Rigidity and Exotic Models for the K-local Stable Homotopy Category

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Chapter 1

Introduction

When two model categories $\mathcal{C}$ and $\mathcal{D}$ are Quillen equivalent, then their homotopy categories $\mathrm{Ho}(\mathcal{C})$ and $\mathrm{Ho}(\mathcal{D})$ are equivalent. But on the other hand, if there is an equivalence between the homotopy categories of two model categories, can anything be said about the underlying model structures?

For the stable homotopy category $\mathrm{Ho}(\mathcal{S})$, i.e., the homotopy category of spectra, there is the following result of [Sch05]:

Rigidity Theorem(Schwede [Sch05]) Let $\mathcal{C}$ be a stable model category, and

$$\Phi : \mathrm{Ho}(\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

an equivalence of triangulated categories. Then the underlying model categories $\mathcal{S}$ and $\mathcal{C}$ are Quillen equivalent.

Usually, when passing from the model category level to the homotopy level, information can be lost, as "higher homotopy information" like mapping spaces or algebraic K-theory is defined via the model structure of the underlying model category. However, the Rigidity Theorem says that for spectra, all such secondary homotopy information is encoded in the triangulated structure of the stable homotopy category.

Now the next question could be if there is a similar result for Bousfield localisations of the stable homotopy category with respect to certain homology theories. In this thesis, we consider localisation with respect to 2-local complex K-theory $K_{(2)}$ with

$$K_{(2)*} = \mathbb{Z}_{(2)}[v_1, v_1^{-1}], \quad |v_1| = 2.$$

The $K_{(2)}$-local model structure is a model structure on the category of spectra where the weak equivalences are the $K_{(2)*}$-isomorphisms (see Definition 3.1). For the resulting $K_{(2)}$-local stable homotopy category we present the following positive answer to the rigidity question which is the main result of this thesis:

$K_{(2)}$-local Rigidity Theorem Let $\mathcal{C}$ be a stable model category, and let $L_1\mathcal{S}$ denote the K-local category of spectra at the prime 2, and let

$$\Phi : \mathrm{Ho}(L_1\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

be any equivalence of triangulated categories. Then $L_1\mathcal{S}$ and $\mathcal{C}$ are Quillen equivalent.

Remark. The notation L_1 for $K_{(p)}$-localisation for a prime p referes to the general context of chromatic localisation: the notation L_n is often used to denote Bousfield localisation with respect to the Johnson-Wilson theories $E(n)$ with

$$E(n)_* = \mathbb{Z}_{(p)}[v_1, v_2, ..., v_n, v_n^{-1}], \quad |v_i| = 2p^i - 2,$$

and therefore, $K_{(p)} = E(1)$.

The proof divides into two main parts: First, we modify the Universal Property of Spectra introduced by Schwede and Shipley in [SS02] 5.1 to obtain a Quillen functor pair

$$X \wedge - : L_1\mathcal{S} \rightleftarrows \mathcal{C} : \mathrm{Hom}(X, -)$$

for $X = \Phi(L_1S^0)$.

The left derived functor $X \wedge^L -$ composed with the inverse of Φ

$$\mathrm{Ho}(L_1\mathcal{S}) \xrightarrow{X\wedge^L -} \mathrm{Ho}(\mathcal{C}) \xrightarrow{\Phi^{-1}} \mathrm{Ho}(L_1\mathcal{S})$$

is an exact endofunctor of the homotopy category of $K_{(2)}$-local spectra, mapping the $K_{(2)}$-local sphere L_1S^0 to itself. The spectrum L_1S^0 is a so-called small weak generator of $L_1\mathcal{S}$. Any exact endofunctor fixing this small weak generator must be a self-equivalence, thus $X \wedge^L -$ is an equivalence of categories induced by a left Quillen functor. This means that $L_1\mathcal{S}$ and $\mathcal{C}$ are Quillen equivalent. The details of this will be explained in the fourth chapter.

As we explain in the last chapter, there cannot be an odd primary version of the $K_{(p)}$-local Rigidity Theorem: For odd primes, Jens Franke constructs an equivalence of triangulated categories

$$\mathcal{R} : \mathcal{D}^1(\mathcal{A}) \longrightarrow \mathrm{Ho}(L_1\mathcal{S})$$

between the homotopy category of K-local spectra at an odd prime p and the derived category of so-called quasi-periodic cochain complexes over a certain abelian category $\mathcal{A}$ (see [Fra96] 3.1). However, the underlying model categories $\mathcal{C}^1(\mathcal{A})$ and $L_1\mathcal{S}$ are *not* Quillen equivalent. This means that $\mathcal{C}^1(\mathcal{A})$ is a so-called "exotic model" for $L_1\mathcal{S}$.

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Chapter 2

Stable model categories - a review

Model categories were introduced in the 1960s by Quillen to provide a set-theoretically clean device to describe homotopy ([Qui67]). A *model category* is a category equipped with classes of morphisms called weak equivalences, fibrations and cofibrations satisfying certain axioms (see e.g. [Hov99] 1.1). These axioms enable us to define a notion of homotopy between morphisms.

Very roughly speaking, one then obtains the *homotopy category* $\mathrm{Ho}(\mathcal{C})$ of a model category $\mathcal{C}$ by formally inverting the weak equivalences, while the model category axioms ensure that the result is indeed a category.

In order to compare model categories, one studies morphisms of model categories, so-called Quillen functors:

Definition 2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be two model categories. An adjoint pair of functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is called a *Quillen functor pair* if F preserves cofibrations and trivial cofibrations (i.e., cofibrations that are also weak equivalences), or equivalently, if G preserves fibrations and trivial fibrations (i.e., fibrations that are also weak equivalences).

Notation. Throughout this thesis, we use the following convention: for an adjoint functor pair $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$, the top arrow denotes the left adjoint and the bottom arrow the right adjoint.

Cofibrations are marked $\rightarrowtail$, fibrations are marked $\twoheadrightarrow$ and weak equivalences $\xrightarrow{\sim}$.

If an adjoint pair of functors is a Quillen pair, it induces an adjoint pair of functors $LF : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}) : RG$ ([Hov99], Lemma 1.3.10).

Definition 2.2. A Quillen functor pair is called a *Quillen equivalence* if in addition, for all cofibrant $X \in \mathcal{C}$ and fibrant $Y \in \mathcal{D}$, a morphism $f : FX \to Y$ is a weak equivalence if and only if its adjoint $\bar{f} : X \to GY$ is.

One can conclude that a Quillen functor pair is a Quillen equivalence if and only if it induces an equivalence of homotopy categories ([Hov99], Prop. 1.3.13). But not only do Quillen equivalent model categories have equivalent homotopy categories, they also have the "same homotopy theory" in the sense that the higher homotopy information mentioned in the introduction such as mapping spaces is preserved by Quillen equivalences.

For a pointed model category $\mathcal{C}$, one can define an adjoint pair of *suspension and loop functors*

$$\Sigma : \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{C}) : \Omega.$$

Without loss of generality let $X \in \mathcal{C}$ be fibrant and cofibrant. We choose a factorisation $X \rightarrowtail C \xrightarrow{\sim} *$ of the unique morphism from X into the terminal object. The suspension ΣX of X is defined as the pushout of the diagram

$$* \longleftarrow X \rightarrowtail C.$$

Dually, choosing a factorisation $* \xrightarrow{\sim} A \longrightarrow X$, the loop functor ΩX of X is defined as the pullback of the diagram

$$* \longrightarrow X \twoheadleftarrow A.$$

Definition 2.3. A pointed, complete and cocomplete model category $\mathcal{C}$ is called *stable* if Σ and Ω are inverse equivalences of categories.

Examples for stable model categories are provided by the category of spectra $\mathcal{S}$ (see the beginning of Chapter 3) or chain complexes $\mathcal{C}(\mathcal{A})$ for certain abelian categories $\mathcal{A}$.

The homotopy category $\mathrm{Ho}(\mathcal{C})$ of a stable model category $\mathcal{C}$ carries the structure of a triangulated category, where the exact triangles are given by the fiber and cofiber sequences ([Hov99] 7.1.6).

In particular, the stable homotopy category $\mathrm{Ho}(\mathcal{S})$ and the derived category $\mathcal{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ are triangulated categories.

Furthermore, note the following: given a Quillen pair $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ with $\mathcal{C}$ and $\mathcal{D}$ being stable model categories, the left derived and right derived functors LF and RG are exact functors, i.e., preserve exact triangles. This justifies the general rigidity question for stable model categories, namely, if two stable model categories whose homotopy categories are equivalent as triangulated categories, are Quillen equivalent. However, the Rigidity Theorems for $\mathrm{Ho}(\mathcal{S})$ and $\mathrm{Ho}(L_1\mathcal{S})$ do not claim that the given equivalence Φ is induced by a Quillen functor, they just claim that the given stable model categories are linked by some Quillen equivalence. The question about possible uniqueness of this Quillen functor is still unanswered.

Chapter 3

The Quillen functor pair

3.1 Universal property of spectra

In this chapter, we construct a Quillen functor pair between the category of spectra equipped with the $K_{(2)}$-local model structure $L_1\mathcal{S}$ and our given stable model category $\mathcal{C}$.

Throughout this thesis, $\mathcal{S}$ denotes the category of *spectra* with the stable Bousfield-Friedlander model structure ([BF78]). Here a spectrum X is a sequence of simplicial sets $(X_0, X_1, ...)$ together with structure maps $\sigma_n^X : \Sigma X_n \to X_{n+1}$. A morphism $f : X \to Y$ of spectra is a collection of morphisms of pointed simplicial sets $f_n : X_n \to Y_n$ that are compatible with the structure maps, i.e., $f_{n+1} \circ \sigma_n^X = \sigma_n^Y \circ \Sigma f_n$ for all $n \geq 0$. The $K_{(2)}$-local model structure on the category of spectra is a localisation of the Bousfield-Friedlander model structure:

Definition 3.1. ($K_{(2)}$*-local model structure for spectra*)
A morphism of spectra $f : A \longrightarrow B$ is called a

- weak equivalence if $K_{(2)*}f : K_{(2)*}(A) \longrightarrow K_{(2)*}(B)$ is an isomorphism
- cofibration, if the induced map

$$\Sigma B_n \cup_{\Sigma A_n} A_{n+1} \longrightarrow B_{n+1}$$

  is a cofibration of simplicial sets for all $n \geq 1$ and $A_0 \longrightarrow B_0$ is a cofibration of simplicial sets.

- fibration if f has the right lifting property with respect to trivial cofibrations, i.e., cofibrations that are also $K_{(2)*}$-isomorphisms.

Remark. A spectrum X is fibrant with respect to this model structure if and only if it is $K_{(2)}$-local in $\mathrm{Ho}(\mathcal{S})$ and an Ω-spectrum.

With the above choices, the category of spectra becomes a stable model category, denoted by $L_1\mathcal{S}$. (For the definition of generalised Bousfield localisations see [Hir03] Definition 3.3.1. For the existence of such localisations, see Theorem 4.1.1 of the same book. The author believes that this theorem can be applied to this special case by using the set-theoretical methods of [Bou75], §10-11. However, the author does not know of any reference for such a proof.)

Now, to construct our desired Quillen functor pair between $L_1\mathcal{S}$ and $\mathcal{C}$, we use

Universal Property of Spectra (Schwede-Shipley [SS02])

Let $\mathcal{C}$ be a stable model category, $X \in \mathcal{C}$ fibrant and cofibrant. Then there is a Quillen adjoint functor pair

$$X \wedge - : \mathcal{S} \rightleftarrows \mathcal{C} : \mathrm{Hom}(X, -)$$

sending the sphere spectrum S^0 to X.

Forgetting their model structures, $\mathcal{S}$ and $L_1\mathcal{S}$ are the same categories, so the above property gives us an adjoint pair of functors

$$X \wedge - : L_1\mathcal{S} \rightleftarrows \mathcal{C} : \mathrm{Hom}(X, -)$$

for any X. However, it is not obvious under which conditions this functor pair is a Quillen functor pair.

Before we answer this, let us briefly summarize the construction of the functor

$$\mathrm{Hom}(X, -) : \mathcal{C} \longrightarrow \mathcal{S}.$$

For simplicity, let us assume $\mathcal{C}$ to be a pointed *simplicial* model category, i.e., a category equipped with three functors

$$\begin{aligned} - \otimes - : \mathcal{C} \times \mathrm{sSet}_* &\longrightarrow \mathcal{C} \\ (-)^{(-)} : \mathrm{sSet}_*{}^{op} \times \mathcal{C} &\longrightarrow \mathcal{C} \\ \mathrm{map}_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} &\longrightarrow \mathrm{sSet}_* \end{aligned}$$

satisfying certain adjunction properties. (For details, see [GJ99], Definition II.2.1.)

Notation. In the pointed case, the first functor is usually denoted $- \wedge -$ instead of $- \otimes -$. However, we choose to write $- \otimes -$ to avoid confusion with the functor $X \wedge -$ as in the Universal Property of Spectra.

For $Y \in \mathcal{C}$, we define the n^{th} level space of the spectrum $\operatorname{Hom}(X, Y)$ to be

$$\operatorname{Hom}(X,Y)_n := \operatorname{map}_{\mathcal{C}}(\omega^n X, Y) \in \mathrm{sSet}_*$$

where $\omega^n X$ is a cofibrant replacement of the n^{th} desuspension of X: We define $\omega^n X$ inductively by setting $\omega^0 X = X$ and for $n \geq 1$ by choosing a factorisation

$$* \rightarrowtail \omega^n X \xrightarrow[\varphi_n]{\sim} \twoheadrightarrow \Omega(\omega^{n-1} X)\ .$$

By $\tilde{\varphi_n}$ we denote the morphism $\Sigma \omega^n X \longrightarrow \omega^{n-1} X$ that is adjoint to φ_n. The structure map $\Sigma \operatorname{Hom}(X,Y)_{n-1} \longrightarrow \operatorname{Hom}(X,Y)_n$ of the spectrum $\operatorname{Hom}(X,Y)$ is now given by the adjoint of the map

$$\operatorname{map}_{\mathcal{C}}(\omega^{n-1}X, Y) \xrightarrow{\operatorname{map}_{\mathcal{C}}(\tilde{\varphi_n}, Y)} \operatorname{map}_{\mathcal{C}}(\Sigma\omega^n X, Y) \simeq \Omega \operatorname{map}_{\mathcal{C}}(\omega^n X, Y).$$

As $\omega^n X$ is cofibrant in $\mathcal{C}$, the functor $\operatorname{map}_{\mathcal{C}}(\omega^n X, -) : \mathcal{C} \longrightarrow \mathrm{sSet}_*$ preserves fibrations and trivial fibrations (see [Hov99], section 5). One can conclude from this that the functor $\operatorname{Hom}(X, -) : \mathcal{C} \longrightarrow \mathcal{S}$ preserves fibrations and trivial fibrations (as shown in [SS02] 6.2). In particular, $\operatorname{Hom}(X,Y)$ is an Ω-spectrum for fibrant Y, which is something we are going to make use of in the proof of the next proposition:

Proposition 3.2. Let $\Phi : \operatorname{Ho}(L_1\mathcal{S}) \longrightarrow \operatorname{Ho}(\mathcal{C})$ be an equivalence of triangulated categories, and let X be a cofibrant and fibrant object in $\mathcal{C}$ isomorphic to $\Phi(L_1 S^0)$. Then

$$X \wedge - : L_1\mathcal{S} \rightleftarrows \mathcal{C} : \operatorname{Hom}(X, -)$$

is a Quillen functor pair with respect to the $K_{(2)}$-local model structure on the left side.

Notation. Throughout the rest of this thesis, X will be a fixed fibrant and cofibrant replacement of $\Phi(L_1 S^0)$. For a stable model category $\mathcal{D}$, and A, B in $\mathcal{D}$, $[A,B]_*^{\mathcal{D}}$ denotes the graded group of morphisms in the homotopy category of $\mathcal{D}$. All spectra are assumed to be 2-local, in particular $S^0 = S^0_{(2)}$. By M we denote the mod-2 Moore spectrum $M(\mathbb{Z}/2)$.

3.2 v_1-periodicity

The key ingredient in the proof of the proposition is showing that the spectra $\mathrm{Hom}(X,Y)$ are $K_{(2)}$-local for all fibrant $Y \in \mathcal{C}$. A spectrum A is $K_{(2)}$-local if and only if v_1^4 induces an isomorphism of its mod-2 homotopy groups $[M,A]_*^{\mathcal{S}}$ ([Bou79] §4). To be more precise:

Let $K(n)$ denote the n^{th} Morava K-theory with

$$K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}], \quad |v_n| = 2p^n - 2, \quad K(0) := H\mathbb{Q}.$$

Any p-local finite spectrum A with $K(0)_*(A) = 0$ but $K(1)_*(A) \neq 0$ possesses a v_1-self map ([HS98] §3), i.e., a map $v_1^{p^i}$ such that

$$v_1^{p^i} : \Sigma^{p^i(2p-2)} A \longrightarrow A$$

induces an isomorphism in $K(1)$-homology. The notation is standard but slightly misleading since it implies that $v_1^{p^j}$ is a power of an existing morphism v_1. However, this need not be the case:

In the case $p = 2$, the mod 2-Moore spectrum M has a v_1-self map

$$v_1^4 : \Sigma^8 M \longrightarrow M$$

that induces an isomorphism in $K(1)$-homology, or in this case equivalently, $K_{(2)}$-homology. This is the smallest degree v_1-self map that can be realised on M. Also, this v_1-self map v_1^4 need not be unique, so we fix one possible v_1^4 for the rest of this thesis which will be specified in the proof of Lemma 5.10.

Lemma 3.3. The map

$$(v_1^4)^* : [M, \mathrm{Hom}(X,X)]_n^{\mathcal{S}} \longrightarrow [M, \mathrm{Hom}(X,X)]_{n+8}^{\mathcal{S}}$$

is an isomorphism for all $n \in \mathbb{Z}$, thus, $\mathrm{Hom}(X,X)$ is $K_{(2)}$-local.

Before we prove this lemma, we have to look at the image of certain elements in $\pi_* L_1 S^0$ under the functor $X \wedge -$, namely the Hopf elements $\eta \in \pi_1 L_1 S^0$, $\nu \in \pi_3 L_1 S^0$ and $\sigma \in \pi_7 L_1 S^0$, and further, the elements $y_0 \in \pi_0 L_1 S^0, y_1 \in \pi_1 L_1 S^0$, $\mu \in \pi_9 L_1 S^0$ $\rho \in \pi_{15} L_1 S^0$ and $\mu_{17} \in \pi_{17} L_1 S^0$.

(For details about the generators of the stable homotopy groups of the $K_{(2)}$-local sphere and their multiplicative relations see the table of generators of $\pi_* L_1 S^0$ on page 36.)

Lemma 3.4. For $\eta, \nu, \sigma, y_0, y_1, \mu, \rho$ and μ_{17} in $\pi_* L_1 S^0$ as before, we have

- $X \wedge \eta = \Phi(\eta) \quad$ or $\quad = \Phi(\eta) + \Phi(y_1)$
- $X \wedge \nu = \bar{u}\Phi(\nu)$, $\bar{u} \in \mathbb{Z}$ odd
- $X \wedge \sigma = \bar{u}\Phi(\sigma)$, $\bar{u} \in \mathbb{Z}$ odd
- $X \wedge \mu = \Phi(\mu) \quad$ or $\quad = \Phi(\mu) + \Phi(\eta^2\sigma)$
- $X \wedge y_0 = \Phi(y_0)$
- $X \wedge y_1 = \Phi(y_1)$.
- $X \wedge \rho = \bar{\bar{u}}\Phi(\rho)$, $\bar{\bar{u}} \in \mathbb{Z}$ odd
- $X \wedge \mu_{17} = \Phi(\mu_{17}) \quad$ or $\quad = \Phi(\mu_{17}) + \Phi(\eta^2\rho)$

Proof. $\boxed{X \wedge \eta}$

On the mod-2 Moore spectrum M, $2\,\mathrm{Id}_M$ factors as

$$M \xrightarrow{\text{pinch}} S^1 \xrightarrow{\eta} S^0 \xrightarrow{\text{incl}} M,$$

and this composition is nonzero. Here, pinch denotes the map that "pinches" off the bottom cell of M, and incl denotes the inclusion of the zero-sphere into the bottom cell of M. Consequently, $2\,\mathrm{Id}_{L_1M}$ factors as

$$L_1M \xrightarrow{\text{pinch}} L_1S^1 \xrightarrow{\eta} L_1S^0 \xrightarrow{\text{incl}} L_1M.$$

Recall that η survives $K_{(2)}$-localisation. As the left derived $X \wedge^L -$ of $X \wedge - : \mathcal{S} \longrightarrow \mathcal{C}$ is exact on homotopy level and as Φ is exact and furthermore $X \wedge S^0 \cong \Phi(L_1S^0)$, it follows that

$$X \wedge M \cong \Phi(L_1M), \quad X \wedge \text{pinch} = \Phi(\text{pinch}), \quad X \wedge \text{incl} = \Phi(\text{incl}),$$

coming from the exact triangle

$$S^0 \xrightarrow{2} S^0 \xrightarrow{\text{incl}} M \xrightarrow{\text{pinch}} S^1$$

in $\mathrm{Ho}(\mathcal{S})$. Throughout this thesis, we fix an isomorphism $X \wedge M \cong \Phi(L_1M)$.

The functor $X \wedge^L -$ is additive, so

$$X \wedge 2\,\mathrm{Id}_{L_1 M} = 2\,\mathrm{Id}_{X\wedge M} \neq 0,$$

and $2\,\mathrm{Id}_{X\wedge M}$ factors as

$$X \wedge M \xrightarrow{X\wedge \mathrm{pinch}} X \wedge S^1 \xrightarrow{X\wedge \eta} X \wedge S^0 \xrightarrow{X\wedge \mathrm{incl}} X \wedge M.$$

Consequently,

$$X \wedge \eta \in [X, X]_1^{\mathcal{C}} \cong \mathbb{Z}/2\{\Phi(\eta), \Phi(y_1)\}$$

cannot be zero.

Also, $X \wedge \eta$ cannot be $\Phi(y_1)$ either: the composition

$$L_1 M \xrightarrow{\mathrm{pinch}} L_1 S^1 \xrightarrow{y_1} L_1 S^0 \xrightarrow{\mathrm{incl}} L_1 M$$

is zero by equation (5.1). So in the case $X \wedge \eta = \Phi(y_1)$ we would have

$$\begin{aligned} 2\,\mathrm{Id}_{X\wedge M} &= (X \wedge \mathrm{incl}) \circ (X \wedge \eta) \circ (X \wedge \mathrm{pinch}) \\ &= \Phi(\mathrm{incl}) \circ \Phi(y_1) \circ \Phi(\mathrm{pinch}) \\ &= \Phi(\mathrm{incl} \circ y_1 \circ \mathrm{pinch}) = \Phi(0) = 0, \end{aligned} \tag{3.1}$$

which is a contradiction. It follows that either

$$X \wedge \eta = \Phi(\eta) \quad \text{or} \quad X \wedge \eta = \Phi(\eta) + \Phi(y_1). \tag{3.2}$$

$\boxed{X \wedge \nu}$

In either case,

$$X \wedge \eta^3 = (X \wedge \eta)^3 = \Phi(\eta)^3.$$

as ηy_1 and y_1^2 are both zero in $\pi_* L_1 S^0$. Since in $\pi_3 L_1 S^0$ there is the relation $\eta^3 = 4\nu$, we have

$$4(X \wedge \nu) = X \wedge \eta^3 = 4\Phi(\nu).$$

As $4\Phi(\nu) \neq 0$ in $[X, X]_3^{\mathcal{C}} \cong \mathbb{Z}/8\{\Phi(\nu)\}$, $X \wedge \nu$ has order eight in this group and is therefore a generator. Consequently

$$X \wedge \nu = u\Phi(\nu), \quad \text{for some odd integer } u \in \mathbb{Z}. \tag{3.3}$$

$\boxed{X \wedge \sigma}$

For $X \wedge \sigma$ we look at the Toda bracket relation

$$8\sigma = \langle \nu, 8, \nu \rangle .$$

So we obtain

$$X \wedge 8\sigma \in \langle X \wedge \nu, X \wedge 8, X \wedge \nu \rangle .$$

The indeterminacy of this Toda bracket is zero, thus, equality holds. By the computations above, we get

$$8(X \wedge \sigma) = X \wedge 8\sigma = \langle u\Phi(\nu), \Phi(8), u\Phi(\nu) \rangle = u^2\Phi(8\sigma)$$

which is nonzero in $[X, X]_7^{\mathcal{C}} \cong \mathbb{Z}/16\{\Phi(\sigma)\}$. We conclude that $X \wedge \sigma$ has order 16 in this group, so

$$X \wedge \sigma = \bar{u}\Phi(\sigma), \quad \bar{u} \in \mathbb{Z} \quad \text{odd}. \tag{3.4}$$

$\boxed{X \wedge \mu}$

Next, we use that $\mu \in \langle 2, 8\sigma, \eta \rangle$ with indeterminacy $\eta^2\sigma$. It follows that

$$X \wedge \mu \in \langle X \wedge 2, X \wedge 8\sigma, X \wedge \eta \rangle .$$

Using our previous computations, this bracket either equals $\langle 2, \Phi(8\sigma), \Phi(\eta) \rangle$ or $\langle 2, \Phi(8\sigma), \Phi(\eta) + \Phi(y_1) \rangle$.

In the first case, $\langle 2, \Phi(8\sigma), \Phi(\eta) \rangle = \Phi(\langle 2, 8\sigma, \eta \rangle) = \{\Phi(\mu), \Phi(\mu) + \Phi(\eta^2\sigma)\}$ which is what we want.

For the second case, we compute

$$\langle 2, 8\sigma, \eta + y_1 \rangle \supseteq \langle 2, 8\sigma, \eta \rangle + \langle 2, 8\sigma, y_1 \rangle . \tag{3.5}$$

The bracket on the left side has indeterminacy $\eta^2\sigma$, which is the same as the intederminacy of the first bracket on the right side. The last bracket has indeterminacy zero and contains the set $\langle 2, 8\sigma, \eta \rangle\, y_0 = \{\mu y_0, \mu y_0 + \eta^2\sigma y_0\} = \{\eta^2\sigma\}$, so it equals $\{\eta^2\sigma\}$. Thus, equality holds in (3.5), and we can also conclude in this case that

$$X \wedge \mu \in \{\Phi(\mu), \Phi(\mu) + \Phi(\eta^2\sigma)\}.$$

$\boxed{X \wedge y_0}$

Next, we look at $X \wedge y_0$. Since y_0 is the only nonzero torsion element in $\pi_0 L_0 S^0 = \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$, the element $X \wedge y_0$ must be a torsion element as well because the functor $X \wedge -$ is additive. Consequently, $X \wedge y_0$ either equals $\Phi(y_0)$ or zero.

We now make use of the multiplicative relation $\mu y_0 = \eta^2\sigma$ (see [Rav84] 8.15.(d)). We have already seen that $X \wedge \eta^2\sigma = \bar{u}\Phi(\eta^2\sigma) \neq 0$, so $X \wedge y_0$ cannot be zero. Consequently,

$$X \wedge y_0 = \Phi(y_0).$$

$\boxed{X \wedge y_1}$

Now determining $X \wedge y_1$ is easy: we have $y_1 = \eta y_0$, so

$$X \wedge y_1 = (X \wedge \eta)(X \wedge y_0) = \Phi(\eta)\Phi(y_0) \quad \text{or} \quad = \Phi(\eta)\Phi(y_0) + \Phi(y_0^2)$$

which in either case equals $\Phi(y_1)$ since $y_0^2 = 0$.

$\boxed{X \wedge \rho}$

Here, we use the Toda bracket relation $\rho \in \langle \sigma, 2\sigma, 8 \rangle$ with indeterminacy $8\pi_{15}L_1S^0$. From this, we get

$$X \wedge \rho \in \langle X \wedge \sigma, X \wedge 2\sigma, X \wedge 8 \rangle .$$

By the previous computations, we obtain

$$\begin{aligned}(X \wedge \rho) \in \langle \bar{u}\Phi(\sigma), 2\bar{u}\Phi(\sigma), 8 \rangle &= \bar{u}^2 \langle \Phi(\sigma), 2\Phi(\sigma), 8 \rangle \\ &= \bar{u}^2\Phi(\{\rho + 8\pi_{15}L_1S^0\}) = \bar{u}^2\Phi(\{\rho, 9\rho, 17\rho, 25\rho\}).\end{aligned}$$

Consequently, $X \wedge \rho \in [X, X]^{\mathcal{C}}_{15} \cong \mathbb{Z}/32\{\Phi(\rho)\}$ has order 32 in this group, so

$$X \wedge \rho = \bar{\bar{u}}\Phi(\rho), \quad \bar{\bar{u}} \in \mathbb{Z} \quad \text{odd}.$$

$\boxed{X \wedge \mu_{17}}$

This is very similar to the computation of $X \wedge \mu$. We use that $\mu_{17} \in \langle \eta, 16\rho, 2 \rangle$ with indeterminacy $\eta^2\rho$ (see Lemma 5.7). Consequently,

$$X \wedge \mu_{17} \in \langle X \wedge \eta, X \wedge 16\rho, X \wedge 2 \rangle ,$$

which is either $\langle \Phi(\eta), \Phi(16\rho), 2\rangle$ or $\langle \Phi(\eta)+\Phi(y_1), \Phi(16\rho), 2\rangle$. In the first case, $\langle \Phi(\eta), \Phi(16\rho), 2\rangle = \Phi(\langle \eta, 16\rho, 2\rangle) = \{\Phi(\mu_{17}), \Phi(\mu_{17})+\Phi(\eta^2\rho)\}$.

In the second case, we look at

$$\langle \eta + y_1, 16\rho, 2\rangle \supseteq \langle \eta, 16\rho, 2\rangle + \langle y_1, 16\rho, 2\rangle . \tag{3.6}$$

The bracket on the left side has indeterminacy $\eta^2\rho$, which is the same as the intederminacy of the first bracket on the right side. The last bracket has indeterminacy zero and contains the set $\langle \eta, 16\rho, 2\rangle\, y_0 = \{\mu_{17}y_0, \mu_{17}y_0 + \eta^2\rho y_0\} = \{\eta^2\rho\}$, so it equals $\{\eta^2\rho\}$. Thus, equality holds in (3.6), and we can also conclude in this case that

$$X \wedge \rho \in \{\Phi(\mu_{17}), \Phi(\mu_{17}) + \Phi(\eta^2\rho)\}.$$

□

Proof. We now prove Lemma 3.3. It suffices to prove that the mod-2 homotopy groups of $\mathrm{Hom}(X,X)$ are v_1^8-periodic instead of v_1^4-periodic, as $(v_1^4)^*$ is an isomorphism if and only if $((v_1^4)^2)^* = (v_1^8)^*$ is. We choose to work with v_1^8 instead of v_1^4 as we need our v_1-self map in this lemma to commute with certain elements of $[M,M]_*^{L_1\mathcal{S}}$, and we know from Lemma 5.10, that v_1^8 lies in the centre of $[M,M]_*^{L_1\mathcal{S}}$. However, we do not know of a proof for v_1^4 being central.

By adjunction, it suffices to prove that

$$(X \wedge v_1^8)^* : [X \wedge M, X]_n^{\mathcal{C}} \longrightarrow [X \wedge \Sigma^{16} M, X]_n^{\mathcal{C}}$$

is an isomorphism for all integers n. Via the equivalence Φ, the left and right side is isomorphic to $[M, S^0]_n^{L_1\mathcal{S}}$ and $[\Sigma^{16}M, S^0]_n^{L_1\mathcal{S}}$, respectively. Since $(v_1^8)^*$ is an isomorphism between these two groups, and therefore

$$\Phi(v_1^8)^* : [X \wedge M, X]_n^{\mathcal{C}} \longrightarrow [X \wedge \Sigma^{16} M, X]_n^{\mathcal{C}}$$

is an isomorphism, we will now investigate how $X \wedge v_1^8$ differs from $\Phi(v_1^8)$ by making use of the preceding lemma and the computations in Chapter 5.

The element $X \wedge v_1^8$ lies in $[X \wedge \Sigma^{16} M, X \wedge M]_0^{\mathcal{C}}$, which, via Φ and Computation 5.8, is isomorphic to

$$[M,M]_{16}^{L_1\mathcal{S}} \cong \mathbb{Z}/4\{v_1^8\} \oplus \mathbb{Z}/2\{\tilde{\eta}\rho \circ \mathrm{pinch}, \mathrm{Id}_{L_1M} \wedge \eta\rho\}.$$

By Corollary 5.9, $2v_1^8 = \mathrm{incl} \circ \mu_{17} \circ \mathrm{pinch}$, so by Lemma 3.4,

$$\begin{aligned} 2(X \wedge v_1^8) &= (X \wedge \mathrm{incl}) \circ (X \wedge \mu_{17}) \circ (X \wedge \mathrm{pinch}) \\ &= \Phi(\mathrm{incl}) \circ \Phi(\mu_{17}) \circ \Phi(\mathrm{pinch}) \\ &= \Phi(\mathrm{incl} \circ \mu_{17} \circ \mathrm{pinch}) = \Phi(2v_1^8) \\ &= 2\Phi(v_1^8). \end{aligned}$$

or

$$\begin{aligned}
2(X \wedge v_1^8) &= (X \wedge \mathrm{incl}) \circ (X \wedge \mu_{17}) \circ (X \wedge \mathrm{pinch}) \\
&= \Phi(\mathrm{incl}) \circ (\Phi(\mu_{17}) + \Phi(\eta^2\rho)) \circ \Phi(\mathrm{pinch}) \\
&= \Phi(\mathrm{incl} \circ \mu_{17} \circ \mathrm{pinch}) + 0 = \Phi(2v_1^8) \\
&= 2\Phi(v_1^8).
\end{aligned}$$

This means that $X \wedge v_1^8$ can only differ from $\Phi(v_1^8)$ by an element of order at most two, i.e.,

$$X \wedge v_1^8 = \Phi(v_1^8) + \Phi(T), \quad \text{for some} \quad T \in [M, M]_{16}^{L_1\mathcal{S}}, \quad 2T = 0.$$

However, all such T are nilpotent in $[M, M]_*^{L_1\mathcal{S}}$ by Lemma 5.11, so

$$(X \wedge v_1^8)^* = \Phi(v_1^8)^* + \Phi(T)^*$$

is the sum of an isomorphism and a nilpotent map. As v_1^8 commutes with all such T by Lemma 5.10, this sum is again an isomorphism. Hence, by adjunction

$$(v_1^8)^* : [M, \mathrm{Hom}(X, X)]_n^{\mathcal{S}} \longrightarrow [M, \mathrm{Hom}(X, X)]_{n+16}^{\mathcal{S}}$$

is an isomorphism for all n, so $\mathrm{Hom}(X, X)$ is a $K_{(2)}$-local spectrum. □

To prove that every $\mathrm{Hom}(X, Y)$ is $K_{(2)}$-local, we make use of the fact that $\mathrm{Ho}(L_1\mathcal{S})$ is a compactly generated triangulated category:

The $K_{(2)}$-local sphere is a small weak generator in $\mathrm{Ho}(L_1\mathcal{S})$, i.e., $[S^0, -]^{L_1\mathcal{S}}$ commutes with coproducts and detects isomorphisms. So by [Kel94] 4.2, any triangulated subcategory of $\mathrm{Ho}(L_1\mathcal{S})$ closed under coproducts and containing the sphere must already be $\mathrm{Ho}(L_1\mathcal{S})$ itself. Since Φ is an equivalence of triangulated categories, $\Phi(L_1S^0) = X$ is a small weak generator for $\mathrm{Ho}(\mathcal{C})$, i.e., any triangulated subcategory of $\mathrm{Ho}(\mathcal{C})$ closed under coproducts and containing X is again $\mathrm{Ho}(\mathcal{C})$ itself.

Lemma 3.5. For $Y \in \mathcal{C}$, $\mathrm{Hom}(X, Y)$ is a $K_{(2)}$-local spectrum.

Proof. Again we use the criterion in [Bou79] §4, that a spectrum is $K_{(2)}$-local iff v_1^4 induces an isomorphism of its mod-2 homotopy groups. As $\mathrm{Ho}(\mathcal{S})$ is triangulated, this is equivalent to being acyclic with respect to $M(2, v_1^4)$, the cofiber of $v_1^4 : \Sigma^8 M \to M$. This means $A \in \mathcal{S}$ is $K_{(2)}$-local iff $[M(2, v_1^4), A]_*^{\mathcal{S}} = 0$.

We now prove this for $\mathrm{Hom}(X, Y), Y \in \mathcal{C}$: Let $\mathcal{T}$ be the full subcategory of $\mathrm{Ho}(\mathcal{C})$ containing all $Y \in \mathcal{C}$ such that $[M(2, v_1^4), \mathrm{Hom}(X, Y)]_*^{\mathcal{S}} = 0$. To prove that

$\mathcal{T} = \mathrm{Ho}(\mathcal{C})$, we show that $\mathcal{T}$ is triangulated, contains X and is closed under coproducts.

The category $\mathcal{T}$ is triangulated because $\mathrm{Hom}(X,-)$ is exact on homotopy level and

$$[M(2,v_1^4),-]_*^{\mathcal{S}} : \mathrm{Ho}(\mathcal{S}) \longrightarrow Ab_*$$

is a homological functor, i.e., sends exact triangles to long exact sequences of abelian groups. Also, $\mathcal{T}$ contains the weak generator X by Lemma 3.3. Now let $Y_i, i \in I$ be a family of objects in $\mathcal{T}$. By adjunction,

$$[M(2,v_1^4), \mathrm{Hom}(X, \coprod_i Y_i)]_*^{\mathcal{S}} \cong [X \wedge M(2,v_1^4), \coprod_i Y_i]_*^{\mathcal{C}}.$$

The object $X \wedge M$ is small in $\mathcal{C}$ as X is small and $X \wedge M$ is the cofiber of multiplication by two on X. As the cofiber of a map between small objects, $X \wedge M(2,v_1^4)$ is again small. Consequently, $[X \wedge M(2,v_1^4),-]_*^{\mathcal{C}}$ commutes with coproducts. Hence, we have

$$[X \wedge M(2,v_1^4), \coprod_i Y_i]_*^{\mathcal{C}} \cong \bigoplus_i [X \wedge M(2,v_1^4), Y_i]_*^{\mathcal{C}}$$

which is zero because $Y_i \in \mathcal{T}$ for all $i \in I$. It follows that

$$\mathcal{T} = \{Y \in \mathcal{C} \quad | \quad [M(2,v_1^4), \mathrm{Hom}(X,Y)]_*^{\mathcal{S}} = 0\} = \mathrm{Ho}(\mathcal{C}),$$

i.e., $\mathrm{Hom}(X,Y)$ is $K_{(2)}$-local for all $Y \in \mathcal{C}$. $\square$

Finally, we can prove Proposition 3.2, which says that for $X \cong \Phi(L_1 S^0)$, the Universal Property of Spectra provides a Quillen functor pair between $L_1\mathcal{S}$ and $\mathcal{C}$.

Proof. We show that the functor

$$\mathrm{Hom}(X,-) : \mathcal{C} \longrightarrow L_1\mathcal{S}$$

is a right Quillen functor, i.e., preserves fibrations and trivial fibrations.

Since the cofibrations in $\mathcal{S}$ are the same as in $L_1\mathcal{S}$, the left adjoint

$$X \wedge - : L_1\mathcal{S} \longrightarrow \mathcal{C}$$

preserves cofibrations because $X \wedge - : \mathcal{S} \longrightarrow \mathcal{C}$ is already a Quillen functor by the Universal Property of Spectra.

Via adjunction it follows that

$$\mathrm{Hom}(X, -) : \mathcal{C} \longrightarrow L_1\mathcal{S}$$

preserves trivial fibrations.

Now it is left to show that $\mathrm{Hom}(X, -)$ preserves fibrations. By [Dug01] A.2 it suffices to show that $\mathrm{Hom}(X, -)$ preserves fibrations between fibrant objects. We do this in the following steps:

- for $Y \in \mathcal{C}$ fibrant, $\mathrm{Hom}(X, Y)$ is fibrant in $L_1\mathcal{S}$
- $\mathrm{Hom}(X, -)$ sends fibrations to level fibrations
- level fibrations between fibrant objects in $L_1\mathcal{S}$ are fibrations.

Let $Y \in \mathcal{C}$ be fibrant. Then, by [SS02] 6.2 the spectrum $\mathrm{Hom}(X, Y)$ is an Ω-spectrum, as also described at the beginning of Section 3.1. By Lemma 3.5, $\mathrm{Hom}(X, Y)$ is $K_{(2)}$-local. So since in $L_1\mathcal{S}$ the fibrant objects are exactly the $K_{(2)}$-local Ω-spectra, $\mathrm{Hom}(X, Y)$ is fibrant for fibrant Y.

By construction, the functor $\mathrm{Hom}(X, -)$ sends fibrations to level fibrations, see [SS02] 6.2. But level fibrations between fibrant objects are fibrations in $L_1\mathcal{S}$:

Let $A, B \in L_1\mathcal{S}$ be fibrant, $f : A \longrightarrow B$ a level fibration. As both A and B are fibrant, f is a fibration in $\mathcal{S}$. In $L_1\mathcal{S}$ we use the Factorisation Axiom to factor f as the composite of a fibration and a trivial cofibration:

$$\begin{tikzcd} A \arrow[r, tail, "\sim", "i"'] & C \arrow[r, two heads, "p"'] & B \end{tikzcd}$$

By assumption, B is fibrant, and so must be C. So i is an $K_{(2)*}$-isomorphism between $K_{(2)*}$-local spectra and therefore a π_*-isomorphism. Also, i is a cofibration in $\mathcal{S}$ since it is a cofibration in $L_1\mathcal{S}$, so i is a trivial cofibration in $\mathcal{S}$.

Consequently, i has the left lifting property in $\mathcal{S}$ with respect to the level fibration f:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, tail, "i"'] & A \arrow[d, two heads, "f"] \\ C \arrow[r, "p"'] \arrow[ur, dotted, "h"] & B \end{tikzcd}$$

This gives us a commutative diagram in $L_1\mathcal{S}$

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[rr, bend left, "id"] \arrow[d, "f"] & C \arrow[r, "h"] \arrow[d, two heads, "p"] & A \arrow[d, "f"] \\ B \arrow[r, equal] & B \arrow[r, equal] & B \end{tikzcd}$$

which says that f is a retract of the $L_1\mathcal{S}$-fibration p and therefore a fibration in $L_1\mathcal{S}$ by the Retract Axiom of model categories.

Putting these steps together, we showed that $\mathrm{Hom}(X, -)$ is a right Quillen functor, which proves the proposition. $\square$

Chapter 4

The Quillen equivalence

Again, let $\Phi : \mathrm{Ho}(L_1\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$ be an equivalence of triangulated categories and X a fibrant and cofibrant replacement of $\Phi(L_1 S^0)$. In the last chapter we used the Universal Property of Spectra ([SS02]) to construct a Quillen functor pair

$$X \wedge - : L_1\mathcal{S} \rightleftarrows \mathcal{C} : \mathrm{Hom}(X, -).$$

This chapter will be devoted to showing that $(X \wedge -, \mathrm{Hom}(X, -))$ is a Quillen equivalence.

4.1 Homotopy type of $\mathrm{Hom}(X, X)$

Our first goal is to show that $\mathrm{Hom}(X, X)$ is stably equivalent to the $K_{(2)}$-local sphere spectrum. Define

$$\iota : S^0 \longrightarrow \mathrm{Hom}(X, X)$$

to be the morphism adjoint to the isomorphism $X \wedge S^0 \cong X$. Since $\mathrm{Hom}(X, X)$ is $K_{(2)}$-local by Lemma 3.3, ι factors over the $K_{(2)}$-local sphere:

$$\begin{tikzcd} S^0 \arrow[r, "\iota"] \arrow[d, "L_1"'] & \mathrm{Hom}(X, X) \\ L_1 S^0 \arrow[ur, dotted, "\lambda"'] & \end{tikzcd}$$

Proposition 4.1. The map λ is a π_n-isomorphism for $n \geq 0$.

Proof. We have the following diagram:

$$\begin{tikzcd} \pi_n L_1 S^0 \cong [S^0, S^0]_n^{L_1\mathcal{S}} \arrow[r, "\lambda_*"] \arrow[d, "X\wedge^L -"'] & {[S^0, \mathrm{Hom}(X,X)]_n^{\mathcal{S}}} \\ {[X,X]_n^{\mathcal{C}}} \arrow[ur, "\cong", "adj"'] & \end{tikzcd}$$

It is commutative because, by definition of λ, for $\alpha \in \pi_* L_1 S^0$ the image of $X \wedge^L \alpha$ under the adjunction isomorphism is precisely $\lambda \circ \alpha$. Hence, λ_* is an isomorphism if and only if

$$X \wedge^L - : [S^0, S^0]_n^{L_1\mathcal{S}} \longrightarrow [X,X]_n^{\mathcal{C}}$$

is an isomorphism for all n. We show that

$$\Psi : [S^0, S^0]_n^{L_1\mathcal{S}} \xrightarrow{X\wedge^L -} [X,X]_n^{\mathcal{C}} \xrightarrow{\Phi^{-1}} [S^0, S^0]_n^{L_1\mathcal{S}}$$

is an isomorphism for all $n \geq 0$.

The statement that Ψ is an isomorphism in degree 0,...,9 follows directly from Lemma 3.4 (see also the table on page 36 in Chapter 5).

Using the exact triangle

$$L_1 S^0 \xrightarrow{2} L_1 S^0 \xrightarrow{\text{incl}} L_1 M \xrightarrow{\text{pinch}} L_1 S^1$$

together with the 5-lemma, it follows that

$$\Psi : [M, S^0]_n^{L_1\mathcal{S}} \longrightarrow [M, S^0]_n^{L_1\mathcal{S}}$$

is an isomorphism for $n = 0, ..., 8$. (We are still using our fixed isomorphism $X \wedge M \cong \Phi(L_1 M)$ which will be omitted from our notation.)

Now M has a v_1-self map v_1^4 that induces an isomorphism in $K_{(2)}$-homology, so

$$v_1^4 : \Sigma^8 L_1 M \longrightarrow L_1 M$$

is an isomorphism in $\mathrm{Ho}(L_1\mathcal{S})$. Using the commutative diagram

$$\begin{tikzcd} {[M, S^0]_n^{L_1\mathcal{S}}} \arrow[r, "\Psi"] \arrow[d, "(v_1^4)^*"', "\cong"] & {[M, S^0]_n^{L_1\mathcal{S}}} \arrow[d, "(\Psi(v_1^4))^*"', "\cong"] \\ {[M, S^0]_{n+8}^{L_1\mathcal{S}}} \arrow[r, "\Psi"] & {[M, S^0]_{n+8}^{L_1\mathcal{S}}} \end{tikzcd}$$

we obtain by induction that

$$\Psi : [M, S^0]_n^{L_1\mathcal{S}} \longrightarrow [M, S^0]_n^{L_1\mathcal{S}}$$

is an isomorphism for all $n \geq 0$. (Note that the commutativity of above diagram does not depend on the choice of the fixed isomorphisms $X \wedge M \cong \Phi(L_1 M)$ and $X \wedge S^0 \cong \Phi(L_1 S^0)$.)

There is an exact triangle of Moore spectra

$$L_1 M = L_1 M(\mathbb{Z}/2) \longrightarrow L_1 M(\mathbb{Z}/2^{k+1}) \longrightarrow L_1 M(\mathbb{Z}/2^k) \longrightarrow \Sigma L_1 M.$$

Again, we conclude by induction on k using the 5-lemma, that

$$\Psi : [M(\mathbb{Z}/2^k), S^0]_n^{L_1\mathcal{S}} \longrightarrow [M(\mathbb{Z}/2^k), S^0]_n^{L_1\mathcal{S}}$$

is an isomorphism for all $n, k \geq 0$.

The Moore spectra $M(\mathbb{Z}/2^k)$ are self-dual with respect to Spanier-Whitehead duality up to suspension: this follows directly from the fact that the dualization functor $D : \mathrm{Ho}(\mathcal{S})^{op} \longrightarrow \mathrm{Ho}(\mathcal{S})$ is exact and sends the sphere and multiplication by 2^k to itself. Thus, we have $\Sigma DM(\mathbb{Z}/2^k) \cong M(\mathbb{Z}/2^k)$, and

$$[M(\mathbb{Z}/2^k), S^0]_n^{L_1\mathcal{S}} = [S^0, M(\mathbb{Z}/2^k)]_{n+1}^{L_1\mathcal{S}}.$$

So Ψ is also an automorphism of $[S^0, M(\mathbb{Z}/2^k)]_n^{L_1\mathcal{S}}$ for all $k \geq 0, n \geq 1$.

The $K(1)$-local sphere $L_{K(1)}S^0$ is equivalent in $L_1\mathcal{S}$ to $\mathrm{holim}_k\, L_1 M(\mathbb{Z}/2^k)$. (For this, use [HS99] Proposition 7.10.(e) with $X = S^0$. The functor $\hat{L}$ denotes Bousfield localisation with respect to $K(1)$, $L = L_1$ and S/I the mod-2^k Moore spectra, see Notation 1.1 of [HS99].) Thus we have

$$[S^0, \mathrm{holim}_k\, M(\mathbb{Z}/2^k)]_n^{L_1\mathcal{S}} = \lim_k [S^0, M(\mathbb{Z}/2^k)]_n^{L_1\mathcal{S}}$$

as $\lim^1_k [S^0, M(\mathbb{Z}/2^k)]_n^{L_1\mathcal{S}} = 0$ since all groups over which the limit is taken are finite.

We obtain that

$$\Psi : [S^0, L_{K(1)}S^0]_n^{L_1\mathcal{S}} \longrightarrow [S^0, L_{K(1)}S^0]_n^{L_1\mathcal{S}}$$

is an isomorphism for all $n \geq 0$.

The $K(1)$-local sphere only differs from the $K_{(2)}$-local sphere by some rational parts in the dimension range $-2, ..., 1$, so

$$\pi_k L_1 S^0 \longrightarrow \pi_k L_{K(1)} S^0$$

is an isomorphism for $k \geq 2$, and it follows that

$$\Psi : [S^0, S^0]_n^{L_1\mathcal{S}} \longrightarrow [S^0, S^0]_n^{L_1\mathcal{S}}$$

is an isomorphism for all $n \geq 2$. But we already showed that Ψ is an isomorphism for $n = 0$ and $n = 1$, so we get the desired result that

$$\Psi : [S^0, S^0]_n^{L_1\mathcal{S}} \longrightarrow [S^0, S^0]_n^{L_1\mathcal{S}}$$

is an isomorphism for all $n \geq 0$. Thus,

$$\lambda_* : \pi_n L_1 S^0 \longrightarrow \pi_n \operatorname{Hom}(X, X)$$

is an isomorphism for all $n \geq 0$, so the connective covers of $L_1 S^0$ and $\operatorname{Hom}(X, X)$ are stably equivalent via λ. □

Lemma 4.2. Let $f : A \longrightarrow B$ be a morphism of $K_{(2)}$-local spectra. Further, assume that $\pi_k A$ and $\pi_k B$ are torsion groups for $k < 0$. If f induces a weak homotopy equivalence between the connective covers A^c and B^c of A and B, then f is a weak homotopy equivalence.

Proof. We can complete the covering map $c_A : A^c \longrightarrow A$ to an exact triangle

$$A^{-\infty} \longrightarrow A^c \xrightarrow{c_A} A \longrightarrow \Sigma A^{-\infty}$$

in $\mathrm{Ho}(\mathcal{S})$. Now we look at the tower of Postnikov sections of $A^{-\infty}$:

$$\begin{tikzcd}
A^{-\infty} = X_{\leq -1} \arrow[r] & X_{\leq -2} \arrow[r] & X_{\leq -3} \arrow[r] & \cdots \\
H\pi_{-1}A^{-\infty} \arrow[u] & \Sigma^{-1}H\pi_{-2}A^{-\infty} \arrow[u] & \Sigma^{-2}H\pi_{-3}A^{-\infty} \arrow[u] &
\end{tikzcd}$$

with

$$\pi_k X_{\leq -i} = \begin{cases} \pi_k A^{-\infty}, & k \leq -i \\ 0, & \text{else} \end{cases}$$

and

$$\Sigma^{-k+1} H\pi_{-k+1} A^{-\infty} \longrightarrow X_{\leq -k} \longrightarrow X_{\leq -k+1}$$

being a fiber sequence in $\mathcal{S}$ where HG denotes the Eilenberg-MacLane spectrum of the group G.

The $K_{(2)}$-homology of a torsion Eilenberg-MacLane spectrum is trivial ([AH68]), so the morphisms

$$X_{\leq -k} \longrightarrow X_{\leq -k+1}$$

are $K_{(2)*}$-isomorphisms. Consequently,

$$K_{(2)*}(A^{-\infty}) \simeq K_{(2)*}(\operatorname{hocolim}_n X_{\leq -n}).$$

However, $\operatorname{hocolim}_n X_{\leq -n} \simeq *$: Applying π_k to the directed system of the $X_{\leq n}$'s, one obtains a directed system of abelian groups that becomes stationary 0 after the $-(k+1)^{th}$ step. So

$$\pi_k \operatorname{colim}_n X_{\leq -n} = 0 \quad \text{for all} \quad k,$$

and thus $\operatorname{hocolim}_n X_{\leq -n} \simeq *$.

It follows that

$$K_{(2)*}(A^{-\infty}) = K_{(2)*}(\operatorname{hocolim}_n X_{\leq -n}) = 0.$$

We now arrive at a morphism of exact triangles in $\operatorname{Ho}(\mathcal{S})$:

$$\begin{tikzcd}
A^{-\infty} \arrow[r] \arrow[d] & A^c \arrow[r, "c_A"] \arrow[d, "f^c"] & A \arrow[r] \arrow[d, "f"] & \Sigma A^{-\infty} \arrow[d] \\
B^{-\infty} \arrow[r] & B^c \arrow[r, "c_B"] & B \arrow[r] & \Sigma B^{-\infty}
\end{tikzcd}$$

The map f^c is a weak homotopy equivalence by assumption. The spectra $A^{-\infty}$ and $B^{-\infty}$ have trivial $K_{(2)}$-homology, so c_A and c_B are $K_{(2)*}$-isomorphisms. Therefore, f must be an $K_{(2)*}$-isomorphism, and since it is an $K_{(2)*}$-isomorphism between $K_{(2)*}$-local spectra, it is a π_*-equivalence. $\square$

Corollary 4.3. Let A be a $K_{(2)}$-local spectrum with $\pi_k A$ being torsion groups for $n < 0$. Then the covering map $c_A : A^c \longrightarrow A$ is a $K_{(2)}$-localisation.

Corollary 4.4. The map $\lambda : L_1 S^0 \longrightarrow \operatorname{Hom}(X, X)$ is a π_*-equivalence.

With this, we can now prove the $K_{(2)}$-local Rigidity Theorem:

4.2 Proof of the Main Theorem

Theorem 4.5. Let $\Phi : \operatorname{Ho}(L_1\mathcal{S}) \longrightarrow \operatorname{Ho}(\mathcal{C})$ be an equivalence of triangulated categories, X a fibrant and cofibrant replacement of $\Phi(L_1 S^0)$, and

$$X \wedge - : L_1\mathcal{S} \rightleftarrows \mathcal{C} : \operatorname{Hom}(X, -)$$

the Quillen functor pair from Proposition 3.2. Then $(X\wedge -, \mathrm{Hom}(X,-))$ is a Quillen equivalence.

Proof. By [Hov99] 1.3.16, it suffices to show the following:

1. $\mathrm{RHom}(X,-) : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(L_1\mathcal{S})$ reflects isomorphisms
2. The map $A \longrightarrow \mathrm{RHom}(X, X\wedge^L A)$ is an isomorphism for all $A \in \mathrm{Ho}(L_1\mathcal{S})$.

Let $Y \longrightarrow Z$ be an isomorphism in $\mathrm{Ho}(\mathcal{C})$. As $X \cong \Phi(L_1S^0)$ with Φ being an equivalence and $L_1S^0 \in \mathrm{Ho}(L_1\mathcal{S})$ a weak generator, X is a weak generator, and so

$$[X,Y]_*^{\mathcal{C}} \longrightarrow [X,Z]_*^{\mathcal{C}}$$

is an isomorphism. By adjunction it follows that

$$[S^0, \mathrm{RHom}(X,Y)]_*^{L_1\mathcal{S}} \longrightarrow [S^0, \mathrm{RHom}(X,Z)]_*^{L_1\mathcal{S}}$$

is an isomorphism. But as the sphere is a generator in $\mathrm{Ho}(L_1\mathcal{S})$, it detects isomorphisms, so

$$\mathrm{RHom}(X,Y) \longrightarrow \mathrm{RHom}(X,Z)$$

is an isomorphism in $\mathrm{Ho}(L_1\mathcal{S})$ which proves the first point.

To prove the second point we define $\mathcal{T}$ to be the full subcategory of $\mathrm{Ho}(L_1\mathcal{S})$ containing those $A \in \mathrm{Ho}(L_1\mathcal{S})$ such that

$$A \longrightarrow \mathrm{RHom}(X, X\wedge^L A)$$

is an isomorphism. We want to prove that $\mathcal{T} = \mathrm{Ho}(L_1\mathcal{S})$.

Since $\mathrm{RHom}(X,-)$ and $X\wedge^L -$ are exact functors, $\mathcal{T}$ is triangulated. By Corollary 4.4, $L_1S^0 \in \mathcal{T}$. Now let $A_i, i \in I$, be a family of objects in $\mathcal{T}$. We want to prove that $\coprod_{i\in I} A_i \in \mathcal{T}$. By adjunction,

$$[S^0, \mathrm{RHom}(X, X\wedge^L (\coprod_i A_i))]_*^{L_1\mathcal{S}} \cong [X, X\wedge^L (\coprod_i A_i)]_*^{\mathcal{C}}.$$

As a left adjoint, $X\wedge^L -$ commutes with coproducts, so

$$[X, X\wedge^L (\coprod_i A_i)]_*^{\mathcal{C}} \cong [X, \coprod_i (X\wedge^L A_i)]_*^{\mathcal{C}}.$$

Since $X \cong \Phi(L_1 S^0)$ is small, we have

$$\begin{aligned}[X, \coprod_i (X \wedge^L A_i)]_*^{\mathcal{C}} &\cong \bigoplus_i [X, X \wedge^L A_i]_*^{\mathcal{C}} \\ &\cong \bigoplus_i [S^0, \mathrm{RHom}(X, X \wedge^L A_i)]_*^{L_1\mathcal{S}}.\end{aligned}$$

As $A_i \in \mathcal{T}$ for all i,

$$[S^0, A_i]_*^{L_1\mathcal{S}} \cong [S^0, \mathrm{RHom}(X, X \wedge^L A_i)]_*^{L_1\mathcal{S}},$$

induced by

$$A_i \xrightarrow{\cong} \mathrm{RHom}(X, X \wedge^L A_i).$$

So by naturality of the preceding isomorphisms,

$$[S^0, \coprod_i A_i]_*^{L_1\mathcal{S}} \cong [S^0, \mathrm{RHom}(X, X \wedge^L (\coprod_i A_i))]_*^{L_1\mathcal{S}}$$

is an isomorphism induced by the map

$$\coprod_i A_i \longrightarrow \mathrm{RHom}(X, X \wedge^L (\coprod_i A_i)).$$

Since the $K_{(2)}$-local sphere detects isomorphisms, this map is an isomorphism in $\mathrm{Ho}(L_1\mathcal{S})$.

So we have seen that $\mathcal{T}$ is triangulated, contains $L_1 S^0$ and is closed under coproducts, therefore $\mathcal{T}$ must be $\mathrm{Ho}(L_1\mathcal{S})$. This means that

$$A \longrightarrow \mathrm{RHom}(X, X \wedge^L A)$$

is an isomorphism for all $A \in \mathrm{Ho}(L_1\mathcal{S})$.

We can now conlude that $(X \wedge -, \mathrm{Hom}(X, -))$ is a Quillen equivalence for $X \cong \Phi(L_1 S^0)$. So, given an equivalence of triangulated categories

$$\Phi : \mathrm{Ho}(L_1\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

we have proved that $\mathcal{C}$ and $L_1\mathcal{S}$ are Quillen equivalent, which proves the K-local Rigidity Theorem at the prime 2. □

Chapter 5

Computations

For our main proofs we need information about $[M, M]_*^{L_1\mathcal{S}}$ and $[S^0, M]_*^{L_1\mathcal{S}}$ in certain degrees. The necessary computations will be summarized in this chapter.

5.1 Generators and relations of $\pi_* L_1 S^0$

First, let us look at the homotopy groups of the $K_{(2)}$-local sphere, see e.g. [Bou79] Proposition 4.5 or [Rav84] 8.15. The ring homomorphism

$$L_1 : \pi_* S^0 \longrightarrow \pi_* L_1 S^0$$

induced by $K_{(2)}$-localisation is surjective in degrees ≥ 2, and it has a cokernel isomorphic to $\mathbb{Z}/2$ in degrees 0 and 1. There is a unique order 2 element of $\pi_0 L_1 S^0$ called y_0, and $y_1 = \eta y_0$ is a generator of the second $\mathbb{Z}/2$ summand in $\pi_1 L_1 S^0$. The other elements of $\pi_* L_1 S^0$ are given the names of their (not necessarily unique) preimage in $\pi_* S^0$. So in low degrees we have

k	$\pi_k L_1 S^0$
0	$\mathbb{Z}_{(2)}\{\iota\} \oplus \mathbb{Z}/2\{y_0\}$
1	$\mathbb{Z}/2\{\eta, y_1\}$
2	$\mathbb{Z}/2\{\eta^2\}$
3	$\mathbb{Z}/8\{\nu\}$
4	0
5	0
6	0
7	$\mathbb{Z}/16\{\sigma\}$
8	$\mathbb{Z}/2\{\eta\sigma\}$
9	$\mathbb{Z}/2\{\eta^2\sigma, \mu\}$
...	...
15	$\mathbb{Z}/32\{\rho\}$
16	$\mathbb{Z}/2\{\eta\rho\}$
17	$\mathbb{Z}/2\{\eta^2\rho, \mu_{17}\}$

Moreover, we have $4\nu = \eta^3, \eta y_1 = 0, y_1^2 = 0$, $\mu y_0 = \eta^2\sigma$ and $\mu_{17} y_0 = \eta^2\rho$ ([Rav84] 8.15.(d)). Furthermore, we make use of the following Toda bracket releations:

$$\begin{aligned} 8\sigma &= \langle \nu, 8, \nu \rangle \\ \mu &\in \langle 2, 8\sigma, \eta \rangle \qquad && \text{(indeterminacy: } \eta^2\sigma) \\ \rho &\in \langle \sigma, 2\sigma, 8 \rangle \qquad && \text{(indeterminacy: } 8\pi_{15} L_1 S^0) \\ \mu_{17} &\in \langle \eta, 16\rho, 2 \rangle \qquad && \text{(indeterminacy: } \eta^2\rho) \end{aligned}$$

The element μ is the unique element of the second Toda bracket with Adams filtration five, μ_{17} is the unique element of the last bracket with Adams filtration nine. For a reference of the first three bracket relations, see [Tod62], Lemma 5.13, Lemma 10.9 and the tables in Chapter XIV. For the last bracket, see Lemma 5.7 of this thesis.

Notation. Throughout this thesis, we read Toda brackets from right to left, i.e., in the same direction as the composition of morphisms.

5.2 Homotopy groups and endomorphisms of $L_1 M$

We will now compute some homotopy groups of the $K_{(2)}$-local mod-2 Moore spectrum. The long exact homotopy sequence of the exact triangle

$$L_1S^0 \xrightarrow{2} L_1S^0 \xrightarrow{\text{incl}} L_1M \xrightarrow{\text{pinch}} L_1S^1$$

splits into short exact sequences of the form

$$0 \longrightarrow \pi_{m+1}L_1S^0/(2) \xrightarrow{\text{incl}_*} \pi_{m+1}L_1M \xrightarrow{\text{pinch}_*} \{\pi_m L_1S^0\}_2 \longrightarrow 0.$$

Here, $\{\pi_m L_1S^0\}_2$ denotes the 2-torsion of the group $\pi_m L_1S^0$, i.e., all $x \in \pi_m L_1S^0$ with $2x = 0$.

Let $x \in \{\pi_m L_1S^0\}_2$, and $\tilde{x} \in \pi_{m+1}L_1M$ a lift of x, i.e., an element with $\text{pinch} \circ \tilde{x} = x$. We have $\text{pinch}_*(2\tilde{x}) = 0$, so $2\tilde{x}$ has a unique preimage under the map incl_*. This preimage is $\eta x \in \pi_{m+1}L_1S^0/(2)$, as

$$\text{incl}_*(\eta x) = (\text{incl} \circ \eta \circ \text{pinch}) \circ \tilde{x} = 2\tilde{x},$$

remembering $\text{incl} \circ \eta \circ \text{pinch} = 2\,\text{Id}_{L_1M}$.

Notation. A preimage of an element x under the pinch map will be denoted by $\tilde{x}$. This $\tilde{x}$ need not be unique, but the following computations do not depend on the choice of such an $\tilde{x}$ unless stated.

For some particular examples this gives us

Computation 5.1. $\pi_0 L_1M \cong \mathbb{Z}/2\{\text{incl}, \text{incl} \circ y_0\}$

Computation 5.2. $\pi_1 L_1M \cong \mathbb{Z}/4\{\tilde{y_0}\} \oplus \mathbb{Z}/2\{\text{incl} \circ \eta\}$

Computation 5.3. $\pi_{16} L_1M \cong \mathbb{Z}/2\{\text{incl} \circ \eta\rho, \widetilde{16\rho}\}$

Computation 5.4. $\pi_{17} L_1M \cong \mathbb{Z}/4\{\widetilde{\eta\rho}\} \oplus \mathbb{Z}/2\{\text{incl} \circ \mu_{17}\}$

Also, note that

$$\text{incl} \circ y_1 \circ \text{pinch} = \text{incl} \circ \eta y_0\, \text{pinch} = 2\tilde{y_0}\, \text{pinch} = \tilde{y_0}(2\, \text{pinch}) = 0. \tag{5.1}$$

To specify the element $\widetilde{16\rho}$ in Computation 5.3 and for further applications we need the following:

Lemma 5.5. $16\rho = \text{pinch} \circ v_1^8 \circ \text{incl}$ in $\text{Ho}(L_1\mathcal{S})$.

Proof. The element $\mathrm{pinch}\circ v_1^8\circ\mathrm{incl}$ lies in $\pi_{15}L_1S^0$. Since $2\,\mathrm{pinch}=0$, it has order at most two, so the element in question is either 16ρ or 0.

Assume that $\mathrm{pinch}\circ v_1^8\circ\mathrm{incl}=0$, then $v_1^8\circ\mathrm{incl}$ factors over the fiber of the pinch map, which, after $K_{(2)}$-localisation, gives us the commutative diagram

$$\begin{tikzcd} \Sigma^{16}L_1M \arrow[r, "v_1^8"] & L_1M \\ L_1S^8 \arrow[u, "\mathrm{incl}"] \arrow[r, "\varphi"] & L_1S^0 \arrow[u, "\mathrm{incl}"'] \end{tikzcd}$$

The element φ lies in $\pi_{16}L_1S^0\cong\mathbb{Z}/2\{\eta\rho\}$, so $\varphi=\eta a$ for either $a=0$ or $a=\rho$. We now apply the m^{th} $K(1)$-homology to above diagram, using this factorisation of φ:

$$\begin{tikzcd} K(1)_m(L_1S^{16}) \arrow[r, "\mathrm{incl}_*"] \arrow[d, "\eta_*"'] & K(1)_m(\Sigma^{16}L_1M) \arrow[r, "(v_1^8)_*"] & K(1)_m(L_1M) \\ K(1)_m(L_1S^{15}) \arrow[rr, "a_*"] & & K(1)_m(L_1S^0) \arrow[u, "\mathrm{incl}_*"'] \end{tikzcd}$$

For even m, $\mathrm{incl}_*=K(1)_m(\mathrm{incl})$ is an isomorphism, the map v_1^8 is a $K(1)_*$-isomorphism, so the upper row is an isomorphism for even m. However, η_* lowers the degree by one, so it must be zero since the $K(1)$-homology of the sphere is concentrated in even degrees. Thus, we have arrived at a contradiction.

So since there is no $\varphi\in\pi_{16}L_1S^0$ with $\mathrm{incl}\circ\varphi=v_1^8\circ\mathrm{incl}$, the composition $\mathrm{pinch}\circ v_1^8\circ\mathrm{incl}\in\pi_{15}S^0$ is nonzero, has order two and therefore must be 16ρ. □

Corollary 5.6. $\pi_8L_1M\cong\mathbb{Z}/2\{\mathrm{incl}\circ\eta\rho, v_1^8\circ\mathrm{incl}\}$

Lemma 5.7. In $\mathrm{Ho}(L_1\mathcal{S})$, $\mu_{17}\in\langle\eta,16\rho,2\rangle$.

Proof. Since $16\rho\cdot 2=32\rho=0$, there is a lift $R:\Sigma^{16}L_1M\longrightarrow L_1M$ such that $R\circ\mathrm{incl}=16\rho$. We have

$$R\in[M,S^0]_{15}^{L_1\mathcal{S}}=\mathbb{Z}/2\{\eta\rho\circ\mathrm{pinch},\mathrm{pinch}\circ v_1^8\},$$

so by Lemma 5.5, R can either be $\mathrm{pinch}\circ v_1^8$ or $\eta\rho\circ\mathrm{pinch}+\mathrm{pinch}\circ v_1^8$.

As $\eta\cdot 16\rho = \eta\circ R\circ \mathrm{incl} = 0$, $\eta\circ R$ has a lift $Q : L_1S^{17} \longrightarrow L_1S^0$ with $Q\circ\mathrm{pinch} = \eta\circ R$:

$$\begin{tikzcd}
L_1S^{16} \arrow[r, "2"] & L_1S^{16} \arrow[r, "\mathrm{incl}"] \arrow[d, "16\rho"'] & \Sigma^{16}L_1M \arrow[r, "\mathrm{pinch}"] \arrow[dl, "R"] & L_1S^{17} \arrow[ddll, "Q"] \\
 & L_1S^1 \arrow[d, "\eta"'] & & \\
 & L_1S^1 & &
\end{tikzcd}$$

The Toda bracket $\langle \eta, 16\rho, 2\rangle$ consists of all such lifts Q and has indeterminacy $\eta^2\rho$, so for a fixed Q, it equals the set $\{Q, Q+\eta^2\rho\}$. It does not contain 0 as $\eta\circ R = Q\circ\mathrm{pinch}$, and, hence, for $Q = 0$ we would have $\eta\circ R = 0$. This is not the case since

$$\mathrm{incl}\circ\eta\circ R = \mathrm{incl}\circ\eta\circ \mathrm{pinch}\circ v_1^8 = 2v_1^8 \neq 0.$$

Consequently, the only possible elements to be contained in this bracket are μ_{17} and $\mu_{17} + \eta^2\rho$. $\square$

Computation 5.8. $[M, M]_{16}^{L_1\mathcal{S}} \cong \mathbb{Z}/4\{v_1^8\} \oplus \mathbb{Z}/2\{\widetilde{\eta\rho}\circ\mathrm{pinch}, \mathrm{Id}_{L_1M}\wedge\eta\rho\}$

Proof. We consider the short exact sequence

$$0 \longrightarrow \pi_{17}L_1M/(2) \xrightarrow{\mathrm{pinch}^*} [M, M]_{16}^{L_1\mathcal{S}} \xrightarrow{\mathrm{incl}^*} \{\pi_{16}L_1M\}_2 \longrightarrow 0. \tag{5.2}$$

Let $x \in \{\pi_{16}L_1M\}_2$, $\bar{x} \in [M, M]_{16}^{L_1\mathcal{S}}$ with $\mathrm{incl}^*(\bar{x}) = \bar{x}\circ\mathrm{incl} = x$. Since $2\,\mathrm{Id}_{L_1M}\, x = 0$, the element $2\bar{x}$ has a unique preimage $q \in \pi_{17}L_1M/(2)$. This q lies in the Toda bracket $\langle 2\,\mathrm{Id}_{L_1M}, x, 2\rangle$:

$$\begin{tikzcd}
L_1S^{16} \arrow[r, "2"] & L_1S^{16} \arrow[r, "x"] \arrow[dr, "\mathrm{incl}"'] & L_1M \arrow[r, "2\,\mathrm{Id}_{L_1M}"] & L_1M \\
 & & \Sigma^{16}L_1M \arrow[u, "\bar{x}"'] \arrow[dr, "\mathrm{pinch}"'] & \\
 & & & L_1S^{17} \arrow[uu, "q\in\langle 2\,\mathrm{Id}_{L_1M}, x, 2\rangle"']
\end{tikzcd}$$

So to determine whether $2\bar{x} = 0$ for any $x \in \{\pi_{16}L_1M\}_2$, we have to compute the brackets $\langle 2\,\mathrm{Id}_{L_1M}, v_1^8\circ\mathrm{incl}, 2\rangle$ and $\langle 2\,\mathrm{Id}_{L_1M}, \mathrm{incl}\circ\eta\rho, 2\rangle$.

Since $2\,\mathrm{Id}_{L_1M} = \mathrm{incl} \circ \eta \circ \mathrm{pinch}$, the first bracket writes as

$$\begin{aligned}\left\langle 2\,\mathrm{Id}_{L_1M}, v_1^8 \circ \mathrm{incl}, 2\right\rangle &= \left\langle \mathrm{incl} \circ \eta \circ \mathrm{pinch}, v_1^8 \circ \mathrm{incl}, 2\right\rangle \\ &= \left\langle \mathrm{incl} \circ \eta, \mathrm{pinch} \circ v_1^8 \circ \mathrm{incl}, 2\right\rangle \\ &= \langle \mathrm{incl} \circ \eta, 16\rho, 2\rangle \\ &= \mathrm{incl} \circ \langle \eta, 16\rho, 2\rangle\end{aligned}$$

where the second and fourth equality are due to the Juggling Theorem ([Rav86] A1.4.6) and the third equality due to Lemma 5.5. This means that $2v_1^8$ is hit in the short exact sequence by $\mathrm{incl} \circ \mu_{17}$ or $\mathrm{incl} \circ (\mu_{17} + \eta^2\rho)$, as $\mu_{17} \in \langle \eta, 16\rho, 2\rangle$ with indeterminacy $\eta^2\rho$. Since

$$\mathrm{incl} \circ \eta^2\rho \circ \mathrm{pinch} = 2\widetilde{\eta\rho}\,\mathrm{pinch} = 0,$$

we have in either case

$$2v_1^8 = \mathrm{incl} \circ \mu_{17} \circ \mathrm{pinch} \neq 0.$$

The second bracket gives us

$$\begin{aligned}\langle 2\,\mathrm{Id}_{L_1M}, \mathrm{incl} \circ \eta\rho, 2\rangle &= \langle \mathrm{incl} \circ \eta, \mathrm{pinch} \circ \mathrm{incl} \circ \eta\rho, 2\rangle \\ &= \langle \mathrm{incl} \circ \eta, 0, 2\rangle = 0.\end{aligned}$$

The indeterminacy here is $2\pi_{17}L_1M$, i.e., zero in $\pi_{17}L_1M/(2)$. Applying these computations to the short exact sequence (5.2) gives us now the desired result: we now know that $[M, M]_{16}^{L_1\mathcal{S}} \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$ with v_1^8 generating the $\mathbb{Z}/4$ summand, and $\tilde{\eta}\rho \circ \mathrm{pinch}$ generating one of the $\mathbb{Z}/2$ summands. Looking at the sequence (5.2) again we see that any element $P \in [M, M]_8^{L_1\mathcal{S}}$ with $P \circ \mathrm{incl} = \mathrm{incl} \circ \eta\rho$ can be taken to be a generator of the other $\mathbb{Z}/2$ summand, so we choose $P = \mathrm{Id}_{L_1M} \wedge \eta\rho$ (where $\wedge$ denotes the smash product in the homotopy category of $\mathcal{S}$). □

Corollary 5.9. $2v_1^8 = \mathrm{incl} \circ \mu_{17} \circ \mathrm{pinch} \neq 0$.

Next, we are going to provide some computations that are made use of in the proof of Lemma 3.3. There, we need that v_1^8 commutes with all elements of $[M, M]_{16}^{L_1\mathcal{S}}$, and that all elements of additive order at most two in $[M, M]_{16}^{L_1\mathcal{S}}$ are nilpotent.

Lemma 5.10. Every element of $[M, M]_{16}^{L_1\mathcal{S}}$ commutes with v_1^8.

Proof. In their paper [CK88], Crabb and Knapp describe the construction of a self map B for certain topological spaces X, that induces an isomorphism in real topological K-theory KO_* and that can be chosen to be central in the stable endomorphism ring of X. In the fifth section, they apply this construction to the mod-2 Moore spectrum to obtain a map $B_2^2 : \Sigma^{16}M \longrightarrow M$ that is a KO_*-isomorphism. Proposition 5.1 of [CK88] now says that this B_2^2 is central in the stable endomorphism ring of M.

This map B_2^2 is a v_1-self map for M: By [Bou90] 1.11, the complex K-theory spectrum K is the cofiber of multiplication by η on KO. Using the 5-lemma, we obtain that a KO_*-isomorphism is also a K_*-isomorphism. So in our case, the map B_2^2 is a K_*-isomorphism, and as M is 2-local, also a $K_{(2)}$-isomorphism. Hence, we can choose our v_1-self map $v_1^8 : \Sigma^{16}M \longrightarrow M$ to be this map B_2^2.

So we now know that our v_1^8 is central in the ring $[M,M]_*^{\mathcal{S}}$. By Corollary F of [DMM87],

$$(v_1^8)^{-1}M := \operatorname{colim}(M \xrightarrow{v_1^8} \Sigma^{-16}M \xrightarrow{v_1^8} ...),$$

is a $K_{(2)}$-localisation of M, so the endomorphisms of the $K_{(2)}$-local Moore spectrum are the endomorphisms of M in $\mathrm{Ho}(\mathcal{S})$ made v_1^8-periodic:

$$[M,M]_*^{L_1\mathcal{S}} = \operatorname{colim}\left([M,M]_*^{\mathcal{S}} \xrightarrow{(v_1^8)^*} [M,M]_{*+16}^{\mathcal{S}} \xrightarrow{(v_1^8)^*} ...\right) = (v_1^8)^{-1}[M,M]_*^{\mathcal{S}},$$

so if v_1^8 is central in $[M,M]_*^{\mathcal{S}}$, then it also is in $[M,M]_*^{L_1\mathcal{S}}$. In particular, v_1^8 commutes with all elements of $[M,M]_{16}^{L_1\mathcal{S}}$. □

Lemma 5.11. Every $T \in [M,M]_{16}^{L_1\mathcal{S}}$ with $2T = 0$ is nilpotent.

Proof. The element $\eta\rho \in \pi_{16}L_1S^0$ satisfies $(\eta\rho)^2 = \eta^2\rho^2 = 0$. Consequently,

$$(\mathrm{Id}_{L_1M} \wedge \eta\rho)^2 = \mathrm{Id}_{L_1M} \wedge (\eta\rho^2) = \mathrm{Id}_{L_1M} \wedge 0 = 0.$$

Next,

$$(\widetilde{\eta\rho} \circ \mathrm{pinch})^3 = \widetilde{\eta\rho} \circ (\mathrm{pinch} \circ (\widetilde{\eta\rho}))^2 \circ \mathrm{pinch} = \widetilde{\eta\rho} \circ (\eta\rho)^2 \circ \mathrm{pinch} = \widetilde{\eta\rho} \circ 0 \circ \mathrm{pinch} = 0$$

and $(2v_1^8)^2 = (4v_1^8)v_1^8 = 0$. These three elements commute, so their sums are also nilpotent. □

Chapter 6

The case against odd primes

As mentioned in the introduction, in the case of $p > 2$, rigidity for $K_{(p)}$-local spectra cannot hold because of a counterexample constructed by Franke in [Fra96]. In this chapter, we give a brief review of this exotic model and explain where the proof of the $K_{(2)}$-local Rigidity Theorem must fail when replacing 2 by an odd prime p.

6.1 Franke's exotic models

Throughout the rest of this chapter, let p denote an odd prime. Franke proves that the homotopy category of $K_{(p)}$-local spectra is triangulated equivalent to the derived category of $2p-2$-twisted chochain complexes over a certain abelian category $\mathcal{B}$:

Theorem(Franke [Fra96]) There is an equivalence of categories

$$\mathcal{R} : \mathcal{D}^{2p-2}(\mathcal{B}) \longrightarrow \mathrm{Ho}(L_1\mathcal{S})$$

where $\mathcal{D}^{2p-2}(\mathcal{B})$ denotes the derived category of twisted cochain complexes over an abelian category $\mathcal{B}$, and $\mathrm{Ho}(L_1\mathcal{S})$ the homotopy category of $K_{(p)}$-local spectra for an odd prime p.

We are now going to explain the ingredients of this theorem. We begin with certain abelian categories $\mathcal{A}$ and $\mathcal{B}$: The category $\mathcal{B}$ consists of $\mathbb{Z}_{(p)}$-modules together with Adams operations $\psi^k, k \in \mathbb{Z}^*_{(p)}$, satisfying some further conditions. (Details can be found in [Bou85] or [Fra96] 3.1.)

To build the category $\mathcal{A}$ out of the above category, we additionally need the following: Let $T : \mathcal{B} \longrightarrow \mathcal{B}$, denote the following self-equivalence:

For all $M \in \mathcal{B}$, $T(M) = M$ as a $\mathbb{Z}_{(p)}$-module, but on $T(M)$, the Adams operation ψ^k now equals $k^{p-1}\psi^k : M \longrightarrow M$ for all $k \in \mathbb{Z}$.

An object $\mathcal{M} \in \mathcal{A}$ is defined as a collection of modules $\mathcal{M} = (M_n)_{n\in\mathbb{Z}}$, $M_n \in \mathcal{B}$, together with isomorphisms

$$T(M_n) \xrightarrow{\cong} M_{n+2p-2} \quad \text{for all} \quad n \in \mathbb{Z}.$$

The resulting category $\mathcal{A}$ is equivalent to the category of $K_{(p)*}K_{(p)}$-comodules.

Note the following: Let X be a spectrum. Then the $K_{(p)*}K_{(p)}$-comodule $K_{(p)*}(X)$ is an object of $\mathcal{A}$ in the above sense by taking $M_n := (K_{(p)})_n(X)$, and the operations ψ^k being the usual Adams operations.

From now on $\mathcal{B}$ will be viewed as the subcategory of $\mathcal{A}$ consisting of those objects $(M_n)_{n\in\mathbb{Z}}$ such that

$$M_n = \begin{cases} M & : \quad n \equiv 0 \mod 2p-2 \\ 0 & : \quad \text{else} \end{cases}$$

for a $\mathbb{Z}_{(p)}$-module M with Adams operations as before. This describes a so-called split of period $2p-2$ of $\mathcal{A}$: $\mathcal{B} \subset \mathcal{A}$ is a Serre class such that

$$\bigoplus_{0\le i<2p-2} \mathcal{B} \longrightarrow \mathcal{A}$$

$$(B_i)_{0\le i<2p-2} \longmapsto \bigoplus_{0\le i<2p-2} B_i[i]$$

is an equivalence of categories, where $[i]$ denotes the i-fold internal shift in the grading, i.e., $M[i]_n = M_{i-n}$.

Now we describe the source of Franke's equivalence. Let $\mathcal{A}$ for the next paragraphs denote an arbitrary abelian category, N a natural number and $\Theta : \mathcal{A} \longrightarrow \mathcal{A}$ a self-equivalence.

Definition 6.1. The *category* $\mathcal{C}^{(\Theta,N)}(\mathcal{A})$ *of* (Θ, N)*-twisted cochain complexes* with values in $\mathcal{A}$ is defined as follows:

The objects are cochain complexes C^* with $C^i \in \mathcal{A}$ for all i together with an isomorphism of cochain complexes

$$\alpha_C : \Theta(C^*) \longrightarrow C^*[N] = C^{*+N}.$$

The morphisms are those morphisms of cochain complexes $f : C^* \to D^*$ that are compatible with the periodicity isomorphims, i.e., the following diagram commutes:

$$\begin{tikzcd} \Theta(C^*) \arrow[r, "\alpha_C"] \arrow[d, "\Theta(f)"'] & C^*[N] \arrow[d, "f[N]"] \\ \Theta(D^*) \arrow[r, "\alpha_D"] & D^*[N]. \end{tikzcd}$$

Such a cochain complex C^* is called injective if each C^i is injective in $\mathcal{A}$. A morphism in $\mathcal{C}^{(\Theta,N)}(\mathcal{A})$ is called a quasi-isomorphism if it induces an isomorphims in cohomology. C^* is called strictly injective if it is injective, and, for each acyclic complex D^*, the cochain complex $\mathrm{Hom}^*_{\mathcal{C}^{(\Theta,N)}(\mathcal{A})}(D^*, C^*)$ is again acyclic.

Notation. In our particular case, let $\mathcal{A}$ be again the category equivalent to $K_{(p)*}K_{(p)}$-comodules described in the previous paragraphs. The self-equivalence of $\mathcal{A}$ we work with from now on is the twisting of Adams operations T^{p-1} described earlier in this section. We denote the category $\mathcal{C}^{(T,1)}(\mathcal{A})$ by $\mathcal{C}^1(\mathcal{A})$.

Secondly, we are interested in the category $\mathcal{C}^{(T^{(2p-2)},2p-2)}(\mathcal{B})$, where $\mathcal{B}$ denotes again the split of $\mathcal{A}$ introduced earlier in this section. This category of cochain complexes will be denoted by $\mathcal{C}^{2p-2}(\mathcal{B})$.

Proposition 6.2. ([Fra96] 1.3.3, Prop.3) There is a model structure on $\mathcal{C}^1(\mathcal{A})$ resp. $\mathcal{C}^{2p-2}(\mathcal{B})$ such that

- weak equivalences are the quasi-isomorphisms
- cofibrations are the monomorphisms
- fibrations are the componentwise split epimorphisms with strictly injective kernel.

Remark. This model structure exists on arbitrary $\mathcal{C}^{(\Theta,N)}(\mathcal{A})$, given that there are enough injectives in $\mathcal{A}$, see [Fra96] 1.3.3.

Notation. $\mathcal{D}^1(\mathcal{A})$ resp. $\mathcal{D}^{2p-2}(\mathcal{B})$ denotes the derived category of $\mathcal{C}^1(\mathcal{A})$ resp. $\mathcal{C}^{2p-2}(\mathcal{B})$, i.e., the homotopy category of these model categories with respect to the above model structure.

Remark. The categories $\mathcal{C}^1(\mathcal{A})$ and $\mathcal{C}^{2p-2}(\mathcal{B})$ are equivalent categories, see [Roi05] 1.4..

Franke's functor $\mathcal{R} : \mathcal{D}^{2p-2}(\mathcal{B}) \longrightarrow \mathrm{Ho}(L_1\mathcal{S})$ now reconstructs a spectrum from the algebraic data given by C^* for each twisted cochain complex C^* over $\mathcal{B}$. The idea is to first associate a spectrum each to the boundaries of C^* and the quotients of C^* by the boundaries. These spectra X_{β_i} and X_{γ_i} $(1 \leq i \leq 2p-2)$ are put into a diagram

$$\begin{tikzcd}
X_{\beta_1} & \cdots & X_{\beta_{i-1}} & X_{\beta_i} & & X_{\beta_{2p-2}} \\
X_{\gamma_1} \arrow[u] \arrow[urrrrr, dotted] & & X_{\gamma_{i-1}} \arrow[u] \arrow[ul, dotted] & X_{\gamma_i} \arrow[u] \arrow[ul] & \cdots \arrow[ul, dotted] & X_{\gamma_{2p-2}}. \arrow[u]
\end{tikzcd}$$

In the next step, the X_{β_i}'s and X_{γ_i}'s are pasted together by the homotopy colimit of this diagram. So all in all, the result is a spectrum $X = \mathcal{R}(C^*) \in \mathrm{Ho}(L_1\mathcal{S})$ assigned to a twisted cochain complex $C^* \in \mathcal{D}^{2p-2}(\mathcal{B})$.

The condition that p is odd is a special case of the condition that the splitting index of $\mathcal{A}$ (into $2p-2$ shifted copies of $\mathcal{B}$) is bigger than the injective dimension of $\mathcal{A}$, which is 2. This ensures sparseness in certain Adams spectral sequences which the proof of Franke's theorem relies on. For details, see [Fra96] section 2 and [Roi05] section 1-3.

Next, we note that

Proposition 6.3. The categories $\mathcal{D}^{2p-2}(\mathcal{B})$ and $\mathrm{Ho}(L_1\mathcal{S})$ are not Quillen equivalent. In particular, $\mathcal{R}$ is not derived from a Quillen equivalence.

Proof. To prove this, we compare the homotopy types of certain mapping spaces for each category. Let us first collect the necessary definitions. For a pointed simplicial model category $\mathcal{C}$ there is a mapping space functor

$$\mathrm{map}_{\mathcal{C}}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathrm{sSet}_*$$

to the category of pointed simplicial sets satisfying

$$\mathrm{map}_{\mathcal{C}}(X,Y)_0 = \mathrm{Hom}_{\mathcal{C}}(X,Y)$$

for all $X, Y \in \mathcal{C}$ and certain adjointness properties (see e.g. [GJ99], Definition II.2.1). However, $\mathcal{D}^1(\mathcal{A})$ and $\mathcal{D}^{2p-2}(\mathcal{B})$ are not simplicial categories. The next best thing we can achieve is a notion of a mapping space that is well-defined up to homotopy, which will do for our purposes.

To achieve this, we look at the category $\mathcal{C}^{\Delta}$ of cosimplicial objects in $\mathcal{C}$ and view X as constant object in $\mathcal{C}^{\Delta}$. The category $\mathcal{C}^{\Delta}$ of cosimplicial objects in a model category $\mathcal{C}$ can be given a model structure, the so-called Reedy model structure. For details of this, see [Hov99] Section 5.2. We now define a special replacement of X in $\mathcal{C}^{\Delta}$, so-called *frames*. To do this, we first need the following:

Definition 6.4. Via the methods of [Hov99], Remark 5.2.3. and Example 5.2.4., there are functors $\mathbf{l}^\bullet, \mathbf{r}^\bullet : \mathcal{C} \longrightarrow \mathcal{C}^{\Delta}$ with the following properties:
Let $X \in \mathcal{C}$:

- the n^{th} level space of the object $\mathbf{l}^\bullet X$ is the $n+1$-fold coproduct of A
- $\mathbf{l}^\bullet : \mathcal{C} \longrightarrow \mathcal{C}^{\Delta}$ is a left adjoint to the evaluation functor $ev_0 : \mathcal{C}^{\Delta} \longrightarrow \mathcal{C}$ that sends $A^\bullet$ to $A^\bullet[0]$
- the n^{th} level space of the object $\mathbf{r}^\bullet X$ is X itself
- $\mathbf{r}^\bullet : \mathcal{C} \longrightarrow \mathcal{C}^{\Delta}$ is a right adjoint to $ev_0 : \mathcal{C}^{\Delta} \longrightarrow \mathcal{C}$

Remark. One can prove that $\mathbf{r}^\bullet$ is the constant cosimplicial functor. There is a natural transformation $\mathbf{l}^\bullet \longrightarrow \mathbf{r}^\bullet$ that is the identity in degree zero and the fold map in higher degrees.

With these functors, we can now define cosimplicial frames:

Definition 6.5. Let $\mathcal{C}$ be a model category, X an object of $\mathcal{C}$. A *cosimplicial frame* for X is a cosimplicial object $X^\bullet \in \mathcal{C}^{\Delta}$ together with a factorisation of the map $\mathbf{l}^\bullet X \longrightarrow \mathbf{r}^\bullet X$ in $\mathcal{C}^{\Delta}$

$$\mathbf{l}^\bullet X \rightarrowtail X^\bullet \xrightarrow{\sim} \mathbf{r}^\bullet X$$

where the weak equivalence $X^\bullet \xrightarrow{\sim} \mathbf{r}^\bullet X$ in degree zero induces a weak equivalence in $\mathcal{C}$.

For the existence of such framings, see [Hov99], Theorem 5.2.8.
We now use this definition to define mapping spaces:

Definition 6.6. Let X, Y be objects of $\mathcal{C}$, $X^\bullet$ a cosimplicial frame for X and

$$Y \overset{\sim}{\rightarrowtail} Y^{\text{fib}} \twoheadrightarrow *$$

a factorisation of $Y \to *$. Then the *(left) mapping space* for X and Y is defined via

$$\text{map}_{\mathcal{C}}(X, Y) := \mathcal{C}(X^\bullet, Y^{\text{fib}}) \in \text{sSet}_*,$$

where $\mathcal{C}(X^\bullet, Y^{\mathrm{fib}})$ is the simplicial set with

$$\mathcal{C}(X^\bullet, Y^{\mathrm{fib}})_n := \mathrm{Hom}_{\mathcal{C}}(X^\bullet[n], Y^{\mathrm{fib}}).$$

However, it is not clear whether this definition actually deserves to be called a definition since it depends on two choices: firstly, the cosimplicial frame for X and secondly, the fibrant replacement for Y. So, for this definition to make sense we need the following:

Lemma 6.7. Let $X_1^\bullet, X_2^\bullet$ be two cosimplicial frames for cofibrant X in $\mathcal{C}$, and let $Y_1^{\mathrm{fib}}, Y_2^{\mathrm{fib}}$ be two fibrant replacements for Y. Then

$$\mathcal{C}(X_1^\bullet, Y_1^{\mathrm{fib}}) \simeq \mathcal{C}(X_2^\bullet, Y_2^{\mathrm{fib}})$$

in sSet$_*$.

Proof. First, let $X_1^\bullet$ and $X_2^\bullet$ be two cosimplicial frames for X. By definition, the frames $X_1^\bullet$ and $X_2^\bullet$ are linked by a zig-zag of weak equivalences

$$X_1^\bullet \xrightarrow{\sim} \mathbf{r}^\bullet X \xleftarrow{\sim} X_2^\bullet.$$

For fibrant Y, the functor $\mathcal{C}(-, Y)$ preserves weak equivalences ([SS02] Lemma 6.3), so for fibrant Y and $X_1^\bullet$, $X_2^\bullet$ as above, we have

$$\mathcal{C}(X_1^\bullet, Y) \simeq \mathcal{C}(X_2^\bullet, Y).$$

For the second part we quote [Hov99], Corollary 5.4.4, which says that for fibrant X in $\mathcal{C}$, the functor

$$\mathcal{C}(X^\bullet, ...) : \mathcal{C} \longrightarrow \mathrm{sSet}_*$$

preserves fibrations and acyclic fibrations, in particular between fibrant objects. So Ken Brown's lemma applies (see e.g. [Hov99], Lemma 1.1.12), and it follows that $\mathcal{C}(X^\bullet, ...)$ takes weak equivalences between fibrant objects in $\mathcal{C}$ to weak equivalences in sSet$_*$ which proves the claim of our lemma. $\square$

Now we look at the behaviour of mapping spaces under Quillen functors and Quillen equivalences.

Lemma 6.8. Let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be a Quillen equivalence, $X, X' \in \mathcal{C}$ both cofibrant. Then

$$\mathrm{map}_{\mathcal{C}}(X, X') \cong \mathrm{map}_{\mathcal{D}}(LX, LX')$$

in Ho(sSet$_*$).

Proof. First of all, let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be a Quillen adjoint functor pair, $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. Then

$$\operatorname{map}_{\mathcal{D}}(LX, Y) = \mathcal{D}((LX)^\bullet, Y^{\mathrm{fib}})$$

by definition. Since L is a left Quillen functor, $L(X^\bullet) \in \mathcal{D}^\Delta$ is also a cosimplicial frame for LX ([Hov99], Lemma 5.6.1), so

$$\mathcal{D}((LX)^\bullet, Y^{\mathrm{fib}}) \cong \mathcal{D}(L(X^\bullet), Y^{\mathrm{fib}})$$

by Lemma 6.7. By adjointness,

$$\mathrm{Hom}_{\mathcal{D}}(L(X^\bullet)[n], Y^{\mathrm{fib}}) \cong \mathrm{Hom}_{\mathcal{C}}(X^\bullet[n], R(Y^{\mathrm{fib}})),$$

so

$$\mathcal{D}(L(X^\bullet), Y^{\mathrm{fib}}) \cong \mathcal{C}(X^\bullet, R(Y^{\mathrm{fib}})).$$

Since R is a right Quillen functor, $R(Y^{\mathrm{fib}})$ is a fibrant replacement for RY, consequently by Lemma 6.7,

$$\mathcal{C}(X^\bullet, R(Y^{\mathrm{fib}})) \simeq \mathcal{C}(X^\bullet, (RY)^{\mathrm{fib}}) = \operatorname{map}_{\mathcal{C}}(X, RY).$$

Thus, altogether we have

$$\operatorname{map}_{\mathcal{C}}(X, RY) \simeq \operatorname{map}_{\mathcal{D}}(LX, Y). \tag{6.1}$$

Next, let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be a Quillen equivalence and $X' \in \mathcal{C}$ cofibrant. Then

$$LX' \xrightarrow{\sim} (LX')^{\mathrm{fib}}$$

is a weak equivalence in $\mathcal{D}$ with cofibrant source and fibrant target, so by definition of a Quillen equivalence, the adjoint map

$$X' \xrightarrow{\sim} R((LX')^{\mathrm{fib}})$$

is a weak equivalence in $\mathcal{C}$. Since R is a right Quillen functor, $R((LX')^{\mathrm{fib}})$ is fibrant in $\mathcal{C}$. Consequently, $R((LX')^{\mathrm{fib}})$ is a fibrant replacement for X' in $\mathcal{C}$. By Lemma 6.7 and above adjointness result for mapping spaces (6.1), it follows that

$$\operatorname{map}_{\mathcal{C}}(X, X') \simeq \operatorname{map}_{\mathcal{C}}(X, R((LX')^{\mathrm{fib}})) \simeq \operatorname{map}_{\mathcal{D}}(LX, LX')$$

in sSet$_*$ which proves the lemma. □

Back to our special case: We will see that for all $C, D \in \mathcal{C}^{2p-2}(\mathcal{B})$, $\mathrm{map}_{\mathcal{C}^{2p-2}(\mathcal{B})}(C, D)$ is weakly equivalent to a product of Eilenberg-MacLane spaces. However, the mapping space $\mathrm{map}_{L_1\mathcal{S}}(S^0, S^0)$ is not a product of Eilenberg-MacLane spaces, so as a consequence of Lemma 6.8, there is no Quillen equivalence between those two model categories which was the claim of the proposition.

The category $\mathcal{C}^{2p-2}(\mathcal{B})$ is abelian, so for all $C_1, C_2 \in \mathcal{C}^{2p-2}(\mathcal{B})$, the n-simplices of $\mathrm{map}_{\mathcal{C}^{2p-2}(\mathcal{B})}(C_1, C_2)$

$$\mathcal{C}(C_1^\bullet, C_2^{\mathrm{fib}})_n = \mathrm{Hom}(C_1^\bullet[n], C_2)$$

form an abelian group, and the simplicial structure maps are group homomorphisms, so

$$\mathcal{C}(C_1^\bullet, C_2^{\mathrm{fib}}) = \mathrm{map}_{\mathcal{C}^{2p-2}(\mathcal{B})}(C_1, C_2)$$

is not just a simplicial set but a simplicial abelian group. From Proposition III.2.20 of [GJ99], it follows that

$$\mathrm{map}_{\mathcal{C}^{2p-2}(\mathcal{B})}(C_1, C_2) \cong \prod_{n \geq 0} K(\pi_n \, \mathrm{map}_{\mathcal{C}^{2p-2}(\mathcal{B})}(C_1, C_2)_n, n)$$

where $K(G, n)$ denotes the n^{th} Eilenberg-MacLane space for the abelian group G.

However, there are spectra for which the mapping spaces over $L_1\mathcal{S}$ are not products of Eilenberg-MacLane spaces, for example $\mathrm{map}_{L_1\mathcal{S}}(S^0, S^0) \cong QL_1S^0 = \mathrm{colim}_n \, \Omega^n L_1 S^n$. Thus, $\mathcal{C}^{2p-2}(\mathcal{B})$ and $L_1\mathcal{S}$ cannot be Quillen equivalent and $\mathcal{C}^{2p-2}(\mathcal{B})$ provides an exotic model for $L_1\mathcal{S}$.

□

6.2 Universal Property of $K_{(p)}$-local spectra

In this section, let $L_1\mathcal{S}$ denote the model category of spectra with the $K_{(p)}$-local model structure for $p > 2$. How can we check in general if a stable model category $\mathcal{C}$ provides an exotic model for $L_1\mathcal{S}$ or not?

Let us return to the proof of the Universal Property for $K_{(p)}$-local spectra and its first step, the question if the spectrum $\mathrm{Hom}(X, X)$ is $K_{(p)}$-local. This is again equivalent to the mod-p homotopy groups of $\mathrm{Hom}(X, X)$ being v_1-periodic ([Bou79], §4):

Let $M = M(\mathbb{Z}/p)$ denote the mod-p Moore spectrum. For odd primes, the v_1-self map of M of smallest existing degree is not $v_1^{p^2}$ as in the case $p = 2$, but

$$v_1 : \Sigma^{2p-2} M \longrightarrow M$$

itself. So the spectrum $\mathrm{Hom}(X, X)$ is $K_{(p)}$-local if and only if the precomposition morphism

$$(v_1)^* : [M, \mathrm{Hom}(X, X)]_*^{\mathcal{S}} \longrightarrow [M, \mathrm{Hom}(X, X)]_*^{\mathcal{S}}$$

is an isomorphism. By adjunction, this is equivalent to

$$(X \wedge v_1)^* : [X \wedge M, X]_*^{\mathcal{C}} \longrightarrow [X \wedge M, X]_*^{\mathcal{C}}$$

being an isomorphism.

The morphism $X \wedge v_1$ lies in

$$[X \wedge M, X \wedge M]_{2p-2}^{\mathcal{C}} \cong [\Phi(L_1 M), \Phi(L_1 M)]_{2p-2}^{\mathcal{C}} \cong [M, M]_{2p-2}^{L_1\mathcal{S}} = \mathbb{Z}/p\{v_1\}.$$

So $(X \wedge v_1)^*$ is either an isomorphism or the zero map.

The element $v_1 \in [M, M]_{2p-2}^{L_1\mathcal{S}}$ factors as

$$\alpha_1 = \mathrm{pinch} \circ v_1 \circ \mathrm{incl}, \qquad \alpha_1 \in \pi_{2p-3} L_1 S^0 = \mathbb{Z}/p\{\alpha_1\}$$

which can be computed by similar methods to those in Chapter 5. It follows that $X \wedge v_1 = 0$ if and only if $X \wedge \alpha_1 = 0$. Let us investigate those two cases seperately.

$$\boxed{X \wedge \alpha_1 = 0}$$

For this case, we look at the action of $\pi_* S^0$ on the morphism sets of a stable homotopy category $\mathcal{C}$. Let us look again at the technique of framings used in the last section. In addition to the mapping space functor

$$\mathrm{map}_{\mathcal{C}}(-,-) : \mathrm{Ho}(\mathcal{C}^{op}) \times \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathrm{sSet}_*)$$

introduced in Definition 6.6, we define a functor

$$- \otimes - : \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathrm{sSet}_*) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

such that

$$A \otimes - : \mathrm{Ho}(\mathrm{sSet}_*) \rightleftarrows \mathrm{Ho}(\mathcal{C}) : \mathrm{map}_{\mathcal{C}}(A, -)$$

is an adjoint functor pair for $A \in \mathcal{C}$:

Let $Y \in \mathcal{C}, K \in \mathrm{sSet}_*$ and $Y^\bullet \in \mathcal{C}^\Delta$ be a cosimplicial frame for Y. Further, let ΔK denote the category of simplices in K ([Hov99], 3.1). Now $Y \otimes K$ is defined to be the image of a framing $Y^\bullet$ under the functor

$$\mathcal{C}^\Delta \longrightarrow \mathcal{C}^{\Delta K} \xrightarrow{\mathrm{colim}} \mathcal{C},$$

(cf [Hov99] 3.1.5). (Again, one has to check that the homotopy type of $Y \otimes K$ does not depend on the choice of the frame for Y.) This functor $- \otimes -$ now makes $\mathrm{Ho}(\mathcal{C})$ a module over the symmetric monoidal category Ho(sSet*) ([Hov99], 4.1.6).

For $Y, Z \in \mathcal{C}$, one can now use this functor to define an action

$$\pi_n S^0 \otimes [Y,Z]_k^{\mathcal{C}} \xrightarrow{\mu} [Y,Z]_{n+k}^{\mathcal{C}} :$$

Let $f \in [Y,Z]_k^{\mathcal{C}} = \mathrm{Hom}^0_{\mathrm{Ho}(\mathcal{C})}(\Sigma^k Y, Z) = \mathrm{Hom}^0_{\mathrm{Ho}(\mathcal{C})}(Y \otimes \mathbb{S}^k, Z)$. Note that the suspension functor Σ defined in Chapter 2 is isomorphic to the functor $- \otimes \mathbb{S}^1$, with $\mathbb{S}^1$ denoting the simplicial 1-sphere.

For $\alpha \in \pi_n S^0$ we choose a representative $a : \mathbb{S}^{n+l} \longrightarrow \mathbb{S}^l$ in Ho(sSet*). The element $f \otimes a$ now lies in $[Y \otimes \mathbb{S}^k \otimes \mathbb{S}^{n+l}, Z \otimes \mathbb{S}^l]_0^{\mathcal{C}}$ which is isomorphic to $[Y,Z]_{n+k}^{\mathcal{C}}$ since $\mathcal{C}$ is stable.

Definition 6.9. We now define $\mu(\alpha, f) := f \cdot \alpha$ to be the unique element such that $(f \cdot \alpha) \otimes \mathrm{id}_{\mathbb{S}^l} = f \otimes \alpha$ in $[Y \otimes \mathbb{S}^{n+k+l}, Z \otimes \mathbb{S}^l]_0^{\mathcal{C}}$. (For details, see [SS02], Construction 2.4.)

Definition 6.10. Let $\mathcal{C}$ and $\mathcal{D}$ be stable model categories. A functor

$$\Lambda : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{D})$$

is called $\pi_* S^0$-*exact* (cf [SS02], Definition 2.2) if Λ is exact and $\pi_* S^0$-linear, i.e., compatible with the $\pi_* S^0$-action.

We are now going to use the following important example of this: If $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a left Quillen functor, then its left derived functor $LF : \mathrm{Ho}(\mathcal{C}) \longrightarrow \mathrm{Ho}(\mathcal{D})$ is $\pi_* S^0$-exact ([SS02], Lemma 6.1).

In particular, for the functor $X \wedge - : \mathcal{S} \longrightarrow \mathcal{C}$ from the Universal Property of Spectra, the diagram

$$\begin{tikzcd}
\pi_* S^0 \otimes [S^0, S^0]_*^{\mathcal{S}} \arrow[r, "\mu"] \arrow[d, "\mathrm{id} \otimes (X \wedge -)"'] & {[S^0, S^0]_*^{\mathcal{S}}} \arrow[d, "X \wedge -"] \\
\pi_* S^0 \otimes [X, X]_*^{\mathcal{C}} \arrow[r, "\mu"] & {[X, X]_*^{\mathcal{C}}}
\end{tikzcd}$$

commutes as $X \wedge - : \mathcal{S} \longrightarrow \mathcal{C}$ is a left Quillen functor. Consequently, for $\alpha \in \pi_* S^0$ we have

$$
\begin{aligned}
X \wedge \alpha &= X \wedge (\mathrm{Id}_{S^0} \cdot \alpha) \\
&= (X \wedge -) \circ \mu(\alpha, \mathrm{Id}_{S^0}) \\
&= \mu \circ (\mathrm{id} \otimes (X \wedge -))(\alpha, \mathrm{Id}_{S^0}) \\
&= \mu(\alpha, X \wedge \mathrm{Id}_{S^0}) \\
&= \mu(\alpha, \mathrm{Id}_X) \\
&= \alpha \cdot \mathrm{Id}_X \, . \qquad (6.2)
\end{aligned}
$$

Let us return to the situation that $\mathcal{C}$ is a stable model category with an equivalence of triangulated categories

$$\Phi : \mathrm{Ho}(L_1\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C}).$$

Also, we assume again that $X \wedge \alpha_1 = 0$ for $X = \Phi(L_1 S^0)$, $\alpha_1 \in \pi_{2p-3} L_1 S^0 = \pi_{2p-3} S^0$. With the help of the previously introduced $\pi_* S^0$-operation we can now show that $L_1\mathcal{S}$ and $\mathcal{C}$ are not Quillen equivalent:

Let us assume that there is a functor

$$F : L_1\mathcal{S} \longrightarrow \mathcal{C}$$

that is part of a Quillen equivalence. (Without loss of generality, let F be a left Quillen functor.) So then its derived functor LF would be a $\pi_* S^0$-exact equivalence, in particular $[S^0, S^0]_*^{L_1\mathcal{S}}$ and $[X, X]_*^{\mathcal{C}}$ would be isomorphic as $\pi_* S^0$-modules. However, this cannot be the case as

$$\alpha_1 \cdot \mathrm{Id}_X = X \wedge \alpha_1 = 0 \qquad \text{and} \qquad \alpha_1 \cdot \mathrm{Id}_{S^0} = \alpha_1 \neq 0.$$

So we have shown

Proposition 6.11. If $X \wedge \alpha_1 = 0$, then $L_1\mathcal{S}$ and $\mathcal{C}$ are not Quillen equivalent.

Next, we will see that the condition $X \wedge \alpha_1 \neq 0$ is both necessary and sufficient for Quillen equivalence:

$\boxed{X \wedge \alpha_1 \neq 0}$

We have seen at the beginning of this section that $X \wedge \alpha_1 \neq 0$ implies that the mod-p homotopy groups of the spectrum $\mathrm{Hom}(X, X)$ are v_1-periodic. Therefore, $\mathrm{Hom}(X, X)$ is $K_{(p)}$-local. With the methods of Proposition 3.2 it now follows that

$$X \wedge - : L_1\mathcal{S} \rightleftarrows \mathcal{C} : \mathrm{Hom}(X, -)$$

is a Quillen functor pair for $X = \Phi(L_1S^0)$.

Analogously to Chapter 4, we now show that $\mathrm{Hom}(X, X)$ is equivalent to the $K_{(p)}$-local sphere by showing that the map

$$\lambda : L_1S^0 \longrightarrow \mathrm{Hom}(X, X)$$

is a π_*-isomorphism.

Again, this is the case if and only if

$$\Psi : [S^0, S^0]_n^{L_1\mathcal{S}} \xrightarrow{X \wedge^L -} [X, X]_n^{\mathcal{C}} \xrightarrow{\Phi^{-1}} [S^0, S^0]_n^{L_1\mathcal{S}}$$

is an isomorphism for all $n \in \mathbb{Z}$. For $n = -1, ..., 2p-1$ this follows easily as the only nontrivial homotopy groups of L_1S^0 in this range are $\pi_0 L_1S^0 = \mathbb{Z}_{(p)}\{\iota\}$ and $\pi_{2p-3}L_1S^0 = \mathbb{Z}/p\{\alpha_1\}$ ([Rav84] 8.10.(b)): By our assumption $X \wedge \alpha_1 \neq 0$ we can conclude that $\Psi(\alpha_1)$ is a nonzero multiple of α_1 again. With the 5-lemma it follows that

$$\Psi : [M, S^0]_n^{L_1\mathcal{S}} \longrightarrow [M, S^0]_n^{L_1\mathcal{S}}$$

is an isomorphism for $n = 0, ..., 2p-2$.

We now use that $v_1 : \Sigma^{2p-2}M \longrightarrow M$ is an isomorphism in $\mathrm{Ho}(L_1\mathcal{S})$, so by proceeding with exactly the same method as in Proposition 4.1 we conclude by induction that

$$\Psi : [S^0, S^0]_n^{L_1\mathcal{S}} \longrightarrow [S^0, S^0]_n^{L_1\mathcal{S}}$$

is an isomorphism for all $n \geq 0$, and thus, $\lambda : L_1S^0 \longrightarrow \mathrm{Hom}(X, X)$ induces a π_*-isomorphism between the connective covers of L_1S^0 and $\mathrm{Hom}(X, X)$. Since Lemma 4.2 also holds for odd primes, we see that λ is a π_*-isomorphism indeed, and so, analogously to Theorem 4.5,

$$X \wedge - : L_1\mathcal{S} \rightleftarrows \mathcal{C} : \mathrm{Hom}(X, -)$$

is a Quillen equivalence for odd primes.

We summarize this section in the following theorem:

Theorem 6.12. Let $\Phi : \mathrm{Ho}(L_1\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$ be an equivalence of triangulated categories, where $L_1\mathcal{S}$ denotes the category of spectra with the $K_{(p)}$-local model structure for p odd. Then $L_1\mathcal{S}$ and $\mathcal{C}$ are Quillen equivalent if and only if

$$X \wedge \alpha_1 \neq 0, \qquad \text{for} \quad X = \Phi(L_1S^0), \quad \alpha_1 \in \pi_{2p-3}S^0 = \mathbb{Z}/p\{\alpha_1\}.$$

However, while this Theorem might say if a model $\mathcal{C}$ for $L_1\mathcal{S}$ is exotic or not, it does not answer the question whether two exotic models are Quillen equivalent. In particular, it would be interesting to find out if two algebraic models (i.e., a model that is also an abelian category, such as Franke's example $\mathcal{C} = \mathcal{C}^{2p-2}(\mathcal{B})$) are automatically Quillen equivalent or not.

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