EVOLUTION IN STRUCTURED POPULATIONS

Inaugural–Dissertation
zur Erlangung des Grades eines Doktors
der Wirtschafts– und Gesellschaftswissenschaften
durch die
Rechts– und Staatswissenschaftliche Fakultät
der Rheinischen Friedrich–Wilhelms–Universität Bonn

vorgelegt von
LARS PETER KOCH
aus Marburg an der Lahn

Bonn 2008

Dekan: Prof. Dr. Erik Theissen
Erstreferent: Prof. Dr. Frank Riedel
Zweitreferent: Prof. Tymon Tatur, Ph.D.

Tag der mündlichen Prüfung: 20. Juni 2008 to my parents

ACKNOWLEDGEMENTS

At first place, I thank Frank Riedel. He spent unmeasurable many efforts discussing my proposals and scrutinizing my reasons. Whenever I asked him for time, he donated some of this precious good. Whenever I left his room, I went with some of his inspiration. I am also indebted to many of his colleagues in Bonn. Paul Heidhues and Tymon Tatur contributed immensely by disassembling and rearranging my models from first line to last line. Without Paul Heidhues' support, this thesis would never have come into being as it is. I owe parts of the characterizition of key components in my third chapter to Tymon Tatur. I am grateful to Martin Hellwig, Urs Schweizer, Benny Moldovanu, Thomas Gall and Tymofiy Mylovanov for constructive questions in seminar presentations at Bonn University, hitting the crucial points of my papers.

I am grateful for helpful comments, stimulating remarks and inspiring suggestions at seminar presentations at the UPF (Barcelona), at CORE (Lovain) the NASM 07 at Duke and the ESEM 07 in Budapest. I am indebted to Fabrizio Germano, Andreu Mas-Colell, Bill Sandholm, Ross Cressman, Dieter Balkenborg, Karl Schlag and Burkhard C. Schipper. Part of this work was written at Universitat Pompeu Fabra, Barcelona, I express my gratitude for the hospitality received there.

I thank my parents for their dedication, unrestricted support and understanding for the plans and projects I pursue.

Michał Paluch, Marc Schiffbauer, Michael Zähringer, Benjamin Huck, Matthias Wibral, Steffen Altmann, Johannes Abeler, An Chen, Xia Su, Simon Junker, Oliver Fries, Sotirios 'wir sollten' Georganas and Jürgen Gaul were the best 'year 2003 graduate class' I could have possibly been part of. They made me enjoy this time! I thank Christina Matzke for sustaining the chaos I produced in our office and for giving the right clues at the right time.

The most I owe to Marion. She gives sense to all I do.

Contents

Introduction

This thesis divides into four parts. The first chapter introduces my part of contribution to economic theory. I describe and motivate the questions that this branch in general aims to give answers to and those questions that I ask in the chapters that follow. I present the methodology I apply and the results I obtain. I close the first part by relating my contributions to the literature. The remaining three chapters contain my contributions that are all selfcontained and independent, but connected by the same evolutionary perspective.

I.1 Game Theory

The subject of this thesis is Evolutionary Game Theory, which is a section of Game Theory. To give a brief introduction to the theory of games I describe the methodology and motivate the questions to which a response can be revealed by game theory.

A *game* is a setting in which several agents strategically interact. An agent represents an economic or social entity that takes decisions, balances reasons for distinct alternatives, or calculates the risk of accepting or rejecting an offer. An entity can differ with respect to the abstraction of the model under consideration: in one setting, the agent can represent a firm, which competes with other firms – the agent then seeks to maximize the profits of the firm given the output-, quality- or price-choices of the other firms. In a different setting, the agent may represent a firm's manager has to decide on the level of efforts to exert, given a known reward scheme and the choices of his colleagues. In a completely different setting, an agent can represent an individual who attaches a value to the completion of a public project – as for example a bridge. The agent has to determine the amount of money she is willing to pay for that bridge, given that she knows other individuals who also would value the bridge. For any setting, the game theorist specifies the relevant and plausible level of aggregation and defines the set of decision taking entities: the *players*. In terms of game theory, the alternative options among the player can choose is called the set of *strategies*. Several players strategically interact, if their choices have joint consequences: when evaluating, which strategy to choose, a player has to consider the strategies chosen by other players, because they also are relevant for the consequence. A complete description of a game in *normal-* or *strategic* form consists of the set of players, the set of their strategies and a representation of the players' preferences over the set of consequences. It is a common assumption, that the preferences satisfy three standard axioms[1] and hereby can be represented by a utility function that assigns a real number to each possible tupel of strategies for each player. The preferences, and therefore the utility functions of the players derive from some plausible economic or social interpretation of the situation that is to be modelled. However, game theoretic methodology does not require that these preferences relfect or neglect particular emotions or ways of thinking, as selfishness or altruism, spite or goodwill, as long as the preferences satisfy the three beforementioned axioms. However, *given* the preferences, the concept of game theory requires different degrees of rationality of the agents. Before describing the methodology of game theory and the questions that game theory can respond to, I mention games in *extensive form.* If a player knows the strategic decision of some other players before she takes her own decision, the game exhibits a *sequential structure.* Games with and without sequential structure can yield strongly diverse predictions. My thesis is focussed on simultaneous decisions.

Suppose agents choose the optimal strategy given their information on the strategy choice of all other players. A game theorist calls such an agent *rational.* A further assumption is that each agent knows that all other agents are rational, and

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[1] The three axioms are *Completeness* ($x \succeq y \vee y \succeq x \ \forall x, y \in X$), *Reflexivity* ($x \succeq x \ \forall x \in X$) and *Transitivity* ($x \succeq y, y \succeq z \Rightarrow x \succeq z \ \forall x, y, z \in X$).

that each other agent knows that all other agents know that the other agents are rational – and so on. This informational structure is referred to as *common knowledge of rationality* (Aumann (1976)). Given a game in normal form, game theory asks “given that each player chooses a particular strategy, does any player have an incentive to deviate from her decision?” A tupel of strategies in which each player is confident with the selected decision is called *Nash-equilibrium.* This is a situation in which a game theorist expects that the economic entities maintain the selected alternatives. If agents have access to randomized strategies, a Nash equilibrium exists for any finite game in normal form.[2] Moreover, a question of interest concerns stability: at a given equilibrium, do the players support this equilibrium, if they expect other players to make an error with small probability? Do players persist to the equilibrium, if the expected consequences are slightly different? Implicetely, to solve these problems, even more rationality is imposed on the players. It is assumed that players sophisticatedly form their expectations (on the strategical choice of the co-players) on the resulting consequences. These and other questions led to a variety of *equilibrium refinements*,[3] each imposing assumptions upon the rationality of the players of different quality. Evo-

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[2] ‘Finite’ refers to the number of (pure, or deterministic) strategies available to each player.

[3] An excellent compendium is van Damme (1991).

lutionary Game Theory, which is introduced in the following section, adresses whether certain situations are stable. However, unlike in the literature on equilibrium refinement, the requirements on the rationality of the agents are drastically *reduced*. The contribution of this approach results in: *Even if agents are extremely boundedly rational, some outcomes that would be predicted by models of rational agents display stability properties.*

The next section defines what is meant by *extreme bounded rationality*.

I.2 Evolutionary Game Theory

Methods of game theory have also been applied to promote the analysis of animal conflicts. An intensively discussed setting is that of the Hawk-Dove-Game. In a population of two species, a large number individuals compete for a scarce resource: breeding area. One species, the dove, acts cooperatively and shares the available area with other entities of the population. The other species, the hawk, is aggressive and fights for its territory. This situation is modelled as follows: at each point in time two exemplars of the population are selected at random. The probability of one exemplar being a member of a particular species is proportional to the number of its members in the population. Each of the two exemplars are drawn independently. The pair can either consist of two doves, two hawks or one dove and one hawk. If two doves meet, they share resources and each individual enjoys half of its value. If two hawks encounter, they fight and subsequently one of them conquere the full resource but has to bear costs in terms of energy and time. If a hawk and a dove meet, the dove immediately withdraws and the hawk enjoys the full resource without bearing costs. The value of the acquired resources balanced by the cost of fighting is employed to produce offsprings. In this model, the type of the species is referred to as a strategy and the consequence is identified with a number that represents the number of produced offsprings. This number is also called *fitness*. In the biological context, each individual is a member of a particular species that fully prescribes the behavior of that individual. The individual does not "choose" a particular strategy but behaves exactly as its parents and it produces offsprings that behave in exactly the same manner as the parents. A species that is successful, has a high fitness (produces a high number of offspring) and the subpopulation of the members of that species grows. For game theory, relative statements are of interest. Therefore the model captures only relative sizes: a species that is *relatively* successful shows a relative high fitness and the representation of that species grows within the entire population. Methodoligically, the growth of each species is captured by a differential equation. If the number of offspring is proportional to the fitness, the relative growth rate of a particular species is linear and equals the difference of its fitness and the average fitness of the population. In this case, the differential equation is termed *replicator equation.* Given this apparatus, game theorists analyze rest points of this dynamic and their stability properties (see section I.4 for a discussion). To provide a conclusion, all Nash-equilibria are rest points of the replicator dynamic and all interior rest-points of the replicator dynamic are Nash-equilibria. Evolutionary game theory can hereby contribute to economics: even if interacting agents are extremely boundedly rational, an outcome that would remain unchanged if agents were rational, remains unchanged. Proceeding to stability issues, the analogy continues: In asymmetric settings, an outcome that is stable with respect to the replicator dynamics is also stable for rational agents in the sense that a vast variety of equilibrium refinements would predict that outcome.

How does a concept of animal behavior in which 'choices' or 'strategic interaction' are omitted, provide a plausible foundation to analyze economic or social problems? Dawkins (1990, chap. 11) illuminates the parallels between genetic and cultural transmission. To illustrate this parallel, he refers to *language*: "Language seems to 'evolve' by non-genetic means, and at a rate which is orders of magnitudes faster than genetic evolution." (p.189) Obviously, language is not inherited. Children learn a language from their parents by imitation. Dawkins calls the cultural replicator, the 'unit of cultural transmission' a *meme*, that underlies the same logic of spreading within a population as a gene.

> "Examples of memes are tunes, ideas, catch-

> phrases, clothes fashions, ways of making pots or of building arches. Just as genes propagate themselves in the gene pool by leaping from body to body via sperms or eggs, so memes propagate themselves in the meme pool by leaping from brain to brain via a process which, in the broad sense, can be called imitation." (Dawkins (1990, chap. 11, p. 192))

Of course, biological evolution cannot provide a suitable explanation for the change of human behavior, simply because the genome essentially remains unmodified within one lifespan. But, in concert with the abovementioned interpretation, the mechanics of evolution is helpful tp develop models in which agents learn by imitation (see section I.4).

A concept, that is common among almost all papers on evolution in game theoryis described by Selten (1991) (p. 21) as follows:

> "We have identified a hierarchy of dynamic processes which shape economic behavior. I name these processes in the order of increasing speed:
>
> 1. (the slowest process) gene substition by mutation,

> 2. adaptation of genotype frequencies without mutation,
> 3. cultural transmission from generation to generation,
> 4. learning (including imitation).
>
> The speed differences are so great that for many purposes an adiabatic approximation seems to be justified. Adiabatic approximation means that if we look at one of the four processes, results of lower processes can be taken as fixed and quicker processes can be assumed to reach equilibrium instantly."

In this thesis, two main types of processes are distinguished that roughly correspond to numbers 1. and 4. in the list above. One type of process, models the selection of strategic choice and is based on the payoffs of the game. A process of this type includes the classes of best- or better-reply dynamics as well as imitation dynamics and others. All processes of this type are subsumed as *selection processes* (see next section). The second process type models 'mistakes' or 'errors' – in economic language (or 'mutations' with a biological interpretation) – and is assumed to be completely independent of the underlying payoffs. The first type of processes incorporates all factors that affect strategic choice, processes of the second type are defined by residual factors that are exogenous to the model. I regard the second type to be slower than the selection process and analyse the extreme case in which it is infinitely slower. I study dynamical systems that – depending on the model under consideration – are defined on a continuous or discrete state space with continuous or discrete time.

I.3 Literature on Stochastic Stability

Freidlin and Wentzell (1998)[4] developed methods that have been applied in dynamic game theoretic models of imitation, best response-behavior and local interaction. They specify a Markov chain that is disturbed by a noise term and derive results for the limit case when the noise vanishes. Although Freidlin and Wentzell (1998) analyse stochastic differential equations defined on continuous phase spaces, they specify a discrete concept of *resistance minimizing graphs* which are defined on a finite index set, each index representing one absorbing subset of the continuous phase space. Hereby, their method can conveniently be applied to models with discrete state spaces. Foster and Young (1990) who introduced these methods into game theory. Their definition of stochastic

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[4] The seminal work originally cited in the literature is the book published in 1984 of which I cite the second edition. I gained also very much from Karlin and Taylor (1975) as an excellent reference for stochastic processes.

stability is widely accepted: A state (which can represent a tupel of strategies or a distribution of strategies within a population) is *stochastically stable*, if it receives positive probability under the limiting invariant distribution. The limiting invariant distribution is the long run distribution of the perturbed Markov process as the perturbation vanishes to zero. The limiting invariant distribution has two interpretations: i) as time goes to infinity, the density on a state is proportional to the share of the time the process spends in the state, and ii) if one stops the process in the far future, the limiting invariant distribution specifies the probability with which the process is in a particular state. The contribution of Foster and Young (1990) to economics is that the number of invariant distributions is reduced to one, if one considers perturbed processes and that even arbitrary small stochastic effects may qualitatively change the asymptotic behavior of the system. This methodology was taken up by Kandori et al. (1993), Young (1993) and Ellison (1993). Kandori et al. (1993) and Young (1993) study best-response dynamics and show that in 2×2-coordination games, only the risk dominant Nash-equilibrium (Harsanyi and Selten (1988)) is stochastically stable. The underlying criteria of stochastic stability can be critizised for various reasons. One criterium is that the process must be active for a very long time. If evolution is slow, the implications of stochastic stability may be uninteresting for settings in which agents interact, who have a finite lifespan. From this point of departure, Ellison (1993) focuses on the speed of convergence of such Markov processes and shows that evolution is fast (and therefore relevant for economic and social applications), if interaction is local. Another point of criticism is the model of mutations, that defines the perturbed process. Bergin and Lipman (1996) show that if the model of mutations is not chosen appropriately, *any* state can be implemented to be stochastically stable. They identify a necessary condition for a non-arbitrary selection as the following: for any two states, the ratio of the probabilities of a mutation, mistake or error, must have an upper bound as the perturbation goes to zero. This means that the probability of a mutation given a particular state must be of the same order of magnitude for all states. The fact that events with very low probability, in fact arbitrary improbable events, play a crucial role for determining stochastic stability is the point of critique implicetly put forward in Binmore and Samuelson (1994) or Samuelson (1997). Suppose that the unperturbed process has two absorbing sets A and B. If the perturbed process transits from B to A, if $n \in \mathbb{N}$ mutations occur and from A to B, if $n+1$ mutations occur, the set A is stochastically stable. In the limit, as the probability of a mutation goes to zero, the probability of the transit from A to B is infinitely smaller than the probability of the reverse transit. It may be difficult to provide a plausible economic explanation why A is stable and B is not, if n is a large number. Binmore and Samuelson (1994) propose to consider *recurrent* sets that consist of a collection of absorbing sets of the unperturbed process that have overlapping ‘one mutation neighborhoods’. A one mutation neighborhood of a state x is the subset of all states that can be reached from x if one mutation occurs. Binmore and Samuelson (1994) coin the term *drift* as the movement between two different absorbing sets that are adjacent in the sense that one mutation can cause the process to transit from one absorbing set to the other.

I.4 Literature on Evolutionary Selection Dynamics

A selection dynamic specifies which kind of behavior or strategy-choice spreads in a population of many agents. Selection dynamics differ with respect to various criteria. One important distinction is

Individual versus Group Selection. The representative model of group selection is the hay stack model (Maynard Smith (1964)). Translated to economic terms, the hay stack model is a setting in which economic agents are arranged in groups. Interaction is possible only within a group, there is no inter-group activity. For a long period of time, selection of relative successful behavior takes place in each group. The avarage success of each group determines its weight in the total population. Once in a while a randomly chosen group creates an exact copy, where the each group is chosen with probability equal to its weight in the total population.[5] In his comparison of models of individual- and group selection, Bergstrom (2002) points out the importance of non-assortative matching in prisoners' dilemma games. Matching is non-assortative, if the probability to be matched with an opponent of a particular type is independent of one's own type. Bergstrom (2002) provides an extensive list of models of assortative matching, I refer to a particular one that is related to my thesis: Eshel and Shaked (1999) model an infinite population whose agents interact with and imitate their neighbors. It is the local interaction structure that induces assortative matching; two neighbors are likely to use the same strategy. Eshel and Shaked (1999) show that in this model agents display *as if*-kinship behavior. Kinship behaviour is described as behavior among relatives who care for the payoffs of their siblings, 'inclusive fitness' according to Hamilton (1964). A particular gene of an individual is likely to be present in a close relative – hence the gene is more likely to survive if it induces behavior of the indivudual that partially internalizes the fitness of its closes relatives. By means of a local interaction structure, Eshel and Shaked (1999) offer a socio-economic explanation of the primary biological phenomenon of kinship behavior.

The results implied by selection dynamics also crucially depend on **finiteness or infiniteness** of key components of the model. Taylor et al. (2004) provide an example (Exam-

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[5] The original model is about mice that reproduce in hay stacks during summer and regroup after each winter.

ple 1) in which selection favors a dominated strategy in small populations, although their dynamic depends (positively) on individual payoffs. The intuition for this result is that if the population is small, the weight of an individuum cannot be neglected as in infinite populations, and as Schaffer (1980) finds: if the population size is small, 'spite' can be a driving determinant of most successful behavior. His finite version of evolutionary stable strategies (ESS, Maynard Smith (1964)) can apply for dominated strategies in small populations. As the population size gets large, Taylor et al. (2004)s' and Schaffer (1980)'s results coincide with those of models with infinite populations. I present the results on stability for large populations in the section on evolutionary stability below. Oechssler and Riedel (2002) show that if the set of strategies is continuous, an ESS may not be asymptotically stable for the replicator dynamic in doubly symmetric games, opposing results of models with a finite set of strategies (Taylor and Jonker (1978)). They propose the stronger concept of *evolutionary robustness*, if asymptotic stability is a desired property. I apply Oechssler and Riedel (2001)'s existence result[6] and their variational norm in chapter 3.

Selection dynamics differ with respect to the rationality imposed on the agents. In any dynamic system considered

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[6] Oechssler and Riedel (2001) provide conditions for the existence of a solution of a differential equation defined on continuous strategy spaces. This result is analogous to the Picard-Lindelöf-Theorem for finite dimensional state spaces.

here, agents are shortsighted. Among those myopic agents, **optimizers** are the most rational ones and **imitators** require less rationality. The most extreme class are *evolutionary* selection dynamics, in which agents must ‘choose’ the inherited strategy. The most famous representative of this class is the *replicator dynamic.* Taylor and Jonker (1978) define the replicator dynamic such that the growth rate of a strategies equals the difference of its payoff and the population average. Dawkins (1990) provides convincing social and economic interpretations of the term replicator. Björnerstedt and Weibull (1996) and also Schlag (1998) show that a model of imitation can induce the replicator dynamic. Hofbauer and Sigmund (1988) relate the Lotka-Volterra equation and its mathematical insights to the symmetric replicator dynamics. In particular, they show that if each strategy receives a positive mass under the symmetric replicator dynamic at all times, there exists a unique interior restpoint. Furthermore, they derive conditions on the parameters of the underlying game to characterize the volume of the image of the solutions of the replicator dynamics. This can be seen as the key argument for the results on asymptotic stability that I discuss in the next subsection. The literature on optimizing agents is vast and since they do not play a central role in my thesis, I mention only very few models: in Kandori et al. (1993), Hofbauer and Sandholm (2002), Fudenberg and Levine (1995) or Hofbauer and Sandholm (2002) agents choose best replies to their beliefs. In Berger and Hofbauer (2005) or Hofbauer et al. (2005) agents choose better replies. An important difference to imitation dynamics is that best- or better-reply-dynamics are *innovative* in the sense that strategies that are absent in the population can well receive positive probability at a particular time in the future. In contrast, for imitation dynamics all homogenous states are steady. In my thesis I apply general imitation dynamics that include the replicator as a special case. Generally, an imitation dynamic must satisfy a continuity condition, it must maintain constant measure and the growth rates must be finite for any strategy at any state. As a consequence, an imitation dynamic has a unique and continuous solution for any initial condition. Furthermore, a strategy is present at all times if and only if it is present at time zero. If for any two strategies the one that yields higher payoffs has the higher growth rate, the dynamics are (payoff-)monotonic (Samuelson and Zhang (1992)). Another class of selection processes are called convex monotonic dynamics (Hofbauer and Weibull (1996)). Both convex– or payoff-monotonicity is implied by aggregate monotonicity (Samuelson and Zhang (1992)).

In the following section I summarize results on stability for imitation dynamics.

I.5 Literature on Evolutionary and Dynamic Stability

In this section I consider only games with finitely many strategies. Maynard Smith (1974) or Maynard Smith and Price (1973) define the static concept of evolutionary stable strategies in symmetric contexts. A strategy is evolutionary stable, if agents applying that strategy in the presence of a small fraction of (homogeneous) mutants fare better than the mutants. Equivalently, a tupel of strategies is evolutionary stable, if i) it is a Nash equilibrium and if ii) it fares better against alternative best replies than the alternative best reply against itself. Selten (1980) considers asymmetric context and assumes that each agent is randomly assign to a player-role before interaction takes place. He shows that an ESS in this symmetrized game must consist of the choice of a strict Nash equilibrium strategy in the asymmetric game. Swinkels (1992) provides a definition for an ESS in asymmetric games with the same underlying intuition as in the ESS for symmetric games and shows that in asymmetric games an ESS is equivalent to a strict Nash equilibrium. Taylor and Jonker (1978) show that for symmetric games a strategy is an ESS if and only if it is asymptotically stable in the replicator dynamics. If the ESS is in mixed strategies, stability is an artefact of the single-population model applied for symmetric games. Ritzberger and Weibull (1995) establish – analogous to Selten (1980)'s or Swinkels (1992)'s results that in multi-population models a tupel of strategies is asymptotically stable in the n-replicator dynamic if and only if it is a strict Nash equilibrium. It turns out that strict Nash equilibria are the only asymptotically stable states for general evolutionary selection dynamics. I discuss analogous results for the stability of sets in the next subsection.

Setwise Stability

In the previous subsection, I reviewed the stability properties of strict Nash equilibrium. Balkenborg and Schlag (2007) generalizes the definition of a strict Nash equilibrium to a setwise concept: *strict equilibrium set*, SEset. A set F is a SEset if it contains only Nash equilibria and if it is closed with respect to mixed strategy best replies. Balkenborg and Schlag (2007) and Cressman (2003) show asymptotic stability of SEsets for general evolutionary selection dynamics. If only two populations are considered, they prove the according if and only if result. Thomas (1985) and Balkenborg and Schlag (2001) define more demanding Evolutionary stable sets.

The following three chapters contain my essays on evolutionary game theory. Deviating from the models presented above, I analyze situations in which the population of agents exhibits a particular structure. In chapters 1 and 2, the structure is given by non-uniform matching of the agents. In ‘Aniticipated Stability in Social and Economic Networks’, agents actively choose with whom to match. I model link formation with agents that have constrained capabilities to foresee the consequences of their decisions. I show that under these circumstances a permanent change of matching decisions can be stable. I use the methods of Freidlin and Wentzell (1998) to characterize stochastic stability of random graphs.

In my second contribution, the deviation of the uniform-matching is more subtle. Agents receive signals and choose a strategy conditionally. As Mailath et al. (1997) remark, a signal generating process can have the interpretation of a matching technology. One section is devoted to emphasize this matter. The case of uniform matching is equivalent to independent signals. I show that behavior can be stable with respect to imitation dynamics even if strategy choices lead to non-Nash outcomes due to correlated signals. I show for generic games that only strict Nash equilibria are robust against manipulation of signals. I apply methods and results of Hofbauer and Sigmund (1988), Ritzberger and Weibull (1995) and Balkenborg and Schlag (2007).

In chapter 3, I consider agents that hold non-verifiable information on the evolutionary relevant payoffs of the game. The structure of the population refers to the distribution of this information. I show that for any stable situation in 2× 2 games there can exist a positive mass of agents that optimize given wrong information who are *not* selected against by evolutionary payoff monotonic selection dynamics. I base my model partially on Sandholm (2001).

Chapter 1

Anticipated Stability in Social and Economic Networks

Jackson and Wolinsky (1996) introduce the concept of *pairwise stability* for graphs that represent social and economic networks, Jackson and Watts (2002) provide a dynamic model of myopic network formation that incorporates the static concept of pairwise stability. Dutta et al. (2005) provide a model of farsighted agents: players know the actions of their coplayers for any possible network, are able to foresee the complete future of play and act optimally accordingly. I introduce agents that are neither myopic nor completely farsighted. The agents establish an *anticipation* of how the own decision influences

the behavior of the other players when choosing whether to form, maintain or sever a link, for a subset of all networks. An anticipation of player i given network g is a probability measure over the possible paths that leave g, but stay in the subset of networks. Agents optimize given their anticipation. I provide examples of small cycles à la Jackson and Watts (2002) that include a network that is stable in anticipation. The main contribution of this chapter is hereby to pinpoint that a cycle is an artificial concept resulting from the assumption of myopia.

1.1 Introduction

When modelling social or economic interactions, some economists focus on the issue of interaction patterns, in other words, on the structure of agents interacting with each other. If there is no such structure, groups of agents meet randomly, as it is the case in simple evolutionary models. In a basic structure agents are located on a line and interact with their neighbors. Given a specific model, for any agent in the population, a structure defines a subpopulation of agents with whom interaction is possible. Ellison (1993) or Eshel and Shaked (1999) analyze the impact of various interaction structures on the outcome of play. However, they assume that agents cannot influence the interaction structure, that is, agents lack the ability of choosing whom to interact with. Aumann and Myerson (1988) are among the first authors studying a model of link formation. Starting in the empty network, pair by pair the players involved are asked whether they would want to form a link. Such a finite sequence of pairs is called a *rule of order*. The authors show that for each rule of order there exists a subgame perfect equilibrium. They raise the point of forward looking reasoning, from the perspective of a player considering forming a link with another one: "*Suppose we form this new link, will other players be motivated to form further new links that were not worthwile for them before?*"[1] I analyse games in which the rule of order is neither finite nor deterministic. I model the structure of beliefs explicitely. This paper is close to Jackson and Wolinsky (1996), who define the structure of interaction as a network. Any node in such a network represents an agent, only agents that are linked are able to interact. Interaction is voluntary, which means that no subgroup of agents can be linked if not all members agree to interact. Jackson and Wolinsky (1996) consider coalitions of size two and introduce the concept of pairwise stability. They find that for any anonymous and component balanced allocation rule, there is an open set of value functions such that no strongly efficient graph is pairwise stable. Their framework is static; it can characterise different types of networks as stable but

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[1] Aumann and Myerson (1988), p.178

it cannot explain how these types of networks emerge.

Jackson and Watts (2002) construct a dynamic model of network formation that builds upon the concept of pairwise stability. At each point in time, nature draws a pair of agents. Those two agents agree to form or maintain a link if and only if both are better off. If the network does not change, no matter which link is under revision, it is pairwise stable. In this dynamic model, agents are myopic. When creating or cutting a link, agents do not take into account that this decision can trigger a sequence of link creations or destructions. In other words, they optimise under ceteris paribus assumptions. These assumptions can make sense, if the agents face a very complex world. The number of potential links within a network increases 'very fast'[2] with the number of nodes. If the model consists of many agents, one could argue that if their capability of computing optimal decisions is limited, the best they can do is to assume that everyone else will continue as they did before. This myopia assumption can lead to cycles. A cycle is a finite sequence of networks, such that if the last network is reached, the process starts again at the first network.

Watts (2001) and Dutta et al. (2005) assume the other extreme of rationality: in equilibrium, every agent knows what every other agent will do, for any possible network. Given some strategy-tupel, agents expect a sequence of networks

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[2] of order $\mathcal{O}(2^{n(n-1)/2})$

to evolve and optimise, discounting future payoffs along this expected sequence. This approach is intuitive if the number of agents within the model is small, that is: small enough such that it is plausible that each agent knows about the strategic plans of all other agents, at least in equilibrium.

In this chapter, I introduce a dynamic model that has a flexible boundary for the rationality of the agents. First, I define the distance between any two networks. I assume, that agents can optimise their expected utility along discounted sequences of networks, if these sequences consist only of networks that are closer than some boundary κ. In settings with many agents, this boundary can be set rather small, in models with very few agents, κ can be set rather large. Further, I analise stochastic properties of the model, where I apply the methods of Freidlin and Wentzell (1998) on a continous state space.

This paper is organised as follows: in section 2, I construct the model and define anticipated stability. In section 3 I define stochastically stable states and show that if a state is stochastically stable, it must be stable in anticipation. In section 4, I illustrate some properties of this concept by three examples. In particular I show that a small myopic cycle may include a network that is stable in anticipation. In section 5, I conclude.

1.2 Model

1.2.1 Definitions

I closely follow Jackson and Wolinsky (1996). Let there be $2 < n < \infty$ players and each player i be identified with a node of some undirected graph g. A graph is undirected, if a link from node i to node j is the same as a link from j to i. The graph consists of a set of nodes, $\mathcal{I} = \{1, ..., n\}$ and a set of edges $\mathcal{E}$. An edge or link between players i and j is denoted by ij. A node i cannot be linked to itself, ie $ii \notin g \; \forall \; i \; \in \mathcal{I}$. g is *complete*, if each player i is linked to each other player j, $\mathcal{E}$ is the set of all possible edges $\mathcal{E}^c$. I denote the complete graph by g^c. For all other graphs g, $\mathcal{E}$ is an element of the powerset of $\mathcal{E}^c$, $\mathcal{P}(\mathcal{E}^c) = \{\mathcal{E} | \mathcal{E} \subseteq \mathcal{E}^c\}$. g is *empty*, if no player i is linked to some other player j and $\mathcal{E} = \emptyset$. I denote the empty graph by g^e. I denote the set of undirected graphs with n nodes by $\mathcal{G}$. Since I do not vary the number of players n, I often will identify a graph g directly with its set of edges $\mathcal{E}$.

A *permutation* is a bijection $\pi : \mathcal{I} \rightarrow \mathcal{I}$. The graph $g^\pi = \{ij | i = \pi(k), j = \pi(l) \; \forall \; kl \in g\}$ is the network obtained from graph g by relabelling the players according to the permutation π. The set $\mathcal{G}^\pi(g) = \{\tilde{g} \in \mathcal{G} | \exists \; \pi$ such that $\tilde{g} = g^\pi\}$ is the set of networks with the same *architecture* as g.

An allocation rule $Y : \mathcal{G} \rightarrow \mathbb{R}^n$ assigns a real number to each player for a given graph g. Since $\mathcal{G}$ is finite, the image of Y is bounded.

Pairwise stability is defined as in Jackson and Wolinsky (1996):
The graph g is *pairwise stable* if for all players $i \in \mathcal{I}$ and all links $ij \in \mathcal{E}^{\mathrm{c}}$

(i) $$Y_i(g) \geq Y_i(g \setminus \{ij\})$$

(no unilateral destruction)

(ii) $$Y_i(g) < Y_i(g \cup \{ij\}) \Rightarrow Y_j(g) > Y_j(g \cup \{ij\}).$$

(no bilateral formation)

Two networks $g \neq g'$ are *adjacent* if they differ only with respect to one link ij, so either $g = g' \cup \{ij\}$ or $g = g' \setminus \{ij\}$.

A *path* $p(g, g') = \{g_0, ..., g_L\}$ of length L is a sequence of adjacent or equal graphs, with $g = g_0$ and $g' = g_L$.
Associated to each path $\{g_0, \ldots, g_L\}$ denote by $\{ij_l\}_{l=1}^{L}$ the set of links to which respect the graphs $g_0, \ldots, g_L$ are adjacent: $ij_l = g_l \ominus g_{l-1}$.[3]

Unlike in Jackson and Watts (2002), here a path needs not to be finite, i.e. I do not exclude the possibility of the same network appearing infinitely often in one path.

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[3] $A \ominus B = (A \setminus B) \cup (B \setminus A)$ (symmetric difference)

A path $p(g, g')$ is *improving* if for each $l = 1, ..., L$ it holds that ${ij}_l \neq \emptyset$ and
if $g_l = g_{l-1} \setminus \{{ij}_l\} \Rightarrow Y_i(g_l) > Y_i(g_{l-1}, v) \vee Y_j(g_l) > Y_j(g_{l-1})$
if $g_l = g_{l-1} \cup \{{ij}_l\} \Rightarrow Y_i(g_l) \geq Y_i(g_{l-1}, v) \wedge Y_j(g_l) \geq Y_j(g_{l-1})$, one inequality being strict.

A *cycle* is a nonempty set of networks $C = \{g_l\}$ such that there is an improving path $p(g, g')$ for all $g, g' \in C$. A cycle C is maximal if there is no cycle $C' \neq C$ such that $C \subset C'$. A cycle C is closed if there exists no pair of graphs $g \in C$ and $g' \notin C$ such that there is an improving path $p(g, g')$.

Jackson and Watts (2002) show that for any Y there exists at least one pairwise stable network or closed cycle of networks.

1.2.2 Dynamics without Mutations

At each point in time, one and only one link $ij \in \mathcal{E}^{\mathrm{c}}$ is randomly and independently drawn with uniform probability $f = \frac{2}{n(n-1)}$.[4]

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[4] Uniform probability is used to simplify the calculations of the transit-probability along a sequence of networks. One could also allow for heterogeneous positive and constant probabilities.

The two agents i and j decide simultaneously whether to form or keep the link ij or not. I call the respective action *specific willingness to form links*[5] of player i, given network g:

$$a_i(g) = (a_{i1}(g), \ldots, a_{i,i-1}(g), 0, a_{i,i+1}(g), \ldots, a_{in}(g)).$$

$a_{ij}(g) = 1$ if i is willing to form or maintain a link to j, given g and $a_{ij}(g) = 0$ if i strictly prefers not to be linked to j. However, $a_{ij}(g)$ can be any real in $[0,1]$ if i is indifferent and denotes the probability with which i wants to form a link to j. Since a player cannot be linked to herself, $a_{ii}(g) \equiv 0 \; \forall \; i \in \mathcal{I}, g \in \mathcal{G}$. $a_i(g)$ is an *action.* A strategy for player i is a collection $a_i = \{a_i(g)\}_{g \in \mathcal{G}}$. $a(g) = \{a_i(g)\}_{i \in \mathcal{I}}$ denotes the set of actions at network g. a denotes the set $\{\{a_i(g)\}_{g \in \mathcal{G}}\}_{i \in \mathcal{I}}$. For any given network g the space of actions for player i is $[0,1]^{n-1}$. Since $\#\mathcal{G} = 2^{\frac{n(n-1)}{2}}$, the strategy space for player i is $[0,1]^{(n-1)2^{\frac{n(n-1)}{2}}}$. Obviously, this space expands extremely fast with n, the number of players. This is the reason why I introduce the notion of a *bounded strategy* below.

If $ij \notin g$, the probability of moving from g to $g' = g \cup \{ij\}$ is $f \cdot a_{ij}(g) \cdot a_{ji}(g)$. Analogous, if $ij \in g$, the probability of moving from g to $g' = g \backslash \{ij\}$ is $f \cdot (1 - a_{ij}(g) \cdot a_{ji}(g))$. In each of both cases, there is only one event that makes it possible

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[5] See Durieu et al. (2004) for a model of nonspecific networking.

to move from g to g', namely the event that $\{ij\} = g \ominus g'$ is drawn. To calculate the probability of remaining in a specific graph g, I need to consider more events. Suppose a link $ij \in g$ is drawn. Then, the probability to stay in g is $a_{ij}(g) \cdot a_{ji}(g)$. Suppose a link $ij \notin g$ is drawn, then the probability that g remains unaltered is $1 - a_{ij}(g) \cdot a_{ji}(g)$. Hence, there is one possibility for each potential link $ij \in \mathcal{E}^{\mathrm{c}}$ to stay in a particular graph. Call $f \cdot \left[g \overset{a(g)}{\to} g\right]$ the probability to remain in g, given $a(g)$, where

$$\left[g \overset{a(g)}{\to} g\right] \quad = \quad \textstyle\sum_{ij \in g} a_{ij} a_{ji} + \sum_{ij \notin g} 1 - a_{ij} a_{ji} \quad .$$

The probability of observing the path $p(g, g') = (g_0, \ldots, g_L)$ of adjacent or equal networks in the next L points of time, given $a = \{\{a_i(g)\}_{g \in \mathcal{G}}\}_{i \in \mathcal{I}}$ and $a_{ij_l} = a_{ij_l}(g_l)$, is

$$\mathrm{Prob}\{p(g, g') | a\} =$$

$$f^L \cdot \prod_{l=1}^{L} \left\{ \mathbb{1}_{(\cup)} a_{ij_l} \cdot a_{ji_l} + \mathbb{1}_{(\setminus)} (1 - a_{ij_l} \cdot a_{ji_l}) + \mathbb{1}_{(=)} \left[g_l \overset{a(g)}{\to} g_l \right] \right\}, \tag{1.1}$$

where $\mathbb{1}_{(\cup)} = 1$ if $ij_l \in g_l$ and 0 otherwise,

$\mathbb{1}_{(\setminus)} = 1$ if $ij_l \notin g_l$ and 0 otherwise,

$\mathbb{1}_{(=)} = 1$ if $ij_l = \{\emptyset\}$ and 0 otherwise.

In the following, I modify the concept of pairwise stability. As already indicated by Jackson and Watts (2002), the concept of pairwise stability is myopic, that is players do not consider future changes of the network which they may trigger when severing or forming a link. Jackson and Watts (2002) argue that if the network includes many other players it may be not too strong a restriction to assume that players do not have the capacity to foresee the succeeding changes of the network. Nevertheless, in some cases it would suffice if a player could anticipate the next few alterations he triggers to give that player an incentive to deviate from myopic behavior. Dutta et al. (2005) consider the other polar case of rationality: in equilibrium, each player has some complete plan of action for each possible network and this plan of action is common knowledge among all players. In contrast to Jackson and Watts (2002), the requirements for the rationality of the involved agents are extremely high if there exist more than a few agents. Note that there are $2^{\frac{n(n-1)}{2}}$ distinct networks with n nodes. If – for example – there are only 5 players, each player has to know the 4 times 1024 strategic plans of his coplayers. Therefore, in the current chapter I propose a flexible concept that is applicable to settings with an arbitrary number of agents. Players form beliefs concerning behavior in a subset of networks "around" the current network. Their degree of rationality is captured by some parameter κ. If $\kappa = 1$, our model is similar to that of Jackson and Watts (2002) , if κ is higher than some finite number I approximate the case studied by Dutta et al. (2005) .

Anticipation

Under pairwise stability, each player maximizes his utility under the hypothesis that all other players behave as in the last period. Alternatively, players could have more sophisticated beliefs concerning the future behavior of their coplayers. I assume that each player anticipates the behavior of all players up to κ networks away from the current one. More precisely, $\tilde{g}$ is up to κ networks away from g, if $\tilde{g}$ is included in a path of length κ of adjacent networks starting at g. In equilibrium, such anticipations need to be correctly specified only for those states that are actually reached – I am interested in equilibria that confirm the beliefs.[6]

A belief $\mathcal{B}^i$ of player i is a set of matrices $\{\mathcal{B}^i(g)\}_{g\in\mathcal{G}} =$

$$\left\{ \begin{bmatrix} 0 & \beta^i_{12}(g) & \dots & \beta^i_{1n}(g) \\ \beta^i_{21}(g) & 0 & \dots & \beta^i_{2n}(g) \\ \vdots & \vdots & \ddots & \vdots \\ \beta^i_{n1}(g) & \beta^i_{n2}(g) & \dots & 0 \end{bmatrix} \right\}_{g\in\mathcal{G}} .$$

The element $\beta^i_{jk}(g)$ denotes i's belief of the willingness $a_{jk}(g)$ of player j to form or maintain the link jk with k, given g. I assume that beliefs about own actions $\beta^i_{i\cdot}(g)$ coincide with the respective willingness $a_{i\cdot}$ to form a specific

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[6] See Fudenberg and Levine (1993) and Noeldeke and Samuelson (1993)

link. Denote by $\mathcal{G}(g)_\kappa$ the set of networks that have distance of at most κ from g.

Let $\mathcal{P}_\kappa(g)$ denote the set of all paths of infinite length that start at graph g and consist only of networks in $\mathcal{G}_\kappa(g)$

$\mathcal{P}_\kappa(g, g')$ is the set of all paths in $\mathcal{P}_\kappa(g)$ that have g' as a direct successor of g, $\mathcal{P}_\kappa(g, g') = \big\{\{g_l\}_l \in \mathcal{P}_\kappa(g) \bigm| g_1 = g'\big\}$.

Given $\mathcal{B}^i(g)$, player i believes that the network g will remain unaltered with probability $f \cdot \left[g \overset{\mathcal{B}^i(g)}{\rightarrow} g\right]$, where

$$\left[g \overset{\mathcal{B}^i(g)}{\rightarrow} g\right] = \sum_{jk \in g} \overset{\text{maintain}}{\beta^i_{jk}(g) \cdot \beta^i_{kj}(g)} + \sum_{jk \notin g} \overset{\text{don't form new}}{\left(1 - \beta^i_{jk}(g) \cdot \beta^i_{kj}(g)\right)} .$$

To give an example, consider the case in which player i believes that the process will stay in network g. That is, $\beta^i_{kl} = 1$, if $kl \in g$ and $\beta^i_{kl} = 0$, if $kl \notin g$. In this case, $\left[g \overset{\mathcal{B}^i(g)}{\rightarrow} g\right] = \sum_{kl \in g} 1 + \sum_{kl \notin g} 1 = \frac{n(n-1)}{2}$ and since $f = \frac{2}{n(n-1)}$ represents the unifom distribution over all links, $f \cdot \left[g \overset{\mathcal{B}^i(g)}{\rightarrow} g\right] = 1$.

An *anticipation* of player k starting at g, $\alpha^k_g = \{\alpha^k(p)\}_{p \in \mathcal{P}_\kappa(g)}$ is a probability measure over the set of paths p starting at g,

$\mathcal{P}_\kappa(g)$, induced by the belief $\mathcal{B}^k$. Define $\forall\, p \in \mathcal{P}_\kappa(g), \alpha_k(p) :=$

$$\prod_{l=1}^{\infty} \left\{ \mathbb{1}_{(\cup)} f \cdot \beta^k_{ij_l} \cdot \beta^k_{ji_l} + \mathbb{1}_{(\setminus)} f \cdot (1 - \beta^k_{ij_l} \cdot \beta^k_{ji_l}) + \mathbb{1}_{(=)} f \cdot \left[g \overset{\mathcal{B}^k(g)}{\rightarrow} g \right] \right\} \tag{1.2}$$

A class of (infinitely many) paths may receive positive probability. I give a simple example: Suppose that $g = \{ij\}$ and $g' = \{\emptyset\}$, $g' \in \mathcal{G}(g)_\kappa$. Define $\mathcal{B}^i(g)$ as $\beta^i_{ij}(g) = \beta^i_{ij}(g') = \frac{1}{2}$ and $\beta^i_{ji}(g) = \beta^i_{ji}(g') = 1$, all other elements being equal to zero. Given these beliefs, there are four transitions, which occur with positive probability: $g \rightarrow g, g \rightarrow g', g' \rightarrow g'$ and $g' \rightarrow g$. The probability for each transition is strictly smaller than 1, but the process will follow a path that consist only of g or g' with probability one, if it starts at g.

Definition *unbiased anticipation*
An anticipation $\alpha^i_g = \{\alpha^i(p)\}_{p \in \mathcal{P}_\kappa(g)}$ is *unbiased* if $\alpha^i(p) = \text{Prob}\{p \mid a\}\ \forall p \in \mathcal{P}_\kappa(g)$, where $\text{Prob}\{p \mid a\}$ is defined in (1.1).

The set of beliefs that imply unbiased anticipations is closed. Given network g, suppose the link ij will occur with probability $a = a_{ij}(g) \cdot a_{ji}(g)$. If $a > 0$, the set of unbiased beliefs $(\beta^k_{ij}, \beta^k_{ji})$ is $\{\beta^k_{ij}, \beta^k_{ji} \in (0, 1] : \beta^k_{ji} = a/\beta^k_{ij}\}$. If $a = 0$, the set of unbiased beliefs $(\beta^k_{ij}, \beta^k_{ji})$ is $\{\beta^k_{ij}, \beta^k_{ji} \in [0, 1] : \min\{\beta^k_{ji}, \beta^k_{ij}\} = 0\}$.

Discounted Utility

Let $u_i(p)$ denote the discounted utility of player i along the path $p = \{g_0, g_1, ...\}$, given discount factor $\delta \in [0, 1)$.[7]

$$u_i(p) = \sum_{l=0}^{\infty} \delta^l Y_i(g_l, v).$$

Denote by $U_i(g|\alpha_g^i)$ the expected discounted utility of player i along paths starting at g given anticipation α_g^i.

$$U_i(g|\alpha_g^i) = \sum_{p \in \mathcal{P}_\kappa(g)} \alpha_g^i(p) u_i(p)$$

Since α_g^i is a probability measure over $\mathcal{P}_\kappa(g)$, I have $\sum_{p \in \mathcal{P}_\kappa(g)} \alpha_g^i(p) = 1$. Since $Y_i(\cdot)$ is bounded and $\delta < 1$, $u_i(\cdot)$ is bounded and $U_i(\cdot)$ is well defined.

Given network g, beliefs $\mathcal{B}^i$ and actions $a_{i\cdot}(\cdot)$, I can calculate the anticipation α_g^i. Note that beliefs for networks that are not in $\mathcal{G}_\kappa(g)$ are irrelevant for α_g^i. Since the space of all $a_{i\cdot}(\cdot)$, $[0, 1]^{(n-1)\#\mathcal{G}_\kappa(g)} \subseteq [0, 1]^{(n-1)2^{\frac{n(n-1)}{2}}}$ is compact and α_g^i is continous (linear, in particular) in a_i, there exists an a_i^* that maximizes $U_i(g|\alpha_g^i)$ and the set of all maximizing a_i^* is compact.

Definition *optimal anticipation*

I call α_g^i *optimal* if it consists of a maximizing a_i^*.

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[7] If $\delta = 0$, the model is similar to the one of Jackson and Watts (2002) (with the convention $0^0 = 1$) .

An anticipation is induced by beliefs about the actions of other players and own actions. If the anticipation is optimal, own actions are chosen optimally given the beliefs about the actions of other players. For each g, there may be multiple optimal anticipations α_g^i supported by different sets of beliefs $\mathcal{B}^i$.

Suppose the current network is g and that the link $ij \in$ g is drawn. Suppose G' can be obtained by removing ij from g. I say that g' *defeats* g, or $g \blacktriangleright g'$, if for $l \in \{i, j\}$ there exist an optimal anticipation α_g^l such that $\sum_{p \in \mathcal{P}_\kappa(g,g')} \alpha_g^l(p) > 0$. This means that g' follows g with positive probability, if l chooses a_l optimally. If $ij \notin g$, ie if both i and j must agree to form the link ij, g' defeats g or $g \blacktriangleright g'$ if there exist optimal anticipations α_g^l for $l = i \wedge j$ such that $\sum_{p \in \mathcal{P}_\kappa(g,g')} \alpha_g^l(p) > 0$ for $l = i \wedge j$.
If there is a graph g with optimal anticipations $\alpha_i(g)\ \forall\ i$ such that $\sum_{p \in \mathcal{P}_\kappa(g,g)} \alpha_g^i(p) = 1\ \forall i$, then $g \blacktriangleright g$.

Definition *improving in anticipation path* or $\alpha-$*path*
Given an action profile a and a set of beliefs $\{\mathcal{B}^i\}_i$, a path $p = \{g_l\}_l$ is *improving in anticipation* if the beliefs induce optimal and unbiased anticipations such that $g_l \blacktriangleright g_{l+1}\ \forall l = 1, 2, \dots$.

Definition *stable in anticipation graph* or $\alpha-$*stable graph*

A graph g^* is *stable in anticipation* if there is no $\alpha-$path $p = \{g_l\}_l$ leaving g^*.

$$g^* \ \alpha - stable \Rightarrow \exists \alpha - path\{g^*, g^*, \ldots\}$$

If there is no $\alpha-$path leaving g^*, there is no network g' defeating g^*, ie $\sum_{p \in \mathcal{P}_\kappa(g^*, g^*)} = 1 \ \forall \ i$ for all optimal anticipations.

Analogous to Jackson and Watts (2002) I define a closed anticipated cycle.

An *anticipated cycle* is a set of networks $\mathcal{C} = \{g_l\}$ such that there is an $\alpha-$path $p(g, g')$ for all $g, g' \in \mathcal{C}$. An anticipated cycle $\mathcal{C}$ is maximal if there is no anticipated cycle $\mathcal{C}'$ such that $\mathcal{C} \subset \mathcal{C}'$. An anticipated cycle $\mathcal{C}$ is closed if there exists no pair of graphs $g \in \mathcal{C}$ and $g' \notin \mathcal{C}$ such that there is an $\alpha-$path $p(g, g')$.

Lemma

There exists either a network that is stable in anticipation or there exists a closed anticipated cycle.

Proof

Suppose there is no network that is stable in anticipation. This means that for any network $g \in \mathcal{G}$ there exists a α-path starting at g. Since the number of nodes n is finite, so must be the set of networks $\mathcal{G}$. Fix for any network

g one particular α-path that starts at g and consider only the first step of that path. Since $\mathcal{G}$ is finite, any network g must be reached again after finitely many such steps, hence there is an anticipated cycle. Since $\mathcal{G}$ is finite, the union of any such anticipated cycles is finite. Therefore, there is a maximal anticipated cycle. A maximal anticipated cycle necessarily is a closed aniticipated cycle. □

Remark
The concept of beliefs appears quite demanding, concerning the capabilities of the players. Note, for all that, if the process is in a closed anticipated cycle $\mathcal{C}$, any beliefs $\{\mathcal{B}(\tilde{g})\}_{\tilde{g} \notin \mathcal{G}(\mathcal{C})_\kappa}$[8] are irrelevant to the unbiasedness of α_g $\forall g \in \mathcal{G}(\mathcal{C})$. Furthermore, there may be multiple beliefs $\{\mathcal{B}(g)\}_{g \in \mathcal{G}(\mathcal{C})}$ that yield unbiased anticipations given some profile of actions a.

1.3 Stochastic Stability

I define stochastic stability as in Foster and Young (1990) who build upon the work of Freidlin and Wentzell (1998). Firstly I define a state and the continous set of all states. Thereafter I construct a non-ergodic Markov chain which represents the anticipated dynamics of the game without

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[8] $\mathcal{G}(\mathcal{C})_\kappa$ denotes all networks that have distance of at most κ to the set $\mathcal{C}$, ie $\mathcal{G}(\mathcal{C})_\kappa = \underset{g \in \mathcal{C}}{\cup} \mathcal{G}(g)_\kappa$.

mutations. Then I set up an appropriate model of mutations that generates an ergodic Markov chain and allows us to define the resistance of each state. As in any ergodic process, there is a unique invariant distribution. I analyze this distribution as mutations become very rare. If a state receives positive probability in the invariant distribution as the probability of a mutation goes to zero, I call this state stochastically stable. I show that among the absorbing states of the undisturbed process those who have minimum resistance in the process with mutations are stochastically stable. Finally, I show that a state is stochastically stable only if it is stable in anticipation.

State Space Θ

A state θ consists of a network $g \in \mathcal{G}$ and a set of beliefs $\mathcal{B} = \left\{ \{\mathcal{B}_i(g')\}_{g' \in \mathcal{G}} \right\}_{i \in \mathcal{N}} \in [0,1]^{n^2(n-1)2^{\frac{n(n-1)}{2}}} \equiv B$. $a(\theta)$ is the profile of actions included in $\mathcal{B}$, $a(\theta) = \{\{\beta^i_{i\cdot}(g')\}_{g' \in \mathcal{G}}\}_{i \in \mathcal{N}}$. Each agent i has some belief $\beta^i_{kl}(g')$ about k's willingness to form a link with l for each network g' and each pair of players k and l. Nevertheless, to calculate optimal actions, each player i does only need to consider $\{\mathcal{B}^i(g')\}_{g' \in \mathcal{G}_\kappa(g)}$, which is small compared to $\mathcal{B}$, if κ is small. Call $B^* \subset B$ the subset of optimal and unbiased beliefs. That is, for each $g \in \mathcal{G}$, only those anticipations α^i_g generated by beliefs in B^* are unbiased and optimal. I will define a finite partition of Θ below and apply the methods to its index-set.

Transitions

I assume that players update their beliefs infinitely faster than they change their actions. I further assume that players update their beliefs such that they do not contradict the observed action profiles. Players choose any such beliefs with uniform probability. This implies that I can model the transitions of the process such that at each point in time the beliefs yield unbiased anticipations. Given these unbiased anticipations, players choose their actions optimally, the process follows $\alpha-$paths.

Absorbing Sets

A closed maximal anticipated cycle $\mathcal{C}$ together with unbiased and optimal beliefs $\mathcal{B} \subset B^*$ is absorbing. If $\mathcal{C}$ is singleton, then it is a graph g^* that is stable in anticipation. Call $\mathcal{A}$ the set of states that represent graphs that are contained in closed maximal anticipated cycles, together with their unbiased and optimal beliefs. Since both the set of unbiased beliefs and the set of optimal beliefs are compact, so is $\mathcal{A}$. Let $\{K_1, \ldots, K_L\}$ be a partition of $\mathcal{A}$, let $K_0 = \Theta \setminus \mathcal{A}$ and define $\mathcal{K} = \{K_0, \ldots, K_L\}$. Define the partition such that two graphs g and g' together with their unbiased and optimal beliefs belong to the same K_l, $l > 0$, if and only if they belong to the same closed maximal anticipated cycle $\mathcal{C}$. If g^* is stable in anticipation and $< g^*, \mathcal{B} > \in K_l$, there is no other graph belonging to K_l. Since $\mathcal{G}$ is finite and the set of optimal and unbiased beliefs is compact, so is $\mathcal{K}_l$, in particular, the cardinality of $\mathcal{K}$ is finite, $L < \infty$.

Mutations

Suppose that in each period, each player follows the transition process described above with probability $1 - \epsilon \in (0, 1)$ and behaves differently, say 'mutates', with probability $\epsilon > 0$. If some agent i derived some optimal anticipation and after mixing according to his specific willingnes to form a link with j, i actually wants to form a link with j, a mutation causes him to sever or deny that link and vice versa. Concerning the beliefs, a mutation has more drastic consequences. A mutating player chooses any beliefs with uniform probability and chooses optimal actions with respect to these new beliefs.[9]

It is possible to reach any state θ' from any other state θ only by means of (multiple) mutations.[10] For example, if θ represents the complete network, one mutation for each of the $\frac{n(n-1)}{2}$ links can cause the process to transit to a state θ' which represents the empty network. Another n mutations would be necessary to change all

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[9] Alternatively, one could consider a weaker model of mutations: instead of replacing the whole set of beliefs $\{\mathcal{B}^i(g)\}_{g \in \mathcal{G}}$, a mutation changes only one entry β^i_{kl} of $\{\mathcal{B}^i(g)\}_{g \in \mathcal{G}}$.

[10] There is one difference to the model of mutations of Jackson and Watts (2002): to build up a link that should not exist without mutations, I need two mutations instead of one. This difference is conceptually negligible.

beliefs. However, to reach the complete network only by means of mutations if the current network is empty, the number of necessary mutations is $2 \cdot \frac{n(n-1)}{2}$, because for any link that is to be formed, both agents need to mutate.

The essential properties of this model of mutations, are that (*i*) it is possible to reach any state θ' from any state θ with a finite number of mutations in finite time[11] ($\Rightarrow$ irreducibility) and that (*ii*) the probability of a single mutation is of the same order in ϵ for any state $\theta \in \Theta$ (see Bergin and Lipman (1996)). Given this model of mutations, there exists a stochastic kernel $p^\epsilon : \Theta \times \Theta \rightarrow \mathbb{R}$ with the property that for any pair of states $\theta, \theta' \in \Theta$ the probability of a transition from θ to θ' is positive, $p^\epsilon(\theta, \theta') > 0$, for any $\epsilon > 0$.

$l-$**trees**

Referring to Freidlin and Wentzell (1998), chp. 6, I define a tree on the index set $\mathcal{L} = \{0, \ldots, L\}$ of the partition $\mathcal{K}$. An $l-$tree is a set of directed edges $m \rightarrow n$, $m \in \mathcal{L} \setminus \{l\}, n \in \mathcal{L}$ such that any index $m \in \mathcal{L} \setminus \{l\}$ has a unique successor n and such that there are no closed cycles of the form $m_1 \rightarrow m_2 \rightarrow \ldots \rightarrow m_k \rightarrow m_1$. Consequently, in such a graph, there is a unique sequence from any state $k \in \mathcal{L} \setminus \{l\}$ to l. Denote by $H(l)$ all $l-$trees.

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[11] In my model of mutations such a transition is possible in one step in which multiple mutations occur.

Volume of $l-$trees

For numbers $p_{mn} \in \mathbb{R}_+, m, n \in \mathcal{L}$ define for any $l-$tree h_l the volume $V(h_l) = \prod\limits_{(m \to n) \in h_l} p_{mn}$.

Lemma 2.1: invariant measure μ^ϵ on $\mathcal{K}$ (Freidlin and Wentzell (1998))

Let us be given an irreducible Markov chain on space Θ devided into disjoint sets $\{K_l\}_{l\in\mathcal{L}}$ with kernel $p^\epsilon : \Theta \times \Theta \to \mathbb{R}$. Suppose there exist non-negative numbers $p_{lm}(l \neq m, l, m \in \mathcal{L})$ and a number $c > 1$ such that $c^{-1}p_{lm} \leq p^\epsilon(\theta, K_m) \leq cp_{lm} \; \forall \; \theta \in K_l, l \neq m$ for the transition probabilities of our chain. Then

$$c^{2-2L}\left(\sum_{m\in\mathcal{L}} Q_m\right)^{-1} Q_l \leq \mu^\epsilon(K_l) \leq c^{2L-2}\left(\sum_{m\in\mathcal{L}} Q_m\right)^{-1} Q_l$$

for any normalized invariant measure μ^ϵ of our chain, where $Q_l = \sum_{h_l \in H(l)} V(h_l)$.

Proof:

I apply Lemma 3.2 of Freidlin and Wentzell (1998), p.178. Our $V(h)$ is their $\pi(h)$ for any tree h. The Markov chain is irreducible for any positive ϵ since $p^\epsilon(\theta, \theta') > 0 \; \forall \; \theta, \theta' \in \Theta$. Given the stochastic kernel $p^\epsilon : \Theta \times \Theta \to \mathbb{R}$, define $p^\epsilon(\theta, K_n) = \int_{K_n} p^\epsilon(\theta, \theta')d\theta'$ and $p_{mn} = \sup\limits_{\theta\in K_m} p^\epsilon(\theta, K_n)$ and $c = \frac{\sup\limits_{\theta\in K_m} p^\epsilon(\theta,K_n)}{\inf\limits_{\theta\in K_m} p^\epsilon(\theta,K_n)} > 1$. Then

$c^{-1}p_{mn} \leq p^{\epsilon}(\theta, K_n) \leq cp_{mn} \; \forall \; \theta \in K_m.$

Resistance

Define the resistance of a pair of states (θ, θ') as the number of mutations necessary for the process to move from θ to θ' with positive probability. For two adjacent states $\theta =< g, \mathcal{B} >$ and $\theta' =< g', \mathcal{B}' >$ define

$$R_{\theta,\theta'} = \begin{cases} 0 & \text{if } \mathcal{B}' \in B^* \text{ and } g \blacktriangleright g' \\ & \text{in some } \alpha - \text{path induced by } \mathcal{B}' \\ & \\ & \text{if } \mathcal{B}' \notin B^* \text{ and } g \blacktriangleright g' \\ & \text{in some } \alpha - \text{path induced by } \mathcal{B}' \\ 1 & \text{or} \\ & \text{if } \mathcal{B}' \in B^* \text{ and } g \blacktriangleright g' \\ & \text{in no } \alpha - \text{path induced by } \mathcal{B}' \\ & \text{and } g \supset g' \\ & \\ 2 & \text{if there is no } \alpha - \text{path} \\ & \text{such that } g \blacktriangleright g' \text{ and } g \subseteq g' \end{cases}$$

For two non-adjacent states $\theta =< g, \mathcal{B} >$ and $\theta' =< g', \mathcal{B}' >$ define as $P_{\theta}^{\theta'}$ the set of shortest paths of adjacent networks that start at g and end at g'. For any $p = (\theta_1, \ldots, \theta_K) \in P_{\theta}^{\theta'}$, define $R(p) = \sum_{k=1}^{K-1} R_{\theta_l,\theta_{l+1}}$. Define $R_{\theta,\theta'} = \min\limits_{p \in P_{\theta}^{\theta'}} R(p)$.

Observe that for any two states $\theta, \theta' \in K_l$ for any $l > 0$, the resistance of a transition from θ to θ' is zero.

Resistance of $l-$trees

Define the resistance of a transition from set K_l to set K_k by $R_{l,k} = \min_{\theta \in K_l} \min_{\theta' \in K_k} R_{\theta,\theta'}$. The resistance of a tree h_l is the sum of the resistance of its elements: $R(h_l) = \sum_{(m \to n) \in h_l} R_{m,n}$. Finally, the resistance of an index $l \in \mathcal{L}$ is defined by the minimal resistance of an l-tree: $R_l = \min_{h_l \in H(l)} R(h_l)$.

Lemma 2.2

If $k = \arg\min_{l \in \mathcal{L}} R_l$, then any $\theta \in K_k$ represents a graph g and beliefs $\mathcal{B}$ such that either g is stable in anticipation with respect to $\mathcal{B}$ or g belongs to a closed anticipated cycle $\mathcal{C}$.

Proof:

If a graph g is not stable in anticipation or does not belong to a closed maximal cycle, it belongs to a state $\theta \in K_0$, by definition of the set K_0. I showed that there exists at least one network that is stable in anticipation or one closed maximal anticipated cycle, i.e. $\Theta \setminus \{K_0\} \neq \emptyset$. Therefore it remains to show that $R_0 > R_l$ for some $l \in \{1, \ldots, L\}$. Consider the tree h_0 that minimizes the resistance over all $0-$trees in $H(0)$. Since in K_0 there exists no graph that is stable in anticipation or a closed maximal anticipated cycle, there must be a state $\theta \in K_0$ with beliefs B^* such that they generate an $\alpha-$path that starts at θ and leads to a state $\theta' \in K_l$, $l > 0$. By the definition of resistance, the edge $0 \to l$ has resistance zero. Consider the edge $l \to m \in h_0$. Since K_l is an absorbing set, the edge $l \to m$ must contain a transition wich is not part of an $\alpha-$path. Therefore, the resistance of $l \to m$ must be greater or equal to 1. Now remove the edge $l \to m$ and add the edge $0 \to l$. The constructed tree has resistance of at most $R_0 - 1$. $\square$

Definition
A function $f(\epsilon) : \mathbb{R}_+ \to \mathbb{R}_+$ is called of order $\epsilon^{=k}$, denoted by $O(\epsilon^{=k})$, if (i) $\lim_{\epsilon \to 0} \sup f(\epsilon)/\epsilon^k < \infty$ and (ii) $\lim_{\epsilon \to 0} \sup f(\epsilon)/\epsilon^{k+1} = \infty$. If (i) holds, $f(\epsilon)$ is called $O(\epsilon^{\geq k})$, and if (ii) holds, $f(\epsilon)$ is called $O(\epsilon^{\leq k})$.

Lemma 2.3 $Q_l = O(\epsilon^{R_l})\ \forall\ l \in \mathcal{L}$.

Proof
For two functions $f(\epsilon) = \mathcal{O}(\epsilon^a)$ and $g(\epsilon) = \mathcal{O}(\epsilon^b)$, $a < b$, I have $f(\epsilon)g(\epsilon) = \mathcal{O}(\epsilon^{a+b})$ and $f(\epsilon) + g(\epsilon) = \mathcal{O}(\epsilon^a)$.
Consider the $l-$tree h_l, that minimizes the resistance in $H(l)$. For any arrow $(i \to j) \in h_l$, $R_{i,j}$ is the miminum number of mutations that is necessary to transit from a state in K_i to a state in K_j. Hence $p_{ij} = \sup_{\theta \in K_i} \int_{K_j} p^\epsilon(\theta, \theta')d\theta' = \mathcal{O}(\epsilon^{R_{i,j}})$.
Hence $V(h_l) = \prod_{(i \to j) \in h_l)} p_{ij} = \mathcal{O}(\epsilon^{\sum_{(i \to j) \in h_l} R_{i,j}}) = \mathcal{O}(\epsilon^{R_l})$, where $R_l = \sum_{(i \to j) \in h_l} R_{i,j}$ is the minimum number of

mutations necessary to move along all arrows in h_l. Since $R_l \leq R(h) \; \forall \; h \in H(l)$, I have $\sum_{h \in H(l)} V(h) = \mathcal{O}(\epsilon^{R_l})$, which completes the proof. $\square$

Theorem 2.1 $\mu^{\epsilon}(K_0) \underset{\epsilon \to 0}{\longrightarrow} 0$.

Proof
Since $k = \arg \min\limits_{l \in \mathcal{L}} R_l \neq 0$ and $\sum_{l \in \mathcal{L}} Q_l = O(\epsilon^{R_k})$ it follows that $O_0 / \sum_{l \in \mathcal{L}} O_l = O(\epsilon^{R_0 - R_k}) \underset{\epsilon \to 0}{\longrightarrow} 0$. It follows directly from lemma 2.1 that $0 \leq \lim_{\epsilon \to 0} \mu^{\epsilon}(K_0) \leq 0$. $\square$

I call states that receive positive probability by the invariant distribution μ as the rate of mutations ϵ goes to zero *stochastically stable.*

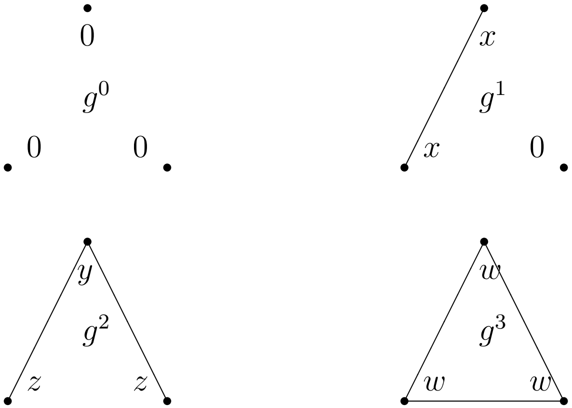
To conclude, I state the contraposition of the theorem:

Corrollary 2.1
If a state is stochastically stable, it must represent a part of a maximal closed anticipated cycle or it represents a network that is stable in anticipation.

1.4 Examples

1.4.1 Example 1

Let the allocation rule be defined as follows, and $\kappa = 2$:



I assume the following inequalities: $w > y > 0 > x > z$. Here, the allocation rule is anonymous, i.e. invariant to the labels of the players. If players behave as in Jackson and Watts (2002), two types of networks are pairwise stable: g^0 and g^3, the latter pareto dominating the former. One mutation in g^3 causes the process to reach g^0 with positive probability while two mutations are necessary to cause the process to transit from g^0 to g^3, hence g^0 is the only network that is stochastically stable, given the process is myopic. If players are nonmyopic and follow the anticipated dynamics presented in this paper with $\kappa = 2$ both g^0 and g^3 are $\alpha-$stable (with appropriate anticipations). I show that *i*) the process can move from g^0 to g^3 with positive probability, if one mutation occurred and that ii) one mutation suffices to reach g^0 from g^3. Therefore both g^0 and g^3 are stochastically stable.

To start with i), suppose players have myopic beliefs à la Jackson and Watts (2002) of the following form, where the first row is i's, the second is j's and the last is k's.

$$\mathcal{B}(g^0) = \mathcal{B}(\{ij\}) = \mathcal{B}(\{ik\}) = \mathcal{B}(\{jk\}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{B}(\{ij, jk\}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \mathcal{B}(\{ij, ik\}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\mathcal{B}(\{ik, jk\}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{B}(g^3) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

These beliefs support both g^0 and g^3 as stable in anticipation. I show that the process can drift[12] to a state that still supports g^0 as $\alpha-$stable, but from which one mutation causes a transit from g^0 to g^3.

Suppose that by mutation, player i thinks that both j and k are willing to form or maintain the link with i in the networks $\{ij\}$ and $\{ij, ik\}$. As long as the process is only in

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[12] Drift is a sequence of single mutations with the property that after each mutation, there is infinite time for the selection process to settle at a rest point ((Samuelson, 1997, chap. 6)).

g^0 these beliefs cannot be falsified and hence are unbiased. In $\{ij, ik\}$, it is optimal for i to maintain the links to j and k, since i receives positive payoffs in $\{ij, ik\}$ and non-positive payoffs in g^0 or $\{ij\}$. For the same reason it is optimal for i to accept the link to k in $\{ij\}$. Given $\mathcal{B}_i^d$, there is only one type of path leaving $\{ij\}$: $\left(\frac{K}{\{ij\}} \frac{\infty}{\{ij,ik\}}\right)$, where $\frac{K}{g}$ denotes a K-fold sequence of g. This path occurs with probability $\left(\frac{2}{3}\right)^{K-1} \frac{1}{3}$ and generates discounted utility $u_i(p) = \frac{x+(y-x)\delta^K}{1-\delta}$. Therefore, $U_i(\{ij\}) = \sum_{K=1}^{\infty} \left(\frac{2}{3}\right)^{K-1} \frac{1}{3} \frac{x+(y-x)\delta^K}{1-\delta} = \frac{3x(1-\delta)+y\delta}{(1-\delta)(3-2\delta)}$ and if $3x(1-\delta) + y\delta > 0$, it is optimal for i to maintain the link with j in $\{ij\}$. Furthermore, i is indifferent between being willing to form a link with j in g^0 since he believes that j would reject this link anyway. Therefore, the beliefs $\mathcal{B}_i^d$ listed below are unbiased and optimal, can be reached by drift and still support g^0 as $\alpha-$stable.

$$\mathcal{B}_i^d(g) = \mathcal{B}(g) \ \forall \ g \in \mathcal{G} \setminus \{g^0, \{ij\}, \{ij, ik\}\}$$

$$\mathcal{B}_i^d(\{g^0\}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \mathcal{B}_i^d(\{ij\}) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\mathcal{B}_i^M(\{ij, ik\}) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

If player k believes that player i rejects the link ik in the network ij, it is optimal for k to be willing to form the link ik with i in ij. Hence the state, where k believes $\mathcal{B}_k^d$ as defined below can be reached by drift and still g^0 would be $\alpha-$stable.

$$\mathcal{B}_k^d(\{ij\}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ , \ \mathcal{B}_k^d(g) = \mathcal{B}_k(g) \ \forall \ g \in \mathcal{G} \setminus \{ij\}$$

Now suppose that in the empty network player j mutates and is willing to form a link to i. Suppose that the next three links that are drawn by nature are, in this specific 'rule of order', ij, ik and jk. In g^0, players i and j form the link ij, i because it is optimal given $\mathcal{B}_i^d$, j because of the mutation. Next period in the just established network $\{ij\}$, i and k form the link ik because it is optimal given $\mathcal{B}_i^d$ and $\mathcal{B}_k^d$ and the process transits to $\{ij, ik\}$. There, it is optimal for j and k to form the link jk to reach g^3 in which both receive maximal payoffs.

Since the observed path $(g^0, \{ij\}, \{ij, ik\}, g^3)$ contradicts the beliefs $\left(\mathcal{B}_i^d, \mathcal{B}_j, \mathcal{B}_k^d\right)$ the players need to update and optimize their beliefs to $\left(\mathcal{B}'_i, \mathcal{B}'_j, \mathcal{B}'_k\right)$. These new beliefs induce anticipations such that g^3 is stable in anticipation.

There are many more beliefs that support g^3 as $\alpha-$stable. Each of these beliefs could have been reached by drift. One of these is $\mathcal{B}^0$, as specified below:

$$\mathcal{B}^0(g^3) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \mathcal{B}^0(g) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ \forall \, g \in \mathcal{G} \backslash \{g^3\}$$

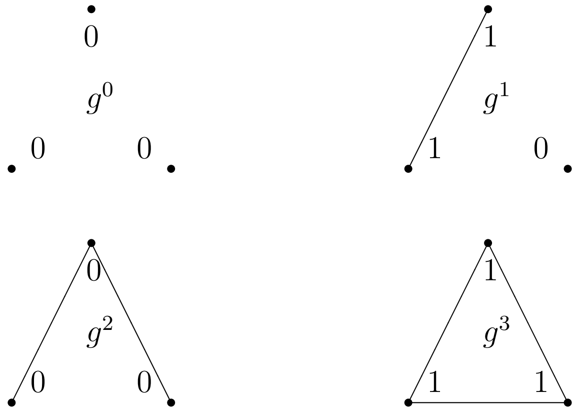
$\mathcal{B}^0$ is the one that makes a transition from g^3 to g^0 most likely. $\mathcal{B}^0$ are quite pessimistic beliefs, hoIver they are unbiased and optimal.

I now show that one mutation suffices to trigger the process to transit from g^3 to g^0. Suppose the process drifted to a state in which each player beliefs $\mathcal{B}^0$ and that player k mutates in g^3 and rejects to maintain the link ik and suppose the link ij is drawn in $\{ij, jk\}$ in the next period. Given $\mathcal{B}^0$, i's anticipation attaches positive probability to two types of graphs: $\left(\frac{K}{\{jk\}}\frac{\infty}{g^0}\right)$ and $\left(\frac{K}{\{ij\}}\frac{\infty}{g^0}\right)$. While i receives negative discounted utility in paths of the latter type, the utility in paths of the first type is zero. Hence in $\{ij, jk\}$ i rejects the link to j. In jk, both j and k will refuse to maintain the link and therefore the process transits to g^0.

To summarize, I showed that from a state in which g^0 is stable in anticipation, after drifting by single mutations, the process needs one mutation to transit to a state in which g^3 is stable in anticipation. Thereafter, after drift, the process needs one mutation to transit back to a state in which g^0 is stable in anticipation. In the words of Samuelson (1997), chap. 7, both g^0 and g^3 are in a one-mutation-neighborhood and hence are both stochastically stable.

1.4.2 Example 2

In the next example I show that inefficient outcomes may be stable in anticipation. Suppose the allocation rule is as follows:[13]



Both g^0 and g^3 are pairwise stable and stable in anticipation. g^2 is in the basin of attraction of both graphs and it is only "one mutation away" from each of the two absorbing graphs.

1.5 Conclusions

In this chapter, I construct a model of forward looking behaviour, however, I restrict the capability of the players. Agents believe that the state of the game changes not too much from the current state. I analyse a very special model of network formation. I give examples in which boundedly

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[13] This example is taken from Dutta et al. (2005)

rational players are willing to forgo potential current payoffs in order to wait for a more attractive network to evolve. In our setup the beliefs of the agents are a part of the state of the game. It could be interesting to generalise this concept to infinitely repeated games of incomplete information, as, for example, a bargaining game in which the share and the beliefs about acceptance levels and outside options define the state.

Chapter 2

Evolution and Correlated Equilibrium

In this chapter I show that a set of outcomes that lies outside the convex hull of Nash equilibria can be asymptotically stable with respect to an evolutionary dynamic that satisfies convex monotonicity. In the model presented here, boundedly rational agents receive signals and condition the choice of strategies on the signals. A set of conditional strategies is asymptotically stable only if it represents a strict (correlated-)equilibrium set. There are correlated equilibria that cannot be represented by an asymptotically stable signal contingent strategy. For generic games I show that if signals are endogenous but no player has an incentive to manipulate the signal generating

process and if the signal contingent strategy is asymptotically stable, then and only then, the outcome must be a strict Nash equilibrium.

2.1 Introduction and Related Literature

Consider a situation of strategic interaction in which agents perceive signals before they choose their strateg. Restricting the general setup of Aumann (1974), I demand that all agents share common prior on the distribution of the signals. Given their own signal and given the conditional distribution of their opponents' signals, each agent optimally chooses a strategy. Finally, suppose that there is common knowledge of rationality. According to Aumann (1987), a resulting outcome must be a correlated equilibrium. Due to the potential correlation between signals, a correlated equilibrium does not need to be a Nash equilibrium. Indeed, a situation of strategic interaction without signals seems artificial – signals are all around us in the real world, we can hardly avoid perceiving them and then condition our behavior on them in many situations. For example, in a financial market agents may receive signals on the value of some asset that are correlated. Several firms competing on a market for some consumption good may receive correlated information on the parametrization of the demand function. Consumers observe signals displaying information on the quality of some good when planning their consumption. Football fans perceive signals concerning the success of their favorite team within some tournament and condition their betting behavior on this information.

Rationality in the sense of Aumann (1987) requires that agents understand the underlying probability space and that this is commonly known. I support the concept of correlated equilibrium from the perspective of bounded rationality. I assume evolutionary dynamics on the game in which agents receive signals and show that states persisting over time in the presence of small mutations are correlated equilibria – and therefore may be non-Nash outcomes. Before I describe our model in detail in the next section, I discuss the concept of evolution. In his survey on adaptive heuristics, Hart (2005) describes evolutionary dynamics as one extreme of bounded rationality: individuals' behavior is completely deterministic. The concept of evolutionary game theory originates from biology; see Dawkins (1990) or Björnerstedt and Weibull (1996) for socio-economic interpretations. Rationality is imposed on an aggregate level: strategies with higher relative success spread faster. Evolutionary game theory contributes by showing that *even if* agents are boundedly rational, certain outcomes predicted by concepts requiring rationality persist over time.

This chapter characterizes the set of correlated equilibria that persist over time, given boundedly rational agents. The first part of the chapter assumes an exogenous and stationary process of signal generation. A set of signal contingent strategies is asymptotically stable with respect to convex monotonic dynamics[1], if it is a strict equilibrium set[2] of the game with signals. Given this selection, I consider endogenous signals. A signal generating process is robust, if no population has an incentive to manipulate the process, given equilibrium choice of the signal contingent strategies. I show for generic games that a signal contingent strategy is asymptotically stable and the signal generating processs is robust, if and only if the induced outcome is a strict Nash equilibrium. For the special case of the traditional example that has an equilibrium outcome with payoffs outside the convex hull of Nash-payoffs, the Chicken game, I show that a correlated equilibrium has robust signals if and only if it induces payoffs that lie *inside* the convex hull of Nash-payoffs.

In the remainder of this section, I relate this chapter to the literature. It is well understood that the aggregate can display some rationality. Ritzberger and Weibull (1995) show that only strict Nash equilibria are asymptotically stable in the multipopulation replicator dynamics. For asymmetric games (animal conflicts), Selten (1980) shows that evolutionary stable strategies must be strict Nash equilibria. I make use of a concept introduced by Balkenborg (1994), *strict equilibrium set.* Each element of a strict equilibrium set is a

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[1] Hofbauer and Weibull (1996)

[2] Balkenborg (1994)

Nash equilibrium, the set is closed under mixed best replies. Balkenborg and Schlag (2007) show asymptotic stability of restpoints within this set for general asymmetric games.[3] I rely on the concept of strict equilibrium set to characterize sets of correlated equilibria that are asymptotically stable. Lenzo and Sarver (2006) build up a model of subpopulations in which agents are matched according to a distribution over the set of subpopulations. They show that every interior[4] Lyapunov stable state is equivalent to a correlated equilibrium. Their model is inspired by the work of Mailath et al. (1997) who show that equilibria in a static model of local interactions coincide with correlated equilibria in the original game. In both models the correlation device is a "matching technology" with which agents of different populations are matched non-uniformly. I show that Lenzo and Sarver (2006) is a special case of our model if one chooses a particular signal generating process. Cripps (1991) analyzes a two player model in which in a first stage nature randomly allocates row or column to the players and in a second step assigns one role of a finite set of roles to each player. He shows that an ESS in the symmetric game yields a distri-

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[3] Other setwise concepts defined for symmetric one population games are introduced by Balkenborg and Schlag (2001) and Thomas (1985). Cressman (2003) also elaborates on the strict equilibrium set.

[4] Interiority in the subpopulation model means that the state is interior for each subpopulation. It is more stringent than interiority in our case.

bution over the set of outcomes that is a strict correlated equilibrium. I abstain from analyzing the symmetrization and extend his model to dynamic analysis. Kim and Wong (2007) define evolutionary stable correlation for symmetric 2×2-games. They apply a special signal space, I discuss this matter after introducing the static model. Finally, I consider exogenous signals. I imagine situations, in which some agents exercise control over the generation of signals. I do not restrict attention to situations in which a signal consists of a message of each player. In such a case, the player can manipulate a part of the signal. I consider players who can replace a signal entirely and model this by considering the choice of probability distributions over the set of signal generating elementary events. I characterize the set of strategies in the original game for evolutionary dynamics of signalcontingent strategies, if no population has an incentive to manipulate the signal generating process.

The remainder of this chapter is structured as follows: section 2 sets up the model, section 3 lists propositions which are already available in the literature and which I transfer to my model to characterize asymptotically stable sets of correlated equilibria. Section 4 gives some examples. Section 5 shows the generalization of the subpopulation model of Lenzo and Sarver (2006), section 6 characterizes the set of stable outcomes that have a robust signal generating process and the appendix collects the remaining proofs.

2.2 Model

2.2.1 Static Model

We give a brief description of the model before I proceed to define it formally. At each point of time, nature randomly draws a tuple of agents from a fixed set of populations. A signal generating process reveals information to each of the active agents, this information may be correlated. Each agent chooses a strategy to interact with the other agents in a normal form game. Each agent is characterized by a *rule* that prescribes the strategic choice given the received signal. The resulting payoff determines wether the applied rule spreads in the population.

Let $\Gamma = \{\mathcal{N}, S, f\}$ be a finite game in normal form where $\mathcal{N} = \{1, \ldots, N\}$ is the set of population, $S = \times_{i\in\mathcal{N}} S^i$ and $S^i = \{s^i_1, \ldots, s^i_{m_i}\}$ is population i's finite set of pure strategies and $f : S \to \mathbb{R}^N$ is a utility or fitness function. Each population consists of infinitely many agents. Let $\Sigma^i = \Delta(S^i)$ be the set of probability measures on S^i and let $\hat{\Sigma}^i$ be a finite subset of Σ^i that contains the vertices of Σ^i. Let $\Sigma = \times_{i\in\mathcal{I}} \Sigma^i$ be the set of product measures on S, define $\hat{\Sigma} = \times_{i\in\mathcal{I}} \hat{\Sigma}^i$ accordingly. $\Delta = \Delta(S)$ is the set of all probability measures on S. Denote by $s^{-i} = (s^1, \ldots, s^{i-1}, s^{i+1}, \ldots, s^N)$ a vector of strategies without the one of population i and by $S^{-i} = \times_{j\in\mathcal{N}\setminus\{i\}} S^j$ the Cartesian product of all but i's strategy spaces. Define

$\Sigma^{-i} = \times_{j \in \mathcal{N} \setminus \{i\}} \Sigma^j$ and $\Delta^{-i} = \times_{j \in \mathcal{N} \setminus \{i\}} \Delta(S^j)$. I extend f to the space of mixed strategies, $f : \Sigma \rightarrow \mathbb{R}$, defined by $f^i(\sigma^i, \sigma^{-i}) = \sum_{s \in S} f^i(s) \prod_{j \in \mathcal{N}} \sigma^j(s^j) \; \forall \; i \in \mathcal{N}$.
A Strategy $s^i \in S^i$ is *dominated* if there exists some mixed strategy $\sigma^i \in \Sigma^i$ such that $f^i(s^i, \sigma^{-i}) \leq f^i(\sigma^i, \sigma^{-i}) \; \forall \; \sigma^{-i} \in \Sigma^{-i}$, with strict inequality for at least one σ^{-i}. If the inequality is strict for all σ^{-i}, s^i is *strictly dominated.* It is immediate to show that if s^i is dominated then there is a dominating strategy σ^i with $\sigma^i(s^i) = 0$.

A strategy tuple $\sigma = (\sigma^i, \sigma^{-i})$ is a Nash Equilibrium (NE) in Γ, if $\forall \; i \in \mathcal{N}, \; f^i(\sigma^i, \sigma^{-i}) - f^i(s^i_h, \sigma^{-i}) \geq 0 \; \forall \; s^i_h \; \in S^i$.

Following Aumann (1987), I define a probability space $\langle \Omega, \mathcal{A}, P \rangle$ which generates signals (that are potentially correlated) on which agents can condition their strategic choices. Both, the original game Γ and the probability space constitute the primitives of my model. Assume Ω to be a nonempty and finite set of generic elements ω. Let $\mathcal{A}$ be the powerset of Ω and let $\{\mathcal{A}^i\}_{i \in \mathcal{N}}$ be a collection of partitions of Ω. $\mathcal{A}^i$ represents an information structure for population i; if nature draws an elementary event $\omega \in \Omega$, population i knows $A^i \in \mathcal{A}^i$ if and only if $\omega \in A^i$. Since for each population i there may be events ω, ω' that i cannot distinguish, the agents are not able to 'learn' P. Therefore, I need to assume P to be a *common prior* on $(\Omega, \mathcal{A})$. I regard P as an objective statistic environment. Without loss of generality, I assume that $P(\omega) > 0 \; \forall \; \omega \in \Omega$. All subjectivity enters the model via the set of partitions $\{\mathcal{A}^i\}_{i \in \mathcal{N}}$. I define the signaling structure $\mathcal{I} = \left\{ \langle \Omega, \mathcal{A}, P \rangle, \{\mathcal{A}^i\}_{i \in \mathcal{N}} \right\}$. I refer to an element $\omega \in \Omega$ as a complete description of a state of the world while I call an element $A^i \in \mathcal{A}^i$ a signal for the true state of the world. I assume that each agent has access to some private randomization device that allows for independent mixing, such that any mixed strategy in $\hat{\Sigma}^i$ is available. Wherever necessary, I assume that $\hat{\Sigma}^i$ is rich enough. Define $A^i(\omega) = \{A^i \in \mathcal{A}^i \mid \omega \in A^i\}$ the information set available to an agent in population i if nature draws ω. Throughout the model I make the assumption that the populations' fitnesses (represented by $f : S \Rightarrow \mathbb{R}^N$) do not depend on any ω. This is because I want to show that *even if information is payoff-irrelevant*, outcomes that are no Nash-equilibra of Γ can be stable under boundedly rational behavior, if agents perceive correlated signals.

Let a *rule* be a mapping from the set of states to strategies, $r^i : \Omega \rightarrow \hat{\Sigma}^i$. I assume for all i that r^i is $\mathcal{A}^i$-measurable, that is if for some ω, $r^i(\omega) = \sigma^i$ then $r^i(\omega') = \sigma^i \; \forall \; \omega' \in A^i(\omega)$. In words, agents cannot distinguish states that are in the same information set A. Define as $r^i_{s^i}(\omega)$ the probability with which an agent that uses rule r^i chooses strategy s^i given event ω, that is $r^i_{s^i}(\omega) = \sigma^i(s^i)$, where $\sigma^i = r^i(\omega)$. Denote the finite set of all rule-profiles by $\mathcal{R}$. I denote the share of agents in population i applying rule r^i by

$\rho^i(r^i)$, the set of all shares in population i, ρ^i by $\Delta(\mathcal{R}^i)$, the set of all population shares ρ by $\Delta_{\mathcal{R}} = \underset{i\in\mathcal{N}}{\times}\Delta(\mathcal{R}^i)$. As before, I denote by r^{-i} the vector r without the element r^i, and by ρ^{-i} the vector ρ without the element ρ^i. Denote by $\mathcal{F} : \Delta_{\mathcal{R}} \to \mathbb{R}^N$ the expected fitness from the choice of the rules, where the components are defined as follows: $\mathcal{F}^i(\rho) = \sum\limits_{\omega\in\Omega} P(\omega) \sum\limits_{r\in\mathcal{R}} f^i(r(w)) \prod\limits_{j\in\mathcal{N}} \rho^j(r^j)$. Given the signaling structure $\mathcal{I}$ and the normal form game Γ, I call $\mathcal{G}_{(\mathcal{I},\Gamma)} = \{\mathcal{N}, \mathcal{R}, \mathcal{F}\}$ the *expansion* of Γ.

A rule $r^i \in \mathcal{R}^i$ is strictly dominated if there exists some population share $\rho^i \in \Delta(\mathcal{R}^i)$ such that $\mathcal{F}^i(r^i, \rho^{-i}) < \mathcal{F}^i(\rho^i, \rho^{-i}) \; \forall \; \rho^{-i} \in \Delta(\mathcal{R}^{-i})$.

To get a flavor of the model, I begin the analysis with a very straight forward result that is helpful to show the extinction of dominated strategies.

Lemma 1

If strategy s^i is strictly dominated in Γ by some mixed strategy $\hat{\sigma}^i \in \hat{\Sigma}^i$, any rule r^i with $r^i_{s^i}(\omega) > 0$ for some ω is strictly dominated in the game $\mathcal{G}_{(\mathcal{I},\Gamma)}$, if $\hat{\Sigma}^i$ is rich enough.

Proof

Assume without loss of generality that $\hat{\sigma}^i(s^i) = 0$. Define for each $\omega \in \Omega$ the new rule $\hat{r}^i_{\tilde{s}^i}(\omega) = r^i_{\tilde{s}^i}(\omega) + r^i_{s^i}(\omega) \cdot \hat{\sigma}^i(\tilde{s}^i) \; \forall \; \tilde{s}^i \neq$

s^i and $\hat{r}^i_{s^i}(\omega) = 0$. It is easy to verify that $\hat{r}^i(\omega) \in \Sigma^i \ \forall\ \omega$, however I need to assume that $\hat{\Sigma}^i$ is rich enough such that $\hat{r}^i(\omega) \in \hat{\Sigma}^i \ \forall\ \omega$. For convenience I define $f^i(r^i(\omega), \rho^{-i}(\omega)) = \sum_{r^{-i}} f^i(r^i(\omega), r^{-i}(\omega)) \prod_{j \neq i} \rho^j(r^j)$. We then have $\forall\ \rho^{-i} \in \Delta_{\mathcal{R}^{-i}}$:

$$\begin{aligned}\mathcal{F}^i(\hat{r}^i, \rho^{-i}) &= \mathcal{F}^i(r^i, \rho^{-i}) + \\ &\underbrace{\sum_{\substack{\omega\in\Omega \\ r^i_{s^i}(\omega)>0}} P(\omega) r^i_{s^i} \left[f^i(\hat{\sigma}^i, \rho^{-i}(\omega)) - f^i(s^i, \rho^{-i}(\omega)) \right]}_{>0}\end{aligned}$$

□

A strategy s^i is iteratively strictly dominated in Γ if there exists a sequence $\{s^{i_t}, \Gamma_t\}_{t=0}^n$ such that s^{i_t} is strictly dominated in Γ_t, where Γ_t is obtained from Γ_{t-1} by removing $s^{i_{t-1}}$ from i_{t-1}'s set of pure strategies in Γ_{t-1}, $\Gamma = \Gamma_0$ and $s^i = s^{i_n}$. The same definition applies for a rule r^i in the game $\mathcal{G}$.

As a consequence of Lemma 1 one can state an analogous statement for iteratively strictly dominated rules:

Lemma 2
If strategy s^i is iteratively strictly dominated in Γ by some mixed strategy $\hat{\sigma}^i \in \hat{\Sigma}^i$, any rule r^i with $r^i_{s^i}(\omega) > 0$ for some ω is iteratively strictly dominated in the game $\mathcal{G}_{(\mathcal{I},\Gamma)}$, if $\hat{\Sigma}^i$ is rich enough.

Definition CORRELATED EQUILIBRIUM (*c.e.*)
Given $\mathcal{I}$, a *correlated equilibrium* in Γ is a mixed rule $\rho \in \Delta_{\mathcal{R}}$ such that for all i, $\mathcal{F}^i(\rho) \geq \mathcal{F}^i(\tilde{\rho}^i, \rho^{-i}) \ \forall \ \tilde{\rho}^i \in \Delta(\mathcal{R}^i)$. A *c.e.* is *strict*, if inequalities hold strictly for all $\tilde{\rho}^i \neq \rho^i$ and $i \in \mathcal{N}$.

Here, an equilibrium is a point in the set of rules. The original definition by Aumann (1987) is for distributions on the space of strategies. See Fudenberg and Tirole (1991, pp. 56) for a short discussion. I regard the distribution over the set of strategies as a result of the model, not a primitive.

Definition INDUCED DISTRIBUTION
Let $\rho \in \Delta_{\mathcal{R}}$ be some distribution over the set of rules. Then $\mathcal{I}$ and ρ induce a distribution over the set of outcomes. I define $\forall \ s \in S$:

$$\lambda(s) = \sum_{\omega} P(\omega) \prod_{i \in \mathcal{N}} \sum_{r^i \in \mathcal{R}^i} \rho^i(r^i) \cdot r^i_{s^i}(\omega)$$

Definition CORRELATED EQUILIBRIUM DISTRIBUTION (*c.e.d.*)
A distribution $\lambda \in \Delta$ induced by $\mathcal{I}$ and a *c.e.* ρ is a *correlated equilibrium distribution.*

Given some expanded game $\mathcal{G}_{(\mathcal{I},\Gamma)}$, there may exist multiple *c.e.* ρ, some being strict and some other being non-strict. See Example 5.1 .

Fix some signal generating process $\mathcal{I}$. Then, a mixed rule $\rho \in \Delta_{\mathcal{R}}$ is a *c.e.* in Γ, if and only if ρ is a Nash equilibrium of expanded game $\mathcal{G}_{(\mathcal{I},\Gamma)}$. I discuss my choice of the signal space. In an earlier version of this chapter, I had $\Omega = S$ and $\mathcal{A}^i = \{\{s^i \times S^{-i}\}_{s^i}\}$, that is each population gets a recommendation to play a particular strategy. I call this signal space *direct*, Kim and Wong (2007) use this direct signal space. With direct signals, it is optimal to follow the recommendation if the signals are distributed according to a *c.e.d.* . However, two problems come with this approach: firstly, even if the distribution of signals P is a *c.e.d.*, it might still be an equilibrium if the agents deviate from the recommendation (see example 4.1). Secondly, if one pins down a special signal generating process, one can always construct a meta game in which agents can condition their choice of rules on some extra signals they might receive. The general formulation of the signal space includes such extra signals.

Definition EVOLUTIONARY STABILITY (Swinkels (1992)) $\rho \in \Delta_{\mathcal{R}}$ is evolutionary stable in $\mathcal{G}_{(\mathcal{I},\Gamma)}$, if for every rule profile $\tilde{\rho} \neq \rho$ there exists some $\bar{\epsilon}_{\tilde{\rho}} \in (0,1)$ such that for all $\epsilon \in (0, \bar{\epsilon}_{\tilde{\rho}})$, and with $\tilde{\tilde{\rho}} = \epsilon\tilde{\rho} + (1-\epsilon)\rho$,

$$\mathcal{F}^i(\rho^i, \tilde{\tilde{\rho}}^{-i}) > \mathcal{F}^i(\tilde{\rho}^i, \tilde{\tilde{\rho}}^{-i}) \text{ for some } i \in \mathcal{N}.$$

It follows immediately that a rule is evolutionary stable if and only if it is a strict Nash equilibrium of $\mathcal{G}$.[5] Note that the above definition is for multi-population games.

The definitions of evolutionary stable *sets* by Thomas (1985), Balkenborg and Schlag (2001) and Cressman (2003) are all specified for symmetric one population games. Therefore I do not list them but state a concept for general asymmetric games:

Definition STRICT EQUILIBRIUM SET (SEset) (Balkenborg (1994))
A nonempty set $F \subset \Delta_{\mathcal{R}}$ is a *strict equilibrium set* if it is a set of Nash equilibria of $\mathcal{G}$ that is closed under mixed-rule best replies by each population i, i.e. if for some $\rho \in F$, $(\tilde{\rho}^i, \rho^{-i}) \in F$ whenever $\mathcal{F}^i(\tilde{\rho}^i, \rho^{-i}) = \mathcal{F}^i(\rho)$ for each population i.

Such a set does not need to exist, see Example 5.4.

Theorem (Balkenborg and Schlag (2007)[6], Cressman (2003)[7])
If F is an SESet of $\mathcal{G}$, then F is a finite union of faces of

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[5] See Swinkels (1992), Theorem 2.

[6] Proposition 2, p.299

[7] Theorem 3.1.2, p.71

$\Delta_{\mathcal{R}}$. In particular, F is closed and contains at least one pure rule $r \in \mathcal{R}$.

2.2.2 Dynamic Model

We assume that at each point in time, agents update their behavior such that the population shares $\rho = (\rho^1, \ldots, \rho^n)$ change according to the regular differential equation

$$\dot{\rho}^i(r^i) = g^i(r^i, \rho) \cdot \rho^i(r^i), \ \forall \ r^i \in \mathcal{R}^i, \ \forall \ i \in \mathcal{N}, \tag{2.1}$$

where regularity presumes that $g = \times_{i \in \mathcal{N}} g^i$ is (i) Lipschitz continuous on $\Delta_{\mathcal{R}} = \times_{i \in \mathcal{N}} \Delta(\mathcal{R}^i)$ and (ii) $g^i(\cdot, \rho) \cdot \rho^i = 0 \ \forall \rho \in \Delta_{\mathcal{R}}$. By the Picard-Lindelöf Theorem[8], there exists a unique solution $\hat{\rho}(\cdot, \rho)$ for each initial condition $\rho \in \Delta_{\mathcal{R}}$.

The following definition is taken from Hofbauer and Weibull (1996):[9]

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[8] A function $\phi : X \rightarrow \mathbb{R}^k$, where $X \subset \mathbb{R}^k$, is (locally) Lipschitz continuous if for every compact subset $C \subset X$ there exists some real number λ such that it holds for all $x, y \in C$: $||\phi(x) - \phi(y)|| \leq \lambda ||x - y||$. If $X \subset \mathbb{R}^k$ is open and the vector field $\phi : X \rightarrow \mathbb{R}^k$ is Lipschitz continuous, then the system $\dot{x} = \phi(x)$ has a unique solution $\hat{x}(\cdot, x^0) : T \rightarrow X$ through every state $x^0 \in X$. Moreover, $\hat{x}(t, x^0)$ is continuous in $t \in T$ and $x^0 \in X$. (Weibull (1995) pp.232)

[9] Convex monotonicity is implied by aggregate monotonicity, it is not implied by and does not imply monotonicity (both Samuelson and Zhang (1992), Definition 3, p.369)

(2.1) is *convex monotonic* (CM), if it satisfies

$$\mathcal{F}^i(r^i, \rho^{-i}) < \mathcal{F}^i(\rho^i_k, \rho^{-i}) \Leftrightarrow g^i(r^i, \rho) < g^i(\cdot, \rho) \cdot \rho^i_k \ \forall\ i \in \mathcal{N}\ .$$

(2.1) is the *replicator dynamics*, if

$$g^i(r^i, \rho) = \mathcal{F}^i(r^i, \rho^{-i}) - \mathcal{F}^i(\rho) \ \forall\ r^i \in \mathcal{R}^i \text{ and } i \in \mathcal{N}.$$

Define $\rho^+ = \{\rho' \in \Delta_{\mathcal{R}} \mid \exists\ t \ \in \mathbb{R}_+, \rho' = \hat{\rho}(t, \rho)\}$, as the subset of $\Delta_{\mathcal{R}}$ that is reached if the dynamics start at ρ.

Definition STABILITY

A closed set $\Lambda \subseteq \Delta_{\mathcal{R}}$ is *Lyapunov stable* if for every neighborhood $\mathcal{U}'$ of Λ there exists a neighborhood $\mathcal{U}''$ such that $\rho^+ \subset \mathcal{U}' \ \forall\ \rho \in \mathcal{U}'' \cap \Delta_{\mathcal{R}}$.

A closed set $A \subseteq \Delta_{\mathcal{R}}$ is *asymptotically stable* if it is Lyapunov stable and if there exists a neighborhood $\mathcal{U}$ of A such that $\hat{\rho}(t, \rho) \underset{t\to\infty}{\longrightarrow} A$ for all $\rho \in \mathcal{U} \cap \Delta_{\mathcal{R}}$.

2.3 Propositions

This section collects the propositions.

Proposition 1
Let g be convex monotonic. If $F \subset \Delta_{\mathcal{R}}$ is a Lyapunov stable set of rest points, then each $\rho \in F$ is a *c.e.* .

Proof: see Appendix A.

The converse of Proposition 1 is not true in general:
Fix some $\mathcal{I}$ and *c.e.* ρ in which for a population i, the rule $r^i \in \mathcal{R}^i : \rho^i(r^i) > 0$ is weakly dominated by some mixed rule $\tilde{\rho}^i$. Then there exists a neighborhood $\mathcal{U}$ of ρ such that $\forall\ \tilde{\tilde{\rho}} = (\rho^i, \tilde{\tilde{\rho}}^{-i}) \in \mathcal{U}, \tilde{\tilde{\rho}}^{-i} \in \text{int}(\Delta_{\mathcal{R}^{-i}})$ it holds that $\mathcal{F}^i(r^i, \tilde{\tilde{\rho}}^{-i}) < \mathcal{F}^i(\tilde{\rho}^i, \tilde{\tilde{\rho}}^{-i})$. Therefore, for some $\tilde{\tilde{\rho}}^{-i}$ there exists some $r^i_h \in \mathcal{R}^i$ with $\tilde{\rho}^i(r^i_h) > 0$ such that $\mathcal{F}^i(r^i_h, \tilde{\tilde{\rho}}^{-i}) > \mathcal{F}^i(r^i, \tilde{\tilde{\rho}}^{-i})$. Since (2.1) is monotonic, $g^i(r^i_h, \tilde{\tilde{\rho}}) > g^i(r^i, \tilde{\tilde{\rho}})$, contradicting Lyapunov stability.

The next propositions specify the relationship of asymptotic stability and correlated equilibrium:

Proposition 2 (cf Balkenborg and Schlag (2007), Theorem 6 and Cressman (2003), Theorem 4.5.3)
If a non-empty set $F \subset \Delta(\mathcal{R})$ of rules ρ is an asymptotically stable set of rest points under the standard replicator dynamic, F is a SEset.

Balkenborg and Schlag (2007) and Cressman (2003) actually show equivalence, if (2.1) is the replicator dynamic. Balkenborg and Schlag (2007) also show the reverse for a wide class of other dynamics. I show the reverse for the distinct class of convex monotone dynamics.[10]

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[10] In Proposition 13 Balkenborg and Schlag (2007) demand (A) that $g^i(r^i, \rho) \geq 0$ whenever r^i is a best response to ρ^{-i}, (B) that $g^i(r^i, \rho) > 0$ whenever r^i is a best response to ρ^{-i} but ρ^i is not and (C) that

Proposition 3
Let (2.1) be convex monotonic. If a set F is a SEset, then F is an asymptotically stable set of rest points.

Proof: see Appendix A.

If the process does not start in the interior of $\Delta_{\mathcal{R}}$, there may exist some $\rho_0 \in \Delta_{\mathcal{R}}$ such that $\lambda(\hat{\rho}(t, \rho_0))$ is not a *c.e.d.* for all $t > 0$, even if an asymptotically stable set exists.

Proposition 4 (Hofbauer and Weibull (1996) Theorem 1)
If a rule $r^i \in \mathcal{R}^i$ is iteratively strictly dominated and the process starts in the interior of the rulespace and if the selection dynamics (2.1) is convex monotonic, r^i gets eliminated.

We do not give a statement whether induced distribution over outcomes converges. Viossat (2004) shows for symmetric 3 × 3-games that the multipopulation replicator dynamics eliminates all strategies not used in a correlated

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$g^i(r^i, \rho) < 0$ whenever ρ^i is a best response to ρ^{-i} but r^i is not. Neither does convex monotonicity imply (A),(B) and (C) nor vice versa. Consider some ρ, $\tilde{\rho}^i$ and r^i such that $\mathcal{F}^i(\tilde{\rho}^i, \rho^{-i}) > \mathcal{F}^i(\rho) > \mathcal{F}^i(r^i, \rho^{-i})$. (A),(B),(C) imply that $g^i(r^i, \rho^{-i}) = 0$. From regularity I have $\rho^i \cdot g^i(\cdot, \rho) = 0$, hence g cannot be convex monotonic.

equilibrium (with interior initial conditions), however Viossat (2007) gives an example of a class of symmetric 4×4 games for which the replicator dynamics eliminates *all* strategies used in correlated equilibrium along interior solutions. Lemma 2 and Proposition 4 allow me to pin down a weaker result, namely to rule out iteratively strictly dominated outcomes in the induced distribution in the long run.

Corollary
If the process starts in the interior of the rulespace and if the selection dynamics (2.1) is convex monotonic, then $\lim_{t\to\infty} \lambda(t)$ attaches zero probability to outcomes s that involve strategies that are iteratively strictly dominated, if $\hat{\Sigma}$ is rich enough.

2.4 Examples

This section demonstrates how the model can be applied to various examples. The examples are complementary to the propositions of the preceeding section.

2.4.1 A Coordination Game

This example illustrates that one signal generating process $\mathcal{I}$ allows for multiple stable rules r, r' that do not induce the same distribution λ over outcomes S. Even if the signal generating process $\mathcal{I}$ itself is a distribution over S and is regarded as a 'recommendation', other strategy choices can well be stable.

Let the game Γ be defined by $\mathcal{N} = \{1, 2\}$, $S = \{u, d\} \times \{l, r\}$ and

$$f = \begin{array}{c|c|c|} & l & r \\ \hline u & (1{,}1) & (0{,}0) \\ \hline d & (0{,}0) & (1{,}1) \\ \hline \end{array} \; .$$

We specify $\mathcal{I}$ with $\Omega = S$, $\mathcal{A}^1 = \big\{\{ul, ur\}, \{dl, dr\}\big\}$, $\mathcal{A}^2 = \big\{\{lu, ld\}, \{ru, rd\}\big\}$. A rule for population 1 (row) assigns a strategy for the first and the second element of $\mathcal{A}^1$ respectively. UD means "choose u if $\omega \in \{ul, ur\}$ and choose d if $\omega \in \{dl, dr\}$". I analogously denote the rules of population 2.

$\mathcal{F}_i =$

	LL	LR	RL	RR
UU	1	$P(\{ul, dl\})$	$P(\{ur, dr\})$	0
UD	$P(\{ul, ur\})$	$P(\{ul, dr\})$	$P(\{ur, dl\})$	$P(\{dl, dr\})$
DU	$P(\{dl, dr\})$	$P(\{dl, ur\})$	$P(\{ul, dr\})$	$P(\{ul, ur\})$
DD	0	$P(\{ur, dr\})$	$P(\{ul, dl\})$	1

for $i = 1, 2$

The rules (UU, LL) and (DD, RR) are the strict correlated equilibria that correspond to the Nash equilibria of the original game for any P with full support. Consider P to be a uniform measure over Ω. Then, (UD, LR) is a non-strict *c.e.*. Even if the signal space Ω equals the set of outcomes S, the induced distribution λ does not need to coincide with P, although it still may be a *c.e.d* . Suppose $P(\{ul\}) = p$ and $P(\{dr\}) = 1-p$. The pair (DU, RL) is a strict *c.e.* and induces the following distribution over the set of outcomes: $\lambda(ul) = 1-p$ and $\lambda(dr) = p$.

2.4.2 Chicken

A non-Nash outcome may be asymptotically stable.

Consider the "chicken game" originally presented in Aumann (1974):

$$\Gamma = \begin{array}{c|c|c|}
 & l & r \\ \cline{2-3}
u & (6,6) & (2,7) \\ \cline{2-3}
d & (7,2) & (0,0) \\ \cline{2-3}
\end{array}$$

Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and let $\mathcal{A}^1 = \big\{\{\omega_1, \omega_2\}, \{\omega_3\}\big\}$ and $\mathcal{A}^2 = \big\{\{\omega_1\}, \{\omega_2, \omega_3\}\big\}$, let $P(\omega) \equiv \frac{1}{3}$. Given this $\mathcal{I}$, the resulting expanded game is

$$\mathcal{G}_{(\Gamma,\mathcal{I})} = \begin{array}{c|c|c|c|c|}
 & \text{LL} & \text{RL} & \text{LR} & \text{RR} \\ \cline{2-5}
\text{UU} & (6,6) & (4\frac{2}{3}, 6\frac{1}{3}) & (3\frac{1}{3}, 6\frac{2}{3}) & (2,7) \\ \cline{2-5}
\text{UD} & (6\frac{1}{3}, 4\frac{2}{3}) & (5,5) & (2\frac{2}{3}, 4\frac{1}{3}) & (1\frac{1}{3}, 4\frac{2}{3}) \\ \cline{2-5}
\text{DU} & (6\frac{2}{3}, 3\frac{1}{3}) & (4\frac{1}{3}, 2\frac{2}{3}) & (3,3) & (\frac{2}{3}, 2\frac{1}{3}) \\ \cline{2-5}
\text{DD} & (7,2) & (4\frac{2}{3}, 1\frac{1}{3}) & (2\frac{1}{3}, \frac{2}{3}) & (0,0) \\ \cline{2-5}
\end{array}$$

$r = (UD, LR)$ is a strict *c.e.*, hence r is a singleton evolutionary stable rule and therefore asymptotically stable in any convex monotonic dynamics. As is well known, the payoffs generated by r lie outside the convex hull of the Nash equilibria of the original game Γ.

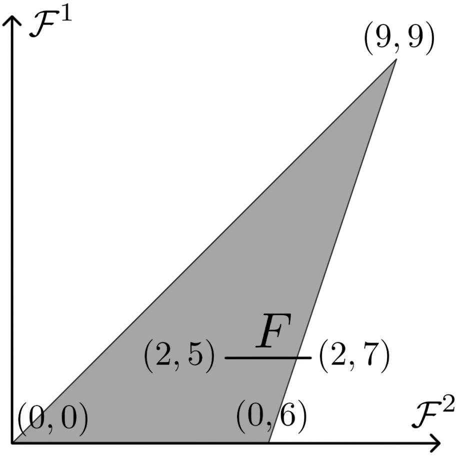
2.4.3 A SEset of correlated equilibria

The chicken example above shows that a single outcome can be asymptotically stable producing payoffs that lie outside the convex hull of the Nash equilibrium outcomes. This example does the same for a *set* of outcomes. Consider the following game

$$\Gamma = \begin{array}{c|c|c|} & l & r \\ \hline u & (0,0) & (0,6) \\ \hline m & (3,-6) & (0,0) \\ \hline d & (9,9) & (-3,6) \\ \hline \end{array}.$$

The pure Nash equilibria are (u, r), (m, r) and (d, l), the unique mixed Nash equilibrium is $(\sigma^1(m) = \frac{1}{3}, \sigma^1(d) = \frac{2}{3}, \sigma^2(l) = \frac{1}{3})$. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\mathcal{A}^1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}, \mathcal{A}^2 = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}$, $P(\omega) \equiv 3$. Each population has two signals, the row population therefore has 9 rules, column has 4 rules. The payoff matrix of the expanded game is given by

$$\mathcal{F} = \begin{array}{c|c|c|c|c|}
 & LL & LR & RL & RR \\ \hline
UU & (0,0) & (0,4) & (0,2) & (0,6) \\ \hline
UM & (1,\text{-}2) & (0,2) & (1,0) & (0,4) \\ \hline
UD & (3,3) & (\text{-}1,4) & (3,5) & (\text{-}1,6) \\ \hline
MU & (2,\text{-}4) & (1,0) & (0,2) & (0,2) \\ \hline
MM & (3,\text{-}6) & (1,\text{-}2) & (2,\text{-}4) & (0,0) \\ \hline
MD & (5,\text{-}1) & (0,0) & (4,1) & (\text{-}1,2) \\ \hline
DU & (6,6) & (\mathbf{2},\mathbf{7}) & (2,5) & (\text{-}2,6) \\ \hline
DM & (7,4) & (\mathbf{2},\mathbf{5}) & (3,3) & (\text{-}2,4) \\ \hline
DD & (9,9) & (1,7) & (5,8) & (\text{-}3,6) \\ \hline
\end{array}.$$



In the figure, the shaded triangle is the convex hull of the Nash equilibrium payoffs of Γ, the thick line connecting the points $(2,5)$ and $(2,7)$ represents the SEset $F = \{\rho \in \Delta(\mathcal{R}) \mid \rho^1(DM) = 1 - \rho^1(DU), \rho^2(LR) = 1\}$, which is not fully contained in the convex hull.

2.4.4 Matching Pennies

Non-existence of SEset.

Consider the original two population game with strategies $\{h,t\}^2$ and payoff matrix

$$\Gamma = \begin{array}{c|c|c|} & h & t \\ \cline{2-3} h & (1,-1) & (-1,1) \\ \cline{2-3} t & (-1,1) & (1,-1) \\ \cline{2-3} \end{array}$$

Let the information structure be given by a singleton $\Omega = \{\omega\}$, in other words let there be no signals. Therefore the rules coincide with the strategies. The set of Nash equilibria of $\mathcal{G}$ has only one element which is not strict (and hence is not closed under mixed-rule best replies). In fact, any information structure $\mathcal{I} = \left\{ \langle \Omega, \mathcal{A}, P \rangle, \{\mathcal{A}^i\}_{i\in\mathcal{N}} \right\}$ that has a common prior induces an expanded game $\mathcal{G}_{\{\mathcal{I},\Gamma\}}$ which has no SEset. If instead of P there would exist some subjective priors $\{P^i\}_{i\in\mathcal{I}}$ with $P^i : \mathcal{A}^i \to \mathbb{R}$ violating the common prior assumption, it would be straightforward to construct an expansion of Γ with strict equilibria, see Aumann and Dreze (2005), example 6.5 .

2.5 Subpopulations

In this section, I illustrate that the model of Lenzo and Sarver (2006) can be expressed as a special case of the general formulation of the model presented in this thesis. I give a special interpretation of the signals: a signal assigns one of finitely many subpopulations to each agent. Let each population i have a set of subpopulations $M^i = \{m^i_1, \dots, m^i_{|M^i|}\}$, defining $M = \bigtimes_{i \in \mathcal{N}} M^i$. Denote by $x^{m^i}_{s^i}$ the share of agents in subpopulation m^i that choose strategy s^i. Let $\eta \in \Delta(M)$ be a probability distribution over M, with $\eta(m^i, \cdot) > 0\ \forall m^i \in M^i$ and $i \in \mathcal{N}$. Note that this distribution may be correlated and that there may be matches $m \in M$ that receive zero-probability.

We show that given a game Γ, for any M, η with state x, there is an $\mathcal{I}$ and a state ρ such that the induced distributions are the same. I therefore can represent any state x of the subpopulations model by a state ρ of our model if I give a particular specification of the signalling structure. Furthermore I show that ρ needs not to be unique and that the dynamic properties of x and ρ need not be the same.

Let $\Omega = M$, $\mathcal{A}^i = \big\{\{m^i \times M^{-i}\}_{m^i \in M^i}\big\}$, $P = \eta$ and $\rho^i(r^i) = \prod_{m^i \in M^i} x^{m^i}_{r^i(m^i)}$.[11] First I show that

---

[11] More precisely: $r^i(m^i) = r^i(m^i, m^{-i})$ for some m^{-i} ($r^i(m^i, m^{-i})$ has the same value $\forall\ m^{-i}$).

$\sum_{r^i \in \mathcal{R}^i} \rho^i(r^i) = 1$. Note that $\sum\limits_{\substack{r^i \in \mathcal{R}^i \\ r^i(m_1^i) = s^i}} \prod\limits_{\substack{m^i \in M^i \\ m^i \neq m_1^i}} x^{m^i}_{r^i(m^i)} = \sum\limits_{\substack{r^i \in \mathcal{R}^i \\ r^i(m_1^i) = s_h^i}} \prod\limits_{\substack{m^i \in M^i \\ m^i \neq m_1^i}} x^{m^i}_{r^i(m^i)} \ \forall \ s^i, s_h^i \in S^i$.

$$
\begin{aligned}
\sum_{r^i \in \mathcal{R}^i} \rho^i(r^i) &= \sum_{r^i \in \mathcal{R}^i} \prod_{m^i \in M^i} x^{m^i}_{r^i(m^i)} \\
&= \sum_{s^i \in S^i} \sum_{\substack{r^i \in \mathcal{R}^i \\ r^i(m_1^i) = s^i}} x^{m_1^i}_{s^i} \prod_{\substack{m^i \in M^i \\ m^i \neq m_1^i}} x^{m^i}_{r^i(m^i)} \\
&= \underbrace{\left(\sum_{s^i \in S^i} x^{m_1^i}_{s^i} \right)}_{=1} \sum_{\substack{r^i \in \mathcal{R}^i \\ r^i(m_1^i) = s_h^i}} \prod_{\substack{m^i \in M^i \\ m^i \neq m_1^i}} x^{m^i}_{r^i(m^i)} \\
&\vdots \\
&= \sum_{s^i \in S^i} \sum_{\substack{r^i \in \mathcal{R}^i \\ r^i(m_1^i) = s_h^i \\ \vdots \\ r^i(m^i_{|M^i|-1}) = s_h^i \\ r^i(m^i_{|M^i|}) = s^i}} \prod_{\substack{m^i \in M^i \\ m^i \neq m_1^i \\ \vdots \\ m^i \neq m^i_{|M^i|-1} \\ m^i = m^i_{|M^i|}}} x^{m^i_{|M^i|}}_{s^i} \\
&= \sum_{s^i \in S^i} x^{m^i_{|M^i|}}_{s^i} \cdot 1
\end{aligned}
$$

Now I calculate $\lambda(s)$ for some $s \in S$:

$$\begin{aligned}
\lambda(s) &= \sum_{\omega\in\Omega} P(\omega) \prod_{i\in\mathcal{N}} \sum_{\substack{r^i\in\mathcal{R}^i \\ r^i(\omega)=s^i}} \rho^i(r^i) \\
&= \sum_{m\in M} \eta(m) \prod_{i\in\mathcal{N}} \sum_{\substack{r^i\in\mathcal{R}^i \\ r^i(m^i)=s^i}} \prod_{m^i_k\in M^i} x^{m^i_k}_{r^i(m^i_k)} \\
&= \sum_{m\in M} \eta(m) \prod_{i\in\mathcal{N}} x^{m^i}_{s^i} \sum_{\substack{r^i\in\mathcal{R}^i \\ r^i(m^i)=s^i}} \prod_{\substack{m^i_k\in M^i \\ m^i_k\neq m^i}} x^{m^i_k}_{r^i(m^i_k)}
\end{aligned}$$

From the third line of the calculation of $\sum\limits_{r^i\in\mathcal{R}^i} \rho^i(r^i)$ I know that $\sum\limits_{\substack{r^i\in\mathcal{R}^i \\ r^i(m^i)=s^i}} \prod\limits_{\substack{m^i_k\in M^i \\ m^i_k\neq m^i}} x^{m^i_k}_{r^i(m^i_k)} = 1$ and have the desired result that the distributions over outcomes are the same. However, there is no one-to-one mapping from one model to the other model. Consider the following simple example with $M = M^1 \times M^2 = \{m^1_1, m^1_2\} \times \{m^2_1, m^2_2\}$, $S = \{s^1_1, s^1_2\} \times \{s^2_1, s^2_2\}$, $\eta(m) = \frac{1}{4}\ \forall\, m \in M$, $x^{m^i_1}_{s^i_1} = x^{m^i_2}_{s^i_2} = 1$, in words: for each population i all agents of subpopulation 1 choose their strategy 1 and all agents from subpopulation 2 choose their strategy 2. If $\Omega = M$, $\mathcal{A}^1 = \big\{\{(m^1_1, m^2_1), (m^1_1, m^2_2)\}, \{(m^1_2, m^2_1), (m^1_2, m^2_2)\}\big\}$ (and $\mathcal{A}^2$ analogous), $P = \eta$ and ρ as constructed above, I have probability mass one on the rule $r^i : r^i(m^i_1) = s^i_1, r^i(m^i_2) = s^i_2$. Alternatively, but for the same Ω, $\mathcal{A}^i$, P, I could assign $\tilde{\rho}^i(r^i) = \frac{1}{4}\ \forall\ r^i \in \mathcal{R}^i$. Both ρ and $\tilde{\rho}$ induce the same distribution λ but while ρ is pure, $\tilde{\rho}$ is completely mixed and therefore ρ and $\tilde{\rho}$ have different dynamic properties.

2.6 Robust Signals

Until now, I assumed that the signal generating process is stationary. This is plausible, if the signals originate from an object that is completely exogenous, i.e. if they are independent from interaction – a somehow polar case. The other polar case would be that the agents themselves can choose messages that serve as signals. I regard situations in which one population i can alter the complete signal and consider the case in which population i can choose a particular probability distribution P. I offer the following interpretation: suppose some institution determines P. Every population knows the design of the institution and therefore has access to the information how the institution determines P. Population i can influence the institution, because – for example – some key positions within the institution are held by members of population i. In this section I derive conditions such that population i does not have an incentive to change P in a stable state ρ. Suppose nature draws a certain elementary event $\omega \in \Omega$. Then, for a given distribution of rules $\rho = \{\rho^i\}_i$, population i's expost payoff is $f^i(\rho(\omega))$. Population i has an incentive to change P if there is some other event ω' with $f^i(\rho(\omega)) < f^i(\rho(\omega'))$. This leads to the following definition:

Definition *Robust to Manipulation*
Given ρ, $P \in \Delta(\Omega)$ is *robust to manipulation* if for all populations i

$$P(\omega) > 0 \Rightarrow f^i(\rho(\omega)) \geq f^i(\rho(\omega')) \ \forall \omega' \in \Omega \ .$$

If a distribution P is robust to manipulation given ρ, no population (regardless wether it has the capability to change P or not) has an incentive to manipulate P. I do not demand that any population *can* change P. I characterize those pairs (P, ρ) such that no population *wants* to change P given ρ. Nevertheless, I implicitly assumed some constrained reasoning. Suppose there is some mapping $g : \Delta(\Omega) \rightrightarrows \Delta(\mathcal{R})$ such that given distribution P, agents play an equilibrium $\rho \in g(P)$. In the approach above, agents believe g to be singlevalued and constant and agents compare the ex-post payoffs. Alternatively, one could argue that population i does not have an incentive to change P to P' if $\mathcal{F}^i_P(\rho) \geq \mathcal{F}^i_{P'}(\rho') \ \forall \ \rho' \in g(P')$. That is, no population has an incentive to change P, if P maximizes ex-ante payoffs for all equilibrium choices ρ', where the equilibrium choice well depends on the distribution P. The consequences of this definition are more exclusive in the sense that it is easy to find a game such that no stable state ρ has a robust distribution P.[12]

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[12] For example, the chicken game, the battle of the sexes game,...

2.6.1 Results

Consider again the general setting, with $\Gamma = \{\mathcal{N}, S, f\}$, $\mathcal{I} = \{\{\Omega, \mathcal{P}(\Omega), P\}, \{\mathcal{A}\}_{i\in\mathcal{N}}\}$ yielding the expanded game $\mathcal{G}_P = \{\mathcal{N}, \mathcal{R}, \mathcal{F}_P\}$ (making the dependence on P explicit). Define $\Delta_P^{\sim} \subset \Delta(\mathcal{R})$ as the set of rules ρ such that P is robust to manipulation. Define $\Delta_P^{CE} \subset \Delta(\mathcal{R})$ as the set of correlated equilibria given P. Our first result is immediate:

Proposition 5
$\Delta_P^{\sim} \cap \Delta_P^{CE} \neq \emptyset \; \forall \; P \; \in \Delta(\Omega)$.

Proof
Consider a Nash equilibrium $\sigma \in \Delta(S)$ of the original game Γ. Define the rule ρ such that for all $i \in \mathcal{N}$ and $s^i \in S^i$, $\rho^i(r^i) = \sigma^i(s^i)$ for $r^i : r^i(\omega) \equiv s^i$. Clearly, ρ is a correlated equilibrium of Γ given P, hence $\rho \in \Delta_P^{CE}$. No population conditions the choice of strategies on signals, hence $f^i(\rho(\omega)) = f^i(\sigma) \; \forall \omega \in \Omega$ and therefore no population has an incentive to manipulate the generation of the signals. Hence, P is robust given ρ, $\rho \in \Delta_P^{\sim}$. $\square$

We argue that there always is a trivial correlated equilibrium in which agents choose Nash equilibrium strategies ignoring any signals. Since all agents ignore any signals, no agent has an incentive to manipulate the signals.

We cannot give a full characterization of $\Delta_P^{\sim} \cap \Delta_P^{CE}$, the set of correlated equilibria given P that induce P to be robust against manipulation. However, I suspect that it is a subset of rules that induce a distribution λ on the set of outcomes S that lies in the convex hull of Nash equilibria (see the Chicken Game example below). I leave this characterization to future work.

We can give a full characterization of $\Delta_P^{\sim} \cap F_P$, F_P being an asymptotically stable set of rules given P if I impose a further assumption on the payoffs of the original game Γ. Suppose Γ is generic[13]. Then all asymptotically stable sets are singleton. In this case, I can state that a probability measure P is robust to manipulation given ρ if and only if ρ puts probability one on rules that choose one strict Nash equilibrium.

Proposition 6
Suppose for each population i, the frequency $\rho^i(r^i)$ updates according to (2.1) and that (2.1) is convex monotonic. Suppose further that the original game Γ is generic. A set $F_P \subset \Delta(\mathcal{R})$ is asymptotically stable given (2.1) and a distribution P with full support on Ω, and P is robust given a $\rho \in F_P$, if and only if ρ attaches probability one to a rule that maps all signals to the same strict Nash equilibrium.

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[13] $f_i(s) \neq f_i(s') \ \forall s, s' \in S, s \neq s'$.

Proof

Suppose F_P is asymptotically stable and suppose P is robust given any $\rho \in F_P$. According to Proposition 2, F_P is a SEset, from genericity follows that F_P is singleton, i.e. $\rho = F_P$ puts probability one to a strict correlated equilibrium $r \in \mathcal{R}$. Because no population has an incentive to manipulate P given r, it must be that $f^i(r(\omega)) = f^i(r(\omega')) \, \forall \omega, \omega' \in \Omega, \forall \, i$. Since Γ is generic, it must be $r(\omega) = r(\omega') = s \; \forall \; \omega, \omega' \in \Omega$ and some $s \in S$. Since r is a strict correlated equilibrium, I have $\mathcal{F}^i(r(\omega)) > \mathcal{F}^i(\tilde{r}^i(\omega), r^{-i}(\omega)) \; \forall \; \tilde{r}^i \neq r^i, \forall \; \omega, \forall \; i \Rightarrow f^i(s) > f^i(\tilde{s}^i, s^{-i}) \; \forall \; \tilde{s}^i \in S^i, \; \forall \; i$. I conclude that s is a strict Nash equilbrium of Γ.

Suppose ρ attaches probability one to a rule $r \in \mathcal{R}$ that maps all signals to a strict Nash equilibrium $s \in S$ of Γ, $r(\omega) = s \; \forall \; \omega \in \Omega$. Then $f^i(r(\omega)) = f^i(r(\omega')) \; \forall \; \omega, \omega' \in \Omega, \forall \; i$ and no population has an incentive to manipulate P. Further $\mathcal{F}^i(r) > \mathcal{F}^i(\tilde{r}^i, r^{-i})) \; \forall \tilde{r}^i \in \mathcal{R}^i \; \forall \; i$, ie r is a strict correlated equilbrium. From Proposition 3, r is asymptotically stable. $\Box$

Proposition 6 claims that if the game Γ is generic, i.e. if one considers the payoffs as random draws and disregards those payoffs that appear with probability zero, if the agents update their rules boundedly rational and if no population would have an incentive to change the signal generating process if it could, then there is nothing I can learn from the concept of correlated equilibrium. Strict Nash equilibria sufficiently explain behavior under such conditions. The proof makes use of the fact that in generic games no two outcomes provide the same payoff. If a population has the capacity to choose certain signals at will, the population will do so as to maximize ex post payoffs.

2.6.2 Example: Chicken Game

We elaborate on this subject for the Chicken example, for which I can characterize $\Delta_P^{\sim} \cap \Delta_P^{CE}$. Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and $\mathcal{A}^1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$ and $\mathcal{A}^2 = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}$. Consider a $P \in \Delta(\Omega)$ with full support. The Chicken game is generic, all asymptotically stable sets are singleton and therefore strict correlated equilibria. I list the expected payoffs and the best replies in Appendix B. For any P, there exist two strict correlated equilibria: (uu, rr) and (dd, ll). If $P(\omega_1) > \max\left\{1 - 3P(\omega_3), \frac{1}{3} - \frac{1}{3}P(\omega_3)\right\}$, also (ud, rl) is a strict correlated equilibrium. If further $P(\omega_1) > \max\left\{1 - \frac{3}{2}P(\omega_3), \frac{2}{3} - \frac{2}{3}P(\omega_3)\right\}$, there exists a fourth strict correlated equilibrium: (du, lr). The equilibria (uu, rr) and (dd, ll) correspond to the two strict Nash equilibria of Γ. In these equilibria, ex ante payoffs $\mathcal{F}$ equal ex post payoffs $f(\omega)$ for any signal ω, no population has an incentive to manipulate the generation of signals. Suppose P is such that (ud, rl) is a strict equilibrium. Then, the ex ante payoffs for population 1 are $6 - 4P(\omega_1) + P(\omega_3)$. Population 1 has an incentive to increase $P(\omega_3)$ at the expense of $P(\omega_1)$. After the manipulation, either P is outside the region in which (ud, rl) is a strict correlated equilibrium or the incentives to manipulate P are still intact. Note that population 2 also has incentives to manipulate P in the equilibrium (ud, rl). Analogous arguments hold for the equilibrium (du, lr). Wrapping up I get that P is robust given the rules $r \in \{(uu, ll), (dd, rr)\}$ and that these are the only states that are asymptotically stable. Note that there are other correlated equilibria, that are not asymptotically stable, that are generated by a robust P: $\Delta_P^{\sim} \cap \Delta_P^{CE} = \{\rho \in \Delta(\mathcal{R})\ |\rho^1 = (\frac{1}{3} + \rho^1(rr), \frac{1}{3} - \rho^1(dd), \frac{1}{3} - \rho^1(dd), \rho^1(dd)), \rho^2 = (\frac{1}{3} + \rho^2(rr), \frac{1}{3} - \rho^2(rr), \frac{1}{3} - \rho^2(rr), \rho^2(rr)),\ \rho^1(dd), \rho^2(rr) \in [0, \frac{1}{3}]\} \cup \{(uu, rr), (dd, ll)\}$. For any mixed correlated equilibrium ρ with a robust P, each population gets a payoff of $4\frac{2}{3}$, which is the outcome of the mixed Nash-equilibrium of the original Chicken game. I conclude for the Chicken game: if agents have the capability to influence the signal generating process, and if the distriution P and the distribution of rules ρ is such that agents do neither have an incentive to manipulate the signals nor to change their behavior, the outcome is a Nash outcome.

2.7 Conclusions

In Aumann (1987), section 3, a player receives a signal and conditions her strategic choice within a normal form game on this signal. She takes into account that other players receive signals that are potentially correlated to hers and calculates conditional beliefs. Aumann (1987) shows that, if players have a common prior on the signal space and if players choose strategies optimally given their beliefs, the equilibrium outcome is a correlated equilibrium. There are correlated equilibrium outcomes that lie outside the convex hull of the Nash outcomes. In this chapter I pursue the question whether agents can achieve a correlated equilibrium without being capable to calculate conditional expectations, indeed even without being able to optimize. For this purpose, given a signal generating process and a game in strategic form, I define an “expanded game” whose strategies are mappings from the set of the signals to the set of the strategies of the original game. For this expanded game I transfer existing and well established results on regular monotonic dynamics, including the replicator dynamic. Applying a result of Samuelson and Zhang (1992), it follows that an outcome which supports a strictly dominated strategy of the original game receives zero weight in the limit. Analogous to results of Weibull (1995) and Ritzberger and Weibull (1995) I show that a Lyapunov stable state of the expanded game represents a correlated equilibrium of the original game and that such a state is asymptotically stable if and only if it represents a strict correlated equilibrium (also Swinkels (1992)). Furthermore, I make use of the setwise concept "strict equilibrium set" introduced by Balkenborg (1994) and transfer the result from Cressman (2003) and Balkenborg and Schlag (2007) to my model that a set of restpoints is asymptotically stable if and only if it is a strict correlated equilibrium set. Therefore I can give a positive answer to our initial question: even if agents are extreme boundedly rational a non-Nash outcome can be robust to random perturbations if agents use simple rules that condition their behavior on observed signals. Finally I discuss endogenous signals. If behavior of the agents can be modelled by convex monotonic dynamics and if the game is generic, I show that an asymptotically stable state has a robust distribution of signals if and only if it corresponds to a strict Nash equilibrium of the original game. I suspect that if the (potentially only Lyapunov stable) state is a correlated equilibrium and if the distribution of signals is robust, then the expected payoffs lie in the convex hull of those produced by Nash equilibria. I illustrate this claim for the Chicken game.

This is not the first attempt linking evolutionary concepts to that of correlated equilibria. Cripps (1991) constructs a model in which nature randomly assigns roles to players in bi-matrix games. Analyzing the statics of the model, he shows that ESS in the symmetrized game represent strict correlated equilibria. Lenzo and Sarver (2006) define a model of subpopulations in which an agent of some subpopulation is non-uniformly matched to agents in other subpopulations. I show that any kind of their subpopulation matching may be represented by a particular signalling structure of our model. Kim and Wong (2007) define an evolutionary stable correlated strategy for symmetric 2×2 games.

Chapter 3

Persistent Ideologies

I analyse finite normal form games with two players in which agents are unable to observe payoffs. Instead of knowing the true payoffs, an agent believes in an ideology that specifies a virtual payoff matrix of the game. I define the set of ideologies that are (strongly) equivalent to the true payoffs. I may, but do not need to assume the presence of ideologies that are equivalent to the true payoffs. There may be infinitely many different ideologies present at the same time. Given a diversity of ideologies, agents maximize and choose actions. I define an equilibrium concept and prove existence. I assume equilibrium play at each point in time, however I refrain from assuming a particular equilibrium selection. Based on this setup, I define an evolutionary dynamic on the distribution of ideologies within the population. In this meta game agents adapt new ideologies. I assume a payoff monotonic imitation dynamic, i.e. ideologies that lead to actions which are relatively successful in terms of true payoffs spread faster in the population than ideologies that recommend relatively unsuccessful actions. I characterize the set of stable distributions on the space of ideologies. For general finite 2 player normal form games with a Nash equilibrium in undominated strategies I show that there is an open set of ideologies being not equivalent to the true payoffs that is not selected against by evolutionary payoff monotonic dynamics based on the true payoffs of the game. I illustrate these results for generic 2×2-games.

3.1 Introduction and Related Literature

Suppose rational agents meet to interact in a strategic situation and none of them has verifiable information about the payoffs of the game. Instead, each agent has been socialized by some ideology that specifies a payoff matrix. Agents then play the game as to maximise the believed payoffs. To give an example, consider a measure taken to prevent terrorist attacks in airplanes: passengers are not allowed to bring flasks with their hand luggage that can contain more than 100 ml of liquid. Since this measure was taken until now, no catastrophe was caused by a large amount of liquid in the hand luggage. The reason for this could be that the measure is indeed successful. Another reasoning could be that 150 ml of liquid in a single bottle is no threat to an airplane, in other words that the measure is useless. In this paper the view is taken that there is no objective knowledge on this matter, because the real payoffs are not observable, maybe not even available. Instead, interacting agents *believe* in certain payoffs. To stay in the example, some agents might feel more secure and therefore more comfortable while airborne; other agents might be annoyed due to the security restrictions and additional waiting times. I do not claim that a particular position is *true* or *false* (actually, as the modeller of the problem I *assume* that certain payoffs are true). I aim to show that even if some perception of reality is wrong, there might be no selection against this perception based on the true payoffs.

In this paper I model an ideology as a bias by which the true payoffs are perturbed. I allow for an unbounded continuum of biases, that is for any two player game with finite strategy space there may exist several groups of agents at the same time, each group believing to face a different strategic setting. Once in a while, agents change their ideology, however I assume they do so boundedly rational and model the adaptation of ideologies with an evolutionary payoff monotonic process: ideologies that result in behavior that produces relatively high *real* payoffs spread faster within the population. Sandholm (2001) constructs a similar model for symmetric 2×2-games in which the payoffs of some *action* are biased, I build upon this model by allowing for biases for *outcomes* of the game. I focus less on the dynamic that yields equilibrium behavior, but assume equilibrium behavior explicitly. I hereby gain tractability of the model and can study general finite normal form games instead of symmetric $2{\times}2$-games. Ok and Vega-Redondo (2001) construct a model in which agents either know the true payoffs of a symmetric game or maximize some alternative symmetric utility function, while they allow for the presence of only one such alternative utility function. For symmetric two player games they show global stability of states in which *all* agents maximize the true payoffs, if – among other conditions – the payoff function is strictly concave. It is exactly this "standard property" that turns out to be crucial for their results. As a special result for the general model I show that for *any* $2{\times}2$-game there is an open set of ideologies that survive in an evolutionary scenario and represent preferences that are not equivalent to the preferences represented by the true payoffs of the game. I give an example for a symmetric 3×3 game that confirms this result. Symmetric 2-player games with a unique pure strategy equilibrium were also studied by Heifetz et al. (2007) within in a similar framework. Ok and Vega-Redondo (2001) state that behavioral distinguishability is necessary for the fact that agents who maximize the evolutionary relevant payoffs have an evolutionary advantage. However, this property is not sufficient: for some asymmetric games I show survival of nonindividualistic preferences, even if they are behaviorally distinguishable. I do so right away for a simple example, to provide a flavor of my model.

Consider the well known matching pennies game. Suppose that the row population has agents of three types. One third knows the true payoffs. Another third has a bias towards "(head, head)", the last third is biased towards "(tail, tail)". I depict the biases below:

	H	T
H	(-1,1)	(1,-1)
T	(1,-1)	(-1,1)

original payoffs

	H	T
H	(-1+**1**,1)	(1,-1)
T	(1,-1)	(-1,1)

bias for $\frac{1}{3}$ of row

	H	T
H	(-1,1)	(1,-1)
T	(1,-1)	(-1+**1**,1)

bias for $\frac{1}{3}$ of row

Each agent knows the sum of the true payoffs and her bias, and knows the distribution of biases within the population. Suppose the column population mixes equally between heads and tails. Then, head-biased agents optimally choose head, tail-biased agents optimally choose tail while those agents who know the true payoffs are just indifferent (assume they mix equally between heads and tails). In this situation no agent has an incentive to deviate. Any agent gets the same real payoffs on average. Moreover, any type has different equilibrium behavior.

Ely and Yilankaya (2001) study which set of outcomes is supported by stable preferences in normal form games. As they rely on static concepts to infer stability properties,[1] I define a dynamic process explicitely to analyze stability issues. In their model, any set of alternative payoff specifications that is robust to exogenous shocks implies equilibrium behavior that produces a probability distribution over the set of Nash equilibria. However, Ely and Yilankaya (2001) do not have results for the case in which there is zero mass on ideologies that represent the true preferences. I do not need to assume the presence of "true" ideologies to prove my results.

3.2 Model

3.2.1 The Stage Game

We consider two infinite populations of agents from which at each point of continuous time, a pair (one agent from each population) is uniformly and independently randomly drawn to play a finite two player normal form game G. An agent

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[1] We feel uncomfortable accepting their definition 3.1 of stability which I would rather call a steady state property.

of population i chooses an element s^i of the finite strategy set S^i with cardinality n^i. The true payoffs are represented by the $n^i \times n^j$ matrix U^i, where $U^i(s^i, s^j)$ denotes i's utility from the outcome generated by (s^i, s^j). I assume that agents believe to be unable to observe the true payoffs. For each population i, each agent is characterized by a vector of parameters $\theta^i = \{\theta^i(s)\}_{s \in S}$ that describes how an agent of type θ^i perceives the structure of the game. An agent θ^i then believes to play a game that specifies her payoffs as $u^i(s, \theta^i) = U^i(s^i, s^j) + \theta^i(s^i, s^j),\ s^i \in S^i, s^j \in S^j$. Let u be extended to the space of mixed strategies and define $u^i((s^i, \sigma^j), \theta^i) = \sum_{s^j \in S^j} \sigma^j(s^j)(U^i(s^i, s^j) + \theta^i(s^i, s^j))$.

I will show that depending on the game, some ideologies that specify non-equivalent payoffs won't have a long-run evolutionary disadvantage in comparison with true payoff equivalent ideologies.

We assume that for each population i types are distributed among the agents by some atomless density $f_i : \mathbb{R}^{(n_i \cdot n_j)} \rightarrow \mathbb{R}_+$ with cumulative distribution function $F_i : \mathbb{R}^{(n_i \cdot n_j)} \rightarrow [0, 1]$. The types are independently distributed across populations. We assume that $\{F\}_i$ is common knowledge.

Equivalent Ideologies

An agent with parameters $\theta^i(s) = 0\ \forall\ s \in S$ plays the game given the true payoffs. These payoffs represent the evolutionary relevant payoffs.

There is an affine subset of $\mathbb{R}^{(n_i \cdot n_j)}$ of parameters θ^i that also represent these evolutionary relevant preferences. Fix some real payoffs U^i. Any positive affine transformation $\hat{U}^i(s) = \alpha^i + \beta^i \cdot U^i(s), \beta_i > 0$ of these payoffs yields other equivalent payoffs.

We call an ideology θ^i *equivalent* to the payoffs U^i of the original unbiased game, if $sign\{U^i(s) - U^i(s')\} = sign\{u^i(s,\theta^i) - u^i(s',\theta^i)\} \ \forall \ s, s' \in S$. I call some ideology θ^i *strongly equivalent* to U^i, if there is some α^i and some positive β^i such that $u^i(s,\theta^i) = \alpha^i + \beta^i \cdot U^i(s) \ \forall \ s \in S$. If all agents had equivalent ideologies, the Nash equilibria would be qualitatively maintained, that is any strict equilibrium would be the same, a mixed Nash equilibrium would be offset. If all agents followed strongly equivalent ideologies, the original Nash equilibria of the game would be exactly maintained.

Equilibrium

Suppose some agent of population i believes σ^j to be the probability distribution over the strategies of population $j \neq i$. An agent of type θ^i chooses s^i iff $s^i \in \arg\max u^i((\tilde{s}^i, \sigma^j), \theta^i)$. I assume that in equilibrium, for all i, all agents of population i hold the same belief σ^j. This assumption is critical to the concept I develop in the following and therefore also to the results of this chapter. Nevertheless, note that it is not as ad hoc as it might seem at first sight. The beliefs are about something that is unobservable – like preferences, ideologies or tastes. The belief is defined as a probability measure on something that is verifiable: actual actions taken by opponents. Since all agents observe this information, homogeneous beliefs seem natural.

The set of types $\Theta_{s^i}(\sigma^j)$ that choose strategy s^i are defined as

$$\Theta_{s^i}(\sigma^j) := \{\theta^i \in \mathbb{R}^{(n_i \cdot n_j)} \mid s^i \in \arg\max u^i((\tilde{s}^i, \sigma^j), \theta_i)\}$$

Any belief σ^j defines such a set $\Theta_{s^i}(\sigma^j)$ uniquely.

Lemma for $i, j = 1, 2, i \neq j$, given any $\sigma^j \in \Delta(S^j)$ there is no $s^i \in S^i$ such that $\Theta_{s^i}(\sigma^j) = \emptyset$.

Proof

Fix a constant $K > \max_{s^i \in S^i} |U^i(s^i, \sigma^j)|$. The max is well defined since S is finite. Then the bias θ^i with $\theta^i(s^i, s^j) = K$ and $\theta^i(\tilde{s}^i, s^j) = -K$ for $\tilde{s}^i \neq s^i$ imposes the choice of s^i, since $u^i((s^i, \sigma^j), \theta^i) = U^i(s^i, \sigma^j) + K > U^i(\tilde{s}^i, \sigma^j) - K = u^i((\tilde{s}^i, \sigma^j), \theta^i)\ \forall\ \tilde{s}^i \neq s^i$. There is an open set $\tilde{\Theta}^i$ around θ^i such that s^i is optimal for all $\theta^i \in \tilde{\Theta}^i$. Therefore, $\Theta_{s^i}(\sigma^j)$ cannot be empty. $\square$

The lemma states that for any strategy s^i an agents virtual payoff matrix can be biased so strongly towards s^i such that the agent believes to maximize his payoffs by choosing s^i.

If all agents of population j share the same belief σ^i, the true probability of facing some opponent of population j choosing s^j is:

$$F_j(\Theta_{s^j}(\sigma^i)) := \int_{\Theta_{s^j}(\sigma^i)} F_j(d\theta^j)$$

Lemma
$F_j(\Theta_{s^j}(\sigma^i))$ is continuous in σ^i $\forall$ s^j, i.
Proof: see Appendix C.

Proposition Existence
There exists at least one pair of beliefs $\sigma = (\sigma^i, \sigma^j)$ such that $\sigma^i(s^i) = F_i(\Theta_{s^i}(\sigma^j))$ $\forall s^i \in S^i, i \neq j = 1, 2$.

Proof:
To prove the result I apply Brouwer's Fixed Point Theorem: any continuous function $G : \Sigma \to \Sigma$ has a fixed point $b^* \in \Sigma$ such that $G(b^*) = b^*$, where $\Sigma = \Delta(S^1) \times \Delta(S^2)$. $\sigma = (\sigma^1, \sigma^2) \in \Sigma$. Define $g^i_{s^i}(\sigma^j) = F_i(\Theta_{s^i}(\sigma^j))$ $\forall$ $\sigma^j \in \Sigma^j$, $\forall$ $s^i \in S^i$ and $\forall$ $i, j, i \neq j$. The function $g(\sigma) = (g^1(\sigma^2), g^2(\sigma^1))$ maps from Σ to Σ. The lemma above states that $g^i_{s^i}(\sigma^j)$ is continuous in σ^j for all s^i. Therefore, all requirements are met and I can apply Brouwer's Fixed Point Theorem to proof existence. $\square$

We assume that in equilibrium j's belief that i plays s^i, $\sigma^i(s^i)$, and the true probability that i plays s^i, $F_i(\Theta_{s^i}(\sigma^j))$, coincide: $\sigma^i(s^i) = F_i(\Theta_{s^i}(\sigma^j))\ \forall\ s^i \in S^i, i = 1, 2$. Given a cdf F, I call a collection $\{\Theta_{s^i}\}_{s^i \in S^i, i=1,2}$ that satisfies this equilibrium condition *equilibrium set*. There may be many $\sigma = (\sigma^1, \sigma^2)$ that satisfy such equalities, the assumption is not as innocent as it might seem: in fact I assume that the agents manage to coordinate on one of potentially many equilibria. However, I do not select any equilibrium; my results require only that the agents coordinate on some, but not on which equilibrium.[2] Note that a distribution on the set of outcomes S induced by an equilibrium set Θ_s and a cdf F does not need to be a Nash equilibrium distribution.

Evolutionary Dynamics

Given some density f with cdf F and some equilibrium set $\{\Theta_s\}_s$, agents of population i with type $\theta^i \in \Theta_{s^i}$ receive true payoffs $U^i(s^i, F_j) = \sum_{s^j} U^i(s^i, s^j) F_j(\Theta_{s^j})$. I assume that once in a while agents adopt different ideologies and that this dynamic process can be captured by the deterministic differential equation

$$\dot{F}_i(\Theta^i) = \int_{\Theta^i} g_i(\theta^i, F, \Theta_s(F)) f_i(\theta^i) d\theta^i\ ,\ \forall \Theta^i \in \mathcal{B}^i,\ i = 1, 2$$

where $g_i(\theta^i, F, \Theta_s)$ is the growth rate of the marginal density f_i at θ^i given cdf F and some equilibrium set Θ_s and

---

[2] Implicitly, I assume that behavior adjusts with infinite speed. Sandholm (2001) shows in a symmetric 2×2-games setting, that the infinite speed dynamics can be expressed as the limit of finite speed dynamics.

$\mathcal{B}^i$ is the Borel σ-algebra of $\mathbb{R}^{n_i}$. I require $g_i(\theta^i, F, \Theta_s)$ to maintain the probability property of the marginals (f_1, f_2), that is $\int_{\mathbb{R}^2} g_i(\theta^i, F, \Theta_s) f_i(\theta^i) d\theta^i = 0$, $i = 1, 2$ and I require $g_i(\theta^i, F, \Theta_s)$ to be Lipschitz continuous in F. Then, as Oechssler and Riedel (2001) show, for any $F(0)$ a solution $F(t)$ exists. If $g_i(\theta^i, F, \Theta_s) = U^i(s^i, F_j) - \sum_{\hat{s}^i} F_i(\Theta_{\hat{s}^i}) \cdot U^i(\hat{s}^i, F_j)$, the dynamics is the well known replicator dynamics. I assume that $g(\theta, F, \Theta_s)$ fulfills the less demanding requirement of payoff monotonicity, that is $g_i(\theta^i, F, \Theta_s) > g_i(\hat{\theta}^i, F, \Theta_s) \Leftrightarrow U^i(s^i, F_j) > U^i(\hat{s}^i, F_j)$ for $\theta^i \in \Theta_{s^i}$ and $\hat{\theta}^i \in \Theta_{\hat{s}^i}$.

Let $||F - F'||$ denote the variational norm: $||F - F'|| = \sup_h \left| \int_{\mathbb{R}^{n_i \cdot n_j}} h(\theta)(f(\theta) - f'(\theta)) d\theta \right|$, where the supremum is taken over all measurable functions $h : \mathbb{R}^{n_i \cdot n_j} \to \mathbb{R}$ satisfying $\sup_\theta |h(\theta)| \leq \frac{1}{2}$.[3]

As in Oechssler and Riedel (2001)'s remark, I discuss the relation of the variational norm to the concept of mutations in evolutionary game theory. Consider a cdf F that has measure zero on some set Θ. Now suppose a small group (of size $\frac{\epsilon}{1+\epsilon} > 0$), that has a cdf F' with measure one on Θ enters the population. The new distribution is $F^\epsilon = (1-\epsilon) \cdot F + \epsilon \cdot F'$ and has distance ϵ from F. Therefore,

---

[3] We take the definition of the variational norm from Oechssler and Riedel (2001).

if the group of entering mutants is relatively small, the distribution changes only little. Consider instead that the original distribution F has mass one on a single point θ and that each agent changes his belief of the payoffs only a little, say to $\theta + \epsilon \cdot \iota$ such that the new distribution F^ϵ has measure one on $\theta + \epsilon \cdot \iota$.[4] Then, F and F^ϵ have distance 1, which is the maximal distance. That means, if all agents mutate, even very little, the measure of change of the distribution is maximal; if only a very small fraction of the population mutates, even very starkly, the measure of change is very small.

3.2.2 Results

Definition
Let F^* be a restpoint satisfying $\dot{F}_i^*(\Theta) = 0 \;\forall\; \Theta \in \mathcal{B}, i = 1, 2$. Then F^* is called Lyapunov stable if $\forall\; \epsilon > 0 \;\exists\; \eta > 0 : ||F_i(0) - F_i^*|| < \eta \;\forall\; i \Rightarrow ||F_i(t) - F_i^*|| < \epsilon \;\forall\; t > 0, \forall\; i$.

Definition An equilibrium set $\{\Theta_s\}_s$ is dynamically stable with respect to some cdf F, if F is Lyapunov stable and $\{\Theta_s\}_s$ is an equilibrium set given F.

Theorem
Let $\{\Theta_s\}_s$ be dynamically stable with respect to some cdf F. Then σ with $\sigma_i(s^i) = F_i(\Theta_{s^i}) \;\forall\; s^i \in S^i$ is a Nash equilibrium.

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[4] $\iota = (1, 1, \ldots, 1)$

Proof
F is Lyapunov stable. Therefore, $\dot{F}_i(\Theta) = 0\ \forall \Theta \in \mathcal{B}(\mathbb{R}^{n^i \cdot n^j})$ and $F_i(\Theta_{s^i}) > 0 \Rightarrow g^i(\theta^i, F, \{\Theta_s\}) = 0\ \forall\ \theta^i \in \Theta_{s^i}$. Hence $U^i(s^i, \sigma^j) = U^i(\tilde{s}^i, \sigma^j)\ \forall\ s^i, \tilde{s}^j : F_i(\Theta_{s^i}), F_i(\Theta_{\tilde{s}^i}) > 0$, with $\sigma^j(s^j) = F_j(\Theta_{s^j})\ \forall\ s^j \in S^j$. It remains to show: $U^i(s^i, \sigma^j) \geq U^i(\tilde{s}^i, \sigma^j)\ \forall\ s^i : F_i(\Theta_{s^i}) > 0, \tilde{s}^i : F(\Theta_{\tilde{s}^i}) = 0 \Leftrightarrow g^i(\theta^i, F, \{\Theta_s\}) \geq g^i(\tilde{\theta}^i, F, \{\Theta_s\})\ \forall\ \theta^i \in \Theta_{s^i} : F_i(\Theta_{s^i}) > 0, \forall\ \tilde{\theta}^i \in \Theta_{\tilde{s}^i} : F_i(\Theta_{\tilde{s}^i}) = 0$. I prove the claim by contradiction. Suppose instead $\exists \Theta_{\tilde{s}^i} : F_i(\Theta_{\tilde{s}^i}) = 0$ and $g^i(\tilde{\theta}^i, F, \{\Theta_s\}) > 0\ \forall \tilde{\theta}^i \in \Theta_{\tilde{s}^i}$. Define $\mathcal{F}^{\epsilon}_{\tilde{s}^i}$ as the set of cdf's F^η satisfying $F^\eta_i(\Theta) = (1-\eta) \cdot F_i(\Theta)\ \forall \Theta \subset \mathbb{R}^{n_i \cdot n_j} \setminus \Theta_{\tilde{s}^i}$ and $F^\eta_i(\Theta_{\tilde{s}^i}) = \eta$, $\epsilon \geq \eta > 0$ and F^η_j such that $||F_j - F^\eta_j|| \leq \eta$. I note that $||F_i - F^\eta_i|| = \eta$ and that $\mathcal{F}^\epsilon$ is a connected subset of cdf's satisfying $||F - \tilde{F}|| \leq \epsilon\ \forall\ \tilde{F} \in \mathcal{F}^\epsilon$ and that there is a sequence $\{F^\eta\}_\eta$ in $\mathcal{F}^\epsilon$ with $\lim_{\eta \to 0} F^\eta = F$. Since $g^i(\cdot, F, \cdot)$ is (Lipschitz-) continuous in F, I have $g^i(\tilde{\theta}^i, \tilde{F}, \{\Theta_s\}_s) > 0\ \forall \tilde{\theta}^i \in \Theta_{\tilde{s}^i},\ \forall \tilde{F} \in \mathcal{F}^{\epsilon}_{\tilde{s}^i}$ if I choose $\epsilon > 0$ small enough. Consider some $F^\eta \in \mathcal{F}^{\epsilon}_{\tilde{s}^i}$. I have $\dot{F}^\eta_i(\Theta_{\tilde{s}^i}) > 0$. Therefore, for some $t > 0$ there is some $\eta' > \eta$ such that $F^\eta(t) = F^{\eta'}$. Since $\eta \leq \epsilon$, there is some $t > 0$ such that for some $\eta < \epsilon$, $F^\eta(t) = F^{\eta'}$ with $\eta' > \epsilon$. This contradicts the property of Lyapunov stability of F and therefore it must be that $g^i(\theta^i, F, \{\Theta_s\}_s) \leq 0\ \forall\ \theta^i \in \Theta_{\tilde{s}^i}$. Therefore $U^i(s^i, \sigma^j) \geq U^i(\tilde{s}^i, \sigma^j)\ \forall\ s^i : F_i(\Theta_{s^i}) > 0, \tilde{s}^i : F(\Theta_{\tilde{s}^i}) = 0$ and I have established that σ is a Nash equilibrium, with $\sigma^i(s^i) = F_i(\Theta_{s^i})\ \forall\ s^i \in S^i$ and $i = 1, 2$. □

We note that the reverse statement, namely that a Nash equilibrium implies dynamic stability is generally not true. Consider, for example, a Nash equilibrium involving strategies that are weakly dominated. It is clear, that for an open environment of distributions with full support around the Nash equilibrium distribution there are ideologies that induce the choice of strategies that gain strictly higher (real) payoffs than the Nash equilibrium strategies.

Main Theorem
For any game that has a Nash equilibrium σ without dominant strategies there exists an equilibrium set $\{\Theta_s\}_s$ with respect to a cdf F such that $F_i(\Theta^i_s) = \sigma^i(s^i)\ \forall\ s^i \in S^i$ and no $\theta^i \in \Theta^i_{s^i}$ is equivalent to the bias 0 for any s^i in the support of σ^i.

Proof
Since σ is a Nash equilibrium without dominant strategies, for any s^i there exist strategies $(\tilde{s}^i, s^j)$ such that $U^i(s^i, s^j) < U^i(\tilde{s}^i, s^j)$. Define for $K > \max_{\tilde{s}^i \in S^i} |U^i(\tilde{s}^i, \sigma^j)|$ the set $\Theta_{s^i} = \{\theta^i \in \mathbb{R}^{n_i \cdot n_j} | \theta^i(s^i, s^j) \geq K, \theta^i(\tilde{s}^i, s^j) \leq -K\ \forall \tilde{s}^i \neq s^i, \forall s^j\}$. Since for any $\theta^i \in \Theta_{s^i}$, $\tilde{s}^i \neq s^i$ and s^j it holds that $U^i(s^i, s^j) + \theta^i(s^i, s^j) > 0$ and $U^i(\tilde{s}^i, s^j) + \theta^i(\tilde{s}^i, s^j) < 0$, θ^i is not equivalent to the origin, $\theta^i_0 = 0$. Since $s^i \in \arg\max_{\tilde{s}^i \in S^i} U^i(\tilde{s}^i, \sigma^j) + \theta^i(\tilde{s}^i, \sigma^j)$ for any $\theta^i \in \Theta_{s^i}$ and $\sigma^j \in \Delta(S^j)$, it holds that $\Theta_{s^i} \subset \Theta_{s^i}(\sigma^j)$.

Set $F_i(\Theta_{s^i}) = \sigma^i(s^i) \ \forall \ s^i \in S^i$. Since the density of F is atomless and the sets $\Theta_{s^i}(\sigma^j), \Theta_{s^{i\prime}}(\sigma^j)$ overlap only at their borders, $F_i(\Theta_{s^i}(\sigma^j)) = \sigma^i(s^i) \ \forall s^i \in S^i$. The argument holds for $i = 1, 2$.[5] By definition, $\{\Theta_s\}_s$ is an equilibrium set given F. $\square$

3.3 Dynamically Stable Sets in 2×2 Games

In the next two sections, we accompany this result by an illustration for 2 × 2 games and for a 3 × 3 game.

We normalize an asymmetric 2 × 2-game to a symmteric game with off-diagonal payoffs of zero (payoffs for player i):

$$\begin{array}{c|c|c|}
 & A & B \\ \hline
A & a^i_1 + \theta^i_{a_1} & a^i_2 + \theta^i_{a_2} \\ \hline
B & b^i_2 + \theta^i_{b_1} & b^i_1 + \theta^i_{b_2} \\ \hline
\end{array}
\rightarrow
\begin{array}{c|c|c|}
 & A & B \\ \hline
A & a^i + \theta^i_a & 0 \\ \hline
B & 0 & b^i + \theta^i_b \\ \hline
\end{array}$$

Assume without loss of generality $a^i > b^i$. A bias θ^i is equivalent to the true payoffs, if the bias does not alter the sign of the true payoff, that is if $sign\{a^i\} = sign\{a^i + \theta^i_a\}$ and $sign\{b^i\} = sign\{b^i + \theta^i_b\}$. For any $\sigma^j \in \Delta(\{A, B\})$, $\Theta^i_A(\sigma^j) = \{\theta \in \mathbb{R}^2 \mid \sigma^j(A)(\theta^i_a + a^i) \geq \sigma^j(B)(b^i + \theta^i_b)\}$ and $F_i(\Theta^i_A) = \int_{\Theta^i_A(\sigma^j} f_i(\theta^i)d\theta^i$. We can write $U^i(A, \theta^i) = a^i \cdot F_j(\Theta^j_A)$ and $U^i(B, \theta^i) = b^i \cdot (1 - F_j(\Theta^j_A))$. Therefore

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[5] I suspect that it holds for games with an arbitrary number of populations.

$g^i(\theta^i, F, \{\Theta_s\})$ can take only two values for any given F and $\{\Theta_s\}$; $g^i(\theta^i, F, \{\Theta_s\}) = g^i_A(F, \{\Theta_s\})$ if $\theta^i \in \Theta^i_A$ and $g^i(\tilde{\theta}^i, F, \{\Theta_s\}) = g^i_B(F, \{\Theta_s\})$ if $\tilde{\theta}^i \in \Theta^i_B$. This implies then since $\dot{F}_i(\mathbb{R}^2) = g^i_A(F, \{\Theta_s\}) \cdot F_i(\Theta^i_A) + g^i_B(F, \{\Theta_s\}) \cdot (1 - F_i(\Theta^i_A)) = 0$ that $g^i_A(F, \{\Theta_s\}) = 0$ if $F_i(\Theta^i_A) = 1$ and that $g^i_B(F, \{\Theta_s\}) = 0$ if $F_i(\Theta^i_A) = 0$. I conclude that $F_i(\Theta^i_A) = 1$ and $F_i(\Theta^i_A) = 0$ are restpoints of the dynamic of population i. If instead $F_i(\Theta^i_A) \in (0, 1)$, I can substitute $g^i_A(F, \{\Theta_s\}) = -g^i_B(F, \{\Theta_s\})\frac{F_i(\Theta^i_A)}{F_i(\Theta^i_B)}$ and the dynamics simplify for all $\Theta^i \in \mathcal{B}$ to

$$\dot{F}_i(\Theta^i) = g^i_A(F, \{\Theta_s\}) \left(F_i(\Theta^i|\Theta^i_A) - F_i(\Theta^i|\Theta^i_B)\right) F_i(\Theta^i_A) \ .$$

An interior steady state ($0 < F_i(\Theta^i_A) < 1$) has the following characteristics: $g^i_A(F, \{\Theta_s\}) = g^i_B(F, \{\Theta_s\}) = 0$ which implies that $U^i_A(F, \{\Theta_s\}) = U^i_B(F, \{\Theta_s\}) \Leftrightarrow F_j(\Theta^j_A) = \frac{b^i}{a^i+b^i}$, which can only be in $(0, 1)$ if $a^i(a^i+b^i) > 0$ and $b^i(a^i+b^i) > 0$. To summarize, there always exist four restpoints: $\{F_i(\Theta^i_A) = 1\}_{i=1,2}; \{F_i(\Theta^i_A) = 0\}_{i=1,2}; \{F_i(\Theta^i_A) = 2 - i\}_{i=1,2}; \{F_i(\Theta^i_A) = i - 1\}_{i=1,2}$. Additionally, if $a^i(a^i + b^i) > 0 \wedge b^i(a^i + b^i) > 0,\ i = 1, 2$, then $\{F_i(\Theta^i_A) = \frac{b^j}{a^j+b^j}\}_{i=1,2,j\neq i}$ is also a restpoint.

3.3.1 Classification of 2×2 Games

So far, I did not specify the true payoffs of the game, $\{a^i, b^i\}^2_{i=1}$. There exist five qualitative categories to characterize the Nash equilibria of all 2×2-games.[6]

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[6] We ignore the cases in which a true payoff equals zero.

	$a^2 > b^2 > 0$
$a^1 > b^1 > 0$	$(A_1, A_2), (B_1, B_2), (p_1, p_2)$
$a^1 > 0 > b^1$	(A_1, A_2)
$0 > a^1 > b^1$	(p_1, p_2)

	$a^2 > 0 > b^2$
$a^1 > b^1 > 0$	(A_1, A_2)
$a^1 > 0 > b^1$	(A_1, A_2)
$0 > a^1 > b^1$	(B_1, A_2)

	$0 > a^2 > b^2$
$a^1 > b^1 > 0$	(p_1, p_2)
$a^1 > 0 > b^1$	(A_1, B_2)
$0 > a^1 > b^1$	$(A_1, B_2), (B_1, A_2), (p_1, p_2)$

Nash equilibria in nine different cases

(p_1, p_2) denotes a mixed Nash equilibrium in which a fraction $p_1 = \frac{b^2}{a^2+b^2}$ of population 1 and a fraction $p_2 = \frac{b^1}{a^1+b^1}$ of population 2 choose A. In the next subsections, I explore these nine cases. In any case, the distribution of play converges to a Nash equilibrium, which was already observed by Ely and Yilankaya (2001). In contrast to Ely and Yilankaya (2001), I do not need to assume that the initial measure of ideologies has positive mass on payoffs that are equivalent to the true payoffs. I show that for any parameters a, b there exist initial measures such that ideologies persist even if they are not equivalent to the true payoffs.

$(a^1 > 0 > b^1), (a^2 > b^2 > 0)$**;** $(a^1 > 0 > b^1), (a^2 > 0 > b^2)$ **and** $(a^1 > b^1 > 0), (a^2 > 0 > b^2)$

$(a^1 > 0 > b^1), (a^2 > b^2 > 0)$:
Population 1 has A_1 as a dominant strategy, population 2's best response to A_1 is A_2, the unique Nash equilibrium is (A_1, A_2). There is no interior rest point, the set of restpoints consists of the four pure strategy restpoints.

Since from payoff monotonicity I have $g_{A_1}(F', \Theta'_A) > g_{B_1}(F', \Theta'_A)$ for any $F'_1(\Theta'_{A_1}) < 1$ and F_2, I have that $F^*_1(\Theta^*_{A_1}) = 1$ is uniquely stable.

For any $F^{*\cdot} = (F^*_1, F_2)$, $U_{A_2}(F^{*\cdot}, \Theta^{*\cdot}_A) = a_2 > 0 = U_{B_2}(F^{*\cdot}, \Theta^{*\cdot})$ and therefore $g_{A_2}(F^{*\cdot}, \Theta^{*\cdot}) > g_{B_2}(F^{*\cdot}, \Theta^{*\cdot})$ for any $F_2(\Theta_{A_2}) < 1$. Hence $F^*_2(\Theta^*_{A_2}) = 1$ is uniquely stable.
The remaining two cases $(a_1 > 0 > b_1), (a_2 > 0 > b_2)$ and $(a_1 > b_1 > 0), (a_2 > 0 > b_2)$ imply the same uniquely stable restpoint, the argument is the same as above.

$(a_1 > 0 > b_1), (0 > a_2 > b_2)$ **and** $(0 > a_1 > b_1), (a_2 > 0 > b_2)$

$(a_1 > 0 > b_1), (0 > a_2 > b_2)$:
Population 1 has A_1 as a dominant strategy, population 2's best response to A_1 is B_2, the unique Nash equilibrium is (A_1, B_2). There is no interior rest point, the set of restpoints consists of the four pure strategy restpoints.

Since from payoff monotonicity I have $g_{A_1}(F', \Theta'_A) > g_{B_1}(F', \Theta'_A)$ for any $F'_1(\Theta'_{A_1}) < 1$ and F_2, I have that $F^*_1(\Theta^*_{A_1}) = 1$ is uniquely stable.

For any $F^{*\cdot} = (F_1^*, F_2)$, $U_{A_2}(F^{*\cdot}, \Theta_A^{*\cdot}) = a_2 < 0 = U_{B_2}(F^{*\cdot}, \Theta^{*\cdot})$ and therefore $g_{A_2}(F^{*\cdot}, \Theta^{*\cdot}) < g_{B_2}(F^{*\cdot}, \Theta^{*\cdot})$ for any $F^*(*\cdot)$ with $F_2(\Theta_{A_2}) < 1$. Hence $F_2^*(\Theta_{A_2}^*) = 0$ is uniquely stable.

The case $(0 > a_1 > b_1), (a_2 > 0 > b_2)$ is analogous.

$(0 > a_1 > b_1), (0 > a_2 > b_2)$

There are two strict Nash equilibria (A_1, B_2), (A_2, B_1) and one mixed Nash equilibrium $(p_1, p_2) = (\frac{b_2}{a_2+b_2}, \frac{a_1}{a_1+b_1})$. The set of restpoints consists of the four pure restpoints plus the unique restpoint in which $F_1(\Theta_{A_1}) = \frac{b_2}{a_2+b_2}$ and $F_2(\Theta_{A_2}) = \frac{b_1}{a_1+b_1}$.

$(0 > a_1 > b_1), (a_2 > b_2 > 0)$ **and** $(a_1 > a_2 > 0), (0 > a_1 > a_2)$ **(matching pennies)**

$(0 > a_1 > b_1), (a_2 > b_2 > 0)$:
The only Nash equilibrium is the mixed equilibrium $(p_1, p_2) = (\frac{b_2}{a_2+b_2}, \frac{b_1}{a_1+b_1})$, which is a steady state, additionally to the four pure steady states. $U_{A_1}(F, \Theta_A) = a_1 \cdot F_2(\Theta_{A_2})$, $U_{B_1}(F, \Theta_A) = b_1 \cdot (1 - F_2(\Theta_{A_2}))$, from payoff monotonicity follows $g_{A_1}(F, \Theta_A) > g_{B_1}(F, \Theta_A) \Leftrightarrow F_2(\Theta_{A_2}) < \frac{b_1}{a_1+b_1}$. For population 2 I have $U_{A_2}(F, \Theta_A) = a_2 \cdot F_1(\Theta_{A_1})$, $U_{B_2}(F, \Theta_A) = b_2 \cdot (1 - F_1(\Theta_{A_1}))$, from payoff monotonicity follows $g_{A_2}(F, \Theta_A) > g_{B_2}(F, \Theta_A) \Leftrightarrow F_1(\Theta_{A_1}) > \frac{b_2}{a_2+b_2}$. Therefore a pure steady state cannot be Lyapunov stable. The mixed steady state is Lyapunov stable if $\dot{F}(\Theta)$ is the above mentioned replicator

dynamic, however it is not Lyapunov stable for all payoff monotonic dynamics.

The case $(a_1 > b_1 > 0), (0 > a_2 > b_2)$ is analogous.

$(a_1 > b_1 > 0), (a_2 > b_2 > 0)$ **(coordination games)**

This class of games has three Nash equilibria, two in pure strategies (A_1, A_2), (B_1, B_2) and one in mixed strategies $(p_1, p_2) = (\frac{b_1}{a_1+b_1}, \frac{b_2}{a_2+b_2})$. From $U_{A_1}(F, \Theta_A) = a_1 \cdot F_2(\Theta_{A_2})$, $U_{B_1}(F, \Theta_A) = b_1 \cdot (1 - F_2(\Theta_{A_2}))$ and payoff monotonicity it follows $g_{A_1}(F, \Theta_A) > g_{B_1}(F, \Theta_A) \Leftrightarrow F_2(\Theta_{A_1}) > \frac{b_1}{a_1+b_1}$, for population 2 analogously. Therefore, only the strict Nash equilibria are Lyapunov stable.

3.4 Conclusions

We study a model in which strategically interacting agents follow an ideology that specifies the unobservable payoffs of a symmetric 2×2-game. When drawn from an infinite population, an agent knows his own ideology, but has incomplete information of the ideology that his opponent follows. I assume that the distribution of ideologies within the population is common knowledge. I allow for a continuous variety of ideologies that each specifies payoffs for games which can be of any class of symmetric 2×2-games. Given any distribution of ideologies, I define equilibrium sets and show the existence thereof. I assume that agents coordinate on such an equilibrium set. My results do not depend upon the selection of such an equilibrium set in the case of multiplicity. My focus is on the dynamics of ideologies. I assume that evolution selects against ideologies that induce behaviour which yields relatively low evolutionary relevant *true* payoffs. I characterize distributions that are stable with respect to small changes (Lyapunov stability). My main result is that for any two player game with a Nash equilibrium without dominant strategies, there exists equilbrium sets that consist only of non-equivalent ideologies. For any generic 2×2-game there exist stable distributions putting positive mass only on those ideologies that represent preferences not equivalent to preferences represented by the true payoffs. I provide an example of 3×3-games that confirms this result.

Appendix A

Proposition 1
Let g be convex monotonic. If $F \subset \Delta_{\mathcal{R}}$ is a Lyapunov stable set of rest points, then each $\rho \in F$ is a *c.e.* .

Proof: Since $\rho \in F$ is a restpoint, $g^i(r^i, \rho) = 0 \ \forall \ r^i \in$ supp (ρ^i). Suppose $\exists r^i_l, r^i_k \in \text{supp}(\rho^i)$ such that $\mathcal{F}^i(r^i_l, \rho^{-i}) > \mathcal{F}^i(r^i_k, \rho^{-i})$. Then, by convex monotonicity, $g^i(r^i_k, \rho) < g^i(r^i_l, \rho) \cdot 1$, a contradiction. Therefore $\mathcal{F}^i(r^i_l, \rho^{-i}) = \mathcal{F}^i(r^i_k, \rho^{-i}) \ \forall \ r^i_l, r^i_k \in \text{supp}(\rho^i)$. If ρ is in the interior of F with respect to $\Delta_{\mathcal{R}}$, I are done. Suppose instead that for some i there exists $r^i_k \notin \text{supp}(\rho^i)$ and suppose that $\mathcal{F}^i(r^i_k, \rho^{-i}) > \mathcal{F}^i(\rho)$. Then, again by convex monotonicity, $g^i(r^i_k, \rho) > \sum_{r^i \in \mathcal{R}^i} g^i(r^i, \rho) \cdot \rho^i(r^i) = 0$. Since g is (Lipschitz–)continuous, there exists a neighborhood $\mathcal{U}$ of ρ such that $g^i(r^i_k, \tilde{\rho}) > 0 \ \forall \ \tilde{\rho} \in \mathcal{U} \cap \Delta_{\mathcal{R}}$. Define $\mathcal{U}' = \{\tilde{\rho} \in \mathcal{U} \cap \Delta_{\mathcal{R}} \mid \tilde{\rho}^i(r^i_k) > 0\}$. It holds that $\hat{\rho}^i_{r^i_k}(t, \tilde{\rho})$ is strictly increasing in t for any $\tilde{\rho} \in \mathcal{U}'$. However, Lyapunov stability implies that $\hat{\rho}(t, \tilde{\rho}) \in \mathcal{U}' \ \forall \ t \geq 0$ and $\tilde{\rho} \in \mathcal{U}''$ for some neighborhood $\mathcal{U}''$, which can only be the case if

$\dot{\tilde{\rho}}^i(r^i_k) \leq 0$ for some $\tilde{\rho} \in \mathcal{U}'$, because $\rho^i(r^i_k) = 0$. Since $g^i(r^i_k, \tilde{\rho}) > 0 \; \forall \; \tilde{\rho} \in \mathcal{U}'$ this is not true for any subset of $\mathcal{U}'$ and $\mathcal{U}''$ does not exist. Therefore, the existence of some $r^i_k \in \mathcal{R}^i$ for some $i \in \mathcal{N}$ such that $\mathcal{F}^i(r^i_k, \rho^{-i}) > \mathcal{F}^i(\rho)$ contradicts Lyapunov stability of F and I have established the claim. $\square$

Proposition 3
Let (2.1) be convex monotonic. If a set F is a SEset, then F is an asymptotically stable set of rest points.

Proof: Suppose F is an SEset and suppose that $F \neq \Delta_{\mathcal{R}}$. Each point in F is a restpoint of (2.1). Further I have that F is a finite union of faces of $\Delta_{\mathcal{R}}$ and therefore is closed. Consider some ρ_* on the boundary of F with respect to $\Delta_{\mathcal{R}}$. For some population there is a pure rule r^i such that $\mathcal{F}^i(r^i, \rho_*^{-i}) < \mathcal{F}^i(\rho_*)$. Since $g^i(\cdot, \rho) \cdot \rho^i = 0 \; \forall \; \rho \in \Delta_{\mathcal{R}}$ it follows that $g^i(r^i, \rho_*) = 0 \; \forall \; r^i \in \text{supp}(\rho^i_*)$. From convex monotonicity I have that $g^i(r^i, \rho_*) < 0 \; \forall \; r^i \notin \text{supp}(\rho^i_*)$ and from continuity follows that there exists some neighborhood $\mathcal{U} : \mathcal{U} \cap int(\Delta_{\mathcal{R}}) \neq \emptyset$ of ρ_* such that $g^i(r^i, \rho) < 0 \; \forall \; r^i \notin \text{supp}(\rho^i_*), \rho \in \mathcal{U}$. Therefore $\dot{\rho}^i_{r^i}(\rho) < 0 \; \forall r^i \notin \text{supp}(\rho^i_*), \forall \; \rho \in \mathcal{U} \setminus F$ and from $g^i(\cdot, \rho) \cdot \rho^i = 0$ I have for at least one $r^i \in \text{supp}(\rho^i_*)$ that $\dot{\rho}^i_{r^i}(\rho) > 0 \; \forall \; \rho \in \mathcal{U} \setminus F$, which establishes the result. $\square$

Appendix B

Row's expected payoffs in the expansion of the Chicken game

	ll	rl
uu	6	$6-4P(\omega_1)$
ud	$6+P(\omega_3)$	$6-4P(\omega_1)+P(\omega_3)$
du	$7-P(\omega_3)$	$7-7P(\omega_1)-P(\omega_3)$
dd	7	$7-7P(\omega_1)$

	lr	rr
uu	$2+4P(\omega_1)$	2
ud	$2+4P(\omega_1)-2P(\omega_3)$	$2(1-P(\omega_3))$
du	$7P(\omega_1)+2P(\omega_3)$	$2P(\omega_3)$
dd	$7P(\omega_1)$	0

Column's expected payoffs in the expansion of the Chicken game

	ll	rl
uu	6	$6+P(\omega_1)$
ud	$6-4P(\omega_3)$	$6-4P(\omega_1)+P(\omega_3)$
du	$2+4P(\omega_3)$	$2-2P(\omega_1)+4P(\omega_3)$
dd	2	$2(1-P(\omega_1))$

	lr	rr
uu	$7-P(\omega_1)$	7
ud	$7-P(\omega_1)-7P(\omega_3)$	$7-7P(\omega_3)$
du	$2P(\omega_1)+7P(\omega_3)$	$7P(\omega_3)$
dd	$2P(\omega_1)$	0

Best replies for P with full support:

$$BR^1(ll) \equiv dd \qquad , \; BR^1(rr) = uu$$

$$BR^1(rl) = \begin{cases} ud & \text{, if } 3P(\omega_1) + P(\omega_3) > 1 \\ \{ud, dd\} & \text{, if } 3P(\omega_1) + P(\omega_3) = 1 \\ dd & \text{, if } 3P(\omega_1) + P(\omega_3) < 1 \end{cases}$$

$$BR^1(lr) = \begin{cases} uu & \text{, if } 3P(\omega_1) + 2P(\omega_3) < 2 \\ \{uu, du\} & \text{, if } 3P(\omega_1) + 2P(\omega_3) = 2 \\ du & \text{, if } 3P(\omega_1) + 2P(\omega_3) > 2 \end{cases}$$

$$BR^2(uu) \equiv rr \qquad , \; BR^2(dd) = ll$$

$$BR^2(ud) = \begin{cases} rl & \text{, if } 3P(\omega_3) + P(\omega_1) > 1 \\ \{rl, rr\} & \text{, if } 3P(\omega_3) + P(\omega_1) = 1 \\ rr & \text{, if } 3P(\omega_3) + P(\omega_1) < 1 \end{cases}$$

$$BR^2(du) = \begin{cases} ll & \text{, if } 2P(\omega_1) + 3P(\omega_3) < 2 \\ \{ll, lr\} & \text{, if } 2P(\omega_1) + 3P(\omega_3) = 2 \\ lr & \text{, if } 2P(\omega_1) + 3P(\omega_3) > 2 \end{cases}$$

Appendix C

Lemma $F_i(\Theta_{s^i}(\sigma^j))$ is continuous in σ^j $\forall$ i, j.

Proof:

Outline:

Given a convergent sequence $\{\sigma_n\}_n$ in $\Delta(S^j)$ and associated halfspaces $\{H_n\}_n$ we show $H_n \cup H \downarrow H$ and $H_n \cap H \uparrow H$ and prove with Lemmata C1 & C2 continuity from below and above. We define H_n such that $\Theta_{s^i}(\sigma^j)$ is the intersection of finitely many halfspaces and therefore is also continuous from above and below.

For any $\sigma, \hat{\sigma} \in \Delta(S^j)$ define $\sigma_n = \sigma\frac{n-1}{n} + \hat{\sigma}\frac{1}{n}$, $n \in \mathbb{N}$. For any $s^i, \tilde{s}^i \in S^i$ and $\sigma_n \in \Delta(S^j)$ define $H_n = \{\theta^i \in \mathbb{R}^{n_i \cdot n_j} \,|U_i(s^i, \sigma_n) + \theta^i(s^i, \sigma_n) \geq U_i(\tilde{s}^i, \sigma_n) + \theta^i(\tilde{s}^i, \sigma_n)\}$, the set of types that weakly prefer strategy s^i over strategy $\tilde{s}^i$ given belief σ_n.[7]

<u>$H_n \cap H \uparrow H$</u>:

We show $\theta^i \in H_n \cap H \Rightarrow \theta^i \in H_{n+1}$. Multiplying the inequality implied by $\theta^i \in H_n$ with $\frac{n}{n+1}$, the inequality

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[7] The proofs for $\tilde{s}^i = s^i$ are trivially valid, however, only the cases $\tilde{s}^i \neq s^i$ are relevant.

implied by $\theta^i \in H$ with $\frac{1}{n+1}$ and summing up yields

$$\begin{aligned}
& U_i(s^i, \sigma_n \frac{n}{n+1} + \sigma \frac{1}{n+1}) + \theta^i(s^i, \sigma_n \frac{n}{n+1} + \sigma \frac{1}{n+1}) \\
\geq \; & U_i(\tilde{s}^i, \sigma_n \frac{n}{n+1} + \sigma \frac{1}{n+1}) + \theta^i(\tilde{s}^i, \sigma_n \frac{n}{n+1} + \sigma \frac{1}{n+1}) \\
\Leftrightarrow \; & U_i(s^i, \sigma_{n+1}) + \theta^i(s^i, \sigma_{n+1}) \geq U_i(\tilde{s}^i, \sigma_{n+1}) + \theta^i(\tilde{s}^i, \sigma_{n+1}) \ ,
\end{aligned}$$

which implies $\theta^i \in H_{n+1}$. $\theta^i \in H$ is trivially implied, we conclude $H_{n+1} \cap H \subset H_n \cap H \ \forall \ n \in \mathbb{N}$. Since $H \cap H = H$, we have shown $H_n \cap H \uparrow H$.

<u>$H_n \cup H \downarrow H$</u>:

We start by showing $\theta^i \in H_{n+1} \cap H_1 \Rightarrow \theta \in H_n$. Multiplying the inequality implied by $\theta^i \in H_{n+1}$ with $\frac{n^2-1}{n^2}$, the inequality implied by $\theta^i \in H_1$ with $\frac{1}{n^2}$ and summing up yields

$$\begin{aligned}
& U_i(s^i, \sigma_{n+1} \frac{n^2-1}{n^2} + \hat{\sigma} \frac{1}{n^2}) + \theta^i(s^i, \sigma_{n+1} \frac{n^2-1}{n^2} + \hat{\sigma} \frac{1}{n^2}) \\
\geq \; & U_i(\tilde{s}^i, \sigma_{n+1} \frac{n^2-1}{n^2} + \hat{\sigma} \frac{1}{n^2}) + \theta^i(\tilde{s}^i, \sigma_{n+1} \frac{n^2-1}{n^2} + \hat{\sigma} \frac{1}{n^2})
\end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned}
& U_i(s^i, \sigma \frac{n-1}{n} + \hat{\sigma} \frac{1}{n}) + \theta^i(s^i, \sigma \frac{n-1}{n} + \hat{\sigma} \frac{1}{n}) \\
\geq \; & U_i(\tilde{s}^i, \sigma \frac{n-1}{n} + \hat{\sigma} \frac{1}{n}) + \theta^i(\tilde{s}^i, \sigma \frac{n-1}{n} + \hat{\sigma} \frac{1}{n}) \ ,
\end{aligned}$$

which implies $\theta^i \in H_n$. We conclude $\theta^i \in H_{n+1} \setminus H_n \Rightarrow \theta^i \notin H_1$.

We proceed by showing $\theta^i \in H_{n+1} \setminus H_n \Rightarrow \theta^i \in H$.

Multiplying the inequality implied by $\theta^i \in H_{n+1}$ with $\frac{n^2-1}{n^2}$

and adding the inequality implied by $\theta^i \notin H_n$ yields

$$\begin{array}{rl} & U_i(s^i,\sigma) + \theta^i(s^i,\sigma) - U_i(\tilde{s}^i,\sigma) - \theta^i(\tilde{s}^i,\sigma) \\ > & \dfrac{2n}{n+1}\left[U_i(\tilde{s}^i,\hat{\sigma}) + \theta^i(\tilde{s}^i,\hat{\sigma}) - U_i(s^i,\hat{\sigma}) - \theta^i(s^i,\hat{\sigma})\right] \end{array}$$

As we have shown above, $\theta^i \notin H_1$ and therefore the right hand side of the inequality is positive. Therefore $\theta^i \in H$. Since $H_{n+1} = (H_{n+1} \setminus H_n) \cup (H_{n+1} \cap H_n)$ we conclude $H_{n+1} \subset H_n \cup H \; \forall \, n \in \mathbb{N}$. As $H \cup H = H$, we have shown $H_n \cup H \downarrow H$.

We apply Lemmata C1 & C2 below with $A_n = H_n \cap H$ and $B_n = H_n \cup H$ and conlude that F is continuous from above and below. From the definition of H, $\Theta_{s^i}(\sigma^j) = \cap_{\tilde{s}^i \in S^i} H_n^{\tilde{s}^i}$ is a finite intersection and the desired properties of F carry over for $\Theta_{s^i}(\sigma^j)$. $\square$

Lemmata C1 & C2 are taken from and proved in Bauer (1992).

Lemma C1 (continuity from below)
Consider a sequence $\{A_n\}_n$ of subsets of $\mathbb{R}^{n_i \cdot n_j}$ with $A_n \uparrow A \subset \mathbb{R}^{n_i \cdot n_j}$.
Then $\lim_{n\to\infty} F(A_n) = F(A)$.
Proof:
Define $A_0 := \emptyset$ and $a_n := A_n \setminus A_{n-1}, n \in \mathbb{N}$. We have $A_n = \cup_{i=1}^n a_i$ and $A = \cup_{n=1}^\infty a_n$. Since $a_n \cap a_m = \emptyset \; \forall n \neq m$ and $\sigma-$additivity of the measure F we have

$$F(A) = \sum_{n=1}^{\infty} F(a_n) = \lim_{n\to\infty} \sum_{i=1}^{n} F(a_i) = \lim_{n\to\infty} F(A_n) \;. \square$$

Lemma C2 (continuity from above)
Consider a sequence $\{B_n\}_n$ of subsets of $\mathbb{R}^{n_i \cdot n_j}$ with $B_n \downarrow B \subset \mathbb{R}^{n_i \cdot n_j}$.
Then $\lim_{n\to\infty} F(B_n) = F(B)$.

Proof:
Since $B_n \subset B_1 \; \forall \; n \in \mathbb{N}$ it holds that $F(B_1 \setminus B_n) = F(B_1) - F(B_n) \; \forall \; n \in \mathbb{N}$. Clearly, $B_1 \setminus B_n \uparrow B_1 \setminus B$. From Lemma C1 we know that $F(B_1 \setminus B) = \lim_{n\to\infty} F(B_1 \setminus B_n)$ and therefore $F(B_1 \setminus B) = F(B_1) - \lim_{n\to\infty} F(B_n)$. We also have $B \subset B_1$ and therefore $F(B_1 \setminus B) = F(B_1) - F(B)$, which establishes $F(B) = \lim_{n\to\infty} F(B_n)$. $\square$

REMARKS

Symmetric difference $A \ominus B = A \setminus B \cup B \setminus A$*:*
Note since we do not require $F(\Theta) > 0 \; \forall \Theta \subset \mathbb{R}^{n_i \cdot n_j}$, $F(A \ominus B) = 0$ does not imply $A = B$. Therefore $d_F(A, B) = F(A \ominus B)$ is only a pseudo metric (satisfying symmetry and the triangle inequality).

Upper hemi continuity:
Note further that $\Theta(\sigma)$ is not upper hemi continuous. To give an example, consider the doubly symmetric 2×2 coordination game:

	s_1^j	s_2^j
s_1^i	(1,1)	(0,0)
s_2^i	(0,0)	(1,1)

Then each $\theta^i \in \Theta_{s_1^i}\left(\frac{1}{3}\right)$ satisfies

$$[1+\theta^i(s_1^i,s_1^j)]\frac{1}{3}+\theta^i(s_1^i,s_2^j)\frac{2}{3} \geq \theta^i(s_2^i,s_1^j)\frac{1}{3}+[1+\theta^i(s_2^i,s_2^j)]\frac{2}{3}$$

Define E as the set of θ^i that satisfy

$$[1+\theta^i(s_1^i,s_1^j)]\frac{1}{3}+\theta^i(s_1^i,s_2^j)\frac{2}{3} > \theta^i(s_2^i,s_1^j)\frac{1}{3}+[1+\theta^i(s_2^i,s_2^j)]\frac{2}{3}-1$$

Clearly, $\Theta_{s_1^i}\left(\frac{1}{3}\right) \subset E$. Consider some $\hat{\sigma}^j(s_1^j)$ arbitrarily close to $\frac{1}{3}$, for example $\hat{\sigma}^j(s_1^j) = \frac{1}{3}+\epsilon$ for $\epsilon \in (0,\frac{2}{3})$. Any $\theta^i \in \Theta_{s^i}(\hat{\sigma}^j)$ satisfies

$$[1+\theta^i(s_1^i,s_1^j)](\frac{1}{3}+\epsilon)+\theta^i(s_1^i,s_2^j)(\frac{2}{3}-\epsilon)$$
$$\geq$$
$$\theta^i(s_2^i,s_1^j)(\frac{1}{3}+\epsilon)+[1+\theta^i(s_2^i,s_2^j)](\frac{2}{3}-\epsilon)\ .$$

The reader can verify that $\theta^i = (\theta^i(s_1^i,s_1^j),\ldots,\theta^i(s_2^i,s_2^j)) = \left(2\frac{1-3\epsilon}{3\epsilon},0,0,\frac{1}{3\epsilon}\right)$ does belong to $\Theta_{s^i}(\hat{\sigma}^j)$ but not to E. Therefore, $\Theta_{s^i}(\cdot)$ is not upper hemi continuous.

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