Stochastic Interacting Particle Systems and Nonlinear Partial Differential Equations from Fluid Mechanics

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1. Referent: Prof. Dr. Karl-Theodor Sturm
2. Referent: Prof. Dr. Sergio Albeverio

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Abstract

We derive stochastic particle approximations for two nonlinear partial differential equations from fluid mechanics: the porous medium equation and the three-dimensional Navier-Stokes equation. We associate interacting particle systems with these equations and obtain, when the number of particles tends to infinity, laws of large numbers for the empirical measures.

In the first chapter we study a system of interacting diffusions and show that the empirical measure of the particle system tends to the solution of the porous medium equation when the number of particles tends to infinity. Moreover we prove propagation of chaos for this system: if initially the positions of the particles are independent and identically distributed, then they remain so – at least approximately – in the course of time.

In the second chapter we consider a sequence of nonlinear stochastic differential equations and show that the distributions of the solutions converge to the solution of the viscous porous medium equation.

The third chapter deals with a stochastic particle approximation for the three-dimensional Navier-Stokes equation. This equation is of a completely different type than the porous medium equation, so that it seems difficult to treat it with the methods of the first chapter. Nevertheless this is possible: we do not consider the Navier-Stokes equation directly, but instead the equation satisfied by the vorticity, and use the fact that the velocity can be recovered from the vorticity.

Contents

Introduction

A fluid is usually modelled as a continuous medium and described by macroscopic quantities such as density, velocity, pressure and temperature. These quantities are then related by partial differential equations. However, mechanics is a physical science that describes the behaviour of matter (solids, liquids, or gases), and therefore its mathematical formulation relies on experience and theory. In view of this the fundamental concept of a continuous medium is an abstraction which is, strictly speaking, against the universally accepted atomic theory, which describes reality at scales which are smaller than nanometers; for example, the radius of the smallest atom is about $4 \cdot 10^{-11}$m. Nevertheless, the mathematical theory of fluid mechanics is based on precisely this concept. This needs an explanation, which is as follows: the task consists in constructing a mathematical theory that serves as a model for *one part* of reality. This model must be judged from the mathematical point of view, taking into account the beauty, extension and profoundness of the involved mathematics; and from the physical point of view, taking into account how efficiently it reflects and explains the underlying reality and allows to predict its future evolution.

In this sense, although the hypothesis of a continuum is rigorously false at microscopic levels, it turns out to be extremely efficient and adequate when one studies phenomena which occur at macroscopic scales; to fix ideas, lenghts greater than 10^{-7}m.

The approximation by the continuous medium turns out to be so efficient that one often forgets that it is just a model. It is nevertheless important to take into account the starting hypotheses. In this way, the consideration of the fluid as a continuous medium is based on the assumption that it consists of an aggregate of particles in chaotic motion and that the characteristic distance of this motion, the so called mean free path, is much smaller than the experimental lengths, so that we only observe a certain average of the individual processes between particles.

Having specified that one works on scales which are much larger than the mean free path of the particles one can forget the fine details of their individual motion and consider around each point of space and at each time a representative elementary volume δV of mesoscopic size, i.e. much larger than the mean free path and much smaller than the macroscopic lengths. This elementary volume, also called fluid particle, is considered as a continuous and homogeneous medium; in this volume one defines a mean velocity of the motion of this element, which is then the point velocity in this point and at this time. More precisely, one supposes that there exists a limit of the averages when δV becomes very small at the intermediate scale, i.e. very small but still much above the atomic scale. In the same way, one speaks of the other macroscopic quantities, such as density, which is the mass per unit of volume in the sense of the limit described above, and pressure, which is the normal force per unit of area exerced by the fluid on an ideal surface which is immersed in it or encloses it. These three quantities are complemented by others, such as e.g. temperature, internal energy and viscosity. The existence of these average values for the fundamental quantities in each fluid particle is what is called the continuum hypothesis[1]. It

[1] Not to be confused with the continuum hypothesis of set theory.

is precisely this hypothesis which allows to describe the motion of a fluid by partial differential equations. For general introductions to fluid mechanics we refer to the classical book by Landau and Lifshitz [19] and to the lecture notes by Vázquez [33].

As we have said, despite its usefulness and success, the continuum hypothesis is strictly speaking false. It is therefore desirable to find rigorous connections between the microscale and the macroscale. More precisely: suppose we know that on the macroscale the motion of a fluid is described by a certain partial differential equation, then we want to find a microscopic model which allows us, when the number of particles tends to infinity, to derive that partial differential equation as limit equation. This is a very important project in mathematics to which many people have contributed. For general introductions (and many references) to this subject we refer to the books by Kipnis and Landim [17] and Spohn [29]. In the last years interesting connections to optimal transportation have been discovered, see e.g. Bolley's PhD thesis [4] and also Problem 15 in Villani's book [34].

In this thesis we study stochastic particle approximations for the following two equations of fluid mechanics, both posed in the whole space: the well-known three-dimensional Navier-Stokes equation

$$\begin{aligned} \frac{\partial u}{\partial t} + (u \cdot \nabla) u &= -\nabla p + \nu \Delta u \\ \operatorname{div} u &= 0 \\ u(t,x) &\to 0 \text{ for } |x| \to \infty \end{aligned} \tag{1}$$

and the less prominent, but also very important porous medium equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\,\Delta(u^2) \tag{2}$$

which describes the density u of a gas flowing through a porous medium.

Let us now explain our stochastic approach at the example of the porous medium equation. Our goal is to find for each $N \in \mathbb{N}$ a system of N particles with positions $(X_t^{N,i})_{i=1}^N$ with the following property: as $N \to \infty$ the *empirical measure* $\frac{1}{N}\sum_{i=1}^N \delta_{X_t^{N,i}}$ of the particle system converges weakly to the measure with density $u(t,\cdot)$:

$$\frac{1}{N}\sum_{i=1}^N \delta_{X_t^{N,i}} \to u(t,x)dx \qquad \text{for } N \to \infty. \tag{3}$$

Of course this convergence can only hold if for each $t \geq 0$ $u(t,\cdot)$ is a probability density, but for the porous medium equation it is well known that this is true, provided that the initial datum u_0 has this property.

The following general fact (see Sznitman [30], Chapter I.2, Proposition 2.2) is very useful: if for each $N \in \mathbb{N}$ the joint distribution of the positions of all N particles is symmetric, then the convergence of the empirical measure is equivalent to the so called *propagation of chaos* property, namely for each fixed $m \in \mathbb{N}$ the convergence of the joint distribution of the positions of the first m particles towards the m-fold product measure $(u(t,x)dx)^{\otimes m}$. Heuristically this means that in the limit $N \to \infty$ the positions of any fixed number of particles become independent, and that the distribution of the position of each particle converges to the measure with density $u(t,\cdot)$.

In the case of the McKean-Vlasov equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u - \operatorname{div}((b * u)u) \tag{4}$$

with a bounded and Lipschitz-continuous function $b : \mathbb{R}^d \to \mathbb{R}^d$ this approximation problem can be solved quite easily (see [30], Chapter I.1): namely, one takes the $(X_t^{N,i})_{i=1}^N$ as solutions of the following system of coupled stochastic differential equations:

$$\begin{aligned} dX_t^{N,i} &= \frac{1}{N}\sum_{j=1}^{N} b(X_t^{N,i} - X_t^{N,j})dt + dB_t^i \\ X_0^{N,i} &= \zeta^i, \end{aligned} \tag{5}$$

where $(B^i)_{i\in\mathbb{N}}$ is a sequence of independent standard Brownian motions, and $(\zeta^i)_{i\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables, independent of the Brownian motions and whose distribution has the density u_0 with respect to Lebesgue measure. One associates with this particle system the following system of nonlinear processes:

$$\begin{aligned} d\overline{X}_t^i &= (b*u)(X_t^{N,i})dt + dB_t^i \\ \overline{X}_0^i &= \zeta^i \\ u(t,dx) &= P\left[\overline{X}_t^i \in dx\right]. \end{aligned}$$

It is then possible to show that $\lim_{N\to\infty} E\left[\left|X_t^{N,i} - \overline{X}_t^i\right|\right] = 0$. Moreover, using Itô's formula and taking expectations one easily obtains that the distribution u of the process $\overline{X}_t^i$ solves the McKean-Vlasov equation (4), and the convergence (3) follows.

There is a fundamental difference between the porous medium equation (2) and the McKean-Vlasov equation (4): while in (4) the interaction is nonlocal due to the convolution $b*u$, it is completely local in (2). This makes the probabilistic interpretation much more difficult.

Our approach, which we present in detail in Chapter 1, consists in approximating the porous medium equation (2) by the equation

$$\frac{\partial u^{\varepsilon,\delta}}{\partial t} = \frac{\delta^2}{2}\Delta u^{\varepsilon,\delta} + \operatorname{div}((\nabla V^\varepsilon * u)u), \tag{6}$$

where $V^\varepsilon(x) := \frac{1}{\varepsilon^d}V(x/\varepsilon)$ is a mollifier and $\varepsilon,\delta > 0$ are small parameters. This equation is of McKean-Vlasov type, and moreover, when in (6) ε and δ tend to 0, one formally obtains as limit equation

$$\frac{\partial u}{\partial t} = \operatorname{div}(\nabla u\, u) = \frac{1}{2}\,\Delta(u^2),$$

i.e. the porous medium equation. The consideration of (6) as intermediate step leads to the following interacting particle system:

$$\begin{aligned} dX_t^{N,i,\varepsilon,\delta} &= -\frac{1}{N}\sum_{j=1}^{N} \nabla V^\varepsilon(X_t^{N,i,\varepsilon,\delta} - X_t^{N,j,\varepsilon,\delta})dt + \delta dB_t^i \\ X_0^{N,i,\varepsilon,\delta} &= \zeta^i, \end{aligned} \tag{7}$$

where $(B^i)_{i\in\mathbb{N}}$ and $(\zeta^i)_{i\in\mathbb{N}}$ are as in (5).

In Chapter 1 we prove the following theorem: If $N\to\infty$, $\varepsilon\to 0$ and $\delta\to 0$ in such a way that $N \gg 1/\varepsilon$ and $\varepsilon \ll \delta$, then the empirical measure of the particle system (7) converges weakly to the measure $u(t,x)dx$ where $u(t,\cdot)$ is the solution of the porous medium equation.

A crucial step of the proof consists in showing that for $\varepsilon,\delta\to 0$ $u^{\varepsilon,\delta}$ does indeed converge to u. However, the rigorous derivation of such a convergence result is much more difficult than it might seem to be. Let us introduce as intermediate object the partial differential equation

$$\frac{\partial u^\delta}{\partial t} = \frac{\delta^2}{2}\Delta u^\delta - \frac{1}{2}\,\Delta((u^\delta)^2).$$

Then it has been known for a long time that $u^\delta \to u$ for $\delta \to 0$ (see [3]). The convergence $u^{\varepsilon,\delta} \to u^\delta$ for $\varepsilon \to 0$ is much more difficult to establish. A first result in this direction was obtained by Oelschläger [27], but only under very restrictive assumptions on the initial datum u_0 (in particular he needs $u_0 \in \mathcal{C}_b^\infty(\mathbb{R}^d)$). Moreover his proof is very complicated. Therefore in Chapter 2 we prove a similar result, where we only require that $u_0 \in L^2(\mathbb{R}^d)$.

Chapter 3 is devoted to a stochastic particle approxiamtion of the three-dimensional Navier-Stokes equation (1). This equation is of a completely different kind than equations (4) or (2), so that it seems difficult to treat it with a similar approach. Nevertheless this is possible: we do not study the Navier-Stokes equation directly, but instead the equation satisfied by the vorticity $w := \operatorname{curl} u$:

$$\frac{\partial w}{\partial t} = -(u \cdot \nabla)w + (w \cdot \nabla)u + \nu \Delta w. \tag{8}$$

Physically the three terms on the right-hand side of (8) mean that vorticity is transported with the fluid, it is stretched and it undergoes diffusion. In two dimensions the vortex stretching term $(w \cdot \nabla)u$ vanishes, and the situation becomes much easier.

An important observation is that the velocity u can be recovered from the vorticity w: Let $K(x) := -\frac{x}{4\pi|x|^3}$. Then

$$u(x) = \int_{\mathbb{R}^3} K(x-y) \times w(y)dy. \tag{9}$$

We now want to approximate the vorticity w by a system of discrete vortices. Since vorticity is a vector-valued quantity, it does not suffice to keep track just of the positions of the vortices; one must also consider their intensities. Therefore we model each discrete vortex by a couple $(X_t^i, a_t^i) \in \mathbb{R}^3 \times \mathbb{R}^3$, where X_t^i represents its position and a_t^i its intensity. Now we define the discrete vorticity $\overline{w}_t$ as the weighted empirical measure of the vortex system:

$$\overline{w}_t := \frac{1}{N} \sum_{i=1}^N a_t^i \delta_{X_t^i}.$$

In analogy to (9) we define the discrete velocity $\overline{u}_t$ as

$$\overline{u}_t(x) := \frac{1}{N} \sum_{i=1}^N K(x - X_t^i) \times a_t^i.$$

Now the question arises: by which equations should the system $(X_t^i, a_t^i)_{i=1}^N$ of discrete vortices be governed? Looking at (8) a natural approach would be:

$$\begin{aligned} dX_t^i &= \overline{u}_t(X_t^i)dt + \sqrt{2\nu}\, dB_t^i \\ da_t^i &= \nabla \overline{u}_t(X_t^i) a_t^i \, dt, \end{aligned}$$

or explicitly

$$dX_t^i = \left[\frac{1}{N} \sum_{j=1}^N K(X_t^i - X_t^j) \times a_t^j \right] dt + \sqrt{2\nu}\, dB_t^i \tag{10}$$

$$da_t^i = \left[\frac{1}{N} \sum_{j=1}^N \nabla K(X_t^i - X_t^j) \times a_t^j \right] a_t^i \, dt. \tag{11}$$

Here the two terms on the right-hand side of (10) correspond to the transport term $(u \cdot \nabla)w$ and the diffusion term $\nu \Delta w$ in (8), while the term on the right-hand side of (11) corresponds to the vortex stretching term $(w \cdot \nabla)u$. However this approach poses two problems: First, the right-hand sides of (10) and (11) might be not well-defined because of the singularity of K at 0. Moreover, a_t appears quadratically on the right-hand side of (11) and therefore might explode in finite time.

To overcome these problems we replace K with $K^\varepsilon := K * \varphi^\varepsilon$, where $\varphi^\varepsilon(x) := \frac{1}{\varepsilon^3}\varphi(x/\varepsilon)$ is a mollifier, introduce the cutoff

$$\chi_R(a_t^i) := \begin{cases} a_t^i & \text{if } |a_t^i| \le R \\ \frac{R}{|a_t^i|} a_t^i & \text{if } |a_t^i| > R \end{cases}$$

and consider the following vortex system:

$$dX_t^i = \left[\frac{1}{N} \sum_{j=1}^N K^\varepsilon(X_t^i - X_t^j) \times \chi_R(a_t^j) \right] dt + \sqrt{2\nu}\, dB_t^i$$
$$da_t^i = \left[\frac{1}{N} \sum_{j=1}^N \nabla K^\varepsilon(X_t^i - X_t^j) \times \chi_R(a_t^j) \right] \chi_R(a_t^i)\, dt.$$

It remains to find a good choice for the initial values X_0^i and a_0^i. To this end we decompose the initial vorticity w_0 in the form $w_0(x) = p(x)h(x)$, where p is a probability density and h is a bounded $\mathbb{R}^3$-valued function. If $w_0 \in L^1(\mathbb{R}^3)$ this is always possible, e.g. $p(x) = \frac{|w_0(x)|}{\|w_0\|_{L^1}}$, $h(x) = \frac{w_0(x)}{|w_0(x)|}\|w_0\|_{L^1}$. In the decomposition $w_0(x) = p(x)h(x)$ we interpret $p(x)$ as density and $h(x)$ as intensity of vortices at x. Therefore we choose the X_0^i to be independent with $P\big[X_0^i \in dx\big] = p(x)dx$, and we set $a_0^i := h(X_0^i)$.

Now we choose a large (but fixed) cutoff parameter $R \gg 0$ and let $N \to \infty$ and $\varepsilon \to 0$ with the constraint $N \gg 1/\varepsilon$. Then we have the following theorem: There exists a strictly positive time $T^* > 0$ such that the discrete vorticity $\overline{w}_t$ converges to the continuous vorticity $w(t,\cdot)$, uniformly in $t \in [0, T^*]$:

$$\overline{w}_t = \frac{1}{N} \sum_{i=1}^N a_t^i \delta_{X_t^i} \;\to\; w(t,\cdot).$$

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Chapter 1

Interacting diffusions approximating the porous medium equation and propagation of chaos

We study a system of interacting diffusions and show that for a large number of particles its empirical measure approximates the solution of the porous medium equation. Furthermore we prove propagation of chaos.

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1.1 Introduction

We study the following system of interacting particles in $\mathbb{R}^d$:

$$\begin{aligned} dX_t^{N,i,\varepsilon,\delta} &= -\frac{1}{N}\sum_{j=1}^{N} \nabla V^\varepsilon(X_t^{N,i,\varepsilon,\delta} - X_t^{N,j,\varepsilon,\delta})dt + \delta dB_t^i, \quad i = 1, \dots, N \\ X_0^{N,i,\varepsilon,\delta} &= \zeta^i. \end{aligned} \tag{1.1}$$

Here V^ε is a smooth interaction kernel which is obtained from a function V by the scaling

$$V^\varepsilon(x) := \frac{1}{\varepsilon^d} V(x/\varepsilon), \tag{1.2}$$

$(B^i)_{i\in\mathbb{N}}$ is a sequence of independent standard Brownian motions, and $(\zeta^i)_{i\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables, independent of the Brownian motions and whose distribution has a given smooth density u_0 with respect to Lebesgue measure.

The particle system (1.1) depends on three parameters: $N \in \mathbb{N}$, $\varepsilon > 0$ and $\delta > 0$. N is the number of particles, ε measures the range of interaction, and δ measures the strength of the additional diffusion caused by the Brownian motions.

Now we let $N \to \infty$, $\varepsilon \to 0$, $\delta \to 0$ in such a way that $N \gg 1/\varepsilon$ and $\varepsilon \ll \delta$. We shall show that then the following hold:

1. For each $t \geq 0$ the empirical measure $\mu_t^{N,\varepsilon,\delta} := \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{N,i,\varepsilon,\delta}}$ of the particle system converges weakly to a deterministic measure P_t on $\mathbb{R}^d$. This measure has a density $u(t,\cdot)$

which solves the porous medium equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\,\Delta(u^2) \quad \text{in } \mathbb{R}_{>0}\times\mathbb{R}^d \\ u(0,\cdot) &= u_0 \end{aligned}$$

with initial datum u_0.

2. The distribution of the position $X_t^{N,i,\varepsilon,\delta}$ of each particle also converges weakly to P_t.

3. Any fixed number of particles remains approximately independent in the course of time, in spite of the interaction.

The third statement is known as *propagation of chaos*. In this context the word "chaotic" is used as a synonym for "independent and identically distributed". By definition the situation at time $t = 0$ is chaotic (because the initial positions ζ^i of the particles are independent and identically distributed), and we claim that at later times the situation is approximately chaotic, too: the chaos propagates. For an introduction to propagation of chaos we refer to Sznitman [30], and for an introduction to the theory of the porous medium equation to Vázquez [31].

1.2 Assumptions on the initial datum and the interaction kernel

We assume that u_0, the common density of the distributions of the initial positions ζ^i of the particles, belongs to the weighted Sobolev space $W^2_{n,1}(\mathbb{R}^d)$ for all $n\in\mathbb{N}$. This space consists of all n times weakly differentiable functions $f:\mathbb{R}^d\to\mathbb{R}$ for which the weighted Sobolev norm

$$\|f\|_{(n,1)} := \left(\sum_{k=0}^{n}\sum_{i_1,\dots,i_k=1}^{d}\int_{\mathbb{R}^d}(1+|x|)\left|\frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}(x)\right|^2 dx\right)^{1/2}$$

is finite. The Sobolev embedding theorems imply that then $u_0\in\mathcal{C}_b^\infty(\mathbb{R}^d)$, where $\mathcal{C}_b^\infty(\mathbb{R}^d)$ is the space of smooth functions which are bounded together with all their partial derivatives. Conversely, if u_0 is smooth with compact support, it belongs to $W^2_{n,1}(\mathbb{R}^d)$ for all $n\in\mathbb{N}$.

The function V is supposed to have the following properties:

1. There exists a function $W\in\mathcal{C}_b^\infty(\mathbb{R}^d)$ with $W\geq 0$, $\int_{\mathbb{R}^d}W(x)dx=1$ and $W(-x)=W(x)$ for all $x\in\mathbb{R}^d$, such that $V=W*W$.

2. All moments of V are finite, i.e. for all $n\in\mathbb{N}$ we have:

$$\int_{\mathbb{R}^d}|x|^n V(x)dx<\infty. \tag{1.3}$$

The first property implies in particular: V is symmetric, i.e. $V(-x)=V(x)$ for all $x\in\mathbb{R}^d$, and ∇V is Lipschitz-continuous and bounded. Let L be a Lipschitz-constant for ∇V, and let $K:=\|\nabla V\|_{L^\infty(\mathbb{R}^d,\mathbb{R}^d)}$. It follows that ∇V^ε is Lipschitz-continuous with Lipschitz-constant $L_\varepsilon := L/\varepsilon^{d+2}$ and bounded by $K_\varepsilon := K/\varepsilon^{d+1}$.

1.3 Statement of the main result

Let u be the unique strong L^1-solution (see next section) of the Cauchy problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\,\Delta(u^2) \quad \text{in } \mathbb{R}_{>0} \times \mathbb{R}^d \\ u(0,\cdot) &= u_0 \end{aligned}$$

for the porous medium equation with exponent 2 and initial datum u_0, and let P_t be the probability measure on $\mathbb{R}^d$ with density $u(t,\cdot)$. Let m be a fixed natural number, and let $P_t^{N,m,\varepsilon,\delta}$ be the joint distribution of the random variables $X_t^{N,i,\varepsilon,\delta}$, $i = 1,\dots,m$. Now we let $N \to \infty$, $\varepsilon \to 0$, $\delta \to 0$, where in addition to that we require N, ε and δ to satisfy the relations

$$N \geq \exp(\varepsilon^{-2d-5}) \tag{1.4}$$

and

$$\varepsilon \leq C(V,T,\delta)^{-1/3}, \tag{1.5}$$

where $C(V,T,\delta)$ is defined in (1.28). We denote this convergence by $(N,\varepsilon,\delta) \xrightarrow{B} (\infty,0,0)$. Then we have the following theorem:

Theorem 1 (Propagation of chaos). *When $(N,\varepsilon,\delta) \xrightarrow{B} (\infty,0,0)$, $P_t^{N,m,\varepsilon,\delta}$ converges weakly to $P_t^{\otimes m}$:*

$$P_t^{N,m,\varepsilon,\delta} \rightharpoonup P_t^{\otimes m},$$

locally uniformly in t.

This result implies the statements made in the introduction:

Corollary 1.1.

1. *The empirical measure $\mu_t^{N,\varepsilon,\delta} = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{N,i,\varepsilon,\delta}}$ converges weakly to P_t.*
2. *The distribution of the position of each particle also converges to P_t.*
3. *A fixed number of particles remains approximately independent in the course of time.*

Proof. The second and the third statement follow immediately from the theorem. The first statement follows from the theorem and the general fact (see [30], Chapter I.2, Proposition 2.2) that propagation of chaos is equivalent to weak convergence of the empirical measure to a deterministic measure. □

Remark 1.1. Conditions (1.4) and (1.5), which are the precise versions of $N \gg 1/\varepsilon$ and $\varepsilon \ll \delta$, are crucial: Condition $N \gg 1/\varepsilon$ ensures that even when ε, which measures the range of interaction, is small, each particle interacts with many other particles. Condition $\varepsilon \ll \delta$ ensures that the stochastic effects, whose strength is measured by δ, are strong enough.

1.4 Remarks concerning the porous medium equation and related work

The classical application of the porous medium equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\,\Delta(u^2)$$

with exponent 2 concerns the density of an ideal gas flowing isothermally through a homogeneous porous medium (see [31], Chapter I.1 or [32], Section 1.9). Let u be the density of the gas, v its velocity and p the pressure. Then we have the following physical laws:

1. Conservation of mass: $\frac{\partial u}{\partial t} + \operatorname{div}(uv) = 0$
2. Equation of state: $p \sim u$
3. Darcy's law: $v \sim -\nabla p$

Combining these equations we see that (up to a positive constant factor that can be scaled away)

$$\frac{\partial u}{\partial t} = \operatorname{div}(u \nabla u) = \frac{1}{2}\,\Delta(u^2),$$

so that the density of the gas satisfies the porous medium equation. For an introduction to flows in porous media we refer to Vázquez [32].

We now turn to mathematical properties of the Cauchy problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\,\Delta(u^2) \quad \text{in } \mathbb{R}_{>0} \times \mathbb{R}^d \\ u(0, \cdot) &= u_0. \end{aligned} \tag{1.6}$$

In general this Cauchy problem does not admit a classical solution. We therefore have to introduce a suitable notion of weak solution. Following Vázquez [31] we define:

Definition 1.1. A nonnegative function $u \in \mathcal{C}(\mathbb{R}_{\geq 0}, L^1(\mathbb{R}^d))$ is a *strong L^1-solution* of the Cauchy problem (1.6) if:

1. $u^2, \frac{\partial u}{\partial t}, \Delta(u^2) \in L^1_{loc}(\mathbb{R}_{\geq 0}, L^1(\mathbb{R}^d))$
2. $\frac{\partial u}{\partial t} = \Delta(u^2)$ in $\mathbb{R}_{\geq 0} \times \mathbb{R}^d$
3. $u(0, \cdot) = u_0$.

It is known (see e.g. [31], Chapter III, Theorems 2 and 3 and Proposition 4) that the Cauchy problem (1.6) has a unique strong L^1-solution u and that for every $t \geq 0$ $u(t, \cdot)$ is a probability density.

We have given a physical derivation of the porous medium equation based on the hypotheses of continuum mechanics. But strictly speaking, a gas is not a continuum, but consists of atoms and molecules. Therefore it is desirable to find rigorous connections between this microscale and the macroscale. We know that on the macroscale the behaviour of the gas is described by the porous medium equation. So our goal is to find a microscopic model which allows us, when the number of particles tends to infinity, to derive the porous medium equation as limit equation.

Until now, in particular lattice models have been studied, i.e. systems of particles evolving on $\mathbb{Z}^d$ or on a discrete torus $(\mathbb{Z}/N\mathbb{Z})^d$, see [7], [10], [13], [14] and chapter 5 of [17]. In particular, Inoue [14] has proved propagation of chaos for his lattice model.

It is therefore natural to ask whether there exist systems of interacting diffusion processes having the same property. Quite surprisingly, until now no completely satisfactory result has been found. There have been only partial solutions to this problem:

Benachour, Chassaing, Roynette and Vallois [2] consider a system of interacting diffusions which converges to the solution of the equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\,\Delta\big(u(\sigma * u)^2\big)\,,$$

where σ is a Lipschitz-continuous and bounded interaction kernel. If we could replace σ with δ_0, the Dirac measure at 0, we would get the porous medium equation with exponent 3, but there is no convergence result for $\sigma \to \delta_0$.

Jourdain [15] studies a system related to a Cauchy problem for the porous medium equation where the initial datum is the distribution function of a probability measure. But thisapproach is limited to dimension $d = 1$ and does not cover the interesting case where the initial datum is a probability measure.

Oelschläger [26] studies a deterministic interacting particle system given by a system of coupled ordinary differential equations and quite similar to our system. In his system the Brownian motion term δdB_t^i is not present. His approach might seem to be more natural because δ tends to 0 anyway, but Oelschläger does not benefit from the regularizing effect induced by the Brownian motions. Therefore he is able to prove convergence to the porous medium equation only under very restrictive conditions on u_0: only in dimension $d = 1$ he allows quite general initial data, while in dimension $d \geq 2$ he requires u_0 to be strictly positive everywhere, a condition which guarantees that an initially smooth solution remains smooth. Thus Oelschläger does not cover the physically most relevant case where $d = 3$ and the initial datum has compact support.

The same author studies in [28] the numerical simulation of the so called viscous porous medium equation $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta(u^2) + \frac{1}{2}\Delta u$ by a stochastic particle method. That paper illustrates the applicability and importance of interacting particle systems for numerical purposes. One should however note that the viscous porous medium equation is much easier to handle than its nonviscous counterpart thanks to the smoothing effect of the extra term $\frac{1}{2}\Delta u$.

1.5 Proof of Theorem 1

1.5.1 Overview of the proof and preliminary results

As intermediate objects between the particle system (1.1) and the porous medium equation we introduce *non-linear processes* $\overline{X}^{i,\varepsilon,\delta}$ ($i \in \mathbb{N}$, $\varepsilon, \delta > 0$) and $\overline{X}^{i,\delta}$ ($i \in \mathbb{N}$, $\delta > 0$) defined as solutions of the following non-linear stochastic differential equations:

$$
\begin{aligned}
d\overline{X}_t^{i,\varepsilon,\delta} &= -(\nabla V^\varepsilon * u^{\varepsilon,\delta})(t, \overline{X}_t^{i,\varepsilon,\delta})dt + \delta dB_t^i, && (1.7)\\
\overline{X}_0^{i,\varepsilon,\delta} &= \zeta^i && (1.8)\\
P\left[\overline{X}_t^{i,\varepsilon,\delta} \in dx\right] &= u^{\varepsilon,\delta}(t, dx) && (1.9)
\end{aligned}
$$

and

$$
\begin{aligned}
d\overline{X}_t^{i,\delta} &= -\nabla u^\delta(t, \overline{X}_t^{i,\delta})dt + \delta dB_t^i && (1.10)\\
\overline{X}_0^{i,\delta} &= \zeta^i && (1.11)\\
P\left[\overline{X}_t^{i,\delta} \in dx\right] &= u^\delta(t, x)dx && (1.12)\\
u^\delta &\in \mathcal{C}_b^{1,2}([0,T] \times \mathbb{R}^d) \quad \forall T \geq 0. && (1.13)
\end{aligned}
$$

Note that the processes $\overline{X}^{i,\varepsilon,\delta}$ and $\overline{X}^{i,\delta}$ are driven by the *same* Brownian motion B^i and have the *same* initial value ζ^i as the i-th particle of the system (1.1).

A solution of (1.7) - (1.9) is a couple $(\overline{X}^{i,\varepsilon,\delta}, u^{\varepsilon,\delta})$ consisting of a stochastic process $\overline{X}^{i,\varepsilon,\delta}$ and a probability measure $u^{\varepsilon,\delta}$ on $\mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^d)$, the space of continuous functions from $\mathbb{R}_{\geq 0}$ to $\mathbb{R}^d$, such that:

1. The stochastic differential equation (1.7) - (1.8) is satisfied.

2. The distribution of $\overline{X}_t^{i,\varepsilon,\delta}$ is given by $u^{\varepsilon,\delta}(t,\cdot)$.

A solution of (1.10)-(1.13) is a couple $(\overline{X}^{i,\delta}, u^\delta)$ consisting of a stochastic process $\overline{X}^{i,\delta}$ and a function $u^\delta : \mathbb{R}_{\geq 0} \times \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ with $u^\delta \in \mathcal{C}_b^{1,2}([0,T] \times \mathbb{R}^d) \quad \forall T \geq 0$ such that:

1. The stochastic differential equation (1.10)-(1.11) is satisfied.

2. The distribution of $\overline{X}_t^{i,\delta}$ is given by the measure with density $u^\delta(t,\cdot)$.

Remark 1.2.

1. We will show (Propositions 1.2 and 1.3) that both non-linear stochastic differential equations have a unique solution.

2. The processes $\overline{X}^{i,\varepsilon,\delta}$ $(i \in \mathbb{N})$ are independent copies of each other: the initial positions ζ^i $(i \in \mathbb{N})$ are independent and identically distributed, and $\overline{X}^{i,\varepsilon,\delta}$ does not interact with $\overline{X}^{j,\varepsilon,\delta}$ for $i \neq j$. The same holds for the processes $\overline{X}^{i,\delta}$ $(i \in \mathbb{N})$. We can therefore omit the index i.

3. The Itô formula implies that $u^{\varepsilon,\delta}$ is a solution of the integro-differential equation

$$\begin{aligned} \frac{\partial u^{\varepsilon,\delta}}{\partial t} &= \frac{\delta^2}{2}\Delta u^{\varepsilon,\delta} + \operatorname{div}\Big((\nabla V^\varepsilon * u^{\varepsilon,\delta})u^{\varepsilon,\delta}\Big) \\ u^{\varepsilon,\delta}(0,\cdot) &= u_0, \end{aligned} \tag{1.14}$$

while u^δ is a solution of the *viscous porous medium equation*

$$\begin{aligned} \frac{\partial u^\delta}{\partial t} &= \frac{\delta^2}{2}\Delta u^\delta + \operatorname{div}\Big(\nabla u^\delta u^\delta\Big) \\ &= \frac{\delta^2}{2}\Delta u^\delta + \frac{1}{2}\Delta\Big((u^\delta)^2\Big) \\ u^\delta(0,\cdot) &= u_0. \end{aligned} \tag{1.15}$$

The proof of Theorem 1 now consists of the following three parts: in the first part we show (for fixed $\varepsilon, \delta > 0$ and $N \to \infty$) convergence of $X^{N,i,\varepsilon,\delta}$ to $\overline{X}^{i,\varepsilon,\delta}$, in the second part we show (for fixed $\delta > 0$ and $\varepsilon \to 0$) convergence of $\overline{X}^{i,\varepsilon,\delta}$ to $\overline{X}^{i,\delta}$, and in the third part we show (for $\delta \to 0$) convergence of u^δ to u.

However, before we can show convergence we must show existence, uniqueness and regularity results for the measure $u^{\varepsilon,\delta}$, the function u^δ and the processes $X^{i,\varepsilon,\delta}$ and $X^{i,\delta}$. We first study $u^{\varepsilon,\delta}$ and u^δ:

Proposition 1.1.

1. *The viscous porous medium equation* (1.15) *has a unique classical solution* u^δ *(classical means:* $u^\delta \in \mathcal{C}_b^{1,2}([0,T] \times \mathbb{R}^d)$ *for all* $T \geq 0$*), and for each* $T \geq 0$ *we have:*

$$u^\delta \in \mathcal{C}_b^\infty([0,T] \times \mathbb{R}^d).$$

2. *The integro-differential equation* (1.14) *has a unique measure-valued (weak) solution.*

3. *This solution is in fact a classical solution: for each* $t \geq 0$ *the measure* $u^{\varepsilon,\delta}(t,\cdot)$ *has a density with respect to Lebesgue measure (which we also denote by* $u^{\varepsilon,\delta}(t,\cdot)$*), and for each* $T \geq 0$ *we have*

$$u^{\varepsilon,\delta} \in \mathcal{C}_b^\infty([0,T] \times \mathbb{R}^d).$$

4. *For each $\delta > 0$, $j, k \in \mathbb{N}_0, i_1, \ldots, i_k \in \{1, \ldots, d\}$ and $T \geq 0$ we have:*

$$\sup_{\varepsilon>0} \frac{1}{\varepsilon^2} \left\| \frac{\partial^{j+k}}{\partial t^j \partial x_{i_1} \cdots \partial x_{i_k}} (u^{\varepsilon,\delta} - u^\delta) \right\|_{L^\infty([0,T]\times\mathbb{R}^d)} < \infty.$$

Corollary 1.2.

1. *For each $T \geq 0$ and each $\delta > 0$ there is a constant $C_1(T, \delta) < \infty$ such that*

$$\left| \nabla u^\delta(s, x) - \nabla u^\delta(s, y) \right| \leq C_1(T, \delta)|x - y| \tag{1.16}$$

   for all $s \in [0, T]$ and all $x, y \in \mathbb{R}^d$.

2. *For each $T \geq 0$ and each $\delta > 0$ there is a constant $C_2(T, \delta) < \infty$ such that*

$$\|D^3 u^{\varepsilon,\delta}\|_{L^\infty([0,T]\times\mathbb{R}^d,\mathbb{R}^{d^3})} \leq C_2(T, \delta) \tag{1.17}$$

   for all $\varepsilon > 0$.

   Here we use the following notation: for $f \in \mathcal{C}^3([0, T] \times \mathbb{R}^d)$ we write

$$D^3 f := \left(\frac{\partial^3 f}{\partial x_{i_1} \partial x_{i_2} \partial x_{i_3}} \right)_{i_1,i_2,i_3=1}^d$$

   and

$$\begin{aligned} &\|D^3 f\|_{L^\infty([0,T]\times\mathbb{R}^d,\mathbb{R}^{d^3})} \\ := \ &\sup \left\{ \left| \frac{\partial^3 f}{\partial x_{i_1} \partial x_{i_2} \partial x_{i_3}} (t, x) \right| \ \middle| \ t \in [0, T], x \in \mathbb{R}^d, i_1, i_2, i_3 \in \{1, \ldots, d\} \right\}. \end{aligned}$$

3. *For each $T \geq 0$ and each $\delta > 0$ there is a constant $C_3(T, \delta) < \infty$ such that*

$$\|\nabla u^{\varepsilon,\delta} - \nabla u^\delta\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leq C_3(T, \delta)\varepsilon^2 \tag{1.18}$$

   for all $\varepsilon > 0$.

Proof of Proposition 1.1. For $\delta = 1$ this has been proved in [27], Theorems 1 and 2. The general case follows from the case $\delta = 1$ with the following scaling argument:

We first show the existence of a classical solution $u^{\varepsilon,\delta}$ of the integro-differential equation (1.14). We will show that it has the form

$$u^{\varepsilon,\delta}(t, x) = \alpha \tilde{u}^\varepsilon(\beta t, \gamma x). \tag{1.19}$$

Here α, β and γ are strictly positive parameters which are still to be determined, and $\overline{u}^\varepsilon$ is the unique classical solution of the Cauchy problem

$$\begin{aligned} \frac{\partial \tilde{u}^\varepsilon}{\partial t} &= \frac{1}{2} \Delta \tilde{u}^\varepsilon + \operatorname{div}\left((\nabla V^{\varepsilon\gamma} * \tilde{u}^\varepsilon)\tilde{u}^\varepsilon\right) \\ \tilde{u}^\varepsilon(0, \cdot) &= \tilde{u}_0. \end{aligned} \tag{1.20}$$

Its existence and uniqueness is proved in [27], Theorem 1. As we want $u^{\varepsilon,\delta}(0, \cdot)$ to be equal to u_0, we must choose

$$\overline{u}_0(x) := \frac{1}{\alpha} u_0(x/\gamma).$$

(1.2) and (1.19) imply:

$$\begin{aligned}\frac{\partial u^{\varepsilon,\delta}}{\partial t}(t,x) &= \alpha\beta\frac{\partial \tilde{u}^{\varepsilon}}{\partial t}(\beta t,\gamma x)\\ \frac{\delta^2}{2}\Delta u^{\varepsilon,\delta}(t,x) &= \alpha\gamma^2\delta^2\frac{1}{2}\Delta\tilde{u}^{\varepsilon}(\beta t,\gamma x)\\ \operatorname{div}\left((\nabla V^{\varepsilon} * u^{\varepsilon,\delta})u^{\varepsilon,\delta}\right)(t,x) &= \alpha^2\gamma^2\operatorname{div}\left((\nabla V^{\varepsilon\gamma} * \tilde{u}^{\varepsilon})\tilde{u}^{\varepsilon}\right)(\beta t,\gamma x).\end{aligned}$$

We now determine α, β and γ in such a way that

$$\alpha\beta = \alpha\gamma^2\delta^2 = \alpha^2\gamma^2 \tag{1.21}$$

and

$$\alpha\gamma^{-d} = 1 \tag{1.22}$$

hold; (1.22) ensures that $\int_{\mathbb{R}^d}\overline{u}_0(x)dx = 1$. The unique solution of (1.21) and (1.22) is

$$\alpha = \delta^2, \quad \beta = \delta^{2+4/d}, \quad \gamma = \delta^{2/d}.$$

We therefore set

$$\overline{u}_0(x) := \frac{1}{\delta^2}u_0(x/\delta^{2/d})$$

and

$$u^{\varepsilon,\delta}(t,x) := \delta^2\tilde{u}^{\varepsilon}(\delta^{2+4/d}t,\delta^{2/d}x) \tag{1.23}$$

and see that the function $u^{\varepsilon,\delta}$ defined in this way is a classical solution of the Cauchy problem (1.14).

Uniqueness of a weak solution of (1.14) follows with the inverse scaling: Two different weak solutions of (1.14) would lead to two different weak solutions of (1.20) with the same initial values, which would contradict the results of [27]. We have thus proved the second and the third statement of the proposition.

We now prove the first and the fourth statement. To this end let $\overline{u}$ be the unique classical solution of the Cauchy problem

$$\begin{aligned}\frac{\partial \tilde{u}}{\partial t} &= \frac{1}{2}\Delta\tilde{u} + \frac{1}{2}\Delta\left(\overline{u}^2\right)\\ \tilde{u}(0,\cdot) &= \tilde{u}_0.\end{aligned}$$

According to [27], formula (2.20), for each $T \geq 0$:

$$\tilde{u} \in \mathcal{C}_b^{\infty}([0,T]\times\mathbb{R}^d).$$

The same scaling argument as above implies that (1.15) has a unique classical solution $u^{\delta} \in \bigcap_{T\geq 0}\mathcal{C}_b^{1,2}([0,T]\times\mathbb{R}^d)$, which is given by

$$u^{\delta}(t,x) = \delta^2\overline{u}(\delta^{2+4/d}t,\delta^{2/d}x) \tag{1.24}$$

and therefore even belongs to $\bigcap_{T\geq 0}\mathcal{C}_b^{\infty}([0,T]\times\mathbb{R}^d)$.

According to [27], Theorem 2, formula (2.19) with $L = 0$ we have:

$$\sup_{\varepsilon>0}\frac{1}{\varepsilon^2}\left\|\frac{\partial^{j+k}}{\partial t^j\partial x_{i_1}\cdots\partial x_{i_k}}(\overline{u}^{\varepsilon}-\overline{u})\right\|_{L^{\infty}([0,T]\times\mathbb{R}^d)} < \infty,$$

and (1.23) and (1.24) imply:

$$\begin{aligned}
& \sup_{\varepsilon>0} \frac{1}{\varepsilon^2} \left\| \frac{\partial^{j+k}}{\partial t^j \partial x_{i_1} \cdots \partial x_{i_k}} (u^{\varepsilon,\delta} - u^\delta) \right\|_{L^\infty([0,T]\times\mathbb{R}^d)} \\
= \; & \delta^2 \delta^{j(2+4/d)} \delta^{k\,2/d} \sup_{\varepsilon>0} \frac{1}{\varepsilon^2} \left\| \frac{\partial^{j+k}}{\partial t^j \partial x_{i_1} \cdots \partial x_{i_k}} (\overline{u}^\varepsilon - \overline{u}) \right\|_{L^\infty([0,T]\times\mathbb{R}^d)} \\
< \; & \infty,
\end{aligned}$$

which concludes the proof of the proposition. □

We now study the process $(\overline{X}_t^\delta)_{t\geq 0}$:

Proposition 1.2. *The stochastic differential equation (1.10)-(1.13) has a unique solution* $(\overline{X}^\delta, u^\delta)$.

Proof. We follow ideas of Jourdain and Méléard [16]. First we show uniqueness. To this end let $(\overline{X}^\delta, u^\delta)$ be a solution. As we have already mentioned, the Itô formula implies that u^δ is a weak solution of the viscous porous medium equation (1.15). Because of $u^\delta \in \mathcal{C}_b^{1,2}([0,T]\times\mathbb{R}^d)\ \forall T \geq 0$ it is even a classical solution of (1.15) and therefore (according to the first statement of Proposition 1.1) unique. This implies immediately that $\overline{X}^\delta$ is unique, too.

We now prove existence of a solution. To this end let $u^\delta \in \bigcap_{T\geq 0} \mathcal{C}_b^\infty([0,T]\times\mathbb{R}^d)$ be the unique classical solution of (1.15) (see Proposition 1.1), and let $\overline{X}^\delta$ be the unique solution of (1.10) and (1.11). This means that we insert u^δ in (1.10) without caring whether the distribution of the process $\overline{X}^\delta$ is really given by u^δ. According to [12], Chapter 6.5, Theorem 5.4, the distribution of $\overline{X}_t^\delta$ has for each $t \geq 0$ a density $v(t,\cdot)$, and v is a classical solution of the uniformly parabolic linear partial differential equation

$$\begin{aligned}
\frac{\partial v}{\partial t} &= \frac{\delta^2}{2}\Delta v + \operatorname{div}(\nabla u^\delta v) \qquad (1.25)\\
v(0,\cdot) &= u_0.
\end{aligned}$$

According to [18], Chapter I.2, Theorem 2.6, this property determines v uniquely. But as also u^δ is a classical solution of (1.25), it follows that $v = u^\delta$, and the proposition is proved. □

Now we study the process $(\overline{X}_t^{\varepsilon,\delta})_{t\geq 0}$:

Proposition 1.3. *The stochastic differential equation* (1.7) - (1.9) *has a unique solution* $(\overline{X}^{\varepsilon,\delta}, u^{\varepsilon,\delta})$.

Proof. Using Proposition 1.1 this can be proved in the same way as Proposition 1.2. An alternative proof (independent of Proposition 1.1) can be given using a fixed point argument in the space of probability measures on $\mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^d)$ (cf. [30], Chapter I.1, Theorem 1.1). □

1.5.2 First step: $N \to \infty$ (ε, δ fixed)

We can now start to prove Theorem 1. As we have already said, we first show convergence of $X^{N,i,\varepsilon,\delta}$ to $\overline{X}^{i,\varepsilon,\delta}$ for $N \to \infty$ with ε and δ fixed:

Proposition 1.4. *For each* $T \geq 0$ *and each* $i \in \{1,\ldots,N\}$ *we have:*

$$E\left[\sup_{0\leq s\leq T} \left| X_s^{N,i,\varepsilon,\delta} - \overline{X}_s^{i,\varepsilon,\delta} \right|^2 \right] \leq 2K^2 L^{-2} \varepsilon^2 \exp(6L^2T^2\varepsilon^{-2d-4}) \frac{1}{N}.$$

Remark 1.3. This estimate obviously implies that for fixed $\varepsilon, \delta > 0$ $X^{N,i,\varepsilon,\delta}$ converges in L^2 to $\overline{X}^{i,\varepsilon,\delta}$, locally uniformly in t. But due to (1.4), even for variable ε and δ we have

$$E\left[\sup_{0\le s\le T}\left|X_s^{N,i,\varepsilon,\delta} - \overline{X}_s^{i,\varepsilon,\delta}\right|^2\right] \to 0 \qquad ((N,\varepsilon,\delta) \overset{B}{\to} (\infty,0,0)).$$

This is important because we will combine Proposition 1.4 with other convergence results (Corollary 1.3 and Proposition 1.6) where $\varepsilon \to 0$ and $\delta \to 0$.

Proof of Proposition 1.4. Since a similar result is proved in [16], Proposition 2.3, we only sketch the proof: Let us first recall that ∇V^ε is Lipschitz-continuous with Lipschitz-constant $L_\varepsilon = L/\varepsilon^{d+2}$ and bounded by $K_\varepsilon = K/\varepsilon^{d+1}$. Let

$$\Phi(t) := E\left[\sup_{0\le s\le t}\left|X_s^{N,i,\varepsilon,\delta} - \overline{X}_s^{i,\varepsilon,\delta}\right|^2\right].$$

Using (1.1) and (1.7) one can easily show that

$$\begin{aligned}\Phi(t) \quad \le \quad & \frac{3t}{N^2}\left\{E\left[\int_0^t\left|\sum_{j=1}^N\left[(\nabla V^\varepsilon * u^{\varepsilon,\delta})(s,\overline{X}_s^{i,\varepsilon,\delta}) - \nabla V^\varepsilon(\overline{X}_s^{i,\varepsilon,\delta} - \overline{X}_s^{j,\varepsilon,\delta})\right]\right|^2 ds\right]\right. \qquad (1.26)\\ & + E\left[\int_0^t\left|\sum_{j=1}^N\left[\nabla V^\varepsilon(\overline{X}_s^{i,\varepsilon,\delta} - \overline{X}_s^{j,\varepsilon,\delta}) - \nabla V^\varepsilon(X_s^{N,i,\varepsilon,\delta} - \overline{X}_s^{j,\varepsilon,\delta})\right]\right|^2 ds\right]\\ & \left. + E\left[\int_0^t\left|\sum_{j=1}^N\left[\nabla V^\varepsilon(X_s^{N,i,\varepsilon,\delta} - \overline{X}_s^{j,\varepsilon,\delta}) - \nabla V^\varepsilon(X_s^{N,i,\varepsilon,\delta} - X_s^{N,j,\varepsilon,\delta})\right]\right|^2 ds\right]\right\}.\end{aligned}$$

Using the Lipschitz-continuity of ∇V^ε one obtains that the second and the third term are both bounded by $N^2 L_\varepsilon^2 \int_0^t \Phi(s)ds$. For the first term we get

$$\begin{aligned} & E\left[\int_0^t\left|\sum_{j=1}^N\left[(\nabla V^\varepsilon * u^{\varepsilon,\delta})(s,\overline{X}_s^{i,\varepsilon,\delta}) - \nabla V^\varepsilon(\overline{X}_s^{i,\varepsilon,\delta} - \overline{X}_s^{j,\varepsilon,\delta})\right]\right|^2 ds\right]\\ = \quad & \sum_{j,k=1}^N \int_0^t E\left\{\left[(\nabla V^\varepsilon * u^{\varepsilon,\delta})(s,\overline{X}_s^{i,\varepsilon,\delta}) - \nabla V^\varepsilon(\overline{X}_s^{i,\varepsilon,\delta} - \overline{X}_s^{j,\varepsilon,\delta})\right]\right.\\ & \qquad \left.\cdot\left[(\nabla V^\varepsilon * u^{\varepsilon,\delta})(s,\overline{X}_s^{i,\varepsilon,\delta}) - \nabla V^\varepsilon(\overline{X}_s^{i,\varepsilon,\delta} - \overline{X}_s^{k,\varepsilon,\delta})\right]\right\} ds.\end{aligned}$$

If $j \ne k$ this expectation vanishes, and otherwise it is bounded by $4K_\varepsilon^2$ due to the boundedness of ∇V^ε. Therefore the first term on the right-hand side of (1.26) is bounded by $4NtK_\varepsilon^2$.

Summarizing these estimates we get:

$$\Phi(t) \le \frac{3t}{N^2}\left\{4NtK_\varepsilon^2 + 2N^2L_\varepsilon^2\int_0^t \Phi(s)ds\right\} \le 12TtK_\varepsilon^2\frac{1}{N} + 6L_\varepsilon^2 T\int_0^t \Phi(s)ds.$$

Now let $\Psi(t) := \Phi(t) + \frac{4K_\varepsilon^2 t}{NL_\varepsilon^2}$. It follows that $\Psi(t) \le 6L_\varepsilon^2 T\int_0^t \Psi(s)ds + \frac{2K_\varepsilon^2}{NL_\varepsilon^2}$. Gronwall's lemma now implies that $\Psi(t) \le \frac{2K_\varepsilon^2}{NL_\varepsilon^2}\exp(6L_\varepsilon^2 Tt)$ which concludes the proof. □

1.5.3 Second step: $\varepsilon \to 0$ (δ fixed)

We now show that $\overline{X}_t^{\varepsilon,\delta}$ converges to $\overline{X}_t^{\delta}$ for $\varepsilon \to 0$ and δ fixed:

Proposition 1.5. *Let*

$$C(V) := \frac{d^{3/2}}{2} \int_{\mathbb{R}^d} |y|^2 V(y) dy$$

($C(V)$ is finite because of (1.3)*), and*

$$K(V,T,\delta) := [C(V)C_2(T,\delta) + C_3(T,\delta)]\, T \exp(C_1(T,\delta)T).$$

Then for each $T \geq 0$:

$$\sup_{0\leq s\leq T} \left| \overline{X}_s^{\varepsilon,\delta} - \overline{X}_s^{\delta} \right| \leq K(V,T,\delta)\varepsilon^2.$$

Corollary 1.3.

$$E\left[\sup_{0\leq s\leq T} \left| \overline{X}_s^{\varepsilon,\delta} - \overline{X}_s^{\delta} \right|^2 \right] \leq K(V,T,\delta)^2 \varepsilon^4.$$

The main ingredient to the proof is Proposition 1.1. We also need the following lemma:

Lemma 1.1. *For every $g \in \mathcal{C}_b^3([0,T]\times\mathbb{R}^d)$:*

$$\|V^\varepsilon * \nabla g - \nabla g\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leq C(V) \|D^3 g\|_{L^\infty([0,T]\times\mathbb{R}^d,\mathbb{R}^{d^3})} \varepsilon^2 \tag{1.27}$$

Proof. This is a simple computation using the symmetry of V. □

Proof of Proposition 1.5. Let

$$\Psi(t) := \sup_{0\leq s\leq t} \left| \overline{X}_s^{\varepsilon,\delta} - \overline{X}_s^{\delta} \right|.$$

(1.7), (1.8), (1.10), (1.11), (1.16), (1.17), (1.18) and (1.27) imply:

$$\begin{aligned}
\Psi(t) &\overset{(1.7),(1.8),(1.10),(1.11)}{\leq} \int_0^t \left| (\nabla V^\varepsilon * u^{\varepsilon,\delta})(s, \overline{X}_s^{\varepsilon,\delta}) - \nabla u^\delta(s, \overline{X}_s^{\delta}) \right| ds \\
&\leq \int_0^t \left| (V^\varepsilon * \nabla u^{\varepsilon,\delta})(s, \overline{X}_s^{\varepsilon,\delta}) - \nabla u^{\varepsilon,\delta}(s, \overline{X}_s^{\varepsilon,\delta}) \right| ds \\
&\quad + \int_0^t \left| \nabla u^{\varepsilon,\delta}(s, \overline{X}_s^{\varepsilon,\delta}) - \nabla u^{\delta}(s, \overline{X}_s^{\varepsilon,\delta}) \right| ds \\
&\quad + \int_0^t \left| \nabla u^{\delta}(s, \overline{X}_s^{\varepsilon,\delta}) - \nabla u^{\delta}(s, \overline{X}_s^{\delta}) \right| ds \\
&\overset{(1.27),(1.18),(1.16)}{\leq} tC(V) \left\| D^3 u^{\varepsilon,\delta} \right\|_{L^\infty([0,T]\times\mathbb{R}^d,\mathbb{R}^{d^3})} \varepsilon^2 \\
&\quad + tC_3(T,\delta)\varepsilon^2 \\
&\quad + C_1(T,\delta) \int_0^t \left| \overline{X}_s^{\varepsilon,\delta} - \overline{X}_s^{\delta} \right| ds \\
&\overset{(1.17)}{\leq} t\left[C(V)C_2(T,\delta) + C_3(T,\delta)\right] \varepsilon^2 + C_1(T,\delta) \int_0^t \Psi(s) ds.
\end{aligned}$$

Gronwall's lemma now implies:

$$\Psi(t) \leq t\left[C(V)C_2(T,\delta) + C_3(T,\delta)\right] \varepsilon^2 \exp(C_1(T,\delta)t),$$

in particular

$$\begin{aligned}\sup_{0\le s\le T}\left|\overline{X}_s^{\varepsilon,\delta}-\overline{X}_s^{\delta}\right| &= \Psi(T)\\ &\le [C(V)C_2(T,\delta)+C_3(T,\delta)]\,T\exp(C_1(T,\delta)T)\varepsilon^2\\ &= K(V,T,\delta)\varepsilon^2,\end{aligned}$$

q.e.d. □

We can now combine Proposition 1.4 and Corollary 1.3 to get the following result:

Corollary 1.4. *Let*

$$C(V,T,\delta) := 2K(V,T,\delta)^2. \tag{1.28}$$

Then for each $T\ge 0$ and each $i\in\{1,\dots,N\}$:

$$\begin{aligned}& E\left[\sup_{0\le s\le T}\left|X_s^{N,i,\varepsilon,\delta}-\overline{X}_s^{i,\delta}\right|^2\right]\\ \le\ & 72K^2T^2\varepsilon^{-2d-2}\exp(6L^2T^2\varepsilon^{-2d-4})\frac{1}{N}+C(V,T,\delta)\varepsilon^4.\end{aligned}$$

Combining this estimate with (1.4) and (1.5) we get:

$$E\left[\sup_{0\le s\le T}\left|X_s^{N,i,\varepsilon,\delta}-\overline{X}_s^{i,\delta}\right|^2\right]\to 0 \qquad ((N,\varepsilon,\delta)\overset{B}{\to}(\infty,0,0)). \tag{1.29}$$

1.5.4 Third step: $\delta\to 0$

We use the following analytical result due to Bénilan and Crandall [3]:

Proposition 1.6. *For each $T\ge 0$:*

$$\sup_{0\le t\le T}\|u^\delta(t,\cdot)-u(t,\cdot)\|_{L^1(\mathbb{R}^d)}\to 0 \qquad (\delta\to 0). \tag{1.30}$$

Proof. Apply the theorem on page 162 of Bénilan and Crandall [3] to the functions $\varphi_n(r) := r^2\operatorname{sign} r/2+\delta_n^2 r/2$ and $\varphi_\infty(r) := r^2\operatorname{sign} r/2$ for a sequence $(\delta_n)_{n\in\mathbb{N}}$ converging to 0. (The factor $\operatorname{sign} r$ makes the functions φ_n and φ_∞ nondecreasing, as required in the formulation of Bénilan's and Crandall's theorem, but since u^δ and u are nonnegative, $\operatorname{sign} u^\delta=\operatorname{sign} u=1$, and these choices of φ_n and φ_∞ do correspond to the (viscous) porous medium equation.) □

We can now easily prove Theorem 1 by combining Corollary 1.4 and Proposition 1.6:

Proof of Theorem 1. Let P_t^δ be the distribution of $\overline{X}_t^\delta$. (1.29) implies that for $(N,\varepsilon,\delta)\overset{B}{\to}(\infty,0,0)$ the measure $P_t^{N,m,\varepsilon,\delta}-P_t^{\delta\otimes m}$ converges weakly to 0, locally uniformly in t. (1.30) implies for $\delta\to 0$ weak convergence of P_t^δ to P_t (locally uniformly in t). It follows that for $(N,\varepsilon,\delta)\overset{B}{\to}(\infty,0,0)$ the measure $P_t^{N,m,\varepsilon,\delta}$ converges weakly to $P_t^{\otimes m}$ (locally uniformly in t). □

Chapter 2

Nonlinear stochastic differential equations and the viscous porous medium equation

2.1 Introduction

In the previous chapter we showed that the solution u of the porous medium equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\,\Delta(u^2) \quad \text{in } \mathbb{R}_{>0}\times\mathbb{R}^d \\ u(0,\cdot) &= u_0 \end{aligned}$$

can be approximated by the interacting particle system

$$\begin{aligned} dX_t^{N,i,\varepsilon,\delta} &= -\frac{1}{N}\sum_{j=1}^{N}\nabla V^\varepsilon(X_t^{N,i,\varepsilon,\delta}-X_t^{N,j,\varepsilon,\delta})dt+\delta dB_t^i, \quad i=1,\ldots,N \\ X_0^{N,i,\varepsilon,\delta} &= \zeta^i \end{aligned}$$

(see Theorem 1). The proof essentially consisted of the following three steps: In a first step we showed that $X_t^{N,i,\varepsilon,\delta}$ converges (for $N\to\infty$) to the process $\overline{X}_t^{i,\varepsilon,\delta}$. In a second step we showed, using a result of Oelschläger [27], that the density $u^{\varepsilon,\delta}$ of the distribution of $\overline{X}_t^{i,\varepsilon,\delta}$ converges (for $\varepsilon\to 0$) to the solution u^δ of the viscous porous medium equation (with viscosity $\delta^2/2$)

$$\begin{aligned} \frac{\partial u^\delta}{\partial t} &= \frac{\delta^2}{2}\Delta u^\delta+\frac{1}{2}\,\Delta\Big((u^\delta)^2\Big) \\ u^\delta(0,\cdot) &= u_0. \end{aligned}$$

In a third step we finally showed that u^δ converges (for $\delta\to 0$) to the solution u of the porous medium equation.

But as in the second step of our proof we used the result of [27], our result had the disadvantage to require the very strong assumption $u_0\in W_{n,1}^2(\mathbb{R}^d)$ (which is even stronger than $u_0\in\mathcal{C}_b^\infty(\mathbb{R}^d)$). In this chapter we will therefore prove an analogous result under the much weaker assumption $u_0\in L^2(\mathbb{R}^d)$.

In order to simplify the formulas we use a slightly different notation than in Chapter 1, namely we consider the following sequence of nonlinear stochastic differential equations in $\mathbb{R}^d$:

$$\begin{aligned} dX_t^N &= -(\nabla V^N * u^N(t))(X_t^N)dt + \delta dB_t \\ X_0^N &= \zeta \\ u^N(t) &= \text{Law}(X_t^N). \end{aligned} \tag{2.1}$$

Here V^N is obtained from a fixed probability density V by the scaling $V^N(x) := \chi_N^d V(\chi_N x)$, where $\lim_{N\to\infty} \chi_N = \infty$, $(B_t)_{t\geq 0}$ is a d-dimensional Brownian motion, and ζ is a random variable whose distribution has the density u_0. So χ_N in (2.1) corresponds to $1/\varepsilon$ in Chapter 1, V^N corresponds to V^ε, and u^N corresponds to $u^{\varepsilon,\delta}$. We omit the factor δ in front of the Brownian motion term dB_t because it can be scaled away as shown in the proof of Proposition 1.1. Alternatively, one can easily see that the proofs of the results in this chapter work in the same way if in (2.1) we replace dB_t with δdB_t.

We will show that u^N converges, as $N \to \infty$, to the solution u of the *viscous porous medium equation*

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\Delta u + \frac{1}{2}\Delta(u^2) \\ u(\cdot,0) &= u_0. \end{aligned}$$

2.2 Assumptions and Notation

As in Chapter 1 we assume the existence of a function $W : \mathbb{R}^d \to \mathbb{R}$ with $W \geq 0$, $\int_{\mathbb{R}^d} W(x)dx = 1$ and $W(-x) = W(x)$ for all $x \in \mathbb{R}^d$ such that $V = W * W$. However, in contrast to Chapter 1 we do not require $W \in \mathcal{C}_b^\infty(\mathbb{R}^d)$, but instead $W \in \mathcal{C}_c^2(\mathbb{R}^d)$ (twice continuously differentiable with compact support). In analogy to the definition of V^N we set $W^N(x) := \chi_N^d W(\chi_N x)$ (so that $V^N = W^N * W^N$). We introduce the following smoothed versions of u^N: $g^N(x,t) := (u^N(t) * V^N)(x)$ and $g_1^N(x,t) := (u^N(t) * W^N)(x)$.

Let $\mathcal{M}(\mathbb{R}^d)$ be the the space of probability measures on $\mathbb{R}^d$, equipped with the following metric:

$$d(\mu,\nu) := \sup_{f\in BL} \left| \int_{\mathbb{R}^d} f(x)\mu(dx) - \int_{\mathbb{R}^d} f(x)\nu(dx) \right|,$$

where BL is the set of all Lipschitz continuous functions on $\mathbb{R}^d$ which are bounded by 1 and have Lipschitz constant 1. It is well known (see e.g. [6]) that d metrizes the weak convergence in $\mathcal{M}(\mathbb{R}^d)$.

Since we allow quite general initial data (recall that we only require $u_0 \in L^2(\mathbb{R}^d)$) we need a suitable notion of weak solution:

Definition 2.1. A weak solution of the viscous porous medium equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\Delta u + \frac{1}{2}\Delta(u^2) \\ u(\cdot,0) &= u_0 \end{aligned}$$

on the time interval $[0,T]$ with initial datum u_0 is a measure-valued function $u \in \mathcal{C}([0,T],\mathcal{M}(\mathbb{R}^d))$ with the following properties:

1. For almost every $t \in [0,T]$ the measure $u(t)$ has a density $g(\cdot,t)$ with respect to Lebesgue measure, and $g \in L^2(\mathbb{R}^d \times [0,T])$.

2. For all $f \in \mathcal{C}_b^2(\mathbb{R}^d)$ and all $t \in [0,T]$:

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)u(t)(dx) &= \int_{\mathbb{R}^d} f(x)u_0(dx) + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \Delta f(x)u(s)(dx)ds \\ &\quad + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \Delta f(x)g(x,s)^2 dxds.\end{aligned}$$

2.3 Main result

Theorem 2. *The sequence $(u^N)_{N\in\mathbb{N}}$ converges in $\mathcal{C}([0,T],\mathcal{M}(\mathbb{R}^d))$ to a weak solution u^∞ of the viscous porous medium equation with initial datum u_0. This solution belongs to the class of those weak solutions u for which the density g belongs to $L^3(\mathbb{R}^d \times [0,T])$ (and not only to $L^2(\mathbb{R}^d \times [0,T])$), and is unique in this class.*

Remark 2.1. As we have already mentioned, Oelschläger [27] proved a similar result, but under the much stronger assumption $u_0 \in \mathcal{C}_b^\infty(\mathbb{R}^d)$. (Recall that we only require $u_0 \in L^2(\mathbb{R}^d)$.) Moreover, his proof is much more complicated.

2.4 Proof of Theorem 2

We first study the dynamics of u^N:

Lemma 2.1. *For any $f \in \mathcal{C}_b^2(\mathbb{R}^d)$:*

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)u^N(t)(dx) &= \int_{\mathbb{R}^d} f(x)u_0(x)dx + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \Delta f(x)u^N(s)(dx)ds \\ &\quad - \int_0^t \int_{\mathbb{R}^d} \nabla f(x)\cdot\nabla g^N(x,s)u^N(s)(dx)ds. \qquad (2.2)\end{aligned}$$

This means that u^N is a weak solution of the integro-differential equation

$$\begin{aligned}\frac{\partial u^N}{\partial t} &= \frac{1}{2}\Delta u^N + \mathrm{div}(\nabla g^N u^N) \\ &= \frac{1}{2}\Delta u^N + \mathrm{div}((\nabla V^N * u^N)u^N) \qquad (2.3) \\ u^N(\cdot,0) &= u_0.\end{aligned}$$

Proof. This follows by applying Itô's formula to (2.1) and taking expectations. □

We also need a version of Lemma 2.1 for functions of two variables:

Lemma 2.2. *For any $f \in \mathcal{C}_b^2(\mathbb{R}^d \times \mathbb{R}^d)$ we have:*

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y,z)u^N(t)(dy)u^N(t)(dz) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y,z)u_0(y)dyu_0(z)dz \\ &- \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_y f(y,z)\cdot\nabla g^N(y,s)u^N(s)(dy)u^N(s)(dz)ds \\ &- \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_z f(y,z)\cdot\nabla g^N(z,s)u^N(s)(dy)u^N(s)(dz)ds \\ &+ \frac{1}{2}\int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta_y f(y,z)u^N(s)(dy)u^N(s)(dz)ds \\ &+ \frac{1}{2}\int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta_z f(y,z)u^N(s)(dy)u^N(s)(dz)ds.\end{aligned}$$

Proof. This can be proved by applying Itô's formula to two independent copies of (2.1) and taking expectations. □

An important step of the proof of Theorem 2 is the following lemma:

Lemma 2.3. *For each $t \geq 0$:*

$$\|g_1^N(\cdot,t)\|_2^2 + 2\int_0^t \int_{\mathbb{R}^d} |\nabla g^N(x,s)|^2 u^N(s)(dx)ds + \int_0^t \|\nabla g_1^N(\cdot,s)\|_2^2\, ds = \|g_1^N(\cdot,0)\|_2^2. \tag{2.4}$$

Remark 2.2. Because of $g_1^N(\cdot,0) = u_0 * V^N$, $\|V^N\|_{L^1(\mathbb{R}^d)} = 1$ and $u_0 \in L^2(\mathbb{R}^d)$ we have

$$\|g_1^N(\cdot,0)\|_2^2 \leq \|u_0\|_2^2 < \infty.$$

Therefore Lemma 2.3 implies that each of the three terms on the left hand side of (2.4) is bounded uniformly in N and t.

Proof of Lemma 2.3. We temporarily fix $x \in \mathbb{R}^d$ and obtain, by applying Lemma 2.2 to the function $f(y,z) := W^N(x-y)W^N(x-z)$:

$$\begin{aligned}
g_1^N(x,t)^2 &= \left[\left(W^N * u^N(t)\right)(x)\right]^2 \\
&= \int_{\mathbb{R}^d} W^N(x-y)u^N(t)(dy) \int_{\mathbb{R}^d} W^N(x-z)u^N(t)(dz) \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W^N(x-y)W^N(x-z)u^N(t)(dy)u^N(t)(dz) \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W^N(x-y)W^N(x-z)u_0(y)dyu_0(z)dz \\
&\quad + \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla W^N(x-y)W^N(x-z)\cdot \nabla g^N(y,s)u^N(s)(dy)u^N(s)(dz)ds \\
&\quad + \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W^N(x-y)\nabla W^N(x-z)\cdot \nabla g^N(z,s)u^N(s)(dy)u^N(s)(dz)ds \\
&\quad + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta W^N(x-y)W^N(x-z)u^N(s)(dy)u^N(s)(dz)ds \\
&\quad + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W^N(x-y)\Delta W^N(x-z)u^N(s)(dy)u^N(s)(dz)ds.
\end{aligned}$$

Now we observe that the second and third, as well as the fourth and fifth term are equal, so that we obtain:

$$\begin{aligned}
g_1^N(x,t)^2 &= g_1^N(x,0)^2 + 2\int_0^t \left[\int_{\mathbb{R}^d} \nabla W^N(x-y)\cdot\nabla g^N(y,s)u^N(s)(dy)\right] g_1^N(x,s)ds \\
&\quad + \int_0^t \left[\int_{\mathbb{R}^d} \Delta W^N(x-y)u^N(s)(dy)\right] g_1^N(x,s)ds.
\end{aligned}$$

We now integrate this over $x \in \mathbb{R}^d$. Using the fact that $\nabla W^N(x-y) = -\nabla W^N(y-x)$ (because of the symmetry of W^N) we obtain:

$$\begin{aligned}
\int_{\mathbb{R}^d} g_1^N(x,t)^2dx &= \int_{\mathbb{R}^d} g_1^N(x,0)^2dx \\
&\quad -2\int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla W^N(y-x)\cdot\nabla g^N(y,s)u^N(s)(dy)g_1^N(x,s)dxds \\
&\quad + \int_0^t \int_{\mathbb{R}^d}\left[\int_{\mathbb{R}^d} \Delta W^N(x-y)g_1^N(x,s)dx\right] u^N(s)(dy)ds.
\end{aligned}$$

As W^N has compact support, we can integrate by parts and obtain:

$$\begin{aligned}\int_{\mathbb{R}^d} g_1^N(x,t)^2 dx &= \int_{\mathbb{R}^d} g_1^N(x,0)^2 dx \\ &\quad -2\int_0^t \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \nabla W^N(y-x) g_1^N(x,s)dx\right] \cdot \nabla g^N(y,s) u^N(s)(dy)ds \\ &\quad -\int_0^t \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \nabla W^N(x-y) \cdot \nabla g_1^N(x,s)dx\right] u^N(s)(dy)ds \\ &= \int_{\mathbb{R}^d} g_1^N(x,0)^2 dx - 2\int_0^t \int_{\mathbb{R}^d} |\nabla g^N(y,s)|^2 u^N(s)(dy)ds - \int_0^t \int_{\mathbb{R}^d} |\nabla g_1^N(x,s)|^2 dxds,\end{aligned}$$

q.e.d. □

Proposition 2.1. *The set $\{u^N \mid N \in \mathbb{N}\}$ is relatively compact in $\mathcal{C}([0,T], \mathcal{M}(\mathbb{R}^d))$.*

Proof. In order to apply the Arzelà-Ascoli theorem we have to show:

1. There is a compact set $\mathcal{K} \subset \mathcal{M}(\mathbb{R}^d)$ with $u^N(t) \in \mathcal{K}$ for all $N \in \mathbb{N}$ and all $t \in [0,T]$.

2. The set $\{u^N \mid N \in \mathbb{N}\}$ is equicontinuous, i.e. for each $\varepsilon > 0$ there exists $\delta > 0$ such that for all $N \in \mathbb{N}$ and all $s,t \in [0,T]$ with $|s-t| \leq \delta$: $d(u^N(s), u^N(t)) \leq \varepsilon$.

We start with the first statement. As a subset $\mathcal{K}$ of $\mathcal{M}(\mathbb{R}^d)$ is relatively compact if and only if it is tight, we have to show that for each $\varepsilon > 0$ there is a compact set $K \subset \mathbb{R}^d$ with $u^N(t)(K) \geq 1-\varepsilon$ or, equivalently, $P\left[X_t^N \in K^c\right] \leq \varepsilon$ for all $N \in \mathbb{N}$ and all $t \in [0,T]$. Let $R > 0$. Then we have:

$$\begin{aligned}P\left[|X_t^N| > R\right] &= P\left[\left|\zeta - \int_0^t \nabla g^N(X_s^N, s)ds + B_t\right| > R\right] \\ &\leq P\left[|\zeta| > \frac{R}{3}\right] + P\left[\left|\int_0^t \nabla g^N(X_s^N, s)ds\right| > \frac{R}{3}\right] + P\left[|B_t| > \frac{R}{3}\right].\end{aligned}$$

The first and the third term tend (for $R \to \infty$) to 0, uniformly in N and $t \in [0,T]$. For the second term we obtain, using Chebyshev's inequality:

$$\begin{aligned}P\left[\left|\int_0^t \nabla g^N(X_s^N, s)ds\right| > \frac{R}{3}\right] &\leq \frac{9}{R^2} E\left[\left|\int_0^t \nabla g^N(X_s^N, s)ds\right|^2\right] \\ &\leq \frac{9t}{R^2} E\left[\int_0^t \left|\nabla g^N(X_s^N, s)\right|^2 ds\right] \\ &\leq \frac{9T}{R^2} \int_0^t \int_{\mathbb{R}^d} \left|\nabla g^N(x,s)\right|^2 u^N(s)(dx)ds,\end{aligned}$$

and due to Lemma 2.3 this also tends (for $R \to \infty$) to 0, uniformly in N and t. This completes the proof of the first statement.

We now prove the second statement. For $s, t \in [0,T]$ we obtain (using Lemma 2.3):

$$\begin{aligned}
d(u^N(s), u^N(t)) &= \sup_{f\in BL} \left| \int_{\mathbb{R}^d} f(x) u^N(t)(dx) - \int_{\mathbb{R}^d} f(x) u^N(s)(dx) \right| \\
&= \sup_{f\in BL} \left| E\left[f(X_t^N)\right] - E\left[f(X_s^N)\right] \right| \\
&\leq E\left[|X_t^N - X_s^N|^2\right]^{1/2} \\
&= E\left[\left| - \int_s^t \nabla g^N(X_r^N, r) dr + B_t - B_s \right|^2\right]^{1/2} \\
&\leq \sqrt{2} \left(E\left[\left| - \int_s^t \nabla g^N(X_r^N, r) dr \right|^2\right] + E\left[|B_t - B_s|^2\right] \right)^{1/2} \\
&\leq \sqrt{2} \left(E\left[|t-s| \int_s^t |\nabla g^N(X_r^N, r)|^2 dr\right] + |t-s| \right)^{1/2} \\
&\leq \sqrt{2} |t-s|^{1/2} \left(\int_s^t \int_{\mathbb{R}^d} |\nabla g^N(x,r)|^2 u^N(r)(dx) dr + 1 \right)^{1/2} \\
&\leq C|t-s|^{1/2}.
\end{aligned}$$

This means that $\{u^N \mid N \in \mathbb{N}\}$ is equicontinuous, so that the lemma is proved. □

We have shown that the sequence $(u^N)_{N\in\mathbb{N}}$ has a convergent subsequence. We now fix such a convergent subsequence and also denote it by $(u^N)_{N\in\mathbb{N}}$. Let $u^\infty \in \mathcal{C}([0,T], \mathcal{M}(\mathbb{R}^d))$ be its limit.

Lemma 2.4. *The sequence $(g_1^N)_{N\in\mathbb{N}}$ also converges in $\mathcal{C}([0,T], \mathcal{M}(\mathbb{R}^d))$ to u^∞.*

Proof. We have to show:

$$\sup_{0\leq t\leq T} \sup_{f\in BL} \left| \int_{\mathbb{R}^d} f(x) g_1^N(x,t) dx - \int_{\mathbb{R}^d} f(x) u^\infty(t)(dx) \right| \to 0 \quad (N \to \infty).$$

We estimate as follows:

$$\begin{aligned}
&\sup_{0\leq t\leq T} \sup_{f\in BL} \left| \int_{\mathbb{R}^d} f(x) g_1^N(x,t) dx - \int_{\mathbb{R}^d} f(x) u^\infty(t)(dx) \right| \\
= &\sup_{0\leq t\leq T} \sup_{f\in BL} \left| \int_{\mathbb{R}^d} f(x) (u^N(t) * W^N)(x) dx - \int_{\mathbb{R}^d} f(x) u^\infty(t)(dx) \right| \\
\leq &\sup_{0\leq t\leq T} \sup_{f\in BL} \left| \int_{\mathbb{R}^d} f(x) (u^N(t) * W^N)(x) dx - \int_{\mathbb{R}^d} f(x) u^N(t)(dx) \right| \\
&+ \sup_{0\leq t\leq T} \sup_{f\in BL} \left| \int_{\mathbb{R}^d} f(x) u^N(t)(dx) - \int_{\mathbb{R}^d} f(x) u^\infty(t)(dx) \right|.
\end{aligned}$$

The second term tends to 0 because $u^N \to u^\infty$. For the first term we obtain (uniformly in

$t \in [0,T]$ and $f \in BL$):

$$
\begin{aligned}
& \left| \int_{\mathbb{R}^d} f(x)(u^N(t) * W^N)(x)dx - \int_{\mathbb{R}^d} f(x)u^N(t)(dx) \right| \\
= & \left| \int_{\mathbb{R}^d} f(x) \int_{\mathbb{R}^d} W^N(x-y)u^N(t)(dy)dx - \int_{\mathbb{R}^d} f(x)u^N(t)(dx) \right| \\
= & \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x+y)W^N(x)dxu^N(t)(dy) - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(y)W^N(x)dxu^N(t)(dy) \right| \\
\le & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x+y) - f(y)| \, W^N(x)dxu^N(t)(dy) \\
\le & \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |x| W^N(x)dxu^N(t)(dy) \\
= & \int_{\mathbb{R}^d} |x| \chi_N^d W(\chi_N x)dx \\
= & \chi_N^{-1} \int_{\mathbb{R}^d} |y| W(y)dy \ \to 0 \quad (N \to \infty),
\end{aligned}
$$

q.e.d. □

Lemma 2.5. *For almost every $t \in [0,T]$ the measure $u^\infty(t)$ has a density $g^\infty(\cdot,t)$, and $g_1^N \to g^\infty$ in $L^2(\mathbb{R}^d \times [0,T])$ (in particular $g^\infty \in L^2(\mathbb{R}^d \times [0,T])$).*

Proof. We first show that the sequence $(g_1^N)_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{R}^d \times [0,T])$. For each $K > 0$ we have (using the L^2-isometry for the Fourier transform):

$$
\begin{aligned}
\limsup_{N,N'\to\infty} \|g_1^N - g_1^{N'}\|^2_{L^2(\mathbb{R}^d\times[0,T])} &= \limsup_{N,N'\to\infty} \int_0^T \int_{\mathbb{R}^d} |\widehat{g_1^N}(\lambda,t) - \widehat{g_1^{N'}}(\lambda,t)|^2 d\lambda dt \\
&\le \limsup_{N,N'\to\infty} \int_0^T \int_{|\lambda|\le K} |\widehat{g_1^N}(\lambda,t) - \widehat{g_1^{N'}}(\lambda,t)|^2 d\lambda dt \\
&\quad + \limsup_{N,N'\to\infty} \int_0^T \int_{|\lambda|> K} |\widehat{g_1^N}(\lambda,t) - \widehat{g_1^{N'}}(\lambda,t)|^2 d\lambda dt.
\end{aligned}
$$

We first consider the first term: According to Lemma 2.4 the integrand converges pointwise to 0. Moreover the integrand and the integration domain are bounded so that the first term converges to 0 according to Lebesgue's dominated convergence theorem. For the second term we obtain:

$$
\begin{aligned}
& \limsup_{N,N'\to\infty} \int_0^T \int_{|\lambda|>K} |\widehat{g_1^N}(\lambda,t) - \widehat{g_1^{N'}}(\lambda,t)|^2 d\lambda dt \le 4 \sup_{N\in\mathbb{N}} \int_0^T \int_{|\lambda|>K} |\widehat{g_1^N}(\lambda,t)|^2 d\lambda dt \\
\le & \ \frac{4}{K^2} \sup_{N\in\mathbb{N}} \int_0^T \int_{|\lambda|>K} |\lambda|^2 |\widehat{g_1^N}(\lambda,t)|^2 d\lambda dt \ \le \frac{4}{K^2} \sup_{N\in\mathbb{N}} \int_0^T \int_{\mathbb{R}^d} |\lambda|^2 |\widehat{g_1^N}(\lambda,t)|^2 d\lambda dt \\
= & \ \frac{4}{K^2} \sup_{N\in\mathbb{N}} \int_0^T \int_{\mathbb{R}^d} |\widehat{\nabla g_1^N}(\lambda,t)|^2 d\lambda dt \ = \frac{4}{K^2} \sup_{N\in\mathbb{N}} \int_0^T \int_{\mathbb{R}^d} |\nabla g_1^N(x,t)|^2 dx dt,
\end{aligned}
$$

and this tends to 0 for $K \to \infty$ because of Lemma 2.3.

We have shown that the sequence $(g_1^N)_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{R}^d \times [0,T])$. Let $g^\infty \in L^2(\mathbb{R}^d \times [0,T])$ be its limit. Because of Lemma 2.4 we already know that g_1^N converges in $\mathcal{C}([0,T], \mathcal{M}(\mathbb{R}^d))$ to u^∞. Therefore the measures $u^\infty(t)(dx)dt$ and $g^\infty(x,t)dxdt$ coincide, and therefore the measure $u^\infty(t)$ has for almost every $t \in [0,T]$ the density $g^\infty(\cdot,t)$. □

Proposition 2.2. *For all $t \in [0,T]$ and all $f \in \mathcal{C}_b^2(\mathbb{R}^d)$ we have:*

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)u^\infty(t)(dx) &= \int_{\mathbb{R}^d} f(x)u_0(x)dx + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} \Delta f(x)u^\infty(s)(dx)dxds \\ &\quad + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} \Delta f(x)g^\infty(x,s)^2dxds. \end{aligned} \tag{2.5}$$

Remark 2.3. As we already know that for almost every t the measure $u^\infty(t)$ has the density $g^\infty(\cdot,t)$ and that $g^\infty \in L^2(\mathbb{R}^d \times [0,T])$, Proposition 2.2 means that u^∞ is a weak solution of the viscous porous medium equation with initial datum u_0.

Proof. According to Lemma 2.1 we have:

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)u^N(t)(dx) &= \int_{\mathbb{R}^d} f(x)u_0(x)dx + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} \Delta f(x)u^N(s)(dx)ds \\ &\quad - \int_0^t\int_{\mathbb{R}^d} \nabla f(x)\cdot\nabla g^N(x,s)u^N(s)(dx)ds. \end{aligned} \tag{2.6}$$

Because of $u^N \to u^\infty$ it suffices to show that the third term of the right hand side of (2.6) converges to the corresponding term in (2.5):

$$\left|\int_0^t\int_{\mathbb{R}^d} \nabla f(x)\cdot\nabla g^N(x,s)u^N(s)(dx)ds + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} \Delta f(x)g^\infty(x,s)^2dxds\right| \tag{2.7}$$

$$\begin{aligned}\leq\; & \left|\int_0^t\int_{\mathbb{R}^d} \nabla f(x)\cdot\nabla g^N(x,s)u^N(s)(dx)ds - \int_0^t\int_{\mathbb{R}^d} \nabla f(x)\cdot\nabla g_1^N(x,s)g_1^N(x,s)dxds\right| \\ &+\left|\int_0^t\int_{\mathbb{R}^d} \nabla f(x)\cdot\nabla g_1^N(x,s)g_1^N(x,s)dxds + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} \Delta f(x)g_1^N(x,s)^2dxds\right| \\ &+\left|\frac{1}{2}\int_0^t\int_{\mathbb{R}^d} \Delta f(x)g_1^N(x,s)^2dxds - \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} \Delta f(x)g^\infty(x,s)^2dxds\right|. \end{aligned} \tag{2.8}$$

The second term in (2.8) vanishes (integration by parts). For the third term we obtain:

$$\begin{aligned}&\left|\frac{1}{2}\int_0^t\int_{\mathbb{R}^d} \Delta f(x)g_1^N(x,s)^2dxds - \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} \Delta f(x)g^\infty(x,s)^2dxds\right| \\ \leq\;& \frac{\|\Delta f\|_\infty}{2}\int_0^t\int_{\mathbb{R}^d} |g_1^N(x,s)^2 - g^\infty(x,s)^2|dxds \\ =\;& \frac{\|\Delta f\|_\infty}{2}\int_0^t\int_{\mathbb{R}^d} |g_1^N(x,s)+g^\infty(x,s)||g_1^N(x,s)-g^\infty(x,s)|dxds \\ \leq\;& \frac{\|\Delta f\|_\infty}{2}\left(\int_0^t\int_{\mathbb{R}^d} |g_1^N(x,s)+g^\infty(x,s)|^2dxds\right)^{1/2}\left(\int_0^t\int_{\mathbb{R}^d} |g_1^N(x,s)-g^\infty(x,s)|^2dxds\right)^{1/2}. \end{aligned}$$

Because of $g_1^N \to g^\infty$ in $L^2(\mathbb{R}^d \times [0,T])$ the second factor is bounded, and the third factor tends to 0.

For the first term in (2.8) we obtain:

$$
\begin{aligned}
&\left|\int_0^t \int_{\mathbb{R}^d} \nabla f(x)\cdot\nabla g^N(x,s)u^N(s)(dx)ds - \int_0^t\int_{\mathbb{R}^d}\nabla f(x)\cdot\nabla g_1^N(x,s)g_1^N(x,s)dxds\right| \\
= &\left|\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\nabla f(x)\cdot\nabla g_1^N(y,s)W^N(x-y)u^N(s)(dx)dyds\right. \\
&\left.-\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\nabla f(y)\cdot\nabla g_1^N(y,s)W^N(x-y)u^N(s)(dx)dyds\right| \\
= &\left|\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}[\nabla f(x)-\nabla f(y)]\cdot\nabla g_1^N(y,s)W^N(x-y)u^N(s)(dx)dyds\right| \\
\leq &\|D^2 f\|_\infty \int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|x-y||\nabla g_1^N(y,s)|W^N(x-y)u^N(s)(dx)dyds.
\end{aligned}
$$

Using the fact that $\operatorname{diam}(\operatorname{supp}(W^N)) = \chi_N^{-1}\operatorname{diam}(\operatorname{supp}(W))$ (recall that W has compact support) we see that this is bounded by

$$
\begin{aligned}
&\chi_N^{-1}\|D^2 f\|_\infty \operatorname{diam}(\operatorname{supp}(W))\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|\nabla g_1^N(y,s)|W^N(x-y)u^N(s)(dx)dyds \\
= &\chi_N^{-1}\|D^2 f\|_\infty \operatorname{diam}(\operatorname{supp}(W))\int_0^t\int_{\mathbb{R}^d} g_1^N(y,s)|\nabla g_1^N(y,s)|dyds \\
\leq &\chi_N^{-1}\|D^2 f\|_\infty \operatorname{diam}(\operatorname{supp}(W))\left(\int_0^t\int_{\mathbb{R}^d} g_1^N(y,s)^2 dyds\right)^{1/2}\left(\int_0^t\int_{\mathbb{R}^d}|\nabla g_1^N(y,s)|^2 dyds\right)^{1/2}.
\end{aligned}
$$

The two last factors are bounded uniformly in N according to Lemma 2.3, so that this expression tends to 0. □

We have now shown that the sequence $(u^N)_{N\in\mathbb{N}}$ is relatively compact (Proposition 2.1) and that any limit point u^∞ of a subsequence is a weak solution of the viscous porous medium equation (Proposition 2.2).

It remains to show uniqueness of weak solutions u of this equation. But to do so we need the additional assumption that the density g belongs to $L^3(\mathbb{R}^d\times[0,T])$ (and not only to $L^2(\mathbb{R}^d\times[0,T])$). We will therefore show that $g^\infty\in L^3(\mathbb{R}^d\times[0,T])$. To achieve this goal, we need the following lemma:

Lemma 2.6. *For all $t\in[0,T]$ and all $f\in\mathcal{C}_b^2(\mathbb{R}^d\times\mathbb{R}^d)$ we have:*

$$
\begin{aligned}
\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x,y)u^\infty(t)(dx)u^\infty(t)(dy) = &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x,y)u_0(x)dxu_0(y)dy \\
&+\frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Delta_x f(x,y)\cdot g^\infty(x,s)^2 g^\infty(y,s)dxdyds \\
&+\frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Delta_y f(x,y)\cdot g^\infty(x,s)g^\infty(y,s)^2 dxdyds \\
&+\frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Delta_x f(x,y)g^\infty(x,s)g^\infty(y,s)dxdyds \\
&+\frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Delta_y f(x,y)g^\infty(x,s)g^\infty(y,s)dxdyds.
\end{aligned}
\tag{2.9}
$$

Proof. According to Lemma 2.2 we have:

$$
\begin{aligned}
\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x,y)u^N(t)(dx)u^N(t)(dy) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x,y)u_0(x)dxu_0(y)dy \\
&- \int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_x f(x,y)\cdot\nabla g^N(x,s)u^N(s)(dx)u^N(s)(dy)ds \\
&- \int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_y f(x,y)\cdot\nabla g^N(y,s)u^N(s)(dx)u^N(s)(dy)ds \\
&+ \frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta_x f(x,y)u^N(s)(dx)u^N(s)(dy)ds \\
&+ \frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta_y f(x,y)u^N(s)(dx)u^N(s)(dy)ds.
\end{aligned}
\tag{2.10}
$$

Because of $u^N \to u^\infty$ it suffices to show that (2.10) tends to (2.9):

$$
\begin{aligned}
&\Bigg|\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_x f(x,y)\cdot\nabla g^N(x,s)u^N(s)(dx)u^N(s)(dy)ds \\
&\quad + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta_x f(x,y)\cdot g^\infty(x,s)^2 g^\infty(y,s)dxdyds\Bigg| \\
\leq\ &\Bigg|\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_x f(x,y)\cdot\nabla g^N(x,s)u^N(s)(dx)u^N(s)(dy)ds \\
&\quad + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta_x f(x,y)\cdot g^\infty(x,s)^2 dxu^N(s)(dy)ds\Bigg| \qquad (2.11) \\
&+ \Bigg|-\frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta_x f(x,y)\cdot g^\infty(x,s)^2 dxu^N(s)(dy)ds \\
&\quad + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta_x f(x,y)\cdot g^\infty(x,s)^2 g^\infty(y,s)dxdyds\Bigg|. \qquad (2.12)
\end{aligned}
$$

For (2.11) we obtain:

$$
\begin{aligned}
&\Bigg|\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_x f(x,y)\cdot\nabla g^N(x,s)u^N(s)(dx)u^N(s)(dy)ds \\
&\quad + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta_x f(x,y)\cdot g^\infty(x,s)^2 dxu^N(s)(dy)ds\Bigg| \\
\leq\ &\int_{\mathbb{R}^d}\Bigg|\int_0^t\int_{\mathbb{R}^d} \nabla_x f(x,y)\cdot\nabla g^N(x,s)u^N(s)(dx)ds \\
&\quad + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} \Delta_x f(x,y)\cdot g^\infty(x,s)^2 dxds\Bigg| u^N(s)(dy) \\
\leq\ &\sup_{y\in\mathbb{R}^d}\left|\int_0^t\int_{\mathbb{R}^d} \nabla_x f(x,y)\cdot\nabla g^N(x,s)u^N(s)(dx)ds + \frac{1}{2}\int_0^t\int_{\mathbb{R}^d} \Delta_x f(x,y)\cdot g^\infty(x,s)^2 dxds\right|,
\end{aligned}
$$

and this tends to 0 (compare with the estimation of (2.7)). For (2.12) we get:

$$\begin{aligned}
&\Big| -\frac{1}{2}\int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta_x f(x,y) g^\infty(x,s)^2 dx u^N(s)(dy) ds \\
&\quad + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta_x f(x,y) g^\infty(x,s)^2 g^\infty(y,s) dx dy ds \Big| \\
\leq\ & \frac{1}{2}\int_0^t \Big| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta_x f(x,y) g^\infty(x,s)^2 dx\, u^N(s)(dy) \\
&\quad - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta_x f(x,y) g^\infty(x,s)^2 dx u^\infty(s)(dy) \Big| ds \\
\leq\ & \frac{1}{2}\sup_{0\leq s\leq T} \Big| \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta_x f(x,y) g^\infty(x,s)^2 dx\, u^N(s)(dy) \\
&\quad - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta_x f(x,y) g^\infty(x,s)^2 dx u^\infty(s)(dy) \Big| ds,
\end{aligned}$$

and this also tends to 0 because of $u^N \to u^\infty$. □

Proposition 2.3. $g^\infty \in L^3(\mathbb{R}^d \times [0,T])$.

Proof. We use ideas from [25]. According to Lemma 2.6 we have for all $f \in \mathcal{C}_b^2(\mathbb{R}^d \times \mathbb{R}^d)$:

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x,y) u^\infty(T)(dx) u^\infty(T)(dy) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(x,y) u_0(x) dx u_0(y) dy \\
&\quad + \frac{1}{2}\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta_x f(x,y) g^\infty(x,s)[1+g^\infty(x,s)] g^\infty(y,s) dx dy ds \\
&\quad + \frac{1}{2}\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta_y f(x,y) g^\infty(x,s)[1+g^\infty(y,s)] g^\infty(y,s) dx dy ds.
\end{aligned}$$

We apply this for $0 < \delta < r$ and $\varepsilon > 0$ to the function $f(x,y) = q_{r,\delta,\varepsilon}(x-y)$, where

$$\begin{aligned}
q_{r,\delta,\varepsilon}(x) &:= \frac{1}{2\delta}\int_{r-\delta}^{r+\delta}\int_{\mathbb{R}^d} q_\eta(x-z)\sigma_\varepsilon(z) dz d\eta, \\
q_\eta(x) &:= \frac{2d}{\eta^2}\left[G_d(|x|) - G_d(\eta)\right]_+, \\
G_d(u) &:= \begin{cases} \frac{1}{d(d-2)\omega_d} u^{2-d} & \text{for } d \neq 2 \\ -\frac{1}{2\pi}\log u & \text{for } d = 2, \end{cases} \\
\omega_d &:= \text{volume of the unit ball in } \mathbb{R}^d, \\
\sigma_\varepsilon(y) &:= (2\pi\varepsilon)^{-d/2} e^{-\frac{|y|^2}{2\varepsilon}}.
\end{aligned}$$

(Remark: r and δ are fixed, and their precise values do not matter. One could e.g. choose $r = 1$ and $\delta = 1/2$. ε, though, will later tend to 0.) It follows that

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} q_{r,\delta,\varepsilon}(x-y) u^\infty(T)(dx) u^\infty(T)(dy) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} q_{r,\delta,\varepsilon}(x-y) u_0(x) dx u_0(y) dy \\
&\quad + \int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta q_{r,\delta,\varepsilon}(x-y) g^\infty(x,s)[1+g^\infty(x,s)] g^\infty(y,s) dx dy ds \\
=\ & \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} q_{r,\delta,\varepsilon}(x-y) u_0(x) dx u_0(y) dy \\
&\quad + \int_0^T \int_{\mathbb{R}^d} g^\infty(x,s)[1+g^\infty(x,s)]\, \Delta(q_{r,\delta,\varepsilon} * g^\infty(\cdot,s))(x) dx ds. \qquad (2.13)
\end{aligned}$$

To compute $\Delta(q_{r,\delta,\varepsilon} * g^\infty(\cdot,s))(x)$ we need the following lemma:

Lemma 2.7. *In the sense of distributions we have*

$$\Delta q_\eta = \frac{2d}{\eta^2}\left[\mu_\eta^d - \delta_0\right],$$

where μ_η^d denotes the normalized surface measure on $\partial B_\eta(0) \subset \mathbb{R}^d$.

Proof. We give the proof only for the case $d \geq 3$, for $d \leq 2$ the proof is essentially the same. We have to show that for any function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} \Delta\varphi(x)\left[|x|^{2-d} - \eta^{2-d}\right]_+ dx = d(d-2)\omega_d\left[\frac{1}{d\omega_d\eta^{d-1}}\int_{\partial B_\eta(0)} \varphi(x)dx - \varphi(0)\right]. \tag{2.14}$$

To this end we choose $\varepsilon < \eta/2$ (this is not the same ε as in the proof of Proposition 2.3). We decompose the integral on the left hand side of (2.14) into four integrals:

$$\int_{\mathbb{R}^d} \Delta\varphi(x)\left[|x|^{2-d} - \eta^{2-d}\right]_+ dx = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}
I_1 &:= \int_{B_\varepsilon(0)} \Delta\varphi(x)\left[|x|^{2-d} - \eta^{2-d}\right]_+ dx \\
I_2 &:= \int_{B_{\eta-\varepsilon}(0)\setminus B_\varepsilon(0)} \Delta\varphi(x)\left[|x|^{2-d} - \eta^{2-d}\right]_+ dx \\
I_3 &:= \int_{B_{\eta+\varepsilon}(0)\setminus B_{\eta-\varepsilon}(0)} \Delta\varphi(x)\left[|x|^{2-d} - \eta^{2-d}\right]_+ dx \\
I_4 &:= \int_{\mathbb{R}^d\setminus B_{\eta+\varepsilon}(0)} \Delta\varphi(x)\left[|x|^{2-d} - \eta^{2-d}\right]_+ dx.
\end{aligned}$$

Clearly $I_4 = 0$ (because the integrand vanishes), and I_1 and I_3 tend to 0 for $\varepsilon \to 0$. To compute I_2 we use Green's formulas, the fact that the function $|x|^{2-d}$ is harmonic and that its gradient is given by $(2-d)|x|^{-d}x$, and obtain (ν = outward normal):

$$\begin{aligned}
&\int_{B_{\eta-\varepsilon}(0)\setminus B_\varepsilon(0)} \Delta\varphi(x)\left[|x|^{2-d} - \eta^{2-d}\right]_+ dx \\
= &\int_{\partial B_{\eta-\varepsilon}(0)\cup\partial B_\varepsilon(0)} \left\{\left[|x|^{2-d} - \eta^{2-d}\right]\nabla\varphi(x) - \varphi(x)(2-d)\frac{x}{|x|^d}\right\}\cdot\nu(x)dx \\
= &\int_{\partial B_{\eta-\varepsilon}(0)} \left[(\eta-\varepsilon)^{2-d} - \eta^{2-d}\right]\nabla\varphi(x)\cdot\frac{x}{\eta-\varepsilon}dx - (2-d)\int_{\partial B_{\eta-\varepsilon}(0)} \varphi(x)(\eta-\varepsilon)^{1-d}dx \\
&- \int_{\partial B_\varepsilon(0)} \left[\varepsilon^{2-d} - \eta^{2-d}\right]\nabla\varphi(x)\cdot\frac{x}{\varepsilon}dx + (2-d)\int_{\partial B_\varepsilon(0)} \varphi(x)\varepsilon^{1-d}dx.
\end{aligned}$$

The first and the third term tend to 0 for $\varepsilon \to 0$, the second term tends to $\frac{d-2}{\eta^{d-1}}\int_{\partial B_\eta(0)} \varphi(x)dx$, and the fourth term tends to $-d(d-2)\omega_d\varphi(0)$. This completes the proof of the lemma. □

Using Lemma 2.7 we can now compute $\Delta(q_{r,\delta,\varepsilon} * g^\infty(\cdot,s))(x)$: Let us first remark that

$$\begin{aligned}
(q_{r,\delta,\varepsilon} * g^\infty(\cdot,s))(x) &= \int_{\mathbb{R}^d} \frac{1}{2\delta}\int_{r-\delta}^{r+\delta} (q_\eta * \sigma_\varepsilon)(x-y)d\eta\, g^\infty(y,s)dy \\
&= \frac{1}{2\delta}\int_{r-\delta}^{r+\delta} (q_\eta * \sigma_\varepsilon * g^\infty(\cdot,s))(x)d\eta.
\end{aligned}$$

Now Lemma 2.7 implies:

$$\Delta(q_{r,\delta,\varepsilon} * g^\infty(\cdot,s))(x) = \frac{1}{2\delta}\int_{r-\delta}^{r+\delta}\int_{\mathbb{R}^d}\Delta(\sigma_\varepsilon * g^\infty(\cdot,s))(x-z)q_\eta(z)dzd\eta$$

$$= \frac{1}{2\delta}\int_{r-\delta}^{r+\delta}\frac{2d}{\eta^2}\frac{1}{\mathrm{vol}(\partial B_\eta(0))}\int_{\partial B_\eta(0)}(\sigma_\varepsilon * g^\infty(\cdot,s))(x-z)dzd\eta - \frac{1}{2\delta}\int_{r-\delta}^{r+\delta}\frac{2d}{\eta^2}(\sigma_\varepsilon * g^\infty(\cdot,s))(x)d\eta$$

$$= \int_{\mathbb{R}^d}\frac{d}{\delta}\int_{r-\delta}^{r+\delta}\frac{1}{\eta^2}\frac{1}{\mathrm{vol}(\partial B_\eta(0))}\int_{\partial B_\eta(0)} g^\infty(x-y-z,s)\sigma_\varepsilon(y)dzd\eta dy$$
$$-\int_{\mathbb{R}^d} g^\infty(x-y,s)\sigma_\varepsilon(y)dy\,\frac{d}{\delta}\int_{r-\delta}^{r+\delta}\frac{1}{\eta^2}d\eta.$$

Inserting this into (2.13) we obtain:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} q_{r,\delta,\varepsilon}(x-y)u^\infty(T)(dx)u^\infty(T)(dy) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} q_{r,\delta,\varepsilon}(x-y)u_0(x)dxu_0(y)dy$$
$$+\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g^\infty(x,s)[1+g^\infty(x,s)]$$
$$\frac{d}{\delta}\int_{r-\delta}^{r+\delta}\frac{1}{\eta^2}\frac{1}{\mathrm{vol}(\partial B_\eta(0))}\int_{\partial B_\eta(0)} g^\infty(x-y-z,s)\sigma_\varepsilon(y)dzd\eta dxdyds$$
$$-\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g^\infty(x,s)g^\infty(x-y,s)\sigma_\varepsilon(y)dxdyds\left[\frac{d}{\delta}\int_{r-\delta}^{r+\delta}\frac{1}{\eta^2}\,d\eta\right]$$
$$-\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g^\infty(x,s)^2 g^\infty(x-y,s)\sigma_\varepsilon(y)dxdyds\left[\frac{d}{\delta}\int_{r-\delta}^{r+\delta}\frac{1}{\eta^2}\,d\eta\right],$$

so that

$$\int_0^T\int_{\mathbb{R}^d} g^\infty(x,s)^2(g^\infty(\cdot,s)*\sigma_\varepsilon)(x)dxds\left[\frac{d}{\delta}\int_{r-\delta}^{r+\delta}\frac{1}{\eta^2}\,d\eta\right] \tag{2.15}$$
$$= \int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g^\infty(x,s)^2 g^\infty(x-y,s)\sigma_\varepsilon(y)dxdyds\left[\frac{d}{\delta}\int_{r-\delta}^{r+\delta}\frac{1}{\eta^2}\,d\eta\right]$$
$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} q_{r,\delta,\varepsilon}(x-y)u_0(x)dxu_0(y)dy$$
$$+\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g^\infty(x,s)[1+g^\infty(x,s)]$$
$$\frac{d}{\delta}\int_{r-\delta}^{r+\delta}\frac{1}{\eta^2}\frac{1}{\mathrm{vol}(\partial B_\eta(0))}\int_{\partial B_\eta(0)} g^\infty(x-y-z,s)\sigma_\varepsilon(y)dzd\eta dxdyds$$
$$-\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g^\infty(x,s)g^\infty(x-y,s)\sigma_\varepsilon(y)dxdyds\left[\frac{d}{\delta}\int_{r-\delta}^{r+\delta}\frac{1}{\eta^2}\,d\eta\right]$$
$$-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} q_{r,\delta,\varepsilon}(x-y)u^\infty(T)(dx)u^\infty(T)(dy)$$
$$\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} q_{r,\delta,\varepsilon}(x-y)u_0(x)dxu_0(y)dy \tag{2.16}$$
$$+\int_0^T\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g^\infty(x,s)[1+g^\infty(x,s)]$$
$$\left[\frac{d}{\delta}\int_{r-\delta}^{r+\delta}\frac{1}{\eta^2}\frac{1}{\mathrm{vol}(\partial B_\eta(0))}\int_{\partial B_\eta(0)} g^\infty(x-y-z,s)\sigma_\varepsilon(y)dzd\eta\right]dydxds. \tag{2.17}$$

We now want to show that

$$\int_0^T \int_{\mathbb{R}^d} g^\infty(x,s)^2 (g^\infty(\cdot,s) * \sigma_\varepsilon)(x) dx ds \tag{2.18}$$

is bounded uniformly in ε. As the factor $\frac{d}{\delta}\int_{r-\delta}^{r+\delta} \frac{1}{\eta^2} d\eta$ in (2.15) is independent of ε, it suffices to show that (2.16) and (2.17) are bounded uniformly in ε. For (2.16) we obtain using $\|q_{r,\delta,\varepsilon}\|_{L^1(\mathbb{R}^d)} = 1$ and $u_0 \in L^2(\mathbb{R}^d)$:

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} q_{r,\delta,\varepsilon}(x-y) u_0(x) dx u_0(y) dy &\leq \|q_{r,\delta,\varepsilon} * u_0\|_{L^2(\mathbb{R}^d)} \|u_0\|_{L^2(\mathbb{R}^d)} \\ &\leq \|u_0\|^2_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Concerning (2.17) we first consider the integral in brackets. Because of $\eta \geq r - \delta$ we have $\frac{1}{\eta^2}\frac{1}{\text{vol}(\partial B_\eta(0))} \leq C_1(r,\delta)$ and therefore

$$\begin{aligned}& \frac{d}{\delta}\int_{r-\delta}^{r+\delta} \frac{1}{\eta^2}\frac{1}{\text{vol}(\partial B_\eta(0))} \int_{\partial B_\eta(0)} g^\infty(x-y-z,s)\sigma_\varepsilon(y) dz d\eta \\ \leq\ & \frac{d}{\delta} C_2(r,\delta) \int_{\mathbb{R}^d} g^\infty(x-y-z,s)\sigma_\varepsilon(y) dz \\ \leq\ & \frac{d}{\delta} C_2(r,\delta) \|g^\infty(\cdot,s)\|_{L^1(\mathbb{R}^d)} \sigma_\varepsilon(y) \\ =\ & \frac{d}{\delta} C_2(r,\delta)\sigma_\varepsilon(y)\end{aligned}$$

for almost all s. (For almost all $s \in [0,T]$ $g^\infty(\cdot,s)$ is a probability density.) Since $g^\infty \in L^1(\mathbb{R}^d \times [0,T]) \cap L^2(\mathbb{R}^d \times [0,T])$ and $\|\sigma_\varepsilon\|_{L^1(\mathbb{R}^d)} = 1$, this implies that (2.17) is bounded uniformly in ε.

By these arguments we have shown that (2.18) is bounded uniformly in ε:

$$\int_0^T \int_{\mathbb{R}^d} g^\infty(x,s)^2 (g^\infty(\cdot,s) * \sigma_\varepsilon)(x) dx ds \leq C_4(r,\delta). \tag{2.19}$$

As $g^\infty(\cdot,s) * \sigma_\varepsilon$ converges for $\varepsilon \to 0$ to $g^\infty(\cdot,s)$ for almost all $s \in [0,T]$ in $L^1(\mathbb{R}^d)$, there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that for almost all $(x,s) \in \mathbb{R}^d \times [0,T]$:

$$(g^\infty(\cdot,s) * \sigma_{\varepsilon_k})(x) \to g^\infty(x,s) \qquad (k \to \infty).$$

Fatou's lemma together with (2.19) now implies:

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} g^\infty(x,s)^3 dx ds &= \int_0^T \int_{\mathbb{R}^d} g^\infty(x,s)^2 \lim_{k\to\infty} (g^\infty(\cdot,s) * \sigma_{\varepsilon_k})(x) dx ds \\ &\leq \liminf_{k\to\infty} \int_0^T \int_{\mathbb{R}^d} g^\infty(x,s)^2 (g^\infty(\cdot,s) * \sigma_{\varepsilon_k})(x) dx ds \\ &\leq C_4(r,\delta),\end{aligned}$$

so that $g^\infty \in L^3(\mathbb{R}^d \times [0,T])$. □

Proposition 2.4. *Let u and $\overline{u}$ be two weak solutions of the viscous porous medium equation on $[0,T]$ (in the sense of Definition 2.1) such that additionally the densities g and $\tilde{g}$ satisfy $g, \tilde{g} \in L^3(\mathbb{R}^d \times [0,T])$. Then $u = \overline{u}$.*

Proof. We follow ideas of the proof of Lemma 3.15 in [7]. According to Definition 2.1 we have for all $f \in \mathcal{C}_b^2(\mathbb{R}^d)$ and all $t \in [0,T]$:

$$\int_{\mathbb{R}^d} f(x)u(t)(dx) = \int_{\mathbb{R}^d} f(x)u_0(x)dx + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \Delta f(x)g(x,s)dxds + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \Delta f(x)g(x,s)^2 dxds \tag{2.20}$$

and

$$\int_{\mathbb{R}^d} f(x)\overline{u}(t)(dx) = \int_{\mathbb{R}^d} f(x)u_0(x)dx + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \Delta f(x)\tilde{g}(x,s)dxds + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \Delta f(x)\tilde{g}(x,s)^2 dxds.$$

Let $\phi_\varepsilon(x) := \varepsilon^{-d}\phi(x/\varepsilon)$ with a symmetric function $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$, and let

$$\begin{aligned} u_\varepsilon(x,t) &:= (u(t) * \phi_\varepsilon)(x), \\ \overline{u}_\varepsilon(x,t) &:= (\overline{u}(t) * \phi_\varepsilon)(x). \end{aligned}$$

We now apply (2.20) to $f * \phi_\varepsilon$ instead of f. Because of the symmetry of ϕ_ε we have

$$\begin{aligned} \int_{\mathbb{R}^d} (f * \phi_\varepsilon)(x)u(t)(dx) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(y)\phi_\varepsilon(x-y)u(t)(dx)dy \\ &= \int_{\mathbb{R}^d} f(y)(u(t) * \phi_\varepsilon)(y)dy \\ &= \int_{\mathbb{R}^d} f(x)u_\varepsilon(x,t)dx, \end{aligned}$$

and similar statements hold for the terms on the right hand side of (2.20). It follows:

$$\begin{aligned} \int_{\mathbb{R}^d} f(x)u_\varepsilon(x,t)dx &= \int_{\mathbb{R}^d} f(x)u_\varepsilon(x,0)dx + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \Delta f(x)u_\varepsilon(x,s)dxds \\ &\quad + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \Delta f(x)\left[g(\cdot,s)^2 * \phi_\varepsilon\right](x)dxds, \end{aligned}$$

and a similar equation for $\overline{u}$. Subtracting these two equations we obtain:

$$\begin{aligned} \int_{\mathbb{R}^d} f(x)\left[u_\varepsilon(x,t) - \overline{u}_\varepsilon(x,t)\right]dx &= \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \Delta f(x)\left[u_\varepsilon(x,s) - \overline{u}_\varepsilon(x,s)\right]dxds \\ &\quad + \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \Delta f(x)\left[\left[g(\cdot,s)^2 - \tilde{g}(\cdot,s)^2\right] * \phi_\varepsilon\right](x)dxds. \end{aligned}$$

Let $\alpha_s(x) := u_\varepsilon(x,s) - \overline{u}_\varepsilon(x,s) + \left[\left[g(\cdot,s)^2 - \tilde{g}(\cdot,s)^2\right] * \phi_\varepsilon\right](x)$. It follows

$$\int_{\mathbb{R}^d} f(x)\left[u_\varepsilon(x,t) - \overline{u}_\varepsilon(x,t)\right]dx = \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \Delta f(x)\alpha_s(x)dxds.$$

As for almost all $t \in [0,T]$ we have $g(\cdot,t) \in L^2(\mathbb{R}^d)$, it follows that for almost all $t \in [0,T]$ the function α_t is smooth. For these t we choose $f = \alpha_t$ and obtain:

$$\int_{\mathbb{R}^d} \alpha_t(x)\left[u_\varepsilon(x,t) - \overline{u}_\varepsilon(x,t)\right]dx = \frac{1}{2}\int_0^t \int_{\mathbb{R}^d} \Delta\alpha_t(x)\alpha_s(x)dxds.$$

By integrating this equation over $t \in [0,T]$ we obtain:

$$
\begin{aligned}
\int_0^T \int_{\mathbb{R}^d} \alpha_t(x) \left[u_\varepsilon(x,t) - \overline{u}_\varepsilon(x,t)\right] dxdt &= -\frac{1}{2} \int_0^T \int_0^t \int_{\mathbb{R}^d} \Delta\alpha_t(x)\alpha_s(x) dxdsdt \\
&= -\frac{1}{2} \int_0^T \int_0^t \int_{\mathbb{R}^d} \nabla\alpha_t(x) \cdot \nabla\alpha_s(x) dxdsdt \\
&= -\frac{1}{4} \int_0^T \int_0^T \int_{\mathbb{R}^d} \nabla\alpha_t(x) \cdot \nabla\alpha_s(x) dxdsdt \\
&= -\frac{1}{4} \int_{\mathbb{R}^d} \left| \int_0^T \nabla\alpha_t(x) dt \right|^2 dx \\
&\leq 0,
\end{aligned}
$$

so that

$$
\int_0^T \int_{\mathbb{R}^d} \big(u_\varepsilon(x,t) - \overline{u}_\varepsilon(x,t) + \left[\left[g(\cdot,t)^2 - \tilde{g}(\cdot,t)^2\right] * \phi_\varepsilon\right](x)\big) \left[u_\varepsilon(x,t) - \overline{u}_\varepsilon(x,t)\right] dxdt \leq 0,
$$

and therefore

$$
\int_0^T \int_{\mathbb{R}^d} \left[\left[g(\cdot,t)^2 - \tilde{g}(\cdot,t)^2\right] * \phi_\varepsilon\right](x) \left[\left[g(\cdot,t) - \tilde{g}(\cdot,t)\right] * \phi_\varepsilon\right](x) dxdt \leq 0.
$$

Since $g, \tilde{g} \in L^3(\mathbb{R}^d \times [0,T])$ and therefore $g^2, \tilde{g}^2 \in L^{3/2}(\mathbb{R}^d \times [0,T])$ we can now pass to the limit $\varepsilon \to 0$ and obtain

$$
\int_0^T \int_{\mathbb{R}^d} \left[g(x,t)^2 - \tilde{g}(x,t)^2\right] \left[g(x,t) - \tilde{g}(x,t)\right] dxdt \leq 0.
$$

As the integrand is nonnegative everywhere, it follows that $g = \tilde{g}$ almost everywhere and therefore $u = \overline{u}$. □

Chapter 3

Microscopic derivation of the three-dimensional Navier-Stokes equation from a stochastic interacting particle system

We study a system of stochastically interacting particles (vortices) and show that for a large number of vortices the weighted empirical measure of the system approximates the solution of the three-dimensional Navier-Stokes equation in vorticity form.

3.1 Introduction

We consider the three-dimensional Navier-Stokes equation posed in the whole space:

$$\begin{aligned} \frac{\partial u}{\partial t} + (u \cdot \nabla) u &= -\nabla p + \nu \Delta u \qquad (3.1) \\ \operatorname{div} u &= 0 \\ u(t,x) &\to 0 \text{ for } |x| \to \infty. \end{aligned}$$

Here u is the velocity field, p is the pressure, and $\nu > 0$ is the viscosity. We are interested in the evolution of the vorticity $w := \operatorname{curl} u$. By taking the curl of equation (3.1) we obtain the vorticity equation

$$\frac{\partial w}{\partial t} + (u \cdot \nabla) w = (w \cdot \nabla) u + \nu \Delta w. \qquad (3.2)$$

The velocity u can be recovered from the vorticity w (see e.g. [21], Proposition 2.16): Let $K(x) := -\frac{x}{4\pi|x|^3}$. Then $u(x) = \int_{\mathbb{R}^3} K(x-y) \times w(y) dy$.

The initial vorticity w_0 is supposed to satisfy $\operatorname{div} w_0 = 0$ (in the sense of distributions) and $w_0 \in L^1(\mathbb{R}^3, \mathbb{R}^3) \cap L^p(\mathbb{R}^3, \mathbb{R}^3)$ for some $p \in (\frac{3}{2}, 3)$. It is known (see Proposition 3.1 below) that under these assumptions there is a $T^* > 0$ such that the vorticity equation (3.2) has a unique mild solution w on $[0, T^*]$.

We now approximate equation (3.2) by the following system $(X_t^{N,i,\varepsilon,R}, a_t^{N,i,\varepsilon,R})_{i=1}^N$ of interacting discrete vortices:

$$
\begin{aligned}
X_t^{N,i,\varepsilon,R} &= \xi^i + \int_0^t \left\{ \frac{1}{N} \sum_{j=1}^N K^\varepsilon(X_s^{N,i,\varepsilon,R} - X_s^{N,j,\varepsilon,R}) \times \chi_R(a_s^{N,j,\varepsilon,R}) \right\} ds + \sqrt{2\nu} W_t^i \\
a_t^{N,i,\varepsilon,R} &= \alpha^i + \int_0^t \left\{ \frac{1}{N} \sum_{j=1}^N \nabla K^\varepsilon(X_s^{N,i,\varepsilon,R} - X_s^{N,j,\varepsilon,R}) \times \chi_R(a_s^{N,j,\varepsilon,R}) \right\} \chi_R(a_s^{N,i,\varepsilon,R}) ds.
\end{aligned}
\tag{3.3}
$$

Here $X_t^{N,i,\varepsilon,R} \in \mathbb{R}^3$ represents the position and $a_t^{N,i,\varepsilon,R} \in \mathbb{R}^3$ the intensity of the i-th vortex. $N \in \mathbb{N}$ is the number of vortices, $\varepsilon > 0$ is a smoothing parameter, and $R > 0$ is a cutoff parameter. K^ε is a smoothed version of the kernel K, defined by

$$
K^\varepsilon := \varphi^\varepsilon * K, \qquad \text{where} \qquad \varphi^\varepsilon(x) := \frac{1}{\varepsilon^3} \varphi(x/\varepsilon)
$$

for a function $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ with $\varphi \geq 0$ and $\int_{\mathbb{R}^3} \varphi(x)dx = 1$. Moreover the cutoff function χ_R is defined by

$$
\chi_R(x) := \begin{cases} x & \text{if } |x| \leq R \\ \frac{R}{|x|} x & \text{if } |x| > R. \end{cases}
$$

Finally the $(W^i)_{i=1}^N$ are independent standard Brownian motions.

We choose the initial positions ξ^i and the initial intensities α^i $(i = 1, \ldots, N)$ of the discrete vortices in the following way: we first decompose the initial vorticity w_0 in the form $w_0(x) = p(x)h(x)$, where p is a probability density and h is a bounded $\mathbb{R}^3$-valued weight function. This is possible thanks to our assumption that $w_0 \in L^1(\mathbb{R}^3, \mathbb{R}^3)$. For instance one can choose $p(x) := \frac{|w_0(x)|}{\|w_0\|_{L^1}}$ and $h(x) := \frac{w_0(x)}{|w_0(x)|} \|w_0\|_{L^1}$ (with the convention $\frac{0}{0} := 0$). Then we choose the ξ^i to be independent of each other and of the Brownian motions, and identically distributed with

$$
P\big[\xi^i \in dx\big] = p(x)dx, \qquad \text{and we set} \qquad \alpha^i := h(\xi^i). \tag{3.4}
$$

Now we choose $R > 0$ large enough (but fixed!), and we let $N \to \infty$ and $\varepsilon \to 0$ in such a way that $\varepsilon \gg 1/N$. We will show that then the following holds:

Main result. *For each $t \in [0, T^*]$ the weighted empirical measure*

$$
\mu_t^{N,\varepsilon,R} := \frac{1}{N} \sum_{i=1}^N a_t^{N,i,\varepsilon,R} \delta_{X_t^{N,i,\varepsilon,R}}
$$

of the system (3.3) *converges to the measure with density $w(t, \cdot)$.*

The precise statement of this theorem is given in Section 3.3.

3.2 Analytical properties of the vorticity equation

In this section we discuss analytical properties of the vorticity equation (3.2) and its smoothed version

$$
\begin{aligned}
\frac{\partial w^\varepsilon}{\partial t} + (\mathcal{K}^\varepsilon(w^\varepsilon) \cdot \nabla) w^\varepsilon &= (w^\varepsilon \cdot \nabla) \mathcal{K}^\varepsilon(w^\varepsilon) + \nu \Delta w^\varepsilon \\
w^\varepsilon(0, \cdot) &= w_0.
\end{aligned}
\tag{3.5}
$$

Here $\mathcal{K}^\varepsilon(f)(x) := \int_{\mathbb{R}^3} K^\varepsilon(x - y) \times f(y) dy$. By setting $K^0 := K$ we obtain (3.2) as a special case of (3.5) (namely for $\varepsilon = 0$). We can therefore study these two equations simultaneously.

Definition 3.1. Let

$$F_{0,p,T} := \left\{ w : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3 \mid w \text{ measurable and } \|w\|_{0,p,T} < \infty \right\}$$
$$\text{and } F_{1,p,T} := \left\{ w : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3 \mid w \text{ measurable and } \|w\|_{1,p,T} < \infty \right\},$$

where

$$\|w\|_{0,p,T} := \sup_{0 \le t \le T} \|w(t)\|_{L^p}$$
$$\text{and } \|w\|_{1,p,T} := \sup_{0 \le t \le T} \left\{ \|w(t)\|_{L^p} + t^{1/2} \|\nabla w(t)\|_{L^p} \right\}.$$

Definition 3.2. Let $\varepsilon \ge 0$. A function $w^\varepsilon \in F_{0,p,T}$ is called *mild solution* of the (smoothed) vorticity equation (3.5) with initial data w_0 if for each $t \in [0,T]$

$$\begin{aligned} w^\varepsilon(t,x) &= \int_{\mathbb{R}^3} G_t^\nu(x-y) w_0(y) dy \\ &+ \int_0^t \int_{\mathbb{R}^3} \left\{ \mathcal{K}^\varepsilon(w^\varepsilon) \otimes w^\varepsilon - w^\varepsilon \otimes \mathcal{K}^\varepsilon(w^\varepsilon) \right\}(s,y) \nabla G_{t-s}^\nu(x-y) dy ds. \end{aligned} \tag{3.6}$$

Here $G_t^\nu(x) := (4\pi\nu t)^{-3/2} \exp(-\frac{|x|^2}{4\nu t})$ is the heat kernel.

Proposition 3.1 (Fontbona)**.** *For the mild equation* (3.6) *the following hold:*

1. *Global uniqueness: For each $\varepsilon \ge 0$ and each $T > 0$ equation* (3.6) *has at most one solution in the class $F_{0,p,T}$.*
2. *Local existence: There exists a number $T^* = T^*(\|w_0\|_{L^p}, \nu) > 0$ with the following properties:*
   (a) For each $\varepsilon \ge 0$ equation (3.6) *has a solution in the class F_{0,p,T^*}.*
   (b) There is a constant $C_1 = C_1(T^*, \|w_0\|_{L^p}, \nu) < \infty$ such that
   $$\|w^\varepsilon\|_{0,p,T^*} \le C_1 \tag{3.7}$$
   for each $\varepsilon \ge 0$.
3. *Regularity in the L^p-sense: Let $T^* > 0$ be as above. Then $w^\varepsilon \in F_{1,p,T^*}$ for each $\varepsilon \ge 0$, and there is a constant $C_2 = C_2(T^*, \|w_0\|_{L^p}, \nu) < \infty$ such that*
   $$\|w^\varepsilon\|_{1,p,T^*} \le C_2 \tag{3.8}$$
   for each $\varepsilon \ge 0$.
4. *Regularity in the L^∞-sense: Let $T^* > 0$ be as above. Define $u^\varepsilon := \mathcal{K}^\varepsilon(w^\varepsilon)$. Then there are constants $C_3 = C_3(T^*, \|w_0\|_{L^p}, \nu) < \infty$ and $C_4 = C_4(T^*, \|w_0\|_{L^p}, \nu) < \infty$ such that*
   $$\begin{aligned} &\sup_{t\in[0,T]} t^{1/2} \|u^\varepsilon(t,\cdot)\|_{L^\infty} \le C_3 \\ \text{and } &\sup_{t\in[0,T]} t^{1/2} \|\nabla u^\varepsilon(t,\cdot)\|_{L^\infty} \le C_4 \end{aligned} \tag{3.9}$$
   for each $\varepsilon \ge 0$.

Proof. See [11], Theorems 3.1 and 3.2, Corollary 3.1 and Remark 6.3. □

Remark 3.1. In addition to the results summarized in Proposition 3.1, Fontbona ([11], Proposition 6.1) also proved a convergence result ($w^\varepsilon \to w$ for $\varepsilon \to 0$), but he did not succeed in estimating the speed of convergence (see [11], Remark 6.5). We will therefore ourselves prove such a result (see Proposition 3.5 below).

3.3 Main result

In order to measure convergence of the random $\mathbb{R}^3$-valued measure $\mu_t^{N,\varepsilon,R}$ to $w(t,\cdot)$ we define

$$H := \{f \in \mathcal{C}^{0,1}(\mathbb{R}^3) \cap L^{p'}(\mathbb{R}^3) \mid \|f\|_{L^\infty} \leq 1, |f|_{lip} \leq 1, \|f\|_{L^{p'}} \leq 1\},$$

where p' is such that $1/p + 1/p' = 1$.

Theorem 3. *Let $T^* > 0$ be as in Proposition 3.1. Fix $R \geq \max(1, \exp(2C_4\sqrt{T^*})\|h\|_{L^\infty})$. Then there are constants $A_1, A_2, A_3 < \infty$ (depending on T^*, $\|w_0\|_{L^p}$, R and ν) such that for each $N \in \mathbb{N}$ and each $\varepsilon \in (0,1]$:*

$$\sup_{t\in[0,T^*]} \sup_{f\in H} E\left[|< \mu_t^{N,\varepsilon,R}, f > - < w(t,\cdot), f >|^2\right] \leq A_1\varepsilon^{12}\exp(A_2\varepsilon^{-10})\frac{1}{N} + A_3\varepsilon.$$

Corollary 3.1. *Let $(\varepsilon_N)_{N\in\mathbb{N}}$ be a sequence converging to 0 such that $\varepsilon_N^{12}\exp(A_2\varepsilon_N^{-10})\frac{1}{N} \to 0$ for $N \to \infty$. Then for each $t \in [0,T^*]$ the weighted empirical measure $\mu_t^{N,\varepsilon_N,R}$ of the particle system converges to the measure with density $w(t,\cdot)$.*

3.4 Remarks concerning related work

We have been inspired by the work of Esposito and Pulvirenti [9] and Fontbona [11]. Compared to those papers our approach offers the following advantages:

1. As already remarked by Fontbona, the proof in [9] is not sufficient. Fontbona ([11], Introduction) comments about this as follows: "They did not give rigorous mathematical proofs of crucial facts."

2. Our particle system is simpler than the one studied by Fontbona [11]: While we only need six real numbers to represent each discrete vortex (three numbers for the position, and three numbers for the intensity), Fontbona needs twelve numbers for each vortex (also three numbers for the position, but nine numbers for the vortex stretching matrix). For numerical purposes our system is hence preferable.

3. In contrast to [9] and [11] we estimate the speed of convergence.

In two dimensions it is much easier to prove that the Navier-Stokes equation can be approximated by a system of interacting discrete vortices (because then the *vortex stretching term* $(w\cdot\nabla)u$ in the vorticity equation (3.2) vanishes), and this problem was solved more than twenty years ago by Marchioro and Pulvirenti [22] (see also Méléard [23], [24]).

For other probabilistic approaches to the three-dimensional Navier-Stokes equation see Busnello, Flandoli and Romito [5], Esposito, Marra, Pulvirenti and Sciarretta [8] and Le Jan and Sznitman [20]. Note however that in none of these papers a stochastic particle approximation for the Navier-Stokes equation is given.

3.5 Proof of Theorem 3

3.5.1 Overview of the proof

As intermediate objects between the system of discrete vortices (3.3) and the vorticity equation (3.2) we introduce processes $(\overline{X}_t^{i,\varepsilon}, \overline{a}_t^{i,\varepsilon})_{0\leq t\leq T^*}$ $(i \in \{1,\ldots,N\},\ \varepsilon > 0)$ defined by the following stochastic differential equations:

$$\begin{aligned} \overline{X}_t^{i,\varepsilon} &= \xi^i + \int_0^t u^\varepsilon(s, \overline{X}_s^{i,\varepsilon}) ds + \sqrt{2\nu} W_t^i \\ \overline{a}_t^{i,\varepsilon} &= \alpha^i + \int_0^t \nabla u^\varepsilon(s, \overline{X}_s^{i,\varepsilon}) \overline{a}_s^{i,\varepsilon} ds. \end{aligned} \tag{3.10}$$

Because of the Lipschitz continuity and boundedness of $u^\varepsilon = \mathcal{K}^\varepsilon(w^\varepsilon)$ and ∇u^ε these stochastic differential equations have a unique strong solution. Note that the i-th process $(\overline{X}^{i,\varepsilon}, \overline{a}^{i,\varepsilon})$ has the *same* initial value (ξ^i, α^i) and is driven by the *same* Brownian motion W^i as the i-th discrete vortex of the system (3.3). For different i these processes are independent copies of each other: the initial values are independent and identically distributed, and there is no interaction.

In the next subsection we show the following crucial properties:

Proposition 3.2. *The processes $(\overline{a}_t^{i,\varepsilon})_{0 \le t \le T^*}$ are uniformly bounded. More precisely,*

$$|\overline{a}_t^{i,\varepsilon}| \le \|h\|_{L^\infty} \exp(2C_4\sqrt{T^*}) \tag{3.11}$$

for all $t \in [0, T^*]$, all $N \in \mathbb{N}$, all $i \in \{1, \dots, N\}$ and all $\varepsilon > 0$.

Proposition 3.3. *The $\mathbb{R}^3$-valued measure μ_t^ε defined by*

$$< \mu_t^\varepsilon, f > := E\left[f(\overline{X}_t^{i,\varepsilon})\overline{a}_t^{i,\varepsilon}\right] \tag{3.12}$$

coincides with the measure with density $w^\varepsilon(t, \cdot)$, i.e.

$$E\left[f(\overline{X}_t^{i,\varepsilon})\overline{a}_t^{i,\varepsilon}\right] = \int_{\mathbb{R}^3} f(x) w^\varepsilon(t, x) dx \tag{3.13}$$

for all test functions f.

Corollary 3.2. *The function u^ε can be recovered from the process $(\overline{X}^{i,\varepsilon}, \overline{a}^{i,\varepsilon})$ in the following way:*

$$u^\varepsilon(t, x) = E\left[K^\varepsilon(x - \overline{X}_t^{i,\varepsilon}) \times \overline{a}_t^{i,\varepsilon}\right]. \tag{3.14}$$

Combining Proposition 3.2 and Corollary 3.2 one obtains that for each $R \ge \|h\|_{L^\infty} \exp(2C_4\sqrt{T^*})$ the process $(\overline{X}^{i,\varepsilon}, \overline{a}^{i,\varepsilon})$ satisfies the following *non-linear* stochastic differential equation:

$$\begin{aligned} \overline{X}_t^{i,\varepsilon} &= \xi^i + \int_0^t u^\varepsilon(s, \overline{X}_s^{i,\varepsilon}) ds + \sqrt{2\nu} W_t^i \\ \overline{a}_t^{i,\varepsilon} &= \alpha^i + \int_0^t \nabla u^\varepsilon(s, \overline{X}_s^{i,\varepsilon}) \chi_R(\overline{a}_s^{i,\varepsilon}) ds \\ u^\varepsilon(s, x) &= E\left[K^\varepsilon(x - \overline{X}_s^{i,\varepsilon}) \times \chi_R(\overline{a}_s^{i,\varepsilon})\right]. \end{aligned} \tag{3.15}$$

This equation is called non-linear because the function u^ε, which appears in the equation, is obtained as an expectation of a quantity related to the solution process.

The proof of Theorem 3 now consists of the following parts: in subsection 3.5.3 we show (for $N \to \infty$) pathwise convergence of $(X^{N,i,\varepsilon,R}, a^{N,i,\varepsilon,R})$ to $(\overline{X}^{i,\varepsilon}, \overline{a}^{i,\varepsilon})$:

Proposition 3.4. *There are constants $C_5, C_6 < \infty$ which only depend on φ such that for every $N \in \mathbb{N}$, every $\varepsilon \in (0, 1]$, every $R \ge \max(1, \exp(2C_4\sqrt{T^*})\|h\|_{L^\infty})$, every $T \le T^*$ and every $i \in \{1, \dots, N\}$:*

$$E\left[\sup_{0 \le t \le T}\left|X_t^{N,i,\varepsilon,R} - \overline{X}_t^{i,\varepsilon}\right|^2 + \sup_{0 \le t \le T}\left|a_t^{N,i,\varepsilon,R} - \overline{a}_t^{i,\varepsilon}\right|^2\right] \le C_5 \varepsilon^{12} R^{-4} T^{-2} \exp(C_6 \varepsilon^{-10} R^4 T^2) \frac{1}{N}.$$

Then, in subsection 3.5.4, we show (for $\varepsilon \to 0$) convergence of w^ε to w:

Proposition 3.5. *There is a constant $C_7 = C_7(T^*, \|w_0\|_{L^p}, \nu, \varphi) < \infty$ such that*

$$\|w(t,\cdot) - w^\varepsilon(t,\cdot)\|_{L^p} \leq C_7 \varepsilon$$

for all $t \in [0, T^*]$.

Finally, in subsection 3.5.5 we combine Propositions 3.3, 3.4 and 3.5 and obtain Theorem 3.

3.5.2 Proofs of Propositions 3.2 and 3.3

As we have already remarked, the processes $(\overline{X}^{i,\varepsilon}, \overline{a}^{i,\varepsilon})$ are (for different i) independent copies of each other. When we study their properties, we can therefore omit the index i.

Proof of Proposition 3.2. By (3.9)

$$\int_0^{T^*} \|\nabla u^\varepsilon(s,\cdot)\|_{L^\infty} \leq 2C_4 \sqrt{T^*}.$$

Therefore (3.10) implies

$$|\overline{a}_t^\varepsilon| \leq \|h\|_{L^\infty} + \int_0^t \|\nabla u^\varepsilon(s,\cdot)\|_{L^\infty} |a_s^\varepsilon| ds.$$

Gronwall's lemma now implies

$$|\overline{a}_t^\varepsilon| \leq \|h\|_{L^\infty} \exp\left(\int_0^t \|\nabla u^\varepsilon(s,\cdot)\|_{L^\infty} ds \right) \leq \|h\|_{L^\infty} \exp(2C_4 \sqrt{T^*}),$$

q.e.d. □

For the proof of Proposition 3.3 we need some lemmas.

Lemma 3.1. *For each $t \geq 0$ the measure μ_t^ε has a density (that we also denote by μ_t^ε) and solves the linear mild equation*

$$\begin{aligned} \mu_t^\varepsilon(x) &= \int_{\mathbb{R}^3} G_t^\nu(x-y) w_0(y) dy - \int_0^t \int_{\mathbb{R}^3} \left[\nabla G_{t-s}^\nu(x-y) \cdot u^\varepsilon(s,y) \right] \mu_s^\varepsilon(y) dy ds \\ &\quad + \int_0^t \int_{\mathbb{R}^3} G_{t-s}^\nu(x-y) \nabla u^\varepsilon(s,y) \mu_s^\varepsilon(y) dy ds. \end{aligned}$$

Proof. Let $f \in \mathcal{C}_b^{1,2}([0,T^*] \times \mathbb{R}^3)$. We apply Itô's formula to (3.10) and the function $g(s,x,a) := f(s,x)a$ and obtain:

$$\begin{aligned} f(t, \overline{X}_t^\varepsilon) \overline{a}_t^\varepsilon &= f(0,\xi)\alpha + \int_0^t \frac{\partial f}{\partial s}(s, \overline{X}_s^\varepsilon) \overline{a}_s^\varepsilon ds + \int_0^t \left[\nabla f(s, \overline{X}_s^\varepsilon) \cdot u^\varepsilon(s, \overline{X}_s^\varepsilon) \right] \overline{a}_s^\varepsilon ds \\ &\quad + \sqrt{2\nu} \int_0^t \overline{a}_s^\varepsilon \otimes \nabla f(s, \overline{X}_s^\varepsilon)\, dW_s + \int_0^t f(s, \overline{X}_s^\varepsilon) \nabla u^\varepsilon(s, \overline{X}_s^\varepsilon) \overline{a}_s^\varepsilon ds \\ &\quad + \nu \int_0^t \Delta f(s, \overline{X}_s^\varepsilon) \overline{a}_s^\varepsilon ds. \end{aligned}$$

We now take expectations and obtain using (3.4):

$$
\begin{aligned}
< \mu_t^\varepsilon, f(t,\cdot) > &= < w_0, f(0,\cdot) > + \int_0^t < \mu_s^\varepsilon, \frac{\partial f}{\partial s}(s,\cdot) > ds + \int_0^t < \mu_s^\varepsilon, \nabla f(s,\cdot)\cdot u^\varepsilon(s,\cdot) > ds \\
&\quad + \int_0^t < \mu_s^\varepsilon, f(s,\cdot)\nabla u^\varepsilon(s,\cdot) > ds + \nu \int_0^t < \mu_s^\varepsilon, \Delta f(s,\cdot) > ds. \qquad (3.16)
\end{aligned}
$$

We now fix $t \in [0, T^*]$ and $\psi \in \mathcal{C}_b^2(\mathbb{R}^3)$ and choose for f the function

$$
f(s,x) := (G_{t-s}^\nu * \psi)(x) = \int_{\mathbb{R}^3} G_{t-s}^\nu (x-y)\psi(y)dy
$$

(so that $\frac{\partial f}{\partial s} = -\nu\Delta f$). It follows:

$$
\begin{aligned}
\int_{\mathbb{R}^3} \psi(x)\mu_t^\varepsilon(dx) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G_t^\nu(x-y)\psi(y)dy w_0(x)dx \\
&\quad + \int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \nabla G_{t-s}^\nu(x-y)\psi(y)dy \cdot u^\varepsilon(s,x)\mu_s^\varepsilon(dx)ds \\
&\quad + \int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} G_{t-s}^\nu(x-y)\psi(y)dy \nabla u^\varepsilon(s,x)\mu_s^\varepsilon(dx)ds \\
&= \int_{\mathbb{R}^3} \psi(y)\Big[\int_{\mathbb{R}^3} G_t^\nu(x-y)w_0(x)dx + \int_0^t\int_{\mathbb{R}^3}\left[\nabla G_{t-s}^\nu(x-y)\cdot u^\varepsilon(s,x)\right]\mu_s^\varepsilon(dx)ds \\
&\qquad + \int_0^t\int_{\mathbb{R}^3} G_{t-s}^\nu(x-y)\nabla u^\varepsilon(s,x)\mu_s^\varepsilon(dx)ds\Big]dy \\
&= \int_{\mathbb{R}^3} \psi(x)\Big[\int_{\mathbb{R}^3} G_t^\nu(x-y)w_0(y)dy - \int_0^t\int_{\mathbb{R}^3}\left[\nabla G_{t-s}^\nu(x-y)\cdot u^\varepsilon(s,y)\right]\mu_s^\varepsilon(dy)ds \\
&\qquad + \int_0^t\int_{\mathbb{R}^3} G_{t-s}^\nu(x-y)\nabla u^\varepsilon(s,y)\mu_s^\varepsilon(dy)ds\Big]dx.
\end{aligned}
$$

As this holds for all $\psi \in \mathcal{C}_b^2(\mathbb{R}^3)$, it follows that the measure μ_t^ε has a density (that we also denote μ_t^ε) and that

$$
\begin{aligned}
\mu_t^\varepsilon(x) &= \int_{\mathbb{R}^3} G_t^\nu(x-y)w_0(y)dy - \int_0^t\int_{\mathbb{R}^3}\left[\nabla G_{t-s}^\nu(x-y)\cdot u^\varepsilon(s,y)\right]\mu_s^\varepsilon(y)dyds \\
&\quad + \int_0^t\int_{\mathbb{R}^3} G_{t-s}^\nu(x-y)\nabla u^\varepsilon(s,y)\mu_s^\varepsilon(y)dyds,
\end{aligned}
$$

q.e.d. □

Lemma 3.2. *For each $t \in [0,T^*]$ we have* div $\mu_t^\varepsilon = 0$ *in the sense of distributions.*

Proof. We need the following version of (3.16) for vector-valued functions: for each test function $\mathbf{f} \in \mathcal{C}_b^{1,2}([0,T^*]\times\mathbb{R}^3,\mathbb{R}^3)$

$$
\begin{aligned}
< \mu_t^\varepsilon, \mathbf{f}(t,\cdot) > &= < w_0, \mathbf{f}(0,\cdot) > \\
&\quad + \int_0^t < \mu_s^\varepsilon, \frac{\partial \mathbf{f}}{\partial s}(s,\cdot) + \nabla\mathbf{f}(s,\cdot)u^\varepsilon(s,\cdot) + \nabla u^\varepsilon(s,\cdot)^{\mathrm{T}}\mathbf{f}(s,\cdot) + \nu\Delta\mathbf{f}(s,\cdot) > ds.
\end{aligned}
$$

Now let $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$, and let f be the solution of the terminal value problem

$$
\begin{aligned}
\frac{\partial f}{\partial t} + \nu\Delta f + u^\varepsilon\cdot\nabla f &= 0 \\
f(t,\cdot) &= \psi.
\end{aligned}
$$

By choosing ∇f in place of $\mathbf{f}$ we obtain:

$$\begin{aligned}
& < \mu_t^\varepsilon, \nabla f(t,\cdot) > - < w_0, \nabla f(0,\cdot) > \\
= & \int_0^t < \mu_s^\varepsilon, \nabla \frac{\partial f}{\partial s}(s,\cdot) + \text{Hess}\, f(s,\cdot) u^\varepsilon(s,\cdot) + \nabla u^\varepsilon(s,\cdot)^{\mathrm{T}} \nabla f(s,\cdot) + \nu \nabla \Delta f(s,\cdot) > ds \\
= & \int_0^t < \mu_s^\varepsilon, \nabla \frac{\partial f}{\partial s}(s,\cdot) + \nabla(u^\varepsilon \cdot \nabla f) + \nu \nabla \Delta f(s,\cdot) > ds \\
= & \int_0^t < \mu_s^\varepsilon, \nabla(\frac{\partial f}{\partial s}(s,\cdot) + u^\varepsilon \cdot \nabla f + \nu \Delta f(s,\cdot)) > ds \\
= & \ 0.
\end{aligned}$$

Since moreover $\operatorname{div} w_0 = 0$ (in the sense of distributions), $< w_0, \nabla f(0,\cdot) >$ vanishes, and therefore

$$< \mu_t^\varepsilon, \nabla \psi(t,\cdot) > = 0.$$

Since $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ was arbitrary, it follows that $\operatorname{div} \mu_t^\varepsilon = 0$. ☐

Lemma 3.3. *μ_t^ε also solves the following linear mild equation:*

$$\mu_t^\varepsilon(x) = \int_{\mathbb{R}^3} G_t^\nu(x-y) w_0(y) dy + \int_0^t \int_{\mathbb{R}^3} \left\{ u^\varepsilon \otimes \mu_s^\varepsilon - \mu_s^\varepsilon \otimes u^\varepsilon \right\}(s,y) \nabla G_{t-s}^\nu(x-y) dy ds. \quad (3.17)$$

Proof. According to Lemma 3.1 and Lemma 3.2:

$$\begin{aligned}
\mu_t^\varepsilon(x) = & \int_{\mathbb{R}^3} G_t^\nu(x-y) w_0(y) dy - \int_0^t \int_{\mathbb{R}^3} \left[\nabla G_{t-s}^\nu(x-y) \cdot u^\varepsilon(s,y) \right] \mu_s^\varepsilon(y) dy ds \\
& + \int_0^t \int_{\mathbb{R}^3} G_{t-s}^\nu(x-y) \nabla u^\varepsilon(s,y) \mu_s^\varepsilon(y) dy ds \\
= & \int_{\mathbb{R}^3} G_t^\nu(x-y) w_0(y) dy - \int_0^t \int_{\mathbb{R}^3} \{ \mu_s^\varepsilon(y) \otimes u^\varepsilon(s,y) \} \nabla G_{t-s}^\nu(x-y) dy ds \\
& - \int_0^t \int_{\mathbb{R}^3} \operatorname{div}_y (G_{t-s}^\nu(x-y) \mu_s^\varepsilon(y)) \nabla u^\varepsilon(s,y) dy ds \\
= & \int_{\mathbb{R}^3} G_t^\nu(x-y) w_0(y) dy - \int_0^t \int_{\mathbb{R}^3} \{ \mu_s^\varepsilon(y) \otimes u^\varepsilon(s,y) \} \nabla G_{t-s}^\nu(x-y) dy ds \\
& + \int_0^t \int_{\mathbb{R}^3} [\nabla G_{t-s}^\nu(x-y) \cdot \nabla \mu_s^\varepsilon(y)] \nabla u^\varepsilon(s,y) dy ds \\
= & \int_{\mathbb{R}^3} G_t^\nu(x-y) w_0(y) dy + \int_0^t \int_{\mathbb{R}^3} \left\{ u^\varepsilon \otimes \mu_s^\varepsilon - \mu_s^\varepsilon \otimes u^\varepsilon \right\}(s,y) \nabla G_{t-s}^\nu(x-y) dy ds,
\end{aligned}$$

q.e.d. ☐

Lemma 3.4. $\mu^\varepsilon \in L^\infty([0,T^*], L^1(\mathbb{R}^3,\mathbb{R}^3))$.

Proof. According to (3.12) and (3.11) we have $| < \mu_t^\varepsilon, f > | \leq \exp(2C_4\sqrt{T^*}) \|h\|_{L^\infty} \|f\|_{L^\infty}$ for each $t \in [0,T^*]$ and each $f \in L^\infty(\mathbb{R}^3)$, and the claim follows. ☐

Lemma 3.5. *For each $m \in [1, \frac{3}{2})$ we have $\mu^\varepsilon \in F_{0,m,T^*}$.*

Proof. Lemmas 3.3 and 3.4 imply:

$$\begin{aligned}
\|\mu_t^\varepsilon\|_{L^m} \leq & \ \|G_t^\nu * w_0\|_{L^m} + 2\int_0^t \|\nabla G_{t-s}^\nu\|_{L^m} \|u_s^\varepsilon\|_{L^\infty} \|\mu_s^\varepsilon\|_{L^1} ds \\
\leq & \ \|w_0\|_{L^m} + 2C_3 \|\mu^\varepsilon\|_{L^\infty([0,T^*],L^1)} \|\nabla G_1^1\|_{L^m} \int_0^t (\nu(t-s))^{\frac{3}{2m}-2} s^{-1/2} ds,
\end{aligned}$$

and since $\frac{3}{2m} - 2 > -1$ (because $m < \frac{3}{2}$) this is bounded uniformly in $t \in [0, T^*]$. Thus $\mu^\varepsilon \in F_{0,m,T^*}$. □

Lemma 3.6. $\mu^\varepsilon \in F_{0,p,T^*}$.

Proof. Choose $m, r \in [1, \frac{3}{2})$ such that $\frac{1}{m} + \frac{1}{r} = 1 + \frac{1}{p}$ (e.g. $m = r = \frac{2p}{p+1}$). Lemmas 3.3 and 3.5 and Young's inequality (Lemma 3.9, Appendix) imply:

$$\begin{aligned}
\|\mu_t^\varepsilon\|_{L^p} &\leq \|G_t^\nu * w_0\|_{L^p} + 2\int_0^t \|\nabla G_{t-s}^\nu\|_{L^r}\|u_s^\varepsilon\|_{L^\infty}\|\mu_s^\varepsilon\|_{L^m} ds \\
&\leq \|w_0\|_{L^m} + 2C_3\|\mu^\varepsilon\|_{0,p,T^*}\|\nabla G_1^1\|_{L^r}\int_0^t (\nu(t-s))^{\frac{3}{2r}-2}s^{-1/2}ds,
\end{aligned}$$

and since $\frac{3}{2r} - 2 > -1$ (because $r < \frac{3}{2}$) this is bounded uniformly in $t \in [0, T^*]$. Thus $\mu^\varepsilon \in F_{0,p,T^*}$. □

Proof of Proposition 3.3. The functions μ^ε and w^ε both belong to the class F_{0,p,T^*} and solve the mild equation (3.17). Let $m(t,x) := \mu_t^\varepsilon(x) - w^\varepsilon(t,x)$. (3.17) implies:

$$m(t,x) = \int_0^t \int_{\mathbb{R}^3} \{u^\varepsilon \otimes m - m \otimes u^\varepsilon\}(s,y)\nabla G_{t-s}^\nu(x-y)dyds.$$

We now introduce the following exponents: $m := \frac{3p}{4p-3}$, $k := \frac{3p}{6-p}$, $q := \frac{3p}{3-p}$. Because of $p \in (\frac{3}{2}, 3)$ we have $m \in (1, \frac{3}{2})$, $k \in (1,3)$ and $q \in (3, \infty)$; in particular all these exponents are finite and strictly greater than 1. Moreover $\frac{1}{m} + \frac{1}{k} - 1 = \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{k}$, so that using Young's inequality (Lemma 3.9, Appendix), Hölder's inequality and Lemma 3.14 (Appendix) we obtain:

$$\begin{aligned}
\|m(t,\cdot)\|_{L^p} &\leq \int_0^t \|\nabla G_{t-s}^\nu\|_{L^m}\|\{u^\varepsilon \otimes m - m \otimes u^\varepsilon\}(s,\cdot)\|_{L^k} ds \\
&\leq 2\int_0^t \|\nabla G_{t-s}^\nu\|_{L^m}\|u^\varepsilon(s,\cdot)\|_{L^q}\|m(s,\cdot)\|_{L^p} ds \\
&\leq 2C_{p,q}\int_0^t \|\nabla G_{t-s}^\nu\|_{L^m}\|w^\varepsilon(s,\cdot)\|_{L^p}\|m(s,\cdot)\|_{L^p} ds.
\end{aligned}$$

We conclude using Gronwall's lemma. □

3.5.3 Proof of Proposition 3.4

We need the following lemma:

Lemma 3.7. *There is a constant $L = L(\varphi) < \infty$ which only depends on φ such that for each $\varepsilon \in (0,1]$:*

1. $\|K^\varepsilon\|_{L^\infty} \leq L\varepsilon^{-3}$.
2. $|K^\varepsilon|_{lip} = \|\nabla K^\varepsilon\|_{L^\infty} \leq L\varepsilon^{-4}$.
3. $|\nabla K^\varepsilon|_{lip} \leq L\varepsilon^{-5}$.

Proof. Use $K \in L^1(\mathbb{R}^3, \mathbb{R}^3) + L^\infty(\mathbb{R}^3, \mathbb{R}^3)$. □

Proof of Proposition 3.4. The proof uses standard arguments from the theory of propagation of chaos (see [30] for an introduction), but since the situation here is more complicated we present the calculations. In order to simplify the notation we omit the indices N, ε and R. For $t \in [0, T^*]$ we set

$$\Psi(t) := E\left[\sup_{0\le s\le t}\left|X_s^i - \overline{X}_s^i\right|^2 + \sup_{0\le s\le t}\left|a_s^i - \overline{a}_s^i\right|^2\right].$$

Using (3.3) and (3.15) one can easily show that $\Psi(t)$ is bounded by

$$\begin{aligned}
&\frac{4t}{N}\int_0^t \sum_{j=1}^N E\left[\left|K^\varepsilon(X_s^i - X_s^j)\times\chi_R(a_s^j) - K^\varepsilon(X_s^i - X_s^j)\times\chi_R(\overline{a}_s^j)\right|^2\right] ds \\
&+\frac{4t}{N}\int_0^t \sum_{j=1}^N E\left[\left|K^\varepsilon(X_s^i - X_s^j)\times\chi_R(\overline{a}_s^j) - K^\varepsilon(X_s^i - \overline{X}_s^j)\times\chi_R(\overline{a}_s^j)\right|^2\right] ds \\
&+\frac{4t}{N}\int_0^t \sum_{j=1}^N E\left[\left|K^\varepsilon(X_s^i - \overline{X}_s^j)\times\chi_R(\overline{a}_s^j) - K^\varepsilon(\overline{X}_s^i - \overline{X}_s^j)\times\chi_R(\overline{a}_s^j)\right|^2\right] ds \\
&+\frac{5t}{N}\int_0^t \sum_{j=1}^N E\left[\left|\left\{\nabla K^\varepsilon(X_s^i - X_s^j)\times\chi_R(a_s^j)\right\}\chi_R(a_s^i) - \left\{\nabla K^\varepsilon(X_s^i - X_s^j)\times\chi_R(a_s^j)\right\}\chi_R(\overline{a}_s^i)\right|^2\right] ds \\
&+\frac{5t}{N}\int_0^t \sum_{j=1}^N E\left[\left|\left\{\nabla K^\varepsilon(X_s^i - X_s^j)\times\chi_R(a_s^j)\right\}\chi_R(\overline{a}_s^i) - \left\{\nabla K^\varepsilon(X_s^i - X_s^j)\times\chi_R(\overline{a}_s^j)\right\}\chi_R(\overline{a}_s^i)\right|^2\right] ds \\
&+\frac{5t}{N}\int_0^t \sum_{j=1}^N E\left[\left|\left\{\nabla K^\varepsilon(X_s^i - X_s^j)\times\chi_R(\overline{a}_s^j)\right\}\chi_R(\overline{a}_s^i) - \left\{\nabla K^\varepsilon(X_s^i - \overline{X}_s^j)\times\chi_R(\overline{a}_s^j)\right\}\chi_R(\overline{a}_s^i)\right|\right] ds \\
&+\frac{5t}{N}\int_0^t \sum_{j=1}^N E\left[\left|\left\{\nabla K^\varepsilon(X_s^i - \overline{X}_s^j)\times\chi_R(\overline{a}_s^j)\right\}\chi_R(\overline{a}_s^i) - \left\{\nabla K^\varepsilon(\overline{X}_s^i - \overline{X}_s^j)\times\chi_R(\overline{a}_s^j)\right\}\chi_R(\overline{a}_s^i)\right|^2\right] ds \\
&+\frac{4t}{N^2}\int_0^t E\left[\left|\sum_{j=1}^N\left[K^\varepsilon(\overline{X}_s^i - \overline{X}_s^j)\times\chi_R(\overline{a}_s^j) - u^\varepsilon(s,\overline{X}_s^i)\right]\right|^2\right] ds \\
&+\frac{5t}{N^2}\int_0^t E\left[\left|\sum_{j=1}^N\left[\left\{\nabla K^\varepsilon(\overline{X}_s^i - \overline{X}_s^j)\times\chi_R(\overline{a}_s^j)\right\}\chi_R(\overline{a}_s^i) - \nabla u^\varepsilon(s,\overline{X}_s^i)\chi_R(\overline{a}_s^i)\right]\right|^2\right] ds.
\end{aligned}$$

We estimate the first seven terms using the Lipschitz continuity and boundedness of K^ε, ∇K^ε and χ_R: According to Lemma 3.7 $\|K^\varepsilon\|_{L^\infty} \le L\varepsilon^{-3}$, $|K^\varepsilon|_{lip} = \|\nabla K^\varepsilon\|_{L^\infty} \le L\varepsilon^{-4}$ and $|\nabla K^\varepsilon|_{lip} \le L\varepsilon^{-5}$. Moreover it is clear that $\|\chi_R\|_{L^\infty} = R$ and $|\chi_R|_{lip} = 1$. Therefore there exists a constant $C_6 < \infty$ which only depends on φ such that

$$\begin{aligned}
\Psi(t) \le\ & C_6 t\varepsilon^{-10}R^4\int_0^t \Psi(s)ds \\
&+\frac{4t}{N^2}\int_0^t E\left[\left|\sum_{j=1}^N\left[K^\varepsilon(\overline{X}_s^i - \overline{X}_s^j)\times\chi_R(\overline{a}_s^j) - u^\varepsilon(s,\overline{X}_s^i)\right]\right|^2\right] ds \qquad (3.18)\\
&+\frac{5t}{N^2}\int_0^t E\left[\left|\sum_{j=1}^N\left[\left\{\nabla K^\varepsilon(\overline{X}_s^i - \overline{X}_s^j)\times\chi_R(\overline{a}_s^j)\right\}\chi_R(\overline{a}_s^i) - \nabla u^\varepsilon(s,\overline{X}_s^i)\chi_R(\overline{a}_s^i)\right]\right|^2\right] ds.
\end{aligned}$$

We now estimate the second and the third term in (3.18). For the second term we obtain:

$$\begin{aligned} & E\left[\left|\sum_{j=1}^{N}\left[K^{\varepsilon}(\overline{X}_s^i-\overline{X}_s^j)\times\chi_R(\overline{a}_s^j)-u^{\varepsilon}(s,\overline{X}_s^i)\right]\right|^2\right] \qquad (3.19)\\ = & \sum_{j,k=1}^{N} E\left\{\left[K^{\varepsilon}(\overline{X}_s^i-\overline{X}_s^j)\times\chi_R(\overline{a}_s^j)-u^{\varepsilon}(s,\overline{X}_s^i)\right]\cdot\left[K^{\varepsilon}(\overline{X}_s^i-\overline{X}_s^k)\times\chi_R(\overline{a}_s^k)-u^{\varepsilon}(s,\overline{X}_s^i)\right]\right\}. \end{aligned}$$

If $j \neq k$ this expectation vanishes according to the following Lemma 3.8, and otherwise it is bounded by $(2L\varepsilon^{-3}R)^2$ thanks to Corollary 3.2 and Lemma 3.7. Therefore

$$E\left[\left|\sum_{j=1}^{N}\left[K^{\varepsilon}(\overline{X}_s^i-\overline{X}_s^j)\times\chi_R(\overline{a}_s^j)-u^{\varepsilon}(s,\overline{X}_s^i)\right]\right|^2\right] \le N(2L\varepsilon^{-3}R)^2 = 4L\varepsilon^{-6}R^2N. \qquad (3.20)$$

The third term in (3.18) is treated analogously:

$$\begin{aligned} & E\left[\left|\sum_{j=1}^{N}\left[\left\{\nabla K^{\varepsilon}(\overline{X}_s^i-\overline{X}_s^j)\times\chi_R(\overline{a}_s^j)\right\}\chi_R(\overline{a}_s^i)-\nabla u^{\varepsilon}(s,\overline{X}_s^i)\chi_R(\overline{a}_s^i)\right]\right|^2\right]\\ = & \sum_{j,k=1}^{N} E\left\{\left[\left\{\nabla K^{\varepsilon}(\overline{X}_s^i-\overline{X}_s^j)\times\chi_R(\overline{a}_s^j)\right\}\chi_R(\overline{a}_s^i)-\nabla u^{\varepsilon}(s,\overline{X}_s^i)\chi_R(\overline{a}_s^i)\right]\right.\\ & \left.\cdot\left[\left\{\nabla K^{\varepsilon}(\overline{X}_s^i-\overline{X}_s^k)\times\chi_R(\overline{a}_s^k)\right\}\chi_R(\overline{a}_s^i)-\nabla u^{\varepsilon}(s,\overline{X}_s^i)\chi_R(\overline{a}_s^i)\right]\right\}. \end{aligned}$$

Also this expectation vanishes if $j \neq k$ and is bounded by $(2L\varepsilon^{-4}R^2)^2$ otherwise. Therefore

$$E\left[\left|\sum_{j=1}^{N}\left[\left\{\nabla K^{\varepsilon}(\overline{X}_s^i-\overline{X}_s^j)\times\chi_R(\overline{a}_s^j)\right\}\chi_R(\overline{a}_s^i)-\nabla u^{\varepsilon}(s,\overline{X}_s^i)\chi_R(\overline{a}_s^i)\right]\right|^2\right] = 4L\varepsilon^{-8}R^4N. \qquad (3.21)$$

By inserting (3.20) and (3.21) into (3.18) we obtain

$$\Psi(t) \le C_6T\varepsilon^{-10}R^4\int_0^t \Psi(s)ds + C\varepsilon^{-8}R^4t^2\frac{1}{N}$$

with a constant $C < \infty$ which only depends on φ. Now let

$$\tilde{\Psi}(t) := \Psi(t) + \frac{2C\varepsilon^2 t}{C_6TN} + \frac{2C\varepsilon^{12}}{C_6^2T^2R^4N}.$$

It follows that

$$\begin{aligned} \tilde{\Psi}(t) &\le C_6T\varepsilon^{-10}R^4\int_0^t \Psi(s)ds + C\varepsilon^{-8}R^4t^2\frac{1}{N} + \frac{2C\varepsilon^2 t}{C_6TN} + \frac{2C\varepsilon^{12}}{C_6^2T^2R^4N}\\ &= C_6T\varepsilon^{-10}R^4\int_0^t\left(\tilde{\Psi}(s) - \frac{2C\varepsilon^2 s}{C_6TN} - \frac{2C\varepsilon^{12}}{C_6^2T^2R^4N}\right)ds + C\varepsilon^{-8}R^4t^2\frac{1}{N} + \frac{2C\varepsilon^2 t}{C_6TN} + \frac{2C\varepsilon^{12}}{C_6^2T^2R^4N}\\ &= C_6T\varepsilon^{-10}R^4\int_0^t \tilde{\Psi}(s)ds + \frac{2C\varepsilon^{12}}{C_6^2T^2R^4N}. \end{aligned}$$

Gronwall's lemma now implies

$$\Psi(t) \le \tilde{\Psi}(t) \le \frac{2C\varepsilon^{12}}{C_6^2T^2R^4}\exp(C_6\varepsilon^{-10}R^4Tt)\frac{1}{N},$$

and the claim follows with $C_5 := 2CC_6^{-2}$. □

Lemma 3.8. *If $j \neq k$, then*

$$E\left\{\left[K^\varepsilon(\overline{X}_s^i - \overline{X}_s^j) \times \chi_R(\overline{a}_s^j) - u^\varepsilon(s, \overline{X}_s^i)\right] \cdot \left[K^\varepsilon(\overline{X}_s^i - \overline{X}_s^k) \times \chi_R(\overline{a}_s^k) - u^\varepsilon(s, \overline{X}_s^i)\right]\right\} = 0.$$

Proof. If $j \neq k$, then $j \neq i$ or $k \neq i$. If e.g. $k \neq i$, we take the conditional expectation with respect to the σ-field generated by $\overline{X}_s^i$, $\overline{X}_s^j$ and $\overline{a}_s^j$ and get

$$\begin{aligned} & E\left\{\left[K^\varepsilon(\overline{X}_s^i - \overline{X}_s^j) \times \chi_R(\overline{a}_s^j) - u^\varepsilon(s, \overline{X}_s^i)\right] \cdot \left[K^\varepsilon(\overline{X}_s^i - \overline{X}_s^k) \times \chi_R(\overline{a}_s^k) - u^\varepsilon(s, \overline{X}_s^i)\right]\right\} \\ = \; & E\left\{\left[K^\varepsilon(\overline{X}_s^i - \overline{X}_s^j) \times \chi_R(\overline{a}_s^j) - u^\varepsilon(s, \overline{X}_s^i)\right]\right. \\ & \left. \cdot\, E\left[K^\varepsilon(\overline{X}_s^i - \overline{X}_s^k) \times \chi_R(\overline{a}_s^k) - u^\varepsilon(s, \overline{X}_s^i) \,\middle|\, \sigma(\overline{X}_s^i, \overline{X}_s^j, \overline{a}_s^j)\right]\right\}. \end{aligned}$$

Using (3.14) and the fact that the couple $(\overline{X}_s^k, \overline{a}_s^k)$ is independent of the quadruple $(\overline{X}_s^i, \overline{a}_s^i, \overline{X}_s^j, \overline{a}_s^j)$ (because $k \neq i$ and $k \neq j$) we obtain that

$$E\left[K^\varepsilon(\overline{X}_s^i - \overline{X}_s^k) \times \chi_R(\overline{a}_s^k) \,\middle|\, \sigma(\overline{X}_s^i, \overline{X}_s^j, \overline{a}_s^j)\right] = u^\varepsilon(s, \overline{X}_s^i),$$

and the claim follows. □

3.5.4 Proof of Proposition 3.5

Proof of Proposition 3.5. Equation (3.6) implies that

$$\begin{aligned} w(t,x) - w^\varepsilon(t,x) &= \int_0^t \int_{\mathbb{R}^3} \{\mathcal{K}(w) \otimes w - \mathcal{K}^\varepsilon(w^\varepsilon) \otimes w^\varepsilon\}(s,y) \nabla G_{t-s}^\nu(x-y) dy ds \\ &\quad - \int_0^t \int_{\mathbb{R}^3} \{w \otimes \mathcal{K}(w) - w^\varepsilon \otimes \mathcal{K}^\varepsilon(w^\varepsilon)\}(s,y) \nabla G_{t-s}^\nu(x-y) dy ds. \end{aligned}$$

We now introduce (as in the proof of Proposition 3.3) the exponents $m := \frac{3p}{4p-3}$, $k := \frac{3p}{6-p}$ and $q := \frac{3p}{3-p}$. Because of $\frac{1}{m} + \frac{1}{k} - 1 = \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{k}$ we can apply Young's inequality (Lemma 3.9, Appendix) and Hölder's inequality. Young's inequality implies

$$\begin{aligned} \|w(t,\cdot) - w^\varepsilon(t,\cdot)\|_{L^p} &\leq \int_0^t \|\nabla G_{t-s}^\nu\|_{L^m} \{\|\mathcal{K}(w(s,\cdot)) \otimes w(s,\cdot) - \mathcal{K}^\varepsilon(w^\varepsilon(s,\cdot)) \otimes w^\varepsilon(s,\cdot)\|_{L^k} \\ &\quad + \|w(s,\cdot) \otimes \mathcal{K}(w(s,\cdot)) - w^\varepsilon(s,\cdot) \otimes \mathcal{K}^\varepsilon(w^\varepsilon(s,\cdot))\|_{L^k}\} ds. \quad (3.22) \end{aligned}$$

By a simple calculation,

$$\|\nabla G_{t-s}^\nu\|_{L^m} = \|\nabla G_1^1\|_{L^m} (\nu(t-s))^{\frac{3}{2m}-2}. \quad (3.23)$$

Since $m \in (1, \frac{3}{2})$ we have $\frac{3}{2m} - 2 \in (-1, -\frac{1}{2})$, in particular this exponent is strictly greater than -1. We now consider the first term in brackets in (3.22). By the triangle inequality we obtain:

$$\begin{aligned} & \|\mathcal{K}(w(s,\cdot)) \otimes w(s,\cdot) - \mathcal{K}^\varepsilon(w^\varepsilon(s,\cdot)) \otimes w^\varepsilon(s,\cdot)\|_{L^k} \\ \leq \; & \|\mathcal{K}(w(s,\cdot)) \otimes w(s,\cdot) - \mathcal{K}^\varepsilon(w(s,\cdot)) \otimes w(s,\cdot)\|_{L^k} \quad (3.24) \\ & + \|\mathcal{K}^\varepsilon(w(s,\cdot)) \otimes w(s,\cdot) - \mathcal{K}^\varepsilon(w^\varepsilon(s,\cdot)) \otimes w(s,\cdot)\|_{L^k} \quad (3.25) \\ & + \|\mathcal{K}^\varepsilon(w^\varepsilon(s,\cdot)) \otimes w(s,\cdot) - \mathcal{K}^\varepsilon(w^\varepsilon(s,\cdot)) \otimes w^\varepsilon(s,\cdot)\|_{L^k}. \quad (3.26) \end{aligned}$$

For (3.24) we obtain using Hölder's inequality and Lemmas 3.13 and 3.14 (Appendix):

$$\begin{aligned}\|\mathcal{K}(w(s,\cdot))\otimes w(s,\cdot)-\mathcal{K}^{\varepsilon}(w(s,\cdot))\otimes w(s,\cdot)\|_{L^k} &\leq \|w(s,\cdot)\|_{L^p}\|\mathcal{K}(w(s,\cdot))-\mathcal{K}^{\varepsilon}(w(s,\cdot)\|_{L^q}\\ &= \|w(s,\cdot)\|_{L^p}\|\mathcal{K}(w(s,\cdot))-\varphi^{\varepsilon}*\mathcal{K}(w(s,\cdot)\|_{L^q}\\ &\leq C(\varphi)\|w(s,\cdot)\|_{L^p}\|\nabla\mathcal{K}(w(s,\cdot))\|_{L^q}\varepsilon\\ &\leq C(\varphi)\widetilde{C}_{p,q}\|w(s,\cdot)\|_{L^p}\|\nabla w(s,\cdot)\|_{L^p}\varepsilon. \qquad (3.27)\end{aligned}$$

For (3.25) we obtain:

$$\begin{aligned}&\|\mathcal{K}^{\varepsilon}(w(s,\cdot))\otimes w(s,\cdot)-\mathcal{K}^{\varepsilon}(w^{\varepsilon}(s,\cdot))\otimes w(s,\cdot)\|_{L^k}\\ \leq\ &\|w(s,\cdot)\|_{L^p}\|\mathcal{K}^{\varepsilon}(w(s,\cdot)-w^{\varepsilon}(s,\cdot))\|_{L^p}\\ =\ &C_{p,q}\|w(s,\cdot)\|_{L^p}\|w(s,\cdot)-w^{\varepsilon}(s,\cdot)\|_{L^p}. \qquad (3.28)\end{aligned}$$

For (3.26) we obtain:

$$\begin{aligned}&\|\mathcal{K}^{\varepsilon}(w^{\varepsilon}(s,\cdot))\otimes w(s,\cdot)-\mathcal{K}^{\varepsilon}(w^{\varepsilon}(s,\cdot))\otimes w^{\varepsilon}(s,\cdot)\|_{L^k}\\ \leq\ &\|\mathcal{K}^{\varepsilon}(w^{\varepsilon}(s,\cdot))\|_{L^q}\,\|w(s,\cdot)-w^{\varepsilon}(s,\cdot)\|_{L^p}\\ =\ &C_{p,q}\|w^{\varepsilon}(s,\cdot))\|_{L^p}\,\|w(s,\cdot)-w^{\varepsilon}(s,\cdot)\|_{L^p}\,. \qquad (3.29)\end{aligned}$$

Since we can estimate the second term in the brackets in (3.22) in the same way, we obtain using (3.22), (3.23), (3.27), (3.28) and (3.29):

$$\begin{aligned}&\|w(t,\cdot)-w^{\varepsilon}(t,\cdot)\|_{L^p}\\ \leq\ &2C(\varphi)\widetilde{C}_{p,q}\|\nabla G_1^1\|_{L^m}\nu^{\frac{3}{2m}-2}\varepsilon\int_0^t(t-s)^{\frac{3}{2m}-2}\|w(s,\cdot)\|_{L^p}\|\nabla w(s,\cdot)\|_{L^p}ds\\ &+2C_{p,q}\|\nabla G_1^1\|_{L^m}\nu^{\frac{3}{2m}-2}\int_0^t(t-s)^{\frac{3}{2m}-2}\left\{\|w^{\varepsilon}(s,\cdot))\|_{L^p}+\|w(s,\cdot))\|_{L^p}\right\}\|w(s,\cdot)-w^{\varepsilon}(s,\cdot)\|_{L^p}\,ds.\end{aligned}$$

We now apply Proposition 3.1: (3.7) implies $\|w(s,\cdot)\|_{L^p}\leq C_1$ and $\|w^{\varepsilon}(s,\cdot)\|_{L^p}\leq C_1$, and (3.8) implies $\|\nabla w(s,\cdot)\|_{L^p}\leq C_2s^{-1/2}$. Therefore

$$\begin{aligned}\|w(t,\cdot)-w^{\varepsilon}(t,\cdot)\|_{L^p} &\leq 2C_1C_2C(\varphi)\widetilde{C}_{p,q}\|\nabla G_1^1\|_{L^m}\nu^{\frac{3}{2m}-2}\varepsilon\int_0^t(t-s)^{\frac{3}{2m}-2}s^{-1/2}ds\\ &\quad+4C_1C_{p,q}\|\nabla G_1^1\|_{L^m}\nu^{\frac{3}{2m}-2}\int_0^t(t-s)^{\frac{3}{2m}-2}\,\|w(s,\cdot)-w^{\varepsilon}(s,\cdot)\|_{L^p}\,ds\\ &=: C'\varepsilon+K\int_0^t(t-s)^{\frac{3}{2m}-2}\,\|w(s,\cdot)-w^{\varepsilon}(s,\cdot)\|_{L^p}\,ds.\end{aligned}$$

Since $\frac{3}{2m}-2>-1$ the generalized Gronwall lemma (Lemma 3.12, Appendix) now implies the existence of a constant $C_7=C_7(T^*,\|w_0\|_{L^p},\nu,\varphi)<\infty$ such that $\|w(t,\cdot)-w^{\varepsilon}(t,\cdot)\|_{L^p}\leq C_7\varepsilon$ for all $t\in[0,T^*]$. □

3.5.5 End of the proof of Theorem 3

Proof of Theorem 3. Fix $t\in[0,T^*]$ and $f\in H$. We recall the definition of the weighted empirical measure of the particle system:

$$\mu_t^{N,\varepsilon,R}:=\frac{1}{N}\sum_{i=1}^{N}a_t^{N,i,\varepsilon,R}\delta_{X_t^{N,i,\varepsilon,R}}.$$

In analogy to that definition we define

$$\overline{\mu}_t^{N,\varepsilon} := \frac{1}{N}\sum_{i=1}^N \overline{a}_t^{i,\varepsilon}\delta_{\overline{X}_t^{i,\varepsilon}}.$$

It follows that

$$\begin{aligned} E\left[|< \mu_t^{N,\varepsilon,R}, f> - < w(t,\cdot), f>|^2\right] &\leq 3E\left[|< \mu_t^{N,\varepsilon,R}, f> - < \overline{\mu}_t^{N,\varepsilon}, f>|^2\right] \\ &\quad + 3E\left[|< \overline{\mu}_t^{N,\varepsilon}, f> - < w^\varepsilon(t,\cdot), f>|^2\right] \\ &\quad + 3|< w^\varepsilon(t,\cdot), f> - < w(t,\cdot), f>|^2. \end{aligned}$$

For the first term we obtain using Proposition 3.4:

$$\begin{aligned} & E\left[|< \mu_t^{N,\varepsilon,R}, f> - < \overline{\mu}_t^{N,\varepsilon,R}, f>|^2\right] \\ = & E\left[\left|\frac{1}{N}\sum_{i=1}^N f(X_t^{N,i,\varepsilon,R})a_t^{N,i,\varepsilon,R} - \frac{1}{N}\sum_{i=1}^N f(\overline{X}_t^{i,\varepsilon})\overline{a}_t^{i,\varepsilon}\right|^2\right] \\ \leq & \frac{1}{N}\sum_{i=1}^N E\left[\left|f(X_t^{N,i,\varepsilon,R})a_t^{N,i,\varepsilon,R} - f(\overline{X}_t^{i,\varepsilon})\overline{a}_t^{i,\varepsilon}\right|^2\right] \\ \leq & \frac{2}{N}\sum_{i=1}^N E\left[\left|f(X_t^{N,i,\varepsilon,R})a_t^{N,i,\varepsilon,R} - f(X_t^{N,i,\varepsilon,R})\overline{a}_t^{i,\varepsilon}\right|^2\right] \\ & + \frac{2}{N}\sum_{i=1}^N E\left[\left|f(X_t^{N,i,\varepsilon,R})\overline{a}_t^{i,\varepsilon} - f(\overline{X}_t^{i,\varepsilon})\overline{a}_t^{i,\varepsilon}\right|^2\right] \\ \leq & \frac{2}{N}\|f\|_{L^\infty}\sum_{i=1}^N E\left[\left|a_t^{N,i,\varepsilon,R} - \overline{a}_t^{i,\varepsilon}\right|^2\right] + \frac{2}{N}R\sum_{i=1}^N E\left[\left|f(X_t^{N,i,\varepsilon,R}) - f(\overline{X}_t^{i,\varepsilon})\right|^2\right] \\ \leq & 2C_5\varepsilon^{12}R^{-3}T^{*-2}\exp(C_6\varepsilon^{-10}R^4T^{*2})\frac{1}{N}. \end{aligned}$$

For the second term we obtain using (3.13):

$$\begin{aligned} & E\left[|< \overline{\mu}_t^{N,\varepsilon}, f> - < w^\varepsilon(t,\cdot), f>|^2\right] = E\left[\left|\frac{1}{N}\sum_{i=1}^N f(\overline{X}_t^{i,\varepsilon})\overline{a}_t^{i,\varepsilon} - < w^\varepsilon(t,\cdot), f>\right|^2\right] \\ = & E\left[\frac{1}{N^2}\sum_{i,j=1}^N f(\overline{X}_t^{i,\varepsilon})\overline{a}_t^{i,\varepsilon} f(\overline{X}_t^{j,\varepsilon})\overline{a}_t^{j,\varepsilon} - \frac{2}{N}\sum_{i=1}^N f(\overline{X}_t^{i,\varepsilon})\overline{a}_t^{i,\varepsilon} < w^\varepsilon(t,\cdot), f> + < w^\varepsilon(t,\cdot), f>^2\right] \\ = & \frac{1}{N}E\left[\left|f(\overline{X}_t^{1,\varepsilon})\overline{a}_t^{1,\varepsilon}\right|^2\right] - \frac{1}{N} < w^\varepsilon(t,\cdot), f>^2 \quad \leq \quad \frac{R^2}{N}. \end{aligned}$$

For the third term we obtain using Hölder's inequality and Proposition 3.5:

$$|< w^\varepsilon(t,\cdot), f> - < w(t,\cdot), f>|^2 \quad \leq \quad \|f\|_{L^{p'}}\|w(t,\cdot) - w^\varepsilon(t,\cdot)\|_{L^p} \leq C_7\varepsilon.$$

Altogether we obtain

$$\begin{aligned} & E\left[|< \mu_t^{N,\varepsilon,R}, f> - < w(t,\cdot), f>|^2\right] \\ \leq & \left(6C_5\varepsilon^{12}R^{-3}T^{*-2}\exp(C_6\varepsilon^{-10}R^4T^{*2}) + 3R^2\right)\frac{1}{N} + 3C_7\varepsilon, \end{aligned}$$

and Theorem 3 follows. □

3.6 Appendix (Inequalities)

In this appendix we summarize the inequalities used in this paper:

Lemma 3.9 (Young's inequality)**.** *Let $m, k, p \geq 1$ and $1/p = 1/m + 1/k - 1$. Let $f \in L^m(\mathbb{R}^d)$ and $g \in L^k(\mathbb{R}^d)$. Then $f * g \in L^p(\mathbb{R}^d)$, and*

$$\|f * g\|_{L^p} \leq \|f\|_{L^m} \|g\|_{L^k}.$$

Proof. See e.g. [1]. □

Lemma 3.10 (Gronwall's lemma)**.** *Let $f, g : [0, T] \to \mathbb{R}_+$ with*

$$f(t) \leq C + \int_0^t f(s)g(s)ds \qquad \text{for all } t \in [0, T].$$

Then

$$f(t) \leq C \exp\left(\int_0^t g(s)ds\right) \qquad \text{for all } t \in [0, T].$$

Lemma 3.11. *Let $\alpha > -1$ and $f : [0, T] \to \mathbb{R}_+$ with*

$$f(t) \leq C + K \int_0^t (t-s)^\alpha f(s)ds \qquad \text{for all } t \in [0, T]. \tag{3.30}$$

Then

$$f(t) \leq C\left(1 + \frac{K}{\alpha+1} T^{\alpha+1}\right) + K^2 \overline{\alpha} \int_0^t (t-s)^{2\alpha+1} f(s)ds \qquad \text{for all } t \in [0, T],$$

where $\overline{\alpha} := \int_0^1 (1-s)^\alpha s^\alpha ds$. ($\overline{\alpha}$ is finite because of $\alpha > -1$.)

Proof. By applying (3.30) twice one obtains

$$\begin{aligned}
f(t) &\leq C + K \int_0^t (t-s)^\alpha \left[C + K \int_0^s (s-r)^\alpha f(r)dr\right] ds \\
&= C + CK \int_0^t (t-s)^\alpha ds + K^2 \int_0^t \int_0^s (t-s)^\alpha (s-r)^\alpha f(r)drds \\
&= C + CK \frac{t^{\alpha+1}}{\alpha+1} + K^2 \int_0^t \left[\int_r^t (t-s)^\alpha (s-r)^\alpha ds\right] f(r)dr \\
&\leq C + CK \frac{T^{\alpha+1}}{\alpha+1} + K^2 \int_0^1 (1-s)^\alpha s^\alpha ds \int_0^t (t-r)^{2\alpha+1} f(r)dr,
\end{aligned}$$

q.e.d. □

Lemma 3.12 (Generalized Gronwall lemma)**.** *Let $\alpha > -1$ and $f : [0, T] \to \mathbb{R}_+$ with*

$$f(t) \leq C + K \int_0^t (t-s)^\alpha f(s)ds \qquad \text{for all } t \in [0, T].$$

Then there is a constant $K' = K'(\alpha, T, K) < \infty$ which only depends on α, T and K such that

$$f(t) \leq CK'(\alpha, T, K) \qquad \text{for all } t \in [0, T].$$

In the special case $\alpha = -1/2$ we have:

$$f(t) \leq C\left(1 + 2KT^{1/2}\right) \exp(K^2 \beta t) \qquad \text{for all } t \in [0, T],$$

where $\beta := \int_0^1 s^{-1/2}(1-s)^{-1/2} ds < \infty$.

Proof. Apply Lemma 3.11 n times, where n is the smallest integer such that $n \geq \log_2(\frac{1}{1+\alpha})$. One then obtains:

$$f(t) \leq C\left(1 + \frac{K}{\alpha+1}T^{\alpha+1}\right)^n + K^{2^n}\overline{\alpha}^{2^n-1}\int_0^t (t-s)^{2^n\alpha+2^n-1}f(s)ds \qquad \text{for all } t \in [0,T].$$

Since by construction the exponent $2^n\alpha + 2^n - 1$ is greater than or equal to 0, the claim now follows from Lemma 3.10. If $\alpha = -1/2$ it suffices to apply Lemma 3.11 once. □

Lemma 3.13. *Let $f \in H^{1,q}(\mathbb{R}^d)$. Then*

$$\|\varphi^\varepsilon * f - f\|_{L^q} \leq C(\varphi)\|\nabla f\|_{L^q}\varepsilon,$$

where

$$C(\varphi) := \left[\text{vol}(\text{supp}(\varphi))^{q-1}\int_{\mathbb{R}^d}|y|^q\varphi(y)^q dy\right]^{1/q}.$$

Proof. Since smooth functions are dense in $H^{1,q}(\mathbb{R}^d)$, it suffices to prove the claim for $f \in \mathcal{C}^1(\mathbb{R}^d) \cap H^{1,q}(\mathbb{R}^d)$:

$$\begin{aligned}
\|\varphi^\varepsilon * f - f\|^q_{L^q} &= \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}[f(x-y) - f(x)]\,\frac{1}{\varepsilon^3}\varphi(y/\varepsilon)dy\right|^q dx \\
&= \int_{\mathbb{R}^d}\left|\int_{\text{supp}(\varphi)}\left[f(x-\varepsilon y') - f(x)\right]\varphi(y')dy'\right|^q dx \\
&\leq \text{vol}(\text{supp}(\varphi))^{q-1}\int_{\mathbb{R}^d}\int_{\text{supp}(\varphi)}|f(x-\varepsilon y) - f(x)|^q\,\varphi(y)^q dy dx \\
&= \text{vol}(\text{supp}(\varphi))^{q-1}\int_{\text{supp}(\varphi)}\left[\int_{\mathbb{R}^d}|f(x-\varepsilon y) - f(x)|^q\,dx\right]\varphi(y)^q dy.
\end{aligned}$$

Moreover

$$\begin{aligned}
\int_{\mathbb{R}^d}|f(x-\varepsilon y) - f(x)|^q\,dx &= \int_{\mathbb{R}^d}\left|\int_0^1 \nabla f(x - s\varepsilon y)\cdot\varepsilon y ds\right|^q dx \\
&\leq \varepsilon^q|y|^q\int_0^1\int_{\mathbb{R}^d}|\nabla f(x-s\varepsilon y)|^q\,dxds \\
&= \varepsilon^q|y|^q\|\nabla f\|^q_{L^q}.
\end{aligned}$$

Therefore

$$\|\varphi^\varepsilon * f - f\|^q_{L^q} \leq \left[\text{vol}(\text{supp}(\varphi))^{q-1}\int_{\mathbb{R}^d}|y|^q\varphi(y)^q dy\right]\|\nabla f\|^q_{L^q}\varepsilon^q,$$

q.e.d. □

Lemma 3.14. *Let $p \in (1,3)$ and $q := \frac{3p}{3-p}$. For each $\varepsilon \geq 0$ $\mathcal{K}^\varepsilon$ is a continuous linear operator from $L^p(\mathbb{R}^3,\mathbb{R}^3)$ to $L^q(\mathbb{R}^3,\mathbb{R}^3)$ as well as from $H^{1,p}(\mathbb{R}^3,\mathbb{R}^3)$ to $H^{1,q}(\mathbb{R}^3,\mathbb{R}^3)$. Moreover there are constants $C_{p,q} < \infty$ and $\widetilde{C}_{p,q} < \infty$ such that for all $\varepsilon \geq 0$ and all $w \in L^p(\mathbb{R}^3,\mathbb{R}^3)$ resp. all $w \in H^{1,p}(\mathbb{R}^3,\mathbb{R}^3)$:*

1. $\|K^\varepsilon(w)\|_{L^q} \leq C_{p,q}\|w\|_{L^p}$.
2. $\|\nabla K^\varepsilon(w)\|_{L^q} \leq \widetilde{C}_{p,q}\|\nabla w\|_{L^p}$.

Proof. See [11], Lemma 2.2 and Remark 4.3. □

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