Complex Multiplication, Rationality and Mirror Symmetry for Abelian Varieties and K3 Surfaces

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Summary

This Ph.D. thesis consists of three parts. In the first part (Section 1) we study abelian varieties and K3 surfaces of CM-type (i.e. their Hodge group is commutative), aiming at a characterization of complex multiplication via the existence of special Kähler metrics. We find out that an abelian variety X is of CM-type if and only if it admits a constant rational Kähler metric (see Theorem 1.2.17). The latter is a constant Kähler metric which only takes values in $\mathbb{Q}$ on the lattice $H_1(X,\mathbb{Z})$ of X. Here we identify the tangent space of X with $H_1(X,\mathbb{R})$. For a K3 surface Y with period $\sigma \in H^{2,0}(Y)$ we interpret a positive definite 3-dimensional subspace $V \subset H^2(Y,\mathbb{R})$ containing $\mathbb{R}\operatorname{Re}\sigma \oplus \mathbb{R}\operatorname{Im}\sigma$ as a Kähler metric. We also find some special V if Y is of CM-type and has high Picard number (≥ 10). This and a converse statement can be found in Theorem 1.3.18.

In the second part (Section 2) we apply the characterizations we found above to give sufficient conditions under which a mirror of an abelian variety or of a K3 surface of CM-type is of CM-type as well. We show that if a triple (X', G', B') is a mirror abelian variety of (X, G, B), where X is an abelian variety of CM-type, G a constant rational Kähler metric on X (which does exist in view of Theorem 1.2.17) and B is a rational B-field (i.e. an arbitrary element of $H^2(X,\mathbb{Q})$), then X' is also of CM-type (or even stronger, X' is isogenous to X). This is the contents of Theorem 2.2.13. For K3 surfaces Theorem 2.4.4 answers the question. Conditions on (Y, ω, B), where Y is a K3 surface of CM-type, $\omega \in H^{1,1}(Y,\mathbb{R})$ has square $\langle \omega, \omega \rangle > 0$ and B is a B-field (i.e. an arbitrary element of $H^2(Y,\mathbb{R})$), and on a mirror map are given such that a mirror (Y', ω', B') is of CM-type.

In the third part (Section 3) we construct a lattice OPE-algebra, which is a generalization of Kac's lattice vertex algebra, in the sense that a lattice OPE-algebra may contain a fermionic and an anti-holomorphic $\bar{z}$-part. Associated to a torus it is isomorphic to Kapustin and Orlov's construction. We define the notion of a rational lattice OPE-algebra and we show that the N=2 lattice OPE-algebra $V(T, G, B)$ associated to a complex torus T together with a constant Kähler metric G and a B-field $B \in H^2(X,\mathbb{R})$ is rational if and only if both G and B are rational. This allows us to conclude in view of Theorem 1.2.17 that if an abelian variety X is of CM-type, then there is a rational N=2 lattice OPE-algebra $V(X, G, B)$ associated to it, and conversely, if a N=2 lattice OPE-algebra $V(X, G, B)$ is rational, then X is of CM-type. In Corollary 3.5.5 we also relate this to mirror symmetry.

Contents

Introduction

We give successively our goal, a review of previous works, our results and finally the organization of our work.

Our goal: This work is inspired by Gukov and Vafa's paper [GV], where they shared their insight on a surprising and intriguing interplay between complex multiplication, rational conformal field theory (CFT) and mirror symmetry on Calabi-Yau varieties. They made the following observation: Let $(E' \cong \mathbb{C}/\mathbb{Z} \oplus \tau'\mathbb{Z}, G', B')$ be a mirror elliptic curve of $(E \cong \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}, G, B)$, where G is a constant Kähler metric and $B \in H^2(E, \mathbb{R})$. Then the N=2 CFT $\mathcal{C}(E, G, B)$ is rational if and only if both E and E' are of CM-type over the same imaginary quadratic field, i.e. $\tau, \tau' \in \mathbb{Q}(\sqrt{-D})$. In this case, E and E' are in particular isogenous. One then naturally asks whether similar relations hold for abelian varieties of arbitrary dimension. We will formulate N=2 CFT in terms of a lattice OPE-algebra based on Kapustin and Orlov's construction in [KO] (OPE stands for Operator Product Expansion, OPE-algebras are generalizations of vertex algebras). More precisely, our goal is to answer the following question:

Let (X', G', B') be a mirror partner of (X, G, B), where X is an abelian variety endowed with a constant Kähler metric G and a B-field B in $H^2(X, \mathbb{R})$. Is the N=2 superconformal lattice OPE-algebra $V(X, G, B)$ rational if and only if X and X' are isogenous and both of CM-type? (QAV)

Our work also contains an extension to K3 surfaces. Unfortunately, in this case, a formulation of the CFT in terms of OPE-algebras is still missing, hence we could only make a few steps towards the problem by answering the following questions:

Let Y be a (projective) K3 surface of CM-type with period σ. How to characterize Y by a "Kähler metric" (we mean a three dimensional positive definite subspace $V \subset H^2(Y, \mathbb{R})$ which contains $\mathbb{R}\operatorname{Re}\sigma \oplus \mathbb{R}\operatorname{Im}\sigma$ and a real (1,1)-form ω with $\langle \omega, \omega \rangle > 0$)? How to choose ω and a B-field $B \in H^2(Y, \mathbb{R})$ such that a mirror partner (Y', ω', B') of (Y, ω, B) is again of CM-type? (QK3)

Previous works: We first review previous works on complex multiplication, rational CFT and mirror symmetry for abelian varieties. We will only mention those works which are of direct relevance to us.

Complex multiplication on abelian varieties has been extensively studied in geometry as well as in number theory (see e.g. [Sh2],[L]). Historically, one says that a simple abelian variety X (i.e contains no abelian subvarieties) is of CM-type if its endomorphism algebra $\operatorname{End}_{\mathbb{Q}} X$ is as big as possible, i.e. is of rank $2\dim X$. In this case $\operatorname{End}_{\mathbb{Q}} X$ is isomorphic to a CM-field which is by definition a totally complex quadratic extension of a totally real number field. For example, in dimension one, an elliptic curve $E \cong \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$ is of CM-type if and only if τ lies in an imaginary quadratic number field $\mathbb{Q}(\sqrt{-D})$. In this case we have $\operatorname{End}_{\mathbb{Q}} X = \mathbb{Q}(\sqrt{-D})$, which is bigger than in the generic case where the endomorphism algebra is equal to $\mathbb{Q}$. The name "complex multiplication" comes from the action of $\sqrt{-D}$ on E. It is the multiplication by $\sqrt{-D}$ on $\mathbb{C}$ which preserves the lattice $\mathbb{Z} \oplus \tau\mathbb{Z}$ (see Example 1.2.8).

In physics, toroidal CFTs are also very familiar objects (see e.g. [W] and [En] and the references therein). From a physicist's point of view, a CFT is called rational, if its partition function can be written as a finite sum of the product of holomorphic and anti-holomorphic characters (see [GV, §2]), and this is the case when its chiral part is maximal.

The interplay between CM and rational CFT having a real 2-torus as target space is already known in Moore's paper [Mo, §10], and it is generalized by Wendland to real tori of arbitrary dimension. We can rephrase her Theorem 4.5.5 in [W] as follows. Let $\mathbb{T}$ be a real torus. We say that a constant (Riemann) metric G on $\mathbb{T}$ is rational if it only takes rational values if restricted on $H_1(\mathbb{T},\mathbb{Z})$. We say moreover that a B-field B on $\mathbb{T}$ is rational if B lies in $H^2(\mathbb{T},\mathbb{Q})$. Then we have: (a) A real torus $\mathbb{T}$ endowed with a rational constant (Riemann) metric G admits a complex structure I compatible with G such that the complex torus $(\mathbb{T}, I)$ is isogenous to a product of elliptic curves of CM-type. (b) A CFT $\mathcal{C}(\mathbb{T}, G, B)$ associated to a real torus $\mathbb{T}$ endowed with a constant metric G and a B-field $B \in H^2(\mathbb{T},\mathbb{R})$ is rational if and only if both G and B are rational. (c) Combining (a) and (b) one can say that a real torus $\mathbb{T}$ endowed with a rational CFT $\mathcal{C}(\mathbb{T}, G, B)$ admits a complex structure I compatible with G such that the complex torus $(\mathbb{T}, I)$ is isogenous to a product of elliptic curves of CM-type.

In [GV] a novelty is that Gukov and Vafa relate complex multiplication and rationality of CFT to mirror symmetry. We would like to mention the paper [GLO] where the authors used certain generalized complex structures to formulate mirror symmetry for abelian varieties although the notion of generalized complex geometry was invented only later by Hitchin in [Hi]. We will also use generalized structures to formulate mirror symmetry which shall incorporate [GLO]'s definition (see Definition 2.2.5).

We already mentioned Gukov and Vafa's observation about elliptic curves. The difference between their statement and Wendland's result is that Gukov and Vafa start with an elliptic curve with a *given* complex structure endowed with a *Kähler* metric, and use an additional N=2 structure on the CFT to encode the complex structure of the target space, and hence link the complex geometry (e.g. complex multiplication) of the mirror pair.

For abelian varieties, along with complex multiplication and mirror symmetry, CFTs and non-linear sigma model received a mathematical treatment by Kapustin and Orlov in terms of OPE-algebras (see [KO]). This is the terminology given by Rosellen (see [Ros]). In [KO] they are still called "vertex algebras" although they generalize the vertex algebras studied by Kac [Kac] among many other authors. In our work we adopt Rosellen's terminology.

Up to now we have described our starting point to answer the question (QAV). Our results are given later. The question (QK3) is an attempt to investigate the case of K3 surfaces. In order to introduce the notion of a K3 surface of CM-type we have to mention Mumford's work. In [Mm2] he proves that a simple abelian variety X is of CM-type (i.e. $\operatorname{rk}\operatorname{End}_{\mathbb{Q}} X = 2\dim X$) if and only if its Hodge group $\operatorname{Hg}(X)$ is commutative. For a not necessarily simple abelian variety X, if $\operatorname{Hg}(X)$ is commutative, then X is isogenous to a product of simple abelian varieties of CM-type (see Propositions 1.2.6 and 1.2.7 for more complete statements). This leads to the definition of Hodge structures of CM-type (see Section 1.1 for generalities about Hodge structures). A Hodge structure V is of CM-type if V is polarizable and its Hodge group $\operatorname{Hg}(V)$ is commutative (see Definition 1.1.7). This allows us to extend complex multiplication to any smooth projective variety carrying a Hodge structure, in particular to projective K3 surfaces. We say that a projective K3 surface Y is of CM-type if the Hodge structure on its transcendental lattice T (or equivalently on

$H^2(Y,\mathbb{Q})$) is of CM-type (see Definition 1.3.3). We will essentially use results from three papers [Za], [Bor] and [PS]. In the first paper Zarhin determines all possible Hodge groups of Y. In the second Borcea shows that Y is of CM-type if and only if $\mathrm{End}_{\mathrm{Hg}(T)} T$ is a CM-field of the same degree over $\mathbb{Q}$ as $\dim_{\mathbb{Q}} T$ (see Proposition 1.3.4). The third paper presents a construction of K3 surfaces of CM-type (see Proposition 1.3.9).

Mirror symmetry for K3 surfaces has also been studied by many authors. Let us mention Huybrechts' paper [H3, §6.4] where the mirror under a particular mirror map is calculated. In our work we formulate mirror symmetry for K3 surfaces in terms of generalized Calabi-Yau structures (GCYSs). This simplifies the formulation given in [H3]. GCYSs are studied in [Hi] and more specially on K3 surfaces in [H2] (see Section 2.1 for some generalities about them). Recall that mirror symmetry for abelian varieties is formulated using generalized complex structures. One can show that they can be induced by GCYSs (see Section 2.2). Thus GCYSs provide a more general framework for mirror symmetry. Schematically the passage from abelian varieties to K3 surfaces is an abstraction of structures as follows

complex multiplication on abelian varieties in terms of endomorphism algebras	$\rightsquigarrow$	Hodge structure of CM-type	$\rightsquigarrow$	K3 surfaces of CM-type

and

mirror symmetry for abelian varieties in terms generalized complex structures	$\rightsquigarrow$	GCYS	$\rightsquigarrow$	mirror symmetry for K3 surfaces.

This gives a unified view to our work on abelian varieties and K3 surfaces.

As to CFT on K3 surfaces there are among many other works several papers by Wendland. But unfortunately a formulation in terms of OPE-algebras is still missing. This prevents us from defining rigorously rational CFT and hence from completely answering a similar question as (QAV) for K3 surfaces.

Our results: It turns out that only one direction of the question (QAV) holds. More precisely we answer (QAV) by

Corollary 3.5.5. *Let (X,G,B) and (X',G',B') be mirror abelian varieties. If the N=2 superconformal lattice OPE-algebra $V(X,G,B)$ is rational, then X and X' are isogenous and both of CM-type. Conversely, however, there exist mirror abelian varieties (X,G,B) and (X',G',B') such that X and X' are isogenous and both of CM-type, but neither $V(X,G,B)$ nor $V(X',G',B')$ is rational.*

We explain a few important steps which lead us to this statement. Our construction of lattice OPE-algebras (given in Section 3.2) is inspired by Kapustin and Orlov's toroidal OPE-algebra, it is actually isomorphic to a toroidal lattice OPE-algebra (see Section 3.3 and Appendix B). However, our construction has the advantage of exhibiting more clearly the role of the lattice of the underlying torus. Hence makes the definition and the study of rationality easier.

Another key step is the following theorem. It shows that complex multiplication, which is *a priori* determined solely by the complex structure of the abelian variety, turns out to be equivalent to the existence of a rational Kähler metric (i.e. a Kähler metric which takes solely rational values on the lattice of the abelian variety). More precisely we have

Theorem 1.2.17. *An abelian variety X is of CM-type if and only if X admits a constant rational Kähler metric.*
Note that this theorem differs from Wendland's result (a) in the fact that the complex structure of X is given beforehand.

The role of a rational Kähler metric for a lattice OPE-algebra becomes evident through the following theorem.

Theorem 3.4.3. *The N=2 superconformal lattice OPE-algebra $V(T,G,B)$ associated to a complex torus T endowed with a constant Kähler metric G and a B-field B is rational if and only if G and B are both rational.*

A few comments are due here. The rationality of a N=2 superconformal lattice OPE-algebra is defined on the underlying lattice OPE-algebra (i.e. without the N=2 and superconformal structures, see Definition 3.4.1). Hence this theorem is in complete accordance with Wendland's result (b). Moreover, our definition is also in accordance with rationality defined in terms of the partition function, this is explained in Remarks 3.2.2 and 3.4.5. Combining the last two theorems we have

Theorem 3.5.4. *An abelian variety X is of CM-type if and only if X admits a rational N=2 superconformal lattice OPE-algebra $V(X,G,B)$.*

Now we make the link between complex multiplication and mirror symmetry for tori. We have

Theorem 2.2.13. *Let (X,G,B) and (X',G',B') be mirror abelian varieties. Suppose X is of CM-type. If both G and B are rational, then X and X' are isogenous. In particular, X' is also of CM-type.*

Combined with Theorem 3.4.3 we obtain one direction of Corollary 3.5.5. The converse is however not true by

Proposition 2.3.1. *There are mirror abelian varieties (X,G,B) and (X',G',B'), such that X and X' are isogenous and of CM-type, but neither $\mathcal{I}\mathcal{J}$ nor $\mathcal{I}'\mathcal{J}'$ is defined over $\mathbb{Q}$, where $(\mathcal{I},\mathcal{J})$ and $(\mathcal{I}',\mathcal{J}')$ denote their induced generalized Kähler structure (GKS).*

Refer to Section 2.2 for GKSs. This proposition leads to the second part of Corollary 3.5.5 which says that on this mirror pair neither $V(X,G,B)$ nor $V(X',G',B')$ is rational.

Now we turn to K3 surfaces. We answer the first part of the question (QK3) by

Theorem 1.3.18. *Denote $L := H^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. Let $V \subset L_{\mathbb{R}}$ be a positive definite 3-dimensional subspace of the form*

$$V = \mathbb{R}v_1 \oplus \mathbb{R}v_2 \oplus \mathbb{R}v_3,$$

with

(i) *$\{v_1, v_2, v_3\}$ is an orthogonal basis.*
(ii) *All v_i's are fully defined over some totally real number field K_0 under an embedding $\epsilon : K_0 \hookrightarrow \mathbb{R}$.*
(iii) *$E(v_i) \perp E(v_j)$ for $i \neq j$.*
(iv) *$\epsilon^{-1}(\frac{\langle v_i, v_i\rangle}{\langle v_j, v_j\rangle}) \in K_0$ is totally positive.*

If a point

$$[\sigma] \in \mathbb{P}(V_{\mathbb{C}}) \cap \Omega = \{[\sigma] \in \mathbb{P}(V_{\mathbb{C}}) \mid \langle \sigma, \sigma\rangle = 0, \langle \sigma, \bar{\sigma}\rangle > 0\}$$

defines a projective K3 surface, then a multiple of σ is fully defined over a CM-field.

Conversely, any K3 surface of CM-type Y with period σ and $\rho(Y) \geq 10$ possesses a 3-dimensional positive definite subspace $V \subset H^2(Y,\mathbb{R})$ as above which contains $\mathbb{R}\operatorname{Re}\sigma \oplus \mathbb{R}\operatorname{Im}\sigma$.

The key notion here is being fully defined over a CM-field. It is an arithmetic property. We show in Proposition 1.3.14 that if a K3 surface Y is of CM-type over a CM-field K, then there is an element $\sigma \in H^{2,0}(Y)$ which is fully defined over K. Unfortunately, the converse does not hold, so the conditions on V given in the theorem above are not strong enough to characterize K3 surfaces of CM-type.

The second part of the question (QK3) is answered by

Theorem 2.4.4. *Let (Y', ω', B') be an involutive mirror of (Y, ω, B) under a mirror map χ, where Y is a K3 surface of CM-type with $\rho(Y) \geq 10$. Suppose further the following conditions:*

(i) $\langle U', U \rangle \equiv 0$ *and* $\chi\big|_{L'} = \mathrm{Id}_{L'}$.

(ii) ω *is as constructed on page 47.*

(iii) $B = -v^* + \operatorname{Re}\sigma$, *where* $\sigma \in H^{2,0}(Y)$ *is fully defined over* K.

(iv) $\langle \omega, v \rangle = \langle B, v \rangle = 0$.

Then Y' is also of CM-type over K.

The notion of involutive mirror is introduced in Definition 2.4.2. This theorem is based on explicit calculations of mirror partners given in Proposition 2.4.3.

Organization of our work: Section 1 presents abelian varieties and K3 surfaces of CM-type (in Section 1.2 respectively 1.3) from the point of view of Hodge structures of CM-type (reviewed in Section 1.1), and contains characterization of complex multiplication by certain properties (e.g. rationality) of a Kähler metric.

Section 2 presents mirror symmetry for abelian varieties and K3 surfaces (in Section 2.2 respectively 2.4) from the point of view of generalized Calabi-Yau structures (reviewed in Section 2.1), and contains sufficient conditions for complex multiplication to be transmitted to mirror partners. Section 2.3 contains the construction of the pair of mirror isogenous abelian varieties of CM-type without their lattice OPE-algebra being rational.

Section 3 deals with lattice OPE-algebras on abelian varieties and their rationality. Section 3.5 contains a complete answer to the question (QAV).

Appendix A gives calculations of the partition function of lattice OPE-algebra. Appendix B shows that toroidal lattice OPE-algebra is isomorphic to Kapustin and Orlov's toroidal OPE-algebra.

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1. Complex multiplication, Hodge structures and rational Kähler metric

Summary of results: The goal of this section is to prove Proposition 1.2.16 which leads to Theorem 1.2.17 and Proposition 1.3.14 which leads to Theorem 1.3.18. The first two claims deal with abelian varieties. They show the equivalence of the following three conditions: complex multiplication, the compactness of the set of real points $\mathrm{Hg}(X)(\mathbb{R})$ of the Hodge group and the existence of a rational Kähler metric. The latter two claims treat K3 surfaces. They characterize complex multiplication with an arithmetic property of the period and with a 3-dimensional positive definite subspace $V \subset H^2(Y, \mathbb{R})$. These are the basis for our treatment of mirror symmetry and OPE-algebras later on.

This section is organized as follows. Section 1.1 contains a reminder of notions (e.g. CM-field and Hodge structure of CM-type) which we need in the sequel. We also give a few useful propositions. We omit their proof if a good reference is available.

In Section 1.2 we specialize to Hodge structures of weight 1 which correspond to abelian varieties. We recall a construction of abelian varieties of CM-type (due to Shimura) and two ways of considering them: one with Hodge group, the other with endomorphism algebra. These two approaches lead to two very different proves of Theorem 1.2.17.

In Section 1.3 we describe the Hodge structure on H^*, H^2 and the transcendental lattice T of a K3 surface. We present a construction of K3 surfaces of CM-type (due to Pjateckii-Shapiro and Shafarevich) and introduce the notion of being fully defined over a number field. This allows us to formulate Theorem 1.3.18.

We give when possible examples which we constructed or calculated. They are helpful to get a concrete understanding of these structures.

1.1. **Hodge structures.**

Let us first recollect a few facts about CM-fields and then about Hodge structures of CM-type. Their relationship is expressed in Proposition 1.1.9.

Definition 1.1.1. *A* CM-field K *is a number field which is a totally complex quadratic extension of a totally real field* K_0.

A *totally real field* is a number field which has only real embeddings. In other words, all embeddings factorize over $\mathbb{R}$. A *totally complex field* is a number field, which in contrast has only complex embeddings, i.e. none of its embeddings into $\mathbb{C}$ factorizes over $\mathbb{R}$. The embeddings of such a field come in complex conjugated pairs, i.e. the set S of embeddings of K is a union of two disjoint parts $S = \Phi \cup \overline{\Phi}$, if $\sigma \in \Phi$ then $\bar{\sigma} \in \overline{\Phi}$. The choice of Φ is called a *CM-type of* K. This determines the complex structure of the abelian variety constructed from a CM-field as we will see later.

Besides, a CM-field has an involution induced by the complex conjugation on $\mathbb{C}$, namely, any embedding σ defines

$$x \longmapsto \bar{x} := \sigma^{-1} \circ \bar{\sigma}(x).$$

It is easy to see that $x \mapsto \bar{x}$ does not depend on the choice of the embedding. Indeed, for any embedding $\sigma : K \hookrightarrow \mathbb{C}$, the composition $\sigma^{-1} \circ \bar{\sigma}$ is a non-trivial automorphism of K which preserves K_0. Since $\operatorname{Gal} K/K_0$ is of order 2, $\sigma^{-1} \circ \bar{\sigma}$ must be independent of the choice of σ. For simplicity we call this automorphism of K *complex conjugation*. Obviously, K_0 is the fixed subfield. The following lemma gives a method to construct CM-fields and will be used numerous times.

Lemma 1.1.2. *A number field K is a CM-field if and only if K is generated by an element $\eta \in K$ over a totally real subfield K_0 with $\eta^2 \in K_0$ totally negative (i.e. η^2 is negative under any embedding of K_0).*

Proof. Let K be a CM-field, so it is of the form $K = K_0(\beta)$ for an element $\beta \in K$ of degree 2 over K_0. Set

$$\eta := \beta - \bar{\beta}.$$

Then $\eta^2 = \beta^2 + \bar{\beta}^2 - 2\beta\bar{\beta}$ lies in K_0 as it is invariant under complex conjugation. Let $\varphi : K_0 \hookrightarrow \mathbb{R}$ be any embedding of K_0 and $\sigma : K \hookrightarrow \mathbb{C}$ be an extension of φ to K. We have

$$\varphi(\eta^2) = \sigma(\eta^2) = \sigma(\eta)^2 = (\sigma(\beta) - \overline{\sigma(\beta)})^2 = (2i \operatorname{Im} \sigma(\beta))^2 < 0.$$

Hence η^2 is totally negative.

Conversely, one can extend any embedding φ of K_0 to K by setting

$$\sigma(\eta) := \sqrt{\varphi(\eta^2)} \quad \text{and} \quad \bar{\sigma}(\eta) := -\sqrt{\varphi(\eta^2)}.$$

This shows that K is totally complex and ends the proof. □

As an example of CM-fields we can give the cyclotomic fields $\mathbb{Q}(\xi), \xi^n = 1, \xi \neq 1$. Indeed we can write them as

$$\mathbb{Q}(\xi) = \mathbb{Q}(\xi + \bar{\xi})(\xi - \bar{\xi}).$$

Then $K_0 = \mathbb{Q}(\xi + \bar{\xi})$ and $\eta = \xi - \bar{\xi}$ in the notation of the lemma above. We will use a cyclotomic field for the construction of an abelian variety of CM-type in Example 1.2.13.

Now we turn to Hodge structures.

Definition 1.1.3. *Let V be a finite dimensional $\mathbb{Q}$-vector space. A* (rational) Hodge structure of weight n *on V is a decomposition*

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q} \tag{1.1.1}$$

over $\mathbb{C}$, where $V^{p,q} = \overline{V^{q,p}}$. For simplicity we use the same letter V for the Hodge structure on it. We call a Hodge structure V irreducible *if V does not contain any non-zero proper Hodge substructure, meaning a $\mathbb{Q}$-vector subspace $0 \neq W \subsetneq V$ with*

$$W_{\mathbb{C}} = \bigoplus_{p+q=n} W^{p,q} \quad \textit{and} \quad W^{p,q} = W_{\mathbb{C}} \cap V^{p,q}.$$

Irreducible Hodge structures of weight 1 characterize simple complex tori as we will see in Proposition 1.2.1. The Hodge structure on the transcendental lattice of a K3 surface is also irreducible, shown in Proposition 1.3.1. A *morphism* of Hodge structures is a $\mathbb{Q}$-linear map $f : V \to W$ which, after $\mathbb{C}$-linear extension, preserves the decomposition (1.1.1), i.e.

$$f(V^{p,q}) \subset W^{p,q}.$$

We say V and W are *isomorphic* if f is an isomorphism.

The theory of Hodge structures is based heavily on the study of Hodge groups. The Hodge group is a linear algebraic group defined over $\mathbb{Q}$ which characterizes the

Hodge structure in a subtle way. We first recall the definition of a linear algebraic group. Let V be a finite dimensional vector space over a field k of characteristic 0 and K be an algebraically closed extension of k. A subgroup G of $GL(V, K)$ which is also an algebraic variety is called a *linear algebraic group*. If the ideal $I(G)$ is generated by polynomials with coefficients in a subfield $k' \subset K$, we say that G is *defined over* k' . For a subfield $F \subset K$, the *F(-rational)-points* of G are $G(F) := G \cap GL(V, F)$. For simplicity we will henceforth use the term *algebraic group* for linear algebraic group (although the first notion is more general than the second one).

A Hodge structure as in (1.1.1) induces a homomorphism of real algebraic groups (i.e. the $\mathbb{R}$-points of an algebraic group defined over $\mathbb{R}$):

$$\begin{aligned} h : \mathbb{S}^1 &\longrightarrow SL(V, \mathbb{R}) \\ z &\longmapsto (h(z) : v = \sum v^{p,q} \mapsto \sum z^p \bar{z}^q v^{p,q}), \end{aligned}$$

where $\mathbb{S}^1$ is the unit circle. Note that an $a \in \mathbb{S}^1$ can be considered as an element of $SL(V, \mathbb{R})$ with $a_{11} = a_{22}, a_{12} = -a_{21}, a_{11}^2 + a_{12}^2 = 1$. Hence $\mathbb{S}^1$ is indeed a real algebraic group.

Definition 1.1.4. *The* Hodge group $\mathrm{Hg}(h)$ *or* $\mathrm{Hg}(V)$ *of a Hodge structure V is the smallest algebraic subgroup of $GL(V, \mathbb{C})$ defined over $\mathbb{Q}$, whose $\mathbb{R}$-points contain the image of $\mathbb{S}^1$ under h, i.e.*

$$h(\mathbb{S}^1) \subseteq \mathrm{Hg}(h)(\mathbb{R}).$$

If $\mathrm{Hg}(h)$ *consists trivially of the identity element, we say that the Hodge structure V is* trivial.

A Hodge structure may be polarizable. This property is closely related to the projectivity of the corresponding variety, see Proposition 1.2.1 for complex tori and Proposition 1.3.2 for K3 surfaces for more precise statements.

Definition 1.1.5. *A* polarization *of a Hodge structure V of weight n is a $\mathbb{Q}$-bilinear form:*

$$E : V \times V \longrightarrow \mathbb{Q},$$

which is symmetric if n is even, anti-symmetric if n is odd, and extended $\mathbb{C}$-bilinearly satisfies

- $E(V^{p,q}, V^{r,s}) = 0$ *unless $r = q$ and $s = p$,*
- $E(\cdot, h(i)\cdot)$ *is positive definite, i.e. $i^{q-p}E(x, \bar{x}) > 0$ for any $x \neq 0$ in $V^{p,q}$.*

A Hodge structure is called polarizable *if one can define a polarization on it.*

We see immediately that $\mathrm{Hg}(V)$ preserves the polarization if V is polarizable. Indeed, the elements of $GL(V, \mathbb{C})$ which preserve the polarization form an algebraic group G defined over $\mathbb{Q}$ and its $\mathbb{R}$-points contain $h(\mathbb{S}^1)$, hence G contains $\mathrm{Hg}(V)$. Moreover, it is well known that the Hodge group is connected, and it is reductive if the Hodge structure is polarizable. These properties will be used in the proof of Proposition 1.2.16, so we recall these notions.

The *identity component* of an algebraic group G is the irreducible component (in Zariski topology) of G containing the identity element. If G consists of the identity component only, we call G *connected*. In order to define reductiveness we need the notion of an algebraic torus. An *algebraic torus over k* is an algebraic group over k which is isomorphic over the algebraic closure of k to a product of the multiplicative algebraic group $\mathbb{G}_m$. It is in particular commutative. An algebraic group G over k is called *reductive* if all k-representations of G are fully reducible or, equivalently, if G is an almost direct product of a k-torus and a semi-simple subgroup (i.e. $G = G^t G^s, G^t \cap G^s = \{e\}$). Now let us state precisely the following

Proposition 1.1.6 (Mumford). *(i) The Hodge group of any Hodge structure is connected.*
(ii) If a Hodge structure is polarizable, then its Hodge group is reductive and preserves polarizations.

It is easy to see (i). Indeed, since $\mathbb{S}^1$ is connected, $h(\mathbb{S}^1)$ is connected and lies in the identity component of $\mathrm{Hg}(h)$. On the other hand, an irreducible component of an algebraic group defined over $\mathbb{Q}$ is again defined over $\mathbb{Q}$. Hence from the minimality of the Hodge group, it must coincide with its identity component. As for the reductiveness, consult [LB, Prop.17.3.6]. The proof therein generalizes to any polarizable Hodge structure.

We give a name to those polarizable Hodge structures whose Hodge group has trivial semi-simple part.

Definition 1.1.7. *A Hodge structure is* of CM-type *if it is polarizable and its Hodge group is an algebraic torus, i.e. commutative.*

We will be very interested in the algebra $F := \mathrm{End}_{\mathrm{Hg}(V)} V$ in the sequel. These are the $\mathbb{Q}$-endomorphisms of V which commute with $\mathrm{Hg}(V)$. The reason for our interest is that F turns out to be the endomorphism algebra of an abelian variety X if one considers the Hodge structure on $H^1(X, \mathbb{Q})$. We extend elements of F $\mathbb{C}$-linearly on $V_{\mathbb{C}}$. A characterization of F is the following easy fact.

Proposition 1.1.8. *Let V be a Hodge structure and $f \in \mathrm{End}_{\mathbb{Q}} V$. Then f commutes with* $\mathrm{Hg}(V)$ *if and only if f preserves the Hodge structure on V.*

Proof. If f commutes with $\mathrm{Hg}(V)$, then f also commutes with $h(\mathbb{S}^1)$. Hence for any $v^{p,q} \in V^{p,q}$ and $z \in \mathbb{S}^1$ we have

$$h(z) \circ f(v^{p,q}) = f \circ h(z)(v^{p,q}) = f(z^p \bar{z}^q v^{p,q}) = z^p \bar{z}^q f(v^{p,q}).$$

It follows that $f(v^{p,q}) \in V^{p,q}$.

Conversely, consider the centralizer M of f in $GL(V, \mathbb{C})$. Obviously, M is an algebraic group defined over $\mathbb{Q}$ as f is defined over $\mathbb{Q}$. Moreover, $M(\mathbb{R})$ contains $h(\mathbb{S}^1)$, since f preserves the Hodge structure on V. It follows that $\mathrm{Hg}(V)$ is contained in M, which ends the proof. □

The following fact establishes the relationship between irreducible Hodge structures and CM-fields.

Proposition 1.1.9. *Let V be a non-trivial irreducible Hodge structure. Then it is of CM-type if and only if $F := \mathrm{End}_{\mathrm{Hg}(V)} V$ is a CM-field and* $\dim_F V = 1$. *In this case we say that the Hodge structure V is* of CM-type over F .

Proof. In [Abd] it is shown that if V is a non-trivial (see Definition 1.1.4) irreducible Hodge structure of CM-type, then F is a CM-field. Further, the commutativity of $\mathrm{Hg}(V)$ implies that all elements of $\mathrm{Hg}(V)$ are simultaneously diagonalizable over $\mathbb{C}$. Hence its centralizer in $\mathrm{End}_{\mathbb{C}} V_{\mathbb{C}}$ contains all diagonal matrices. It follows that its centralizer in $\mathrm{End}_{\mathbb{Q}} V$ must be a $\mathbb{Q}$-algebra of dimension equal to $\dim_{\mathbb{Q}} V$. This shows $\dim_F V = 1$.

Conversely, let $V = Fv_0$ be any isomorphism. Then the action of $\mathrm{Hg}(V)$ on V is determined by its action on v_0. Hence there is an inclusion $\mathrm{Hg}(V)(\mathbb{Q}) \hookrightarrow F$, and $\mathrm{Hg}(V)(\mathbb{Q})$ is commutative. Now we show that $\mathrm{Hg}(V)(\mathbb{Q})$ lies in the center of $\mathrm{Hg}(V)(\mathbb{C})$. Indeed, by definition, F commutes with $h(\mathbb{S}^1)$, hence $h(\mathbb{S}^1)$ in turn commutes with $\mathrm{Hg}(V)(\mathbb{Q})$. On the other hand, the centralizer of $\mathrm{Hg}(V)(\mathbb{Q})$ in $GL(V, \mathbb{C})$ is defined over $\mathbb{Q}$, and by what we just said contains $h(\mathbb{S}^1)$, hence the whole $\mathrm{Hg}(V)$ is contained in the centralizer of $\mathrm{Hg}(V)(\mathbb{Q})$ by the very definition of the Hodge group, i.e.

$$\mathrm{Hg}(V)(\mathbb{C}) \subset \{M \in GL(V, \mathbb{C}) \mid [M, \mathrm{Hg}(V)(\mathbb{Q})] = 0\}.$$

This shows

$$\mathrm{Hg}(V)(\mathbb{Q}) \subset \mathrm{Center}(\mathrm{Hg}(V)(\mathbb{C})) = \mathrm{Center}(\mathrm{Hg}(V))(\mathbb{C}).$$

As well known, $\mathrm{Hg}(V)(\mathbb{Q})$ is dense in $\mathrm{Hg}(V)(\mathbb{C})$ (see [Sp, Cor. 13.3.9] or [Hm, Thm. in §34.4]), and $\mathrm{Center}(\mathrm{Hg}(V))(\mathbb{C})$ is a closed algebraic group defined over $\mathbb{Q}$ (see in [Hm, Cor. §8.2]), hence

$$\mathrm{Hg}(V)(\mathbb{C}) = \overline{\mathrm{Hg}(V)(\mathbb{Q})} \subset \mathrm{Center}(\mathrm{Hg}(V))(\mathbb{C}).$$

This shows the commutativity of $\mathrm{Hg}(V)$. □

More generally for not necessarily irreducible Hodge structures we have the following

Proposition 1.1.10. *Let V be a polarizable Hodge structure which decomposes into a direct sum of polarizable irreducible Hodge substructures as follows:*

$$V = V_1^{r_1} \oplus \cdots \oplus V_n^{r_n}, \tag{1.1.2}$$

where $V_i^{r_i}$ is r_i copies of V_i and all V_i's are pairwise non-isomorphic. Then V is of CM-type if and only if all V_i's are of CM-type.

Proof. Put

$$V' := V_1 \oplus \cdots \oplus V_n,$$

the decomposition (1.1.2) implies an isomorphism and an inclusion

$$\mathrm{Hg}(V) \cong \mathrm{Hg}(V') \hookrightarrow \mathrm{Hg}(V_1) \times \cdots \times \mathrm{Hg}(V_n).$$

Now it is obvious that if all V_i's are of CM-type, then $\mathrm{Hg}(V')$ must be commutative. This proves one direction of the claim.

Conversely, let $\mathrm{Hg}(V')$ be commutative. The following composition of group homomorphisms:

$$\mathrm{Hg}(V') \hookrightarrow \mathrm{Hg}(V_1) \times \cdots \times \mathrm{Hg}(V_n) \xrightarrow{\pi_i} \mathrm{Hg}(V_i)$$

is surjective for all i, where π_i denotes the projection on the i-th factor. This is because the restriction of the action of $\mathbb{S}^1$ on V' to each V_i is exactly the Hodge structure on V_i. It follows that all $\mathrm{Hg}(V_i)$ are commutative. □

This proposition has as consequence Proposition 1.2.6 in the next section.

1.2. **Weight 1: abelian varieties.**

In this paragraph we specialize to Hodge structures of weight 1. Our goal is to prove Proposition 1.2.16 and Theorem 1.2.17. Hodge structures of weight 1 have a beautiful geometric meaning (see [H4, Prop.3.C.10,3.C.11]):

Proposition 1.2.1. *There are bijections*

$$\{\textit{rational Hodge structures of weight 1}\}/\{\textit{isom.}\} \xrightarrow{1:1} \{\textit{complex tori}\}/\{\textit{isogenies}\}$$
$$V \longmapsto V^{1,0}/W$$
$$H^1(T,\mathbb{Q}) \longleftarrow T,$$

where W is a maximal $\mathbb{Z}$-module in V and projects injectively into $V^{1,0}$.

$$\{\textit{polarizable rational Hodge structures of weight 1}\}/\{\textit{isom.}\}$$
$$\xrightarrow{1:1} \{\textit{abelian varieties}\}/\{\textit{isogenies}\}.$$

We write for a complex torus T

$$\mathrm{Hg}(T) := \mathrm{Hg}(H^1(T,\mathbb{Q}))$$

and define

Definition 1.2.2. *An abelian variety X is* of CM-type *if the Hodge structure on $H^1(X,\mathbb{Q})$ is of CM-type.*

This is not the first definition of abelian variety of CM-type. Historically, complex multiplication is a term to designate those abelian varieties X whose endomorphism algebra $\mathrm{End}_{\mathbb{Q}} X$ is as big as possible. There are a few equivalent ways to view the endomorphism algebra of a complex torus $T \cong \mathbb{C}^n/\Gamma$:

$$\begin{aligned}\mathrm{End}(T) := \{f : T \to T \mid\ & f \text{ holomorphic and} \\ & \text{preserves the (additive) group structure of } T\} \\ = \{f : \mathbb{C}^n \to \mathbb{C}^n \mid\ & f \text{ is a } \mathbb{C}\text{-linear map and } f(\Gamma) \subset \Gamma\} \qquad (1.2.1) \\ = \{f : \Gamma \to \Gamma \mid\ & fI = If, \text{ where } I \text{ is the complex structure of } T, \\ & \text{considered as an element of } \mathrm{End}_{\mathbb{R}} \Gamma_{\mathbb{R}} \text{ with } I^2 = -\mathrm{Id}\}\end{aligned}$$

Put $\mathrm{End}_{\mathbb{Q}} T := \mathrm{End}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. The relationship between $\mathrm{End}_{\mathbb{Q}} T$ and the Hodge structure on $V := H^1(T,\mathbb{Q})$ is established by the following

Proposition 1.2.3 (Torelli Theorem)**.** *Let T be a complex torus. Denote $V := H^1(T,\mathbb{Q})$. We have*

$$\mathrm{End}_{\mathbb{Q}} T \cong \{f \in \mathrm{End}_{\mathbb{Q}} V \mid f \textit{ preserves } V^{1,0}\}.$$

By Proposition 1.1.8 we see the well-known fact

Proposition 1.2.4. *Let T be a complex torus. Denote $V := H^1(T,\mathbb{Q})$. We have*

$$\mathrm{End}_{\mathbb{Q}} T \cong \mathrm{End}_{\mathrm{Hg}(V)} V$$

as $\mathbb{Q}$-algebras.

Together with Proposition 1.1.9 we have easily

Proposition 1.2.5. *A simple abelian variety X is of CM-type if and only if $\mathrm{End}_{\mathbb{Q}} X$ is a CM-field.*

Historically this was the first definition of a simple abelian variety of CM-type (i.e. $\mathrm{End}_{\mathbb{Q}} X$ is a CM-field). Mumford showed its equivalence to the commutativity of the Hodge group. His approach has the advantage to be generalizable to Hodge structures of higher weight which correspond to other geometric objects. More generally for not necessarily simple abelian varieties it is known by [Mm2, §2] or simply by Proposition 1.1.10 that

Proposition 1.2.6 (Mumford)**.** *Let X be an abelian variety. The following conditions are equivalent:*

(i) *X is of CM-type (i.e. $\mathrm{Hg}(X)$ is commutative).*
(ii) *X is isogenous to the product of simple abelian varieties of CM-type.*
(iii) *$\mathrm{End}_{\mathbb{Q}} X$ contains a commutative semi-simple $\mathbb{Q}$-algebra of dimension equal to $2 \cdot \dim_{\mathbb{C}} X$.*

If one strengthens the condition (iii), one gets (see [Sh2, §5] or [L, §2])

Proposition 1.2.7. *If $\mathrm{End}_{\mathbb{Q}} X$ of an abelian variety X of dimension g contains a number field of degree $2g$ over $\mathbb{Q}$, then X is isogenous to a product $B \times \cdots \times B$ with a simple abelian variety B of CM-type.*

However, the case where $\mathrm{End}_{\mathbb{Q}} X$ is a CM-field is rather exceptional. We present Albert's classification. It is necessary for the second proof of Theorem 1.2.17 and also for the understanding of the Hodge structure on the transcendental lattice of a K3 surface. In general $\mathrm{End}_{\mathbb{Q}} X$ of a simple abelian variety X is a division algebra of finite rank over $\mathbb{Q}$ endowed with a positive anti-involution. The latter structure is due to the presence of a Rosati involution. Recall that an involution $f \mapsto f^{\sigma}$ on a division algebra A with center K is called positive, if the quadratic form

$$\mathrm{tr}_{A|\mathbb{Q}} f^{\sigma} f := \mathrm{Tr}_{K|\mathbb{Q}}(\mathrm{tr}_{A|K} f^{\sigma} f) \tag{1.2.2}$$

is positive definite, where $\mathrm{tr}_{A|K}$ denotes the reduced trace of A over K, and $\mathrm{Tr}_{K|\mathbb{Q}}$ denotes the usual trace for the field extension $K|\mathbb{Q}$. Albert gave the classification of such division algebras A (see [LB, Thm 5.5.3, Lemma 5.5.4 and Prop. 5.5.5]):

I. $A =$ totally real number field, left invariant by the positive anti-involution.
II. $A =$ totally indefinite quaternion algebra, there is an element $a \in A$ whose square a^2 is in its center K and is totally negative (i.e. is a negative real number under any embedding $K \hookrightarrow \mathbb{R}$), such that the positive anti-involution $f \mapsto f^{\sigma}$ is given by $f^{\sigma} = a(\mathrm{tr}_{A|K} f - f)a^{-1}$.
III. $A =$ totally definite quaternion algebra, and $f \mapsto f^{\sigma}$ is given by $f^{\sigma} = \mathrm{tr}_{A|K} f - f$.

The first three algebras are of *the first kind*, i.e. the center is a totally real number field and coincides with the subfield fixed by the involution.

IV. The center K of A is a CM-field. The positive anti-involution restricted to K is the complex conjugation.

This is called *the second kind*, since K is not invariant under complex conjugation. Being of CM-type for a simple abelian variety is precisely when $F = K$ is commutative and is of rank $2 \cdot \dim_{\mathbb{C}} X$, and in this case it is a CM-field.

We give two examples:

Example 1.2.8. Let $E := \mathbb{C}/\mathbb{Z}\omega_0 \oplus \mathbb{Z}\omega_1$ be an elliptic curve. We show the well-known fact that E is of CM-type if and only if $\frac{\omega_1}{\omega_0}$ lies in an imaginary quadratic number field $\mathbb{Q}(\sqrt{-D})$. Indeed, an element $\gamma \in \operatorname{End}_{\mathbb{Q}} E$ as in (1.2.1) is of the form

$$\begin{aligned} \gamma : \mathbb{C} &\longrightarrow \mathbb{C} \\ \omega_0 &\longmapsto \gamma\omega_0 = a_1\omega_0 + a_2\omega_1 \\ \omega_1 &\longmapsto \gamma\omega_1 = a_3\omega_0 + a_4\omega_1 \end{aligned}$$

with $\{a_i\} \subset \mathbb{Q}$. Obviously, γ lies in $\mathbb{Q}$ if and only if $a_2 = a_3 = 0, a_1 = a_4 = \gamma$. Suppose now $a_2 \neq 0$ (or equivalently $a_3 \neq 0$). Then we have

$$(\gamma - a_1)\omega_0 = a_2\omega_1 \quad \text{and} \quad (\gamma - a_4)\omega_1 = a_3\omega_0.$$

Hence

$$\frac{\omega_1}{\omega_0} = \frac{\gamma - a_1}{a_2} = \frac{a_3}{\gamma - a_4}.$$

Since $\operatorname{Im} \frac{\omega_1}{\omega_0} \neq 0$, we see that γ lies in some $\mathbb{Q}(\sqrt{-D})$, $D \in \mathbb{N}^*$ with $\operatorname{Im} \gamma \neq 0$, and hence $\frac{\omega_1}{\omega_0} \in \mathbb{Q}(\sqrt{-D})$. Conversely, if $\frac{\omega_1}{\omega_0} \in \mathbb{Q}(\sqrt{-D})$, it is easy to see that $\sqrt{-D}$ preserves $\mathbb{Q}\omega_0 \oplus \mathbb{Q}\omega_1$. It follows that $\mathbb{Q}(\sqrt{-D}) \subseteq \operatorname{End}_{\mathbb{Q}} E$, hence the equality as $\operatorname{End}_{\mathbb{Q}} E$ cannot be of higher rank.

Remark 1.2.9. We give an abelian surface which shows that Gukov and Vafa's definition of complex multiplication (see [GV, §7.1]) is looser than ours. They say that a complex torus $T = \mathbb{C}^g/(1\ \mathcal{T})\mathbb{Z}^{2g}$, where $\mathcal{T}$ is a $g \times g$ complex symmetric matrix, "admits complex multiplication" if there is a non-trivial endomorphism $A \in GL(g, \mathbb{C})$ and integer matrices M', N', M, N such that N has rank g and $A = M + N\mathcal{T}$ and $\mathcal{T}A = M' + N'\mathcal{T}$.

Let us consider the following 2-dimensional complex torus

$$X := \mathbb{C}^2 / \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & b & a \end{pmatrix} \mathbb{Z}^4,$$

where $a = \sqrt{1+\sqrt{2}}i$ and $b = \sqrt[4]{2}i$. We have $a^2 - b^2 = -1$. Set

$$\mathcal{T} = \begin{pmatrix} a & b \\ b & a \end{pmatrix},\ A = \begin{pmatrix} -b & -a \\ a & b \end{pmatrix},\ M = 0,\ N = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ M' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ N' = 0,$$

we see that X "admits complex multiplication" in Gukov-Vafa's sense.

On the other hand we show that X is isogenous to the following product of two elliptic curves:

$$X' := E_1 \times E_2 := \mathbb{C}/\mathbb{Z} \oplus \frac{-1+b}{a}\mathbb{Z} \times \mathbb{C}/\mathbb{Z} \oplus \frac{4+3a}{20a+16b}\mathbb{Z}.$$

In view of Example 1.2.8 neither E_1 nor E_2 is of CM-type, hence X is not of CM-type in our sense.

For an isogeny $X' \to X$ consider the following invertible $\mathbb{C}$-linear map

$$\varphi = \begin{pmatrix} a & 5a+4b \\ b+1 & 3+5b+4a \end{pmatrix} : \mathbb{C}^2 \xrightarrow{\sim} \mathbb{C}^2.$$

Denote by Γ the lattice of X, i.e.

$$\Gamma := \mathbb{Z}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} a \\ b \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} b \\ a \end{pmatrix} =: \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \oplus \mathbb{Z}\omega_3 \oplus \mathbb{Z}\omega_4$$

and by Γ' the lattice of X', i.e.

$$\Gamma' := \mathbb{Z}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} \frac{-1+b}{a} \\ 0 \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} 0 \\ \frac{4+3a}{20a+16b} \end{pmatrix} =: \mathbb{Z}\omega'_1 \oplus \mathbb{Z}\omega'_2 \oplus \mathbb{Z}\omega'_3 \oplus \mathbb{Z}\omega'_4.$$

We see that

$$\begin{array}{ll} \varphi(\omega_1') = \omega_2 + \omega_3, & \varphi(\omega_2') = -\omega_1 + \omega_4, \\ \varphi(\omega_3') = 3\omega_2 + 5\omega_3 + 4\omega_4, & \varphi(\omega_4') = \omega_1 + \frac{5}{4}\omega_2 + \frac{3}{4}\omega_3. \end{array}$$

That is $\varphi(\Gamma'_{\mathbb{Q}}) = \Gamma_{\mathbb{Q}}$. Hence φ is an isogeny. We used Ruppert's method (see [LB, §10.6]) to determine that X is isogenous to a product of elliptic curves. Then an explicit isogeny is easy to find.

The next three propositions can be read from [Mm3, §22]. We first present Shimura's construction of abelian varieties of CM-type from a given CM-field K such that $\operatorname{End}_{\mathbb{Q}} X$ contains K (see [Sh2]). It is used in the proof of Theorem 1.2.17.

Proposition 1.2.10 (Mumford)**.** *For any given CM-field K there is an abelian variety of CM-type X such that there is an inclusion $K \hookrightarrow \operatorname{End}_{\mathbb{Q}} X$.*

Proof. Denote by S the set of embeddings $K \hookrightarrow \mathbb{C}$. We have a decomposition

$$K_{\mathbb{C}} := K \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\rho \in S} K^{\rho}, \tag{1.2.3}$$

where K^{ρ} is the eigenspace of the character ρ under multiplication, in other words

$$K^{\rho} = \{x \in K_{\mathbb{C}} \mid a \cdot x = \rho(a)x, \forall a \in K\}.$$

To each choice of CM-type Φ one can define a Hodge structure of weight 1 on K

$$K_{\mathbb{C}} = \bigoplus_{\rho \in \Phi} K^{\rho} \oplus \bigoplus_{\epsilon \in \overline{\Phi}} K^{\epsilon} =: K^{1,0} \oplus K^{0,1}. \tag{1.2.4}$$

One finds a polarization as follows. We need a generator η of K over K_0 such that $\eta^2 \in K_0$ is totally negative and $\operatorname{Im} \rho(\eta) > 0, \forall \rho \in \Phi$. Such η exists. Indeed, let η be as in Lemma 1.1.2. If η does not already satisfy $\operatorname{Im} \rho(\eta) > 0, \forall \rho \in \Phi$, by the Approximation Theorem (see [Wae]) there is $\alpha \in K_0$ such that $\alpha\eta$ satisfies the condition. Extend all the embeddings $\mathbb{C}$-linearly. For $v, w \in K_{\mathbb{C}}$ define

$$E(v,w) := \sum_{\rho \in \Phi} \rho(\eta)(\rho(v)\bar{\rho}(w) - \bar{\rho}(v)\rho(w)). \tag{1.2.5}$$

It is anti-symmetric. Moreover, as $\rho(\eta) = -\bar{\rho}(\eta)$, we have for $v, w \in K$,

$$E(v,w) = \operatorname{Tr}_{K|\mathbb{Q}}(\eta v \bar{w}) \in \mathbb{Q}.$$

Since $\rho(K^{0,1}) \equiv 0, \forall \rho \in \Phi$, we have

$$E(K^{1,0}, K^{1,0}) \equiv 0 \quad \text{and} \quad E(K^{0,1}, K^{0,1}) \equiv 0.$$

The positivity can be shown for any $v^{1,0} \neq 0 \in K^{1,0}$

$$i^{-1}E(v^{1,0}, v^{0,1}) = \sum_{\rho \in \Phi} \underbrace{(-i)\rho(\eta)}_{>0} \underbrace{|\rho(v^{1,0})|^2}_{>0} > 0$$

by the choice of η. Hence E is a polarization. This polarizable Hodge structure corresponds to an isogeny class of abelian varieties by Proposition 1.2.1. Let X be a representant of this isogeny class. Clearly, $\mathbb{C}$-linearly extended multiplication of K on $K_{\mathbb{C}}$ respects the Hodge structure (1.2.4) as it preserves each K^{ρ}. Hence by Proposition 1.1.8 we have

$$K \subset \operatorname{End}_{\operatorname{Hg}(K)} K \quad \text{or} \quad K \subset \operatorname{End}_{\mathbb{Q}} X.$$

By Proposition 1.2.6 X is of CM-type. $\square$

In general, however, X is not simple, since V may not be an irreducible Hodge structure. Moreover, if X is not simple, it may not be constructible in this way. One can easily show

Proposition 1.2.11 (Mumford)**.** *The abelian variety X constructed above is simple if and only if there is* no *proper subfield L of K satisfying*

(a) *L is a totally complex quadratic extension of $L \cap K_0$, and*
(b) *if $\sigma_i\big|_{L\cap K_0} = \sigma_j\big|_{L\cap K_0}$ then $\sigma_i\big|_L = \sigma_j\big|_L, \forall \sigma_i, \sigma_j \in \Phi$.*

Simple abelian varieties of CM-type are however all constructible in this manner.

Proposition 1.2.12 (Mumford)**.** *Any simple abelian variety of CM-type over a CM-field K is isogenous to one of the type constructed in Proposition 1.2.10.*

Proof. By Propositions 1.1.9 and 1.2.1 we have an isomorphism $V := H^1(X, \mathbb{Q}) \cong K$ where K is a CM-field. The Hodge decomposition induces a decomposition

$$H^{1,0}(X) \oplus H^{0,1}(X) = K^{1,0} \oplus K^{0,1}.$$

Two different isomorphisms $\phi, \psi : V \xrightarrow{\sim} K$ induce isomorphic (rational) Hodge structures on K. This does not change the isogeny class of X. On the other hand, we also have $K_{\mathbb{C}} = \oplus_{\rho\in S} K^\rho$ as in (1.2.3). We show that either $K^\rho \subset K^{1,0}$ or $K^\rho \subset K^{0,1}$ (hence the complex structure of X determines a CM-type Φ). Let $x \neq 0 \in K^\rho$ and write $x = x^{1,0} + x^{0,1}$. Then we have

$$a \cdot x = \rho(a)x = \rho(a)x^{1,0} + \rho(a)x^{0,1}.$$

Since $K = \operatorname{End}_{\mathrm{Hg}(V)} V$, multiplication by K preserves the Hodge structure, hence

$$a \cdot x^{1,0} = \rho(a)x^{1,0} \quad \text{and} \quad a \cdot x^{0,1} = \rho(a)x^{0,1}.$$

It follows that both $x^{1,0}$ and $x^{0,1}$ lie in K^ρ. But K^ρ is one-dimensional, so $x^{1,0}$ and $x^{0,1}$ must be linearly dependent. This implies that either $x^{1,0} = 0$ or $x^{0,1} = 0$. Hence we obtain a CM-type

$$\Phi := \{\rho \in S \mid K^\rho \subset K^{1,0}\}.$$

Now the claim is clear. □

We give an example of this construction which will be again used in Section 2.3.

Example 1.2.13. Consider the cyclotomic field $K := \mathbb{Q}(\xi), \xi^5 = 1, \xi \neq 1$. It is a CM-field and satisfies the conditions in Proposition 1.2.11. Indeed, the only possibility for L would be an imaginary quadratic extension over $\mathbb{Q}$ and $L \cap K_0 = \mathbb{Q}$, then the condition (b) would not be satisfied. Denote $w := e^{\frac{2\pi}{5}i}$, one can write the four embeddings of K into $\mathbb{C}$ as

$$\sigma_k : \xi \longmapsto w^k, \quad k = 1, \ldots, 4.$$

One has $\sigma_1 = \bar{\sigma}_4$ and $\sigma_2 = \bar{\sigma}_3$. Choose the CM-type $\Phi = \{\sigma_1, \sigma_2\}$ and choose the lattice to be the ring of integers

$$\Gamma = \mathcal{O}_K = \mathbb{Z}[\xi].$$

The complex torus $X := \mathbb{C}^2/\Phi(\mathcal{O}_K)$ is then a simple abelian variety of CM-type over K.

Now we are near to the proof of Theorem 1.2.17 which says that on an abelian variety, complex multiplication is equivalent to the existence of a constant rational Kähler metric. The latter notion is defined as follows

Definition 1.2.14. *We call a (constant Riemannian) metric G on a real torus $\mathbb{T}$* rational *if G only takes rational values on the lattice $H_1(\mathbb{T},\mathbb{Z})$ of $\mathbb{T}$. A* rational Kähler metric *G on a complex torus T is a rational metric on the underlying real torus and is compatible with the complex structure I of T, i.e. $G(\cdot,\cdot) = G(I\cdot, I\cdot)$.*

Let us first consider the example of elliptic curves.

Example 1.2.15. In Example 1.2.8 we showed that an elliptic curve $E = \mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z}$ is of CM-type if and only if τ lies in an imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$. We also know that on an elliptic curve there is up to scaling only one Kähler metric. Let us write

$$H_1(E,\mathbb{Z}) =: \mathbb{Z}e_1 \oplus \mathbb{Z}e_2, \quad \text{with } e_2 = \tau e_1.$$

We claim that the following matrix represents a Kähler metric in the basis $\{e_1, e_2\}$:

$$[G] = \begin{pmatrix} 1 & \operatorname{Re}\tau \\ \operatorname{Re}\tau & \tau\bar{\tau} \end{pmatrix}.$$

This matrix is clearly symmetric and positive definite. We show that it is compatible with the complex structure I. Denote the period matrix by $\Pi = (1 \quad \tau)$, then I in the basis $\{e_1, e_2\}$ is given by the matrix

$$[I] = \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}^{-1} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix} = -\frac{1}{\operatorname{Im}\tau} \begin{pmatrix} \operatorname{Re}\tau & \tau\bar{\tau} \\ -1 & -\operatorname{Re}\tau \end{pmatrix}.$$

One verifies easily

$$[I]^t[G][I] = [G].$$

Hence G is Kähler. We see that $[G]$ is a rational matrix if and only if $\tau \in \mathbb{Q}(\sqrt{-D})$. Its corresponding Kähler form is

$$[\omega] = [G][I] = \begin{pmatrix} 0 & -\operatorname{Im}\tau \\ \operatorname{Im}\tau & 0 \end{pmatrix}.$$

It is though not rational. The canonical rational Kähler form which makes an elliptic curve algebraic is a rational multiple of

$$[\omega_0] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We show that this phenomenon is also true for higher dimensional abelian varieties. Let us start with the following proposition. Recall first the definition of a Cartan involution. Let $\mathcal{G}$ be a connected algebraic group defined over $\mathbb{R}$ and denote by $x \mapsto \bar{x}$ the complex conjugation on $\mathcal{G}(\mathbb{C})$. An involution θ of $\mathcal{G}$ is said to be *Cartan* if the group

$$\mathcal{G}^{(\theta)}(\mathbb{R}) := \{x \in \mathcal{G}(\mathbb{C}) \mid x = \theta(\bar{x})\}$$

is compact.

Proposition 1.2.16. *An abelian variety X is of CM-type if and only if* $\operatorname{Hg}(X)(\mathbb{R})$ *is compact.*

Proof. On $\operatorname{Hg}(X)(\mathbb{R})$ we have the conjugation by $h(i)$

$$\begin{aligned} \operatorname{Ad} h(i) : \operatorname{Hg}(X)(\mathbb{R}) &\longrightarrow \operatorname{Hg}(X)(\mathbb{R}) \\ M &\longmapsto h(i)Mh(i)^{-1}, \end{aligned}$$

where h denotes the action of $\mathbb{S}^1$ on $H^1(X,\mathbb{R})$. As shown in (see [Del, §2]), $\operatorname{Ad} h(i)$ is a Cartan involution. If $\operatorname{Hg}(X)$ is commutative, then $\operatorname{Ad} h(i)$ is just the identity map, and hence $\operatorname{Hg}(X)(\mathbb{R})$ is compact by the very definition of Cartan involutions.

Conversely, if $\mathrm{Hg}(X)(\mathbb{R})$ is compact, then the identity map is a Cartan involution. By [Sat, Chap.1 Cor. 4.3] any two Cartan involutions of a connected reductive real algebraic group $\mathcal{G}$ are conjugate to each other by an inner automorphism of $\mathcal{G}$. As mentioned in Proposition 1.1.6 $\mathrm{Hg}(X)$ is connected and reductive. Hence we have $\mathrm{Ad}\, h(i) = \mathrm{Id}$ on $\mathrm{Hg}(X)(\mathbb{R})$. This means in particular that for every $\mathbb{Q}$-point $N \in \mathrm{Hg}(X)(\mathbb{Q})$ we have $Nh(i) = h(i)N$. Since the Hodge structure on $H^1(X, \mathbb{Q})$ is of weight 1, N commuting with $h(i)$ implies that N respects the Hodge decomposition $H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X)$. Hence N commutes with the whole $\mathrm{Hg}(X)$ in view of Proposition 1.1.8. Whence we have

$$\mathrm{Hg}(V)(\mathbb{Q}) \subset \mathrm{Center}(\mathrm{Hg}(V)(\mathbb{C})) = \mathrm{Center}(\mathrm{Hg}(V))(\mathbb{C}).$$

Now repeat the same arguments at the end of the proof of Proposition 1.1.9 to conclude that $\mathrm{Hg}(X)(\mathbb{C})$ is commutative. This shows the claim. □

Now we show

Theorem 1.2.17. *An abelian variety X is of CM-type if and only if X admits a constant rational Kähler metric.*

Proof. $\Leftarrow$: First suppose G is an arbitrary constant Kähler metric on X. Then for all $z = x + yi \in \mathbb{S}^1$ we have

$$G(h(z)v, h(z)w) = G((x + yI)v, (x + yI)w) = (x^2 + y^2)G(v, w) = G(v, w),$$

in other words, $h(S^1) \subset O(G, \mathbb{R})$. If G is moreover rational, then $O(G)$ is an algebraic group defined over $\mathbb{Q}$, whose $\mathbb{R}$-points contain $h(S^1)$. Hence $\mathrm{Hg}(X)$ is an algebraic subgroup of $O(G)$, and in particular $\mathrm{Hg}(X)(\mathbb{R}) \subset O(G, \mathbb{R})$. Therefore $\mathrm{Hg}(X)(\mathbb{R})$ is compact, and X is of CM-type by Proposition 1.2.16.

$\Rightarrow$: If X is of CM-type, then X is isogenous to a product of simple abelian varieties of CM-type by Proposition 1.2.6. So we may assume X is simple. By Proposition 1.2.12, X is then isogenous to a simple abelian variety of CM-type with a Riemann form E as in (1.2.5). It allows us to define the following bilinear form on the tangent space:

$$G(z, w) := E(z, \eta w) = \mathrm{Tr}_{K/\mathbb{Q}}(-\eta^2 z\bar{w}).$$

We see that G is compatible with I (as E is), rational, symmetric, and positive definite (as $-\eta^2$ is totally positive). The existence of a rational Kähler metric is preserved under isogeny, this completes the proof. □

Now we adopt the approach of endomorphism algebras to consider simple abelian varieties of CM-type and eventually give a second proof of $\Leftarrow$ of the last theorem. This approach has the advantage of exhibiting more clearly how a rational metric endows $\mathrm{End}_{\mathbb{Q}} X$ with additional structures which force $\mathrm{End}_{\mathbb{Q}} X$ to be very "big".

Let us put $F := \mathrm{End}_{\mathbb{Q}} X$, $V := H_1(X, \mathbb{Q})$ and denote by ω_0 a (rational) polarization of X, which always exists, since X is algebraic. The index 0 is to distinguish it from the Kähler form $\omega = GI$, which is in general not rational. This was illustrated in Example 1.2.15. Further we denote by $f \mapsto f'$ the Rosati involution with respect to ω_0 and by G_0 the Kähler metric associated to ω_0.

The presence of a rational Kähler metric induces two new structures on F:

- A linear map $\eta \in \mathrm{End}\, V$ determined by

$$G(\cdot, \cdot) = \omega_0(\eta\cdot, \cdot). \tag{1.2.6}$$

- An involution $f \mapsto f^G$ on $\operatorname{End} V$ defined by

$$G(fv, w) = G(v, f^G w).$$

They have the following properties:

Lemma 1.2.18. (i) *$\eta \in F$.*
(ii) *$\eta' = -\eta$.*
(iii) *$f \mapsto f^G$ defines a positive anti-involution on F.*
(iv) *The involution $f \mapsto f^G$ and the Rosati involution are conjugate to each other by η, i.e. $f^G = \eta^{-1} f' \eta$, $\forall f \in F$.*

Proof. (i) Since ω_0 and G are compatible with I, we have for all $v, w \in V$:

$$\omega_0(\eta I v, w) = G(Iv, w) = -G(v, Iw) = -\omega_0(\eta v, Iw) = \omega_0(I\eta v, w),$$

hence $\eta I = I\eta$, i.e. $\eta \in F$ by (1.2.1).

(ii) Since $\omega_0(\eta v, w) = \omega_0(v, \eta' w)$, it suffices to show $\omega_0(\eta v, w) = -\omega_0(v, \eta w)$ for all $v, w \in V$. This follows from

$$\begin{aligned} \omega_0(\eta^{-1} v, w) &= -\omega_0(w, \eta^{-1} v) \\ &= -G(\eta^{-1} w, \eta^{-1} v) \\ &= -G(\eta^{-1} v, \eta^{-1} w) \\ &= -\omega_0(v, \eta^{-1} w). \end{aligned}$$

(iii) If $f \in F$, i.e. $fI = If$, then $(fI)^G = (If)^G$ and hence $I^G f^G = f^G I^G$. As $I^G = -I$ we get $If^G = f^G I$, i.e. $f^G \in F$. Next we show that $f \mapsto f^G$ defines a positive anti-involution, i.e. $\operatorname{tr}_{F|\mathbb{Q}} f^G f > 0$ for all $f \neq 0 \in F$. Since F acts on V, one has $V \cong F^m$, and f acts on F^m by left multiplication on each component. On the other hand, the action of F on itself by left multiplication has trace $d \cdot \operatorname{tr}_{F|K} f$ over its center K, where $\operatorname{tr}_{F|K}$ is the reduced trace and d^2 is the degree of F over K. Denote by $\operatorname{Tr} f$ the trace of $f \in F$, when considered as an endomorphism of V. Then we have

$$\operatorname{Tr} f = m \cdot d \cdot \operatorname{Tr}_{K|\mathbb{Q}}(\operatorname{tr}_{F|K} f) = m \cdot \operatorname{tr}_{F|\mathbb{Q}} f, \quad \forall f \in F. \tag{1.2.7}$$

In an orthonormal basis with respect to G, f^G is just the transposed matrix of f, hence $\operatorname{Tr} f^G f > 0$, $\forall f \neq 0 \in F$. Then (1.2.7) implies that the involution induced by G is positive.

(iv) This follows immediately from

$$\omega_0(f'\eta v, w) = \omega_0(\eta v, fw) = G(v, fw) = G(f^G v, w) = \omega_0(\eta f^G v, w)$$

for all $v, w \in V$, which yields $\eta f^G = f' \eta$. □

Since Albert classified $\operatorname{End}_{\mathbb{Q}} X$ of a *simple* abelian variety X, we first prove

Lemma 1.2.19. *If a simple abelian variety X admits a constant rational Kähler metric, then* $\operatorname{End}_{\mathbb{Q}} X$ *is a CM-field, i.e. X is of CM-type.*

Proof. We proceed by elimination of other types of algebras using Lemma 1.2.18.

Type I is already made impossible by (i) and (ii) of Lemma 1.2.18. On Type III algebras there is a unique positive anti-involution (see [Sh1, Prop. 3]), hence $f' = f^G$. Then Lemma 1.2.18 (iv) implies that η lies in K, in contradiction with Lemma 1.2.18 (ii).

On Type II algebras, positive anti-involutions are not unique and they are all of the form given in Albert's classification mentioned earlier. Let us write $f^\rho := \mathrm{tr}_{F|K} f - f$, then

$$f' = a_1 f^\rho a_1^{-1} \quad \text{and} \quad f^G = a_2 f^\rho a_2^{-1},$$

for some a_1 and a_2 in F. Then $f^G = a_2 a_1^{-1} f' a_1 a_2^{-1}$, and, hence, $\eta = \epsilon a_1 a_2^{-1}$ for some ϵ in K. On the one hand, $\eta' = -\eta$ by Lemma 1.2.18 (ii), and on the other hand, in view of $a_i^\rho = -a_i$ (since $a_i^2 \in K$ and $a_i \notin K$), we have

$$\begin{aligned}\eta' &= \epsilon a_2^{-1\prime} a_1' \\ &= \epsilon a_1 (a_2^{-1})^\rho a_1^{-1} a_1 a_1^\rho a_1^{-1} \\ &= \epsilon a_1 (-a_2^{-1})(-a_1) a_1^{-1} \\ &= \epsilon a_1 a_2^{-1} \\ &= \eta.\end{aligned}$$

A contradiction.

So it remains to deal with the Type IV algebras. Recall that the center K of F is a CM-field. Let us write as before $V := H_1(X, \mathbb{Q})$, and denote by $2m$ the rank of K over $\mathbb{Q}$ and put $n := \frac{g}{m}$. We shall show $n = 1$, which implies that X is of CM-type. With respect to I there is the splitting

$$V_\mathbb{C} \xrightarrow{\sim} V^{1,0} \oplus V^{0,1}. \tag{1.2.8}$$

We extend the action of F on V $\mathbb{R}$-linearly onto $V_\mathbb{R}$ and denote by ρ its action on $V^{1,0}$ under the isomorphism $V^{1,0} \cong V_\mathbb{R}$. Since K is commutative and since there is an isomorphism $V \cong K^n$, the action of K on V diagonalizes on $V_\mathbb{C}$, and the diagonal entries are exactly n copies of the complete set of $2m$ embeddings of K into $\mathbb{C}$. The splitting (1.2.8) then implies that $\rho(K)$ even diagonalizes on $V^{1,0}$, i.e. there is a complex basis $\{e_1, \ldots, e_g\}$ of $V_\mathbb{R}$, with respect to which, for all $x \in K$ we have $\rho(x) e_l = \rho_l(x) e_l$, where $\Psi := \{\rho_1, \ldots, \rho_g\}$ are embeddings of K into $\mathbb{C}$.

We show that ρ_l and $\bar\rho_l$ can not both belong to Ψ. As before let us denote by $f \mapsto f^G$ the involution induced by G and by $f \mapsto f'$ the one induced by ω_0. According to Theorem 5.5.6 in [LB] there is a positive anti-involution $x \mapsto \hat{x}$ of the second kind on F and for every $\sigma : K \hookrightarrow \mathbb{C}$ there is an isomorphism

$$\varphi : F \otimes_\sigma \mathbb{C} \xrightarrow{\sim} \mathrm{M}_d(\mathbb{C})$$

such that $x \mapsto \hat{x}$ extends via φ to the canonical anti-involution $A \mapsto \bar{A}^t$ on $\mathrm{M}_d(\mathbb{C})$ (here d comes from $[F : K] = d^2$). Any other positive anti-involution on F is of the form

$$x \mapsto a \hat{x} a^{-1}$$

with $a \in F, \hat{a} = a$ and such that $\varphi(a \otimes 1)$ is a positive definite hermitian matrix in $\mathrm{M}_d(\mathbb{C})$ for every embedding $\sigma : K \hookrightarrow \mathbb{C}$. So for the two induced involutions which we have, we can write

$$f^G = a_1 \hat{f} a_1^{-1} \quad \text{and} \quad f' = a_2 \hat{f} a_2^{-1}.$$

But since $f' = \eta^{-1} f^G \eta$ as shown in Lemma 1.2.18 (iv), we see that

$$\eta = a_2 a_1^{-1} \epsilon,$$

for an element ϵ in the center K. Further, since $\eta' = -\eta$ we see that $\epsilon' = -\epsilon$. Now because $G(\cdot,\cdot) = \omega_0(\eta\cdot,\cdot)$ is positive definite and $a_2 a_1^{-1}$ is a positive definite hermitian matrix under φ, the bilinear form

$$G'(\cdot,\cdot) := \omega_0(\epsilon\cdot,\cdot)$$

is also positive definite and hence is a rational Kähler metric. Since $\epsilon \in K$ and $\epsilon' = -\epsilon$ it follows that $\rho(\epsilon)$ has only purely imaginary diagonal entries, as the Rosati involution restricted to K is just the complex conjugation. Now suppose $\rho_2 = \bar{\rho}_1$. Since G' is positive definite on $V_{\mathbb{R}}$, we have

$$\begin{aligned} 0 < G'(e_1, e_1) &= \omega_0(\rho(\epsilon)e_1, e_1) \\ &= \omega_0(\rho_1(\epsilon)e_1, e_1) \\ &= \mathrm{Im}(\rho_1(\epsilon))\omega_0(Ie_1, e_1) \\ &= \mathrm{Im}(\rho_1(\epsilon))G_0(e_1, e_1), \end{aligned}$$

and hence $\mathrm{Im}(\rho_1(\epsilon)) > 0$. On the other hand,

$$0 < G'(e_2, e_2) = \omega_0(\rho_2(\epsilon)e_2, e_2) = -\operatorname{Im}\rho_1(\epsilon)G_0(e_2, e_2).$$

A contradiction. It follows that Ψ consists exactly of n copies of a CM-type $\Phi := \{\sigma_1, \ldots, \sigma_m\}$ of K.

Now we show that this implies

$$I \in K \otimes_{\mathbb{Q}} \mathbb{R}, \tag{1.2.9}$$

or equivalently, $\rho(I) \in \rho(K) \otimes_{\mathbb{Q}} \mathbb{R}$, where $K \otimes_{\mathbb{Q}} \mathbb{R}$ is considered as a subspace of $F \otimes_{\mathbb{Q}} \mathbb{R}$. Since $\rho(I)$ is just the multiplication by i on $V^{1,0}$ and as we showed that Ψ consists of n copies of Φ, (1.2.9) amounts to show that there is a unique element $x \in K \otimes_{\mathbb{Q}} \mathbb{R}$, such that for all $\sigma_l \in \Phi$ we have $\sigma_l(x) = i$. This is clear due to the isomorphism:

$$\begin{aligned} K \otimes_{\mathbb{Q}} \mathbb{R} &\xrightarrow{\sim} \mathbb{C}^m \\ a &\longmapsto (\sigma_1(a), \ldots, \sigma_m(a))^t, \end{aligned}$$

where we extended σ_l $\mathbb{R}$-linearly.

Finally, as K acts by left multiplication on each copy of K under the isomorphism $V \cong K^n$ and hence leaves each copy invariant, (1.2.9) implies that I leaves each copy of $K \otimes_{\mathbb{Q}} \mathbb{R}$ invariant under the isomorphism $V_{\mathbb{R}} \cong (K \otimes_{\mathbb{Q}} \mathbb{R})^n$. By the simplicity of X we get $n = 1$. This is what we wanted to show. □

Remark 1.2.20. From the proof above, one sees that ϵ can be taken as β to define the Riemann form E in (1.2.5). Moreover, we could also conclude the proof above by pointing out that (1.2.9) means nothing but that the Hodge group $\mathrm{Hg}(X)$ is contained in K, which implies immediately that $\mathrm{Hg}(X)$ is commutative and hence X is of CM-type.

Now we give

The second proof of $\Leftarrow$ of Theorem 1.2.17. It suffices to reduce the problem to simple abelian varieties. Indeed, according to Poincaré's Complete Reducibility Theorem, any abelian variety X is isogenous to a product of simple abelian varieties. If G is a rational Kähler metric on X, then it is a such restricted on each of the simple factors. Hence every simple factor is of CM-type in view of the last lemma. This implies immediately that X is of CM-type. □

Now we give another interpretation of Theorem 1.2.17 in terms of the twistor space.

Corollary 1.2.21. *Let $\mathbb{T}$ be a real torus with a flat rational Riemannian metric G. Then with respect to any complex structure I which is compatible with G and provided that the complex torus $T := (\mathbb{T}, I)$ is algebraic, T is of CM-type.*

1.3. **Weight 2: K3 surfaces.**

In this section Proposition 1.3.14 and Theorem 1.3.18 provide an answer to the first part of the question (QK3) posed in Introduction. The question will be made clearer after introducing necessary notions. Roughly speaking, we want a similar statement as Corollary 1.2.21 by replacing $\mathbb{T}$ by the underlying differential manifold M of a K3 surface and the Riemannian metric by a three-dimensional positive definite subspace V of $H^2(M,\mathbb{R})$. The question is which conditions on V shall play the role of rationality.

In general, for weight 2 we don't have a such correspondence between Hodge structures and geometric objects as for weight 1. However K3 surfaces are to a large extent determined by the Hodge structure on their second cohomology $H^2(Y,\mathbb{Q})$ (although for mirror symmetry we need the whole cohomology as we will see in Section 2.4). We first review a few well-known facts about K3 surfaces.

By a *K3 surface* we mean a simply connected compact surface with trivial canonical bundle, i.e. $K_Y = \mathcal{O}_Y$ and the first Betti number $b_1(Y) = 0$. As immediate consequences, its cohomology stops at H^4; H^1 and H^3 are trivial (equal to 0); H^0 and H^4 are 1-dimensional; and H^2 is 22-dimensional. The whole cohomology is hence

$$H^*(Y,\mathbb{Q}) = H^0(Y,\mathbb{Q}) \oplus H^2(Y,\mathbb{Q}) \oplus H^4(Y,\mathbb{Q}).$$

The Hodge decomposition on $H^2(Y,\mathbb{Q})$ is due to

$$h^{2,0}(Y) = h^0(Y,K_Y) = h^0(Y,\mathcal{O}_Y) = 1$$

also clear:

$$H^2(Y,\mathbb{C}) = H^{2,0}(Y) \oplus H^{1,1}(Y) \oplus H^{0,2}(Y) = \mathbb{C}\sigma \oplus H^{1,1}(Y) \oplus \mathbb{C}\bar{\sigma}.$$

We extend this Hodge structure on the whole $H^*(Y,\mathbb{Q})$:

$$\begin{aligned} H^*(Y,\mathbb{C}) &= H^{2,0}(Y) \oplus (H^{1,1}(Y) \oplus H^0(Y,\mathbb{C}) \oplus H^4(Y,\mathbb{C})) \oplus H^{0,2}(Y) \\ &=: H^{2,0}(Y) \oplus \tilde{H}^{1,1}(Y) \oplus H^{0,2}(Y). \end{aligned}$$

In general $H^2(Y,\mathbb{Q})$ is not an irreducible Hodge structure. Indeed, put

$$NS(Y) := H^{1,1}(Y) \cap H^2(Y,\mathbb{Z}).$$

It is called the *Néron-Severi group of* Y . Tensored with $\mathbb{Q}$, it is

$$NS(Y)_{\mathbb{Q}} := NS(Y) \otimes_{\mathbb{Z}} \mathbb{Q} = H^{1,1}(Y) \cap H^2(Y,\mathbb{Q}).$$

Its rank $\rho(Y) := \operatorname{rk} NS(Y)$ is called the *Picard number of* Y . Clearly, $NS(Y)_{\mathbb{Q}} \oplus H^0(Y,\mathbb{Q}) \oplus H^4(Y,\mathbb{Q})$ carries a trivial Hodge substructure of weight 2 of $H^*(Y,\mathbb{Q})$ i.e. no (2,0)- and (0,2)-parts, hence the Hodge group is trivial (see Definition 1.1.4).

Besides the Hodge structure, we also have the Mukai pairing on $H^*(Y,\mathbb{Q})$. Consider the following product of $\alpha = \alpha_0 + \alpha_2 + \alpha_4$ and $\beta = \beta_0 + \beta_2 + \beta_4 \in H^*(Y,\mathbb{Z})$:

$$\alpha \wedge_M \beta := -\alpha_0 \wedge \beta_4 + \alpha_2 \wedge \beta_2 - \alpha_4 \wedge \beta_0 \in H^4(Y,\mathbb{Z}).$$

Let w be a generator of $H^4(Y,\mathbb{Z})$ with the orientation determined by $\sigma \wedge \bar{\sigma} = \lambda w, \lambda > 0$. Then the *Mukai pairing* $\langle , \rangle$ is defined by

$$\alpha \wedge_M \beta =: \langle \alpha, \beta \rangle w. \tag{1.3.1}$$

With respect to $\langle , \rangle$ we can consider the orthogonal complement T of $NS(Y)_{\mathbb{Q}} \oplus H^0(Y,\mathbb{Q}) \oplus H^4(Y,\mathbb{Q})$ in $H^*(Y,\mathbb{Q})$. It is called the *(rational) transcendental lattice.* Then T carries a Hodge substructure, namely

$$T_{\mathbb{C}} = T^{2,0} \oplus T^{1,1} \oplus T^{0,2}$$

with $T^{2,0} = H^{2,0}(Y) = \mathbb{C}\sigma$. From [Za] we know

Proposition 1.3.1. *The Hodge structure on the transcendental lattice T is irreducible.*

Proof. Let $0 \neq M \subset T$ be a Hodge substructure. Then

$$M^{2,0} = M_{\mathbb{C}} \cap T^{2,0}.$$

Either $M^{2,0} = 0$, i.e. $M \subset T^{1,1}$, this would imply that M also lies in $H^2(Y, \mathbb{Q})$, and hence in $NS(Y)_{\mathbb{Q}}$, which is impossible; or $M^{2,0} \neq 0$. But since $T^{2,0}$ is one-dimensional, we must have $M^{2,0} = T^{2,0}$. Then the orthogonal complement of M in T must lie in $T^{1,1}$, which is also impossible by the same argument as before. Hence $M = T$ and T is irreducible. □

We want to say that a K3 surface is of CM-type if the Hodge structure on T is of CM-type. In order to make sense of this definition we have to look at the Mukai pairing more closely to find out when T is polarizable.

From the definition of the Mukai pairing (1.3.1) we see that it restricts to the intersection form on $H^2(Y, \mathbb{Z})$, which we also denote by $\langle , \rangle$. By Poincaré duality, $H^2(Y, \mathbb{Z})$ is unimodular with respect to $\langle , \rangle$. Moreover, by [BV, Exposé IV], $\langle , \rangle$ is even (i.e. $\langle x, x\rangle \equiv 0 \mod 2, \forall x \in H^2(Y, \mathbb{Z})$). The classification of even unimodular indefinite lattices gives then an isometry

$$H^2(Y, \mathbb{Z}) \cong H^{\oplus 3} \oplus E_8(-1)^{\oplus 2},$$

where H is the hyperbolic plane and $E_8(-1)$ is the lattice E_8 with its intersection form multiplied by -1. The piece $H^0(Y, \mathbb{Z}) \oplus H^4(Y, \mathbb{Z})$ is orthogonal to $H^2(Y, \mathbb{Z})$ in $H^*(Y, \mathbb{Z})$, and the Mukai pairing makes it into $H(-1)$. We have then an isometry

$$H^*(Y, \mathbb{Z}) \cong H^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus H(-1).$$

The signature of $\langle , \rangle$ is (4,20). Because $\langle \sigma, \sigma\rangle = 0$ and $\langle \sigma, \bar{\sigma}\rangle > 0$, $\langle , \rangle$ is positive definite on the 2-dimensional real vector space

$$\mathbb{R}\,\mathrm{Re}\,\sigma \oplus \mathbb{R}\,\mathrm{Im}\,\sigma = (T^{2,0} \oplus T^{0,2}) \cap T_{\mathbb{R}}.$$

Now we show a well-known fact (see [Za]):

Proposition 1.3.2. *If a K3 surface Y is projective, then the Hodge structure on T is polarizable.*

Proof. The projectivity of Y is equivalent to the existence of a rational Kähler class, i.e. an element $x \in NS(Y)_{\mathbb{Q}}$ with in particular $\langle x, x\rangle > 0$. Then on T, $\langle , \rangle$ has signature $(2, n-2)$. It is positive definite on $(T^{2,0} \oplus T^{0,2}) \cap T_{\mathbb{R}}$ as we already mentioned, and hence negative definite on $T^{1,1} \cap T_{\mathbb{R}}$. Set

$$E(\cdot, \cdot) := -\langle \cdot, \cdot\rangle,$$

we have then

$$E(T^{p,q}, T^{r,s}) \equiv 0 \text{ unless } r = q \text{ and } s = p$$

due to the same property of $\langle , \rangle$. We also have for any $x^{p,q} \neq 0 \in T^{p,q}$

$$i^{-2}E(x^{2,0}, \overline{x^{2,0}}) = \langle x^{2,0}, \overline{x^{2,0}}\rangle > 0, \text{ and}$$
$$i^{-2}E(x^{1,1}, \overline{x^{1,1}}) = \langle x^{1,1}, \overline{x^{1,1}}\rangle > 0.$$

Therefore E is a polarization on T. □

So it makes sense to define

Definition 1.3.3. *Let Y be a projective K3 surface with transcendental lattice T. We say that Y is* of CM-type over (a CM-field) K *if the (irreducible) Hodge structure T is of CM-type over K.*

Henceforth by a K3 surface of CM-type we always mean a *projective* K3 surface. In view of Proposition 1.1.9 we have immediately (see [Bor])

Proposition 1.3.4 (Borcea)**.** *A K3 surface Y with transcendental lattice T is of CM-type if and only if $F := \operatorname{End}_{\mathrm{Hg}(T)} T$ is a CM-field and* $\dim_F T = 1$.

In [Za] Zarhin gives all the possible endomorphism algebras $F := \operatorname{End}_{\mathrm{Hg}(T)} T$. The situation here is simpler than for abelian varieties. Indeed, the irreducibility of the Hodge structure T implies that F is a division algebra (otherwise, the kernel or the image would be a non-trivial Hodge substructure). Moreover, $T^{2,0}$ induces an embedding:

$$\epsilon : F \hookrightarrow \operatorname{End}_{\mathbb{C}} T^{2,0} \cong \mathbb{C}.$$

Hence F is commutative and is a number field. The intersection form $\langle , \rangle$ induces a positive involution $'$ on F by adjunction, i.e. for $a \in F$:

$$\langle ax, y\rangle = \langle x, a'y\rangle, \quad \forall x, y \in T.$$

By Albert's classification given in Section 1.2, the pair $(F,')$ can be

(a) either a totally real number field with the identity as the trivial involution,
(b) or a CM-field with complex conjugation as the involution.

Note that F being a CM-field is not sufficient for T to be of CM-type (i.e. $\mathrm{Hg}(T)$ to be commutative), the condition $\dim_F T = 1$ is necessary.

Example 1.3.5. We show the well-known fact that any attractive K3 surface is of CM-type. A K3 surface Y is called *attractive* or sometimes *supersingular* if $\rho(Y) = 20$ (it is in particular projective). This condition implies that

$$T_{\mathbb{C}} = H^{2,0}(Y) \oplus H^{0,2}(Y)$$

and $\langle , \rangle$ is positive definite on T. We show that this decomposition is defined over a quadratic CM-field.

Let $\{e_1, e_2\}$ be an orthogonal $\mathbb{Q}$-basis of T. Let

$$\sigma := e_1 + \lambda_2 e_2, \quad \lambda_2 \in \mathbb{C} \setminus \mathbb{R}$$

be a basis vector of $H^{2,0}(Y)$. Then

$$0 = \langle \sigma, \sigma\rangle = \langle e_1, e_1\rangle + \lambda_2^2 \langle e_2, e_2\rangle$$
$$\Rightarrow \lambda_2^2 = -\frac{\langle e_1, e_1\rangle}{\langle e_2, e_2\rangle} < 0.$$

Hence $\mathbb{Q}(\lambda_2)$ is a CM-field and σ lies in $T \otimes_{\mathbb{Q}} \mathbb{Q}(\lambda_2)$. There is an action of $\mathbb{Q}(\lambda_2)$ on T as follows:

$$\lambda_2 : T \longrightarrow T, \quad e_1 \longmapsto \lambda_2^2 e_2, \quad e_2 \longmapsto e_1.$$

Then

$$\lambda_2(\sigma) = \lambda_2^2 e_2 + \lambda_2 e_1 = \lambda_2(e_1 + \lambda_2 e_2) = \lambda_2 \sigma.$$

So λ_2 preserves $H^{2,0}(Y)$. Hence

$$\mathbb{Q}(\lambda_2) = \operatorname{End}_{\mathrm{Hg}(T)} T \quad \text{and} \quad \dim_{\mathbb{Q}(\lambda_2)} T = 1.$$

This shows that Y is of CM-type over $\mathbb{Q}(\lambda_2)$. Note the arithmetic property of the coefficients $\{1, \lambda_2\}$ of σ. They generate the CM-field $\mathbb{Q}(\lambda_2)$ as a $\mathbb{Q}$-vector space. This generalizes to K3 surfaces of CM-type with any Picard number. Proposition 1.3.14 gives the exact statement.

Example 1.3.6. We show that the Kummer surface $Y := \mathrm{Km}(X)$ associated to an abelian surface X of CM-type is of CM-type. As well known (see [BPV, Chap.VIII §5]), there is an inclusion of Hodge structure of weight 2

$$\iota : H^2(X,\mathbb{Q}) \hookrightarrow H^2(Y,\mathbb{Q}).$$

The Hodge structure on $H^2(X,\mathbb{Q})$ is induced by the one of $H^1(X,\mathbb{Q})$ as follows. If $h_0 : \mathbb{S}^1 \to SL(H^1(X,\mathbb{R}))$ is the Hodge structure on $H^1(X,\mathbb{R})$, then the Hodge structure on $H^2(X,\mathbb{Q})$ is

$$\begin{aligned} h : \mathbb{S}^1 &\longrightarrow SL(H^2(X,\mathbb{R})) \\ z &\longmapsto h(z) := h_0(z) \wedge h_0(z). \end{aligned}$$

Write

$$T(X) := (H^{1,1}(X) \cap H^2(X,\mathbb{Q}))^{\perp} \quad \text{and} \quad T(Y) := NS(Y)^{\perp}_{\mathbb{Q}},$$

then ι restricts to a Hodge isomorphism

$$\iota : T(X) \xrightarrow{\sim} T(Y)$$

(though not an isometry). So we have

$$\mathrm{Hg}(T(X)) \cong \mathrm{Hg}(T(Y)).$$

Since the Hodge group fixes the algebraic part, we have

$$\mathrm{Hg}(T(X)) \cong \mathrm{Hg}(H^2(X,\mathbb{Q})).$$

By what we said earlier about the Hodge structure on $H^2(X,\mathbb{Q})$, we have an inclusion

$$\mathrm{Hg}(H^2(X,\mathbb{Q})) \subset \mathrm{Hg}(H^1(X,\mathbb{Q})) \times \mathrm{Hg}(H^1(X,\mathbb{Q})).$$

Hence the commutativity on the right hand side implies that $\mathrm{Hg}(H^2(X,\mathbb{Q}))$, $\mathrm{Hg}(T(X))$ and hence $\mathrm{Hg}(T(Y))$ are commutative. This shows the claim.

Example 1.3.7. The Kummer surface associated to the product of two elliptic curves of CM-type over the same CM-field $\mathbb{Q}(\sqrt{-D})$ is attractive and of CM-type over $\mathbb{Q}(\sqrt{-D})$.

The Kummer surface associated to the product of two elliptic curves of CM-type over two different CM-fields $\mathbb{Q}(\sqrt{-D_1})$ respectively $\mathbb{Q}(\sqrt{-D_2})$ is of CM-type over the compositum of these two fields which is again a CM-field. It has Picard number 18.

More generally, the Kummer surface Y associated to any abelian surface X of CM-type has Picard number either $\rho(Y) = 18$ or $\rho(Y) = 20$. The reason is that if X is of CM-type, i.e. the Hodge group $\mathrm{Hg}(H^1(X,\mathbb{Q}))$ is commutative, then as $H^2(X,\mathbb{Q}) = H^1(X,\mathbb{Q}) \wedge H^1(X,\mathbb{Q})$ it follows that $\mathrm{Hg}(H^2(X,\mathbb{Q}))$ is also commutative. If we denote $N := H^{1,1}(X) \cap H^2(X,\mathbb{Q})$ and by U the orthogonal complement of N in $H^2(X,\mathbb{Q})$, then U carries an irreducible (using the same arguments as in Proposition 1.3.1) Hodge structure of CM-type over some CM-field, and hence its dimension is an even number. So the Picard number $\rho(X) = \dim U$ can be either 2 or 4, and for the K3 surface either $\rho(Y) = 18$ or $\rho(Y) = 20$.

In parallel with the construction of abelian varieties of CM-type (see Proposition 1.2.10) we present a construction of a K3 surface of CM-type over a given CM-field of degree ≤ 16 (see [PS]). The reason for the constrain on the degree will be clear later. For the construction we have to state the surjectivity of the period map.

As we already mentioned, $H^2(Y,\mathbb{Z})$ of a K3 surface Y is isometric to the lattice $L := H^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$, but not canonically. Let

$$\varphi : H^2(Y,\mathbb{Z}) \xrightarrow{\sim} L$$

be an isometry. We call a pair (Y,φ) a *marked K3 surface*. After $\mathbb{C}$-linear extension, the image of $H^{2,0}(Y)$ under $\varphi_{\mathbb{C}}$ is a complex line in $L_{\mathbb{C}}$. In the projectivized space $\mathbb{P}(L_{\mathbb{C}}), \alpha := \varphi_{\mathbb{C}}(H^{2,0}(Y))$ is then a point $[\alpha] \in \mathbb{P}(L_{\mathbb{C}})$. We call $[\alpha]$ (and by abuse of language α) the *period* of Y. Morally, the map

$$(Y,\varphi) \longmapsto [\alpha] = [\varphi_{\mathbb{C}}(H^{2,0}(Y)]$$

is called the *period map*. For a more precise definition see [BV] or [BPV]. Denote by Ω the period domain, i.e.

$$\Omega := \{[x] \in \mathbb{P}(L_{\mathbb{C}}) \mid \langle x,x\rangle = 0 \text{ and } \langle x,\bar{x}\rangle > 0\}.$$

Obviously, the period $[\alpha]$ of any marked K3 surface is contained in Ω. The following theorem establishes the surjectivity of the period map (see [BPV, Chap.VIII, Cor. 14.2]).

Theorem 1.3.8 (Surjectivity of the period map)**.** *Every point of Ω occurs as the period of a marked K3 surface.*

Now we give the construction of K3 surfaces of CM-type (see [PS]).

Proposition 1.3.9 (Pjateckii-Shapiro, Shafarevich)**.** *For any CM-field K of degree ≤ 16 there is a K3 surface of CM-type over K.*

Proof. Let K be a CM-field of degree n. Denote by S the set of embeddings of K. The choice of an embedding ϵ determines a Hodge structure of weight 2 on K:

$$K_{\mathbb{C}} = \bigoplus_{\rho\in S} K^{\rho} = K^{\epsilon} \oplus \bigoplus_{\rho\in S\setminus\{\epsilon,\bar{\epsilon}\}} K^{\rho} \oplus K^{\bar{\epsilon}} =: K^{2,0} \oplus K^{1,1} \oplus K^{0,2}. \tag{1.3.2}$$

One can endow K with a bilinear form of signature $(2, n-2)$. Indeed, by the Approximation Theorem one finds an element $\lambda \in K_0$ with $\epsilon(\lambda) > 0$ and $\rho(\lambda) < 0$ for any other embedding $\rho : K_0 \hookrightarrow \mathbb{R}$. Define for any $a, b \in K$:

$$\langle a,b\rangle := \mathrm{Tr}(\lambda a\bar{b}) = \sum_{\rho\in S} \rho(\lambda a\bar{b}) \in \mathbb{Q}. \tag{1.3.3}$$

Then $\langle,\rangle$ is positive definite on the two dimensional real subspace

$$(K^{\epsilon} \oplus K^{\bar{\epsilon}}) \cap K_{\mathbb{R}}$$

and negative definite on the orthogonal complement. Hence the signature is $(2, n-2)$. It is also even and $-\langle,\rangle$ is a polarization. Because $K^{2,0}$ is one-dimensional this Hodge structure is irreducible and of CM-type by Proposition 1.1.9.

It still remains to show that there is a K3 surface with such K as its transcendental lattice. As well known, every indefinite rational quadratic form of rank ≥ 5 represents 0, and hence any rational number. So if $\dim_{\mathbb{Q}} K \leq 16$, one can embed K into $L_{\mathbb{Q}}$. Let N be the orthogonal complement of K in $L_{\mathbb{Q}}$. Now let $\sigma \neq 0$ be a generator of K^{ϵ}, then

$$\langle\sigma,\sigma\rangle = 2\epsilon(\lambda)\epsilon(\sigma)\bar{\epsilon}(\sigma) = 0, \text{ and}$$
$$\langle\sigma,\bar{\sigma}\rangle = 2\epsilon(\lambda)\epsilon(\sigma)\bar{\epsilon}(\bar{\sigma}) > 0.$$

Hence $[\sigma]$ is a point in the period domain Ω. By the surjectivity of the period map there is a K3 surface with period σ, $NS(Y) = N \cap L$ and $T \cong K$. It is of CM-type by construction. This ends the proof. $\square$

There is also a notion of isogeny for K3 surfaces. See [Mk1, Def.1.7] for Mukai's definition. For us, we say that two K3 surfaces S and T are *isogenous* if there is a Hodge isometry between $H^2(S,\mathbb{Q})$ and $H^2(T,\mathbb{Q})$. Theorem 2 in [Mk2] claims the equivalence of these two definitions. We show

Proposition 1.3.10. *Any K3 surface of CM-type is isogenous to one of the type constructed in the proof of Proposition 1.3.9.*

Proof. Let Y be a K3 surface of CM-type with transcendental lattice T. As shown in Proposition 1.3.4, $K := \mathrm{End}_{\mathrm{Hg}(T)} T$ is a CM-field and $\dim_K T = 1$. We show that there is a $\lambda \in K_0$ such that the intersection form $\langle , \rangle$ on T is given by (1.3.3).

By the non-degeneracy of $\mathrm{Tr}_{K|\mathbb{Q}}$ we have a $\mathbb{Q}$-bilinear form

$$\Phi : T \times T \longrightarrow K$$
$$(x,y) \longmapsto \Phi(x,y)$$

such that the intersection form $\langle , \rangle$ on T is

$$\langle ex, y\rangle = \mathrm{Tr}_{K|\mathbb{Q}}(e\Phi(x,y)).$$

Zarhin already used the map Φ in [Za, p.210]. Let $T = Kv_0$ be any isomorphism and set

$$\lambda := \Phi(v_0, v_0) \in K.$$

Then for any $x = av_0, y = bv_0 \in T$ we have

$$\langle x, y\rangle = \langle av_0, bv_0\rangle = \langle a\bar{b}v_0, v_0\rangle = \mathrm{Tr}_{K|\mathbb{Q}}(\lambda a\bar{b}).$$

Due to the symmetry of $\langle , \rangle$ we have

$$\mathrm{Tr}_{K|\mathbb{Q}}(\lambda a\bar{b}) = \mathrm{Tr}_{K|\mathbb{Q}}(\lambda \bar{a}b), \quad \forall a, b \in K.$$

The non-degeneracy of the trace form implies then that $\bar{\lambda} = \lambda$, in other words, $\lambda \in K_0$.

Further, let $\epsilon : K \hookrightarrow \mathbb{C}$ be the embedding of K induced by σ, and if $\{e_i\}$ is an orthogonal basis of T, we write

$$\sigma = \sum \lambda_i e_i = \sum \lambda_i \mu_i v_0, \quad \lambda_i \in \mathbb{C}, \mu_i \in K.$$

We have on the one hand

$$\langle \lambda\sigma, \bar{\sigma}\rangle = \sum_i \lambda_i\bar{\lambda}_i\langle \lambda e_i, e_i\rangle = \sum_i \lambda_i\bar{\lambda}_i\langle \lambda\mu_i\bar{\mu}_i v_0, v_0\rangle = \sum \lambda_i\bar{\lambda}_i \,\mathrm{Tr}_{K|\mathbb{Q}}(\lambda^2\mu_i\bar{\mu}_i) > 0.$$

The last inequality is due to $\lambda \in K_0$ is totally real, hence λ^2 is totally positive, so is $\lambda^2\mu_i\bar{\mu}_i$. On the other hand we have

$$\langle \lambda\sigma, \bar{\sigma}\rangle = \epsilon(\lambda)\langle \sigma, \bar{\sigma}\rangle.$$

We see that $\epsilon(\lambda) > 0$.

Now let $\rho : K \hookrightarrow \mathbb{C}$ be any other embedding. We show $\rho(\lambda) < 0$. Indeed, we have $T^{1,1} = \bigoplus_{\rho \in S\setminus\{\epsilon,\bar{\epsilon}\}} K^\rho$, and $\langle , \rangle$ is negative definite on $T^{1,1} \cap T_{\mathbb{R}}$. Let v^ρ be a generator of K^ρ. We can also write v^ρ as a linear combination of $\{e_i\}$:

$$v^\rho = \sum \alpha_i e_i =: \sum \alpha_i \mu_i v_0, \quad \alpha_i \in \mathbb{C}, \mu_i \in K.$$

Then we have on the one hand

$$\langle \lambda v^\rho, \overline{v^\rho}\rangle = \sum \alpha_i\bar{\alpha}_i\langle \lambda e_i, e_i\rangle = \sum \alpha_i\bar{\alpha}_i\langle \lambda\mu_i\bar{\mu}_i v_0, v_0\rangle = \sum \alpha_i\bar{\alpha}_i \,\mathrm{Tr}_{K|\mathbb{Q}}(\lambda^2\mu_i\bar{\mu}_i) > 0$$

and on the other hand we have

$$\langle \lambda v^\rho, \overline{v^\rho}\rangle = \rho(\lambda) \underbrace{\langle v^\rho, \overline{v^\rho}\rangle}_{<0}.$$

Hence $\rho(\lambda) < 0$ for any embedding $\rho \neq \epsilon$.

Now it is clear that $(T, \langle , \rangle)$ is Hodge isometric to K as a $\mathbb{Q}$-vector space endowed with the bilinear form

$$\begin{aligned} \langle , \rangle_K : K \times K &\longrightarrow \mathbb{Q} \\ (a, b) &\longmapsto \mathrm{Tr}_{K|\mathbb{Q}}(\lambda a \bar{b}) \end{aligned}$$

and the decomposition

$$K_{\mathbb{C}} = K^\epsilon \oplus \bigoplus_{\rho \in S \setminus \{\epsilon, \bar\epsilon\}} K^\rho \oplus K^{\bar\epsilon}.$$

So $(K, \langle , \rangle_K)$ together with $NS(Y)$ defines a K3 surface Y' of CM-type. By construction, $H^2(Y, \mathbb{Q})$ and $H^2(Y', \mathbb{Q})$ are Hodge isometric, hence Y and Y' are isogenous. This completes the proof. □

In order to answer the question (QK3) we try to characterize complex multiplication. We first make a tentative with the elementary methods used in Lemma 1.2.18 and the second proof of Theorem 1.2.17. We showed that on abelian varieties, complex multiplication is equivalent to the existence of a constant rational Kähler metric G. Making abstraction of the geometric meaning of G, it is simply a positive definite rational bilinear form whose hermitian extension satisfies

$$G(V^{p,q}, V^{r,s}) \equiv 0 \quad \text{unless} \quad p = r, q = s,$$

where $p, q, r, s = 0$ or 1. We give the K3 analogue, though it seems to be very formal and offers no geometric interpretations.

Proposition 1.3.11. *A K3 surface Y is of CM-type if and only if the following conditions are simultaneously satisfied:*

(i) *There exists a rational symplectic form ω on T, with*

$$\omega(T^{p,q}, T^{r,s}) \equiv 0, \textit{ unless } r = q \textit{ and } s = p, \tag{1.3.4}$$

after $\mathbb{C}$-bilinear extension.

(ii) *There exists a rational positive definite symmetric bilinear form G on T, such that it extends to a positive definite Hermitian form which is orthogonal to the Hodge decomposition, i.e.*

$$G(T^{p,q}, T^{r,s}) \equiv 0, \textit{ unless } r = p \textit{ and } s = q. \tag{1.3.5}$$

Proof. Let Y be a K3 surface of CM-type over K. Write $K = K_0(\eta)$ with $\eta^2 \in K_0$ totally negative (see Lemma 1.1.2). Define on T:

$$\omega(v, w) := \langle v, \eta w\rangle.$$

We have

$$\omega(v, w) = \langle v, \eta w\rangle = \langle \bar\eta v, w\rangle = -\langle \eta v, w\rangle = -\langle w, \eta v\rangle = -\omega(w, v),$$

hence ω is symplectic. Then (1.3.4) follows directly from the fact that $\eta \in K$, hence preserves the Hodge decomposition.

Further, denote the embedding

$$\epsilon : K \hookrightarrow \mathrm{End}_{\mathbb{C}}\, T^{2,0} \cong \mathbb{C}.$$

By the Approximation Theorem, there is an element λ in K_0 with $\epsilon(\lambda) > 0$, and $\rho(\lambda) < 0$ for any other embedding ρ of K_0. Define on T:

$$G(v, w) := \langle v, \lambda w \rangle.$$

Obviously, it is rational and symmetric (due to $\bar{\lambda} = \lambda$). It is also positive definite, because for any $v \neq 0 \in T$ we have

$$\begin{aligned} G(v, v) &= \langle v, \lambda v \rangle \\ &= \langle v^{2,0}, \lambda v^{0,2} \rangle + \langle v^{1,1}, \lambda v^{1,1} \rangle + \langle v^{0,2}, \lambda v^{2,0} \rangle \\ &= \epsilon(\lambda) \langle v^{2,0}, v^{0,2} \rangle + \langle v^{1,1}, \lambda v^{1,1} \rangle + \bar{\epsilon}(\lambda) \langle v^{0,2}, v^{2,0} \rangle. \end{aligned}$$

The first and the third terms are positive as $\bar{\epsilon}(\lambda) = \epsilon(\lambda) > 0$. The second term is also positive because by its choice, λ has only negative eigenvalues on $T^{1,1} \cap T_{\mathbb{R}}$ and $\langle , \rangle$ is negative definite on it. Hence G is positive definite. Extended hermitianly, G satisfies (1.3.5).

For the converse, denote $F := \operatorname{End}_{\mathrm{Hg}(T)} T$. Recall that F can either be a totally real number field and the involution induced by $\langle , \rangle$ is trivially the identity, or, F is a CM-field with complex conjugation as involution. We use ω to show that F is a CM-field and use G to show $\dim_F T = 1$. First note that each of ω and G induces an endomorphism of T as follows:

$$\omega(v, w) =: \langle v, \eta w \rangle \quad \text{and} \quad G(v, w) =: \langle v, \lambda w \rangle.$$

We claim that both η and λ lie in F. Indeed, in view of (1.3.4), we have for all $v^{1,1} \in T^{1,1}$ and $w^{2,0} \in T^{2,0}$:

$$0 = \omega(v^{1,1}, w^{2,0}) = \langle v^{1,1}, \eta w^{2,0} \rangle.$$

Hence $\eta T^{2,0} \subset T^{2,0} \oplus T^{0,2}$. On the other hand, for all $v^{2,0} \in T^{2,0}$,

$$0 = \omega(v^{2,0}, w^{2,0}) = \langle v^{2,0}, \eta w^{2,0} \rangle.$$

Hence $\eta T^{2,0} = T^{2,0}$. Similarly one shows that η preserves all other components of the Hodge decomposition. Hence η commutes with $\mathrm{Hg}(Y)$ and $\eta \in F$ by Proposition 1.1.8. That $\lambda \in F$ can be shown analogously.

Further, the anti-symmetry of ω implies that $\bar{\eta} = -\eta$, therefore F is a CM-field. Write $F = F_0(\xi)$, where F_0 is a totally real field with $[F : F_0] = 2$. The symmetry of G implies that $\lambda \in F_0$. Next we use λ to show $\dim_F T = 1$.

Let us write $T = \bigoplus_{i=1}^m F$. Under this isomorphism, F acts on each copy of F by left multiplication. Denote by Φ the set of embeddings of F_0 into $\mathbb{R}$, we have the decompositions

$$T = \bigoplus_{i=1}^{2m} F_0 \quad \text{and} \quad T_{\mathbb{R}} = \bigoplus_{\rho \in \Phi} \bigoplus_{i=1}^{2m} T^{\rho},$$

where $\bigoplus_{i=1}^{2m} T^{\rho}$ is the eigenspace of any $p \in F_0$ to the eigenvalue $\rho(p)$. The latter decomposition is orthogonal to $\langle , \rangle$, since for any $p \in F_0$ we have

$$\rho'(p) \langle v^{\rho}, w^{\rho'} \rangle = \langle v^{\rho}, p(w^{\rho'}) \rangle = \langle p(v^{\rho}), w^{\rho'} \rangle = \rho(p) \langle v^{\rho}, w^{\rho'} \rangle.$$

It follows that $\langle , \rangle$ and λ are simultaneously diagonalizable. Recall that $\langle , \rangle$ has signature $(2, n-2)$, where $n = \dim_{\mathbb{Q}} T$, and $G(\cdot, \cdot) = \langle \cdot, \lambda \cdot \rangle$ is positive definite. We see that λ must have exactly 2 positive and $n - 2$ negative eigenvalues, which implies that $m = 1$. This concludes the proof. $\square$

Unfortunately, we could not discern any geometric interpretation of ω and G. Now we turn to the period of a K3 surface of CM-type to gain more geometric insight on complex multiplication. We find an arithmetic property, for that we need some new terminology.

Definition 1.3.12. *Let V be a finite dimensional $\mathbb{Q}$-vector space and $v \in V_{\mathbb{C}}$.*

(i) *Denote by $E(v)$ the smallest subspace V' of V with $v \in V'_{\mathbb{C}}$.*

(ii) *We call the* width $w(v)$ *of v the dimension of $E(v)$, i.e.*

$$w(v) := \dim_{\mathbb{Q}} E(v).$$

(iii) *The* field of definition *of v is the smallest field $F \subset \mathbb{C}$ with $v \in V'_F$.*

(iv) *We say that v is* fully defined over *a number field K under an embedding $\epsilon : K \hookrightarrow \mathbb{C}$ if*

  (a) *$F = \epsilon(K)$ where F is the field of definition of v, and*

  (b) *$w(v) = [K : \mathbb{Q}]$.*

(v) *Let V be additionally endowed with a quadratic form $\langle , \rangle$ and $v_1, v_2 \in V_{\mathbb{C}}$ be two vectors. If $E(v_1) \cap E(v_2) = 0$ and $\langle E(v_1), E(v_2)\rangle \equiv 0$, then we write*

$$E(v_1) \perp E(v_2).$$

The width of a vector can be understood as a measure of its non-rationality. For example, any vector $u \neq 0 \in V$ has width 1, $E(u) = \mathbb{Q}u$, and is fully defined over $\mathbb{Q}$. For more general vectors we give the following explicit description.

Lemma 1.3.13. *Let V be a finite dimensional $\mathbb{Q}$-vector space. A vector $v \in V_{\mathbb{C}}$ is fully defined over a number field K under an embedding $\epsilon : K \hookrightarrow \mathbb{C}$ if and only if v can be written in the form*

$$v = \lambda_1 e_1 + \cdots + \lambda_n e_n, \tag{1.3.6}$$

where $\{e_i\} \subset V$ are $\mathbb{Q}$-linearly independent vectors and $\{\lambda_i\} \subset \mathbb{C}$ are $\mathbb{Q}$-linearly independent numbers which generate $\epsilon(K)$ as a $\mathbb{Q}$-vector space. In this case, we have

$$E(v) = \mathbb{Q}e_1 \oplus \cdots \oplus \mathbb{Q}e_n.$$

Proof. "$\Leftarrow$": It is clear.

"$\Rightarrow$": Denote $n := w(v)$. Let $\{e_i\}_{i=1,\dots,n}$ be a basis of $E(v)$, since $v \in E(v)_{\mathbb{C}}$, we can write

$$v = \lambda_1 e_1 + \cdots + \lambda_n e_n, \quad \lambda_i \in \mathbb{C}.$$

The coefficients $\{\lambda_i\}$ must be $\mathbb{Q}$-linearly independent, otherwise we can group the e_i's and $w(v)$ would be smaller. The field of definition of v is $F = \epsilon(K)$, and has degree n. It follows that $\{\lambda_i\}$ generate F as a $\mathbb{Q}$-vector space. □

Now we give a characterization of, or more exactly a necessary condition for complex multiplication by the period.

Proposition 1.3.14. *Let Y be a K3 surface of CM-type over K, then there is an element $\sigma \in H^{2,0}(Y)$ which is fully defined over K under an embedding $\epsilon : K \hookrightarrow \mathbb{C}$.*

Proof. Let $\{e_i\}_{i=1,\dots,n}$ be a $\mathbb{Q}$-basis of T, then one can write a basis of $H^{2,0}(Y)$ as

$$\sigma = \lambda_1 e_1 + \cdots + \lambda_n e_n, \quad \lambda_i \in \mathbb{C}.$$

We can norm $\lambda_1 = 1$. Write further $\epsilon : K \hookrightarrow \mathbb{C}$ for the embedding induced by σ. As $K = \mathrm{End}_{\mathrm{Hg}(T)} T$ and for any $a \in K$ we have on the one hand

$$a(\sigma) = a(\sum_i \lambda_i e_i) = \sum_i \lambda_i (\sum_j a_{ji} e_j) = \sum_j (\sum_i \lambda_i a_{ji}) e_j,$$

and on the other hand

$$a(\sigma) = \epsilon(a)\sigma = \epsilon(a)\sum_j \lambda_j e_j.$$

Comparing the coefficients of e_1 we obtain the equality

$$\epsilon(a) = \sum_i \lambda_i a_{1i}.$$

This means that any element $\epsilon(a) \in \epsilon(K)$ can be obtained as a $\mathbb{Q}$-linear combination of $\{\lambda_i\}$. That $\{\lambda_i\}$ form moreover a minimal family can be seen from $E(\sigma) = T$ and $w(\sigma) = \dim_{\mathbb{Q}} T$. The claim follows then from Lemma 1.3.13. □

One could ask whether being fully defined over a CM-field is a sufficient property for the period σ to imply that the K3 surface is of CM-type. The answer is negative. Indeed, σ possesses other necessary properties. We explain this.

In any basis $\{e_i\}$ of T one can write $\sigma = \sum_i \lambda_i e_i$ for some $\{\lambda_i\} \subset \epsilon(K)$. While σ does not depend on the choice of the basis, its coefficients $\{\lambda_i\}$ are uniquely determined by $\{e_i\}$. In the following proposition we choose a particular basis in order to determine the relationship between $\{\lambda_i\}$ and $\{e_i\}$. In the arguments we use the proof of Proposition 1.3.10.

Proposition 1.3.15. *Let Y be a K3 surface of CM-type over K and $\{e_i\}$ be an orthogonal basis of the transcendental lattice T with respect to $\langle,\rangle$. Let $\sigma = \sum_i \lambda_i e_i \in H^{2,0}(Y)$ with $\lambda_1 = 1$ be fully defined over K under ϵ as claimed in Proposition 1.3.14. Under the isomorphism $T \xrightarrow{\sim} K, e_1 \mapsto 1$, let us write $e_i = \mu_i e_1$ for a $\mu_i \in K$. Let further $\lambda = \Phi(e_1, e_1)$ be as in the proof of Proposition 1.3.10. Then $\{\mu_i\}$ satisfy*

$$\mathrm{Tr}_{K/\mathbb{Q}}(\lambda \mu_i \bar{\mu}_j) = \langle e_i, e_i\rangle \delta_{ij}. \tag{1.3.7}$$

Moreover, the coefficients $\{\lambda_i\}$ in σ are

$$\lambda_i = \frac{\langle e_1, e_1\rangle}{\langle e_i, e_i\rangle}\epsilon(\bar{\mu}_i). \tag{1.3.8}$$

Proof. Since $\{e_i\}$ is an orthogonal basis we have

$$\langle e_i, e_i\rangle \delta_{ij} = \langle e_i, e_j\rangle = \langle \mu_i e_1, \mu_j e_1\rangle = \langle \mu_i \bar{\mu}_j e_1, e_1\rangle = \mathrm{Tr}_{K/\mathbb{Q}}(\lambda \mu_i \bar{\mu}_j).$$

We refer to the proof of Proposition 1.3.10 for the last equality. This establishes the equality (1.3.7).

As to the coefficients $\{\lambda_i\}$ notice that on the one hand

$$\langle \sigma, e_i\rangle = \langle \sum_j \lambda_j e_j, e_i\rangle = \lambda_i \langle e_i, e_i\rangle,$$

and on the other hand

$$\langle \sigma, e_i\rangle = \langle \sigma, \mu_i e_1\rangle = \langle \bar{\mu}_i \sigma, e_1\rangle = \epsilon(\bar{\mu}_i)\langle \sigma, e_1\rangle = \epsilon(\bar{\mu}_i)\langle e_1, e_1\rangle.$$

Hence

$$\lambda_i = \frac{\langle e_1, e_1\rangle}{\langle e_i, e_i\rangle}\epsilon(\bar{\mu}_i).$$

□

We see that for the K3 surface to be of CM-type, the conditions on σ are stronger than just being fully defined over a CM-field.

Another consequence of Propositions 1.3.14 and 1.3.15 is that we can further decompose T in an orthogonal way according to the arithmetic properties of σ we just found. This is a preparation for Theorem 1.3.18 later.

Corollary 1.3.16. *Let Y, σ, K and ϵ be as in Proposition 1.3.14. Write $K = K_0(\eta)$ with $\eta^2 \in K_0$ totally negative. Then*

(i) $\operatorname{Re}\sigma$ *and* $\frac{i}{\epsilon(\eta)}\operatorname{Im}\sigma$ *are fully defined over* K_0.

(ii) *In the notation of Definition 1.3.12 we have an orthogonal decomposition*

$$T = E(\operatorname{Re}\sigma) \oplus E(\operatorname{Im}\sigma).$$

Proof. Since Y is of CM-type, for any $0 \neq v_0 \in T$ there is an isomorphism $\varphi : T \xrightarrow{\sim} K, v_0 \mapsto 1$. Let $\lambda = \Phi(v_0, v_0)$ as in the proof of Proposition 1.3.10. We saw there that the intersection form $\langle , \rangle$ on T is given by

$$\langle x, y\rangle = \langle av_0, bv_0\rangle = \operatorname{Tr}_{K/\mathbb{Q}}(\lambda a\bar{b}) \quad \text{for any } x = av_0, y = bv_0 \in T.$$

Write $K = K_0 \oplus \eta K_0$. We claim that the decomposition

$$T = \varphi^{-1}(K_0) \oplus \varphi^{-1}(\eta K_0) \tag{1.3.9}$$

is orthogonal with respect to $\langle , \rangle$. Under the isomorphism φ it suffices to show that $K = K_0 \oplus \eta K_0$ is orthogonal with respect to the bilinear form

$$(,) : K \times K \longrightarrow \mathbb{Q}, \quad (a, b) \longmapsto \operatorname{Tr}_{K/\mathbb{Q}}(\lambda a\bar{b}).$$

Denote by S a CM-type of K. We have for any $\alpha \in K_0$ and $\eta\beta \in \eta K_0$

$$\begin{aligned}(\alpha, \eta\beta) &= \operatorname{Tr}_{K/\mathbb{Q}}(\lambda\alpha\bar{\eta}\bar{\beta}) = \operatorname{Tr}_{K/\mathbb{Q}}(\lambda\alpha\beta\bar{\eta}) \\ &= \sum_{\rho\in S\cup\bar{S}} \rho(\lambda\alpha\beta\bar{\eta}) = \sum_{\rho\in S} \rho(\lambda\alpha\beta\bar{\eta}) + \bar{\rho}(\lambda\alpha\beta\bar{\eta}) \\ &= \sum_{\rho\in S} \rho(\lambda\alpha\beta)(\underbrace{\rho(\bar{\eta}) + \bar{\rho}(\bar{\eta})}_{=0}) = 0.\end{aligned}$$

Hence the decomposition (1.3.9) is orthogonal.

Now let $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_n\}$ be an orthogonal basis of $\varphi^{-1}(K_0)$ respectively $\varphi^{-1}(\eta K_0)$. Here we let e_1 play the rôle of v_0 above. Under the isomorphism φ we have

$$e_i = \mu_i e_1 \quad \text{and} \quad f_i = \nu_i e_1 \qquad \text{for some } \mu_i \in K_0 \text{ and } \nu_i \in \eta K_0.$$

While writing $\sigma = \sum_{i=1}^n \lambda_i e_i + \sum_{i=1}^n \gamma_i f_i$ we see in view of (1.3.8) that

$$\lambda_i \in K_0 \quad \text{and} \quad \gamma_i \in \eta K_0.$$

It follows that

$$\operatorname{Re}\sigma = \sum_{i=1}^n \lambda_i e_i \quad \text{and} \quad \operatorname{Im}\sigma = \sum_{i=1}^n \gamma_i f_i,$$
$$E(\operatorname{Re}\sigma) \cong K_0 \quad \text{and} \quad E(\operatorname{Im}\sigma) \cong \eta K_0.$$

This shows at once the claims (i) and (ii). □

Let us now give a more precise meaning to the first part of the question (QK3) posed in Introduction. In contrast with abelian varieties, there is no easy arithmetic properties one can impose on the tangent bundle of a K3 surface. This makes a definition of a rational Kähler metric inappropriate and we can hardly make geometric sense of the structures we found in Proposition 1.3.11. We remedy this problem by considering a positive definite 3-dimensional space $V \subset L_{\mathbb{R}}$, where $L := H^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. The intersection

$$\mathbb{P}(V_{\mathbb{C}}) \cap \Omega = \{[\sigma] \in \mathbb{P}(V_{\mathbb{C}}) \mid \langle \sigma, \sigma \rangle = 0, \langle \sigma, \bar{\sigma} \rangle > 0\}$$

of the projectivized space $\mathbb{P}(V_{\mathbb{C}})$ and the period domain is geometrically an open subset of a quadric in $\mathbb{P}^2$ and is called the *twistor space* associated to V. It is the K3-analog of the twistor space for abelian varieties mentioned in Corollary 1.2.21. One can consider V as a "Riemannian metric" on the underlying manifold M of a K3 surface (all K3 surfaces are diffeomorphic). A point $[\sigma]$ in the twistor space endows M with a complex structure and makes it into a K3 surface Y and V can be written as

$$V = \mathbb{R}\,\mathrm{Re}\,\sigma \oplus \mathbb{R}\,\mathrm{Im}\,\sigma \oplus \mathbb{R}\omega,$$

where ω is a real $(1,1)$-form on Y with $\langle \omega, \omega \rangle > 0$. It is a symplectic structure on Y. Thus V contains information as well about the complex structure of Y (carried by σ) as about a symplectic structure on Y (carried by ω). This resembles a Kähler metric G on a complex manifold as it is defined by $G(\cdot, \cdot) := \omega(I\cdot, \cdot)$, where I is the complex structure and ω is a Kähler form. In this sense V shall play the role of a "Kähler metric" on Y.

The following theorem tends to answer the first part of the question (QK3). It gives conditions on V such that a point $[\sigma]$ in the twistor space, provided that it defines a *projective* K3 surface, is fully defined over a CM-field. In Proposition 1.3.14 we showed that if a K3 surface is of CM-type, then its period is fully defined over a CM-field, but as we already mentioned, the converse does not hold. Since an exact characterization of complex multiplication by the period is missing, our statement in the next theorem is weaker in the sense that the conditions on V are not sufficient to characterize K3 surfaces of CM-type. In the other direction we can only show that K3 surfaces of CM-type with high Picard number ≥ 10 possess such V. First a technical lemma.

Lemma 1.3.17. *Let V be a finite dimensional $\mathbb{Q}$-vector space endowed with a non-degenerate rational quadratic form. Let $v_1, v_2 \in V_{\mathbb{C}}$ be fully defined over a number field K under an embedding ϵ. Suppose $E(v_1) \perp E(v_2)$. Then for any $\lambda \in \epsilon(K)$, the vector*

$$v_1 + \lambda v_2$$

is fully defined over K under ϵ.

Proof. Although λv_2 may not be in general fully defined over K, the condition $E(v_1) \perp E(v_2)$ guarantees that

$$w(v_1 + \lambda v_2) = \max\{w(v_1), w(\lambda v_2)\} = w(v_1).$$

This can be easily seen by writing out $v_1 + \lambda v_2$ in the form (1.3.6). □

Theorem 1.3.18. *Denote $L := H^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. Let $V \subset L_{\mathbb{R}}$ be a positive definite 3-dimensional subspace of the form*

$$V = \mathbb{R}v_1 \oplus \mathbb{R}v_2 \oplus \mathbb{R}v_3,$$

with

(i) *$\{v_1, v_2, v_3\}$ is an orthogonal basis.*
(ii) *All v_i's are fully defined over some totally real number field K_0 under an embedding $\epsilon : K_0 \hookrightarrow \mathbb{R}$.*
(iii) *$E(v_i) \perp E(v_j)$ for $i \neq j$.*

(iv) $\epsilon^{-1}(\frac{\langle v_i, v_i\rangle}{\langle v_j, v_j\rangle}) \in K_0$ *is totally positive.*

If a point

$$[\sigma] \in \mathbb{P}(V_{\mathbb{C}}) \cap \Omega = \{[\sigma] \in \mathbb{P}(V_{\mathbb{C}}) \mid \langle \sigma, \sigma\rangle = 0, \langle \sigma, \bar{\sigma}\rangle > 0\}$$

defines a projective K3 surface, then a multiple of σ *is fully defined over a CM-field.*

Conversely, any K3 surface of CM-type Y *with period* σ *and* $\rho(Y) \geq 10$ *possesses a 3-dimensional positive definite subspace* $V \subset H^2(Y, \mathbb{R})$ *as above which contains* $\mathbb{R}\operatorname{Re}\sigma \oplus \mathbb{R}\operatorname{Im}\sigma$.

Proof. Let $V \subset L_{\mathbb{R}}$ and $[\sigma] \in \mathbb{P}(V_{\mathbb{C}}) \cap \Omega$ as in the claim. We can then write

$$\sigma = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3, \quad \alpha_i \in \mathbb{C}.$$

Let us abbreviate $v_i^2 := \langle v_i, v_i\rangle$. If one of the α_i's, say α_3, is zero, then after scaling we can suppose

$$\sigma = v_1 + \alpha_2 v_2.$$

We have

$$0 = \langle \sigma, \sigma\rangle = v_1^2 + \alpha_2^2 v_2^2$$

$$\Rightarrow \quad \alpha_2^2 = -\frac{v_1^2}{v_2^2} \in \epsilon(K_0),$$

so $\epsilon^{-1}(\alpha_2^2)$ is totally negative by hypothesis (iv). From (iii) it follows that σ is fully defined over $K_0(\alpha_2)$, which is a CM-field by Lemma 1.1.2.

Now let us deal with the case where none of the α_i's is zero. Then we can write

$$\sigma = v_1 + \alpha_2 v_2 + \alpha_3 v_3.$$

In order for σ to define a projective K3 surface we have three conditions on σ:

(a) $\langle \sigma, \sigma\rangle = 0$, i.e. $v_1^2 + \alpha_2^2 v_2^2 + \alpha_3^2 v_3^2 = 0$.
(b) $\langle \sigma, \bar{\sigma}\rangle > 0$, i.e. $v_1^2 + |\alpha_2|^2 v_2^2 + |\alpha_3|^2 v_3^2 > 0$.
(c) There is an x in the orthogonal complement $E(\sigma)^{\perp}$ of $E(\sigma)$ in $L_{\mathbb{Q}}$ with $\langle x, x\rangle > 0$. In particular,

$$0 = \langle \sigma, x\rangle = \langle v_1, x\rangle + \alpha_2\langle v_2, x\rangle + \alpha_3\langle v_3, x\rangle. \tag{1.3.10}$$

Since a vector with positive square cannot be perpendicular to V, at least one of the v_i's satisfies $\langle v_i, x\rangle \neq 0$. Let v_1 be a such vector, then because of (1.3.10) also $\langle v_2, x\rangle \neq 0$ or $\langle v_3, x\rangle \neq 0$. Without restriction to the generality, we can assume $\langle v_2, x\rangle \neq 0$. It follows that

$$\alpha_2 = -\alpha_3 \frac{\langle v_3, x\rangle}{\langle v_2, x\rangle} - \frac{\langle v_1, x\rangle}{\langle v_2, x\rangle}. \tag{1.3.11}$$

Substituting this in (a) we get

$$\alpha_3^2(v_2^2\langle v_3, x\rangle^2 + v_3^2\langle v_2, x\rangle^2) + 2\alpha_3 v_2^2\langle v_3, x\rangle\langle v_1, x\rangle + v_1^2\langle v_2, x\rangle^2 + v_2^2\langle v_1, x\rangle^2 = 0,$$

$$\begin{aligned}\Rightarrow \alpha_3 = &-\frac{v_2^2\langle v_3, x\rangle\langle v_1, x\rangle}{v_2^2\langle v_3, x\rangle^2 + v_3^2\langle v_2, x\rangle^2}\\ &\pm \frac{\langle v_2, x\rangle}{v_2^2\langle v_3, x\rangle^2 + v_3^2\langle v_2, x\rangle^2}\sqrt{-v_1^2 v_2^2\langle v_3, x\rangle^2 - v_1^2 v_3^2\langle v_2, x\rangle^2 - v_2^2 v_3^2\langle v_1, x\rangle^2}.\end{aligned}$$

By hypothesis (iv), $\epsilon^{-1}\left((\operatorname{Im}\alpha_3)^2\right) \neq 0 \in K_0$ is totally positive. It follows that

$$\begin{aligned}\operatorname{Re}\sigma &= v_1 - \left(\frac{\langle v_1, x\rangle}{\langle v_2, x\rangle} + \operatorname{Re}\alpha_3 \frac{\langle v_3, x\rangle}{\langle v_2, x\rangle}\right) v_2 + (\operatorname{Re}\alpha_3) v_3,\\ \operatorname{Im}\sigma &= (\operatorname{Im}\alpha_3)\left(-\frac{\langle v_3, x\rangle}{\langle v_2, x\rangle} v_2 + v_3\right).\end{aligned}$$

By the last lemma and hypothesis (iii), $\operatorname{Re}\sigma$ is fully defined over K_0, and $\frac{1}{\operatorname{Im}\alpha_3}\operatorname{Im}\sigma$ is fully defined over K_0 for the same reason. Writing $\operatorname{Re}\sigma$ and $\operatorname{Im}\sigma$ out as in (1.3.6) we see that because of the presence of v_1 in $\operatorname{Re}\sigma$, we have $E(\operatorname{Re}\sigma) \cap E(\operatorname{Im}\sigma) = 0$. Hence σ is fully defined over $K_0(i \operatorname{Im}\alpha_3)$ which is a CM-field.

Conversely, let Y be a K3 surface of CM-type over K. In view of Corollary 1.3.16 we can set

$$v_1 := \operatorname{Re}\sigma \quad \text{and} \quad v_2 := \frac{i}{\epsilon(\eta)} \operatorname{Im}\sigma.$$

Clearly, $\epsilon^{-1}(\frac{v_1^2}{v_2^2}) = -\eta^2 \in K_0$ is totally positive.

Now it remains to find v_3 with the claimed properties. Denote $n := \frac{1}{2}\dim_{\mathbb{Q}} T$. The hypothesis $\rho(Y) \geq 10$ ensures that $n \leq 6$ and $\rho(Y) - n \geq 4$. As $v_1^2 > 0$, the intersection form has signature $(1, n-1)$ on $E(v_1)$. On the other hand, since Y is projective, the signature of $\langle,\rangle$ on $NS(Y)_{\mathbb{Q}}$ is $(1, \rho(Y)-1)$, hence $\langle,\rangle$ is indefinite on $NS(Y)_{\mathbb{Q}}$. Let $\{e_1, \ldots, e_n\}$ be an orthogonal basis of $E(v_1)$ with $\langle e_n, e_n\rangle > 0$ and $\langle e_i, e_i\rangle < 0, \forall i \neq n$. Since $\rho(Y) \geq 10$ and any indefinite quadratic form over $\mathbb{Q}$ of rank ≥ 5 represents 0 (hence any rational number), one can find a $f_1 \in NS(Y)_{\mathbb{Q}}$ with $\langle f_1, f_1\rangle = \langle e_1, e_1\rangle$. Then consider the orthogonal complement N_1 of $\mathbb{Q}f_1$ in $NS(Y)_{\mathbb{Q}}$. The intersection form $\langle,\rangle$ is still indefinite, then one can choose $f_2 \in N_1$ with $\langle f_2, f_2\rangle = \langle e_2, e_2\rangle$. The condition $\rho(Y) - n \geq 4$ guarantees that one finds thus successively n vectors $\{f_i\}$ in $NS(Y)_{\mathbb{Q}}$ with $\langle f_i, f_i\rangle = \langle e_i, e_i\rangle$. If we write

$$v_1 = \sum_{i=1}^{n} \lambda_i e_i$$

where $\{e_i\}$ is an orthogonal basis of $E(v_1)$, set

$$v_3 := \sum_{i=1}^{n} \lambda_i f_i$$

with the same coefficients. We have then $\langle v_3, v_3\rangle = \langle v_1, v_1\rangle$ and $E(v_1) \perp E(v_3)$, also $E(v_2) \perp E(v_3)$. Now $\{v_1, v_2, v_3\}$ satisfy all the requirements. This completes the proof. $\square$

2. Mirror symmetry, generalized Calabi-Yau structures and complex multiplication

Summary of results: The goal of this section is to show Theorem 2.2.13 and Theorem 2.4.4. They describe when a mirror of an abelian variety or a K3 surface of CM-type is again of CM-type. For abelian varieties it suffices to choose a rational Kähler metric and a rational B-field, but for K3 surfaces we could only find more complicated conditions. Lemma 2.2.10, Lemma 2.2.11 and Proposition 2.4.3 contain detailed calculations of mirror partners and are decisive for the theorems. Moreover, Section 2.3 contains an explicit construction of a pair of isogenous mirror abelian varieties of CM-type, it shows that the converse of Theorem 2.2.13 does not hold.

This section is organized as follows. We recall in Section 2.1 the definition of generalized Calabi-Yau structures which provide a more unified point of view for mirror symmetry for different manifolds.

In Section 2.2 we use the generalized complex structure induced by a generalized Calabi-Yau structure to formulate mirror symmetry for generalized complex tori (i.e. real tori endowed with a generalized Kähler structure). Then we show Theorem 2.2.13.

Section 2.3 contains a pair of isogenous mirror abelian varieties of CM-type which shall refute the converse of Theorem 2.2.13.

In Section 2.4 we formulate mirror symmetry for K3 surfaces and calculate in Proposition 2.4.3 a special kind of mirror partners. Then we prove Theorem 2.4.4.

2.1. Generalized Calabi-Yau structures.

Hitchin introduced generalized complex geometry in his seminal paper [Hi] (see also among others [Gu], [H2] and [Be] for further works). A novelty is that one considers structures on the sum $T \oplus T^*$ of the tangent and cotangent bundles of a smooth manifold. It offers among other advantages a unified point of view for complex and symplectic structures. A fundamental structure is

Definition 2.1.1. *A* generalized Calabi-Yau structure (GCYS) *on a smooth manifold of dimension $2m$ is a closed even complex form or a closed odd complex form φ which is a pure spinor, i.e. its annihilator*

$$E_\varphi := \{v + \xi \in (T \oplus T^*)_{\mathbb{C}} \mid (v + \xi) \cdot \varphi := \iota(v)\varphi + \xi \wedge \varphi = 0\} \tag{2.1.1}$$

is maximal and $E_\varphi \cap \overline{E}_\varphi = 0$ at each point.

We are most interested in the following two GCYSs. If X is a smooth complex manifold of dimension m, denote by σ a holomorphic form of top degree m and let B be a real closed 2-form on X, then

$$\varphi := (\exp B)\sigma := (1 + B + \frac{B^2}{2} + \cdots) \wedge \sigma$$

is a GCYS. It carries information about the complex structure of X. If M is a symplectic manifold of dimension $2m$ with symplectic structure ω and B is again a real closed 2-form, then

$$\psi := \exp(B + i\omega) := 1 + B + i\omega + \frac{1}{2}(B + i\omega)^2 + \cdots$$

is also a GCYS (see [Hi]). It carries information about the symplectic structure of M. The operator $\exp B$ is called the *B-transform*. We will say that B is *rational* if its class $[B]$ lies in $H^2(X,\mathbb{Q})$. Roughly speaking, a mirror map between two manifolds endowed with both complex and symplectic structures shall exchange these two structures. However, technically this goes somewhat differently for K3 surfaces and abelian varieties. We first present mirror symmetry and the behavior of complex multiplication under a mirror map for abelian varieties.

2.2. **Abelian varieties.** The aim of this section is to show Theorem 2.2.13. It gives a sufficient condition which ensures that complex multiplication is transmitted to the mirror partners, namely, it suffices to choose a rational Kähler metric G and a rational B-field for any mirror of (X,G,B) to be of CM-type. Mirror symmetry for abelian varieties was treated in [GLO]. Some of their results can be rephrased more naturally in terms of generalized complex structures (see [Gu, Chap. 6] or [K, §8]).

Definition 2.2.1. *A* generalized complex structure *on a smooth manifold Y is a bundle map $\mathcal{I}: TY\oplus T^*Y \longrightarrow TY\oplus T^*Y$ satisfying*

(i) $\mathcal{I}^2=-id$,

(ii) *$\mathcal{I}$ preserves the pseudo-Euclidean metric q on $TY\oplus T^*Y$, where*

$$q((a_1,a_2),(b_1,b_2)) := -\langle a_1,b_2\rangle - \langle a_2,b_1\rangle,$$

(iii) *$\mathcal{I}$ is integrable with respect to the Courant bracket.*

We will only deal with constant generalized complex structures, so the condition (iii) is not relevant for us. The $J_{A\times\hat{A}}$ and I_{ω_A} which appeared in [GLO] are examples of generalized complex structures. Hitchin shows that some of them may be induced by GCYSs:

Proposition 2.2.2 (Hitchin)**.** *Any GCYS φ induces a generalized complex structure $\mathcal{I}$ by setting*

$$\mathcal{I}\big|_{E_\varphi} := i\cdot \mathrm{Id} \quad \text{and} \quad \mathcal{I}\big|_{\bar{E}_\varphi} := -i\cdot \mathrm{Id},$$

where E_φ respectively $\bar{E}_\varphi$ is the annihilator of φ respectively $\bar{\varphi}$.

For the purpose of mirror symmetry we are more interested in a pair of generalized complex structures.

Definition 2.2.3. *A* generalized Kähler structure (GKS) *on a smooth manifold Y is a pair $(\mathcal{I},\mathcal{J})$ of commuting generalized complex structures such that $\mathcal{G}(\cdot,\cdot):=q(\cdot,\mathcal{I}\mathcal{J}\cdot)$ is a positive definite symmetric bilinear form on $TY\oplus T^*Y$.*

Using GKS, one can define mirror symmetry for a more general class of tori.

Definition 2.2.4. *A* generalized complex torus $(\mathbb{T},\mathcal{I},\mathcal{J})$ *is a real torus $\mathbb{T}$ endowed with a GKS $(\mathcal{I},\mathcal{J})$.*

Definition 2.2.5. *Two generalized complex tori $(\mathbb{T}=V/\Gamma,\mathcal{I},\mathcal{J})$ and $(\mathbb{T}'=V'/\Gamma',\mathcal{I}',\mathcal{J}')$ are* mirror partners *if there is a lattice isomorphism*

$$\varphi:\Gamma\oplus\Gamma^* \longrightarrow \Gamma'\oplus\Gamma'^*,$$

such that $q(\cdot,\cdot)=q'(\varphi\cdot,\varphi\cdot)$, $\mathcal{I}'=\varphi\mathcal{J}\varphi^{-1}$, and $\mathcal{J}'=\varphi\mathcal{I}\varphi^{-1}$. We call φ a mirror map*. We also denote by φ its $\mathbb{R}$-linear extension.*

This generality is suitable for the notion of mirror symmetry for N=2 lattice OPE-algebras which we will construct on any generalized complex torus in Section 3.3. However, for the study of complex multiplication under mirror symmetry we specialize to the following GKS. Let (T, G, B) be a triple consisting of a complex torus $T = \mathbb{C}^g/\Gamma$ with complex structure I considered as an $\mathbb{R}$-linear map on $\Gamma_\mathbb{R}$ with $I^2 = -\operatorname{Id}$. It is also determined by a holomorphic form σ of top degree g by setting the eigenvalue of I to $-i$ on the annihilator of σ and to i on the complement. Further, B is an arbitrary real 2-form in $H^2(T, \mathbb{R})$ and G is a Kähler metric, it induces a Kähler form $\omega(\cdot, \cdot) := G(\cdot, I\cdot)$. We will always assume that B and G are constant. Let us denote by $\mathcal{I}$ and $\mathcal{J}$ the generalized complex structures induced by the GCYSs $\varphi = (\exp B)\sigma$ and $\psi = \exp(B + i\omega)$ (see Proposition 2.2.2). Easy calculations show that

$$\begin{aligned}
\mathcal{I} &= \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -I^t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} = \begin{pmatrix} I & 0 \\ BI + I^t B & -I^t \end{pmatrix}, \\
\mathcal{J} &= \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} = \begin{pmatrix} \omega^{-1}B & -\omega^{-1} \\ \omega + B\omega^{-1}B & -B\omega^{-1} \end{pmatrix} \\
&= \begin{pmatrix} -IG^{-1}B & IG^{-1} \\ GI - BIG^{-1}B & BIG^{-1} \end{pmatrix},
\end{aligned} \tag{2.2.1}$$

and $(\mathcal{I}, \mathcal{J})$ forms a GKS. For the positivity of $q(\cdot, \mathcal{I}\mathcal{J})$ just notice that

$$\mathcal{I}\mathcal{J} = \begin{pmatrix} G^{-1}B & -G^{-1} \\ BG^{-1}B - G & -BG^{-1} \end{pmatrix} \quad \text{and} \quad q\mathcal{I}\mathcal{J} = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix} \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}^t .$$

We say

Definition 2.2.6. *The triple* (T, G, B) induces *a generalized complex torus* $(\mathbb{T}, \mathcal{I}, \mathcal{J})$*, with* $(\mathcal{I}, \mathcal{J})$ *as in (2.2.1).*

Now it is clear that Definition 2.2.5 of mirror symmetry makes sense: morally, for the generalized complex tori induced by (T, G, B) respectively (T', G', B'), a mirror map exchanges the complex structure $\mathcal{I}$ with the symplectic structure $\mathcal{J}'$ and the symplectic structure $\mathcal{J}$ with the complex structure $\mathcal{I}'$. Thus, the language of GKS provides a conceptually clean approach to mirror symmetry. We formulate this more precisely in the following

Definition 2.2.7. *We say that two complex tori* (T, G, B) *and* (T', G', B') *with complex structure* I *respectively* I'*, a constant Kähler metric* G *respectively* G' *and a B-field* B *respectively* B' *are* mirror partners*, if the generalized complex tori they induce as in Definition 2.2.6 are mirror of each other.*

Remark 2.2.8. Suppose $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ and $(\mathbb{T}', \mathcal{I}', \mathcal{J}')$ are mirror generalized complex tori and $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ is induced by a triple (T, G, B). In general however, $(\mathbb{T}', \mathcal{I}', \mathcal{J}')$ may not be induced by some triple (T', G', B'). Definition 2.2.7 concerns only those mirror pairs where both partners are induced.

In order to prove Theorem 2.2.13 we give the next lemma which first studies the rationality of the composition $\mathcal{I}\mathcal{J}$ on a generalized complex torus $(\mathbb{T}, \mathcal{I}, \mathcal{J})$, then links the rationality of G and B to the rationality of $\mathcal{I}\mathcal{J}$ of the GKS they induce. The rationality of $\mathcal{I}\mathcal{J}$ is defined as follows.

Definition 2.2.9. *Let* $(\mathbb{T} = V/\Gamma, \mathcal{I}, \mathcal{J})$ *be a generalized complex torus. Denote*

$$\Lambda := \Gamma \oplus \Gamma^*.$$

We identify the tangent space of $\mathbb{T}$ with $\Gamma_{\mathbb{R}}$. We say that the composition $\mathcal{I}\mathcal{J}$ is defined over $\mathbb{Q}$ *if $\mathcal{I}\mathcal{J}$ preserves the rational lattices:*

$$\mathcal{I}\mathcal{J} : \Lambda_{\mathbb{Q}} \longrightarrow \Lambda_{\mathbb{Q}}.$$

Lemma 2.2.10. *Let $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ be a generalized complex torus. Denote by $C_{\pm} \subset \Lambda_{\mathbb{R}}$ the eigenspace of $\mathcal{I}\mathcal{J}$ with eigenvalue ± 1. Then*

(i) *We have*

$$C_{\pm} = (\pm \operatorname{Id} + \mathcal{I}\mathcal{J})(\Lambda_{\mathbb{R}}) = \operatorname{Image}_{\Lambda_{\mathbb{R}}}(\pm \operatorname{Id} + \mathcal{I}\mathcal{J}),$$

and the decomposition

$$\Lambda_{\mathbb{R}} = C_+ \oplus C_-. \tag{2.2.2}$$

We have $\mathcal{I} = -\mathcal{J}$ on C_+ and $\mathcal{I} = \mathcal{J}$ on C_-. This decomposition is orthogonal with respect to q and it is defined over $\mathbb{Q}$ if and only if $\mathcal{I}\mathcal{J}$ is defined over $\mathbb{Q}$.

(ii) *If $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ is induced by (T, G, B), then we have*

$$C_{\pm} = \operatorname{Graph}_{\Gamma_{\mathbb{R}}}(\mp G + B),$$

and q is positive definite on C_+ and negative definite on C_-. Moreover, the following is equivalent:

(a) *The decomposition (2.2.2) is defined over $\mathbb{Q}$,*
(b) *$\mathcal{I}\mathcal{J}$ is defined over $\mathbb{Q}$,*
(c) *G and B are both rational.*

Proof. The orthogonality of (2.2.2) follows from $\mathcal{I}, \mathcal{J} \in O(q)$. To prove (ii) it suffices to note that

$$\mathcal{I}\mathcal{J}\begin{pmatrix} 1 \\ \mp G + B \end{pmatrix} = \pm \begin{pmatrix} 1 \\ \mp G + B \end{pmatrix}.$$

□

The following lemma shows how the lattice of mirror partners is related to each other.

Lemma 2.2.11. *Let $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ and $(\mathbb{T}', \mathcal{I}', \mathcal{J}')$ be mirror generalized complex tori and φ a mirror map between them. Then*

(i) *φ respects the decomposition (2.2.2), i.e. $\varphi : C_{\pm} \to C'_{\pm}$. In particular, $C_+ \oplus C_-$ is defined over $\mathbb{Q}$ if and only if $C'_+ \oplus C'_-$ is defined over $\mathbb{Q}$.*

(ii) *If $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ and $(\mathbb{T}', \mathcal{I}', \mathcal{J}')$ are induced by (T, G, B) respectively (T', G', B'), then φ induces isomorphisms $\psi_{\pm} : \Gamma_{\mathbb{R}} \xrightarrow{\sim} \Gamma'_{\mathbb{R}}$ by*

$$\varphi(a, (\mp G + B)a) = (\psi_{\pm} a, (\mp G' + B')\psi_{\pm} a), \tag{2.2.3}$$

and we have

(a) *$G(a, b) = G'(\psi_{\pm} a, \psi_{\pm} b)$ for all $a, b \in \Gamma_{\mathbb{R}}$.*
(b) *$I' = \psi_+ \circ I \circ \psi_+^{-1} = \psi_- \circ I \circ \psi_-^{-1}$.*

Proof. (i) is immediate. For (ii)(a) we make an explicit calculation

$$q\left((a, (\mp G + B)a), (b, (\mp G + B)b)\right) = \pm 2G(a, b) \quad \forall a, b \in \Gamma_{\mathbb{R}}.$$

Then use $q(\cdot, \cdot) = q'(\varphi \cdot, \varphi \cdot)$ to get the claim. For (ii)(b) we verify the equality for ψ_-, the case of ψ_+ is similar. Recalling the definition of the generalized Kähler structure from (2.2.1) we have for any $(a', (G' + B')a') \in C'_-$:

$$\mathcal{I}'\begin{pmatrix} a' \\ (G' + B')a' \end{pmatrix} = \begin{pmatrix} I'a' \\ * \end{pmatrix}, \tag{2.2.4}$$

we are only interested in the first component. Using (2.2.3) the left hand side of (2.2.4) is

$$\varphi\mathcal{J}\varphi^{-1}\begin{pmatrix} a' \\ (G'+B')a' \end{pmatrix} = \begin{pmatrix} \psi_- I\psi_-^{-1}a' \\ (G+B)\psi_- I\psi_-^{-1}a' \end{pmatrix}$$

Comparing with the right hand side of (2.2.4), we obtain (ii)(b). □

As mirror symmetry exchanges complex and symplectic structures, two mirror Calabi-Yau manifolds are in general very different as complex manifolds. This remains true for abelian varieties, but surprisingly the lemmas above lead to the following result which shows: it suffices that the composition $\mathcal{I}\mathcal{J}$ is defined over $\mathbb{Q}$ for the mirror to be an isogenous complex torus. If the reader is not familiar with isogeny, see Lemma 2.3.2 for explicit calculations of isogeny.

Proposition 2.2.12. *Let (T, G, B) and (T', G', B') be mirror partners. If G and B are both rational, then T and T' are isogenous.*

Proof. By Lemma 2.2.10 (ii), G and B are both rational if and only if $\mathcal{I}\mathcal{J}$ is defined over $\mathbb{Q}$. Then Lemma 2.2.11 (i) implies that G' and B' are rational. By (2.2.3) the $\psi_\pm$ are then defined over $\mathbb{Q}$. Finally, from (ii)(b) of the same lemma, it follows that some integral multiple of $\psi_\pm$ is actually an isogeny between T and T'. □

This immediately implies the following

Theorem 2.2.13. *Let (X, G, B) and (X', G', B') be mirror abelian varieties. Suppose X is of CM-type. If both G and B are rational, then X and X' are isogenous. In particular, X' is also of CM-type.*

At this stage using Theorem 3.4.3 (iii) we can answer one direction of the question (QAV) posed in Introduction positively. In the next section we will show that the converse of Theorem 2.2.13 however does not hold, which shows that the other direction of (QAV) is not true.

2.3. **An example of mirror abelian varieties of CM-type.**

In this section we show the following proposition. It says that the converse of Theorem 2.2.13 does not hold. This has as consequence that one of the directions of the question (QAV) posed in Introduction is not true. To see this, one has to use Theorem 3.4.3 (ii).

Proposition 2.3.1. *There are mirror abelian varieties (X, G, B) and (X', G', B'), such that X and X' are isogenous and of CM-type, but neither $\mathcal{I}\mathcal{J}$ nor $\mathcal{I}'\mathcal{J}'$ is defined over $\mathbb{Q}$, where $(\mathcal{I}, \mathcal{J})$ and $(\mathcal{I}', \mathcal{J}')$ denote their induced GKS.*

We shall eventually construct an explicit example of such mirror pairs, we need first some preparation.

Lemma 2.3.2. *Let $T \cong \mathbb{C}^g/\Pi\mathbb{Z}^{2g}$ and $T' \cong \mathbb{C}^g/\Pi'\mathbb{Z}^{2g}$ be complex tori with period matrix Π and Π' respectively. Then T and T' are isogenous if and only if there is a complex matrix $C \in GL(g, \mathbb{C})$ and a rational matrix $\gamma \in GL(2g, \mathbb{Q})$, such that*

$$\Pi' = C\Pi\gamma. \tag{2.3.1}$$

In particular, if there is a such matrix γ, then $I' = \gamma^{-1}I\gamma$ and if G is a Kähler metric on T then $G'(\cdot,\cdot) := G(\gamma\cdot, \gamma\cdot)$ is a Kähler metric on T'.

Proof. It is easy to see that the rational respectively analytic representation of any isogeny provides the matrix γ respectively C. For the converse, recall that in general, the rational representation of I of a complex torus with period matrix Π is $I = \left(\begin{smallmatrix}\Pi\\ \overline{\Pi}\end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} i\mathbf{1} & 0\\ 0 & -i\mathbf{1}\end{smallmatrix}\right)\left(\begin{smallmatrix}\Pi\\ \overline{\Pi}\end{smallmatrix}\right)$. Replacing $\left(\begin{smallmatrix}\Pi\\ \overline{\Pi}\end{smallmatrix}\right)$ by $\left(\begin{smallmatrix}\Pi'\\ \overline{\Pi'}\end{smallmatrix}\right) = \left(\begin{smallmatrix} C & 0\\ 0 & \overline{C}\end{smallmatrix}\right)\left(\begin{smallmatrix}\Pi\\ \overline{\Pi}\end{smallmatrix}\right)\gamma$ for I', we get $I' = \gamma^{-1}I\gamma$. It follows that some integral multiple of γ is an isogeny. The rest of the claim is obvious. □

Next we give a special form (see (2.3.2) below) of the period matrix, which makes the construction of an isogenous mirror easier. Later we will give an abelian variety of CM-type over a cyclotomic field, whose period matrix can be written in the form (2.3.2). First a lemma which expresses I explicitly.

Lemma 2.3.3. *Let Γ be the lattice of a complex torus T generated by $e_1, \ldots, e_{2g}$. Suppose that the period matrix Π of T in the complex basis $\{e_1, \ldots, e_g\}$ has the form $\Pi = (\mathbf{1} \quad T_1 + T_2 i)$, where T_1 and T_2 are real $g \times g$ matrices, then in the basis $\{e_1, \ldots, e_{2g}\}$ we have*

$$I = \begin{pmatrix} -T_1T_2^{-1} & -T_1T_2^{-1}T_1 - T_2 \\ T_2^{-1} & T_2^{-1}T_1 \end{pmatrix}.$$

Proof. The proof is a matter of calculating the matrix $I = \left(\begin{smallmatrix}\Pi\\ \overline{\Pi}\end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} i\mathbf{1} & 0\\ 0 & -i\mathbf{1}\end{smallmatrix}\right)\left(\begin{smallmatrix}\Pi\\ \overline{\Pi}\end{smallmatrix}\right)$, where

$$\begin{pmatrix}\Pi\\ \overline{\Pi}\end{pmatrix}^{-1} = \frac{i}{2}\begin{pmatrix} T_1T_2^{-1} - i & -T_1T_2^{-1} - i \\ -T_2^{-1} & T_2^{-1} \end{pmatrix}.$$

□

Proposition 2.3.4. *If an abelian variety X has a period matrix of the form*

$$\Pi = \begin{pmatrix}\mathbf{1} & Ai\end{pmatrix}, \tag{2.3.2}$$

with a real matrix $A \in GL(g, \mathbb{R})$, then by choosing a suitable constant Kähler metric G and by setting $B = 0$, one can find an isogenous mirror abelian variety (X', G', B').

Proof. Suppose that Π has the form in (2.3.2). Then by Lemma 2.3.3 we have $I = \left(\begin{smallmatrix} 0 & -A\\ A^{-1} & 0\end{smallmatrix}\right)$. Let us choose the metric

$$G = \begin{pmatrix} -\rho & 0 \\ 0 & -A^t\rho A \end{pmatrix},$$

where ρ is a symmetric negative definite matrix with integral coefficients. One verifies that G is compatible with I, i.e. $I^tGI = G$, so G is a Kähler metric. Setting $B = 0$, then by (2.2.1) $\mathcal{I}$ and $\mathcal{J}$ have the form

$$\mathcal{I} = \begin{pmatrix} 0 & -A & 0 & 0 \\ A^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -A^{-1t} \\ 0 & 0 & A^t & 0 \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & 0 & 0 & \rho^{-1}A^{-1t} \\ 0 & 0 & -A^{-1}\rho^{-1} & 0 \\ 0 & \rho A & 0 & 0 \\ -A^t\rho & 0 & 0 & 0 \end{pmatrix}.$$

Further, we choose

$$C = \rho \quad \text{and} \quad \gamma = \begin{pmatrix} \rho^{-1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$

to get a new period matrix

$$\Pi' = C\Pi\gamma = \begin{pmatrix}\mathbf{1} & \rho A i\end{pmatrix}.$$

Then by Lemma 2.3.2 the complex torus $X' := \mathbb{C}^g/\Pi'\mathbb{Z}^{2g}$ has complex structure respectively Kähler metric

$$I' = \gamma^{-1} I \gamma = \begin{pmatrix} 0 & -\rho A \\ A^{-1}\rho^{-1} & 0 \end{pmatrix} \quad \text{resp.} \quad G' = \gamma^t G \gamma = \begin{pmatrix} -\rho^{-1} & 0 \\ 0 & -A^t \rho A \end{pmatrix}.$$

Setting $B' = 0$ we get

$$\mathcal{I}' = \begin{pmatrix} 0 & -\rho A & 0 & 0 \\ A^{-1}\rho^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\rho^{-1}A^{-1t} \\ 0 & 0 & A^t\rho & 0 \end{pmatrix}, \quad \mathcal{J}' = \begin{pmatrix} 0 & 0 & 0 & A^{-1t} \\ 0 & 0 & -A^{-1} & 0 \\ 0 & A & 0 & 0 \\ -A^t & 0 & 0 & 0 \end{pmatrix}.$$

By easy calculations, one verifies that (X, G, B) and (X', G', B') as defined above are mirror partners with the mirror map

$$\varphi = \begin{pmatrix} 0 & 0 & \mathbf{1} & 0 \\ 0 & -\mathbf{1} & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathbf{1} \end{pmatrix} : \Gamma \oplus \Gamma^* \longrightarrow \Gamma' \oplus \Gamma'^*.$$

Hence, to any abelian variety with period matrix $\Pi = (\mathbf{1}\ Ai)$ by choosing a suitable G and B-field one can find an isogenous mirror abelian variety. □

In order to obtain a mirror pair of CM-type, let us consider the abelian variety over a cyclotomic field $\mathbb{Q}(\xi)$ of degree 4 which we constructed in Example 1.2.13. Recall the notations. Denote

$$w := e^{\frac{2\pi}{5}i} = \frac{1}{4}(-1+\sqrt{5}) + \frac{i}{2}\sqrt{\frac{1}{2}(5+\sqrt{5})},$$

and the four embeddings are $\sigma_k : \xi \mapsto w^k, k = 1, \ldots, 4$. We chose the CM-type $\Phi = \{\sigma_1, \sigma_2\}$ and the lattice $\Gamma = \mathcal{O}_K = \mathbb{Z}[\xi]$. Let us fix the following generators for Γ:

$$\Gamma = \mathbb{Z}\cdot 1 \oplus \mathbb{Z}(\xi + \xi^{-1}) \oplus \mathbb{Z}(\xi - \xi^{-1}) \oplus \mathbb{Z}(\xi^2 - \xi^{-2}).$$

Then under Φ the lattice is

$$\Phi(\mathcal{O}_K) = \begin{pmatrix} 1 & w + w^{-1} & w - w^{-1} & w^2 - w^{-2} \\ 1 & w^2 + w^{-2} & w^2 - w^{-2} & w^4 - w^{-4} \end{pmatrix} \mathbb{Z} =: \begin{pmatrix} Z & Ai \end{pmatrix} \mathbb{Z}.$$

The left two columns form a real matrix Z, while the right two columns form a purely imaginary matrix which we write as Ai where A is a real matrix. Choosing the first two generators to be a complex basis of $\Phi(\mathcal{O}_K) \otimes_{\mathbb{Z}} \mathbb{R}$, we get the period matrix

$$\Pi = \begin{pmatrix} \mathbf{1} & Z^{-1}Ai \end{pmatrix}$$

in the form (2.3.2) with the real matrix $Z^{-1}A$. Together with

$$G = \begin{pmatrix} -\rho & 0 \\ 0 & -A^t Z^{-1t} \rho Z^{-1} A \end{pmatrix} \quad \text{and} \quad B = 0,$$

the abelian variety (X, G, B) possesses an isogenous mirror (X', G', B') as constructed in Proposition 2.3.4.

The last step to get $(\mathcal{I}, \mathcal{J})$ such that the composition $\mathcal{I}\mathcal{J}$ is not defined over $\mathbb{Q}$ is to choose an appropriate ρ. Indeed, as claimed by Lemma 2.2.10 (ii), $\mathcal{I}\mathcal{J}$ is defined over $\mathbb{Q}$ if and only if G and B are both rational. We set $B = 0$ and choose

$$\rho = \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix},$$

then

$$G = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5+\frac{2}{\sqrt{5}} & 2-\frac{1}{\sqrt{5}} \\ 0 & 0 & 2-\frac{1}{\sqrt{5}} & 3-\frac{2}{\sqrt{5}} \end{pmatrix},$$

which is not rational. Hence $\mathcal{IJ}$ is not defined over $\mathbb{Q}$, although X is of CM-type and has a mirror (X', G', B') of CM-type over the same field K. This shows Proposition 2.3.1.

2.4. **K3 surfaces.**

In this section we define mirror symmetry for K3 surfaces in terms of GCYSs. Then we specialize to a particular kind of mirror maps, namely $\chi^2 = \mathrm{Id}$, we call χ "involutive" (see Definition 2.4.2), and we calculate explicitly an involutive mirror partner (see Proposition 2.4.3). In Theorem 2.4.4 we give sufficient conditions for complex multiplication to be transmitted to an involutive mirror.

Let Y be a (not necessarily projective) K3 surface. Recall the Mukai pairing $\langle , \rangle$ on $H^*(Y, \mathbb{Z})$ from Section 1.3: for $\alpha, \beta \in H^*(Y, \mathbb{Z})$ we have

$$\alpha \wedge_M \beta := -\alpha_0 \wedge \beta_4 + \alpha_2 \wedge \beta_2 - \alpha_4 \wedge \beta_0 =: \langle \alpha, \beta \rangle w,$$

where w is a generator of $H^4(Y, \mathbb{Z})$ with the orientation $\sigma \wedge \bar{\sigma} = \lambda w, \lambda > 0$. Let furthermore $w^* \in H^0(Y, \mathbb{Z})$ be the vector with $\langle w^*, w \rangle = -1$. For a triple (Y, ω, B) where ω is a real (1,1)-form with $\langle \omega, \omega \rangle > 0$ and B is arbitrary in $H^2(Y, \mathbb{R})$, the two GCYSs which we will use are as already mentioned in Section 2.1:

$$\varphi := (\exp B)\sigma = \sigma + \langle B, \sigma \rangle w \quad \text{and}$$

$$\psi := \exp(B + i\omega) = w^* + B + i\omega + \frac{1}{2}(\langle B, B \rangle - \langle \omega, \omega \rangle + 2i\langle B, \omega \rangle)w.$$

Consider the following spaces

$$\begin{aligned} W_1 :&= \mathbb{R}\,\mathrm{Re}\,\varphi \oplus \mathbb{R}\,\mathrm{Im}\,\varphi \\ &= \mathbb{R}(\mathrm{Re}\,\sigma + \langle B, \mathrm{Re}\,\sigma \rangle w) \oplus \mathbb{R}(\mathrm{Im}\,\sigma + \langle B, \mathrm{Im}\,\sigma \rangle w) \\ &=: \mathbb{R}w_{11} \oplus \mathbb{R}w_{12}, \\ W_2 :&= \mathbb{R}\,\mathrm{Re}\,\psi \oplus \mathbb{R}\,\mathrm{Im}\,\psi \\ &= \mathbb{R}(w^* + B + \frac{1}{2}(\langle B, B \rangle - \langle \omega, \omega \rangle)w) \oplus \mathbb{R}(\omega + \langle B, \omega \rangle w) \\ &=: \mathbb{R}w_{21} \oplus \mathbb{R}w_{22}. \end{aligned}$$

A few properties of these spaces are useful for the calculations in the next proposition. One can easily show that the operator $\exp B$ is orthogonal with respect to the Mukai pairing, i.e.

$$\langle (\exp B)\alpha, (\exp B)\beta \rangle = \langle \alpha, \beta \rangle.$$

Hence because of $\langle \sigma, \exp(i\omega) \rangle = 0$ we have immediately $\langle W_1, W_2 \rangle \equiv 0$. Furthermore, the four basis vectors $\{w_{ij}\}$ are orthogonal to each other and we have

$$\langle w_{11}, w_{11} \rangle = \langle w_{12}, w_{12} \rangle = \langle \mathrm{Re}\,\sigma, \mathrm{Re}\,\sigma \rangle \quad \text{and} \quad \langle w_{21}, w_{21} \rangle = \langle w_{22}, w_{22} \rangle = \langle \omega, \omega \rangle.$$

Hence $W_1 \oplus W_2$ is a positive definite 4-dimensional subspace of $H^*(Y, \mathbb{R})$. We fix an orientation by taking the bases $\{w_{11}, w_{12}\}$ and $\{w_{21}, w_{22}\}$ to be *positive oriented bases* of W_1 respectively W_2. For another triple (Y', ω', B'), denote by w'_{ij}, W'_i the corresponding objects. We define mirror symmetry for K3 surfaces as follows.

Definition 2.4.1. *Consider triples like (Y, ω, B), where Y is a (not necessarily projective) K3 surface, ω is a real (1,1)-form with $\langle \omega, \omega \rangle > 0$ and B is an arbitrary element of $H^2(Y, \mathbb{R})$. We say that two triples (Y, ω, B) and (Y', ω', B') are* mirror partners *if there is a (Mukai) lattice isomorphism*

$$\chi : H^*(Y, \mathbb{Z}) \xrightarrow{\sim} H^*(Y', \mathbb{Z}),$$

such that

$$\chi(W_1) = W_2' \quad \textit{and} \quad \chi(W_2) = W_1',$$

while preserving their orientation. We call χ a mirror map.

One sees easily that the inverse map χ^{-1} is also a mirror map. Denote

$$U := H(-1) \quad \text{and} \quad L := H^{\oplus 3} \oplus E_8(-1)^{\oplus 2}.$$

Fixing the isomorphisms

$$H^*(Y, \mathbb{Z}) \cong U \oplus L \quad \text{and} \quad H^*(Y', \mathbb{Z}) \cong U \oplus L,$$

we can consider χ as a lattice isomorphism:

$$\chi : U \oplus L \xrightarrow{\sim} U \oplus L.$$

Hence it makes sense to talk about $\chi^2 = \chi \circ \chi$. For calculations we consider a special kind of mirror map.

Definition 2.4.2. *We say that (Y, ω, B) and (Y', ω', B') are* involutive mirror partners *if their mirror map χ satisfies $\chi^2 = \mathrm{Id}$. Denote furthermore*

$$U' := \chi(U), \quad v := \chi(w) \quad \textit{and} \quad v^* := \chi(w^*).$$

The following proposition gives explicitly the mirror of a triple (Y, ω, B) under an involutive mirror map. Yet some additional assumptions are made.

Proposition 2.4.3. *Let (Y, ω, B) and (Y', ω', B') be involutive mirrors under a mirror map χ. Suppose additionally that*

$$\langle U', U \rangle \equiv 0 \quad \textit{and} \quad \langle B, v \rangle = \langle \omega, v \rangle = 0.$$

Then (Y', ω', B') is given by

$$\begin{aligned}
\operatorname{Re}\sigma' &= v^* + \chi(B + \langle B, v^* \rangle v) + \frac{1}{2}(\langle B, B \rangle - \langle \omega, \omega \rangle) v, \\
\operatorname{Im}\sigma' &= \chi(\omega + \langle \omega, v^* \rangle v) + \langle B, \omega \rangle v, \\
B' &= \chi(\operatorname{Re}\sigma - v^* + \langle \operatorname{Re}\sigma, v^* \rangle v) + \langle B, \operatorname{Re}\sigma \rangle v, \\
\omega' &= \chi(\operatorname{Im}\sigma + \langle \operatorname{Im}\sigma, v^* \rangle v) + \langle B, \operatorname{Im}\sigma \rangle v.
\end{aligned}$$

Proof. The condition $\langle U', U \rangle \equiv 0$ implies $U' \subset L$. Lemma 6.7 in [H3] states that there is an orthogonal decomposition

$$L = U' \oplus {U'}^{\perp} =: U' \oplus L'.$$

Since χ is involutive, we have $\chi(L') = L'$. Then one can write

$$\begin{aligned}
\operatorname{Re}\sigma &= -\langle \operatorname{Re}\sigma, v \rangle v^* - \langle \operatorname{Re}\sigma, v^* \rangle v + \nu_1, \\
\operatorname{Im}\sigma &= -\langle \operatorname{Im}\sigma, v \rangle v^* - \langle \operatorname{Im}\sigma, v^* \rangle v + \nu_2,
\end{aligned}$$

where $\nu_1, \nu_2 \in L'_{\mathbb{R}}$ are given by

$$\begin{aligned}
\nu_1 &:= \operatorname{Re}\sigma + \langle \operatorname{Re}\sigma, v \rangle v^* + \langle \operatorname{Re}\sigma, v^* \rangle v, \\
\nu_2 &:= \operatorname{Im}\sigma + \langle \operatorname{Im}\sigma, v \rangle v^* + \langle \operatorname{Im}\sigma, v^* \rangle v.
\end{aligned}$$

Set $\lambda := \langle \operatorname{Re}\sigma, v\rangle - i\langle \operatorname{Im}\sigma, v\rangle$ and $\sigma' := \lambda\sigma$ one gets $\langle \operatorname{Im}\sigma', v\rangle = 0$. We will see that ω' and B', which depend on σ, are however independent of the scaling of σ. So we can assume that $\langle \operatorname{Im}\sigma, v\rangle = 0$, and hence

$$\begin{aligned}\operatorname{Re}\sigma &= -\langle \operatorname{Re}\sigma, v\rangle v^* - \langle \operatorname{Re}\sigma, v^*\rangle v + \nu_1,\\ \operatorname{Im}\sigma &= -\langle \operatorname{Im}\sigma, v^*\rangle v + \nu_2.\end{aligned}$$

Now we calculate B'. Since $\chi(W_1) = W_2'$, write

$$\chi(w_{11}) = \alpha_1 w_{21}' + \alpha_2 w_{22}' \quad \text{and} \quad \chi(w_{12}) = \beta_1 w_{21}' + \beta_2 w_{22}'.$$

As we supposed $\langle U', U\rangle \equiv 0$, β_1 must equal to 0. Since χ preserves orthogonality, we have $\alpha_2 = 0$. It follows that

$$\chi(w_{11}) = \alpha_1 w_{21}' \quad \text{and} \quad \chi(w_{12}) = \beta_2 w_{22}'.$$

From

$$\langle w_{11}, w_{11}\rangle = \langle w_{12}, w_{12}\rangle = \langle \chi(w_{11}), \chi(w_{11})\rangle = \langle \chi(w_{12}), \chi(w_{12})\rangle$$

we have

$$\alpha_1 = \pm\beta_2.$$

Further, our assumption $\chi^2 = \mathrm{Id}$ implies

$$w_{11} = \alpha_1 \chi(w_{21}') \quad \text{and} \quad w_{12} = \beta_2 \chi(w_{22}').$$

By rescaling σ with α_1 and by the orientation preserving property of χ we can assume

$$\chi(w_{11}) = w_{21}' \quad \text{and} \quad \chi(w_{12}) = w_{22}'. \tag{2.4.1}$$

Writing the first equation out we get

$$-\langle \operatorname{Re}\sigma, v\rangle w^* - \langle \operatorname{Re}\sigma, v^*\rangle w + \chi(\nu_1) + \langle B, \operatorname{Re}\sigma\rangle v = w^* + B' + \frac{1}{2}(\langle B', B'\rangle - \langle \omega', \omega'\rangle)w.$$

The terms in w^* and w on both sides cancel out since we assumed $\langle U', U\rangle \equiv 0$. A closer look at the w^*-terms shows that our scaling implies $\langle \operatorname{Re}\sigma, v\rangle = -1$. Writing $\chi(\nu_1)$ out we get

$$\Rightarrow \quad B' = \chi(\operatorname{Re}\sigma - v^* + \langle \operatorname{Re}\sigma, v^*\rangle v) + \langle B, \operatorname{Re}\sigma\rangle v.$$

From the second equation in (2.4.1) we get

$$\begin{aligned}-\langle \operatorname{Im}\sigma, v^*\rangle w + \chi(\nu_2) + \langle B, \operatorname{Im}\sigma\rangle v &= \omega' + \langle B', \omega'\rangle w\\ \Rightarrow \quad \omega' &= \chi(\operatorname{Im}\sigma + \langle \operatorname{Im}\sigma, v^*\rangle v) + \langle B, \operatorname{Im}\sigma\rangle v.\end{aligned}$$

Let us now turn to W_2. Again, $\chi^2 = \mathrm{Id}$ implies

$$\chi(W_2) = W_1' \quad \Leftrightarrow \quad \chi(W_1') = W_2.$$

Let us write

$$\chi(w_{11}') = \gamma_1 w_{21} + \gamma_2 w_{22} \quad \text{and} \quad \chi(w_{12}') = \delta_1 w_{21} + \delta_2 w_{22}.$$

For the same reason as for σ we can assume $\langle \operatorname{Im}\sigma', v\rangle = 0$. Hence $\delta_1 = \gamma_2 = 0$ and we can assume $\gamma_1 = 1 = \pm\delta_2$. Applying again $\chi = \chi^{-1}$ and by orientation reasons we get

$$\chi(w_{21}) = w_{11}' \quad \text{and} \quad \chi(w_{22}) = w_{12}'. \tag{2.4.2}$$

From the first equation we have

$$v^* + \chi(B) + \frac{1}{2}(\langle B, B\rangle - \langle \omega, \omega\rangle)v = \operatorname{Re}\sigma' + \langle B', \operatorname{Re}\sigma'\rangle w. \tag{2.4.3}$$

We pair both sides with w^* and get

$$\langle \chi(B), w^*\rangle = -\langle B', \operatorname{Re}\sigma'\rangle.$$

Thus (2.4.3) becomes

$$\Rightarrow \quad \operatorname{Re}\sigma' = v^* + \chi(B + \langle B, v^*\rangle v) + \frac{1}{2}(\langle B, B\rangle - \langle \omega, \omega\rangle v).$$

One sees that if $\langle B, v\rangle = 0$, then $\chi(B + \langle B, v^*\rangle v)$ lies in $L'_{\mathbb{R}}$ and will not give unwished terms in w^* (as σ' is a 2-form). This justifies our assumption on B.

We use the second equation of (2.4.2) to determine $\operatorname{Im}\sigma'$. We have

$$\chi(\omega) + \langle B, \omega\rangle v = \operatorname{Im}\sigma' + \langle B', \operatorname{Im}\sigma'\rangle w.$$

Again we pair both sides with w^* and get

$$\langle \chi(\omega), w^*\rangle = -\langle B', \operatorname{Im}\sigma'\rangle.$$

So

$$\Rightarrow \quad \operatorname{Im}\sigma' = \chi(\omega + \langle \omega, v^*\rangle v) + \langle B, \omega\rangle v.$$

Also here we have to impose $\langle \omega, v\rangle = 0$ in order to avoid w^* on the right hand side. This completes the proof. $\square$

For a sufficient condition for complex multiplication to be transmitted via mirror symmetry, we consider a special ω. Let Y be a K3 surface of CM-type over $K = K_0(\eta)$ with $\rho(Y) \geq 10$, $\eta^2 \in K_0$ is totally negative, and let $\sigma \in H^{2,0}(Y)$ be fully defined over K under ϵ (see Proposition 1.3.14). As we already saw in Corollary 1.3.16

$$v_1 := \operatorname{Re}\sigma \quad \text{and} \quad v_2 := \frac{i}{\epsilon(\eta)}\operatorname{Im}\sigma$$

are fully defined over K_0 and

$$E(v_1) \perp E(v_2).$$

Denote $n := \dim_{\mathbb{Q}} E(v_2)$ and let $\{e_i\}$ be an orthogonal basis of $E(v_2)$. Since $\rho(Y) \geq 10, n \leq 6$ and the intersection form is indefinite on $NS(Y)_{\mathbb{Q}}$, one finds successively pairwise orthogonal vectors $\{f_i\} \subset NS(Y)_{\mathbb{Q}}$ with

$$\langle f_i, f_i\rangle = \langle e_i, e_i\rangle.$$

If one writes

$$\operatorname{Im}\sigma = \sum_{i=1}^{n} \lambda_i e_i,$$

then set

$$\omega := \sum_{i=1}^{n} \lambda_i f_i$$

with the same coefficients. Then ω has the properties

(i) There is an isometry

$$\begin{aligned} E(\operatorname{Im}\sigma) &\longrightarrow E(\omega), \\ e_i &\longmapsto f_i. \end{aligned}$$

(ii) $\langle \omega, \omega\rangle = \langle \operatorname{Im}\sigma, \operatorname{Im}\sigma\rangle > 0$.

Theorem 2.4.4. *Let (Y', ω', B') be an involutive mirror of (Y, ω, B) under a mirror map χ, where Y is a K3 surface of CM-type with $\rho(Y) \geq 10$. Suppose further the following conditions:*

(i) $\langle U', U\rangle \equiv 0$ *and* $\chi\big|_{L'} = \mathrm{Id}_{L'}$.
(ii) ω *is as constructed above.*
(iii) $B = -v^* + \operatorname{Re}\sigma$, *where* $\sigma \in H^{2,0}(Y)$ *is fully defined over* K.

(iv) $\langle \omega, v\rangle = \langle B, v\rangle = 0$.

Then Y' is also of CM-type over K.

Proof. In view of the last proposition, conditions (i) and (iv) imply that there is a $\sigma' \in H^{2,0}(Y')$ with

$$\begin{aligned} \operatorname{Re}\sigma' &= v^* + B + (\langle B, v^*\rangle + \frac{1}{2}(\langle B, B\rangle - \langle \omega, \omega\rangle))v \\ \operatorname{Im}\sigma' &= \omega + (\langle \omega, v^*\rangle + \langle B, \omega\rangle)v. \end{aligned}$$

Then, in view of $\langle \omega, \omega\rangle = \langle \operatorname{Im}\sigma, \operatorname{Im}\sigma\rangle = \langle \operatorname{Re}\sigma, \operatorname{Re}\sigma\rangle$, (ii) and (iii) guarantee that

$$\operatorname{Re}\sigma' = \operatorname{Re}\sigma \quad \text{and} \quad \operatorname{Im}\sigma' = \omega.$$

Moreover we have a Hodge isometry

$$T = E(\operatorname{Re}\sigma) \oplus E(\operatorname{Im}\sigma) \xrightarrow{\sim} E(\operatorname{Re}\sigma) \oplus E(\omega).$$

Hence Y' is also of CM-type over K. □

Remark 2.4.5. In the case of the proposition above, we have

$$\operatorname{Re}\sigma' = \operatorname{Re}\sigma, \quad \operatorname{Im}\sigma' = \omega \quad \text{and} \quad \omega' = \operatorname{Im}\sigma.$$

We see that the mirror map χ is in fact a hyperkähler rotation (see [H5] for more details on the latter notation).

3. Lattice OPE-algebra, rationality and complex multiplication for abelian varieties

Summary of results: We construct a lattice OPE-algebra in Section 3.2 which generalizes the lattice vertex algebra in [Kac, §5.4]. We give explicitly its partition function in Remark 3.2.2, calculations are in Appendix A. We explain how to associate a superconformal lattice OPE-algebra to a generalized complex torus in Section 3.3. We show in Appendix B that it is isomorphic to the superconformal OPE-algebra constructed by Kapustin and Orlov in [KO]. In Section 3.4 we define the notion of rationality for lattice OPE-algebras. Applied to the toroidal ones we get Theorem 3.4.3 with which we rephrase in Section 3.5 our results on complex multiplication, rational Kähler metric and mirror symmetry for abelian varieties obtained in Sections 1 and 2 in terms of lattice OPE-algebras. This is given in Corollaries 3.5.3, 3.5.4 and 3.5.5.

We begin with generalities on OPE-algebras. OPE stands for Operator Product Expansion, it is the name for the formulas in (v) of Definition 3.1.1 below.

3.1. **OPE-algebras.**

We first recall the definition and a few facts about OPE-algebras from [Ros]. They generalize the much-studied vertex algebras.

A very general notion is an *E-valued distribution*, wehre E is a vector space. It is a formal sum

$$a(z,\bar z) := \sum_{n,\bar n\in\mathbb{R}} a_{n,\bar n} z^{-n-1}\bar z^{-\bar n-1}, \quad a_{n,\bar n}\in E.$$

Note that the exponents may be real numbers. The space of these distributions is denoted by $E\{z,\bar z\}$. If V is a vector space, we can talk about $\mathrm{End}(V)$-valued distributions. Moreover, for each $v\in V$, there is a map

$$\begin{aligned} \mathrm{End}(V)\{z,\bar z\} &\longrightarrow V\{z,\bar z\} \\ a(z,\bar z) &\longmapsto a(z,\bar z)v := \sum_{n,\bar n\in\mathbb{R}} a_{n,\bar n}(v) z^{-n-1}\bar z^{-\bar n-1}. \end{aligned}$$

A field, which is of interest for us, is a special kind of distributions. If for any $v\in V$, $a(z,\bar z)v$ is of the form

$$a(z,\bar z)v = \sum_{i=1}^{r} p_i(z,\bar z) z^{h_i}\bar z^{\bar h_i},$$

where $h_i,\bar h_i\in\mathbb{R}$ and $p_i(z,\bar z)$ is a power series in z and $\bar z$ with coefficients in V, then we call $a(z,\bar z)$ a *field on* V or an $\mathrm{End}(V)$*-valued field.* Distributions in more variables are defined similarly.

An OPE-algebra is defined as follows.

Definition 3.1.1. *A vector superspace* $V=V_0\oplus V_1$ *together with an even vector* $\mathbf{1}\in V_0$ *and a subspace* $\mathcal{F}\subset\mathrm{End}(V)\{z,\bar z\}$ *of* $\mathrm{End}(V)$*-valued distributions is an* operator product expansion (OPE-)algebra *if there exists a pair* $T,\bar T$ *of commuting even endomorphisms of* V *such that*

(i) $\mathbf{1}$ *is invariant, i.e.* $T\mathbf{1}=\bar T\mathbf{1}=0$.

(ii) *$\mathcal{F}$ is weakly creative, i.e. $a(z,\bar{z})\mathbf{1}$ is a power series in z and in $\bar{z}$, $\forall a(z,\bar{z}) \in \mathcal{F}$.*

(iii) *$\mathcal{F}$ is translation covariant, i.e.*

$$[T, a(z,\bar{z})] = \partial a(z,\bar{z}) \quad \textit{and} \quad [\bar{T}, a(z,\bar{z})] = \bar{\partial} a(z,\bar{z}).$$

(iv) *$\mathcal{F}$ is complete, i.e. the following map is surjective:*

$$\begin{aligned} s : \mathcal{F} &\longrightarrow V \\ a(z,\bar{z}) &\longmapsto a_{-1,-1}\mathbf{1}. \end{aligned} \tag{3.1.1}$$

(v) *$\mathcal{F}$ is local, i.e. for each $a(z,\bar{z}), b(z,\bar{z}) \in \mathcal{F}$ with parity $\tilde{a}, \tilde{b}$ (i.e. $s(a(z,\bar{z})) \in V_{\tilde{a}}$ and $s(b(z,\bar{z})) \in V_{\tilde{b}}$) there exist* $\mathrm{End}(V)$*-valued fields $C^i(z,\bar{z},w,\bar{w})$ and $h_i, \bar{h}_i \in \mathbb{R}$ with $h_i - \bar{h}_i \in \mathbb{Z}$ such that*

$$a(z,\bar{z})b(w,\bar{w}) = \sum_{i=1}^{r} \frac{C^i(z,\bar{z},w,\bar{w})}{(z-w)^{h_i}(\bar{z}-\bar{w})^{\bar{h}_i}}, \quad \textit{and}$$

$$(-1)^{\tilde{a}\tilde{b}} b(w,\bar{w})a(z,\bar{z}) = \sum_{i=1}^{r} (-1)^{\bar{h}_i - h_i} \frac{C^i(z,\bar{z},w,\bar{w})}{(w-z)^{h_i}(\bar{w}-\bar{z})^{\bar{h}_i}},$$

where the denominators are series as follows

$$\frac{1}{(u-v)^h(\bar{u}-\bar{v})^{\bar{h}}} = \sum_{j,\bar{j}=0}^{\infty} \binom{-h}{j}\binom{-\bar{h}}{\bar{j}} (-1)^{j+\bar{j}} v^j u^{-j-h} \bar{v}^{\bar{j}} \bar{u}^{-\bar{j}-\bar{h}}.$$

Let $\mathcal{F} \subset \mathrm{End}(V)\{z,\bar{z}\}$ define an OPE-algebra. A priori, from the definition above it is not clear that $\mathcal{F}$ only contains fields. This is actually a result of Rosellen [Ros, Prop.2 (iii)] Moreover, only powers $z^{-h-1}\bar{z}^{-\bar{h}-1}$ with integral difference $h - \bar{h} \in \mathbb{Z}$ occur. More precisely we have

Proposition 3.1.2 (Rosellen)**.** *If a subspace $\mathcal{F} \subset \mathrm{End}(V)\{z,\bar{z}\}$ defines an OPE-algebra, then $\mathcal{F}$ only contains fields. They are of the form*

$$a(z,\bar{z}) = \sum_{h,\bar{h} \in \mathbb{R}, h-\bar{h} \in \mathbb{Z}} a_{h,\bar{h}} z^{-h-1} \bar{z}^{-\bar{h}-1}.$$

Note that $h - \bar{h} \in \mathbb{Z}$ and field means for any $v \in V$, $a(z,\bar{z})v$ has bounded negative powers. We call V the space of states *and $\mathcal{F}$ the* space of fields .

Till now the algebra structure (e.g. a multiplication) is not yet apparent. We will explain this after giving

Theorem 3.1.3 (Goddard's Uniqueness Theorem)**.** *If $\mathcal{F} \subset \mathrm{End}(V)\{z,\bar{z}\}$ is a creative, complete, local subspace, then the map* (3.1.1) *is an isomorphism. Its inverse is an even linear map*

$$\begin{aligned} Y : V &\longrightarrow \mathcal{F} \\ a &\longmapsto Y(a,z,\bar{z}). \end{aligned}$$

We call it the state-field correspondence*. We have moreover for any $a(z,\bar{z}) \in \mathcal{F}$*

$$a(z,\bar{z}) = Y(a_{-1,-1}\mathbf{1}, z, \bar{z}).$$

We also write $a(z,\bar{z}) = Y(a,z,\bar{z})$.

For our construction of the lattice OPE-algebra in the next section it is more convenient to give explicitly the state-field correspondence Y than to describe the subspace $\mathcal{F}$. Moreover, Y endows V with an $\mathbb{R}^2$-fold algebra structure, namely, any $(m,\bar{m}) \in \mathbb{R}^2$ defines a (not necessarily associative) multiplication

$$\begin{aligned} V \otimes V &\longrightarrow V \\ a \otimes b &\longmapsto a_{(m,\bar{m})} b := a_{m,\bar{m}}(b), \end{aligned} \tag{3.1.2}$$

where $a_{m,\bar{m}} \in \operatorname{End}(V)$ is the coefficient of $z^{-m-1}\bar{z}^{-\bar{m}-1}$ in the field $a(z,\bar{z})$.

In order to define rationality we are interested in the subalgebras V_z and $V_{\bar{z}}$ which contain fields with integral powers in z respectively $\bar{z}$. More precisely

$$V_z := \{v \in V \mid Y(a,z,\bar{z}) \in \operatorname{End}(V)[[z^{\pm 1}]]\},$$

$$V_{\bar{z}} := \{v \in V \mid Y(a,z,\bar{z}) \in \operatorname{End}(V)[[\bar{z}^{\pm 1}]]\}.$$

In [KO] it is shown that V_z and $V_{\bar{z}}$ are vertex algebras in Kac's sense. In Cor.B.6 therein we also find that they supercommute. They act on V with multiplication (3.1.2) restricted to $\mathbb{Z} \times \{-1\}$ and $\{-1\} \times \mathbb{Z}$. This action is compatible with multiplications on the space of fields, namely the following map:

$$\begin{aligned} Y(V_z, z) \times Y(V,z,\bar{z}) &\longrightarrow Y(V,z,\bar{z}) \\ (a(z), b(z,\bar{z})) &\longmapsto a(z)_{(n)} b(z,\bar{z}) := \operatorname{res}_w (w-z)^n [a(w), b(z,\bar{z})]. \end{aligned} \tag{3.1.3}$$

That $a(z)_{(n)} b(z,\bar{z})$ is a field and lies in $Y(V,z,\bar{z})$ is shown in [KO, p.129] by using Goddard's Uniqueness Theorem. The compatibility with the product on the space of states is the fact

$$a(z)_{(n)} b(z,\bar{z}) = Y(a_{(n)} b, z, \bar{z}).$$

For the lattice OPE-algebra we construct in the next section we will determine explicitly V_z and $V_{\bar{z}}$. Besides, we need

Definition 3.1.4. *A* morphism *of OPE-algebras is a linear map of vector superspaces $f : V \to V'$ such that*

(i) $f(\mathbf{1}) = \mathbf{1}'$.
(ii) $fT = T'f, f\bar{T} = \bar{T}'f$.
(iii) *For all $u, v \in V$ we have $Y'(f(u),z,\bar{z})f(v) = f(Y(u,z,\bar{z})v)$.*

One can define additional structures on an OPE-algebra, e.g. superconformal structure, N=1, N=2 structures. We will see these in the next section. The notion of morphism shall be accordingly extended to comply with additional structures.

3.2. **Construction of lattice OPE-algebras.**

In this section we construct a N=2 superconformal lattice OPE-algebra which generalizes the lattice vertex algebra in [Kac, §5.4] in the sense that it contains the z- and bosonic parts as well as the $\bar{z}$- and fermionic parts. In a special case (for a special decomposition of the lattice) it reduces to Kac's lattice vertex algebra, see Remark 3.2.1. Moreover, we give its partition function in Remark 3.2.2 which can shed some light on its physical interpretation in CFT. How to attach it to tori will be explained in the next section.

We begin with an integral lattice (Λ, q), together with a $(z,\bar{z})$-decomposition

$$\Lambda_{\mathbb{R}} = \Lambda_z \oplus \Lambda_{\bar{z}} \tag{3.2.1}$$

over $\mathbb{R}$ which is orthogonal with respect to q. In this section q needs not be non-degenerate, nevertheless we will impose this condition for the discussion on rationality. The data (Λ, q, Λ_z) shall suffice to construct an N=1 OPE-algebra. If moreover we endow the vector space $\Lambda_\mathbb{R}$ with an almost complex structure $\mathcal{I}$, i.e. $\mathcal{I}^2 = -\operatorname{Id}$, then we will get an N=2 OPE-algebra (see [KO, §3] for the definition of N=1,N=2 structures). We shall write $a = a_z - a_{\bar{z}}$ with $a_z \in \Lambda_z$ and $a_{\bar{z}} \in \Lambda_{\bar{z}}$. Introduce two copies $\mathfrak{h}_b = \mathfrak{h}_f = \Lambda_\mathbb{C}$ of $\Lambda_\mathbb{C}$, which both inherit the decomposition

$$\mathfrak{h}_b = \mathfrak{h}_{bz} \oplus \mathfrak{h}_{b\bar{z}} \quad \text{and} \quad \mathfrak{h}_f = \mathfrak{h}_{fz} \oplus \mathfrak{h}_{f\bar{z}}.$$

The affinization is the Lie superalgebra

$$\hat{\mathfrak{h}} := \mathbb{C}[t, t^{-1}] \otimes \mathfrak{h}_b \oplus t^{\frac{1}{2}}\mathbb{C}[t, t^{-1}] \otimes \mathfrak{h}_f \oplus \mathbb{C}K$$

with even $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{h}_b \oplus \mathbb{C}K$, and odd $t^{\frac{1}{2}}\mathbb{C}[t, t^{-1}] \otimes \mathfrak{h}_f$. The supercommutators are described below by (3.2.5). Write $\mathbb{C}[\Lambda]$ for the group algebra of Λ over $\mathbb{C}$, and denote by $e^a, a \in \Lambda$ the basis vectors of $\mathbb{C}[\Lambda]$. Furthermore, write $\mathfrak{h}_b^< := t^{-1}\mathfrak{h}_b[t^{-1}]$, and $\mathfrak{h}_f^< := t^{-\frac{1}{2}}\mathfrak{h}_f[t^{-1}]$. The space of states is the superspace

$$V := \operatorname{Sym} \mathfrak{h}_b^< \otimes \bigwedge \mathfrak{h}_f^< \otimes \mathbb{C}[\Lambda],$$

where $\operatorname{Sym} \mathfrak{h}_b^<$ is the symmetric algebra of $\mathfrak{h}_b^<$ and $\bigwedge \mathfrak{h}_f^<$ is the exterior algebra of $\mathfrak{h}_f^<$. The vacuum vector $\mathbf{1} = |vac\rangle := 1 \otimes 1 \otimes 1$. The parity on V is

$$p(s \otimes e^a) = p(s) + (q(a, a) \mod 2), \tag{3.2.2}$$

for $p(s)$ just note that $p(s_1 \otimes s_2) = p(s_1) + p(s_2)$. The representation π of $\hat{\mathfrak{h}}$ on V is defined as

$$\pi := \pi_1 \otimes 1 + 1 \otimes \pi_2,$$

with π_1 the representation of $\hat{\mathfrak{h}}$ on $\operatorname{Sym} \mathfrak{h}_b^< \otimes \bigwedge \mathfrak{h}_f^<$ determined by

$$\begin{aligned}
& K \longmapsto \operatorname{Id}, \\
& h \longmapsto 0, \quad h \in \mathfrak{h}_b, \\
n < 0, \quad & t^n h \longmapsto \text{ multiplication by } t^n h, h \in \mathfrak{h}_b \text{ or } h \in \mathfrak{h}_f, \\
n > 0, \quad & t^n h \longmapsto (t^{-s}h' \mapsto n\delta_{n,s} q(h, h')), \quad h, h' \in \mathfrak{h}_{bz}, \\
& t^n \bar{h} \longmapsto (t^{-s}\bar{h}' \mapsto -n\delta_{n,s} q(\bar{h}, \bar{h}')), \quad \bar{h}, \bar{h}' \in \mathfrak{h}_{b\bar{z}}, \\
& t^n h \longmapsto (t^{-s}h' \mapsto \delta_{n,s} q(h, h')), \quad h, h' \in \mathfrak{h}_{fz}, \\
& t^n \bar{h} \longmapsto (t^{-s}\bar{h}' \mapsto -\delta_{n,s} q(\bar{h}, \bar{h}')), \quad \bar{h}, \bar{h}' \in \mathfrak{h}_{f\bar{z}},
\end{aligned} \tag{3.2.3}$$

and π_2 the representation of $\hat{\mathfrak{h}}$ on $\mathbb{C}[\Lambda]$ determined by:

$$\begin{aligned}
& K \longmapsto 0, \\
& t^n h \longmapsto (e^a \mapsto \delta_{n,0} q(h, a) e^a), \quad h \in \mathfrak{h}_{bz}, \\
& t^n \bar{h} \longmapsto (e^a \mapsto -\delta_{n,0} q(\bar{h}, a) e^a), \quad \bar{h} \in \mathfrak{h}_{b\bar{z}}, \\
& t^n h \longmapsto 0, \quad h \in \mathfrak{h}_f, \forall n.
\end{aligned} \tag{3.2.4}$$

If we write $h_n := \pi(t^n h)$, then for $h \in \mathfrak{h}_{bz}$, $\bar{h} \in \mathfrak{h}_{b\bar{z}}$, $f \in \mathfrak{h}_{fz}$, $\bar{f} \in \mathfrak{h}_{f\bar{z}}$, and $m, n \in \mathbb{Z}$, $r, s \in \mathbb{Z} + \frac{1}{2}$ the supercommutators are

$$\begin{aligned}
& [h_n, h'_m] = n\delta_{n,-m} q(h, h'), \quad [\bar{h}_n, \bar{h}'_m] = -n\delta_{n,-m} q(\bar{h}, \bar{h}'), \\
& \{f_r, f'_s\} = \delta_{r,-s} q(f, f'), \qquad \{\bar{f}_r, \bar{f}'_s\} = -\delta_{r,-s} q(\bar{f}, \bar{f}'),
\end{aligned} \tag{3.2.5}$$

and all other relations are trivial. The state-field correspondence maps a homogeneous vector

$$v = h^1_{-s_1} \cdots h^n_{-s_n} \bar{h}^1_{-\bar{s}_1} \cdots \bar{h}^{\bar{n}}_{-\bar{s}_{\bar{n}}} f^1_{-r_1} \cdots f^q_{-r_q} \bar{f}^1_{-\bar{r}_1} \cdots \bar{f}^{\bar{q}}_{-\bar{r}_{\bar{q}}} \otimes e^a, \tag{3.2.6}$$

where $s_i, \bar{s}_{\bar{i}}$ are positive integers and $r_i, \bar{r}_{\bar{i}}$ are positive half-integers, to the field

$$\begin{aligned} v(z,\bar{z}) :=& Y(v,z,\bar{z}) = \sum_{b\in\Lambda} \epsilon(a,b) e^a \operatorname{Pr}_b z^{q(a_z,b_z)} \bar{z}^{-q(a_{\bar{z}},b_{\bar{z}})} \\ & \times \exp(-\sum_{n<0} \frac{a_{zn}}{nz^n} + \sum_{n<0} \frac{a_{\bar{z}n}}{n\bar{z}^n}) \\ & \times : \prod_{l=1}^{n} \frac{\partial^{s_l} H^l(z)}{(s_l-1)!} \prod_{\bar{l}=1}^{\bar{n}} \frac{\bar{\partial}^{\bar{s}_{\bar{l}}} \bar{H}^{\bar{l}}(\bar{z})}{(\bar{s}_{\bar{l}}-1)!} \prod_{t=1}^{q} \frac{\partial^{r_t-\frac{1}{2}} F^t(z)}{(r_t-\frac{1}{2})!} \prod_{\bar{t}=1}^{\bar{q}} \frac{\bar{\partial}^{\bar{r}_{\bar{t}}-\frac{1}{2}} \bar{F}^{\bar{t}}(\bar{z})}{(\bar{r}_{\bar{t}}-\frac{1}{2})!} : \\ & \times \exp(-\sum_{n>0} \frac{a_{zn}}{nz^n} + \sum_{n>0} \frac{a_{\bar{z}n}}{n\bar{z}^n}), \end{aligned} \tag{3.2.7}$$

where Pr_b is the projection onto $\operatorname{Sym} \mathfrak{h}_b^< \otimes \bigwedge \mathfrak{h}_f^< \otimes e^b$ and

$$\begin{aligned} \partial H(z) &:= \sum_{m\in\mathbb{Z}} h_m z^{-m-1}, & \bar{\partial}\bar{H}(\bar{z}) &:= \sum_{m\in\mathbb{Z}} \bar{h}_m \bar{z}^{-m-1}, \\ F(z) &:= \sum_{r\in\mathbb{Z}+\frac{1}{2}} f_r z^{-r-\frac{1}{2}}, & \bar{F}(\bar{z}) &:= \sum_{r\in\mathbb{Z}+\frac{1}{2}} \bar{f}_r \bar{z}^{-r-\frac{1}{2}}, \end{aligned}$$

and the factor $\epsilon(a,b)$ satisfies the equations (5.4.14) in [Kac]. We give a few examples:

- $v = 1 \otimes 1 \otimes e^0 = \mathbf{1}$, $Y(v,z,\bar{z}) = \sum_{b\in\Lambda} \operatorname{Pr}_b = id$,
- $v = h_{-s} \otimes e^0$, $Y(v,z,\bar{z}) = \frac{1}{(s-1)!} \partial^s H(z)$,
- $v = f_{-r} \otimes e^0$, $Y(v,z,\bar{z}) = \frac{1}{(r-\frac{1}{2})!} \partial^{r-\frac{1}{2}} F(z)$,
- $v = 1 \otimes 1 \otimes e^a$,

$$\begin{aligned} Y(v,z,\bar{z}) =& \sum_{b\in\Lambda} \epsilon(a,b) e^a \operatorname{Pr}_b z^{q(a_z,b_z)} \bar{z}^{-q(a_{\bar{z}},b_{\bar{z}})} \\ & \times \exp(-\sum_{n<0} \frac{a_{zn}}{nz^n} + \sum_{n<0} \frac{a_{\bar{z}n}}{n\bar{z}^n}) \exp(-\sum_{n>0} \frac{a_{zn}}{nz^n} + \sum_{n>0} \frac{a_{\bar{z}n}}{n\bar{z}^n}). \end{aligned}$$

Next we define the maps T and $\bar{T}$. Let $\{E^i\}$ respectively $\{\bar{E}^i\}$ be a bosonic basis of Λ_z respectively $\Lambda_{\bar{z}}$ and $\{\widetilde{E}^i\}$ respectively $\{\widetilde{\bar{E}}^i\}$ be the dual basis with respect to q, i.e. $q(E^i, \widetilde{E}^j) = \delta^{ij}$ and $q(\bar{E}^i, \widetilde{\bar{E}}^j) = -\delta^{ij}$. The fermionic bases are denoted by $\{F^i\}$, $\{\bar{F}^i\}$ and $\{\widetilde{F}^i\}$, $\{\widetilde{\bar{F}}^i\}$. Then

$$T := \sum_i \left(\sum_{n\geq 0} E^i_{-n-1} \widetilde{E}^i_n + \sum_{r=\frac{1}{2},\frac{3}{2}\dots} (r+\frac{1}{2}) F^i_{-r-1} \widetilde{F}^i_r \right), \quad \bar{T} \text{ is analogous.} \tag{3.2.8}$$

So we have all the ingredients for an OPE-algebra. In Appendix B we show that attached to a torus it is isomorphic to the vertex algebra in [KO] which is an OPE-algebra.

Further, the superconformal structure is:

$$L := \frac{1}{2} \sum_i (E^i_{-1} \widetilde{E}^i_{-1} - F^i_{-\frac{1}{2}} \widetilde{F}^i_{-\frac{3}{2}}) \otimes e^0, \quad \bar{L} \text{ is analogous.}$$

Remark 3.2.1. By inspecting the state-field correspondence (3.2.7) one sees that fields may have non-integral powers in z and $\bar{z}$ due to the term $z^{q(a_z,b_z)}\bar{z}^{-q(a_{\bar{z}},b_{\bar{z}})}$ (recall that the $(z,\bar{z})$-decomposition (3.2.1) is only required to be defined over $\mathbb{R}$). However, the difference of the exponents of z and $\bar{z}$ is always an integer:

$$q(a_z,b_z)+q(a_{\bar{z}},b_{\bar{z}})=q(a,b)\in\mathbb{Z}.$$

This implies that in the special case of a trivial decomposition, i.e. $\Lambda_{\bar{z}}=0$ or in other words $\Lambda_{\mathbb{R}}=\Lambda_z$, the lattice OPE-algebra is nothing but a conformal lattice vertex algebra in the sense of [Kac, §5.4 §5.5] (plus a fermionic part).

The N=1 structure is

$$Q:=\frac{i}{2\sqrt{2}}\sum_i F^i_{-\frac{1}{2}}\widetilde{E}^i_{-1}\otimes e^0.$$

Given an almost complex structure $\mathcal{I}$ on the vector space $\Lambda_{\mathbb{R}}$, the N=2 structure is denoted by

$$\begin{aligned}Q^{\pm}:&=\frac{i}{4\sqrt{2}}\sum_i(F^i_{-\frac{1}{2}}\widetilde{E}^i_{-1}\pm F^i_{-\frac{1}{2}}(\mathcal{I}\widetilde{E}^i)_{-1})\otimes e^0,\\ J:&=-\frac{i}{2}\sum_i F^i_{-\frac{1}{2}}(\mathcal{I}\widetilde{F}^i)_{-\frac{1}{2}}\otimes e^0.\end{aligned}$$

and the analogous $\bar{z}$-part (see [KO, §3] for the definition of these structures).

Superconformal lattice OPE-algebras find an interpretation in CFT once attached to tori (discussed in the next section). Its partition function is central to the physical theory. We give explicitly the partition function for any superconformal lattice OPE-algebra. Since this does not influence our work on rationality we put these results in a remark and the proofs are given in Appendix A.

Remark 3.2.2. Let $\Lambda_{\mathbb{R}}=\Lambda_z\oplus\Lambda_{\bar{z}}$ be an orthogonal decomposition of a lattice (Λ,q). Denote

$$d:=\dim\Lambda_z\quad\text{and}\quad\bar{d}:=\dim\Lambda_{\bar{z}}.$$

Let $V(\Lambda,q,\Lambda_z)$ together with the superconformal vectors $(L,\bar{L})$ be the associated superconformal lattice OPE-algebra. Then the central charges are

$$c=\frac{3d}{2}\quad\text{and}\quad\bar{c}=\frac{3\bar{d}}{2}.$$

From the expansion of $L(z)=Y(L,z)$ we get

$$L_0=\sum_{i=1}^{d}\left(\frac{1}{2}E^i_0\tilde{E}^i_0+\sum_{n\geq 1}E^i_{-n}\tilde{E}^i_n+\sum_{r=\frac{1}{2},\frac{3}{2},\ldots}rF^i_{-r}\tilde{F}^i_r\right)\tag{3.2.9}$$

(analogously for $\bar{L}_0$). The partition function is then

$$\begin{aligned}Z&=\mathrm{Tr}_V\,\mathfrak{q}^{L_0-\frac{c}{24}}\bar{\mathfrak{q}}^{\bar{L}_0-\frac{\bar{c}}{24}}\\ &=\frac{1}{\eta(\tau)^d\eta(\bar{\tau})^{\bar{d}}}\left(\frac{\theta_3(\tau)}{\eta(\tau)}\right)^{\frac{d}{2}}\left(\frac{\theta_3(\bar{\tau})}{\eta(\bar{\tau})}\right)^{\frac{\bar{d}}{2}}\left(\sum_{a\in\Lambda}\mathfrak{q}^{\frac{1}{2}q(a_z,a_z)}\bar{\mathfrak{q}}^{\frac{1}{2}q(a_{\bar{z}},a_{\bar{z}})}\right),\end{aligned}\tag{3.2.10}$$

where we set $\mathfrak{q}=e^{2\pi i\tau}$ and $\bar{\mathfrak{q}}=e^{2\pi i\bar{\tau}}$. The two modular forms are Jacobi theta function θ_3 and the Dedekind eta function η. See Appendix A for proofs. The sum in the last term reveals whether the lattice OPE-algebra is rational, see Remark 3.4.5.

In the next section we attach a superconformal lattice OPE-algebra to tori and we will see the case of a circle in Example 3.4.4 where we also discuss about rationality.

3.3. **Toroidal lattice OPE-algebras.**

Now we explain how tori give rise to lattice OPE-algebras. To a real torus $\mathbb{T}$ together with a constant metric G and a B-field, one can associate a N=1 superconformal lattice OPE-algebra $V(\mathbb{T}, G, B)$ by setting

$$\begin{aligned} \Lambda = \Gamma \oplus \Gamma^*, \quad & q((a_1, a_2), (b_1, b_2)) := -\langle a_1, b_2 \rangle - \langle a_2, b_1 \rangle \\ \text{and} \quad \Lambda_z = \mathrm{Graph}_{\Gamma_\mathbb{R}}(-G + B), \quad & \Lambda_{\bar{z}} = \mathrm{Graph}_{\Gamma_\mathbb{R}}(G + B) \end{aligned} \tag{3.3.1}$$

and define the factor $\epsilon(a, b) := \exp(i\pi\langle a_1, b_2\rangle)$ for $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in $\Gamma \oplus \Gamma^*$.

To a generalized complex torus $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ one can associate two N=2 structures. Set (Λ, q) and $\epsilon(a, b)$ as in (3.3.1) and

$$\Lambda_z = \mathrm{Image}_{\Lambda_\mathbb{R}}(\mathrm{Id} + \mathcal{I}\mathcal{J}), \quad \Lambda_{\bar{z}} = \mathrm{Image}_{\Lambda_\mathbb{R}}(-\mathrm{Id} + \mathcal{I}\mathcal{J})$$

(see Lemma 2.2.10 (i)). Now one can choose either $\mathcal{I}$ or $\mathcal{J}$ to define $Q^\pm$ and J. As $\mathcal{I} = -\mathcal{J}$ on Λ_z and $\mathcal{I} = \mathcal{J}$ on $\Lambda_{\bar{z}}$ the two N=2 structures are related as follows:

$$Q^\pm_\mathcal{I} = Q^\mp_\mathcal{J}, \quad \bar{Q}^\pm_\mathcal{I} = \bar{Q}^\pm_\mathcal{J}, \quad J_\mathcal{I} = -J_\mathcal{J}, \quad \bar{J}_\mathcal{I} = \bar{J}_\mathcal{J}. \tag{3.3.2}$$

For simplicity, we denote by $V(\mathbb{T}, \mathcal{I}, \mathcal{J})$ either the N=2 superconformal lattice OPE-algebra defined by $\mathcal{I}$ or $\mathcal{J}$. If additionally, $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ is induced by (T, G, B) (see Definition 2.2.6), we also write $V(T, G, B)$ for the N=2 superconformal lattice OPE-algebra $V(\mathbb{T}, \mathcal{I}, \mathcal{J})$.

In Appendix B we prove

Proposition 3.3.1. *The N=2 superconformal lattice OPE-algebra $V(T, G, B)$ is isomorphic to the N=2 superconformal vertex algebra constructed in* [KO].

Adopting the viewpoint of lattice OPE-algebras has the advantage of having a basis-free construction. Moreover, it is more apparent how the lattice determines the structure of the OPE-algebra. This leads us to the formulation of rationality for lattice OPE-algebras (see Definition 3.4.1) and eventually to prove that rationality is completely determined by the size of the so-called chiral sublattice Λ_{ch} of Λ (see Proposition 3.4.2). In Example 3.4.4 we look at the case of a circle.

3.4. **Rationality.**

In this section we suppose that the integral bilinear form q is non-degenerate. We phrase rationality of a lattice OPE-algebra $V := V(\Lambda, q, \Lambda_z)$ in terms of the size of its so-called chiral subalgebra (see Definition 3.4.1). It is a subalgebra whose fields only contain integral exponents in z and $\bar{z}$, though it is not necessarily a *lattice* OPE-algebra. We will explicitly construct it. Roughly speaking, we call V rational if its chiral subalgebra is so big that V breaks into a finite direct sum of irreducible representations of it. Theorem 3.4.3 gives necessary and sufficient conditions for rationality of toroidal lattice OPE-algebras.

In Section 3.1 we mentioned that V_z and $V_{\bar{z}}$ which only give rise to fields containing integral powers in z respectively $\bar{z}$ are vertex algebras in the classical sense. For lattice OPE-algebras they are easy to determine. Write $\Lambda_\mathbb{R} = \Lambda_z \oplus \Lambda_{\bar{z}}$ and define

$$\mathfrak{h}_{b,z} := \Lambda_z \otimes_\mathbb{R} \mathbb{C}, \quad \mathfrak{h}^<_{b,z} := t^{-1}\mathfrak{h}_{b,z}[t^{-1}],$$

$$\mathfrak{h}_{f,z} := \Lambda_z \otimes_{\mathbb{R}} \mathbb{C}, \quad \mathfrak{h}^<_{f,z} := t^{-\frac{1}{2}}\mathfrak{h}_{f,z}[t^{-1}],$$
$$\Gamma_z := \{\lambda \in \Lambda \mid \lambda_{\bar{z}} = 0\}.$$

Then we get

$$V_z = \operatorname{Sym}\mathfrak{h}^<_{b,z} \otimes \bigwedge \mathfrak{h}^<_{f,z} \otimes \mathbb{C}[\Gamma_z], \quad V_{\bar{z}} \text{ is analogous.}$$

For this equality we need the non-degeneracy of q. Note that in general V_z and $V_{\bar{z}}$ do not bear the structure of a lattice OPE-algebra because the rank of Γ_z may be smaller than the dimension of Λ_z. Rationality designates precisely the case where they do possess the lattice structure. We explain this.

We build the tensor product

$$V_{ch} := V_z \otimes V_{\bar{z}},$$
$$(u \otimes e^a) \otimes (\bar{u} \otimes e^{\bar{a}}) := (-1)^{p(e^a)p(\bar{u})}(u \otimes \bar{u}) \otimes e^{a+\bar{a}},$$

where the parity $p(\cdot)$ was defined in (3.2.2). Define the *chiral sublattice* as

$$\Lambda_{ch} := \Gamma_z \oplus \Gamma_{\bar{z}}. \tag{3.4.1}$$

We have

$$\Lambda_{ch} = \{\lambda \in \Lambda \mid \lambda_z \in \Lambda \text{ or equivalently } \lambda_{\bar{z}} \in \Lambda\}$$

and

$$V_{ch} = \operatorname{Sym}\mathfrak{h}^<_b \otimes \bigwedge \mathfrak{h}^<_f \otimes \mathbb{C}[\Lambda_{ch}]. \tag{3.4.2}$$

With the restricted operators $T, \bar{T}$ and state-field correspondence Y, V_{ch} is again an OPE-algebra, though not necessarily a lattice OPE-algebra. We see that all fields of V_{ch} only contain integral powers in z and $\bar{z}$. We call V_{ch} the *chiral subalgebra* of V. Conversely, the fields of V which only have integral powers in z and $\bar{z}$ are not all contained in V_{ch}. This is because the following inclusion

$$\Lambda_{ch} \subseteq \{\lambda \in \Lambda \mid q(\lambda_z, a) \in \mathbb{Z}, \forall a \in \Lambda\}$$

is not an equality in general unless we impose unimodularity on the lattice (Λ, q). In other words, if (Λ, q) is unimodular, then the chiral subalgebra V_{ch} consists exactly of the fields of V which only contain integral powers in z and $\bar{z}$.

Furthermore, V_{ch} has an action on V. Recall the structure of $\mathbb{R}^2$-fold algebra of V from Section 3.1. Then V_{ch} acts on V by restricted $\mathbb{Z}^2$-multiplications. They are compatible with multiplications on the space of fields (defined in (3.1.3)). More precisely,

$$\begin{aligned} Y((u \otimes e^a) &\otimes (\bar{u} \otimes e^{\bar{a}})_{(n,\bar{n})}(v \otimes e^b), z, \bar{z}) \\ &= Y(u \otimes e^a, z) \otimes_{(n)} (Y(\bar{u} \otimes e^{\bar{a}}, \bar{z}) \otimes_{(\bar{n})} Y(v \otimes e^b, z, \bar{z})) \\ &= Y(\bar{u} \otimes e^{\bar{a}}, \bar{z}) \otimes_{(\bar{n})} (Y(u \otimes e^a, z) \otimes_{(n)} Y(v \otimes e^b, z, \bar{z})). \end{aligned}$$

Now we give

Definition 3.4.1. *A lattice OPE-algebra $V(\Lambda, q, \Lambda_z)$ is* rational *if it decomposes into a finite sum of irreducible modules over its chiral subalgebra V_{ch}. A N=2 superconformal lattice OPE-algebra is* rational*, if its underlying lattice OPE-algebra (without the N=2 and superconformal structures) is rational.*

It turns out that rationality is determined by the size of Λ_{ch}. Indeed, there is an obvious decomposition of V. For any $\alpha \in \Lambda$, denote by $\chi_\alpha \subset \mathbb{C}[\Lambda]$ the subspace spanned by all vectors $e^{\alpha+\lambda}$ with $\lambda \in \Lambda_{ch}$. Clearly, it is independent of the choice of the representant of $[\alpha] \in \Lambda/\Lambda_{ch}$, i.e. $\chi_\alpha = \chi_{\alpha'}$ for $[\alpha] = [\alpha'] \in \Lambda/\Lambda_{ch}$. Then

$$V = \bigoplus_{[\alpha]\in\Lambda/\Lambda_{ch}} \operatorname{Sym} \mathfrak{h}_b^< \otimes \bigwedge \mathfrak{h}_f^< \otimes \chi_\alpha =: \bigoplus_{[\alpha]\in\Lambda/\Lambda_{ch}} V_\alpha, \tag{3.4.3}$$

and each V_α is a $\mathbb{Z}^2$-fold module over the chiral subalgebra V_{ch}. We show

Theorem 3.4.2. *Let (Λ, q) be an integral lattice endowed with an orthogonal decomposition $\Lambda_\mathbb{R} = \Lambda_z \oplus \Lambda_{\bar z}$ over $\mathbb{R}$. The following is equivalent.*

(i) *The lattice OPE-algebra $V(\Lambda, q, \Lambda_z)$ is rational.*
(ii) *The chiral sublattice Λ_{ch} is of maximal rank in Λ, i.e. $[\Lambda : \Lambda_{ch}] < \infty$.*
(iii) *The decomposition of $\Lambda_\mathbb{R}$ is defined over $\mathbb{Q}$.*
(iv) *The chiral subalgebra V_{ch} is a lattice OPE-algebra.*

Proof. (i)⇔(ii): If Λ_{ch} is of maximal rank, the decomposition (3.4.3) is finite. We show that in this case, each V_α is an irreducible module. Indeed, as Λ_{ch} is of maximal rank, V_{ch} is isomorphic to V_α as vector space, and the action of V_{ch} on V_α is faithful. By [Kac, Prop.5.4] we only need check that if for some $v \in V_\alpha$, we have $(E^i_{-1} \otimes 1)_{(m)}v = 0$ and $(1 \otimes e^a)_{(m)}v = 0, \forall m, \forall i$ and $\forall a \in \Lambda_{ch}$ and similarly for the $\bar z$-part, then $v = 0$. This is clear by inspecting the explicit expressions of the corresponding field of these vectors (see the examples of fields given in Section 3.2).

Conversely, if $V(\Lambda, q, \Lambda_z)$ is rational, then the sum (3.4.3) must be finite, hence Λ_{ch} is of maximal rank.

(ii)⇔(iii): If the decomposition of $\Lambda_\mathbb{R}$ is defined over $\mathbb{Q}$, then we have

$$\Lambda_\mathbb{Q} = (\Lambda_z \cap \Lambda_\mathbb{Q}) \oplus (\Lambda_{\bar z} \cap \Lambda_\mathbb{Q}),$$
$$\Gamma_{z,\mathbb{Q}} = \Lambda_z \cap \Lambda_\mathbb{Q}, \qquad \Gamma_{\bar z,\mathbb{Q}} = \Lambda_{\bar z} \cap \Lambda_\mathbb{Q},$$

hence $\Lambda_\mathbb{Q} = \Lambda_{ch,\mathbb{Q}}$.

Conversely we have $\Lambda_z = \Gamma_{z,\mathbb{R}}$ and $\Lambda_{\bar z} = \Gamma_{\bar z,\mathbb{R}}$. Hence (iii).

(ii)⇔(iv): This is obvious by (3.4.2). □

For toroidal lattice OPE-algebras, in view of Lemma 2.2.10, Theorem 3.4.2 has as consequence the following

Theorem 3.4.3. (i) *The lattice OPE-algebra $V(\mathbb{T}, G, B)$ associated to a real torus with a constant metric G and a B-field B is rational if and only if G and B are both rational.*

(ii) *The N=2 superconformal lattice OPE-algebra $V(\mathbb{T}, \mathcal{I}, \mathcal{J})$ associated to a generalized complex torus is rational if and only if the composition $\mathcal{I}\mathcal{J}$ is defined over $\mathbb{Q}$.*

(iii) *The N=2 superconformal lattice OPE-algebra $V(T, G, B)$ associated to a complex torus T endowed with a constant Kähler metric G and a B-field is rational if and only if G and B are both rational.*

We illustrate this theorem in the simplest case.

Example 3.4.4. We describe the lattice OPE-algebra associated to a circle. Let $C := \mathbb{R}/\mathbb{Z}$ be a circle of radius R, i.e.

$$C \cong \operatorname{Image}(\mathbb{R} \longrightarrow \mathbb{R}^2, t \longmapsto Re^{2\pi i t}).$$

Consider the standard metric on $\mathbb{R}^2$ to be normed such that $(1,0)$ and $(0,1)$ are orthogonal and have norm $\frac{1}{2\pi}$. Thus the pulled-back metric G on C is $G = R^2$. As to the B-field, for dimension reasons we have $B = 0$.

The lattice of the OPE-algebra is

$$\Lambda = H_1(C, \mathbb{Z}) \oplus H^1(C, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

The decomposition is

$$\begin{aligned}\Lambda_{\mathbb{R}} &= \Lambda_z \oplus \Lambda_{\bar{z}} \\ &= \mathrm{Graph}(-G) \oplus \mathrm{Graph}(G) \\ &\cong \{(x, -R^2x) \mid x \in \mathbb{R}\} \oplus \{(x, R^2x) \mid x \in \mathbb{R}\}.\end{aligned}$$

The exponents in z and $\bar{z}$ which may not be rational are $q(a_z, b_z)$ and $q(a_{\bar{z}}, b_{\bar{z}})$, $a, b \in \Lambda$. Let us write

$$\begin{aligned}a = (a_1, a_2) = a_z - a_{\bar{z}} &= (x, -R^2x) - (y, R^2y) \quad \text{for } x, y \in \mathbb{R}, \\ b = (b_1, b_2) = b_z - b_{\bar{z}} &= (u, -R^2u) - (v, R^2v) \quad \text{for } u, v \in \mathbb{R}.\end{aligned}$$

Hence

$$a_1 = x - y, \quad a_2 = -R^2(x+y), \quad b_1 = u - v, \quad b_2 = -R^2(u+v) \tag{3.4.4}$$

and

$$q(a_z, b_z) = 2R^2xu, \quad q(a_{\bar{z}}, b_{\bar{z}}) = -2R^2yv. \tag{3.4.5}$$

Obviously, for $a, b \in \Lambda$, if $R^2 \in \mathbb{Q}$, then in view of (3.4.4), we get $x, y, u, v \in \mathbb{Q}$, hence the numbers in (3.4.5) are rational, and the chiral lattice Λ_{ch} as in (3.4.1) is of maximal rank in Λ. The lattice OPE-algebra is thus rational.

Conversely, for Λ_{ch} to be of maximal rank in Λ, (3.4.5) must be rational numbers for all $a, b \in \Lambda$. Substituting (3.4.4) into (3.4.5) one gets the condition

$$R^2 a_1 b_1 + \frac{a_2 b_2}{R^2} \in \mathbb{Q}, \quad \forall a, b \in \Lambda.$$

Hence R^2 must be rational. Moreover one can also consider the superconformal structure we defined in Section 3.2. The partition function takes an easy form.

Remark 3.4.5. In terms of the partition function Z, we see that the sum in (3.2.10) can be written as a finite sum over the elements of Λ/Λ_{ch} if $V(\mathbb{T}, G, B)$ is rational. Indeed, due to the following decomposition over $\mathbb{Z}$:

$$\Lambda_{ch} = \Gamma_z \oplus \Gamma_{\bar{z}}$$

the sum in Z becomes

$$\sum_{b \in \Lambda/\Lambda_{ch}} \left(\sum_{c \in \Gamma_z} \mathfrak{q}^{\frac{1}{2}q(c+b_z, c+b_z)} \right) \left(\sum_{d \in \Gamma_{\bar{z}}} \bar{\mathfrak{q}}^{\frac{1}{2}q(d+b_{\bar{z}}, d+b_{\bar{z}})} \right).$$

The first summation is finite if Λ_{ch} is of maximal rank in Λ.

In the next section we draw some consequences of Theorem 3.4.3.

3.5. **Complex multiplication, rationality and mirror symmetry.**

In this section we use Theorem 3.4.3 to rephrase results of Sections 1 and 2 about abelian varieties in terms of lattice OPE-algebras. Let us start out with an analogue of [KO, Thm 5.4], which shows that mirror symmetry for generalized complex tori can be alternatively expressed in terms of their associated lattice OPE-algebras. First we recall the definition of mirror symmetry for OPE-algebras from [KO].

Definition 3.5.1. *Two N=2 lattice OPE-algebras are* mirror partners *if there is an isomorphism $f : V \to V'$ of their space of states, such that*

(i) $f(\mathbf{1}) = f(\mathbf{1}')$,
(ii) $fT = T'f, f\bar{T} = \bar{T}'f$,
(iii) *for all $u, v \in V$, we have $Y'(f(u), z, \bar{z})v = f(Y(u, z, \bar{z})v)$,*

with the additional property:

$$f(Q^{\pm}) = Q^{\mp'},\ f(J) = -J',$$
$$f(\bar{Q}^{\pm}) = \bar{Q}^{\pm'},\ f(\bar{J}) = \bar{J}'.$$

Theorem 3.5.2. *Two generalized complex tori $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ and $(\mathbb{T}', \mathcal{I}', \mathcal{J}')$ are mirror partners if and only if the N=2 lattice OPE-algebras $V(\mathbb{T}, \mathcal{I}, \mathcal{J})$ and $V(\mathbb{T}', \mathcal{I}', \mathcal{J}')$ are mirror partners.*

Proof. Let φ be a mirror map between the two generalized complex tori. Since φ preserves q and the decomposition (2.2.2) (see Lemma 2.2.11 (i)), it induces an isomorphism f between the representations π of $\hat{\mathfrak{h}}$ on V and π' of $\hat{\mathfrak{h}}'$ on V', hence f satisfies (i)-(iii) of Definition 3.5.1. As φ sends $\mathcal{I} \mapsto \mathcal{J}'$, we have in view of (3.3.2)

$$Q^{\pm}_{\mathcal{I}} \mapsto Q^{\pm'}_{\mathcal{J}'} = Q^{\mp'}_{\mathcal{I}'} \quad J_{\mathcal{I}} \mapsto J'_{\mathcal{J}'} = -J'_{\mathcal{I}'}$$
$$\bar{Q}^{\pm}_{\mathcal{I}} \mapsto \bar{Q}^{\pm'}_{\mathcal{J}'} = \bar{Q}^{\pm'}_{\mathcal{I}'} \quad \bar{J}_{\mathcal{I}} \mapsto \bar{J}'_{\mathcal{J}'} = \bar{J}'_{\mathcal{I}'}$$

Similarly for $\mathcal{J} \mapsto \mathcal{I}'$, hence the N=2 OPE-algebra mirror morphism for both N=2 structures.

Conversely, the isomorphism between the spaces of states induces a bijective map $\varphi : \Lambda \to \Lambda'$ of the lattices. The requirements (i)-(iii) of Definition 3.5.1 force φ to be compatible with q and q'. Finally, φ maps $\mathcal{J} \mapsto \mathcal{I}'$ and $\mathcal{I} \mapsto \mathcal{J}'$ because of the N=2 structure of the OPE-algebras. This completes the proof. □

Now we draw a direct consequence of Theorem 3.4.3, which shows that mirror symmetry has the virtue of letting the rationality of lattice OPE-algebra to be transmitted:

Corollary 3.5.3. *Suppose $(\mathbb{T}, \mathcal{I}, \mathcal{J})$ and $(\mathbb{T}', \mathcal{I}', \mathcal{J}')$ are mirror generalized complex tori. Then the N=2 superconformal lattice OPE-algebra $V(\mathbb{T}, \mathcal{I}, \mathcal{J})$ is rational if and only if $V(\mathbb{T}', \mathcal{I}', \mathcal{J}')$ is rational.*

Proof. From Theorem 3.4.3 (ii) it follows that $\mathcal{I}\mathcal{J}$ is defined over $\mathbb{Q}$. Lemma 2.2.10 (i) implies that the decomposition (2.2.2) is defined over $\mathbb{Q}$. Hence the same holds for $(\mathbb{T}', \mathcal{I}', \mathcal{J}')$ due to Lemma 2.2.11 (i). □

Further, by combining Theorem 3.4.3 and Theorem 1.2.17 one gets

Corollary 3.5.4. *An abelian variety X is of CM-type if and only if X admits a rational N=2 superconformal lattice OPE-algebra $V(X, G, B)$.*

Proof. If X is of CM-type, one can choose $B=0$ together with the rational Kähler metric claimed by Theorem 1.2.17 to define a rational N=2 superconformal lattice OPE-algebra $V(X,G,B)$ in view of Theorem 3.4.3 (iii). Conversely, the rationality of $V(X,G,B)$ forces G to be rational again by Theorem 3.4.3 (iii), and its N=2 structure means that G is Kähler. Again by Theorem 1.2.17, X is of CM-type. □

Finally, we give an answer to our question (QAV) on the interplay between complex multiplication, rationality of the lattice OPE-algebra and mirror symmetry for abelian varieties. It is actually a reformulation of Theorem 2.2.13 and Proposition 2.3.1.

Corollary 3.5.5. *Let (X,G,B) and (X',G',B') be mirror abelian varieties. If the N=2 superconformal lattice OPE-algebra $V(X,G,B)$ is rational, then X and X' are isogenous and both of CM-type. Conversely, however, there exist mirror abelian varieties (X,G,B) and (X',G',B') such that X and X' are isogenous and both of CM-type, but neither $V(X,G,B)$ nor $V(X',G',B')$ is rational.*

Appendix A: Partition function of superconformal lattice OPE-algebras

We prove equalities (3.2.8) and (3.2.9) and the partition function formula (3.2.10) for a superconformal lattice OPE-algebra associated to a lattice Λ with a decomposition

$$\Lambda_{\mathbb{R}} = \Lambda_z \oplus \Lambda_{\bar{z}}, \quad d := \dim \Lambda_z \quad \text{and} \quad \bar{d} := \dim \Lambda_{\bar{z}}.$$

Recall the notations. The bosonic bases of Λ_z and $\Lambda_{\bar{z}}$ are $\{E^i\}$ respectively $\{\bar{E}^i\}$ with dual bases $\{\tilde{E}^i\}$ respectively $\{\tilde{\bar{E}}^i\}$, i.e.

$$q(E^i, \tilde{E}^j) = \delta^{ij} \quad \text{and} \quad q(\bar{E}^i, \tilde{\bar{E}}^j) = -\delta^{ij}.$$

The fermionic bases are denoted by $\{F^i\}$, $\{\bar{F}^i\}$ with dual bases $\{\tilde{F}^i\}$, $\{\tilde{\bar{E}}^i\}$. Recall the superconformal vector

$$L := \frac{1}{2}\sum_i (E^i_{-1}\widetilde{E}^i_{-1} - F^i_{-\frac{1}{2}}\widetilde{F}^i_{-\frac{3}{2}}) \otimes e^0, \quad \bar{L} \text{ is analogous.}$$

The coefficients L_n and $\bar{L}_n$ in

$$Y(L,z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}, \quad Y(\bar{L},\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{L}_n \bar{z}^{-n-2}$$

generate the Virasoro algebra. In view of the state-field correspondence (3.2.7) we have

$$Y(L,z) = \frac{1}{2}\sum_i (: \partial E^i(z)\partial\tilde{E}^i(z) : - : F^i(z)\partial\tilde{F}^i(z) :).$$

We calculate the coefficients L_{-1} and L_0. The operators T and $\bar{T}$ turn out to be $T = L_{-1}$ and $\bar{T} = \bar{L}_{-1}$.

Claim A.6. *We have*

$$L_{-1} = \sum_{i=1}^{d}\left(\sum_{n\geq 0} E^i_{-n-1}\widetilde{E}^i_n + \sum_{r=\frac{1}{2},\frac{3}{2}\cdots}(r+\frac{1}{2})F^i_{-r-1}\widetilde{F}^i_r\right) \quad \textit{and}$$

$$L_0 = \sum_{i=1}^{d}\left(\frac{1}{2}E^i_0\tilde{E}^i_0 + \sum_{n\geq 1} E^i_{-n}\tilde{E}^i_n + \sum_{r=\frac{1}{2},\frac{3}{2},\ldots} rF^i_{-r}\tilde{F}^i_r\right).$$

Proof. Note that

$$\partial E^i(z) = \sum_{m\in\mathbb{Z}} E^i_m z^{-m-1} = \underbrace{\sum_{m\leq -1} E^i_m z^{-m-1}}_{=\partial E^i(z)_+} + \underbrace{\sum_{m\geq 0} E^i_m z^{-m-1}}_{=\partial E^i(z)_-},$$

$$F^i(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} F^i_r z^{-r-\frac{1}{2}} = \underbrace{\sum_{r=-\frac{1}{2},-\frac{3}{2},\ldots} F^i_r z^{-r-\frac{1}{2}}}_{=F^i(z)_+} + \underbrace{\sum_{r=\frac{1}{2},\frac{3}{2},\ldots} F^i_r z^{-r-\frac{1}{2}}}_{=F^i(z)_-}.$$

Hence

$$
\begin{aligned}
:\partial E^i(z)\partial\tilde{E}^i(z): &= \partial E^i(z)_+\partial\tilde{E}^i(z) + (-1)^{p(E^i)p(\tilde{E}^i)}\partial\tilde{E}^i(z)\partial E^i(z)_- \\
&= \left(\sum_{m\le -1} E^i_m z^{-m-1}\right)\left(\sum_{n\in\mathbb{Z}}\tilde{E}^i_n z^{-n-1}\right) \\
&\qquad + \left(\sum_{n\in\mathbb{Z}}\tilde{E}^i_n z^{-n-1}\right)\left(\sum_{m\ge 0} E^i_m z^{-m-1}\right) \\
:F^i(z)\partial\tilde{F}^i(z): &= F^i(z)_+\partial\tilde{F}^i(z) + (-1)^{p(F^i)p(\tilde{F}^i)}\partial\tilde{F}^i(z)F^i(z)_- \\
&= \left(\sum_{r=-\frac{1}{2},-\frac{3}{2},\dots} F^i_r z^{-r-\frac{1}{2}}\right)\left(\sum_{s\in\mathbb{Z}+\frac{1}{2}}(-s-\frac{1}{2})\tilde{F}^i_s z^{-s-\frac{3}{2}}\right) \\
&\qquad - \left(\sum_{s\in\mathbb{Z}+\frac{1}{2}}(-s-\frac{1}{2})\tilde{F}^i_s z^{-s-\frac{3}{2}}\right)\left(\sum_{r=\frac{1}{2},\frac{3}{2},\dots} F^i_r z^{-r-\frac{1}{2}}\right).
\end{aligned}
$$

For L_{-1} we need look at z^{-1}, we get

$$
\begin{aligned}
L_{-1} &= \frac{1}{2}\sum_i\Bigg(\sum_{m\le -1} E^i_m\tilde{E}^i_{-m-1} + \sum_{m\ge 0}\tilde{E}^i_{-m-1}E^i_m \\
&\qquad - \sum_{r=-\frac{1}{2},-\frac{3}{2},\dots}(r+\frac{1}{2})F^i_r\tilde{F}^i_{-r-1} + \sum_{r=\frac{1}{2},\frac{3}{2},\dots}(r+\frac{1}{2})\tilde{F}^i_{-r-1}F^i_r\Bigg) \\
&= \frac{1}{2}\sum_i\Bigg(\sum_{n\ge 0} E^i_{-n-1}\tilde{E}^i_n + \sum_{m\ge 0} E^i_{-m-1}\tilde{E}^i_m \\
&\qquad - \sum_{s=\frac{1}{2},\frac{3}{2},\dots}(-s-\frac{1}{2})F^i_{-s-1}\tilde{F}^i_s + \sum_{r=\frac{1}{2},\frac{3}{2},\dots}(r+\frac{1}{2})F^i_{-r-1}\tilde{F}^i_r\Bigg) \\
&= \sum_i\left(\sum_{n\ge 0} E^i_{-n-1}\tilde{E}^i_n + \sum_{r=\frac{1}{2},\frac{3}{2},\dots}(r+\frac{1}{2})F^i_{-r-1}\tilde{F}^i_r\right).
\end{aligned}
$$

In the second equality above we used

$$\sum_i \tilde{E}^i_{-m-1}E^i_m = \sum_i E^i_{-m-1}\tilde{E}^i_m \quad \text{and} \quad \sum_i \tilde{F}^i_{-r-1}F^i_r = \sum_i F^i_{-r-1}\tilde{F}^i_r$$

whence the equality for L_{-1}. The calculations for L_0 are similar. □

For the partition function

$$Z = \mathrm{Tr}_V\, \mathfrak{q}^{L_0-\frac{c}{24}}\bar{\mathfrak{q}}^{\bar{L}_0-\frac{\bar{c}}{24}}, \quad \text{where} \quad q = e^{2\pi i\tau} \quad \text{and} \quad \bar{\mathfrak{q}} = e^{2\pi i\bar{\tau}}$$

we determine first c and $\bar{c}$.

Claim A.7. *We have*

$$c = \frac{3d}{2} \quad \textit{and} \quad \bar{c} = \frac{3\bar{d}}{2}.$$

Proof. Recall that c is the central charge, it is the number acting by multiplication in

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}c.$$

We show the equality for c (analogous for $\bar{c}$).

From the proof of [Kac, Thm 5.7] we know that

$$L_2 L = \frac{c}{2}$$

for vertex algebras. This is also true for lattice OPE-algebras. We have

$$L_2 = \frac{1}{2}\sum_{i=1}^{d}\left(\sum_{p\in\mathbb{Z}} E_p^i \tilde{E}_{2-p}^i + \sum_{r\in\mathbb{Z}+\frac{1}{2}} (r-\frac{5}{2})F_r^i \tilde{F}_{-r+2}^i\right).$$

By easy calculations we get

$$L_2 L = \frac{3d}{4}.$$

Hence $c = \frac{3d}{2}$. □

Now we calculate the eigenvalues of L_0.

Claim A.8. *The eigenvalues of L_0 are*

$$\{\sum_{i=1}^{d}\sum_{k=1}^{l} p_{ki}s_k + \sum_{i=1}^{d}\sum_{k=1}^{m} q_{ki}t_k + \frac{1}{2}q(a_z,a_z) \mid l,m\geq 1, s_k < s_{k+1}, p_{ki}\in\mathbb{N}$$

$$t_k < t_{k+1} \in \mathbb{N}+\frac{1}{2}, q_{ki}\in\{0,1\}, a\in\Lambda\}.$$

Proof. The term $\frac{1}{2}q(a_z,a_z)$ is due to $\frac{1}{2}\sum_i E_0^i\tilde{E}_0^i$ in L_0 which only acts on $\mathbb{C}[\Lambda]$. More precisely, in view of (3.2.4) we have

$$\frac{1}{2}\sum_i E_0^i\tilde{E}_0^i e^a = \frac{1}{2}\sum_i E_0^i q(\tilde{E}_0^i, a)e^a = \frac{1}{2}a_{z0}e^a = \frac{1}{2}q(a_z,a)e^a = \frac{1}{2}q(a_z,a_z)e^a.$$

The term $\sum_{i=1}^{d}\sum_{k=1}^{l} p_{ki}s_k$ is due to $\sum_{i=1}^{d}\sum_{n\geq 1} E_{-n}^i\tilde{E}_n^i$. Indeed, any bosonic state is a sum of vectors of the form

$$v = \prod_{i=1}^{d}(E_{-s_1}^i)^{p_{1i}}\cdots\prod_{i=1}^{d}(E_{-s_l}^i)^{p_{li}}\otimes e^a, \quad s_i < s_{i+1}, p_{ki}\in\mathbb{N}.$$

This gives

$$\sum_{i=1}^{d}\sum_{n\geq 1} E_{-n}^i\tilde{E}_n^i v = \sum_{i=1}^{d}\sum_{k=1}^{l} p_{ki}s_k v. \tag{A.1}$$

For the proof we show

$$\sum_{n\geq 1} E_{-n}^j\tilde{E}_n^j v = \sum_{k=1}^{l} p_{ki}s_k v.$$

Indeed,

$$lhs = \sum_{k=1}^{l} E_{-s_k}^j\tilde{E}_{s_k}^j v,$$

each summand is

$$\begin{aligned}E^j_{-s_k}\tilde{E}^j_{s_k}v &= \prod_{i=1}^{d}(E^i_{-s_1})^{p_{1i}}\cdots E^j_{-s_k}\tilde{E}^j_{s_k}\prod_{i=1}^{d}(E^i_{-s_k})^{p_{ki}}\otimes e^a\\ &= \prod_{i=1}^{d}(E^i_{-s_1})^{p_{1i}}\cdots\prod_{i\neq j}(E^i_{-s_k})^{p_{ki}}E^j_{-s_k}(s_k+E^j_{-s_k}\tilde{E}^j_{s_k})^{p_{ki}-1}\otimes e^a\\ &= p_{ki}s_kv.\end{aligned}$$

This shows (A.1) after summing over k and i.

For fermions the space of states is an exterior algebra, this means that the same term $F^i_{-t_k}$ cannot occur more than once. So any fermionic state is a sum of vectors of the form

$$u=\prod_{i=1}^{d}(F^i_{-t_1})^{q_{1i}}\cdots\prod_{i=1}^{d}(F^i_{-t_l})^{q_{li}}\otimes e^a,\quad s_i<s_{i+1}\in\mathbb{N}+\frac{1}{2}, q_{ki}\in\{0,1\}.$$

This gives

$$\sum_{i=1}^{d}\sum_{r=\frac{1}{2},\frac{3}{2},\dots} rF^i_{-r}\tilde{F}^i_r u=\sum_{i=1}^{d}\sum_{k=1}^{m}q_{ki}t_ku$$

similarly to bosons. □

In order to express explicitly the partition function we make a reminder about the Dedekind eta function and the Jacobi theta function. Write $\mathfrak{q}=e^{2\pi i\tau}$, they are defined as follows

$$\eta(\tau)=\mathfrak{q}^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-\mathfrak{q}^n),$$

$$\theta_3(\tau)=\prod_{n=1}^{\infty}(1-\mathfrak{q}^n)\prod_{r\in\mathbb{N}+\frac{1}{2}}(1+\mathfrak{q}^r)^2.$$

We see

$$\frac{1}{\eta(\tau)}=\mathfrak{q}^{-\frac{1}{24}}\prod_{n=1}^{\infty}(1+\mathfrak{q}^n+\mathfrak{q}^{2n}+\cdots)$$

$$\frac{\theta_3(\tau)}{\eta(\tau)}=\left(\mathfrak{q}^{-\frac{1}{48}}\prod_{r\in\mathbb{N}+\frac{1}{2}}(1+\mathfrak{q}^r)\right)^2.$$

We show

Claim A.9. *The partition function is*

$$Z=\frac{1}{\eta(\tau)^d\eta(\bar{\tau})^{\bar{d}}}\left(\frac{\theta_3(\tau)}{\eta(\tau)}\right)^{\frac{d}{2}}\left(\frac{\theta_3(\bar{\tau})}{\eta(\bar{\tau})}\right)^{\frac{\bar{d}}{2}}\left(\sum_{a\in\Lambda}\mathfrak{q}^{\frac{1}{2}q(a_z,a_z)}\bar{\mathfrak{q}}^{\frac{1}{2}q(a_{\bar{z}},a_{\bar{z}})}\right),$$

Proof. In the partition function the bosonic contribution is

$$\prod_{i=1}^{d}\mathfrak{q}^{-\frac{1}{24}}\left(\sum_{l=1}^{\infty}\sum_{p_{i1},\dots,p_{il}\in\mathbb{N}\setminus\{0\}}\sum_{s_1<\dots<s_l\in\mathbb{N}\setminus\{0\}}\mathfrak{q}^{\sum_{k=1}^{l}s_kp_{ik}}\right)\times\bar{z}\text{-part}+1=\frac{1}{\eta(\tau)^d}\frac{1}{\eta(\bar{\tau})^{\bar{d}}}.$$

The fermionic contribution is

$$\prod_{i=1}^{d} \mathfrak{q}^{-\frac{1}{48}} \left(\sum_{m=1}^{\infty} \sum_{t_1 < \cdots < t_m \in \mathbb{N} + \frac{1}{2}} \mathfrak{q}^{\sum_{k=1}^{m} t_k} \right) \times \bar{z}\text{-part} + 1 = \left(\frac{\theta_3(\tau)}{\eta(\tau)} \right)^{\frac{d}{2}} \left(\frac{\theta_3(\bar{\tau})}{\eta(\bar{\tau})} \right)^{\frac{\bar{d}}{2}}.$$

The term +1 is due to the vacuum state. The contribution of the lattice is

$$\sum_{a \in \Lambda} \mathfrak{q}^{\frac{1}{2} q(a_z, a_z)} \bar{\mathfrak{q}}^{\frac{1}{2} q(a_{\bar{z}}, a_{\bar{z}})}.$$

The shows the claim. $\square$

Appendix B: An isomorphism to Kapustin-Orlov's N=2 superconformal OPE-algebra

We show Proposition 3.3.1. Recall the

Definition A.10. *Two N=2 superconformal OPE-algebras are* isomorphic *if there is an isomorphism $f : V \to V'$ of their space of states, such that*

(i) $f(\mathbf{1}) = f(\mathbf{1}')$
(ii) $fT = T'f, f\bar{T} = \bar{T}'f$
(iii) *For all $u, v \in V$, we have $Y'(f(u), z, \bar{z})f(v) = f(Y(u, z, \bar{z})v)$*
(iv) *$fL = L'$, $fQ^{\pm} = Q'^{\pm}$, $fJ = J'$, similarly for the $\bar{z}$-part.*

Let (T, G, B) be a complex torus. Recall the charge lattice isomorphism from [H1]:

$$\phi := \frac{1}{\sqrt{2}} \begin{pmatrix} -G-B & 1 \\ G-B & 1 \end{pmatrix} : \Gamma_{\mathbb{R}} \oplus \Gamma^*_{\mathbb{R}} \longrightarrow \Gamma^*_{\mathbb{R}} \oplus \Gamma^*_{\mathbb{R}}$$

with inverse

$$\phi^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} -G^{-1} & G^{-1} \\ 1-BG^{-1} & 1+BG^{-1} \end{pmatrix}.$$

Elementary calculations show that ϕ^{-1} is an isometry, i.e.:

$$q(\phi^{-1}\cdot, \phi^{-1}\cdot) = \begin{pmatrix} G^{-1} & 0 \\ 0 & -G^{-1} \end{pmatrix}. \tag{A.2}$$

Writing $\phi(a) =: (\phi_1(a), \phi_2(a))$, the decomposition $\Lambda_{\mathbb{R}} = \Lambda_z \oplus \Lambda_{\bar{z}} = \mathrm{Graph}_{\Gamma_{\mathbb{R}}}(-G + B) \oplus \mathrm{Graph}_{\Gamma_{\mathbb{R}}}(G + B)$ from (2.2.2) corresponds to

$$a \mapsto a_z := \phi^{-1} \circ \phi_1(a) \quad \text{and} \quad a \mapsto a_{\bar{z}} := -\phi^{-1} \circ \phi_2(a).$$

Write $f := \phi^{-1}$. We show that f is the isomorphism we are looking for. Here we only give the calculation for the bosonic part. The fermionic part is similar.

We interpret [KO]'s choice of bases as follows: $\{\alpha^i\}_{i=1\dots 2g}$ as the basis of the *first* component of $\Gamma^*_{\mathbb{R}} \oplus \Gamma^*_{\mathbb{R}}$ and $\{\bar{\alpha}^i\}_{i=1\dots 2g}$ as the basis of the *second* component of $\Gamma^*_{\mathbb{R}} \oplus \Gamma^*_{\mathbb{R}}$. Set

$$E^i := f(\alpha^i) \quad \text{and} \quad \bar{E}^i := f(\bar{\alpha}^i).$$

Then for the dual bases, i.e. $\{\widetilde{\alpha}^j\} \in \Gamma^*_{\mathbb{R}} \oplus 0$ and $\{\widetilde{\bar{\alpha}}^j\} \in 0 \oplus \Gamma^*_{\mathbb{R}}$ with $G^{-1}(\alpha^i, \widetilde{\alpha}^j) = G^{-1}(\bar{\alpha}^i, \widetilde{\bar{\alpha}}^j) = \delta^{ij}$ set

$$\widetilde{E}^i := f(\widetilde{\alpha}^i) \quad \text{and} \quad \widetilde{\bar{E}}^i := f(\widetilde{\bar{\alpha}}^i).$$

Then in view of (A.2) we have $q(E^i, \widetilde{E}^j) = \delta^{ij}$ and $q(\bar{E}^i, \widetilde{\bar{E}}^j) = -\delta^{ij}$. On the representations, f induces the correspondence:

$$\alpha^i_s \mapsto E^i_s,\ \widetilde{\alpha}^i_s \mapsto \widetilde{E}^i_s \text{ for } s \in \mathbb{Z}^* \quad \text{and} \quad (G^{-1})^{kj} P_k \mapsto E^j_0.$$

Then the commutators are

$$\begin{aligned} [E^i_s, E^j_p] &\overset{(3.2.5)}{=} s\delta_{s,-p} q(E^i, E^j) \\ &\overset{(A.2)}{=} s\delta_{s,-p} (G^{-1})^{ij} \\ &= [\alpha^i_s, \alpha^j_p], \end{aligned}$$

where $s, p \in \mathbb{Z}^*$. Similarly for the $\bar{z}$-part.

At this stage it is clear that f possesses the properties (i), (ii) and (iv) of Definition A.10. For (iii) we first translate the notations in [KO] into ours. For $(w,m) \in \Gamma \oplus \Gamma^*$:

$$\begin{aligned}
&P_i(w,m) = \phi_{1i}(w,m) \quad \text{is the } i\text{-th coordinate of } \phi_1(w,m) \in \Gamma^*,\\
&\bar{P}_i(w,m) = \phi_{2i}(w,m) \quad \text{is the } i\text{-th coordinate of } \phi_2(w,m) \in \Gamma^*,\\
&k = \phi_1(w,m),\\
&\bar{k} = \phi_2(w,m).
\end{aligned}$$

Then for $a = (w,m), b = (w',m') \in \Gamma \oplus \Gamma^*$, we have again by (A.2) $q(a_z, b_z) = G^{-1}(k,k')$ and $q(a_{\bar{z}}, b_{\bar{z}}) = -G^{-1}(\bar{k}, \bar{k}')$, and

$$\partial^s X(z) = \partial^{s-1}(G^{-1})^{jk} P_k \frac{1}{z} - \partial^s Y^j(z) \longmapsto \partial^{s-1} \sum_{m \in \mathbb{Z}} E_m z^{-m-1},$$

$$\begin{aligned}
k_j Y^j(z)_+ = k_j \sum_{m<0} \frac{\alpha_m^j}{m z^m} \longmapsto \phi_{1j}(a) \sum_{m<0} \frac{E_m^j}{m z^m} &= \sum_{m<0} \frac{\phi^{-1}(\sum_j \phi_{1j}(a)\alpha^j)_m}{m z^m}\\
&= \sum_{m<0} \frac{a_{zm}}{m z^m}.
\end{aligned}$$

Compare the state-field correspondence (3.2.7) with the one given in [KO], the isomorphism is then obvious.

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