# MANIFOLDS WITH KILLING SPINORS AND PINCHING OF FIRST DIRAC EIGENVALUES

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VORGELEGT VON
ANDRÉS VARGAS DOMÍNGUEZ
AUS
BOGOTÁ



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Erstgutachter (Betreuer): Prof. Dr. Werner Ballmann
Zweitgutachter: Prof. Dr. Bernd Ammann

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*To my parents and my brother*

# Introduction

Spinors in geometry have been widely studied in the past few decades motivated by the fundamental role that the Dirac operator plays in many areas at the cross-border between geometry, global analysis, topology and of course physics, from where the original motivation came. Previously, the Laplacian operator was the focus of most of the developments in the study of differential operators and from the profound knowledge gained about its properties, some guidance has been available in the search for interesting questions and methods to consider in the case of the Dirac operator.

One particular example, in which we are interested here, is the existence of lower and upper bounds on the spectrum of the Dirac operator under different sets of conditions. Moreover, the explicit spectrum on compact manifolds has been calculated for particularly "nice" cases where available algebraic or analytical techniques are fruitful, but this is not the most common situation.

On the other hand, the relation of this spectrum with the geometry of the manifold, which is another instance of the problem of "hearing the shape" of a space, is one of the fundamental questions of the subject. Pinching theorems for the eigenvalues of geometric operators offer some insight into this questions. For the eigenvalues of the Laplace operator many different results have been found in various contexts.

As an example we recall here the following cases (see [Pet99], [PS99], [PS03]). Consider the Hodge Laplacian $\Delta\omega = (d^*d + dd^*)\omega$, and the connection (Bochner) Laplacian $\nabla^*\nabla\omega = -\operatorname{tr}\nabla^2\omega$ acting on forms. For the eigenvalues of 1-forms on compact Riemannian manifolds there are pinching results stating that under certain conditions, smallness of the first $n$ eigenvalues implies that the manifold is the quotient of a nilpotent Lie group by a discrete group of isometries i.e. a *nilmanifold*.

Let us denote by $\lambda_i$ the eigenvalues of the Hodge Laplacian and with $\bar{\lambda}_i$ the eigenvalues of the connection Laplacian.

**Theorem.** (Petersen-Sprouse 1999). *Let $M^n$ be a compact manifold with $|\sec| < K$, $\operatorname{diam}(M) < d$. Then there is an $\varepsilon = \varepsilon(n,K,d)$ such that if $\bar{\lambda}_n < \varepsilon$, then $M$ is a nilmanifold.*

Motivated by this and by similar results for the Dirac operator obtained recently by Ammann and Sprouse, we focused on the study of convergence and regularity techniques to obtain pinching restrictions for the eigenvalues of the Dirac operator close to the Friedrich lower bound ($\lambda^2 \geq n^2/4$, cf. chapter 5) under appropriate geometrical conditions.

In this thesis, we introduce spaces and metrics of $C^{1,\alpha}$ regularity and employ the theory of convergence of Riemannian manifolds developed after Gromov, to study spin manifolds converging in the $C^{1,\alpha}$-topology. In particular, we are concerned with the existence of Killing spinors in these limits and the improving of the regularity of the metric for such cases.

Now we outline briefly the contents. In chapter 1, we review basic but important results from the theory of partial differential equations in the context of compact manifolds, and establish some elliptic regularity theorems and estimates needed in chapters 3 and 4.

In chapter 2, a brief discussion on spin geometry appears, focusing at the end on the notions of Killing spinors, Einstein manifolds and their relationship on smooth Riemannian manifolds.

Chapter 3 is devoted to the study of Riemannian manifolds with $C^{1,\alpha}$-metrics, making use of the well-known optimal regularity properties of harmonic coordinates. Since curvature is not well-defined without second derivatives of the metric, we introduce appropriate weak notions of Ricci curvature and the Einstein condition for $C^{1,\alpha}$-metrics. Then, it is shown that $C^{1,\alpha}$-metrics satisfying the Einstein condition weakly in harmonic coordinates are actually smooth and Einstein in the usual sense.

Then, weak spinorial Riemann curvature is introduced and with that, it is proved that manifolds with $C^{1,\alpha}$-metrics carrying a Killing spinor satisfy the weak Einstein condition and therefore, the smoothness of those metrics is obtained.

The most important results of chapter 3 are:

**Theorem 3.3.2.** *Let $M$ be an $n$-dimensional manifold with a Riemannian metric $g$ of class $C^{1,\alpha}$. If the components of $g$ are weak solutions of the Einstein equation $\mathrm{Ric}_{ab} = \kappa(n-1)g_{ab}$ in a harmonic coordinate chart $(U,\phi)$, where $\phi : U \subset M \longrightarrow V \subset \mathbb{R}^n$ is invertible, then viewed as functions $g_{ab} \circ \phi^{-1} : V \longrightarrow \mathbb{R}$, they are smooth. We say in this case that the components of $g$ are smooth with respect to the harmonic coordinate chart $(U,\phi)$.*

**Corollary 3.3.3.** *If $g$ is weakly Einstein with respect to all charts in a harmonic coordinate atlas $\mathscr{A} = \{\phi_\gamma : U_\gamma \longrightarrow V_\gamma\}$ of $M$, then $\mathscr{A}$ is actually a smooth atlas, and $g$ is a smooth Riemannian metric with respect to it.*

**Theorem 3.4.2.** *If $M$ is a compact spin manifold with a $C^{1,\alpha}$-metric $g$ such that* *there is a non-trivial Killing spinor, then the manifold satisfies weakly the Einstein condition in any local coordinate chart.*

**Theorem 3.4.3.** *If M is a compact spin manifold with a $C^{1,\alpha}$-metric g such that there is a non-trivial Killing spinor, then g is smooth with respect to a harmonic coordinate atlas for M. Furthermore g is Einstein (and hence analytic) in this atlas.*

In chapter 4, the identification procedure for spinor fields associated to different metrics on the base manifold is reviewed as well as the essential aspects of the theory of convergence of Riemannian manifolds under appropriate diameter, curvature and volume bounds. Then we study sequences of manifolds carrying "almost Killing spinors" and consider their convergence in the space of $L^2$-spinors. Existence of a Killing spinor in the limit, with appropriate regularity, is proved.

The main result here can be summarized as:

**Theorem 4.5.2.** *A sequence of almost Killing spinors $(\psi_i)_{i\geq 1}$ where $\psi_i \in L^2(\Sigma^{g_i}M)$, has a subsequence, denoted again by $(\psi_i)_{i\geq 1}$, such that $(A_i\psi_i)_{i\geq 1}$ converge strongly in $L^2(\Sigma^g M)$ to a non-trivial Killing spinor $\psi$ of class $C^{1,\alpha}$.*

In chapter 5 all the material presented before is used to find pinching results for eigenvalues of the Dirac operator close to the Friedrich bound on $n$-manifolds with bounded diameter diam $< d$, volume vol $> V$ and sectional curvature $|\sec| < K$ (whose class is denoted by $\mathscr{M}(n,d,K,V)$). In particular the existence of a Killing spinor in such manifolds allows to characterize the sphere for many dimensions. Additionally, uniform lower bounds on manifolds no diffeomorphic to the sphere and with the same geometrical restrictions are presented.

Denoting $I_\varepsilon^+ := \mathrm{spec}(D) \cap [0, \frac{n}{2} + \varepsilon]$, and $I_\varepsilon^- := \mathrm{spec}(D) \cap [-\frac{n}{2} - \varepsilon, 0]$, we find:

**Theorem 5.4.1.** *Suppose $(M,g,\chi)$ is a compact n-dimensional Riemannian spin manifold in $\mathscr{M}(n,d,K,V)$ and $\mathrm{scal}_g \geq n(n-1)$. For every $\delta > 0$ there is an $\varepsilon = \varepsilon(n,K,d,\delta) > 0$ such that if any of the following conditions hold:*

1. $\#(I_\varepsilon^+ \cup I_\varepsilon^-) \geq 1$ *if n is even and $n \neq 6$,*
2. $\#(I_\varepsilon^+) \geq 2$ *or* $\#(I_\varepsilon^-) \geq 2$ *if $n = 6$ or $n \equiv 1 (\mathrm{mod}\, 4)$,*
3. $\#(I_\varepsilon^+) \geq \frac{n+9}{4}$ *or* $\#(I_\varepsilon^-) \geq \frac{n+9}{4}$ *or* $\#(I_\varepsilon^+) \geq 1, \#(I_\varepsilon^-) \geq 1$ *if $n \equiv 3 (\mathrm{mod}\, 4)$,*

*with M being simply-connected in the cases 2 and 3 if $n > 3$, then $(M,g)$ has $C^{1,\alpha}$-distance $\leq \delta$ to the sphere $\mathbb{S}^n$ with the standard metric of constant sectional curvature* $\sec = 1$.

**Corollary 5.4.4.** *Suppose $(M,\chi)$ is a simply-connected, n-dimensional, compact spin manifold not diffeomorphic with the sphere $\mathbb{S}^n$. Then, among all metrics with* *bounded diameter, volume and curvature in* $\mathscr{M}(n,d,K,V)$ *and with scalar curvature* $\mathrm{scal}_g \geq n(n-1)$*, there exists an* $\varepsilon = \varepsilon(n,K,d,V) > 0$ *and an integer* $r \in \mathbb{Z}^+$ *yielding a uniform lower bound on the r-th eigenvalue of* $D^2$*:*

$$\lambda_r(D^2) \geq \frac{n^2}{4} + \varepsilon,$$

*where*

$$r = \begin{cases} 3 & \textit{if } n = 6 \textit{ or } n \equiv 1 (\mathrm{mod}\, 4). \\ \frac{n+9}{4} & \textit{if } n \equiv 3 (\mathrm{mod}\, 4). \end{cases} \tag{1}$$

When $n$ is even but $n \neq 6$ it is known that only spheres carry Killing spinors.

For manifolds with positive scalar curvature, it was shown in [BD03] that proximity of the first eigenvalue of the Dirac operator to the Friedrich estimate does not impose topological restrictions on the manifold. Nevertheless, if in addition the manifolds belong to the class $\mathscr{M}(n,d,K,V)$ our result shows that proximity of enough eigenvalues to this estimate do impose some restrictions. In particular for $n = 6$ or $n \equiv 1(\mathrm{mod}\, 4)$, only 3 eigenvalues are needed for the manifold to be the sphere, which interestingly is independent of $n$. Even in the remaining case $n \equiv 3(\mathrm{mod}\, 4)$, the dependance on $n$ is linear.

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# Contents

# Chapter 1

# Preliminaries on PDE's

Here we present a brief summary of important tools from the theory of partial differential equations which are necessary for the regularity theorems required in chapter 3. Since we are going to work on compact Riemannian manifolds, the theory will be described in this setting. A good exposition of this material in the context of manifolds and vector bundles can be found for example in [Kaz93], [Aub98] and the first chapter of [Joy00].

For what follows, let $(M,g)$ denote an $n$-dimensional Riemannian manifold. In this chapter we suppose that $M$ has a smooth differentiable structure and $g$ is a smooth Riemannian metric defined on it. Later, and specially for the applications in following chapters, we will weaken this hypothesis as necessary. In particular the degree of differentiability of $g$ will be considered.

## 1.1 Hölder Spaces

The familiar spaces $C^k(M)$ of continuous, bounded functions on $M$ with $k$ continuous and bounded derivatives do not behave nicely enough to satisfy the kind of regularity properties that one would like to have in the study of partial differential equations. For this reason it is customary to introduce additional function spaces.

Recall that the space $C^k(M)$ for $k \geq 0$ carries a Banach space structure with the supremum norm

$$\|f\|_{C^k} := \sum_{j=0}^{k} \sup_M |\nabla^j f|, \quad \text{where } \nabla^j f := \underbrace{\nabla \cdots \nabla}_{j-\text{times}} f$$

denotes the Levi-Civita connection acting $j$ times on $f$. Here we use by convention $\nabla^0 f := f$. However, with this norm, the inclusion $C^k \hookrightarrow C^{k-1}$ does not send $C^k$ into a closed subspace of $C^{k-1}$. To avoid this problem the so-called $C^\alpha$*-seminorm* $[\cdot]_\alpha$ is introduced. For each $\alpha \in (0,1)$, it is given by

$$[f]_\alpha := \sup_{x \neq y \in M} \frac{|f(x) - f(y)|}{d_g(x,y)^\alpha}.$$

Here $d_g(x,y)$ denotes the distance between two points $x, y \in M$ calculated using the metric $g$. A function $f$ on $M$ is said to be *Hölder continuous* with exponent $\alpha$ if it has a finite $C^\alpha$-seminorm. With this definition any Hölder continuous function with exponent $\alpha \in (0,1)$ is continuous on $M$.

To define the Hölder spaces $C^{k,\alpha}(M)$ for an integer $k \geq 0$ and $\alpha \in (0,1)$ we consider first the case $k = 0$.

**Definition 1.1.1.** The vector space $C^{0,\alpha}(M)$ is the set of $C^0(M)$ functions which are Hölder continuous of exponent $\alpha$, and is endowed with the norm

$$\|f\|_{C^{0,\alpha}} := \|f\|_{C^0} + [f]_\alpha.$$

The Hölder spaces $C^{k,\alpha}(M)$ of functions on manifolds for an arbitrary integer $k$ will be generalized in section 1.3 to sections of vector bundles. Nevertheless, the global approach we follow here requires the notion of a connection to define covariant derivatives of functions and, because of this, it is important to talk about sections of vector bundles already at this point. Therefore, we extend first the $C^\alpha$-seminorm to sections, which will allow to take the norm of covariant derivatives of a function $f$, needed in the expression for the total $C^{k,\alpha}$-norm.

Let $E \longrightarrow M$ be a vector bundle over $M$, with Euclidean metrics in the fibers, and a connection $\nabla^E$ preserving these metrics. Denote by $\iota_g := \operatorname{inj rad}(g)$ the injectivity radius of the metric $g$ on $M$, which we suppose to be positive (as it happens in particular when $M$ is compact), and for a given section $u \in \Gamma(E)$ define

$$[u]_\alpha := \sup_{\substack{x \neq y \in M \\ d_g(x,y) < \iota_g}} \frac{|u(x) - u(y)|}{d_g(x,y)^\alpha},$$

in case the supremum exists. Since the elements $u(x)$ and $u(y)$ lie in different fibers (i.e. different vector spaces), to make sense of the expression $|u(x) - u(y)|$, we attach it to the following prescription. First, given two different points $x, y \in M$ for which the value $d_g(x,y) < \operatorname{inj rad}(g)$, there is a unique geodesic $\gamma$ of length $d_g(x,y)$ joining $x$ and $y$ in $M$. Then, parallel translation along $\gamma$ using $\nabla^E$ identifies $E_x$ and $E_y$, the fibers of $E$ over $x$ and $y$ respectively, and their corresponding metrics. In this way the expression is well-defined.

**Definition 1.1.2.** The *Hölder space* $C^{k,\alpha}(M)$ is the set of functions in $C^k(M)$ with finite $C^\alpha$-seminorm for all their covariant derivatives $\nabla^k f \in \Gamma(T^*M^{\otimes k})$ of order $k$. To calculate these seminorms $[\nabla^k f]_\alpha$, the natural metric and connection inherited from $T^*M$ are required. Hence, for Hölder spaces $C^{k,\alpha}(M)$ on a manifold $M$ we define the norm

$$\|f\|_{C^{k,\alpha}} := \|f\|_{C^k} + [\nabla^k f]_\alpha,$$

which provides a Banach space structure. Furthermore, for $\alpha > \beta$ there are inclusions $C^{k,\alpha}(M) \hookrightarrow C^{k,\beta}(M)$ which are compact maps when $M$ itself is compact, i.e. closed bounded sets are mapped to compact sets. See also section 1.4 below.

## 1.2 Sobolev Spaces of Integer Order

Recall that the *Lebesgue spaces* $L^p(M)$ for a Riemannian manifold $(M,g)$ and $p \geq 1$ are defined as the set of locally integrable functions $f$ on $M$ for which the $L^p$*-norm*

$$\|f\|_{L^p} := \left(\int_M |f|^p \,\mathrm{dvol}_g\right)^{1/p}$$

is finite. The expression $\mathrm{dvol}_g$ denotes the volume form on the manifold associated with the metric $g$, i.e. in a local coordinate system $(x^1,\dots,x^n)$ we have $\mathrm{dvol}_g := \sqrt{g}\,dx^1\cdots dx^n$.

**Definition 1.2.1.** For every $f \in C^\infty(M)$, $p \geq 1$ and an integer $k \geq 0$, define the *Sobolev* $H^p_k$*-norm* as

$$\|f\|_{H^p_k} := \left(\int_M \sum_{j=0}^{k} |\nabla^j f|^p \,\mathrm{dvol}_g\right)^{1/p}, \tag{1.1}$$

where $|\nabla^j f|$ is the pointwise norm of the $j$-th covariant derivative of $f$. The *Sobolev space* $H^p_k(M)$ is the completion of $C^\infty(M)$ in this norm.

Alternatively, the Sobolev space $H^p_k$ can be defined as the set of all $f \in L^p(M)$ such that $f$ is $k$ times weakly differentiable and $|\nabla^j f| \in L^p(M)$ for $j \leq k$. It is, of course, an important theorem that these two different descriptions coincide, but we will use one or the other indistinctly, according to the convenience of the context. Locally, the Sobolev spaces are given by equivalence classes of measurable functions whose partial (possibly weak) derivatives up to order $k$ are in $L^p$. A partition of unity argument suffices to paste this local viewpoint into the global one just explained.

The Sobolev spaces $H^p_k(M)$ are Banach spaces with respect to the Sobolev norm and are reflexive for $1 < p < \infty$. For $p = 2$ the spaces $H^2_k(M)$ are Hilbert spaces with the obvious inner product.

**Remark 1.2.1.** Under a change of metric $g$, assuming that the differentiable structure on the manifold $M$ is fixed, the norms of the spaces $C^{k,\alpha}(M)$ and $H_k^p(M)$ change, but if $M$ is compact the new norms are still equivalent and hence the induced topologies are the same. Moreover, the assertion is also true in the compact case if different connections are used to define the norms of these spaces. This observation will be fundamental (but implicitly assumed) for our applications in chapter 4 where we need to work with sequences of metrics (one particular case appears in the proof of Lemma 4.5.1).

## 1.3 Function Spaces on Vector Bundles

The previous spaces $C^{k,\alpha}(M)$, $L^p(M)$ and $H_k^p(M)$ are vector spaces of real functions on a manifold $M$. They can be generalized to vector spaces of sections of a vector bundle in a straightforward way. Let $E \longrightarrow M$ be a vector bundle on $M$ as before, endowed with Euclidean metrics on its fibers and a connection $\nabla^E$ (denoted simply as $\nabla$ if the context is clear) on $E$ preserving these metrics. The generalization of the Hölder spaces $C^{k,\alpha}(M)$ on $M$ to the Hölder spaces $C^{k,\alpha}(E)$ of sections of $E$ is done, for example, by replacing the occurrence of functions on $M$ with the corresponding sections needed in each case. It is important to note that, once again, any substraction of values of sections $u \in \Gamma(E)$ in different points of $M$ is understood using the parallel transport prescription, as mentioned in section 1.1.

*Notation.* In reference to an arbitrary section $u$ of a vector bundle $E \longrightarrow M$ we write $u \in \Gamma(E)$, but when we want to emphasize its differentiability we write explicitly, for example, $u \in C^k(E)$. In the same manner, we will denote by $C^{k,\alpha}(E)$ and $H_k^p(E)$ the corresponding Hölder and Sobolev sections of $E$.

**Definition 1.3.1.** For $p \geq 1$, we define the *Lebesgue space* $L^p(E)$ to be the set of locally integrable sections $u \in \Gamma(E)$ over $M$ for which the norm

$$\|u\|_{L^p} = \left( \int_M |u|^p \, \mathrm{dvol}_g \right)^{1/p}$$

is finite. Similarly, the *Sobolev space* $H_k^p(E)$ is the completion of the set of smooth sections $u \in C^\infty(E)$ in the corresponding Sobolev norm for this case. It can be also seen as the set of $u \in L^p(E)$ such that $u$ is $k$ times weakly differentiable and $|\nabla^j u| \in L^p(M)$ for $j \leq k$.

## 1.4 Sobolev Inequalities and Embedding Theorems

Now let us state important results concerning the relations between the Hölder spaces $C^{k,\alpha}$, the Sobolev spaces $H_k^p$ and the usual spaces of $k$ times differentiable functions $C^k$. In addition to standard references as [GT77], a complete exposition of embedding theorems is found in [Ada75]. For the next statement, recall that given Banach spaces $B_1, B_2$, a continuous map $T : B_1 \longrightarrow B_2$ is compact if for any bounded set $A \subset B_1$, the closure of its image $\overline{T(A)} \subset B_2$ is compact.

**Theorem 1.4.1** (Sobolev Embedding and Kondrakov Theorems). *Suppose $M$ is a compact Riemannian $n$-manifold, let $k,l \in \mathbb{Z}$ with $0 \leq l \leq k$ and $p,q \in \mathbb{R}$ such that $p,q \geq 1$. Given a function $f \in H_k^p(M)$ we have:*

1. *If $k-l \leq n/p$ and an integer $q$ satisfies*

$$\frac{1}{p} - \frac{k-l}{n} \leq \frac{1}{q},$$

*then there is a constant $c > 0$ independent of $f$ such that*

$$\|f\|_{H_l^q} \leq c\|f\|_{H_k^p}. \tag{1.2}$$

*Thus there is a continuous inclusion $H_k^p(M) \hookrightarrow H_l^q(M)$. Furthermore, if $l < k$ and the inequality holds strictly, this inclusion is a compact operator.*

2. *If $k-l-1 < n/p < k-l$, define $\alpha := k-l-n/p$ so that $\alpha \in (0,1)$. Then there is a constant $c$ independent of $f$ such that*

$$\|f\|_{C^{l,\alpha}} \leq c\|f\|_{H_k^p}.$$

*Thus, there is a continuous inclusion $H_k^p(M) \hookrightarrow C^{l,\alpha}$ and a compact inclusion $H_k^p(M) \hookrightarrow C^{l,\beta}$ for any $\beta$ with $0 < \beta < \alpha = k-l-n/p$. Also $C^{k,\alpha}(M) \hookrightarrow C^k(M)$ is compact.*

**Remark 1.4.1.** Some particular and important consequences of the previous theorem are:

1. if $f \in H_k^p(M)$ and $p > n$, then $f \in C^{k-1}(M)$,
2. the inclusion $H_{k+1}^p(M) \hookrightarrow H_k^p(M)$ is compact,
3. if $f \in H_k^p(M)$ and $k > n/p$, then $f \in C^0(M)$,
4. $C^\infty = \bigcap_k H_k^p$ for any $1 < p < \infty$,

5. if $f \in H_1^2(M)$, then $f \in L^{2n/(n-2)}(M)$ for $n \geq 3$, and there are constants $A, B > 0$ independent of $f$ such that

$$\|f\|_{L^{2n/(n-2)}} \leq A\|\nabla f\|_{L^2} + B\|f\|_{L^2}.$$

The value $q = 2n/(n-2)$ is the largest value for which (1.2) holds in the case $k = 1$, $p = 2$. In fact, it is a limit case of the Sobolev inequality.

## 1.5 Elliptic Differential Operators

Let $M$ be an $n$-dimensional Riemannian manifold, the following definitions extend the usual notion of partial differential operators on domains in the Euclidean space to partial differential operators on $M$.

**Definition 1.5.1.** A *partial differential operator $P$ on $M$ of order $k$* is an operator taking real functions on $M$ to real functions on $M$, that depends on $u$ and its first $k$ derivatives. For a real function $u$ on $M$ whose first $k$ derivatives $\nabla u, \ldots \nabla^k u$ exist (even in some weak sense), then $Pu$ as a function on $M$ is written as

$$Pu(x) = f\big(x, u(x), \nabla u(x) \ldots, \nabla^k u(x)\big), \tag{1.3}$$

where $f$ is a real function of its arguments, generally at least continuous. The differential operator $P$ is called smooth when $f$ is a smooth function of its arguments and linear when it is linear in $u$.

*Notation.* More explicitly, a linear differential operator of order $k$ on $M$, acting on a function $u \in C^k(M)$, is written in a local chart $(U, \phi)$ as an expression of the form:

$$Pu = \sum_{l=0}^{k} a_l^{i_1 \ldots i_l} \nabla_{i_1 \ldots i_l} u =: \sum_{l=0}^{k} a_l \nabla^l u, \tag{1.4}$$

where the $a_l$'s are symmetric $l$-tensors called the coefficients of $P$ and in the case $l = 0$, the coefficient $a_0$ is understood to be a real function on $M$. In addition the leading coefficients $a_k^{i_1 \ldots i_k}$ are presumed to be non-zero. In this notation, a summation convention is understood on repeated upper and lower indices, this will appear in general expressions for differential operators and in the definition of the symbol below. The right hand side of (1.4) must be understood as an abbreviated (although a bit abusive) notation just for simplicity.

**Remark 1.5.1.** When $P$ is smooth of order $k$ and $u \in C^\infty(M)$, then $Pu \in C^\infty(M)$ so we have a map $P : C^\infty(M) \longrightarrow C^\infty(M)$. However $P$ is not necessarily required to be smooth. If, for example, $P$ is linear of order $k$ and the coefficients of $P$ are just bounded, we have a linear map $P : H_{k+l}^p(M) \longrightarrow H_l^p(M)$, and if the coefficients of $P$ are at least $C^{l,\alpha}$, then $P : C^{k+l,\alpha} \longrightarrow C^{l,\alpha}(M)$ is again linear.

**Definition 1.5.2.** Let $P$ be a partial differential operator of order $k$. For each point $x \in M$ and each $\xi \in T_x^*M$, define $\sigma_\xi(P;x) := a_k^{i_1\dots i_k}\xi_{i_1}\cdots\xi_{i_k}$. Let us denote by $\sigma(P): T^*M \longrightarrow \mathbb{R}$ the function $\xi \mapsto \sigma_\xi(P;x)$. Then $\sigma(P)$ is called the *symbol (or principal symbol) of P*. It is a homogeneous polynomial of degree $k$ on each cotangent space.

**Definition 1.5.3.** A linear differential operator $P$ of order $k$ on $M$ is said to be *elliptic* if for each $x \in M$ and each non-zero $\xi \in T_x^*M$, we have $\sigma_\xi(P;x) \neq 0$.

If follows from the previous definitions that the principal symbol $\sigma(P)$ of an elliptic operator $P$ must be non-zero on the complement of the zero section of $T^*M$, i.e. on each $T_x^*M \setminus \{0\}$. Using this, one can conclude that every elliptic operator defined on a manifold $M$ with $\dim M > 1$ has necessarily even degree.

When this is the case, an operator $P$ of degree $k = 2m$ is said to be given in *divergence form* if we can write $Pu$ in the following way,

$$Pu = \sum_{l,l'=0}^{m} \nabla_{i_1\dots i_l}(a_{l,l'}^{i_1\dots i_l j_1\dots j_{l'}}\nabla_{j_1\dots j_{l'}}u) + \sum_{l=0}^{m} b_l^{i_1\dots i_l}\nabla_{i_1\dots i_l}u,$$

where the coefficients $a_{l,l'}$ are $(l+l')$-tensors and $b_l$ are $l$-tensors.

In a local coordinate chart $(U,\phi)$ of $M$, the condition of ellipticity of the operator $P$ is equivalent to saying that at a point $x \in U$, there exists $\lambda(x) \geq 1$ such that for all vectors $\xi \in \mathbb{R}^n$ the following inequalities are satisfied:

$$\|\xi\|^k\lambda^{-1}(x) \leq a_k^{i_1\dots i_k}(x)\xi_{i_1}\dots\xi_{i_k} \leq \lambda(x)\|\xi\|^k.$$

**Definition 1.5.4.** Let $P$ be a non-linear differential operator of order $k$ defined by a function $f$ that is at least $C^1$ in its arguments $u, \nabla u, \dots, \nabla^k u$, according to the notation used in (1.3). Also, suppose $u$ is a real function with $k$ derivatives. The *linearization $L_uP$ of P at u* is defined as the derivative of $P(v)$ with respect to $v$ at $u$, that is

$$L_uPv = \lim_{\delta\to 0}\left(\frac{P(u+\delta v) - P(u)}{\delta}\right). \tag{1.5}$$

This $L_uP$ is a linear differential operator of order $k$. If $P$ is linear then $L_uP = P$, but note that even if $P$ is smooth, $L_uP$ does not need to be smooth if $u$ is not smooth. For instance, if $P$ is of order $k$ and $u \in C^{k+l}(M)$, then $L_uP$ will have $C^l$ coefficients in general, since they depend on the $k$-th derivatives of $u$.

**Remark 1.5.2.** A non-linear differential operator $P$ of degree $k$ on $M$ is elliptic at a function $u$ (assumed to have $k$ derivatives) if the linearization $L_uP$ of $P$ at $u$ is elliptic. According to this, $P$ may be elliptic at some functions $u$ and not at others.

### 1.5.1 Elliptic Operators on Vector Bundles

The same previous discussion of differential operators on manifolds can be generalized to differential operators on vector bundles in a natural way, where functions on $M$ are replaced by sections of the vector bundle. Let $E$ and $F$ be vector bundles over $M$, denote by $\nabla$ a connection on $TM$ and by $\nabla^E$ a connection on $E$. Given a section $u \in \Gamma(E)$ one can form repeated derivatives of $u$ by coupling the connections $\nabla$ and $\nabla^E$. Let us denote by $\nabla^E_{i_1 \dots i_k} u$ the $k$-th derivative of $u$ defined in this manner.

**Definition 1.5.5.** A differential operator $P$ of order $k$ taking sections of $E$ to sections of $F$ is explicitly given, for a $k$ times differentiable section $u$ of $E$ by

$$Pu(x) = f\big(x, u(x), \nabla^E_{i_1} u(x) \dots, \nabla^E_{i_1 \dots i_k} u(x)\big) \in F_x$$

for $x \in M$. Again $P$ is called *smooth*, *linear* or *non-linear* according to the smoothness of $f$ and the linearity or non-linearity of this expression with respect to $u$.

For instance, if $P$ is a smooth, linear differential operator of order $k$ from $E$ to $F$, then $P$ can act as follows: $P : C^\infty(E) \longrightarrow C^\infty(F)$, $P : C^{k+l,\alpha}(E) \longrightarrow C^{l,\alpha}(F)$ and $P : H^p_{k+l}(E) \longrightarrow H^p_l(F)$.

*Notation.* Let $P$ be a linear differential operator of order $k$ from $E$ to $F$. In index notation, we write

$$Pu = \sum_{l=0}^{k} a_l^{i_1 \dots i_l} \nabla_{i_1 \dots i_l} u =: \sum_{l=0}^{k} a_l \nabla^l u,$$

where the quantities $a_l^{i_1, \dots, i_l}$ are tensors with values in $E^* \otimes F$ (the subindices for $F$ and $E^*$ are omitted), called the coefficients of $P$. Again, the summation convention for repeated indices is used. The right-hand side abbreviation is used for simplicity as before in case no confusion arises from the context.

The next step is to generalize the notions of the symbol and ellipticity for differential operators on functions given before. For each point $x \in M$ and each $\xi \in T^*_x M$, we define a linear map from $E_x$ to $F_x$ given by

$$\sigma_\xi(P; x) = a_k^{i_1 \dots i_k} \xi_{i_1} \dots \xi_{i_k}.$$

Let $\sigma(P) : T^*M \times E \longrightarrow F$ be the bundle map defined by $\sigma(P)(\xi, u) = \sigma_\xi(P; x) u \in F_x$ for every $x \in M$, $\xi \in T^*_x M$ and $u \in F_x$. Then $\sigma(P)$ is called the *symbol* (or *principal symbol*) of $P$, and $\sigma(P)(\xi, u)$ is homogeneous of degree $k$ in $\xi$ and linear in $u$.

**Definition 1.5.6.** A linear differential operator $P$ of degree $k$ from $E$ to $F$ is called an *elliptic operator* if for each $x \in M$ and each non-zero $\xi \in T^*_x M$, the linear map $\sigma_\xi(P; x) : E_x \longrightarrow F_x$ is invertible, where $\sigma(P)$ is the principal symbol of $P$.

In case of ellipticity, $\sigma_\xi(P;x)$ determines an isomorphism of vector spaces between the fibers, so the vector bundles $E$ and $F$ have the same rank. If $P$ is nonlinear and $u$ is a section of $E$ with $k$ derivatives, then $P$ is said to be elliptic at $u$ if the linearization $L_uP$ of $P$ at $u$ is elliptic.

### 1.5.2 Formal Adjoint of an Operator

Suppose that $E$ and $F$ are vector bundles endowed with corresponding metrics on the fibers over a compact Riemannian manifold without boundary $(M,g)$ and let $P$ be a linear differential operator of order $k$ from $E$ to $F$, with coefficients at least $k$ times differentiable. Then there is a unique linear differential operator $P^*$ of order $k$ from $F$ to $E$, with continuous coefficients, such that

$$\langle Pu, v\rangle_{L^2(F)} = \langle u, P^*v\rangle_{L^2(E)}$$

for all sections $u \in H_k^2(E)$ and $v \in H_k^2(F)$. Here, the inner products on the Hilbert spaces of sections $H_k^2(E)$ and $H_k^2(F)$ are the natural ones, defined for example in $E$ by $\langle u,v\rangle_{L^2(E)} := \int_M (u,v)\,\mathrm{dvol}_g$, where $(\cdot,\cdot)$ represents here the metrics on the fibers of $E$. This operator $P^*$ is called the *adjoint* or *formal adjoint* of $P$. Additionally, the following properties are satisfied: $(P^*)^* = P$ for any $P$ and if $P$ is smooth (or elliptic) then $P^*$ is also smooth (or elliptic respectively). When $E = F$ and $P = P^*$ then $P$ is called *self-adjoint*.

## 1.6 Weak Solutions on Manifolds

Given a linear differential operator $P$ of order $k$ defined on a Riemannian manifold $(M,g)$, a solution of the equation $Pu = f$ is a function $u \in C^k(M)$ that satisfies the equation pointwise. Nevertheless, this equation can also be satisfied in a "weak" sense when $u$ is allowed to be an element of $H_k^p(M)$, i.e. a distribution.

**Definitions 1.6.1.** The following definitions deal with these more general solutions for different cases (see [Aub98]):

1. If $f \in L^p$ and if $P$ has locally bounded and measurable coefficients, then an element $u \in H_k^p(M)$ is a *strong solution* of $Pu = f$ in the $L^p$ sense if there is a sequence of smooth functions $\{u_i\}_{i\geq 1}$ on $M$ such that $u_i \to u$ in $H_k^p(M)$ and $Pu_i \to f$ in $L^p(M)$. In this case, the weak (distributional) derivatives of $u$ up to order $k$ are functions in $L^p(M)$ and the equation $Pu = f$ is satisfied almost everywhere.

2. Suppose $Pu = \sum_{l=0}^{k} a_l^{i_1\dots i_l}\nabla_{i_1\dots i_l} u$. If the tensors $a_l^{i_1\dots i_l} \in C^l(M)$, for $0 \le l \le k$, then the *formal adjoint* of $P$, acting on a function $v \in C^l(M)$, is given by

$$P^* v = \sum_{l=0}^{k} (-1)^l \nabla_{i_1\dots i_l}(v\, a_l^{i_1\dots i_l}). \tag{1.6}$$

We say that an element $u \in L^1(M)$ satisfies $Pu = f$ in the sense of distributions (or weak sense), if for all compactly supported test functions $\eta \in C_c^\infty(M)$:

$$\int_M u P^* \eta \,\mathrm{dvol}_g = \int_M f\eta \,\mathrm{dvol}_g\,.$$

If $P$ has smooth coefficients $a_l^{i_1\dots i_l} \in C^\infty(M)$, we say that a distribution $u \in H_r^2(M)$ satisfies $Pu = f$ if for all $\eta \in C_c^\infty(M)$:

$$\langle u, P^*\eta\rangle_{L^2} = \langle f, \eta\rangle_{L^2}.$$

Since a distribution $u \in H_r^2(M)$ involves a weak derivative of order $r$ of a locally integrable function, the product $\langle u, P^*\eta\rangle_{L^2}$ makes sense even when the coefficients $a_l^{i_1\dots i_l} \in C^{l+r}(M)$ and not only for smooth ones, as the terms in the expression (1.6) for $P^*$ indicate.

3. If a differential operator $P$ of degree $k = 2m$ can be written in *divergence form*, i.e.

$$Pu = \sum_{l,l'=0}^{m} \nabla_{i_1\dots i_l}(a_{l,l'}^{i_1\dots i_l j_1\dots j_{l'}} \nabla_{j_1\dots j_{l'}} u) + \sum_{l=0}^{m} b_l^{i_1\dots i_l}\nabla_{i_1\dots i_l} u,$$

then $u \in H_m^p(M)$ is said to be a *weak solution* of $Pu = f$ with $f \in L^1(M)$ if for all $\eta \in C_c^\infty(M)$:

$$\sum_{l,l'=0}^{m} (-1)^k \int_M a_{l,l'} \nabla^{l'} u \nabla^l \eta \,\mathrm{dvol}_g + \sum_{l=0}^{m} \int_M \eta\, b_l \nabla^l u \,\mathrm{dvol}_g = \int_M f\eta \,\mathrm{dvol}_g,$$

where the only requirement needed is that the coefficients $a_{l,l'}$ must be measurable and locally bounded for all pairs $(l, l')$. These definitions depend on the particular properties of the coefficients and hence the generalized solutions are not equivalent.

4. More generally, given a non-linear differential operator $P$ of degree $k$ with the form

$$Pu = \sum_{l=0}^{k} (-1)^l \nabla_{i_1\dots i_l} A_l^{i_1\dots i_l}(x, u, \nabla u, \dots \nabla^k u) =: \sum_{l=0}^{k} (-1)^l \nabla^l A_l,$$

where $A_l$ are $l$-tensors on $M$, a function $u \in C^k(M)$ is said to be a *weak solution* of $Pu = 0$, if for all test functions $\eta \in C_c^\infty(M)$:

$$\sum_{l=0}^{k} \int_M A_l^{i_1 \dots i_l} \nabla_{i_1 \dots i_l} \eta \, \mathrm{dvol}_g = 0.$$

**Remark 1.6.1.** Given that the previous definitions of weak solutions for elliptic PDE's are not equivalent and in particular depend strongly on the regularity of the coefficients of the partial differential operator $P$, we want to draw attention to the case 2 in the list of Definitions 1.6.1 where a weak solution is defined for coefficients of $P$ of class $C^l(M)$. When $P$ is the Laplacian operator on manifolds $\Delta_g$ and the metric $g$ is at least $C^2$, this is the natural case to consider (with $l = 2$) to define weak solutions of the Poisson equation. Nevertheless for a metric of less regularity like $C^{1,\alpha}$ which we will study in chapter 3, a different definition will be given to deal with the fact that the coefficients of $\Delta_g$, which depend on $g$, cannot be differentiated twice. This will require the solutions to be at least $C^1$ from the beginning (compare with Definition 3.3.2).

# 1.7 Regularity for Linear Elliptic Equations

The results in this section, devoted to the general setting of vector bundles, can be found in [Joy00], [Bes87] and [Kaz93]. Recommended references for a complete treatment of the theory of partial differential equations, where proofs of the results appearing below for domains in $\mathbb{R}^n$ can be found, are for example [Eva98] and the classical [GT77].

## 1.7.1 Existence of Solutions to Elliptic PDE's

Here we concentrate already on partial elliptic operators with $C^{k,\alpha}$ coefficients, which are required for our applications. It is a slight generalization of the corresponding case for smooth coefficients. Existence results for equations $Pu = v$, which require a simple condition $v \perp \ker P^*$ for $u$ to exist, are known as the *Fredholm alternative*. For more details on this (and the Sobolev spaces case), see [Joy00, section 1.5].

**Theorem 1.7.1.** *Let $k, l \in \mathbb{Z}$ with $0 < k \leq l$, and $\alpha \in (0,1)$. Denote by $E$ and $F$ two vector bundles over a compact Riemannian manifold $(M, g)$, endowed with corresponding metrics in their fibers, and suppose $P$ is a linear elliptic operator of order $k$ from $E$ to $F$ with $C^{l,\alpha}$ coefficients. Then $P^*$ is elliptic with $C^{l-k,\alpha}$ coefficients, and the kernels* $\ker P$, $\ker P^*$ *are finite-dimensional subspaces of $C^{k+l,\alpha}(E)$ and $C^{l,\alpha}(F)$*

*respectively. If* $v \in C^{l,\alpha}(F)$ *then there exists* $u \in C^{k+l,\alpha}(E)$ *with* $Pu = v$ *if and only if* $v \perp \ker P^*$*, and if it is required that* $u \perp \ker P$ *then* $v$ *is unique.*

### 1.7.2 Global Elliptic Estimates

Let us assume here that $(M,g)$ is a compact Riemannian manifold and let $E,F$ be vector bundles over $M$ with the same rank. First we state the following regularity result for smooth linear elliptic differential operators in the context of vector bundles.

**Theorem 1.7.2** (Global elliptic estimates)**.** *Suppose* $P : C^\infty(E) \longrightarrow C^\infty(F)$ *is a smooth linear elliptic differential operator of order* $k$ *from* $E$ *to* $F$*. Let* $\alpha \in (0,1)$*,* $p > 1$*, and* $l \geq 0$ *be an integer. Furthermore, assume* $Pu = v$ *holds weakly with* $u \in L^1(E)$ *and* $v \in L^1(F)$*.*

1. *($L^p$ estimates) If* $v \in H^p_l(F)$ *then* $u \in H^p_{k+l}(E)$*, and*

$$\|u\|_{H^p_{k+l}(E)} \leq C\big(\|Pu\|_{H^p_k(F)} + \|u\|_{L^1(E)}\big), \tag{1.7}$$

*for some* $C > 0$ *independent of* $u$ *and* $v$*. In particular, if* $v \in C^\infty(F)$*, then together with Remark 1.4.1 we get* $u \in C^\infty(E)$*.*

2. *(Schauder estimates) If* $v \in C^{l,\alpha}(F)$*, then* $u \in C^{k+l,\alpha}(E)$*, and*

$$\|u\|_{C^{k+l,\alpha}(E)} \leq C\big(\|Pu\|_{C^{l,\alpha}(F)} + \|u\|_{C^0(E)}\big), \tag{1.8}$$

*for some* $C > 0$ *independent of* $u$ *and* $v$*.*

The previous estimates can be generalized to the case where $P$ has Hölder continuous coefficients instead of smooth ones. In the next theorem the explicit Schauder estimates for this more general case are presented. They will be useful for the study of the Einstein condition for $C^{1,\alpha}$ metrics that we consider later.

**Theorem 1.7.3.** *Let* $P$ *be a linear elliptic differential operator of order* $k$ *from* $E$ *to* $F$*. Let* $\alpha \in (0,1)$ *and* $l \geq 0$ *be an integer. Suppose that the coefficients of* $P$ *are in* $C^{l,\alpha}(M)$ *and that,* $Pu = v$ *(almost everywhere) for some* $u \in C^{k,\alpha}(E)$*,* $v \in C^{l,\alpha}(F)$*. Then* $u \in C^{k+l,\alpha}(E)$ *and*

$$\|u\|_{C^{k+l,\alpha}(E)} \leq C\big(\|Pu\|_{C^{l,\alpha}(F)} + \|u\|_{C^0(E)}\big), \tag{1.9}$$

*for some* $C > 0$ *independent of* $u$ *and* $v$*.*

### 1.7.3 Regularity for Weak Solutions

Weak solutions of linear elliptic equations of second order have also useful regularity properties from which we select the following relevant one (taken from [Aub98, p. 86]), that can be easily applied to the interior of a manifold. It will play an important role in our regularity results of chapter 3.

**Theorem 1.7.4** (Ladyzhenskaya and Uraltseva)**.** *Let $U$ be an open set of $\mathbb{R}^n$ and $P$ a second order linear elliptic operator, with $C^{k,\alpha}$ coefficients for an integer $k \geq 0$ and $\alpha \in (0,1)$. If a bounded function $u \in H_2^2(U)$ satisfies the relation $Pu = f$ almost everywhere for a function $f \in C^{k,\alpha}(U)$, then $u \in C^{k+2,\alpha}(U)$. The same conclusion is true when $u \in H_1^2(U)$ is a weak solution of $Pu = f$ and the operator $P$ can be written in divergence form.*

As mentioned in the statement, this theorem can be traced back to [LU68, p. 195]. Some related results follow from general theorems for weak solutions appearing in [Mor66, chapters 5 and 6]. We emphasize here that a second order elliptic operator $P$, acting on functions on a manifold $M$, is given in local coordinates $(x^1, \ldots, x^n) : U \subset M \longrightarrow \mathbb{R}^n$ by a expression of the type

$$P = \sum_{i,j=1}^{n} a^{ij} \partial_i \partial_j + \sum_{i=1}^{n} a^i \partial_i.$$

According to the previous theorem, this operator is required to admit a divergence form (Definition 1.6.1), which in this case is

$$P = \sum_{i,j=1}^{n} \partial_i (a^{ij} \partial_j) + \sum_{i=1}^{n} b^i \partial_i,$$

in order to preserve the appropriate regularity for the case of weak solutions. This forces the coefficients to be at least differentiable, i.e. $a^{ij} \in C^1(U)$.

# Chapter 2

# Spin Manifolds and Killing Spinors

The aim of this chapter is to introduce the required tools from Riemannian spin geometry to study the classical Dirac operator and consider its relation to Killing spinors and the Einstein curvature condition. Good references for these topics are [LM89], [Fri00] and the lecture notes [Hij99]. Additionally, a compendium of this and related material has recently appeared in [BBC]. Other references will be provided on the way for particular details. Throughout this chapter, we implicitly assume that the Riemannian manifolds considered have a smooth differentiable structure and the corresponding metrics are either smooth or their degree of differentiability is enough to define all the objects involved in the discussion.

## 2.1 Metric and Topological Spin Structures

It is useful for our purposes to discuss about the concept of metric and topological spin structures separately and consider afterwards their relationship. This notions will be relevant in chapter 4, in the context of the identification procedure introduced in [BG92].

Given a Riemannian $n$-manifold $(M,g)$, let $P_{GL}M$ be the $GL(n,\mathbb{R})$-principal frame bundle of $M$ and denote by $P_{GL^+}M$ its reduction to the bundle of positively oriented frames, for which the structural group has been restricted to orientation preserving automorphisms $GL^+(n,\mathbb{R}) \subset GL(n,\mathbb{R})$.

We know that there exists a unique connected double-covering homomorphism $\widetilde{GL}^+(n,\mathbb{R}) \longrightarrow GL^+(n,\mathbb{R})$, which for $n \geq 3$ is in fact a universal covering (i.e. simply-connected). The restriction of this map to $Spin(n) \subset \widetilde{GL}^+(n,\mathbb{R})$ induces a two-fold homomorphism onto $SO(n) \subset GL^+(n,\mathbb{R})$, which we will denote by $\mathrm{Ad} : Spin(n) \longrightarrow SO(n)$. In conclusion, we have the following diagram where the horizontal levels are short exact sequences

$$\begin{array}{ccccccccc}
\{0\} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \widetilde{GL}^+(n,\mathbb{R}) & \xrightarrow{\times 2} & GL^+(n,\mathbb{R}) & \longrightarrow & \{1\} \\
 & & & & \uparrow & & \uparrow & & \\
\{0\} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & Spin(n) & \xrightarrow[\mathrm{Ad}]{} & SO(n) & \longrightarrow & \{1\}.
\end{array}$$

**Definition 2.1.1.** A *topological spin structure* for $M$ is a lifting of $P_{GL^+}M$ to a $\widetilde{GL}^+(n,\mathbb{R})$-principal bundle $P_{\widetilde{GL}^+}M$ by means of a double covering $\chi$

$$\begin{array}{ccc}
 & & P_{\widetilde{GL}^+}M \\
 & \overset{\tilde{s}}{\nearrow} & \downarrow \chi \\
U \subset M & \xrightarrow[s]{} & P_{GL^+}M
\end{array}$$

that respects the fiberwise action of the corresponding structure groups on the bundles. In a similar way, a *metric spin structure* for $(M,g)$ is a lifting of the $SO(n)$-principal bundle of $g$-orthonormal frames $P_{SO}(M,g)$ to a $Spin(n)$-principal bundle $P_{Spin}(M,g)$, such that the covering application $\hat{\chi} : P_{Spin}(M,g) \longrightarrow P_{SO}(M,g)$ preserve the actions of the respective groups involved.

**Definition 2.1.2.** Two topological spin structures given by $\chi_1 : P^1_{\widetilde{GL}^+}M \longrightarrow P_{GL^+}M$ and $\chi_2 : P^2_{\widetilde{GL}^+}M \longrightarrow P_{GL^+}M$ are *equivalent* if there is a $GL^+(n,\mathbb{R})$-equivariant map $\Upsilon$ that completes the diagram

$$\begin{array}{ccc}
P^1_{\widetilde{GL}^+}M & \xrightarrow{\Upsilon} & P^2_{\widetilde{GL}^+}M \\
 {\scriptstyle \chi_1}\searrow & & \swarrow{\scriptstyle \chi_2} \\
 & P_{GL^+}M. &
\end{array}$$

Analogously, we say that two metric spin structures $\hat{\chi}_1 : P^1_{Spin}(M,g) \longrightarrow P_{SO}(M,g)$ and $\hat{\chi}_2 : P^2_{Spin}(M,g) \longrightarrow P_{SO}(M,g)$ are *equivalent* if there is a $Spin(n)$-equivariant application $\hat{\Upsilon} : P^1_{Spin}(M,g) \longrightarrow P^2_{Spin}(M,g)$ which factorizes the coverings as in the previous diagram, i.e., $\hat{\chi}_1 = \hat{\chi}_2 \circ \hat{\Upsilon}$.

*Notation.* When the metric $g$ is clear from the context we will denote a metric spin structure $P_{Spin}(M,g)$ simply by $P_{Spin}M$ and will refer to it as a $g$-spin structure.

**Remark 2.1.1.** Note that the definition of equivalence between metric spin structures just given assumes that the base manifold has the same Riemannian metric $g$. This assumption will be now improved to include the case of different metrics. Given a topological spin structure $\chi : P_{\widetilde{GL^+}}M \longrightarrow P_{GL^+}M$ on a Riemannian manifold $(M,g)$, the inclusions $P_{Spin}M \subset P_{\widetilde{GL^+}}M$ and $P_{SO}M \subset P_{GL^+}M$, imply that the restricted map $\hat{\chi} := \chi|_{P_{Spin}M} : P_{Spin}M \longrightarrow P_{SO}M$ defines a metric spin structure for $(M,g)$. Conversely, any metric spin structure $\hat{\chi}$ can be extended into a topological one $\chi$. In this way, there is an induced $Spin(n)$-equivariant bijection between equivalence classes of metric and topological spin structures (cf. [Swi93] for details).

**Theorem 2.1.1.** *Let $g_1$, $g_2$ be two Riemannian metrics on a spin manifold $M$ with fixed metric spin structures $\hat{\chi}_1$ and $\hat{\chi}_2$, respectively. Assume additionally that there exists an $SO(n)$-equivariant map $\zeta : P_{SO}(M,g_1) \longrightarrow P_{SO}(M,g_2)$. Then, the underlying topological spin structures $\chi_1$ and $\chi_2$ are equivalent, if and only if the induced bundle $\zeta^*\hat{\chi}_2$ defines a metric spin structure on $(M,g_1)$ equivalent to $\hat{\chi}_1$.*

*Proof.* Suppose that the topological spin structures $\chi_1$ and $\chi_2$ are equivalent. Then, the $SO(n)$-equivariant application $\zeta : P_{SO}(M,g_1) \longrightarrow P_{SO}(M,g_2)$ can be lifted to a $Spin(n)$-equivariant map $\widetilde{\zeta} : P_{Spin}(M,g_1) \longrightarrow P_{Spin}(M,g_2)$ such that the following diagram commutes,

$$\begin{array}{ccc} P_{Spin}(M,g_1) & \xrightarrow{\widetilde{\zeta}} & P_{Spin}(M,g_2) \\ \Big\downarrow \hat{\chi}_1 & & \Big\downarrow \hat{\chi}_2 \\ P_{SO}(M,g_1) & \xrightarrow[\zeta]{} & P_{SO}(M,g_2). \end{array}$$

Recall, with reference to the diagram below, that the induced bundle $\zeta^*\hat{\chi}_2$, defined by the pullback of $\hat{\chi}_2$ with respect to $\zeta$, has total space

$$\zeta^*P_{Spin}(M,g_2) := \{(s,B) \in P_{Spin}(M,g_2) \times P_{SO}(M,g_1) \mid \hat{\chi}_2(s) = \zeta(B)\},$$

with the covering $\zeta^*\hat{\chi}_2 : \zeta^*P_{Spin}(M,g_2) \longrightarrow P_{SO}(M,g_1)$ given by the projection $\mathrm{pr}_2$ onto the second component. The action of $Spin(n)$ associated to $g_1$ on this bundle is given by

$$\begin{aligned} \zeta^*P_{Spin}(M,g_2) \times Spin(n) &\longrightarrow \zeta^*P_{Spin}(M,g_2) \\ ((s,B),\varsigma) &\longmapsto (s,B)\cdot_1 \varsigma := (s\cdot_2 \varsigma, B\cdot_1 \mathrm{Ad}(\varsigma)), \end{aligned}$$

where we denote by $\cdot_i$ the corresponding action (of $Spin(n)$ or $SO(n)$ according to the context), associated to the metric $g_i$. The next diagram ilustrates the construction,

$$\begin{array}{ccc} \zeta^* P_{Spin}(M,g_2) & \xrightarrow{\mathrm{pr}_1} & P_{Spin}(M,g_2) \\ {\scriptstyle \zeta^*\hat{\chi}_2 := \mathrm{pr}_2}\downarrow & & \downarrow{\scriptstyle \hat{\chi}_2} \\ P_{SO}(M,g_1) & \xrightarrow[\zeta]{} & P_{SO}(M,g_2). \end{array}$$

Since $\hat{\chi}_2(s \cdot_2 \varsigma) = \hat{\chi}_2(s) \cdot_2 \mathrm{Ad}(\varsigma)$ and $\zeta(B \cdot_1 \mathrm{Ad}(\varsigma)) = \zeta(B) \cdot_2 \mathrm{Ad}(\varsigma)$, this action is well-defined on the induced bundle, and is projected under $\zeta^*\hat{\chi}_2$ to the action of $SO(n)$ on $P_{SO}(M,g_1)$. In this manner, the induced bundle construction defines a metric spin structure for $(M,g_1)$.

Now, we show explicitly that $\hat{\chi}_1$ and $\zeta^*\hat{\chi}_2$ yield equivalent metric spin structures for $(M,g_1)$. Let us choose an arbitrary element $s \in P_{Spin}(M,g_1)$, then the pair $(\widetilde{\zeta}(s), \hat{\chi}_1(s)) \in P_{Spin}(M,g_2) \times P_{SO}(M,g_1)$, and since $\hat{\chi}_2 \circ \widetilde{\zeta} = \zeta \circ \hat{\chi}_1$, it follows that $(\widetilde{\zeta}(s), \hat{\chi}_1(s)) \in \zeta^* P_{Spin}(M,g_2)$. This means that the map $\Upsilon(s) := (\widetilde{\zeta}(s), \hat{\chi}_1(s))$ defines an application which makes the following diagram commutative

$$\begin{array}{ccc} P_{Spin}(M,g_1) & \xrightarrow{\Upsilon} & \zeta^* P_{Spin}(M,g_2) \\ & {\scriptstyle \hat{\chi}_1}\searrow \quad \swarrow{\scriptstyle \zeta^*\hat{\chi}_2} & \\ & P_{SO}(M,g_1). & \end{array}$$

By similar arguments as invoked before, $\Upsilon$ is $Spin(n)$-equivariant and behaves appropriately under the projection to $P_{SO}(M,g_1)$. Then it is a bundle morphism that determines the claimed equivalence.

The converse part of this theorem follows by an analogous procedure, using that the equivalence of the metric spin structures should come from appropriate restrictions of the corresponding topological spin structures. □

As a conclusion we have shown that there is a bijection between equivalence classes of metric and topological spin structures and the use of one or another description is a matter of convenience according to the context. Moreover, the induced bundle construction provides a natural way to pass from equivalent topological spin structures to equivalent metric spin structures when the identification of metric spinors introduced in [BG92] is possible.

**Definition 2.1.3.** In case the conditions of Theorem 2.1.1 are satisfied, we say that for a spin manifold $M$, the metric spin structures $\hat{\chi}_1 : P_{Spin}(M,g_1) \longrightarrow P_{SO}(M,g_1)$ and $\hat{\chi}_2 : P_{Spin}(M,g_2) \longrightarrow P_{SO}(M,g_2)$ coming from different metrics $g_1$ and $g_2$ are equivalent.

## 2.2 The Connection on Spinor Bundles

As a reminder of the construction of spinor fields we collect here some important facts that are well detailed in the references cited at the beginning of this chapter.

First, recall that for a complex vector space $\Sigma_n \cong \mathbb{C}^{2^{[\frac{n}{2}]}}$ the *complex spin representation* is given by a map $\sigma : Spin(n) \longrightarrow \mathrm{Aut}(\Sigma_n)$ which is induced by any irreducible complex finite dimensional representation of the whole Clifford algebra $C\ell(\mathbb{R}^n)$. It is important to emphasize here that the metric $g$ appears in the definition of the Clifford algebra and hence, is implicitly carried in these notions.

**Definition 2.2.1.** The *complex spinor bundle* associated to a (metric) spin structure $P_{Spin}(M,g)$ of a Riemannian spin manifold $(M,g)$ is the complex vector bundle

$$\Sigma^g M := P_{Spin}(M,g) \times_\sigma \Sigma_n,$$

endowed with a *Hermitian metric* $(\cdot\,,\cdot)$ defined fiberwise as the natural Hermitian product on $\Sigma_n$. A section $\psi \in \Gamma(\Sigma^g M)$ is called a *spinor field* over $M$. When the metric in question is clear from the context we denote the spinor bundle simply by $\Sigma M$.

In the case $n = 2m$, there is a splitting $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$ of the complex spin representation, denoted by $\sigma^\pm$, that induces a splitting of the spinor bundle $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$, where $\Sigma^\pm M := P_{Spin}M \times_{\sigma^\pm} \Sigma_n^\pm$.

Since the tangent bundle $TM$ is a vector bundle associated to the $SO(n)$-principal bundle of orthonormal frames $P_{SO}M$ by means of the $SO(n)$-action on $\mathbb{R}^n$ (i.e., $TM = P_{SO}M \times_{SO(n)} \mathbb{R}^n$), we can compose this action with the double-covering homomorphism $\mathrm{Ad} : Spin(n) \longrightarrow SO(n)$ to obtain an identification

$$TM \simeq P_{Spin}M \times_{\mathrm{Ad}} \mathbb{R}^n.$$

Seeing $TM$ in this manner we have that *Clifford multiplication* $c : \mathbb{R}^n \otimes_{\mathbb{R}} \Sigma_n \longrightarrow \Sigma_n$ induces an action

$$c : TM \otimes \Sigma^g M \longrightarrow \Sigma^g M$$

which is given by Clifford multiplication on each fiber. Thus, for any $X \in \Gamma(TM)$, $\psi \in \Gamma(\Sigma M)$ and $p \in M$ this action is $(X \cdot \psi)_p := (X_p \cdot \psi_p)$.

For all $X \in \Gamma(TM)$ and $\psi, \varphi \in \Gamma(\Sigma M)$ the Clifford multiplication and the Hermitian metric on $\Sigma M$ satisfy

$$(X \cdot \psi, \varphi) = -(\psi, X \cdot \varphi). \tag{2.1}$$

Any connection over the principal bundle $P_{SO}M$ of a spin manifold $M$ has a natural lifting to a connection over $P_{Spin}M$. If $M$ is also a Riemannian manifold, the Levi-Civita connection lifts to a *spinorial Levi-Civita* connection over $P_{Spin}M$.

To ilustrate this procedure, first consider a given simply connected open subset $U \subset M$ and a local section $s \in \Gamma_U(P_{SO}M)$, so we have a lifting to a section $\tilde{s} \in \Gamma_U(P_{Spin}M)$, i.e.,

$$\begin{array}{ccc} & & P_{Spin}M \\ & \overset{\tilde{s}}{\nearrow} & \downarrow \chi \\ U \subset M & \xrightarrow[s]{} & P_{SO}M. \end{array}$$

Let us denote by $\mathrm{Ad}_* : \mathfrak{spin}(n) \longrightarrow \mathfrak{so}(n)$ the Lie algebra homomorphism induced by the pushforward of the double-covering $\mathrm{Ad} : Spin(n) \longrightarrow SO(n)$. The connection 1-form $\omega \in \Omega^1(P_{SO}M, \mathfrak{so}(n))$ defined on $P_{SO}M$ and taking values in the Lie algebra $\mathfrak{so}(n)$, lifts to a 1-form $\widetilde{\omega} \in \Omega^1(P_{Spin}M, \mathfrak{spin}(n))$ over $P_{Spin}M$, which means that the following diagram commutes

$$\begin{array}{ccccc} & & TP_{Spin}M & \xrightarrow{\widetilde{\omega}} & \mathfrak{spin}(n) \\ & \overset{\tilde{s}_*}{\nearrow} & \downarrow \chi_* & & \downarrow \mathrm{Ad}_* \\ TU \subset TM & \xrightarrow[s_*]{} & TP_{SO}M & \xrightarrow[\omega]{} & \mathfrak{so}(n). \end{array}$$

On a Riemannian spin manifold $(M,g)$, the spinorial Levi-Civita connection associated to the Riemannian structure allows us to define a covariant derivative

$$\nabla^{\Sigma} : \Gamma(\Sigma^g M) \longrightarrow \Gamma(T^*M \otimes \Sigma^g M)$$

acting on sections of the spinor bundle $\Sigma^g M$.

Let $(e_1, \ldots, e_n)$ be a local oriented orthonormal frame. The Levi-Civita connection $\nabla^{\Sigma}$ on the spinor bundle $\Sigma^g M$ can be expressed locally in terms of the Levi-Civita connection $\nabla$ on the tangent bundle $TM$ by

$$\nabla^{\Sigma}_X \psi = X(\psi) + \frac{1}{4} \sum_{i,j=1}^{n} g(\nabla_X e_i, e_j) e_i \cdot e_j \cdot \psi, \tag{2.2}$$

where $\psi \in \Gamma(\Sigma M)$, $X \in \Gamma(TM)$. In the same way, the corresponding Riemann curvature tensors $R^{\Sigma} : \Lambda^2(TM) \longrightarrow \mathrm{End}(\Gamma(\Sigma^g M))$ and $R : \Lambda^2(TM) \longrightarrow \mathrm{End}(\Gamma(TM))$ of the spinor and tangent bundles, respectively, are related by the formula

$$R^{\Sigma}(X,Y)\psi = \frac{1}{4} \sum_{i,j=1}^{n} g(R(X,Y)e_i, e_j) e_i \cdot e_j \cdot \psi. \tag{2.3}$$

Finally, we summarize the following compatibility conditions between Clifford multiplication, the Hermitian metric $(\cdot,\cdot)$ and the spinorial covariant derivative $\nabla^\Sigma$:

$$X(\psi,\varphi) = (\nabla^\Sigma_X \psi,\varphi) + (\psi,\nabla^\Sigma_X \varphi), \tag{2.4}$$

$$\nabla^\Sigma_Y (X\cdot\psi) = (\nabla_Y X)\cdot\psi + X\cdot\nabla^\Sigma_Y \psi, \tag{2.5}$$

for all $X,Y \in \Gamma(TM)$ and $\psi,\varphi \in \Gamma(\Sigma M)$. Equation (2.4) says that $\nabla^\Sigma$ is metric and (2.5) establishes an appropriate Leibniz rule for the Clifford product.

## 2.3 The Spinor Laplacian and the Dirac Operator

Given a Riemannian spin manifold $(M,g)$, denote by $\nabla^\Sigma$ the Levi-Civita spinorial covariant derivative with respect to the metric $g$.

**Lemma 2.3.1.** *The spinorial Levi-Civita connection $\nabla^\Sigma$ admits a formal adjoint $(\nabla^\Sigma)^* : \Gamma(T^*M\otimes\Sigma M) \longrightarrow \Gamma(\Sigma^g M)$, which in local normal coordinates $(e_1,\dots,e_n)$ can be written as*

$$(\nabla^\Sigma)^*\nabla^\Sigma\psi = -\sum_{i=1}^n \nabla^\Sigma_{e_i}\nabla^\Sigma_{e_i}\psi$$

*for all $\psi\in\Gamma(\Sigma M)$. More generally, this lemma is also true for any metric connection $\nabla' : \Gamma(\Sigma^g M) \longrightarrow \Gamma(T^*M\otimes\Sigma M)$ acting on spinor fields.*

**Definition 2.3.1** (Laplacian on spinors)**.** If $\psi\in\Gamma(\Sigma M)$ is a spinor field and we choose a local orthonormal frame field $(e_1,\dots,e_n)$ on an open set $U\subset M$, then the Laplace operator acting on $\psi$ is defined on $U$ by

$$\Delta^\Sigma\psi := -\sum_{i=1}^n \nabla^\Sigma_{e_i}\nabla^\Sigma_{e_i}\psi - \sum_{i=1}^n \mathrm{div}(e_i)\nabla^\Sigma_{e_i}\psi,$$

where $\mathrm{div}(X)$ is the trace of the homomorphism $Y\mapsto\nabla_Y X$. Since this definition is independent on the choice of local frame, the Laplacian is well-defined in this way everywhere on $M$.

For two spinor fields $\psi,\varphi$ over a closed (i.e. compact without boundary) manifold $M$, the Laplace operator satisfies

$$\int_M (\Delta^\Sigma\psi,\varphi)\,\mathrm{dvol}_g = \int_M (\nabla^\Sigma\psi,\nabla^\Sigma\varphi)\,\mathrm{dvol}_g = \int_M (\psi,\Delta^\Sigma\varphi)\,\mathrm{dvol}_g,$$

where $(\nabla^\Sigma\psi,\nabla^\Sigma\varphi)$ denotes the scalar product on 1-forms. This property is a direct application of Stokes' theorem (using that the boundary $\partial M=\varnothing$) and the compatibility condition (2.4) of $\nabla^\Sigma$ with the Hermitian metric.

Now we use Clifford multiplication to define the Dirac operator.

**Definition 2.3.2.** The *Dirac operator* is the composition of covariant differentiation with Clifford multiplication $c$:

$$D := c \circ \nabla^{\Sigma} : \Gamma(\Sigma^g M) \xrightarrow{\nabla^{\Sigma}} \Gamma(T^*M \otimes \Sigma^g M) \xrightarrow{c} \Gamma(\Sigma^g M).$$

With respect to a local orthonormal frame $(e_1, \ldots, e_n)$ on an open set $U \subset M$ we have

$$D\psi = \sum_{i=1}^{n} e_i \cdot \nabla^{\Sigma}_{e_i}\psi, \quad \text{for } \psi \in \Gamma(\Sigma M). \tag{2.6}$$

As before, this expression is invariant under change of frame as a straightforward calculation shows. Let us summarize in the following theorem a few important properties of the Dirac operator.

**Theorem 2.3.2.** *Let $(M, g, \chi)$ be an n-dimensional Riemannian spin manifold and denote by D its corresponding Dirac operator, then*

1. *$D : \Gamma(\Sigma M) \longrightarrow \Gamma(\Sigma M)$ is a first order elliptic partial differential operator.*

2. *If M is a compact manifold, D is formally self-adjoint with respect to the product in $L^2(\Sigma^g M)$ and its spectrum is discrete.*

3. *If $n = 2m$, the Dirac operator interchange the splitting of the spinor bundle, i.e. $D : \Gamma(\Sigma^{\pm} M) \longrightarrow \Gamma(\Sigma^{\mp} M)$, and the eigenvalues of D are symmetric with respect to the origin.*

There is a well-known relation between the square of the Dirac operator with the spinor Laplacian over a spin manifold.

**Theorem 2.3.3** (Schrödinger-Lichnerowicz formula)**.** *If* $\mathrm{scal}_g$ *denotes the scalar curvature of $(M, g)$, then*

$$D^2\psi = (\nabla^{\Sigma})^*\nabla^{\Sigma}\psi + \frac{1}{4}\,\mathrm{scal}_g\,\psi, \quad \textit{for all } \psi \in \Gamma(\Sigma M). \tag{2.7}$$

## 2.4 Killing Spinors and Einstein Manifolds

Now we concentrate on a special kind of spinor fields known as Killing spinors and their geometric relation to the Einstein condition for a Riemannian metric on spin manifolds. Let us recall these notions first and then consider their relationship.

**Definition 2.4.1.** An $n$-dimensional Riemannian manifold $(M,g)$ is said to have *constant Ricci curvature* $\mathrm{Ric}_g$ if for some constant $\kappa \in \mathbb{R}$,

$$\mathrm{Ric}_g = \kappa g. \tag{2.8}$$

Riemannian metrics (manifolds) having constant Ricci curvature are called *Einstein metrics (manifolds)* and the corresponding constant of proportionality $\kappa$ is called *Einstein constant*.

It turns out that for $n \geq 3$, the manifold $(M,g)$ is Einstein if and only if

$$\mathrm{Ric}_g = \frac{1}{n}\,\mathrm{scal}_g\, g, \tag{2.9}$$

where, for a given local orthonormal frame $(e_1,\ldots,e_n)$,

$$\mathrm{scal}_g := \mathrm{tr}\,\mathrm{Ric}_g = \sum_{i,j=1}^{n} \sec(e_i,e_j)$$

is the scalar curvature of the metric $g$. In dimension $n=2$, this condition is always true, but $\mathrm{scal}_g$ may not be constant and hence the metric is not necessarily Einstein.

**Definition 2.4.2.** A spinor field $\psi$ defined on an $n$-dimensional Riemannian spin manifold $(M,g)$ is called a *Killing spinor*, if there exists a (complex) number $\mu$ for which the equation

$$\nabla^{\Sigma}_X \psi = \mu X \cdot \psi, \tag{2.10}$$

is satisfied for all $X \in \Gamma(TM)$. In that case, $\mu$ is called the *Killing number* associated to $\psi$.

If $\psi$ is a Killing spinor with Killing number $\mu$, then it follows that $\psi$ is an eigenspinor of the Dirac operator

$$D\psi = \sum_i e_i \cdot \nabla^{\Sigma}_{e_i}\psi = \sum_i \mu\, e_i \cdot e_i \cdot \psi = -\mu\, n\psi,$$

with eigenvalue $\lambda = -\mu\, n$.

Killing spinors were first introduced in the context of general relativity to construct quadratic first integrals of free geodesic motion (see [HPSW] and [PR86]), but their use has extended to more areas in physics like supergravity and supersymmetry. In mathematics, Killing spinors were shown in [ACDS] to correspond to Killing vector fields in Riemannian supergeometry and, on compact manifolds, they play a fundamental role in the limiting case for the lowest eigenvalue of the Dirac operator. Is this latter aspect the one we are mostly interested in this thesis.

Let us recall some useful properties of Killing spinors whose proofs can be found, for example, in [Fri00]. A deeper discussion of Killing spinors in Riemannian geometry appears in [BFGK].

**Theorem 2.4.1.** *Let $(M,g)$ be an n-dimensional Riemannian spin manifold which we suppose to be connected.*

1. *A non-trivial Killing spinor never vanishes.*
2. *For a Killing spinor $\psi$ with Killing constant $\mu$, the associated vector field defined by $X_\psi = i\sum_{j=1}^n (\psi, e_j\cdot\psi)e_j$ is a Killing vector field, i.e. the Lie derivative of the metric in the direction of $X_\psi$ vanishes.*

It is worth mentioning that the second property in the theorem is the reason why Killing spinors deserve that name. Nevertheless the Killing vector field associated to a Killing spinor can vanish, depending on the Clifford action and the Hermitian product on the spinor bundle $\Sigma^g M$.

In Riemannian spin manifolds, the presence of a Killing spinor is enough to guarantee the Einstein condition defined above. Although in this section we are generally concerned about smooth Riemannian metrics, the statement holds for any metric of class at least $C^2$, for which the curvature tensor and in particular the Ricci curvature are well-defined. For this and additional closely related results, see [Fri00, section 5.2].

**Theorem 2.4.2.** *If $(M,g)$ is a connected n-dimensional Riemannian spin manifold (whose metric g is at least of class $C^2$) with a non-trivial Killing spinor, then the manifold is Einstein.*

*Proof.* Let $\psi\in\Gamma(\Sigma M)$ be a non-trivial Killing spinor with Killing constant $\mu$ so that $\nabla^\Sigma_X\psi = \mu X\cdot\psi$ for every vector field $X\in\Gamma(TM)$. Let us calculate for the curvature tensor of the spinor bundle using a local orthonormal basis $(e_1,\dots,e_n)$:

$$
\begin{aligned}
4\sum_{i=1}^n e_i\cdot R^\Sigma(X,e_i)\psi &= \sum_{i,j,k=1}^n \langle R(X,e_i)e_j,e_k\rangle e_i\cdot e_j\cdot e_k\cdot\psi \\
&= \sum_{i,j,k=1}^n \langle R(e_k,e_j)e_i,X\rangle e_i\cdot e_j\cdot e_k\cdot\psi
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{3}\sum_{\substack{i,j,k=1\\ \text{distinct}}}^n &\langle R(e_k,e_j)e_i + R(e_j,e_i)e_k + R(e_i,e_k)e_j, X\rangle e_i\cdot e_j\cdot e_k\cdot\psi \\
&+\sum_{i,k=1}^n \langle R(e_k,e_i)e_i,X\rangle e_i\cdot e_i\cdot e_k\cdot\psi \\
&\qquad+\sum_{i,j=1}^n \langle R(e_i,e_j)e_i,X\rangle e_i\cdot e_j\cdot e_i\cdot\psi
\end{aligned}
$$

$$= -\sum_{k=1}^{n} \mathrm{Ric}(e_k, X) e_k \cdot \psi - \sum_{j=1}^{n} \mathrm{Ric}(e_j, X) e_j \cdot \psi$$
$$= -2\,\mathrm{Ric}(X) \cdot \psi,$$

hence we get the identity

$$\sum_{i=0}^{n} e_i \cdot R^{\Sigma}(X, e_i)\psi = -\frac{1}{2}\,\mathrm{Ric}(X) \cdot \psi.$$

Now we use that $\psi$ is a Killing spinor to simplify the expression for the spinorial curvature tensor acting on $\psi$:

$$\begin{aligned} R^{\Sigma}(X,Y)\psi &= \nabla^{\Sigma}_X \nabla^{\Sigma}_Y \psi - \nabla^{\Sigma}_Y \nabla^{\Sigma}_X \psi - \nabla^{\Sigma}_{[X,Y]}\psi \\ &= \nabla^{\Sigma}_X(\mu Y \cdot \psi) - \nabla^{\Sigma}_Y(\mu X \cdot \psi) - \mu [X,Y] \cdot \psi \\ &= \mu(\nabla_X Y - \nabla_Y X - [X,Y]) \cdot \psi + \mu(Y \cdot \nabla^{\Sigma}_X \psi - X \cdot \nabla^{\Sigma}_Y \psi) \\ &= \mu^2 (Y \cdot X - X \cdot Y) \cdot \psi. \end{aligned}$$

Finally we calculate the Ricci tensor by replacing the action of the spinorial curvature tensor $R^{\Sigma}$ on $\psi$ with the expression above.

$$\begin{aligned} \mathrm{Ric}(X) \cdot \psi &= -2 \sum_{i=1}^{n} e_i \cdot R^{\Sigma}(X, e_i)\psi \\ &= -2\mu^2 \sum_{i=1}^{n} e_i \cdot (e_i \cdot X - X \cdot e_i) \\ &= -4\mu^2 \sum_{i=1}^{n} e_i \cdot (e_i \cdot X + g(X, e_i)) \cdot \psi \\ &= 4\mu^2 (n-1) X \cdot \psi. \end{aligned}$$

Since $\psi$ is a non-trivial Killing spinor, it does not have any zeros, then necessarily

$$\mathrm{Ric}(X) = 4\mu^2 (n-1) X. \tag{2.11}$$

This last expression is coordinate independent, so it holds in all of $M$. It follows that Riemannian manifold $(M, g)$ is an Einstein manifold of constant scalar curvature $\mathrm{scal}_g = 4\mu^2 n(n-1)$. □

### 2.4.1 Einstein Condition in Local Coordinates

The Einstein equation (2.8) defines a set of partial differential equations that can be written in local coordinates, using the formalism of $\gamma$-matrices commonly employed in physics. The calculation used for the proof of Theorem 2.4.2, including a brief introduction to this formalism, can be found in [CGLS] from the local coordinate point of view. Below we sketch the main steps of this procedure needed to obtain the Einstein condition, but avoiding the algebraic arguments needed for the simplification. Einstein's summation convention is used here (and in following chapters).

Using a vector field frame $(e_1,\dots,e_n)$ on an open chart $U\subset M$ and abbreviating $\nabla_a := \nabla_{e_a}$, for $a=1,\dots,n$, equation (2.10) takes the form,

$$\nabla^{\Sigma}_{a}\psi = \mu\gamma_a\psi. \tag{2.12}$$

In index notation, the relation between the curvature tensor of the spinor bundle (defined by means of the spinorial covariant derivative) and the usual Riemann curvature tensor of the tangent bundle can be written as

$$\nabla^{\Sigma}_{a}(\nabla^{\Sigma}_{b}\psi) - \nabla^{\Sigma}_{b}(\nabla^{\Sigma}_{a}\psi) - T^{d}_{ab}\nabla^{\Sigma}_{d}\psi = -\frac{1}{4}R^{cd}_{\ \ ab}\gamma_c\gamma_d\psi,$$

where $R^{cd}_{ab}$ denotes the component $g(R(e_a,e_b)e_c,e_d)$ of the curvature tensor. On the other hand $T^{d}_{ab}$ is the torsion tensor for the spin connection, that vanishes in our case by the Levi-Civita condition. Using twice equation (2.12), the fact that $\nabla\gamma=0$ and the previous comments, we get after some algebra

$$\mu^2(\gamma_b\gamma_a - \gamma_a\gamma_b) = -\frac{1}{4}R^{cd}_{\ \ ab}\gamma_c\gamma_d\psi.$$

Finally, using some useful identities from the algebra of $\gamma$-matrices one finds the expression for the Ricci curvature of $M$,

$$\mathrm{Ric}_{ab} = 4\mu^2(n-1)g_{ab}. \tag{2.13}$$

which is exactly the local form of (2.11).

# Chapter 3

# Hölder Metrics and Weak Einstein Condition

## 3.1 Regularity of the Riemannian structure

We will be working in this and the following chapters in a Riemannian setting where the degree of differentiability of the metric is a fundamental aspect. Usually, we assume that the underlying differentiable manifold $M$ has a smooth structure, but in fact, for a metric $g$ of class $C^{k,\alpha}$ to be well-defined, it is enough to suppose that $M$ carries a $C^{k+1,\alpha}$ differentiable structure. The reason for this lies in the fact that from a $C^{k+1,\alpha}$-atlas for $M$ we can construct $C^{k,\alpha}$ local trivializations of the tangent bundle $TM$ and hence, define a $C^{k,\alpha}$-atlas for it. Thus, sections of $TM$ which are entries of the metric $g$ under consideration will have well-defined $C^{k,\alpha}$ regularity.

*Notation.* Through this chapter, to calculate in local coordinates we will use freely Einstein's summation convention as necessary, over repeated apparitions of upper and lower indices. Nevertheless, for convenience the indices $i, j, k, l$ will be left out of this convention. Instead, other lowercase characters like $a, b, r, s$ will denote components of tensors and for them the summation applies. The notation $\partial_a := \partial/\partial x^a$ for partial derivatives and coordinate vector fields will be adopted.

Recall briefly that vector fields act as derivations on functions defined over a manifold. Let $\mathscr{A} = \{\phi_\gamma : U_\gamma \longrightarrow V_\gamma\}$ be the underlying atlas of a smooth manifold $M$. Then, for an arbitrary coordinate chart $\phi = (x^1, \ldots, x^n) : U \subset M \longrightarrow \mathbb{R}^n$ (not necessarily part of $\mathscr{A}$) and some function $f \in C^\infty(M)$ (with respect to $\mathscr{A}$) we know that, at each $p = \phi^{-1}(x) \in M$, a coordinate vector field $\partial_a$, for $a = 1, \ldots, n$, acts upon $f$ by

$$\partial_a(p) f = \frac{\partial}{\partial x^a}(f \circ \phi^{-1})(x). \tag{3.1}$$

If the coordinate chart $\phi$ is at least $C^{k+1,\alpha}$ (with respect to $\mathscr{A}$) it means that all the compositions $\phi \circ \phi_\gamma^{-1}$ are at least of class $C^{k+1,\alpha}$. In that case the coordinate vector fields of the chart $\phi$ acting on $f$ as in (3.1) will have at least $C^{k,\alpha}$ regularity. In addition, assuming that the metric $g$ is $C^{1,\alpha}$ in this chart, the Christoffel symbols $\Gamma^a_{bc}$ of the Riemannian connection associated to $g$, will be of class $C^{k-1,\alpha}$, as clearly follows from the relation,

$$\Gamma^a_{bc} = \frac{1}{2} g^{ad}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}). \tag{3.2}$$

The next theorem asserts that the converse is also true.

**Theorem 3.1.1.** *Let $\Gamma$ be the connection of a $C^1$ metric $g$. If in some local coordinates $\Gamma^a_{bc}$ is of class $C^{k,\alpha}$ for $k \geq 0$, then in these coordinates the metric $g$ is of class $C^{k+1,\alpha}$.*

*Proof.* This is a generalization of Theorem 3.4 in [DK81], where the case when $g$ is $C^2$ is considered. The proof is similar and obtained by invoking regularity for weak solutions (see [Mor66, Theorem 6.4.3.]) instead of the standard elliptic regularity arguments. □

## 3.2 Ricci Curvature and Harmonic Coordinates

Contrary to what one would expect, the most appropriate coordinates to work locally with partial differential equations on manifolds are not the common geodesic normal coordinates, specially when regularity of geometrical quantities as the metric or the curvature are involved. Here we introduce a system of harmonic coordinates for which the components of the metric tensor and other related objects have optimal behavior with respect to regularity, as was shown by DeTurck and Kazdan in [DK81].

**Definition 3.2.1.** A *harmonic coordinate system* on an $n$-dimensional Riemannian manifold $(M,g)$ is a system of coordinates $(U,\phi)$, where $\phi = (x^1,\dots,x^n) : U \longrightarrow \mathbb{R}^n$, for which the coordinate functions $x^a$ are harmonic with respect to the Laplacian on $(M,g)$, i.e. $\Delta_g x^a = 0$, for $a = 1,\dots,n$.

*Notation.* When working with geometric objects it is important for us to keep their dependence on $g$ in mind, hence we use a subscript $g$, like $\Delta_g$ for the Laplacian, $\mathrm{dvol}_g$ for the Riemannian volume form, and so on.

**Definition 3.2.2.** We say that a tensor field (like a metric tensor $g$) on a coordinate chart $(U,\phi)$ of a manifold $M$ is of class $C^{k,\alpha}$ if its components ($g_{ab} : U \longrightarrow \mathbb{R}$ in the case of the metric) are $C^{k,\alpha}$ functions on $U$. The tensor field will be $C^{k,\alpha}$ globally on $M$ if it is $C^{k,\alpha}$ on every chart $U \subset M$.

Recall that the *Laplace-Beltrami operator* (or simply the *Laplacian* on functions) $\Delta_g$ acting on $u \in C^2(M)$ in a Riemannian manifold $(M,g)$, depends on the metric $g$ and can be written locally as

$$\begin{aligned}\Delta_g u &= \frac{1}{\sqrt{g}}\partial_a\big(g^{ab}\sqrt{g}\,\partial_b u\big) \\ &= g^{ab}\partial_a\partial_b u + \frac{1}{\sqrt{g}}\partial_a\big(\sqrt{g}g^{ab}\big)\partial_b u,\end{aligned} \tag{3.3}$$

where $|g| := \det(g_{ab})$. It is possible to write the Laplacian directly in terms of the Christoffel symbols $\Gamma^a_{bc}$ in the following way. First, using the identity

$$\operatorname{tr}\left\{A^{-1}(x)\frac{d}{dx}A(x)\right\} = \frac{d}{dx}\ln\det A(x)$$

where $A(x)$ is an arbitrary invertible matrix whose entries are functions of $x$, we get for the metric tensor $g$,

$$\frac{1}{2}g^{ab}\partial_c g_{ab} = \frac{1}{\sqrt{g}}\,\partial_c\big(\sqrt{g}\big). \tag{3.4}$$

This last expression implies immediately from (3.2),

$$\Gamma^a_{ac} = \frac{1}{2}g^{ad}(\partial_a g_{cd} + \partial_c g_{ad} - \partial_d g_{ac}) = \frac{1}{2}g^{ad}\partial_c g_{ad} = \frac{1}{\sqrt{g}}\partial_a\big(\sqrt{g}\big).$$

Now, let us denote $\Gamma^a := g^{bc}\Gamma^a_{bc}$. Using the relation $g^{ab}\partial_r g_{bc} + g_{bc}\partial_r g^{ab} = 0$ and the previos calculation to simplify, we find

$$\Gamma^a = -\frac{1}{\sqrt{g}}\partial_b\big(\sqrt{g}g^{ba}\big).$$

Then, the Laplacian can be rewritten in terms of the Christoffel symbols as

$$\Delta_g u = g^{ab}\partial_a\partial_b u - \Gamma^b\partial_b u.$$

Applied to the coordinate functions $x^a$, the Laplacian reads now shortly as: $\Delta_g x^a = -\Gamma^a$. We conclude that the coordinate system $(x^1,\ldots,x^n)$ is harmonic if and only if for every $a = 1,\ldots,n$,

$$\Gamma^a = g^{bc}\Gamma^a_{bc} = 0 \tag{3.5}$$

and that in such harmonic charts, the Laplacian takes the simpler form

$$\Delta_g u = g^{ab}\partial_a\partial_b u. \tag{3.6}$$

The existence of harmonic coordinate systems on any neighborhood of every point of a Riemannian manifold is guaranteed. See [DK81] and [JK82] for a complete account of harmonic coordinates in this context and the proofs of the following results.

**Theorem 3.2.1** (DeTurck, Kazdan)**.** *Let $M$ be a Riemannian manifold endowed with a $C^{k,\alpha}$-metric $g$ for $k \geq 1$ in a local coordinate chart containing some point $p$. Then, there is a neighborhood of $p$ in which harmonic coordinates exist and are $C^{k+1,\alpha}$ functions of the original coordinates. Furthermore,*

1. *this harmonic coordinate system can be chosen such that $g_{ab}(x) = \delta_{ab}$ for any $a,b \in \{1,\dots,n\}$,*
2. *all harmonic coordinate charts about $p$ have the same $C^{k+1,\alpha}$ regularity,*
3. *the metric $g$ is of class $C^{k,\alpha}$ in any harmonic coordinate chart about $p$, while it is at least of class $C^{k-2,\alpha}$ in geodesic normal coordinates.*
4. *any tensor which in the original coordinates is of class $C^{l,\beta}$, for $l \geq k$ and $\beta \geq \alpha$, is at least of class $C^{k,\alpha}$ in harmonic coordinates.*

The optimal regularity of the metric in harmonic coordinates follows from the last property in this theorem. Briefly, the result asserts that in changing from arbitrary coordinates to a harmonic coordinate chart the regularity of the metric is preserved, while changing to normal coordinates involves always the loss of at least two derivatives.

Finally, the existence of uniform bounds on the structure of a compact Riemannian manifold is asserted in the next theorem (see [JK82]). In particular they are useful to prove one version of the compactness theorems for Riemannian manifolds (cf. [Pts87]).

**Theorem 3.2.2** (Jost, Karcher)**.** *Let $M$ be a compact Riemannian manifold and $\alpha \in (0,1)$. About any point $p \in M$ there exists a ball $B_p(r)$ of fixed radius $r$, on which harmonic coordinates exist and have the following properties:*

1. *There exists a uniform $C^{2,\alpha}$-Hölder bound for the transition functions,*
2. *a uniform $C^{1,\alpha}$-bound for the metric, and*

3. *a uniform $C^{0,\alpha}$-bound for the Christoffel symbols, where the radius and the Hölder bounds depend on the dimension, the injectivity radius and the curvature bounds of $M$.*

In conclusion one can always find a harmonic coordinate system on balls of a-priori fixed radius about any point of the manifold. In these coordinates, the transition functions $\phi_U \circ \phi_V^{-1}$ are $C^{2,\alpha}$, the metric $g_{ab}$ is $C^{1,\alpha}$ and the Christoffel symbols $\Gamma^a_{bc}$ are $C^{0,\alpha}$. In fact, as follows from Theorem 3.1.1, the metric (even if it is only of class $C^1$) has always one degree of differentiability more than the Christoffel symbols in any local coordinate chart.

For our purposes, the principal reason in using harmonic coordinates comes from the fact that the local expression for the Ricci curvature has a simpler form, in which derivatives of the Christoffel symbols do not appear (see [DK81, Lemma 4.1]).

**Theorem 3.2.3** (Lanczos, [Lan22])**.** *In an arbitrary system of local coordinates, the Ricci curvature can be written in terms of the metric $g$ and the Christoffel symbols $\Gamma^a = g^{bc}\Gamma^a_{bc}$ as*

$$\mathrm{Ric}_{ab} = -\frac{1}{2}g^{rs}\partial_r\partial_s g_{ab} + \frac{1}{2}(g_{ar}\partial_b\Gamma^r + g_{br}\partial_a\Gamma^r) + Q_{ab}(g,\partial g), \tag{3.7}$$

*where $Q(g,\partial g)$ is a quadratic form depending only on $g$ and its first partial derivatives $\partial g$. Moreover, in a harmonic coordinate system the Ricci curvature takes the simpler form,*

$$\mathrm{Ric}_{ab} = -\frac{1}{2}g^{rs}\partial_r\partial_s g_{ab} + Q_{ab}(g,\partial g). \tag{3.8}$$

*Proof.* The standard formula for the curvature tensor in terms of the Christoffel symbols says

$$R^d_{abc} = \partial_a\Gamma^d_{bc} - \partial_b\Gamma^d_{ac} + (\Gamma^r_{ac}\Gamma^d_{br} - \Gamma^r_{bc}\Gamma^d_{ar}), \tag{3.9}$$

Then, replacing (3.2) in the first two terms of the previous expression for the Riemann tensor we get for the Ricci curvature,

$$\mathrm{Ric}_{ab} = R^c_{acb} = \frac{1}{2}g^{rs}(\partial_b\partial_s g_{ar} + \partial_a\partial_s g_{br} - \partial_r\partial_s g_{ab} - \partial_a\partial_b g_{rs}) + \widetilde{Q}_{ab}(g,\partial g) \tag{3.10}$$

where $\widetilde{Q}_{ab}(g,\partial g) := (\Gamma^r_{ac}\Gamma^c_{br} - \Gamma^r_{bc}\Gamma^c_{ar})$ is a function depending only on $g$ and its first partial derivatives $\partial g$. In fact, $\widetilde{Q}$ is homogeneous of degree 2 in the first derivatives

$\partial g$. Now, according to the definitions we calculate

$$\begin{aligned} g_{ac}\Gamma^c &= g_{ac}g^{rs}\Gamma^c_{rs} = \frac{1}{2}g_{ac}g^{rs}g^{cd}(\partial_r g_{sd} + \partial_s g_{rd} - \partial_d g_{rs}) \\ &= \frac{1}{2}g^{rs}(\partial_r g_{sa} + \partial_s g_{ra} - \partial_a g_{rs}) \\ &= g^{rs}(\partial_r g_{sa} - \frac{1}{2}\partial_a g_{rs}). \end{aligned}$$

Taking partial derivative $\partial_b$ of the previous equality and write it down together with a similar one where the indices $a$ and $b$ are interchanged,

$$\begin{aligned} \partial_b(g_{ac}\Gamma^c) &= \partial_b g^{rs}\Big(\partial_s g_{ar} - \frac{1}{2}\partial_a g_{rs}\Big) + g^{rs}\Big(\partial_b\partial_s g_{ar} - \frac{1}{2}\partial_b\partial_a g_{rs}\Big), \\ \partial_a(g_{bc}\Gamma^c) &= \partial_a g^{rs}\Big(\partial_s g_{br} - \frac{1}{2}\partial_b g_{rs}\Big) + g^{rs}\Big(\partial_a\partial_s g_{br} - \frac{1}{2}\partial_a\partial_b g_{rs}\Big). \end{aligned}$$

Adding the last two expressions and reordering terms we get the relation

$$\begin{aligned} g^{rs}(\partial_b\partial_s g_{ar} + \partial_a\partial_s g_{br} - \partial_a\partial_b g_{rs}) &= \partial_b(g_{ac}\Gamma^c) + \partial_a(g_{bc}\Gamma^c) \\ &\quad -\partial_b g^{rs}\partial_s g_{ar} - \partial_a g^{rs}\partial_s g_{br} + \frac{1}{2}(\partial_b g^{rs}\partial_a g_{rs} + \partial_a g^{rs}\partial_b g_{rs}). \end{aligned}$$

Finally, replacing all this directly in the formula for the Ricci curvature (3.10) yields, after additional rearrangements

$$\begin{aligned} \mathrm{Ric}_{ab} = -\frac{1}{2}g^{rs}\partial_r\partial_s g_{ab} &+ \frac{1}{2}(g_{ac}\partial_b\Gamma^c + g_{bc}\partial_a\Gamma^c) \\ &+ \frac{1}{2}(\partial_b g_{ac} + \partial_a g_{bc})\Gamma^c - \frac{1}{2}(\partial_b g^{rs}\partial_s g_{ar} - \partial_a g^{rs}\partial_s g_{br}) \\ &\qquad + \frac{1}{4}(\partial_b g^{rs}\partial_a g_{rs} + \partial_a g^{rs}\partial_b g_{rs}) + \widetilde{Q}_{ab}(g, \partial g). \end{aligned}$$

The third, fourth and fifth terms in the right hand side of this equality depend only on $g$ and its first derivatives, as the function $\widetilde{Q}_{ab}$ does. We denote these last four terms simply by $Q_{ab}(g, \partial g)$, obtaining in this way the desired expression (3.7) for the Ricci tensor in a local coordinate system. Since in harmonic coordinates $\Gamma^c = 0$, it is a straightforward consequence that Ricci curvature takes the form (3.8) in this case. □

## 3.3 Einstein Equation and $C^{1,\alpha}$ Metrics

In the previous section we saw that the Laplacian $\Delta_g$ adopts the short expression $\Delta_g = g^{rs}\partial_r\partial_s$ in any harmonic coordinate chart, therefore the Ricci curvature reads shortly as

$$\mathrm{Ric}_{ab} = -\tfrac{1}{2}\Delta_g g_{ab} + Q_{ab}(g, \partial g) \tag{3.11}$$

in such coordinates. This will be a central point in what follows because in this way the Ricci curvature is shown to be a quasilinear operator, where $\Delta_g g_{ab}$ is elliptic and the estimates of section 1.7 apply to it.

Concerning regularity, the following theorem asserts that in harmonic coordinates the metric has always two degrees of differentiability more than the Ricci tensor. This is not true in general as one could naïvely expect, see [DK81, Theorem 4.5]. In fact, a smooth Ricci tensor can even have a non-smooth metric in arbitrary coordinates.

**Theorem 3.3.1.** *If in harmonic coordinates $g \in C^2$ is a Riemannian metric with $\mathrm{Ric}_g \in C^{k,\alpha}$ for some $k \geq 0$, then in these coordinates $g \in C^{k+2,\alpha}$.*

Recall that the Einstein condition for the Ricci curvature in local coordinates, as it was presented in section 2.4.1, can be written as

$$\mathrm{Ric}_{ab} = \kappa(n-1)g_{ab}, \tag{3.12}$$

where $\kappa$ is some fixed real constant.

Now, to define the Ricci curvature in an appropriate weak sense, we use the formula (3.7) for the Ricci tensor in local coordinates, rewritten below for convenience:

$$\mathrm{Ric}_{ab} = -\frac{1}{2}g^{rs}\partial_r\partial_s g_{ab} + \frac{1}{2}\left(g_{ra}\partial_b\Gamma^r + g_{rb}\partial_a\Gamma^r\right) + Q_{ab}(g, \partial g),$$

where $\Gamma^r = g^{ab}\Gamma^r_{ab}$.

**Definition 3.3.1.** Given a coordinate chart $(U, \phi)$ on a Riemannian manifold $M$ with a $C^{1,\alpha}$-metric $g$, we define the *weak Ricci curvature* componentwise as a functional $\mathrm{Ric}_{ab}$ acting on a compactly supported test function $\eta \in C^1_c(U)$ by

$$\begin{aligned}\left\langle\!\left\langle \mathrm{Ric}_{ab}, \eta \right\rangle\!\right\rangle_U := \frac{1}{2}\int_U \partial_s g_{ab}\partial_r(g^{rs}\eta)\,\mathrm{dvol}_g - \frac{1}{2}\int_U \Gamma^r\partial_b(g_{ra}\eta)\,\mathrm{dvol}_g \\ -\frac{1}{2}\int_U \Gamma^r\partial_a(g_{rb}\eta)\,\mathrm{dvol}_g + \int_U Q_{ab}\,\eta\,\mathrm{dvol}_g\,.\end{aligned} \tag{3.13}$$

In this last expression all the terms are well-defined for a $C^{1,\alpha}$-metric since no second derivatives are involved. Furthermore, for $X, Y \in C^{1,\alpha}(TM)$ on the open chart $U \subset M$ and a local frame field $(e_1, \ldots, e_n)$ such that $X = X^a e_a$ and $Y = Y^b e_b$, we define

$$\left\langle\!\left\langle \mathrm{Ric}(X,Y), \eta \right\rangle\!\right\rangle_U := \sum_{a,b=1}^{n} \left\langle\!\left\langle \mathrm{Ric}_{ab}, X^a Y^b \eta \right\rangle\!\right\rangle_U.$$

*Notation.* For weak objects like the components of the Ricci curvature just defined acting on test functions, we introduce the notation $\langle\!\langle \mathrm{Ric}_{ab}, \eta \rangle\!\rangle_U$ instead of the possible but a bit misleading $\int_U \mathrm{Ric}_{ab}\, \eta$ which does really make sense only when Ric is defined in the classical way.

The definition just given suffices to introduce weak solutions of the Einstein equation and although it is not needed here for our arguments, in section A we will show how this weak Ricci curvature arises from a natural weak notion for the Riemann tensor of the tangent bundle.

Having the Ricci curvature defined in this weak (distributional) sense, we can introduce a weak Einstein condition for the metric of a Riemannian manifold.

**Definition 3.3.2.** A $C^{1,\alpha}$-metric $g$ of a Riemannian manifold $M$ is *weakly Einstein* in a local coordinate chart $(U,\phi)$ if the equality

$$\langle\!\langle \mathrm{Ric}_{ab}, \eta \rangle\!\rangle_U = \kappa \int_U g_{ab}\, \eta \;\mathrm{dvol}_g, \tag{3.14}$$

holds for any compactly supported test function $\eta \in C_c^{1,\alpha}(U)$, an appropriate constant $\kappa \in \mathbb{R}$ and all $a,b \in \{1,\dots,n\}$. In index-free notation, this can be written for $X,Y \in \Gamma_U(TM)$ as

$$\langle\!\langle \mathrm{Ric}(X,Y), \eta \rangle\!\rangle_U = \kappa \int_U g(X,Y)\, \eta \;\mathrm{dvol}_g\,. \tag{3.15}$$

If this condition is satisfied on any chart of an atlas of $M$ then the manifold is called *weakly Einstein* in this atlas.

A generalization of Theorem 3.3.1 for weak solutions of the Einstein equation is possible as we show now.

**Theorem 3.3.2.** *Let $M$ be an n-dimensional manifold with a Riemannian metric $g$ of class $C^{1,\alpha}$. If the components of $g$ are weak solutions of* (3.12) *in a harmonic coordinate chart $(U,\phi)$, where $\phi : U \subset M \longrightarrow V \subset \mathbb{R}^n$ is invertible, then viewed as functions $g_{ab} \circ \phi^{-1} : V \longrightarrow \mathbb{R}$, they are smooth. We say in this case that the components of g are smooth with respect to the harmonic coordinate chart $(U,\phi)$.*

*Proof.* First we replace the expression (3.8) for the Ricci curvature in harmonic coordinates into (3.12) to obtain

$$\Delta_g g_{ab} = g^{rs}\partial_r\partial_s g_{ab} = -2\kappa(n-1)g_{ab} + 2Q_{ab}(g,\partial g). \tag{3.16}$$

Given that $g$ is $C^{1,\alpha}$ we know that $Q(g,\partial g)$, which does not depend on second or higher derivatives of $g$, has to be at least of class $C^{0,\alpha}$. It means that the whole right hand side of equation (3.16) is at least a $C^{0,\alpha}$ function. Let us denote it by $f_{ab} := -2\kappa(n-1)g_{ab} + 2Q_{ab}(g,\partial g) \in C^{0,\alpha}$.

The assumption of the theorem is that $g \in C^{1,\alpha}$ is a weak solution to the equation $g^{rs}\partial_r\partial_s g_{ab} = f_{ab}$, which requires that $\partial_s g_{ab} \in H_1^2$ and $f_{ab} \in L^2$. In other words, for any compactly supported test function $\eta \in C_c^1(U) \cap H_1^2$ the following integral equation in harmonic coordinates is satisfied

$$\langle\!\langle \Delta_g g_{ab}, \eta \rangle\!\rangle_U := \int_U \partial_s(g_{ab})\, \partial_r(g^{rs}\eta)\, \mathrm{dvol}_g = \int_U f_{ab}\eta\, \mathrm{dvol}_g\,. \tag{3.17}$$

Comparing the left hand and right hand sides of (3.17), we conclude that $g \in C^{1,\alpha}$ admits $C^{0,\alpha}$ (and in particular continuous) weak second partial derivatives or Laplacian. Since weak derivatives are unique up to sets of measure zero, and the space $\{u \in C^1(U) \mid \|u\|_{H_1^2} < \infty\}$ is dense in $H_1^2$, then $\partial_s g_{ab}$ has usual partial derivatives $\partial_r\partial_s g_{ab}$ almost everywhere. Said briefly, the Laplacian of $g_{ab}$ is equal to $f_{ab} \in C^{0,\alpha}(U) \cap L^2$ except possibly on a set of null measure. Applying the regularity Theorem 1.7.4 we get that $g_{ab}$ is actually a $C^{2,\alpha}$ function. Hence, from (3.16) the function $f_{ab}$ must be $C^{1,\alpha}$ and the process can be iterated again any number of times in what is known as a bootstrap argument, to obtain the smoothness conclusion claimed by the theorem. □

Now, let $\mathscr{A} = \{\phi_\gamma : U_\gamma \longrightarrow V_\gamma\}$ be an atlas consisting of harmonic coordinate charts on $M$. Due to part 1 of Theorem 3.2.2 there are $C^{2,\alpha}$ transition functions for this atlas, i.e., $\mathscr{A}$ is a $C^{2,\alpha}$-atlas for $M$. Then, we obtain the following.

**Corollary 3.3.3.** *If $g$ is weakly Einstein with respect to all charts in a harmonic coordinate atlas $\mathscr{A} = \{\phi_\gamma : U_\gamma \longrightarrow V_\gamma\}$ of $M$, then $\mathscr{A}$ is actually a smooth atlas, and $g$ is a smooth Riemannian metric with respect to it.*

*Proof.* The $C^{2,\alpha}$ differentiable structure given by the atlas $\mathscr{A}$ define a $C^{1,\alpha}$ structure on the tangent bundle $TM$, associated to the harmonic charts of $\mathscr{A}$, where coordinate frame fields of class $C^{1,\alpha}$ can be defined. This allows the components $g_{ab}$ of the metric to be well-defined of class $C^{1,\alpha}$ in this atlas. But from Theorem 3.3.2 it follows that indeed, the functions $g_{ab} \circ \phi_\gamma^{-1} : V \longrightarrow \mathbb{R}$ must be smooth for every $\phi_\gamma$, which necessarily requires that the transition functions be also smooth and hence, the atlas itself is a smooth one. The smoothness of the components of the metric in every harmonic chart means that $g$ is smooth with respect to this atlas. □

**Remark 3.3.1.** If $g$ is a $C^{1,\alpha}$-metric that is weakly Einstein on $M$, then the harmonic coordinate atlas $\mathscr{A}$ just discussed might define a different differentiable structure on $M$. But in case these differentiable structures are compatible, the transition functions between charts of one atlas to charts of the other are smooth and, therefore, smoothness of $g$ in the harmonic atlas imply also its smoothness in the original atlas of $M$.

Given the ellipticity of the Laplacian $\Delta_g$, the elliptic estimates (1.9) in Theorem 1.7.3

$$\|g\|_{C^{2,\alpha}(M)} \leq C\big(\|g\|_{C^0(M)} + \|f\|_{C^{0,\alpha}(M)}\big),$$

where $C > 0$ is independent of $g$ and $f$, require in fact that $g$ is bounded in $C^{2,\alpha}$ and by a bootstrap, actually in every Hölder space $C^{k,\alpha}$. Nevertheless this conclusion and the relation with the harmonic coordinate atlas is rather obscure without the previous analysis.

Finally we summarize two additional well-known properties about analyticity and unique continuation of Einstein metrics, whose proofs appear in [DK81, section 5].

**Theorem 3.3.4.** *Any Einstein metric $g$ of class $C^2$ on a connected manifold $M$ with* $\dim M \geq 3$ *is real analytic in harmonic and geodesic normal coordinates.*

**Theorem 3.3.5.** *On a simply connected manifold $M$, any two Einstein metrics $g_1$ and $g_2$ which coincide locally (on some open set $U \subset M$) are globally diffeomorphic, i.e., if $g_1|_U = g_2|_U$, there is a diffeomorphism $\Phi : M \longrightarrow M$ such that $g_1 = \Phi^* g_2$.*

As a consequence of our results, Theorem 3.3.2 implies the following corollary.

**Corollary 3.3.6.** *On a connected manifold $M$, with* $\dim M \geq 3$*, any $C^{1,\alpha}$-metric which is weakly Einstein in harmonic coordinates (in the sense of Theorem 3.3.2) is actually real analytic in these coordinates.*

## 3.4 Weak Solutions on the Spinor Bundle

According to item 2 from the list of Definitions 1.6.1, when the coefficients of a partial differential operator $P = \sum_l a_l \nabla^l$ of order $k$ acting on spinors, are at least of class $C^k$, the natural way to define weak solutions should be given as follows (compare also with Remark 1.6.1).

**Definition 3.4.1.** The equation $P\psi_1 = \psi_2$ holds *weakly* on a local chart $U \subset M$ if and only if for every compactly supported test spinor $\varphi \in C^k_c(\Sigma M)$, with $\operatorname{supp}\varphi \subset U$, holds

$$\int_U (\psi_1, P^*\varphi)\,\mathrm{dvol}_g = \int_U (\psi_2, \varphi)\,\mathrm{dvol}_g\,. \tag{3.18}$$

where $P^*$ is the formal adjoint of $P$ on the spinor bundle $\Sigma^g M$ (see subsection 1.5.2).

This definition is clearly useful when $\psi_1$ is not differentiable enough for $P\psi_1$ to be well-defined. In particular, we can try to apply it to the Riemann tensor of $\Sigma^g M$. Since the Levi-Civita spin connection $\nabla^\Sigma : \Gamma(\Sigma^g M) \longrightarrow \Gamma(T^*M \otimes \Sigma^g M)$ is metric, the spinorial curvature viewed as a map $R^\Sigma : \Gamma(\Sigma^g M) \longrightarrow \Gamma(\Lambda^2(T^*M)\otimes \Sigma^g M)$ has a formal adjoint $(R^\Sigma)^* : \Gamma(\Lambda^2(T^*M)\otimes\Sigma^g M) \longrightarrow \Gamma(\Sigma^g M)$. It can be shown[1] that the identity

$$(R^\Sigma)^*((dx^a \wedge dx^b)\otimes\varphi) = -\frac{1}{\sqrt{g}} R^\Sigma(\partial_a,\partial_b)\varphi,$$

holds for a compactly supported $\varphi \in C^2_c(\Sigma M)$ and some local coordinate vector fields $\partial_a, \partial_b \in C^2(TM)$, i.e. vector fields coming from the coordinate functions $(x^1,\dots,x^n) : U \longrightarrow \mathbb{R}^n$ in the local chart. Thus, we could define for a non-differentiable spinor $\psi \in \Gamma(\Sigma^g M)$,

$$\int_U \big(R^\Sigma(\partial_a,\partial_b)\psi,\varphi\big)\,\mathrm{dvol}_g := -\int_U \frac{1}{\sqrt{g}}\big(\psi, R^\Sigma(\partial_a,\partial_b)\varphi\big)\,\mathrm{dvol}_g\,. \tag{3.19}$$

Nevertheless, it only makes sense when $R^\Sigma$ on the right hand side of (3.19) is well-defined. This requires the existence of two derivatives of the metric, i.e., $g$ must be at least of class $C^2$, which obviously means that Definition 3.4.1 and relation (3.19) are not useful to define weak curvatures for $C^{1,\alpha}$ metrics.

Our primary goal is to generalize in an appropriate weak sense the identity,

$$-\frac{1}{2}\operatorname{Ric}(X)\cdot\psi = \sum_{i=1}^n e_i\cdot R^\Sigma(X,e_i)\psi, \tag{3.20}$$

which holds on spin manifolds with smooth metrics, as shown in the proof of Theorem 2.4.2. To this end, we introduce new definitions below.

First, let $\varphi \in C^1_c(\Sigma M)$ be a compactly supported test spinor, with $\mathrm{supp}(\varphi) \subset U$ for an open chart $U \subset M$. Assuming for a moment that the spinorial curvature is well-defined, we take the Hermitian product of $R^\Sigma$ acting on a spinor $\psi$ with $\varphi$,

$$\big(R^\Sigma(X,Y)\psi,\varphi\big) = \big(\nabla^\Sigma_X\nabla^\Sigma_Y\psi - \nabla^\Sigma_Y\nabla^\Sigma_X\psi - \nabla^\Sigma_{[X,Y]}\psi,\varphi\big)$$

and integrate both sides locally. Then, partial integration and Stokes' theorem simplify the result. To establish this last step, local coordinate vector fields, denoted by $\partial_a, \partial_b \in C^2(TM)$, are needed as the first two entries of the spinorial curvature tensor.

For the complete calculation, remember that $\mathrm{dvol}_g = \sqrt{g}\,dx$, where we abbreviate $dx = dx^1\cdots dx^n$ for coordinate functions $(x^1,\dots,x^n)$ on the local chart,

[1]We do not present the proof here but half of the necessary calculation appears in the next page before the introduction of the weak spinorial curvature.

$\nabla^\Sigma_c := \nabla^\Sigma_{\partial_c}$ for any $c = 1,\dots,n$, and denote $\hat{\varphi} = \sqrt{g}\,\varphi$. Thus, we can write

$$\begin{aligned}\int_U \big(R^\Sigma(\partial_a,\partial_b)\psi,\varphi\big)\,\mathrm{dvol}_g &= \int_U \big(\nabla^\Sigma_a\nabla^\Sigma_b\psi - \nabla^\Sigma_b\nabla^\Sigma_a\psi - \nabla^\Sigma_{[\partial_a,\partial_b]}\psi,\hat{\varphi}\big)\,dx \\ &= \int_U \big\{\partial_a\big(\nabla^\Sigma_b\psi,\hat{\varphi}\big) - \big(\nabla^\Sigma_b\psi,\nabla^\Sigma_a\hat{\varphi}\big) \\ &\qquad - \partial_b\big(\nabla^\Sigma_a\psi,\hat{\varphi}\big) + \big(\nabla^\Sigma_a\psi,\nabla^\Sigma_b\hat{\varphi}\big)\big\}\,dx \\ &= \int_U \big(\nabla^\Sigma_a\psi,\nabla^\Sigma_b\hat{\varphi}\big)\,dx - \int_U \big(\nabla^\Sigma_b\psi,\nabla^\Sigma_a\hat{\varphi}\big)\,dx.\end{aligned}$$

This procedure indicates that the weak definition we need is what comes out after the integration.

*Notation.* To simplify, we use from now on $\langle X,Y\rangle := g(X,Y)$, but the reader should not confuse the Riemannian metric $\langle\cdot,\cdot\rangle$ with our notation $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ used to distinguish distributions.

**Definition 3.4.2.** On a Riemannian spin manifold with a $C^{1,\alpha}$-metric $g$ we define the *weak spinorial curvature tensor* acting on local coordinate vector fields $\partial_a,\partial_b \in \Gamma_U(TM)$ and spinor fields $\psi,\varphi \in C^{1,\alpha}(M)$ by

$$\langle\!\langle R^\Sigma(\partial_a,\partial_b)\psi,\varphi\rangle\!\rangle_U := \int_U \big(\nabla^\Sigma_a\psi,\nabla^\Sigma_b\hat{\varphi}\big)\,dx - \int_U \big(\nabla^\Sigma_b\psi,\nabla^\Sigma_a\hat{\varphi}\big)\,dx$$

with $\hat{\varphi} = \sqrt{g}\,\varphi$ and $dx = dx^1\cdots dx^n$ as before.

By the above calculation is clear that for a Riemannian metric $g$ of class $C^2$, the weak and the classic expressions coincide

$$\langle\!\langle R^\Sigma(\partial_a,\partial_b)\psi,\varphi\rangle\!\rangle_U = \int_U \big(R^\Sigma(\partial_a,\partial_b)\psi,\varphi\big)\,\mathrm{dvol}_g\,.$$

**Remark 3.4.1.** Given a local orthonormal frame $(e_1,\dots,e_n)$, let us assume that we can write each vector in terms of a basis of coordinate vectors $(\partial_1,\dots,\partial_n)$, as $e_i = \sum_b E_i^b\partial_b$. We point out here, as it was mentioned at the beginning of section 3.1, that the derivations $\partial_a$ act in a $C^{1,\alpha}$ way if the differentiable structure of the manifold is at least $C^{2,\alpha}$. This is enough for the metric $g$ of class $C^{1,\alpha}$ to be well-defined. Any other local vector field frame can be obtained from $(\partial_1,\dots,\partial_n)$ by a orthonormalization process and a rotation on $T_pM$ for each point $p\in M$, i.e., an action of $SO_g(n)$ on each basis. This two steps are linear and depend only on the metric and not on its derivatives. If the metric is $C^{1,\alpha}$ the result is that the new vector frame will act also as a $C^{1,\alpha}$ derivation and, in particular, the coefficients $E_i^b$ in the expression for $e_i$ in terms of $\partial_b$ are in fact $C^{1,\alpha}$ functions on the manifold.

**Lemma 3.4.1.** *In a Riemannian spin manifold M with a $C^{1,\alpha}$-metric g, the relation*

$$-\frac{1}{2}\operatorname{Ric}(\partial_a)\cdot\psi = \sum_{i=1}^{n} e_i \cdot R^{\Sigma}(\partial_a, e_i)\psi, \tag{3.21}$$

*is satisfied weakly, for any coordinate vector field $\partial_a$, and any local orthonormal frame field $(e_1,\dots,e_n)$ on an open chart $U\subset M$. This means that the following equality, which holds for metrics of class at least $C^2$ and any test spinor $\varphi\in C^1_c(\Sigma M)$ with* $\operatorname{supp}\varphi\subset U$*,*

$$\frac{1}{2}\int_U \big(\operatorname{Ric}(\partial_a)\cdot\psi,\varphi\big)\,\mathrm{dvol}_g = \sum_{i,b=1}^{n}\int_U \big(R^{\Sigma}(\partial_a,\partial_b)\psi, E_i^b e_i\cdot\varphi\big)\,\mathrm{dvol}_g \tag{3.22}$$

*is well-defined in the weak sense established above for $C^{1,\alpha}$ metrics. Here, we assume $e_i=\sum_{b=1}^n E_i^b\partial_b$ is the expression for the vector field $e_i$ in terms of the local coordinate field basis $(\partial_1,\dots,\partial_n)$.*

*Proof.* First, we prove that indeed (3.22) holds for any $C^2$ Riemannian metric $g$, a spinor $\psi\in C^2(\Sigma M)$, and a test spinor $\varphi\in C^1_c(\Sigma M)$ with $\operatorname{supp}\varphi\subset U$. For doing this, we integrate both sides of (3.21), which holds under the previous assumptions, after taking Hermitian product with $\varphi$:

$$\frac{1}{2}\int_U \big(\operatorname{Ric}(\partial_a)\cdot\psi,\varphi\big)\,\mathrm{dvol}_g = \sum_{i=1}^{n}\int_U \big(e_i\cdot R^{\Sigma}(\partial_a,e_i)\psi,\varphi\big)\,\mathrm{dvol}_g\,.$$

Passing one $e_i$ to the right of the Hermitian product and replacing the other by $e_i=\sum_b E_i^b\partial_b$ we get,

$$\frac{1}{2}\int_U \big(\operatorname{Ric}(\partial_a)\cdot\psi,\varphi\big)\,\mathrm{dvol}_g = \sum_{i,b=1}^{n}\int_U \big(R^{\Sigma}(\partial_a,\partial_b)\psi, E_i^b e_i\cdot\varphi\big)\,\mathrm{dvol}_g\,.$$

Thus, we get (3.22), which in fact will hold for smooth metrics. To translate this equation to a weak interpretation for $C^{1,\alpha}$-metrics, we need to consider both of its sides separately. Beginning from the left, when Ric is defined in the classical sense, consider the calculation

$$\begin{aligned}\int_U \big(\operatorname{Ric}(\partial_a)\cdot\psi,\varphi\big)\,\mathrm{dvol}_g &= \int_U \Big(\sum_i \operatorname{Ric}(\partial_a,e_i)e_i\cdot\psi,\varphi\Big)\,\mathrm{dvol}_g\\ &= \sum_{i=1}^{n}\int_U \operatorname{Ric}(\partial_a,e_i)(e_i\cdot\psi,\varphi)\,\mathrm{dvol}_g\\ &= \sum_{i,b=1}^{n}\int_U \operatorname{Ric}(\partial_a,\partial_b)E_i^b(e_i\cdot\psi,\varphi)\,\mathrm{dvol}_g,\end{aligned}$$

where $e_i = \sum_i E_i^b \partial_b$. After passing this equality to the weak context, the factor $E_i^b(e_i \cdot \psi, \varphi)$ play the roll of a test function, dependent on the test spinor $\varphi$, and we find that the left-hand side of (3.22) should be understood as,

$$\begin{aligned}\left\langle\!\left\langle \operatorname{Ric}(\partial_a)\cdot\psi,\varphi \right\rangle\!\right\rangle_U &:= \sum_i^n \left\langle\!\left\langle \operatorname{Ric}(\partial_a, e_i), (e_i\cdot\psi,\varphi) \right\rangle\!\right\rangle_U \\ &:= \sum_{i,b=1}^n \left\langle\!\left\langle \operatorname{Ric}(\partial_a,\partial_b), E_i^b(e_i\cdot\psi,\varphi) \right\rangle\!\right\rangle_U \end{aligned} \tag{3.23}$$

where the last expression to the right is given according to Definition 3.3.1. Similarly, the right-hand side of (3.22) admits also a weak sense for $C^{1,\alpha}$-metrics using directly Definition 3.4.2. Summing all this up, the weak version of (3.22) can be written as

$$\frac{1}{2}\sum_{i,b=1}^n \left\langle\!\left\langle \operatorname{Ric}(\partial_a,\partial_b), E_i^b(e_i\cdot\psi,\varphi) \right\rangle\!\right\rangle_U = \sum_{i,b=1}^n \left\langle\!\left\langle R^\Sigma(\partial_a,\partial_b)\psi, E_i^b e_i\cdot\varphi \right\rangle\!\right\rangle_U, \tag{3.24}$$

although we have not proved yet the validity of this equality. Clearly, the weak expressions at both sides of (3.24) coincide with the corresponding "classical" ones when the metric $g$ and the spinors $\varphi, \psi$ are already differentiable enough for (3.22) to be well-defined.

Let us suppose now that $\psi \in C^{1,\alpha}(\Sigma M)$ and $g \in C^{1,\alpha}(TM^{\odot 2})$. Then, if the test spinor $\varphi \in C_c^{1,\alpha}(\Sigma M)$, we have that $\nabla_c^\Sigma \psi$ and $\nabla_c^\Sigma \hat{\varphi} := \nabla_c^\Sigma(\sqrt{g}\varphi)$ are of class $C^{0,\alpha}$ for every $c = 1,\ldots,n$. Indeed, equation (3.4) shows that the partial derivatives $\partial_c(\sqrt{g}) = \frac{1}{2}\sqrt{g}\, g^{ab}\partial_c g_{ab}$ so they really are $C^{0,\alpha}$ functions and therefore the spinorial covariant derivatives $\nabla_c^\Sigma \hat{\varphi}$ also are. From this, Definition 3.4.2 implies that for any $\partial_a, \partial_b$, the function $\left\langle\!\left\langle R^\Sigma(\partial_a,\partial_b)\psi,\varphi \right\rangle\!\right\rangle_U$ has $C^{0,\alpha}$ regularity on $U \subset M$ and is, in particular, continuous.

Similarly, checking all the terms in Definition 3.3.1 we see that $\left\langle\!\left\langle \operatorname{Ric}(\partial_a,\partial_b),\varphi \right\rangle\!\right\rangle_U$ is also of class $C^{0,\alpha}$ on $U$ in this case. Moreover, we have that the weak Ricci and weak spinorial curvatures viewed as applications

$$\begin{aligned}\mathscr{W}_{R^\Sigma} : C^{1,\alpha}(\Sigma M)\times C^{1,\alpha}(\Sigma M)\times C^{1,\alpha}(T^*M^{\odot 2}) &\longrightarrow \quad \mathbb{C} \\ (\psi,\varphi,g) &\longmapsto \left\langle\!\left\langle R^\Sigma(\partial_a,\partial_b)\psi,\varphi \right\rangle\!\right\rangle_U,\end{aligned}$$

and the weak Ricci curvature as

$$\begin{aligned}\mathscr{W}_{\operatorname{Ric}} : C^{1,\alpha}(\Sigma M)\times C^{1,\alpha}(\Sigma M)\times C^{1,\alpha}(T^*M^{\odot 2}) &\longrightarrow \quad \mathbb{C} \\ (\psi,\varphi,g) &\longmapsto \left\langle\!\left\langle \operatorname{Ric}(\partial_a)\cdot\psi,\varphi \right\rangle\!\right\rangle_U,\end{aligned}$$

are continuous in the variables $\psi, \varphi$ and $g$ in the $C^{1,\alpha}$-topology. For a detailed proof of this, which requires long calculations and identification of spinors that will be introduced in chapter 4 we refer the reader to Appendix B.

As was explained in Remark 3.4.1, if the vector fields $e_i$ are at least $C^{1,\alpha}$, the coefficients $E_i^b$ have also this regularity and then the expressions $E_i^b(e_i \cdot \psi, \varphi)$ and $E_i^b e_i \cdot \varphi$ at the left and right-hand sides of (3.24), respectively, can be seen as test functions of class $C^{1,\alpha}$. Then, replacing $\varphi$ for these expressions in what was just said concerning $W_{R^\Sigma}$ and $W_{\mathrm{Ric}}$ guarantees that both sides of (3.24) are continuous in the $C^{1,\alpha}$-topology.

Since smooth sections $\psi, \varphi$ and $g$ are dense in the domain of $\mathscr{W}_{R^\Sigma}$ and $\mathscr{W}_{\mathrm{Ric}}$, i.e., $C^{1,\alpha}(\Sigma M) \times C^{1,\alpha}(\Sigma M) \times C^{1,\alpha}(T^*M^{\odot 2})$, if (3.22) holds in the smooth case, the corresponding weak version (3.24) will hold when $g$ and $\varphi, \psi$ have only $C^{1,\alpha}$ regularity. □

**Theorem 3.4.2.** *If $M$ is a compact spin manifold with a $C^{1,\alpha}$-metric $g$ such that there is a (non-trivial) Killing spinor, then the manifold satisfies weakly the Einstein condition in any local coordinate chart.*

*Proof.* Let $\psi \in \Gamma(\Sigma M)$ be a Killing spinor, $\nabla_X^\Sigma \psi = \mu X \cdot \psi$, for any $X \in \Gamma(TM)$. First, given a local basis of coordinate vector fields $(\partial_1, \dots, \partial_n)$ and a spinor field $\varphi \in \Gamma(\Sigma M)$ we get,

$$\begin{aligned}\langle\!\langle R^\Sigma(\partial_a, \partial_b)\psi, \varphi \rangle\!\rangle_U &= \int_U \left(\nabla_a^\Sigma \psi, \nabla_b^\Sigma \hat{\varphi}\right) dx - \int_U \left(\nabla_b^\Sigma \psi, \nabla_a^\Sigma \hat{\varphi}\right) dx \\ &= \int_U \left(\mu \partial_a \cdot \psi, \nabla_b^\Sigma \hat{\varphi}\right) dx - \int_U \left(\mu \partial_b \cdot \psi, \nabla_a^\Sigma \hat{\varphi}\right) dx.\end{aligned}$$

Now we pass the covariant derivatives from the right-hand side to the left-hand side of the hermitian products and collect the integrands,

$$= \int_U \left(\nabla_a^\Sigma(\mu \partial_b \cdot \psi) - \nabla_b^\Sigma(\mu \partial_a \cdot \psi), \varphi\right) \mathrm{dvol}_g\,.$$

Finally, the product formula and the Killing equation once again are used to simplify the remaining spinorial derivatives,

$$\begin{aligned}&= \int_U \left(\mu(\nabla_k \partial_b - \nabla_l \partial_a) \cdot \psi + \mu(\partial_b \cdot \nabla_a^\Sigma \psi - \partial_a \cdot \nabla_b^\Sigma \psi), \varphi\right) \mathrm{dvol}_g \\ &= \int_U \left(\mu(\Gamma_{kl}^m - \Gamma_{lk}^m)\partial_m \cdot \psi + \mu^2(\partial_b \cdot \partial_a - \partial_a \cdot \partial_b) \cdot \psi, \varphi\right) \mathrm{dvol}_g \\ &= \int_U \left(\mu^2(\partial_b \cdot \partial_a - \partial_a \cdot \partial_b) \cdot \psi, \varphi\right) \mathrm{dvol}_g\,.\end{aligned}$$

In the previous result, let us replace now the test spinor $\varphi$ in the right hand side of the hermitian product by $E_i^b e_i \cdot \varphi$ and choose a particular $\partial_a$ to find,

$$\begin{aligned}
&\sum_{i,b=1}^{n} \left\langle\!\left\langle R^{\Sigma}(\partial_a,\partial_b)\psi, E_i^b e_i\cdot\varphi \right\rangle\!\right\rangle_U \\
&\qquad = \sum_{i,b=1}^{n} \int_U \big(\mu^2(\partial_b\cdot\partial_a - \partial_a\cdot\partial_b)\cdot\psi, E_i^b e_i\cdot\varphi\big)\,\mathrm{dvol}_g\,. \\
&\qquad = \sum_{i=1}^{n} \int_U \big(\mu^2(e_i\cdot\partial_a - \partial_a\cdot e_i)\cdot\psi\,, e_i\cdot\varphi\big)\,\mathrm{dvol}_g\,. \\
&\qquad = -\sum_{i=1}^{n} \int_U \big(\mu^2(e_i\cdot e_i\cdot\partial_a - e_i\cdot\partial_a\cdot e_i)\cdot\psi,\varphi\big)\,\mathrm{dvol}_g\,. \\
&\qquad = \sum_{i=1}^{n} \int_U \big(\mu^2(\partial_a - (\partial_a\cdot e_i + 2g(\partial_a,e_i))\cdot e_i)\cdot\psi,\varphi\big)\,\mathrm{dvol}_g\,. \\
&\qquad = \sum_{i=1}^{n} \int_U \big(\mu^2(2\partial_a - 2g(\partial_a,e_i)\,e_i)\cdot\psi\,,\varphi\big)\,\mathrm{dvol}_g\,. \\
&\qquad = 2\mu^2 \int_U \big((n\partial_a - \textstyle\sum_i g(\partial_a,e_i)\,e_i)\cdot\psi,\varphi\big)\,\mathrm{dvol}_g\,.
\end{aligned}$$

Since $(e_1,\ldots,e_n)$ is orthonormal then $\partial_a = \sum_{i=1}^n g(\partial_a,e_i)e_i$, so we arrive at

$$\sum_{i,b=1}^{n} \left\langle\!\left\langle R^{\Sigma}(\partial_a,\partial_b)\psi, E_i^b e_i\cdot\varphi \right\rangle\!\right\rangle_U = 2\mu^2(n-1)\int_U (\partial_a\cdot\psi,\varphi)\,\mathrm{dvol}_g\,. \tag{3.25}$$

With this, the previous lemma implies that the Einstein condition is satisfied locally in a weak sense, i.e.,

$$\left\langle\!\left\langle \mathrm{Ric}(\partial_a)\cdot\psi,\varphi \right\rangle\!\right\rangle_U = 4\mu^2(n-1)\int_U (\partial_a\cdot\psi,\varphi)\,\mathrm{dvol}_g\,. \tag{3.26}$$

Both sides of this equation can be rewritten in the form of Definition 3.3.2, but since we have not proved any linearity in the entries of $\mathrm{Ric}(\cdot,\cdot)$ when working in the weak sense, we have to stick to Definition 3.3.1 to proceed. Let us use (3.23) at the left-hand side of (3.26) and the substitution $\partial_a = \sum_i g(\partial_a,e_i)e_i$ to introduce the metric at the right-hand side, then

$$\sum_{c=1}^{n} \left\langle\!\left\langle \mathrm{Ric}(\partial_a,\partial_c), \textstyle\sum_i E_i^c(e_i\cdot\psi,\varphi) \right\rangle\!\right\rangle_U = 4\mu^2(n-1)\int_U (\textstyle\sum_i g(\partial_a,e_i)e_i\cdot\psi,\varphi)\,\mathrm{dvol}_g,$$

and replace $e_i$ inside $g(\partial_a,e_i)$ by $e_i = \sum_c E_i^c\partial_c$ to obtain

$$= 4\mu^2(n-1)\sum_{c=1}^{n}\int_U g(\partial_a,\partial_c)(\textstyle\sum_i E_i^c e_i\cdot\psi,\varphi)\,\mathrm{dvol}_g\,.$$

Since this holds for every $\varphi \in C_c(\Sigma M)$, take any test function $\eta \in C_c(M,\mathbb{R})$ with $\operatorname{supp}\eta \subset U$ and choose for any fixed $b \in \{1,\dots,n\}$, a test spinor $\varphi_b := g(\partial_b, e_i)\frac{e_i\cdot\psi}{\|\psi\|^2}\eta$. Thus, for each $\varphi_b$ we can write (note that $\sum_i E_i^c g(\partial_b, e_i) = \delta_{bc}$),

$$\sum_{c=1}^{n} \left\langle\!\left\langle \mathrm{Ric}(\partial_a,\partial_c), \textstyle\sum_i E_i^c g(\partial_b,e_i)\frac{(e_i\cdot\psi, e_i\cdot\psi)}{\|\psi\|^2}\eta \right\rangle\!\right\rangle_U$$
$$= 4\mu^2(n-1)\sum_{c=1}^{n}\int_U g(\partial_a,\partial_c)\textstyle\sum_i E_i^c g(\partial_b,e_i)\frac{(e_i\cdot\psi, e_i\cdot\psi)}{\|\psi\|^2}\eta \,\mathrm{dvol}_g$$

$$\sum_{c=1}^{n} \left\langle\!\left\langle \mathrm{Ric}(\partial_a,\partial_c), \delta_{bc}\eta \right\rangle\!\right\rangle_U = 4\mu^2(n-1)\sum_{c=1}^{n}\int_U g(\partial_a,\partial_c)\delta_{bc}\eta \,\mathrm{dvol}_g,$$

so we finally get the Einstein condition (3.14) for an arbitrary compactly supported test function $\eta$ and all $a,b = 1,\dots,n$:

$$\left\langle\!\left\langle \mathrm{Ric}(\partial_a,\partial_b), \eta \right\rangle\!\right\rangle_U = 4\mu^2(n-1)\int_U g(\partial_a,\partial_b)\eta \,\mathrm{dvol}_g\,.$$

□

Finally, we arrive to our main result of this chapter taking a global point of view in the previous theorem.

**Theorem 3.4.3.** *If $M$ is a compact spin manifold with a $C^{1,\alpha}$-metric $g$ carrying a non-trivial Killing spinor, then $g$ is smooth with respect to a harmonic coordinate atlas for $M$ or to any compatible one. Furthermore, $g$ is Einstein (and hence analytic) in these atlases.*

*Proof.* By Theorem 3.3.2 a spin manifold with a $C^{1,\alpha}$-metric $g$ carrying a non-trivial Killing spinor satisfies weakly the Einstein equation (3.14) on any coordinate chart, in particular on any chart of a harmonic coordinate atlas for $M$. But by Theorem 2.4.2 any $C^{1,\alpha}$-metric satisfying the Einstein equation in the weak sense on a harmonic chart is smooth in this chart. This means that $g$ is smooth with respect to a harmonic coordinate atlas for $M$ and therefore the Einstein equation is satisfied in the classical sense. Theorem 3.3.4 (or Corollary 3.3.6) implies that the metric is analytic for this atlas. □

# Chapter 4

# Spin Manifolds and Convergence

Now, we want to consider the convergence of sequences of Riemannian spin manifolds (with their corresponding metrics) under appropriate geometrical conditions, and study the behavior of spinor fields and the Dirac operator in the limit. First we recall the basic machinery from the theory of convergence of manifolds in a Riemannian setting (see, for instance [Pet97] and [HH97]).

## 4.1 Convergence of Manifolds and $C^{k,\alpha}$ Metrics

We begin by defining the notion of $C^{k,\alpha}$ convergence of functions with domain in $\mathbb{R}^n$ and then extend it to the manifold case, which will be needed to formulate the main result of this chapter.

**Definition 4.1.1.** Let $f_i$ be a sequence of $C^{k,\alpha}$ functions defined on an open set $U \subset \mathbb{R}^n$, then $f_i$ *converges in* $C^{k,\alpha}(U)$ to a $C^{k,\alpha}$ function $f$, defined also on $U$, if $\lim_{i\to\infty} \|f_i - f\|_{C^{k,\alpha}(U)} = 0$.

Alternatively, this kind of convergence is also called convergence in the $C^{k,\alpha}$-topology for obvious reasons.

**Definition 4.1.2.** We say that $f_i$ *converge in* $C^{k,\alpha}_{\text{loc}}(U)$ *to* $f$, for an open set $U \subset R^n$, if for every compact set $K \subset U$ it holds that $\lim_{i\to\infty} \|f_i - f\|_{C^{k,\alpha}(K)} = 0$.

**Remark 4.1.1.** Here $C^{k,\alpha}_{\text{loc}}(U)$ denote the set of locally $C^{k,\alpha}$ functions on $U$. Using the usual definition of the $\alpha$-seminorm one can show that for $U \subset \mathbb{R}^n$, a fixed $k \in \mathbb{N}$ and $\alpha \in (0,1)$, a function $f$ belongs to $C^{k,\alpha}_{\text{loc}}(U)$ whenever $f \in C^k(U)$ and for all multi-indices $\beta \in \mathbb{N}^n$ with $|\beta| = k$ and all open balls $B(x_0, r) \subset U$,

$$\sup_{\substack{x,y \in B(x_0,r) \\ x \neq y}} \frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x-y|^\alpha} \leq C(x_0, r),$$

for some real bound $C(x_0, r)$ depending only on the center $x_0 \in U$ and radius $r$ of the corresponding ball.

When working on a manifold $M$ there is a natural notion of $C^{k,\alpha}$ convergence of tensors on $M$ defined below.

**Definition 4.1.3.** A sequence $T_i$ of tensors on a given manifold $M$ is said to *converge* to a tensor $T$ in the *$C^{k,\alpha}$-topology* if there is a sub-atlas of the complete atlas of $M$ whose coordinate charts $\phi_\gamma : U_\gamma \longrightarrow \mathbb{R}^n$ define transition functions of class at least $C^{k+1,\alpha}$, and all the components of the tensors $T_i$ converge in the $C^{k,\alpha}$-topology to the components of $T$, on any chart of this sub-atlas (viewed as functions on $\phi_\gamma(U_\gamma) \subset \mathbb{R}^n$).

The previous definition clearly requires that $M$ carries at least a $C^{k+1,\alpha}$ differentiable structure since the components of tensors are computed by evaluating them on $C^{k,\alpha}$ objects as vector fields $\partial_a := \partial/\partial x^a$ and forms $dx^a$. However, this notion is not the same as the next one, with which we want to study the convergence of metrics in our setting.

**Definition 4.1.4.** Let $(M_i, g_i)$ be a sequence of smooth compact Riemannian $n$-manifolds, $M$ a smooth compact differentiable $n$-manifold, and $g$ a $C^{k,\alpha}$ Riemannian metric on $M$, for an integer $k$ and a real $\alpha \in (0,1)$. We say that $(M_i, g_i)$ *converge to* $(M, g)$ *in the $C^{k,\alpha}$-topology* if there exists $j_0$ such that for every $i \geq j_0$ we can find $C^{k+1,\alpha}$ diffeomorphisms $\theta_i : M \longrightarrow M_i$ for which the components of $\theta_i^* g_i$ converge in $C^{k,\alpha}_{\mathrm{loc}}(U_\gamma)$ to the components of $g$, on any chart $U_\gamma$ of the smooth complete atlas of $M$.

This convergence is generally valid in the framework of $C^{k+1,\alpha}$ manifolds but we restrict ourselves to the case of smooth manifolds which is enough for our purposes. The general definition is straightforward from what we said for the convergence of tensors.

**Remark 4.1.2.** $C^{k,\alpha}_{\mathrm{loc}}$ convergence on any chart of the complete atlas of $M$ implies the existence of a (smooth) sub-atlas for which the components of the metrics $\theta_i^* g_i$ converge in the $C^{k,\alpha}$-topology to the components of $g$ on any of its charts (viewed again as functions on $\mathbb{R}^n$).

**Definition 4.1.5.** A set $\mathscr{M}$ of smooth compact Riemannian $n$-manifolds is *precompact* in the $C^{k,\alpha}$-topology if any sequence in $\mathscr{M}$ has a subsequence which is convergent in this topology.

### 4.1.1 Compactness Theorems

There are many compactness theorems in Riemannian geometry having their origin in the seminal work of M. Gromov in 1981, who proved that the space of compact Riemannian $n$-manifolds with sectional curvature and diameter bounded from above and with lower bound for the volume, is precompact in the Lipschitz topology (cf. [Gro99]). Since then, several extensions have been settled in different directions. Of particular interest to us is the generalization of this theorem to the $C^{1,\alpha}$-topology. Related results extend these properties to the case of lower bounds on the Ricci curvature as it is done in the work of M. Anderson and J. Cheeger (e.g. [And90], [AC92]) and even further in [HH97].

**Definition 4.1.6.** Let $\mathscr{M}(n,d,K,V)$ denote the class of compact $n$-dimensional Riemannian manifolds $(M,g)$ with diameter $\mathrm{diam}(M) \leq d$, volume $\mathrm{vol}(M) \geq V$, and sectional curvature $|\sec| \leq K$. (Alternatively, the condition $\mathrm{vol}(M) \geq V$ can be replaced by a lower bound on the injectivity radius $\mathrm{inj\,rad}(g)$ of $M$, so we could take $\mathrm{inj\,rad}(g) \geq i_0$ for some real constant $i_0 > 0$.)

**Theorem 4.1.1** (Compactness Theorem)**.** *The class $\mathscr{M}(n,d,K,V)$ is precompact in the $C^{1,\alpha}$-topology for any $\alpha \in (0,1)$, and contains only finitely many diffeomorphism types.*

The following reformulation of the previous theorem that appears in [Pts87] summarize very well the convergence and compactness properties we have defined.

**Theorem 4.1.2.** *Let $(M_i, g_i)$ denote a sequence of Riemannian manifolds in the class $\mathscr{M}(n,d,K,V)$ and $\alpha \in (0,1)$. Then, there exists a subsequence denoted also by $(M_i, g_i)$ with the properties:*

1. *Each $M_i$ is diffeomorphic to a single manifold $M$.*
2. *There exist diffeomorphisms $\theta_i : M \longrightarrow M_i$ such that $\theta_i^* g_i$ converges in the $C^{1,\alpha}$-topology to a $C^{1,\alpha}$-metric $g$ on $M$.*
3. *For the injectivity radius we have* $\limsup_i \mathrm{inj\,rad}(g_i) \leq \mathrm{inj\,rad}(g)$.
4. *Let* $\exp_i$ *be the exponential map of $M_i$,* $\exp$ *that of $M$ and* $\overline{\exp}_i = \theta_i^* \exp_i$*, then* $\overline{\exp}_i$ *converges to* $\exp$ *uniformly on compact subsets of $T_pM$, and* $\exp(p)$ *is Lipschitz.*

**Theorem 4.1.3** (Nikolaev)**.** *In the previous compactness theorem, the components of the limit metric $g$, expressed in harmonic coordinates, are contained in the Sobolev spaces $H_2^p$ for any $p \leq 1$. In particular, all notions of curvature are almost everywhere defined.*

**Remark 4.1.3.** If a sequence of Riemannian manifolds $(M_i, g_i)$ converge in the sense just defined to a limit manifold $(M, g)$, the first property of Theorem 4.1.2 says that the actual manifolds in question are diffeomorphic, therefore we can choose the limit one as representative and concentrate in the convergence of the pullbacks of the corresponding metrics.

### 4.1.2 Norms and Convergence of Metrics

Let $M$ be a smooth compact spin manifold and denote by $h$ a fixed Riemannian metric over $M$ with corresponding Levi-Civita connection $\nabla^h$. Let

$$\mathbb{S}M = \bigsqcup_{p\in M} \{X_p \in T_pM \mid \|X_p\|_h = 1\} \subset TM,$$

be the sphere bundle of unit vectors on $M$ associated to $h$. For an operator $T \in \mathrm{End}(TM)$ there are natural norms with respect to the Riemannian metric $h$:

$$\|T\|_h := \sup_{X\in\mathbb{S}M} |TX|_h,$$
$$\|\nabla^h T\|_h := \sup_{X,Y\in\mathbb{S}M} |(\nabla^h_X T)(Y)|_h.$$

For a Riemannian metric $g$ we can define, as well, a norm of the covariant derivative of $g$ with respect to $h$ as,

$$\|\nabla^h g\|_h := \sup_{p\in M} \left\{ |(\nabla^h_X g)(Y,Z)|_h \mid X,Y,Z \in \mathbb{S}_pM \right\}. \tag{4.1}$$

**Definition 4.1.7.** The $C^1$-distance with respect to $h$ between two Riemannian metrics $g$ and $g'$ is given by

$$d_h^{C^1}(g,g') := \|g-g'\|_h + \|\nabla^h(g-g')\|_h.$$

The following two properties related to this distance in the $C^1$-topology can be found in [Pfä03].

**Lemma 4.1.4.** *Let $(g_i)_{i\geq 1}$ denote a sequence of Riemannian metrics and $g$ some fixed Riemannian metric on a manifold $M$. The sequence $(g_i)_{i\geq 1}$ converge to $g$ in the $C^1$-topology if and only if in every chart $U \subset M$ of local coordinates $(x^1 \ldots, x^n)$, the components of the metrics and its derivatives converge uniformly,*

$$(g_i)_{ab} \xrightarrow[i\to\infty]{} g_{ab} \qquad \text{and} \qquad \partial_c (g_i)_{ab} \xrightarrow[i\to\infty]{} \partial_c g_{ab},$$

*for every $a,b,c \in \{1,\ldots,n\}$.*

**Theorem 4.1.5.** *If a sequence of Riemannian metrics $(g_i)_{i\geq 1}$ on $M$ converge in the $C^1$-topology to a Riemannian metric $g$, then for the $C^1$-distance we have*

$$d_g(g_i) := d_g^{C^1}(g, g_i) \xrightarrow[i\to\infty]{} 0.$$

## 4.2 Identification of Metric Spinors

Using the procedure introduced in [BG92] (see also [Pfä03] and [ADH06]) we explain below how spinors coming from spin structures associated to different metrics are going to be identified. Specifically, this will give us a way to associate spinors on a sequence of spin manifolds to spinors in the limit manifold. In fact, given that the actual manifolds in a convergent sequence are diffeomorphic to the limit one, the importance of this construction is how it relates different metrics and corresponding spin structures, providing an identification of vector and spinor fields.

We should point out here that there is another viewpoint to identify spinors for different metrics given by J. Lott in [Lot00]. As explained there, one obtains from a topological spin structure on $M$ a $Spin(n)$-principal bundle $P_{Spin}(M)$ without using explicitly a metric. This $P_{Spin}(M)$ is well-defined up to gauge transformations and the spinor bundle is the corresponding associated bundle. In Lott's approach, the choice of a metric on $M$ defines a connection-1-form on $P_{Spin}(M)$ and hence a connection on spinors.

However, the above mentioned gauge transformations may cause several troubles when one writes down analytical arguments explicitly. In order to be as concrete as possible, we have chosen in this thesis not to follow this presentation. From the viewpoint of J. Lott's approach what we do is to fix one metric (namely the limit metric) as a reference metric, and then this gauge ambiguity does no longer exist.

Let $(g_i)_{i\geq 1}$ denote a sequence of Riemannian metrics on an $n$-dimensional spin manifold $M$, converging to a fixed Riemannian metric $g$. We assume by now that all these metrics have at least $C^{1,\alpha}$ regularity. In addition, we suppose that there is a fixed (metric) spin structure $\chi$ for $(M,g)$ which can be pulled-back as shown in section 2.1 to define compatible metric spin structures $\chi_i$ for the metrics $g_i$ in the sequence. This compatibility amounts to the equivalence of the background topological spin structures, but the metric aspect is necessary to construct associated spinor bundles.

We want to relate (and identify) vector and spinor fields coming from the metrics in the sequence, to the ones coming from the metric in the limit. For this, let us take any metric $g_i$ with the corresponding metric spin structure $\chi_i$ for $(M, g_i)$ and consider the relation to $(M,g)$ with the spin structure $\chi$. There exists a unique positive definite transformation $A_i \in \mathrm{End}(TM)$, which is symmetric with respect to

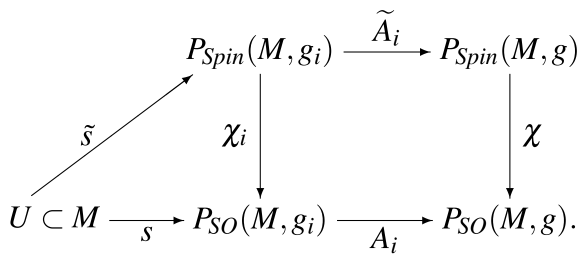
$g$ and such that for every $X, Y \in TM$

$$g(A_iX, A_iY) = g_i(X, Y). \tag{4.2}$$

Because of this property, $A_i$ sends orthonormal frames for $g$ to orthonormal frames for $g_i$, and thus induces a map denoted again by $A_i$ between the $SO(n)$-principal frame bundles associated to $g$ and $g_i$,

$$\begin{aligned} A_i : P_{SO}(M, g_i) &\longrightarrow P_{SO}(M, g) \\ s = (e_1, \dots, e_n) &\longmapsto A_i(s) = (A_ie_1, \dots, A_ie_n). \end{aligned}$$

Given that the spin structures $\chi$ and $\chi_i$ are equivalent, as it was mentioned earlier, the endomorphism $A_i$ can be lifted to a map $\widetilde{A}_i : P_{Spin}(M, g) \longrightarrow P_{Spin}(M, g_i)$. Thus, we have the following diagram, where we have included two arbitrary sections $s \in \Gamma(P_{Spin}(M, g))$ and $s \in \Gamma(P_{SO}(M, g))$,




The associated spinor bundles inherit from this construction a natural identification $A_i : \Sigma^{g_i}M \longrightarrow \Sigma^g M$ labeled for convenience with the same symbol $A_i$ as the one for vectors,

$$\begin{aligned} \Sigma^{g_i}M := P_{Spin}(M, g_i) \times_\sigma \Sigma_n &\longrightarrow \Sigma^g M := P_{Spin}(M, g) \times_\sigma \Sigma_n \\ \psi := [\tilde{s}, \Psi] &\longmapsto A_i\psi := [\widetilde{A}_i(\tilde{s}), \Psi]. \end{aligned}$$

This map is $Spin(n)$-equivariant and respects Clifford multiplication by vector fields defined through the identification $TM \simeq P_{Spin}(M, g_i) \times_{\mathrm{Ad}} \mathbb{R}^n$, so for $X \in \Gamma(TM)$ we can write,

$$A_i(X \cdot \psi_i) = (A_iX) \cdot (A_i\psi_i).$$

Since the metric on the spinor bundles is given by a fixed Hermitian inner product on $\Sigma^{g_i}M$ and $\Sigma^g M$, the endomorphism $A_i$ defines a fiberwise isometry between the bundles, i.e., the norms of the spinors are preserved.

We will now look at the relation between the canonical covariant derivatives for $(M, g)$ and $(M, g_i)$. Let $\nabla$ and $\nabla^i$ be the Levi-Civita connections for $g$ and $g_i$. To compare $\nabla$ and $\nabla^i$ on the frame and spin bundles for $g$ we define a connection $\overline{\nabla}^i : \Gamma(TM) \longrightarrow \Gamma(T^*M \otimes TM)$ by

$$\overline{\nabla}^i X := A_i^{-1}(\nabla A_iX). \tag{4.3}$$

The connection $\overline{\nabla}^i$ is metric with respect to $g_i$, thanks to the metricity of $\nabla$ with respect to $g$ and the properties of $A_i$,

$$\begin{aligned} g_i(\overline{\nabla}^i_Z X, Y) + g_i(X, \overline{\nabla}^i_Z Y) \\ &= g_i(A_i^{-1}\nabla_Z(A_iX), A_i^{-1}(A_iY)) + g_i(A_i^{-1}(A_iX), A_i^{-1}\nabla_Z(A_iY)) \\ &= g(\nabla_Z(A_iX), A_iY) + g(A_i^{-1}X, \nabla_Z(A_iY)) \\ &= Zg(A_iX, A_iY) \\ &= Zg_i(X,Y), \end{aligned} \tag{4.4}$$

and has torsion given by

$$\begin{aligned} \overline{T}^i(X,Y) :&= \overline{\nabla}^i_X Y - \overline{\nabla}^i_Y X - [X,Y] \\ &= A_i^{-1}\nabla_X(A_iY) - A_i^{-1}\nabla_Y(A_iX) - [X,Y] \\ &= \nabla_X(A_i^{-1}A_iY) - \nabla_Y(A_i^{-1}A_iX) - [X,Y] \\ &\qquad - ((\nabla_X A_i^{-1})A_iY - (\nabla_Y A_i^{-1})A_iX) \\ &= (\nabla_Y A_i^{-1})A_iX - (\nabla_X A_i^{-1})A_iY, \end{aligned} \tag{4.5}$$

where we have used that $\nabla$ is torsion free.

Writing the covariant derivative in terms of the Lie bracket and the metric we arrive at the following identity,

$$2g_i(\overline{\nabla}^i_X Y - \nabla^i_X Y, Z) = g_i(\overline{T}^i(X,Y),Z) - g_i(\overline{T}^i(X,Z),Y) - g_i(\overline{T}^i(Y,Z),X). \tag{4.6}$$

Next we compare $\nabla^i$, $\overline{\nabla}^i$ when lifted to the spinor bundle $\Sigma^{g_i}M$. To avoid complicating the notation even more we drop the superscript in $\nabla^\Sigma$ for the covariant derivatives on the spinor bundles. Let $(e_1,\dots,e_n)$ be a local $g$-orthonormal frame on $M$, and let $\{\varphi_\alpha\}$ be the corresponding local orthonormal frame of the spinor bundle. Denote by $\omega^i, \overline{\omega}^i$ the connection one-forms for $\nabla^i, \overline{\nabla}^i$, whose components with respect to $(e_1,\dots,e_n)$ can be written as,

$$\begin{aligned} \omega^i_{ab} &= g_i(\nabla^i e_a, e_b), \\ \overline{\omega}^i_{ab} &= g_i(\overline{\nabla}^i e_a, e_b). \end{aligned}$$

Using (4.5) and (4.6) we can estimate

$$\|(\overline{\omega}^i_{ab} - \omega^i_{ab})(e_k)\|_{g_i} \le C\|A_i\|_{g_i}\|\nabla(A_i^{-1})\|_{g_i}.$$

Thus, the covariant derivatives of $\psi_i \in \Gamma(\Sigma^{g_i}M)$ with respect to a vector field $X \in \Gamma(TM)$ are

$$\nabla^i_X \psi_i = X(\psi_i) + \frac{1}{4}\sum_{a,b=1}^{n} \omega^i_{ab}(X)\, e_a \cdot e_b \cdot \psi_i,$$
$$\overline{\nabla}^i_X \psi_i = X(\psi_i) + \frac{1}{4}\sum_{a,b=1}^{n} \overline{\omega}^i_{ab}(X)\, e_a \cdot e_b \cdot \psi_i$$

and the difference between $\overline{\nabla}^i$ and $\nabla^i$ acting on $\psi_i$ is

$$\overline{\nabla}^i_X \psi - \nabla^i_X \psi_i = \frac{1}{4}\sum_{a,b=1}^{n} (\overline{\omega}^i_{ab} - \omega^i_{ab})(X)\, e_a \cdot e_b \cdot \psi_i. \tag{4.7}$$

With the previous calculations we can state the following lemma

**Lemma 4.2.1.** *Let $X, Y \in \Gamma(TM)$ be vector fields and $\psi_i \in \Gamma(\Sigma^{g_i}M)$ a spinor field, then we have the bounds*

$$|\overline{\nabla}^i_X Y - \nabla^i_X Y|_{g_i} \leq C\|A_i\|_{g_i}\|\nabla(A_i^{-1})\|_{g_i}\,|X|_{g_i}\,|Y|_{g_i}, \tag{4.8}$$
$$|\overline{\nabla}^i_X \psi_i - \nabla^i_X \psi_i| \leq C\|A_i\|_{g_i}\|\nabla(A_i^{-1})\|_{g_i}\,|X|_{g_i}\,|\psi_i|, \tag{4.9}$$
$$|\overline{D}^i \psi_i - D^i \psi_i| \leq C\|A_i\|_{g_i}\|\nabla(A_i^{-1})\|_{g_i}\,|\psi_i|, \tag{4.10}$$

*where $D^i$ and $\overline{D}^i$ denote the Dirac operators associated to the connections $\nabla^i$ and $\overline{\nabla}^i$ respectively.*

## 4.3 Spectral Closeness

Now we study the behavior of the eigenvalues of the Dirac operator under changes of the underlying metric $g$ of the manifold. This will be useful to understand the consequences arising from the identification of spinors for different $C^{1,\alpha}$-metrics on the manifold that we will need.

**Definition 4.3.1.** Let $\varepsilon > 0$ and $\Lambda > 0$. Two operators with discrete spectrum are said to be $(\Lambda, \varepsilon)$-*spectral close* if

1. $\pm\Lambda$ are not eigenvalues of either operator.

2. Both operators have the same total number $m$ of eigenvalues in the interval $(-\Lambda, \Lambda)$.

3. If the eigenvalues in $(-\Lambda,\Lambda)$ are denoted by $\lambda_1 \leq \cdots \leq \lambda_m$ and $\mu_1 \leq \cdots \leq \mu_m$ respectively (each eigenvalue repeated according to its multiplicity), then $|\lambda_j - \mu_j| < \varepsilon$ for $j = 1,\ldots,m$.

The following theorem states that on a fixed closed spin manifold the convergence of Riemannian metrics in the $C^1$-topology implies the convergence of the Dirac spectra (see [Bär96b] and [Pfä03] for proofs and careful exposition).

**Theorem 4.3.1** (Bär). *Let $(M,g)$ be a closed spin manifold and let $\varepsilon > 0$ and $\Lambda > 0$ with $\pm\Lambda \notin \mathrm{spec}(D_g)$. Then, there exists $\delta > 0$ such that for all Riemannian metrics $g'$ with $d_g(g') < \delta$ the Dirac operators $D_g$ and $D_{g'}$ are $(\Lambda,\varepsilon)$-spectral close.*

*Notation.* Let $P$ be a self-adjoint operator and $r < s$ be real numbers such that $\mathrm{ess\,spec}(P) \cap [r,s] = \varnothing$. We denote by $E_\lambda(P)$ the eigenspace of $P$ associated to the eigenvalue $\lambda$ and define

$$E_{[r,s]}(P) := \bigoplus_{\substack{r\leq\lambda\leq s\\ \lambda\in\mathrm{spec}(P)}} E_\lambda(P).$$

**Corollary 4.3.2.** *Let $S > 0$ be a real constant and $(M,g)$ be an n-dimensional compact Riemannian spin manifold (possibly with boundary), with scalar curvature $\mathrm{scal}_g \geq -S$. For fixed numbers $\Lambda > 0$, $\varepsilon \geq 0$ and $\nu > 0$ there exists some $\delta > 0$ such that for any Riemannian metric $g'$ on $M$ with $d_g(g') < \delta$ and every $\mu \in [0,\Lambda]$, holds:*

$$\dim E_{[\mu-\nu,\mu+\nu]}(D_g^2) \leq \dim E_{[\mu-\nu-\varepsilon,\mu+\nu+\varepsilon]}(D_{g'}^2) \qquad \textit{and}$$
$$\dim E_{[\mu-\nu,\mu+\nu]}(D_{g'}^2) \leq \dim E_{[\mu-\nu-\varepsilon,\mu+\nu+\varepsilon]}(D_g^2).$$

## 4.4 Modified Connections

Let $(M,g)$ be a Riemannian spin manifold with a fixed (metric) spin structure $\chi$. Recall that the spinor bundle $\Sigma^g M$ determined by $\chi$ depends intrinsically on $g$ and in particular, the Clifford multiplication on the fibers of $\Sigma^g M$ also does.

**Definition 4.4.1.** Denote by $\nabla^{\Sigma^g}$ the Levi-Civita connection on the spinor bundle $\Sigma^g M$ and let $\mu \in \mathbb{C}$ be fixed. The so-called *Friedrich connection* on $\Sigma^g M$ corresponding to this $\mu$ is defined as

$$\widehat{\nabla}^g_X \psi := \nabla^{\Sigma^g}_X \psi - \mu X \cdot \psi,$$

where $X \in \Gamma(TM)$ and $\psi \in \Gamma(\Sigma^g M)$.

In this way, we regard Killing spinors as parallel sections with respect to $\widehat{\nabla}^g$.

The following calculation shows that if $\mu$ is real then the Friedrich connection $\widehat{\nabla}$ is metric with respect to the hermitian product of $\Sigma^g M$.

$$\begin{aligned}(\widehat{\nabla}_X\psi,\varphi)+(\psi,\widehat{\nabla}_X\varphi) &= (\nabla^\Sigma_X\psi,\varphi)-(\mu X\cdot\psi,\varphi)+(\psi,\nabla^\Sigma_X\varphi)-(\psi,\mu X\cdot\varphi)\\ &= (\nabla^\Sigma_X,\varphi)+(\psi,\nabla^\Sigma_X\varphi)-\mu(X\cdot\psi,\varphi)+\overline{\mu}(X\cdot\psi,\varphi)\\ &= X(\psi,\varphi)-2(X\cdot\psi,\varphi)\operatorname{Im}\mu.\end{aligned}$$

This implies that Lemma 2.3.1, which is needed for the proof that follows, holds also for $\widehat{\nabla}$ for $\mu\in\mathbb{R}$.

**Theorem 4.4.1.** *The Friedrich connection for a Killing constant $\mu\in\mathbb{R}$ satisfies the following* Weitzenböck formula *for $\psi\in\Gamma(\Sigma^g M)$,*

$$(D+\mu)^2\psi=\widehat{\nabla}^*\widehat{\nabla}\psi+\frac{1}{4}\operatorname{scal}_g\psi-\mu^2(n-1)\psi. \tag{4.11}$$

*Proof.* As mentioned above, since $\widehat{\nabla}$ is metric for $\mu\in\mathbb{R}$, we can use Lemma 2.3.1 and the Schrödinger-Lichnerowicz formula (2.7) to calculate, taking a normal coordinate frame field $(e_1,\dots,e_n)$ on a local chart $U\subset M$,

$$\begin{aligned}\widehat{\nabla}^*\widehat{\nabla}\psi &= -\sum_{i=1}^n\widehat{\nabla}_{e_i}\widehat{\nabla}_{e_i}\psi\\ &= -\sum_{i=1}^n(\nabla^\Sigma_{e_i}-\mu e_i\cdot)(\nabla^\Sigma_{e_i}\psi-\mu e_i\cdot\psi)\\ &= -\sum_{i=1}^n(\nabla^\Sigma_{e_i}\nabla^\Sigma_{e_i}\psi-2\mu e_i\cdot\nabla^\Sigma_{e_i}\psi-\mu^2\psi)\\ &= (\nabla^\Sigma)^*\nabla^\Sigma\psi+2\mu D\psi+\mu^2 n\psi\\ &= D^2\psi-\frac{1}{4}\operatorname{scal}_g\psi+2\mu D\psi+\mu^2 n\psi\\ &= (D+\mu)^2\psi-\frac{1}{4}\operatorname{scal}_g\psi+\mu^2(n-1)\psi.\end{aligned}$$

□

**Lemma 4.4.2.** *Let $\alpha\in(0,1)$, $k\geq 1$ be fixed and assume $\psi\in L^2(\Sigma^g M)$ is a Killing spinor on a spin manifold $M$ with $C^{k,\alpha}$-metric $g$. Then, $\psi$ is actually of Hölder class $C^{k-1,\alpha}$.*

*Proof.* Since $\psi$ is a Killing spinor we have, for all $X\in C^{k,\alpha}(TM)$ and certain $\mu\in\mathbb{C}$,

$$\widehat{\nabla}^g_X\psi=\nabla^{\Sigma^g}_X\psi-\mu X\cdot\psi=0.$$

So if we choose a $C^{k,\alpha}$-coordinate frame field $(\partial_1,\dots,\partial_n)$ and express the connection using (2.2) we get

$$\begin{aligned}\widehat{\nabla}^g_a\psi := \widehat{\nabla}^g_{\partial_a}\psi &= \partial_a\psi + \frac{1}{4}\sum_{b,c=1}^{n} g(\nabla_a e_b, e_c) e_b\cdot e_c\cdot\psi - \mu e_a\cdot\psi \\ &= \partial_a\psi + \frac{1}{4}\sum_{b,c=1}^{n}\Gamma^c_{ab}\gamma_b\gamma_c\psi - \mu\gamma_a\psi = 0,\end{aligned}$$

where $\gamma_a := \rho(e_a)$. Since $g$ is a $C^{k,\alpha}$ metric, the Christoffel symbols $\Gamma^c_{ab}$ are of class $C^{k-1,\alpha}(U)$ and the Clifford multiplication, that depends intrinsically on $g$ and is represented componentwise by the $\gamma$ matrices, has also $C^{k,\alpha}$ regularity. In conclusion, we get

$$\widehat{\nabla}^g_a\psi = \Big(\partial_a + \frac{1}{4}\sum_{b,c=1}^{n}\Gamma^c_{ab}\gamma_b\gamma_c - \mu\gamma_a\Big)\psi = 0,$$

which shows that in coordinates $\widehat{\nabla}^g$ is given by a first order differential operator (in parenthesis) whose coefficients are at least $C^{k-1,\alpha}$. The equation $\widehat{\nabla}^g_a\psi = 0$ is then an elliptic differential equation whose right-hand side is smooth. Elliptic regularity theory (see Theorem 1.7.3) guarantees now that $\psi$ has to be at least of class $C^{k,\alpha}$. □

From the previous lemma it follows immediately this useful reformulation.

**Corollary 4.4.3.** *For a spin manifold $M$ with a Riemannian metric $g$ of class $C^{1,\alpha}$, the Friedrich connection $\widehat{\nabla}^g$ is well-defined as an application*

$$\widehat{\nabla}^g : C^{k,\alpha}(\Sigma^g M) \longrightarrow C^{k,\alpha}(T^*M)\otimes C^{k-1,\alpha}(\Sigma^g M)$$

*and given $\psi\in C^{k-1,\alpha}(\Sigma^g M)$, the covariant derivative $\widehat{\nabla}^g\psi$ is bounded in the $C^{k-1,\alpha}$-norm defined before, i.e., for a real $K>0$,*

$$\|\widehat{\nabla}^g\psi\|_{C^{k-1,\alpha}(\Sigma^g M)} \le K.$$

## 4.5 Convergence of Almost Killing Spinors

In this section we denote by $\tau(\varepsilon; r_1,\dots,r_n)$ a function depending on certain (real) parameters $\varepsilon, r_1,\dots,r_n$, that goes to zero when $\varepsilon$ goes to zero for fixed $r_1,\dots r_n$. The precise form of the function $\tau$ may vary during the steps of a calculation but, when used, we will be interested only in its vanishing behavior depending on $\varepsilon$.

Now, we want to introduce the notion of *almost Killing spinors* to mean that the $L^2$-norm of any $\psi\in L^2(\Sigma^g M)$ from a sequence of spinors, on a Riemannian spin manifold $(M,g)$, is bounded by a function of type $\tau(\varepsilon)$. Of course a genuine Killing spinor has $\|\widehat{\nabla}^g\psi\|_{L^2}=0$. We can formalize that notion, in the context of sequences of spinors for different metrics, in the following way.

**Definition 4.5.1.** Let $\mu\in\mathbb{C}$ be fixed. Let $(g_i)_{i\geq 1}$ be a sequence of Riemannian metrics on a spin manifold $M$, converging in the $C^{1,\alpha}$-topology to a limit Riemannian metric $g$, and suppose the manifold carries equivalent (metric) spin structures $\chi_i$ for each $g_i$. We say that a sequence $(\psi_i)_{i\geq 1}$ of spinors $\psi_i\in H_1^2(\Sigma^{g_i}M)$, associated to each $(M,g_i,\chi_i)$, is an *almost Killing spinor solution* with constant $\mu$ if there exist a vanishing real sequence $(\varepsilon_i)_{i\geq 1}$, i.e., $\lim_{i\to\infty}\varepsilon_i=0$, such that the inequality $\|\widehat{\nabla}^{g_i}\psi_i\|_{L^2(\Sigma^{g_i}M)}\leq\tau(\varepsilon_i)\|\psi_i\|_{L^2(\Sigma^{g_i}M)}$ holds for every $i\geq 1$. An almost Killing spinor solution is *$L^2$-normalized* if $\|\psi_i\|_{L^2(\Sigma^{g_i}M)}=1$ for all $i\geq 1$.

Clearly this notion makes sense in the particular case when all the metrics in the sequence reduce to a fixed one, say $g$. In this case we speak about an almost Killing spinor solution on the spinor bundle $\Sigma^g M$.

**Lemma 4.5.1.** *Let $(\psi_i)_{i\geq 1}$, with $\psi_i\in H_1^2(\Sigma^{g_i}M)$ be an $L^2$-normalized almost Killing spinor solution associated to a convergent sequence of Riemannian metrics $(g_i)_{i\geq 1}$ whose limit (in the $C^{1,\alpha}$-topology) is a Riemannian metric $g$. Then the sequence $(A_i\psi_i)_{i\geq 1}$ is bounded in $H_1^2(\Sigma^g M)$ and $\lim_{i\to\infty}\|A_i\psi_i\|_{\widehat{H}_1^2(\Sigma^g M)}=1$.*

*Proof.* First we recall that the relevant norms on $\Sigma^{g_i}M$ for each $i\geq 0$ are,

$$\|\psi_i\|^2_{L^2(\Sigma^{g_i}M)}:=\int_M(\psi_i,\psi_i)_{g_i}\,\mathrm{dvol}_{g_i},$$
$$\|\psi_i\|^2_{\widehat{H}_1^2(\Sigma^{g_i}M)}:=\|\psi_i\|^2_{L^2(\Sigma^{g_i}M)}+\|\widehat{\nabla}^{g_i}\psi_i\|^2_{L^2(\Sigma^{g_i}M)}.$$

Similar expressions hold for $A_i\psi_i$ in $\Sigma^g M$, where $g$ appears instead of $g_i$. Notice, however, that $\widehat{H}_1^2(\Sigma^g M)$ refers to the Sobolev space defined through the Friedrich connection $\widehat{\nabla}$, while $H_1^2(\Sigma^g M)$ refers to the one defined by the usual Levi-Civita spinorial connection abbreviated for simplicity as $\nabla^g:=\nabla^{\Sigma^g}$. These two norms are actually equivalent, i.e., there exist a real constant $R>0$ such that for any $\psi\in\Gamma(\Sigma^{g_i}M)$, they satisfy

$$R^{-1}\|\psi\|_{H_1^2(\Sigma^{g_i}M)}\leq\|\psi\|_{\widehat{H}_1^2(\Sigma^{g_i}M)}\leq R\,\|\psi\|_{H_1^2(\Sigma^{g_i}M)}.$$

*Notation.* In what follows, we employ the standard notation $o(\varepsilon_i)$ for refering to terms going to zero when $i\to\infty$. We avoid to mix this notation with the $\tau(\varepsilon_i)$ of the almost Killing spinors for the sake of clarity.

Since the endomorphism $A_i : \Sigma^{g_i}M \longrightarrow \Sigma^{g}M$ preserves the norms of the spinors, i.e. $(\psi_i,\psi_i)_{g_i} = (A_i\psi_i, A_i\psi_i)_g$, we can write:

$$\begin{aligned}\|A_i\psi_i\|^2_{L^2(\Sigma^g M)} &= \int_M (A_i\psi_i, A_i\psi_i)_g \,\mathrm{dvol}_g = \int_M (\psi_i,\psi_i)_{g_i}\,\mathrm{dvol}_g \\ &= (1+o(\varepsilon_i))\int_M (\psi_i,\psi_i)_{g_i}\,\mathrm{dvol}_{g_i} \\ &= (1+o(\varepsilon_i))\|\psi_i\|^2_{L^2(\Sigma^{g_i}M)}. \end{aligned} \tag{4.12}$$

Using the almost Killing spinor condition $\|\widehat{\nabla}^{g_i}\psi_i\|_{L^2(\Sigma^{g_i}M)} \leq \tau(\varepsilon_i)\|\psi_i\|_{L^2(\Sigma^{g_i}M)}$, and defining $\widetilde{\nabla}^g\psi_i := A_i^{-1}\widehat{\nabla}^g(A_i\psi_i)$ while $\overline{\nabla}^g\psi_i := A^{-1}\nabla^g(A_i\psi_i)$ we conclude,

$$\begin{aligned}\|\widehat{\nabla}^g(A_i\psi_i)\|^2_{L^2(\Sigma^g M)} &= \int_M (\widehat{\nabla}^g(A_i\psi_i), \widehat{\nabla}^g(A_i\psi_i))_g\,\mathrm{dvol}_g \\ &= \int_M (A_i^{-1}\widehat{\nabla}^g(A_i\psi_i), A_i^{-1}\widehat{\nabla}^g(A_i\psi_i))_{g_i}\,\mathrm{dvol}_g \\ &= \int_M (\widetilde{\nabla}^g\psi_i, \widetilde{\nabla}^g\psi_i)_{g_i}\,\mathrm{dvol}_g \\ &= (1+o(\varepsilon_i))\int_M (\widetilde{\nabla}^g\psi_i, \widetilde{\nabla}^g\psi_i)_{g_i}\,\mathrm{dvol}_{g_i} \\ &= (1+o(\varepsilon_i))\|\widetilde{\nabla}^g\psi_i\|^2_{L^2(\Sigma^{g_i}M)}. \end{aligned}$$

Estimating further,

$$\begin{aligned}\|\widetilde{\nabla}^g\psi_i\|^2_{L^2(\Sigma^{g_i}M)} &= \|\widetilde{\nabla}^g\psi_i - \widehat{\nabla}^{g_i}\psi_i + \widehat{\nabla}^{g_i}\psi_i\|^2_{L^2(\Sigma^{g_i}M)} \\ &\leq 2\|\widetilde{\nabla}^g\psi_i - \widehat{\nabla}^g\psi_i\|^2_{L^2(\Sigma^{g_i}M)} + 2\|\widetilde{\nabla}^g\psi_i\|^2_{L^2(\Sigma^{g_i}M)} \\ &\leq 2\|\overline{\nabla}^g\psi_i - \nabla^g\psi_i\|^2_{L^2(\Sigma^{g_i}M)} + 2\|\widetilde{\nabla}^g\psi_i\|^2_{L^2(\Sigma^{g_i}M)} \\ &\leq 2C\,\|\nabla^g(A_i^{-1})\|^2_{g_i}\|A_i\|^2_{g_i}\|\psi_i\|^2_{L^2(\Sigma^{g_i}M)} + \tau(\varepsilon_i)\|\psi_i\|^2_{L^2(\Sigma^{g_i}M)}. \end{aligned}$$

The norm $\|A_i\|^2_{g_i}$ is bounded and $\|\nabla^g(A_i^{-1})\|^2_{g_i}$ actually goes to zero when $i \to \infty$, so we can write simply $\|\widetilde{\nabla}^g\psi_i\|^2_{L^2(\Sigma^{g_i}M)} \leq \tau(\varepsilon_i)\|\psi_i\|^2_{L^2(\Sigma^{g_i}M)}$. Summing up these estimates we have now

$$\begin{aligned}\|A_i\psi_i\|^2_{\widehat{H}^2_1(\Sigma^g M)} &= \|A_i\psi_i\|^2_{L^2(\Sigma^g M)} + \|\widehat{\nabla}^{g_i}(A_i\psi_i)\|^2_{L^2(\Sigma^g M)} \\ &\leq (1+o(\varepsilon_i))\|\psi_i\|_{L^2(\Sigma^{g_i}M)} + (1+o(\varepsilon_i))\|\widetilde{\nabla}^g\psi_i\|^2_{L^2(\Sigma^{g_i}M)} \\ &\leq (1+o(\varepsilon_i))\big\{\|\psi_i\|_{L^2(\Sigma^{g_i}M)} + \|\widetilde{\nabla}^g\psi_i\|^2_{L^2(\Sigma^{g_i}M)}\big\} \\ &\leq (1+o(\varepsilon_i))\,(1+\tau(\varepsilon_i))\,\|\psi_i\|^2_{L^2(\Sigma^{g_i}M)}. \end{aligned} \tag{4.13}$$

Given that the almost Killing spinor solution is $L^2$-normalized, $\|\psi_i\|_{L^2(\Sigma^{g_i}M)} = 1$ for all $i \geq 1$, the sequence $(A_i\psi_i)_{i\geq 1}$ is bounded in $\widehat{H}^2_1(\Sigma^g M)$ and, in fact, (4.13) implies $\lim_{i\to\infty} \|A_i\psi_i\|_{\widehat{H}^2_1(\Sigma^g M)} = 1$. Since the norms in $\widehat{H}^2_1(\Sigma^g M)$ and $H^2_1(\Sigma^g M)$ are equivalent we conclude that $(A_i\psi_i)_{i\geq 1}$ is indeed bounded in $H^2_1(\Sigma^g M)$, as claimed by the lemma. □

**Theorem 4.5.2.** *Assume the same conditions of the previous lemma. Then, the sequence of almost Killing spinors $(\psi_i)_{i\geq 1}$ where $\psi_i \in L^2(\Sigma^{g_i}M)$, has a subsequence, denoted again by $(\psi_i)_{i\geq 1}$, such that $(A_i\psi_i)_{i\geq 1}$ converge strongly in $L^2(\Sigma^g M)$ to a non-trivial Killing spinor $\psi$ of class $C^{1,\alpha}$.*

*Proof.* First, the sequence $(A_i\psi_i)_{i\geq 1}$ is bounded in $H^2_1(\Sigma^g M)$ according to the preceding lemma, so there exists a subsequence that is weakly convergent to some spinor $\psi \in H^2_1(\Sigma^g M)$. Then, weak convergence in $H^2_1(\Sigma^g M)$ and the compactness of the embedding $H^2_1 \hookrightarrow L^2$ implies that there is a further subsequence, which we can denote again as $(A_i\psi_i)_{i\geq 1}$, that is strongly convergent in $L^2(\Sigma^g M)$ to the limit spinor $\psi$.

Strong convergence means $\lim_{i\to\infty} \|A_i\psi_i - \psi\|_{L^2(\Sigma^g M)} = 0$. By (4.12) and the normalization of the almost Killing spinors we find

$$\begin{aligned}
\|\psi\|_{L^2(\Sigma^g M)} &= \lim_{i\to\infty} \|A_i\psi_i - \psi\|_{L^2(\Sigma^g M)} + \|\psi\|_{L^2(\Sigma^g M)} \\
&\geq \lim_{i\to\infty} \|A_i\psi_i - \psi + \psi\|_{L^2(\Sigma^g M)} \\
&= \lim_{i\to\infty} \|A_i\psi_i\|_{L^2(\Sigma^g M)} = 1,
\end{aligned}$$

therefore, $\psi$ cannot be a trivial spinor. By construction, according to Lemma 4.5.1 and the lower semicontinuity of the weakly convergence in $H^2_1$ we know

$$\|\psi\|_{\widehat{H}^2_1(\Sigma^g M)} \leq \liminf_{i\to\infty} \|A_i\psi_i\|_{\widehat{H}^2_1(\Sigma^g M)} = 1.$$

On the other side,

$$1 + \|\widehat{\nabla}^g\psi\|_{L^2(\Sigma^g M)} \leq \|\psi\|_{L^2(\Sigma^g M)} + \|\widehat{\nabla}^g\psi\|_{L^2(\Sigma^g M)} = \|\psi\|_{\widehat{H}^2_1(\Sigma^g M)}.$$

From these inequalities it follows that $\|\widehat{\nabla}^g\psi\|_{L^2(\Sigma^g M)} = 0$, thus $\widehat{\nabla}^g\psi = 0$ (almost everywhere). It means $\psi \in L^2(\Sigma^g M)$ is a Killing spinor and by Lemma 4.4.2 it has $C^{1,\alpha}$ regularity. □

# Chapter 5

# Pinching of Dirac Eigenvalues and Holonomy

## 5.1 First Dirac Eigenvalues

The following is a classical estimate for the eigenvalues of the Dirac operator, which is central for our discussion.

**Theorem 5.1.1** (Friedrich)**.** *On a compact n-dimensional Riemannian spin manifold $(M,g)$ the eigenvalues $\lambda$ of the Dirac operator D satisfy*

$$\lambda^2 \geq \frac{1}{4}\frac{n}{n-1}\min_M \mathrm{scal}_g$$

In particular, for the $n$-dimensional sphere $S^n$ of constant scalar curvature scal $= n(n-1)$ the theorem says $\lambda^2 \geq \frac{n^2}{4}$.

Recently Bär and Dahl showed that on compact spin manifolds with positive scalar curvature, Friedrich's lower bound estimate for the eigenvalues of the Dirac operator can be made as sharp as needed making appropriate choice of metrics.

**Theorem 5.1.2** (Bär and Dahl, 2003)**.** *Let M be a compact n-dimensional Riemannian spin manifold with positive scalar curvature. Then there is a smooth one-parameter family of Riemannian metrics $g_\varepsilon$ on M, for a real $\varepsilon \in (0, \varepsilon_0]$, such that*

1. $\mathrm{scal}_{g_\varepsilon} \geq n(n-1)$
2. $\frac{n^2}{4} \leq \lambda_1(D^2_{g_\varepsilon}) \leq \frac{n^2}{4} + \varepsilon$,

*where $\lambda_1(D^2_{g_\varepsilon})$ is the smallest eigenvalue of the square of the Dirac operator $D^2_{g_\varepsilon}$ on $(M, g_\varepsilon)$.*

This interesting result shows that the lower bound cannot be improved with additional topological conditions on the space. Additionally, this shows that there is a big difference between actually reaching the bound, which force the manifold to have a real Killing spinor and therefore be Einstein, and being arbitrarily close to it, which the theorem shows does not impose additional topological conditions.

### 5.1.1 Small Eigenvalues of $D^2$

Denote by $\lambda_i(D^2)$ the $i$-th eigenvalue of the square of the Dirac Operator and by $\lambda_i(\nabla^*\nabla)$ the $i$-th eigenvalue of the connection Laplacian on spinors. Also, define $r(n) = 1$ for $n \leq 3$ and $r(n) = 2^{[\frac{n}{2}]-1} + 1$ for $n \geq 4$.

**Theorem 5.1.3** (Ammann–Sprouse, 2003)**.** *Let $(M, g, \chi)$ be an n-dimensional compact Riemannian spin manifold with sectional curvature $|\sec| < K$ and diameter* diam $< d$. *Then there is $\varepsilon = \varepsilon(n, K, d) > 0$, such that if $\lambda_r(\nabla^*\nabla) < \varepsilon$, then M is diffeomorphic to a nilmanifold. Furthermore, $\chi$ is the trivial spin structure on M.*

Using the Schrödinger-Lichnerowicz formula $D^2 = \nabla^*\nabla + \frac{1}{4}\operatorname{scal}$ it implies

**Corollary 5.1.4.** *Let $(M, g, \chi)$ be an n-dimensional compact Riemannian spin manifold satisfying $|\sec| < K$ and* diam $< d$. *There is $\varepsilon = \varepsilon(n, K, d) > 0$, such that if* scal $> -\varepsilon$ *and $\lambda_r(D^2) < \varepsilon$, then M is diffeomorphic to a nilmanifold. Furthermore, $\chi$ is the trivial spin structure on M.*

## 5.2 Riemannian Holonomies and Spinors

We want to summarize briefly important results as the Berger classification of Riemannian holonomies and its related implications for real Killing spinors, found by Bär in [Bär93].

### 5.2.1 The classification of Riemannian Holonomy Groups

**Theorem 5.2.1** (Berger)**.** *Suppose M is a simply-connected manifold of dimension n, and that g is a Riemannian metric on M, that is irreducible and non-symmetric. Then exactly one of the following seven cases holds.*

1. $\mathrm{Hol}_g(M) = SO(n)$,
2. $n = 2m$ *with* $m \geq 2$, *and* $\mathrm{Hol}_g(M) = U(m)$ *in* $SO(2m)$,
3. $n = 2m$ *with* $m \geq 2$, *and* $\mathrm{Hol}_g(M) = SU(m)$ *in* $SO(2m)$,

4. $n = 4m$ *with* $m \geq 2$, *and* $\mathrm{Hol}_g(M) = Sp(m)$ *in* $SO(4m)$,

5. $n = 4m$ *with* $m \geq 2$, *and* $\mathrm{Hol}_g(M) = Sp(m)Sp(1)$ *in* $SO(4m)$,

6. $n = 7$ *and* $\mathrm{Hol}_g(M) = G_2$ *in* $SO(7)$, *or*

7. $n = 8$ *and* $\mathrm{Hol}_g(M) = Spin(7)$ *in* $SO(8)$.

For a proof of the previous theorem and a careful exposition of Riemannian holonomies, see [Joy00].

### 5.2.2 Holonomies with Real Killing Spinors

Let $M$ be a compact simply connected Riemannian spin manifold of dimension $n$ carrying real Killing spinors.

**Definition 5.2.1.** $M$ is said to be of *Killing-type* $(p,q)$ if $M$ carries exactly $p$ linearly independent Killing spinors with Killing constant $\mu = \frac{1}{2}$ and exactly $q$ linear independent Killing spinors of Killing constant $\mu = -\frac{1}{2}$.

Let $\overline{M} = M \times_{r^2} \mathbb{R}^+$ be the Euclidean cone constructed over $M$. The classification of the possible holonomies admitting Killing spinors is now summarized.

**Theorem 5.2.2** (Bär)**.** *Let $M$ be a complete simply-connected Riemannian spin manifold of dimension $n$ carrying a Killing spinor with Killing constant $\mu = \frac{1}{2}$ or $\mu = -\frac{1}{2}$.*

1. *If $n > 3$ is even and $n \neq 6$, then $M$ is isometric to the standard sphere. This still holds if $M$ is not simply-connected.*

2. *If $n = 2m-1$ and $m \geq 3$ is odd, then either $M = S^n$ or $M$ is of type $(1,1)$ and $\overline{M}$ is Kähler.*

3. *If $n = 4m-1$ with $m \geq 3$, then there are three possibilities*

   *(a) $M = S^n$*

   *(b) $M$ is of type $(2,0)$ and $\overline{M}$ is a Kähler manifold, but not hyperkähler.*

   *(c) $M$ is of type $(m+1,0)$ and $\overline{M}$ is hyperkähler.*

4. *If $n = 6$ then either $M = S^6$ or $M$ is of type $(1,1)$ and $\overline{M} = M \times_{r^2} \mathbb{R}^+$ has holonomy $G_2$.*

5. *If $n = 7$ there are four different possibilities*

*(a)* $M = S^7$

*(b)* *M is of type* $(1,0)$ *and* $\overline{M}$ *has holonomy group* $Spin(7)$.

*(c)* *M is of type* $(2,0)$ *and* $\overline{M}$ *is Kähler, but not hyperkäher.*

*(d)* *M is of type* $(3,0)$ *and* $\overline{M}$ *is hyperkähler.*

In the following table we collect some important facts from the previous classification that we use for our pinching results below.

| $n = \dim M$ | $m \in \mathbb{Z}$ | Hol $(\overline{M})$ | Killing type |
|---|---|---|---|
| $n$ arbitrary | — | trivial | $S^n$ |
| $n = 2m-1$ | odd, $m \geq 3$ | $SU(m)$ | $(1,1)$ |
| $n = 4m-1$ | any, $m \geq 3$ | $SU(2m)$ | $(2,0)$ |
| $n = 4m-1$ | any, $m \geq 3$ | $Sp(m)$ | $(m+1,0)$ |
| $n = 6$ | — | $G_2$ | $(1,1)$ |
| $n = 7$ | — | $Spin(7)$ | $(1,0)$ |

Table 5.1: Holonomies with real Killing spinors.

This classification can in fact be made more specific, as the detailed study of each case in [Bär93] shows, where geometric conditions for the manifold $M$ instead of those for the cone $\overline{M}$ are obtained.

It is known (cf. [Bär96a]) that only spheres (or possibly projective spaces $\mathbb{RP}^n$) carry the maximum possible number of linearly independent Killing spinors.

**Theorem 5.2.3** (Bär). *Let M be a closed connected Riemannian spin manifold of dimension* $n > 3$, *carrying* $2^{[\frac{n}{2}]}$ *linearly independent Killing spinors with the same Killing constant* $\mu = \pm\frac{1}{2}$. *Then M is isometric to the standard sphere* $\mathbb{S}^n$ *or* $n \equiv 3 (\mathrm{mod}\ 4)$ *and M is isometric to* $\mathbb{RP}^n$.

## 5.3 Almost Killing spinors and convergence

Using the convergence theorem of Riemannian manifolds and the almost Killing spinors introduced in chapter 4 we prove the following lemma which contains the core of the proof of our main theorem on pinching of Dirac eigenvalues for the standard sphere.

**Lemma 5.3.1.** *Assume we have a sequence* $(M_i, g_i)_{i\geq 1}$ *of compact Riemannian spin manifolds in* $\mathscr{M}(n,d,K,V)$*, with* $\mathrm{scal}_{g_i} \geq n(n-1)$ *for every* $i \geq 1$*. In addition, suppose that each* $(M_i, g_i)$ *carries a Dirac eigenspinor* $\psi_i \in \Gamma(\Sigma^{g_i} M_i)$ *whose eigenvalue satisfy* $|\lambda_i| \in [0, \frac{n}{2} + \varepsilon_i]$ *for a vanishing positive real sequence* $(\varepsilon_i)_{i\geq 1}$*. Then* $(M_i, g_i)_{i\geq 1}$ *has a subsequence that converges in the* $C^{1,\alpha}$*-topology to a limit Riemannian Einstein manifold* $(M, g)$ *carrying a non-trivial Killing spinor* $\psi \in L^2(\Sigma^g M)$ *of class* $C^{1,\alpha}$ *with Killing constant* $\mu = \pm\frac{1}{2}$*.*

*Proof.* Since $(M_i, g_i)_{i\geq 1}$ is contained in $\mathscr{M}(n,K,d,V)$, the compactness theorem guarantees the existence of a subsequence, relabeled again as $(M_i, g_i)_{i\geq 1}$, that converge in the $C^{1,\alpha}$-topology to a limit manifold $(M, g)$. Moreover, the convergence implies that the manifolds $M_i$ are diffeomorphic to the limit $M$, so the sequence can be regarded as being simply $(M, g_i)_{i\geq 1}$. In general, we may choose a fixed topological spin structure in the limit and it will be inherited, under these diffeomorphisms, by the manifolds in the sequence. Then, the identification of spinors described in section 4.2 provides equivalent metric spin structures for each $(M_i, g_i)$ to which the spinors will be attached.

By the Friedrich lower bound on the square of the eigenvalues of the Dirac operator we actually have, under the conditions of the lemma, that the eigenvalues must satisfy $\frac{n}{2} \leq |\lambda_i| \leq \frac{n}{2} + \varepsilon_i$. Now, if $\lambda_i > 0$, set $\mu = -\frac{1}{2}$ or if $\lambda_i < 0$, take $\mu = \frac{1}{2}$. Applying Weitzenböck formula for the Friedrich connection (4.11) with the appropriate $\mu$ defined above and invoking the bound $\mathrm{scal}_{g_i} \geq n(n-1)$, we get the following estimate for the $L^2$-norm of the Friedrich covariant derivative of $\psi_i$,

$$
\begin{aligned}
(\widehat{\nabla}^{g_i}\psi_i, \widehat{\nabla}^{g_i}\psi_i)_{L^2(\Sigma^{g_i}M)} &= \big((\widehat{\nabla}^{g_i})^*\widehat{\nabla}^{g_i}\psi_i, \psi_i\big)_{L^2(\Sigma^{g_i}M)} \\
&\leq \big((D_{g_i} \mp \tfrac{1}{2})^2\psi_i - \tfrac{1}{4}\,\mathrm{scal}_{g_i}\psi_i + \tfrac{1}{4}(n-1)\psi_i, \psi_i\big)_{L^2(\Sigma^{g_i}M)} \\
&\leq \sup_M \big\{(\pm|\lambda_i| \mp \tfrac{1}{2})^2 - \tfrac{1}{4}\,\mathrm{scal}_{g_i} + \tfrac{1}{4}(n-1)\big\}\,(\psi_i, \psi_i)_{L^2(\Sigma^{g_i}M)} \\
&\leq \sup_M \big\{(|\lambda_i| - \tfrac{1}{2})^2 - \tfrac{1}{4}\,\mathrm{scal}_{g_i} + \tfrac{1}{4}(n-1)\big\}\,(\psi_i, \psi_i)_{L^2(\Sigma^{g_i}M)} \\
&\leq \big\{(\tfrac{1}{2}(n-1) + \varepsilon_i)^2 - \tfrac{1}{4}n(n-1) + \tfrac{1}{4}(n-1)\big\}\,\|\psi_i\|^2_{L^2(\Sigma^{g_i}M)} \\
&\leq \big\{(n-1)\varepsilon_i + \varepsilon_i^2\big\}\,\|\psi_i\|^2_{L^2(\Sigma^{g_i}M)} \\
&\leq \tau(\varepsilon_i)\,\|\psi_i\|^2_{L^2(\Sigma^{g_i}M)}.
\end{aligned}
$$

Then, $(\psi_i)_{i\geq 1}$ is an almost Killing spinor solution. By Theorem 4.5.2, $(\psi_i)_{i\geq 1}$ has a subsequence which converge strongly in $L^2(\Sigma^g M)$ to a non-trivial Killing spinor $\psi$ of class $C^{1,\alpha}$ in the limit manifold $(M, g)$. From Theorem 3.4.3 we conclude that the limit manifold $(M, g)$ is actually an Einstein manifold with smooth metric $g$ in a harmonic (or compatible) atlas.

□

## 5.4 Dirac Eigenvalue Pinching for the Sphere

Let us divide the intersection of the spectrum of the Dirac operator with the interval $[-\frac{n}{2}-\varepsilon, \frac{n}{2}+\varepsilon]$ in two parts, by defining

$$I_\varepsilon^+ := \operatorname{spec}(D) \cap [0, \tfrac{n}{2}+\varepsilon],$$
$$I_\varepsilon^- := \operatorname{spec}(D) \cap [-\tfrac{n}{2}-\varepsilon, 0].$$

Recall that, by the Friedrich bound on the first eigenvalue of the Dirac operator $D$ for manifolds with positive scalar curvature, any eigenvalue $\lambda$ of $D$ is actually $\lambda \leq -\frac{n}{2}$ or $\lambda \geq \frac{n}{2}$. Hence, there is a spectral gap in the interval $(-\frac{n}{2}, \frac{n}{2})$ where $\operatorname{spec}(D)$ has no elements.

**Theorem 5.4.1.** *Suppose $(M, g, \chi)$ is a compact n-dimensional Riemannian spin manifold in $\mathscr{M}(n, d, K, V)$ and $\operatorname{scal}_g \geq n(n-1)$. For every $\delta > 0$ there is an $\varepsilon = \varepsilon(n, K, d, \delta) > 0$ such that if any of the following conditions hold:*

1. $\#(I_\varepsilon^+ \cup I_\varepsilon^-) \geq 1$ *if n is even and $n \neq 6$,*
2. $\#(I_\varepsilon^+) \geq 2$ *or* $\#(I_\varepsilon^-) \geq 2$ *if $n = 6$ or $n \equiv 1 (\mathrm{mod}\, 4)$,*
3. $\#(I_\varepsilon^+) \geq \frac{n+9}{4}$ *or* $\#(I_\varepsilon^-) \geq \frac{n+9}{4}$ *or* $\#(I_\varepsilon^+) \geq 1$, $\#(I_\varepsilon^-) \geq 1$ *if $n \equiv 3 (\mathrm{mod}\, 4)$,*

*with M being simply-connected in the cases 2 and 3 if $n > 3$, then $(M, g)$ has $C^{1,\alpha}$-distance $\leq \delta$ to the sphere $\mathbb{S}^n$ with the standard metric of constant sectional curvature* $\sec = 1$.

*Proof.* First, we show that under the assumptions of the theorem $(M, g)$ is $C^{1,\alpha}$-close to an Einstein manifold with Killing spinor. Assume, by contradiction, that this is not the case. Then for any $\delta > 0$, conditions 1, 2 and 3 in the theorem guarantee that the Dirac operator on $M$ has always at least one eigenvalue $\lambda$ with $|\lambda| \in [\frac{n}{2}, \frac{n}{2}+\varepsilon]$, for some $\varepsilon > 0$. Therefore, we can construct a sequence of manifolds $(M_i, g_i)_{i \geq 1}$ in $\mathscr{M}(n, d, K, V)$ carrying eigenspinors $\psi_i \in \Gamma(\Sigma^{g_i} M)$ of $D_{g_i}$, with eigenvalue $|\lambda_i| \in [0, \frac{n}{2}+\varepsilon_i]$, for each $i \geq 1$. Now, Lemma 5.3.1 implies that $(M_i, g_i)_{i \geq 1}$ converges in the $C^{1,\alpha}$-topology to an Einstein manifold $(M', g')$ carrying a real Killing spinor, contrary to our assumption that prevents any of the manifolds $(M_i, g_i)$ to be $C^{1,\alpha}$-close to such an $(M', g')$. In conclusion, $(M, g)$ has indeed $C^{1,\alpha}$-distance $< \delta$ to an Einstein manifold with real Killing spinor.

Now, we go to Bär's list of Riemannian holonomies admitting real Killing spinors to extract all the cases where enough number of Killling spinors force the manifold to be the sphere. Recall that a manifold is of Killing-type $(p, q)$ if the number of linearly independent Killing spinors on $M$ is exactly $p$ for $\mu = \frac{1}{2}$ and $q$

for $\mu = -\frac{1}{2}$ or viceversa. Also, note that for $\varepsilon \to 0$ each eigenvalue in $I_\varepsilon^+$ or $I_\varepsilon^-$ goes to $\pm\frac{n}{2}$ and therefore produce a Killing spinor in the limit. Finally, Theorem 5.2.2 implies that if $(M,g)$ admits a Killing spinor, it is isometric to the standard sphere $\mathbb{S}^n$ in any of the following cases,

- $n$ even and $n \neq 6$: then in this case Killing-type $(1,0)$ is enough.
- $n = 6$ or $n \geq 5$ with $n \equiv 1 (\mathrm{mod}\, 4)$: when $M$ is simply connected and it is not of Killing-type $(1,1)$, so type (1,0) or (2,0) is enough.
- $n = 3$: when $M$ is of Killing-type $(1,1)$ or $(3,0)$.
- $n = 7$: when $M$ is simply connected and not of Killing-type $(1,0)$, $(2,0)$ or $(3,0)$, hence type $(1,1)$ or $(4,0)$ suffices.
- $n \geq 11$ with $n \equiv 3 (\mathrm{mod}\, 4)$: when $M$ is simply connected and it is not of Killing-type $(2,0)$ or $(\frac{n+5}{4},0)$, then type $(1,1)$ or $(\frac{n+9}{4},0)$ is enough.

Passing these considerations into conditions on the number of positive and/or negative eigenvalues yields the result of the theorem. Notice that for $n = 3$ the types $(1,1)$ or $(3,0)$ required and for $n = 7$ the types $(1,1)$ or $(4,0)$, fit into the same type-condition $(1,1)$ or $(\frac{n+9}{4},0)$ for $n \geq 11$. □

From Lemma 5.3.1 and the first part of the previous proof, we extract the following corollary.

**Corollary 5.4.2.** *Suppose $(M,g,\chi)$ is a compact n-dimensional Riemannian spin manifold in $\mathscr{M}(n,d,K,V)$ and $\mathrm{scal}_g \geq n(n-1)$. For every $\delta > 0$ there is an $\varepsilon = \varepsilon(n,K,d,\delta) > 0$ such that if the Dirac operator has an eigenvalue $\lambda$ with $|\lambda| \in [0, \frac{n}{2} + \varepsilon]$ then $(M,g)$ has $C^{1,\alpha}$-distance $\leq \delta$ to an Einstein manifold $(M,g)$ admitting a Killing spinor.*

We can reformulate Theorem 5.4.1 to establish uniform lower bounds for particular eigenvalues of the Dirac operator on Riemannian spin manifolds not diffeomorphic to the standard sphere $\mathbb{S}^n$.

**Corollary 5.4.3.** *Let n be even and $n \neq 6$. Assume $(M,\chi)$ is an n-dimensional compact spin manifold not diffeomorphic to the standard sphere $\mathbb{S}^n$ (with spin structure). Then, among all metrics with bounded diameter, volume and curvature as in $\mathscr{M}(n,d,K,V)$ and with $\mathrm{scal}_g \geq n(n-1)$, there exists an $\varepsilon = \varepsilon(n,K,d,V) > 0$ which provides a uniform lower bound on the first eigenvalue of $D^2$:*

$$\lambda_1(D^2) \geq \frac{n^2}{4} + \varepsilon.$$

**Corollary 5.4.4.** *Suppose* $(M,\chi)$ *is a simply-connected, n-dimensional, compact spin manifold not diffeomorphic with the sphere* $\mathbb{S}^n$*. Then, among all metrics with bounded diameter, volume and curvature in* $\mathscr{M}(n,d,K,V)$ *and with scalar curvature* $\mathrm{scal}_g \geq n(n-1)$*, there exists an* $\varepsilon = \varepsilon(n,K,d,V) > 0$ *and a number* $r \in \mathbb{Z}^+$ *yielding a uniform bound on the r-th eigenvalue of* $D^2$*:*

$$\lambda_r(D^2) \geq \frac{n^2}{4} + \varepsilon,$$

*where*

$$r = \begin{cases} 3 & \text{if } n = 6 \text{ or } n \equiv 1 (\mathrm{mod}\, 4). \\ \frac{n+9}{4} & \text{if } n \equiv 3 (\mathrm{mod}\, 4). \end{cases} \tag{5.1}$$

Under the given conditions this $r$ is generally optimal (minimum possible), but from the list of manifolds with real Killing spinors one can see that with additional assumptions on $M$, like being or not Kähler, hyperkähler, Sasaki, etc., this $r$ can be made smaller. Nevertheless this can imply restricting too much the geometry of the manifold.

# Appendix A

# Weak Riemann Curvature

In this section we show how the weak Ricci curvature introduced in 3.3.1 comes from a very natural weak definition of the Riemann tensor acting on compactly supported test functions on the manifold $M$.

To introduce this definition suppose first that we have a smooth (or at least $C^2$) Riemannian metric $g$ on $M$ and choose four local vector fields $X,Y,Z,W \in \Gamma(TM)$. Then we know that the Riemann tensor is well-defined and we can write,

$$\begin{aligned} R(X,Y,Z,W) :&= g(R(X,Y)Z,W) = \langle R(X,Y)Z,W\rangle \\ &= \langle \nabla_X \nabla_Y Z, W\rangle - \langle \nabla_Y \nabla_X Z, W\rangle - \langle \nabla_{[X,Y]} Z, W\rangle \\ &= X\langle \nabla_Y Z, W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle - Y\langle \nabla_X Z, W\rangle \\ &\qquad + \langle \nabla_X Z, \nabla_Y W\rangle - \langle \nabla_{[X,Y]} Z, W\rangle. \end{aligned}$$

Given a compactly supported test function $\eta \in C^1_c(M)$ with $\operatorname{supp}\eta \subset U$ for an open chart $U \subset M$, we can integrate the product of the components of the Riemann tensor $R$ with $\eta$ on $U$. Recall $\mathrm{dvol}_g = \sqrt{g}\,dx = \sqrt{g}\,dx_1 \cdots dx_n$ and define $\hat{\eta} := \sqrt{g}\,\eta$, so we have

$$\begin{aligned} \int_U \langle R(X,Y)Z,W\rangle\, \eta \,\mathrm{dvol}_g = &- \int_U \left\{ \langle \nabla_Y Z, \nabla_X W\rangle - \langle \nabla_X Z, \nabla_Y W\rangle \right\} \hat{\eta}\, dx \\ &+ \int_U \left\{ \langle \nabla_{[X,Y]} Z, W\rangle + X\langle \nabla_Y Z, W\rangle - Y\langle \nabla_X Z, W\rangle \right\} \hat{\eta}\, dx \end{aligned}$$

$$\begin{aligned} = \int_U & \left\{ \langle \nabla_X Z, \nabla_Y W\rangle - \langle \nabla_Y Z, \nabla_X W\rangle + \langle \nabla_{[X,Y]} Z, W\rangle \right\} \hat{\eta}\, dx \\ &+ \int_U \left\{ X(\langle \nabla_Y Z, W\rangle \hat{\eta}) - Y(\langle \nabla_X Z, W\rangle \hat{\eta}) \right\} dx \\ &\qquad + \int_U \left\{ \langle \nabla_X Z, W\rangle Y\hat{\eta} - \langle \nabla_Y Z, W\rangle X\hat{\eta} \right\} dx \end{aligned}$$

This expression can be simplified further if we choose $X = \partial_a$ and $Y = \partial_b$ for some local coordinate vector field frame $(\partial_1, \ldots, \partial_n)$ defined on $U$. In this case, after Stokes' theorem and $\operatorname{supp}\hat{\eta} \subset U$, the second integral vanishes and $[\partial_a, \partial_b] = 0$, so this simplifies to

$$= \int_U \left\{ \langle \nabla_X Z, \nabla_Y W \rangle - \langle \nabla_Y Z, \nabla_X W \rangle \right\} \hat{\eta}\, dx \\ + \int_U \left\{ \langle \nabla_X Z, W \rangle Y\hat{\eta} - \langle \nabla_Y Z, W \rangle X\hat{\eta} \right\} dx$$

The remaining terms do not depend on second derivatives of the metric, so we take this equation as a weak definition for the curvature tensor.

**Definition A.1.** Let $\eta \in C_c^{1,\alpha}(M)$ be a compactly supported test function with $\operatorname{supp}\eta \subset U$ and $(\partial_1, \ldots, \partial_n)$ a local basis of coordinate vector fields for a chart $(U, \phi)$ of $M$. We define locally the *weak Riemann curvature tensor* acting on $\partial_a, \partial_b, Z, W \in \Gamma(TM)$ as

$$\langle\!\langle R(\partial_a, \partial_b, Z, W), \eta \rangle\!\rangle_U := \int_U \left\{ \langle \nabla_{\partial_a} Z, \nabla_{\partial_b} W \rangle - \langle \nabla_{\partial_b} Z, \nabla_{\partial_a} W \rangle \right\} \hat{\eta}\, dx \\ + \int_U \left\{ \langle \nabla_{\partial_a} Z, W \rangle \partial_b \hat{\eta} - \langle \nabla_{\partial_b} Z, W \rangle \partial_a \hat{\eta} \right\} dx \quad \text{(A.1)}$$

Given the definition and the linearity properties of the metric $g$, the Levi-Civita connection $\nabla$ and of derivation and integration in general, we see that this weak definition is linear in $X$, $Y$, $Z$ and $W$ with respect to addition of vector fields and multiplication by constants. Moreover, similarly to the smooth case, this weak definition have some tensorial-like properties as the next theorem asserts.

**Theorem A.5.** *The local weak curvature is $C^1(M)$-linear in the last two entries (corresponding to $Z$ and $W$). Namely, for a function $f \in C^1(M)$ we have:*

1. *The $Z$ entry satisfies*

$$\langle\!\langle R(\partial_a, \partial_b, fZ, W), \eta \rangle\!\rangle_U = \langle\!\langle R(\partial_a, \partial_b, Z, W), f\eta \rangle\!\rangle_U. \quad \text{(A.2)}$$

2. *and for the $W$ entry, it holds also*

$$\langle\!\langle R(\partial_a, \partial_b, Z, fW), \eta \rangle\!\rangle_U = \langle\!\langle R(\partial_a, \partial_b, Z, W), f\eta \rangle\!\rangle_U. \quad \text{(A.3)}$$

*Proof.* The first property is easily proved using the second one, so we prove 2 first.

$$\langle\!\langle R(\partial_a, \partial_b, Z, fW), \eta \rangle\!\rangle_U = \int_U \left\{ \langle \nabla_{\partial_a} Z, \nabla_{\partial_b} fW \rangle - \langle \nabla_{\partial_b} Z, \nabla_{\partial_a} fW \rangle \right\} \hat{\eta}\, dx \\ + \int_U \left\{ \langle \nabla_{\partial_a} Z, fW \rangle \partial_b \hat{\eta} - \langle \nabla_{\partial_b} Z, fW \rangle \partial_a \hat{\eta} \right\} dx$$

$$
\begin{aligned}
&= \int_U \big\{\langle \nabla_{\partial_a} Z, \nabla_{\partial_b} W\rangle - \langle \nabla_{\partial_b} Z, \nabla_{\partial_a} W\rangle\big\} f\hat{\eta}\, dx \\
&\qquad + \int_U \big\{\langle \nabla_{\partial_a} Z, W\rangle \partial_b f - \langle \nabla_{\partial_b} Z, W\rangle \partial_a f\big\}\hat{\eta}\, dx \\
&\qquad\qquad + \int_U \big\{\langle \nabla_{\partial_a} Z, W\rangle f(\partial_b \hat{\eta}) - \langle \nabla_{\partial_b} Z, W\rangle f(\partial_a \hat{\eta})\big\}\, dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_U \big\{\langle \nabla_{\partial_a} Z, \nabla_{\partial_b} W\rangle - \langle \nabla_{\partial_b} Z, \nabla_{\partial_a} W\rangle\big\} f\hat{\eta}\, dx \\
&\qquad + \int_U \langle \nabla_{\partial_a} Z, W\rangle \{(\partial_b f)\hat{\eta} + f(\partial_b \hat{\eta})\}\, dx \\
&\qquad\qquad - \int_U \langle \nabla_{\partial_b} Z, W\rangle \{(\partial_a f)\hat{\eta} + f(\partial_a \hat{\eta})\}\, dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_U \big\{\langle \nabla_{\partial_a} Z, \nabla_{\partial_b} W\rangle - \langle \nabla_{\partial_b} Z, \nabla_{\partial_a} W\rangle\big\} f\hat{\eta}\, dx \\
&\qquad\qquad + \int_U \big\{\langle \nabla_{\partial_a} Z, W\rangle \partial_b (f\hat{\eta}) - \langle \nabla_{\partial_b} Z, W\rangle \partial_a (f\hat{\eta})\big\}\, dx
\end{aligned}
$$

$$
= \big\langle\!\big\langle R(\partial_a, \partial_b, Z, W), f\eta \big\rangle\!\big\rangle_U .
$$

Here we have only used the Leibniz property of the covariant derivative and the action of vector fields on functions. What remains to proof is the behavior under multiplication by a function $f \in C^1(M)$ on the entry $W$. For this we calculate,

$$
\begin{aligned}
\big\langle\!\big\langle R(\partial_a, \partial_b, fZ, W), \eta \big\rangle\!\big\rangle_U &= \int_U \big\{\langle \nabla_{\partial_a} fZ, \nabla_{\partial_b} W\rangle - \langle \nabla_{\partial_b} fZ, \nabla_{\partial_a} W\rangle\big\}\hat{\eta}\, dx \\
&\qquad + \int_U \big\{\langle \nabla_{\partial_a} fZ, W\rangle \partial_b \hat{\eta} - \langle \nabla_{\partial_b} fZ, W\rangle \partial_a \hat{\eta}\big\}\, dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_U \big\{\langle \nabla_{\partial_a} Z, \nabla_{\partial_b} W\rangle - \langle \nabla_{\partial_b} Z, \nabla_{\partial_a} W\rangle\big\} f\hat{\eta}\, dx \\
&\qquad + \int_U \big\{\langle Z, \nabla_{\partial_b} W\rangle (\partial_a f)\hat{\eta} - \langle Z, \nabla_{\partial_a} W\rangle (\partial_b f)\hat{\eta}\big\}\, dx \\
&\qquad + \int_U \big\{\langle \nabla_{\partial_a} Z, W\rangle f(\partial_b \hat{\eta}) - \langle \nabla_{\partial_b} Z, W\rangle f(\partial_a \hat{\eta})\big\}\, dx \\
&\qquad\qquad + \int_U \big\{\langle Z, W\rangle (\partial_a f)(\partial_b \hat{\eta}) - \langle Z, W\rangle (\partial_b f)(\partial_a \hat{\eta})\big\}\, dx
\end{aligned}
$$

$$= \int_U \left\{ \langle \nabla_{\partial_a} Z, \nabla_{\partial_b}(\hat{\eta} W)\rangle - \langle \nabla_{\partial_b} Z, \nabla_{\partial_a}(\hat{\eta} W)\rangle \right\} f\, dx + \int_U \left\{ \langle Z, \nabla_{\partial_b}(\hat{\eta} W)\rangle \partial_a f - \langle Z, \nabla_{\partial_a}(\hat{\eta} W)\rangle \partial_b f \right\} dx$$

$$= \left\langle\!\left\langle R(\partial_a, \partial_b, Z, \eta W), f \right\rangle\!\right\rangle_U = \left\langle\!\left\langle R(\partial_a, \partial_b, Z, W), f\eta \right\rangle\!\right\rangle_U.$$

In the last step we have used the property 1 just proved. Then the $C^1(M)$-linearity in this weak sense is proved also for the Z entry. □

**Remark A.1.** Similar properties to the ones just proved but for the first and second entries of the weak curvature do not hold, due to the terms containing derivatives of the test function $\partial_a \hat{\eta}$ and $\partial_b \hat{\eta}$, in (A.1), which are not $C^1(M)$-linear but satisfy the Leibniz formula.

It can be shown that the weak Riemann curvature verify some analogous versions of the symmetries of the usual Riemann tensor, but only for local coordinate vector fields instead of arbitrary ones.

**Theorem A.6.** *For any compactly supported test function $\eta \in C^1_c(M)$ the weak Riemann curvature satisfies the following properties:*

1. *it is antisymmetric in the first two and last two entries independently, i.e., for $Z, W \in \Gamma(TM)$,*

$$\begin{aligned} \left\langle\!\left\langle R(\partial_a, \partial_b, Z, W), \eta \right\rangle\!\right\rangle_U &= -\left\langle\!\left\langle R(\partial_b, \partial_a, Z, W), \eta \right\rangle\!\right\rangle_U \\ &= -\left\langle\!\left\langle R(\partial_a, \partial_b, W, Z), \eta \right\rangle\!\right\rangle_U. \end{aligned} \tag{A.4}$$

2. *If the third and fourth entries are local coordinate vector fields $\partial_c, \partial_d \in \Gamma(TM)$ as the first two entries are, then it is symmetric between the first two and last two entries,*

$$\left\langle\!\left\langle R(\partial_a, \partial_b, \partial_c, \partial_d), \eta \right\rangle\!\right\rangle_U = \left\langle\!\left\langle R(\partial_c, \partial_d, \partial_a, \partial_b), f\eta \right\rangle\!\right\rangle_U. \tag{A.5}$$

3. *It satisfies a cyclic permutation property (i.e. Bianchi's first identity) if the first three entries correspond to local coordinate vector fields and the fourth to an arbitrary vector field $W \in \Gamma(TM)$,*

$$\left\langle\!\left\langle R(\partial_a, \partial_b, \partial_c, W) + R(\partial_b, \partial_c, \partial_a, W) + R(\partial_c, \partial_a, \partial_b, W), \eta \right\rangle\!\right\rangle_U = 0. \tag{A.6}$$

*Proof.* Property 1 is obvious from (A.1) and the symmetry of the metric, and since 2 follows from 3 we prove first this last one. The left-hand side of (A.6) should be understood using linearity. With this and abbreviating $\nabla_i := \nabla_{\partial_i}$ we get,

$$\begin{aligned}
&\langle\!\langle R(\partial_a,\partial_b,\partial_c,W)+R(\partial_b,\partial_c,\partial_a,W)+R(\partial_c,\partial_a,\partial_b,W),\eta\rangle\!\rangle_U = \\
&\quad \langle\!\langle R(\partial_a,\partial_b,\partial_c,W),\eta\rangle\!\rangle_U + \langle\!\langle R(\partial_b,\partial_c,\partial_a,W),\eta\rangle\!\rangle_U + \langle\!\langle R(\partial_c,\partial_a,\partial_b,W),\eta\rangle\!\rangle_U
\end{aligned}$$

$$\begin{aligned}
&= \int_U \left\{\langle\nabla_a\partial_c,\nabla_b W\rangle - \langle\nabla_b\partial_c,\nabla_a W\rangle\right\}\hat{\eta}\,dx \\
&\qquad + \int_U \left\{\langle\nabla_a\partial_c,W\rangle\partial_b\hat{\eta} - \langle\nabla_b\partial_c,W\rangle\partial_a\hat{\eta}\right\}dx \\
&\qquad\qquad + \int_U \left\{\langle\nabla_b\partial_a,\nabla_c W\rangle - \langle\nabla_b W,\nabla_c\partial_a\rangle\right\}\hat{\eta}\,dx \\
&\quad + \int_U \left\{\langle\nabla_c\partial_a,W\rangle\partial_b\hat{\eta} - \langle\nabla_c\partial_b,W\rangle\partial_a\hat{\eta}\right\}dx \\
&\qquad + \int_U \left\{\langle\nabla_c\partial_b,\nabla_a W\rangle - \langle\nabla_c W,\nabla_a\partial_b\rangle\right\}\hat{\eta}\,dx \\
&\qquad\qquad + \int_U \left\{\langle\nabla_a\partial_b,W\rangle\partial_c\hat{\eta} - \langle\nabla_c\partial_b,W\rangle\partial_a\hat{\eta}\right\}dx,
\end{aligned}$$

$$\begin{aligned}
&= \int_U \langle\nabla_a\partial_c - \nabla_c\partial_a,\nabla_b W\rangle\hat{\eta}\,dx + \int_U \langle\nabla_b\partial_a - \nabla_a\partial_b,\nabla_c W\rangle\hat{\eta}\,dx \\
&\quad + \int_U \langle\nabla_c\partial_b - \nabla_b\partial_c,\nabla_a W\rangle\hat{\eta}\,dx + \int_U \langle\nabla_b\partial_c - \nabla_c\partial_b,W\rangle\partial_a\hat{\eta}\,dx \\
&\qquad + \int_U \langle\nabla_c\partial_a - \nabla_a\partial_c,W\rangle\partial_b\hat{\eta}\,dx + \int_U \langle\nabla_a\partial_b - \nabla_b\partial_a,W\rangle\partial_c\hat{\eta}\,dx.
\end{aligned}$$

Now using that the Riemannian connection is torsion-free, i.e., $\nabla_i\partial_j - \nabla_j\partial_i = [\partial_i,\partial_j]$, we get

$$\begin{aligned}
&= \int_U \langle[\partial_a,\partial_c],\nabla_b W\rangle\hat{\eta}\,dx + \int_U \langle[\partial_a,\partial_c],\nabla_b W\rangle\hat{\eta}\,dx + \int_U \langle[\partial_a,\partial_c],\nabla_b W\rangle\hat{\eta}\,dx \\
&\quad + \int_U \langle[\partial_b,\partial_c],W\rangle\partial_a\hat{\eta}\,dx + \int_U \langle[\partial_c,\partial_a],W\rangle\partial_b\hat{\eta}\,dx + \int_U \langle[\partial_a,\partial_b],W\rangle\partial_c\hat{\eta}\,dx,
\end{aligned}$$

but since local vector fields commute, $[\partial_i,\partial_j] = 0$ for every $i,j = 1,\ldots,n$ and the whole expression vanishes. Part 2 of the theorem, as in the smooth case, is just an algebraic consequence of properties 1 and 2 and holds choosing another local coordinate vector field as the fourth entry of the weak curvature, instead of the arbitrary vector $W$. □

These properties show that the weak curvature introduced here has similar behavior to the classical one, but only for vector fields coming from local coordinates. The corresponding second Bianchi identity in this context would require a covariant derivative of the curvature which implicitly needs one more (weak) derivative of the metric, and we do not consider it here.

From the weak Riemann curvature defined in this appendix, if follows a natural definition of weak Ricci curvature.

**Definition A.2.** Let $\eta \in C^{1,\alpha}_{\mathrm{c}}(M)$ be a compactly supported test function with $\operatorname{supp}\eta \subset U$ and $(\partial_1, \ldots, \partial_n)$ a local basis of coordinate vector fields for a chart $(U, \phi)$ of $M$. We define locally the *weak Riemann curvature tensor* acting on $\partial_a, \partial_b, Z, W \in \Gamma(TM)$ as

$$\left\langle\!\left\langle R(\partial_a, \partial_b), \eta \right\rangle\!\right\rangle_U := \sum_{c=1}^{n} \left\langle\!\left\langle R(\partial_a, \partial_c, \partial_c, \partial_b), \eta \right\rangle\!\right\rangle_U$$

It is possible to show that this definition is equivalent to Definition 3.3.1. In particular it is clear that in the smooth metric case both expressions should coincide.

# Appendix B

# Continuity of Weak Curvatures

Recall that for a Riemannian metric $g$ we use the notation $\sqrt{g} := \sqrt{\det(g)}$. Given another metric $g'$ we will denote $\sqrt{g}' := \sqrt{\det(g')}$.

**Theorem B.7.** *Let $M$ be a differentiable manifold endowed with a $C^{1,\alpha}$ Riemannian metric $g$ on it. For fixed coordinate vector fields $\partial_a, \partial_b \in C^1(TM)$ on an open chart $U \subset M$, the application*

$$\begin{aligned} \mathscr{W}_{\mathrm{Ric}_{ab}} : C^{1,\alpha}(\Sigma M) \times C^{1,\alpha}(T^*M^{\odot 2}) &\longrightarrow \quad \mathbb{R} \\ (\eta, g) &\longmapsto \left\langle\!\left\langle \mathrm{Ric}(\partial_a, \partial_b), \eta \right\rangle\!\right\rangle_U, \end{aligned}$$

*given by Definition 3.3.1 is continuous in the $C^1$-topology.*

*Proof.* The weak Ricci curvature is easily seen to be linear in $\eta$. Then, to show continuity in this variable we just need to prove that $|\mathscr{W}_{\mathrm{Ric}_{ab}}(\eta, g)|$ is bounded by $\|\eta\|_{C^0}$ and $\|\partial\eta\|_{C^0}$.

$$\begin{aligned} \left|\left\langle\!\left\langle \mathrm{Ric}(\partial_a, \partial_b), \eta \right\rangle\!\right\rangle_U\right| &\le \frac{1}{2}\left| \int_U \partial_s g_{ab} \partial_r (g^{rs}\eta)\, \mathrm{dvol}_g - \int_U \Gamma^r \partial_b (g_{ra}\eta)\, \mathrm{dvol}_g \right. \\ &\qquad \left. - \int_U \Gamma^r \partial_a (g_{rb}\eta)\, \mathrm{dvol}_g + 2\int_U Q_{ab}\,\eta\, \mathrm{dvol}_g \right| \\ &\le \frac{1}{2}\int_U \left|\partial_s g_{ab}(\partial_r g^{rs}\eta + g^{rs}\partial_r\eta) - \Gamma^r(\partial_b g_{ra}\eta + g_{ra}\partial_b\eta)\right. \\ &\qquad \left. - \Gamma^r(\partial_a g_{rb}\eta + g_{rb}\partial_a\eta) + 2Q_{ab}\,\eta\right| \mathrm{dvol}_g \\ &\le \frac{1}{2}\int_U \left|\partial_s g_{ab}\partial_r g^{rs} - \Gamma^r(\partial_b g_{ra} - \partial_a g_{rb}) + 2Q_{ab}\right| |\eta|\, \mathrm{dvol}_g \\ &\qquad + \int_U \left|g^{rs}\partial_s g_{ab}\partial_r\eta - \Gamma^r(g_{ra}\partial_b\eta - g_{rb}\partial_a\eta)\right| \mathrm{dvol}_g \\ &\le C\left\{\|\partial_s g_{ab}\partial_r g^{rs}\|_{C^0} + \|\Gamma^r(\partial_b g_{ra} + \partial_a g_{rb})\|_{C^0} + \|2Q_{ab}\|_{C^0}\right\}\|\eta\|_{C^0} \\ &\qquad + \left\|g^{rs}\partial_s g_{ab}\partial_r\eta - \Gamma^r(g_{ra}\partial_b\eta + g_{rb}\partial_a\eta)\right\|_{C^0} \end{aligned}$$

$$
\begin{aligned}
&\leq C\big\{\|\partial_s g_{ab}\partial_r g^{rs}\| + \|g^{cd}\Gamma^r_{cd}(\partial_b g_{ra} + \partial_a g_{rb})\| + \|2Q_{ab}\|\big\}\|\eta\| \\
&\qquad\qquad + \big\|g^{rs}\partial_s g_{ab}\partial_r\eta - g^{cd}\Gamma^r_{cd}(g_{ra}\partial_b\eta + g_{rb}\partial_a\eta)\big\| \\
&\leq C\big\{n^2\|\partial g\|_{C^0}\|\partial(g^{-1})\|_{C^0} + 2n^3\|g\|_{C^0}\|\Gamma\|_{C^0}\|\partial g\|_{C^0} + 2\|Q\|_{C^0}\big\}\|\eta\|_{C^0} \\
&\qquad\qquad + \|g^{-1}\|_{C^0}\big(n^2\|\partial g\|_{C^0} + 2n^3\|\Gamma\|_{C^0}\|g\|_{C^0}\big)\|\partial\eta\|_{C^0}.
\end{aligned}
$$

To prove the continuity in $g$, we calculate:

$$
\begin{aligned}
&\big|\mathscr{W}_{\mathrm{Ric}}(\eta,g) - \mathscr{W}_{\mathrm{Ric}'}(\eta,g')\big| = \big|\big\langle\!\big\langle \mathrm{Ric}_g(\partial_a,\partial_b),\eta\big\rangle\!\big\rangle_U - \big\langle\!\big\langle \mathrm{Ric}_{g'}(\partial_a,\partial_b),\eta\big\rangle\!\big\rangle_U\big| \\
&\leq \frac{1}{2}\bigg|\int_U \big\{\partial_s g_{ab}\partial_r(g^{rs}\eta)\,\mathrm{dvol}_g - \Gamma^r(\partial_b(g_{ra}\eta) - \partial_a(g_{rb}\eta)) + 2Q_{ab}\eta\big\}\,\mathrm{dvol}_g \\
&\qquad - \int_U \big\{\partial_s g_{ab}\partial_r(g'^{rs}\eta) - \Gamma^r(\partial_b(g'_{ra}\eta) + \partial_a(g'_{rb}\eta)) + 2Q'_{ab}\eta\big\}\,\mathrm{dvol}_{g'}\bigg| \\
&\leq \frac{1}{2}\bigg|\int_U \big(\partial_s g_{ab}\partial_r g^{rs} - \Gamma^r(\partial_b g_{ra} - \partial_a g_{rb}) + 2Q_{ab}\big)\eta\sqrt{g}\,dx \\
&\qquad\qquad + \int_U \big(g^{rs}\partial_s g_{ab}\partial_r\eta - \Gamma^r(g_{ra}\partial_b\eta - g_{rb}\partial_a\eta)\big)\sqrt{g}\,dx \\
&\qquad - \int_U \big(\partial_s g'_{ab}\partial_r g'^{rs} - \Gamma'^r(\partial_b g'_{ra} - \partial_a g'_{rb}) + 2Q'_{ab}\big)\eta\sqrt{g}'\,dx \\
&\qquad\qquad\quad + \int_U \big(g'^{rs}\partial_s g'_{ab}\partial_r\eta - \Gamma'^r(g'_{ra}\partial_b\eta - g'_{rb}\partial_a\eta)\big)\sqrt{g}'\,dx\bigg| \\
&\leq \frac{1}{2}\int_U \Big|\big\{\sqrt{g}\,(\partial_s g_{ab}\partial_r(g-g')^{rs} + \partial_s(g-g')_{ab}\partial_r g^{rs}) + (\sqrt{g}-\sqrt{g}')\partial_s g'_{ab}\partial_r g'^{rs} \\
&\qquad - \sqrt{g}\,(\Gamma^r - \Gamma'^r)(\partial_b g_{ra} - \partial_a g_{rb}) - \sqrt{g}\,\Gamma'^r(\partial_b(g-g')_{ra} - \partial_a(g-g')_{rb}) \\
&\quad - (\sqrt{g}-\sqrt{g}')\Gamma'^r(\partial_b g'_{ra} - \partial_a g_{rb}) + 2\sqrt{g}(Q-Q')_{ab} + 2(\sqrt{g}-\sqrt{g}')Q'_{ab}\big\}\eta \\
&\qquad + \big\{\sqrt{g}\,((g-g')^{rs}\partial_s g_{ab} + g^{rs}\partial_s(g-g')_{ab}) + (\sqrt{g}-\sqrt{g}')g'^{rs}\partial_s g'_{ab}\big\}\partial_r\eta \\
&\quad + \sqrt{g}\,(\Gamma^r - \Gamma'^r)(g_{ra}\partial_b\eta - g_{rb}\partial_a\eta) - \sqrt{g}\,\Gamma'^r((g-g')_{ra}\partial_b\eta - (g-g')_{rb}\partial_a\eta) \\
&\qquad\qquad - (\sqrt{g}-\sqrt{g}')\Gamma'^r(g'_{ra}\partial_b\eta - g_{rb}\partial_a\eta)\Big|\,dx \\
&\leq C\,\|\eta\|_{C^0}\Big\{n^2\|\sqrt{g}\|_{C^0}\big(\|\partial g\|_{C^0}\|\partial(g-g')^{-1}\|_{C^0} + \|\partial(g-g')\|_{C^0}\|\partial g^{-1}\|_{C^0}\big) \\
&\qquad + n^2\|\sqrt{g}-\sqrt{g}'\|_{C^0}\|\partial g'\|_{C^0}\|\partial(g')^{-1}\|_{C^0} + 2n\|\sqrt{g}\|_{C^0}\|\Gamma-\Gamma'\|_{C^0}\|\partial g\|_{C^0} \\
&\quad + 2n\|\sqrt{g}\|_{C^0}\|\Gamma'\|_{C^0}\|\partial(g-g')\|_{C^0} + n\|\sqrt{g}-\sqrt{g}'\|_{C^0}\|\Gamma'\|_{C^0}\big(\|\partial g'\|_{C^0} + \|\partial g\|_{C^0}\big) \\
&\qquad\qquad + 2\|\sqrt{g}\|_{C^0}\|Q-Q'\|_{C^0} + 2\|\sqrt{g}-\sqrt{g}'\|_{C^0}\|Q'\|_{C^0}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+C\|\partial\eta\|_{C^0}\Big\{n^2\|\sqrt{g}\|_{C^0}\big(\|(g-g')^{-1}\|_{C^0}\|\partial g\|_{C^0}+\|g^{-1}\|_{C^0}\|\partial(g-g')\|_{C^0}\big)\\
&\quad+n^2\|\sqrt{g}-\sqrt{g}'\|_{C^0}\|(g')^{-1}\|_{C^0}\|\partial g'\|_{C^0}+2n\|\sqrt{g}\|_{C^0}\|\Gamma-\Gamma'\|_{C^0}\|g\|_{C^0}\\
&\quad+2n\|\sqrt{g}\|_{C^0}\|\Gamma'\|_{C^0}\|g-g'\|_{C^0}+n\|\sqrt{g}-\sqrt{g}'\|_{C^0}\|\Gamma'\|_{C^0}\big(\|g'\|_{C^0}+\|g\|_{C^0}\big)\Big\}.
\end{aligned}
$$

□

**Theorem B.8.** *Let $M$ be a spin manifold and denote by $C^1(T^*M^{\odot 2})$ the space of $C^1$ symmetric (2,0)-tensors on it. For fixed coordinate vector fields $\partial_a,\partial_b\in C^1(TM)$ on an open chart $U\subset M$, the application*

$$
\begin{aligned}
\mathscr{W}_{R^\Sigma_{ab}}: C^{1,\alpha}(\Sigma M)\times C^{1,\alpha}(\Sigma M)\times C^{1,\alpha}(T^*M^{\odot 2}) &\longrightarrow \quad \mathbb{C}\\
(\psi,\varphi,g) &\longmapsto \langle\!\langle R^\Sigma(\partial_a,\partial_b)\psi,\varphi\rangle\!\rangle_U,
\end{aligned}
$$

*given by Definition 3.4.2 is continuous in the $C^1$-topology.*

*Proof.* First, let us suppose we have a fixed Riemannian $C^1$ metric $g$, with respect to which there is a (metric) spin structure $P_{Spin}(M,g)$ on $M$ and an associated spinor bundle $\Sigma^g M$. By definition of the weak spinorial curvature and the properties of $\nabla^\Sigma$, we know that $\mathscr{W}_{R^\Sigma_{ab}}(\psi,\varphi,g)$ is $\mathbb{C}$-linear in $\psi$ and $\varphi$. Thus, to check continuity in these first two variables a boundedness argument suffices.

$$
\begin{aligned}
\big|\langle\!\langle R^\Sigma(\partial_a,\partial_b)\psi,\varphi\rangle\!\rangle_U\big| &\le \left|\int_U \big(\nabla^\Sigma_a\psi,\nabla^\Sigma_b\hat{\varphi}\big)\,dx-\int_U\big(\nabla^\Sigma_b\psi,\nabla^\Sigma_a\hat{\varphi}\big)\,dx\right|\\
&\le 2\max_{a,b}\int_U \big|\big(\nabla^\Sigma_a\psi,\nabla^\Sigma_b\hat{\varphi}\big)\big|\,dx\\
&\le 2\max_{a,b}\sup_U\left\{\frac{|\nabla^\Sigma_a\psi|_g}{|\partial_a|_g}\frac{|\nabla^\Sigma_b\hat{\varphi}|_g}{|\partial_b|_g}|\partial_a|_g|\partial_b|_g\right\}\int_U dx\\
&\le C\max_{a,b}\sup_U\{|\partial_a|_g|\partial_b|_g\}\sup_{X,Y\in\Gamma(TM)}\left\{\frac{|\nabla^\Sigma_X\psi|_g}{|X|_g}\frac{|\nabla^\Sigma_Y\hat{\varphi}|_g}{|Y|_g}\right\}\\
&\le C\max_{a,b}\sup_U\{|\partial_a|_g|\partial_b|_g\}\|\nabla^\Sigma\psi\|_{C^0(U)}\|\nabla^\Sigma\hat{\varphi}\|_{C^0(U)}.
\end{aligned}
$$

Since the coordinate vector fields $\partial_a$ are of class $C^0$, their norms are appropriately bounded on $U$, in fact, they are uniformly bounded, so there is a constant $K$ such that $\max_{a,b}\sup_U|\partial_a|_g\le K$,

$$
\begin{aligned}
\big|\langle\!\langle R^\Sigma(\partial_a,\partial_b)\psi,\varphi\rangle\!\rangle_U\big| &\le CK^2\|\psi\|_{C^1(U)}\|\hat{\varphi}\|_{C^1(U)}\\
&\le CK^2\|\sqrt{g}\|_{C^1(U)}\|\psi\|_{C^1(U)}\|\varphi\|_{C^1(U)}.
\end{aligned}
$$

This shows that $\mathscr{W}_{R^\Sigma}(\psi,\varphi,g)$ is bounded by $\|\psi\|_{C^1(U)}\|$ and $\|\varphi\|_{C^1(U)}$ and therefore continuous in the $C^1$-topology for each of these variables.

Continuity in $g$ is more involved to prove since a change in $g$ affects all the metric structure of the manifold and relevant bundles. In particular, and assuming that the new metric spin structure for $g'$ is equivalent to the one for $g$, we have to identify the spinors in the spinor bundles corresponding to each metric to be able to calculate. Let $\psi' := A\psi$, $\varphi' := A\varphi$, and for readability, denote $\nabla_a\psi := \nabla^\Sigma_{\partial_a}\psi$, $\nabla'_a\psi' := \nabla^{\Sigma'}_{\partial_a}(A\psi)$ and $\nabla'_a\hat{\varphi}' := \nabla^{\Sigma'}_{\partial_a}(\sqrt{g}'A\varphi)$:

$$
\begin{aligned}
\big|\mathscr{W}_{R^\Sigma}(\psi,\varphi,g) - \mathscr{W}_{R^{\Sigma'}}(\psi',\varphi',g')\big| &= \big|\langle\!\langle R^\Sigma(\partial_a,\partial_b)\psi,\varphi\rangle\!\rangle_U - \langle\!\langle R^{\Sigma'}(\partial'_a,\partial'_b)\psi',\varphi'\rangle\!\rangle_U\big| \\
&\leq \bigg|\int_U (\nabla_a\psi,\nabla_b\hat{\varphi}_g)_g\,dx - \int_U (\nabla_b\psi,\nabla_a\hat{\varphi}_g)_g\,dx \\
&\qquad - \int_U (\nabla'_a\psi',\nabla'_b\hat{\varphi}'_{g'})_{g'}\,dx + \int_U (\nabla'_b\psi',\nabla'_a\hat{\varphi}'_{g'})_{g'}\,dx\bigg| \\
&\leq \bigg|\int_U (\nabla_a\psi,\nabla_b\hat{\varphi}_g)_g\,dx - \int_U (A^{-1}\nabla'_a\psi', A^{-1}\nabla'_b\hat{\varphi}'_{g'})_g\,dx\bigg| \\
&\qquad + \bigg|\int_U (\nabla_b\psi,\nabla_a\hat{\varphi}_g)_g\,dx - \int_U (A^{-1}\nabla'_b\psi', A^{-1}\nabla'_a\hat{\varphi}'_{g'})_g\,dx\bigg| \\
&\leq 2\max_{a,b}\bigg|\int_U (\nabla_a\psi,\nabla_b\hat{\varphi}_g)_g\,dx - \int_U (\overline{\nabla}_a\psi,\overline{\nabla}_b\hat{\varphi}_{g'})_g\,dx\bigg| \\
&\leq 2\max_{a,b}\int_U \Big|(\nabla_a\psi - \overline{\nabla}_a\psi,\nabla_b\hat{\varphi}_g)_g + (\overline{\nabla}_a\psi,\nabla_b\hat{\varphi}_g - \overline{\nabla}_b\hat{\varphi}_{g'})_g\Big|\,dx \\
&\leq 2\max_{a,b}\int_U \big\{|\nabla_a\psi - \overline{\nabla}_a\psi|_g\,|\nabla_b(\sqrt{g}\varphi)|_g \\
&\qquad\qquad + |\overline{\nabla}_a\psi|_g\,|\nabla_b(\sqrt{g}\varphi) - \overline{\nabla}_b(\sqrt{g}'\varphi)|_g\big\}\,dx \\
&\leq C_1\max_{a,b}\sup_U \big\{|\nabla_a\psi - \overline{\nabla}_a\psi|_g\,|\nabla_b(\sqrt{g}\varphi)|_g \\
&\qquad + |\overline{\nabla}_a\psi|_g\,|\nabla_b(\sqrt{g}'\varphi) - \overline{\nabla}_b(\sqrt{g}'\varphi)|_g \\
&\qquad\qquad + |\overline{\nabla}_a\psi|_g\,|\nabla_b(\sqrt{g}\varphi) - \nabla_b(\sqrt{g}'\varphi)|_g\big\} \\
&\leq C_1\max_{a,b}\sup_U \big\{C_2\|\nabla'(A^{-1})\|_g\,\|A\|_g\,\big(|\psi|_g\,|\partial_a|_g\,|\nabla_b(\sqrt{g}\varphi)|_g \\
&\qquad + |\overline{\nabla}_a\psi|_g\,|\sqrt{g}'\varphi|_g\,|\partial_b|_g\big) + |\overline{\nabla}_a\psi|_g\,|\sqrt{g} - \sqrt{g}'|_g\,|\nabla_b\varphi|_g \\
&\qquad\qquad + |\overline{\nabla}_a\psi|_g\,|\partial_b(\sqrt{g} - \sqrt{g}')|_g\,|\varphi|_g\big\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C_1 \max_{a,b}\sup_U\{|\partial_a|_g\,|\partial_b|_g\}\sup_U\big\{C_2\|\nabla'(A^{-1})\|_g\,\|A\|_g\,\big(|\psi|_g\,|\nabla(\sqrt{g}\varphi)|_g \\
&\qquad\qquad + |\overline{\nabla}\psi|_g\,|\sqrt{g}'\varphi|_g\big) + |\overline{\nabla}\psi|_g\,|\sqrt{g}-\sqrt{g}'|_g\,|\nabla\varphi|_g\big\} \\
&\qquad\qquad\qquad + \max_{a,b}\sup_U\big\{|\partial_a|_g\,|\overline{\nabla}\psi|_g\,|\partial_b(\sqrt{g}-\sqrt{g}')|_g\,|\varphi|_g\big\} \\
&\leq C_3K^2\|\nabla'(A^{-1})\|_g\,\|A\|_g\,\big\{\|\psi\|_{C^0}\|\nabla(\sqrt{g}\varphi)\|_{C^0} + \|\overline{\nabla}\psi\|_{C^0}\|\sqrt{g}'\varphi\|_{C^0}\big\} \\
&\qquad\qquad + C_1K^2\|\overline{\nabla}\psi\|_{C^0}\|\sqrt{g}-\sqrt{g}'\|_{C^0}\|\nabla\varphi\|_{C^0} \\
&\qquad\qquad\qquad + C_1K\|\overline{\nabla}\psi\|_{C^0}\|\varphi\|_{C^0}\max_b\|\partial_b(\sqrt{g}-\sqrt{g}')\|_{C^0},
\end{aligned}
$$

where we have used that there is a $K$ such that $\max_a \sup_U |\partial_a|_g \leq K$, since the coordinate vector fields $\partial_a$ are $C^{0,\alpha}$.

The spinors $\psi$ and $\varphi$ are $C^{1,\alpha}$, therefore their norms and the norms of their covariant derivatives are bounded. We also know, that the norms $\|A\|_g$, $\|\nabla'(A^{-1})\|_g$ and $\|\sqrt{g}-\sqrt{g}'\|$ are appropriately bounded (see [Pfä03, chapter 1]) and

$$
\begin{aligned}
\|\partial_b(\sqrt{g}-\sqrt{g}')\|_{C^0} &\leq \tfrac{1}{2}\|\sqrt{g}(g^{cd}\partial_b g_{cd} - (g')^{cd}\partial_b g'_{cd})\|_{C^0} \\
&\qquad\qquad + \|(\sqrt{g}-\sqrt{g}')(g')^{cd}\partial_b g'_{cd}\|_{C^0} \\
&\leq \tfrac{1}{2}\|\sqrt{g}\|_{C^0}\big(\|g^{cd}\partial_b(g-g')_{cd}\|_{C^0} + \|(g-g')^{cd}\partial_b g'_{cd}\|_{C^0}\big) \\
&\qquad + \|(\sqrt{g}-\sqrt{g}')(g')^{cd}\partial_b g'_{cd}\|_{C^0} \\
&\leq \tfrac{1}{2}\|\sqrt{g}\|_{C^0}\big(\|g^{cd}\partial_b(g-g')_{cd}\|_{C^0} + \|(g-g')^{cd}\partial_b g'_{cd}\|_{C^0}\big) \\
&\qquad + \|(\sqrt{g}-\sqrt{g}')(g')^{cd}\partial_b g'_{cd}\|_{C^0} \\
&\leq \tfrac{1}{2}n^2\|\sqrt{g}\|_{C^0}\big(\|g\|_{C^0}\|\partial_b(g-g')\|_{C^0} + \|g-g'\|_{C^0}\|\partial_b g'\|_{C^0}\big) \\
&\qquad + n^2\|\sqrt{g}-\sqrt{g}'\|_{C^0}\|g'\|_{C^0}\|\partial_b g'\|_{C^0}.
\end{aligned}
$$

□

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# Summary

## Manifolds with Killing spinors and pinching of first Dirac Eigenvalues

by Andrés Vargas

In this thesis, we introduce spaces and metrics of Hölder class and employ the theory of convergence of Riemannian manifolds in the $C^{1,\alpha}$-regularity case, to study spin manifolds converging in the $C^{1,\alpha}$-topology. The regularity of the metric and the convergence of Killing spinor fields is considered. Pinching results are found for the first eigenvalues of the Dirac operator using this techniques.

In chapter 1, we review basic but important results from the theory of partial differential equations in the context of compact manifolds, and establish some elliptic regularity theorems and estimates needed in chapters 3 and 4.

In chapter 2, a brief discussion on spin geometry appears, to introduce the notions of Killing spinors, Einstein manifolds and their relationship on smooth Riemannian manifolds.

Chapter 3 is devoted to the study of Riemannian manifolds with $C^{1,\alpha}$-metrics, making use of the well-known optimal regularity properties of harmonic coordinates. Since curvature is not well-defined without second derivatives of the metric, we introduce appropriate weak notions of Ricci curvature and the Einstein condition for $C^{1,\alpha}$-metrics. Then it is shown that $C^{1,\alpha}$-metrics satisfying the Einstein condition weakly, in harmonic coordinates, are actually smooth and Einstein in the usual sense.

Then, weak spinorial curvature is introduced and with that, it is proved that manifolds with $C^{1,\alpha}$-metrics carrying a Killing spinor satisfy the weak Einstein condition and therefore, the smoothness of those metrics is obtained.

In chapter 4, the identification procedure for spinor fields associated to different metrics on the base manifold is reviewed, as well as the essential aspects of the theory of convergence of Riemannian manifolds under appropriate diameter, curvature and volume bounds. Then we study sequences of manifolds carrying "almost Killing spinors" and consider their convergence in the space of $L^2$-spinors. Existence of a Killing spinor in the limit, with appropriate regularity, is proved.

In chapter 5 all the material presented before is used to find pinching results for the first eigenvalues of the Dirac operator on manifolds with upper bounds on diameter and sectional curvature, and with volume bounded from below. In particular, the existece of a Killing spinor in such manifolds allows to characterize the sphere for many dimensions. Additionaly, uniform lower bounds on manifolds no diffeomorphic to the sphere and with the same geometrical restrictions are presented.

For manifolds with positive scalar curvature, it was known that proximity of the first eigenvalue of the Dirac operator to the Friedrich estimate does not impose topological restrictions on the manifold. Nevertheless, if in addition the manifolds have the previously mentioned bounds, our result shows that proximity of enough eigenvalues to this estimate do impose some restrictions.