Arithmetic structures on noncommutative tori with real multiplication

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Jorge Plazas

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Erster Referent: Prof. Dr. Matilde Marcolli
Zweiter Referent: Prof. Dr. Daniel Huybrechts

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For Paula and Mateo

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Contents

Introduction

Noncommutative tori are standard prototypes of noncommutative spaces. Since the early stages of noncommutative geometry these spaces have been central examples arising naturally in various context. In [**16**] Manin proposed the use of noncommutative tori as a geometric framework for the study of abelian class field theory of real quadratic fields. This is the so called "real multiplication program". The main idea is that noncommutative tori may play a role in the study of real quadratic fields analogous to the role played by elliptic curves in the study of imaginary quadratic fields. The relation between the endomorphism rings of noncommutative tori and orders in real quadratic fields is a good evidence supporting this point of view. If a noncommutative torus admits nontrivial Morita autoequivalences then the ring of such autoequivalences is an order in a real quadratic field. A noncommutative torus having this property is called a real multiplication noncommutative torus.

The fact that noncommutative geometry may be relevant for addressing questions in number theory is also supported by various results and relations that have emerged in the last years. In particular explicit class field theory of $\mathbb{Q}$ (Kronecker-Weber theorem) and explicit class field theory of imaginary quadratic fields (complex multiplication) can both be recovered from the dynamics of certain quantum statistical mechanical systems ([**2, 7, 8**]). The existence of quantum statistical mechanical systems with rich arithmetical properties opens a new approach to the study of explicit class field theory using the tools of quantum statistical mechanics. The first case for which there is not yet a complete solution to the explicit class field theory problem is the case of real quadratic fields, $K = \mathbb{Q}(\sqrt{D})$, where $D \in \mathbb{N}^+$ is a square free positive integer.

Noncommutative tori are, a priori, analytical objects. In order to achieve arithmetical applications it is important to find appropriate algebraic structures underlying these spaces. Rings admitting models algebraic over the corresponding base fields have proved to be essential for the analysis of quantum statistical mechanical systems of arithmetic nature.

In [**26**] Polishchuk defined homogeneous coordinate rings for real multiplication noncommutative tori endowed with a complex structure. These rings seem to be good candidates for the applications described above. Our aim is to understand in which sense these rings provide an arithmetic structure on noncommutative tori. Starting from the explicit formulas defining these rings we study their rationality properties and their dependence on the complex structure. Some examples related to particular quadratic fields are treated.

Homogeneous coordinate rings for real multiplication noncommutative tori can be viewed as a family of algebras varying with the parameter defining the complex structure on the noncommutative torus. We give a presentation of these rings in terms of modular forms and use this modularity to define rings which do not depend on the choice of a complex structure. This is done by an averaging process over (limiting) modular symbols.

The fact that modular forms play an essential role in our analysis points to deep relations with the quantum thermodynamical system introduced by Connes and Marcolli in [**6**]. This quantum mechanical system recovers the class field theory of the modular field and provides a two dimensional analog of the dynamical system corresponding to the Kronecker-Weber theorem.

In the case of real quadratic fields explicit class field theory is conjecturally given in terms of special values of L-functions, this is the content of Stark's famous conjectures [**39**]. In order to apply our results on noncommutative tori in this direction using the tools of quantum statistical mechanics we still need to find C^*-completions for the homogeneous coordinate rings of real multiplication noncommutative tori. As a preliminary step in this direction we describe the geometric data associated to the homogeneous coordinate rings of noncommutative tori. We expect that at a future stage the use of the techniques developed in [**5**] will make it possible to obtain suitable C^*-completions.

In a different but related perspective arithmetic structures on noncommutative tori have been recently studied by Vlasenko in [**40**] where the theory of rings of quantum theta functions is developed.

CHAPTER 1

Homogeneous coordinate rings on noncommutative tori with real multiplication

In this chapter we introduce the main notions and notations used in the present work. In the first sections we introduce noncommutative tori with real multiplication and their homogeneous coordinate rings as defined by Polishchuk in [**26**]. In Section 5 we use the explicit formulas involved in the definition of these rings in order to obtain a presentation in terms of generators and relations, this section is the core of the chapter. We end with some particular examples in order to illustrate the behavior of the rings corresponding to different tori.

1. Noncommutative tori

On what follows we will denote by $\mathbb{T}$ the two dimensional torus $S^1 \times S^1$. As a topological space $\mathbb{T}$ it is characterized by its algebra of continuous functions $C(\mathbb{T})$. This algebra is a unital commutative C^*-algebra. $C(\mathbb{T})$ can be realized as the universal C^*-algebra generated by two commuting unitaries U and V. Any element of $C(\mathbb{T})$ admits a Fourier expansion in terms of powers of these unitaries and smooth functions are characterized as those functions whose coefficients in the corresponding Fourier expansion decay rapidly at infinity.

Noncommutative tori are defined by their function algebras which are noncommutative deformations of $C(\mathbb{T})$ and $C^\infty(\mathbb{T})$.

Definition 1.1. Given $\theta \in \mathbb{R}$ we define $A_\theta = C(\mathbb{T}_\theta)$, the *algebra of continuous functions on the noncommutative torus* $\mathbb{T}_\theta$, as the universal C^*-algebra generated by two unitaries U and V subject to the relations:

$$UV = e^{2\pi i\theta} VU. \tag{1}$$

Definition 1.2. Given $\theta \in \mathbb{R}$ we define $\mathcal{A}_\theta$, the *algebra of smooth functions on the noncommutative torus* $\mathbb{T}_\theta$, as the algebra of formal power series in two unitaries U and V with rapidly decreasing coefficients and multiplication given by the relation $UV = e^{2\pi i\theta} VU$:

$$\begin{aligned} \mathcal{A}_\theta &= C^\infty(\mathbb{T}_\theta) \\ &= \{a = \sum_{n,m\in\mathbb{Z}} a_{n,m} U^n V^m \mid \{a_{n,m}\} \in \mathcal{S}(\mathbb{Z}^2)\} \end{aligned}$$

The compact group $\mathbb{T}$ acts on the algebras A_θ and $\mathcal{A}_\theta$. Much of the structure of these algebras is determined by this action. The action of $\mathbb{T}$ induces an action of its Lie algebra $L = \mathbb{R}^2$ given by the derivations:

$$\delta_1(U) = 2\pi\imath U; \qquad \delta_1(U) = 0 \tag{2}$$

$$\delta_2(U) = 0; \qquad \delta_2(V) = 2\pi\imath V \tag{3}$$

The algebra $\mathcal{A}_\theta$ is a pre-C^*-algebra whose C^*-completion is isomorphic to A_θ. The corresponding Frechet structure is determined by the derivations δ_1 and δ_2. Likewise one can obtain $\mathcal{A}_\theta$ as the algebra of smooth elements of A_θ determined by these derivations. The relation between these algebras parallels the classical situation which corresponds to the value $\theta = 0$ for which one recovers $C(\mathbb{T})$ and $C^\infty(\mathbb{T})$. We refer the reader to the seminal paper [**3**] and the survey [**31**] for the main results about the algebras $A_\theta = C(\mathbb{T}_\theta)$ and $\mathcal{A}_\theta = C^\infty(\mathbb{T}_\theta)$.

On what follows we will restrict to the case were θ is an irrational number. In this case the algebra A_θ is simple and admits a unique normalized trace χ invariant under the action of $\mathbb{T}$. In the algebra $\mathcal{A}_\theta$ this trace is given by

$$\chi(\sum a_{n,m} U^n V^m) = a_{0,0}. \tag{4}$$

2. Morita equivalences and real multiplication

By the Serre-Swan theorem the theory of vector bundles over $\mathbb{T}$ is equivalent to to the theory of finite type projective modules over the algebra $C(\mathbb{T})$. To each complex vector bundle over $\mathbb{T}$ one associates the $C(\mathbb{T})$-module of its global sections. Smooth bundles correspond to finite type projective modules over $C^\infty(\mathbb{T})$ and every vector bundle over $\mathbb{T}$ is equivalent to a smooth one.

We consider projective finite type right $\mathcal{A}_\theta$-modules as vector bundles over $\mathbb{T}_\theta$. If $\tilde{E}$ is a projective finite type right A_θ-module then there exists a projective finite type right $\mathcal{A}_\theta$-module E such that one has an isomorphism of right A_θ-modules:

$$\tilde{E} \simeq E \otimes_{\mathcal{A}_\theta} A_\theta.$$

Therefore, as in the commutative case, the categories of smooth and continuous vector bundles over $\mathbb{T}_\theta$ are equivalent (c.f. [**3**]). On what follows we will restrict to $\mathcal{A}_\theta$-modules.

The trace χ defined in (4) can be extended to a trace Tr_χ on the matrix algebra $M_n(\mathcal{A}_\theta) = End(\mathcal{A}_\theta^n)$. A right $\mathcal{A}_\theta$-module E is projective of finite type if and only if there exists an idempotent $e = e^2 = e^*$ in $M_n(\mathcal{A}_\theta)$ such that $E \simeq e\mathcal{A}_\theta^n$, thus we can define the rank of E by

$$rk(E) = Tr_\chi(e) \tag{5}$$

Unless otherwise stated a right (resp. left) $\mathcal{A}_\theta$-module will always mean a right (resp. left) projective finite type $\mathcal{A}_\theta$-module.

Let $\theta \in \mathbb{R}$ be irrational. Following [**3**] we define, for any pair $c, d \in \mathbb{Z}$, $c > 0$, a right $\mathcal{A}_\theta$-module $E_{d,c}(\theta)$ given by the following action of $\mathcal{A}_\theta$ on the Schwartz space $\mathcal{S}(\mathbb{R} \times \mathbb{Z}/c\mathbb{Z}) = \mathcal{S}(\mathbb{R})^c$:

$$(fU)(x,\alpha) = f(x - \frac{c\theta + d}{c}, \alpha - 1) \tag{6}$$

$$(fV)(x,\alpha) = \exp(2\pi i(x - \frac{\alpha d}{c}))f(x,\alpha) \tag{7}$$

The rank of $E_{d,c}(\theta)$ is $|c\theta + d|$ and if E is any right $\mathcal{A}_\theta$-module with $rk(E) = |c\theta + d|$ then $E \simeq E_{d,c}(\theta)$. The K_0 group of $\mathcal{A}_\theta$, $K_0(\mathcal{A}_\theta)$, is by definition the enveloping group of the abelian semigroup given by isomorphism classes of right $\mathcal{A}_\theta$-modules together with direct sum. The rank function rk extends to a injective morphism

$$rk : K_0(\mathcal{A}_\theta) \to \mathbb{R} \tag{8}$$

whose image is $\mathbb{Z} \oplus \theta\mathbb{Z}$. Therefore one gets a ordered structure on $K_0(\mathcal{A}_\theta)$ given by the isomorphism

$$K_0(\mathcal{A}_\theta) \simeq \mathbb{Z} \oplus \theta\mathbb{Z} \subset \mathbb{R} \tag{9}$$

The fact that rk is injective is the content of the cancellation theorem due to Rieffel (c.f. [**32**]). From this theorem it follows that right $\mathcal{A}_\theta$-modules are classified up to isomorphism by their rank and that any finite type projective right $\mathcal{A}_\theta$-module is either free or isomorphic to a right module of the form $E_{d,c}(\theta)$.

If c and d are relatively prime we say that $E_{d,c}(\theta)$ is a *basic* $\mathcal{A}_\theta$-module. Being this the case the pair d, c can be completed to a matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \tag{10}$$

We write $E_g(\theta)$ for the module $E_{d,c}(\theta)$. By definition the degree of $E_g(\theta)$ is taken to be c. We also define the degree of a matrix $g \in SL_2(\mathbb{Z})$, given as above, by $deg(g) = c$.

Let $SL_2(\mathbb{Z})$ act on $\mathbb{R}$ by fractional linear transformations. Let $g \in SL_2(\mathbb{Z})$ be as above and denote by U' and V' two generating unitaries of the algebra $\mathcal{A}_{g\theta}$. We can define a left action of the algebra $\mathcal{A}_{g\theta}$ on E_g by:

$$(U'f)(x,\alpha) = f\left(x - \frac{1}{c}, \alpha - a\right) \tag{11}$$

$$(V'f)(x,\alpha) = \exp(2\pi i(\frac{x}{c\theta + d} - \frac{\alpha}{c}))f(x,\alpha) \tag{12}$$

This action gives an identification:

$$End_{\mathcal{A}_\theta}(E_g(\theta)) \simeq \mathcal{A}_{g\theta}. \tag{13}$$

The tensor product of basic modules is again a basic module. More precisely, given $g_1, g_2 \in SL_2(\mathbb{Z})$, there is a well defined pairing of right $\mathcal{A}_\theta$-modules:

$$t_{g_1,g_2} : E_{g_1}(g_2\theta) \otimes_{\mathbb{C}} E_{g_2}(\theta) \to E_{g_1g_2}(\theta) \tag{14}$$

This map gives rise to an isomorphism of $\mathcal{A}_{g_1g_2\theta} - \mathcal{A}_\theta$ bimodules:

$$E_{g_1}(g_2\theta) \otimes_{\mathcal{A}_{g_2\theta}} E_{g_2}(\theta) \to E_{g_1g_2}(\theta). \tag{15}$$

In particular, if $g\theta = \theta$ one has an isomorphism

$$\underbrace{E_g(\theta) \otimes_{\mathcal{A}_\theta} \cdots \otimes_{\mathcal{A}_\theta} E_g(\theta)}_{n} \simeq E_{g^n}(\theta). \tag{16}$$

We say that two noncommutative tori $\mathbb{T}_{\theta'}$ and $\mathbb{T}_\theta$ are Morita equivalent if there exist a $\mathcal{A}_{\theta'}$-$\mathcal{A}_\theta$-bimodule which is projective and of finite type both as a left $\mathcal{A}_{\theta'}$-module and as a right $\mathcal{A}_\theta$-module. We will consider the category whose objects are noncommutative tori and whose morphisms are given by isomorphism classes of finite type projective bimodules over the corresponding algebras of smooth functions. Composition is provided by tensor product over the corresponding algebra. A isomorphism in this category is called a *Morita equivalence*. From the discussion above we see that given a real number θ and a matrix $g \in SL_2(\mathbb{Z})$ the noncommutative tori $\mathbb{T}_{g\theta}$ and $\mathbb{T}_\theta$ are Morita equivalent. The inverse of the morphism represented by the $\mathcal{A}_{g\theta}$-$\mathcal{A}_\theta$-bimodule $E_g(\theta)$ is the morphism represented by the $\mathcal{A}_\theta$-$\mathcal{A}_{g\theta}$-bimodule$E_{g^{-1}}(g\theta)$. By a result of Rieffel these are the only possible Morita equivalences in the category of noncommutative tori (c.f. [**30**]). More precisely, two noncommutative tori $\mathbb{T}'_\theta$ and $\mathbb{T}_\theta$ are Morita equivalent if and only if there exist $g \in SL_2(\mathbb{Z})$ such that $\theta' = g\theta$.

If $g\theta = \theta$ then $E_g(\theta)$ has the structure of $\mathcal{A}_\theta$-bimodule. An irrational number $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is a fixed point of a fractional linear transformation $g \in SL_2(\mathbb{Z})$ if and only if it generates a quadratic extension of $\mathbb{Q}$.

Definition 2.1. The noncommutative torus $\mathbb{T}_\theta$ with algebra of smooth functions $\mathcal{A}_\theta$ is a *real multiplication noncommutative torus* if the parameter θ generates a quadratic extension of $\mathbb{Q}$.

Thus $\mathbb{T}_\theta$ is a real multiplication noncommutative torus if and only if it has nontrivial Morita autoequivalences.

In [**16**] Manin proposed the use of noncommutative tori as a geometric framework under which to attack the explicit class field theory problem for real quadratic extensions of $\mathbb{Q}$. The explicit class field theory problem ask for explicit generators of the maximal abelian extension of a given field and the corresponding Galois action of the abelianization of the absolute Galois group on these generators. The only number fields for which a complete solution of this problem is known are the imaginary quadratic extensions of $\mathbb{Q}$ and $\mathbb{Q}$ itself. Elliptic curves whose endomorphism ring is nontrivial correspond to lattices generated by imaginary quadratic irrationalities and play a central role in the solution of the explicit class field theory problem for the corresponding imaginary quadratic extensions. It is believed that noncommutative tori with real multiplication may play analogous role in the study of real quadratic extensions of $\mathbb{Q}$. In order to achieve arithmetical applications it is important to realize algebraic structures underlying noncommutative tori. This is our main motivation for the study of the homogeneous coordinate rings described below.

3. Complex structures on tori and holomorphic connections

A complex structure on the noncommutative torus $\mathbb{T}_\theta$ is determined through the the derivations δ_1 and δ_2 by choosing a complex structure on the Lie algebra $L = \mathbb{R}^2$ of $\mathbb{T}$. For this we make a decomposition of the complexification of L into two complex conjugate subspaces. This can be done by choosing a complex parameter τ with nonzero imaginary part and taking $\{1, \tau\}$ as a basis for the holomorphic part of this decomposition. The resulting derivation $\delta_\tau = \tau\delta_1 + \delta_2$ is a complex structure $\mathbb{T}_\theta$. Explicitly we have:

$$\delta_\tau : \sum_{n,m\in\mathbb{Z}} a_{n,m} U^n V^m \mapsto (2\pi\imath) \sum_{n,m\in\mathbb{Z}} (n\tau + m) a_{n,m} U^n V^m \tag{17}$$

This derivation should be viewed as an analog of the operator $\bar{\partial}$ on a complex elliptic curve. We will denote by $\mathbb{T}_{\theta,\tau}$ the noncommutative torus $\mathbb{T}_\theta$ equipped with this complex structure. In what follows we will assume that $Im(\tau) < 0$. We will also freely refer to τ as the complex structure on $\mathbb{T}_{\theta,\tau}$.

Complex structures on noncommutative tori were introduced by Connes in relation with the Yang Mills equation and positivity in Hochschild cohomology for noncommutative tori (c.f. [**4**]). The study of the structure of the space of connections associated to the above derivations was carried out in [**9**]. An approach through noncommutative analogs of theta functions was developed in [**33, 10**] were these are viewed as holomorphic sections on noncommutative tori. The resulting categories were studied throughly in [**29**]and [**27**].

A holomorphic structure on a right $\mathcal{A}_\theta$-module E is given by an operator $\bar{\nabla} : E \to E$ which is compatible with the complex structure δ_τ in the sense that it satisfies the following Leibniz rule:

$$\bar{\nabla}(ea) = \bar{\nabla}(e)a + e\delta_\tau(a), \quad e \in E, a \in \mathcal{A}_\theta \tag{18}$$

Given a holomorphic structure $\bar{\nabla}$ on a right $\mathcal{A}_\theta$-module E the corresponding set of holomorphic sections is the space

$$H^0(\mathbb{T}_{\theta,\tau}, E_{\bar{\nabla}}) := Ker(\bar{\nabla}) \tag{19}$$

On every basic module $E_{d,c}$ one can define a family of holomorphic structures $\{\bar{\nabla}_z\}$ depending on a complex parameter $z \in \mathbb{C}$:

$$\bar{\nabla}_z(f) = \frac{\partial f}{\partial x} + 2\pi i \left(\frac{d\tau}{c\theta + d} x + z \right) f. \tag{20}$$

By definition a *standard holomorphic vector bundle on* $\mathbb{T}_{\theta,\tau}$ is given by a basic module $E_{d,c} = E_g$ together with one of the holomorphic structures $\bar{\nabla}_z$.

The spaces of holomorphic sections of a standard holomorphic vector bundles on $\mathbb{T}_{\theta,\tau}$ are finite dimensional (c.f. [**29**], Section 2). If $c\theta + d > 0$ then $\dim H^0(E_g, \bar{\nabla}_0) = c$. On what follows we will consider the spaces of holomorphic sections corresponding to $\bar{\nabla}_0$:

$$\mathcal{H}_g := H^0(\mathbb{T}_{\theta,\tau}, E_{g,\bar{\nabla}_0}). \tag{21}$$

A basis of $\mathcal{H}_g$ is given by the Schwartz functions:

$$\varphi_\alpha(x,\beta) = \exp(-\frac{c\tau}{c\theta+d}\frac{x^2}{2})\delta_\alpha^\beta \qquad \alpha = 1, ..., c. \tag{22}$$

The tensor product of holomorphic sections is again holomorphic. Using the above basis the product can be written in terms of the corresponding structure constants.

Theorem 3.1. ([**29**] Section 2) *Suppose g_1 and g_2 have positive degree. Then $g_1 g_2$ has positive degree and t_{g_1,g_2} induces a well defined linear map*

$$t_{g_1,g_2} : \mathcal{H}_{g_1}(g_2\theta) \otimes_{\mathbb{C}} \mathcal{H}_{g_2}(\theta) \to \mathcal{H}_{g_1 g_2}(\theta). \tag{23}$$

Let g_1, g_2 and $g_1 g_2$ be given by

$$g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \quad g_1 g_2 = \begin{pmatrix} a_{12} & b_{12} \\ c_{12} & d_{12} \end{pmatrix}$$

and let $\{\varphi_\alpha\}$, $\{\varphi'_\beta\}$ and $\{\psi_\gamma\}$ be respectively the basis of $\mathcal{H}_{g_1}(g_2\theta)$, $\mathcal{H}_{g_2}(\theta)$ and $\mathcal{H}_{g_1 g_2}(\theta)$ as given in (22). Then

$$t_{g_1,g_2} : \varphi_\alpha \otimes \varphi'_\beta \mapsto C^\gamma_{\alpha,\beta}\psi_\gamma \tag{24}$$

Where for $\alpha = 1, ..., c_1$, $\beta = 1, ..., c_2$ and $\gamma = 1, ..., c_{12}$ we have:

$$C^\gamma_{\alpha,\beta} = \sum_{m \in I_{g_1,g_2}(\alpha,\beta,\gamma)} \exp[\pi\imath\frac{-\tau m^2}{2c_1c_2c_{12}}] \tag{25}$$

with

$$\begin{aligned} I_{g_1,g_2}(\alpha,\beta,\gamma) = \{n \in \mathbb{Z} \quad | \quad & n \equiv -c_1\gamma + c_{12}\alpha \mod c_{12}c_1, \\ & n \equiv c_2 d_{12}\gamma - c_{12}d_2\beta \mod c_{12}c_2\} \end{aligned}$$

Notation 3.2. *Throughout we use the convention of summing over repeated indexes.*

4. Homogeneous coordinate rings

Given a projective scheme Y over a field k together with an ample line bundle $\mathcal{L}$ on Y one can construct the homogeneous coordinate ring

$$B = \bigoplus_{n \geq 0} H^0(Y, \mathcal{L}^{\otimes n}).$$

This ring plays a prominent role in the study of the geometry of Y (c.f.[**34**]).

In [**26**] Polishchuk proposed an analogous definition of the homogeneous coordinate ring of a real multiplication noncommutative torus $\mathbb{T}_\theta$ in terms of holomorphic sections of tensor powers of a standard holomorphic vector bundle on $\mathbb{T}_{\theta,\tau}$. As mentioned above the real multiplication condition is fundamental in order to be able to perform the tensor power operation.

Assume that $\theta \in \mathbb{R}$ is a quadratic irrationality. So there exist some $g \in SL_2(\mathbb{Z})$ with $g\theta = \theta$ and $\mathbb{T}_\theta$ has real multiplication. Fix a complex structure τ on $\mathbb{T}_\theta$. In the case $E = E_g(\theta)$ we can extend a holomorphic structure on E_g to a holomorphic structure on the tensor powers $E_g^{\otimes n}$. Following [**26**] we define a *homogeneous coordinate ring for* $\mathbb{T}_{\theta,\tau}$ by:

$$\begin{aligned} B_g(\theta,\tau) &= \bigoplus_{n\geq 0} H^0(\mathbb{T}_{\theta,\tau}, E_{\nabla_0}^{\otimes n}) \\ &= \bigoplus_{n\geq 0} \mathcal{H}_{g^n} \end{aligned} \tag{26}$$

The category of holomorphic vector bundles on $\mathbb{T}_{\theta,\tau}$ is equivalent to the heart $\mathcal{C}^\theta$ of a t-structure on $\mathcal{D}^b(E_\tau)$, the derived category of the elliptic curve $E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$. In [**26**] Polishchuk exploits this equivalence in order to study the properties of the algebra $B_g(\theta,\tau)$ by studying the the corresponding image under this equivalence. The following result characterizes some structural properties of $B_g(\theta,\tau)$ in terms of the matrix elements of g:

THEOREM 4.1. ([**26**] Theorem 3.5) *Assume $g \in SL_2(\mathbb{Z})$ has positive real eigenvalues*

- *If $c \geq a + d$ then $B_g(\theta,\tau)$ is generated over $\mathbb{C}$ by $\mathcal{H}_g$.*
- *If $c \geq a + d + 1$ then $B_g(\theta,\tau)$ is a quadratic algebra.*
- *If $c \geq a + d + 2$ then $B_g(\theta,\tau)$ is a Koszul algebra.*

Let us briefly recall these definitions. If $A = \bigoplus_{n\geq 0} A_n$ is a connected graded algebra over a field k generated by its degree one piece A_1 then A is isomorphic to a quotient $T(A_1)/\mathcal{I}$ where $T(A_1) = \bigoplus_{n\geq 0} A_1^{\otimes n}$ is the tensor algebra of the vector space A_1 and $\mathcal{I}$ is a two sided ideal in $T(A_1)$. The algebra A is a *quadratic algebra* if the ideal $\mathcal{I}$ can be generated by homogeneous elements of degree two. Since A is connected we can consider $A_0 = k$ as a left module over A. A quadratic A algebra is a *Koszul algebra* if the graded k-algebra $\bigoplus_{n\geq 0} Ext_A^n(k,k)$ is generated by $Ext_A^1(k,k) \simeq A_1^*$.

We will fix some notations and conventions for the rest of the paper. As above g will denote a matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

We will always assume that the following inequalities hold

$$Tr(g) = a+d > 2 \quad (g \text{ is hyperbolic}) \tag{27}$$

$$c \geq Tr(g)+2 = a+d+2 \tag{28}$$

The first inequality implies that g has positive real eigenvalues. By Theorem 4.1 the second inequality implies that $B_g(\theta,\tau)$ is a quadratic Koszul algebra generated in degree 1.

We denote by λ^+ and λ^- the eigenvalues of g with $0 < \lambda^+ < 1$ and $1 < \lambda^-$. It is important to note that λ^- is a fundamental unit for the quadratic extension it generates. We also take

$$\theta = \frac{\lambda^+ - d}{c} \quad , \quad \theta' = \frac{\lambda^- - d}{c} \tag{29}$$

These are the fixed points of g. By definition the n-graded part of $B_g(\theta,\tau)$ is $\mathcal{H}_{g^n}$. The dimension of $\mathcal{H}_{g^n}$ is $\deg(g^n)$. Accordingly the Hilbert series for $B_g(\tau,\theta)$ is given by (c.f [**26**]):

$$h_{B_g(\tau,\theta)}(t) = \frac{1+(c-a-d)\ t+t^2}{1-(a+d)\ t+t^2}. \tag{30}$$

Proposition 4.2. *Let $\alpha \in \mathbb{R}$ be a quadratic irrationality. Then there exist g and θ satisfying (27) and (28) such that $\mathbb{Q}(\alpha) = \mathbb{Q}(\theta)$.*

Proof. Suppose $\alpha \in \mathbb{R}$ is a quadratic irrationality. Then there exist a hyperbolic element

$$h = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL_2(\mathbb{Z})$$

having α as one of its two fixed points. Being a fixed point of h, α satisfies the quadratic equation $c'\alpha^2 + (d'-a')\alpha - b' = 0$. Since $a'd' - b'c' = 1$ we can write the discriminant of this equation as $D = (a'+d')^2 - 4 = Tr(h)^2 - 4$. α and $\sqrt{D}$ generate the same field extension of $\mathbb{Q}$. $|Tr(h)| > 2$ and we may assume $Tr(h) > 2$ since multiplying h by -1 does not change D. Define

$$g = \begin{pmatrix} Tr(h)+1 & -1 \\ Tr(h)+2 & -1 \end{pmatrix}$$

Then $g \in SL_2(\mathbb{Z})$ and $Tr(g) = Tr(h)$ so the fixed points of g generate the same extension of $\mathbb{Q}$ than α. By construction g satisfies (27) and (28). $\square$

5. A presentation in terms of generators and relations

We want to describe $B_g(\theta,\tau)$ in terms of generators and relations. Let $\{\varphi_\alpha | \alpha = 1,...,c\}$ be the basis for $\mathcal{H}_g$ and $\{\psi_\gamma | \gamma = 1,...,c(a+d)\}$ the corresponding basis of $\mathcal{H}_{g^2}$ as given in (22).

The multiplication map $m : \mathcal{H}_g \otimes \mathcal{H}_g \to \mathcal{H}_{g^2}$ of the algebra $B_g(\theta,\tau)$ is $t_{g,g}$. It is given in the above basis as in Theorem 3.1 by the structure constants:

$$m : \varphi_\alpha \otimes \varphi_\beta \mapsto C^\gamma_{\alpha,\beta}\psi_\gamma \tag{31}$$

Our computation of the relations defining $B_g(\theta,\tau)$ is based on the following observation:

LEMMA 5.1. *Denote by $\mathcal{T}$ the tensor algebra of $\mathcal{H}_g$. Then $B_g(\theta,\tau)$ is isomorphic to the quotient of $\mathcal{T}$ by the homogeneous ideal $\mathcal{R}$ generated by $Ker(m) \subset \mathcal{H}_g \otimes \mathcal{H}_g$.*

PROOF. The algebra graded $B_g(\theta,\tau)$ is generated by its degree one part $\mathcal{H}_g$ therefore it is a quotient of $\mathcal{T}$. Being quadratic the ideal of relations $\mathcal{R} = Ker(m)$ consist of degree two elements in $\mathcal{T}$. □

Let $v^{\alpha,\beta}\varphi_\alpha \otimes \varphi_\beta \in \mathcal{H}_g \otimes \mathcal{H}_g$, $v^{\alpha,\beta} \in \mathbb{C}$ be an arbitrary homogeneous element of $\mathcal{T}$ of degree 2. Since $m : v^{\alpha,\beta}\varphi_\alpha \otimes \varphi_\beta \mapsto v^{\alpha,\beta}C^\gamma_{\alpha,\beta}\psi_\gamma$ we have that $v^{\alpha,\beta}\varphi_\alpha \otimes \varphi_\beta$ belongs to $Ker(m)$ if and only if $v^{\alpha,\beta}C^\gamma_{\alpha,\beta} = 0$ for all $\gamma = 1, ..., c(a+d)$. Using the bases $\{\varphi_\alpha\}$ and $\{\psi_\gamma\}$ we identify $\mathcal{H}_g \otimes \mathcal{H}_g$ and $\mathcal{H}_{g^2}$ with $\mathbb{C}^{c^2}$ and $\mathbb{C}^{c(a+d)}$ respectively. Finding a set of defining relations of $B_g(\theta,\tau)$ for the generators $\{\varphi_\alpha | \alpha = 1, ..., c\}$ amounts to finding a basis for the kernel of the linear map $M : \mathbb{C}^{c^2} \to \mathbb{C}^{c(a+d)}$ given by the $C^\gamma_{\alpha,\beta}$.

LEMMA 5.2. *The structure constant $C^\gamma_{\alpha,\beta}$ is different from zero if and only if $\alpha \equiv d(\gamma - \beta) \mod c$.*

PROOF. The formula for the structure constants (25) in this case is:

$$C^\gamma_{\alpha,\beta} = \sum_{m \in I_{g,g}(\alpha,\beta,\gamma)} \exp[\pi\imath \frac{-\tau m^2}{c^3(a+d)}] \tag{32}$$

The index set of the series is nonempty only when $\alpha \equiv d(\gamma - \beta) \mod c$.

$$\begin{aligned}
I_{g,g}(\alpha,\beta,\gamma) \neq \emptyset \quad & \Longleftrightarrow \quad -c\gamma + c(a+d)\alpha \equiv c(d^2+bc)\gamma - c(a+d)d\beta \mod c^2(a+d) \\
& \Longleftrightarrow \quad -\gamma + (a+d)\alpha \equiv (d^2+bc)\gamma - (a+d)d\beta \mod c(a+d) \\
& \Longleftrightarrow \quad (a+d)\alpha \equiv (d^2+bc+1)\gamma - (a+d)d\beta \mod c(a+d) \\
& \Longleftrightarrow \quad (a+d)\alpha \equiv (d^2+da)\gamma - (a+d)d\beta \mod c(a+d) \\
& \Longleftrightarrow \quad (a+d)\alpha \equiv d(d+a)\gamma - (a+d)d\beta \mod c(a+d) \\
& \Longleftrightarrow \quad \alpha \equiv d(\gamma - \beta) \mod c
\end{aligned}$$

Thus $C^\gamma_{\alpha,\beta} = 0$ if $\alpha \not\equiv d(\gamma - \beta) \mod c$.

Conversely if $\alpha \equiv d(\gamma - \beta) \mod c$ then

$$\begin{aligned}
I_{g,g}(\alpha,\beta,\gamma) &= \{n \in \mathbb{Z} | n \equiv -c\gamma + c(a+d)\alpha \mod c^2(a+d)\} \\
&= \{n \in \mathbb{Z} | n = -c\gamma + c(a+d)\alpha + mc^2(a+d) \text{ for some } m \in \mathbb{Z}\}
\end{aligned}$$

And thus

$$\begin{aligned} C_{\alpha,\beta}^{\gamma} &= \sum_{n\in\mathbb{Z}} \exp[-\pi\imath\tau \frac{(-c\gamma + c(a+d)\alpha + nc^2(a+d))^2}{c^3(a+d)}] \\ &= \sum_{n\in\mathbb{Z}} \exp[-\pi\imath\tau(\frac{(-c\gamma + c(a+d)\alpha)^2}{c^3(a+d)} + \frac{2n(-c\gamma + c(a+d)\alpha)}{c} + c(a+d)n^2)] \\ &= \exp[-\pi\imath\tau \frac{((a+d)\alpha - \gamma)^2}{c(a+d)}] \sum_{n\in\mathbb{Z}} \exp[-2\pi\imath\tau((a+d)\alpha - \gamma)n - \pi\imath\tau c(a+d)n^2] \end{aligned}$$

The last series is a theta series (73):

$$C_{\alpha,\beta}^{\gamma} = \exp[-\pi\imath\tau \frac{[(a+d)\alpha - \gamma]^2}{c(a+d)}]\vartheta(-\tau((a+d)\alpha - \gamma), -\tau c(a+d))$$

And we can write it as a theta constant with rational coefficients by taking, $\tau' = -\tau c(a+d)$ and $l = c(a+d)$.

$$\begin{aligned} C_{\alpha,\beta}^{\gamma} &= \exp[-\pi\imath\tau c(a+d)[\frac{(a+d)\alpha - \gamma}{c(a+d)}]^2]\vartheta(-\tau c(a+d)\frac{(a+d)\alpha - \gamma}{c(a+d)}, -\tau c(a+d)) \\ &= \exp[\pi\imath\tau'[\frac{(a+d)\alpha - \gamma}{l}]^2]\vartheta(\tau'\frac{(a+d)\alpha - \gamma}{l}, \tau') \\ &= \vartheta_{\frac{(a+d)\alpha-\gamma}{l}}(\tau'). \end{aligned}$$

Now, by Lemma 0.5 for $r, s \in \frac{1}{l}\mathbb{Z}$ the zeroes of $\vartheta_{r,s}(z, \tau')$ occur at the points of the form $(r+p+\frac{1}{2})\tau' + (s+q+\frac{1}{2})$ for $p, q \in \mathbb{Z}$. In particular the zeroes of $\vartheta_{r,0}$ are at points $(r+p+\frac{1}{2})\tau' + (q+\frac{1}{2})$. Thus $\vartheta_{r,0}(0, \tau') \neq 0$ for all $r \in \frac{1}{l}\mathbb{Z}$ which proves the lemma. □

The expression for the nonzero values of the structure constants in the proof of Lemma 5.2 is crucial in all that follows. We state it as a corollary.

COROLLARY 5.3. *The nonzero values of the structure constants* $C_{\alpha,\beta}^{\gamma}$ *are theta constants with rational characteristics depending on* g. *If* $\alpha \equiv d(\gamma - \beta)$ mod c *then*

$$\begin{aligned} C_{\alpha,\beta}^{\gamma} &= \vartheta_{\frac{(a+d)\alpha-\gamma}{l}}(\tau') \\ &= \vartheta_{\frac{(a+d)d(\gamma-\beta)-\gamma}{l}}(\tau') \end{aligned}$$

where $\tau' = -\tau c(a+d)$ *and* $l = c(a+d)$.

PROOF. The first equality follows from the proof of Lemma 5.2. For the second one note that the theta constant $\vartheta_r(\tau')$ only depend on the class of

r in $\frac{1}{l}\mathbb{Z}/\mathbb{Z}$ so we can replace α by $d(\gamma-\beta) \mod c$ in the last formula for the nonzero structure constants. ☐

We can use Lemma 5.2 to write the linear system M corresponding to m as a sum of c independent systems. We do this by grouping the nonzero structure constants. Since c and d are relatively prime it follows that for γ in a fixed congruence class $\mod c$ the value of β determines the unique value of α for which $C^{\gamma}_{\alpha,\beta} \neq 0$.

NOTATION 5.4. *Given $\mu, \beta \in \{1,...,c\}$ we denote by $\alpha(\mu,\beta)$ the unique representative $\mod c$ of $d(\mu-\beta)$ laying in $\in \{1,...,c\}$.*

LEMMA 5.5. *Let $M : \mathbb{C}^{c^2} \to \mathbb{C}^{c(a+d)}$ be given by $C^{\gamma}_{\alpha,\beta}$. Then M is equivalent to c independent systems $M(\mu) : \mathbb{C}^{c} \to \mathbb{C}^{(a+d)}$, $\mu = 1,...,c$, each one of rank $c-(a+d)$.*

PROOF. Fix α and β in $\{1,2,...c\}$. Given $\gamma, \gamma' \in \{1,2,...c(a+d)\}$ we have by Lemma 5.2

$$\begin{aligned} C^{\gamma}_{\alpha,\beta} \neq 0 \text{ and } C^{\gamma'}_{\alpha,\beta} \neq 0 &\iff \alpha + d\beta \equiv d\gamma \text{ and } \alpha + d\beta \equiv d\gamma' \\ &\iff d\gamma \equiv d\gamma' \mod c \\ &\iff \gamma \equiv \gamma' \mod c \end{aligned}$$

Therefore nonzero values of $C^{\gamma}_{\alpha,\beta}$ occur for values of γ in the same congruence class $\mod c$ and we can arrange the system as c independent systems of dimension $c^2 \times (a+d)$, each one corresponding to the values of of γ in the same congruence class $\mod c$. Again by Lemma 5.2 each one of these systems will have only c nontrivial columns corresponding to the values of α and β satisfying $\alpha \equiv d(\gamma-\beta) \mod c$. Leaving aside the zero structure constants we are left with the c independent systems:

$$M(\mu) = \begin{pmatrix} C^{\mu}_{\alpha(\mu,1),1} & C^{\mu}_{\alpha(\mu,2),2} & \cdots & C^{\mu}_{\alpha(\mu,c),c} \\ C^{\mu+c}_{\alpha(\mu,1),1} & C^{\mu+c}_{\alpha(\mu,2),2} & \cdots & C^{\mu+c}_{\alpha(\mu,c),c} \\ . & & & \\ . & & & \\ C^{\mu+(a+d-1)c}_{\alpha(\mu,1),1} & C^{\mu+(a+d-1)c}_{\alpha(\mu,2),2} & \cdots & C^{\mu+(a+d-1)c}_{\alpha(\mu,c),c} \end{pmatrix} \tag{33}$$

where $\mu \in \{1,2,...,c\}$.

By Theorem 4.1 $B_g(\theta,\tau)$ is generated by its degree 1 part (remember we assumed $c \geq a+d+2$). This in particular means that $\mathcal{H}_{g^2}$, its degree two part, is generated by products of elements in $\mathcal{H}_g$ so the multiplication map m is surjective. At the level of the representing matrices this just means that M has maximal rank. Also, since M is the direct sum of the $M(\mu)$ each one of these must have maximal rank. ☐

Componentwise we have

$$M(\mu)_{i,j} = C^{\mu+(i-1)c}_{\alpha(\mu,j),j}.$$

By Lemma 5.2 the denominator in the characteristic of the corresponding theta constant is $(a+d)d(\mu-j)-(\mu+(i-1)c)$ so dividing out by $l = c(a+d)$ one gets $\frac{d\mu}{c} - \frac{dj}{c} - \frac{\mu}{l} - \frac{i}{a+d} + \frac{1}{a+d}$ i.e.

$$M(\mu)_{i,j} = \vartheta_{q(\mu)-\frac{dj}{c}-\frac{i}{a+d}}(\tau') \tag{34}$$

Where $q(\mu) = \frac{d\mu}{c} - \frac{\mu}{l} + \frac{1}{a+d}$ is the term not depending on i or j.

It is important to note that the characteristics giving the theta constants which appear as the coefficients of the $M(\mu)$ are all the same up to a shift. We can arrange these characteristics in a matrix $\Lambda = \Lambda(g) \in \mathcal{M}_{c\times(a+d)}(\frac{1}{l}\mathbb{Z}/\mathbb{Z})$ given by $\Lambda_{i,j} = -\frac{dj}{c} - \frac{i}{(a+d)}$. The matrices $M(\mu)$ are then functions of τ and μ determined by g:

$$(\tau,\mu) \mapsto M(\mu)_{i,j} = \vartheta_{q(\mu)+\Lambda_{i,j}}(\tau') \tag{35}$$

Each one of the kernels of the matrices $M(\mu)$ gives us a set of relations for $B_g(\theta,\tau)$. By the above discussion we see that these sets are independent. We write down a basis for the kernel of each $M(\mu)$ in terms of its minor determinants. First we introduce some notation:

NOTATION 5.6. *Let $n > m$ and let $L : \mathbb{C}^n \to \mathbb{C}^m$ be a surjective linear map. Denote also by $L \in \mathcal{M}_{m\times n}(\mathbb{C})$ its matrix representation in the canonical basis. Given $i_1, i_2, ..., i_m \in \{1,2,...,n\}$, $i_1 < i_2 < ... < i_m$ we write $|L|_{i_1,i_2,...,i_m}$ for the minor determinant corresponding to the columns $i_1, i_2, ..., i_m$.*

LEMMA 5.7. *Let $n > m$ and let $L \in \mathcal{M}_{n\times m}(\mathbb{C})$. Assume the first m columns of L are linearly independent. For each $k = 1, ..., n-m$ let $v^k \in \mathbb{C}^n$ be defined by:*

$$v^k_j = \begin{cases} |L|_{1,2,...,j-1,m+k,j+1,...,n} & if\, 1 \le j \le m \\ -|L|_{1,2,...,m} & if\, j = m+k \\ 0 & otherwise \end{cases} \tag{36}$$

Then $\{v^k | k = 1, ..., n-m\}$ forms a basis for $Ker(L)$.

PROOF. Let L be as above. Denote by $L^1, ..., L^n \in \mathbb{C}^m$ its columns. Denote by $\tilde{L} \in \mathcal{M}_{m\times m}$ the matrix corresponding to the first m columns of L. For $k \in \{1, ..., n-m\}$ let $\tilde{X^k} \in \mathbb{C}^m$ be a solution of $\tilde{L}Y = -L^{m+k}$ we can complete $\tilde{X^k}$ to a vector $X^k \in \mathbb{C}^n$ by taking the remaining coordinates to be 0 except for the $n+k$ coordinate which we set to 1. Then $LX^k = 0$ for all $k \in \{1, ..., n-m\}$. Also, it is clear from the construction that the X^k are linearly independent. In this way we may construct a basis for the kernel of L. Now, we solve each one of the systems $\tilde{L}Y = -L^{m+k}$ using Cramer's rule. After clearing denominators in the solution and completing to a vector in $\mathbb{C}^n$ we get the vectors v^k in (36). $\square$

It will be convenient to view the minor determinants of $M(\mu)$ as functions of τ.

DEFINITION 5.8. Let $M(\mu)$ be given as in 33 let $i_1, ..., i_{a+d} \in \{1, 2, ..., c\}$ with $i_1 < i_2 < ... < i_{a+d}$. Define

$$\begin{aligned} F^{g,\mu}_{i_1,i_2,...,i_{a+d}}(\tau) &= |M(\mu)|_{i_1,i_2,...,i_{a+d}} \\ &= \sum_{\sigma \in S_{a+d}} \operatorname{sgn}(\sigma) \prod_{k=1}^{a+d} M(\mu)_{\sigma(k),i_k} \\ &= \sum_{\sigma \in S_{a+d}} \operatorname{sgn}(\sigma) \prod_{k=1}^{a+d} \vartheta_{q(\mu) - \frac{di_k}{c} - \frac{\sigma(k)}{(a+d)}}(\tau') \end{aligned}$$

We will show later that for each g and μ these are modular functions on τ.

By applying Lemma 5.7 we can write now a explicit presentation of $B_g(\tau, \theta)$:

THEOREM 5.9. *Given $\mu \in \{1, 2, ..., c\}$ and $k \in \{1, ..., c - a - d\}$ let $v^{\mu,k} = v^{\mu,k}(\tau) \in \mathbb{C}^c$ be given by*

$$\begin{aligned} v^{\mu,k}_j &= v^{\mu,k}_j(\tau) \\ &= \begin{cases} F^{g,\mu}_{1,2,...,j-1,a+d+k,j+1,...,a+d}(\tau) & \textit{if } 1 \leq j \leq a+d \\ F^{g,\mu}_{1,2,...,a+d}(\tau) & \textit{if } j = m+k \\ 0 & \textit{otherwise} \end{cases} \end{aligned} \tag{37}$$

Then the algebra $B_g(\tau, \theta)$ is generated by elements $x_1, ..., x_c$ of degree 1 subject to relations $f^{\mu}_k = 0$ where:

$$f^{\mu}_k = v^{\mu,k}_1 x_{\alpha(\mu,1)} x_1 + ... + v^{\mu,k}_c x_{\alpha(\mu,c)} x_c \tag{38}$$

PROOF. Each one of the matrices $M(\mu)$ has maximal rank equal to $a+d$. Therefore there are $a+d$ linearly independent columns and we can reorder them in order to apply Lemma 5.7. We let $x_1, ..., x_c$ be the generators of the tensor algebra $\mathcal{T}$ of $\mathcal{H}_g$ corresponding to the basis $\varphi_1, ..., \varphi_c$. Thus $\mathcal{T} = \mathbb{C}\langle x_1, ..., x_c\rangle$. For each $\mu \in \{1, 2, ..., c\}$ Lemma 5.7 gives us a basis for the kernel of $M(\mu)$ which corresponds by Lemma 5.1 to a set of defining relations for $B_g(\tau, \theta)$. □

6. First examples

In this section we look at the behavior of $B_g(\theta, \tau)$ for some particular values of the matrix g.

EXAMPLE 6.1. Let

$$g = \begin{pmatrix} 4 & -1 \\ 5 & -1 \end{pmatrix}$$

The eigenvalues of g are:

$$\lambda^{+} = \frac{3-\sqrt{5}}{2}, \quad \lambda^{-} = \frac{3+\sqrt{5}}{2}$$

and the fixed points of g are:

$$\theta = \frac{5-\sqrt{5}}{10}, \quad \theta' = \frac{5+\sqrt{5}}{10}$$

Fix now a complex structure τ on $\mathcal{A}_\theta$ and consider the corresponding connection $\bar{\nabla}_0$ on the $\mathcal{A}_\theta$-bimodule $E_g(\theta) = E_{-1,5}(\theta)$. The Hilbert series for $B_g(\tau,\theta)$ is given by (30):

$$\begin{aligned} h_{B_g(\tau,\theta)}(t) &= \frac{1+2\,t+t^2}{1-3\,t+t^2} \\ &= 1+5t+15t^2+40t^3+105t^4+275t^5+\dots \end{aligned}$$

In particular $\mathcal{H}_g \simeq \mathbb{C}^5$ and $\mathcal{H}_{g^2} \simeq \mathbb{C}^{15}$. After choosing a basis the multiplication map $m : \mathcal{H}_g \otimes \mathcal{H}_g \to \mathcal{H}_{g^2}$ is represented by a matrix $M \in \mathcal{M}_{15,25}$. We take as above $\{\varphi_\alpha \otimes \varphi_\beta | \alpha,\beta = 1,\dots,5\}$ as basis for $\mathcal{H}_g \otimes \mathcal{H}_g$ and $\{\psi_\gamma | \gamma = 1,\dots,15\}$ as basis for $\mathcal{H}_{g^2}$ so that M is the matrix corresponding to the structure constants

$$C^{\gamma}_{\alpha,\beta} = \begin{cases} \vartheta_{\frac{3\beta-4\gamma}{15}}(-15\tau) \quad if \quad \alpha \equiv d(\gamma-\beta) \mod c \\ 0 \quad \text{otherwise} \end{cases}$$

We write it as $M \simeq M(1) \oplus M(2) \oplus M(3) \oplus M(4) \oplus M(5)$ where the elements of $M(\mu) \in \mathcal{M}_{3,5}(\mathbb{C})$ are given by

$$M(\mu)_{i,j} = \vartheta_{q(\mu)+\Lambda_{i,j}}(-15\tau)$$

With $q(\mu) = \frac{5-4\mu}{15}$ and

$$\Lambda = \frac{1}{15}\begin{pmatrix} 2 & 14 & 11 & 8 & 5 \\ 7 & 4 & 1 & 13 & 10 \\ 12 & 9 & 6 & 3 & 0 \end{pmatrix}$$

Each $\mu \in \{1,\dots,5\}$ gives us a set of 2 relations corresponding to a basis for the kernel of $M(\mu)$. In this case $B_g(\theta,\tau)$ is a quadratic algebra with 5 generators of degree 1 and 10 quadratic relations.

The minors of $M(\mu)$ give the functions of τ appearing as coefficients of the defining relations of $B_g(\theta,\tau)$. For each ordered triple $i_1,i_2,i_3 \in \{1,2,3,4,5\}$ we have:

$$\begin{aligned} F^{g,\mu}_{i_1,i_2,i_3}(\tau) &= |M(\mu)|_{i_1,i_2,i_3} \\ &= \sum_{\sigma\in S_3} \mathrm{sgn}(\sigma) \prod_{k=1}^{3} \vartheta_{\frac{5-4\mu}{15}-\frac{di_k}{5}-\frac{\sigma(k)}{3}}(-15\tau) \end{aligned}$$

Thus each one of the coefficients of the defining relations is a sum of triple products of theta constants. For example, taking $\mu = 1$ we get the two relations:

$$f_1^1 = F^{g,1}_{4,2,3}(\tau)x_5x_1 + F^{g,1}_{1,4,3}(\tau)x_1x_2 + F^{g,1}_{1,2,4}(\tau)x_2x_3 - F^{g,1}_{1,2,3}(\tau)x_3x_4$$

and

$$f_2^1 = F^{g,1}_{5,2,3}(\tau)x_4x_1 + F^{g,1}_{1,5,3}(\tau)x_5x_2 + F^{g,1}_{1,2,5}(\tau)x_1x_3 - F^{g,1}_{1,2,3}(\tau)x_3x_5.$$

Example 6.2. Let

$$g = \begin{pmatrix} 5 & -1 \\ 6 & -1 \end{pmatrix}$$

The eigenvalues of g are

$$\lambda^+ = 2 - \sqrt{3}, \quad \lambda^- = 2 + \sqrt{3}$$

and the fixed points of g are

$$\theta = \frac{3-\sqrt{3}}{6}, \quad \theta' = \frac{3+\sqrt{3}}{6}.$$

Fix now a complex structure τ on $\mathcal{A}_\theta$ and consider the corresponding connection $\bar{\nabla}_0$ on the $\mathcal{A}_\theta$-bimodule $E_g(\theta) = E_{-1,6}(\theta)$. The Hilbert series for $B_g(\tau, \theta)$ is given by (30):

$$\begin{aligned} h_{B_g(\tau,\theta)}(t) &= \frac{1 + 2\,t + t^2}{1 - 4\,t + t^2} \\ &= 1 + 6t + 24t^2 + 90t^3 + 336t^4 + 1254t^5 + \ldots \end{aligned}$$

In particular $\mathcal{H}_g \simeq \mathbb{C}^6$ and $\mathcal{H}_{g^2} \simeq \mathbb{C}^{24}$. After choosing a basis the multiplication map $m : \mathcal{H}_g \otimes \mathcal{H}_g \to \mathcal{H}_{g^2}$ is represented by a matrix $M \in \mathcal{M}_{24,36}$. We take as above $\{\varphi_\alpha \otimes \varphi_\beta | \alpha, \beta = 1, ..., 6\}$ as basis for $\mathcal{H}_g \otimes \mathcal{H}_g$ and $\{\psi_\gamma | \gamma = 1, ..., 24\}$ as basis for $\mathcal{H}_{g^2}$ so that M is the matrix corresponding to the structure constants

$$C^\gamma_{\alpha,\beta} = \begin{cases} \vartheta_{\frac{4\beta-5\gamma}{24}}(-24\tau) & if \quad \alpha \equiv d(\gamma-\beta) \mod 6 \\ 0 \quad \text{otherwise} \end{cases}$$

We write it as $M \simeq M(1) \oplus M(2) \oplus M(3) \oplus M(4) \oplus M(5) \oplus M(6)$ where the elements of $M(\mu) \in \mathcal{M}_{4,6}(\mathbb{C})$ are given by

$$M(\mu)_{i,j} = \vartheta_{q(\mu)+\Lambda_{i,j}}(-24\tau)$$

With $q(\mu) = \frac{6-5\mu}{24}$ and

$$\Lambda = \frac{1}{24}\begin{pmatrix} 2 & 22 & 18 & 14 & 10 & 6 \\ 8 & 4 & 0 & 20 & 16 & 12 \\ 14 & 10 & 6 & 2 & 22 & 18 \\ 20 & 16 & 12 & 8 & 4 & 0 \end{pmatrix}$$

Each $\mu \in \{1, ..., 6\}$ gives us a set of 2 relations corresponding to a basis for the kernel of $M(\mu)$. In this case $B_g(\theta, \tau)$ is a quadratic algebra with 6 generators of degree 1 and 12 quadratic relations.

Example 6.3. Let

$$g = \begin{pmatrix} 3 & 1 \\ 8 & 3 \end{pmatrix}, \qquad \theta = \frac{\sqrt{2}}{4}$$

Fix now a complex structure τ on $\mathcal{A}_\theta$ and consider the corresponding connection $\bar{\nabla}_0$ on the $\mathcal{A}_\theta$-bimodule $E_g(\theta) = E_{3,8}(\theta)$. The Hilbert series for $B_g(\tau, \theta)$ is given by (30):

$$\begin{aligned} h_{B_g(\tau,\theta)}(t) &= \frac{1 + 2\,t + t^2}{1 - 6\,t + t^2} \\ &= 1 + 8t + 48t^2 + 280t^3 + 1632t^4 + 9512t^5 + + \dots \end{aligned}$$

In particular $\mathcal{H}_g \simeq \mathbb{C}^8$ and $\mathcal{H}_{g^2} \simeq \mathbb{C}^{48}$. After choosing a basis the multiplication map $m : \mathcal{H}_g \otimes \mathcal{H}_g \to \mathcal{H}_{g^2}$ is represented by a matrix $M \in \mathcal{M}_{48,64}$. We take as above $\{\varphi_\alpha \otimes \varphi_\beta | \alpha, \beta = 1, ..., 8\}$ as basis for $\mathcal{H}_g \otimes \mathcal{H}_g$ and $\{\psi_\gamma | \gamma = 1, ..., 48\}$ as basis for $\mathcal{H}_{g^2}$. We write the matrix corresponding to the structure constants as $M \simeq M(1) \oplus \cdots \oplus M(8)$ where the elements of $M(\mu) \in \mathcal{M}_{6,8}(\mathbb{C})$ are given by

$$M(\mu)_{i,j} = \vartheta_{q(\mu)+\Lambda_{i,j}}(-48\tau)$$

With $q(\mu) = \frac{17\mu}{48} + \frac{1}{6}$ and

$$\Lambda = \frac{1}{48} \begin{pmatrix} 26 & 44 & 14 & 32 & 2 & 20 & 38 & 8 \\ 34 & 4 & 22 & 40 & 10 & 28 & 46 & 16 \\ 42 & 12 & 30 & 0 & 18 & 36 & 6 & 24 \\ 2 & 20 & 38 & 8 & 26 & 44 & 14 & 32 \\ 10 & 28 & 46 & 16 & 34 & 4 & 22 & 40 \\ 18 & 36 & 6 & 24 & 42 & 12 & 30 & 0 \end{pmatrix}$$

Each $\mu \in \{1, ..., 8\}$ gives us a set of 2 relations corresponding to a basis for the kernel of $M(\mu)$. In this case $B_g(\theta, \tau)$ is a quadratic algebra with 8 generators of degree 1 and 16 quadratic relations.

Example 6.4. Let

$$g = \begin{pmatrix} -1 & -1 \\ 7 & 6 \end{pmatrix}, \qquad \theta = \frac{\sqrt{21} - 7}{14}$$

Fix now a complex structure τ on $\mathcal{A}_\theta$ and consider the corresponding connection $\bar{\nabla}_0$ on the $\mathcal{A}_\theta$-bimodule $E_g(\theta) = E_{6,7}(\theta)$. The Hilbert series for $B_g(\tau, \theta)$ is given by (30):

$$\begin{aligned} h_{B_g(\tau,\theta)}(t) &= \frac{1+2\,t+t^2}{1-5\,t+t^2} \\ &= 1+7t+35t^2+168t^3+805t^4+3857t^5+\dots \end{aligned}$$

In particular $\mathcal{H}_g \simeq \mathbb{C}^7$ and $\mathcal{H}_{g^2} \simeq \mathbb{C}^{35}$. After choosing a basis the multiplication map $m : \mathcal{H}_g \otimes \mathcal{H}_g \to \mathcal{H}_{g^2}$ is represented by a matrix $M \in \mathcal{M}_{35,49}$. We take as above $\{\varphi_\alpha \otimes \varphi_\beta | \alpha, \beta = 1, ..., 7\}$ as basis for $\mathcal{H}_g \otimes \mathcal{H}_g$ and $\{\psi_\gamma | \gamma = 1, ..., 35\}$ as basis for $\mathcal{H}_{g^2}$. We write the matrix corresponding to the structure constants as $M \simeq M(1) \oplus \cdots \oplus M(7)$ where the elements of $M(\mu) \in \mathcal{M}_{5,7}(\mathbb{C})$ are given by

$$M(\mu)_{i,j} = \vartheta_{q(\mu)+\Lambda_{i,j}}(-48\tau)$$

With $q(\mu) = \frac{17\mu}{48} + \frac{1}{6}$ and

$$\Lambda = \frac{1}{35}\begin{pmatrix} 2 & 32 & 27 & 22 & 17 & 12 & 7 \\ 9 & 4 & 34 & 29 & 24 & 19 & 14 \\ 16 & 11 & 6 & 1 & 31 & 26 & 21 \\ 23 & 18 & 13 & 8 & 3 & 33 & 28 \\ 30 & 25 & 20 & 15 & 10 & 5 & 0 \end{pmatrix}$$

Each $\mu \in \{1, ..., 7\}$ gives us a set of 2 relations corresponding to a basis for the kernel of $M(\mu)$. In this case $B_g(\theta, \tau)$ is a quadratic algebra with 7 generators of degree 1 and 14 quadratic relations.

Example 6.5. Let

$$g = \begin{pmatrix} 4 & 1 \\ 15 & 4 \end{pmatrix}, \qquad \theta = -\frac{\sqrt{15}}{15}$$

Fix now a complex structure τ on $\mathcal{A}_\theta$ and consider the corresponding connection $\bar{\nabla}_0$ on the $\mathcal{A}_\theta$-bimodule $E_g(\theta) = E_{4,15}(\theta)$. The Hilbert series for $B_g(\tau, \theta)$ is given by (30):

$$\begin{aligned} h_{B_g(\tau,\theta)}(t) &= \frac{1+7\,t+t^2}{1-8\,t+t^2} \\ &= 1+15t+120t^2+945t^3+7440t^4+58575t^5+\dots \end{aligned}$$

In particular $\mathcal{H}_g \simeq \mathbb{C}^{15}$ and $\mathcal{H}_{g^2} \simeq \mathbb{C}^{120}$. After choosing a basis the multiplication map $m : \mathcal{H}_g \otimes \mathcal{H}_g \to \mathcal{H}_{g^2}$ is represented by a matrix $M \in \mathcal{M}_{120,225}$. We take as above $\{\varphi_\alpha \otimes \varphi_\beta | \alpha, \beta = 1, ..., 15\}$ as basis for $\mathcal{H}_g \otimes \mathcal{H}_g$ and $\{\psi_\gamma | \gamma = 1, ..., 120\}$ as basis for $\mathcal{H}_{g^2}$. We write the matrix corresponding to the structure constants as $M \simeq M(1) \oplus \cdots \oplus M(15)$ where the elements of $M(\mu) \in \mathcal{M}_{8,15}(\mathbb{C})$ are given by

$$M(\mu)_{i,j} = \vartheta_{q(\mu)+\Lambda_{i,j}}(-120\tau)$$

With $q(\mu) = \frac{31\mu}{120} + \frac{1}{8}$ and $\Lambda \in \mathcal{M}_{15\times 8}(\frac{1}{120}\mathbb{Z}/\mathbb{Z})$ is given by

$$\Lambda_{i,j} = -\frac{4j}{15} - \frac{i}{8}; \qquad i = 1, \dots, 8;\ j = 1, \dots, 15$$

Each $\mu \in \{1, ..., 15\}$ gives us a set of 7 relations corresponding to a basis for the kernel of $M(\mu)$. In this case $B_g(\theta, \tau)$ is a quadratic algebra with 15 generators of degree 1 and 105 quadratic relations.

Example 6.6. Let

$$g = \begin{pmatrix} -1 & -1 \\ 11 & 10 \end{pmatrix}, \qquad \theta = -\frac{\sqrt{77} - 11}{22}$$

Fix now a complex structure τ on $\mathcal{A}_\theta$ and consider the corresponding connection $\bar{\nabla}_0$ on the $\mathcal{A}_\theta$-bimodule $E_g(\theta) = E_{11,10}(\theta)$. The Hilbert series for $B_g(\tau, \theta)$ is given by (30):

$$\begin{aligned} h_{B_g(\tau,\theta)}(t) &= \frac{1 + 2\,t + t^2}{1 - 9\,t + t^2} \\ &= 1 + 11t + 99t^2 + 880t^3 + 7821t^4 + 69509t^5 + \dots \end{aligned}$$

In particular $\mathcal{H}_g \simeq \mathbb{C}^{11}$ and $\mathcal{H}_{g^2} \simeq \mathbb{C}^{99}$. After choosing a basis the multiplication map $m : \mathcal{H}_g \otimes \mathcal{H}_g \to \mathcal{H}_{g^2}$ is represented by a matrix $M \in \mathcal{M}_{99,121}$. We take as above $\{\varphi_\alpha \otimes \varphi_\beta | \alpha, \beta = 1, ..., 11\}$ as basis for $\mathcal{H}_g \otimes \mathcal{H}_g$ and $\{\psi_\gamma | \gamma = 1, ..., 99\}$ as basis for $\mathcal{H}_{g^2}$. We write the matrix corresponding to the structure constants as $M \simeq M(1) \oplus \dots \oplus M(11)$ where the elements of $M(\mu) \in \mathcal{M}_{9,11}(\mathbb{C})$ are given by

$$M(\mu)_{i,j} = \vartheta_{q(\mu)+\Lambda_{i,j}}(-99\tau)$$

With $q(\mu) = \frac{89\mu}{99} + \frac{1}{9}$ and $\Lambda \in \mathcal{M}_{11\times 9}(\frac{1}{99}\mathbb{Z}/\mathbb{Z})$ is given by

$$\Lambda_{i,j} = -\frac{10j}{11} - \frac{i}{9}; \qquad i = 1, \dots, 9;\ j = 1, \dots, 11$$

Each $\mu \in \{1, ..., 11\}$ gives us a set of 2 relations corresponding to a basis for the kernel of $M(\mu)$. In this case $B_g(\theta, \tau)$ is a quadratic algebra with 11 generators of degree 1 and 22 quadratic relations.

Example 6.7. Let

$$g = \begin{pmatrix} 6 & 1 \\ 35 & 6 \end{pmatrix}, \qquad \theta = -\frac{\sqrt{35}}{35}$$

Fix now a complex structure τ on $\mathcal{A}_\theta$ and consider the corresponding connection $\bar{\nabla}_0$ on the $\mathcal{A}_\theta$-bimodule $E_g(\theta) = E_{6,35}(\theta)$. The Hilbert series for $B_g(\tau, \theta)$ is given by (30):

$$\begin{aligned} h_{B_g(\tau,\theta)}(t) &= \frac{1+23\,t+t^2}{1-12\,t+t^2} \\ &= 1+35t+420t^2+5005t^3+59640t^4+710675t^5+\dots \end{aligned}$$

In particular $\mathcal{H}_g \simeq \mathbb{C}^{35}$ and $\mathcal{H}_{g^2} \simeq \mathbb{C}^{420}$. After choosing a basis the multiplication map $m : \mathcal{H}_g \otimes \mathcal{H}_g \to \mathcal{H}_{g^2}$ is represented by a matrix $M \in \mathcal{M}_{420,1225}$. We take as above $\{\varphi_\alpha \otimes \varphi_\beta | \alpha, \beta = 1, ..., 35\}$ as basis for $\mathcal{H}_g \otimes \mathcal{H}_g$ and $\{\psi_\gamma | \gamma = 1, ..., 420\}$ as basis for $\mathcal{H}_{g^2}$. We write the matrix corresponding to the structure constants as $M \simeq M(1) \oplus \cdots \oplus M(35)$ where the elements of $M(\mu) \in \mathcal{M}_{12,35}(\mathbb{C})$ are given by

$$M(\mu)_{i,j} = \vartheta_{q(\mu)+\Lambda_{i,j}}(-120\tau)$$

With $q(\mu) = \frac{71\mu}{420} + \frac{1}{12}$ and $\Lambda \in \mathcal{M}_{35\times 12}(\frac{1}{420}\mathbb{Z}/\mathbb{Z})$ is given by

$$\Lambda_{i,j} = -\frac{6j}{35} - \frac{i}{12}; \qquad i = 1, \dots, 12;\ j = 1, \dots, 35.$$

Each $\mu \in \{1, ..., 35\}$ gives us a set of 23 relations corresponding to a basis for the kernel of $M(\mu)$. In this case $B_g(\theta, \tau)$ is a quadratic algebra with 35 generators of degree 1 and 805 quadratic relations.

CHAPTER 2

Rationality properties of $B_g(\tau, \theta)$

In this chapter we use the presentation of $B_g(\tau, \theta)$ in terms of theta constants given in the last chapter to derive some results about the rationality of this algebra. The algebra $B_g(\tau, \theta)$ is a priori only defined over $\mathbb{C}$. It is important to know in which cases $B_g(\tau, \theta)$ can be defined over smaller fields. In order to do this we will look for particular presentations of these algebras in which the coefficients of the defining relations belong to fields smaller than $\mathbb{C}$. We begin by studying the rationality properties of special values of homogeneous rational combinations of theta functions. By an appropriate rescaling of the relations defining $B_g(\tau, \theta)$ we can then find a presentation whose coefficients are given in terms of these special values. The main result of this chapter relates the field of definition of the algebra $B_g(\tau, \theta)$ to the field of definition of the elliptic curve defined by the complex structure τ.

1. Rationality properties of theta constants

Let $g \in SL_2(\mathbb{Z})$ and assume g satisfies the conditions (27) and (28). Let θ be the corresponding quadratic irrationality. Fix a complex structure τ on $\mathbb{T}_\theta$ and let $\tau' = -c(a+d)\tau$. Consider the elliptic curve $E_{\tau'}$ with complex points given by:

$$E_{\tau'}(\mathbb{C}) = \mathbb{C}/(\mathbb{Z} \oplus \tau'\mathbb{Z}) \tag{39}$$

Let $j(\tau')$ denote the absolute invariant of the elliptic curve $E_{\tau'}$. The minimal field of definition of $E_{\tau'}$ is

$$k' = \mathbb{Q}(j(\tau')) \tag{40}$$

Since the group structure on the elliptic curve $E_{\tau'}$ is given in terms of algebraic maps any torsion point on $E_{\tau'}$ is defined over a finite algebraic extension of the field k'. If we denote by $E_{\tau'}[N]$ the set of N torsion points of $E_{\tau'}$ and let k'_N be the field extension of k' over which they are defined we obtain a Galois extension $k' \hookrightarrow k'_N$. In general, given a extension $k' \hookrightarrow K$ of k' we denote by $E_{\tau'}(K)$ the elliptic curve over K obtained by the base extension $Spec(K) \to Spec(k')$. We refer the reader to [**15**] and [**12**] for these and other standard results about elliptic curves used in this chapter.

In the previous chapter we showed that the structure constants of the algebra $B_g(\tau, \theta)$ and the corresponding presentation could be given in terms of theta constants of the form $\vartheta_r(\tau')$. From (79) it follows that

$$\vartheta_r(\tau') = \vartheta^{\alpha}_{r,0}(\tau') \qquad \forall r \in \frac{1}{l}\mathbb{Z}$$

Thus we can apply Theorem 0.6 to study the rationality of these values. Given $r \in \frac{1}{l}\mathbb{Z}$ the point $x(r,0)_{2l}$ is a torsion point of order dividing $2l^2$ and is therefore rational over k'_{2l^2}. Since $E_{\tau'}$ and its basic line bundle $\mathcal{L}$ are both defined over k'_{2l^2} we get:

Theorem 1.1. *For all* $r \in \frac{1}{l}\mathbb{Z}$

$$\frac{\vartheta_r(\tau')}{\vartheta(\tau')} \in k'_{2l^2} \tag{41}$$

Now let $l = c(a+d)$ and let $\tau' = -l\tau$, denote by E_τ the corresponding elliptic curve $\mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z})$. The minimal field of definition of the elliptic curve E_τ is

$$k = \mathbb{Q}(j(\tau)). \tag{42}$$

Since k'_{2l^2} is a finite algebraic extension of k' Theorem 1.1 implies that $\frac{\vartheta_r(\tau')}{\vartheta(\tau')}$ is algebraic over k'. We want to show that the values $\frac{\vartheta_r(\tau')}{\vartheta(\tau')}$ are also algebraic over k. For this we must study the relation between k and k'_{2l^2}. The main point here is that the absolute invariants $j(\tau)$ and $j(\tau')$ are related by an algebraic equation with integer coefficients.

For a positive integer n denote by Δ^*_n the set of matrices in $\mathcal{M}_2(\mathbb{Z})$ with determinant n and relatively prime components. Multiplication on the right by elements in $Sl_2(\mathbb{Z})$ maps Δ^*_n to into itself. We denoe by $\psi(n)$ the number of right coset of Δ^*_n.

Definition 1.2. Let $\Xi = \{\alpha_1, \ldots, \alpha_{\psi(n)}\}$ be a complete set of coset representatives for of Δ^*_n. Let j denote the absolute invariant modular function. We define the *modular polynomial of order n* $\Phi_n(T,j) \in \mathbb{Z}[T,j]$ by:

$$\Phi_n(T,j) = \prod_{\alpha_i \in \Xi} (T - j \circ \alpha_i) \tag{43}$$

We are now ready to apply the the following result (c.f [**15**] Chapter5 §3):

Theorem 1.3. *Let* $\Phi_n(T,j)$ *be the modular polynomial of order* n *and let* E_1, E_2 *be two elliptic curves over* $\mathbb{C}$. *Then* $\Phi_n(j(E_1), j(E_2)) = 0$ *if and only if there exist an isogeny* $E_1 \to E_2$ *with cyclic kernel of degree* n.

Corollary 1.4. *Let* $l > 1$ *be an integer and let* $r \in \frac{1}{l}(\mathbb{Z})$. *Then*

$$\frac{\vartheta_r(\tau')}{\vartheta(\tau')} = \frac{\vartheta_r(-l\tau)}{\vartheta(-l\tau)}$$

is algebraic over k.

PROOF. Let E_τ and $E_{\tau'}$ be as above. Multiplication by l induces an isogeny $E_\tau \to E_{\tau'}$. The kernel of this isogeny is $(\mathbb{Z}\oplus\tau\mathbb{Z})/(\mathbb{Z}\oplus l\tau\mathbb{Z}) \simeq \mathbb{Z}/l\mathbb{Z}$. From Theorem 1.3 it follows that $j(\tau')$ is algebraic over $k = \mathbb{Q}(j(\tau))$. Taking into account that k'_{2l^2} is algebraic over $\mathbb{Q}(j(\tau'))$ we get a tower of algebraic extensions:

$$\mathbb{Q}(j(\tau)) \hookrightarrow \mathbb{Q}(j(\tau'), j(\tau)) \hookrightarrow k'_{2l^2}(j(\tau)).$$

The theorem then follows from Theorem 1.1 □

2. A rational presentation of $B_g(\tau,\theta)$

In this section we will rescale the relations defining $B_g(\tau,\theta)$ by appropriate factors that will allow the study of their rationality properties. As we saw in the preceding chapter the relations defining $B_g(\tau,\theta)$ are given by a linearly independent generating set for the kernel of the multiplication map. Multiplying each element of this basis by a nonzero constant we still obtain a basis. Therefore we are free to multiply each one of the relations in Theorem 5.9 by a nonzero constant and still get a set of defining relations for $B_g(\tau,\theta)$. Using the results about rationality of theta constants obtained in the last section we can prove the main result of this chapter:

THEOREM 2.1. *Let E_τ be the elliptic curve $\mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z})$. Let k be its minimal field of definition. Then the algebra $B_g(\theta,\tau)$ admits a rational presentation over a finite algebraic extension of k.*

PROOF. Multiplying each one of the relations in Theorem 5.9 by

$$[\vartheta(-l\tau)]^{-(a+d)}$$

we get a new basis $\{\tilde{f}_k^\mu\}$ for $\mathcal{R}$, the ideal giving the relations of the $B_g(\tau,\theta)$:

$$\tilde{f}_k^\mu = \tilde{v}_1^{\mu,k} x_{\alpha(\mu,1)} x_1 + \dots + \tilde{v}_c^{\mu,k} x_{\alpha(\mu,c)} x_c \tag{44}$$

where

$$\tilde{v}_j^{\mu,k} = [\vartheta(-l\tau)]^{-(a+d)}\, v_c^{\mu,k} \tag{45}$$

Each $\tilde{v}_j^{\mu,k}$ has the form:

$$\begin{aligned} &[\vartheta(-l\tau)]^{-(a+d)}\, F^{g,\mu}_{i_1,\dots,i_{a+d}}(\tau) \\ = {}&[\vartheta(-l\tau)]^{-(a+d)} \sum_{\sigma\in S_{a+d}} \operatorname{sgn}(\sigma) \prod_{k=1}^{a+d} \vartheta_{q(\mu)-\frac{di_k}{c}-\frac{\sigma(k)}{(a+d)}}(-l\tau) \end{aligned}$$

for some $i_1,\dots,i_{a+d} \in \{1,\dots,a+d\}$. Therefore $\tilde{v}_j^{\mu,k}$ belongs to the field generated over $\mathbb{Q}$ by the values of the form $\frac{\vartheta_r(-l\tau)}{\vartheta(-l\tau')}$ which are algebraic over k by Corollary 1.4 . □

From Corollary 1.4 and Theorem 2.1 we see that $B_g(\theta,\tau)$ is defined over the field

$$\mathbb{K} = k'_{2l^2}(j(\tau)) \tag{46}$$

A minimal model of the elliptic curve E_τ is given by its Weierstrass equation. In particular the field k'_{2l^2} is generated by the values of the Weierstrass functions $\wp$ and $\wp'$ at the points in the complex plane of the form $\frac{1}{2l^2}p\tau' + \frac{1}{2l^2}q$ with $p, q \in \mathbb{Z}$. The field $\mathbb{K}$ is therefore generated over $\mathbb{Q}$ by these values of $\wp$ and $\wp'$ together with $j(\tau)$ and $j(\tau')$.

We can restrict our field of scalars to $\mathbb{K}$ to obtain a noncommutative $\mathbb{K}$-algebra:

$$B_g(\tau,\theta)_{\mathbb{K}} = \mathbb{K}\langle x_1, ..., x_c\rangle/\mathcal{R} \tag{47}$$

An important question is whether the Galois group $Gal(\mathbb{K}/k)$ leaves $\mathcal{R}$ invariant and being this the case how does it act on $B_g(\tau,\theta)_{\mathbb{K}}$.

3. Special values of τ

Starting with Theorem 2.1 we can study the properties of $B_g(\tau,\theta)$ for special values of τ giving interesting fields of definition. We use the same notation as in the previous sections.

3.1. Subfields of $\mathbb{R}$. We start with a simple, yet useful, remark:

Proposition 3.1. *Let $\tau \in \imath\mathbb{R}$. Then $B_g(\tau,\theta)$ is defined over a subfield of $\mathbb{R}$.*

Proof. $B_g(\tau,\theta)$ is defined over the field generated over $\mathbb{Q}$ by the values of the theta constants $\vartheta_{r_i}(\tau')$. By the series expression (73) we know that the theta constant ϑ_{r_i} takes real values on $\imath\mathbb{R}$. Since $\tau \in \imath\mathbb{R}$ implies $\tau' \in \imath\mathbb{R}$ the proposition follows. □

3.2. Number fields. The most interesting family of examples comes from elliptic curves defined over number fields. In the case $j(\tau)$ is algebraic over $\mathbb{Q}$ the field $\mathbb{K} = k'_{2l^2}(j(\tau))$ is a number field. As before we consider the algebra

$$B_g(\tau,\theta)_{\mathbb{K}} = \mathbb{K}\langle x_1, ..., x_c\rangle/\mathcal{R}$$

obtained by restriction of scalars from $\mathbb{C}$ to $\mathbb{K}$. Let now $\mathcal{O}_{\mathbb{K}}$ be the ring of integers of $\mathbb{K}$. Since $\mathbb{K}$ is the field of fractions of $\mathcal{O}_{\mathbb{K}}$ we can clear denominators in each one of the defining relations of $B_g(\tau,\theta)_{\mathbb{K}}$ and obtain a basis $\{\bar{f}_k^\mu\}$ of $\mathcal{R}$ of the form

$$\bar{f}_k^\mu = \bar{v}_1^{\mu,k} x_{\alpha(\mu,1)}x_1 + ... + \bar{v}_c^{\mu,k} x_{\alpha(\mu,c)}x_c$$

with $\bar{v}_j^{\mu,k}$ in $\mathcal{O}_{\mathbb{K}}$. In particular we can consider the various

reductions corresponding to different finite places $\mathfrak{P}$ of $\mathbb{K}$. If the coefficient $\bar{v}_j^{\mu,k}$ are nonzero modulo $\mathfrak{P}$ it makes sense to talk about the $\mathcal{O}_{\mathbb{K}}/\mathfrak{P}$ algebra given by the relations $\bar{f}_k^\mu \mod \mathfrak{P}$.

Among number fields some cases deserve particular attention. The first case comes from taking $\tau = \sqrt{-D}$ where D is a positive integer. Since the

j invariant of an elliptic curve with complex multiplication is algebraic we get from Proposition 3.1:

COROLLARY 3.2. *Let* $\tau = \sqrt{-D}$ *be a generator in* $\imath\mathbb{R}$ *of the quadratic imaginary field* $\mathbb{Q}(\sqrt{-D})$. *Then* $B_g(\tau, \theta)$ *is defined over a real algebraic extension of* $\mathbb{Q}$.

3.3. Canonical choices. Our original objects of study are noncommutative tori with real multiplication and the corresponding quadratic extensions of $\mathbb{Q}$. The complex structure τ should be viewed as an auxiliary tool for the definition o the corresponding homogeneous coordinate rings. It becomes then important two know whether there are some canonical complex structures associated to a given a real multiplication noncommutative tori $\mathbb{T}_\theta$ and whether we can define homogeneous coordinate rings that do not depend on a particular choice of a complex structure. We will address the second question in later chapters. As for the first question There are at least two complex structures that have some arithmetical meaning and can be associated to $\mathbb{T}_\theta$:

- Choose $\tau = \sqrt{-D}$ where D is a positive integer such that $\mathbb{Q}(\theta) = \mathbb{Q}(\sqrt{D})$.
- Choose τ with absolute invariant $j(\tau) = \theta$. Then elliptic curve E_τ is defined over $\mathbb{Q}(\theta)$ and the defining relations of the $B_g(\tau, \theta)$ are algebraic over this field.

Once relation between the rings $B_g(\tau, \theta)$ and the complex torus $\mathbb{T}_\theta$ is better understood it will be nice to address the question of a canonical complex structure from a variational point of view.

CHAPTER 3

A linear basis for $B_g(\tau,\theta)$

Let $\mathbb{K}$ be a subfield of $\mathbb{C}$ over which $B_g(\tau,\theta)$ is defined. We would like to construct interesting linear functionals on $B_g(\tau,\theta)_{\mathbb{K}} = \mathbb{K}\langle x_1,...,x_c\rangle/\mathcal{R}$ with values on $\mathbb{K}$. Provided we are given a linear basis for $B_g(\tau,\theta)_{\mathbb{K}}$ over $\mathbb{K}$ we can define linear functionals by prescribing the values of the elements in such a basis. The aim of this chapter is to study the natural linear basis corresponding to the presentations of $B_g(\tau,\theta)$ in terms of generators and relations given in Theorem 2.1. For this purpose the theory of noncommutative Gröbner basis for two sided ideals on the free algebra $\mathbb{K}\langle x_1,...,x_c\rangle$ provides the right framework. The general idea is an extrapolation of Gaussian reduction to infinite dimensions (c.f. [**22**], see also [**28**]).

We view $\mathbb{K}\langle x_1,...,x_c\rangle$ as the semigroup algebra corresponding to the free semigroup $\mathbf{S}$ generated by $\{x_1,...,x_c\}$. The semigroup $\mathbf{S}$ is graded by total degree and it becomes an ordered semigroup by imposing on it the *deglex* order. Thus given $t_1,t_2\in\mathbf{S}$ we say that $t_1<t_2$ if either $deg(t_1)<deg(t_2)$ or $deg(t_1)=deg(t_2)$ and there exist $l,r_1,r_2\in S$ such that $t_1=lx_jr_1$ and $t_2=lx_ir_2$ with $j<i$. Once this order is given every element f of $\mathbb{K}\langle x_1,...,x_c\rangle$ has a well defined leading term $T(f)\in\mathbf{S}$ and a leading coefficients $lc(f)\in\mathbb{K}$.

As in the previous chapters let

$$g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$$

be a matrix satisfying 27 and 28. For $\mu,j\in\{1,2,...,c\}$ and $k\in\{1,...,c-a-d\}$ let $\tilde{v}_j^{\mu,k}$ be defined as in Theorem 2.1. In order to keep track of the ordering of the terms in the relations defining $B_g(\tau,\theta)$ we interchange the roles of the two factors in the degree two part of the free algebra $\mathbb{K}\langle x_1,...,x_c\rangle$. Also, we divide each one of the relations $\{\tilde{f}_k^\mu\}$ by its leading coefficient. Accordingly we define a basis $\{\hat{f}_k^\mu\}$ for $\mathcal{R}$, the ideal of relations of $B_g(\tau,\theta)$, by:

$$\hat{f}_k^\mu \quad=\quad x_{a+d+k}x_{\alpha(\mu,a+d+k)}+\frac{\tilde{v}_{a+d}^{\mu,k}}{\tilde{v}_{a+d+k}^{\mu,k}}x_{a+d}x_{\alpha(\mu,a+d)}+...+\frac{\tilde{v}_{1}^{\mu,k}}{\tilde{v}_{a+d+k}^{\mu,k}}x_1x_{\alpha(\mu,1)}.$$

In the above expression the terms of the relations appear in decreasing order with $T(\hat{f}_k^\mu)=x_{a+d+k}x_{\alpha(\mu,a+d+k)}$ and $lc(\hat{f}_k^\mu)=1$.

Consider now the following decomposition of $\mathbf{S}_2$, the set of degree two elements in $\mathbf{S}$:

$$\begin{aligned}\mathbf{S}'_2 &= \{x_{i_1}x_{i_2} \in \mathbf{S}_2 \mid i_1 \leq a+d\}\\ \mathbf{S}''_2 &= \{x_{i_1}x_{i_2} \in \mathbf{S}_2 \mid i_1 > a+d\}.\end{aligned}$$

Then $t' < t''$ for any $t' \in \mathbf{S}'_2, t'' \in \mathbf{S}''_2$ and the defining relations of $B_g(\tau, \theta)$ have the form

$$t^{\mu}_k \;=\; \sum_{s=1}^{a+d} c^{\mu,k}_s t'^{\mu}_s \qquad t'^{\mu}_s \in \mathbf{S}'_2 \;, t^{\mu}_k \in \mathbf{S}''_2 \tag{48}$$

With $t^{\mu}_k = x_{a+d+k}x_{\alpha(\mu,a+d+k)}$, $t'^{\mu}_s = x_s x_{\alpha(\mu,s)}$ and $c^{\mu,k}_s = -\frac{\tilde{v}^{\mu,k}_s}{\tilde{v}^{\mu,k}_{a+d+k}}$. In particular $t'^{\mu}_1 < t'^{\mu}_2 < ... < t'^{\mu}_{a+d} < t^{\mu}_k$.

Denote by $\mathbf{S}_n$ the set of elements of degree n in $\mathbf{S}$ and set

$$\mathbf{S}'_n \;=\; \{x_{i_1}x_{i_2}...x_{i_n} \in \mathbf{S}_n \mid i_1, i_2, ..., i_{n-1} \leq a+d\} \tag{49}$$

The set $\mathbf{S}'_n = \cup_n \mathbf{S}'_n$ linearly spans $B_g(\tau, \theta)$ since any element $t = x_{j_1}x_{j_2}...x_{j_n} \in \mathbf{S}_n \setminus \mathbf{S}'_n$ can be expressed by smaller elements modulo $\mathcal{R}_n$.

We want to extract a basis from the set $\mathbf{S}'_n = \cup_n \mathbf{S}'_n$. By cardinality conditions we can see that this set is redundant. The cardinality of $\mathbf{S}'_n$ is $(a+d)^{n-1}c$ while the n-th graded component of $B_g(\tau, \theta)$ has dimension $deg(g^n) = dim\mathcal{H}_{g^n} < (a+d)^{n-1}c$. The Hilbert series for $B_g(\tau, \theta)$ is given by (30):

$$h_{B_g(\tau,\theta)}(t) = \frac{1 + (c-a-d)\ t + t^2}{1 - (a+d)\ t + t^2}.$$

Starting with the linear generating set $\mathbf{S}'$ and the Hilbert series $h_{B_g(\tau,\theta)}(t)$ we can find a linear basis for each graded piece of $B_g(\tau, \theta)$ by extracting a minimal set of linearly dependent elements from each $\mathbf{S}'_n$. The redundant elements will correspond to leading terms of elements of the ideal $\mathcal{R}$. We will obtain a basis for the semigroup ideal $T(\mathcal{R}) = \{T(f) \mid f \in \mathcal{R}\}$ by computing a Gröbner basis for the ideal $\mathcal{R}$. We state the main parts of this formalism in our context (c.f. [**22**]).

In general, given a two sided ideal $\mathcal{I}$ of $\mathbb{K}\langle x_1, ..., x_c\rangle$ we can consider the semigroup ideal formed by its leading terms $T(\mathcal{I}) \subset \mathbf{S}$ and its complement on $\mathbf{S}$, $O(\mathcal{I}) := \mathbf{S} \setminus T(\mathcal{I})$. Then we have:

- $\mathbb{K}\langle x_1, ..., x_c\rangle = \mathcal{I} \oplus Span_{\mathbb{K}}(O(\mathcal{I}))$
- $\mathbb{K}\langle x_1, ..., x_c\rangle / \mathcal{I} \simeq Span_{\mathbb{K}}(O(\mathcal{I}))$

DEFINITION 0.3. A generating set $G \subset \mathcal{I}$ such that the semigroup ideal $T(G)$ generated by $\{T(g)|g \in G\}$ coincides with $T(\mathcal{I})$ is called a Gröbner basis for $\mathcal{I}$.

For the ideal $\mathcal{R}$ we have that $O(\mathcal{R})_2 = \mathbf{S}'_2$ and $O(\mathcal{R})_n \subset \mathbf{S}'_n$. Given a Gröbner basis G for $\mathcal{R}$ the set of redundant elements of $\mathbf{S}'_n$ will be $T(G)_n \cap \mathbf{S}'_n$. Buchberger's algorithm provides a way to find a Gröbner basis G for $\mathcal{R}$ starting with the set of generators $\{f_k^\mu\}$.

REMARK 0.4. Since $B_g(\tau,\theta)$ has exponential growth we should not expect $\mathcal{R}$ to have a finite Gröbner basis. Still an infinite Gröbner basis G exist and Buchberger's algorithm provides a way to compute its elements of a given degree.

Buchberger's algorithm is a infinite dimensional analog of Gaussian reduction. First one reduces the problem of finding a Gröbner basis to a linear algebra problem. Then the computation of the corresponding linear basis is done by a combinatorial manipulation of the vectors that takes into account order of the generators in $\mathbf{S}$. Buchberger's algorithm is explained below, the implementation of the algorithm is done in Appendix B.

Let V be a linear subspace of $\mathbb{K}\langle x_1, ..., x_c\rangle$. A linearly generating set B of V such that $T(V) = T(B)$ is called a Gauss generating set. A linearly basis B of V such that $T(V) = T(B)$ is called a Gauss Basis. G is a Gröbner basis of a two sided ideal $\mathcal{I}$ of $\mathbb{K}\langle x_1, ..., x_c\rangle$ if and only if $\mathcal{G} = \{lfr \mid l, r \in S;\ f \in G\}$ is a Gauss basis for $\mathcal{I}$.

Assume we start with a ordered set of generators $G' = \{f_1, ..., f_s\}$ of the ideal $\mathcal{I}$. Then $\mathcal{G}' = \{lfr \mid l, r \in \mathbf{S};\ f \in G\}$ is an ordered generating set for $\mathcal{I}$. From $\mathcal{G}'$ we can extract a canonical Gauss basis for $\mathcal{I}$ by taking a set of linearly independent elements $\mathcal{G}$ such that:

- $T(\mathcal{G}) = T(\mathcal{G}')$
- If $v \in \mathcal{G}$, $w \in \mathcal{G}'$ are such that $T(v) = T(w)$ then $v \leq w$

One has to consider the fact that different elements in $\mathcal{G}'$ may have the same leading term. This self obstructions have to be taken into account inductively. Given $f \in \mathbb{K}\langle x_1, ..., x_c\rangle$ we say that h is a *n*ormal form of f with respect to G' if:

- $f - h \in \mathcal{I}$.
- Either $h = 0$ or $T(h) \notin T(G')$.

Given $j = 1, ..., s$ for each pair $(l, r) \in \mathbf{S} \times \mathbf{S}$ such that $lT(g_j)r \in (T(g_1), T(g_2), ..., T(g_{j-1}))$ or $lT(g_j) = T(g_j)r$ the normal form of the element $lT(g_j)r$ with respect to G' must be added to the set G'. Choosing at each stage a minimal iredundant set of pairs (l, r) and adding the corresponding normal forms to G' we get a Gröbner basis G of $\mathcal{I}$ (See Appendix B).

When we apply this procedure to $\{\tilde{f}_k^\mu\}$ we can algorithmically compute for each degree n a set of obstructions in

$$\{lfr \mid l, r \in \mathbf{S};\ f \in \mathcal{R}\ , deg(lfr) = n\}$$

together with their normal forms having degree n. The leading terms of this normal forms correspond to the elements of $\mathbf{S}'_n$ which are redundant.

EXAMPLE 0.5. To illustrate the above ideas we will compute the linear basis of the degree three part of the ring considered in Example 6.2 above. We start with the following basis for the ideal of relations $\mathcal{R}$:

$$\begin{aligned}
\hat{f}_1^1 &= x_5x_1 + \frac{1}{\tilde{v}_5^{1,1}}(\tilde{v}_4^{1,1}x_4x_2 + \tilde{v}_3^{1,1}x_3x_3 + \tilde{v}_2^{1,1}x_2x_4 + \tilde{v}_1^{1,1}x_1x_5)\\
\hat{f}_1^2 &= x_5x_2 + \frac{1}{\tilde{v}_5^{2,1}}(\tilde{v}_4^{2,1}x_4x_3 + \tilde{v}_3^{2,1}x_3x_4 + \tilde{v}_2^{2,1}x_2x_5 + \tilde{v}_1^{2,1}x_1x_6)\\
\hat{f}_1^3 &= x_5x_3 + \frac{1}{\tilde{v}_5^{3,1}}(\tilde{v}_4^{3,1}x_4x_4 + \tilde{v}_3^{3,1}x_3x_5 + \tilde{v}_2^{3,1}x_2x_6 + \tilde{v}_1^{3,1}x_1x_1)\\
\hat{f}_1^4 &= x_5x_4 + \frac{1}{\tilde{v}_5^{4,1}}(\tilde{v}_4^{4,1}x_4x_5 + \tilde{v}_3^{4,1}x_3x_6 + \tilde{v}_2^{4,1}x_2x_1 + \tilde{v}_1^{4,1}x_1x_2)\\
\hat{f}_1^5 &= x_5x_5 + \frac{1}{\tilde{v}_5^{5,1}}(\tilde{v}_4^{5,1}x_4x_6 + \tilde{v}_3^{5,1}x_3x_1 + \tilde{v}_2^{5,1}x_2x_2 + \tilde{v}_1^{5,1}x_1x_3)\\
\hat{f}_1^6 &= x_5x_6 + \frac{1}{\tilde{v}_5^{6,1}}(\tilde{v}_4^{6,1}x_4x_1 + \tilde{v}_3^{6,1}x_3x_2 + \tilde{v}_2^{6,1}x_2x_3 + \tilde{v}_1^{6,1}x_1x_4)\\
\hat{f}_2^1 &= x_6x_1 + \frac{1}{\tilde{v}_5^{1,2}}(\tilde{v}_4^{1,2}x_4x_3 + \tilde{v}_3^{1,2}x_3x_4 + \tilde{v}_2^{1,2}x_2x_5 + \tilde{v}_1^{1,2}x_1x_6)\\
\hat{f}_2^2 &= x_6x_2 + \frac{1}{\tilde{v}_5^{2,2}}(\tilde{v}_4^{2,2}x_4x_4 + \tilde{v}_3^{2,2}x_3x_5 + \tilde{v}_2^{2,2}x_2x_6 + \tilde{v}_1^{2,2}x_1x_1)\\
\hat{f}_2^3 &= x_6x_3 + \frac{1}{\tilde{v}_5^{3,2}}(\tilde{v}_4^{3,2}x_4x_5 + \tilde{v}_3^{3,2}x_3x_6 + \tilde{v}_2^{3,2}x_2x_1 + \tilde{v}_1^{3,2}x_1x_2)\\
\hat{f}_2^4 &= x_6x_4 + \frac{1}{\tilde{v}_5^{4,2}}(\tilde{v}_4^{4,2}x_4x_2 + \tilde{v}_3^{4,2}x_3x_3 + \tilde{v}_2^{4,2}x_2x_4 + \tilde{v}_1^{4,2}x_1x_5)\\
\hat{f}_2^5 &= x_6x_5 + \frac{1}{\tilde{v}_5^{5,2}}(\tilde{v}_4^{5,2}x_4x_1 + \tilde{v}_3^{5,2}x_3x_2 + \tilde{v}_2^{5,2}x_2x_3 + \tilde{v}_1^{5,2}x_1x_4)\\
\hat{f}_2^6 &= x_6x_6 + \frac{1}{\tilde{v}_5^{6,2}}(\tilde{v}_4^{6,2}x_4x_2 + \tilde{v}_3^{6,2}x_3x_3 + \tilde{v}_2^{6,2}x_2x_4 + \tilde{v}_1^{6,2}x_1x_5)
\end{aligned}$$

From the relations one immediately sees that

$$\mathbf{S}'_2 \quad = \quad \{x_{i_1}x_{i_2} \in \mathbf{S}_2 \mid i_1 \leq 4\}$$

spans $\mathbb{K}\langle x_1, ..., x_c\rangle_2$ modulo $\mathcal{R}$. Equivalently $\mathbf{S}'_2$ is a linear basis for $B_g(\tau, \theta)_2$. Also, from the discussion above we see that

$$\mathbf{S}'_3 \quad = \quad \{x_{i_1}x_{i_2}x_{i_3} \in \mathbf{S}_3 \mid i_1, i_2 \leq 4\}$$

spans $\mathbb{K}\langle x_1, ..., x_c\rangle_3$ modulo $\mathcal{R}$. $|\mathbf{S}'_3| = (a+d)^2c = 96$ while $B_g(\tau, \theta)_3$ has dimension $((a+d)^2-1)c = 90$ thus there are 6 redundant elements in $\mathbf{S}'_3$.

For each one of the relations we look at the obstructions coming from lower relations:

(1) For $\hat{f}_1^5$:

$$\begin{array}{lcl} T(\hat{f}_1^5)x_1 = x_1T(\hat{f}_1^1) = x_5x_5x_1 & ; & T(\hat{f}_1^5)x_2 = x_5T(\hat{f}_1^2) = x_5x_5x_2 \\ T(\hat{f}_1^5)x_3 = x_5T(\hat{f}_1^3) = x_5x_5x_3 & ; & T(\hat{f}_1^5)x_4 = x_5T(\hat{f}_1^4) = x_5x_5x_3 \\ T(\hat{f}_1^5)x_5 = x_5T(\hat{f}_1^5) = x_5x_5x_5 & & \end{array}$$

(2) For $\hat{f}_1^6$:

$$T(\hat{f}_1^5)x_6 = x_5T(\hat{f}_1^6) \quad = \quad x_5x_5x_6$$

(3) For $\hat{f}_2^1$:

$$T(\hat{f}_1^5)x_1 \quad = x_5T(\hat{f}_1^1) \quad = x_5x_6x_1$$

(4) For $\hat{f}_2^2$:

$$T(\hat{f}_1^5)x_2 \quad = x_5T(\hat{f}_2^2) \quad = x_5x_6x_2$$

(5) For $\hat{f}_2^3$:

$$T(\hat{f}_1^5)x_3 \quad = x_5T(\hat{f}_2^3) \quad = x_5x_6x_3$$

(6) For $\hat{f}_2^4$:

$$T(\hat{f}_1^5)x_4 \quad = x_5T(\hat{f}_2^4) \quad = x_5x_6x_4$$

(7) For $\hat{f}_2^5$:

$$\begin{array}{lcl} T(\hat{f}_2^5)x_1 = x_6T(\hat{f}_1^1) = x_6x_5x_1 & ; & T(\hat{f}_2^5)x_2 = x_6T(\hat{f}_1^2) = x_6x_5x_2 \\ T(\hat{f}_2^5)x_3 = x_6T(\hat{f}_1^3) = x_6x_5x_3 & ; & T(\hat{f}_2^5)x_4 = x_6T(\hat{f}_1^4) = x_6x_5x_4 \\ T(\hat{f}_2^5)x_5 = x_6T(\hat{f}_1^5) = x_6x_5x_5 & ; & T(\hat{f}_2^5)x_6 = x_6T(\hat{f}_1^6) = x_6x_5x_6 \\ T(\hat{f}_1^6)x_5 = x_5T(\hat{f}_2^5) = x_5x_6x_5. & & \end{array}$$

(8) For $\hat{f}_2^6$:

$$\begin{array}{lcl} T(\hat{f}_2^6)x_1 = x_6T(\hat{f}_2^1) = x_6x_6x_1 & ; & T(\hat{f}_2^6)x_2 = x_6T(\hat{f}_2^2) = x_6x_6x_2 \\ T(\hat{f}_2^6)x_3 = x_6T(\hat{f}_2^3) = x_6x_6x_3 & ; & T(\hat{f}_2^6)x_4 = x_6T(\hat{f}_2^4) = x_6x_6x_4 \\ T(\hat{f}_2^6)x_5 = x_6T(\hat{f}_2^5) = x_6x_6x_5 & ; & T(\hat{f}_2^6)x_6 = x_6T(\hat{f}_2^6) = x_6x_6x_6 \\ T(\hat{f}_1^6)x_5 = x_5T(\hat{f}_2^6) = x_5x_6x_6. & & \end{array}$$

For each obstruction of $\hat{f}_k^\mu$; $T(\hat{f}_k^\mu)x_{i_3} = x_{i_1}T(\hat{f}_{k'}^{\mu'}) = x_{i_1}x_{i_2}x_{i_3} \in \mathbf{S}_3$ we must look at the normal form of:

$$\hat{f}_k^\mu x_{i_3} - x_{i_1}\hat{f}_{k'}^{\mu'}$$

with respect to the ideal generated by the relations which are lower than $\hat{f}_k^\mu$. The following elements give nontrivial normal forms:

$$\begin{array}{lcl} \hat{f}_2^5x_1 - x_6\hat{f}_1^1 & ; & \hat{f}_2^5x_2 - x_6\hat{f}_1^2 \\ \hat{f}_2^5x_3 - x_6\hat{f}_1^3 & ; & \hat{f}_2^5x_4 - x_6\hat{f}_1^4 \\ \hat{f}_2^5x_5 - x_6\hat{f}_1^5 & ; & \hat{f}_2^5x_6 - x_6\hat{f}_1^6. \end{array}$$

The leading terms of the corresponding normal forms must be removed from $\mathbf{S}_3'$. These terms are:

$$\begin{array}{lll} x_4x_1x_1, & x_4x_1x_2, & x_4x_1x_3, \\ x_4x_1x_4, & x_4x_1x_5, & x_4x_1x_6. \end{array}$$

Thus a linear basis for $B_g(\tau,\theta)_3$ is given by:

$$\{x_{i_1}x_{i_2}x_{i_3} \in \mathbf{S}_3 \mid i_1, i_2 \le 4,\ x_{i_1}x_{i_2} \ne x_1x_4\}. \tag{50}$$

CHAPTER 4

Modularity properties of $B_g(\tau, \theta)$

Let $g \in Sl_2(\mathbb{Z})$ satisfy (27) and (28) and let θ be the corresponding quadratic irrationality. In this section we consider $B_g(\tau, \theta)$ as a family of algebras parametrized by τ. Theta constants can be viewed as modular forms of weight $\frac{1}{2}$ and certain level. We will use the fact that the relations defining $B_g(\tau, \theta)$ are given in terms of theta constants to define a presentation in which the defining relations have coefficients which are modular functions of the complex structure τ. We will then exploit the modularity of $B_g(\tau, \theta)$ in order to define quadratic algebras associated to $\mathbb{T}_\theta$ which do not depend on a particular choice of a complex structure.

1. A presentation in terms of modular functions

Lets start by considering the presentation of $B_g(\tau, \theta)$ given in Theorem 5.9 of Chapter 1. Each one of the coefficients $v_i^{\mu,k}$ in the defining relations f_k^μ is given by one of the functions $F^{g,\mu}_{i_1,i_2,\ldots,i_{a+d}}(\tau)$. The function $F^{g,\mu}_{i_1,i_2,\ldots,i_{a+d}}(\tau)$ was defined as a determinant of a matrix consisting of theta constants in $\tau' = -l\tau$ therefore it belongs to the ring generated by this functions. Being more precise, let l be a positive integer and let $\mathfrak{C}_l$ be the ring generated over $\mathbb{C}$ by the functions $\vartheta_{r,s}$ with $(r, s) \in (\frac{1}{l}\mathbb{Z})^2$. Since $\vartheta_{r,s}$ depends on the characteristics (r, s) only modulo integers we see that $\mathfrak{C}_l$ is of finite type over $\mathbb{C}$; it is generated by $\{\vartheta_{r_i,s_i}\}$ where (r_i, s_i) runs though a set of representatives in $(\frac{1}{r}\mathbb{Z}/\mathbb{Z})^2$. If we take $l = c(a + d)$ and view the coefficients as functions of $\tau' = -l\tau$ then our algebras will be naturally defined over $\mathfrak{C}_l$. $\mathfrak{C}_l$ becomes a graded ring by assigning to each $\vartheta_{r,s} \in \mathfrak{C}_l$ weight $\frac{1}{2}$.

The rings $\mathfrak{C}_l$ of theta constants were studied by Igusa in [**13**] and [**14**]. The main results relate these rings to rings of modular forms for even values of l.

For a positive integer n we define

$$\Gamma_{n,2n} = \{\gamma = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in SL_2(\mathbb{Z}) \mid \gamma \equiv 1_2 \mod n; xy \equiv zw \equiv 0 \mod 2n\}$$

If we denote by Γ_N the principal congruence subgroup of level N in $Sl_2(\mathbb{Z})$ then we see that

$$\Gamma_{2n} \subset \Gamma_{n,2n} \subset \Gamma_n \tag{51}$$

In particular $\Gamma_{n,2n}$ is a congruence subgroup of $Sl_2(\mathbb{Z})$.

Let $\mathfrak{B}_l := \mathfrak{C}_l^{(2)}$ be the subring of elements with homogeneous components of even degree i.e. the ring generated over $\mathbb{C}$ by the double products $\vartheta_{r,s}\vartheta_{r',s'}$ with $(r,s),(r',s')$ in $(\frac{1}{l}\mathbb{Z})^2$. The transformation law for theta constants (78) shows that $Sl_2(\mathbb{Z})$ acts on $\mathfrak{B}_l$ thus one gets a homomorphism from $Sl_2(\mathbb{Z})$ to the group of degree preserving automorphisms of the algebra $\mathfrak{B}_l$. If l is even the kernel of this morphism is precisely $\Gamma_{l,2l^2}$. Thus, the normal subgroup of $Sl_2(\mathbb{Z})$ consisting of elements which keep $\mathfrak{B}_l$ element wise invariant is $\Gamma_{l,2l^2}$. It follows then that $\tau' \mapsto \vartheta_{r,s}(\tau')\vartheta_{r',s'}(\tau')$ is a modular form of weight 1 and level $\Gamma_{l^2,2l^2}$. We estate this classical result together with a theorem due to Igusa ([**13, 14**]).

Theorem 1.1. *Let l be a positive even integer. Given a congruence subgroup Γ of $Sl_2(\mathbb{Z})$ denote by $\mathfrak{G}(\Gamma)$ the ring of all holomorphic modular forms of level Γ. Then:*

1. *$\mathfrak{B}_l$ is a subring of $\mathfrak{G}(\Gamma_{l,2l^2})$.*
2. *The integral closure of $\mathfrak{B}_l$ in its field of fractions is $\mathfrak{G}(\Gamma_{l,2l^2})$.*

In order to study the behavior of the coefficients of the defining relations of $B_g(\tau,\theta)$ it is useful at this point to make some remarks about the structure of $\Gamma_{l,2l^2}$. They follow from the general theory of discrete subgroups of $Gl_2^+(\mathbb{R})$ (c.f. [**36**]). As above we will consider the action of subgroups of $Gl_2(\mathbb{C})$ on $\mathbb{C}\cup\{\infty\}$ by fractional linear transformations. In particular $Gl_2^+(\mathbb{R})$ is identified with the group of holomorphic automorphisms of the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} | Im(\tau) > 0\}$.

Two subgroups of a group G are said to be commensurable if their intersection has finite index in both of them. If two discrete subgroups Γ and Γ' of $Sl_2(\mathbb{R})$ are commensurable then their sets of cusps in $\mathbb{C}\cup\{\infty\}$ are the same i.e. the set of points which are fixed points of some parabolic transformation in Γ coincides with the corresponding set for Γ'. In particular, since $\Gamma_{n,2n}$ is a congruence subgroup of $Sl_2(\mathbb{Z})$ it is commensurable with it and so the set of cusp of $\Gamma_{n,2n}$ is $\mathbb{Q}\cup\{\infty\}$. Denote by $\mathbb{H}^* = \mathbb{H}\cup\mathbb{Q}\cup\{\infty\}$ the upper half plane with the cusps added. It follows from commensurability with $Sl_2(\mathbb{Z})$ that the quotient

$$X_{\Gamma_{n,2n}} = \mathbb{H}^*/\Gamma_{n,2n} \tag{52}$$

is a compact Riemann surface. In the terminology of Shimura, $\Gamma_{n,2n}$ is a Fuchsian group of the first kind (c.f. [**36**]).

Notation 1.2. *On what follows we will change the sign on our complex structure and take τ with $Im(\tau) > 0$. τ' is then given by $\tau' = l\tau$.*

In order to study the modularity of the defining relations of $B_g(\tau,\theta)$ as functions of τ we have to take care of the scaling by $l = c(a+d)$. For this we introduce some notation.

Definition 1.3. let m be a even positive integer, define

$$\Gamma^{[m]}_{n,2n} = \Gamma_{1,2} \cap \{\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma_{n,2n} \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}\} \tag{53}$$

This subgroups are the levels for our relations:

Theorem 1.4. *Let $g \in Sl_2(\mathbb{Z})$ satisfy (27) and (28) and assume $l = c(a+d)$ is even. Denote by $w = \lfloor \frac{a+d+1}{2} \rfloor$ the integer part of $\frac{a+d+1}{2}$. Let $\mu \in \{1, 2, ..., c\}$, $k \in \{1, ..., c-a-d\}$ and let $\tau \mapsto v^{\mu,k}(\tau)$ be given as in Theorem 5.9. Define $\hat{v}^{\mu,k} = \hat{v}^{\mu,k}(\tau) \in \mathbb{C}^c$ by*

$$\hat{v}^{\mu,k}(\tau) = v^{\mu,k}(\tau) \quad \text{if } a+d \text{ is even}$$
$$\hat{v}^{\mu,k}(\tau) = \vartheta(l\tau) v^{\mu,k}(\tau) \quad \text{if } a+d \text{ is odd.}$$

Then:

1. *The algebra $B_g(\tau, \theta)$ is generated by elements $x_1, ..., x_c$ of degree 1 subject to relations $\hat{f}^{\mu}_k = 0$ where:*
$$\hat{f}^{\mu}_k = \hat{v}^{\mu,k}_1 x_{\alpha(\mu,1)} x_1 + ... + \hat{v}^{\mu,k}_c x_{\alpha(\mu,c)} x_c$$
2. *Each one of the functions $\tau \mapsto \hat{v}^{\mu,k}_j(\tau)$ is a modular form of weight w and level $\Gamma^{[l]}_{l^2,2l^2}$.*

Proof. The first part of the theorem follows from Theorem 5.9.

For the second part note that each $\hat{v}^{\mu,k}_j(\tau)$ has the form:

$$h^{\mu,k}_j(\tau') = \vartheta(\tau')^{\epsilon} \sum_{\sigma \in S_{a+d}} \mathrm{sgn}(\sigma) \prod_{k=1}^{a+d} \vartheta_{q(\mu) - \frac{di_k}{c} - \frac{\sigma(k)}{(a+d)}, 0}(\tau') \tag{54}$$

for some $i_1, ..., i_{a+d} \in \{1, ..., a+d\}$ where $\epsilon = 0, 1$.

The factor on the right just makes the number of theta constants in the products even so, as function of $\tau' = l\tau$, $h^{\mu,k}_j$ is a homogeneous element of $\mathfrak{B}_l$ of weight w. In particular, $h^{\mu,k}_j$ is a modular form of weight w and level $\Gamma_{l,2l^2}$. Since $\hat{v}^{\mu,k}_j(\tau) = h^{\mu,k}_j(l\tau)$ we see that whenever

$$\begin{pmatrix} x & ly \\ l^{-1}z & w \end{pmatrix} \in \Gamma_{l,2l^2}$$

we have

$$\hat{v}^{\mu,k}_j(\frac{x\tau + y}{z\tau + w}) = h^{\mu,k}_j(l\frac{x\tau + y}{z\tau + w}) = (z\tau + w)^k h^{\mu,k}_j(l\tau) = (z\tau + w)^k \hat{v}^{\mu,k}_j(\tau).$$

Therefore $\tau \mapsto \hat{v}^{\mu,k}_j(\tau)$ is a modular form of weight k and level

$$\Gamma_{1,2} \cap \{\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma_{l,2l^2} \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}\}$$

□

REMARK 1.5. If l is odd we only have modularity "up to a sign". This fact is a consequence of the appearance of an eight root of unity in the functional equation of ϑ.

The above theorem will allow us to average the values of the coefficients of the defining relations of $B_g(\tau,\theta)$. For this purpose we are interested in determining whether the modular forms in the defining relations of $B_g(\tau,\theta)$ are modular forms of cusp type; that is, their Fourier expansions at each one of the cusp should have constant term equal to 0. In what follows we will show that this is the case.

PROPOSITION 1.6. *For* $\mu, j \in \{1,2,...,c\}$ *and* $k \in \{1,...,c-a-d\}$ *let* $\hat{v}_j^{\mu,k}(\tau)$ *be defined as in Theorem 1.4. Then* $\hat{v}_j^{\mu,k}(\tau)$ *is modular form of cusp type for* $\Gamma^{[l]}_{l^2,2l^2}$.

PROOF. By conjugating with $\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}$ it is enough to prove that $h_j^{\mu,k}(\tau')$ in (54) is a cusp form for $\Gamma_{l,2l^2}$.

We must look at the Fourier series expansions around the cusps of each $h_j^{\mu,k}(\tau')$. Since $h_j^{\mu,k}(\tau')$ is given as a homogeneous combination of products of theta constants we are interested in the behavior of ϑ_r around the cusps. Moreover, taking into account that the set of cusp forms is an ideal of the graded ring $\mathfrak{G}(\Gamma_{l,2l^2})$ of holomorphic modular forms we see that once we show that in each term of $h_j^{\mu,k}(\tau')$ some factor is a cusp form the result will follow.

Let l be a positive even integer and let $r \in \frac{\mathbb{Z}}{l}$. First we look at the behavior of ϑ_r at ∞. We can assume $r = \frac{k}{l}$ with $k \in \{0,...l-1\}$. For $\tau' \in \mathbb{H}$ let $\tilde{q}_{\tau'} = \exp(\frac{\pi\imath\tau'}{l^2})$. Then the series defining ϑ_r is given by:

$$\begin{aligned} \vartheta_r(\tau') &= \sum_{n\in\mathbb{Z}} \tilde{q}_{\tau'}^{(nl+rl)^2} \\ &= \sum_{m\geq 0} a_m \tilde{q}_{\tau'}^{m} \end{aligned}$$

where in the last sum the coefficient a_m takes the values 1, 2 or 0 depending on whether m is the square of one integer of the form $(nl+rl)^2$ for some $n \in \mathbb{Z}$, for two integers of this form or for none. Since we choose $r = \frac{k}{l}$ with $k \in \{0,...l-1\}$ for the constant term we have that $a_0 \neq 0$ only if $nl + rl = 0$ for some $n \in \mathbb{Z}$. This can only happen if $n = r = 0$. Therefore the constant term in the $\tilde{q}_{\tau'}$ series expansion of any product of theta constants $\prod_{i=1}^{s} \vartheta_{r_i}$ will vanish provided that at least one of the factors has a nonzero characteristic $r_j \neq 0$. Finally, since any cusp can be carried to ∞ by an element in Sl_2 the result follows. □

Example 1.7. Consider Example 6.2 of Chapter 1:

$$g = \begin{pmatrix} 5 & -1 \\ 6 & -1 \end{pmatrix} \quad ; \quad \theta = \frac{3-\sqrt{3}}{6}$$

In this case $B_g(\theta,\tau)$ is a quadratic algebra with 6 generators of degree 1 and 12 quadratic relations. After choosing a basis the multiplication map $m : \mathcal{H}_g \otimes \mathcal{H}_g \to \mathcal{H}_{g^2}$ is represented by a 36×24 matrix whose coefficients belong to $\mathfrak{C}_{24}$ when viewed as functions of $\tau' = 24\tau$. The corresponding functions of τ determining the coefficients of the relations are the minors of each $M(\mu)$, $\mu \in \{1,...,6\}$. Each ordered 4-tuple $i_1, i_2, i_3, i_4 \in \{1,2,3,4,5,6\}$ determines a weight 2 modular form of τ' which belongs to $\mathfrak{B}_4$:

$$F^{g,\mu}_{i_1,i_2,i_3,i_4}(\tau) \quad = \quad \sum_{\sigma \in S_4} \operatorname{sgn}(\sigma) \prod_{k=1}^{4} \vartheta_{\frac{6-5\mu}{24}-\frac{di_k}{6}-\frac{\sigma(k)}{4}}(24\tau)$$

Considered as functions of τ each $F^{g,\mu}_{i_1,i_2,i_3,i_4}$ is a modular form of weight 2 and level $\Gamma^{[24]}_{24^2,2(24^2)}$. In particular, each $F^{g,\mu}_{i_1,i_2,i_3,i_4}(\tau)$ defines a differential of the first kind in the modular curve $X_{\Gamma^{[24]}_{24^2,2(24^2)}}$.

Example 1.8. Consider Example 6.5 of Chapter 1:

$$g = \begin{pmatrix} 4 & 1 \\ 15 & 4 \end{pmatrix}, \qquad \theta = -\frac{\sqrt{15}}{15}$$

In this case $B_g(\theta,\tau)$ is a quadratic algebra with 15 generators of degree 1 and 105 quadratic relations. After choosing a basis the multiplication map $m : \mathcal{H}_g \otimes \mathcal{H}_g \to \mathcal{H}_{g^2}$ is represented by a 120×225 matrix whose coefficients belong to $\mathfrak{C}_{120}$ when viewed as functions of $\tau' = 120\tau$. The corresponding functions of τ determining the coefficients of the relations are the minors of each $M(\mu)$, $\mu \in \{1,...,15\}$. Each ordered 8-tuple $i_1, \ldots, i_8 \in \{1, \ldots, 15\}$ determines a weight 4 modular form of τ' which belongs to $\mathfrak{B}_{120}$:

$$F^{g,\mu}_{i_1,\ldots,i_8}(\tau) \quad = \quad \sum_{\sigma \in S_8} \operatorname{sgn}(\sigma) \prod_{k=1}^{8} \vartheta_{\frac{15+31\mu}{120}-\frac{4i_k}{15}-\frac{\sigma(k)}{8}}(120\tau)$$

Each $F^{g,\mu}_{i_1,\ldots,i_8}$ is a modular form of weight 4 and level $\Gamma^{[120]}_{120^2,2(120^2)}$.

Example 1.9. Consider Example 6.7 of Chapter 1:

$$g = \begin{pmatrix} 6 & 1 \\ 35 & 6 \end{pmatrix}, \qquad \theta = -\frac{\sqrt{35}}{35}$$

In this case $B_g(\theta,\tau)$ is a quadratic algebra with 35 generators of degree 1 and 805 quadratic relations. After choosing a basis the multiplication map $m : \mathcal{H}_g \otimes \mathcal{H}_g \to \mathcal{H}_{g^2}$ is represented by a 420×1225 matrix whose coefficients belong to $\mathfrak{C}_{120}$ when viewed as functions of $\tau' = 120\tau$. The corresponding functions of τ determining the coefficients of the relations are the minors of each $M(\mu)$, $\mu \in \{1, ..., 35\}$. Each ordered 12-tuple $i_1, \ldots, i_{12} \in \{1, \ldots, 35\}$ determines a weight 6 modular form of τ' which belongs to $\mathfrak{B}_{420}$:

$$F^{g,\mu}_{i_1,\ldots,i_{12}}(\tau) \quad = \quad \sum_{\sigma \in S_{12}} \operatorname{sgn}(\sigma) \prod_{k=1}^{12} \vartheta_{\frac{20+71\mu}{120} - \frac{6i_k}{35} - \frac{\sigma(k)}{12}}(420\tau)$$

Each $F^{g,\mu}_{i_1,\ldots,i_{12}}$ is a modular form of weight 6 and level $\Gamma^{[420]}_{420^2,2(420^2)}$.

2. Modular symbols and averaged algebras

In this section we use the results about modularity obtained in the Section in order to define algebras which do not depend on a particular choice of the complex structure τ. Fix g, l and θ as in the previous sections. Let $\Gamma = \Gamma^{[l]}_{l^2,2l^2}$, by Theorem 1.4 one can take a presentation of $B_g(\tau,\theta)$ in which each one of the coefficients $v(\tau)$ in the defining relations is a modular form for Γ. If $v(\tau)$ is a modular form of even weight $w = 2r$ we can consider it as a r-fold differential on $\mathbb{H}$ invariant under Γ. That is v is a function in $(dz)^{-r}((\Omega^1_{\mathbb{H}})^{\otimes r})^{\Gamma}$. A Γ invariant k-fold differential can be pushed down to a differential form in the $(2r-2)$ fibered power of the universal curve over X_Γ. The corresponding integrals along homology classes can be realized as values of line integrals along geodesics in $\mathbb{H}$. This formalism was developed by Manin in [**19**] for modular forms of weight 2 and extended to arbitrary weights by Shokurov in [**37**].

Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$. Let k be a positive integer and consider the action of the crossed product $\Gamma \ltimes (\mathbb{Z}^k \times \mathbb{Z}^k)$ on $\mathbb{H} \times \mathbb{C}^k$ given by:

$$(\gamma; n, m) : (\tau, z) \mapsto (\gamma\tau, \frac{z + \tau n + m}{C\tau + D}) \tag{55}$$

where $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma; n, m \in \mathbb{Z}^k; \tau \in \mathbb{H}$ and $z \in \mathbb{C}^k$.

The quotient $\Gamma \ltimes (\mathbb{Z}^k \times \mathbb{Z}^k) \backslash \mathbb{H} \times \mathbb{C}^k$ admits a canonical smooth compactification $\mathcal{Z}^k_\Gamma$ called the Kuga-Sato variety. Let $\delta, \rho \in \mathbb{P}^1(\mathbb{Q})$ be two cusps and let $n, m \in \mathbb{Z}^k$. This data defines a homology class $\{\delta, \rho, n, m\}_\Gamma \in H_{k+1}(\mathcal{Z}^k_\Gamma)$ called a modular symbol (c.f. [**37**]).

Now let $v(\tau)$ be a cusp modular form of even weight $w = 2r$ and level Γ. Then $\omega = v(\tau) d\tau \wedge d\zeta \wedge d\zeta_1 \wedge ... \wedge d\zeta_{w-2}$ is a $\Gamma \times \mathbb{Z}^{w-2} \times \mathbb{Z}^{w-2}$ invariant holomorphic volume form on $\mathbb{H} \times \mathbb{C}^{w-2}$ so it can be pushed down to a holomorphic volume form $\hat{\omega}$ in the Kuga-Sato variety $\mathcal{Z}^{w-2}_\Gamma$. The pairing of this form with the modular symbol $\{\delta, \rho, n, m\}_\Gamma$ is given by (c.f. [**18**]):

$$\int_{\delta}^{\rho} v(\tau) \sum_{i=1}^{2r-2} (n_i\tau + m_i) d\tau = \int_{\{\delta,\rho,n,m\}_\Gamma} \hat{\omega}. \tag{56}$$

Where the integral on the left is the line integral of the holomorphic differential $v(\tau)\sum_{i=1}^{2r-2}(n_i\tau + m_i)d\tau$ along the geodesic in $\mathbb{H}$ joining δ and ρ.

Definition 2.1. Let g satisfy (27) and (28) with $l = c(a+d)$ and $w = \lfloor\frac{a+d+1}{2}\rfloor$ even. Let $\hat{v}_j^{\mu,k}(\tau)$ be given as in Theorem 1.4. Let $\{\delta,\rho,n,m\}_\Gamma$ be a modular symbol. We define

$$B_g(\theta)\{\delta,\rho,n,m\}_\Gamma, \tag{57}$$

the *averaged homogeneous coordinate ring of* $\mathbb{T}_\theta$ *with respect to* $\{\delta,\rho,n,m\}_\Gamma$ as the quadratic algebra generated by elements $x_1,\dots,x_c$ of degree 1 subject to relations $\xi_k^\mu = 0$, $\mu \in \{1,2,\dots,c\}$ and $k \in \{1,\dots,c-a-d\}$ where

$$\xi_k^\mu = \nu_1^{\mu,k} x_{\alpha(\mu,1)} x_1 + \dots + \nu_c^{\mu,k} x_{\alpha(\mu,c)} x_c \tag{58}$$

with

$$\nu_j^{\mu,k} = \nu_j^{\mu,k}(\{\delta,\rho,n,m\}_\Gamma) := \int_{\delta}^{\rho} \hat{v}_j^{\mu,k}(\tau) \sum_{i=1}^{2r-2} (n_i\tau + m_i) d\tau \tag{59}$$

Example 2.2. Lets look at Example 1.7 in this setting. So we take

$$g = \begin{pmatrix} 5 & -1 \\ 6 & -1 \end{pmatrix} \quad ; \quad \theta = \frac{3-\sqrt{3}}{6}$$

We are working with modular forms of weight 2 for the groups $\Gamma_{24^2,2(24^2)}^{[24]}$ and $\Gamma_{24^2,2(24^2)}$. As remarked in Example 1.7, having weight 2, the modular forms appearing as coefficients of the defining relations of $B_g(\theta,\tau)$ correspond to differentials of the first kind in the modular curve $X_{\Gamma_{24^2,2(24^2)}^{[24]}}$. The corresponding modular symbols

$$\{\delta,\rho\}_{\Gamma_{24^2,2(24^2)}^{[24]}} \in H^1(X_{\Gamma_{24^2,2(24^2)}^{[24]}}, \mathbb{Q})$$

are classical and given any two cusp $\delta,\rho \in \mathbb{Q}\cup\{\infty\}$ we get a averaged homogeneous coordinate ring $B_g(\theta)\{\delta,\rho\}_{\Gamma_{24^2,2(24^2)}^{[24]}}$ whose defining relations have coefficients

$$\nu_j^{\mu,k}(\{\delta,\rho\}_{\Gamma_{24^2,2(24^2)}^{[24]}}) := \int_{\delta}^{\rho} \hat{v}_j^{\mu,k}(\tau) d\tau.$$

3. Limiting modular symbols and averaged algebras

In the definition of the the averaged homogeneous coordinate ring

$$B_g(\theta)\{\delta,\rho,n,m\}_\Gamma$$

we have a non canonical choice coming from the cusps δ and ρ in the limits of the integration (59). We will use the ideas developed by Manin and

Marcolli in [**20**] to get canonical set of cusps associated to θ. The modular symbols corresponding to these cusps will then define a averaged homogeneous coordinate ring canonically associated to $\mathbb{T}_\theta$.

One can not naively replace one of the cusp in $\{\delta,\rho\}_\Gamma$ by a irrational number $\beta\in\mathbb{R}\backslash\mathbb{Q}$ since corresponding integral (59) would then diverge. Still, it makes sense to look at the asymptotic behavior of the integrals along infinite geodesics in $\mathbb{H}$ having a irrational endpoint β. If this asymptotic limit exist it defines a *limiting modular symbol* (c.f. [**20**]):

$$\{\{*,\beta\}\}_\Gamma\in H^1(X_\Gamma,\mathbb{R}). \tag{60}$$

If β is a real quadratic irrationality then the corresponding limiting modular symbol exist and can be computed as a combination of classical modular symbols.

EXAMPLE 3.1. Consider again the situation in Example 1.7. Thus $\theta=\frac{3-\sqrt{3}}{6}$ and g is given by (55). Let

$$\tilde{g}=g^{4l^2}=g^{4(24^2)}$$

Then we have that $\tilde{g}$ is a hyperbolic element of $\Gamma^{[24]}_{24^2,2(24^2)}$ and θ is one of its fixed points. Let also $\tilde{\lambda}^-=(\lambda^-)^{4l^2}>1$ denote the corresponding eigenvalue of $\tilde{g}$. The limiting modular symbol defined by θ is given in this case by (c.f. [**20**]):

$$\{\{*,\theta\}\}_{\Gamma^{[24]}_{24^2,2(24^2)}}=\frac{\{0,\tilde{g}(0)\}_{\Gamma^{[24]}_{24^2,2(24^2)}}}{\log\tilde{\lambda}^-}\in H^1(X_{\Gamma^{[24]}_{24^2,2(24^2)}},\mathbb{R}). \tag{61}$$

We can now integrate along this homology class the modular forms defining $B_g(\tau,\theta)$. Taking

$$\nu_j^{\mu,k}(\theta):=\int_{\{\{*,\theta\}\}_{\Gamma^{[24]}_{24^2,2(24^2)}}}\hat{v}_j^{\mu,k}(\tau)d\tau$$

and imposing the relations $\nu_1^{\mu,k}(\theta)x_{\alpha(\mu,1)}x_1+...+\nu_c^{\mu,k}(\theta)x_{\alpha(\mu,c)}x_c$ on $\mathbb{C}\langle x_1,...,x_c\rangle$ we get a set of relations for a quadratic algebra $B_g(\theta)$ canonically associated with θ and g.

The limiting modular symbol defined by a real quadratic irrationality $\theta\in(0,1)$ can be computed using its continued fraction expansion. Let $\{k_n(\theta)\mid n=1,2,...\}\subset\mathbb{N}$ be the eventually periodic sequence corresponding to the continued fraction expansion of θ. The corresponding convergents are:

$$[k_1(\theta),k_2(\theta),...,k_n(\theta)]=\frac{1}{k_1(\theta)+\frac{1}{k_2(\theta)+\cdots\frac{1}{k_n(\theta)}}}=\frac{p_n(\theta)}{q_n(\theta)}. \tag{62}$$

Let also

$$g_n(\theta) := \begin{pmatrix} p_{n-1}(\theta) & p_n(\theta) \\ q_{n-1}(\theta) & q_n(\theta) \end{pmatrix} \in GL_2(\mathbb{R}) \tag{63}$$

and take $\lambda(\theta) = \lim_{n\to\infty} \frac{\log q_n(\theta)}{n}$. Then the modular symbol defined by θ can be computed by the following formula (c.f.[**21**]):

$$\{\{*, \theta\}\}_\Gamma \;=\; \sum_{n=1}^{m} \frac{\{g_n^{-1}(\theta)0\,,\, g_n^{-1}(\theta)\imath\infty\}_\Gamma}{m\lambda(\theta)}. \tag{64}$$

Where m is the period of the continued fraction expansion $\{k_n(\theta) \mid n = 1, 2, ...\}$.

At present the theory of limiting modular symbols has been developed only for weight $w = 2$. It is expected that an analogous theory for higher weight can be developed. For our purposes it is enough to consider (64) as providing a canonical choice of cusps defining modular symbols over which to average the coefficients of the defining relations of the homogeneous coordinate ring $B_g(\tau, \theta)$:

Definition 3.2. Let g satisfy (27) and (28) with $l = c(a + d)$ and $w = \lfloor \frac{a+d+1}{2} \rfloor$ even. Take $\Gamma = \Gamma^{[l]}_{l^2,2l^2}$ and let $\hat{v}^{\mu,k}_j(\tau)$ be given as in Theorem 1.4. Let also $g_n(\theta), \lambda(\theta)$ and m be as in (64). We define $B_g(\theta)$ the *averaged homogeneous coordinate ring of* $\mathbb{T}_\theta$ as the quadratic algebra generated by elements $x_1, ..., x_c$ of degree 1 subject to relations $\hat{\xi}^\mu_k = 0$, $\mu \in \{1, 2, ..., c\}$ and $k \in \{1, ..., c - a - d\}$ where

$$\hat{\xi}^\mu_k = \hat{\nu}^{\mu,k}_1 x_{\alpha(\mu,1)} x_1 + ... + \hat{\nu}^{\mu,k}_c x_{\alpha(\mu,c)} x_c \tag{65}$$

with

$$\hat{\nu}^{\mu,k}_j \;=\; \frac{1}{m\lambda(\theta)} \sum_{n=1}^{m} \int_{g_n^{-1}(\theta)0}^{g_n^{-1}(\theta)\imath\infty} \hat{v}^{\mu,k}_j(\tau) d\tau. \tag{66}$$

In this way we obtain a finitely generated finitely presented homogeneous coordinate ring canonically associated to $\mathbb{T}_\theta$.

CHAPTER 5

Further developments

The role played by modular forms in the above discussion points to deep relations with the quantum thermodynamical system introduced by Connes and Marcolli in [**6**] in relation with the class field theory of the modular field. Several results point to the fact that quantum statistical mechanics provides the right framework under which the tools of noncommutative geometry may be applied to class field theory. In this chapter we provide what we believe may be the first steps in the construction of a quantum statistical mechanical system associated to a real multiplication noncommutative torus $\mathbb{T}_\theta$.

1. The geometric data

In the case of real quadratic fields explicit class field theory is conjecturally given in terms of special values of L-functions, this is the content of Stark's conjectures [**39**]. In order to apply our results on noncommutative tori in this direction using the tools of quantum statistical mechanics we still need to find C^*-completions of the algebras $B_g(\tau,\theta)$, $B_g(\theta)\{\delta,\rho,n,m\}_\Gamma$ and $B_g(\theta)$. The constructions of [**5**] seem to provide the right tools in for this purpose.

The first step in this direction is the construction of the geometric data corresponding to the algebra $B_g(\theta,\tau)$. The geometric data associates to a finitely generated graded algebra $A=\oplus A_n$ a triple $T=(Y,\sigma,\mathcal{L})$ where Y is a projective variety, σ is an automorphism of Y and $\mathcal{L}$ is an ample line bundle over Y. Starting from such a triple one can construct the graded algebra:

$$B(T)=\bigoplus_{n\geq 0} H^0(Y,\mathcal{L}\otimes\mathcal{L}^{\sigma}\otimes\ldots\otimes\mathcal{L}^{\sigma^{n-1}}) \tag{67}$$

where $\mathcal{L}^\sigma := \sigma^*\mathcal{L}$ and the multiplication of two sections $s_1\in B(T)_n, s_2\in B(T)_m$ is given by $s_1s_2 := s_1\otimes s_2^{\sigma^n}$.

The construction is made in such a way that one gets a morphism $A\to B(T)$. This construction was introduced by Artin, Tate and Van den Bergh in [**1**] in order to study regular algebras of dimension 3.

Let $\mathcal{T}=\mathbb{K}\langle x_1,\ldots,x_c\rangle$ be the free associative algebra on c generators of degree one over $\mathbb{K}$. So $\mathcal{T}_1=\sum\mathbb{K}x_i\simeq\mathbb{K}^c$ and $\mathcal{T}$ is the tensor algebra

$$\mathcal{T}=\bigoplus_{n\geq 0}(\mathcal{T}_1)^{\otimes n}\simeq\bigoplus_{n\geq 0}(\mathbb{K}^c)^{\otimes n}. \tag{68}$$

To each homogeneous element $f \in \mathcal{T}_n = \mathcal{T}_1^{\otimes n}$ we associate the corresponding n-multilinear form $\check{f} : \mathcal{T}_1^* \times ... \times \mathcal{T}_1^* \to \mathbb{K}$ acting on the n-th Cartesian product of the dual space $\mathcal{T}_1^*$. We call $\check{f}$ the multi linearization of f. Since $\check{f}$ is multihomogeneous its zero locus defines a hypersurface in $(\mathbb{P}^{c-1}(\mathbb{K}))^n$.

Let now A be a finitely generated quadratic algebra over $\mathbb{K}$. Assume A is generated in degree one so that

$$A \simeq \mathcal{T}/\mathcal{R} \tag{69}$$

where $\mathcal{R} = (f_1, ..., f_r)$, $f_i \in \mathcal{T}_1 \otimes \mathcal{T}_1$, is the homogeneous ideal generated by the defining relations of the algebra A. The locus of common zeroes of the multilinearizations of the elements of $\mathcal{R}$ defines a variety $\{\check{f}_i = 0\} = \Gamma \subset \mathbb{P}^{c-1} \times \mathbb{P}^{c-1}$. Let Y_1 and Y_2 be the corresponding projections and $\sigma : Y_1 \to Y_2$ be the correspondence with graph Γ. Assume we can make an identification $Y_1 = Y_2 = Y$. If σ is an isomorphism we consider it as an automorphism of Y. Letting then $i : Y \hookrightarrow \mathbb{P}^{c-1}$ be the inclusion and taking $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}^{c-1}}(1)$ we get the corresponding geometric data $T = (Y, \sigma, \mathcal{L})$. Taking $B(T)$ as above we have that the canonical map $A_1 \to H^0(Y, \mathcal{L})$ extends to a morphism of graded algebras $A \to B(T)$. We call Y the characteristic variety of A.

Consider now the algebra $B_g(\theta, \tau)$. We start with its presentation in terms of generators and relations given in Theorem 1.4

For $\mu \in \{1, 2, ..., c\}$ and $k \in \{1, ..., c-a-d\}$ let

$$f_k^\mu = v_1^{\mu,k} x_{\alpha(\mu,1)} x_1 + ... + v_c^{\mu,k} x_{\alpha(\mu,c)} x_c \tag{70}$$

be the corresponding quadratic relation. Denote by $(x_i)_1(x_j)_2$ the map $\mathbb{C}^c \times \mathbb{C}^c \to \mathbb{C}$ given by $(v, w) \mapsto v_i w_j$. The multilinearization of f_k^μ is then

$$\check{f}_k^\mu = v_1^{\mu,k} (x_{\alpha(\mu,1)})_1 (x_1)_2 + ... + v_c^{\mu,k} (x_{\alpha(\mu,c)})_1 (x_c)_2 \tag{71}$$

By definition the graph $\Gamma \subset \mathbb{P}^{c-1} \times \mathbb{P}^{c-1}$ of the correspondence σ in the geometric data of $B_g(\theta, \tau)$ is the common zero locus of the $c(c-a-d)$ bihomogeneous forms $\check{f}_k^\mu$. Let $\Omega \in \mathcal{M}_{c,c(c-a-d)}(\mathcal{T}_1)$ be the matrix defined by

$$f_k^\mu = \Omega_{k,i}^\mu x_i, \quad i = 1, ..., c;\ \mu = 1, 2, ..., c;\ k = 1, ..., c-a-d. \tag{72}$$

If the images of Γ under the two projections $\pi_1 : \mathbb{P}^{c-1} \times \mathbb{P}^{c-1} \to \mathbb{P}^{c-1}$ and $\pi_2 : \mathbb{P}^{c-1} \times \mathbb{P}^{c-1} \to \mathbb{P}^{c-1}$ are equal then σ is an automorphism of $Y = \pi_1(\Gamma) = \pi_2(\Gamma)$. Y is given in this case by the vanishing of the $c \times c$ minor determinants of the matrix:

$$\check{\Omega} = \begin{pmatrix} v_1^{1,1}(x_{\alpha(1,1)})_1 & v_2^{1,1}(x_{\alpha(1,2)})_1 & \cdots & v_c^{1,1}(x_{\alpha(1,c)})_1 \\ v_1^{1,2}(x_{\alpha(1,1)})_1 & v_2^{1,2}(x_{\alpha(1,2)})_1 & \cdots & v_c^{1,2}(x_{\alpha(1,c)})_1 \\ \vdots & \vdots & \cdots & \vdots \\ v_1^{1,c-a-d}(x_{\alpha(1,1)})_1 & v_2^{1,c-a-d}(x_{\alpha(1,2)})_1 & \cdots & v_c^{1,c-a-d}(x_{\alpha(1,c)})_1 \\ v_1^{2,1}(x_{\alpha(2,1)})_1 & v_2^{2,1}(x_{\alpha(2,2)})_1 & \cdots & v_c^{2,1}(x_{\alpha(2,c)})_1 \\ \vdots & \vdots & \cdots & \vdots \\ v_1^{2,c-a-d}(x_{\alpha(2,1)})_1 & v_2^{2,c-a-d}(x_{\alpha(2,2)})_1 & \cdots & v_c^{2,c-a-d}(x_{\alpha(2,c)})_1 \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ v_1^{c,1}(x_{\alpha(c,1)})_1 & v_2^{c,1}(x_{\alpha(c,2)})_1 & \cdots & v_c^{c,1}(x_{\alpha(c,c)})_1 \\ \vdots & \vdots & \cdots & \vdots \\ v_1^{c,c-a-d}(x_{\alpha(c,1)})_1 & v_2^{c,c-a-d}(x_{\alpha(c,2)})_1 & \cdots & v_c^{c,c-a-d}(x_{\alpha(c,c)})_1 \end{pmatrix}.$$

From this we see that $Y \subset \mathbb{P}^{c-1}$ is defined by $\binom{c(c-a-d)}{c}$ homogeneous equations of degree c. Each one of these equations has as coefficients values of theta constants in $\mathfrak{B}_l$. The explicit form of these determinantial varieties depends in each particular case on the relations satisfied by the corresponding theta constants.

Remark 1.1. In [**38**] Smith and Stafford analyzed the geometric data of a class of graded rings known as Sklyanin algebras. These are quadratic algebras whose relations have coefficients which can be realised as values of theta functions with characteristics in $\frac{1}{2}\mathbb{Z}$. The relations satisfied by $\vartheta_{0,0}, \vartheta_{0,\frac{1}{2}}, \vartheta_{\frac{1}{2},0}$ and $\vartheta_{\frac{1}{2},\frac{1}{2}}$ play a central role in their analysis and are responsible for the appearance of an elliptic curve as part of the characteristic variety. At a first glance it seems like our situation is analogous to that on [**38**] and either the modular curves or $Proj\mathfrak{B}_l$ could be playing the role that elliptic curves played for the Sklyanin algebras.

We shall return to this point in future work where a detailed analysis of the geometric data associated to the various rings considered in this thesis will be carried out. We expect then the use of the techniques developed in [**5**] will then make it possible to obtain C^*-completions suitable for arithmetic applications.

APPENDIX A

Theta functions and theta constants with rational characteristics

In this appendix we recall the main facts about theta functions and theta constants with rational characteristics that are used in this thesis. Our treatment follows closely [**23**], [**24**] and [**25**].

Let $(r,s)\in\mathbb{Q}^2$. The series

$$\vartheta_{r,s}(z,\tau)=\sum_{n\in\mathbb{Z}}\exp[\pi\imath(n+r)^2\tau+2\pi\imath(n+r)(z+s)] \tag{73}$$

defines a holomorphic function of $(z,\tau)\in\mathbb{C}\times\mathbb{H}$. We call $\vartheta_{r,s}(z,\tau)$ the *theta function with rational characteristics* (r,s). Note that changing the characteristics by integer values does not affect the values of $\vartheta_{r,s}(z,\tau)$. Taking $(r,s)=(0,0)$ we get Riemann's theta function $\vartheta(z,\tau):=\vartheta_{0,0}(z,\tau)$. For $(r,s)\in\mathbb{Q}^2$ we have

$$\vartheta_{r,s}(z,\tau)=\exp[\pi\imath r^2\tau+2\pi\imath r(z+s)]\vartheta(z+r\tau+s,\tau). \tag{74}$$

One of the most important and useful result about ϑ is the functional equation it satisfies:

Theorem 0.2. *Let* $\gamma=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\Gamma_{1,2}$. *Then*

$$\vartheta\left(\frac{z}{C\tau+D},\frac{A\tau+D}{C\tau+D}\right)=\kappa(\gamma)(C\tau+D)^{\frac{1}{2}}\exp\left[\frac{\pi\imath Cz^2}{C\tau+D}\right]\vartheta(z,\tau)$$

where the constant $\kappa(\gamma)$ *satisfies* $\kappa(\gamma)^8=1$.

For a fixed $\tau\in\mathbb{H}$ consider the lattice $\Lambda_\tau=\mathbb{Z}\oplus\tau\mathbb{Z}$. Let $(r,s)\in\mathbb{Q}^2$ and define for $\lambda=m'+m\tau\in\Lambda_\tau$

$$e_\lambda(z):=\exp[-\pi\imath m^2\tau+2\pi\imath(rm'-m(z+s)]$$

Then

$$\vartheta_{r,s}(z+\lambda,\tau)=e_\lambda\vartheta_{r,s}(z,\tau)$$

and the values e_λ satisfy:

$$e_{\lambda_1+\lambda_2}(z)=e_{\lambda_1}(z+\lambda_2)e_{\lambda_2}(z).$$

Thus the function $\vartheta_{r,s}(z,\tau)$ is an entire Λ_τ-automorphic function of $z\in\mathbb{C}$. The functions $\{e_\lambda\}_{\lambda\in\Lambda_\tau}$ are called automorphy factors. The above conditions just mean that each $\vartheta_{r,s}(z,\tau)$ gives a section of a holomorphic line bundle on the elliptic curve $E_\tau = \mathbb{C}/\Lambda_\tau$.

We will denote by $\mathcal{L}$ the *basic line bundle over* E_τ. This is the bundle obtained as the quotient of $\mathbb{C}\times\mathbb{C}$ by the action of Λ_τ given by

$$\lambda\cdot(\zeta,z) = (\exp[-\pi\imath m^2\tau + 2\pi\imath m(z-m')]\zeta, z-\lambda)$$

where $\lambda = m' + m\tau \in \Lambda_\tau$, $(\zeta,z)\in\mathbb{C}\times\mathbb{C}$, and we view $\mathbb{C}\times\mathbb{C}$ as the total space of the trivial line bundle over $\mathbb{C}$. For an integer $l>1$ the functions $z\mapsto\vartheta_{r,s}(lz,\tau)$ with $(r,s)\in\frac{1}{l}\mathbb{Z}$ give sections of a line bundle whose l tensor power is isomorphic to $\mathcal{L}$. In this case the Lefschetz embedding theorem gives us an embedding to projective space:

Theorem 0.3. *Let $l>1$ be an integer and let $(r_i,s_i)\in(\frac{1}{l}\mathbb{Z})^2$, $i=0,...,l^2-1$, be a complete set of representatives of $(\frac{1}{l}\mathbb{Z}/\mathbb{Z})^2$. Then the holomorphic map $\phi_l:\mathbb{C}/\Lambda_\tau\to\mathbb{P}^{l^2-1}\mathbb{C}$ given by*

$$\phi_l: \quad z\mapsto(\vartheta_{r_0,s_0}(lz,\tau),...,\vartheta_{r_{l^2-1},s_{l^2-1}}(lz,\tau)) \tag{75}$$

is an embedding. In particular, $\phi_l(\mathbb{C}/\Lambda_\tau)$ is an algebraic subvariety of $\mathbb{P}^{l^2-1}\mathbb{C}$.

Remark 0.4. The algebraic equations defining $\phi_l(\mathbb{C}/\Lambda_\tau)$ can be determined in some cases by relations satisfied by the $\vartheta_{r,s}$. For instance, Riemann's 16 theta relations between $\vartheta_{0,0},\vartheta_{0,\frac{1}{2}},\vartheta_{\frac{1}{2},0}$ and $\vartheta_{\frac{1}{2},\frac{1}{2}}$ can be used to realize $\phi_2(\mathbb{C}/\Lambda_\tau)$ as the curve in $\mathbb{P}^3\mathbb{C}$ defined by the quadratic equations

$$\begin{aligned}\vartheta_{0,0}(0)^2X_0^2 &= \vartheta_{0,\frac{1}{2}}(0)^2X_1^2+\vartheta_{\frac{1}{2},0}(0)^2X_2^2\\ \vartheta_{0,0}(0)^2X_3^2 &= \vartheta_{\frac{1}{2},0}(0)^2X_1^2-\vartheta_{0,\frac{1}{2}}(0)^2X_2^2\end{aligned}$$

The zeroes of $z\mapsto\vartheta_{r,s}(z,\tau)$ are characterized by the following lemma:

Lemma 0.5. *Fix $\tau\in\mathbb{H}$, let $l>1$ be an integer and let $(r,s)\in\mathbb{Q}$. Then, the zeroes of $z\mapsto\vartheta_{r,s}(z,\tau)$ are the points $(r+p+\frac{1}{2})\tau+(s+q+\frac{1}{2})$ with $p,q\in\mathbb{Z}$.*

The holomorphic function on $\mathbb{H}$ obtained when we restrict $\vartheta_{r,s}(z,\tau)$ to $z=0$ will be refereed as the *theta constant with rational characteristics* (r,s). We will use the following notations:

$$\vartheta_{r,s}(\tau) := \vartheta_{r,s}(0,\tau). \tag{76}$$

$$\vartheta_r(\tau) := \vartheta_{r,0}(0,\tau). \tag{77}$$

The functional equation in Theorem 0.2 becomes in this case:

$$\vartheta(\frac{A\tau+D}{C\tau+D}) = \kappa(\gamma)(C\tau+D)^{\frac{1}{2}}\vartheta(\tau). \tag{78}$$

A very important variant of these functions is given by the algebraic theta constants:

$$\vartheta^\alpha_{r,s}(\tau) := e^{-\pi\imath rs}\vartheta_{r,s}(0,\tau) \qquad (r,s)\in\mathbb{Q}^2. \tag{79}$$

The rationality behavior of these functions plays an essential role for us. Let $\tau \in \mathbb{H}$. Consider as above the elliptic curve E_τ and the basic line bundle $\mathcal{L}$ on it. For all $a, b \in \mathbb{Q}$ we take

$$x(a,b)_m \quad = \quad \text{Image in } E_\tau \text{ of } \frac{1}{m}(a\tau + b)$$

Then we have:

THEOREM 0.6. ([**24**] Corollary 5.12) *Suppose that $a, b \in \frac{1}{l}\mathbb{Z}$ and $k \hookrightarrow \mathbb{C}$ is a subfield such that E_τ and $\mathcal{L}$ can be defined over k and $x(a,b)_{2l}$ is rational over k. Then*

$$\frac{\vartheta^\alpha_{a,b}(\tau)}{\vartheta^\alpha_{0,0}(\tau)} \quad \in \quad k.$$

APPENDIX B

Buchberger's algorithm

In this appendix we describe the implementation of Buchberger's algorithm for the computation of a Gröbner basis for a finitely generated two sided ideal $\mathcal{I}$ in $\mathbb{K}\langle x_1, ..., x_c\rangle$. We will work on the setting of Chapter 3, in particular we use the same notation and conventions. We assume that the ideal $\mathcal{I}$ is a homogeneous. In this case Buchberger's algorithm provides for each given degree the set of elements of this degree in the corresponding Gröbner basis. The auxiliary algorithms providing the normal form of an element in a set of generators and a minimal set of obstructions are also given. We follow the approach of [**22**].

The first algorithm computes a minimal set of obstructions of the element g_j in the finite ordered set $\{g_1, \ldots, g_s\}$. The elements belonging to this set take into account the fact that two elements in the ideal generated by $\{g_1, \ldots, g_s\}$ can have the same leading term. This happens if there exists elements $l, r, l', r' \in \mathbf{S}$ such that $lT(g_j)r = l'T(g_i)r'$ for some $i, j \in \{1, \ldots, s\}$. The corresponding obstruction is the element (j, l, r, i, l', r') in the Cartesian product:

$$(\{1, \ldots, s\} \times \mathbf{S} \times \mathbf{S}) \times (\{1, \ldots, s\} \times \mathbf{S} \times \mathbf{S}) \tag{80}$$

Given g_j in the finite ordered set $\{g_1, \ldots, g_s\}$ the output of the algorithm $OBS(j)$ is the minimal set of obstructions of g_j:

```
$OBS(j) = \emptyset$;
For each   $l, r, w \in \mathbf{S} \setminus \{1\}$   s.t. :   $T(g_j) = lw = wr$:
        $OBS(j) = OBS(j) \cup \{(j, 1, r, j, l, 1)\}$;
For   $i = 1, i < j$
    For each   $l, r, w \in \mathbf{S} \setminus \{1\}$   s.t. :   $T(g_j) = lw,\ T(g_i) = wr$:
            $OBS(j) = OBS(j) \cup \{j, 1, r, i, l, 1)\}$;
    For each   $l, r, w \in \mathbf{S} \setminus \{1\}$   s.t.   $T(g_j) = wr,\ T(g_i) = lw$:
            $OBS(j) = OBS(j) \cup \{(j, l, 1, i, 1, r\}$;
Let $\mathbf{s} = \{(l, r) \mid (j, l, r, k, l', r') \in OBS(j)\}$
For each $s = (l, r) \in \mathbf{s}$ Choose $(j, l_s, r_s, k_s, l'_s, r'_s) \in OBS(j)$
such that $l_s, r_s = (l, r)$
$OBS(j) = \{(j, l_s, r_s, k_s, l'_s, r'_s) \mid s \in \mathbf{s}\}$.
```

The following simple algorithm computes the normal form of an element $f \in \mathbb{K}\langle x_1, ..., x_c\rangle$ with respect to finite set $G \subset \mathbb{K}\langle x_1, ..., x_c\rangle$:

```
$h \ = \ \mathsf{NormalForm}(f, G)$:
$h \ = \ f$;
**While** $h \neq 0$ or $T(h) \in T(G)$;
    Choose $g \in G;\ l,\ r \in \mathbf{S}$ s.t. $lT(g_i)r \ = \ T(h)$;
    $h \ = \ h - lc(h)lc(g)^{-1}lgr$.
```

Let $\mathcal{I}$ be a finitely generated homogeneous two sided ideal in $\mathbb{K}\langle x_1, ..., x_c\rangle$ and let F a finite basis for $\mathcal{I}$. Assume F is given as an ordered set $\{g_1, \dots, g_s\}$ such that $lc(g_i) = 1$ and that all $T(g_i)$ are different. The following algorithm computes a Gröbner basis for $\mathcal{I}$.[1]

```
$G = \mathsf{Gr\ddot{o}bner}(\mathcal{I})$

$G = \emptyset$;
$OBS = \emptyset$;
**For** $t = 1,\ t \leq s$:
    $RED = \{(k, l, r, i, l', r') \in OBS :\ \exists\, w_l, w_r\ \in \mathbf{S} \setminus \{1\}$ s.t.
                                          $w_l T(g_t) w_r = lT(g_k)r\}$;
    $OBS = OBS \setminus RED$;
    $G = G \cup \{g_t\}$;
    $OBS = OBS \cup OBS(t)$.

**While** $OBS \neq \emptyset$;
    Choose $(k, l, r, i, l', r') \in OBS$;
    $OBS = OBS \setminus \{(k, l, r, i, l', r')\}$;
    $h = \mathsf{NormalForm}(lg_k r - l' g_i r',\ G)$;
    **If** $h \neq 0$ **then**:
       $t = t + 1$;
       $g_t = h$;
       $G = G \cup \{g_t\}$;
       $RED = \{(k, l, r, i, l', r') \in OBS : \exists w_l, w_r \in \mathbf{S} \setminus \{1\}$ s.t.
                                          $w_l T(g_t) w_r = lT(g_k)r\}$;
       $OBS = OBS \setminus RED$;
       $G = G \cup \{g_t\}$;
       $OBS = OBS \cup OBS(t)$.
```

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Summary

Arithmetic structures on noncommutative tori with real multiplication

by Jorge Plazas

In this thesis we study homogeneous coordinate rings of real multiplication noncommutative tori. Our aim is to understand how these rings give rise to an arithmetic structure on the noncommutative torus. The main motivation for the study of arithmetic structures on noncommutative tori comes from possible applications to the explicit class field theory problem for real quadratic extensions of $\mathbb{Q}$. This relation is discussed in the introduction.

The first part of Chapter 1 discusses the setup of the rest of the thesis. Noncommutative tori with real multiplication and their homogeneous coordinate rings are introduced. This definition due to Polishchuk depends on the choice of a complex structure on the noncommutative torus, these structures are also treated in this part. In the second part of Chapter 1 we use the explicit formulas defining homogeneous coordinate rings in order to obtain their presentation in terms of generators and relation. This presentation is given in terms of theta constants with rational characteristics and is the starting point of the subsequent analysis.

In Chapter 2 we exploit the relation between theta constants and elliptic curves in order to study the rationality properties of the homogeneous coordinate rings of real multiplication noncommutative tori. It is shown that these rings admit a rational presentation which is algebraic over the field of definition of the elliptic curve corresponding to the given complex structure on the noncommutative torus.

In Chapter 3 we study the natural linear basis of the homogeneous coordinate rings given in Chapters 1 and 2. The theory of noncommutative Gröbner basis for two sided ideals on the free algebra $\mathbb{K}\langle x_1, ..., x_c\rangle$ plays a central role in this computations.

In Chapter 4 we shift our point of view and look at homogeneous coordinate rings as a family of algebras varying with the parameter defining the complex structure on the noncommutative torus. For this we give a presentation of the rings in terms of modular forms. We use the modularity of these coordinate rings to define new rings by an averaging process over (limiting) modular symbols. In this way we obtain a homogeneous ring associated with the noncommutative torus which does not depend on the choice of a particular complex structure.

In the last Chapter we analyze the geometric data corresponding to the homogeneous rings discussed in the previous chapters. This is the first step towards the construction of C^*-completions which are fundamental in order to recast the above ideas in the context of quantum statistical mechanical systems of arithmetic nature.