta M) + \left[G^B(v) - G^B\left(p^0\right)\right] / M & \text{if } v > p^0, \end{cases}$$

and from $\Phi^B\left(1|p^0\right) = 1$ we get

$$M\left(p^0\right) = G^B\left(p^0\right) / \delta + 1 - G^B\left(p^0\right).$$

Finally, let $p\left(0|p^0\right) = p^0$ and for $c > 0$ let $p\left(c|p^0\right)$ be any optimal price for type c. (The prices set by these sellers do not matter since they have zero measure). Taken together, for every p^0, the other components of any equilibrium vector are fixed via $\sigma(\cdot)$. We have found an equilibrium if and only if $p^0 \in \arg\max U^S\left(\cdot, 0|\sigma\left(p^0\right)\right)$. Thus, the problem of finding an equilibrium σ^* in Σ is reduced to the relatively straightforward one-dimensional fixed point problem for the correspondence $\arg\max U^S\left(\cdot, 0|\sigma\left(p^0\right)\right)$ in $[0,1]$.

To tackle the maximization problem, we derive the trading probability per period, $D\left(p|\sigma\left(p^0\right)\right)$ and its derivative $\frac{\partial}{\partial p} D\left(p|\sigma\left(p^0\right)\right)$. Reservation prices are $r\left(v|p^0\right) = p^0 + \delta\left(v - p^0\right)$ for $v \geq p^0$. So the reservation price is increasing in v and for $p \in \left[p^0, r(1)\right]$, $r(v) \geq p$ if and only if $v \geq v\left(p|p^0\right) \equiv p^0 + \left(p - p^0\right)/\delta$ (the inverse of $r\left(v|p^0\right)$). We set $v\left(p|p^0\right) \equiv 1$ for all $p > r(1)$. For $v \leq p^0$ reservation prices are $r\left(v|p^0\right) = v$ and so $v\left(p|p^0\right) \equiv p$ for $p \in \left[0, p^0\right]$. Note that $v\left(p|p^0\right)$ is continuous at p^0. The trading probability is

$$D\left(p|\sigma\left(p^0\right)\right) = 1 - \Phi^B\left(v\left(p|p^0\right)|p^0\right),$$

and thus for $p \in \left[p^0, r(1)\right]$

$$\begin{aligned}
\frac{\partial}{\partial p} D\left(p|\sigma\left(p^0\right)\right) &= \frac{\partial}{\partial p}\left[1-\left(G^B\left(p^0\right)/(\delta M)+\left[G^B\left(v\left(p|p^0\right)\right)-G^B\left(p^0\right)\right]\right)/M\right], \\
&= -g^B\left(v\left(p|p^0\right)\right)/(\delta M),
\end{aligned}$$

with $M = M\left(p^0\right)$. For $p \in [0, r(1))$, with $d\left(p|\sigma\left(p^0\right)\right) \equiv \frac{\partial}{\partial p} D\left(p|\sigma\left(p^0\right)\right)$

$$\begin{aligned}
d\left(p|\sigma\left(p^0\right)\right) &= \frac{\partial}{\partial p}\left(1-G^B\left(v\left(p|p^0\right)\right)/(\delta M)\right) \\
&= -g^B\left(v\left(p|p^0\right)\right)/(\delta M), \qquad (2.10)
\end{aligned}$$

and for $p > r(1)$, $d\left(p|\sigma\left(p^0\right)\right) \equiv 0$. This implies that $d\left(p|\sigma\left(p^0\right)\right)$ is continuous at p^0 and hence $U^S\left(\cdot, 0|\sigma\left(p^0\right)\right)$ is continuously differentiable on $\left[0, r\left(1|p^0\right)\right)$. To derive $q^S\left(p^0|\sigma\left(p^0\right)\right)$, note that $D\left(p^0|\sigma\left(p^0\right)\right) = \left(1-G^B\left(p^0\right)\right) M\left(p^0\right)^{-1}$. Substituting this and the definition of $M\left(p^0\right)$ into the definition of q^S in (3.3) yields

$$q^S\left(p^0|\sigma\left(p^0\right)\right) = 1 - G^B\left(p^0\right). \qquad (2.11)$$

For the intuition behind (2.11), note that $q^S\left(p^0|\sigma\left(p^0\right)\right)$ is equal to the share of entering sellers who will be able to conduct a trade. In a steady state this share is equal to the entering share of buyers who can actually trade at p^0 (see Lemma 2). This share of buyers is $\left(1-G\left(p^0\right)\right)$, and thus, $q^S\left(p^0|p^0\right) = 1 - G\left(p^0\right)$.

Existence. We want to find some p^0 such that $p^0 \in \arg\max U^S\left(\cdot, 0|\sigma\left(p^0\right)\right)$. From $U^S\left(p, 0|\sigma\left(p^0\right)\right) = \frac{D(p)}{D(p)+\delta-\delta D(p)} p$ we get

$$\frac{\partial}{\partial p} U^S\left(p, 0|\sigma\left(p^0\right)\right) = \left[D(p)+\delta-\delta D(p)\right]^{-2}\left[d(p)\,\delta p + D(p) - (1-\delta)\, d(p)\, q(p)\, p\right],$$

with $d(p) = \frac{\partial}{\partial p} D\left(p|\sigma\left(p^0\right)\right)$. In the appendix, we show that $\frac{\partial}{\partial p} U^S\left(p, 0|\sigma\left(p^0\right)\right)$ is strictly concave on $\left[0, \left(1|p^0\right)\right)$. Together with $U^S\left(r\left(1|p^0\right), 0|\sigma\left(p^0\right)\right)$, $\frac{\partial}{\partial p} U^S\left(p, 0|\sigma\left(p^0\right)\right) = 0$ is therefore a necessary and sufficient condition for an optimum. We use the intermediate value theorem to show that there exists some p^* such that $\frac{\partial}{\partial p} U^S\left(p|p^*\right)|_{p=p^*} = 0$. Let $X\left(p^0\right)$ be the second term of the derivative of $\frac{\partial}{\partial p} U^S\left(\cdot|p^0\right)$ at p^0:

$$\begin{aligned}
X\left(p^0\right) &= \left[d\left(p^0\right)\delta p^0 + D\left(p^0\right) - (1-\delta)\, d\left(p^0\right) q\left(p^0\right) p^0\right] \qquad (2.12) \\
&= \left[p^0\left(1-(1-\delta)\left(1-G^B\left(p^0\right)\right)\right) - \frac{\left(1-G^B\left(p^0\right)\right)\delta}{g^B\left(p^0\right)}\right] d\left(p^0|\sigma\left(p^0\right)\right).
\end{aligned}$$

Because the first term of $\frac{\partial}{\partial p} U^S\left(p, 0|\sigma\left(p^0\right)\right)$ is strictly positive for all p, $\frac{\partial}{\partial p} U^S\left(\cdot|p^0\right)|_{p=p^0} = 0$ if and only if $X\left(p^0\right) = 0$. The function $X(\cdot)$ is continuous by $G^B(\cdot)$ and $g^B(\cdot)$ being continuous. Inspection of (2.10) shows that $d\left(p^0|\sigma\left(p^0\right)\right) < 0$ for all p^0 which implies

$$\begin{aligned}
X(0) &= \left[0(-\delta) - \frac{\delta}{g^B(0)}\right] d(0|\sigma(0)) > 0, \\
\text{and} \quad X(1) &= \left[1 - \frac{0}{g^B(1)}\right] d(1|\sigma(1)) < 0.
\end{aligned}$$

So we can apply the intermediate value theorem and conclude that there is some p^* such that $X(p^*) = 0$. By definition of $X(\cdot)$, this implies that $\frac{\partial}{\partial p} U^S(p|p^*)|_{p=p^*} = 0$ and hence, $p^* \in \arg\max U^S(p, 0|\sigma(p^*))$. Thus, there is an equilibrium σ^*, with $\sigma^* \equiv \sigma(p^*)$.

Uniqueness. Rewriting (2.12) with $p^0 = p^*$ we get as a necessary condition for an equilibrium

$$p^* \left(\delta + (1-\delta) G^B(p^*)\right) = \frac{\left(1 - G^B(p^*)\right)\delta}{g^B(p^*)}. \tag{2.13}$$

Note that the left hand side is strictly increasing in p^*, while the right hand side is strictly decreasing since $g'(p^*) > 1$. So there can be at most one price p^* satisfying this equality.

Decreasing Prices. Rewriting (2.12) further shows that $X\left(p^0\right) = 0$ requires

$$1 = \frac{1 - G\left(p^0\right)}{G\left(p^0\right)} \delta \left[\frac{1}{g\left(p^0\right) p^0} - 1\right]. \tag{2.14}$$

The right hand side of this condition is strictly increasing in δ and strictly decreasing in p^0. So if δ increases, $p^0 = p^*_\delta$ has to increase for compensation.

Convergence. Let $p^*_k = p^*_{\delta_k}$. Rewrite (2.13) to get the necessary condition

$$\left(1 - (1-\delta)\left(1 - G^B(p^*_k)\right)\right) = \frac{\left(1 - G^B(p^*_k)\right)\delta}{g^B\left(p^*_k\right) p^*_k}, \tag{2.15}$$

which is equivalent to the Lerner formula (2.8).[14] If $\lim_{k\to\infty} p^*_k \neq 0$, then for some (sub-)sequence, $p^*_{k'} \to p^N > 0$, and the right hand side becomes

$$\lim_{k'\to\infty} \frac{1 - G^B\left(p^*_{k'}\right)}{g^B\left(p^*_{k'}\right) p^*_{k'}} \delta = \frac{1 - G^B\left(p^N\right)}{g^B\left(p^N\right) p^N} 0 = 0,$$

by continuity of $G^B(\cdot)$ and $g^B(\cdot)$. The left hand side becomes

$$\lim_{k'\to\infty} \left(1 - (1-\delta)\left(1 - G^B(p^*_{k'})\right)\right) = G^B\left(p^N\right) > 0,$$

and thus, the necessary condition (2.13) cannot hold for k' large enough, a contradiction to $p^*_{k'}$ being part of an equilibrium. This completes the proof.

2.4 Heterogeneous Sellers: Existence and Consistency

In this section, we prepare the analysis of the model with heterogeneous sellers. Here and in the following, we assume that $G^S(\cdot)$ and $G^B(\cdot)$ are strictly increasing and have continuous, strictly positive densities $g^S(\cdot)$ and $g^B(\cdot)$, respectively. In particular, we do no longer assume that $\left(1 - G^B(\cdot)\right)$ is concave. We prove first that we can restrict attention to equilibria in which prices $p(\cdot)$ and reservation prices $r(\cdot)$ are monotone. Then we show that for every monotone strategy combination $p(\cdot)$ and $r(\cdot)$ (not just equilibrium strategies) there exists a steady-state pool of traders. We also show that for

[14] Multiplying the left hand side by $\frac{p^*_k}{p^*_k}$ and observing that the right hand side is the elasticity $\tilde{\varepsilon}$ because $\frac{\partial}{\partial p} D(p^*_k|p^*_k) p^*_k D(p^*_k|p^*_k)^{-1} = g^B(p^*_k)\delta^{-1} p^*_k \left(1 - G^B(p^*_k)\right)^{-1}$.

every monotone strategy combination and for every corresponding steady-state pool, the transfers collectively made by sellers are equal to the transfers received by buyers. And finally, we prove that an equilibrium exists for all δ.

First, we prove that we can restrict attention to a subset of Σ when analyzing equilibrium outcomes. We use this to ease notational burden, for the proof of Proposition 3, and to prepare the existence proof by restricting the set of equilibrium candidates. Let $\sigma^* = \left[p^*, r^*, \Phi^{S*}, \Phi^{S*}, M^*\right] \in \Sigma$ be an equilibrium. Then the equilibrium conditions imply a restriction on these functions: We show that by $r^*(\tilde{v}) = \tilde{v} - (1-\delta) V(\tilde{v}|\sigma^*)$ from condition (3.5), $r^*(\cdot)$ must have a slope in $[\delta, 1]$, i.e., $r^*(\cdot)$ is in the set

$$\Sigma_{r(\cdot)} = \{f : [0,1] \to [0,1] \,|\, f(a) - f(b) \in [\delta(a-b), (a-b)]\}.$$

For this, note that if the value function is differentiable at some point $\tilde{v}$, then $V'(\tilde{v}|\sigma^*) = Q^B(r^*(\tilde{v})|\sigma^*)$ by the envelope theorem (see e.g., Milgrom and Segal (2002)). Therefore, $r(\cdot|\sigma^*)$ is differentiable at $\tilde{v}$ as well and $r'(\tilde{v}|\sigma^*) = 1 - (1-\delta) Q^B(r^*(\tilde{v})|\sigma^*)$; Hence, $r'(\tilde{v}|\sigma^*) \in [\delta, 1]$ at all differentiability points. This can be generalized to all points.[15]

Inspecting the steady-state conditions (3.6) and (3.7), shows that M^* must be in $\left[1, \delta^{-1}\right]$: Rewriting (3.7) at $v = 1$, with $\Phi^{B*}(1) = 1$ and observing that $(S(r(\tau)|\sigma^*) + \delta(1 - S(r(\tau)|\sigma^*))) \in [\delta, 1]$:

$$M^* = \int_0^1 \frac{dG^B(\tau)}{(S(r^*(\tau)|\sigma^*) + \delta(1 - S(r^*(\tau)|\sigma^*)))} d\tau \in \left[1, \delta^{-1}\right].$$

The distribution functions are strictly increasing with a slope in $\left[\delta g_L, g_H \delta^{-1}\right]$: For $\Phi^B(\cdot)$, note that the density $dG^B(v)$ is strictly positive and continuous by assumption, so $dG^B(v) \in [g_L, g_H]$ for some $(g_L, g_H) \in (0,1)^2$. Rewriting (3.7) shows that

$$\begin{aligned}\Phi^{B*}(a) - \Phi^{B*}(b) &= \int_b^a \frac{dG^B(\tau)}{M^*(S(r^*(\tau)|\sigma^*) + \delta(1 - S(r^*(\tau)|\sigma^*)))} d\tau \\ &\in \left[\delta g_L, g_H \delta^{-1}\right][a-b].\end{aligned}$$

where we used already that $M^* \in \left[1, \delta^{-1}\right]$. So $\Phi^{B*}(\cdot)$ and $\Phi^{S*}(\cdot)$ (by analogous reasoning) are in the set

$$\Sigma_\Phi = \left\{f : [0,1] \to [0,1] \,|\, f(a) - f(b) \in \left[\delta g_L(a-b), g_H \delta^{-1}(a-b)\right]\right\}.$$

Given monotonicity of reservation prices, we want to show monotonicity of prices $p^*(\cdot)$. For this, we use that payoffs satisfy the strict single crossing property. To show that this is true, note that a seller who offers a price p trades with all buyers with a valuation above $v(p|\sigma^*) \equiv \inf\{v, 1|r^*(v) \geq p\}$ by monotonicity of $r(\cdot)$. Therefore $D(p|\sigma^*) = 1 - \Phi^{B*}(v(p))$. By $\Phi^{B*}(\cdot)$ and $r^*(\cdot)$ being both continuous and strictly increasing, the trading probability $D(\cdot|\sigma^*)$ is strictly positive at all prices below the highest reservation price $r(1)$: for all $p < r(1)$, we have $v(p|\sigma^*) < 1$ and therefore $1 - \Phi^B(v(p|\sigma^*)) = D(p|\sigma^*) > 0$. Hence, for these prices the lifetime trading probability $q^S(p|\sigma^*)$ is strictly positive. For all prices above $r^*(1)$ trading probabilities are zero: by

[15] By rewriting the optimality condition $V^B(a) - V^B(b) \geq U^B(r(b), a) - U^B(r(b), b)$ and its symmetric analogue and by using the definition of $U^B(\cdot,\cdot)$.

$v(p) = 1$ for all $p \geq r^*(1)$ we have $1 - \Phi^{B*}(1) = 0$. So the relevant range of optimal prices for sellers with costs below $r^*(1)$ is $[0, r(1))$. Now we show that profits $U^S(\cdot, \cdot|\sigma^*)$ satisfy the *strict single crossing condition* on the domain $[0, r^*(1))^2$, i.e., we show that for all $p_H > p_L$ and $c_H > c_L$ with $(p_L, c_L, p_H, c_H) \in [0, r^*(1))^4$:

$$U^S(p_H, c_L|\sigma^*) - U^S(p_L, c_L|\sigma^*) \geq 0 \quad \Rightarrow \quad U^S(p_H, c_H|\sigma^*) - U^S(p_L, c_H|\sigma^*) > 0.$$

Rewriting shows that the left hand side is equivalent to

$$c_L\left(q^S(p_L|\sigma^*) - q^S(p_H|\sigma^*)\right) \geq q^S(p_L|\sigma^*)\, p_L - q^S(p_H|\sigma^*)\, p_H.$$

Now, observe that $q^S(\cdot|\sigma^*)$ is strictly decreasing in p on $[0, r^*(1))$ by $v(\cdot|\sigma^*)$ and $\Phi^{B*}(\cdot)$ being strictly increasing. Therefore, $\left(q^S(p_L|\sigma^*) - q^S(p_H|\sigma^*)\right) > 0$, which implies that the left hand side is strictly increasing in costs and hence

$$c_H\left(q^S(p_L|\sigma^*) - q^S(p_H|\sigma^*)\right) > q^S(p_L|\sigma^*)\, p_L - q^S(p_H|\sigma^*)\, p_H,$$

which implies $U^S(p_H, c_H|\sigma^*) - U^S(p_L, c_H|\sigma^*) > 0$ as claimed. By the monotone selection principle of Milgrom and Shannon (1994), the strict single crossing property implies that all selections from the maximum correspondence $\arg\max_p U^S(p, c|\sigma^*)$ are weakly increasing. Therefore, $p^*(\cdot)$ is weakly increasing on $[0, r^*(1))$. But we cannot use optimality conditions to extend monotonicity of $p^*(\cdot)$ to types beyond $r^*(1)$: *Every* price $p \geq r^*(1)$ is optimal for a type $c \geq r^*(1)$ since at every such price trading probabilities and profits are zero while at every price $p < r^*(1)$ profits would be strictly negative. Nevertheless, we may simply assume that these types set monotone prices and without further loss of generality, we may assume that they set prices equal to their costs: This will not change the steady-state distribution (because trading probabilities are unchanged for all traders) and this will not change the optimality condition of buyers (because given the prices set by the other sellers and the distribution of their price offers, accepting any $p \geq r^*(1)$ would make any buyer $v \in (0, 1)$ strictly worse off).

Let Σ_+ be the set of weakly increasing functions and define the set $\tilde{\Sigma} \subset \Sigma$

$$\tilde{\Sigma} \equiv \Sigma_+ \times \Sigma_{r(\cdot)} \times \Sigma_\Phi \times \Sigma_\Phi \times \left[1, \delta^{-1}\right].$$

We summarize our findings in a proposition. It states that every equilibrium σ^* is *equivalent* to an equilibrium $\tilde{\sigma}$ which is in the set $\tilde{\Sigma}$, changing $p^*(\cdot)$ to $\tilde{p}(\cdot)$ on $[r^*(1), 1]$ as described before:

Lemma 1 *If $\sigma^* = \left[p^*, r^*, \Phi^{S*}, \Phi^{S*}, M^*\right]$ is a steady-state equilibrium, then with*

$$\tilde{p}(c) \equiv \begin{cases} p^*(c) & \forall c \in [0, r(1)) \\ c & \forall c \in [r^*(1), 1], \end{cases}$$

$\tilde{\sigma} = \left[\tilde{p}, r^*, \Phi^{S*}, \Phi^{S*}, M^*\right]$ is a steady-state equilibrium and $\tilde{\sigma} \in \tilde{\Sigma}$.

Remark 1 *The outcome of the equilibrium $\tilde{\sigma}$ is* equivalent *to the outcome of σ^*: For all c, $q^S(p^*(c)|\sigma^*) = q^S(\tilde{p}(c)|\tilde{\sigma})$ and $V^S(c|\sigma^*) = V^S(c|\tilde{\sigma})$ and similarly for buyers. Hence, if we have proven convergence to efficiency for outcomes of equilibria $\tilde{\sigma} \in \tilde{\Sigma}$, we have proven convergence to efficiency for outcomes of equilibria $\sigma \in \Sigma$.*

Now we show that for every strategy combination $(p(\cdot)\, r(\cdot)) \in \left(\Sigma_+, \Sigma_{r(\cdot)}\right)$ we can find a pool characterized by $\left(\Phi^B(\cdot), \Phi^S(\cdot), M\right)$ such that the steady-state conditions (3.6) and (3.7) hold. So these steady-state conditions do not restrict the strategy set $\left(\Sigma_+, \Sigma_{r(\cdot)}\right)$ further. In models without an exit rate, this is not true and for some strategies a steady-state pool fails to exist. If this is the case, the steady-state assumption implies a restriction on the strategies (see the discussion of models with infinitely lived players in Section 2.6.1).

Proposition 3 *For every strategy combination $(p(\cdot), r(\cdot)) \in \Sigma_{p(\cdot)} \times \Sigma_{r(\cdot)}$ we can find are monotone functions $\Phi^B(\cdot) \in \Sigma_{\Phi^B}$ and $\Phi^S(\cdot) \in \Sigma_{\Phi^S}$, and $M \in \left[1, \delta^{-1}\right]$ such that the steady-state conditions (3.6) and (3.7) hold.*

Proof: See the remark following Lemma (8) on page 27 ■

This theorem suggests to define the subset $\bar{\Sigma}$ of $\tilde{\Sigma}$, consisting of all σ such that Φ^S, Φ^B and M are a steady state given $p(\cdot)$ and $r(\cdot)$, i.e.,

$$\bar{\Sigma} \equiv \left\{\sigma \in \tilde{\Sigma} | \text{Conditions (3.6), (3.7)hold}\right\}.$$

Take some $\sigma \in \bar{\Sigma}$, i.e., some strategy combination and some steady-state pool that is consistent with it. Intuition suggests, that the expected payments made by buyers is equal to the expected payments received by sellers. In addition, the mass of buyers who expect to trade is equal to the mass of sellers. Indeed, straightforward algebraic manipulation of the conditions (3.7) and (3.6) show that this is the case (see the appendix, Section A.3 for details):

Lemma 2 ***Mass Balance.*** *Expected payments and the mass of expected trades are equal on both sides of the market, i.e., for all $\sigma \in \bar{\Sigma}$:*

$$\begin{aligned} \int_0^1 q^S(p(c)|\sigma)\, p(c)\, g^S(c)\, dc &= \int_0^1 q^B(r(v)|\sigma)\, E[p|p \le r(v), \sigma]\, g^B(v)\, dv \\ \textit{and} \quad \int_0^1 q^S(p(c)|\sigma)\, g^S(c)\, dc &= \int_0^1 q^B(r(v)|\sigma)\, g^B(v)\, dv. \end{aligned}$$

Finally, we show that for every δ an equilibrium exists. With heterogeneous sellers, we cannot reduce the existence problem to a one-dimensional fixed point problem; we have to prove the existence of a fixed point in the function space. Instead of the intermediate value theorem, we therefore use the Kakutani-Fan-Glicksberg theorem. We first introduce some notation: We describe the pool by the *mass* of buyers with valuations *above* v, $M^B(v)$ and the *mass* of sellers with costs below c, $M^S(c)$. The total mass of buyers is $M^B(0)$ and the total mass of sellers is $M^S(1)$. Throughout the proof we do not require $M^B(0)$ to be equal to $M^S(1)$ but we show that if we have found an equilibrium such that the steady-state conditions hold, these two masses must be equal. With this new notation, the market is characterized by a quartuple of functions $\omega = \left[r(\cdot), p(\cdot), M^S(\cdot), M^B(\cdot)\right]$. Given ω, we now derive a response operator K. This operator consists of the best response (correspondence) for sellers and buyers, $K^p[\omega]$ and $K^r[\omega]$, and the *pool response* $K^S[\omega]$ and $K^B[\omega]$. The former will consist of ex ante optimal strategies $p(\cdot)$ and $r(\cdot)$. The *pool responses* are the masses of traders in the pool that will result in period $t+1$, if the pool in period t is described by $M^S(\cdot)$, $M^B(\cdot)$ and if they trade according to $p(\cdot)$ and $r(\cdot)$. If ω^* is a fixed-point of K, $\omega^* \in K[\omega^*]$, traders play mutual best responses and the pool is in a steady state, i.e., ω^* is an equilibrium. We show that such a fixed point exist and show that ω^* corresponds to an equilibrium σ^* as originally defined in 2. The main technical difficulty is the proof of continuity of the pool-response operators.

To prepare for the fixed point theorem, we restrict the set of candidate strategies and distributions under consideration. The restrictions on distributions of types becomes now a restriction on masses by observing that $M^S(\cdot)$ corresponds to $\Phi^S(\cdot)M$. So we have

$$\begin{aligned}\Sigma_{M^S} &\equiv \left\{M^S(\cdot):[0,1]\to\left[0,\delta^{-1}\right] | \frac{M^S(a)-M^S(b)}{a-b}\in\left[g_l, g_h\delta^{-1}\right], \forall a\neq b\right\}\\ \Sigma_{M^B} &\equiv \left\{M^B(\cdot):[0,1]\to\left[0,\delta^{-1}\right] | \frac{M^B(b)-M^B(a)}{a-b}\in\left[g_l, g_h\delta^{-1}\right], \forall a\neq b\right\},\end{aligned}$$

and the domain of K is

$$\bar{\Sigma}\equiv\Sigma_{p(\cdot)}\times\Sigma_{r(\cdot)}\times\Sigma_{M^S}\times\Sigma_{M^B}. \tag{2.16}$$

Because all functions in $\bar{\Sigma}$ are integrable, we use the integral norm $\|f(\cdot)\|_1=\int_0^1|f(t)|\,dt$ such that $\bar{\Sigma}$ becomes a compact subspace of L_1.[16]

To define payoffs, note that the share of buyers with valuations above v is given by the expression $M^B(v)M^B(0)^{-1}$ and we may define $D(p|\omega)\equiv M^B(v(p))M(0)^{-1}$ and similarly $S(r|\omega)\equiv M^S(c(r))M^S(1)^{-1}$ with the generalized inverses $c(\cdot)$ and $v(\cdot)$ defined as before. Lifetime trading probabilities for a given ω are

$$q^B(r|\omega)\equiv\frac{S(r|\omega)}{1-(1-S(r|\omega))(1-\delta)} \text{ and } q^S(p|\omega)\equiv\frac{D(p|\omega)}{1-(1-D(p|\omega))(1-\delta)};$$

and payoffs are

$$U^S(p,c|\omega)\equiv q^S(p|\omega)(p-c) \text{ and } U^B(r,v|\omega)\equiv q^B(r|\omega)(v-E[p|p\leq r,\omega]).$$

Ex ante expected payoffs to sellers are $\Pi(p(\cdot)|\omega)$ and interim maximized payoffs to buyers are $V^B(v)$:

$$\begin{aligned}\Pi(p(\cdot)|\omega) &\equiv \int_0^1 q^S(p(c)|\omega)(p(c)-c)g(c)\\ \text{and}\qquad V^B(v) &\equiv \max_r U^B(r,v|\omega).\end{aligned}$$

Now we define the operator K. The sellers' best response correspondence is defined as

$$K^p[\omega]\equiv\arg\max_{p(\cdot)\in\Sigma_{p(\cdot)}}\Pi(p(\cdot)|\omega),$$

[16] As usual, we continue working with the function space itself, rather than the corresponding space of equivalence classes. Two functions are equivalent under $\|\cdot\|_1$ if they are equal almost everywhere.

and $K^p[\omega] \in \Sigma_{p(\cdot)}$ by definition. With $r(v|\omega) = v - (1-\delta) V^B(v|\omega)$, buyers' best response is given by

$$K^r[\omega] \equiv r(\cdot).$$

Inspection of $V^B(v|\omega) = q^B(r(v)|\omega)(v - E[p|p \leq r(v)|\omega])$ shows that $r(\cdot|\omega)$ must have a slope between δ (if $q^B(r(v)|\omega) = 1$) and 1 (if $q^B(r(v)|\omega) = 0$), i.e., $K^r[\omega] \in \Sigma_{r(\cdot)}$.

Define pool response operators by

$$K^S(c|\omega) \equiv G^S(c) + \int_0^c (1-\delta)(1 - D(p(t)|\omega))\, dM^S(t) \tag{2.17}$$

$$K^B(p|\omega) \equiv \left(1 - G^B(v)\right) + \int_v^1 (1-\delta)(1 - S(r(t)|\omega)) - dM^B(t), \tag{2.18}$$

where $K^S(c|\omega)$ is the mass of sellers at the end of the period, consisting of the inflow $G^S(c)$ and the remaining mass of sellers from the beginning of the period, i.e., those sellers who neither trade nor die. Similarly, $K^B(v|\omega)$ is the mass of buyers at the end of the period, consisting of the new inflow and the remaining buyers from the mass at the beginning. To check that K^S maps Σ_{M^S} into itself, note that $K^S(1|\omega)$ attains its maximal value if no sellers trades and then $K^S(1|\omega) \leq 1 + (1-\delta) M^S(1) \leq \delta^{-1}$. The slope of $K^S(\cdot|\omega)$ is maximal at $dK^S(t) = g_h \delta^{-1}$ and minimal at $dK^S(t) = g_l + 0$. Therefore, $K^S[\omega] \in \Sigma_{M^S}$. Reasoning similarly for buyers and adding our observations on the best response operators, we have that $K[\cdot]$ is a self map of $\bar{\Sigma}$:

$$K[\omega] \equiv K^p \times K^r \times K^S \times K^B : \bar{\Sigma} \rightrightarrows \bar{\Sigma}.$$

We want to prove that K has a fixed point ω^* using the Kakutani-Fan-Glicksberg fixed point theorem. The theorem states that if Ω is a non-empty, convex, and compact subset of a locally convex Hausdorff space, and if K has a closed graph and non-empty, convex values, then K has a fixed point (see Aliprantis, Border, 1994, p484). In the following lemmas, we show that the functions K^r, K^S, K^B are continuous in ω and that the correspondence K^p has convex values and a closed graph.

We apply Berge's Maximum Theorem to show that the best response correspondence K^p is upper hemicontinuous with compact non-empty values. This implies that K^p has a closed graph (see Aliprantis, Border, p. 473 and p. 465, respectively). To apply Berge's Theorem, we need to show that expected profits are continuous in $p(\cdot)$ and ω, which will follow from reservation prices being continuous and strictly increasing. Then we show convexity, utilizing that the best response correspondence is essentially unique for all $c \in [0, r(1)]$ (because payoffs satisfy the strict single crossing condition) and for $c \in (r(1), 1]$, all elements and all convex combinations of elements of K^p yield zero profits:

Lemma 3 $K^p[\cdot]$ *has a closed graph and is non-empty, and convex valued.*

Proof: Trading probabilities $q^S(p|\cdot)$ are continuous for all p in ω, because $M^B(v(p)|\cdot)$ is continuous in ω. By the dominated convergence theorem, expected payoffs $\Pi(p(\cdot)|\cdot)$ are therefore continuous in ω for given $p(\cdot)$. Similarly, expected payoffs are continuous in the own price function $p(\cdot)$. Therefore Berge's maximum theorem applies.

Now we want to show convexity. Let the highest type who can possibly trade with positive payoffs be $\bar{c} = \sup\left\{c|q^S(c|\omega) > 0\right\}$. As a first step, we show that every two interim optimal pricing functions are equivalent (a.e. identical) on $[0, \bar{c}]$. Take any $p(\cdot) \in K^p[\omega]$. Then $p(c)$ is ex ante optimal for almost every type, i.e., $p(c) \in \arg\max U^S(c, p|\omega)$ for almost every c, and in particular, at every point of continuity: Suppose not, then for some pair p', and c', $U^S(c', p(c')|\omega) < U^S(c', p'|\omega)$. By continuity of $U^S(\cdot,\cdot|\omega)$ in p and c, and by continuity of $p(\cdot)$ at c', there is a neighborhood $B_\delta(c')$ of c' such that $U^S(c, p(c)|\omega) < U^S(c, p'|\omega)$ for all $c \in B_\delta(c')$. Because $B_\delta(c')$ has strictly positive mass, this implies that $p(\cdot)$ is not profit maximizing ex ante, a contradiction.

Now take two functions $p_1 \in K^p[\omega]$ and $p_2 \in K^p[\omega]$ and some $c' < \bar{c}$ such that $p_1(c')$ and $p_2(c')$ are (interim) optimal, i.e., $p_i(c') \in \arg\max U^S(p, c'|\omega)$. Suppose that $p_1(c') \neq p_2(c')$ and without loss of generality, suppose $p_1(c') \equiv p_1 < p_2 \equiv p_2(c')$. We show that c' must be a jump point of both functions: For all prices p^+ above p_1, $p^+ > p_1$, the optimality of p_1 implies $U^S(p_1, c'|\omega) \geq U^S(p^+, c'|\omega)$. Since payoffs have the strict single crossing property, all types c^- below c' strictly prefer p_1 to any such p^+, i.e., $U^S(p_1, c^-|\omega) > U^S(p^+, c^-|\omega)$. Similarly, the optimality of p_2 for c' implies that all types $c^+ \in (c', \bar{c})$ prefer p_2 strictly to any $p^- < p_2$. Hence, optimal prices to the left of c' are below p_1 and optimal prices to the right are above p_2. Finally, if some (single) type $c^- < c'$ plays a suboptimal price $p_2(c^-) > p_1$, all $c \in (c^-, c')$ must play prices above p_1 by monotonicity of $p_2(\cdot)$. But then a strictly positive mass of types sets a strictly suboptimal price and $p_2(\cdot)$ fails ex ante optimality. Therefore $p_i(c'-0) \leq p_1 < p_2 \leq p_i(c'+0)$, $i \in \{1,2\}$ and c' is a jump point as claimed. Note that the set of jump points has zero measure and together with $p_i(c) \in \arg\max U^S(p, c|\omega)$ a.e., we conclude that $p_1(c) = p_2(c)$ for almost all $c \in [0, \bar{c}]$. Therefore, every convex combination $p_\alpha(\cdot) \equiv \alpha p_1(\cdot) + (1-\alpha) p_2(\cdot)$ will be equivalent on $c \in [0, \bar{c}]$. For $c \in (\bar{c}, 1]$, note that by prices and ultimate trading probabilities being monotone, it must be that for all such c, $q^S(p_1(c)) = 0$ and $q^S(p_2(c)) = 0$ and so $q^S(p_\alpha(c)) = 0$ for all $\alpha \in [0,1]$ and for all $c \in (\bar{c}, 1]$ Therefore, we have $\Pi(p_\alpha(\cdot)|\omega) = \Pi(p_1(\cdot)|\omega)$, i.e., $p_\alpha(\cdot) \in K^p[\omega]$. ■

The next lemma states that reservation prices are continuous in ω. With $r(v|\omega) = v - (1-\delta) V^B(v|\omega)$ we need to show continuity of the value function $V^B(\cdot|\omega)$ which basically follows from Berge's Maximum theorem. Payoffs $U(r, v|\omega)$, however, do not need to be continuous in ω, because, for given r, the mass of sellers who offer $p \leq r$, $M^S(c(r)|\omega)$, can have a discontinuity. Therefore, we use the following *trick*: instead of choosing a reservation price r, buyers are thought to choose a threshold seller c_x and trade with all sellers with $c \leq c_x$:

Lemma 4 $K^r[\cdot]$ *is continuous in* ω.

Proof: Given $c_x \in [0,1]$, let the ultimate trading probability be defined as the function $q_x^B(c_x, \omega) \equiv \frac{M^S(c_x) M^S(1)^{-1}}{1-(1-\delta)\left(1-M^S(c_x) M^S(1)^{-1}\right)}$ and let expected prices be given by $E_x[p|c_x, \omega] \equiv \frac{1}{M^S(c_x)} \int_0^{c_x} p(c)\, dM^S(c)$. Then payoffs from trading with all $c \leq c_x$ are

$$U_x^B(c_x, v|\omega) \equiv q_x^B(c_x, \omega)\left(v - E_x[p|c_x, \omega]\right),$$

and clearly $U_x^B(c_x, v|\omega)$ is continuous in c_x, v, and ω. Thus, $V_x^B(v|\omega) = \max_{c_x} U_x^B(c_x, v|\omega)$ is continuous in ω by the Maximum theorem. In addition, payoffs from maximizing over cutoff types c_x are equal to payoffs from maximizing over cutoff prices r, i.e., $V_x^B(v|\omega) = V^B(v|\omega)$: Whenever $p^S(\cdot)$ is increasing at c_x, this follows immediately by setting $r(v) = p(c_x)$; if $p^S(\cdot)$ is flat at c_x, the buyer is indifferent between accepting and rejecting $p^S(c_x)$. Thus, continuity of $V_x^B(\cdot|\cdot)$ carries over to $V^B(\cdot|\cdot)$ ■

Now we want to show that the pool responses K^S and K^B are continuous. This is the main technical problem of the existence proof. The problem here is that we need to evaluate composite functions. In particular, to calculate the trading probability of a type v, we need to evaluate the share $M^S(c(r(v))|\omega) M^S(1|\omega)$. However, the type c who trades with v, $c(r(v))$, does not need to be continuous in ω. Therefore, we need to state first three auxiliary lemmas to deal with the problem of composite (inverse) functions. The first lemma states a partial converse to Lebesgue's bounded convergence theorem:

Lemma 5 *Let $\{f_N\}$ be a sequence in Σ_M such that $f_N \to \bar{f}$ in L_1. Then $f_N(x) \to \bar{f}(x)$ pointwise for almost all $x \in [0,1]$ if a) all f_N and $\bar{f}$ are weakly increasing or if b) the family $\{f_N\}$ is equicontinuous.*

Proof: For the first part. We show convergence at all interior continuity points $\bar{f}$ which implies the statement. Let x' be such a point and suppose there is some subsequence such that $\lim f_{N'}(x') \equiv f_H \neq \bar{f}(x') \equiv \bar{f}$. Suppose $f_H > \bar{f}$ (the other case is symmetric). Choose μ such that for all $x \in B_\mu(x')$, $\left|\bar{f}(x') - \bar{f}(x)\right| \leq \left|f_H - \bar{f}\right|/2$, and choose any $x_H \in B_\mu(x')$ such that $x_H > x'$. By monotonicity of each element $f_{N'}$, $f_{N'}(x) \geq f_{N'}(x')$ for all $x \in [x', x_H]$ and for all N'. Thus, $\liminf f_{N'}(x) \geq f_H$. Hence, $\liminf \int_{x'}^{x_H} \left|f_{N'}(x) - \bar{f}(x)\right| \geq (x_H - x')\left|f_H - \bar{f}\right|/2 > 0$, contradicting $f_N \to \bar{f}$. The second part is immediate ■

Take a sequence $\omega_N = \left[p_N, r_N, M_N^S, M_N^B\right]$ with $\omega_N \to \bar{\omega}$: we want to show convergence of the composite functions $M_N^B(v_N(p_N(\cdot)))$ and $M_N^S(c_N(r_N(\cdot)))$. The former composite function is not a problem: the families $\left\{M_N^B(\cdot)\right\}$ and $\{v_N(\cdot)\}$ are equicontinuous by the assumptions that $M_N^B \in \Sigma_{M^B}$ and $r_N \in \Sigma_r$, which imply that they are globally Lipschitz continuous. Lipschitz continuity of r_N carries over to its inverse v_N, by r_N being strictly increasing. Therefore, the family of composite functions $\left\{M_N^B(v_N(\cdot))\right\}$ is equicontinuous and hence $M_N^B(v_N(p_N(c)))$ converges pointwise. The next lemma states that $c_N(r_N(v))$ converges pointwise almost everywhere, which is sufficient for pointwise convergence of $M_N^S(c_N(r_N(v)))$:

Lemma 6 *Given a sequence $\{p_N(\cdot)\}_{N=1}^\infty$ with $p_N \in \Sigma_{p(\cdot)}$ and a sequence $\{r_N(\cdot)\}_{N=1}^\infty$ with $r_N(\cdot) \in \Sigma_{r(\cdot)}$. Suppose $p_N(c) \to \bar{p}(c)$ and $r_N(v) \to \bar{r}(v)$ pointwise for almost every c and v. Then $c_N(p) \equiv c(p|p_N(\cdot)) \to \bar{c}(p) \equiv c(p|\bar{p}(\cdot))$ a.e. and $c(r_N(v)|p_N(\cdot)) \to c(\bar{r}(v)|\bar{p}(\cdot))$ a.e..*

Proof: See Appendix.

Pool response operators K^S and K^B are integrals over functions of ω and integrals are taken with respect to measures induced by ω. To prove continuity of this operator, we need the following technical lemma which combines the idea of Lebesgue's convergence result for a sequence of functions with Helly's convergence result for a sequence of measures:

Lemma 7 *Let F_N be a sequence of c.d.f.s converging to some c.d.f. F almost everywhere and let $g_N : [0,1] \to [0,1]$ be a sequence of measurable functions converging almost everywhere to some function $g(\cdot)$. Then $\int_0^1 g_N dF_N \to \int_0^1 g dF$*

Proof: Note that $\int_0^1 g_N dF_N - \int_0^1 g dF = \int_0^1 (g_N - g)\, dF_N + \int g dF - \int g dF_N$. The second term converges to zero by Helly's convergence theorem (Kolmogorov, Fomin p.370), noting that monotone functions are of totally bounded variation. For the first term note that by Egorov's theorem (Kolmogorov, Fomin, p.290), for every $\varepsilon > 0$, g_N converges to g uniformly on a measurable subset $A \subset [0,1]$ such that the remaining set is of measure $\varepsilon/2$, i.e., $\varepsilon/2 \geq \int_{[0,1]-A} dF(x)$. By Helly's convergence theorem, $\int_{x\in[0,1]-A} dF_N(x) \to \int_{x\in[0,1]-A} dF(x)$ and thus for some N', $\varepsilon \geq \int_{x\in[0,1]-A} dF_N(x)$ which is larger than $\int_{x\in[0,1]-A} |g_N(x) - g(x)|\, dF_N(x)$ for all $N \geq N'$. By $g_N \to g$ uniformly on A, there is some N'' such that $|g_N(x) - g(x)| \leq \varepsilon$ for all $x \in A$ and $N \geq N''$. Thus, for all $N \geq \max\{N', N''\}$, $\int_0^1 (g_N - g)\, dF_N = \int_A (g_N - g)\, dF_N + \int_{[0,1]-A} (g_N - g)\, dF_N \leq 2\varepsilon$ and by ε being arbitrary the claim follows ■

From the latter two lemmas, we get

Lemma 8 *K^S and K^B are continuous.*

Proof: From Lemma 6, we know that $D(p(c)|\cdot) \equiv M^B(v(p(c)|\cdot))\, M^B(1)$ and that $S(r(v)|\cdot) \equiv M^S(c(r(v)|\cdot))\, M^S(1)$ are pointwise continuous in ω. From Lemma 7, this carries over to $K^S[\cdot]$ and $K^B[\cdot]$ ■

Theorem 1 *For every δ there exists an equilibrium σ^*.*

Proof: Monotone functions are compact in the locally convex Hausdorff space L_1 by Helly's selection principle (Kolmogorov Fomin, p. 372) and sequential compactness is sufficient for compactness in metric spaces (see Aliprantis, Border, p. 84). Monotone functions with a bounded slope as defined here form a closed subset of the monotone functions, and hence, these sets are compact. Therefore, the set $\Omega \subset L_1$ endowed with the integral norm is compact. Convexity of Ω is immediate. Together with the above Lemmas, the correspondence K satisfies the conditions of the Kakutani-Fan-Glicksberg fixed point theorem. Thus, there exists some $\omega^* \in \Omega$ such that $\omega^* \subset K[\omega^*]$.

The fixed point ω^* corresponds to a steady-state equilibrium σ^*: Let $M^* \equiv M^{S*}(1)$, $\Phi^{S*}(c) \equiv M^S(c)\, M^{*-1}$, and $\Phi^{B*}(v) \equiv \left(M^* - M^B(v)\right) M^{*-1}$.[17] The identity $M^{S*}(1) = M^{B*}(1)$ follows similar to Lemma (2), see Section A.5 in the appendix for details. Let $\bar{p}(\cdot)$ be a monotone function which is equal to $p^*(\cdot)$ whenever $p^*(c)$ is interim optimal. Since $p^*(c)$ is interim optimal almost everywhere, $\bar{p}(\cdot)$ is equivalent to $p^*(\cdot)$. For all other points $c \in [0,1)$, take the right limit, $\bar{p}(c) = \lim_{\varepsilon\to 0, \varepsilon>0} p^*(c+\varepsilon)$ which preserves monotonicity and optimality by continuity of $U^S(\cdot,\cdot)$. Let $\bar{p}(1) = \lim_{\varepsilon\to 0, \varepsilon>0} p^*(1-\varepsilon)$. Changing prices for a measure zero set of sellers does not change the distribution of price offers; so neither steady-state conditions nor buyers' optimality conditions are affected and hence, $\sigma^* \equiv \left[\bar{p}(\cdot), r^*(\cdot), \Phi^{S*}(\cdot), \Phi^{B*}(\cdot), M^*\right]$ is a steady-state equilibrium ■■

Remark 2 *Lemma 8 implies Proposition 3 together with the reasoning in the proof of the theorem: For fixed $p(\cdot)$ and $r(\cdot)$, the joint pool response $\left(K^S, K^B\right)[p(\cdot), r(\cdot), \cdot, \cdot]$ defines a function which maps the compact subset $\Sigma_{M^S} \times \Sigma_{M^B} \subset L_1$ into itself. In particular, for fixed $p(\cdot)$ and $r(\cdot)$, the pool response $\left(K^S, K^B\right)[p(\cdot), r(\cdot), \cdot, \cdot]$ is continuous in M^S and M^B. Thus, according to the Kakutani-Fan-Glicksberg theorem, there exists a fixed point,*

[17] Note that $\Phi^B(v)$ is the share of types below v while $M^B(v)$ is the mass of types above v.

i.e., some $\left(\bar{M}^B, \bar{M}^S\right)$ *such that* $\left(K^S, K^B\right)\left[p\left(\cdot\right), r\left(\cdot\right), \bar{M}^B, \bar{M}^S\right] = \left(\bar{M}^B, \bar{M}^S\right)$*. As shown in the preceding proof, if* $K^S\left[\bar{M}^B, \bar{M}^S\right] = \bar{M}^S$ *and* $K^B\left[p\left(\cdot\right), r\left(\cdot\right), \bar{M}^B, \bar{M}^S\right] = \bar{M}^B$*, then* $\left(\bar{M}, \bar{\Phi}^S, \bar{\Phi}^B\right)$ *- defined by* $\bar{M} \equiv \bar{M}^S(1)$*,* $\bar{\Phi}^S(c) \equiv \bar{M}^S(c)\,\bar{M}$*, and* $\bar{\Phi}^B(v) \equiv \bar{M}^B(v)\,\bar{M}$ *- satisfy the steady-state conditions (3.6) and (3.7) for given* $p(\cdot)$ *and* $r(\cdot)$*.*

2.5 Main Result

We want to characterize the set of equilibria with $\delta \to 0$. For this, we will look at a strictly decreasing sequence of exit rates $\{\delta_k\}_{k=1}^{\infty}$ with $\lim_{k\to\infty}\delta_k = 0$. For each δ_k, we know from the preceding section that at least one equilibrium $\sigma_k^* \in \bar{\Sigma}$ as defined in (2.16) exists. We select one equilibrium for each k, which gives us a sequence $\{\sigma_k^*\}_{k=1}^{\infty}$with elements $(p_k(\cdot), r_k(\cdot), S_k(\cdot), D_k(\cdot), M_k)$. We will show that for every such sequence the support of prices at which trade happens, shrinks to a singleton, i.e., a *law of one price* holds. This is the first theorem. Given that this *law* holds, we then show that the *one price* must be the Walrasian price p^w, which is stated in the second theorem. During this section, we often refer to the lifetime trading probabilities of a *type* and we denote these probabilities by capital letters, i.e., we define $Q^S(c|\sigma) \equiv q^S(p(c)|\sigma)$ and similarly $Q^B(v|\sigma) = q^B(r(v)|\sigma)$. For notational convenience, we abbreviate $Q^S(c|\sigma_k^*) \equiv Q_k^S(c)$ and $Q^B(v|\sigma_k^*) \equiv Q_k^B(v)$.

2.5.1 The Law of One Price

To define the support of trading prices, let h_k be the highest accepted price, $h_k \equiv r_k(1)$ and let l_k be the lowest offered price, $l_k \equiv p_k(0)$. Now we state the *law of one price*:

Theorem 2 *For every sequence of steady-state equilibria with* $h_k \equiv r_k(1)$ *and* $l_k \equiv p_k(0)$ *:*

$$\lim_{k\to\infty}(h_k - l_k) = 0.$$

We prove the theorem by contradiction: Suppose that, contrary to the theorem, there is some (sub-)sequence along which the cutoff prices h_k and l_k converge to two different limits, i.e., $\lim_{k\to\infty} h_k = h$ and $\lim_{k\to\infty} l_k = l$ with $h > l$. As said in the introduction, to sustain price dispersion in equilibrium, two opposing conditions must hold: For $r_k(1) = h_k$ to be optimal (i.e., for h_k to be *incentive compatible*), intermediate prices $p \in (l, h)$ must be *rare* so that the buyer $v = 1$ accepts $r_k(1) = h_k$ instead of waiting for better prices. For $p_k(0) = l_k$ to be optimal for sellers, intermediate prices must be offered *frequently* enough: Otherwise, buyers with intermediate valuations do not find trading opportunities and accumulate in the market. And then, a seller of type $c = 0$ would deviate from the low price l_k to offering some intermediate price. When δ_k is small, the two incentive constraints for the buyers and for the sellers cannot both be satisfied. The following three lemmas formalize this idea and together they imply the theorem.

The first lemma is based on the following intuition which starts with the observation that buyers trade only at prices below their valuations. In particular, types $v' \in [l, h)$ trade only at prices $p < h$ and thus, their expected trading price $E[p|p \le r_{k'}(v')]$ will be strictly below h. So if such a buyer v' could trade with certainty in the limit, then a buyer with a valuation $v = 1$ would be strictly better off *mimicking* v' by using this lower type's reservation price $r_{k'}(v')$. Since $\limsup r_{k'}(v') \le v' < h$, contradicting the definition of h as the limiting reservation price for $v = 1$, it has to be that trading probabilities are bounded away from one for all types $v' \in [l, h)$:

Lemma 9 *For all* $v \in [l, h)$ *there is some* $\bar{Q}^B(v) < 1$ *such that*

$$\lim_{k' \to \infty} \sup Q^B_{k'}(v) \leq \bar{Q}^B(v) < 1.$$

Proof: The payoff to $v = 1$ from mimicking some $v' \in [l, h)$ by using the reservation price $r_{k'}(v')$ are

$$\begin{aligned} U^B_{k'}\left(r_{k'}(v'), 1\right) &= Q^B_{k'}(v')\left(1 - E\left[p | p \leq r_{k'}(v')\right]\right) \\ &\geq Q^B_{k'}(v')\left(1 - v'\right), \end{aligned}$$

since $E[p|p \leq r_{k'}(v')] \leq v'$. If $Q^B_{k''}(v') \to 1$ along some subsubsequence k'', then

$$\begin{aligned} \lim_{k'' \to \infty} \inf U^B_{k''}\left(r_{k''}(v'), 1\right) &\geq 1 - v' \\ &> 1 - h \end{aligned}$$

Because $V^B_{k''}(1) \geq U^B_{k''}(r_{k''}(v'), 1)$ for all k'' by definition, the above inequality implies $\liminf V^B_{k''}(1) > 1 - h$. Thus, we have

$$\lim_{k'' \to \infty} \inf \left(1 - (1 - \delta_{k''}) V^B_{k''}(1)\right) > h$$

Recall that $h = \lim_{k''} r_{k''}(1)$. Hence, for k'' large enough, the equilibrium requirement $r_{k''}(1) = 1 - (1 - \delta_{k''}) V^B_{k''}(1)$ is violated, a contradiction ■

As trading probabilities at the very best prices are bounded away from one, some buyers also accept intermediate prices strictly above l:

Lemma 10 *For all* $v \in [l, h)$*, there is some* $\bar{r}(v) > l$ *such that*

$$\lim_{k' \to \infty} \inf r_{k'}(v) \geq \bar{r}(v) > l.$$

Proof: From the definition of reservation prices

$$\begin{aligned} r_{k'}(v) &= v - (1 - \delta_{k'}) Q^B_{k'}(v)\left(v - E\left[p | p \leq r_{k'}(v)\right]\right) \\ &\geq l_{k'} + \left(1 - Q^B_{k'}\right)(v)\left(v - l_{k'}\right), \end{aligned}$$

where we used that $E[p|p \leq r_{k'}(v)] \geq l_{k'}$ to derive the second line. Thus,

$$\begin{aligned} \lim_{k' \to \infty} \inf r_{k'}(v) &\geq l + \liminf \left(1 - Q^B_{k'}(v)\right)(v - l) \\ &\geq l + \left(1 - \bar{Q}(v)\right)(v - l) > l \end{aligned}$$ ■

The distribution of types in the pool is proportional to their probability of not trading: Buyers who trade less frequently make up a larger share, i.e., $\Phi^B(v)$ depends positively on $\left(1 - q^B(r(v))\right)$. Rewriting the steady-state condition by substituting $S(r(v))$ by

$q^B(r(v))$ shows[18]

$$\Phi^B(v) = \int_0^v \frac{1 - q^B(r(\tau)) + \delta q^B(r(\tau))}{M\delta} dG^B(\tau),$$

and by $\delta M \leq 1$ we get as lower bound

$$\Phi^B(v) \geq \int_0^v \left[1 - q^B(r(\tau))\right] dG^B(\tau). \tag{2.19}$$

Using our knowledge about $Q^B(v)$ for $v \in [l, h)$ from the last lemma this implies

Lemma 11 *For all $v \in [l, h)$, there is some $\bar{D}(v) > 0$ such that*

$$\lim_{k' \to \infty} \inf \left(1 - \Phi^B_{k'}(v)\right) \geq \bar{D}(v) > 0.$$

Proof: Take any $v' \in [l, h)$ and $\varepsilon \in (0, h - v')$. Then

$$1 - \Phi^B_{k'}(v') \geq \int_{v'}^{v'+\varepsilon} d\Phi^B_{k'}(v), \tag{2.20}$$

and, by (2.19), we have $d\Phi^B_{k'}(v) \geq \left[1 - Q^B_{k'}(v)\right] g^B(v)$ a.e. Let $\bar{Q}$ be the bound on $Q^B_{k'}(\cdot)$ for the type $v' + \varepsilon$ from the first lemma, $\bar{Q} \equiv \bar{Q}^B(v' + \varepsilon) < 1$. With monotonicity and $g^B(v) \geq g_l$ we can bound the limit of $d\Phi^B_{k'}(v)$:

$$\lim_{k' \to \infty} \inf d\Phi^B_{k'}(v) \geq \left[1 - \bar{Q}\right] g_l \quad \forall v \leq v' + \varepsilon,$$

and thus

$$\lim_{k' \to \infty} \inf \left[1 - \Phi^B_{k'}(v')\right] \geq \int_{v'}^{v'+\varepsilon} \left[1 - \bar{Q}\right] g_l = \varepsilon \left[1 - \bar{Q}\right] g_l > 0,$$

so with $\bar{D}(v') \equiv \varepsilon \left[1 - \bar{Q}\right] g_l$ we have proven the lemma ■

Now, we connect to three lemmas to prove the theorem:

Proof of Theorem 2: By contradiction. If the theorem does not hold, then there is some subsequence, indexed by k', such that $\lim_{k' \to \infty} l_{k'} = l < \lim_{k' \to \infty} h_{k'} = h$.[19] Then Lemma 10 and Lemma 11 imply that sellers would want to deviate from $p_{k'}(0) = l_{k'}$ to some intermediate $p' \in (l, h)$ for k' large enough: Take any $v' \in [l, h)$ and choose any $p' \in (l, \bar{r}(v'))$ with $\bar{r}(v')$ as defined in Lemma 10. Then for some K_1 large enough, we have $r_{k'}(v') \geq p'$ for all $k' \geq K_1$ by Lemma 10. The probability of trading at p' is

$$D_{k'}(p') \geq 1 - \Phi^B_{k'}(v') \quad \text{for } k' \geq K_1,$$

and thus by definition

$$q^S_{k'}(p') \geq \frac{1 - \Phi^B_{k'}(v')}{1 - \left(1 - \left(1 - \Phi^B_{k'}(v')\right)\right)(1 - \delta_{k'})},$$

[18] The definition of $q^B(\cdot)$ implies $\frac{q^B(r(v))}{S(r(v))} = \frac{1}{S(r(v)) + \delta - \delta S(r(v))}$ and $S(r(v)) = \frac{\delta q^B(r(v))}{\left(1 - q^B(r(v)) + \delta q^B(r(v))\right)}$. Together, $\frac{1}{S(r(v)) + \delta - \delta S(r(v))} = \frac{q^B(r(v))\left(1 - q^B(r(v)) + \delta q^B(r(v))\right)}{\delta q^B(r(v))}$.

[19] If $\lim(h_k - l_k) \neq 0$, then for some subsequence k' and some $C > 0$, $(h_{k'} - l_{k'}) \to C$. According to the Bolzano-Weierstrass theorem, there is some convergent subsubsequence k'' of the cutoff prices. With $h \equiv \lim_{k'' \to \infty} h_{k''}$, $\lim_{k'' \to \infty} l_{k''} = h - C$, from the hypothesis above.

and by Lemma 9 and Lemma 11:

$$\lim_{k'\to\infty} \inf q_{k'}^{S}\left(p'\right) \geq \frac{1-\varepsilon\left[1-\bar{Q}\right]g_l}{1-\left(1-\varepsilon\left[1-\bar{Q}\right]g_l\right)} = 1.$$

So the limiting payoffs at the price p' are

$$\lim_{k'\to\infty} U_{k'}^{S}\left(p',0\right) = p' > l \geq \lim_{k'\to\infty} \sup U_{k'}^{S}\left(l_{k'},0\right),$$

contradicting $l_{k'} \in \arg\max_p U_{k'}^{S}(p,0)$ for k' large enough ■■

2.5.2 Convergence to the Walrasian Price

Theorem 3 *For every sequence of steady-state equilibria, prices converge to the Walrasian Price:*

$$\lim_{k\to\infty} p_k(c) = p^w \ \forall c < p^w \quad \text{and} \quad \lim_{k\to\infty} r_k(v) = p^w \ \forall v > p^w$$

With h_k being the highest accepted price in equilibrium no seller of type $c < h_k$ offers a price $p > h_k$: Such an offer would yield a profit of zero while any price below h_k yields strictly positive profits. So $p_k(c) \in [p_k(0), h_k]$ and by $p_k(0) = l_k \to h_k$, it is sufficient for proving the theorem to prove that $\lim_{k\to\infty} h_k = p^w$. To do so, we take some convergent subsequence of $\{h_k\}_{k=1}^{\infty}$, indexed by k', with $\lim_{k'\to\infty} h_{k'} = p^c$. First, we show that all sellers with costs below p^c must be able to trade for sure in the limit. Second, all buyers with valuations above p^c must be able to trade for sure in the limit. Finally, the only price at which *all* buyers with valuations above p^c and *all* sellers with costs below p^c can trade is p^w. Therefore, it must be that the limit p^c is equal to p^w for every convergent subsequence. And thus, p^w must be the limit for the sequence itself.[20]

For the first lemma, observe that along this subsequence, for any $p' < p^c$, buyers with $v \in (p', l_k)$ do not trade but accumulate in the market (this and the following statements are trivial if $p^c = 0$). Hence, they have a strictly positive share in the pool and they accept a price $p' \leq v$. Therefore, with $\delta_{k'} \to 0$, a seller offering any p' below p^c can be sure to trade in the limit, i.e., $q_{k'}^{S}(p') \to 1 \ \forall p' < p^c$. Thus, the trading probability at the equilibrium price $p_k(c) \cong p^c$ must converge to one as well and $Q_{k'}^{S}(c) \to 1$ for all $c < p^c$. For prices p'' strictly above p^c, even the highest reservation price is below p'' when k' is large enough, since $\lim r_{k'}(1) = p^c < p''$. Therefore, $\lim\sup q_{k'}^{S}(p'') = 0$ for all $p'' > p^c$, which implies that types $c > p^c$ cannot trade and $Q_{k'}^{S}(c) \to 0$ for all $c > p^c$. Together:

Lemma 12 *For every convergent subsequence $\{h_{k'}\}_{k'=1}^{\infty}$ with $\lim_{k'\to\infty} h_{k'} = p^c$:*

$$\lim_{k'\to\infty} Q_{k'}^{S}(c) = 1 \ \forall c < p^c \quad \text{and} \quad \lim_{k'\to\infty} Q_{k'}^{S}(c) = 0 \ \forall c > p^c.$$

Similarly, we can show that the trading probabilities of buyers with valuations $v > p^c$ must converge to one. If not, some buyers would be willing to accept prices strictly above p^c, contradicting the definition of $h_{k'}$:

[20] By standard reasoning: If $\lim_{k\to\infty} p_k \neq p^w$, then there is some $\varepsilon > 0$ and some subsequence such that $(p_{k'} - p^w) \geq \varepsilon$. Take any converging subsubsequence k''. By the reasoning before, its limit $\lim_{k''} p_{k''}$ is p^w. Contradiction!

Lemma 13 *For every convergent subsequence with* $\{h_{k'}\}_{k'=1}^{\infty}$, $\lim h_{k'} = p^c$*:*

$$\lim_{k'\to\infty} Q_{k'}^B(v) = 1 \qquad \forall v > p^c \qquad \textit{and} \qquad \lim_{k'\to\infty} Q_{k'}^B(v) = 0 \qquad \forall v < p^c.$$

Proof: From $l_{k'} \to p^c$, the trading probability at all $r' < p^c$ is zero for k' large enough. Therefore, for all $v > p^c$, $\liminf r_{k'}(v) \geq p^c$. From its definition, $r_{k'}(v) \leq h_{k'}$ so $\lim_{k'\to\infty} r_{k'}(v) = p^c$ for all $v > p^c$. Together with $p_{k'}(c) \geq l_{k'}$, $E[p|p \leq r_{k'}(v)] \to p^c$ for all $v > p^c$. Recall the definition of $r_{k'}(v)$:

$$r_{k'}(v) = v - (1-\delta_{k'}) Q_{k'}^B(v)(v - E[p|p \leq r_{k'}(v)]),$$

and note that $r_{k'}(v) \to p^c$ if and only if $Q_{k'}^B(v) \to 1$ because:

$$\begin{aligned}\lim_{k'\to\infty} r_{k'}(v) &= \lim_{k'\to\infty} v - (1-\delta_{k'}) Q_{k'}^B(v)(v - E[p|p \leq r_{k'}(v)]) \\ &= v - p^c - v \lim_{k'\to\infty} Q_{k'}^B(v),\end{aligned}$$

and therefore

$$\lim_{k'\to\infty} r_{k'}(v) = p^c \Leftrightarrow \lim_{k'\to\infty} Q_{k'}^B(v) = 1 \qquad \blacksquare$$

Together, the two lemmas imply the theorem:

Proof of Theorem 3: Because trading probabilities of sellers and buyers in the limit are given by the step functions $1_{c\leq p^w}$ and $1_{v\geq p^w}$ we know the mass of players who trade in the limit:

$$\begin{aligned}\lim_{k'\to\infty} \int_0^1 Q_{k'}^S(c) g^S(c) &= \int_0^1 1_{c\leq p^w} g^S(c) = G^S(p^c) \\ \lim_{k'\to\infty} \int_0^1 Q_{k'}^B(v) g^B(v) &= \int_0^1 1_{v\geq p^w} g^B(v) = 1 - G^B(p^c).\end{aligned}$$

In every equilibrium σ_k^*, the mass of buyers who trade must be equal to the mass of sellers who trade, as shown in Lemma 2. Therefore, $\int_0^1 Q_{k'}^S(c) g^S(c) = \int_0^1 Q_{k'}^B(v) g^B(v)$ for all k, and thus,

$$\lim_{k'\to\infty} \int_0^1 Q_{k'}^S(c) g^S(c) = \lim_{k'\to\infty} \int_0^1 Q_{k'}^B(v) g^B(v),$$

which implies $G^S(p^c) = 1 - G^B(p^c)$ at p^c. The unique price which satisfies this is p^w and hence all sequences h_k and l_k converge to p^w $\blacksquare\blacksquare$

An immediate corollary of Lemma 12 and 13 is this

Corollary 1 *For every sequence of steady-state equilibria, the outcome converge to the* Walrasian allocation*, i.e.,*

$$\begin{aligned}\lim_{k'\to\infty} Q_{k'}^S(c) &= 1 \quad \forall c < p^c \quad \textit{and} \quad \lim_{k'\to\infty} Q_{k'}^S(c) = 0 \quad \forall c > p^c, \\ \lim_{k'\to\infty} Q_{k'}^B(v) &= 1 \quad \forall v > p^c \quad \textit{and} \quad \lim_{k'\to\infty} Q_{k'}^B(v) = 0 \quad \forall v < p^c.\end{aligned}$$

2.6 Discussion

2.6.1 Existing Literature and Other Market Clearing Forces

We know that the incentives to reach out for additional buyers is all that is needed to guarantee the efficient outcome.[21] In the existing literature, one can find two assumption that give sellers additional incentives to decrease their prices. In the main strand of the literature[22], within each pair *both* sides of the market have a chance to make an offer. In recent models only sellers can make the offer but in addition, buyers have the chance to simultaneously receive several offers from *competing sellers*.[23]

For illustration, take the model with homogeneous sellers where the market clearing price is zero. Suppose that sellers would set a common price $p' > 0$ even for small δ. At this price, not all sellers will be able to trade so their ultimate trading probability is bounded away from one. This implies that their profits are strictly smaller than p'. Now consider a model with a positive chance that a seller competes directly against the offer of another seller. Then we have additional pressure on prices: given the common price level p', any incremental decrease below p' increases the trading probability strictly by undercutting rivals' prices. And because profits are strictly below p', this is profitable. Again, assuming all sellers offer p', consider a model in which buyers can make offers with some probability themselves. First, note that a seller would accept a price offer p'' from a buyer even if it is considerably less than p' in order to avoid rationing. Therefore, buyers have the possibility to trade at a price p'' in the future when it is their turn to propose, and given δ is close enough to zero, they become certain that they can do so. This makes them unwilling to accept an offer p' from the seller. Therefore, sellers will decrease their price offer to make it acceptable.[24]

So we can distinguish three forces towards the competitive price level, the incentive to reach out for additional buyers analyzed here, the incentive to undercut the competitors, and the better outside option for buyers if they have some bargaining power. Rationing on the sellers' side is their common starting point. But there is an important qualitative difference between the three forces: While the existence of additional buyers at lower prices is a basic feature shared by most markets, the possibility of directly competing offers or the distribution of bargaining power between traders depends on the fine details of the situation and of the model. By showing that the convergence result does not depend on these latter details, we provide evidence for the robustness of the prediction that decentralized trading is efficient.

When modelling the evolution of the pool of traders we follow McAfee (1993) and assume that there is some death rate, letting the rate converge to zero. This seems a natural condition when analyzing steady states. The main alternative would be to follow

[21] Strictly speaking, asymmetric information is also needed as demonstrated in chapter one. However, asymmetric information might be considered so omnipresent in situations of economic transactions as to be part of every decentralized market.

[22] Mortensen (1982), Rubinstein&Wolinsky (1985) and Gale (1986, 1987) initiated the analysis for complete information. Serrano (2002), Moreno&Wooders (2002) and Inderst (2001) extended it towards incomplete information.

[23] See Satterthwaite and Shneyerov (2004, 2005). Though without convergence results, this structure can be found in the literature on nosiy search, e.g., in Burdett and Judd (1983) .

[24] Of course, in both kinds of models it has to be proven that there is no price dispersion. Our proof of the *law of one price* can be applied to both situations to yield this conclusion.

Gale (1987) and later authors who assume that traders are literally infinitely lived. With this assumptions, agents can leave the market only through trading and agents who do not trade would accumulate in the market and have mass infinity. Therefore, to ensure the existence of a steady state with finite masses, these authors include an entry decision. This assumption of infinitely lived agents has direct implications for the set of possible equilibria by introducing a sort of *zero profit condition* for sellers: To ensure a steady state, the inflows of buyers and sellers must be identical with infinitely lived agents. In addition, all traders who decide to enter the market must trade at some point. Now there are two possible equilibrium scenarios: Either all buyers enter, including those with zero valuation. Because even these buyers must be able to trade, sellers must offer prices close to zero. Since sellers would not do so otherwise, this requires that sellers earn zero profits. In the other case, not all buyers enter but only a mass strictly below one. Then some of the sellers must also choose to stay out of the market to balance the inflows of sellers and buyers. But sellers will stay out only if their equilibrium profits are zero. So for both cases the seemingly minor assumption of infinitely lived agents implies immediately that sellers must make zero profits. Note well, that this observation is independent of any further strategic considerations and it is true even away from the limit for *all* levels of frictions. Models with infinitely lived agents can therefore not include the idea that frictions allow traders to enjoy market power in a decentralized market and that therefore trading is inefficient unless the market becomes frictionless.[25] Formally, with infinitely lived agents, the analogue of Proposition 3 does not hold, so that a steady state exists only for some strategies. This restriction on strategies turns out to be a force towards market clearance.

With infinitely lived agents, the inclusion of an entry decision is necessary for technical reasons. We can dispense with it here. If, however, we would include such a decision, we could have sustained multiple equilibria:[26] For example, there will always be a *trivial* equilibrium in which no trader enters and no trade takes place. In addition, for δ small enough there will be equilibria in which only buyers above some arbitrary threshold $p^c \in (0,1)$ enter: Given that only such buyers are available in the pool, sellers would have no incentives to decrease their price below such a threshold p^c since they cannot increase their revenue by selling to addition buyers. Such an equilibrium, however, might be considered unstable because it relies on the literal impossibility of sellers to reach those inactive buyers with valuations $v < p^c$ who accumulate outside the market. If sellers could for example *advertise* their prices at some cost k per ad to buyers and $k \to 0$ we could restore the convergence result.

Finally, our model might be considered as going back to the very roots of the dynamic matching and bargaining literature. Already in the seventies and long before Wolinsky and Rubinstein (1985) published their seminal paper on dynamic matching and bargaining, Gerard Butters (1977) began working on a model where only sellers make price offers and other market clearing forces are absent. He also allowed for non-stationary inflows, a more general matching technique and two-sided heterogeneity. Thus, relative to later

[25] In the same spirit, Satterthwaite and Shneyerov (2006, footnote 3) note that their companion paper, that features positive exit rates, also "[...] demonstrates more clearly [than a model without an exit rate] the power of supply and demand to force price to converge to p^w."

[26] Note, that we would need to change the matching technology to accomodate the possibility that the masses of entering traders on each side of the market are not identical. A simple way would be to follow Gale (1987) and to assume that matching probabilities are proportional to shares among *all* traders

authors building upon his unfinished typescript, his analysis was much more ambitious. But it remained cumbersome and was not published. Although we have not reached the level of generality envisioned by Butters yet, this note might be a step in the direction of the analysis he had in mind.

2.6.2 Conclusion

In our analysis, we checked the robustness of the market clearing hypothesis and the underlying intuition. We were able to prove asymptotic efficiency of decentralized trading by appealing to the basic economic forces of rationing of traders at non-market clearing prices. We have shown that with homogeneous sellers the optimality condition for pricing looks like the familiar Lerner formula, i.e., the mark up of prices over costs is proportional to the inverse elasticity of demand. Different to the static case, costs of trading for sellers include foregone future profits, and the elasticity of demand includes the possibility of intertemporal substitution. When frictions become small, we have seen that the increasing elasticity of demand implies that prices must converge to their competitive level.

Nevo and Hendel (2006) have shown in a recent study of demand for laundry detergents that indeed the dynamic elasticity of demand is much higher than the static elasticity. They point out the relevance of this difference for example when using the Lerner formula for policy problems like merger analysis: They warn that if firms set prices relative to the static demand elasticity, then predictions of the mark-up that use estimates of the dynamic elasticity might be misleading. In our model, however, firms actually use the dynamic elasticity. Although we look at the extreme case of a many competing firms, our result suggests that oligopolistic pricing decisions might incorporate the dynamic elasticity as well.

Clearly, open questions about decentralized markets remain: In chapter two and in Merzyn (2006), we analyze a dynamic matching and bargaining game in which we relax the assumption that traders know the aggregate *supply.* We characterize the equilibrium behavior of buyers who have to learn about the state of the market. It would be interesting to include such aggregate uncertainty into the present model, in order to analyze whether decentralized trading becomes efficient even in this case. Another issue is raised by Gale (2000), who suggests that a complete model of decentralized trading should also include the problem of coordination across markets (and time) for different products, e.g., between the market for labor and the market for consumption goods. This problem is central to a market economy. The simple dynamic matching and bargaining game that was analysed in this paper might provide a useful framework for this question.

3 A General Approach to Decentralized Markets

Dynamic matching and bargaining games provide models of decentralized markets with trading frictions. A central objective of the literature is to investigate how equilibrium outcomes depend on the size of the frictions. In particular, will the outcome become efficient when frictions become small? Existing specifications of such games give different answers. To investigate what causes these differences, we identify four simple conditions on trading outcomes. We show that for every game which satisfies these conditions, the equilibrium outcome must become efficient when frictions are small. We demonstrate that our conditions hold under several specifications in the literature, suggesting a common cause behind their convergence results. These specifications include, for example, the recent contribution by Satterthwaite and Shneyerov (Econometrica, forthcoming.) For those specifications in the literature for which outcomes do not become efficient, we show exactly which of our conditions do not hold. These specifications include, for example, Serrano (2002, JME) and DeFraja and Sakovics (2001, JPE).

3.1 Introduction

In a dynamic matching and bargaining game, a large population of traders interacts repeatedly in a decentralized market. Every trading period, traders are *matched* to form small groups where they *bargain* over the terms of trade. If they fail to reach an agreement, at some cost they can wait until the next period to be rematched into a new group. These waiting costs are the *frictions* of trading in the decentralized market. A major question in the literature concerns the trading outcome when frictions become small: Will the outcome become efficient? Ideally, one would like not only to find answers for particular trading institutions but also to gain a general understanding of the conditions under which trading with vanishing frictions has this property and the conditions under which it does not. This is the task of this paper. Its primary goal is to provide a general, "detail free" framework for the analysis of decentralized markets. Recent contributions that fall into the framework of this paper include papers by Moreno and Wooders (2001), Satterthwaite and Shneyerov (2007), and De Fraja and Sakovics (2001).

As a basic setup we use the following dynamic matching and bargaining environment, similar to the one used by Gale (1987):[27] There is a continuum of buyers who have unit demand and valuations $v \in [0, 1]$ for an indivisible good, and there is a continuum of sellers who have unit capacity and costs $c \in [0, 1]$. These traders are matched into small groups. In these groups they bargain, and if they reach an agreement, they trade. The groups are connected to form a large market by allowing unsuccessful traders to be matched into new groups in the next period. Integration, however, is imperfect because there is a probability $\delta \in (0, 1)$ that a trader will die while waiting. These are the *frictions* of trading. Finally, at the end of each period, there is an exogeneous inflow of new buyers and sellers.

This framework is general with respect to both the matching technology and the bargaining protocol, i.e., we do not specify how traders are matched into groups. Also,

[27] The main difference is our assumption that traders have a finite life expectancy; see Section 3.6.2 for the case of infinitely lived traders.

we do not specify how bargaining within the groups takes place and what information is released before and during bargaining. We will see how existing models in the literature differ in how they fill in these details. But no matter how this is done, every specification of the model will give rise to an *outcome* which consists of (a) probabilities of trading for entering types and (b) expected equilibrium payoffs. Let $Q^S(c)$ denote the probability that a seller of type c sells his good, and let $Q^B(v)$ denote the probability that a buyer of type v gets the good. Similarly, let $V^S(c)$ and $V^B(v)$ be the payoffs to these types. Taken together, an outcome is a vector $A = \left[Q^S, Q^B, V^S, V^B\right]$.

Now, suppose there is some sequence of exit rates $\{\delta_k\}_{k=1}^{\infty}$, which converges to zero, $\delta_k \to 0$. In addition, suppose for each δ_k, we take an equilibrium outcome of a specific trading game. This gives us a sequence of outcomes $\{A_k\}_{k=1}^{\infty}$, with $A_k = \left[Q_k^S, Q_k^B, V_k^S, V_k^B\right]$. We state four conditions on this sequence that will ensure that its limit is efficient. The first condition, *monotonicity*, requires that trading probabilities are monotone, i.e., buyers with higher valuations are more likely to trade while sellers with higher costs are less likely to trade. The second condition, *no rent extraction*, requires that traders receive some part of the surplus that they generate. Technically, this is a condition on the slope of the payoffs. The third condition, *availability*, requires that a trader is matched frequently with those traders who do not trade with certainty and who remain in the pool for many periods. These traders are said to be available. The fourth condition, *weak pairwise efficiency*, requires that for all pairs of buyers and sellers who are available, i.e., for all pairs who are matched frequently, the joint surplus is at least their private surplus and $V^S(c) + V^B(v) \geq (v - c)$. Note that the third condition relates to the matching technology, while the other conditions relate to the bargaining protocol. As we will see, these conditions hold for a wide range of models.

Next, let $S(A_k)$ be the surplus of an outcome A_k, and let S^* be its maximum over the set of all outcomes satisfying a mass balance condition (see Section 3.3.2). Our main result is this: Every sequence of outcomes $\{A_k\}_{k=1}^{\infty}$ which satisfies the four conditions becomes efficient when δ becomes small, i.e.,

$$\lim_{k \to \infty} S(A_k) = S^*.$$

In the second part of the paper, we discuss specific dynamic matching and bargaining games. In particular, we use these games to discuss the economic meaning of our conditions. To keep this part consistent, we introduce the games by varying a *basic model* which we take from chapter two. This model is particularly simple: Groups consist of just one seller and one buyer. Bargaining takes place by the seller making a price offer to the buyer, which the buyer can either accept or reject. The seller cannot observe the valuation of the buyer, i.e., information is asymmetric in the basic model. The setup of this model is introduced in the next section - and before the general framework - to allow the reader to familiarize himself with the environment.

We introduce the first variant of our model to show that our conditions do not only hold with asymmetric information, but also in a model similar to Douglas Gale's own specification with *symmetric* information. We then provide some intuition that the analysis is not confined to steady states by considering the non-steady-state model of Moreno and

Wooders (2001), where traders enter only in the first period. We also provide some intuition for the case of bargaining between one seller and many buyers, assuming that the seller holds a second price auction.

Auctions are also used to specify the bargaining protocol in the model of Satterthwaite and Shneyerov (2007). But, like Gale (1987), they include an entry stage and assume infinitely lived traders. To show that our approach is also valid with these additional complications, we extend our general framework by including an entry decision and by considering the case where the exit rate δ is set equal to zero. In this new framework, we need stronger conditions to ensure convergence to efficiency. This is discussed in detail in Section 3.6.

Whenever convergence to efficiency fails in some model, at least one of our conditions must be violated. By pointing out exactly which conditions are violated, we show which assumptions of the model are the reasons for the non-convergence results. In particular, we show that the failure in chapter one can be traced back to rent extraction (a failure of the second condition), the failure in Serrano (2002), to the failure of weak efficiency (the fourth condition), and the failure in De Fraja and Sakovics (2001), to a failure of a fundamental mass balance condition.

The rest of the paper is structured as follows. First, we introduce the basic model as an example in Section 3.2. Then we provide the general framework in Section 3.3.1, and in Section 3.3.2 we discuss necessary and sufficient conditions for outcomes to be efficient. In Section 3.3.3 we introduce the four conditions on outcomes. We prove our main result in Section 3.4: every sequence of outcomes that meets the four conditions becomes efficient. In Section 3.5 we demonstrate that the conditions are met in some examples. We introduce some variations of the general framework by adding an entry stage (Section 3.6.1) and assuming that traders are infinitely lived (Section 3.6.2). Failures of convergence to efficiency are discussed in Section 3.7.

3.2 The Basic Model - An Example of a Dynamic Matching and Bargaining Game

In this section, we introduce a specific dynamic matching and bargaining game. We will use this game to motivate and illustrate the general framework.

We assume that there is a continuum of buyers and sellers who interact in a repeated market over an infinite number of periods, with time running from minus to plus infinity. Sellers have one unit of an indivisible good, and their costs of trading are $c \in [0,1]$. Buyers want to acquire one unit of the good, and their valuation for the good is $v \in [0,1]$. At the beginning of each period, there is some pool of buyers and sellers. The traders from this pool are matched into pairs consisting of one seller and one buyer. Within each pair, the seller announces a price offer $p \in [0,1]$ and the buyer announces whether he rejects or accepts the offer. If he accepts, the seller receives $p - c$, while the buyer receives $v - p$. Next, all buyers and sellers who have traded exit the pool. Likewise, a share δ of all those traders who failed to trade exits. Finally, new players enter the market and the period ends. The next period starts, using the same rules. We will look at the steady-state equilibrium of this market.

The inflow of buyers and the inflow of sellers each have mass one. The distribution of valuations among buyers in the inflow is exogeneously given by some c.d.f. $G^B(\cdot)$ and, similarly, the distribution of costs is given by some distribution $G^S(\cdot)$. We assume that $G^B(\cdot)$ and $G^S(\cdot)$ have continuous and strictly positive densities. Let p^w be the price at which the mass of sellers in the inflow with costs below p^w is exactly equal to the mass of buyers with valuations above p^w:

$$G^S(p^w) = 1 - G^B(p^w). \tag{3.1}$$

Since the left hand side is strictly increasing while the right hand side is strictly decreasing, the solution to (3.1) is unique. The function $G^S(\cdot)$ can be interpreted as *supply,* and $1 - G^B(p^w)$ can be interpreted as *demand.* So p^w is the price at which supply equals demand, i.e., p^w is the *Walrasian* market clearing price relative to the inflow.

The market constellation is given by a vector $\sigma = \left[p(\cdot), r(\cdot), \Phi^S(\cdot), \Phi^B(\cdot), M\right]$, where $p(c) \in [0,1]$ is the price offered by a seller of type c, $r(v) \in [0,1]$ is the highest price accepted by a buyer of type v, $\Phi^S(\cdot)$ is the cumulative distribution function of costs in the pool of sellers, $\Phi^B(\cdot)$ is the corresponding distribution function for buyers, and M is the total mass of buyers in the pool, which is equal to the total mass of sellers in a steady state. For the analysis, we assume that all functions under consideration are measurable. With Σ_M being the set of measurable functions $f : [0,1] \to [0,1]$, σ is an element of $\Sigma \equiv \Sigma_M^4 \times \mathbb{R}$.

We say that a vector σ constitutes an *equilibrium* if strategies are mutually optimal given the distribution of types and if the distribution of types in the pool is consistent with the trading strategies and the exogeneous inflow. These conditions are now spelled out in detail.

First we turn to the sellers. If the seller offers a price p, let us denote by $D(p|\sigma)$ the probability that the buyer will accept the offer. Buyers accept a price p if $p \leq r(v)$ (see below), so $D(p|\sigma)$ is

$$D(p|\sigma) \equiv \int_{\{v|p\leq r(v)\}} d\Phi^B(v) \ . \tag{3.2}$$

Let $q^S(p|\sigma,\delta)$ be the probability that a seller can trade some time during his lifetime

$$q^S(p|\sigma,\delta) \equiv \frac{D(p|\sigma)}{1-(1-D(p|\sigma))(1-\delta)}, \tag{3.3}$$

which we also call the *lifetime trading probability*, and which is derived from the recursive formula

$$q^S(p|\sigma,\delta) = D(p|\sigma) + (1-D(p|\sigma))(1-\delta)\,q^S(p|\sigma,\delta).$$

The expected payoff to a seller when offering a price p is his trading probability times his profit, i.e.,

$$U^S(p,c|\sigma,\delta) \equiv q^S(p|\sigma,\delta)(p-c),$$

and we require that $p(c) \in \arg\max U^S(\cdot,c|\sigma,\delta)$ for all c in equilibrium.

For buyers, let $S(r|\sigma)$ denote the probability of receiving an offer $p \leq r$ in any period.

Again, we define the lifetime trading probability by

$$q^B(r|\sigma,\delta) \equiv \frac{S(r|\sigma)}{1-(1-S(r|\sigma))(1-\delta)}.$$

The expected price offer conditional on $p \leq r$ is denoted by $E[p|p \leq r,\sigma]$.[28] Payoffs when accepting all $p \leq r$ are given by

$$U^B(r,v|\sigma,\delta) \equiv q^B(r|\sigma,\delta)(v - E[p|p \leq r,\sigma]). \tag{3.4}$$

Let $V^B(v|\sigma) \equiv \sup_r U^B(r,v|\sigma,\delta)$. Suppose that the following condition holds

$$r(v) = v - (1-\delta)V^B(v|\sigma,\delta). \tag{3.5}$$

Then a buyer who receives an offer $p = r(v)$ is just indifferent about accepting and rejecting the offer: his payoff if accepting the offer, $v - r(v)$, is equal to his payoff if rejecting it, which is the continuation payoff $(1-\delta)V^B(v|\sigma,\delta)$. If $p < r(v)$, the buyer is strictly better off when accepting the offer, and when $p > r(v)$, the buyer is strictly better off rejecting the offer. Hence, it is optimal for a buyer to accept a price if it is below $r(v)$ and to reject the price otherwise.[29]

We restrict attention to steady-state equilibria in which the pool does not change over time. If the distribution at the beginning of a period is given by $\Phi_t^S(\cdot)$ and the trading strategies are $r(\cdot)$ and $p(\cdot)$, then the distribution of sellers at the end of the period is the sum of the entering and the initial sellers who have neither traded nor died:

$$\Phi_{t+1}^S(c|\sigma) = G^S(c) + (1-\delta)\int_0^c (1 - D(p(\tau)))\,d\Phi_t^S(\tau).$$

The pool of traders is in a steady state if and only if the distribution of types does not change over time. For sellers it is necessary that $\Phi_{t+1}^S(c|\sigma) = \Phi_t^S(c) = \Phi^S(c)$ for all c. This condition can be written as[30]

$$\Phi^S(c) = \int_0^c \frac{dG^S(\tau)}{M(D(p(\tau)|\sigma) + \delta(1 - D(p(\tau)|\sigma)))} \quad \text{for all } c. \tag{3.6}$$

A similar condition can be obtained for buyers:

$$\Phi^B(v) = \int_0^v \frac{dG^B(\tau)}{M(S(r(\tau)|\sigma) + \delta(1 - S(r(\tau)|\sigma)))} \quad \text{for all } v. \tag{3.7}$$

Summing up, we say σ^* is an equilibrium if it satisfies the above conditions:

Definition 2 *A steady-state equilibrium vector $\sigma^* \in \Sigma$ consists of an optimal pair of strategies and a corresponding steady-state pool, i.e., σ^* is a vector $[p(\cdot), r(\cdot), \Phi^S(\cdot), \Phi^B(\cdot), M]$ for which*

[28] If $Q^B(r) = 0$, then $E[p|p \leq r] \equiv r$.

[29] In general, reservation price strategies are the unique optimal sequentially rational strategies when sampling without recall from a known stationary distribution of prices; see McMillan and Rothschild (1994). In our model, we nevertheless simply *assume* that buyers use reservation prices to avoid unnecessary notation.

[30] We get this by an algebraic manipulation of $\Phi^S(c) = \Phi_{t+1}^S(c)$, by observing that for all c, $\int_0^c d\Phi^S(\tau) = \int_0^c dG^S(\tau) + \int_0^c (1-\delta)(1-D(p(\tau)))\,d\Phi^S(c)$ and then $\int_0^c d\Phi^S(\tau) - \frac{dG^S(\tau)}{1-(1-\delta)(1-D(p(\tau)))} = 0$.

- $p(c) \in \arg\max U^S(p, c|\sigma^*, \delta)$ *for all c*
- $r(v) = v - (1-\delta) V^B(v|\sigma^*, \delta)$ *for all v*
- $\Phi^S(\cdot)$, $\Phi^B(\cdot)$, *and M satisfy the steady-state conditions (3.6), (3.7).*

Every equilibrium σ^* and exit rate δ, gives rise to a trading *outcome* as follows. Let $V^S(c|\sigma^*, \delta) \equiv U^S(p(c), c|\sigma^*, \delta)$ and $V^B(v|\sigma^*, \delta) \equiv U^B(r(v), v|\sigma^*, \delta)$ be the equilibrium payoffs and let $Q^S(c|\sigma^*, \delta) = q^S(p(c)|\sigma^*, \delta)$ and $Q^B(v|\sigma^*, \delta) = q^B(r(v)|\sigma^*, \delta)$ be the equilibrium trading probabilities. Then the outcome $A = [V^S, V^B, Q^S, Q^B]$ of the equilibrium σ^* is given by the mapping $A(\cdot,\cdot) : \Sigma \times [0,1] \to \Sigma_M^4$, i.e., $A(\sigma^*, \delta) = [V^S(\cdot|\sigma^*, \delta), V^B(\cdot|\sigma^*, \delta), Q^S(\cdot|\sigma^*, \delta), Q^B(\cdot|\sigma^*, \delta)]$. As we will see in Section 3.5, outcomes can be identified across a wide range of specifications. Therefore, we now turn to a general discussion of outcomes.

3.3 The General Approach

For the general approach, we introduce the basic notation and make some preliminary observations in the first subsection. In the following subsection, we show that outcomes are efficient if they are "Walrasian," and we derive a sufficient condition for efficiency. Finally, we introduce the four conditions that we want to use to characterize outcomes.

3.3.1 Outcomes

An *outcome* is a vector $A = [V^S(\cdot), V^B(\cdot), Q^S(\cdot), Q^B(\cdot)]$, where $V^S(c)$ is the expected payoff of an entering seller of type c and $Q^S(c)$ is his (lifetime) trading probability. Similarly, $V^B(v)$ is the expected payoff of an entering buyer of type v and $Q^B(v)$ is his (lifetime) trading probability. We define $T^S(\cdot)$ and $T^B(\cdot)$ implicitly by

$$V^S(c) = T^S(c) - c\,Q^S(c) \quad \text{and} \quad V^B(v) = v\,Q^B(v) - T^B(v). \tag{3.8}$$

Because we assume that there is no discounting,[31] $T^S(\cdot)$ and $T^B(\cdot)$ can be interpreted as expected transfers. Given an outcome A, the surplus of entering traders is defined as

$$S\left(A|G^S(\cdot), G^B(\cdot)\right) \equiv \int_0^1 V^B(v)\, dG^B(v) + \int_0^1 V^S(c)\, dG^S(c).$$

The distribution of types $G^S(\cdot)$ and $G^B(\cdot)$, together with the size of the friction $\delta \in (0,1)$, describe our economy.[32] We assume that both distributions are smooth and strictly increasing. In particular, we assume that $G^S(\cdot)$ and $G^B(\cdot)$ have continuous and strictly positive densities. This assumption is identical to the one made in the basic model, and it ensures that there is a unique Walrasian price p^w which satisfies $G^S(p^w) = 1 - G^B(p^w)$. In the sequel, we take $G^S(\cdot)$ and $G^B(\cdot)$ as fixed and drop them from the argument.

[31] When we consider infinitely lived agents in Section 3.6.2, we also discuss the implications of a model with discounting. We show that the basic insights still apply.

[32] The parameters δ and G^S and G^B can be interpreted more broadly. In Section 3.6.2 the exit rate δ is replaced by a discount factor β, and in Section 3.5.3 we suggest that one can interpret G^S and G^B as the population at the beginning of a non-steady state dynamic matching and bargaining game that has no further entry.

For our analysis, we assume that all components of A are elements of the set of measurable functions, i.e., $A \in \Sigma_M^4$ and $S(\cdot) : \Sigma_M^4 \to \mathbb{R}$. With the Lebesgue integral, we can define a *distance* between two functions, $d(\cdot,\cdot) : \Sigma_M^2 \to [0,1]$ with $d(f_1, f_2) = \int_0^1 |f_1(x) - f_2(x)|\, dx$.[33] We use $d(\cdot,\cdot)$ to define convergence in Σ_M. In many cases, we can find conditions that ensure that the set of functions is restricted to the set of monotone functions. This will turn out to be helpful because every sequence of monotone functions has a convergent subsequence, i.e., sets of monotone functions are sequentially compact.[34] For future references, let $\Sigma_+ \subset \Sigma_M$ be the subset of weakly increasing functions and let $\Sigma_- \subset \Sigma_M$ be the subset of weakly decreasing functions.

A natural consistency requirement on an outcome is that total transfers collectively made by all buyers are equal to total transfers received by all sellers, $\int_0^1 T^B(v)\, dG^B(v) = \int_0^1 T^S(c)\, dG^S(c)$. From (3.8), it follows that this is equivalent to the following condition on A:

$$\int_0^1 \left(v\, Q^B(v) - V^B(v)\right) dG^B(v) = \int_0^1 \left(V^S(c) + c\, Q^S(c)\right) dG^S(c). \tag{3.9}$$

Define

$$S_Q(A) \equiv \int_0^1 v\, Q^B(v)\, dG^B(v) - \int_0^1 c\, Q^S(c)\, dG^S(c).$$

Condition (3.9) is equivalent to

$$S(A) = S_Q(A). \tag{3.10}$$

This equality reflects the idea that, for the purpose of welfare analysis, only the allocation of the good matters while transfers cancel.

Similar to the balance of transfers, the total mass of buyers who trade is required to be equal to the total mass of sellers who trade:

$$\int_0^1 Q^S(c)\, dG^S(c) = \int_0^1 Q^B(v)\, dG^B(v). \tag{3.11}$$

Economically, this condition corresponds to the scarcity of the good: For every buyer who enjoys consumption, there must be some seller who incurs costs. We define the set $\hat{Q}$ of all trading outcomes satisfying the balance of total trades:

$$\hat{Q} \equiv \left\{Q^S(\cdot), Q^B(\cdot) \in \Sigma_M^2 \mid \text{condition (3.11) holds}\right\}.$$

An outcome A satisfies *mass balance* if it satisfies the two consistency conditions:

Definition 3 ***Mass balance****.* *An outcome $A = \left[V^S(\cdot), V^B(\cdot), Q^S(\cdot), Q^B(\cdot)\right]$ is said to satisfies mass balance if*

$$A \in \hat{A} \equiv \left\{A \mid Q \in \hat{Q} \text{ and } S(A) = S_Q(A)\right\}.$$

[33] Note that $d(\cdot,\cdot)$ is only a semimetric: $d(f_1, f_2) = 0$ does not imply $f_1 = f_2$. Still $d(f_1, f_2)$ is non-negative and symmetric, and it satifies $d(f_1, f_1) = 0$ and the triangle inequality. We endow Σ_M with the semimetric topology (see Aliprantis, Border (1994, p. 23)), defined in the usual way by using open $\varepsilon-$balls $B_\varepsilon(f_1) = \{f \in \Sigma_M | d(f_1, f) < \varepsilon\}$, to define open sets just as in a metric space.

[34] According to Helly's selection theorem (see Kolmogorov, Fomin (1970, p. 372)), every sequence $\{f_N\}_{N=1}^\infty$ of monotone functions has a pointwise convergent subsequence $\{f_{N'}\}_{N=1}^\infty$. Lebesgue's bounded convergence theorem implies $d\left(f_{N'}, \bar{f}\right) \to 0$ for some $\bar{f}$. The limit $\bar{f}$ is clearly monotone itself.

We say that a *sequence* of outcomes $\{A_k\}_{k=1}^{\infty}$satisfies mass balance if each of its members A_k is in $\hat{A}$.

3.3.2 Efficiency

Our object of interest is the maximal surplus that can be reached subject to the resource constraint $Q \in \hat{Q}$:

$$S^* \equiv \sup_{A \in \hat{A}} S_Q(\cdot).$$

Basic economic intuition suggests that the optimal allocation is the following: All buyers with valuations above the market clearing price p^w get the good, while all sellers with costs below p^w sell theirs; buyers with lower valuations and sellers with higher costs do not trade. Let Q^W be the set of *Walrasian* allocations of the good that are equivalent[35] to this rule:

$$Q^W \equiv \left\{ Q \in \hat{Q} \mid \int_0^1 \left| Q^S(c) - 1_{c \le p^w}(c) \right| dc = 0,\ \int_0^1 \left| Q^B(v) - 1_{v \ge p^w}(v) \right| dv = 0 \right\}. \tag{3.12}$$

It is straightforward to prove that indeed an outcome is efficient if and only it is in Q^W (see the appendix for details)[36]:

Lemma 14 *For all outcomes that satisfy mass balance, i.e., for all $A \in \hat{A}$: $S(A) = S^*$ if and only if $Q \in Q^W$.*

Accordingly, the maximal surplus S^* is given by:

$$S^* = \int_{p^w}^1 v dG^B(v) - \int_0^{p^w} c dG^S(c). \tag{3.13}$$

Let $\hat{Q}_+$ be the set of trading probabilities which are monotone and which satisfy mass balance of trades i.e., $\hat{Q}_+ \equiv \left\{ Q \in \hat{Q} \mid Q^S \in \Sigma_-,\ Q^B \in \Sigma_+ \right\}$. Because $\hat{Q}_+$ is sequentially compact, we can show that the former lemma also holds in the limit: a sequence of outcomes $\{Q_k\}_{k=1}^{\infty}$ becomes efficient if and only if it converges to the set Q^W. (We say that a sequence $\{Q_k\}_{k=1}^{\infty}$ converges to Q^W if its distance to any element of Q^W becomes zero in every component.) The proof of the following lemma is relegated to the appendix:

Lemma 15 *For every sequence $\{A_k\}_{k=1}^{\infty}$with $A_k \in \hat{A}$ and with $Q_k \in \hat{Q}_+$:*

$$\lim_{k \to \infty} S_Q(A_k) = S^* \quad \textit{if and only if} \quad \lim_{k \to \infty} Q_k = Q^W.$$

Next, we derive a simple sufficient condition for the efficiency of an outcome: Suppose an outcome A is such that, for any cost c and for any valuation v, joint payoffs $V^S(c) + V^B(v)$ are weakly larger than the private surplus $v - c$. It is easy to show that this implies that A is efficient, i.e., $S(A) = S^*$:

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[35] Two functions Q_1 and Q_2 are equivalent if $d(Q_1^s, Q_2^s) = d(Q_1^B, Q_2^B) = 0$.

[36] Note, however, that in many models of the literature, this relation between efficiency and the Walrasian allocation is less straigthforward: With infinitely lived agents, as, e.g., in Gale (1987) we need to take care of "Ponzi-Schemes" and with cloning, as e.g., in the model by DeFraja and Sakovics (2001) the definition of surplus itself becomes problematic (see Section 3.6.2 and 3.7.3, respectively).

Lemma 16 ***Sufficiency****. If some outcome satisfies mass balance, i.e., if $A \in \hat{A}$ and if for all v and c, $V^S(c) + V^B(v) \geq v - c$, then $S(A) = S^*$.*

Proof: Let $\bar{p} \equiv \inf_{c \leq p^w} \left(V^S(c) + c\right)$. Then $V^S(c) + V^B(v) \geq v - c$ for all v and c implies $V^B(v) \geq v - \inf_{c \leq p^w} \left(V^S(c) + c\right)$ for all v. Together with the definition of $\bar{p}$, we use this to bound $S(A)$:

$$\begin{aligned} S(A) &\geq \int_{p^w}^{1} V^B(v)\, dG^B(v) + \int_0^{p^w} V^S(c)\, dG^S(c) \\ &\geq \int_{p^w}^{1} (v - \bar{p})\, dG^B(v) + \int_0^{p^w} (\bar{p} - c)\, dG^S(c) \\ &= S^* + \bar{p}\left(G^S(p^w) - \left(1 - G^B(p^w)\right)\right) = S^*, \end{aligned}$$

where the last line follows from the definition of p^w. By the restriction $A \in \hat{A}$ and by the definition of S^*, $S(A) \leq S^*$. Therefore, $S(A) \geq S^*$ implies $S(A) = S^*$. ■

By continuity of $S(\cdot)$, the last lemma carries over to sequences (see the appendix for details). For technical reasons, we restrict the elements of A_k to be in the set of outcomes which satisfy mass balance and which are monotone in each component, $\hat{A}_+ \equiv \hat{A} \cap [\Sigma_- \times \Sigma_+ \times \Sigma_- \times \Sigma_+]$:

Lemma 17 *For every sequence $\{A_k\}_{k=1}^{\infty}$ with $A_k \in \hat{A}_+$*

$$\lim_{k \to \infty} S(A_k) = S^* \qquad \text{if} \qquad \lim_{k \to \infty} \inf \left[V_k^S(c) + V_k^B(v)\right] \geq v - c \quad \text{for all } v,\ c.$$

3.3.3 General Conditions

We take some sequence of exit rates $\{\delta_k\}_{k=1}^{\infty}$, and for each exit rate δ_k we take some outcome A_k. This gives us a sequence $\{A_k\}_{k=1}^{\infty}$. We now define four conditions for this sequence. In the next section, we show that if the sequence satisfies these conditions, then its limit is efficient.

In the following, we denote pointwise limits by upper bars. For sequences of trading probabilities, we define

$$\bar{Q}^S(c) \equiv \lim_{k \to \infty} Q_k^S(c), \text{ and } \bar{Q}^B(v) \equiv \lim_{k \to \infty} Q_k^B(v),$$

whenever these limits exist. For sequences of payoffs, we define analogously

$$\bar{V}^S(c) \equiv \lim_{k \to \infty} V_k^S(c) \text{ and } \bar{V}^B(v) \equiv \lim_{k \to \infty} V_k^B(v).$$

We motivate the first two conditions by the trading situation with asymmetric information in the basic model. While we provide here only a sketch of the idea, we prove in Section 3.5.1 in detail that the conditions hold. In Section 3.5.2 we show that the conditions also hold with symmetric information when bargaining power is intermediate (see also remark 5).[37]

[37] See also Remark 9 in Section 3.7.1 for the case of noisy information.

The main observation for the basic model is that, with asymmetric information, the revelation principle requires that the trading outcome is incentive compatible. Intuitively, a type c can *mimic* the strategy of another type c_x. If he does so, he receives a transfer $T_k^S(c_x)$ and trades with probability $Q_k^S(c_x)$. Thus, for $V_k^S(c)$ to be the equilibrium payoff for type c, $V_k^S(c)$ must be at least as large as $T_k^S(c_x) - c\,Q_k^S(c_x)$. The same observations apply to buyers.

It is standard to verify that incentive compatibility requires that trading probabilities are monotone (for details, see Section 3.5.1):

Condition 1 ***Monotonicity***. *A sequence $\{A_k\}_{k=1}^{\infty}$ satisfies monotonicity if every member A_k has monotone trading probabilities:*

$$Q_k^S(\cdot) \in \Sigma_- \quad \text{and} \quad Q_k^B(\cdot) \in \Sigma_+.$$

Also, by standard reasoning, incentive compatibility imposes a restriction on the *slopes* of payoff functions: The difference between the payoffs of two types cannot be too large for otherwise one of these types would have an incentive to mimic the other (see Section 3.5.1, equation (3.18) for details). This is reflected in the following condition, which requires that the slope is bounded between zero and one. We require a tighter bound if a type c_x trades with certainty in the limit. In this case, every other type could mimic him and receive at least the same revenue. The payoff difference would be entirely due to the difference in their costs, i.e., payoffs from mimicking the type c_x change with a slope of one.

Condition 2 ***No Rent Extraction***. *A sequence $\{A_k\}_{k=1}^{\infty}$ satisfies no rent extraction if for every member A_k of the sequence $\{A_k\}_{k=1}^{\infty}$ and for every c, $c_x \in [0,1]$ and $v, v_x \in [0,1]$ there is some $a \in [0,1]$ such that*

$$V_k^S(c) \geq V_k^S(c_x) + a(c_x - c) \quad \text{and} \quad V_k^B(v) \geq V_k^B(v_x) + a(v - v_x).$$

In addition, whenever $\bar{Q}^S(c_x)$ and $\bar{V}^S(c_x)$ exist and $\bar{Q}^S(c_x) = 1$, then $\liminf V_k^S(c) \geq \bar{V}^S(c_x) + (c_x - c)$ for all c. Symmetrically, whenever $\bar{Q}^B(v_x)$ and $\bar{V}^B(v_x)$ exists and $\bar{Q}^B(v_x) = 1$ then $\liminf V_k^B(v) \geq \bar{V}^B(v_x) + (v - v_x)$ for all v.

The no rent extraction property implies monotonicity and continuity of the payoff functions, something we will utilize in the proof. In particular, monotonicity and continuity carry over to the limiting functions $\bar{V}^S$ and $\bar{V}^B$.[38]

Remark 3 *With asymmetric information, it is well known from the proof of the envelope theorem (see, e.g., Milgrom and Segal (2002)) that we can state the bound more tightly as $V_k^S(c) \geq V_k^S(c_x) + Q_k^S(c_x)(c_x - c)$ (see inequality (3.18)). The same applies to buyers, of course. However, here we want to find conditions which are just strong enough to imply the convergence result, but still weak enough so that they hold in a wide range of models. In particular, we want to include the possibility of symmetric information, and, in this case, we can only require the weaker bounds that we stated in the condition.*

[38] According to the condition, all payoff functions must be Lipschitz continuous with Lipschitz constant 1, since $\left|V^S(c) - V^S(c_x)\right| \leq |c - c_x|$. Therefore, every sequence of such functions is equicontinuous; hence, its limit must be continuous whenever it exists (see Kolmogorov, Fomin (1970, p. 102)).

For the next two conditions, we introduce the concept of *availability*. This is formalized by the introduction of an operator $L^B(\cdot) : [0,1]^2 \times \Sigma_M^4 \to [0,1]$. $L^B(v'', \delta_k, A_k)$ is interpreted as the probability that a seller who is just passively waiting in the pool will be matched at least once with a buyer of type $v \geq v''$ before he dies, given the exit rate δ_k and the outcome A_k.[39] Let $L_k^B(v) \equiv L^B(v, \delta_k, A_k)$ and let $\bar{L}^B(v) = \liminf_{k\to\infty} L_k^B(v)$. As we will demonstrate in the basic model, whenever some set of buyers does not trade with certainty in the limit, then this set is available, i.e., $L^B = 1$. Introducing a similar function $L^S(\cdot)$ for sellers, we state:

Condition 3 ***Availability.*** *A sequence $\{A_k\}_{k=1}^{\infty}$ satisfies availability relative to some pair of functions L^B and L^S if, whenever $\bar{Q}^B(v')$ exists for some v' and $\bar{Q}^B(v') < 1$, then $\bar{L}^B(v'')$ exists and $\bar{L}^B(v'') = 1$ for all $v'' < v'$; And if $\bar{Q}^S(c')$ exists for some c' and if $\bar{Q}^S(c') < 1$, then $\bar{L}^S(c'')$ exists and $\bar{L}^S(c'') = 1$ for all $c'' > c'$.*

Now, suppose it is commonly known that types c_x and v_x are *available*, i.e., buyers and sellers are mutually sure to meet some $c \leq c_x$ and some $v \geq v_x$, respectively. Then, intuitively, their joint payoffs should be ex ante pairwise efficient. Otherwise, their joint payoffs is below the surplus they could realize by trading. So it becomes certain that (a) between these types there is still "money left on the table," and (b) these types are certain to meet each other so that they can realize this additional surplus. This observation motivates the final condition:

Condition 4 ***Weak pairwise efficiency.*** *A sequence $\{A_k\}_{k=1}^{\infty}$ satisfies weak pairwise efficiency relative to some pair of functions L^B and L^S, if $\bar{L}^S(c_x) = 1$ and $\bar{L}^B(v_x) = 1$ for any pair of types c_x and v_x implies*

$$\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq v_x - c_x.$$

But note well that the condition requires pairwise efficiency only with respect to those types which do not trade with certainty and which are available. Payoffs might still be inefficient for those pairs of types which trade with certainty and which are not available.

Remark 4 *Instead of giving two conditions, we could have stated a single condition that requires that whenever for some pair v_x and c_x, $\bar{Q}^B(v_x) < 1$ and $\bar{Q}^S(c_x) < 1$, then for all pairs v and c with $v < v_x$ and $c > c_x$, $V^S(c) + \bar{V}^B(v) \geq v - c$. Mathematically, the functions L^S and L^B are just arbitrary indicator functions that connect the two conditions. However, we stated them separately, since in the applications the first of the two conditions can be formulated as a condition on the matching technology while the second condition refers to the bilateral bargaining outcomes (see Section 3.5 and the definition of the functions L^S and L^B in (3.17) and (3.16)). As discussed in Section 3.7.2, these two condition can fail separately, i.e., economically, they are separate.*

3.4 Main Result

In this section, we state and prove our main result: Suppose there is a pair of functions L^S, L^B and a sequence of exit rates $\{\delta_k\}_{k=1}^{\infty}$ such that a given sequence of outcomes $\{A_k\}_{k=1}^{\infty}$ satisfies the conditions stated before. Then outcomes along this sequence become efficient:

[39] For an example, see the definition of $L^B(\cdot)$ in the basic model in equation (3.16).

Proposition 4 *Suppose some sequence $\{A_k\}_{k=1}^{\infty}$ satisfies mass balance, monotonicity, no rent extraction, and suppose it satisfies availability and weak pairwise efficiency relative to some sequence of frictions $\{\delta_k\}_{k=1}^{\infty}$ and to some pair of functions L^B, L^S. Then the outcome becomes efficient, i.e.,*

$$\lim_{k\to\infty} S(A_k) = S^*.$$

Proof: The monotonicity condition and the no rent extraction condition require that all components of each element A_k are monotone, i.e., $A_k \in \hat{A}_+$. By Helly's selection principle (see Kolmogorov, Fomin 1970), we can find a pointwise convergent subsequence $\{A_{k'}\}_{k'=1}^{\infty}$. Let its limit be denoted by $(\bar{V}^S, \bar{V}^B, \bar{Q}^S, \bar{Q}^B)$. We first show that $(\bar{Q}^S, \bar{Q}^B)$ is in the set of Walrasian allocations Q^W for every such subsequence. Then we show that this is sufficient for $\lim_{k\to\infty} S_Q(A_k) = S^*$ for the sequence itself.

Given the subsequence $\{A_{k'}\}_{k'=1}^{\infty}$, define cutoff types c_x and v_x as the lowest cost and highest valuation, such that traders with these types do not trade with certainty in the limit, i.e.,

$$c_x \equiv \inf\left\{c, 1 | \bar{Q}^S(c) < 1\right\} \quad \text{and} \quad v_x \equiv \sup\left\{v, 0 | \bar{Q}^B(v) < 1\right\}.$$

First, we show that the no rent extraction conditions implies

$$\begin{aligned} & \bar{V}^S(c) \geq \bar{V}^S(c_x) + (c_x - c) \quad \text{for all } c, \\ \text{and} \quad & \bar{V}^B(v) \geq \bar{V}^B(v_x) + (v - v_x) \quad \text{for all } v. \end{aligned}$$

So the payoffs to all types can be bounded from below once we know the payoffs of the cutoff types. The first inequality follows directly for all types $c \in [c_x, 1]$ by the no rent extraction condition, observing that $(c_x - c)$ is negative. For types $[0, c_x]$, the inequality is trivially true if $c_x = 0$; if $c_x > 0$, choose some $\varepsilon \in (0, c_x)$ and note that $\bar{Q}^S(c_x - \varepsilon) = 1$ by definition of c_x and by monotonicity of $\bar{Q}(\cdot)$ (which is implied by the monotonicity of each element $Q_{k'}$). Hence, for all $c \leq c_x - \varepsilon$, the no rent extraction condition implies that $\bar{V}^S(c) \geq \bar{V}^S(c_x) + (c_x - c) - \varepsilon$. Because $\bar{V}(\cdot)$ is continuous (see the statements following the no rent extraction condition), and because ε was chosen arbitrary, we get $\bar{V}^S(c) \geq \bar{V}^S(c_x) + (c_x - c)$. So the first inequality holds for all $c \in [0, 1]$. The second inequality follows for all buyers by symmetric reasoning.

Adding the two inequalities yields a lower bound on the joint surplus of all c and v:

$$\bar{V}^S(c) + \bar{V}^B(v) \geq v - c + \bar{V}^S(c_x) + \bar{V}^B(v_x) - (v_x - c_x). \tag{3.14}$$

We use the availability and the weak efficiency conditions to show that the right hand side is at least $(v - c)$:

We consider two cases for the ordering of c_x and v_x. First, suppose $c_x < v_x$. Take some $\varepsilon \in (0, v_x - c_x)$. By definition of c_x and v_x, and by monotonicity of $\bar{Q}^S(\cdot)$ and $\bar{Q}^B(\cdot)$, we have $\bar{Q}^S(c_x + 0.5\varepsilon) < 1$ and $\bar{Q}^B(v_x - 0.5\varepsilon) < 1$. The availability condition implies that $\bar{L}^S(c_x + \varepsilon) = \bar{L}^B(v_x - \varepsilon) = 1$. By the weak efficiency condition:

$$\bar{V}^S(c_x + \varepsilon) + \bar{V}^B(v_x - \varepsilon) \geq v_x - c_x - 2\varepsilon.$$

Since the sum $\bar{V}^S(\cdot) + \bar{V}^B(\cdot)$ is continuous and ε is arbitrary:

$$\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq v_x - c_x.$$

Now consider the case $v_x \leq c_x$. Since $(v_x - c_x)$ is non-positive and since payoffs are non-negative, we get

$$\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq v_x - c_x.$$

So for both possible orderings of c_x and v_x, the sum of the last four terms in (3.14) is positive. Hence, payoffs are pairwise efficient, i.e., for all v and for all c:

$$\bar{V}^S(c) + \bar{V}^B(v) \geq v - c.$$

According to Lemma 16, pairwise efficiency is a sufficient condition for the subsequence to become efficient since payoffs V_k^S and V_k^B are monotone. Therefore $\lim_{k\to\infty} S(A_{k'}) = S^*$ along the subsequence. Hence, the Lemma 15 implies that limiting trading probabilities must necessarily be Walrasian, i.e., $(\bar{Q}^S, \bar{Q}^B)$ must be in Q^W.

Because the choice of the subsequence was arbitrary, this implies that the limit of *every* convergent subsequence is in Q^W. Because $\hat{A}$ is sequentially compact, this implies $\lim_{k\to\infty}(Q_k^S, Q_k^B) = Q^W$ for the original sequence.[40] According to Lemma 15 this is sufficient for the sequence to become efficient and $\lim_{k\to\infty} S(A_k) = S^*$, as claimed. ■

3.5 Application of the Main Result

In this section we discuss four specifications of dynamic matching and bargaining games to show how to apply and check our conditions: First, we show that the conditions hold in the basic example introduced already in Section 3.2. Second, we consider a variant of this specification where we assume symmetric information and intermediate bargaining power. This variant is essentially a version of the steady-state model in Gale (1987). The last two specifications are only briefly sketched: We consider a variant where entry occurs only in the first period and another variant where sellers conduct auctions. These variants similar to the setups in Moreno and Wooders (2002) and in Satterthwaite and Shneyerov (2007), respectively.

3.5.1 Basic Model

Take a decreasing sequence of exit rates $\{\delta_k\}_{k=1}^{\infty}$ with $\delta_k \to 0$. As shown in chapter two, for every k there exists an equilibrium σ_k^*. Fixing one equilibrium for each k yields a sequence $\{\sigma_k^*\}_{k=0}^{\infty}$. With every equilibrium σ_k^*, we associate an outcome A_k, using the map $A(\cdot,\cdot) : \Sigma \times [0,1] \to \Sigma_M^4$ defined as in Section 3.2.

Now, we want to derive the functions L^B and L^S. For this, we first show how trading probabilities relate to the distributions of types in the pool; then we show how the distribution of types in the pool translates into the matching probabilities L^B and L^S. First, we substitute Q_k^B into the steady-state condition (3.7), so that we can write the

[40] In a sequentially compact space, if all convergent subsequences of some sequence have a common limit, then the sequence itself converges to that limit (see also Lemma 20 in the Appendix.)

distribution function Φ^B as a function of A_k:

$$\Phi^B(v|\delta_k, A_k) = \int_0^v \frac{1 - Q_k^B(\tau) + \delta Q_k^B(\tau)}{M_k \delta_k} dG^B(\tau), \tag{3.15}$$

where M_k is derived from $\Phi^B(1|\delta_k, A_k) = 1$. $\Phi^S(c|\delta_k, A_k)$ can be defined similarly. Now, we define L^S as the solution to the recursive matching formula:

$$L^S(c|\delta_k, A_k) = \Phi^S(c|\delta_k, A_k) + (1-\delta_k)\left(1 - \Phi^S(c|\delta_k, A_k)\right) L^S(c|\delta_k, A_k)$$

and together with the equivalent formula for L^B we get:

$$L^B(v|\delta_k, A_k) = \frac{1 - \Phi^B(v|\delta_k, A_k)}{1 - \Phi^B(v|\delta_k, A_k)(1-\delta_k)} \tag{3.16}$$

$$\text{and } L^S(c|\delta_k, A_k) = \frac{\Phi^S(c|\delta_k, A_k)}{1 - (1 - \Phi^S(c|\delta_k, A_k))(1-\delta_k)}. \tag{3.17}$$

Now we prove that our conditions hold:

Lemma 18 *Given any sequence of exits rates $\{\delta_k\}_{k=1}^{\infty}$ with $\delta_k \to 0$, every sequence of equilibrium outcomes $\{A_k\}_{k=0}^{\infty}$ satisfies mass balance, monotonicity, no rent extraction, and it satisfies availability and weak pairwise efficiency with respect to L^S and L^B as defined in (3.17) and (3.16).*

Proof:

Mass Balance: Proven in chapter two (Lemma 2).

Monotonicity. For $Q_k^S(\cdot)$: Suppose the function is not monotone decreasing for some k and for some $c_H > c_L$, $Q_k^S(c_H) \equiv Q_H > Q_L \equiv Q_k^S(c_L)$ by optimality. Then with $p_L \equiv p_k(c_L)$ and $p_H \equiv p_k(c_H)$, it must be that $U^S(p_H, c_H|\sigma_k, \delta_k) \geq U^S(p_L, c_H|\sigma_k, \delta_k)$. This is equivalent to

$$Q_H(p_H - c_H) \geq Q_L(p_L - c_H),$$

and this implies that for costs $c_L < c_H$

$$Q_H(p_H - c_L) > Q_L(p_L - c_L),$$

and thus, $U^S(p_H, c_L|\sigma_k, \delta_k) > U^S(p_L, c_L|\sigma_k, \delta_k)$; This contradicts the optimality of $p_L \equiv p_k(c_L)$ for c_L. Similar reasoning holds for $Q_k^S(\cdot)$.

No Rent Extraction. For $V_k^S(\cdot)$: Again, we use a direct implication of optimality:

$$V_k^S(c) - V_k^S(c_x) \geq U^S(p_k(c_x), c|\sigma_k, \delta_k) - U^S(p_k(c_x), c_x|\sigma_k, \delta_k),$$

which implies that for all c the condition holds, since by definition of $U^S(\cdot, \cdot|\sigma_k, \delta_k)$, the above inequality is equivalent to

$$V_k^S(c) \geq V_k^S(c_x) + q^S(p_k(c_x)|\delta_k, \tilde{\sigma}_k^*)(c_x - c), \tag{3.18}$$

and similarly for V_k^B.

Availability. For $\{L_k^B\}_{k=1}^\infty \equiv \{L^B(\cdot|\delta_k, A_k)\}_{k=1}^\infty$. Evaluating the steady-state condition (3.15) at $\Phi_k^B(1|\delta_k, A_k)$ shows $M_k\delta_k \leq 1$. Choosing any $v' < v$, we get a lower bound on $1 - \Phi^B(v'|\delta_k, A_k)$:

$$1 - \Phi^B(v'|\delta_k, A_k) \geq \int_{v'}^{v} \left[1 - Q_k^B(\tau)\right] dG^B(\tau).$$

Since Q_k^B is monotone, $Q_k^B(v') \leq Q_k^B(v)$ for all $v' < v$. By assumption, the density $dG^B(\tau)$ is continuous and strictly positive, so there is some $g_L > 0$ such that $dG^B(v) \geq g_l$ for all v. Together:

$$1 - \Phi^B(v'|\delta_k, A_k) \geq \left(1 - Q_k^B(v)\right)(v - v')\, g_l,$$

and so for all sequences of $Q_k^B(v)$ with a limit $\bar{Q}^B(v) < 1$:

$$\lim_{k\to\infty} \inf L_k^B(v') \geq \frac{\left(1 - \bar{Q}_k^B(v)\right)(v - v')\, g_l}{1 - \left(1 - \left(1 - Q_k^B(v)\right)(v - v')\, g_l\right)} = 1,$$

and similarly for $\{L_k^S\}_{k=1}^\infty$.

Weak Efficiency: Suppose for some c_x and v_x, $\bar{L}^S(c_x) = \bar{L}^B(v_x) = 1$. By the no rent extraction condition, $V_k^B(\cdot)$ is increasing with a slope in $[0,1]$. Thus, $r_k(v)$ is increasing by $r_k(v) = v - (1 - \delta_k)V_k(v)$. Therefore, the set of types accepting a price $p = r_k(v_x)$ is at least the set $[v_x, 1]$. Therefore, the trading probability $D(r_k(v_x)|\sigma_k)$ is at least $1 - \Phi^B(v_x)$. By definition of L_k^B and q^S:

$$q_k^S(r_k(v_x)|\delta_k, \tilde{\sigma}_k^*) \geq L_k^B(v_x),$$

and therefore

$$\begin{aligned} U_k^S(r_k(v_x), c_x|\delta_k, \tilde{\sigma}_k^*) &\geq L_k^B(v_x)(r_k(v_x) - c_x) \\ &= L_k^B(v_x)\left(v_x - (1 - \delta_k)V_k^B(v_x) - c_x\right), \end{aligned}$$

where the last line follows from the equilibrium condition for $r_k(v_x)$. Given the equilibrium conditions, $V_k^S(c_x) \geq U_k^S(r_k(v_x), c_x|\delta_k, \tilde{\sigma}_k^*)$ for all k. Therefore,

$$\begin{aligned} \lim_{k\to\infty} \inf V_k^S(c_x) &\geq \lim_{k\to\infty} \inf L_k^B(v_x)\left(v_x - (1 - \delta_k)V_k^B(v_x) - c_x\right) \\ &= \left(v_x - \bar{V}_k^B(v_x) - c_x\right), \end{aligned}$$

which implies $\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq v_x - c_x$ (whenever they exist), as claimed. ∎

So $\{A_k\}_{k=1}^\infty$ satisfies our conditions and thus we have:

Corollary 2 *For every sequence of exit rates $\{\delta_k\}_{k=1}^\infty$ with $\delta_k \to 0$ and for every sequence of associated equilibrium outcomes $\{A_k\}_{k=1}^\infty$ of the basic model:*

$$\lim_{k\to\infty} S(A_k) = S^*.$$

3.5.2 Symmetric Information and Intermediate Bargaining Power

We change the basic model by assuming that traders in each match observe what type of player they are up against and that both of them, the buyer and the seller, have a chance to propose a price. Let $\alpha \in (0,1)$ be the probability that the seller is chosen to propose a price, and let $(1-\alpha)$ be the probability that the buyer is chosen. This is similar to the analysis in Gale (1987).[41] We will first derive the formal setup and provide an equilibrium definition. Then we sketch why our conditions hold in this setup. A remark at the end of this section summarizes the intuition.

Strategies now account for the role of the trader and for the type of the opponent. Let Σ_{M^2} be the set of measurable functions $f : [0,1]^2 \to [0,1]$. Strategies are $\left[p^S, p^B, r^S, r^B\right] \in \Sigma^2_{M^2} \times \Sigma^2_M$, where $p^B(v,c)$ is the price proposed by a buyer of type v to a seller of type c, and $r^S(c)$ is the reservation price of a seller c. $p^S(c,v)$ and $r^B(v)$ are the corresponding proposals and reservation prices of sellers and buyers. A market constellation is a vector $\sigma_F \in \Sigma_F$ with $\Sigma_F \equiv \Sigma^2_{M^2} \times \Sigma^4_M \times \mathbb{R}_+$ and with a typical element $\sigma_F = \left[p^S, p^B, r^S, r^B, \Phi^S, \Phi^B, M\right]$.

Let $P^S\left(p^S, c|\sigma_F\right)$ be the probability that a seller who is chosen as a proposer will trade in a given period when using $p^S = p^S(\cdot,\cdot)$, defined as

$$P^S\left(p^S, c|\sigma_F\right) \equiv \int_{v|r^B(v)\geq p^S(c,v)} d\Phi^B(v),$$

and let $R^S\left(r^S, c|\sigma_F\right)$ be the probability that the seller will trade when chosen to respond:

$$R^S\left(r^S, c|\sigma_F\right) \equiv \int_{v|p^B(v,c)\geq r^S(c)} d\Phi^B(v);$$

then the per period probability of trading is $D^S\left(p^S, r^S, c|\sigma_F\right)$ given by the expression $\alpha P^S\left(p^S, c|\sigma_F\right) + (1-\alpha) R^S\left(r^S, c|\sigma_F\right)$. Let $E^{RS}\left[p|p \geq r^S(c), c, \sigma_F\right]$ be the expected price conditional on trade when responding, and let $E^{PS}\left[p|p \leq r^B(v), \sigma_F\right]$ be the expected price conditional on trade when proposing. Expected payoffs are implicitly defined via

$$\begin{aligned} U^S\left(p^S, r^S, c|\sigma_F\right) &= \alpha P^S\left(p^S, c|\sigma_F\right)\left(E^{PS}[p] - c\right) + (1-\alpha) R^S\left(r^S, c|\sigma_F\right)\left(E^{RS}[p] - c\right) \\ &\quad + (1-\delta)\left(1 - D^S\left(p^S, r^S, c|\sigma_F\right)\right) U^S\left(p^S, r^S, c\right), \end{aligned} \tag{3.19}$$

with $E^{PS}[p] = E^{PS}\left[p|p \leq r^B(v), \sigma_F\right]$ and $E^{RS}[p] = E^{RS}\left[p|p \geq r^S(c), c, \sigma_F\right]$. Let $U^{PS}\left(p, v|p^S, r^S, c, \sigma_F\right)$ be the payoff when matched with a type v, proposing p and continuing according to $\left(p^S, r^S\right)$:

$$U^{PS}\left(p, v, c|p^S, r^S, \sigma_F\right) = \begin{cases} p - c & \text{if } p \leq r^B(v), \\ (1-\delta) U^S\left(p^S, r^S, c|\sigma_F\right) & \text{otherwise.} \end{cases}$$

We define the corresponding functions for buyers analogously.

[41] Different from Gale we consider a continuum of types. He also assumes that traders are infinitely lived and that (therefore) there is an entry stage. In Section 3.6 we cover the latter cases.

The steady-state conditions do not change. They are

$$\Phi^S(c) = \int_0^c \frac{dG^S(\tau)}{M\left(D^S\left(p^S, r^S, \tau|\sigma_F\right) + \delta\left(1 - D^S\left(p^S, r^S, \tau|\sigma_F\right)\right)\right)} \quad \forall c \quad (3.20)$$

$$\text{and } \Phi^B(v) = \int_0^v \frac{dG^B(\tau)}{M\left(D^B\left(p^B, r^B, \tau|\sigma_F\right) + \delta\left(1 - D^B\left(p^B, r^B, \tau|\sigma_F\right)\right)\right)} \quad \forall v. (3.21)$$

We define an equilibrium, with x and y denoting types of traders. We require that the price offered by the proposer must be optimal for every possible type of responder, and we require that the reservation price has the same properties as derived in the basic model. These requirements incorporate the idea of sequential rationality:

Definition 4 *A steady-state equilibrium vector with full information, $\sigma_F^* \in \Sigma_F$, consists of an optimal pair of strategies and a corresponding steady-state pool such that*

1. $\left(p^j, r^j\right) \in \arg\max U^j\left(\cdot, \cdot, x|\sigma_F\right) \quad \forall x$ *and* $j \in \{B, S\}$
2. $p^j(x) \in \arg\max U^{Pj}\left(\cdot, x, y|p^j, r^j, \sigma_F\right) \; \forall x, y$ *and* $j \in \{B, S\}$
3. $r^B(v) = v - (1-\delta) U^B\left(p^B, r^B, v|\sigma_F\right)$ *and* $r^S(c) = (1-\delta) U^S\left(p^S, r^S, c|\sigma_F\right) + c$ $\forall v,\, c$
4. $\Phi^S(\cdot)$, $\Phi^B(\cdot)$, M *satisfy the steady-state conditions (3.20), (3.21).*

We show that payoffs can be rewritten very compactly. First, the optimal price offer of a buyer v to a seller of type c is clearly never strictly above $r^S(c)$, but is either equal to the reservation price or equal to some unacceptable price below, $p < r^S(c)$. Hence, the expected price offer to the seller, conditional upon acceptance, is $E^{RS}\left[p|p \geq r^S(c), c, \sigma_F\right] = r^S(c)$. This also applies to buyers. This implies in particular that a responder is indifferent about accepting or rejecting an offer. Therefore, expected payoffs do not change if a trader plans to simply reject *all* offers. Thus, payoffs depend only on the price offers a trader makes when he is a proposer. To derive this payoff, let q^{PS} be the lifetime trading probability conditional on trading only as a proposer and using the offer strategy p^S. We can derive $q^{PS}(\cdot, \cdot|\cdot)$ as the solution to

$$q^{PS}\left(p^S, c|\sigma_F\right) = \alpha P^S\left(p^S, c\right) + (1-\delta)\left(1 - \alpha P^S\left(p^S, c\right)\right) q^{PS}\left(p^S, r^S, c|\sigma_F\right).$$

where $P^S\left(p^S, c\right) = P^S\left(p^S, c|\sigma_F\right)$. Rewriting the payoff definition (3.19), using q^{PS} and the observation that
$E^{RS}\left[p|p \geq r^S(c), c, \sigma_F\right] = r^S(c)$, yields

$$U^S\left(p^S, r^S, c|\sigma_F\right) = q^{PS}\left(p^S, c|\sigma_F\right)\left(E^{PS}\left[p|p \leq r^B(v), \sigma_F\right] - c\right), \quad (3.22)$$

and similarly for buyers,

$$U^B\left(p^B, r^B, v|\sigma_F\right) = q^{PB}\left(p^B, c|\sigma_F\right)\left(v - E^{PB}\left[p|p \geq r^S(c), \sigma_F\right]\right). \quad (3.23)$$

Now take a sequence of exit rates $\{\delta_k\}_{k=1}^{\infty}$, with $\delta_k \to 0$, as before, and assume that for every δ_k, there is some equilibrium. Let this be σ_{Fk}, which gives us a sequence $\{\sigma_{Fk}\}_{k=1}^{\infty}$.

Let A_k be the outcome of equilibrium σ_{Fk}, with $A_k = A\left(\sigma_k^F, \delta_k\right)$ defined in the obvious way. We check only the no rent Extraction condition, because the other conditions are immediate. For this, let $U^S\left(p_k^S(\cdot,\cdot), 1, c|\sigma_{Fk}, \delta_k\right)$ be the payoff to a seller of type c if offering a price $p_k^S(\cdot, c)$ when chosen to propose, while rejecting any price offer if chosen to respond. From (3.22):

$$V_k^S(c) = U^S\left(p_k^S(\cdot,\cdot), 1, c|\sigma_{Fk}, \delta_k\right),$$

and from optimality

$$\begin{aligned} V_k^S(c_x) - V_k^S(c) &\geq U^S\left(p_k^S(\cdot,c), 1, c_x|\sigma_{Fk}, \delta_k\right) - U^S\left(p_k^S(\cdot,c), 1, c|\sigma_{Fk}, \delta_k\right) \\ &\geq q^{PS}\left(p_k^S(\cdot,\cdot), c|\sigma_F\right)(c - c_x), \end{aligned} \tag{3.24}$$

and together with symmetric reasoning for buyers, the first parts of the condition hold. For the limiting part, we show that if the lifetime trading probability Q_k^S converges to one, then $q^{PS}\left(p_k^S(\cdot,\cdot), c|\sigma_F\right)$ converges to one as well. This is proven in detail in the appendix, Section A.10. Therefore, (3.24) implies that whenever $Q_k^S \to 1$, we get $V_k^S(c_x) - V_k^S(c) \geq (c - c_x)$.

Now the other conditions follow, and we sketch out the idea: Given the no rent extraction condition, payoffs $V^S(\cdot)$ and $V^B(\cdot)$ are monotone. From the equilibrium conditions it follows that two matched traders v_x and c_x trade if and only if their joint trading surplus $v_x - c_x$ is larger than their joint continuation payoff $(1-\delta)\left[V^S(c_x) + V^B(v_x)\right]$. This, together with $V^B(\cdot)$ being increasing at a rate smaller than one (from the no rent extraction condition), implies that a buyer with a higher valuation trades with a larger set of sellers, and hence, the trading probability $Q_k^B(\cdot)$ is monotone increasing in v. Analogous reasoning implies the same for sellers. Weak pairwise efficiency is a direct implication of the above observation. Finally, availability follows by the same reasoning as in the basic model, because we are using exactly the same matching technology. Hence:

Corollary 3 *For every sequence of exit rates $\{\delta_k\}_{k=1}^{\infty}$ and equilibrium outcomes $\{A_k\}_{k=1}^{\infty}$ of the full information model with intermediate bargaining power $\alpha \in (0,1)$,*

$$\lim_{k\to\infty} S(A_k) = S^*.$$

Remark 5 *The crucial step for proving convergence with symmetric information is the following observation: Although it is true that a trader of type c does not need to* ***receive*** *the same offers as a trader of type c_x, he can* ***make*** *the same offers when chosen as the proposer. Even more to the point: As we have seen in (3.22) and (3.23), payoffs depend* ***only*** *on the offers made as a proposer. Therefore, a seller of type c can mimic the strategy of another type c_x in much the same way as a seller in our basic model can mimic the pricing strategy of another seller. When $\alpha = 1$, i.e., when buyers are never chosen to be the proposer, this reasoning breaks down. In chapter one, I look at this case and I show that convergence to efficiency does not hold, see also Section 3.7.1.*

3.5.3 Further Applications

Two more variants of the basic model that can be analyzed as before include one-time entry and second-price auctions with reservation prices. Suppose, in the basic model, we assume that time runs from $t = 0$ to infinity. In period zero, a unit mass of buyers and a unit mass of sellers arrive with types distributed according to distribution functions G^S and G^B, with the same properties as in the basic model. There is no further inflow in the subsequent periods. Thus, the pool in $t \geq 1$ consists only of those who did not trade before and who did not die before. So the pool depletes over time.[42] Otherwise, we assume that matching is pairwise, information is asymmetric and sellers make price offers, just as in the basic model.[43] We can characterize outcomes by considering the *ex ante* trading probabilities and payoffs to the entering traders in the first period. Their joint expected surplus is the natural welfare criterion. Clearly, mass balance should hold with respect to the ex ante outcome. With Q_0^S, Q_0^B denoting the first period expected lifetime trading probabilities, $Q_0^S, Q_0^B \in Q^W$ is a necessary and sufficient condition for efficiency, with Q^W as defined in (3.12). For this model, one can show that our conditions hold: By asymmetric information, traders can mimic each other. Just as in the basic model, this implies that trading probabilities are monotone and ex ante payoffs have a bounded slope. For the availability condition, note that if the ex ante trading probability of some buyers is not one, then these buyers will stay in the market for many periods. One can show that this implies that a seller is certain to be matched with them some time, i.e., availability holds. Finally, weak efficiency holds by similar reasoning to that found in the basic model. Thus, our main result applies even to non-steady-state markets, and the outcome will become efficient with δ converging to 0.

We can include auctions similar to Satterthwaite and Shneyerov (2007) in the basic model as follows: Suppose matches consist of one seller and a random number of buyers, where the number of buyers ("bidders") per seller is Poisson distributed with parameter one. Further, suppose the seller conducts a second price auction among the bidders: Upon observing the number of buyers in his match, he announces a reservation price p. Then the buyers submit their bids r. Restricting attention to equilibria in dominant strategies, these bids are equal to the reservation prices derived before. This allows a simple characterization of the equilibrium. Suppose we keep the basic model otherwise - that is, we retain the assumption that there is an exogeneous inflow and that there is some death rate δ. Our conditions hold in this model as well: Monotonicity and no rent extraction follow from asymmetric information, and availability and weak efficiency follow by reasoning familiar from the basic model. Therefore, if sellers can use auctions to sell their goods, with vanishing δ, the outcome becomes efficient.

3.6 Extensions

To show that our analysis also extends to the original setups by Gale (1987) and Satterthwaite and Shneyerov (2007), we first include an entry stage in the next subsection and then we assume that traders are infinitely lived.

3.6.1 Including an Entry Stage

Suppose we include an entry stage into the basic model, i.e., suppose that new traders must decide whether they want to enter the pool or not. If they enter the pool, they must

[42] The pool cannot deplete fully if prices are individually rational.

[43] This model with one-time entry would be different in two aspects: First, instead of a stationary pool, the pool would depend on the time via some law-of-motion condition. Second, price offers and reservation prices would depend on time.

pay some entry costs $e \in (0,1)$. Let $Z^S(\cdot) : [0,1] \to \{0,1\}$ denote the entry decision, with $Z^S(c) = 1$ indicating the decision of type c to become active. Let $V^S(\cdot)$ denote the expected payoffs to a seller if he enters, gross of e. ($V^S(\cdot)$ is also calculated for those who do not actually become active.) Let $Z^B(\cdot)$ and $V^B(\cdot)$ be the corresponding functions for buyers.

We assume that sellers enter whenever this is profitable, i.e., $Z^S(c) = 1$ whenever $V^S(c) \geq e$, and we assume $Z^S(c) = 0$ otherwise. For buyers, we assume the same: $Z^B(v) = 1$ whenever $V^B(v) \geq e$. Let c_0^e be the highest type of a seller for whom entry is profitable, $c_0^e \equiv \sup\{c, 0|V^S(c) \geq e\}$, and let v_0^e be the lowest type of a buyer for whom entry is profitable, $v_0^e \equiv \inf\{v, 1|V^B(v) \geq e\}$. By this definition, types $c > c_0^e$ or $v < v_0^e$ do not enter.[44]

Given the entry stage, the matching technology of the basic model has to be changed to account for the possibility that the masses of the two sides of the market are not identical. But no matter how this is done, types who do not enter are not available. Therefore, the probability to match some set of buyers might be zero, even though the lifetime trading probability of these types is strictly below one. One can show that this failure of availability leads to a failure of convergence to efficiency in the basic model (see Section 3.7.) Therefore, stronger forces towards efficiency are needed. In the models by Gale (1987), as well as in Satterthwaite and Shneyerov (2007), these forces come from curtailing the bargaining power of the seller. Formally, these models satisfy a stronger condition than Condition 4 (weak efficiency). Sequences of trading outcomes that satisfy this stronger condition converge to efficiency even though they satisfy only a weaker availability condition, due to the entry stage.

An outcome A^E of a model with an entry stage is a vector $[V^S, V^B, Q^S, Q^B, Z^S, Z^B]$. We assume that all components are measurable, i.e., $A^E \subset \Sigma_M^6$. Surplus conditional on (Q^S, Q^B, Z^S, Z^B) and gross of entry costs (which will become zero) is

$$S_Q^E(A^E) = \int_0^1 vZQ^B(v)\,dG^B(v) - \int_0^1 cZQ^S(c)\,dG^S(c),$$

with $ZB^B(v) \equiv Z^B(v)\,Q^B(v)$ and $ZQ^S(c) \equiv Z^S(c)\,Q^S(c)$. These latter functions are the *effective* trading probabilities, and we work with them throughout this section. Mass balance with entry is satisfied if the transfers collectively made by all buyers are equal to the expected transfers collectively received by all entering sellers. Equivalently, the mass of sellers who trade must be equal to the mass of buyers who trade:

Definition 5 ***Mass Balance with Entry***. *An outcome A^E satisfies mass balance with entry if*

$$S^E(A^E) = \int_0^1 Z^S(c)\,V^S(c)\,dG^S(c) + \int_0^1 Z^B(v)\,V^B(v)\,dG^B(v) = S_Q^E(A^E), \quad (3.25)$$

and if

$$\int_0^1 ZQ^S(c)\,dG^S(c) = \int_0^1 ZQ^B(v)\,dG^B(v). \quad (3.26)$$

[44]If payoffs are monotone, all types c below c_0 and all type v above v_0 enter.

We say that an outcome A^E is *Walrasian* if the effective trading probabilities are in Q^W, i.e., if $\left(ZQ^S, ZQ^B\right) \in Q^W$. Reasoning analogously to the case without entry, we find that an outcome is efficient if and only if it is Walrasian:

Lemma 19 *For all outcomes that satisfy mass balance with entry, $S^E\left(A^E\right) = S^*$ if and only if $ZQ \in Q^W$. For every sequence $\left\{A_k^E\right\}_{k=1}^{\infty}$ which satisfies mass balance with entry and which has monotone trading probabilities, $ZQ \in \Sigma_- \times \Sigma_+$:*

$$\lim_{k\to\infty} S_Q^E\left(A_k^E\right) = S^* \quad \text{if and only if} \quad \lim_{k\to\infty}\left(ZQ_k\right) = Q^W.$$

Take any sequence of outcomes $\left\{A_k^E\right\}_{k=1}^{\infty}$ and a sequence of frictions $\{\delta_k, e_k\}_{k=1}^{\infty}$ with $(\delta_k, e_k) \to (0,0)$. If the limits of effective trading probabilities exist, we denote them by $\overline{ZQ}^S$ and $\overline{ZQ}^B$, and if the limits of the cutoff types $c_0^{e_k}$ and $v_0^{e_k}$ exist, we denote them by c_0 and v_0. Now we restate the conditions. The monotonicity condition becomes a condition regarding effective trading probabilities:

Condition 5 ***Monotonicity with Entry****. For every member A_k^E,*

$$\overline{ZQ}_k^S \in \Sigma_- \quad \text{and} \quad \overline{ZQ}_k^B \in \Sigma_+.$$

The no rent extraction condition remains unchanged. But as said in the introduction, we weaken availability and we assume that it holds only for those types below (above) the cutoffs, i.e., for those $c \leq c_0$ and $v \geq v_0$. With $L_E^j : [0,1]^2 \times \Sigma_M^6 \to [0,1]$, $j \in \{S, B\}$:

Condition 6 ***Weak Availability****. If $\overline{ZQ}^S(c')$ and c_0 exist, and if $\overline{ZQ}^S(c') < 1$ for some $c' < c_0$, then*

$$\bar{L}_E^S(c) = 1 \quad \text{for all } c \in \left(c', c_0\right).$$

If $\overline{ZQ}^B(v')$ and v_0 exist, and if $\overline{ZQ}^B(v') < 1$ for some $v' > v_0$, then

$$\bar{L}_E^S(v) = 1 \quad \text{for all } v \in \left(v_0, v'\right).$$

We strengthen weak pairwise efficiency by requiring availability only on one side of the market. But it has to hold only for pairs involving either v_0 or c_0. As we will see, the limiting payoffs of these cutoff types are zero. Therefore, $\bar{V}^S(c') + \bar{V}^B(v_0) \geq v_0 - c'$ implies $\bar{V}^S(c') \geq v_0 - c'$. The following condition is formulated such that it is met by the models of Satterthwaite and Shneyerov and by the model of Gale:

Condition 7 ***Strong Pairwise Efficiency****. If $\bar{L}^S(c') = 1$ for some c' and if v_0 exists, then*

$$\bar{V}^S\left(c'\right) \geq v_0 - c'.$$

If $\bar{L}^B(v') = 1$ for some v' and if c_0 exists, then

$$\bar{V}^B\left(v'\right) \geq v - c_0.$$

Remark 6 *In the basic model, the first part of this condition does not hold: Suppose there is some cutoff $p^N > p^w$ such that all buyers with $v \geq p^N$ and all sellers with $c \leq p^N$ enter* *and no one else. Suppose in addition that all sellers offer the common price p^N. Since there are more sellers with costs below p^N than there are buyers with valuations above p^N, sellers must be rationed and they do not trade with certainty, i.e., $\overline{ZQ}^S(c') < 1$ and $\bar{L}^S(c') = 1$ for $c' \leq v_0 = p^N$. Payoffs for sellers become $\bar{V}^S(c') = \overline{ZQ}^S(c')\left(p^N - c'\right) < p^N - c' = v_0 - c'$. Nevertheless, they have no incentive to decrease their offers since they cannot increase their revenue if $Z^B(v) = 0$ for all $v < p^N$. Therefore, strong pairwise efficiency fails in the basic model.*

Note that for all models with entry, there exists an equilibrium in which no trader enters. If a sequence of outcomes includes such outcomes as subsequence, its limit cannot become efficient. Hence, we restrict attention to *non-trivial sequences*, where entry does not vanish along any subsequence, i.e.,

$$\lim_{k\to\infty} \sup v_0^{e_k} < 1 \quad \text{and} \quad \lim_{k\to\infty} \inf c_o^{e_k} > 0.$$

Under the stronger efficiency condition, we can state:

Proposition 5 *Suppose some non-trivial sequence $\left\{A_k^E\right\}_{k=1}^{\infty}$ satisfies mass balance and monotonicity with entry, no rent extraction, weak availability and strong pairwise efficiency for some pair of functions L^B and L^S and for some sequence $\{\delta_k\}_{k=1}^{\infty}$ and $\{e_k\}_{k=1}^{\infty}$ with $e_k \to 0$. Then the outcome becomes efficient, i.e.,*

$$\lim_{k\to\infty} S^E\left(A_k^E\right) = S^*.$$

Proof: As before, we take some convergent subsequence of outcomes and denote the limit by $\left(\bar{V}^S, \bar{V}^S, \overline{ZQ}^S, \overline{ZQ}^B, \bar{Q}^S, \bar{Q}^B\right)$. Let v_x be the lowest valuation and c_x the highest cost that does not trade for sure in the limit:

$$v_x = \sup\left\{v, 0|\overline{ZQ}^B(v) < 1\right\} \quad \text{and} \quad c_x = \inf\left\{c, 1|\overline{ZQ}^S(c) < 1\right\}.$$

Let us take a further subsubsequence indexed by k' such that the cutoffs $v_0^{e_{k'}}$ and $c_0^{e_{k'}}$ converge to some v_0 and c_0. Now we want to show that $(ZQ_{k''}) \to Q^W$. We will argue at the end of the proof that this implies $S\left(A_k^E\right) \to S^*$ for the sequence itself.

Noting that $\lim Q_{k'}^S(c) = 1$ whenever $\overline{ZQ}^S(c) = 1$, and, symmetrically, $\lim Q_{k'}^B(v) = 1$ whenever $\overline{ZQ}^S(v) = 1$, the no rent extraction condition has the same implication as in the proof of the main result, i.e.,

$$\bar{V}^S(c) + \bar{V}^B(v) \geq v - c + \bar{V}^S(c_x) + \bar{V}^B(v_x) - (v_x - c_x). \tag{3.27}$$

Now we want to derive again a lower bound on the joint payoff of c_x and v_x by showing that $\bar{V}^S(c_x) + \bar{V}^B(v_x) \geq v_x - c_x$:

If $v_x \leq c_x$, the desired inequality follows immediately because payoffs are non-negative. So suppose $v_x > c_x$. We consider three subcases for the relation between c_x, c_0, v_0, v_x. Subcase 1 is $c_x < c_0 < v_0 < v_x$. Then, for all $\varepsilon \in (0, \min\{c_0 - c_x; v_x - v_0\})$, by definition of c_x, $\overline{ZQ}^S(c_x+0.5\varepsilon)<1$. Thus, $\bar{L}^S(c_x+\varepsilon)=1$ by weak availability and, by symmetric reasoning, $\bar{L}^B(v_x-\varepsilon)=1$. Therefore, strong efficiency implies

$$\bar{V}^S(c_x+\varepsilon)\geq v_0-c_x-\varepsilon \qquad \text{and} \qquad \bar{V}^B(v_x-\varepsilon)\geq v_x-c_0-\varepsilon.$$

Since payoffs are continuous, ε is arbitrary, and $v_0\geq c_0$, we get

$$\bar{V}^S(c_x)+\ \bar{V}^B(v_x)\geq v_x-c_x+v_0-c_0\geq v_x-c_x,$$

as claimed.

Subcase 2a: $c_x=c_0$ and $v_0<v_x$. By definition of v_x, $\overline{ZQ}^B(v_x-0.5\varepsilon)<1$, for all $\varepsilon\in(0,v_x-v_0)$. Then, the availability condition requires that $\bar{L}^B(v_x-\varepsilon)=1$, and the strong efficiency condition implies

$$\bar{V}^B(v_x-\varepsilon)\geq v_x-c_0-\varepsilon$$

Since, again, payoffs are continuous, and since $c_x=c_0$, this implies $\bar{V}^B(v_x)\geq v_x-c_x$. Hence, by $\bar{V}^S(c_x)\geq 0$, we get the desired inequality $\bar{V}^B(v_x)+\bar{V}^S(c_x)\geq v_x-c_x$. Subcase 2b: $c_x<c_0$ and $v_0=v_x$. By analogous reasoning: $\bar{V}^B(v_x)+\bar{V}^S(c_x)\geq v_x-c_x$.

Subcase 3: $c_x=c_0$ and $v_0=v_x$. Note first that marginal types must make zero profits in the limit: If $\lim_{k\to\infty}\sup V_k^S(c_o^{e_k})>0$, the (Lipschitz-) continuity of payoffs implies that for some ε small enough, $\lim_{k\to\infty}\sup V_k^S(c_o^{e_k}+\varepsilon)>0$. This contradicts the definition of the marginal type. Hence $\lim_{k\to\infty}V_k^S(c_o^{e_k})=0$ and, symmetrically, $\lim_{k\to\infty}V_k^B(v_o^{e_k})=0$. With this observation, we show that this subcase leads to a contradiction: If $c_x=c_0$ and $v_0=v_x$, with $c_x<v_x$, then the mass of sellers who trade becomes

$$\lim_{k'\to\infty}\int_0^1 ZQ_{k'}^S(c)\,dG^S(c)=\int_0^{c_x}dG^S(c)=G^S(c_x),$$

and, similarly, the mass of buyers who trade becomes

$$\lim_{k'\to\infty}\int_0^1 ZQ_{k'}^B(v)\,dG^B(v)=\int_{v_x}^1 dG^B(v)=1-G^B(c_x).$$

The mass balance of total trades, see (3.26), requires therefore that the mass of entering sellers becomes equal to the mass of entering buyers, i.e., $G^S(c_x)=1-\left(G^B(v_x)\right)$. Furthermore, since $\bar{Q}^S(c)=1$ for all $c<c_x=c_0$, no rent extraction requires that

$$\bar{V}^S(c_0)\geq\bar{V}^S(c)+(c-c_0),$$

and thus $\bar{V}^S(c)\leq\bar{V}^S(c_0)+(c_0-c)$ for $c<c_0$. From before, we know that $\bar{V}^S(c_0)=0$, so together we have $\bar{V}^S(c)\leq c_0-c$. By symmetric reasoning, $\bar{V}^B(v)\leq v-v_0$. We use

this to get an upper bound on the limit of $S\left(A_{k'}^{E}\right)$:

$$\begin{aligned}
\lim_{k'\to\infty}\inf S^{E}\left(A_{k'}^{E}\right) &\leq \int_{0}^{c_x}\left(c_0-c\right)dG^{S}\left(c\right)+\int_{v_x}^{1}\left(v-v_0\right)dG^{B}\left(v\right)\\
&\leq \int_{v_x}^{1}vdG^{B}\left(v\right)-\int_{0}^{c_x}cdG^{S}\left(c\right)-G^{S}\left(c_x\right)\left(v_x-c_x\right)\\
&< \int_{v_x}^{1}vdG^{B}\left(v\right)-\int_{0}^{c_x}cdG^{S}\left(c\right)=\lim_{k'\to\infty}S_{Q}^{E}\left(A_{k'}^{E}\right),
\end{aligned}$$

where we use that $G^{S}\left(c_x\right)$ is equal to $1-\left(G^{B}\left(v_x\right)\right)$ for the second line and the hypothesis of the subcase, $\left(v_x-c_x\right)>0$, for the third line. Since $S_{Q}^{E}\left(A_{k'}^{E}\right)$ has a limit different from $S^{E}\left(A_{k'}^{E}\right)$, the mass balance identity (3.25), $S^{E}\left(A_{k'}^{E}\right)=S_{Q}^{E}\left(Q^{S}\left(\cdot\right),Q^{B}\left(\cdot\right)\right)$ is violated for k' large enough. As a result, this subcase is impossible, since by choice of the (sub-)sequence $\left\{A_{k'}^{E}\right\}$, each of its elements does satisfy mass balance. (Note, that this subcase is the only place where we need $e_k\to 0$.)

Hence, $\bar{V}^{B}\left(v_x\right)+\bar{V}^{S}\left(c_x\right)\geq v_x-c_x$ in all possible cases. Thus, inequality (3.27) implies that limiting payoffs are pairwise efficient for all types c and v:

$$\bar{V}^{S}\left(c\right)+\bar{V}^{B}\left(v\right)\geq v-c\ . \tag{3.28}$$

By reasoning analogously to the second part of the main result, this implies that the outcomes of the original sequence become efficient and

$$\lim_{k\to\infty}S^{E}\left(A_{k}^{E}\right)=S^{*}. \qquad \blacksquare$$

Remark 7 *Since Gale (1987) assumes infinitely lived agents, he cannot let the entry fee converge to zero simultaneously with δ_k for technical reasons (see the next section.) Note however, that we need the assumption of vanishing entry fees only in Subcase 3 in the proof. As the reader can immediately verify, if entry costs remain constant, i.e., if $e_k=e$, this would imply that cutoff types might be separated by a wedge of size $2e$, i.e., $\left(c_x-v_x\right)\in\left(0,2e\right)$. In this case, the inequality (3.27) would imply that $\bar{V}^{S}\left(c\right)+\bar{V}^{B}\left(v\right)\geq v-c-2e$ for all v and c. Hence, with e being small, the outcome is close to being pairwise efficient. By continuity of $S\left(\cdot\right)$, this implies that when e becomes small, the outcome becomes efficient. Satterthwaite and Shneyerov (2007) assume participation costs κ_k per period instead of one-time entry costs e_k. It can be easily verified that whenever the lifetime trading probability converges to one, accumulated lifetime participation costs become zero. In Subcase 3 this is the case for all entering traders (by definition of c_x and v_x). Thus, absolute entry costs are zero, and Subcase 3 is impossible in the case of participation costs as well.*

3.6.2 Infinitely Lived Traders and Ponzi Schemes

Suppose the exit rate δ is equal to zero. This is a common assumption in the literature, e.g., it is used in Gale (1987) and in Satterthwaite and Shneyerov (2007). We want to know whether our approach is still valid. In the case of $\delta=0$, traders are "infinitely lived" and they can exit the pool only through trading. Time preferences are introduced by assuming the presence of a discount factor $\beta\in\left(0,1\right)$. Discounting then implies the existence of search costs and $\left(1-\beta\right)$ corresponds to the size of the friction. Again the question is whether the trading outcome becomes efficient when frictions vanish and the discount factor becomes one, $(1-\beta) \to 0$.

As we will see, we can indeed apply our approach to such a setup but we first need to take care of the problem of *Ponzi Schemes*. Because of such schemes, the expected payoff to entering traders can in principle be much higher than S^* (as defined in (3.13)): By shifting the timing of the sellers' trade, discounting implies that costs are diminished. Nevertheless, since sellers are infinitely lived, a mere shift in the timing of their trading does not influence how many buyers can trade, i.e., a shift does not influence the feasibility conditions. In the extreme, when shifting the timing of trade for all sellers to "infinity," all costs are discounted to zero. For the general problems of Ponzi schemes in economies with infinitely lived agent (or dynasties) and discounting, see for example Diamond (1965) and the subsequent literature.

Nevertheless, we show that Ponzi schemes are not part of *equilibrium* outcomes if two conditions hold: The first condition is that transfers are made through prices (condition *"prices only"*). The second condition is that a seller who trades receives a price above his cost, while a buyer who trades pays a price below his valuation (condition *"individual rationality"*). In outcomes which satisfy these conditions, Ponzi schemes are ruled out and the maximal surplus that can be attained subject to the conditions is S^*. Finally, since in Gale (1987) and in Satterthwaite and Shneyerov (2007) transfers are actually made through prices and since traders can reject to trade at unfavorable prices, equilibrium outcomes of their games satisfy these conditions.

Now, we go into the details. With infinitely lived agents, every trader who enters the market must ultimately trade. Otherwise a steady state with a finite pool is impossible. This makes the inclusion of an entry stage necessary. As in Section 3.6.1, let $Z^j(\cdot) \in \{0,1\}$ denote the entry decision, with $Z^S(c) = 1$ and $Z^B(v) = 1$ indicating the decision of types c and v to become active. Let $T^S_\infty(\cdot)$ be a measurable function, mapping $[0,1]$ into $\mathbb{R}_+$, where $T^S_\infty(c)$ denotes the undiscounted expected payments received by a seller of type c. Similarly, let $T^B_\infty(v)$ denote the undiscounted payment made by a buyer of type v. The undiscounted trading probabilities are $Q^S_\infty(\cdot) \in \Sigma_M$ and $Q^B_\infty(\cdot) \in \Sigma_M$. *Discounted* transfers and trading probabilities are denoted by $T^S_\beta(\cdot), T^B_\beta(\cdot)$ and by $Q^S_\beta(\cdot)$, $Q^B_\beta(\cdot)$, respectively.[45] Expected payoffs are given by

$$V^S(c) = T^S_\beta(c) - c\,Q^S_\beta(c) \quad \text{and} \quad V^B(v) = vQ^B_\beta(v) - T^B_\beta(v). \tag{3.29}$$

An outcome is given by $A^\infty = \left[V^S, V^B, Q^S_\infty, Q^B_\infty, T^S_\infty, T^B_\infty, Z^S, Z^B, Q^S_\beta, Q^B_\beta\right]$ and surplus is

$$S(A^\infty) = \int_0^1 V^S(c)\,dG^S(c) + \int_0^1 V^B(v)\,dG^B(v).$$

The mass balance condition becomes

Condition 8 ***Mass Balance with Infinitely Lived Players***. *An outcome A^∞ satis-*

[45] Suppose $D(p)$ is the probability of trading per period, then the discounted trading probability is $Q^S(p) = \frac{D(p)}{1-\beta(1-D(p))}$. Similarly, if $t(p)$ is the expected transfer per period, then discounted expected transfers are $\frac{t(p)}{1-\beta(1-D(p))}$.

fies mass balance if

$$\int_0^1 Z^S(c)\, Q_\infty^S(c)\, dG^S(c) = \int_0^1 Z^B(v)\, Q_\infty^B(v)\, dG^B(v) \tag{3.30}$$

$$\text{and} \quad \int_0^1 Z^S(c)\, T_\infty^S(c)\, dG^S(c) = \int_0^1 Z^B(v)\, T_\infty^B(v)\, dG^B(v). \tag{3.31}$$

A surplus maximizing outcome A' which satisfies mass balance is the following: transfers are zero, all sellers and all buyers enter, and discounted trading probabilities are one for buyers and zero for sellers. Then we have

$$S(A') = \int_0^1 v dG^B(v) > S^*.$$

Note that this corresponds to the extreme case of a Ponzi scheme discussed in the beginning of this section: Since in expectation, sellers trade "infinitely" many periods after their entry, costs are discounted to zero.

To rule out outcomes like that, we introduce two conditions that are satisfied by the existing models. First, transfers are made only through prices:

Condition 9 ***Prices only****. There are functions $p^S(\cdot) \in \Sigma_M$ and $p^B(\cdot) \in \Sigma_M$ such that for all c and for all v:*

$$\begin{aligned} T_\beta^S(c) &= Q_\beta^S(c)\, p^S(c) \quad \text{and} \quad T_\beta^B(v) = Q_\beta^B(v)\, p^B(v) \\ T_\infty^S(c) &= Q_\infty^S(c)\, p^S(c) \quad \text{and} \quad T_\infty^B(v) = Q_\infty^B(v)\, p^B(v). \end{aligned} \tag{3.32}$$

Second, we require that for all entering types, prices are *individually rational*:

Condition 10 ***Individual Rationality****. An outcome is individually rational if*

$$\begin{aligned} \forall c \text{ st. } Z^S(c) &= 1: \quad p^S(c) \geq c, \\ \forall v \text{ st. } Z^B(v) &= 1: \quad p^B(v) \leq v. \end{aligned}$$

Let $\hat{A}^{IR}$ be the set of outcomes which satisfy *mass balance*, *prices only*, and *individual rationality*. Together with the definition of payoffs in (3.29), surplus for any $A \in \hat{A}^{IR}$ is given by

$$S^\infty(A) = \int_0^1 Z^S(c)\, Q_\beta^S(c) \left(p^S(c) - c\right) dG^S(c) + \int_0^1 Z^B(v)\, Q_\beta^B(v) \left(v - p^B(v)\right) dG^B(v).$$

Now we demonstrate that S^*, as defined in (3.13), is the constrained maximum. First, note that the terms in the integral are positive, i.e., for all c such that $Z^S(c) = 1$, we have $\left(p^S(c) - c\right) \geq 0$; the same applies to buyers. Hence, in order to maximize $Z^\infty(\cdot)$, all entering traders must trade immediately, i.e., $Q_\beta^S(c) = 1$ for all c st. $Z^S(c) = 1$, and similarly for buyers. In addition, mass balance (3.31) requires $\int_0^1 Z^S(c)\, p^S(c)\, dG^S(c) = \int_0^1 Z^B(v)\, p^B(v)\, dG^B(v)$. Together, a necessary condition for an outcome A to be in $\arg\max_{A \in \hat{A}^{IR}} S^\infty(\cdot)$ is that

$$S^\infty(A) = \int_0^1 v\, Z^B(v)\, dG^B(v) - \int_0^1 c\, Z^S(c)\, dG^S(c).$$

Now the problem of maximizing $S^{\infty}(\cdot)$ is similar to our original problem in Section 3.3.2. Indeed, let Q^{EW} be the set of "Walrasian" outcomes,

$$A^{EW} \equiv \left\{ A \mid \int_0^1 \left| Q_\beta^S(c) Z^S(c) - 1_{c \le p^w}(c) \right| dc, \int_0^1 \left| Q_\beta^B(v) Z^B(v) - 1_{v \ge p^w}(v) \right| dv = 0 \right\},$$

then by reasoning analogously to Lemma 14, A^{EW} is the set of the maximizers of the surplus, $A^{EW} \equiv \arg\max_{A \in \hat{A}^{IR}} S^{\infty}(\cdot)$. Thus

$$S^* = \sup_{A^\infty \in \hat{A}^{IR}} S^{\infty}(\cdot).$$

As mentioned in the beginning of this section, in the models by Gale (1987) and Satterthwaite and Shneyerov (2007) traders are restricted to use prices and bids, respectively. In addition, trade is voluntary so that no seller would agree to trade at a price below costs and no buyer would agree to trade at a price above his valuation. Thus, the set of equilibrium outcomes is a subset of $\hat{A}^{IR}$, and our approach is valid.

To apply our approach to specifications with infinitely lived traders and entry, we need to rewrite the conditions of Section 3.6.1 by simply substituting Q_β^j for Q^j.[46] Then, the proof in Section 3.6.1 would imply that $S^{\infty}(A_k^{\infty}) \to S^*$ for all sequences $\{A_k^{\infty}\}_{k=1}^{\infty}$ that satisfy the four conditions and that contain only elements from $\hat{A}^{IR}$. To check whether our conditions actually hold in the models by Gale (1987) and Satterthwaite and Shneyerov (2007), note that *monotonicity*, *no rent extraction*, and *strong efficiency* are immediate.

The *weak availability condition*, however, is somewhat more subtle with infinitely lived agents. In particular, if the trading probability for some set of types converges to zero, then this set might *flood* the market. And even if for some other set of types the limiting trading probability is below one, this other set might make up only a vanishing fraction of the total pool. To avoid this problem, i.e., to avoid the existence of a set of types that trade with a probability approaching zero, both papers include a variant of nonvanishing absolute search costs. The idea is that whenever the limiting trading probability becomes zero for some types, their expected trading revenues become zero and therefore they cannot recover any positive entry costs. Thus, lifetime trading probabilities must stay strictly positive for all entering types. Specifically, Gale (1987) assumes that even as the discount factor converges to one, the entry cost $e \in (0,1)$ remains constant. Therefore, it is not profitable for agents to enter if they trade only with a probability close to zero. Satterthwaite and Shneyerov (2007) assume a participation cost κ per period that converges to zero at the same rate as the discount factor. One can verify that the accumulated lifetime participation costs are strictly positive whenever the limiting discounted lifetime trading probability becomes zero. Again, this implies that entry is unprofitable when trading probabilities are close to zero.

3.7 Failures

In this section we demonstrate how to use our approach to understand why convergence to efficiency fails in some specifications of dynamic matching and bargaining games. In

[46] Monotonicity would be required of $\bar{Q}_\beta^j$; no rent extraction would be a condition on the slope of $V^j(\cdot)$; weak availability would require that $\bar{L}^j(x) = 1$ whenever $\bar{Q}_\beta^j(x) < 1$ and $Z^j(x) = 1$; and weak efficiency would still be a condition on the joint surplus of available types.

the first subsection, we discuss how the failure of convergence with symmetric information can be attributed to the failure of the **no rent extraction** condition. In the following section, we discuss how the simultaneity of decisions in double auctions can lead to the failure of **weak efficiency**. Finally, we discuss a model with cloning and show that the **mass balance** condition does not hold in this case.

We do not provide a specification in which the **monotonicity** condition is the only condition that fails, because there is no such model in the literature. The failure of **availability** with an entry stage is discussed at the end of the second subsection and interpreted as a coordination failure when traders have to decide simultaneously whether to enter the market.

3.7.1 No Rent Extraction fails with Full Information and Asymmetric Bargaining Power

Suppose sellers in the basic model can observe the valuation of the buyer prior to making an offer. Clearly, this makes trading *within* each pair efficient: They trade whenever the trading surplus $(v-c)$ is larger than the joint continuation payoff $(1-\delta)\left(V^S(c)+V^B(v)\right)$. But as we will see, overall efficiency of trading in the market as a whole decreases: with $\delta\to 0$, the limiting trading outcome is no longer efficient. Here, we want to show which of our conditions is violated to explain why convergence to efficiency fails. A full discussion of the model can be found in the note by chapter one.[47]

For illustration, we use the setup of Section 3.5.2: There, traders in each pair can mutually observe their valuations and costs. With probability α, the seller is chosen to be the proposer of a price offer, while with probability $(1-\alpha)$, the buyer is chosen. While in Section 3.5.2 we assume that α must be interior, i.e., $\alpha\in(0,1)$, here we assume that the seller has all the bargaining power, i.e., $\alpha=1$. Let $\left\{A_k^F\right\}_{k=1}^{\infty}$ be a sequence of equilibrium outcomes of the model of Section 3.5.2, with α set equal to 1. We can characterize the outcomes by two observations. First, sellers appropriate all the trading surplus: no buyer receives strictly positive payoffs and $V^B(v)\equiv 0$. The price offer to a buyer is either equal to his type or too high to be acceptable, i.e., $p^S(c,v)\geq v$ for all c,v.[48] Second, the limiting outcome can be described by a some cutoff $\bar{v}\in(0,1)$ such that the limiting lifetime trading probabilities of a buyer is zero if $v<\bar{v}$ and one if $v>\bar{v}$, i.e., $\bar{Q}^B=1_{v>\bar{v}}$.[49]

While the sequence $\left\{A_k^F\right\}_{k=1}^{\infty}$ can be shown to satisfy monotonicity, availability, and weak efficiency,[50] the no rent extraction condition fails: Since $\bar{Q}^B(v_x)=1$ for any $v_x>\bar{v}$, the condition requires that payoffs increase with a slope of one, i.e., for types $v'>v_x$, it

[47] In chapter one, sellers have homogeneous costs $c\equiv 0$ to ease exposition. Here, sellers are heterogeneous to retain the consistency of the underlying economy across specifications.

[48] Price offers are always larger than or equal to reservation prices, as argued in Section 3.5.2, i.e., $p^S(c,v)\geq r^B(v)$. By definition, $v-r^B(v)=(1-\delta)V^B(v)$, and by $V^B(v)\leq v-r^B(v)$, $v-r^B(v)=0$.

[49] Suppose not. Because trading probabilities $\bar{Q}^B(\cdot)$ can be shown to be monotone, this would imply that for some interval (a,b), $\bar{Q}^B(v)\in(0,1)$ for all $v\in(0,1)$ ($\bar{Q}^B(v)\equiv 0$ (or $\equiv 1$) for all v is never an equilibrium outcome). Then, for any $v'\in(a,b)$, types $v\geq v'$ are *available* and a seller $c=0$ who trades only with $v\geq v'$ at prices $p^S(0,v)=\max\{v',v\}$ would trade with certainty and receive a payoff $\lim_{k\to\infty}U^S\left(0,p^S\right)\geq v'>a$. This is a contradiction.

[50] Weak efficiency is immediate with symmetric information; availability holds because the matching technology is unchanged to the case of $\alpha\in(0,1)$; monotonicity holds essentially because sellers' profits satisfy the strict single crossing condition, i.e., sellers with lower costs prefer to trade with a higher probability at a lower price.

must be that $\bar{V}^B(v') \geq \bar{V}^B(v_x) + (v' - v_x) > 0$. However, the payoff to any such type v' is still zero, and his rent $(v' - v_x)$ is *extracted*: Part of this rent will go to the sellers but part of it is wasted. Because of this, the equilibrium outcome is not efficient in the limit.

Remark 8 *Prices with symmetric information are "monopolistic," i.e., $p^S(c,v) \geq v$, by the same reasoning as in Diamond (1971): Sellers can use the waiting costs $\delta \in (0,1)$ to "hold-up" buyers. However, in the models that are used to derive the familiar Diamond paradox, this outcome is still efficient because buyers and sellers are assumed to be homogeneous.*[51] *Here, with heterogeneous types, inefficiencies first stem from the fact that sellers rather incur rationing than trading at low prices with low valuation buyers, and second they stem from the possibility of trading for sellers who have costs above p^w and who should not trade.*

Remark 9 *One possible way to restore the no rent extraction property while leaving the bargaining power with sellers ($\alpha = 1$) is to assume that buyers' valuations are not perfectly observable (i.e., buyers have some "privacy"): the appendix of chapter one contains an extension where sellers receive only a signal about the valuation of the buyer and where this signal contains noise. With $\delta \to 0$, buyers can patiently wait until their type is misconceived as being very low so that they receive a low price offer. In particular, suppose it becomes certain that some buyer of type v_x can trade at an expected price $p \leq v_x$ in the limit. Then any type $v' > v_x$ can wait until he receives the same offers and he can trade at an expected price $p \leq v_x$ as well. The payoff to v' is therefore at least $(v' - v_x)$ larger than the payoff to v_x. Thus, the no rent extraction condition holds, and the outcome becomes efficient in the limit.*

3.7.2 Weak Efficiency fails without Sequential Rationality

Serrano (2002) is the first to specify the bargaining protocol as a simultaneous double auction.[52] He shows that equilibrium outcomes do not need to become efficient. Without going into the details, we can replicate his result in our framework: Suppose we assume in the basic model that the buyer and the seller *simultaneously* announce a reservation price r and price offer p, respectively. Trade happens at the price p whenever the reservation price is below the price offer. If we leave the rest of the model unchanged, the following is an equilibrium for every δ_k: $p(c) \equiv 1$ and $r(v) \equiv 0$. In the corresponding equilibrium outcome A_k, trading probabilities are zero for all types and $S(A_k) = 0$ for all k.

While the sequence of outcomes satisfies monotonicity, no rent extraction, and availability, weak efficiency fails: For any pair $(v,\ c)$ with $v > c$, the trading surplus $(v - c)$ is strictly larger than their joint limiting payoffs, $\liminf \left(V_k^S(c) + V_k^B(v)\right)$, which is 0. Bargaining is inefficient because of a "coordination failure" between the traders. As observed by Serrano, this failure occurs because we cannot use sequential rationality to rule out such equilibria.[53]

[51] In the original model, individual buyers have elastic demand for multiple units. Sellers, however, are restricted to offer linear prices. Therefore, they distort the trading quantity downwards. This inefficiency disappears once the restriction to linerar prices is dropped.

[52] His interest, however, stems from the prior use of simultaneous auctions in dynamic matching and bargaining games in the context of *common values*, see e.g., Wolinsky (1990).

[53] In our basic model, sequential rationality enter via the assumption that buyers use a reservation price that is equal to the continuation payoff.

Note the similarity to the failure of convergence with an entry stage: Setting a price above the highest valuation (and setting a reservation price below the lowest cost) is similar to deciding to not become an active trader. And just as it is a best response not to take an interior bargaining position if no other trader does so, it is a best response not to become active if no other trader does. But note also that just in the same way as we can restore sequential rationality by introducing "trembles" to the price setting decisions we can restore equilibria with trading when traders tremble at the entry decision stage.[54]

3.7.3 Mass Balance fails with Cloning

Cloning refers to the assumption that every trader who leaves the market is replaced by an exact copy of his type, a *clone*. With this assumption, the inflow depends on the trading outcome and is *endogeneous*. The pool of traders, however, does not change over time and is exogeneous. A model with cloning has been recently used by De Fraja and Sakovics (2001). They use this model to argue that trading outcomes depend sensitively on the exact specifications of the bargaining protocol. Since this contrasts with the view taken in our paper, we want to understand their result. Throughout the first part of this section, we will follow De Fraja and Sakovics and take the exogeneously given pool (the "stocks") as the fundamental of our model, i.e., we define the "Walrasian" price and the surplus both with respect to this distribution. As we will see, with cloning, *equilibrium* outcomes generically yield a surplus strictly above S^*. This is the analogue to prices not being Walrasian, which is what De Fraja and Sakovics concentrate on.

Gale (1987) argued that one should define the Walrasian price and, analogously, the surplus, with respect to the inflows (see his critique of the model by Rubinstein and Wolinsky (1985) who use a cloning assumption). We provide a short comment on how to evaluate the surplus with respect to the inflow at the end.

To understand how it is feasible with cloning that the surplus of an outcome exceeds S^* we first sketch the idea in the following example. In this example, the full consumer surplus for all $v > 0$ is realized while expected costs are zero. (Note the similarity with the Ponzi scheme with infinitely lived traders): Suppose all buyers who are matched with a seller with costs $c \leq \varepsilon$ can trade at a price ε. If ε is close to zero and if all buyers can be certain to be matched with such a seller, then indeed the full consumer surplus for all $v > 0$ is realized while costs are zero. Cloning makes it possible: Because of cloning, the share of sellers in the pool who have costs $c \in [0, \varepsilon]$ is exogeneously fixed and strictly positive. Therefore, buyers have a strictly positive chance to be matched with such a seller in every single period, and with $\delta \to 0$, they become certain to be able to trade with such a seller in the limit.[55]

To understand the result in more depth, we use the symmetric information model of Section 3.5.2.[56] To recall the model: All traders from the pool are matched into pairs. In each pair they observe each others' valuation v and cost c. Then, with probability

[54] See Gale (1987, p. 30), who argues that equilibria without entry are not stable.

[55] Note, however, that within every period almost no trade takes place. Therefore, almost no new traders enter the market and the surplus with respect to the inflow converges to one (see the comment at the end of this section).

[56] The main differences are that DeFraja and Sakovics (2001) include an entry stage and have discounting instead of an exit rate. In addition, they use a noisy search technology, i.e., they assume that a buyer is matched with a random number of sellers. None of these differences affects the main conclusions.

$\alpha \in (0,1)$, the sellers is chosen to be the *proposer* of a price while with probability $(1-\alpha)$ the buyer is chosen to be the proposer. The other trader, the *responder*, can either accept or reject the offer. Afterwards, all those pairs in which the responder accepts the offer, trade and leave the pool. An additional share δ of those who did not trade leaves (dies). Now the new traders enter. But different from the model in Section 3.5.2, the inflow consists of exact clones of the leaving traders. Therefore, independent of who actually traded, the distribution of traders in the pool at the end of the period is always equal to the distribution in the beginning. Let these distributions be $G^S(\cdot)$ and $G^B(\cdot)$.

For every δ_k and for α, we fix an equilibrium outcome $A_k^C(\alpha) = \left[V_k^S, V_k^B, Q_k^S, Q_k^B\right]$ of the cloning variant. Since we are using exactly the same matching and bargaining technology as in Section 3.5.2, our four conditions still hold. Therefore, from the first part of the proof of the main Proposition 4, we know that the outcome must become pairwise efficient for every convergent subsequence, i.e.,

$$\bar{V}^S(c) + \bar{V}^B(v) \geq v - c. \tag{3.33}$$

Actually, the limiting outcome can be fully characterized quite easily: It is standard to verify[57] that there is a price $p^N(\alpha)$ such that, for $\delta_k \to 0$, limiting payoffs become $\bar{V}^S(c) = \max\left\{p^N(\alpha) - c, 0\right\}$ and $\bar{V}^B(v) = \max\left\{v - p^N(\alpha), 0\right\}$. Limiting trading probabilities become $\bar{Q}^S(\cdot) = 1_{c<p^N(\alpha)}$ and $\bar{Q}^B(\cdot) = 1_{v>p^N(\alpha)}$. The price $p^N(\alpha)$ itself is given as the unique solution to the following condition:

$$(1-\alpha)\int_0^{p^N(\alpha)} \left(p^N(\alpha) - c\right) dG^S(c) = \alpha \int_{p^N(\alpha)}^1 \left(v - p^N(\alpha)\right) dG^B(v).$$

Note that for $\alpha = \frac{1}{2}$, the price $p^N\left(\frac{1}{2}\right)$ equates the expected surplus of buyers and sellers.

The price $p^N(\alpha)$ depends on the distribution of bargaining power, and the price is strictly increasing in α. Thus, generically, the limiting outcome fails to be Walrasian:[58] Only for a single point α^* does $p^N(\alpha^*)$ equal p^w, while for all $\alpha \in (0,1)\setminus\alpha^*$, $p^N(\alpha) \neq p^w$ and hence $\left(\bar{Q}^S, \bar{Q}^B\right) \notin Q^W$. This is not necessarily bad: for every $p^N(\alpha)$, the expected limiting payoff is *above* S^*. Simple algebra reveals that for all $p^N(\alpha) \neq p^w$

$$S^* < \int_{p^N(\alpha)}^1 \left(v - p^N(\alpha)\right) dG^B(v) + \int_0^{p^N(\alpha)} \left(p^N(\alpha) - c\right) dG^S(c).$$

To illustrate the failure we look at the extreme case with $\alpha \to 0$, i.e., when buyers enjoy all the bargaining power. In this case, the condition requires that the price must become zero, $\lim_{\alpha\to 0} p^N(\alpha) = 0$. (We take the limit of outcomes with respect to $\delta \to 0$ first and with $\alpha \to 0$ afterwards.) This corresponds to our introductory example, with $\varepsilon = p^N(\alpha)$ being small. Thus, expected equilibrium payoffs among buyers become approximately $\int_0^1 v dG^B(v)$ while expected payoffs to sellers become zero. Hence,

[57] Using the techniques by Gale (1987), see for example the teaching notes by Wright, http://www.ssc.upenn.edu/~rwright/courses/rw.pdf .

[58] DeFraja and Sakovics interpret this and similar results as indicating the importance of "local market conditions" for limiting outcomes, reflected here in the distribution of bargaining power.

$\lim_{\alpha\to 0}\lim_{\delta_k\to 0} S\left(A_k^C(\alpha)\right) = \int_0^1 vdG^B(v) - 0$. This is strictly larger than S^* - how can this happen?

Note that in the limiting outcome, almost all buyers trade with certainty, while almost no seller trades. Thus, the mass buyers who trade converges to $\int_0^1 1dG^B(v) = 1$, while the mass of sellers who trade converges to $\int_0^1 0dG^S(c) = 0$. Therefore, for some δ_k and α small enough, the mass balance condition (3.11) is violated.

Why is it possible with cloning that all buyers can trade? For any $\alpha \in (0,1)$, in any period, the probability that a buyer is matched with a sellers who accepts to trade at the price $p^N(\alpha)$ is strictly positive, since $G^S\left(p^N(\alpha)\right) > 0$. So with $\delta_k \to 0$, it becomes certain that a buyer will be matched with a seller who agrees to trade. In the original model, without cloning, this is not true: If all trade occurs at a price $p^N(\alpha)$ close to zero, sellers with costs $c \leq p^N(\alpha)$ would become scarce, and the share of such sellers in the pool would become zero.

Since our specification of the matching and bargaining protocol in the above example is standard, the peculiar results are only due to the cloning assumption. Nevertheless, De Fraja and Sakovics (2001) explicitly introduce cloning as a *technical* assumption.[59] They do not claim that cloning is meant to reflect underlying economic conditions. But because this assumption has such a strong implication for the results, one might try to use means other than cloning to solve possible technical problems.

Also, one might take the inflows as fundamental objects as argued by Gale in 1987. Let us therefore consider the surplus with respect to the entering traders. For this, let $A^{IN} = \left[V^S, V^B, G^{SC}, G^{BC}, Q^S, Q^B\right]$ denote an outcome where the c.d.f.s $G^{SC}(\cdot)$ and $G^{BC}(\cdot)$, refer to the endogeneous inflows of clones. The expected surplus of the entering traders is given by

$$S^C\left(A^{IN}\right) = \int_0^1 V^S(c)\, dG^{SC}(c) + \int_0^1 V^B(v)\, dG^{BC}(v).$$

Thus, maximization of the surplus requires not only maximization with respect to the expected payoff of each type of the clones, i.e., with respect to V^S and V^B, but also with respect to their endogeneous distributions G^{SC} and G^{BC}. In particular, an outcome which satisfies the condition of Lemma 16, i.e., pairwise efficiency for all types, does not need to be efficient. In fact, the limiting equilibrium outcome associated with small α is pairwise efficient as we know from (3.33) but it is nevertheless quite inefficient when α is close to the zero: Then, the share of sellers who actually trade (i.e., those with costs below α) is almost zero. But then almost no buyer can trade in a given period. Thus, the inflow of clones who replace those buyers who trade, must be almost zero. So with $A_k^{IN}(\alpha)$ being the equilibrium outcome for given α and k, the limiting surplus is zero for

[59] DeFraja and Sakovics have infinitely lived traders and assume an entry stage (see Section 3.6.2 of our paper). They write that they assume cloning to ensure the stationarity of the mass of traders who decide *not* to enter the pool (see p. 846). They do not explicitly state any problem that would arise otherwise. Actually, in Gale (1987, section 6) and Satterthwaite and Shneyerov (2006), the mass of non-entering types is "infinite" without causing problems. (Basically, this mass just plays no role in any of the above papers and is not even explicitly defined.)

$k \to \infty$ and $\alpha \to 0$:

$$\begin{aligned} \lim_{\alpha\to 0}\lim_{k\to\infty} S^C\left(A_k^{IN}(\alpha)\right) &= \lim_{\alpha\to 0}\lim_{k\to\infty} \int_0^1 \underbrace{V_k^S(c)}_{\to 0}\underbrace{dG_k^{SC}(c)}_{\to 0} + \int_0^1 \underbrace{V_k^B(v)}_{\to 1}\underbrace{dG_k^{BC}(v)}_{\to 0} \\ &= 0. \end{aligned}$$

This finding is our analogue to the almost trivial observation that prices are market clearing with respect to the endogeneous inflow.

3.8 Conclusion

We have introduced a new approach to the analysis of decentralized markets with vanishing frictions. By directly characterizing sequences of trading outcomes independently of the fine details of the trading institution, we have shown which conditions imply convergence to efficiency across different models. We then have validated this approach by showing that sequences of equilibrium outcomes for models in the literature satisfy these conditions.

Several open questions remain. First, we assume that p^w is known ex ante. In many markets, however, traders are uncertain about the supply and demand, and p^w is a random variable. Can we expect decentralized markets to converge to efficiency even if traders have to learn the state of the market? Second, when analyzing trading with an entry stage we had to exclude by assumption sequences which are trivial and in which no trader ever enters. Are there conditions on the economic fundamentals that ensure that every sequence is non-trivial? Finally, what can we say about more general preferences and production technologies, in particular, what can we say about the interaction of markets for several different goods, like, for example, the markets for labor and capital?

A Appendix

A.1 Chapter 1: Noisy Signals

Suppose sellers do not observe v directly but receive a signal θ that is correlated with v. We look at three signal distributions, denoted by some parameter $\rho \in \{0, \bar{\rho}, 1\}$: If $\rho = 0$, then the signal is constant and it contains no information. If $\rho = 1$, then $\theta = v$ and the seller learns v . If $\rho = \bar{\rho}$, the signal θ is distributed according to some smooth distribution $F(\theta|v)$ where $F(\cdot|v)$ has full support and a strictly positive density $dF(\theta|v)$. The density is jointly uniform continuous in v and in θ. Otherwise, the signal can be arbitrarily precise, i.e., the posterior variance $var[v|\theta]$ can be close to zero.

We take the model from before. But now the pricing strategy must depend on θ instead of v, $P(p', \theta) = prob[p \leq p'|\theta]$. The per period probability of trading for a type v with a reservation price r is $S(r, v) = \int_0^1 P(r(v), \theta) dF(\theta|v)$. Given the distribution of types in the pool, sellers who receive a signal θ use Bayes' rule to update their belief $B(v'|\theta) = prob[v \leq v'|\theta, \Phi^B, \rho]$. An equilibrium is described by the quintuple $\{P(p', \theta), r(v), \Phi^B, M, B(v'|\theta)\}$, where the functions have to satisfy the appropriate equilibrium conditions described before and in the main text. We assume that for every δ and $\rho \in \{0, \bar{\rho}, 1\}$ there is some equilibrium. Denote by $W(\delta, \rho)$ the expected ex ante welfare of the entering traders in one of these equilibria with

$$W(\delta, \rho) = q^S(P(\cdot, \cdot)) E[p|P(\cdot, \cdot)] + \int_0^1 q^B(r(v))(v - E[p|p \leq r(v), v]) g(v).$$

The maximal welfare is

$$W^{eff} \equiv \int_0^1 v dG(v) = E[v].$$

We show that when frictions are large, welfare is higher with precise information: for δ close enough to one, welfare is higher if $\rho = 1$ than if $\rho = \bar{\rho}$. When frictions are small, welfare is higher when information is imprecise: for δ close enough to zero, welfare is higher if $\rho = \bar{\rho}$ than if $\rho = 1$:

Proposition 6 *There are some δ_H and δ_L with $0 < \delta_L < \delta_H < 1$ such that*

$$\begin{aligned} W(\delta, \bar{\rho}) &< W(\delta, \rho = 1) \quad \forall \delta \geq \delta_H, \\ W(\delta, \bar{\rho}) &> W(\delta, \rho = 1) \quad \forall \delta \leq \delta_L. \end{aligned}$$

In the remainder we sketch the proof. Note first that the outcome of bilateral trade is clearly efficient with symmetric information and $\delta = 0$, i.e., $W(1, 1) = W^{eff}$ and

$$\lim_{\delta \to 1} W(\delta, 1) = W^{eff}.$$

With asymmetric information, the outcome is not efficient, $W(1, \bar{\rho}) < W^{eff}$ and

$$\lim_{\delta \to 1} \sup W(\delta, \bar{\rho}) < W^{eff},$$

and so the first line of the proposition follows. For the second line, note that we have already proven that the limiting outcome is bounded away from efficiency

$$\lim_{\delta\to 0} W(\delta, 1) < W^{eff},$$

and it remains to be proven that every equilibrium with vanishing frictions and with a noisy signaling technology becomes efficient:

$$\lim_{\delta\to 0} W(\delta, \bar{\rho}) = W^{eff}.$$

To do so, recall that the density $dF(\cdot|\cdot)$ is strictly positive and continuous in v and θ. We show that this implies that if the trading probability of any type v' who uses a reservation price $r_k(v')$ converges to one, $q_k(r_k(v'), v') \to 1$, then the trading probability of any type v'' who mimics v' and uses $r_k(v')$ converges to one as well. With this observation in place, one can use the reasoning put forth in chapter two to show convergence to efficiency.

We want to show that for every r and v'', v' the ratio of trading probabilities $\frac{S(r,v'')}{S(r,v')}$ is bounded. For this, we utilize the fact that the ratio of densities is bounded: $\frac{dF(\theta|v')}{dF(\theta|v'')} \geq \bar{S}$ for some $\bar{S} > 0$ by joint continuity of the density $dF(\cdot|\cdot)$ in v and θ. This bound carries over to the ratio of trading probabilities:

$$\begin{aligned}\frac{S(r, v')}{S(r, v'')} &= \frac{\int_0^1 P_k(r, \theta)\, dF(\theta|v')}{\int_0^1 P_k(r, \theta)\, dF(\theta|v'')} \\ &\geq \frac{\int_0^1 P_k(r, \theta)\, \bar{S} dF(\theta|v'')}{\int_0^1 P_k(r, \theta)\, dF(\theta|v'')} = \bar{S} \quad \forall r.\end{aligned}$$

We can derive the lifetime trading probability from the recursive formula

$$q_k^B(r_k(v''), v'') = S_k(r_k(v''), v'') + \left(1 - S_k(r_k(v''), v'')\right)(1-\delta_k)\, q_k^B(r_k(v''), v''),$$

and so

$$\begin{aligned}q_k^B(r_k(v''), v'') &= \frac{S_k(r_k(v''), v'')}{\delta_k + S_k(r_k(v''), v'')(1-\delta_k)} \\ &= \frac{1}{\frac{\delta_k}{S_k(r_k(v''),v'')} + (1-\delta_k)}.\end{aligned}$$

Thus, if the trading probability for v'' converges to one, i.e., if $\lim_{k\to\infty} q_k^B(r_k(v''), v'') = 1$, it must be that the ratio $\frac{\delta_k}{S_k(r_k(v''),v'')}$ converges to 0. Thus, the trading probability $S_k(r_k(v''), v'')$ is "large" relative to δ_k. Now we utilize that the trading probability for type v', who waits for prices $p \leq r_k(v'')$, is at least $\bar{S}S_k(r_k(v''), v')$. Therefore, $S_k(r_k(v''), v'')$ is large relative to δ_k as well: if $\frac{\delta_k}{S_k(r_k(v''),v'')} \to 0$, then

$$\begin{aligned}\lim_{k\to\infty} \frac{\delta_k}{S_k(r_k(v''), v')} &\leq \lim_{k\to\infty} \frac{\delta_k}{\bar{S}S_k(r_k(v''), v'')} \\ &= 0.\end{aligned}$$

Evaluating the limit $q_k^B(r_k(v''), v')$ with this knowledge, our claim follows:

$$\lim_{k\to\infty} q_k^B\left(r_k\left(v''\right), v'\right) = \lim_{k\to\infty} \frac{1}{\underbrace{\frac{\delta_k}{S_k\left(r_k\left(v''\right), v'\right)}}_{\to 0} + (1-\delta_k)} = 1.$$

Now we can use the reasoning of chapter two. We sketch the proof informally: Let $h_k \equiv \sup_v r_k(v)$ be the highest accepted price and $l_k \equiv \inf\{\text{supp}P(\cdot,\cdot)\}$ be the lowest offered price.[60] Then the law of one price holds and $\lim_{k\to\infty}(h_k - l_k) = 0$ because the lemmas supporting that proof hold here as well: Suppose price dispersion did not vanish, then for some h and l with $h > l$, $h_k \to h$ and $l_k \to h$. Along this (sub-)sequence three lemmas hold: First, the trading probability of all types $v < h$ must be bounded away from one in the limit, i.e., Lemma 6 holds and $\lim_{k\to\infty} \inf q_k^B(r_k(v), v) < 1$ for all $v < h$. Suppose not, then some type $v'' < h$ would be able to trade for sure at some expected price $E[p|p \leq r(v'')]$ below v''. But then the type who is supposed to accept h_k would want to mimic v'', a contradiction to the definition of h_k.[61] As prices below h are rare in the limit, types $v \in (l, h)$ accept intermediate prices and $\lim_{k\to\infty} \inf r_k(v) > l$ for all $v \in (l, h)$, corresponding to Lemma 7. And Lemma 8, which states that buyers who do not trade with certainty make up a strictly positive share of the pool, holds as well. Together, these lemmas imply that there is some price $p' > l$ such that a seller who offers this price is able to find a buyer accepting his offer with $\delta_k \to 0$, contradicting the definition of l. Given the law of one price, we can again use the reasoning from chapter two to show that this price must be 0: Suppose not and suppose prices converge to some $p^c > 0$ along some (sub-)sequence instead. Then a seller who offers to trade at any $p' < p^c$ would be able to trade for sure with $\delta_k \to 0$. Hence, by p' being arbitrarily close to p^c, all sellers must be able to trade for sure at p^c as well. But the mass of sellers is one, while at all prices $p^c > 0$ only a mass of buyers strictly smaller than one, $1 - G(p^c) < 1$, can trade. So more sellers than buyers would have to trade - a contradiction. Hence, p^c must be zero ■

A.2 Chapter 2: Proof of Global Concavity

Note that $\frac{\partial^2}{\partial p^2} U^S\left(p, 0|\sigma\left(p^0\right)\right) = q''(p)\,p + q'(p)\,p + q'(p)$, with $q'(p) = \frac{\partial}{\partial p} q^S\left(p|\sigma\left(p^0\right)\right)$ and $q''(p) = \frac{\partial^2}{\partial p^2} q^S\left(p|\sigma\left(p^0\right)\right)$. For the latter, note that

$$\begin{aligned}\frac{\partial}{\partial p} q^S\left(p|\sigma\left(p^0\right)\right) &= \frac{d(p)\left[D(p) + \delta - \delta D(p)\right] - D(p)\, d(p)\left[1-\delta\right]}{\left[D(p) + \delta - \delta D(p)\right]^2} \\ &= \frac{d(p)\, D(p)}{\left[D(p) + \delta - \delta D(p)\right]^2},\end{aligned}$$

[60] Note that we did not prove that reservation prices are increasing in types. So it might be that $r_k(v) > r_k(1)$, for some v and k.

[61] If $r_k(v) < h_k$ for all k, choose any v_k such that $r_k \geq 0.5\,(v'' + h_k)$ and observe that this type would want to deviate for k large enough.

and

$$\frac{\partial^2}{\partial p^2} q^S\left(p|\sigma\left(p^0\right)\right) = \frac{d'(p)\left[D(p)+\delta-\delta D(p)\right]^2 - \delta 2 d^2(p)\left[D(p)+\delta-\delta D(p)\right](1-\delta)}{\left[D(p)+\delta-\delta D(p)\right]^2}$$

with $d'(p) = -g^{B\prime}(v(p))\frac{1}{\delta}$ for $p \in \left(p^0, r\left(1|p^0\right)\right]$ and $d'(p) = -g^{B\prime}(v(p))$ for $p \in [0, p^0)$. By the assumption that $\left(1 - G^B(\cdot)\right)$ is strictly concave, $g^{B\prime}(v) > 0$ for all v. Inspection of the first and second derivative of $q^S(\cdot)$ show that they are strictly negative, which implies that payoffs are strictly concave on the intervals $[0, p^0)$ and $\left(p^0, r\left(1|p^0\right)\right)$. By $\frac{\partial}{\partial p}U^S\left(p, 0|\sigma\left(p^0\right)\right)$ being continuous at p^0 (but not differentiable) this implies that it is globally strictly concave.

A.3 Chapter 2: Proof of Mass Balance

We show the identity of trading masses by algebraic manipulation, dropping the dependency on σ:

$$\begin{aligned}
\int_0^1 q^S(r(c))\,g^S(c)\,dc &= \int_0^1 \frac{D(p(c))}{D(p(c))+\delta-\delta D(p(c))}dG^S(c) \\
&= \int_0^1 MD(p(c))\,d\Phi^S(c) \\
&= \int_0^1 M\left[\int_{v(c)}^1 d\Phi^B(v)\right]d\Phi^S(c) \\
&= \int_0^1 M\left[\int_{v(c)}^1 \frac{g^B(v)}{M\left(S(r(v))+\delta-\delta S(r(v))\right)}dv\right]d\Phi^S(c) \\
&= \int_0^1 \left[\int_0^{c(v)} d\Phi^S(c)\right]\frac{g^B(v)}{\left(S(r(v))+\delta-\delta S(r(v))\right)}dv \\
&= \int_0^1 S(r(v))\frac{g^B(v)}{\left(S(r(v))+\delta-\delta S(r(v))\right)}dv = \int_0^1 q^B(r(v))\,g^B(v)\,dv,
\end{aligned}$$

and similarly the identity of expected payments follows from:

$$\begin{aligned}
\int_0^1 p(c)\,q^S(p(c))\,g^S(c)\,dc &= \int_0^1 p(c)\,MD(p(c))\,d\Phi^S(c) \\
&= \int_0^1 M\left[\int_{v(c)}^1 \frac{g^B(v)}{M\left(S(r(v))+\delta-\delta S(r(v))\right)}dv\right]p(c)\,d\Phi^S(c) \\
&= \int_0^1 \left[\int_{v(c)}^1 \frac{1}{S(r(v))}g^B(v)\,q^B(r(v))\,dv\right]p(c)\,d\Phi^S(c) \\
&= \int_0^1 g^B(v)\,q^B(r(v))\frac{1}{S(r(v))}\left[\int_0^{c(v)} p(c)\,d\Phi^S(c)\right]dv \\
&= \int_0^1 q^B(r(v))\,E\left[p|p\le r(v)\right]g^B(v)\,dv \qquad \blacksquare
\end{aligned}$$

A.4 Chapter 2: Proof of Lemma 6

First, we show convergence of the inverse $c(\cdot|p(\cdot))$ starting by showing that we can disregard points where $p(\cdot)$ is flat because then the inverse function will have a jump and these points have zero measure. At all c where $p(\cdot)$ is not flat, convergence of the inverse function at $p(c)$ is not a problem. Then we prove convergence of the composite making use of its monotonicity and continuity almost everywhere.

Suppose $\bar{p}(\cdot)$ is *flat* at c_f i.e., for some $c_{ff} \neq c_f$, $\bar{p}(c_f) = \bar{p}(c_{ff}) \equiv p_f$ and suppose $p_f \in (0,1)$ and $c_f < c_{ff}$ (wlog). Then $c(p_f - 0|\bar{p}(\cdot)) < c(p_f + 0|\bar{p}(\cdot))$ because $c(p_f - \varepsilon) = \sup\{c, 0|\bar{p}(c) \leq p_f - \varepsilon\} \leq c_f$ and $c(p_f + \varepsilon) = \sup\{c, 0|\bar{p}(c) \leq p_f + \varepsilon\} \geq c_{ff}$. So p_f is a jump point of $\bar{c}(\cdot) = c(\cdot|\bar{p}(\cdot))$. Let $p_l \equiv \bar{p}(0)$ and $p_h \equiv \bar{p}(1)$, then for all $p^+ > p_h$ and $\varepsilon > 0$ we have $\bar{p}(1-\varepsilon) < p^+$. Thus, at all $c' \in (1-\varepsilon, 1)$ such that $p_N(c')$ converges, $\bar{p}(c') < p^+$ and for N large enough, $p_N(c') < p^+$. Therefore $c_N(p^+) \geq c' > 1-\varepsilon$ and with ε arbitrary this implies $c_N(p^+) \to 1$ for all $p^+ > p_h$. Reasoning similarly for $p^- < p_l$ we conclude that $c(\cdot|p_N(\cdot))$ converges on $[0, p_l)$ and $(p_h, 1]$. Now, take any $p' \in (p_l, p_h)$ such that p' is not a jump point of $\bar{c}(\cdot)$. Let $c' = \bar{c}(p')$ and suppose $c' \in (0,1)$. Then $\bar{p}(\cdot)$ is not flat at c', i.e., for all $c^- < c' < c^+$, $\bar{p}(c^-) < p(c') < \bar{p}(c^+)$. Take some c^- and c^+ such that $p_N(\cdot)$ converges pointwise at these points. Then for some ε and N large enough, $p_N(c^-) \leq p' - \varepsilon$ and $p_N(c^+) \geq p' + \varepsilon$ and so $c_N(p') = \sup\{c, 0|p_N(c) \leq p'\} \in [c^-, c^+]$ and by c^- and c^+ being arbitrary, $c_N(p') \to c'$. Suppose $c' = 0$, then by $\bar{p}$ not being flat, $\bar{p}(\varepsilon) > p'$ for all $\varepsilon > 0$. Choose some $\varepsilon_1 < \varepsilon$ such that $p_N(\varepsilon_1) \to \bar{p}(\varepsilon_1)$. Then for N large enough, $c_N(p') \leq \varepsilon_1$ and by ε arbitrary, $c_N(p') \to 0$. Similar reasoning holds for $c' = 1$. Together, $c_N(p) \to \bar{c}(p)$ for almost all p.

Now we want to show $c_N(r_N(\cdot)) \to \bar{c}(\bar{r}(\cdot))$. By $r(\cdot)$ being strictly increasing we can disregard the zero measure set of v where $\bar{c}(\cdot)$ is discontinuous at $\bar{r}(v)$. We also disregard the two points $v \in \{0,1\}$ and $\bar{r}(v) \in (0,1)$ at all $v \in (0,1)$. Note that at all remaining v', $\bar{c}(\cdot)$ will be continuous at $\bar{r}(v')$ and therefore $c_N(\bar{r}(v'))$ converges pointwise to $\bar{c}(\bar{r}(v'))$ by the reasoning in the preceding paragraph. Take such a type $v' \in (0,1)$ and some $\delta_1 < \min\{\bar{r}(v'), 1 - \bar{r}(v')\}$. Then we want to show that for every such δ_1 there is some N large enough such that $c_N(r_N(v'))$ is in an open ball with radius δ_1 around $c' \equiv \bar{c}(\bar{r}(v'))$, i.e., $c_N(r_N(v'))$ converges to $\bar{c}(\bar{r}(v'))$ pointwise: By continuity of $\bar{c}(\cdot)$ at $\bar{r}(v')$ there are some prices p_l, p_h around $\bar{r}(v')$ with $p_l < \bar{r}(v') < p_h$ such that $\bar{c}(p_l) \in B_{\delta_1}(c') \equiv (c' - \delta_1, c' + \delta_1)$ and $\bar{c}(p_h) \in B_{\delta_1}(c')$. In addition, we choose these prices such that $c_N(\cdot)$ converges pointwise at p_l and p_h. Such prices exist within the open ball because $c_N(\cdot)$ converges pointwise almost everywhere. By this choice there is some N_1 large enough st. $c_N(p_l) \in B_{\delta_1}(c')$ and $c_N(p_h) \in B_{\delta_1}(c')$ for $N \geq N_1$. Now choose $N_2 \geq N_1$ such that $r_N(v') \in (p_l, p_h)$ for all $N \geq N_2$ as well. Monotonicity of $c_N(\cdot)$ implies that we have *sandwiched* $c_N(r_N(v'))$, $c_N(p_l) \leq c_N(r_N(v')) \leq c_N(p_h)$ and from $c_N(p_l) \in B_{\delta_1}(c')$ and $c_N(p_h) \in B_{\delta_1}(c')$ we have $c_N(r_N(v')) \in B_{\delta_1}(c')$ with $c' = \bar{c}(\bar{r}(v'))$ for all $N \geq N_2$. By δ_1 being arbitrary, it must be that $c_N(r_N(v')) \to \bar{c}(\bar{r}(v'))$ ■

A.5 Chapter 2: Proof of the Identity of Pool Sizes

Let x^S and x^B be the shares of sellers and buyers, respectively, who trade:

$$\begin{aligned} x^S &= M^S(1)^{-1}\int_0^1 \left[\int_{v(p(c))}^1 \left[-dM^B(v)\right] M^B(0)^{-1}\right] dM^S(c) \\ x^B &= M^B(0)^{-1}\int_0^1 \left[\int_0^{c(r(v))} dM^S(c)\, M^S(1)^{-1}\right] - dM^B(v)\,, \end{aligned}$$

and note that by the same reasoning as before, the share of sellers and buyers who trade must be the same:

$$\begin{aligned} x^S &= M^S(1)^{-1}\int_0^1 \left[\int_{v(p(c))}^1 \left[-dM^B(v)\right] M^B(0)^{-1}\right] dM^S(c) \\ &= \int_0^1 \left[\int_{v(p(c))}^1 \left(\left[-dM^B(v)\right] M^B(0)^{-1}\right)\right] \left(dM^S(c)\, M^S(1)^{-1}\right) \\ &= \int_0^1 \left[\int_0^{c(r(v))} \left(dM^S(c)\, M^S(1)^{-1}\right)\right] \left(\left[-dM^B(v)\right] M^B(0)^{-1}\right) \\ &= x^B, \end{aligned}$$

and by rewriting the steady-state conditions we get

$$\begin{aligned} G^S(1) &= x^S M^S(1) + M^S(1)\left(1-x^S\right)\delta \\ G^B(1) &= x^B M^B(0) + M^B(0)\left(1-x^B\right)\delta, \end{aligned}$$

so that $x^S = x^B$ implies $M^S(1) = M^B(0)$ by rewriting further:

$$\begin{aligned} 1 &= M^S(1)\left(x^S + \left(1-x^S\right)\delta\right) \\ 1 &= M^B(0)\left(x^B + \left(1-x^B\right)\delta\right) \qquad \blacksquare \end{aligned}$$

A.6 Chapter 3: Proof of Lemma 14

Let

$$\begin{aligned} S_M\left(M^T\right) &\equiv \max_{Q\in\hat{Q}} S_Q(\cdot) \\ & st.\ M^T = \int_0^1 Q^S(c)\, dG^S(c) = \int_0^1 Q^B(v)\, dG^B(v)\,, \end{aligned} \qquad \text{(A.1)}$$

and note that

$$\max_{Q\in\hat{Q}} S_Q(\cdot) = \max_{M^T\in[0,1]} S_M(\cdot)\,.$$

Let $p^S\left(M^T\right)$ be such that $G^S\left(p^S\right) = M^T$ and $p^B\left(M^T\right)$ be such that $1-G^B\left(p^B\right) = M^T$. Then clearly

$$\begin{aligned} Q(M) &\equiv \arg\max_{Q\in\hat{Q}} S_Q(\cdot) \quad \text{st. (A.1)} \\ &= \left\{Q\in\hat{Q} | \int_0^1 \left|Q^S(c) - 1_{c\le p^S(M)}(c)\right| dc + \int_0^1 \left|Q^B(v) - 1_{v\ge p^B(M)}(v)\right| dv = 0\right\} \end{aligned}$$

and thus

$$S_M\left(M^T\right) = \int_{p^B(M^T)}^1 v dG^B(v) - \int_0^{p^S(M^T)} c dG^S(v).$$

Note that $S_M(\cdot)$ is continuously differentiable in M^T and the second derivative of $S_M\left(M^T\right)$ is $-\left(\frac{1}{dG^B(p^B(M^T))} + \frac{1}{dG^S(p^S(M^T))}\right)$; so the surplus is strictly concave in M^T. Therefore, a necessary and sufficient condition for $M^* \in \arg\max S_M(\cdot)$ is that the first derivative is zero:

$$p^B(M^*) - p^S(M^*) = 0, \tag{A.2}$$

which implies that the cutoffs must be the market clearing price p^w: By definition of M^T, $M^T = G^S\left(p^S\left(M^T\right)\right) = 1 - G^B\left(p^B\left(M^T\right)\right)$. This is true at $p^B(M^*) = p^S(M^*)$ only for $p^B(M^*) = p^S(M^*) = p^w$. Thus:

$$Q^W = Q\left(G^s\left(p^w\right)\right) = \arg\max_{Q\in\hat{Q}} S_Q(\cdot). \qquad \blacksquare$$

A.7 Chapter 3: Proof of Lemma 15

The "if" part follows directly from continuity of $S_Q(\cdot)$ and from Lemma 14. For the "only if" part, recall that we say $\lim_{k\to\infty} Q_k = Q^W$ if $d(Q_k, Q') \to 0$ for all $Q' \in Q^W$. By Helly's selection theorem, every sequence of monotone functions has a convergent subsequence. Take such a subsequence and let $\bar{Q}$ denote its limit. Lebesgue's bounded convergence theorem implies that $S_Q\left(\bar{Q}\right) = S^*$. Therefore $\bar{Q} \in Q^W$ from Lemma 14. Hence, every convergent subsequence has its limit in Q^W, and thus the sequence itself converges to Q^W(see Lemma 20 in the Appendix) $\blacksquare$

A.8 Chapter 3: Proof of Lemma 16

Suppose the limiting statement does not hold. According to the Bolzano-Weierstrass theorem, this implies that there is some $\varepsilon > 0$ and some subsequence indexed by k', such that $S(A_{k'})$ converges and $\lim_{k'\to\infty} S(A_{k'}) \le S^* - \varepsilon$. Take some subsubsequence indexed by k'' such that $V^S_{k''}$, $V^B_{k''}$ converge pointwise. Such a subsubsequence exists by $\left(V^S_{k''}, V^B_{k''}\right) \in \Sigma_+ \times \Sigma_-$ and Helly's selection theorem. Let $\bar{p}$ be defined as before: $\bar{p} \equiv \inf_{c\le p^w}\left(\bar{V}^S(c) + c\right)$. Along the subsubsequence, $\lim_{k''\to\infty} V^S_{k''}(c) \ge \bar{V}^S(c) \ge \bar{p} - c$ for all $c \le p^w$, and similarly, $\lim_{k''\to\infty} \inf V^B_{k''}(v) \ge v - \bar{p}$ for all $v \ge p^w$, by the condition of the lemma. Hence

$$\lim_{k''\to\infty} \inf S(A_{k''}) \ge \int_{p^w}^1 (v - \bar{p}) dG^B(v) + \int_0^{p^w} (\bar{p} - c) dG^S(c) = S^*,$$

where the last equality follows the observation in the first part of the proof. This contradicts the starting hypothesis $\lim_{k'\to\infty} S(A_{k'}) \le S^* - \varepsilon$. $\blacksquare$

A.9 Chapter 3: Proof of Convergence of Sequences

The sets of monotone functions Σ_+ and Σ_- are sequentially compact and satisfy the conditions of the following lemma (see footnote 3.3.1). The proof of the first part is standard and the second part is a straightforward extension:

Lemma 20 *Let (X, τ) be a sequentially compact topological space. Suppose there is some sequence $\{x_n\} = x_1, x_2, ...$ in X and some $y \in X$ such that every convergent subsequence converges to y. Then*

$$x_n \to y.$$

Similarly, suppose there is some subset $Y \subset X$ such that every convergent subsequence of $\{x_n\}$ converges to some $y \in Y$ (possibly y is different for each subsequence). Then for every neighborhood $G \supset Y$, there is some $N(G)$, st. $x_n \in G$ for all $n \geq N(G)$, and we say $x_n \to Y$.

Proof: We prove the first part by contradiction: Suppose not, then by the definition of convergence, there is some neighborhood G of y that does not contain all elements of the sequence from some index onwards, i.e., for every N there is some $n'(N) \geq N$ such that $x_{n'(N)} \notin G$. This allows the construction of a subsequence $\{x_{n'}\}$ such that $x_{n'} \notin G$ for all n'. By X being sequentially compact, there is some convergent subsubsequence of $\{x_{n'}\}$. By the hypothesis of the lemma, this subsubsequence converges to y. This contradicts $x_{n'} \notin G$ for all n'. The second statement follows similarly: Suppose not, then there would be some neighborhood $G \supset Y$ and some subsequence $\{x_{n'}\}$ such that $x_{n'} \notin G$ for all n'. Again, we can find a convergent subsubsequence by X being sequentially compact; this implies a contradiction to the definition of $\{x_{n'}\}$ ■

A.10 Chapter 3: Proof of $q^{PS} \to 1$

We want to show that $\lim_{k\to\infty} Q_k^S(c) = 1$ implies $\lim_{k\to\infty} q^{PS}\left(p_k^S(\cdot, c), c|\sigma_F, \delta_k\right) = 1$. Note that $V_k^S(\cdot)$ is decreasing and $V_k^B(\cdot)$ is increasing. This implies that reservation prices $r_k^S(\cdot)$ and $r_k^B(\cdot)$ are monotone. Furthermore, if $r_k^B(v) = p_k^S(c, v)$, then $r_k^S(c) = p_k^B(v, c)$ because $p_k^S(c, v) = r_k^B(v)$ if and only if the continuation payoff is below the reservation price, i.e., if and only if

$$\begin{aligned}(1-\delta_k)\, U^S\left(p_k^S, r_k^S, c|\sigma_{kF}\right) &\leq r_k^B(v) - c \\ &= v - (1-\delta)\, U^B\left(p^B, r^B, v|\sigma_F\right) - c\end{aligned}$$

and hence $v - r_k^S(c) \geq (1-\delta)\, U^B\left(p^B, r^B, v|\sigma_F\right)$. Therefore, the probability to trade is independent of whether one is a proposer or a responder. So $D^S\left(p_k^S, r_k^S, c|\sigma_{Fk}, \delta_k\right) = P^S\left(p_k^S, r_k^S, c|\sigma_{Fk}, \delta_k\right)$. This implies that if $\lim_{k\to\infty} Q_k^S(c) = 1$,
then $\lim_{k\to\infty} q^{PS}\left(p_k^S(\cdot, c), c|\sigma_F, \delta_k\right) = 1$: With $P_k^S\left(p_k^S\right) \equiv P^S\left(p_k^S, r_k^S, c|\sigma_{Fk}, \delta_k\right)$

$$\lim_{k\to\infty} Q_k^S(c) = \lim_{k\to\infty} \frac{P_k^S\left(p_k^S\right)}{1-(1-\delta_k)\left(1-P_k^S\left(p_k^S\right)\right)} = 1$$

implies $\lim_{k\to\infty} \delta_k \left[P_k^S\left(p_k^S\right)\right]^{-1} = 0$ and therefore $\lim_{k\to\infty} \delta_k \left[\alpha P_k^S\left(p_k^S\right)\right]^{-1} = 0$. Thus $q^{PS}\left(p_k^S(\cdot, c), c|\sigma_F, \delta_k\right) \to 1$ by

$$q^{PS}\left(p_k^S(\cdot, c), c|\sigma_F, \delta_k\right) = \frac{\alpha P_k^S\left(p_k^S\right)}{1-(1-\delta_k)\left(1-\alpha P_k^S\left(p_k^S\right)\right)}. \quad ■$$

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