Essays on Basket Options Hedging and Irreversible Investment Valuation

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to my parents and Tao

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Preface

Options are used in finance describing contracts that grant the holder the right to purchase or sell a certain underlying asset at a predetermined price. Since they were first traded on the Chicago Board Options Exchange on April 26th, 1973, options became more and more widespread and option trading active over 50 exchange worldwide (cf. Wilmott (1992)). Meanwhile, many non–standard and complex products have been created and are traded over the counter. **Basket options** are one of such newly–generated exotic options. A basket option, as its name implies, is an option on a portfolio of several assets. As the underlying basket offers more diversification, basket options gain increasing popularity in world financial markets as a fundamental instrument to manage portfolio risks. Examples thereof are equity index options which are traded on the exchange and usually contingent on at least 15 stocks, as well as currency basket options traded over the counter and written on over two currencies.

Obviously, the unique feature of basket options is the basket underlying and a complex correlation structure therefore involved. It provides investors a couple of benefits like high diversification, a lower price against a portfolio of single options and so on, and meanwhile complicates the evaluation of basket options. The inherent challenge in pricing and hedging basket options stems primarily from the analytical intractability of the distribution of the basket. If the single underlying asset is as usual assumed to be lognormal distributed, then a weighted sum of correlated lognormals is clearly not. The direct consequence of this absence is the infeasibility of closed–form pricing formula and hedging ratios in the Black and Scholes (1973) framework. Moreover, the correlations between underlying assets are observed to be volatile over time. Due to the lack of standardized basket options traded in the market, the correlation structure can be only estimated from historical time series or from scarce option data. This further prevents us from exactly pricing basket options, and more importantly, perfectly hedging basket options. As a result, a partial– or super–hedge is often pursued in the literature when hedging basket options. Apart from these difficulties, we address another difficulty resulted from a great number of underlying assets in the basket while hedging basket options. If following the standard hedging method, a hedging portfolio for basket options should be related to all underlying assets in the basket. Clearly, if the number of the underlying assets is over 15, such a dynamic hedging strategy would be not only hardly implementable in many practical situations but also create a large transaction cost. In this sense, a static or buy–and–hold hedge strategy has its advantage in cost saving and hence hedge performance. As a result, the first part of this dissertation aims to design a static hedging strategy for European–style basket options and to analyze its hedging result.

The newly developed static hedging strategies consist of traded plain–vanilla options on only subset of underlying assets. The optimal hedge is either super– or partial–replicating, depending on the objective function taken in the numerical optimization. Considering the numerical challenge in the optimization with constraints on the initial capital (or some other hedging requirements) and the maximal number of hedging assets, hedging portfolios are suggested in this thesis to be obtained in two steps, namely pre–selection of the sub–hedge–basket and determination of optimal hedging instruments, more precisely, the optimal strikes of available plain–vanilla options on the chosen subset of the basket. Especially, a multivariate statistical technique, Principal Components Analysis, is introduced to identify dominant assets in the basket by taking into account all the coefficients that greatly influence the basket value, such as weight, volatility and correlation. As demonstrated by numerical examples, such hedging portfolios work satisfactorily, generating a reasonably small hedging error though by using only several assets.

One basic type of options is known as American options. Such options can be exercised (bought or sold) at any date before a predetermined date (expiration date or maturity). In comparison with another type of options, European options, that can be exercised only at the maturity, American options provide an investor with a greater degree of flexibility. The right or flexibility embedded in options is not the sole product of financial markets, but also of capital markets. Physical asset value can be affected directly by the management decision. Their decision is obviously not an obligation, but a right to make an investment or to stop an investment. The only difference here from financial options is that the underlying is a physical asset. These properties of investment decisions are recognized in the late seventies. Since then, they are especially named in the literature as **real options** to describe opportunities of investment in non–financial assets with some degree of freedom in decision making against the underlying uncertainty. As many other researchers, we are also interested in this topic and are going to study irreversible investment valuation in the second part of this dissertation.

Built on the pioneering works by Jorgenson (1963) and Arrow (1968), an extensive literature investigates the irreversible investment problem under uncertainty via different approaches such as the conventional Net Present Value rule and the real options theory. Ever since its appearance, the real options analysis is regarded as a great improvement for the investment theory:

> *"..., real options add a rich economic theory to capital investing under uncertainty."*
>
> —*Bob Jensen*[1]

Despite a high reputation in academics, the real options theory is not widely adopted by corporate managers and practitioners due to the lack of transparency and simplicity of the standard real options approaches, i.e., the contingent claim analysis and the dynamic programming method. The second part of this dissertation first develops a *Shadow Net*

[1] Quoted in Bob Jensen's threads on Real Options, Option Pricing Theory, and Arbitrage Pricing Theory which are available at http://www.trinity.edu/rjensen/realopt.htm.

Present Value rule by using a new approach in the real options theory. The method starts with identifying the expected present value from the investment and comes to the final conclusion via representing the expected present revenue in terms of the expected present value of the running supremum of the *shadow revenue* of the investment. By aiming at the net profit of the investment which is the mere concern of investors, this approach thus facilitates an intuitive understanding of the real options theory and also a wider application into the practice. Meanwhile, it generalizes the elegant explicit characterization of the investment decision rule to all exponential Lévy processes: The optimal investment policy is a trigger strategy such that the investment is initiated at the first time when the value of the investment project comes to a critical threshold. As two extensions, this technique is then applied to two more complicated and practical models taking into consideration gradual capacity generation and risk neutrality, respectively. In each model, both qualitative and quantitative analysis is given on the investment feature and its relationship with related parameters.

Part I

Hedging Basket Options

Chapter 1

Introduction

1.1 Basket Options: Basic Feature and Literature Review

A basket option is an option whose final payoff is linked to a portfolio or "basket" of underlying assets. With the analogous payoff structure to a plain–vanilla option, a basket option grants investors an amount of money equal to the maximum value of zero and the difference between the basket value and the exercise price. Various types of basket options have emerged in the market and become increasingly popular as a tool for reducing risks since the early 1990s. They are either sold separately over–the–counter or sometimes issued as part of complex financial contracts, for instance, as "equity–kickers" in bond–like structures where a large coupon or a certain participation is usually offered conditionally on the performance of a predetermined basket of stocks.

Basket Composition Generally, the basket can be any weighted sum of underlying assets as long as the weights are all positive. The typical underlying of a basket option is a basket consisting of several stocks, indices or currencies. Less frequently, interest rates are also possible. Moreover, as often observed in the market, most of the new contracts are related to a large number of assets. For instance,

Example 1.1.1 (VarioZins IHS Contract). *The VarioZins IHS contract was issued by Deutsche Zentral–Genossenschaftsbank in November,* 2002. *It is basically a bond whose yearly coupon rate is closely related to the performance of an embedded basket option. The basket is an international stock portfolio and composed of* 15 *blue chip stocks. They are* 15 *international top companies from various industries as given in Table* 1.1.1.

Another increasing trend of basket options is the hybrid composition of the underlying basket. The basket is not restricted with basic financial assets like stocks, currencies and so on, but generalized to other products, say commodity prices, which currently offer a high growth rate. The addition of such assets undoubtedly increases the final payoff and also provides a broader diversification benefit. Example 1.1.2 illustrates exactly this property.

Stock	Country	Industry
Citigroup	USA	bank
McDonald's Corporation	USA	fast food gastronomy
IBM	USA	computer
Lockheed Martin	USA	aerospace and defense
Honda	Japan	automobile
AXA	France	insurance
Allianz	Germany	insurance
BNP Paribas	France	bank
L'Oreal	France	cosmetic
Nestlé	Switzerland	food processing
TotalFinaElf	France	oil and gas
E.ON	Germany	utility
Novartis	Switzerland	pharmaceuticals
PSA Peugeot Citroën	France	automobile
BASF	Germany	basic materials

Table 1.1: Basket Composition of the VarioZins IHS Contract

Example 1.1.2 (Multi–Asset Combination Bond). *Ulster Banks Ireland Ltd. introduced in January,* 2005 *a multi–asset combination bond that offers* 100% *capital security at maturity and accesses to high growth potential in a diverse range of asset classes. In detail, the rates of the return of this bond depend on a basket containing three different asset classes as follows:*

Weights	Asset Classes	Composition
40%	metals basket	five of the world's most liquid metals (access to the world's global commodity markets)
40%	equity basket	equally weighted indices of FTSE in UK, Nikkei 225 in Japan, Eurostoxx 50 in Europe and SMI in Switzerland (access to the world's larger stock markets)
20%	EPRA index	European Property Real Estate Association Index tracking the performance of 69 of the largest listed European property companies

Table 1.2: Basket Composition of the Multi–Asset Combination Bond

Advantages of Basket Options Several reasons to trade basket options are reported in the literature. Basket options are regarded as a superior product to plain–vanilla options mainly due to the following points:

- The major advantage of basket options is that they tend to be cheaper than the corresponding portfolio of plain–vanilla options. On one hand, this is due to the fact that the underlying assets in the basket are usually not perfectly correlated. Therefore, the volatility of the basket is in most cases less than the sum of volatilities, unless they are positively perfectly correlated[1]. In this way, a portfolio of plain–vanilla options is exposed to higher risks and hence more expensive than the corresponding basket option. On the other hand, a basket option minimizes transaction costs because an investor has to buy only one option instead of several ones. Thus, basket options become a cost–effective tool for risk managers to hedge a risky position consisting of several assets. For example, Boston–based Gillette Company, Illinois–based McDonald's Corporation and Pittsburgh–based Westinghouse Electric Company use a currency basket option rather than a portfolio of individual options on each currency (see Falloon (1997), Falloon (1998) and Smith (1998) for more examples of basket–linked financial products).

- Basket options are also ideal for clients who have a specific view of the market. They may be interested in diversified risk, or have a view on a particular sector, best expressed by a portfolio of individual stocks. So, the use of a basket of assets as an underlying allows products to be tailored to clients' needs. That is why the most widespread underlying of a basket option is a basket of stocks that represents a certain economy sector, industry or region. Moreover, by using basket options, investors need predict the performance of a particular industry but not one specific company, which is definitely an easier task.

- An additional, though minor, advantage of basket options is less effort and time that will be otherwise required for investors to monitor possible a large number of individual assets. This virtue was already identified by Falloon (1998) as it allows "key executive of the end–user to invest the time they would have spent on (currency) hedging decisions on problems they feel are more important to the company's overall operating performance". It becomes more obvious in index options whose underlying is composed of all the stocks in the index. An index option is therefore one special case of basket options and turns out in the market as an individual financial product. These index options provide investors the opportunity of investing various national as well as industrial equities in a efficient way. It not only reduces the transaction costs as we mentioned above, but also "removes the need for active assets selection" (cf. Beisser (2001), p. 124 and also Nicolls (1997), p. 120). Furthermore, it simplifies the trading in unfamiliar markets that is sometimes impeded due to custody issues, settlement problems and so on.

Difficulties of Valuing Basket Options Although positing these advantages, basket options are much more complicated to evaluate than plain–vanilla options. As well acknowledged in the literature, the inherent challenge in pricing and hedging basket options

[1]Mathematically, we have $Var\left[\sum_{i=1}^{N} X_i\right] = \sum_i \sum_j \sqrt{Var(X_i)}\sqrt{Var(X_j)}Corr(X_i, X_j)$. Assets with positive perfect correlation ($Corr(X_i, X_j) = 1 \ \forall i, j = 1, \cdots, N$) obviously get the highest volatility among all the possible cases with $Corr(X_i, X_j) \in [-1, 1]$.

stems primarily from the analytical intractability of the distribution of the basket. Usually, asset prices are assumed to be log–normal distributed. Practitioners sometimes take the basket itself also as a lognormal distribution[2]. However, it comes out only for simplicity and "looks more like a strategy of the last resort than a genuine solution" (cf. Hunziker and Koch-Medina (1996), p. 163). More importantly, it leads to an inconsistency in the basic assumption: The distribution of a weighted average of correlated lognormals is anything but lognormal. To keep along with the standard financial model, it is then plausible to have lognormality of individual assets and to find out the basket distribution. However, even if the distribution of the sum of lognormals is known by some numerical calculation, the result is exceedingly complicated to be applied in option pricing. The direct consequence of this absence is the infeasibility of the closed–form pricing formula and hedging ratios in the Black and Scholes (1973) (BS) framework.

Another difficulty in evaluating basket options is due to the correlation structure involved in the basket, which is the main feature that distinguishes these products from single–underlying options. Correlation is observed to be volatile over time as is the volatility. However, opposed to the volatility, correlations are not available in the market due to the lack of standardized basket options. In practice, traders heavily reply on a conservative estimation of correlations from historical time series or from sometimes scarce option data. Meanwhile, the current common practice is to assume it constant. In this sense, correlation risk usually cannot be hedged precisely in reality[3].

Literature Review on Basket Options Hedging[4] Basket options are nevertheless intensively studied in the literature. Its pricing is dealt with first by approximating the underlying basket's distribution and then greatly improved with a fairly accurate lower bound by means of the conditional expectation method first suggested by Curran (1994), Rogers and Shi (1995) and Nielsen and Sandmann (2003) for Asian options[5]. This dissertation focuses on another important issue — basket options hedging. A brief literature review is given in the following before presenting our own contributions.

So far, several methods have been proposed for hedging European–style basket options. Basically, they can be classified into three categories.

(a) First, numerical methods such as Monte Carlo simulations are used by Engelmann and Schwendner (2001) to compute Greeks. They assume that the market is com-

[2]Index options are the only exception whose underlying distribution is taken as a lognormal but without great counter argument, because they are in general actively traded as a single asset. One supportive argument is given in Nelken (1999) based on the law of large number: "As we incorporate more and more underlying securities into the index, it begins to have a distribution that resembles lognormal. The more underlying securities we place into the index, the more it looks lognormal".

[3]It is possible to hedge correlation risk of basket options on currencies, since volatilities and correlations of currency pairs are linked via exchange rates, as shown via geometric interpretation in Wystup (2002).

[4]The literature review is only on European–style basket options which are the concern of this dissertation. Hence, all the basket options in the text are European style if without further specification.

[5]A detailed description on basket options pricing methods is referred to Beisser (2001) and the literature therein.

plete and hence basket options can be perfectly hedged by a self–financing portfolio. However, such numerical computations can give only approximate but not exact hedge parameters, due to the lack of the knowledge of the underlying distribution. Thus, it is generally almost impossible to perfectly hedge basket options by buying or selling a portfolio of assets.

(b) In this context, some researchers are endeavored to develop partial hedging strategies. For example, in the second category, some static hedging portfolios are found to minimize the variance of the discrepancy between the final payoffs of the target basket option and the hedging portfolio. Pellizzari (2005) achieves this objective directly with the help of Monte Carlo simulation and Ashraff, Tarczon and Wu (1995) develop a variance–minimizing hedging strategy based on Gamma hedging which additionally considers the cross–gamma effect.

(c) In the absence of a perfect hedge, the next best thing is the least expensive super–replicating strategy. The problem of computing super–hedging portfolios has received a fair amount of attention in recent years. In a Copula framework, an upper bound on a basket option is obtained by Rapuch and Roncalli (2001) and Cherubini and Luciano (2002). It is shown that this bound is equal to the so–called Fréchet bound and corresponds to a particular case where the underlying assets are comonotonic. Chen, Deelstra, Dhaene and Vanmaele (2006) use the related idea based on the theory of stochastic orders and on the theory of comonotonic risks, to derive the largest possible price that occurs when the components assets are comonotomic. Basically, the hedging portfolio involves long position in traded options on all the underlying assets. It is an arbitrage–free universal bound in the sense that it is model independent and consistent with the market prices of related products on the component assets (e.g., futures, stock prices and stock options). One alternative approach to compute the similar static–arbitrage super–replicating strategies is via solving an optimization problem as in d'Aspremont and El-Ghaoui (2006), Laurence and Wang (2004), Laurence and Wang (2005) as well as Peña, Vera and Zuluaga (2006). They come to almost the same result although with various methods of semi–definite programming or linear programming. Of particular relevance to our work is the article by Hobson, Laurence and Wang (2005). The least expensive upper bound is achieved by a Lagrangian programming formulation given market prices of plain–vanilla options on each individual asset with all traded strikes.

In the presence of multiple assets, the calculation of these methods is generally complicated. In fact, a large number of underlying assets poses a challenge for quantitative finance when hedging basket options: With a complex dependence structure, one has great difficulties to calculate hedge ratios even by running Monte Carlo simulation. Besides, these approaches all yield a large hedging portfolio dependent on all the underlying assets in the basket. This is indeed impractical for most of newly–designed basket contracts with a large number of underlying assets. In an even worse situation like in Example (1.1.2), even if some assets (commodity prices) or their related products (their options) may be approximately priced, it is impossible to exploit them as hedging instruments due to illiquidity.

1.2 Motivation and Contributions of the Work

"Hedging of basket options presents a very real problem" (cf. Nelken (1999)). The typical scenario in practice is hedging a basket option that includes, say, 30 stocks. Suppose that all coefficients are estimated and its BS price can be computed. It then allows for a dynamical hedge by buying or selling the delta ratios. At each rebalancing date, hedgers have to adjust the hedging portfolio by indeed 30 independent trades. Undoubtedly, hedging with all the underlying assets would be not only computationally expensive, but also would create high transaction costs which greatly reduce the hedging efficiency. Thus, it is indeed impractical to consider such a strategy based on all the underlying assets. Furthermore, hedging with a subset of assets becomes more practical and essential when some of the underlying assets are illiquid or not even available for trading[6]. Thus, it is desirable to find a strategy to hedge a basket option by using only a subset of assets at a reasonable cost. As a result, the first part of this thesis considers the possibility of hedging basket options by only a subset of constituent assets and designs some hedging strategies on this basis.

The single work in the literature with the same objective is by Lamberton and Lapeyre (1992). They design a dynamic approximate hedging portfolio which consists of plain–vanilla options on the sub–basket, identified by a multiple regression analysis. To be more specific, hedging assets are chosen by minimizing the price difference between the self–financing portfolio which is assumed to be achievable and the hedging portfolio. The minimization is in essence a regression procedure. In turn, the selection of the subset of assets is equivalent to the selection of variables of a multiple regression. Accordingly, the numerical methods, such as forward, backward selection algorithms and stepwise regression methods, are recommended. In practice, hedging basket options with subset of assets is quite popular but lacking of an accurate criterion. According to Nelken (1999), the sub–hedge–basket is most often determined simply according to the liquidity or exposure of the underlying assets. For instance, the Heng Seng index is a market capitalization weighted stock market index, consisting of 40 stocks. To track or hedge this index, a subindex of the 5 largest stocks is usually used and turns out to be pretty accurate. In general, as argued in Nelken (1999), there is nevertheless no perfect solution for sub–hedge–basket selection. A sub–basket can never perfectly track the entire basket and hence leave some risks unhedged. In principle, a subset is chosen to catch the tradeoff between reasonably good duplication of the original basket and the reduced transaction cost.

This thesis aims at introducing another approach, Principal Components Analysis (PCA), to select hedging assets. PCA is one of the classical data mining tools to reduce dimensionality of multivariate data. In PCA, dimension of multivariate data is reduced by transforming the correlated variables into uncorrelated variables. PCA presents the vari-

[6]This is possible when the underlying is a mutual fund.

ation in a correlated multi–attribute to uncorrelated components, each of which is in principle a linear combination of the original variables. These uncorrelated components are usually regarded as the underlying factors influencing the data and are extracted by decomposing the covariance matrix of the original variables into eigenvectors and eigenvalues. Simply speaking, PCA achieves dimension reduction by identifying the smallest number components that account for most of the variation in the original multivariate data and summarize the data with little loss of information. Moreover, this method is quite easy to implement with almost instantaneous calculation as well as reasonable accuracy.

So far, this method is applied in finance mainly to identify the multiple risk factors in portfolio management and to figure out the dominant factor components driving the term structure movements of at–the–money (ATM) implied volatilities (cf. Fengler, Härdle and Schmidt (2002)). Furthermore, it is also applied to find a low–rank correlation matrix nearest to a given correlation matrix. Particularly, Dahl and Benth (2002) develop a method combining PCA and Quasi Monte Carlo simulations for a fast valuation of Asian basket options. The idea is to capture the main or most of the information of the noise term (the covariance structure), which is complicated with a rather large number of dimensions in both time and asset, by considerably reduced dimensions. They call the dimension reduction technique as Singular Value Decomposition, which is equivalent to PCA when the covariance structure is studied.

Similarly, PCA is adopted in the present thesis to find the most effective underlying factors that capture the main information of the basket. Thereafter, one step further is taken to obtain the subset of the underlying assets that are highly correlated to these selected factors. That is, we choose those assets that are significant with the largest contribution to the most effective PCs. It is worth noting that by decomposing the covariance structure of the basket we take into account all the coefficients which more or less affect the basket value, including weights, dividend yields, volatilities and most importantly correlations.

The second contribution of this dissertation (first part) is to design a hedging strategy related to only several underlying assets. It is a static super– or partial–hedging portfolio composed of plain–vanilla options written on a subset of significant assets in the original basket. Meanwhile, it also considers the issue of liquidity. In general, only a small number of options are traded for a single stock. That is, not every theoretically possible strikes exist in the market. Our hedging portfolio is then constructed to take only those available products as hedging instruments. This strategy is inspired by a static super–hedge method which dominates the final payoff of a European–style basket option by using plain–vanilla options on all the underlying assets. As we mentioned in the previous section, an investor holds a basket to reduce the risk exposure compared with exposure to a portfolio of individual assets. Hedging such a position with a set of options on the individual basket components works against the purpose. It therefore over–hedges the risk and costs too much. More specifically, this upper bound works well only in case of high correlation and the hedging performance decreases greatly with the correlation.

Hence, a problem has to be tackled when hedging basket options of how to incorporate correlation structure in the hedging strategy and simultaneously deal with the problem of a large number of underlying assets. Of course, an optimal hedge can be obtained by solving an optimization problem with constraints on the initial capital (or other hedging requirements) and the maximal number of assets. It would be however numerically infeasible. Regarding all these problems, we may find that PCA is the technique to be pursued that allows for pre–selection of hedging assets by decomposing the covariance matrix (of course the correlation matrix) of the original basket.

Basically, the hedging portfolio is accomplished in two steps. In the first step, dominant assets are figured out by means of PCA while taking the correlation structure and other pricing parameters of the basket into consideration. Then, appropriate hedging instruments, more precisely, the optimal strikes of plain–vanilla options on the chosen sub–basket are calculated by solving an optimization problem. Surely, a subset can not perfectly track the original underlying basket and may leave some risk exposure uncovered. In this context, different optimality criteria can be designed to obtain super– or partial–replications. Generally, the criterion depends on the risk attitude of the hedger. He may favor a super–replication to eliminate all risks. An upper bound is undoubtedly favored for the purpose of hedging. It is however sometimes not attainable in our case by using only several assets. Alternatively, with a constraint on the hedging cost at the initial date, optimal strikes are computed by minimizing a particular risk measure, e.g., the variance of the hedging error or the expected shortfall. Due to the lack of the distribution of the underlying basket, hedging portfolios are obtained numerically through Monte Carlo simulations.

Considering a more realistic market situation where only a limited number of options are available in the market, we have to make a proper adjustment on the optimization problem: one condition has to be imposed such that the strikes are confined in a given set. Generally, hedging portfolios can be determined by using a numerical searching algorithm. However, such a numerical optimization is computationally inefficient especially when the (sub–)basket is large and when a large set of strikes is traded in the market for the chosen hedging assets. In this context, a simple calibration procedure, convexity correction method, is developed for super–hedging portfolios. Those optimal but unavailable options are approximated by a linear combination of two options with neighboring strikes. The key feature of this calibration method is the easy and quick implementation, of course. Although the technique maintains the super–replicating property of the hedging portfolio, it is not feasible to prove for generality that it gives the cheapest portfolio attainable in the market. Nevertheless, it is shown by numerical results to be a good approximation.

The first part of this dissertation proceeds as follows: We end **this chapter** after presenting the model framework in Section 1.3. To cope with a large number of underlying assets and complicated correlation structure, the problem of assets selection is first addressed in **Chapter** 2 by means of the PCA method. The technique is briefly outlined and then applied to the basket options hedging context with a geometric interpretation.

On the basis, **Chapter** 3 develops a two–step static hedging strategy by a proper combination of the asset selection technique and a static super–hedging method based on all component assets in the basket. Thus, it starts in Section 3.2 with introducing the static super–replicating portfolio and pointing out three problems to be fixed. Thereafter, Section 3.3 presents our new hedging strategy step by step and particularly solves the problem of hedging with a discrete strike set.

In order to show the effectiveness of the newly–developed hedging strategies, numerical results are reported in **Chapter** 4. The numerical study shows that the hedging error (measured by the expected shortfall) at the maturity date decreases with the optimal strikes and hence the hedging cost. As a result, the newly–proposed static hedging portfolio by a subset of underlying assets achieves a trade–off between reduced hedging costs and overall super–replication. It is also demonstrated that hedging with only several underlying assets gives a satisfactory performance: when the super–hedging portfolio composed of plain–vanilla options on all the underlying assets does not exist or is not easily implementable, hedging with several underlying assets generates only a reasonably small hedging error by investing the same capital as the hedging cost of the super–hedging portfolio. Furthermore, such a hedging portfolio creates far less transaction costs than the super–hedging portfolio based on all the underlying assets if it is available. It enhances in turn the performance of the new hedging strategy by using only several underlying assets. Finally, some remarks are given, analyzing possible reasons for some unsatisfactory results and providing suggestions to remedy.

1.3 Model Framework

To develop a new hedging portfolio for European basket options, we give first basic assumptions and notations used throughout the first part of this thesis. Consider a financial market consisting of a **bank account** B and N **risky assets** S_i, $i = 1, \cdots, N$ (The risky assets can be in general referred to stocks, currencies, indices and even commodities). The dynamics of the bank account, which is continuously compounded with a constant risk free interest rate $r \geq 0$, are given by

$$dB(t) = rB(t)dt$$

for $t \in [0, T]$.

To model the N risky assets, let $W = (W_1(t), \cdots, W_N(t))$ be a standard N–dimensional Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}, \mathcal{Q})$ with the risk–neutral probability measure $\mathcal{Q}$ and an information filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions of completeness and right–continuity, i.e., $\mathcal{F}_0$ contains all the $\mathbb{Q}$–null set of $\mathcal{F}$ and $\mathbb{F}$ is right continuous. These one–dimensional Brownian motions (BM), W_i for $i = 1, \cdots, N$, are correlated with each other according to the following correlation matrix

$$R = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{N1} & \rho_{N2} & \cdots & \rho_{NN} \end{pmatrix},$$

where $\rho_{ij} = \rho_{ji}$, $\rho_{ii} = 1$ and $\rho_{ij} \in [-1, 1]$ for $i, j = 1, \cdots, N$. Moreover, as often assumed in the literature, the correlation structure of assets is constant over time and its determinant is strictly non–zero[7].

On this basis, the price process of each risky asset S_i, $i = 1, \cdots, N$ is supposed to follow a geometric Brownian motion (GBM). More explicitly, under the risk–neutral probability measure $\mathcal{Q}$, the risky assets satisfy the stochastic differential equation

$$\begin{aligned} dS_i(t) &= (r - q_i)S_i(t)dt + \sigma_i S_i(t)dW_i(t) \qquad (1.1) \\ \rho_{ij}dt &= dW_i(t)dW_j(t) \qquad i, j = 1, \cdots, N \end{aligned}$$

or simply

$$S_i(t) = S_i(0)e^{(r - q_i - \frac{1}{2}\sigma_i^2)t + \sigma_i W_i(t)}, \qquad (1.2)$$

where $S_i(0)$, σ_i and q_i are the initial price at time zero, volatility and continuously compounded dividend yield of asset i, respectively.

In addition to the above–mentioned primary assets, there are also derivatives whose value is contingent on the values of some basic assets, like stocks, interest rates and so on. In the objective financial market, we have European–style plain–vanilla calls on each risky asset S_i with strike price $k \in \mathcal{K}^{(i)}$, the set of all strike prices traded in the market, and maturity date T

$$C_T^{(i)}(k) = (S_i(T) - k)^+ \qquad i = 1, \cdots, N,$$

where $(\cdot)^+$ denotes $\max\{\cdot, 0\}$.

We are going to develop a hedging strategy for a European–style basket call on the N risky assets with maturity date T and strike price K

$$BC_T(K) = \left(\sum_{i=1}^{N} \omega_i S_i(T) - K \right)^+,$$

where each risky asset is weighted by a positive constant ω_i, $i = 1, \cdots, N$. That is, if $\sum_{i=1}^{N} \omega_i S_i(T)$, the sum of asset prices S_i weighted by positive constants ω_i at date T, is more than K, the payoff equals the difference; otherwise, the payoff is zero. This hedging strategy for a European basket *call* can be easily translated into one for the corresponding

[7] This implies that the correlation structure is nonsingular (cf. Lipschutz (1991), p. 45).

European basket *put* based on the put–call parity result (see Laurence and Wang (2005) and Deelstra, Liinev and Vanmaele (2004))

$$\left(K - \sum_{i=1}^{N} \omega_i S_i(T)\right)^+ = \left(\sum_{i=1}^{N} \omega_i S_i(T) - K\right)^+ + \left(K - \sum_{i=1}^{N} \omega_i S_i(T)\right).$$

Thus, we concentrate only on basket call options in this dissertation.

Under this construction, the market is definitely complete since the number of uncertainty sources (Wiener processes) is equal to that of the risky assets and furthermore the correlation structure is nonsingular (see Karatzas and Shreve (1998), Theorem 6.6). In such a market, there is no arbitrage (because we have an equivalent martingale measure) such that any contingent claim can be replicated by a self–financing trading strategy. Moreover, the absence of arbitrage opportunities and market completeness is equivalent to the uniqueness of the risk–neutral measure (or equivalent martingale measure), as stated in the fundamental papers of Harrison and Kreps (1979), Harrison and Pliska (1981), Harrison and Pliska (1983) and Back and Pliska (1991). Then, following the argument originated in Cox and Ross (1976), one can always find the fair price of the above–mentioned European–style contingent claims by discounting its expected payoff at the maturity date T under the risk–neutral measure.

Finally, two terms are specified which will be examined in the numerical examples in order to demonstrate the effectiveness of the designed hedging portfolio (HP): Hedging cost (HC) is defined as the price of the hedging portfolio at the initial date 0; meanwhile, hedging error at the maturity date T is simply denoted as HE, giving the difference between the final payoffs of the basket option and the hedging portfolio at time T, i.e., $BC_T(K) - HP_T$.

Chapter 2

Sub–Hedge–Basket Selection

Given the multi–dimensional nature of basket options, the derived hedging strategy is often composed of all the underlying assets. In practice, underlying assets in the contract are differently weighted and sometimes some assets in the basket are with a quite small weight. Thus, one can simply hedge such basket options by neglecting those assets. However, this is rather arbitrary and lacks a theoretical foundation for the general case. This chapter aims to offer a criterion for hedging assets selection based on the PCA technique. First, the method is briefly introduced by giving the mathematical foundation and properties. It is then applied to determine the dominant sub–basket for basket options hedging. Finally, the PCA method is geometrically interpreted in the asset selection context for a better and intuitive understanding.

2.1 Principal Components Analysis

PCA is a popular method for dimensionality reduction in multivariate data analysis. Thus, it is useful in visualizing multidimensional data, and most importantly, identifying the underlying principal factors of the original variables. PCA is originated by Pearson (1901) and proposed later by Hotelling (1933) for the specific adaptations to correlation structure analysis. Its idea has been well described, among others, in Harman (1967), Härdle and Simar (2003) and Srirastava and Khatri (1979). We follow here the lines of Härdle and Simar (2003).

The main objective of PCA is to reduce the dimensionality of a data set without a significant loss of information. This is achieved by decomposing the covariance matrix into a vector of eigenvalues ordered by importance and eigenvectors. To be precise, consider the asset prices vector $\mathcal{S} = (S_1, \cdots, S_N)^T$ with mean vector and variance matrix

$$E(\mathcal{S}) = \mu \qquad \text{and} \qquad Var(\mathcal{S}) = \Sigma = E\left[(\mathcal{S} - \mu)(\mathcal{S} - \mu)^T\right] .$$

PCA decomposes the covariance matrix into its eigenvalues and eigenvectors as

$$\Sigma = \Gamma \Lambda \Gamma^T , \tag{2.1}$$

where $\Lambda = diag(\lambda_1, \cdots, \lambda_N)$ is the diagonal eigenvalue matrix with $\lambda_1 > \cdots > \lambda_N$ and Γ the matrix of the corresponding eigenvectors

$$\Gamma = \begin{pmatrix} \gamma_{11} & \gamma_{12} & \cdots & \gamma_{1N} \\ \gamma_{21} & \gamma_{22} & \cdots & \gamma_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{N1} & \gamma_{N2} & \cdots & \gamma_{NN} \end{pmatrix}$$

or simply $(\gamma_1, \cdots, \gamma_N)$ given by the columns of the matrix. Principal Component (PC) transformation is then defined as the product of the eigenvectors and the original matrix less the mean vector

$$P = \Gamma^T(\mathcal{S} - \mu)\,. \tag{2.2}$$

That is, the PC transformation is a linear transformation of the underlying assets. Its elements $P_1, \cdots, P_N$ are named i–th PCs since they can be considered as the underlying factors that influence the underlying assets with decreasing significance as measured by the size of the corresponding eigenvalues.

The ability of the first n $(n < N)$ PCs to explain the variation in data is measured by the relative proportion of the cumulated sum of eigenvalues

$$\pi_n = \frac{\sum_{j=1}^{n} \lambda_j}{\sum_{j=1}^{N} \lambda_j}\,.$$

If a satisfactory percentage of the total variance has been accounted for by the first few components, the remaining PCs can be ignored as the assets are already well represented without significant loss of information. Usually, the first several n PCs are chosen such that over 75% of the variance are accounted for or simply the first three factors are selected $(n = 3)$ for the convenience of visualizing the data.

The weighting of the PCs, or simply the element of each eigenvector, describes how the original variables are interpreted by the factors. This could be validated by considering the covariance between the PC vector P and the original vector $\mathcal{S}$

$$\begin{aligned} Cov(\mathcal{S}, P) &= E(\mathcal{S}P^T) - E\mathcal{S}EP^T \\ &= E(\mathcal{S}\mathcal{S}^T\Gamma) - \mu\mu^T\Gamma \\ &= \Sigma\Gamma \\ &= \Gamma\Lambda\Gamma^T\Gamma \\ &= \Gamma\Lambda\,. \end{aligned} \tag{2.3}$$

It implies that the correlation $r_{ij} = \rho_{S_i,P_j}$ between the variable S_i and the PC P_j is[1]

$$r_{ij} = \frac{\gamma_{ij}\lambda_j}{(\sigma_i^2\lambda_j)^{1/2}} = \gamma_{ij}\left(\frac{\lambda_j}{\sigma_i^2}\right)^{1/2}.$$

[1] Note that $Var(P_j) = \lambda_j$. The detailed derivation is referred to the related textbooks and literature mentioned above.

Clearly, γ_{ij} is proportional to the covariance of S_i and P_j. The higher it is, the more related is the i–th asset to the j–th PC. Hence, γ_{ij} are usually called factor loadings, characterizing the relationship between the original variables S_i, $i = 1, \cdots, N$ and the derived factors, i.e., P_j's, $j = 1, \cdots, n$. Furthermore, one can easily find

$$\sum_{j=1}^{N} \lambda_j \gamma_{ij}^2 = \gamma_i^T \Lambda \gamma_i$$

is indeed the (i, i)–element of the matrix $\Gamma\Lambda\Gamma^T = \Sigma$. Summing up all r_{ij}^2 yields

$$\sum_{j=1}^{N} r_{ij}^2 = \frac{\sum_{j=1}^{N} \lambda_j \gamma_{ij}^2}{\sigma_i^2} = \frac{\sigma_i^2}{\sigma_i^2} = 1 \, .$$

Thus, r_{ij}^2 is calculated in the standard practice measuring the proportion of variance of S_i explained by P_j.

As the final remark to this technique, it should be noticed that the PCs are not scale invariant, e.g., the PCs derived from the covariance matrix give different results when the variables take different scales. Consequently, instead of the covariance matrix, the correlation matrix is recommended to be decomposed.

2.2 Application to Basket Options Hedging

Now based on the principle of PCA, hedging assets selection can be completed in four steps as follows:

Step I: *Find the covariance matrix of the underlying basket.* As assumed in Section 1.3, each underlying asset follows a GBM with constant drift and volatility. According to the derivation in Appendix $A.2$, the entire basket at the maturity date has the covariance matrix with the diagonal elements for $i = 1, \cdots, N$

$$Var(\omega_i S_i(T)) = \omega_i^2 S_i^2(0) e^{2(r-q_i)T} (e^{\sigma_i^2 T} - 1)$$

and non–diagonal elements for i, $j = 1, \cdots, N, i \neq j$

$$Cov(\omega_i S_i(T), \omega_j S_j(T)) = \omega_i S_i(0) \omega_j S_j(0) e^{(2r-q_i-q_j)T} (e^{\sigma_i \sigma_j \rho_{ij} T} - 1) \, .$$

In practice, this step has to be done by first studying the time series of the asset price to achieve the basic correlation structure and the (ATM) volatility of all the underlying assets[2]. Then combine these further with dividend yields and weights to obtain the covariance structure of the basket at time T. One may argue that the assumption of constant correlation and variance of the underlying assets is not

[2]Due to the complexity of volatility, traders and analysts have used the ATM volatility for each component asset to price basket options as a rule of thumb.

true, therefore may exert an impact on the choice of sub–hedge–basket. As far as we know, it is almost impossible to account for the volatility smile and changeable correlation in one single covariance matrix. In this sense, it is definitely the drawback of this method. However, we argue first that there is no listed trading price and hence no reliable data for pricing parameters such as correlations, as basket options are usually traded over the counter. Thus, the usual practice nowadays to treat basket options is to approximate the correlation as a constant if no further information is available. Moreover, considering the transaction costs involved, it is also not favorable to change the subset of hedging assets often.

Alternatively, the covariance matrix can be dynamically adjusted according to real–time data. In this way, one can observe any change in the significance of the underlying assets. For instance, if a dynamic hedging is preferred by using only subset of assets, we suggest to consider the following covariance structure:

$$Cov(\frac{d\omega\mathcal{S}}{\mathcal{S}}) = \begin{pmatrix} \omega_1^2\sigma_1^2 & \omega_1\omega_2\sigma_1\sigma_2\rho_{12} & \cdots & \omega_1\omega_N\sigma_1\sigma_N\rho_{1N} \\ \omega_1\omega_2\sigma_1\sigma_2\rho_{12} & \omega_2^2\sigma_2^2 & \cdots & \omega_2\omega_N\sigma_2\sigma_N\rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1\omega_N\sigma_1\sigma_N\rho_{1N} & \omega_2\omega_N\sigma_2\sigma_N\rho_{2N} & \cdots & \omega_N^2\sigma_N^2 \end{pmatrix} dt\,.$$

Clearly, it is the covariance of the relative price change of the basket. To keep consistent with the market information, this structure can be updated over time and thus the determination of the sub–hedge–basket. Compared with the former covariance matrix, this covariance structure is such simpler while taking into consideration most of the parameters involved for basket options pricing and hedging (except asset initial prices and dividends). In general, the latter structure should work well except for the case in which spot prices of the underlying assets differ significantly from one another. Hence, one has to additionally pay attention to the effect of asset prices. In such an extreme case, those underlying assets with greatly high prices should be always chosen (even with a relatively low volatility) due to its absolute dominant effect on the basket option price.

Whichever covariance matrix is chosen, the correlation effect is taken into account in the hedging assets selection procedure. In contrast to the usual practice of decomposing the correlation matrix as recommended in PCA textbooks, the covariance is however used in this application. This is simply because weights, individual asset prices as well as volatilities do have a great impact on the basket option price.

Step *II*: *Decompose the covariance matrix into eigenvalues ordered in significance and the corresponding eigenvectors.* This evaluation procedure could be easily done by many programs such as Matlab, Mathematica, C++ etc.

Step *III*: *Choose the first several important PCs according to the cumulative proportion of the explained variance.*

Step *IV*: *Select $N_1 < N$ most dominant underlying assets by examining their cumulative r^2 with the chosen PCs.* The selection can be done in two ways: First, if the number of hedging assets, N_1, is beforehand determined, the list of least important assets is checked out after a comparison of cumulative r^2. If there is no prior requirement on the number of assets, a more careful study of the cumulative r^2 has to be done to find the most effective assets.

Remark 2.2.1. *It is noted that some information of the original basket is lost in the fourth step by taking the underlying assets which are strongly dependent on the first N_1 PCs. Meanwhile, it loses a quantitative measure of the explained variance. This can not be improved by PCA itself and hence may give a lower performance while hedging basket options. Nevertheless, this obstacle will be to some extent overcome in the hedging portfolio construction procedure via for instance introducing an optimization problem to reduce the discrepancy between final payoffs of the basket call option and the hedging portfolio. See the discussion in Chapter* 3.

2.3 Geometrical Interpretation of PCA for Asset Selection

Before ending this chapter, a geometric interpretation is provided in this subsection for a better and intuitive understanding of the PCA technique as well as its application to the selection of hedging assets for a given basket option.

In general, PCA can be geometrically interpreted as a method searching for a low–dimensional subspace to represent as best as possible the information in a data matrix. In this way, the newly defined subspace provides a good fit for the observations and the variables such that the distances between the points in the subspace provide an accurate representation of the distances in the original space.

Put it in the sub–hedge–basket selection framework, m realizations/observations of asset prices $\mathcal{S}$ are available and can be considered as a cloud of points in the N–dimensional space which is defined by the N asset prices. PCA is here taken for the purpose of finding a N_1–dimensional subspace, such that $N_1 << N$ and the configuration of m points in this subspace closely approximates that of m points in the original N–dimensional space. First, we shift the origin of this subspace to the mean of the m points (or simply the mean of the matrix $\mathcal{S}$), then these points correspond $\mathcal{S} - \mu$ in the transformed N–dimensional space. Then we first find a 1–dimensional vector space, i.e., a straight line passing through the new origin O (the mean of the original matrix, μ) to best fit the data. Let γ be a unit vector defining the subspace such that $\gamma\gamma^T = 1$. Consider a vector OV_i pointing the location/direction of i–th observation in the data, its projection on the 1–dimensional subspace is the scalar product of OV_i and γ. These points are fitted to the subspace, using the least square principle, i.e., by minimizing the sum of the squares of the distances:

$$\sum_{i=1}^{m}(VP_i)^2.$$

Also we have

$$\sum_{i=1}^{m}(VP_i)^2 = \sum_{i=1}^{m}(OV_i)^2 - \sum_{i=1}^{m}(OP_i)^2.$$

Since $\sum_{i=1}^{m}(OV_i)^2$ is fixed, the minimization of $\sum_{i=1}^{m}(VP_i)^2$ is equivalent to the maximization of $\sum_{i=1}^{m}(OP_i)^2$. This quantity can be expressed as a function of $\mathcal{S}$ and γ as:

$$\sum_{i=1}^{m}(OP_i)^2 = ((\mathcal{S}-\mu)\gamma)^T(\mathcal{S}-\mu)\gamma = \gamma^T(\mathcal{S}-\mu)^T(\mathcal{S}-\mu)\gamma$$

Then we can determine γ by maximizing the quadratic form $\gamma^T(\mathcal{S}-\mu)^T(\mathcal{S}-\mu)\gamma$, subject to the constraint $\gamma^T\gamma = 1$. This is done by setting the derivative of the Lagrangian equal to zero as

$$\begin{aligned}\mathcal{L} &= \gamma^T(\mathcal{S}-\mu)^T(\mathcal{S}-\mu)\gamma - \lambda(\gamma^T\gamma - 1)\\ \mathcal{L}' &= 2(\mathcal{S}-\mu)^T(\mathcal{S}-\mu)\gamma - 2\lambda\gamma \quad = \; 0\end{aligned}$$

Thus the optimal first subspace, denoted as γ_1, is the solution of

$$(\mathcal{S}-\mu)^T(\mathcal{S}-\mu)\gamma = \lambda\gamma.$$

The solution of this equation is well–known: γ_1 is the eigenvector associated with the largest eigenvalue λ of the matrix $(\mathcal{S}-\mu)^T(\mathcal{S}-\mu)$, which is actually the covariance matrix Σ of $\mathcal{S}$.

Obviously, the two–dimensional subspace that best fits the data contains the subspace defined by γ_1. Then, one can find the second vector γ_2 in this subspace, which is orthogonal to γ_1 and maximizes the quadratic function $\gamma^T(\mathcal{S}-\mu)^T(\mathcal{S}-\mu)\gamma$. Following the same procedure, the N_1–dimensional subspace that is best fit in the least squares sense can be found. As a result, we get orthogonal vectors $\gamma_1, \gamma_2, \cdots, \gamma_{N_1}$ of the covariance matrix $\Sigma = (\mathcal{S}-\mu)^T(\mathcal{S}-\mu)$ corresponding to the first N_1 largest eigenvalues, ranked in the descending order $\lambda_1 > \lambda_2 > \cdots > \lambda_{N_1}$. Clearly, all these are proceeded in the second step of decomposing the identified covariance of the underlying basket. Meanwhile, this geometric interpretation demonstrates that by decomposing the variance matrix of the basket, the m observation points is projected on a new N–dimensional space explained by those recognized γ factors. This new space is uniquely characterized by the decreasing volatility explanation capability of the axes factors. These vectors are indeed the underlying factors that influence the basket options price, such as a downward shift of the interest rate, or an increase in the oil price, etc.. However, the main disadvantage of PCA is that the PCs are usually hard to explain or define. This is not a problem any more in our case, since we only need to tell the significance of the composite assets in the basket. It is determined in our method by means of their contributions to the first several N_1 PCs.

Chapter 3

Static Hedging Strategies with a Subset of Assets

3.1 Introduction

This chapter develops a static hedging strategy for European basket options by using only a subset of underlying assets. The hedging portfolio consists of plain–vanilla options contingent only on the dominant assets in the basket. It is basically completed in two steps by first picking up significant hedging assets as introduced in the previous chapter and then choosing the optimal strikes of the hedging instruments.

The basic idea of this hedging strategy is inspired by a static hedging strategy which is the cheapest portfolio dominating the final payoff of a basket option. It is nevertheless composed of plain–vanilla options on all the composite assets in the basket. As we argued in the motivation, it is indeed impractical to consider such a strategy based on all the underlying assets. It is hence desirable to find a strategy to hedge a basket option by using only a subset of assets at a reasonable cost. Meanwhile, this hedging portfolio by using plain–vanilla options provides a feasible hedging method to cope with inadequate data of correlations, but on the other hand indeed neglects their essential effect on basket options hedging. In most time, it is rather expensive unless in the extreme case where the underlying assets are perfectly correlated.

These two drawbacks of the static super–hedging portfolio are tackled by running PCA which greatly reduces the size of the basket by taking the correlation as an important element to examine. This hence motivates our new two–step static hedging strategy: In the first step, the appropriate set of hedging assets is figured out by means of PCA while taking the correlation structure of the basket into the consideration. Then, the optimal strikes of the options on the chosen sub–basket are calculated by solving an optimization problem. Surely, a subset could not perfectly track the original underlying basket and may leave some risk exposure uncovered. In this context, different optimality criteria can be designed to pursue super– or partial–replications. Basically, the criterion depends on the risk attitude of hedgers. They may favor a super–replication to eliminate all risks.

Although super–hedge is favored without any risks, it is nevertheless not always available by simply using several underlying assets. Alternatively, with a constraint on the hedging cost at the initial date, optimal strikes are computed by minimizing a particular risk measure, e.g., the variance of the hedging error or the expected shortfall. Due to the lack of the distribution of the underlying basket, the hedging portfolios are obtained numerically through Monte Carlo simulations.

Considering that there is only a limited number of options traded in the market, we have to make a proper adjustment on the optimization problem: one condition has to be imposed such that the strikes are restricted in the given set. Generally, the hedging portfolio can be obtained by using a numerical searching algorithm. However, such a numerical optimization is computationally inefficient, especially when the (sub–)basket is large and when a big number of strikes is available in the market for the chosen hedging assets. In this context, a simple calibration procedure, convexity correction method, is developed but only for super–hedging portfolios, if they are obtainable. Those optimal but unavailable options are approximated by a linear combination of two options with neighboring strikes. Clearly, this calibration method maintains the super–replication property in a quick and easy algorithm. A general proof is however not possible to show the robustness of the hedging portfolio in a sense that it is the cheapest super–hedge portfolio attainable in the market. We nevertheless demonstrate through numerical example in the next chapter that it is quite closed to the upper bound computed in the idealized context.

The remainder of the chapter is organized as follows: Section 3.2 presents the basic idea of the static super–hedging strategy and points out three problems remaining to be fixed. On this basis, a two–step static hedging strategy is proposed in Section 3.3 by properly combining the asset selection technique and the static super–hedging strategy.

3.2 Problem Formulation

Theoretically, a perfect hedge is achievable for European basket options in a complete market. However, due to the lack of the distribution of the underlying basket and the knowledge of related parameters and most importantly a large number of the underlying assets, it is indeed impossible and impractical to hedge basket call options perfectly by all the constitute assets. In this context, a static hedging method is developed only related to a subset of underlying assets. This section formulates three problems in a static super–hedging portfolio consisting of plain–vanilla call options on all the constituent assets with optimal strike prices, which is a cornerstone of this work initiating the basic idea.

3.2.1 A Static Super–Hedging Strategy

This static hedging strategy aims at finding the least expensive portfolio whose final payoff always dominates that of a basket call. The idea is stimulated by Jensen's inequality for the final payoff of a basket call:

$$\begin{aligned} BC_T(K) &= \left(\sum_{i=1}^{N} \omega_i S_i(T) - K\right)^+ \\ &= \left[\sum_{i=1}^{N} \omega_i \left(S_i(T) - k_i\right)\right]^+ \\ &\leq \sum_{i=1}^{N} \omega_i \left(S_i(T) - k_i\right)^+ = \sum_{i=1}^{N} \omega_i C_T^{(i)}(k_i)\,. \end{aligned}$$

First, ω_i is taken out of the bracket such that the equality holds if and only if $\sum_{i=1}^{N} \omega_i k_i = K$. The second transformation is due to Jensen's inequality. That is, the payoff of any portfolio consisting of N plain–vanilla calls is never lower than that of the corresponding basket call. Moreover, as a consequence of the no–arbitrage assumption, the price of a financial product is given by the discounted expected final payoff under the risk–neutral measure $\mathcal{Q}$. The corresponding relationship between the prices of a basket call and the hedging portfolio is then obtained as

$$e^{-rT} E^{\mathcal{Q}} \left[\left(\sum_{i=1}^{N} \omega_i S_i(T) - K\right)^+\right] \leq \sum_{i=1}^{N} \omega_i e^{-rT} E^{\mathcal{Q}} \left[\left(S_i(T) - k_i\right)^+\right]\,. \tag{3.1}$$

For the purpose of hedging, one would like to look for a portfolio of plain–vanilla call options with the optimal strike prices such that it is the cheapest hedging strategy to dominate the final payoff of a basket call. As a result, a minimization problem has to be dealt with: minimize the price of a weighted portfolio of standard options with respect to k_i's subject to the condition that the sum of $\omega_i k_i$'s is equal to K, i.e.,

$$\min_{k_i} \quad \sum_{i=1}^{N} \omega_i e^{-rT} E^{\mathcal{Q}} \left[\left(S_i(T) - k_i\right)^+\right] \tag{3.2}$$

$$\text{s.t.} \quad \sum_{i=1}^{N} \omega_i k_i = K\,. \tag{3.3}$$

The optimal sequence of strikes k_i^* is uniquely determined by the following proposition.

Theorem 3.2.1. *Suppose the underlying assets of a basket option follow GBMs and the BS model is valid, then the optimal k_i^*'s satisfying*

$$BC_0(K) \leq \sum_{i=1}^{N} \omega_i e^{-rT} E^{\mathcal{Q}} \left[\left(S_i(T) - k_i^*\right)^+\right]$$

are uniquely obtained by solving a set of equations:

$$k_i = S_i \left(\frac{k_1}{S_1}\right)^{\frac{\sigma_i}{\sigma_1}} \exp\left\{T\left[\left(1-\frac{\sigma_i}{\sigma_1}\right)\left(r+\frac{1}{2}\sigma_1\sigma_i\right)+\left(\frac{\sigma_i}{\sigma_1}q_1-q_i\right)\right]\right\} \quad (3.4)$$

$$\sum_{i=1}^{N}\omega_i k_i = K\,.$$

PROOF: The rigorous proof is provided in Appendix $A.1$. ■

The optimization problem is treated in the BS framework and solved by the corresponding Lagrange function. In this sense, it is similar to Hobson et al. (2005). They analyze the problem in a more general model independent context thus with a focus on proving the existence of a super–replicating strategy.

3.2.2 Discussions

Clearly, this hedging portfolio based on Jensen's inequality is a "buy–and–hold" strategy, ensuring hedger's position with a dominating payoff over the corresponding basket option at the maturity date in any market situations. Nevertheless, there are three problems inherent in the method especially from the practical implementation perspective, which are to be addressed on by one in the following.

A Large Number of Underlying Assets Our first concern is on the number of underlying assets. Adopting this hedging strategy, one has to hold options on all the underlying assets. As we mentioned above, it becomes almost impossible in practice when the basket option is contingent on a large number of assets. This is not only computationally expensive but also creates unfavorable high transaction costs. Moreover, the problem would be much worse when some of the underlying assets are illiquid or even not available for trading. Hence, it is essential and practical to find a strategy to hedge basket options at a reasonable cost but with only a subset of assets.

Correlation Effect This static hedging portfolio is an upper bound. In this way, all the risks are avoided, which is the second best for risk managers as the first best, perfect hedging, is almost impossible or complicated. The similar idea was once applied by Nielsen and Sandmann (2003) to Asian options[1]. It is well–known that Asian options and basket options are similar in structure: Both of them are average options, depending on a weighted sum of lognormally distributed random variables. The analogy is limited though: Asian options are relate to the prices of one unique asset at different moments of time; while basket options depend on the prices of several assets at the maturity date. This makes a big difference in the performance of this super–hedging strategy, which is mainly due to the correlation effect. Following the same idea, a super–replicating portfolio for Asian options is obtained consisting of options on the same underlying asset but

[1]Such a super–hedging portfolio is also developed in Simon, Goovaerts and Dhaene (2000) in a model–independent framework by using some results from risk theory on stop–loss order and comonotone risks.

with different maturity dates. In this case, the correlations are in effect auto–correlations and indeed involved endogenously in calculation. While, the hedging portfolio above for basket options is composed of a portfolio of plain–vanilla options thus completely independent of the correlation structure between assets. This point becomes even clear when observing that the correlation does not appear in the calculation of the optimal k_i's in (3.4). It is clearly an advantage of this method since it alleviates the difficulty of basket options hedging in controlling the correlation structure with scarce reliable data. However as easily observed, the upper bound performs well only when the underlying assets are strongly correlated, for example when all the constituent stocks belong to the same industry. The performance decreases in correlation with too much over–hedge and high hedging cost. In this sense, correlation should not be totally neglected but be properly treated.

It is obvious that the first two issues are already investigated in the asset selection procedure. Only several significant assets are chosen from the original basket by examining each composite asset's contribution to the underlying PCs of the basket. Correlations are indeed one of the essential parameters to analyze when identifying those PCs.

Discrete Set of Strikes Traded So far, this strategy is derived in an idealized situation where all the option on the constituent assets are available with a continuum of strikes. That is, $\mathcal{K}^{(i)}$, the set of all strikes of options traded in the market on the underlying asset S_i, is a continuum interval. With this full information, the portfolio could be obtained by simply computing a Lagrangian function in the BS framework. However, those optimal hedging instruments may be illiquidly traded. In fact, options are available in the market only with a limited number of strikes. Thus, the obtained portfolio has to be calibrated accordingly to the reality.

3.3 New Static Hedging Strategies by Using a Subset of Assets

A number of two–step static hedging methods are proposed in this section by properly combining the static super–hedging portfolio and the asset selection technique. First, the sub–hedge–basket is determined by PCA. Then, the hedging portfolio is designed to be composed of plain–vanilla call options written on these N_1 most important underlying assets with optimal strikes. They are chosen via numerical optimization using Monte Carlo simulation according to an optimality criterion which is defined by a particular risk measure. We furthermore demonstrate that the obtained hedging strategies capture a trade–off between reduced hedging costs and overall super–replication of basket options.

3.3.1 First Step: Hedging Assets Selection

As the first step of the newly–designed static hedging method, PCA is utilized to find the subset of important assets in the basket through a careful study of the modified covariance structure. In this way, all the underlying assets are newly indexed and regrouped into two subsets: one subset of N_1 assets of high significance S_j, where $j = 1, \cdots, N_1$ and one with the other $N - N_1$ assets S_j, where $j = N_1 + 1, \cdots, N$. Then the final payoff of the basket option can be rewritten as

$$\begin{aligned}
\left(\sum_{j=1}^{N} \omega_j S_j(T) - K\right)^+ &\overset{(A)}{=} \left(\sum_{j=1}^{N_1} \omega_j S_j(T) - K_1 + \sum_{j=N_1+1}^{N} \omega_j S_j(T) - K_2\right)^+ \\
&\overset{(B)}{\leq} \underbrace{\left(\sum_{j=1}^{N_1} \omega_j S_j(T) - K_1\right)^+}_{I} + \underbrace{\left(\sum_{j=N_1+1}^{N} \omega_j S_j(T) - K_2\right)^+}_{II} \\
&\overset{(C)}{\leq} \underbrace{\sum_{j=1}^{N_1} \omega_j \left(S_j(T) - k_j\right)^+}_{I'} + \underbrace{\left(\sum_{j=N_1+1}^{N} \omega_j S_j(T) - K_2\right)^+}_{II}, \qquad (3.5)
\end{aligned}$$

where $K = K_1 + K_2$ and $\sum_{j=1}^{N_1} \omega_j k_j = K_1$.

That is, the final payoff if a basket call is always dominated by a portfolios of plain–vanilla call options denoted as I' and a basket option of Part II' in (C). This result is achieved by applying two times Jensen's inequality in (B) and (C). Serving as a trick for the further derivation, the strike of the basket option K is in (A) split into K_1 and K_2 where $K_1, K_2 \in [0, K]$ such that the final payoff of the basket call is first dominated by two basket calls on the two disjoint subsets of the original underlying assets. Then following the same idea as in the previous section, one could find portfolios of plain–vanilla call options to further dominate the first basket option on those dominant assets. Clearly, if $N_1 = N$ and $K_1 = K$, the obtained hedging portfolio consists of all the underlying assets. Thus, hedging with all the assets discussed in Section 3.2.1 is one special case. With the assumption of no arbitrage, we can get the same relationship for the price at the initial date of the objective basket option and of the portfolio of plain–vanilla call options and one basket call on subset of the basket, after taking expectations and discounting their final payoffs under the risk neutral measure $\mathcal{Q}$:

$$BC_0(K) \leq \underbrace{\sum_{j=1}^{N_1} \omega_j e^{-rT} E^{\mathcal{Q}}\left[(S_j(T) - k_j)^+\right]}_{I'} + \underbrace{e^{-rT} E^{\mathcal{Q}}\left[\left(\sum_{j=N_1+1}^{N} \omega_j S_j(T) - K_2\right)^+\right]}_{II}. \qquad (3.6)$$

3.3.2 Second Step: Optimal Strikes Computation

Since the new hedging portfolio is only related to the dominant assets, our concern here is simply on part I'. Thus, in the second step, we have to search for the optimal strike prices k_j^*'s of the plain–vanilla calls in the hedging portfolio to cover as well as possible the risks that basket options are exposed to.

Such hedging portfolios related to only subset of underlying assets could not be a perfect replication. It could be nevertheless a super– or partial–hedge as required by hedgers. In any case, the designed hedging portfolio is derived through an optimization problem satisfying a certain optimality criterion.

Optimality Criteria Basically, the optimality criteria depend on the risk attitude of hedgers and are defined by particular risk measures. For instance, the criteria considered in this paper are designed to achieve super–replication, the minimum variance of the hedging error or the minimum expected shortfall given a certain initial hedging cost.

Criterion 1: Super–Replicate the Basket Option The first constraint is imposed on part I' in order to keep the the hedging portfolio's payoff at the maturity date never lower than that of the basket option. In this way, this hedging portfolio eliminates all the risks of holding a basket option. It is achieved by solving an optimization problem with the constraint of no sub–replication. More explicitly,

$$\min_{k_j} \quad \sum_{j=1}^{N_1} \omega_j e^{-rT} E^{\mathbb{Q}} \left[(S_j(T) - k_j)^+ \right] \tag{3.7}$$

$$s.t. \quad \mathbb{P}^{\mathbb{Q}} \Big[\sum_{j=1}^{N_1} \omega_j \left(S_j(T) - k_j \right)^+ \geq \Big(\sum_{j=1}^{N} \omega_j S_j(T) - K \Big)^+ \Big] = 1 \,. \tag{3.8}$$

Alternatively, one additional parameter K_1 can be involved in the optimization problem as follows:

$$\min_{K_1,\, k_j} \quad \sum_{j=1}^{N_1} \omega_j e^{-rT} E^{\mathbb{Q}} \left[(S_j(T) - k_j)^+ \right] \tag{3.9}$$

$$s.t. \quad \mathbb{P}^{\mathbb{Q}} \Big[\sum_{j=1}^{N_1} \omega_j \left(S_j(T) - k_j \right)^+ \geq \Big(\sum_{j=1}^{N} \omega_j S_j(T) - K \Big)^+ \Big] = 1 \tag{3.10}$$

$$\sum_{j=1}^{N_1} \omega_j k_j = K_1$$

$$0 \leq K_1 \leq K.$$

Although K_1 is not relevant to the hedging portfolio, it greatly simplifies the computation algorithm. The argument is as follows: First, the optimal k_j^* can be determined uniquely by Theorem 3.2.1 to super–replicate the basket option on the dominant assets for any given strike price $K_1 \in [0, K]$. However, K_1 has to be lower enough such that the basket option is well dominated simply by part I'. Consequently, we only need to compute such dominating hedging portfolios with all possible K_1 and then find out the least expensive one. The final hedging portfolio is then composed of only plain–vanilla call options on significant assets with the optimal strikes such that the basket options are super–replicated.

Remark 3.3.1. *It is in general hard to derive the existence and uniqueness (or the necessary condition) of this super–hedging portfolio composed only of options on dominant assets in the basket. In fact, there exists sometimes no such a hedging portfolio. To name one example, we can consider one basket option contingent on two assets with equal weights with strike price* 100*, namely we have* $(\frac{1}{2}S_1 + \frac{1}{2}S_2 - 100)^+$*. Asset One has a relatively large volatility and turns out to be the hedging asset after examining by PCA. At the maturity date, we have the realization that* $S_1(T) = 100$ *and* $S_2(T) = 310$*. Then the basket option gives the payoff of* $(\frac{1}{2} * 100 + \frac{1}{2} * 310 - 100)^+ = 105$*, which is even higher than* S_1 *itself. That means, the hedging portfolio related only on* S_1 *does not posit the feature of super–replication. Surely, this scenario is an extreme case and definitely not that typical. Nevertheless, it happens with positive probability and as a heuristic example it shows the possibility of sub–replication by using only subset assets. In such situations, some amount of cash has to be held to guarantee overall super–hedge.*

Put it more formally, we can understand it by following the idea in Equation (3.5)*. In Step* (C)*, the minimum of the hedging portfolio is obtained for each possible strike* K_1*. The determination of* K_1 *in Step* (B) *is however heavily dependent on the dominant assets and no guarantee can be generally achieved for its existence. Alternatively, the no–sub–replication constraint is sometimes over–demanding when using only subset of underlying assets. Moreover, a general proof for the condition required for the existence is almost impossible since we are lack of the measurement of dominance of sub–hedge–basket while performing PCA.*

From a practical point of view, this super–hedging portfolio may over–hedge too much and hence require a high hedging cost if it is available. This is partly due to the property of super–hedging portfolio whose hedging costs have to be high enough to stay always on the safe side. In addition, since the hedging portfolio is related to the sub–basket, more capital has to be invested because some risks arise from neglecting those insignificant assets. As a result, partial hedging strategies may be considered to achieve the trade–off of reduced hedging costs and overall super–replication.

Criterion 2: Minimize the Variance of Hedging Errors Given a Constraint on the Hedging Cost (HC) With restricted capital, one can only cover risks or minimize shortfall risks as well as possible. The shortfall risk is for instance in this case measured by the variance of hedging errors. Put it in an optimization problem, it is

$$\min_{k_j} \quad E^{\mathbb{Q}}\left[\left(\Big(\sum_{j=1}^{N}\omega_j S_j(T) - K\Big)^+ - \sum_{j=1}^{N_1}\omega_j\left(S_j(T) - k_j\right)^+\right)^2\right] \tag{3.11}$$

$$s.t. \quad \sum_{j=1}^{N_1}\omega_j e^{-rT} E^{\mathbb{Q}}\left[(S_j(T) - k_j)^+\right] \leq V_0 \tag{3.12}$$

$$k_j \geq 0 \quad \forall \quad j = 1, \cdots, N_1.$$

That is, we search for a hedging portfolio which minimizes the variance of hedging errors when the hedging cost is constrained to be lower than V_0, the maximal capital that hedgers would like to invest to hedge the basket option.

Criterion 3: Minimize the Expected Shortfall Given $HC \leq V_0$ One main drawback of the quadratic criterion is that it punishes both positive and negative differences between the payoffs of the hedging portfolio and the basket option. Actually, for the purpose of hedging, only the negative difference is not favored. To avoid such a problem, some other effective risk measures can be considered. The expected shortfall (ES) is in the context of hedging basket options defined by $E[(BC_T - HP_T)^+]$. Obviously, it accounts for only the positive hedging error. Meanwhile as a risk measure, it takes into account not only the probability of exposed risks but also the size. Hence, it is often used in the literature recently as a risk indicator. In this case, the optimization problem becomes

$$\min_{k_j} \quad E^{\mathcal{Q}}\left[\left(\Big(\sum_{j=1}^{N}\omega_j S_j(T) - K\Big)^+ - \sum_{j=1}^{N_1}\omega_j\left(S_j(T)-k_j\right)^+\right)^+\right] \tag{3.13}$$

$$s.t. \quad \sum_{j=1}^{N_1}\omega_j e^{-rT}E^{\mathcal{Q}}\left[(S_j(T)-k_j)^+\right] \leq V_0 \tag{3.14}$$

$$k_j \geq 0 \quad \forall \quad j = 1, \cdots, N_1\,,$$

where V_0 is again the restriction on the hedging cost.

To summarize, our new hedging portfolio is composed of plain–vanilla call options only on the dominant underlying assets in the basket with optimal strikes. This hedging portfolio is achieved by first identifying the subset of hedging assets by means of PCA, and then figuring out the optimal strikes for the call options on these assets based on an optimality criterion, e.g., super–replication, minimum variance or minimum ES given a certain investment into the hedge. The chosen criterion depends on the risk attitude of hedgers. The more risk averse he is, the tighter the criterion on the hedging error is, and the more probable the hedging portfolio with subset assets super–hedges basket options. In this context, the static hedging strategy presented in this paper finds a compromise between reduced hedging costs and overall super–replication. It is worth mentioning that all the optimization problems above are solved numerically by running Monte Carlo simulations because of the lack of the distribution of the underlying basket.

3.3.3 Hedging with a Discrete Set of Strikes

The above optimization problems are constructed in an ideal situation where the optimal strikes are always available in the market. In reality, $\mathcal{K}^{(i)}$, the set of all strike prices of options traded in the market on the underlying asset S_i, is however never a continuum range or interval but a discrete set. It makes a direct impact on the hedging portfolio since the optimal hedging products may not exist. Hence, the optimization problems have to be modified when considering only discrete sets of strikes traded. Generally, they are solved by running a numerical searching optimization, which searches numerically the cheapest portfolio confined in the given strike set and also satisfying the constraint.

More explicitly, the set of traded strikes for asset S_j $(j = 1, \cdots, N_1)$ entails $p+1$ strikes in the increasing order, i.e., $\mathcal{K}^{(j)} = (k_0^{(j)}, k_1^{(j)}, \cdots, k_p^{(j)})$ with $k_i^{(j)} < k_{i+1}^{(j)}$ for $i+1 \leq p$ and $k_0^{(j)} = 0$, namely, the least strike is such that the call option is the asset itself. Take the super–hedging strategy as an example. By restricting the hedging instruments to be those available in the market, the optimization problem for the super–hedging portfolio is modified as

$$\begin{aligned} \min_{k_j \in \mathcal{K}^{(j)}} \quad & \sum_{j=1}^{N_1} \omega_j e^{-rT} E^{\mathbb{Q}} \left[(S_j(T) - k_j)^+ \right] \\ s.t. \quad & \mathbb{P}^{\mathbb{Q}} \Big[\sum_{j=1}^{N_1} \omega_j \left(S_j(T) - k_j \right)^+ \geq \big(\sum_{j=1}^{N} \omega_j S_j(T) - K \big)^+ \Big] = 1 \,. \end{aligned}$$

Although the idea is straightforward, this optimization problem is only solvable by using numerical methods which is computationally intractable for a large number of underlying assets and a wide choice in strikes. Suppose each component asset has p options traded, the numerical search has to be done among all the possible combinations of those options of the order p^{N_1}. It is in general rather large since p is about 10 in reality. To gain computational efficiency, another simple calibration method, convexity correction, is developed for super–hedging portfolios via approximating the option's price with the optimal strike by two traded options with the neighboring strikes.

Recall the main property of a convex function: its value at a particular point is bounded from above by a linear interpolation of two neighboring values. This can be used to maintain the super–replication feature of the desired hedging portfolio since the BS call option price is well–known to be convex with respect to the strike price. Assume some optimal strikes $k_{opt}^{(j)}$'s obtained by solving the optimization problem (3.9) are not always traded in the market. For those assets whose call options with strike price $k_{opt}^{(j)}$ are not traded, one can replace them by a linear combination of two call options with the neighboring strikes $k_i^{(j)}$ and $k_{i+1}^{(j)}$ such that

$$C^{(j)}(k_{opt}^{(j)}) \leq \beta^* C^{(j)}(k_i^{(j)}) + (1-\beta^*) C^{(j)}(k_{i+1}^{(j)}),$$

where $\beta^* = \frac{k_{i+1}^{(j)} - k_{opt}^{(j)}}{k_{i+1}^{(j)} - k_i^{(j)}}$. In this way, the upper bound for a basket call option can be generally expressed for $j = 1, \cdots, N_1$

$$\sum_{k_{opt}^{(j)} \text{ traded}} \omega_j C^{(j)}(k_{opt}^{(j)}) \quad + \sum_{k_{opt}^{(j)} \text{ non–traded}} \omega_j \left(\beta^* C^{(j)}(k_i^{(j)}) + (1-\beta^*) C^{(j)}(k_{i+1}^{(j)}) \right) . \quad (3.15)$$

Consequently by means of convexity correction, a super–hedging strategy is achieved consisting of one or two traded call options on each dominant asset.

Remark 3.3.2. *The similar idea is also used Hobson et al. (2005) for basket option hedging but by using all the composite assets in the model independent framework. They show formally that the calibrated hedging portfolio is sharp in the sense that it is the cheapest arbitrage–free super–hedging portfolio by using the traded assets only. However, the tightness cannot be easily generalized to our case with only subset of assets. Basically, as shown by numerical results, the price of the convexity–corrected hedging portfolio is quite close to the original optimal portfolio.*

Chapter 4

Numerical Illustration of the Hedging Strategy

In this chapter, we give some numerical results for the new two–step static hedging strategy. Here we use the example that is first presented in Milevsky and Posner (1998). Basically, it is an index–linked guaranteed investment certificate offered by Canada Trust Co., fusing a zero coupon bond with a basket option that is stuck at the spot rate of the underlying indices. Here we are interested in hedging the embedded basket option of a weighted average of the renormalized G–7 indices

$$BC_T = \left(\sum_{i=1}^{7} \omega_i \frac{S_i(T)}{S_i(t)} - 1 \right)^+ .$$

That is, effectively, a call option on the rates of return of 7 indices. The necessary pricing parameters are given in Table 4.1 and 4.2. In addition, the risk–free interest rate is assumed to be deterministic and equal to 6.3%[1].

4.1 Asset Selection Through PCA

Given the data above, the covariance structure of the G–7 index–linked guaranteed investment certificate can be easily calculated according to the first covariance matrix specified in Section 2.2. Here, we show only the result for $T = 5$. Although the numbers may differ a little bit for different maturities, the same subset of assets are achieved finally. An implementation of the decomposition on the covariance gives then the eigenvalue vector in the order of significance

$$\lambda = (0.0144, 0.0108, 0.0072, 0.0032, 0.0009, 0.0005, 0.0002)^T,$$

[1]One important issue has to be mentioned for this illustrative example. Since the underlying assets are stock indices of different countries, exchange rate risks between different currencies will be involved in pricing and hedging the basket option. Here, in order to fully focus on the hedging issue, we neglect these risks by simply assuming that all the indices are traded in the market and are denominated in the same currency.

Country	Index	Weight (in %)	Volatility (in %)	Dividend Yield (in %)
Canada	TSE 100	10	11.55	1.69
Germany	DAX	15	14.53	1.36
France	CAC 40	15	20.68	2.39
U.K.	FTSE 100	10	14.62	3.62
Italy	MIB 30	5	17.99	1.92
Japan	Nikkei 225	20	15.59	0.81
U.S.	S&P 500	25	15.68	1.66

Table 4.1: G–7 Index–linked Guaranteed Investment Certificate

	Canada	Germany	France	U.K.	Italy	Japan	U.S.
Canada	1.00	0.35	0.10	0.27	0.04	0.17	0.71
Germany	0.35	1.00	0.39	0.27	0.50	-0.08	0.15
France	0.10	0.39	1.00	0.53	0.70	-0.23	0.09
U.K.	0.27	0.27	0.53	1.00	0.46	-0.22	0.32
Italy	0.04	0.50	0.70	0.46	1.00	-0.29	0.13
Japan	0.17	-0.08	-0.23	-0.22	-0.29	1.00	-0.03
U.S.	0.71	0.15	0.09	0.32	0.13	-0.03	1.00

Table 4.2: Correlation Structure of G–7 Index–linked Guaranteed Investment Certificate

and the eigenvectors γ_j in columns of the matrix

$$\Gamma = \begin{pmatrix} -0.1888 & -0.1022 & 0.0590 & -0.1200 & 0.0543 & -0.8515 & -0.4562 \\ -0.1870 & 0.1878 & 0.3002 & -0.8982 & -0.0291 & 0.0780 & 0.1613 \\ -0.2939 & 0.5790 & 0.5895 & 0.3994 & -0.2172 & -0.0983 & 0.1206 \\ -0.1585 & 0.1231 & 0.0930 & 0.0765 & 0.9680 & 0.0324 & 0.0847 \\ -0.0820 & 0.1401 & 0.1053 & -0.0320 & 0.0204 & 0.4742 & -0.8581 \\ 0.1728 & -0.6657 & 0.7170 & 0.0713 & 0.0353 & 0.0805 & 0.0054 \\ -0.8839 & -0.3756 & -0.1583 & 0.0861 & -0.1018 & 0.1640 & 0.0887 \end{pmatrix}.$$

Based on the knowledge of the eigenvalues and eigenvectors, one can determine the most significant factors according to the (cumulative) proportions of explained variance. As the results in Table 4.3 show, the first PC already explains around 39% of the total variation. An additional 57% is captured by the next three PCs. The remaining three PCs explain a considerably small amount of total volatility. In all, the first four PCs together account for about 96% of the total variation associated with all 7 assets. It suggests that we can capture most of the variability in the data by choosing the first four principal components and neglecting the other three.

The final step is to find the optimal subset of the underlying assets by checking the cumulative r^2 of each asset with the first four components. If two assets are planned to be used in the hedging portfolio, we need only find out the two most important assets

eigenvalue	proportion of variance	cumulated proportion
0.0144	0.3868	0.3868
0.0108	0.2895	0.6763
0.0072	0.1944	0.8707
0.0032	0.0852	0.9559
0.0009	0.0248	0.9807
0.0005	0.0127	0.9934
0.0002	0.0066	1

Table 4.3: Proportion of Variance Explained by PCs

from the basket. To achieve this result, the individual r and cumulative r^2 with the first four PCs are reported in Table 4.4. The assets are ordered in the significance: S_6, S_7, S_2, S_3, S_1, S_5 and S_4. Hence, the subset of optimal hedging assets is composed of S_6 (Japan Nikkei 225) and S_7 (U.S. S&P 500). If the restriction on the number of assets is relaxed, a careful check has to be made on the cumulative r^2. An obvious cut–off can be found between S_3 and S_1, as indicated by the large discrepancy of the cumulative r^2 (the difference between 99.35% and 63.73%). Therefore, we can finally determine the subset of assets for the purpose of hedging consisting of four assets of S_2 (Germany DAX), S_3 (France CAC 40), S_6 (Japan Nikkei 225) and S_7 (U.S. S&P 500).

	r_{i1}	r_{i2}	r_{i3}	r_{i4}	$\sum_{j=1}^{4} r_{ij}^2$
S_1	-0.6852	-0.3209	0.1517	-0.2043	0.6373
S_2	-0.3503	0.3043	0.3986	-0.7895	0.9975
S_3	-0.3960	0.6750	0.5632	0.2526	0.9935
S_4	-0.4953	0.3329	0.2060	0.1122	0.4112
S_5	-0.3772	0.5577	0.3434	-0.0690	0.5760
S_6	0.2192	-0.7306	0.6449	0.0425	0.9995
S_7	-0.9303	-0.3420	-0.1182	0.0425	0.9981

Table 4.4: Correlation Between the Original Variables and the PCs

4.2 Static Hedging with Four Dominant Assets

With the selected assets, the static hedging strategy could be achieved by figuring out the optimal strikes for the call options on these assets. In the following, only the numerical results for the hedging portfolios with four assets are shown. Generally, a hedge with four assets works better than that with two assets due to the importance of S_2 and S_3 in the basket. Moreover, as the weights in the basket are not changed after asset selection, the hedging subset surely better duplicates the original basket when more assets are included in the hedging portfolio. Nevertheless, the proper number of assets should be chosen in practice by comparing the additional hedging cost and the reduced hedging error.

To give a hint on the performance of this new static hedging method, the hedging cost is compared to the basket options price. All the prices of basket options and the corresponding hedging portfolios are obtained numerically by Monte Carlo simulations with a number of simulated paths equal to $500,000$. Such a simulation procedure guarantees that the basket option price of 100 contracts is relatively accurate to the second digit as shown in Table 4.5. In addition to the hedging cost, the expected value of the hedging error at the maturity date is reported for each hedging portfolio to account for hedging performance. Based on the definition in Section 1.3, negative hedging errors are favorable, suggesting that the basket option is well hedged with no risk exposure any more. Meanwhile, a special attention is paid to ES which plays a major role as a risk indicator to measure the hedging result. Especially, the strike of this basket option is varied with different values $K \in \{0.90, 0.95, 1.00, 1.05, 1.10\}$ and the maturity date $T \in \{1, 3, 5, 10\}$ years to gain an overall view of the hedging performance across maturities and strikes. Moreover, the set of strikes traded in the market for each asset is assumed to be $\mathcal{K}^{(i)} = \{0, 0.15, 0.30, 0.45, 0.60, 0.65, 0.70, 0.75, 0.80, 0.85, 0.90, 0.95, 1.00, 1.05, 1.10, 1.15, 1.20\}$ for all $i = 1, \cdots, N$.

	$T = 1$	$T = 3$	$T = 5$	$T = 10$
$K = 1.10$	1.50	7.61	13.75	26.25
	(0.0053)	(0.0177)	(0.0293)	(0.0604)
$K = 1.05$	3.19	10.24	16.47	28.73
	(0.0077)	(0.0197)	(0.0310)	(0.0611)
$K = 1.00$	5.90	13.33	19.50	31.14
	(0.0099)	(0.0215)	(0.0323)	(0.0618)
$K = 0.95$	9.56	16.81	22.74	33.68
	(0.0115)	(0.0227)	(0.0333)	(0.0623)
$K = 0.90$	13.88	20.60	26.17	36.24
	(0.0124)	(0.0236)	(0.0339)	(0.0626)

Table 4.5: MC Simulated Basket Call Prices and Standard Errors (in Bracket) for 100 Contracts with $500,000$ Simulations

Table 4.6 presents the results of static super–hedging portfolio with only four assets based on the first criterion. Convexity correction technique is used when the optimal strikes are not available for trading. Consequently, two options have to be included in the portfolio for one asset as often observed in the table. Super–replication is not available for those options with long maturity of 10 years due to large volatility involved. Otherwise, this hedging strategy well dominates the basket option, as shown by negative expected hedging errors and zero shortfall probability required in the optimization. However, super–replication requires a rather low K_1 and hence a pretty high hedging cost which amounts to even almost 7 times the basket option price for the case $T = 1$ and $K = 1.10$. Es-

K	T	BC_0	K_1	HC	$E[HE]$	k_2	k_3	k_6	k_7
0.90	1	13.88	0.46	30.30	-17.51	0.65/0.70	0.45/0.60	0.60/0.65	0.60/0.65
	3	20.60	0.22	53.75	-40.01	0.30/0.45	0.15/0.30	0.30/0.45	0.30
	5	26.17	0.08	63.52	-51.18	0/0.15	0/0.15	0/0.15	0/0.15
0.95	1	9.56	0.53	23.71	-15.08	0.75	0.60/0.65	0.70/0.75	0.70/0.75
	3	16.81	0.24	51.89	-42.34	0.30/0.45	0.15/0.30	0.30/0.45	0.30/0.45
	5	22.74	0.13	59.76	-50.70	0.15/0.30	0/0.15	0.15/0.30	0.15/0.30
1.00	1	5.90	0.56	21.78	-16.91	0.75/0.80	0.65/0.70	0.75/0.80	0.75
	3	13.33	0.26	50.13	-44.46	0.30/0.45	0.15/0.30	0.30/0.45	0.30/0.45
	5	19.50	0.15	58.84	-53.91	0.15/0.30	0/0.15	0.15/0.30	0.15/0.30
1.05	1	3.19	0.64	14.49	-12.04	0.85/0.90	0.75/0.80	0.85/0.90	0.85/0.90
	3	10.24	0.38	40.17	-36.17	0.45/0.60	0.30/0.45	0.45/0.60	0.45/0.60
	5	16.47	0.18	56.58	-54.96	0.15/0.30	0/0.15	0.15/0.30	0.15/0.30
1.10	1	1.50	0.70	10.01	-9.06	0.95	0.85/0.90	0.90/0.95	0.90/0.95
	3	7.61	0.46	33.42	-31.17	0.65/0.70	0.45/0.60	0.65	0.60/0.65
	5	13.75	0.19	55.78	-57.64	0.30	0.15	0.15/0.30	0.15/0.30

Table 4.6: Super–Hedging Portfolio with Four Dominant Assets

pecially, Figure 4.1 is designed to demonstrate how K_1 influences the hedging cost and the hedging error. Clearly, K_1 has two opposite effects on the hedging performance: a reduction in K_1 decreases the expected shortfall and meanwhile increases the hedging cost. Thus, a higher hedging cost is unavoidable to achieve super–replication. Besides, it demonstrates that the hedging strategy proposed in this thesis is exactly to achieve a trade–off between successful hedges and reduced hedging costs by varying strikes.

When relaxing the strong requirement of super–replication, the hedging cost can be surely decreased, for instance, in the hedging portfolio obtained by taking the second criterion. As formulated in the model, the variance of the hedging error is minimized given a certain hedging cost V_0. Here, two constraints are imposed on the hedging cost: BC_0, the basket option price, and $HP(7)$, the hedging cost of the static super–hedging portfolio with all 7 underlying assets. First as shown in Table 4.7, both constraints lead to sub–replication, leaving some risks uncovered. Even for the case of $T = 1$ and $K = 1.10$, there is no such a portfolio when $V_0 = BC_0$ due to the limited number of strikes traded in the market. Given the strike set, all possible combinations of traded options have higher prices than the basket call. As the ES (the identifier of the hedging performance) decreases with the hedging cost, better results are achieved with the constraint of $HP(7)$: Opposite to the positive expected hedging error and high ES obtained in the case of $V_0 = BC$, the hedging error turns out to be negative on average and the ES decreases greatly to $4\% - 9\%$ across all maturities and strikes of the basket call. This result indicates that hedging with four assets gives a relatively satisfactory performance: only a reasonable low hedging error arises when investing the same capital as the hedging cost of the super–hedging portfolio

Table 4.7: Minimum–Variance Hedging Portfolio with Four Dominant Assets

K	T	BC_0	$V_0 = BC_0$								$HP(7)$	$V_0 = HP(7)$							
			HC	E[HE]	Var[HE]	$ES\%$	k_2	k_3	k_6	k_7		HC	E[HE]	Var[HE]	$ES\%$	k_2	k_3	k_6	k_7
0.90	1	13.88	13.62	0.28	0.0008	9.48	0.85	0.80	0.95	0.85	15.05	14.93	-1.12	0.0007	4.80	0.85	0.70	0.95	0.85
	3	20.60	20.57	0.03	0.0030	10.49	0.85	0.70	1.10	0.75	22.72	22.70	-2.54	0.0026	5.18	0.80	0.65	1.00	0.75
	5	26.17	26.08	0.13	0.0054	11.25	0.80	0.60	1.10	0.75	28.43	28.41	-3.07	0.0051	6.22	0.75	0.60	1.00	0.70
	10	36.24	36.10	0.26	0.0148	13.31	0.70	0.45	1.15	0.60	38.1	38.02	-3.36	0.0146	9.30	0.65	0.30	1.10	0.60
0.95	1	9.56	9.29	0.29	0.0010	14.89	0.95	0.85	1.10	0.90	11.50	11.36	-1.91	0.0008	4.56	0.90	0.80	1.00	0.90
	3	16.81	16.76	0.05	0.0031	13.39	0.90	0.80	1.15	0.85	19.66	19.63	-3.41	0.0026	5.01	0.85	0.70	1.10	0.80
	5	22.74	22.61	0.18	0.0058	13.49	0.90	0.65	1.15	0.85	25.68	25.58	-3.90	0.0051	6.13	0.85	0.60	1.10	0.75
	10	33.68	33.64	0.07	0.0153	14.24	0.90	0.45	1.20	0.65	35.99	35.85	-4.08	0.0146	9.31	0.65	0.45	1.15	0.65
1.00	1	5.90	5.85	0.06	0.0010	21.03	1.00	0.95	1.15	1.00	8.48	8.26	-2.51	0.0007	4.45	0.95	0.90	1.05	0.95
	3	13.33	12.97	0.45	0.0034	18.91	1.00	0.90	1.20	0.95	16.87	16.79	-4.17	0.0025	4.90	0.95	0.75	1.15	0.85
	5	19.50	19.42	0.11	0.0061	15.97	0.95	0.75	1.20	0.95	23.11	23.07	-4.89	0.0050	5.76	0.85	0.65	1.15	0.85
	10	31.14	31.11	0.06	0.0158	15.62	0.85	0.60	1.20	0.80	33.95	33.73	-4.86	0.0146	9.31	0.80	0.45	1.20	0.70
1.05	1	3.19	3.14	0.06	0.0008	31.82	1.05	1.10	1.20	1.10	6.04	5.74	-2.71	0.0006	4.30	1.00	1.00	1.10	1.00
	3	10.24	10.11	0.16	0.0035	22.91	1.10	1.00	1.20	1.05	14.36	14.31	-4.92	0.0025	4.48	0.95	0.80	1.20	0.95
	5	16.47	16.42	0.06	0.0066	19.32	1.10	0.90	1.20	1.00	20.73	20.55	-5.60	0.0050	5.74	0.95	0.70	1.20	0.90
	10	28.73	28.67	0.12	0.0166	17.47	1.00	0.65	1.20	0.90	32.01	31.99	-6.11	0.0148	8.90	0.90	0.60	1.20	0.70
1.10	1	1.50	Given the strike set, all possible portfolios' price is above BC_0.								4.16	2.63	-1.20	0.0005	16.52	1.10	1.15	1.20	1.10
	3	7.61	7.60	0.02	0.0037	29.07	1.20	1.15	1.20	1.15	12.14	11.88	-5.15	0.0024	4.57	1.05	0.90	1.20	1.00
	5	13.75	13.70	0.08	0.0073	23.92	1.20	1.05	1.20	1.10	18.53	18.53	-6.55	0.0049	5.29	1.05	0.85	1.20	0.90
	10	26.25	26.01	0.45	0.0180	20.46	1.10	0.80	1.20	1.00	30.15	30.05	-7.14	0.0148	8.72	0.95	0.65	1.20	0.80

Note: $ES\%$ denotes the relative ES, namely the expected value measured in percentage of shortfalls divided by the corresponding basket option payoffs at the maturity date T.

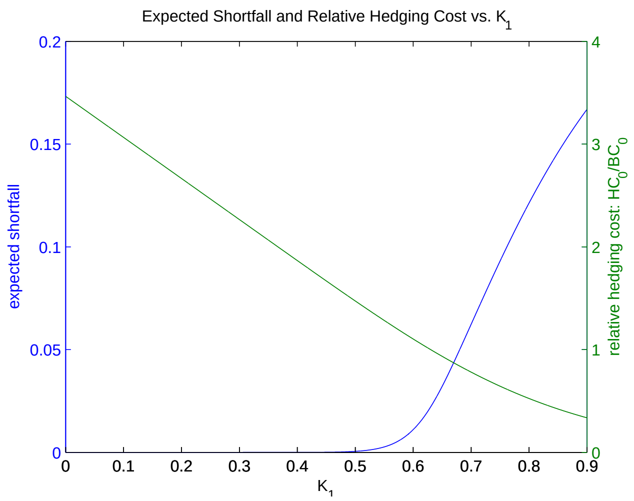

Figure 4.1: Expected Shortfall and Relative Hedging Cost vs. K_1 for the Basket Call with $T = 3$ and $K = 0.9$

composed of plain–vanilla options on all 7 assets which is supposed to be not available or at least not easily implementable. Moreover, such a hedging portfolio creates far less transaction costs than the super–hedging portfolio based on all the underlying assets if it exists. It enhances in turn the performance of the new hedging strategy by using only 4 underlying assets.

In addition, one can easily observe that the hedging portfolio given $V_0 = HP(7)$ performs better for short maturity. The relative ES and the variance of the hedging error of such portfolios always increase with T. Nevertheless, the variance of the hedging error and ES differ insignificantly across the strikes of the basket option. Unfortunately, such general rules can not be summarized in the case of $V_0 = BC$. Especially, the hedging performance surprisingly turns out to be poorest for the shortest maturity $T = 1$. As to the obtained optimal strikes of the hedging portfolio, they generally increase with the strike of the basket option in both cases. However, because S_6 is negatively correlated to the other three assets, k_6 rises with T opposed to the decreasing relation of k_2, k_3, and k_7 to the maturity.

The minimum–expected–shortfall hedging portfolios are demonstrated in Table 4.8 given the same two constraints on the hedging cost, BC_0 and $HP(7)$. With a restricted number of options, the obtained hedging portfolio sometimes coincides with the minimum–variance hedging portfolio. Nevertheless, the risk measure in this case is the expected shortfall, which is in effect a stricter criterion than the second one concerning only the positive difference between the prices of the basket option and the corresponding hedging portfolio. As a result, the hedging cost is generally higher than that of the minimum–variance hedging portfolio. Obviously, it leads to a better performance with lower hedging error and ES. To achieve a even smaller ES, the hedging cost constraint is further raised in Table 4.9 to the Value at Risk at the level 10% of the basket option discounted payoff. Due to the lack of

Table 4.8: Minimum–Expected–Shortfall Hedging Portfolios with Four Dominant Assets (I)

K	T	BC_0	$V_0 = BC_0$								$HP(7)$	$V_0 = HP(7)$							
			HC	E[HE]	Var[HE]	$ES\%$	k_2	k_3	k_6	k_7		HC	E[HE]	Var[HE]	$ES\%$	k_2	k_3	k_6	k_7
0.90	1	13.88	13.82	0.07	0.0009	8.98	0.95	0.75	0.90	0.85	15.05	15.04	-1.23	0.0008	4.59	0.90	0.70	0.90	0.85
	3	20.60	20.57	0.03	0.0030	10.49	0.85	0.70	1.10	0.75	22.72	22.70	-2.54	0.0011	5.18	0.80	0.65	1.00	0.75
	5	26.17	26.17	0.01	0.0055	11.14	0.70	0.65	1.15	0.75	28.43	28.41	-3.07	0.0051	6.22	0.75	0.60	1.00	0.70
	10	36.24	36.24	-0.01	0.0149	12.98	0.65	0.45	1.10	0.65	38.10	38.10	-3.50	0.0147	9.20	0.75	0.30	1.00	0.60
0.95	1	9.56	9.55	0.01	0.0011	13.60	0.90	0.85	1.15	0.90	11.50	11.48	-2.04	0.0009	4.39	0.90	0.75	1.05	0.90
	3	16.81	16.78	0.04	0.0032	13.35	0.95	0.75	1.15	0.85	19.66	19.63	-3.41	0.0026	5.01	0.85	0.70	1.10	0.80
	5	22.74	22.69	0.08	0.0059	13.32	0.90	0.60	1.20	0.85	25.68	25.64	-3.97	0.0052	6.07	0.75	0.65	1.15	0.75
	10	33.68	33.68	-0.02	0.0155	14.19	0.85	0.60	1.10	0.65	35.99	35.95	-4.26	0.0148	9.19	0.80	0.45	1.00	0.65
1.00	1	5.90	5.85	0.06	0.0010	21.03	1.00	0.95	1.15	1.00	8.48	8.40	-2.66	0.0008	4.14	0.95	0.85	1.10	0.95
	3	13.33	13.31	0.03	0.0035	17.46	1.05	0.95	1.15	0.90	16.87	16.85	-4.24	0.0026	4.78	0.90	0.80	1.05	0.90
	5	19.50	19.46	0.06	0.0062	15.90	1.00	0.70	1.20	0.95	23.11	23.10	-4.92	0.0050	5.75	0.90	0.60	1.15	0.85
	10	31.14	31.14	-0.01	0.0159	15.58	0.85	0.65	1.15	0.80	33.95	33.94	-5.26	0.0149	9.02	0.85	0.45	1.05	0.75
1.05	1	3.19	3.14	0.06	0.0008	31.82	1.05	1.10	1.20	1.10	6.04	6.01	-3.00	0.0007	3.68	1.00	0.90	1.10	1.05
	3	10.24	10.17	0.09	0.0036	22.67	1.15	0.95	1.20	1.05	14.36	14.31	-4.92	0.0025	4.48	0.95	0.80	1.20	0.95
	5	16.47	16.42	0.06	0.0066	19.32	1.10	0.90	1.20	1.00	20.73	20.71	-5.81	0.0051	5.55	0.90	0.70	1.15	0.95
	10	28.73	28.73	-0.06	0.0169	17.36	1.10	0.60	1.15	0.90	32.01	32.01	-6.21	0.0149	8.89	0.80	0.70	1.20	0.70
1.10	1	1.50	Given the strike set, all possible portfolios' price is above BC_0.								4.16	4.11	-2.77	0.0006	2.68	1.05	1.05	1.15	1.05
	3	7.61	7.60	0.02	0.0037	29.07	1.20	1.15	1.20	1.15	12.14	12.14	-5.47	0.0025	4.22	1.05	0.80	1.20	1.05
	5	13.75	13.78	-0.03	0.0074	23.73	1.10	1.15	1.20	1.10	18.53	18.53	-6.55	0.0049	5.29	1.05	0.85	1.20	0.90
	10	26.25	26.21	-0.08	0.0181	19.85	1.20	0.85	1.20	0.90	30.15	30.12	-7.27	0.0151	8.70	1.05	0.60	1.15	0.80

the distribution of the underlying basket, this has to be obtained by running simulations. Under this construction, the hedging cost of the hedging portfolio becomes surely higher (about $VaR_{0.10}$). It then gives a quite promising result that the ES is greatly reduced and turns out to be almost zero, except those basket options with a long time to maturity.

K	T	BC_0	$V_0 = VaR_{10\%}$							
			HC	E[HE]	Var[HE]	$ES\%$	k_2	k_3	k_6	k_7
0.90	1	13.88	24.57	-11.38	0.0007	0.0009	0.45	0.65	0.95	0.70
	3	20.60	38.13	-21.17	0.0023	0.0061	0.15	0.30	0.90	0.65
	5	26.17	48.12	-30.07	0.0045	0.0141	0.15	0.45	0.85	0.15
	10	36.24	60.15	-44.90	0.0143	0.1148	0	0.30	0	0.15
0.95	1	9.56	19.91	-11.02	0.0006	0.0012	0.75	0.65	0.90	0.75
	3	16.81	34.19	-21.00	0.0021	0.0065	0.70	0.30	1.00	0.45
	5	22.74	44.49	-29.81	0.0047	0.0200	0.30	0.30	1.20	0.15
	10	33.68	58.95	-47.46	0.0140	0.0950	0.15	0.30	0	0.15
1.00	1	5.90	15.21	-9.92	0.0007	0.0039	0.90	0.75	1.00	0.75
	3	13.33	30.29	-20.48	0.0021	0.0072	0.65	0.30	0.95	0.70
	5	19.50	41.10	-29.59	0.0042	0.0164	0.65	0.30	1.15	0.15
	10	31.14	58.54	-51.44	0.0136	0.0713	0.15	0.15	0.15	0.15
1.05	1	3.19	10.63	-7.92	0.0008	0.0111	0.80	0.85	1.05	0.95
	3	10.24	26.28	-19.37	0.0023	0.0179	0.75	0.65	1.00	0.60
	5	16.47	37.55	-28.88	0.0041	0.0195	0.15	0.30	1.05	0.70
	10	28.73	56.58	-52.30	0.0134	0.0617	0.15	0.15	0.15	0.30
1.10	1	1.50	6.01	-4.80	0.0009	0.1631	1.00	0.90	1.10	1.05
	3	7.61	22.27	-17.71	0.0027	0.0130	0.80	0.60	1.10	0.75
	5	13.75	34.30	-28.16	0.0043	0.0207	0.45	0.30	1.05	0.70
	10	26.25	53.75	-51.62	0.0130	0.0702	0.15	0.15	0.60	0.15

Table 4.9: Minimum–Expected–Shortfall Hedging Portfolios with Four Dominant Assets (II)

As also observed in the results above, relatively lower hedging costs are required for in– and at–the–money basket options to achieve almost the same relative ES compared with out–of–the–money options. Consequently, if aiming at capturing the trade–off between reduced hedging costs and successful replications, the hedging portfolio performs better for in– and at–the–money basket options. To clearly show the regions of sub– and super–replication, the payoffs of the basket option ($T = 3$, $K = 0.9$) and its minimum–ES hedging portfolio given $HC_0 = HP(7)$ are simulated and plotted in Figure 4.2. It can be observed that the basket option is completely hedged if the realized value of the basket is below or around the strike. The possibility of sub–replication rises with the value of the basket being above 1.00. Nevertheless, the hedging error is rather small compared to the basket option.

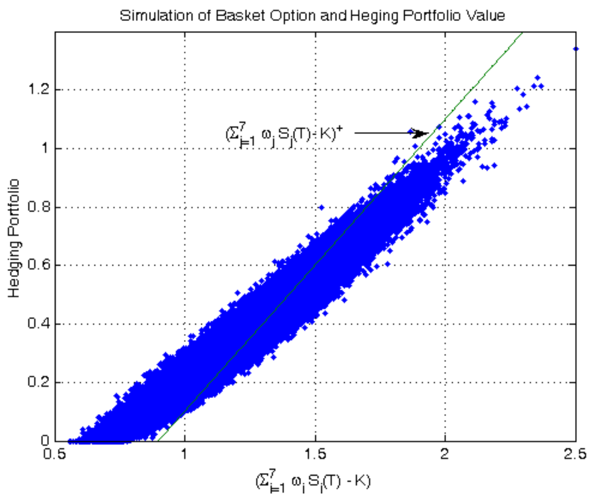

Figure 4.2: Simulation of the Basket Option and Minimum–Expected–Shortfall Hedge Portfolio with Constraint $V_0 = VaR_{0.10}$ (for the Case of $T = 3$, $K = 0.9$)

4.3 Remarks

Sometimes, the hedging performance is not that satisfactory especially for out–of–the–money options. It is mainly due to the following two factors.

- First, the sub–hedge–basket is composed of simply dominant assets without reallocating weights. Therefore, the value of the subset is only part of the original basket. The only tool in the model to match the payoff of the basket option is varying the strikes of the hedging instruments. However, their power to match the distribution is fairly limited since they do not change the shape of the distribution of the sub–hedge–basket, but only shift the distribution closer to the original basket. This can be easily observed in Figure 4.3. By neglecting those insignificant underlying assets, the sub–basket experiences less extreme cases. However, since it is part of the original basket, it is located on the left of the original basket. Therefore, the function of varying strikes is to relocate the distribution of the hedging portfolio to the proper position near the basket option. As shown in the figure, the tighter the hedging criterion is, the further the distribution is shifted to the right.

- In addition, all the hedging portfolios designed in this paper are static. Hence, more capital may be required to well hedge the basket option. However, the model is restricted to be static under the construction of hedging with plain–vanilla options on the significant underlying assets. As the control variables in this model are strikes of these options, frequent trading on options with different strikes would cause great loss and additional transaction costs.

As a result, other control variables have to be considered to improve the hedging effect. One possible instrument is reallocating the weights of the hedging basket such that the new sub–hedge–basket can better match the distribution of the original basket. On this

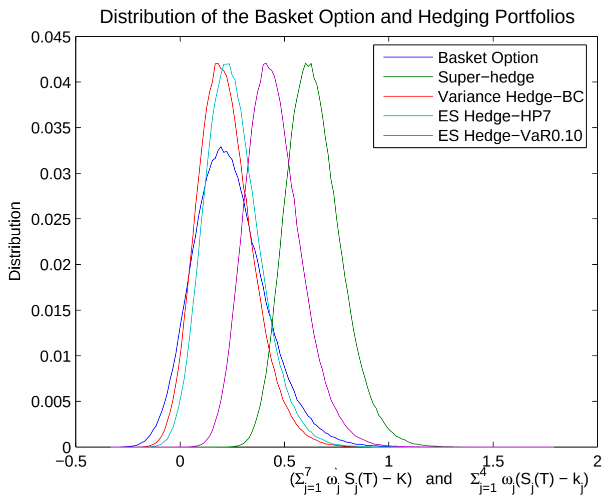

Figure 4.3: Distribution of the Underlying Basket and the Hedging Portfolios (for the Case of $T = 3$, $K = 0.9$)

basis, dynamic hedging would also be possible by duplicating the basket option with the hedging assets. This would be an extension to be considered in future works.

Chapter 5

Conclusion

In summary, the first part of this dissertation investigates how to hedge basket options whose final payoff is related to more than one asset. A basket of underlying assets instead of a single one on one hand benefits investors from the diversification effect, on the other hand poses hedgers a big problem especially when the number of underlying assets is large. Thus, this work is intended to design a hedging portfolio related to only subset of underlying assets. Hedging basket options based on only several assets can not only reduce transaction costs if combined with other hedging strategies, but also become practical and essential when some of the underlying assets are illiquid or not even available for trading.

The newly–designed hedging strategy is a static one and the hedging instruments are plain–vanilla options of those $N_1 < N$ individual assets in the basket with the most significant effect on basket options price. Thus, the hedging portfolio is obtained in two steps, aiming to determine the dominant assets and optimal strikes, respectively. In the first step, Principal Components Analysis, a popular multivariate statistical method for dimension reduction, is applied to basket options hedging to select only a subset of underlying assets. The selection procedure is completed mainly by decomposing the covariance structure of the underlying basket into eigenvalues in the order of significance and eigenvectors. Those eigenvectors in essential specify the underlying factors with decreasing significance on the basket value. By checking the correlation of each underlying asset and the first several important factors, we finally pick out those assets that contribute mostly to the factors and hence the basket. The optimal strikes are in the second step chosen by solving a numerical optimization problem with some economic optimality objective. The optimality criterion depends on the risk attitude of hedgers. As given in the thesis, the first objective is to eliminate all the risks that the basket option is exposed to. It is nevertheless in some cases not possible by using only several assets. Alternatively, optimal strikes are obtained by minimizing a particular risk measure, e.g., the variance of the hedging error or the expected shortfall, given a constraint on the hedging cost.

Considering liquidity of the hedging instruments, the numerical optimizations are modified by imposing another constraint that the strikes are only from the set of traded assets. The optimization problems become then more complicated. A simple and computationally efficient calibration procedure, convexity correction, is therefore designed to achieve the super–replication hedging portfolio. Basically, the optimal options which are not available are approximated by the linear combination of the two options on the same underlying asset with the neighboring strikes.

As observed from the numerical results, the static hedging method achieves the trade–off between reduced hedging cost and overall super–replication. It also demonstrates that hedging with only a subset of assets works quite well even without considering reduced transaction costs, generating a reasonably small hedging error by investing the same capital as the super–hedging portfolio on all the underlying assets which is difficult to construct or is even not available in the market. Actually, its performance will become more satisfactory if the underlying basket is large and illiquid. Since the hedging performance is sensitive to the subset of the selected assets, it is recommended to examine hedging costs, involved transaction costs as well as hedging errors of several subsets. To achieve a better performance, hedging basket options with a subset of assets could be improved by reallocating weights of the sub–hedge–basket to approximately match the distribution of the original basket.

Part II

Irreversible Investment Valuation

Chapter 6

Introduction and Overview

As an essential activity for firms and economic growth, investment has attracted a great deal of academic attention for decades. In general, investment is defined as an action of purchasing some goods (financial or physical) in hope of favorable future returns. It occurs at every moment and everywhere around us. For instance, merchandisers raise an inventory for sales, manufactories install new equipments for producing and firms put up new buildings or new plants. Even when we visit a museum, we are making an investment in the sense that some knowledge or fun is expected as a return.

One of the fundamental issues in the investment theory lies in the decision if and when, if yes, investment should be undertaken for a project. Traditionally, the Net Present Value (NPV) method is utilized to value a potential investment. The NPV of a project is according to Ross, Westerfield and Jaffe (2008) defined as the present value of its expected future incremental cash flows. The investment is then undertaken only when the NPV is nonnegative. However, the NPV rule as widely acknowledged in the literature corresponds to the assumption of zero volatility of the underlying stochastic state variable. Most importantly, it neglects the possibility to delay the project, as well as other alternatives to subsequently expand or contract the project. These flexibilities can be nevertheless valued in the real options theory, which is the topic of this dissertation.

We first provide an introduction of the real options analysis as a solution to the challenges inherent in investment decision problems. This chapter briefly defines the real option and its analysis, and then profiles different types of common real options. Furthermore, we give an overview of two standard real option methods and summarize the difference between these two techniques. The standard irreversible investment model used for the method illustration and the analysis of the approaches serve as the benchmark for the future discussions on irreversible investment.

6.1 Real Options: Problems and Concepts

The phrase *Real options* is attributed to Myers (1977). As specifically defined by Brealey and Myers (2002) as *"opportunities to modify projects as the future unfolds"*, they are invented to describe opportunities of investment in non–financial assets with some degree of freedom in decision making against the underlying uncertainty. Indeed, real option and its analysis first realize and offer dimensions of flexibility required by the irreversibility and uncertainty involved in most of the investment projects.

Characteristics of Investment In many cases, investments are observed to have the following three significant characteristics:

(a) *Irreversibility.* In reality, investment is hindered by many frictions. One particularly significant class of frictions is due to irreversibility. The cost of investment is partially or completely sunk as in many cases. For example, entrepreneurs are unable to recover the capital invested due to sunk costs like machine with specific use, marketing and advertisement, investment and research, adverse selection as well as institutional arrangement.

(b) *Uncertainty.* Levy and Sarnat (1984) (p. 77) define certainty as *"situations when the investor knows with probability* 1 *what the return on his investment is going to be in the future"*. Following this idea, uncertainty refers then to situations when a range of values (at least two) arise with strictly positive probabilities. Alternatively in an often–used economic term, one is uncertain of *states of nature* with several possible results. Despite some theoretical distinction between risk and uncertainty in the literature, uncertainty here means underlying risks investors have to face after launching projects. That is, the investment prospect may turn out to be negative when, for instance, the output price unfortunately declines; it will be probably favorable when the demand for produced outputs rises.

(c) *Flexibility.* Under uncertainty, rational managers make investment decisions not passively but actively by revising investment and its operation in response to fluctuating market conditions in order to maximize the firm's wealth. Surely, they prefer economic *booms* and try to avoid or at least mitigate losses in *busts.* That means, in presence of economic uncertainty, active management can add value to investment opportunity that is however not captured by the conventional NPV approach (see, e.g., Trigeorgis and Mason (1987), p. 15). To be more specific, having an investment opportunity, investors may prefer first waiting to learn the investment prospects, before they make an *irreversible* decision to invest[1]. Investment is made only at the favorable time when collected information while postponing investment reveals profitability.

As one may find in the above analysis, real options are required or take place together with uncertainty, irreversibility and flexibility in investment timing. If there is no uncertainty, or if a decision is reversible without any cost, no additional benefit can be obtained by waiting. All decision making occurs then upfront without any other alternatives. In

[1]There are of course some occasions that firms cannot delay the investment due to some strategic considerations. See Tirole (1988), Chapter 8 for a rigorous literature survey on these strategic effects on investment decisions.

this sense, real options take into account and highlight the significant characteristics of investment and its effect on investment decisions.

Analogy to American options Once invested in the project, a firm obtains a stream of profits and costs that vary over time depending on the unanticipated future economics. The firm who has the opportunity to invest in this project owns the right to make the investment, and to receive the stream of profits of the project if completed. To those who are familiar with financial options, it resembles in structure an American call option. An American–style call option gives the option holder the right at any time before a specified date to obtain a share of stock at the exercise price. Similarly, decision makers who face an investment opportunity have an exercise right in an American option in all future profits the investment will bring. The analogy between an American call option and a real option is shown clearly in the following table:

	American Call Options	Options to Defer
Underlying	a financial common stock	an investment project /physical assets
Uncertainty	stock price	project revenue
Exercise Price	some predetermined amount of money	investment cost
Expiration Time	option maturity date (predetermined time)	closing–down of the opportunity to invest
Cost before Exercise	forgone dividend of the stock	forgone cash flow from the project
Intrinsic Value	the stock price minus the exercise price	the net profit (present value of expected revenues−investment cost)
Tradability	option itself traded on the exchange or over–the–counter in financial markets	produced output traded in capital markets
Nature	a financial product contingent on another financial instrument	a conceptual framework to valuate an investment opportunity

Table 6.1: Analogy between an American Call and an Option to Defer

In one word, an investment opportunity can be regarded as an American call option on project future revenues. The project revenue fluctuates stochastically and real options are in general not a *now or never* opportunity. Then when should it be exercised? This question is trivial for all rational decision makers: only when the benefit of exercising is greater than the incurred cost. As we have already mentioned, there is some value increment by waiting and learning how the project revenue develops. On the other hand, one loses while postponing the investment some possible profits that the delayed investment could have created. Thus, real options are to be exercised only when the project is sufficiently *deep in the money* in order to cover the foregone cost.

Types of Real Options In real business life, there are many different opportunities to exercise managerial flexibility for existing or potential projects. In the following, some real options that are most likely encountered (both theoretically and practically) are classified[2]. Especially, we take the oil industry as an example to illustrate each type option.

(a) Options to defer/timing options: It is the most common real option and also the basic target of this dissertation. When facing an investment opportunity, firms as previously analyzed can wait to learn more about the project. For instance, managers may have an oilfield development plan. The real investment is heavily dependent on the oil price and future demands for oil. Hence, managers now have an option to put through the plan but only when the economic conditions turn out to be favorable.

(b) Abandonment/suspension options: Firms are not obligated to continue any project once after undertaken. Instead, an option is available to them to close down the project if it brings no any profits. The sequential appraisal programs for the oilfield development plan may be abandoned if the information shows a negative sign for the future development. A little bit different from the above mentioned options, it is an American put option on the investment revenue. The strike price is the net liquidation value of the project which is the resale price of the investment less all the closing–down costs. Hence, the strike of this option could be negative due to higher costs or lower liquidation value. However, abandonment options reduce the impact of the poor economic conditions on the investment and therefore increase the value of the investment.

(c) Expansion options: The oilfield is constructed as the world oil market according to appraisal will continue its boom since 1999. Currently, the demand becomes even higher mainly due to some Asian countries like China and India. Consequently, managers consider to drill several new wells to expand the production capacity. In general, an expansion option is an option to make further investment to increase the production output. In the real options theory, it is an American call option on the additional revenue created by reinvestment with the strike price equal to all the costs aroused for the further investment.

(d) Contraction options: As the opposite case of the expansion option, this is an option to reduce the production capacity. Similarly, it is also an American call but underlying on the *lost* revenue by reducing the scale. Its strike price is the expected value of the future expenditures saved by contraction.

(e) Options for temporary suspension: Sometimes, it is not necessary to close down the existing investment completely. Instead, managers can consider an option to

[2]A full and detailed common real options classification can be found in Trigeorgis (1996).

mothball the investment only for a certain time. Take our oilfield example, both oil and gas wells have been drilled in the initial construction. Unfortunately, gas production is not that profitable as expected because of lower scale demanded and hence higher production and transportation cost. In this situation, managers contemplate to suspend the gas production. Nevertheless, the well is kept for future reactivation since gas is according to market research the most potential resource with less pollution.

(f) Other embedded options: The above–mentioned options are all a single decision to enter, to expand, or to exit an investment. In many cases, one investment process involves several decisions. We name all such options in a large category of embedded options. For instance, several decisions may have to be made in a particular sequence of steps. An oilfield is usually built in two stages: In the first stage, reserves of oil should be obtained either through self–exploration or purchase; Wells, pipelines and other drilling equipments have to be built for oil production from the reserve. The second decision is surely not required to follow the first one at once and further measure is again dependent on economic conditions. As to the production capacity, investors as often observed in reality build up the capacity gradually over time instead of once in time. In other words, firms make up a sequence of decisions of incremental investment. Another simple example is the option to reactivation embedded in the option for temporary suspension.

The Real Options Analysis and the Decision Rule By taking the analogy of an investment opportunity to a financial option, the real options analysis applies the option–pricing theory to irreversible investment valuation. By doing so, it first offers significant economic insights into investment theory. Besides, managerial flexibility is captured and quantified as a response to the uncertain market development and the irreversibility of the investment. As one of the most significant results of the real options analysis, it realizes that uncertainty enhances incentives of investors to wait. A project with a considerable positive NPV may be insufficient to immediate investment. The investment is postponed by comparing the benefits and costs of waiting until the project becomes deep in the money. Consider specifically an option to defer, the real options theory comes to the following decision rule: A project is undertaken if and only if the expected discounted investment revenue exceeds the cost and the option premium of waiting for better information relevant to the investment revenue. Clearly, the critical project revenue is not equal to the investment cost as required by the NPV rule but larger than that with an add–up for the option to wait. This significant feature of the threshold value will be addressed analytically later in the method illustration. Currently, the real options analysis is greatly developed in many aspects allowing for strategic consideration, defaults possibility and so on, and hence provides much broader and more practical results. A full systematic introduction and standard reference on irreversible investment can be found in Dixit and Pindyck (1994) and some other recent developments in Trigeorgis (1996), Friedl (2007) and the literature therein.

6.2 Current Real Options Approaches

In order to give a clear overview of the real options theory and also to facilitate easy comparison in the later discussion, we are going to present in this section briefly the standard real options analysis techniques, the contingent claim analysis and the dynamic programming method, respectively. Particularly, the similarity and difference of these approaches are investigated afterwards[3].

The basic model we used throughout this thesis is taken from Pindyck (1988), McDonald and Siegel (1986) and also presented in Dixit and Pindyck (1994), which is regarded in the literature as the standard irreversible investment model or the benchmark. In this framework, the resemblance between investment decision problems and American options can be easily recognized. Before we move on introducing the approaches, the model setup is first given as follows.

Suppose that a firm has an opportunity to invest. This project requires only an initial investment cost I which is supposed to be constant over time and there is no marginal cost. The investment is irreversible in the sense that the investment cost is sunk. The project has a fixed scale producing a commodity good of quantity Q ever after the investment date τ[4]. For convenience, Q is fixed to be 1, i.e., the project yields a unit flow of output forever. The spot price of the firm's output P evolves stochastically conditional on the economic situation. Formally, the stochastic process $(P_t)_{t\in[0,\hat{T}]}$ is modelled as a geometric Brownian motion (GBM) with constant drift μ and volatility σ:

$$P_t = P_0 e^{Y_t} \qquad \text{and} \qquad Y_t = \mu t + \sigma W_t, \tag{6.1}$$

where P_0 is the initial price at time zero, $(W_t)_{t\geq 0}$ is the standard Wiener process defined on a probability space on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq \hat{T}}, \mathbb{P})$ with the physical measure $\mathbb{P}$ and an information filtration $(\mathcal{F}_t)_{0\leq t\leq \hat{T}}$ satisfying the usual conditions of completeness and right–continuity. Alternatively, P_t can also be expressed in the following differential equation

$$\frac{dP_t}{P_t} = (\mu + \frac{1}{2}\sigma^2)dt + \sigma dW_t \,.$$

Here, $\mu + \frac{1}{2}\sigma^2$ can be interpreted as capital gain return.

[3]Detailed and rigorous treatment of these two methods is available in several textbooks: see for instance the pioneer work of Black and Scholes (1973) and advanced discussion of Duffie (1992) for the contingent claim analysis; Fleming and Rishel (1975) as well as Krylov (1980) for the dynamic programming method and the formal existence and uniqueness proof of the continuous–time model. Our illustration is rather heuristic, basically following Dixit and Pindyck (1994).

[4]It implies that this project has an infinite life. This assumption is undoubtedly unrealistic but generally taken to achieve an explicit solution. In this way, the dimensionality of the problem is reduced by removing the dependence on time. Several authors as Grasselli (2006) have studied a finite time horizon and thus have to deal with the valuation problem by numerical methods. Our own work in the next chapters are not restricted to infinity. Here, it is assumed for easy understanding of the methods.

Given these assumptions, the firm has to decide whether to take project or not and most importantly it has to decide on the time to invest, i.e., the time which maximizes the present value of the expected net profit. To be more specific, the firm faces the problem

$$\begin{aligned} F &= \max_{0\leq\tau<\infty} E\left[e^{-\rho\tau}\left(\int_\tau^\infty e^{-\rho(s-\tau)}P_s\,ds - I\right)^+\right] \\ &= \max_{0\leq\tau<\infty} E\left[e^{-\rho\tau}\left(\pi_\tau - I\right)^+\right], \end{aligned} \tag{6.2}$$

where ρ is the constant discount factor measured in the physical measure $\mathbb{P}$, $\pi_t = E\big[\int_t^\infty e^{-\rho(s-t)}P_s\,ds\,|\mathcal{F}_t\big]$ represents the expected operating profit of the project at time t and the expected value is taken with respect to $\mathbb{P}$. In addition, $\mu + \frac{1}{2}\sigma^2 < \rho$ is usually assumed to guarantee that the objective maximization function is mathematically well posed[5].

With the specification of P_t, we can easily calculate the perpetual operating profit as

$$\begin{aligned} \pi_t &= E\Big[\int_t^\infty e^{-\rho(s-t)}P_t e^{\mu(s-t)+\sigma(W_s-W_t)}\,ds\,|\mathcal{F}_t\Big] \\ &= P_t\int_t^\infty E\big[e^{(\mu-\rho)(s-t)+\sigma(W_s-W_t)}\,|\mathcal{F}_t\big]\,ds \\ &= P_t/(\rho-\mu-\frac{1}{2}\sigma^2). \end{aligned}$$

This implies that π_t has the following modification as

$$\begin{aligned} d\pi_t &= \frac{\mu+\frac{1}{2}\sigma^2}{\rho-\mu-\frac{1}{2}\sigma^2}P_t\,dt + \frac{\sigma}{\rho-\mu-\frac{1}{2}\sigma^2}P_t\,dW_t \\ &= (\mu+\frac{1}{2}\sigma^2)\pi_t\,dt + \sigma\pi_t\,dW_t\,, \end{aligned}$$

with the initial value of $\pi_0 = P_0/(\rho-\mu-\frac{1}{2}\sigma^2)$. For convenience and clear illustration, we take π as the underlying uncertainty of the investment.

Clearly seen from (6.2), the profit maximization problem is in structure analogous to a perpetual American call option which is written on the future revenue (expected operating profit) of the investment. That is why an opportunity to invest is usually referred to as a real option, an option contingent on real assets.

6.2.1 Dynamic Programming Method

Dynamic programming method is a mathematical technique for solving sequential planning decisions/optimal control problems in economic analysis and operation research. In general, it regards all decisions into two components: the immediate choice and a valuation

[5] See Section 7.3 where this assumption will be fully explained.

function that encloses the consequences of all subsequent decisions. Then, a comparison is made on the present values of these two parts for each decision. In order to find the optimal sequence of decisions, the method works systematically backwards. Starting from the last decision point, one can take the optimal decision and obtain the future value of the last but one decision. With this value, one moves on optimizing the last but one decision by comparing the discounted future value and the immediate action's value. Repeating the same procedure for all decisions leads to the optimal dynamic decisions. This is basically the essential idea of the dynamic programming method. Indeed, this recursive structure implies the fundamental guideline for making each single decision that all the subsequent decisions are made optimally from this moment onwards. As formally summarized in Bellman's principle, *an optimal policy has the property that, whatever the initial action, the remaining choices constitute an optimal policy with respect to the subproblem starting at the state that results from the initial actions.* (Dixit and Pindyck (1994), p. 100)

To easily elucidate this approach in our (continuous–time) decision problem, let us consider any time t before investment. At each date, the firm can either invest at once and gain the immediate payoff; otherwise it continues waiting and gets ready to invest after the next short time instant dt. This objective is common for all periods even until infinite time, namely, the objective function is independent on a specific time. Without further assumption, the decision problem is solved in the framework as first given in the physical measure. Especially, the discount rate denoted by ρ is exogenously given.

The decision at the general time t depends on the current operating profit, π_t, the so–called state variable. To account for the decision choice of the firm at each time, we define the control variable of this problem as u, which is a scalar binary variable with 0 and 1 representing wait and immediate invest respectively. Furthermore, the strategy $\{u_t\}_{0\leq t<\infty}$ is assumed to be *continuous from the left* opposite to the *continuity from the right* of the uncertainty $\{\pi_t\}_{0\leq t<\infty}$, showing that strategies change accordingly after the economic shock occurrence. In this way, the sequence of the control variables $\{u_t\}_{0\leq t<\infty}$ is going to be determined to maximize the expected present value of the project net profit.

If the investment is undertaken immediately and the firm gets the expected present value of the revenue less the investment cost, $\pi_t - I$. If the investment is postponed, the firm gets no any immediate payment but some discounted payoff related to future optimal decisions up to today's expectation, $e^{-\rho dt}E[F(\pi_{t+dt})|\mathcal{F}_t, u]$. Since the project is not launched and the decision is still going on, let us call it the *continuation value.* Obviously, the firm is going to take the one which yields the larger value. Consequently, we have the well–known *Bellman equation* or *fundamental equation of optimality* for our optimization problem

$$F = \max_u \{\pi_t - I, e^{-\rho dt}E[F(\pi_t + d\pi_t)|\mathcal{F}_t, u]\}\,.$$

In the continuation region (i.e., waiting), it simplifies to the following equation since dt

converges to 0

$$F(\pi_t) \approx (1-\rho dt)E\left[F(\pi_t)+F'(\pi_t)d\pi_t+\frac{1}{2}F''(\pi_t)(d\pi_t)^2\right]$$
$$= E\left[F(\pi_t)+\frac{1}{2}F''(\pi_t)(d\pi_t)^2+F'(\pi_t)d\pi_t-\rho dt\left(\frac{1}{2}F''(\pi_t)(d\pi_t)^2+F'(\pi_t)d\pi_t+F(\pi_t)\right)\right],$$

where the right hand side can be easily calculated with the help of *Itô's Lemma* for the process π_t. Hence, the Bellman equation becomes after simplification:

$$\frac{1}{2}F''(\pi)\sigma^2\pi_t^2+(\mu+\frac{1}{2}\sigma^2)F'(\pi)\pi_t-\rho F(\pi) = 0\,. \tag{6.3}$$

If we want to find a unique solution or one single trigger value for investment, Equation (6.3) itself is not enough. Three boundary conditions of $F(\pi)$ have to be satisfied:

$$F(0) = 0\,, \tag{6.4}$$
$$F(\pi^*) = \pi^*-I\,, \tag{6.5}$$
$$F'(\pi^*) = 1\,, \tag{6.6}$$

where π^* is the critical operating profit that initiates the investment. Condition (6.4) is intuitively understandable: If the net operating profit is zero, then the firm would never invest and get zero as return. The remaining two conditions are related to the threshold and they are called *value–matching* and *smooth–pasting* condition, respectively. At the optimal value π^*, the firm becomes indifferent to investing and no–investing since they give the same value as expressed in Condition (6.5). Equivalently, the difference of the continuation value and immediate value measures the value of managerial flexibility, namely the option to wait. The firm is willing to invest if and only if the difference becomes non–positive. Condition (6.6) requires the continuation value and $F(\pi)$ to have the same slope at the optimal point if the value function $F(\pi)$ is continuous. As an additional auxiliary condition, it ensures that the maximum is achieved for $F(\pi)$ exactly at π^*.

To find $F(\pi)$, we guess it in form of $F(\pi) = A\pi^b$ with two constants A and $b > 1$. It is written in this way because of Condition (6.4) which implies the graphic of $F(\pi)$ increasing and passing the origin. This function then helps determining the critical value π^* as follows by substituting it into Boundary Condition (6.5) and (6.6)

$$\pi^* = \frac{b}{b-1}I \quad \text{or} \quad P^* = (\rho-\mu-\frac{1}{2}\sigma^2)\frac{b}{b-1}I \tag{6.7}$$

and

$$A = \frac{\pi^*-I}{(\pi^*)^b} = \frac{(b-1)^{b-1}}{b^b I^{b-1}}\,. \tag{6.8}$$

Turn back to the Bellman Equation (6.3). Trying the guess $F = A\pi^b$ shows that it is feasible if and only if b is a root of the following equation

$$\frac{1}{2}\sigma^2 b^2+\mu b-\rho = 0\,, \tag{6.9}$$

which is the well–known fundamental quadratic equation of the real options theory. It is clear that it has two roots, one of which is definitely negative equal to $\frac{-\mu-\sqrt{\mu^2+2\sigma^2\rho}}{\sigma^2}$. Note that Equation (6.3) is a second–order differential equation and linear in F and its derivatives. As already figured out in the literature, its general solution is a linear combination of two independent solutions. In this way, the general solution can be written as a sum of two components in form of $A\pi^b$ where b is the root of (6.9) and A a constant to de determined. However, a negative b will give a infinitely large value when $\pi = 0$ instead of 0 required by Condition (6.4). Therefore, the coefficient A of this component must be zero. In this way, the investment value is obtained as

$$F \;=\; A\pi^b\,, \tag{6.10}$$

where $b = \frac{-\mu+\sqrt{\mu^2+2\sigma^2\rho}}{\sigma^2}$, the positive root of Equation (6.9), which can be easily shown to be always larger than 1 and $A = \frac{(b-1)^{b-1}}{b^b I^{b-1}}$.

6.2.2 Contingent Claim Analysis

As one can tell from the name, this method is built up on ideas of financial economics. Assume that the firm's output is itself traded in the market. In addition, the market is sufficiently *complete* so that the investment opportunity, i.e., the real option can be fully replicated by constructing a dynamic portfolio of this traded asset and the risk–free asset (the so–called self–financing portfolio). Even if the output itself is not traded, the same argument follows when there is another existing asset or a portfolio of traded assets which is perfectly correlated to the pattern of returns from the investment. This *spanning* assumption is validated in practice by most of commodities that are traded on spot or futures markets, and by some manufactured goods whose prices are to great extent correlated with values of some traded shares or portfolios. Based upon the *spanning* assumption, investment is then able to be valued via the risk neutral valuation.

Assume in this subsection that uncertainty of the investment π_t can be in principle spanned by existing assets. Let x be the price of such an asset. Since it is perfectly correlated with π_t, its evolvement is generated by the same Wiener process W_t as given in (6.1) under the physical measure. More specifically, it follows

$$dx_t = \mu^x x_t dt + \sigma^x x_t dW_t\,,$$

where μ^x and σ^x is the constant drift and volatility of the replicating asset x. To value the investment opportunity, we construct a portfolio composed of the investment opportunity F and n units of short position in x. It costs $F(\pi) - nx$ at any time t before the option to wait is exercised. The total capital gain from t over a short time interval dt (with a fixed n) is easily identified as $dF(\pi) - ndx_t$. With the help of the *Itô's Lemma* of F, this can be further calculated

$$\begin{aligned} & dF - ndx_t \\ = \;& [(\mu + \frac{1}{2}\sigma^2)\pi_t F'(\pi) + \frac{1}{2}\sigma^2\pi_t^2 F''(\pi) - n\mu^x x_t]dt + (\sigma F'(\pi)\pi_t - n\sigma^x x_t)dW_t\,. \end{aligned}$$

If choosing $n = \frac{\sigma\pi_t}{\sigma^x x_t}F'(\pi)$, we have then a riskless portfolio. In a complete market without any arbitrage possibilities, the rate of return of this portfolio must equal to the risk–free interest rate. As a consequence, we obtain the following governing equation of this problem:

$$\begin{aligned} r(F(\pi) - nx_t)dt &= [(\mu + \frac{1}{2}\sigma^2)\pi_t F'(\pi) + \frac{1}{2}\sigma^2\pi_t^2 F''(\pi) - n\mu^x x_t]dt \\ r(F(\pi) - \frac{\sigma\pi_t}{\sigma^x}F'(\pi)) &= \frac{1}{2}\sigma^2\pi_t^2 F''(\pi) + (\mu + \frac{1}{2}\sigma^2)\pi_t F'(\pi) - \frac{\sigma\pi_t}{\sigma^x}F'(\pi)\mu^x \\ 0 &= \frac{1}{2}\sigma^2\pi_t^2 F'' + \pi_t F'[(\mu + \frac{1}{2}\sigma^2) - \frac{\sigma}{\sigma^x}(\mu^x - r)] - rF. \quad (6.11) \end{aligned}$$

Recall that two perfectly correlated assets are subject to the same risk and hence have the same market price of risk[6]

$$\frac{\mu + \frac{1}{2}\sigma^2 - r}{\sigma} = \frac{\mu^x - r}{\sigma^x}.$$

It helps simplifying the coefficient of Equation (6.11) to

$$\frac{1}{2}\sigma^2\pi_t^2 F''(\pi) + r\pi_t F'(\pi) - rF(\pi) \;=\; 0\,. \quad (6.12)$$

Observe that it is almost identical to Equation (6.3). Besides, the three boundary conditions also work in this approach with the same reasoning. Therefore, it finally comes to the following result

$$\pi^* = \frac{b}{b-1}I \quad \text{and} \quad F = A(\pi^*)^b$$

with $b = \frac{1}{2} - \frac{r}{\sigma^2} + \sqrt{(\frac{r}{\sigma} - \frac{1}{2})^2 + \frac{2r}{\sigma^2}}$ and $A = \frac{(b-1)^{b-1}}{b^b I^{b-1}}$.

Before finishing this subsection, we would like to address the investment rule formally derived here. As easily observed, the critical operating profit to trigger the investment has a mark–up factor, i.e., $\frac{b}{b-1} > 1$ (because $b > 1$). It hints in turn that $\pi^* > I$, confirming our argument above that the conventional NPV method gives an incorrect rule and additional premium is required by uncertainty and flexibility of exercising the option to wait.

6.2.3 Method Comparison

These two approaches are in fact closely related to each other, leading to identical results in many applications (Dixit and Pindyck (1994)). Nevertheless, they are different in the

[6]The market price of risk is also called Sharpe ratio. It is calculated in the context of Capital Asset Pricing Model (CAPM) and defined as the expected extra premium of a portfolio if the portfolio return over the risk–free interest rate divided by the standard deviation of the excess return. Simply speaking, it is a reward–to–risk ratio. For more detailed discussion of CAPM, see Brealey and Myers (1991) and Duffie (1992).

selection of assumptions about financial markets and the discount rates used for evaluating cash flows.

As always argued in the literature, the contingent claim analysis is built up on the *market completeness* (then there is no arbitrage and hence a perfect spanning asset) and *risk–neutrality* towards idiosyncratic risks. No arbitrage argument is indeed the underlying idea of the contingent claim analysis which allows one to find the value of the firm by means of replicating the return and risk characteristics of the project cash flow using traded assets. Precisely speaking, a self–financing trading strategy is constructed in financial markets to replicate the project value. Because of market completeness, we have a *unique* equivalent martingale measure (EMM) for investment valuation. In this sense, the financial economics theory is taken to find the unique EMM which is the well–known risk neutral measure. It is worth remarking that choosing an EMM is somewhat equivalent to choosing a discount factor. It hence implies that risk–free interest rate is the fair discount rate in complete markets. Alternative argument is based on the CAPM. If all risks are able to be diversified, no premium above the risk–free interest rate can be required. With this assumption, an investment opportunity is therefore valued without making further supposition on risk preferences or discount rates.

Apart from the computational tractability, the dynamic programming method is more general. Indeed, it is one of the most popular tool dealing with dynamic optimizations with uncertainty. The project overall payoff is maximized via the backward induction argument given an *exogenous discount rate*. In practice, the discount rate is interpreted as the opportunity cost of capital. Consequently, it is in principle equated with the return that the investor demands for taking the risk of the investment or that he would earn on other investment with comparable risk characteristics (see Dixit and Pindyck (1994), p. 114). Surely, the dynamic programming method obscures the highlight of real options, i.e., the intuition that facing an investment opportunity investors have an option to wait. For example, in the options pricing framework, investment on a multi–stage projet resembles a portfolio of options or simply a compound option. Every has its own value and influences each other in a complicated way towards the value of the whole project. If using the dynamic programming method, this analogy is gone. The problem is dealt with as a sequence of optimal decision problems under uncertainty and solved by backward induction, which lacks of economic intuition.

Although taking different assumptions and following different arguments, these two methods finally reach the final stage of solving a homogenous differential equation. It is noted that the equation is rather complex to compute especially when the problem has high dimensionality due to the presence of many state variables, control variables and/or sources of uncertainty. It hence hindered to generalize the real options theory to other more complicated but practical processes than GBM and to other some nonlinear functions related to profit or utility.

6.3 Overview of the Content

The main part of this dissertation (Part *II*) consists of three chapters. We begin in Chapter 7 with the application of the stochastic representation method into irreversible investment valuation. Most importantly, a new *Shadow* NPV rule is derived which correctly recognizes the pure net profit or NPV of the project. In Chapter 8, we generalize the standard real options model by allowing gradual capacity construction. Finally, we shift our attention to market incompleteness and risk neutrality and study how risk aversion affects the optimal investment strategy in Chapter 9. We then summarize this part in Chapter 10 by collecting main results.

Stochastic Representation Method and Shadow NPV Rule

This part starts in **Chapter** 7 by introducing an alternative method to evaluate irreversible investment in real assets and showing how it improves the current status of the investment theory.

Although the real options analysis has captured attention of both academic and practitioners with its great improvement for the investment decision making, it remains unfamiliar to many corporate managers with restricted application in reality ever since its appearance. According to the 2002 survey of 205 Fortune 1000 CFOs by Patricia Ryan, merely 11.4% used the real options idea, while the NPV method stayed at the top of the list with 96%. It is clear that the framework of financial options pricing enables one to capture the essence of the investment problem. However, it meanwhile limits the scope of applicability. Many managers complain of the complication and obscurity inherent in the method. To them, the traditional NPV rule is more straightforward, focusing on the net profit that they care about. Therefore, is that possible to find another framework which is simpler and more economically intuitive to practitioners for irreversible investment valuation? This dissertation designs to first address this question.

The aim of this chapter is twofold:

- First, we provide an intuitive derivation of the investment policy by means of the stochastic representation method. The formal analysis is based on the well–established model of irreversible investment as presented in the previous section. Due to the analogy between real options and American options, most of the proposed new methods are related to or originated from American options pricing methods (see e.g. Boyarchenko (2004)). As American options can be exercised at any time before maturity, their holders would like to maximize payoffs by choosing a proper exercise time. In this way, pricing an American option is equivalent to solve an optimal stopping problem. The standard approach consists of finding the smallest super–martingale dominating the American option payoff at any time. Alternatively, the optimal stopping time of an American option can be identified by representing the option payoff process by a running supremum process. In the real options framework, it is the present value of expected revenue/operating profit process that is rewritten by a running supremum. This supremum process is interpreted as the *shadow* revenue from the investment and demonstrated to play the key role of signalling the optimal exercise time. In this way, the optimal stopping problem is reduced into a representation problem in terms of *shadow* revenue. Especially, we show the importance of dealing with the expected present value of the project revenue and the derived economically sensible representation. Instead of the money value from the investment, the *shadow* revenue of the investment records the *real* net revenue of the project after extracting the entire cost. Here, the induced cost of the investment encompasses not only the investment cost but also the extra opportunity cost of holding the option to wait. In this sense, our decision rule identifies the proper NPV and hence corrects the conventional NPV.

- Second, we extend the analytical tractability of the standard GBM models to a general framework with, e.g., exponential Lévy processes that better explain fat tails and skewness of probability distributions as often observed for commodity prices, as well as time–inhomogeneous diffusion processes for mean–reversion features. In addition to the general solution characterization, Section 7.3 presents a number of examples to illustrate the application of this approach.

Sequential Irreversible Investment Problems

This thesis then derives in **Chapter** 8 the optimal *dynamic* investment decision of one firm. That is, we study one firm that continuously in time makes up an decision to expand capacity or not. Due to irreversibility, the firm has to carefully select the investment time and also the amount of capacity to build up when considering the unexpected changes in economic conditions. Meanwhile, the purchased capital depreciates continuously over time. Thus, one concern becomes increasingly important in the literature on incremental capital expansion, or sequential irreversible investment in capacity.

Arrow (1968) first comes up with sequential irreversible investment problems under *certainty*. The problem is formulated in a continuous–time but deterministic optimization model with a deterministic interest rate and a profit function which does not incorporate uncertain economic shocks. The optimal solution is fully characterized by means of Pontryagin's principle. The same problem but under uncertainty is studied in Pindyck (1988) by the contingent claim analysis. Adapting the technique for single investment to sequential investment problems, Pindyck (1988) considers the marginal investment decision. Rather than focusing on how much to invest at each time, he identifies the timing of the infinitesimal stock of capital. Generally, models of irreversible investments under uncertainty assume that the firm is subject to a multiplicative economic shock that evolves according to a GBM with constant drift and volatility (e.g., Bertola (1988), Bertola (1998), Pindyck (1988) and Kobila (1993)). In their models, either a Cobb-Douglas or a general (see Kobila (1993)) operating profit function is assumed. Boyarchenko (2004) extends the capital expansion model to the case where the multiplicative economic shock is characterized by an exponential Lévy process. An interesting extension by Guo, Miao and Morellec

(2005) concerns regime shifts where the drift and volatility of the Brownian motion (BM) switch between different states according to a continuous–time Markov chain. By contrast, in order to develop a general theory for sequential irreversible investments in capital when the firm faces uncertain economic situations, a very general model is constructed in this work which is free of any distributional and parametric assumptions. In this way, it covers not only all the previously studied models but also the standard finance model where the uncertainty is usually specified by a semi–martingale process.

With this general model, we first develop in Section 8.2.2 the existence and uniqueness theorem, which has not been studied so far in the literature. It comes to the conclusion that two conditions have to be satisfied to maintain the existence of a unique optimal investment policy. Moreover, we show through examples that these assumptions hold for the currently–adopted models and argue that in general they cannot be relaxed.

Then, we move on to define and characterize the optimal investment policy in a very detailed and intuitive way. The derivation starts with the necessary and sufficient optimality conditions. In case of irreversibility, the marginal profit at any time is composed of the immediate marginal gain and additionally all the future changes in the marginal profit due to the current investment. In principle, investment occurs if and only if the capacity is depreciated or the investment cost declines such that the marginal profit becomes lower than or equal to the cost. However, the first–order condition does not frequently bind as a result of irreversibility. Thus, it is not that useful to find the solution. To remedy, a *base capacity* policy is constructed such that the firm expands the capacity whenever the current capacity is lower than or equal to the base capacity, otherwise just keep the current capacity. This so–called *base capacity* indicates the optimal capacity level the firm is going to take if starting exactly at that time point with zero capacity. With this ansatz, we then have to identify this significant *base capacity* level process. In Section 8.3, two alternative methods are introduced. Either it is characterized through solving our key stochastic backward equation. Or, we can characterize it via optimal stopping problems that investors have for a continuum of American options to next marginal investment. We show that the first–order conditions at any investment time become a strict equality if the firm invests optimally at the constructed *auxiliary levels* and the *base capacity* is the lower envelop of these *auxiliary levels*. Intuitively speaking, the base capacity is found in a "cautious way" to be the lowest capacity level that makes the optimality condition binding so as to gain the maximal flexibility for future decisions.

Using this technique, we not only identify the optimal investment strategy, but also learn more about implications of the optimal investment in the following three aspects. First, it facilitates a general qualitative characterization of the irreversible investment. A thorough analysis is carried out in Section 8.4 and the obtained results are also compared with those of Arrow (1968) for irreversible investments under certainty. To date, we distinguish free and block intervals as done in Arrow (1968). In a free interval, investment arises in an absolutely continuous way at positive rates. Such investment is called *smooth investment*. Besides, it is noted that during free intervals the marginal profit is always equal to the user cost of capital, as in the case of reversible investment. Whereas, the equality is maintained only in expectation on average over time in blocked intervals during which there is no investment. When uncertainty is processed by a diffusion, investment turns out to be singular with respect to Lebesgue measure. We define this type investment as *singular investment*. Its positive increment occurs on a set of Lebesgue measure zero, or say, it takes place like a diffusion in an oscillating way for an infinitesimal time. Investment activity may also be lumpy with a sudden large adjustment. It is a natural and best response to shocks of the future economic conditions. However, we argue that there is no *lump sum investment* at those fixed dates with no surprise. When saying that there are no surprise/jumps at a fixed date, it means that at this time the information flow is continuous and a jump of the underlying uncertainty has zero probability. Moreover, it is shown that the capacity never jumps to an excess capacity with respect to the operating profit.

Furthermore, this method leads to the first results on general comparative statics of the optimal investment. It is basically completed by applying implicit differentiation of the simple equation of the constructed *auxiliary levels*. First, the base capacity is shown to be monotonically increasing in the exogenous shock when the operating profit function has increasing differences in capacity and exogenous economic shock. This result is to our knowledge completely new regarding the stochastic process as one parameter. Another result is related to interest rate and depreciation rate. It is shown that the firm size always decreases with their sum which is the so–called *user cost of capital* in our construction.

The third appealing feature of this approach is that it provides the trigger value which is unique, intuitive and analytical tractable when an infinitely–lived firm is endowed with the operating profit function of Cobb–Douglas type and the underlying uncertainty is modelled by an exponential Lévy process. Section 8.6 specifies the solution to the threshold and also the value of the project. In particular, those identified investment properties are demonstrated through two typical examples, namely, a GBM and a compound Poisson process with exponential distributed jump sizes.

Chapter 9 explains how this approach can be extended to handle the valuation of irreversible investment when considering market incompleteness and risk aversion.

There is always a hot debate on the asset spanning hypothesis of the standard real options theory. Even recent books (see Simit and Trigeorgis (2004)) suppose that *"real–options valuation is still applicable provided we can find a reliable estimate for the market value of the asset"*. Clearly, it adheres quite closely to the argument that *markets are sufficiently complete*. However, as concerned in the literature and quite often by financial experts, most investment decisions have to be made up in the markets which are far from being complete. For instance, the frequently–mentioned R&D investment is in principle connected with a new product which is not traded at all in the market (at least currently). Furthermore, there is generally no effective method of perfectly replicating the cash flows from the investment. Lack of market completeness, the risk neutrality assumption that the discount factor is identified *universally* as the risk–free interest rate becomes immediately invalid. As a matter of fact, the discount factor is a subjective assessment based on the trading prices and outlook for future prospects and hence heavily related to risk preferences of decision makers. Therefore, subjective risk preference has to be considered to correctly value irreversible investment.

In such a context, there are two streams of line to deal with irreversible investment problems under uncertainty in an incomplete market. First, according to the analysis in the previous section (see also Dixit and Pindyck (1994), Chapter 5), the dynamic programming method with an exogenously specified discount rate is still able to solve the problem. The expected rate of return on the investment opportunity is in principle equal to the expected capital appreciation from a project. It is surely different from risk–free interest rate and shows the risk preference of the corporation. As indicated by some literature (see for instance Grasselli (2006)), such an approach has the serious theoretical drawback that nonlinear risk preferences of a corporation can hardly be expressed through a single discount factor. In fact, risk preferences are modelled in the majority of financial economics literature by an expected utility function together with an exogenous discount factor. For instance, Hugonnier and Morellec (2005) relax the assumption of the market completeness and define a power utility to account for risk aversion. This utility maximization model is further extended in Henderson (2005), Miao and Wang (2005), Henderson and Hobson (2002) and Hugonnier and Morellec (2006) by introducing a correlated asset which is traded in the market to account for market incompleteness. Meanwhile, they assume that investors are risk neutral only towards market risks which can be diversified via the correlated asset but risk averse to idiosyncratic risks. Therefore, their works focus more on the wealth allocation in riskless and risky assets. In addition, Grasselli (2006) also studies Henderson's model but in a finite-horizon version with numerical methods. So far, all the work on this topic focus on a GBM and some specific utility function such as a power or exponential utility.

Along the main stream in the literature, we argue that the investor's risk preference should be explicitly used for valuing the option to invest. To this end, the standard real options problem is combined with the utility function. Although the introduction of utility maximization brings about further problems, such as which utility function is to be select, it does generalize the basic model by considering risk aversion. Indeed, such utility–based model includes the profit maximization problem as a special case by assuming a linear (risk–neutral) utility. Our proposed method handles perfectly this economically sound utility–based framework. The obtained investment policy is formulated in terms of *shadow utility* and can be viewed as an extension of the *Shadow NPV rule*. In addition, this decision rule is valid for a wide class of concave and increasing utility functions and semi–martingale processes, provided the objective maximization problem is bounded to be finite.

Applying the optimal investment strategy, we offer a detailed derivation of the *shadow* utility process given a power utility function and an exponential Lévy process in Section

9.3. In this case, the decision maker is risk averse with constant relative risk aversion. The result is in structure quite similar to the profit–maximization case: The critical expected utility is equal to the utility gain by investing the investment cost in risk–free assets multiplied by a mark–up factor. On this basis, Section 9.4 provides a qualitative and quantitative analysis of risk aversion's effect on the investment trigger value. In particular, the combined effect of jumps and risk aversion on the threshold is examined given an estimate of the first two moments of the output price.

Chapter 7

Investment Decision Based on Shadow NPV Rule[1]

7.1 Introduction

Facing with an opportunity to invest, firms have to make the decision whether to invest or not. Traditionally, the investment strategy is determined by the NPV method which, as widely acknowledged (c.f. Dixit and Pindyck (1994) and the literature therein), considers the investment only as being *now or never* and neglects the stochastic nature of the project values. On the contrary, the contemporary real options theory highlights flexibility which is required by the uncertainty in the future economic conditions and the complete or partial irreversibility associated with the investment. It regards an investment opportunity as an option, i.e., a right but not an obligation, to launch the project at the time point when the investment brings the maximal profit. In this sense, the real options theory is more plausible and more advantageous than the NPV method.

Although the real options approach has been viewed for long as a modern and correct means in academics, it has yet to catch on with practitioners. One of the most important factors that lead to such a failure in practical application lies in the lack of transparency and simplicity of the real options method (See Teach (2003)). To many managers, the framework is not easy enough to understand. Moreover, the mere concern of shareholders is on the net profit or revenue of the new investment. In this sense, the NPV is surely the most *meaningful* and *straightforward* measure of the investment performance. It is however not really accounted for by the real options method. To promote the wide application of the real options theory, this paper is going to find an alternative method which provides a correct and economically intuitive decision rule based on the NPV.

As a starting point, the NPV of the investment is formulated as the expected present value (EPV) of all the operating profits after the optimal investment time less the investment cost. In contrast to the EPV that starts accruing at a deterministic time, it

[1] This chapter is based on Su (2006).

is rather difficult to determine the EPV from the investment since the investment time is instead a stopping time when the project revenue comes to a satisfactory level. Thus, another approach is required here in order to calculate or rewrite the EPV term. In this work, we represent the expected discounted revenue in terms of the EPV of the running supremum of another process. This process is extremely useful, signalling the investment in the decision rule: The optimal investment time is identified as the first moment at which the process reaches the investment cost. In this way, we derive the decision rule by solving a representation problem rather than treating it as an optimal stopping problem. Especially, we define the signalling process as *shadow revenue process*. The key reformulation procedure here is based on the stochastic representation method first proposed in Bank and Föllmer (2003) for various stochastic optimization problems.

By specifying the project's NPV in terms of the EPV of the running supremum of another process, this method gives some economic intuition as follows. First, it exactly coincides with the fact that to maximize the investment profit, the investor is not concerned with the instantaneous revenue of the investment at the moment when the investment is done, but with the future profits it creates after the investment. Surely, it would be optimal to invest at the moment when the project starts to create positive profits (net of all the costs). More precisely, it is a *Shadow NPV rule*: The investment is undertaken if and only if the *shadow revenue* rises up to the investment cost. Thus, we finally achieve a simple optimal investment strategy based on the NPV as desired. Moreover, this method extends and corrects the conventional NPV method by determining the *proper* NPV. Second, the *shadow value* is defined in this work in the sense that it is the true or pure value of the investment that the firm gains after compensating total costs. We also demonstrate that the *shadow revenue process* is always lower than the expected revenue at any stopping time. The value difference can be interpreted to account for the opportunity cost of delaying the investment. In other words, the *shadow revenue* records the economic value of the investment by deducting the option premium of waiting from the revenue cash flow. In this way, with the trick of reformulating the expected discounted revenue, the new method derives an investment decision rule consistent with the standard real options theory: in addition to the investment cost, the overall revenue has to cover the option premium of waiting.

Another highlight of this method is its applicability to a wide class of general stochastic processes. In this way, we extend the classical real options theory from a GBM to a Lévy process, to a time–inhomogeneous diffusion process and even to all semi–martingale processes that are economically plausible. The decision problem on a GBM has been fully exploited as in e.g. Dixit and Pindyck (1994). However, the lognormal distribution is contradictory to the well–known empirical evidence (see, for instance, Yang and Brorsen (1992) as well as Deaton and Laroque (1992)). Indeed, commodity prices exhibit significant skewness and kurtosis and sometimes mean–reversion, hence project values which are closely related to the price of the output prices. More importantly, there is a high probability of large random fluctuations such as crashes or sudden upsurges. As a result, a Lévy process which combines a diffusion process and embedded jumps turns out to be a more correct model description to account for fat tails and skewness of probability distributions as well as abrupt large jumps, and some other processes for the mean-reverting feature. Moreover, this method also works when the discount rates are modelled as a strictly positive stochastic process.

More powerful than the standard real option pricing methods, this new approach provides explicit characterizations for the threshold value when the uncertainty is specified as an exponential Lévy process. The exercise threshold is identified as the investment cost multiplied by a correction factor, in the same form as the standard result for the GBM. Hence, this method generalizes the simple decision rule to general exponential Lévy processes, providing a clear qualitative view of the investment strategy. The obtained correction factor is expressed in terms of the supremum process. This result coincides with those of Mordecki (2002) and Boyarchenko and Levendorskiĭ (2002b). Mordecki's work is basically an extension of the discrete–time model on random walks by Darling, Liggert and Taylor (1972) and applies well to the general Lévy process. While, Boyarchenko and Levendorskiĭ's method is based on reducing the optimal stopping problem to a free boundary problem for the generalized Black and Scholes (1973) equation in form of pseudo–differential operators. This representation gives not only an economic interpretation as the EPV of an instantaneous payoff or a stream of payoffs, but also provides the possibility of finding an explicit formula by means of the Wiener–Hopf factorization. Through numerical example, we then demonstrate the result as observed in the literature (e.g., Boyarchenko and Levendorskiĭ (2004a)) that the existence of jump term lowers the investment threshold and hence alleviates the concern of practitioners that the trigger level of investment recommended by the real option approach is too high.

This chapter proceeds as follows: Section 7.2 sets up the model of irreversible investment under uncertainty, analyzes the profit–maximization problem by the stochastic representation method and discusses the inherent economic implications. Particularly, explicit characterizations of investment trigger values are given in Section 7.3 when the output price is modelled by an exponential Lévy process and a time–inhomogeneous diffusion process, respectively. As illustration, several examples are provided for a GBM, a mixed jump–diffusion process, a GBM combined with a compound Poisson process and a Cox–Ingersoll–Ross process. For each case, we derive the optimal investment strategy in an analytical form. Finally, Section 7.4 concludes with a short summary and remark. Technical details are presented in Appendix B.

7.2 Real Options and New Valuation Method

This chapter solves the irreversible investment decision problem under uncertainty with a new approach in the real options literature. In order to facilitate the derivation and interpretation, we continue with the well–established irreversible investment model which is set up in Section 6.2 but generalized here with a life time $\hat{T} \leq \infty$. After a brief recapitulation of the model, the new valuation approach is introduced and interpreted in detail.

7.2.1 Irreversible Investment Decision Problem

Consider a firm that has an opportunity to invest in a project with a fixed scale and no marginal cost. This investment is irreversible and requires only a constant initial investment cost I. The project generates then a continuous stream of cash flows by producing a unit of commodity good ever after the investment until time $\hat{T}$ which can be finite or infinite. The spot price of the firm's output $(P_t)_{t\in[0,\hat{T}]}$ evolves stochastically and is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\le t\le\hat{T}}, \mathbb{P})$. Under the specified physical measure $\mathbb{P}$, the decision problem the firm is facing is formally expressed as

$$F = \max_{0\le\tau\le\hat{T}} E\left[e^{-\rho\tau}\left(\int_\tau^{\hat{T}} e^{-\rho(s-\tau)}P_s\,ds - I\right)^+\right] \quad (7.1)$$

$$= \max_{0\le\tau\le\hat{T}} E\left[e^{-\rho\tau}\left(\pi_\tau - I\right)^+\right], \quad (7.2)$$

where ρ is the exogenously given constant discount factor and $\pi_t = E\left[\int_t^{\hat{T}} e^{-\rho(s-t)}P_s\,ds\,|\mathcal{F}_t\right]$ again the expected overall revenue of the investment at time t. In addition, we assume that

$$E\left[\int_0^{\hat{T}} e^{-\rho t}P_t\,dt\right] < \infty\,, \quad (7.3)$$

and the filtration is quasi–left–continuous, i.e.,

$$\mathcal{F}_\tau = \mathcal{F}_{\tau^-} \quad (7.4)$$

for any predictable stopping time $\tau \in [0, \hat{T}]$.

Obviously, the firm has an optimal stopping problem at hand. The standard real options theory offers two approaches that are both relevant to solving a stochastic differential equation. Alternatively, we are going to deal with the optimal stopping problem by the stochastic representation method as fully explained in the next subsection.

7.2.2 Stochastic Representation Method and Shadow NPV Rule

The EPV of a cash flow accumulated from a deterministic time point is clear and can be easily obtained. The difficulty of our problem lies in the fact that the EPV begins at a stopping time when the project value reaches a satisfactory level. In order to derive a decision rule based on this essential and straightforward concept of NPV, we rewrite the EPV of the project revenue in terms of the EPV of the running supremum of another process. More explicitly, the EPV of the operating profit from investment at time τ, $e^{-\rho\tau}\pi_\tau$, is represented in form of

$$e^{-\rho\tau}\pi_\tau = E\left[\int_\tau^{\hat{T}} \rho e^{-\rho t}\sup_{\tau\le v<t}\xi_v^{pm}\,dt + e^{-\rho\hat{T}}\sup_{\tau\le v\le\hat{T}}\xi_v^{pm}\Big|\mathcal{F}_\tau\right] \quad (7.5)$$

by some progressively measurable process $(\xi_t^{pm})_{t\in[0,\hat{T})}$ with upper-right continuous paths. This representation then allows for a characterization of the optimal stopping time as the first moment at which the obtained exercise signal process ξ_t^{pm} hits the investment cost, I.

Theorem 7.2.1. *Suppose that the decision problem of an irreversible investment specified by* (7.1) *admits the stochastic representation* (7.5) *in terms of the* ***shadow revenue process*** $(\xi_t^{pm})_{t\in[0,\hat{T}]}$ *which is progressively measurable with upper–right continuous paths. Then, the level passage time when the process* ξ *rises up to the investment cost, i.e.,*

$$\tau^* = \inf\{t \geq 0 \mid \xi_t^{pm} \geq I\}$$

maximizes the investment value over all stopping times $\tau \in [0, \hat{T}]$.

Proof: Bank and El Karoui (2004) give a detailed technical analysis of the representation form (7.5). In particular, they show that the representation form is valid whenever $e^{-\rho\tau}\pi_\tau$ is uniformly integrable and upper–semicontinuous in expectation[2]. Hence, we have to show first whether the regularity conditions are satisfied in our construction or not. As assumed in Conditions (7.3) and (7.4), we have an expected discounted revenue that is bounded from above and a quasi–left–continuous filtration. Clearly, the uniform integrability is guaranteed by Condition (7.3). According to the definition, the discounted revenue is

$$\begin{aligned} e^{-\rho t}\pi_t &= E\left[\int_t^{\hat{T}} e^{-\rho s} P_s ds \Big| \mathcal{F}_t\right] \\ &= \underbrace{E\left[\int_0^{\hat{T}} e^{-\rho s} P_s ds \Big| \mathcal{F}_t\right]}_{:=M_t} - \underbrace{\int_0^t e^{-\rho s} P_s ds}_{:=A_t}, \end{aligned}$$

where A_t is predictable and absolutely continuous. Moreover, we can show that M_t is a martingale as follows: for any $u < t$

$$\begin{aligned} E[M_t|\mathcal{F}_u] &= E\left[E\Big[\int_0^{\hat{T}} e^{-\rho s} P_s ds \Big| \mathcal{F}_t\Big]\Big| \mathcal{F}_u\right] \\ &= E\left[\int_0^{\hat{T}} e^{-\rho s} P_s ds \Big| \mathcal{F}_u\right] = M_u\,. \end{aligned}$$

This martingale is cadlag, quasi–left–continuous if the filtration is quasi–left–continuous. As the sum of M_t and A_t, $e^{-\rho t}\pi_t$ is then always quasi–left–continuous, i.e.,

$$\limsup_n e^{-\rho\tau^n}\pi_{\tau^n} = e^{-\rho\tau}\pi_\tau \qquad \text{a.s.}$$

[2]The uniform integrability is one basic condition in the optimal stopping problem to guarantee the existence of a finite solution. Upper–semicontinuity in expectation is precisely stated as follows: $\limsup_n E[X_{\tau^n}] \leq E[X_\tau]$ for any monotone sequence of stopping times τ^n $(n = 1, 2, \cdots)$ converging to some stopping time $\tau \in [0, \hat{T}]$ almost surely.

for any monotone sequence of stopping times τ^n $(n = 1, 2, \cdots)$ converging to some stopping time $\tau \in [0, \hat{T}]$, whenever Condition (7.4) is fulfilled. It hence yields the required upper–semicontinuity in expectation of $e^{-rt}\pi_t$.

In the following, we are going to prove that the project gives a positive net profit at the optimal investment time and some loss would be created for any earlier and later investment. Following the investment policy τ^*, the investment is undertaken at the first time when ξ^{pm} reaches I. It gives then the project in value

$$\begin{aligned} F &= E\big[e^{-\rho\tau^*}\pi_{\tau^*} - e^{-\rho\tau^*}I\big] \\ &= E\left[E\Big[\int_{\tau^*}^{\hat{T}} \rho e^{-\rho t} \sup_{\tau^*\le v<t} \xi_v^{pm}\, dt + e^{-\rho\hat{T}} \sup_{\tau^*\le v\le\hat{T}} \xi_v^{pm}\Big|\mathcal{F}_{\tau*}\Big] - e^{-\rho\tau^*}I\right] \\ &\ge E\left[E\Big[\int_{\tau^*}^{\hat{T}} \rho e^{-\rho t} I\, dt + e^{-\rho\hat{T}} I\Big|\mathcal{F}_{\tau}\Big] - e^{-\rho\tau^*}I\right] \\ &= 0\,. \end{aligned}$$

This shows that the investment at τ^* always brings about a non–negative profit.

Before moving on, it is noted that $\xi_{\tau^*}^{pm} \ge I > \xi_t^{pm}$ whenever $t \in [0, \tau^*)$ and hence

$$\sup_{\hat{\tau}\le v<t} \xi_v^{pm} < I \qquad \text{for} \quad t \in [\hat{\tau}, \tau^*) \tag{7.6}$$

as well as

$$\sup_{\hat{\tau}\le v<t} \xi_v^{pm} = \sup_{\tau^*\le v<t} \xi_v^{pm} \qquad \text{for} \quad t \in [\tau^*, \hat{T})\,. \tag{7.7}$$

On the event of $\{\hat{\tau} < \tau^*\}$, the present value of the project net profit at that moment is obtained as

$$\begin{aligned} \hat{F} &= E\left[E\Big[\int_{\hat{\tau}}^{\hat{T}} \rho e^{-\rho t} \sup_{\hat{\tau}\le v<t} \xi_v^{pm}\, dt + e^{-\rho\hat{T}} \sup_{\hat{\tau}\le v\le\hat{T}} \xi_v^{pm}\Big|\mathcal{F}_{\hat{\tau}}\Big] - e^{-\rho\hat{\tau}}I\right] \\ &= E\left[\int_{\tau^*}^{\hat{T}} \rho e^{-\rho t} \sup_{\hat{\tau}\le v<t} \xi_v^{pm}\, dt + e^{-\rho\hat{T}} \sup_{\hat{\tau}\le v\le\hat{T}} \xi_v^{pm} - e^{-\rho\tau^*}I\right] \\ &\quad + E\left[\int_{\hat{\tau}}^{\tau^*} \rho e^{-\rho t} \sup_{\hat{\tau}\le v<t} \xi_v^{pm}\, dt - e^{-\rho\hat{\tau}}I + e^{-\rho\tau^*}I\right] \\ &= \left[\int_{\tau^*}^{\hat{T}} \rho e^{-\rho t} \sup_{\tau^*\le v<t} \xi_v^{pm}\, dt + e^{-\rho\hat{T}} \sup_{\tau^*\le v\le\hat{T}} \xi_v^{pm} - e^{-\rho\tau^*}I\right] + E\left[\int_{\hat{\tau}}^{\tau^*} \rho e^{-\rho t}(\sup_{\hat{\tau}\le v<t} \xi_v^{pm} - I)\, dt\right] \\ &= F + E\left[\int_{\hat{\tau}}^{\tau^*} \rho e^{-\rho t}(\sup_{\hat{\tau}\le v<t} \xi_v^{pm} - I)\, dt\right], \end{aligned}$$

where the second step is achieved by splitting the integral into two parts. That is, $\hat{F}$ is a sum of F, the present value of the net profit of the project invested at the optimal time

τ^*, and another term which is according to (7.6) definitely negative. Therefore, an earlier investment yields a lower project value.

Consider another event $\{\tau' > \tau\}$. We check in this case the difference in the project value at the optimal time and the moment τ':

$$\begin{aligned}
F - F' &= E\left[\int_{\tau^*}^{\hat{T}} \rho e^{-\rho t} \sup_{\tau^* \le v < t} \xi_v^{pm}\, dt + e^{-\rho \hat{T}} \sup_{\tau^* \le v \le \hat{T}} \xi_v^{pm} - e^{-\rho \tau^*} I\right] \\
&\quad - E\left[\int_{\tau'}^{\hat{T}} \rho e^{-\rho t} \sup_{\tau' \le v < t} \xi_v^{pm}\, dt + e^{-\rho \hat{T}} \sup_{\tau^* \le v \le \hat{T}} \xi_v^{pm} - e^{-\rho \tau'} I\right] \\
&= E\Bigg[\int_{\tau^*}^{\tau'} \rho e^{-\rho t} \sup_{\tau^* \le v < t} \xi_v^{pm}\, dt + \int_{\tau'}^{\hat{T}} \rho e^{-\rho t} \max\{\sup_{\tau^* \le v < \tau'} \xi_v^{pm}, \sup_{\tau' \le v < t} \xi_v^{pm}\}\, dt \\
&\quad - e^{-\rho \tau^*} I\Bigg] - E\left[\int_{\tau'}^{\hat{T}} \rho e^{-\rho t} \sup_{\tau' \le v < t} \xi_v^{pm}\, dt - e^{-\rho \tau'} I\right] \\
&= E\left[\int_{\tau'}^{\hat{T}} \rho e^{-\rho t} \big(\max\{\sup_{\tau^* \le v < \tau'} \xi_v^{pm}, \sup_{\tau' \le v < t} \xi_v^{pm}\} - \sup_{\tau' \le v < t} \xi_v^{pm}\big)\, dt\right] \\
&\quad + E\left[\int_{\tau^*}^{\tau'} \rho e^{-\rho t} \big(\sup_{\tau^* \le v < t} \xi_v^{pm} - I\big)\, dt\right],
\end{aligned}$$

where we write the running supremum in F into the maximum of the two running supremums before and after τ' in the second step. Obviously, the first term is always non-negative no matter a new all time high is achieved before or after time τ'. Furthermore, the second term is also shown to be non–negative since $\sup_{\tau^* \le v < t} \xi_v^{pm} \ge \xi_{\tau^*}^{pm} \ge I$ for any $t \in [\tau^*, \hat{T}]$. In all, we have $F \ge F'$. This completes the proof that τ^* is the optimal investment time for the firm to maximize the project value. ■

In this way, the optimal stopping problem is reduced to a representation problem based on the stochastic representation method first proposed by Bank and Föllmer (2003) for various stochastic optimization problems. The representation form (7.5) is valid whenever the two regularity conditions (7.3) and (7.4) are satisfied. In general, there always exists a unique solution of ξ_t^{pm} to this problem.

Remark 7.2.2. *In general, the regularity conditions* (7.3) *and* (7.4) *are relatively weak and easily satisfied. First, the investment decision problem is well–posed or makes economically sense only when* (7.3) *is true. Moreover, for a semi–martingale process* $(X_t)_{t\in[0,\hat{T}]}$ *and its generated filtration* $\mathcal{F}$,

$$X_\tau = X_{\tau-} \quad \text{for any predictable stopping time } 0 \le \tau \le \hat{T}$$

is the only requirement to achieve the quasi–left–continuity of the filtration. Intuitively speaking, one cannot tell in advance when the jumps of X *will take place. For instance,* *the filtration generated by a BM or a Lévy process is always quasi–left–continuous as the $\sigma-$field of a BM is absolutely continuous and for the latter case the stopping time at which a jump occurs is never predictable (see Protter (2004), Chapter* 3 *p.* 105*).*

The intuition behind this method is as follows. The holder of a real option would like to maximize the EPV of the net profit from the investment. That means, the investor does not care about the instantaneous value of the project at the investment time, but about its future proceeds after the investment. By means of this method, the EPV of all future revenues is then specified by the running supremum of the process ξ_t^{pm}. Formally, it can be expressed as[3]:

$$\begin{aligned} E\Big[e^{-\rho\tau}(\pi_\tau - I)^+\Big] &= E\left[E\left[\int_\tau^{\hat{T}} \rho e^{-\rho t} \sup_{\tau\le v<t} \xi_v^{pm}\, dt + e^{-\rho\hat{T}} \sup_{\tau\le v\le\hat{T}} \xi_v^{pm} \Big| \mathcal{F}_\tau\right] - e^{-\rho\tau} I\right] \\ &= E\left[\int_\tau^{\hat{T}} \rho e^{-\rho t} \sup_{\tau\le v\le t} \xi_v^{pm}\, dt - e^{-\rho\tau} I\right]. \end{aligned}$$

Obviously, the trick of this method is to represent the EPV of the project revenue in terms of the supremum of another process.

According to the optimal investment policy, the investment is undertaken if and only if ξ_t^{pm} becomes equal to or greater than the investment cost. Otherwise, some positive revenues are lost. Earlier exercise, i.e., when $\xi_t^{pm} < I$, is also not optimal since the investment at such a time yields only negative payoff. Although ξ_t^{pm} is not the revenue cash flow recieved from the investment, it takes the role of initiating the investment. Especially, we define ξ_t^{pm} as the *shadow revenue process* and the decision rule on this basis as the *Shadow NPV rule*. It states that the investment is taken if and only if the *shadow revenue* rises up to cover the investment cost. In this sense, this method indeed extends and corrects the conventional NPV method by identifying the proper net present value.

Shadow value is defined in this thesis as the value of the investment that the firm purely gains from the project after compensating all the costs incurred. Thus, the final obtained *Shadow NPV* measures exactly the willingness of the decision maker to give up money and also time for the investment opportunity. Therefore, it is not only the market value or simply the revenue cash flow from the investment less the initial investment cost. Additionally, subjective valuation of the investment should be considered. Under uncertainty, investors are reluctant to invest and prefer waiting for better information. During the waiting process, the firm may be losing other opportunities to gain profit, hence increasing the (opportunity) cost of undertaking the investment. Therefore, we argue that the *shadow revenue* records the true or pure expected benefit embedded in the real revenue of the investment after deducting the full opportunity cost. In other words, the *shadow*

[3]The simple expression in the second line is written by relaxing mathematical strictness in the left continuity of the process ξ^{pm} (It is only right continuous with left limits.). It is nevertheless used here to show the intuition.

NPV accurately captures the economic value of the investment expected present value in an uncertain environment. Furthermore, we show (in Appendix $C.1$) that the *shadow revenue* is always smaller than or at least equal to the created operating profit at any stopping time, i.e., $\pi_\tau \geq \xi_\tau^{pm}$ for all the stopping times $0 \leq \tau \leq \hat{T}$. The discrepancy between these two values accounts exactly for the opportunity cost of delaying the investment. In this sense, the *shadow revenue* measures the expected economic value of the investment which is the real revenue less the option premium of waiting. Thus, it becomes optimal to invest when the *correct* NPV becomes non–zero. Clearly, the investment rule obtained by this method is fully consistent with the established result in the real options theory.

Remark 7.2.3.

(a) *The same argument can be applied to the so–called exit problem (an abandon option). Consider the same firm who has already invested and produced a unit output at price P_t. The firm contemplates scrapping the investment for a value S (the salvage cost), once the price declines and results in loss. In this context, the firm would like to maximize the payoff of the investment abandonment*

$$\max_{0\leq\tau\leq\hat{T}} E\left[e^{-\rho\tau}\left(S - \int_\tau^{\hat{T}} e^{-\rho(s-\tau)} P_s\, ds\right)^+\right].$$

That is, the firm has a put on the investment at hand. By means of this method, the EPV of the future revenues that would be lost after exit is reduced to a representation in terms of the infimum process of the shadow revenue process ξ^{pm} as

$$e^{-\rho\tau}\pi_\tau = E\left[\int_\tau^{\hat{T}} \rho e^{-\rho t} \inf_{\tau\leq v<t} \xi_v^{pm} + e^{-\rho\hat{T}} \inf_{\tau\leq v\leq\hat{T}} \xi_v^{pm}\, dt \middle| \mathcal{F}_\tau\right],$$

where $\pi_t = E\left[\int_t^{\hat{T}} e^{-\rho(s-t)} P_s\, ds \middle| \mathcal{F}_t\right]$. The optimal investment time is then characterized as the first time when the **shadow revenue process** *becomes equal to or lower than the gain of exit S, namely,*

$$\tau^{**} = \inf\{t \geq 0 \mid \xi_t^{pm} \leq S\}.$$

(b) *This method works also when the discount rate is not constant but stochastic with strictly positive values.*

As a short summary, the irreversible investment decision problem is solved by finding the solution of a stochastic representation problem in terms of the running supremum/infimum process of the *shadow revenue process*. Obviously, the *shadow revenue process*, ξ_t^{pm}, is the key process in this method, signalling the optimal exercise rule. In particular, this exercise signal process is universal in the sense that it is the single reference process determining optimal investment times for any possible investment costs. This property would be favorable in more complicated investment decision problems, for instance, sequential investments as well as capital expansion programs that are to be addressed in the coming chapter.

This approach fits all semi–martingale processes which are economically plausible and hence often used in finance, provided that the mild regularity condition is satisfied. Generally, numerical methods have to be used to specify the universal exercise signal process. To some cases, e.g., exponential Lévy processes and time–inhomogeneous diffusion processes, analytical solution formulae are already available for the perpetual investment problem. We will come to its feasibility and derivation in the next section. Particularly, the solution is in a simple and intuitive form such that the expected future operating profit from the investment has to cover not only the investment cost but also the opportunity cost of delaying the investment. Thus, in this sense, this method generalizes the solution of real options in the GBM model and provides additional interpretations even within the GBM model framework.

7.3 Explicit Solution Formulae for Investment Problems

One outstanding advantage of this method is the capability of providing an analytical solution formula of the investment threshold for an infinite investment, i.e., $\hat{T} = \infty$ is assumed for explicit solution derivation, whose underlying uncertainty is modelled by an exponential Lévy process or even a time–inhomogeneous diffusion process. Lévy processes are a general class of Markov processes with independent identically distributed increments and can be decomposed into a continuous Gaussian process and a pure jump process (see, e.g., Bertoin (1996)). Hence, it is frequently used to capture the significant skewness and kurtosis of commodity prices as empirically observed in for instance Yang and Brorsen (1992) as well as Deaton and Laroque (1992). Meanwhile, some other diffusion processes are applied to describe the mean reversion property of the commodity price. In this section, a thorough analysis is provided to characterize analytical solutions to the irreversible investment decision problem where the underlying uncertainty is modelled by a general exponential Lévy process and a time–inhomogeneous diffusion process, respectively.

7.3.1 Explicit Solution Formulae for Exponential Lévy Processes

In this subsection, the uncertainty in the model is described by a general exponential Lévy process: To put it in a formal way, assume that the exogenous output price is generated by the following stochastic process

$$P_t = P_0 e^{Y_t}, \tag{7.8}$$

where $P_0 > 0$ is again the initial price and $Y = (Y_t)_{t\geq 0}$ is a Lévy process[4] defined on the probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_\mathbf{t})_{\mathbf{t}\geq\mathbf{0}}, \mathbf{P})$ with initial value $Y_0 = 0$. The Lévy–Laplace exponent of the Lévy process Y is $\Psi(z)$ satisfying $E[e^{zY_t}] = e^{t\Psi(z)}$ and $\Psi(z) = \rho$ is the corresponding characteristic equation of Y_t for $\rho > 0$. In order to make mathematically well–posed, we additionally assume

$$E[e^{zY_1}] < \infty, \qquad \text{for all} \quad z \in \mathbb{R}. \tag{7.9}$$

Denote $\overline{Y}_t = \sup_{0\leq s\leq t} Y_t$ and $\underline{Y}_t = \inf_{0\leq s\leq t} Y_t$, the running supremum and infimum of Y_t. The main technique for solving the problem in this dissertation is the Wiener–Hopf factorization

$$\frac{\rho}{\rho - \Psi(z)} = E\left[\int_0^\infty \rho e^{-\rho t} e^{z\overline{Y}_t} dt\right] E\left[\int_0^\infty \rho e^{-\rho t} e^{z\underline{Y}_t} dt\right] = \Psi_\rho^+(z)\Psi_\rho^-(z)\,,$$

where $\Psi_\rho^+(z)$ and $\Psi_\rho^-(z)$ are usually called as Wiener–Hopf left and right factor, respectively. It is possible to obtain analytical forms of the two Wiener–Hopf factors, as the factorization is unique. For instance, for the case of a GBM, the characteristic equation has one positive and one negative root as β^+ and β^-. Then the two factors are given by

$$\Psi_\rho^+(z) = \frac{\beta^+}{\beta^+ - z} \qquad \text{and} \qquad \Psi_\rho^-(z) = \frac{\beta^-}{\beta^- - z}.$$

Furthermore, Boyarchenko and Levendorskiǐ (2002a) derive a general solution form to regular Lévy processes of exponential type as we assume here.

The Investment Threshold and Project Value In this context, a closed–form characterization can be found for the critical shadow revenue process identifying the investment initiating time:

Theorem 7.3.1. *Under Assumption* (7.8) *and* (7.9)*, the solution of the representation problem* (7.5)*, namely, the* **shadow revenue process** *is obtained as* $\xi_v^{pm} = P_v/\kappa$ *with*

$$\kappa = (\rho - \Psi(1))\, E\left[e^{\overline{Y}_{\tau(\rho)}}\right]\,,$$

where $\tau(\rho)$ is an independent exponentially distributed time with parameter ρ.

Proof: Based on the specification of P_t, the left–hand side of Equation (7.5) with $\hat{T} = \infty$, the perpetual cash flow starting from p_τ is easily calculated as

$$e^{-\rho\tau}\pi_\tau = e^{-\rho\tau}\frac{P_0 e^{Y_\tau}}{\rho - \log E[e^{Y_1}]}\,,$$

where $\log E[e^{Y_1}]$ records the time increasing rate of the price process and is as defined equal to $\Psi(1)$.

[4] A short review is provided in Appendix B on Lévy processes and some of their mathematical properties. More details on this process are found in Bertoin (1996) and literature cited therein.

Construct the *shadow revenue process* in form of $\xi_v^{pm} = P_v/\kappa$. Then by substituting the general exercise signal, the representation equation is reduced into

$$\begin{aligned} e^{-\rho\tau}\frac{P_0 e^{Y_\tau}}{\rho-\Psi(1)} &= E\left[\int_\tau^\infty \rho e^{-\rho t}\sup_{\tau\le v\le t} P_0\frac{\exp(Y_v)}{\kappa}\,dt\,\Big|\,\mathcal{F}_\tau\right] \\ &= e^{-\rho\tau}P_0 e^{Y_\tau}E\left[\int_\tau^\infty \rho e^{-\rho(t-\tau)}\sup_{\tau\le v\le t} e^{Y_v-Y_\tau}\,dt\,\Big|\,\mathcal{F}_\tau\right]/\kappa\,. \end{aligned}$$

This can be further simplified by using the property of Lévy processes that $Y_v - Y_\tau$ has the same distribution as $Y_{v-\tau}$ and is independent of the σ-field $\mathcal{F}_\tau$

$$\begin{aligned} e^{-\rho\tau}\frac{P_0 e^{Y_\tau}}{\rho-\Psi(1)} &= e^{-\rho\tau}P_0 e^{Y_\tau}E\left[\int_0^\infty \rho e^{-\rho t}\sup_{0\le v\le t} e^{Y_v}\,dt\right]/\kappa \\ &= e^{-\rho\tau}P_0 e^{Y_\tau}E\left[e^{\overline{Y}_{\tau(\rho)}}\right]/\kappa\,, \end{aligned}$$

where $\overline{Y}_t = \sup_{s\le t} Y_s$ and $\tau(\rho)$ is an independent exponentially distributed time with parameter ρ. Clearly, $\xi_v^{pm} = P_v/\kappa$ provides the solution to the representation problem (7.5) if and only if $\kappa = (\rho - \Psi(1))\,E\left[e^{\overline{Y}_{\tau(\rho)}}\right]$. ■

According to the above theorem, the shadow revenue of the investment is determined to be the revenue of the investment divided by a constant factor. The optimal investment time can be then rewritten as

$$\tau^* = \inf\{t \ge 0 \mid P_t \ge \kappa I\}.$$

It suggests that the expected revenue at time τ^* satisfies

$$\pi_{\tau^*} = \frac{P_{\tau^*}}{\rho-\Psi(1)} \ge \frac{\kappa}{\rho-\Psi(1)}I = E\left[e^{\overline{Y}_{\tau(\rho)}}\right]I\,,$$

where the expectation term is always larger than 1 as $e^{\overline{Y}_t} \ge e^{Y_0} = 1$ for all $t \in [0,\infty)$. Thus, it gives the following investment rule: an investor undertakes the investment at the first time when the expected revenue reaches or exceeds the investment cost multiplied by a correction factor. Alternatively, we can obtain

$$P_{\tau^*} \ge \frac{\rho}{E\left[e^{\underline{Y}_{\tau(\rho)}}\right]}I$$

after applying the Wiener–Hopf formula. It is a modified Jorgensonian trigger value which includes a risk premium for the marginal revenue product above the Jorgensonian user cost of capital[5] due to the irreversibility and uncertainty. Indeed, it gives the same form

[5] As first defined by Jorgenson (1963), the user cost of capital is the opportunity cost of holding one unit of capital for a period in the standard neoclassical economics. It consists of three components: the financial cost of the capital measured by the discount rate ρ, the depreciation cost δ and the lost gain in the value of that unit of capital $\frac{E[dp_t]}{p_t}$ where p_t denotes the purchasing price of the capital. Therefore, the Jorgensonian user cost of capital is given in this chapter by ρ since the depreciation cost is according to the model construction zero and the investment cost keeps constant over the time.

of the trigger value as in Dixit and Pindyck (1994). In this sense, this new method is more favorable because it generalizes explicit formulae to an exponential Lévy process. This will be addressed below further with specific examples.

The remaining problem is how to solve κ and the value of the option to invest

$$F = E\Big[e^{-\rho\tau^*}(\pi_{\tau^*} - I)^+\Big].$$

Thanks to some mathematical properties of Lévy processes, they can be obtained in analytical form. Moreover, simple explicit formulae are possible for those Lévy processes with only negative jumps, as stated in the following theorem and shown in Appendix C.2.

Theorem 7.3.2. *κ in the threshold value $P_{\tau^*} = \kappa I$ is calculated in explicit formulae:*

(a) In general, $\kappa = (\rho - \Psi(1))\,\Psi^+_\rho(1)$.

(b) For a Lévy process with no positive jumps, $\kappa = (\rho - \Psi(1))\,\frac{\beta^+}{\beta^+-1}$ where β^+ is the unique positive root of the characteristic equation of Y_t, $\Psi(z) = \rho$.

With the knowledge of κ, the value of the option to invest is given as

(a) $F = I\left[E\left[e^{\overline{Y}_{\tau(\rho)}}\right] E\left[e^{-\rho\tau^* + (Y_{\tau^*} - y^*)}\right] - E\left[e^{-\rho\tau^*}\right]\right]$,
where y^* is the value of Y at the time point τ^* and the Laplace transforms of the two expectations are obtained as follows:

$$\int_0^\infty e^{-qy} E\left[e^{-\rho\tau^* + \,(Y_{\tau^*} - y)}\right] dy = \frac{1}{q+1}\left(1 - \frac{\Psi^+_\rho(-q)}{\Psi^+_\rho(1)}\right)$$

and

$$\int_0^\infty e^{-qy} E\left[e^{-\rho\tau^*}\right] dy = \frac{1 - \Psi^+_\rho(-q)}{q}.$$

(b) In particular, $F = \left(E\left[e^{\overline{Y}_{\tau(\rho)}}\right] - 1\right)\left(\frac{P_0}{\kappa}\right)^{\beta^+} I^{1-\beta^+}$ *for any Lévy process Y_t with no positive jumps.*

It is worth noting that $\kappa > 0$ should be always true to make economic sense. It is satisfied whenever

$$E\left[\int_0^\infty e^{-\rho t} P_t\, dt\right] < \infty,$$

which is exactly the condition required for uniform integrability. In particular, it is valid for the GBM case if and only if $\beta^+ > 1$, i.e., $\mu + \frac{1}{2}\sigma^2 < \rho$ where μ and σ are the drift and volatility of the GBM. Intuitively, the expected growth rate of the revenue is bounded from above by the time cost, namely, the discount factor ρ. Otherwise, the discounted payoff is a submartingale and goes to infinity with increasing time. In this sense, the regularity condition coincides with that in Dixit and Pindyck (1994) and with that in Boyarchenko and Levendorskiĭ (2004b) to guarantee that the EPV of the project is finite as time goes to infinity.

Case Studies Three specific examples of the irreversible investment model are provided in this subsection in order to well illustrate this method. These examples are differentiated by the specifications of the output price P_t which nevertheless all belong to the general category of Lévy processes.

Case I. Geometric Brownian Motion: A GBM is most often used in the irreversible investment model in the literature to characterize the uncertainty. Assume that the output price follows a GBM as defined in (6.1). As is well known, a Lévy process pins down to a GBM when the jump component is absent. In this case, a simple and well–known analytical solution for the investment threshold can be easily achieved to be[6]

$$P_{\tau^*} = \kappa I = (\rho - \mu - \frac{1}{2}\sigma^2)\frac{\beta^+}{\beta^+ - 1}I,$$

where β^+ is the positive root of the characteristic equation

$$\frac{1}{2}\sigma^2 z^2 + \mu z - \rho = 0\,.$$

Referring back to the basic model in Dixit and Pindyck (1994), the trigger value of the investment is the investment cost multiplied by a correction factor $\frac{b}{b-1}$, where $b > 1$ is the positive root of the fundamental quadratic equation which coincides with the characteristic equation. Thus, this new method recovers the standard result for the simplest case of GBM.

Figure 7.1 gives the critical κ value for different drifts and volatilities, given $I = 1$, $V_0 = 0.9$ and $\rho = 10\%$. Clearly, κ increases with both μ and σ. As to the volatility, it is argued in the literature that it has two opposite effects on the threshold value: High volatility increases the expected value of overall net profit of the investment; meanwhile it decreases the threshold value due to the high risk involved. Given the parameters in this example, the positive effect dominates the declining one and hence we have a monotone increasing curve/relationship between κ and σ. Moreover, κ responds more greatly to the change in σ when the drift is high. Therefore, one should consider all the parameters and their combined effect as a whole.

Case II. Mixed Jump–Diffusion Process: A mixed jump–diffusion process is a combination of a GBM and a pure jump process characterized by a Poisson process. In this case, the dynamics of the output price are given by

$$P_t = P_0 e^{Y_t} \qquad \text{and} \qquad Y_t = \mu t + \sigma W_t - \eta N_t, \tag{7.10}$$

[6] The techniques for specifying the characteristic equation and Wiener–Hopf factors in these examples are explained in Appendix B.

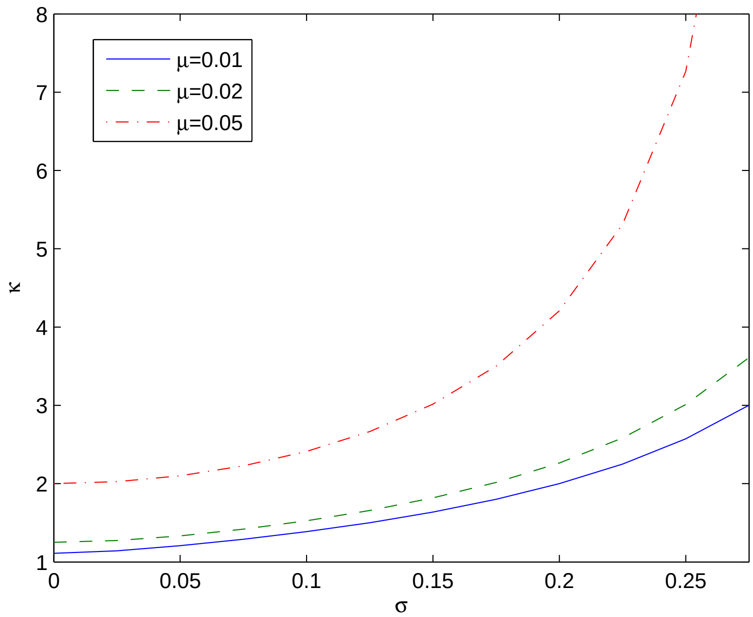

Figure 7.1: Threshold κ Value of a GBM Model

where $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ and $\eta > 0$ combined with the negative sign denotes a constant negative jump size. Under this construction, the project value is a Lévy process with negative jumps only. Following Theorem 3.2, the trigger value can be easily calculated as

$$P_{\tau^*} = \kappa I = \left(\rho - \mu - \frac{1}{2}\sigma^2 - \lambda\left(e^{-\eta} - 1\right)\right)\frac{\beta^+}{\beta^+ - 1}I,$$

where β^+ is the unique positive solution of the characteristic equation $\frac{1}{2}\sigma^2 z^2 + \mu z + \lambda\left(e^{-\eta z} - 1\right) = \rho$.

Obviously, even for a complicated mixed jump–diffusion process, our new method gives an explicit formula to characterize the threshold value and the option to invest, which is not readily derived in Dixit and Pindyck (1994) due to the heavy computation involved in solving a differential equation with one exponential term. Moreover as easily observed in the result, the obtained solution has the same simple form as that in GBM.

In order to show the jump effect, we give a plot showing the relationship of κ and jump parameters λ and η as in Figure 7.2. In case of $\lambda = 0$, the underlying process responds to a GBM. Whatever value η chooses, κ is decreasing in λ, which implies that the resulted threshold values are much lower than those under the construction of the GBM. This result is quite favorable especially to those CEOs who are complaining about higher critical values derived by means of the real options theory (based on a GBM in the standard model). Furthermore, κ declines also with the jump size, η, when fixing a specific value of λ. Such a decreasing impact of jump parameters on threshold values is not surprising since an increase of these parameters for negative jumps not only reduces the expected

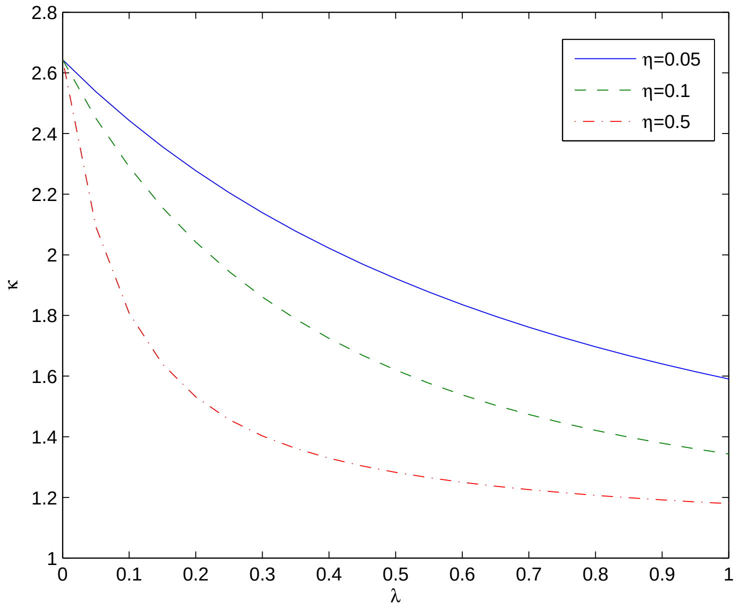

Figure 7.2: Threshold κ Value of a Mixed Jump–Diffusion Process Model with Different Jump Parameters (Parameter Values: $I = 1$, $V_0 = 0.9$, $\rho = 10\%$, $\mu = 0.03$ and $\sigma = 20\%$)

present value of the overall profit but also decreases the threshold value due to more possible (downward) risks or downfalls.

In the following, Figure 7.3 demonstrates how κ changes with σ and λ. It is obvious that σ and λ together define the variance of the underlying uncertainty. As observed, a larger σ and a smaller λ brings a higher threshold value and vice versa. However, there is no distinct dominant effect of one over another on κ. As a result, again their interaction should be considered instead of their separate effects. Moreover, it implies that the parameter choice of GBM and jump components given an estimated variance of the underlying uncertainty (the so–called model misspecification) is so significant, which may lead to a completely different investment decision.

Case III. GBM Combined with a Compound Poisson Process: The price process is modelled as a combination of a GBM and a jump component characterized by a compound Poisson process with random jump sizes. The randomness from the jump component undoubtedly causes complicated computations. As pointed out by Dixit and Pindyck (1994), numerical methods to such cases have to be used when applying the standard real options method. However, explicit formulae can even be found by means of this new method.

Consider the model

$$P_t = P_0 e^{Y_t} \qquad \text{and} \qquad Y_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} J_k, \tag{7.11}$$

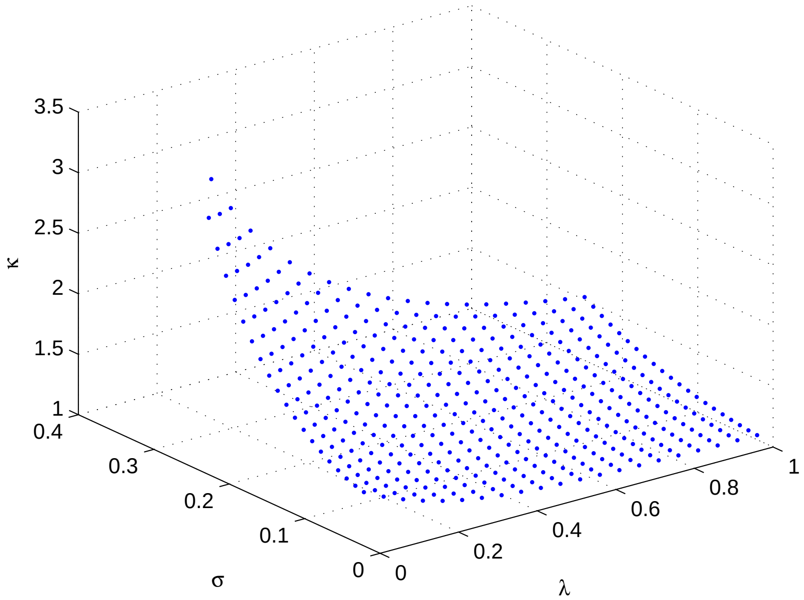

Figure 7.3: Threshold κ Value of a Mixed Jump–Diffusion Process Model vs. Volatility and Jump Intensity (Parameter Values: $I = 1$, $V_0 = 0.9$, $\rho = 10\%$, $\mu = 0.03$ and $\eta = 0.1$)

where $(N_t)_{t\geq 0}$ is again a Poisson process of intensity λ and $J = (J_k)_{k\in\mathbb{N}}$ is a sequence of independent identically distributed random variables with density

$$f(j) = \begin{cases} pc^+e^{-c^+j} & j \geq 0, \\ (1-p)c^-e^{c^-j} & j < 0. \end{cases}$$

where the parameters $c^\pm > 0$ and $0 \leq p \leq 1$. Under this assumption, the project value at time t has in all N_t possible upward and downward jumps which occur with probability p and $1-p$, respectively. Each positive/negative jump is exponentially distributed with the parameter c^+/c^-. This specific model has the Lévy–Laplace exponent

$$\Psi(z) = \mu z + \frac{1}{2}\sigma^2 z^2 + \lambda p \frac{z}{c^+ - z} - \lambda(1-p)\frac{z}{c^- + z}.$$

Accordingly, the optimal investment threshold is determined by solving κ

$$\kappa = \left(\rho - \mu - \frac{1}{2}\sigma^2 - \frac{\lambda p}{c^+ - 1} + \frac{\lambda(1-p)}{c^- + 1}\right)\Psi_\rho^+(1)\,,$$

where the left Wiener–Hopf factor is found to be

$$\Psi_\rho^+(1) = \frac{\beta_1^+}{\beta_1^+ - 1}\frac{\beta_2^+}{\beta_2^+ - 1}\frac{c^+ - 1}{c^+}$$

given the two positive roots $\beta_{1/2}^+$ of the characteristic equation of $\Psi(z) = \rho$.

7.3.2 Explicit Solution Formulae for Cox–Ingersoll–Ross Processes

As often observed empirically, another important feature of the commodity price is mean reversion. Consequently, some time–inhomogeneous diffusion processes are used in the literature for more precise characterization. In such cases, explicit forms of solutions are also possible, although the problem is more computationally involved. To give a clear image, we first consider a specific model of the Cox–Ingersoll–Ross process and then generalize to the general case.

Cox–Ingersoll–Ross (CIR) processes are first proposed and studied by Cox, Ingersoll and Ross (1985) for short interest rates and modelled as follows:

$$dP_t = \gamma(\mu - P_t)dt + \sigma\sqrt{P_t}dW_t\,,$$

where P_t has a long–term mean $\mu > 0$, volatility σ, mean reversion speed $\gamma > 0$ and initial price P_0. As usual, one condition $\gamma\mu - \frac{1}{2}\sigma^2 > 0$ is imposed to maintain that $(P_t)_{t\in[0,\infty)}$ is non–explosive and in particular that the first passage time of 0 is infinite with probability 1.

Under this construction, the solution ξ^{pm} to the representation problem (7.5) admits the form of $\xi^{pm}_t = \kappa(\pi_t)$ where π_t is as previously defined the expected revenue of the project with initial value π_0. The function κ is shown to be

$$\kappa(y) = y - \frac{\varphi_\rho(y)}{\varphi'_\rho(y)} \qquad \forall\, y\, \in (0,\infty)\,,$$

where $\varphi_\rho(y) = {}_1F_1(\frac{\rho}{\gamma}, \frac{2\gamma\mu}{\sigma^2}; \frac{2\gamma}{\sigma^2}y)$ and ${}_1F_1(a,b;x)$ denotes the confluent hypergeometric function[7]. The derivation given in Appendix C.3 in full detail is mainly based on the strong Markov property and the Laplace transform of the first passage time of the CIR process.

Consequently, the optimal investment time can be determined as

$$\tau^* = \inf\left\{t \geq 0 \middle| \pi_t \geq I + \frac{\varphi_\rho(\pi_t)}{\varphi'_\rho(\pi_t)}\right\}.$$

This result coincides with that obtained by the standard real option theory: the critical project value has to cover not only the investment cost but also the option premium of waiting. Moreover in this case, the premium is explicitly specified by the second term $\frac{\varphi_\rho(\pi_{\tau^*})}{\varphi'_\rho(\pi_{\tau^*})}$ which is fully independent of the investment cost. Meanwhile, it can be shown to be always positive with $\lim\limits_{y\to 0}\dfrac{\varphi_\rho(y)}{\varphi'_\rho(y)} = 0$ due to the convexity of $\varphi_\rho(y)$. With the critical

[7] Refer to Abramowitz and Stegun (1969), Chapter 13 for the definition and properties of this function.

project value, the option to invest can be simply calculated as

$$\begin{aligned} F &= E\Big[e^{-\rho\tau^*}(\pi_{\tau^*}-I)^+\Big|\pi_0\Big] \\ &= E\Big[e^{-\rho\tau^*}\Big|\pi_0\Big]\,(\pi_{\tau^*}-I) \\ &= \frac{\varphi_\rho(\pi_0)}{\varphi'_\rho(\pi_{\tau^*})} \end{aligned}$$

In general, the solution and result apply to all time–inhomogeneous diffusion processes

$$d\pi_t = \mu(\pi_t)dt + \sigma(\pi_t)dW_t\,,$$

where $\mu(\pi_t)$ and $\sigma(\pi_t)$ denote the state–dependent drift and volatility, since it is a Markov process and has always the strong Markov property. To each specification, $\varphi_\rho(y)$ has to be derived accordingly and the solution is valid if and only if $\varphi_\rho(y)$ is strictly convex and continuously differentiable.

7.4 Conclusion

The literature treats the irreversible investment decision problem under uncertainty as an option on real assets and solves the optimal stopping problem by means of the contingent claim analysis or the dynamic programming method. Despite its analytical appeal, the real options analysis has yet to take root in practice in the broad–based fashion. One main reason is according to many corporate managers due to the obscure and complication of the standard techniques. In this chapter, we analyze the same real options model but with an alternative approach – the stochastic representation method. This method starts with the EPV of the project, the natural and meaningful definition in economics and represents it in a form of the EPV of the running supremum of another process. By solving the representation problem, the investment decision rule is identified in terms of the *shadow revenue process* such that the investment is initiated at the first moment at which the *shadow net present value* becomes non–negative. The obtained rule is demonstrated to be consistent to that given by the standard real options theory: The critical investment revenue has to be high enough to cover the investment cost plus the option premium of waiting. More importantly, our new method extends and corrects the conventional NPV method by figuring out the *proper* net present value. By doing so, this formulation in terms of EPV gives a clear and intuitive understanding of the investment strategy and then enables a wide application of real options theory in reality.

Compared to the existing standard approaches, this method is advantageous for the applicability to a large class of stochastic processes (all semi–martingale processes) as well as the feasibility of giving an explicit characterization of the solution for an exponential Lévy process and a time–inhomogeneous diffusion process. To illustrate this approach, we consider the irreversible investment decision problem with uncertainty modelled by a fairly flexible family of jump-diffusion processes and mean–reverting CIR processes. It is demonstrated in the paper that the closed–form characterization for exponential Lévy processes is obtained almost as easily as in the Gaussian case by solving the fundamental characteristic equations. Moreover, the result defines the optimal investment timing as the first moment when the underlying project value rises to or exceeds κ times the investment cost, which confirms and generalizes the well–known result in the literature for the case of a GBM. The critical project value for the case of a CIR process is even more economically sensible. It can be decomposed into the investment cost and the opportunity cost of delaying the investment.

In all, the technique used in this work can be applied to many more complicated real option problems. For instance, an investment timing decision problem with the possibility of temporary suspension or abandonment in the later stage when the firm is subjective to poor economic conditions. Following Dixit and Pindyck (1994), this problem can be solved by considering two real option problems (investment and deinvestment) and combining them together under the construction of a compound option. In the next two chapters, we address two extensions on the sequential investments decision problem or the capital expansions problem and the irreversible investment in an incomplete market.

Chapter 8

Sequential Irreversible Investment[1]

8.1 Introduction

A new *shadow* NPV decision rule is derived in the previous chapter. Investment is undertaken if and only if the *shadow* NPV becomes non–negative, i.e., when the *shadow* value from the investment exceeds the investment cost. In order to provide a clear elucidation of the newly–adopted approach, a simple investment model is constructed such that the project is able to be launched ever since the initial investment. However, capacity is usually built up gradually over time rather than once at time. As uncertainty prevents the firm from one–shot investment when taking into consideration the possibility of economic trough. On the other hand, uncertainty also creates new investment opportunities (cf. Henry (1974), Arrow and Fisher (1974)). In this case, the single investment model is relatively restricted and not that relevant to the reality.

This chapter hence develops a general theory of irreversible investment for a firm that sequentially builds up capacity in a risky environment under the constraint that investment into capacity is sunk. This problem is studied by an extensive literature. In the pioneering work, Arrow (1968) deals with the problem of irreversibility under perfect foresight; Pindyck (1988) and Bertola (1988) analyze the benchmark problem of a firm with Cobb–Douglas profit function and stochastic shocks modelled by a GBM. The problem shares close links to the literature on real options and the value of waiting to invest, as emphasized in McDonald and Siegel (1986) and Dixit (1992). Many other authors continue the investigation of the same problem on the basis of their work, e.g., see Davis, Dempster, Sephi and Vermes (1987), Bertola (1998), Kobila (1993), Abel and Eberly (1997), Baldursson and Karatzas (1997), Oksendal (2000), Wang (2003), Chiarolla and Haussmann (2005) and Bank (2005). Recently, the benchmark model has been extended to Markov processes with independent identically distributed increments (Boyarchenko (2004)) and regime shifts (Guo et al. (2005)).

However so far, little work has been done beyond specific classes of models. This thesis

[1] This chapter is based on a joint work with Frank Riedel, c.f. Riedel and Su (2006).

designs to generalize the study of sequential irreversible investment under uncertainty which is free of any distributional or parametric assumptions. Throughout the work, we consider a profit–maximizing single firm which chooses a dynamic capacity expansion plan in a risky environment. The operating profit function depends on the current capacity of the firm and a stochastic process that models the uncertainty. In this way, the model covers not only all the previously studied models in economics but also the standard finance model where the uncertainty is usually specified by a semimartingale process. Based upon the Stochastic Representation method that is originally invented for utility maximization problems in Bank and Riedel (2001b), we develop a qualitative theory of irreversible investment that allows characterization of the investment behavior for any type of profit function and general stochastic processes. Furthermore, general monotone comparative statics is established for the relevant parameters of the model.

First, to have a sound foundation for our theory, a general existence and uniqueness theorem is developed, which is not yet available in the literature. Uniqueness of the optimal policy is easy as usual, given a maximization problem of a strict concave functional. For the proof of existence, the optimal investment policy under perfect reversibility is taken as a benchmark case. As is well known, a firm in this case equates the marginal operating profit with the user cost of capital at all times. It is reasonable and necessary to assume that the problem under reversibility is finite, which in turn guarantees the well–posedness of the irreversible investment problem. On this basis, the existence result is obtained by further assuming that the running maximum of the optimal frictionless policy is integrable. This assumption is required to show that all sensible investment policies are bounded by the running maximum of the optimal frictionless policy. With this integrable upper bound, Komlos' Theorem can be used as a substitute for the lack of compactness in the infinite–dimensional space to identify a candidate optimal policy. Generally, it is impossible to relax our assumptions as the constructed model includes the setup where the optimal policies under reversibility and irreversibility coincide.

Moving on, we study the explicit construction of the optimal investment policy. As the starting point, the first–order condition is derived as done in Bertola (1998)[2]. In contrast to the frictionless model where only the immediate marginal operating profit comes into effect, all the changes in future marginal operating profits due to the current investment have to be taken into account. Consequently in case of irreversibility, the marginal gross profit from the current investment is given by the properly discounted expected present value of future marginal operating profits. The firm aims then to keep it below the cost of current investment at all times. In Bertola's explicit model, it is sufficient to verify the first–order condition by guessing the optimal policy. Nevertheless due to irreversibility, the first–order condition is frequently not binding and hence can not be used to obtain solutions in general. To overcome this difficulty, we borrow an approach which is well known in inventory theory and make the ansatz that the optimal policy is going to be a so–called *base capacity policy*: there exists a base capacity (l_t), a stochastic process indicating the optimal capacity level the firm would like to have if it started with zero

[2]The first–order approach has recently seen a revival in other contexts as well, see Chow (1997), e.g.

capacity at that point in time. The optimal policy is then to expand firm's capacity to the base capacity level if the current capacity is lower, or otherwise to maintain the current level. This base capacity is in this thesis first characterized by a stochastic backward equation, which has been studied in other contexts before: Bank and Riedel (2001a) in the framework of intertemporal utility functions with memory, Bank and Föllmer (2003) and El Karoui and Karatzas (1994) for optimal stopping problems; and a general study of the mathematical properties of this equation is clarified in Bank and El Karoui (2004). As this backward equation can always be solved numerically via backward induction, the irreversible investment problem is completely solved.

In addition to the backward equation, we show that the base capacity can also be characterized via a family of optimal stopping problems. This formalizes in a rigorous way the approach taken by Pindyck (1988) who solves the irreversible investment problem by considering a continuum of American options for the next marginal investment. Starting from the first–order condition, we construct auxiliary levels L_t^τ. These numbers would be the optimal capacity level if it were optimal to invest at time t, wait until the next (possible) investment time τ. It is easy to see that these levels are chosen such that the discounted expected difference between the marginal operating profit and the user cost of capital equals zero. It is then shown that the optimal base capacity l_t is the lower envelope of all these auxiliary levels L_t^τ. The firm thus solves at any point in time an optimal stopping problem that determines the next time of investment.

The auxiliary levels L_t^τ are very useful, because one can infer properties of the optimal investment policy from those of the auxiliary levels. The auxiliary levels solve a simple equation and hence can be easily handled. As a first application, they are used to give a general qualitative characterization of the optimal policy. Following Arrow (1968), we distinguish free and blocked intervals. In a free interval, the firm invests in an absolutely continuous way at strictly positive rates[3]. It is shown that in free intervals the firm always equates the marginal operating profit with the user cost of capital. In this sense, it generalizes Arrow's result for the benchmark case of the frictionless world to the stochastic model. During a blocked interval, no investment occurs as the firm has excess capacity from the past. Using our construction of the auxiliary levels, it follows immediately that the marginal operating profit is equal to the user cost of capital only in expectation on average over time.

Whenever uncertainty is generated by a diffusion, the optimal policy is going to be related to the running maximum of another diffusion. Therefore, investment will be generally singular with respect to Lebesgue measure. This means that positive investment occurs on a set of Lebesgue measure zero. Peculiar as this might seem, it is the well–known standard case in a Brownian model. Diffusions oscillate in such an irregular way that highly irregular action patterns have to be taken to keep the processes below a certain boundary.

[3]Note that diffusion models which are usually studied do not have such free intervals. However, it is perfectly natural to consider other stochastic processes, such as compound Poisson processes. In this case, free intervals exist as shown by the specific examples in Section 8.6.

Under perfect foresight, Arrow shows that the firm usually invests in lumps at time zero to boost the firm's capacity to a good level. Then, no lump sum investments occur afterwards as the firm anticipates the future changes and adjusts capacity in a smooth way. This is essentially due to the fact that Arrow's model is continuous in the sense that his parameters – interest rates and the profit function — are continuous. With stochastic jumps, lump sum investments may be the optimal response to shocks. If demand shocks are generated, e.g., by a Poisson process, then it is optimal to respond to a (favorable) Poisson jump by a lump sum investment. On the other hand, for a Poisson process, the probability of a jump at a fixed time t is zero, and this carries over to more general classes of processes. Inspired by this, we say that the model has no fixed surprise at time t if the information flow is continuous at t and the probability of a jump in demand is zero. We show that at fixed times with no surprise, the optimal policy has no lump sum investment. In addition, we prove that whenever a lump sum investment happens as a reaction to an information surprise, the capacity never jumps to "excess" capacity with respect to the operating profit. Thus, the firm remains cautious in the sense that it usually invests less than it would in a frictionless environment.

Furthermore, our new approach allows for the first results on general monotone comparative statics of the optimal investment. The auxiliary levels L_t^τ form the building block for these results which are determined by applying implicit differentiation of the simple equation of the auxiliary levels. Following the methods and ideas from Topkis (1978) and Milgrom and Shannon (1994), we establish that the base capacity is monotonically increasing in the exogenous shock process when the operating profit function is supermodular, or equivalently, exhibits increasing differences in capacity and exogenous economic shock. To our knowledge, this is the first result in monotone comparative statics which takes a whole stochastic process as a parameter. Another two significant parameters of the model are depreciation and interest rate. In general, no monotone comparative statics hold true for any one of them alone. Instead, their sum, the user cost of capital, is the right quantity to study and we demonstrate that investment is decreasing in the user cost of capital.

Generally, numerical methods have to be used to identify the base capacity according to the algorithm given in the work. Nevertheless, closed–form solutions of the optimal investment policy are possible for an infinite time horizon, separable operating profit functions of Cobb–Douglas type and shocks specified by an exponential of a Markov process with independent identically distributed increments, namely, an exponential Lévy process. We show how to recover the results of Pindyck (1988), Bertola (1998), and Boyarchenko (2004) with our method. Under this construction, the base capacity is given by the exogenous economic shock multiplied by a constant factor expressed in terms of expectation. In this way, the marginal profit under the optimal investment plan is always kept below the user cost of capital times a markup factor.

The remainder of this chapter is organized as follows. Section 8.2 presents the general model and derives the uniqueness and existence theorem. The heuristics and explicit construction of the auxiliary levels and the base capacity are provided in Section 8.3. Section 8.4 characterizes the optimal policy and Section 8.5 gives general comparative statics results for the irreversible investment problem. Explicit solutions are derived in Section 8.6 for the case that the firm is facing a Cobb–Douglas operating profit function and is subject to a multiplicative economic shock modelled by an exponential Lévy process. Finally, Section 8.7 concludes the paper with a short summary and remark.

8.2 Irreversible Investment Model

To develop the sequential irreversible investment theory, a general model is first constructed where a single profit–maximizing firm chooses a dynamic capacity expansion plan in an uncertain environment. This setup encompasses all existing models in the literature. Then, this section moves on to the investigation of existence and uniqueness of the optimal investment strategy.

8.2.1 Irreversible Investment: A General Model

Consider a firm that chooses a dynamic capacity expansion plan over a time horizon $\hat{T} \leq \infty$ which can be finite or infinite. The operating profit flow of the firm is assumed to be summarized by a function $\pi\left(X_t, C_t\right)$ of current capacity C_t and some exogenous *state variable* X_t with values in some complete metric space[4] E. X_t can be regarded as an economic shock, reflecting the changes in, e.g., technologically feasible output, demand and macroeconomic conditions and so on, which have direct or indirect effects on the firm's profit. The stochastic process $(X_t)_{t\in[0,\hat{T}]}$ is formally defined on some underlying filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t,\ 0 \leq t \leq \hat{T}), \mathbb{P})$ with an information filtration $(\mathcal{F}_t)_{0\leq t\leq \hat{T}}$ satisfying the usual conditions of completeness, i.e., $\mathcal{F}_0$ contains all the $\mathbb{P}$-null set of $\mathcal{F}$ and $\mathbb{F}$ is right continuous. In addition, X_t is known at time t, or formally, the process $(X_t)_{t\in[0,\hat{T}]}$ is progressively measurable w.r.t. $\mathcal{F}_t$. Suppose in addition that $\pi : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing and strictly concave in capacity C with derivative $\pi_c\left(x, c\right)$ that satisfies the Inada conditions

$$\lim_{c\to 0} \pi_c\left(x, c\right) = \infty$$

and

$$\lim_{c\to\infty} \pi_c\left(x, c\right) = 0$$

for all $x \in \mathbb{R}$. Moreover, there are no costs as long as no investment has been made, namely, $\pi(0) = 0$. As in Arrow (1968), the price of capital goods used to build up capacity is taken as the numéraire. Thus, the cost of investment is always 1 and the short–term interest rate at time t, r_t, is expressed in terms of capital goods not money[5]. Formally,

[4] Such an assumption can be justified by a microeconomic foundation, see e.g. Bertola (1998).

[5] This assumption is not restrictive. Indeed, it can be always achieved by a change of numéraire if, e.g., the price of capital goods is a bounded semimartingale.

$(r_t)_{t\in[0,\hat{T}]}$ is a nonnegative bounded optional process.

Given the operating profit, the firm chooses a plan $I = (I_t)_{t\in[0,\hat{T}]}$ of cumulative investments, a right–continuous adapted process. The initial investment $I_0 > 0$ indicates the size of the lump sum investment at time 0. As investment is irreversible, I has to be nondecreasing. The investment plan leads to a capacity $C^I = (C_t^I)_{t\in[0,\hat{T}]}$ that starts in $C_{0-}^I = 0$[6] and evolves according to the differential equation

$$dC_t^I = dI_t - \delta_t C_t^I dt \tag{8.1}$$

for some *depreciation rate* $(\delta_t)_{t\in[0,\hat{T}]} \geq 0$, a nonnegative bounded optional process. An investment plan I is *admissible* if its net present value is finite, i.e.,

$$E\left[\int_0^{\hat{T}} e^{-\int_0^t r_s ds} dI_t\right] < \infty\,.$$

In this context, the firm maximizes the expected present value of the future overall net profits

$$\Pi(I) = E\left[\int_0^{\hat{T}} e^{-\int_0^t r_s ds}\Big(\pi\left(X_t, C_t^I\right)\, dt - dI_t\Big)\right] \tag{8.2}$$

over all admissible plans I. The gross profit is for the future use explicitly written as

$$G(I) = E\left[\int_0^{\hat{T}} e^{-\int_0^t r_s ds}\pi\left(X_t, C_t^I\right)\, dt\right]\,.$$

Note that the net profit $\Pi(I)$ is well defined for all admissible plans but potentially infinite. In the next subsection, some conditions are given that ensure finiteness.

Before solving the sequential irreversible investment decision problem, we show that all models studied so far in the literature are included in our setup.

Example 8.2.1. *The general setup includes the deterministic case with an arbitrary deterministic interest rate r and the operating profit $\pi(t, C_t)$ (Here, time is the state variable, i.e., $X_t = t$). This case has been fully analyzed by Arrow (1968) in complete generality by using the calculus of variations, in particular Pontryagin's principle.*

Example 8.2.2. *The best studied special case under uncertainty has a separable operating profit function $\pi(x, c) = e^x\pi(c)$ and an infinite time horizon. Bertola (1998) and Pindyck (1988) take X as a BM with drift,*

$$X_t = \mu t + \sigma W_t, \tag{8.3}$$

[6]It is assumed here that the firm does not come into being with past capacity. All results hold true, however, with some initial capacity $C_{0-} > 0$. We distinguish between time $0-$ and 0 because a lump sum investment usually occurs at time 0 of size I_0 and brings the capacity $C_0 = C_{0-} + I_0$ at time 0.

where W_t is the standard Wiener process, μ and σ are the constant drift and volatility, respectively. Moreover, they assume a constant interest rate $r_t = r$, $\forall\, t \in [0, \infty)$, and a Cobb–Douglas operating profit function $\pi(c) = \frac{1}{1-\alpha} c^{1-\alpha}$ for some $0 < \alpha < 1$. Boyarchenko (2004) allows X to be a Lévy process, a Markov process with i.i.d. increments. An interesting extension concerns regime shifts where the parameters of the BM switch between different states according to a continuous–time Markov chain, see Guo et al. (2005). Kobila (1993) presents the general dynamic programming approach for nonseparable operating profit and diffusion state variables.

8.2.2 Existence and Uniqueness Theorem

Although a number of explicit case studies have been carried out, no general existence and uniqueness theorem is available in the literature. The present subsection provides sufficient conditions that ensure existence and uniqueness of a solution for the case of a finite horizon. Those for an infinite horizon are given in Appendix D.1.

Take an auxiliary function as the starting point. Due to the assumptions of the operating profit function π, the *indirect profit function*

$$\pi^*(x, r, \delta) = \max_{c \geq 0} \; \pi\,(x, c) - (r + \delta)c \tag{8.4}$$

exists for fixed parameters $x, r, \delta \in \mathbb{R}$. The unique maximizer denoted by $c^*(x, r, \delta)$ solves the first–order condition

$$\pi_c\,(x, c) = r + \delta\,.$$

Remark 8.2.3. *This auxiliary function describes the optimal investment under perfect reversibility, the so–called myopic decision rule. In this case, the marginal operating profit has to be equal to the* user cost of capital[7]*, which is given in this chapter by the sum of interest and depreciation rate, $r+\delta$. Compare this result with the discussion on the optimal capacity with perfect reversibility in Section* 8.3.

The following two conditions are imposed for existence and uniqueness of the optimal policies. We assume that the reversible investment problem has a finite value and that the overall maximum of optimal reversible capacity is integrable.

Assumption 8.2.4. *(i)* $E\Big[\pi^*\,(X_t, r_t, \delta_t)\Big] < \infty, \quad \forall\, t \in [0,\, \hat{T}]$*;*

(ii) $K \triangleq E\Big[\sup_{t \leq \hat{T}} \; c^*\,(X_t, r_t, \delta_t)\Big] < \infty\,.$

[7]As first defined by (Jorgenson 1963), the user cost of capital is the opportunity cost of holding one unit of capital for a period in the standard neoclassical economics . It consists of three components: the financial cost of the capital measured by the interest rate r, the depreciation cost δ and the lost gain in the value of that unit of capital $\frac{E[dP_t]}{P_t}$ where P_t denotes the purchasing price of the capital.

The example below shows that these assumptions hold true generally for the widely studied separable operating profit function and Lévy processes with bounded positive jumps.

Example 8.2.5. *The benchmark example in the literature has the operating profit function* $\pi(x,c) = e^x \frac{c^{1-\alpha}}{1-\alpha}$ *with a constant parameter* $\alpha > 0$*. Assumption* (8.2.4) *is satisfied, if* X *is a BM as defined in Equation* (8.3)*. More generally, Assumption* (8.2.4) *holds true for Lévy processes with bounded positive jumps.*

Proof: The maximizer of (8.4) is obtained as

$$c^*(x,r,\delta) = \frac{1}{(r+\delta)^{\frac{1}{\alpha}}} e^{\frac{x}{\alpha}}.$$

It gives then the optimal indirect operating profit

$$\pi^*(x,r,\delta) = \frac{1}{1-\alpha}(r+\delta)^{\frac{\alpha-1}{\alpha}} e^{\frac{x}{\alpha}}.$$

Thus, it is enough to show that $Z_t = \sup_{t \leq \hat{T}} X_t$ satisfies $E\left[e^{\lambda Z_t}\right] < \infty$ for all positive $\lambda > 0$. This always holds true for BM and more generally for Lévy processes with bounded positive jumps (see, e.g., Bertoin (1996), Chapter VII). ■

Given these two assumptions, we are now ready to state and discuss the general existence and uniqueness theorem.

Theorem 8.2.6. *Under Assumption* (8.2.4)*, there always exists a unique optimal irreversible investment plan* I^**.*

The proof of Theorem 8.2.6 is given in full detail in Appendix D.1. To briefly sketch the idea: We have a maximization problem of a concave functional (8.2). In this case, uniqueness is easy due to the strict concavity of the objective function; and existence is usually achieved by continuity and some compactness (subsequence) principle. Given our assumptions, continuity of the profit functional is obtained through dominated convergence. We then restrict our attention only to the investment policies whose corresponding capacity stays below the overall maximal capacity under perfect reversibility. Nevertheless, this restriction does not exclude any promising policies as the firm in general would like to invest less under irreversibility. By Assumption 8.2.4(ii), an integrable upper bound is achieved for all investment policies. It allows one to use *Komlos' theorem* which seems not to be widely used in Economics although it is as Taylor–made for many optimization problems. Komlos' Theorem states that a sequence of random variables $(Z_n)_{n\in\mathbb{N}}$ which is bounded from above in expectation has a subsequence (ζ_n) converging in the sense of the Law of Large Numbers to some random variable ξ. Here, we use a version of Komlos' theorem for increasing processes which is proven in Kabanov (1999) and Balder (1990). The limit identified by Komlos' Theorem turns out to be an optimal policy.

Remark 8.2.7. *In general, Assumption* 8.2.4(i) *and* (ii) *are necessary for existence. To see this, consider the case in which the irreversibility constraint is never binding and the firm invests all the time. In this case, the optimal policy under reversibility also solves the investment problem under irreversibility. Thus, the irreversible problem is well–posed whenever the reversible case is. When there is no depreciation, the overall maximum of capacity is equal to the total investment* $I_{\hat{T}}$. *Then, this policy is admissible if and only if Assumption* 8.2.4(ii) *is satisfied. It follows that Assumption* 8.2.4 *cannot be weakened in general.*

The impossibility of relaxing these two assumptions can also be verified through explicit examples as follows. Suppose $\pi(x,c) = 2x\sqrt{c}$ *with a constant interest rate* $r > 0$ *and zero depreciation rate, i.e.,* $\delta = 0$. *Moreover,* X_t *is modelled such that* X_t *is a strictly increasing stochastic process (e.g.* $X_t = e^{N_t}$ *where* N_t *is a compound Poisson process with positive jumps). Here, the optimal policy in the reversible case is given by (compare our discussion in Section 8.3)*

$$X_t \frac{1}{\sqrt{C_t}} = r\,,$$

or

$$I_t = C_t = X_t^2/r^2\,.$$

Hence, the optimal investment policy under reversible investment is strictly increasing. In this case, an optimal policy under irreversibility exists if and only if Assumption 8.2.4(i) *and* (ii) *are satisfied.*

8.3 Optimal Irreversible Investment Policies

Having established the existence and uniqueness of optimal policies, we are now going to find their explicit construction. For comparison purposes, the optimal investment rule is first briefly introduced when investment is perfectly reversible. Then, we develop the base capacity rule for the investment problem with complete irreversibility and show how to characterize the *base capacity* from the derived first–order condition. Basically, the optimal policy keeps the actual capacity above the *base capacity* in a minimal way. This defined *base capacity* is finally identified as the unique solution to a backward equation that can be numerically solved by backward induction for any general model setup.

Reversible Investment Policies If investment is perfectly reversible, the firm can adjust capacity by selling and purchasing the capital goods freely at every point of time. As is well known (Jorgenson (1963), see also Arrow (1968)), the optimal investment criterion is to equate the marginal operating profit with the user cost of capital, i.e.,

$$\pi_c\left(X_t, C_t^I\right) = r + \delta\,. \tag{8.5}$$

The optimal investment plan has a special "myopic" property in the sense that future expected marginal profits do not appear in Equation (8.5). The firm equates only the immediate marginal operating profit from capacity with the cost of renting a further marginal unit for an infinitesimal period. This cost is given by the interest rate augmented by the cost of replacing the depreciated amount of capacity. However, this does not mean that the firm is myopic. The optimal plan does not consider future marginal profits since the firm can resize its capacity in any desired way by purchasing or selling the capital.

Once investment is irreversible, the optimal investment plan is no longer of myopic nature as today's investment cannot be abandoned later on. The marginal gain from investment is therefore going to be a functional of all future marginal operating profits created by today's investment. To keep the notation simple, the interest and discount rate are assumed from now on to be constant. Nevertheless, the argument is valid for stochastic interest and discount rates as well.

Necessary Optimality Conditions under Irreversibility Before constructing the optimal policy, it is worth to note that some degree of integrability has to be imposed on the process X. Meanwhile, the following inequality is assumed to be true for all values $L > 0$

$$E\left[\int_0^{\hat{T}} e^{-(r+\delta)s}\pi_c\left(X_s, Le^{-\delta s}\right) ds\right] < \infty\,.$$

At any time, installation of any infinitesimal unit of capital will create a stream of marginal profits. At optimum, the marginal gross profit from investing has to be lower than or equal to the cost of investing. The investment cost at t discounted back to the initial time is e^{-rt}. Denote the marginal gross profit at time t following the investment plan I after discounting by $G_t'(I)$. Then, the necessary optimality conditions are given by

$$G_t'(I) \le e^{-rt} \qquad \text{for all times } t \le \hat{T} \tag{8.6}$$

and

$$G_t'(I) = e^{-rt} \qquad \text{whenever } dI_t > 0\,. \tag{8.7}$$

Conditions (8.6) and (8.7) can also be interpreted as the Kuhn–Tucker conditions for the optimality problem (8.2) with an inequality constraint $dI_t \ge 0$.

Marginal Gross Profit The marginal investment at time t first induces an immediate marginal gain of $\pi_c\left(X_t, C_t^I\right)$. As capital accumulation is irreversible, all future profits are increased marginally by

$$\pi_c\left(X_s, C_s^I\right) e^{-\delta(s-t)} \quad \forall\, s \in [t, \hat{T}]\,,$$

where the discount factor $e^{-\delta(s-t)}$ is due to the depreciation of current capital stocks[8]. This marginal gain has to be discounted by the interest rate as well to the initial date

[8] In the case of reversible investment, there is no such effect on future profits because earlier investments can be withdrawn at any time. Thus, it is sufficient to consider the marginal gain at present time t only.

0. Overall, the expected marginal gross profit conditional on the information at time t is given by

$$
\begin{aligned}
G'_t(I) &= E\left[\int_t^{\hat{T}} e^{-rs}\pi_c\left(X_s, C_s^I\right)e^{-\delta(s-t)}ds \,\middle|\, \mathcal{F}_t\right] \\
&= e^{\delta t}E\left[\int_t^{\hat{T}} e^{-(r+\delta)s}\pi_c\left(X_s, C_s^I\right)ds \,\middle|\, \mathcal{F}_t\right]. \qquad (8.8)
\end{aligned}
$$

Remark 8.3.1. *(a) Equation* (8.8) *is used by Bertola (1998) to check the optimality of certain policies. The heuristics that lead to* (8.8) *can also be made rigorous, see e.g. Duffie and Skiadas (1994) or Bank and Riedel (2001b) in the context of intertemporal utility maximization.*

(b) Assume for the moment that the firm is infinitely lived with $\hat{T} = \infty$. The first–order condition can be reformulated as

$$
E\left[\int_t^{\infty} e^{-(r+\delta)s}\left[\pi_c\left(X_s, C_s^I\right) - (r+\delta)\right]ds \,\middle|\, \mathcal{F}_t\right] \leq 0\,,
$$

after multiplying $e^{-\delta t}$ at the both sides of Inequality (8.6) *and rewriting $e^{-(r+\delta)t} = \int_t^\infty (r+\delta)e^{-(r+\delta)s}ds$. In the reversible case, the integrand at the left–hand side is always equal to zero, as the marginal operating profit is always equal to the user cost of capital, $r+\delta$. In the irreversible case, the firm however aims to achieve the equality of the marginal operating profit and the user cost of capital only* in expectation on average in time. *The inequality becomes strict when capacity is excess at time t.*

The Base Capacity Generally, the first–order condition is not that helpful for finding the solution as it is not binding so frequently. Nevertheless, it is of great use for constructing a *base capacity* $(l_t)_{t\in[0,\hat{T}]}$, the capacity level that a firm would choose if it were about to start operating at time t regardless of the past capacity. In the following, we aim to show that the optimal policy is to keep the capacity above the base capacity in a minimal way. As a result, the firm does not invest if current capacity is above the base capacity; and does invest up to the base capacity level if current capacity is below the base capacity[9].

Suppose that the firm follows the optimal investment plan: invest at some (random stopping) time τ_0, wait for a while till $\tau_1 > \tau_0$ and invest again. In this case, the first–order condition is binding at both times, namely,

$$
G'_{\tau_0}(I) = e^{-r\tau_0} \qquad \text{and} \qquad G'_{\tau_1}(I) = e^{-r\tau_1}\,.
$$

[9]Such a policy is well known in operations research, especially inventory theory, see Porteus (1990) for instance.

Multiplying both equations with $e^{-\delta\tau_i}$, $i=0,1$, respectively and subtracting them from each another yields

$$E\left[\int_{\tau_0}^{\tau_1} e^{-(r+\delta)s}\pi_c\left(X_s, C_s^I\right)ds \,\middle|\, \mathcal{F}_{\tau_0}\right] = E\left[e^{-(r+\delta)\tau_0} - e^{-(r+\delta)\tau_1} \,\middle|\, \mathcal{F}_{\tau_0}\right],$$

where the conditional expectation is taken with respect to the information available at time τ_0. The conditional expectation appears also at the right–hand side because τ_1 is generally random. Upon realizing that the difference on the right–hand side can be written as $\int_{\tau_0}^{\tau_1}(r+\delta)e^{-(r+\delta)s}ds$, one arrives at

$$E\left[\int_{\tau_0}^{\tau_1} e^{-(r+\delta)s}\left[\pi_c\left(X_s, C_s^I\right) - (r+\delta)\right]ds \,\middle|\, \mathcal{F}_{\tau_0}\right] = 0\,.$$

As no investment occurs between τ_0 and τ_1, the capacity starts at some level L at time τ_0 and depreciates at the rate δ, i.e.,

$$C_s^I = Le^{-\delta(s-\tau_0)}$$

for $s\in(\tau_0,\tau_1)$. By plugging this back into the equation above one arrives at

$$E\left[\int_{\tau_0}^{\tau_1} e^{-(r+\delta)s}\left[\pi_c\left(X_s, Le^{-\delta(s-\tau_0)}\right) - (r+\delta)\right]ds \,\middle|\, \mathcal{F}_{\tau_0}\right] = 0\,. \quad (8.9)$$

This equation has a unique solution $L_{\tau_0}^{\tau_1}$, a $\mathcal{F}_{\tau_0}$–measurable random variable[10].

The level $L_{\tau_0}^{\tau_1}$ will be the optimal capacity at time τ_0 if a *blocked interval*[11] starts at time τ_0. In general, the firm asks at time τ_0: when and how much should be invested (marginally or in lumps) next time? Taking the whole variety of possible levels $\left(L_{\tau_0}^{\tau_1}\right)_{\tau_1>\tau_0}$ and the irreversibility constraint into consideration, the *lowest* level

$$l_{\tau_0} = \operatorname*{ess\,inf}_{\tau_1>\tau_0} L_{\tau_0}^{\tau_1} \quad (8.10)$$

is defined as the *base capacity*, indicating the optimal capacity to hold at τ_0.

Remark 8.3.2. *One might wonder why the firm would like to take the smallest of all auxiliary levels L_t^τ. The reasoning is given in the following way. Suppose that current capacity exceeds some $L_{\tau_0}^{\tau_1}$ and assume $\delta=0$ for simplicity. From irreversibility, it is clear that $C_s > L_{\tau_0}^{\tau_1}$ for all times $s\in(\tau_0,\tau_1)$. By the definition of $L_{\tau_0}^{\tau_1}$, one obtains*

$$\begin{aligned} E\left[\int_{\tau_0}^{\tau_1} e^{-rs}\pi_c\left(X_s, C_s\right)ds \,\middle|\, \mathcal{F}_{\tau_0}\right] &< E\left[\int_{\tau_0}^{\tau_1} e^{-rs}\pi_c\left(X_s, L_{\tau_0}^{\tau_1}\right)ds \,\middle|\, \mathcal{F}_{\tau_0}\right] \\ &= E\left[e^{-r\tau_0} - e^{-r\tau_1} \,\middle|\, \mathcal{F}_{\tau_0}\right]\,. \end{aligned}$$

[10]The derivation given here is heuristic. Thus, the proof is not provided for the uniqueness of the solution to this implicit equation. This argument can be made rigorous however by considering that the marginal operating profit π_c is strictly decreasing to 0 in capacity.

[11]Please refer to the full discussion in Section 8.4.

It follows that

$$\begin{aligned} G'_{\tau_0}(I) &= E\left[\int_{\tau_0}^{\tau_1} e^{-rs}\pi_c\left(X_s, C_s\right) ds \,\middle|\, \mathcal{F}_{\tau_0}\right] + E\left[\int_{\tau_1}^{\hat{T}} e^{-rs}\pi_c\left(X_s, C_s\right) ds \,\middle|\, \mathcal{F}_{\tau_0}\right] \\ &< E\left[e^{-r\tau_0} - e^{-r\tau_1} \,\middle|\, \mathcal{F}_{\tau_0}\right] + E\left[\int_{\tau_1}^{\hat{T}} e^{-rs}\pi_c\left(X_s, C_s\right) ds \,\middle|\, \mathcal{F}_{\tau_0}\right] \\ &= E\left[e^{-r\tau_0} - e^{-r\tau_1} \,\middle|\, \mathcal{F}_{\tau_0}\right] + G'_{\tau_1}(I) \le e^{-r\tau_0}\,, \end{aligned}$$

where the first–order constraint is used in the last line. Thus, the necessary condition for investment at time τ_0 is that the current capacity has to be always less than or equal to all levels $L_{\tau_0}^{\tau_1}$ for $\tau_1 > \tau_0$, justifying the infimum in our definition of the base capacity.

Characterization of the Optimal Investment Policy: Tracking the Base Capacity Generally, the base capacity l is a widely fluctuating stochastic process. Irreversibility prevents the firm from exactly matching the base capacity at all times, e.g., when downward jumps occur or when the base capacity decreases at a higher rate than δ or when the base capacity decreases in a non–differentiable way as is typical for diffusion models. Therefore, a feasible capacity process C_t has to be found out that tracks the base capacity as closely as possible. According to the base capacity policy, $C_t \ge l_t$ has to hold in a minimal way at all times. Consequently, the correct means is to look for the smallest feasible capacity that dominates the base capacity.

If there is no depreciation, i.e., $\delta = 0$, C must be a nondecreasing process. That is, $C_t \ge C_s$ for $t > s$. Meanwhile, in accordance with the requirement $C_s \ge l_s$, $C_t \ge l_s$ always holds for $s \le t$, or equivalently,

$$C_t \ge \sup_{s \le t} l_s\,.$$

Being the running maximum of the base capacities, $\sup_{s\le t} l_s$ *is* surely a nondecreasing process, and hence can be a feasible capacity. Therefore, the running maximum

$$C_t = \sup_{s \le t} l_s$$

is the smallest feasible capacity that dominates the base capacity. For the general case ($\delta > 0$), it is better to study the nondecreasing process $A_t = C_t e^{\delta t}$. By the same reasoning as above, one shows that A has to satisfy the relationship

$$A_t = \sup_{s \le t}\left(l_s e^{\delta s}\right)\,. \tag{8.11}$$

The feasible capacity becomes then

$$C_t = e^{-\delta t} \sup_{s \le t}\left(l_s e^{\delta s}\right)\,.$$

In the case of no depreciation, the corresponding investment plan is trivially obtained as $C^I = I$. In general, one can derive the investment plan from Equation (8.1), namely, $dI_t = dC_t^I + \delta C_t^I dt$. All these findings are summarized in the following definition.

Definition 8.3.3. *For a given optional process l and depreciation rate $\delta \geq 0$,*

$$C_t^{l,\delta} = e^{-\delta t} \sup_{s \leq t} \left(l_s e^{\delta s} \right) \tag{8.12}$$

is the capacity that tracks l at depreciation rate δ. *The investment plan that finances $C^{l,\delta}$ is denoted by $I^{l,\delta}$ and satisfies*

$$I_0^{l,\delta} = l_0 \quad \text{and} \quad dI_t^{l,\delta} = dC_t^{l,\delta} + \delta C_t^{l,\delta} dt \,.$$

If l is the base capacity as defined in Equation (8.10)*, we call $I^{l,\delta}$ the* base capacity policy with depreciation rate δ.

Remark 8.3.4. *Note that the capacity that tracks the base capacity satisfies*

$$dC_t^{l,\delta} = -\delta C_t^{l,\delta} dt + e^{-\delta t} dA_t^{l,\delta} \,,$$

where $A^{l,\delta}$ is given by (8.11)*. It follows that*

$$dI_t^{l,\delta} = e^{-\delta t} dA_t^{l,\delta} \,. \tag{8.13}$$

As a result, investment takes place if and only if the process $A^{l,\delta}$ increases; this in turn happens whenever the process $(l_s e^{\delta s})$ reaches a new all time high.

Stochastic Backward Equation and Optimality of the Base Capacity Policy It remains to be shown that the constructed base capacity policy is indeed optimal. To this end, an equation is achieved to determine the base capacity via backward induction. This equation is very similar to the first–order condition, but has the advantage of being an equality at all times almost surely. It is thus extremely useful for explicit computations and for qualitative assessments in subsequent sections.

The capacity $C_s^{l,\delta}$ at time $s > \tau$ created by the base capacity policy can be rewritten as

$$C_s^{l,\delta} = e^{-\delta s} \sup_{0 \leq u \leq s} l_u e^{\delta u} = e^{-\delta s} \max \left\{ \sup_{0 \leq u \leq \tau} l_u e^{\delta u}, \sup_{\tau \leq u \leq s} l_u e^{\delta u} \right\} .$$

Plugging it into the first–order inequality yields then

$$E \left[\int_\tau^{\hat{T}} e^{-(r+\delta)s} \pi_c \left(X_s, e^{-\delta s} \max \left\{ \sup_{0 \leq u \leq \tau} l_u e^{\delta u}, \sup_{\tau \leq u \leq s} l_u e^{\delta u} \right\} \right) ds \,\middle|\, \mathcal{F}_\tau \right] \leq e^{-(r+\delta)\tau} .$$

It is a strict inequality if we have excess capacity from the past. However, whenever the past capacity is ignored which is expressed exactly by the term $\sup_{0 \leq u \leq \tau} l_u e^{\delta u}$, it turns out to be an equality. Indeed, this equation is the first–order condition of a firm that starts at time τ with zero capacity.

Theorem 8.3.5 (Optimal Investment). *The base capacity policy $I^{l,\delta}$ as defined in Definition* 8.3.3 *is optimal. Furthermore, the base capacity l which is defined in Equation* (8.10) *is the* unique *solution of the following* modified first–order condition*: for all stopping times $\tau < \hat{T}$*

$$E\left[\int_\tau^{\hat{T}} e^{-(r+\delta)s}\pi_c\left(X_s, e^{-\delta s}\sup_{\tau\le u\le s} l_u e^{\delta u}\right) ds \,\middle|\, \mathcal{F}_\tau\right] = e^{-(r+\delta)\tau}\,. \tag{8.14}$$

The rigorous proof is given in Appendix $D.2$. In addition to proving optimality, the theorem provides also a very useful tool to calculate the optimal policy. As the *unique* solution of the backward equation (8.14), the base capacity can be calculated numerically via backward induction and hence the optimal investment policy that tracks the running supremum of the base capacity. In this way, the optimal policy is fully characterized by the stochastic backward equation (8.14).

Remark 8.3.6. *The same argument can be simply generalized to the case with stochastic interest and discount rates.*

8.4 Qualitative Properties of Irreversible Investments

In the analysis of the deterministic case, Arrow (1968) distinguishes between *free* and *blocked* intervals. In free intervals, the irreversibility constraint is not binding and investment occurs at some rate, i.e., we have $dI_t = i_t dt$ for some investment rate $i_t > 0$. In blocked intervals, the firm would like to disinvest in blocked intervals, namely, $dI_t = 0$. Under uncertainty, the diffusion case studied by Bertola (1998), Pindyck (1988) has such blocked intervals as well. However, due to the special nature of diffusions, there exist no free intervals in the sense of Arrow (1968). Whenever investment occurs, it happens in a *singular* way: the set of time points at which investment occurs is of Lebesgue measure zero; hence there is no rate of investment. In general, all three types of investment can occur. In order to fully characterize the qualitative properties of the optimal investment plan, this section carries out a thorough analysis on the irreversible investment under uncertainty and compares the implications to those in Arrow (1968).

Given the general model discussed in the present thesis, there exist in all three phenomena in irreversible investment: Every investment plan I can be decomposed into three parts,

$$I = I^a + I^j + I^\perp\,,$$

where $I^a_t = \int_0^t i^a_u du$ with $i^a_u > 0$, $\forall\, t \in [0, \hat{T}]$ is the smooth investment with an absolutely continuous plan, $I^j_t = \sum_{n:\tau_n \le t} \Delta_n$, $\forall\, t \in [0, \hat{T}]$ consists of lump sum investments Δ_n that take place at stopping times $(\tau_n)_{n\ge 0}$, and $I^\perp$ describes the singular part of the investment plan.

Free Intervals A random (optional) time interval $[\tau_0, \tau_1]$ is defined as a free interval when smooth investment appears. Throughout the free interval, investment occurs at a strictly positive rate, i.e.,

$$i_u^a > 0\,, \quad \forall\, u \in (\tau_0, \tau_1)\,.$$

The following theorem generalizes the result of Arrow (1968) to the case of investment under uncertainty: The investment rate in free intervals corresponds to the reversible case in the sense of the following theorem.

Theorem 8.4.1. *In free intervals $[\tau_0, \tau_1]$, the marginal operating profit is equal to the user cost of capital, i.e.,*

$$\pi_c\left(X_t, C_t^I\right) = r + \delta \qquad a.s.$$

for all $t \in (\tau_0, \tau_1)$.

Proof: In a free interval where investment occurs continuously, the irreversibility constraint always binds for all $t \in (\tau_0, \tau_1)$ as

$$E\left[\int_t^{\hat{T}} e^{-(r+\delta)s}\pi_c\left(X_s, C_s^I\right)ds \,\middle|\, \mathcal{F}_t\right] = e^{-(r+\delta)t}.$$

Define

$$H = \int_0^{\hat{T}} e^{-(r+\delta)s}\pi_c\left(X_s, C_s^I\right)ds$$

and its conditional expectation given the information at time t as the martingale

$$M_t = E\left[H \,|\, \mathcal{F}_t\right]\,.$$

Note that H is integrable because of the first–order condition at time 0. We can then rewrite the first–order condition in the free interval as

$$M_t = \int_0^t e^{-(r+\delta)s}\pi_c\left(X_s, C_s^I\right)ds + e^{-(r+\delta)t}\,.$$

It follows that M is a martingale with an absolutely continuous sample path on (τ_0, τ_1). Hence, M is continuous on (τ_0, τ_1) a.s. (cf. Protter (2004), Chapter *II*, p. 64 – 65)[12]. Taking derivatives on both sides of the equation yields then

$$\pi_c\left(X_t, C_t^I\right) = r + \delta$$

as desired. ■

Blocked Intervals In blocked intervals where no investment occurs, we have initially excess capacity in comparison with the benchmark reversible case. From the derivation of the base capacity, it is obvious that in blocked intervals the firm tries to equate the marginal operating profit and the user cost of capital in expectation on average over time:

[12]If a martingale has an absolutely continuous sample path, it must have finite variation. As stated in Protter (2004), a continuous martingale with paths of finite variation is constant.

Theorem 8.4.2. *In blocked intervals, the marginal operating profit equals the user cost of capital on average in expectation. Formally, we have the following equation*

$$E\left[\int_{\tau_0}^{\tau_1} e^{-(r+\delta)s}\left[\pi_c\left(X_s, C_s^I\right) - (r+\delta)\right] ds \,\middle|\, \mathcal{F}_{\tau_0}\right] = 0\,.$$

The Set of Singular Investment If uncertainty is generated by a diffusion, singular investment will be generally encountered. Let $S = \{(\omega, t) : dI_t^{\perp}(\omega) > 0\}$ be the support of the random measure $I^{\perp}$. As noted above, this set has, by definition of the singular part $I^{\perp}$, Lebesgue measure zero. The following theorem is a direct consequence of the first–order condition.

Theorem 8.4.3. *On the support of the singular investment part $I^{\perp}$, the first–order condition is binding as*

$$E\left[\int_t^{\hat{T}} e^{-(r+\delta)s}\pi_c\left(X_s, C_s^I\right) ds \,\middle|\, \mathcal{F}_t\right] = e^{-(r+\delta)t} \quad dI^{\perp} - a.s.$$

Remark 8.4.4. *The great difference between singular and smooth investments lies in the fact that singular investment occurs* not *in an absolute way at a measurable rate. This is due to the nature of BMs and diffusions. Generally speaking, if one wants to keep a BM below some boundary, actions have to be taken at very irregular steps. Alternatively, it can be considered in the following way. Since the diffusion process moves continuously with infinite variations, the optimal capacity $l_t = \operatorname{ess\,inf}_{\tau > t} L_t^{\tau}$, $\forall\, \tau \in [t, \hat{T})$ arrives local minimums within any infinitesimal interval. The set of disjoint times when local minima take place is countable but dense almost surely. Thus, investment occurs never in an open interval during which the firm invests all the time. Instead, it takes place at nearly adjacent dates and fluctuates in a nowhere differentiable fashion.*

Lumpy Investment We now discuss the possibility of lump sum investments. It certainly makes sense for the firm to start with a lump sum investment at time 0 in order to bring the firm to a certain size as marginal operating profit at 0 is infinite. Thereafter, intuition suggests that the firm adjusts the capacity continuously *as long as no surprises occur*. Indeed, Arrow has shown in the deterministic model that lump sum investment does not occur except at time zero. The same holds true in the BM case analyzed by Bertola (1998) and Pindyck (1988). In general, jumps are however quite possible, e.g., when a Poisson–like jump occurs.

In the following, we give two theorems on properties of this type investment. First, if lump sum investment occurs, then the firm never jumps to excess capacity in terms of the marginal operating profit function. In other words, at times of jumps, the firm size stays below the size under perfect reversibility. Secondly, we formalize the idea that at fixed times t, the model has a surprise and lump sum investment occurs only with this surprise.

Theorem 8.4.5. *Suppose that the optimal investment plan has a jump at the stopping time τ. Then we have*

$$\pi_c\left(X_\tau, C^I_\tau\right) \geq r+\delta \qquad a.s.\ on\ \{\tau < T\}\,.$$

PROOF: Let τ be a stopping time with $\Delta I_\tau > 0$ a.s on $\{\tau < T\}$. From now on, we work on the set $\{\tau < T\}$ without further mention. For shorter notation, denote the difference of the marginal operating profit and the user cost of capital by $\zeta_t = \pi_c\left(X_t, C^I_t\right) - (r+\delta)$. In this way, it is only necessary to show $\zeta_\tau \geq 0$. Fix $\varepsilon \geq 0$. Let $\nu = \inf\{t \geq \tau : \zeta_t \geq -\varepsilon\}$ be the first time when ζ is greater than or equal to $-\varepsilon$. The first–order conditions, $G'_\tau = e^{-r\tau}$ and $G'_\nu \leq e^{-r\nu}$, are equivalent to

$$E\left[\int_\tau^{\hat{T}} e^{-(r+\delta)s}\zeta_s ds \,\middle|\, \mathcal{F}_\tau\right] = e^{-(r+\delta)\hat{T}}\,,$$

$$E\left[\int_\nu^{\hat{T}} e^{-(r+\delta)s}\zeta_s ds \,\middle|\, \mathcal{F}_\nu\right] \leq e^{-(r+\delta)\hat{T}}\,.$$

We obtain by taking the conditional expectation at time τ of their differences

$$0 \leq E\left[\int_\tau^{\nu} e^{-(r+\delta)s}\zeta_s ds \,\middle|\, \mathcal{F}_\tau\right]\,.$$

By the definition of ν, it follows that

$$0 \leq -\varepsilon\, E\left[\int_\tau^{\nu} e^{-(r+\delta)s} ds \,\middle|\, \mathcal{F}_\tau\right]\,.$$

This is only possible when $\nu = \tau$ almost surely as $\nu \geq \tau$ as defined. Therefore, we have (from right–continuity of X and C^I) $\zeta_\tau \geq -\varepsilon$. As ε is arbitrary, $\zeta_\tau \geq 0$ follows. ■

Example 8.4.6. *Consider a simple infinite horizon model in which there is only one shock taking place at time* 1*. Formally, $X_t = 1$ for $0 \leq t < 1$. At time* 1*, the shock jumps to either a good or a bad state with the same probability, i.e., $\mathbb{P}[X_1 = \xi] = \mathbb{P}[X_1 = \vartheta] = 1/2$ for $\xi > 1 > \vartheta > 0$. After time* 1*, X stays constant, that is, $X_t = X_1$ for $t > 1$. Let $(\mathcal{F}_t)_{t\in[0,\infty)}$ be the filtration generated by X. The profit function is separable in the form of $\pi(x,c) = x\pi(c)$ for some nice function π. In addition, there is no depreciation, i.e., $\delta = 0$.*

It is easy to check that the following investment policy is optimal. The optimal base capacity at time 1 *satisfies $X_1\pi'(l_1) = r$ and stays constant afterwards, namely, $l_t = l_1$ for $t \geq 1$. Let a and b be the optimal base capacities after the good and bad shock, respectively. Then, we have $\xi\pi'(a) = r$ and $\vartheta\pi'(b) = r$ with $a > b$. Between time* 0 *and* 1*, l stays again constant after time* 0*, $l_t = l_0\ \forall\, t \in [0,1)$. Intuitively, it is due to the fact that no new information is released during that interval.*

At time 0*, the optimal level* l_0 *lies between* a *and* b *and gives the first–order condition in equality*

$$1 = E\int_0^\infty e^{-rs} X_s \pi'(\sup_{u\le s} l_u)\, ds\,.$$

After time 1*, the capacity stays constant at* l_0 *all the time afterwards with probability* $1/2$*. Otherwise, it jumps to* a *at time* 1 *when a good shock occurs. It gives then*

$$1 = \frac{1}{r}\pi'(l_0)(1-e^{-r}) + \frac{1}{2r}\left[\xi\pi'(a) + \vartheta\pi'(l_0)\right] e^{-r}$$

or equivalently,

$$\pi'(l_0) = r\frac{1-\frac{1}{2}e^{-r}}{1-(1-\frac{1}{2}\vartheta)e^{-r}}\,.$$

As $0<\vartheta<1$*, it is clear that* $\frac{1-\frac{1}{2}e^{-r}}{1-(1-\frac{1}{2}\vartheta)e^{-r}} > 1$ *and hence*

$$\pi'(l_0) > r\,.$$

The next theorem shows that the occurrence of lumpy investment is closely related to *surprises*. In our general framework, a surprise occurs, e.g., when the exogenous process X jumps. Therefore, we cannot exclude lump sum investment at random times. On the other hand, if X is a diffusion or a Lévy process, X does not jump at fixed times $t>0$, namely $\Delta X_t = X_t - X_{t-} = 0$ with probability 1. Therefore, one might naturally suggest that the probability of a lump sum investment at a fixed time $t>0$ is zero. Another source for a surprise lies in information discontinuity. We say that the information filtration has a fixed surprise at $t>0$ if there exist events at time t that are not known immediately before t. Formally, we define $\mathcal{F}_{t-} = \sigma\left(\bigcup_{s<t}\mathcal{F}_s\right)$ as the information known immediately before t. $(\mathcal{F}_s)_{s\in[0,\hat{T}]}$ has a fixed surprise at t if $\mathcal{F}_{t-}\neq\mathcal{F}_t$. Example (8.4.6) has a fixed surprise at time 1. Finally, we say that the model $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_s),(X_s))$ has no fixed surprise at $t>0$ if $\mathbb{P}[\Delta X_t = 0] = 1$ and $\mathcal{F}_{t-} = \mathcal{F}_t$. Our argument in the following theorem needs an additional integrability condition which is easy to check in applications.

Theorem 8.4.7. *Assume that optimal capacity satisfies*

$$E\left[\sup_{s\le\hat{T}} \pi_c\left(X_s, C_0\right)\right] < \infty\,. \tag{8.15}$$

If the model $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_s))$ *has no fixed surprise at* $t>0$*, then there is no lump sum investment at time* t*, i.e.,* $\mathbb{P}[\Delta I_t = 0] = 1$.

PROOF: Without loss of generality, we assume $\delta = 0$ in the proof. Let $A = \{\Delta I_t > 0\}$. We have to show that $\mathbb{P}(A) = 0$. Assume on the contrary that $\mathbb{P}(A) > 0$. As the first–order condition is binding on A, we have

$$E\left[\int_t^{\hat{T}} e^{-rs}\pi_c\left(X_s, I_s\right) ds \,\middle|\, \mathcal{F}_t\right] = e^{-rt} \qquad \text{on } A\,.$$

Now let $t_n \nearrow t$. The first–order condition yields then

$$E\left[\int_{t_n}^{\hat{T}} e^{-rs}\pi_c\left(X_s, I_s\right) ds \,\middle|\, \mathcal{F}_{t_n}\right] \leq e^{-rt_n}\,.$$

By taking differences and also expectation, we get

$$E\left[1_A E\left[\int_{t_n}^{t} e^{-rs}\pi_c\left(X_s, I_s\right) ds \,\middle|\, \mathcal{F}_{t_n}\right]\right] \leq E\left[1_A\left(e^{-rt_n} - e^{-rt}\right)\right]\,,$$

which is equivalent to

$$E\left[1_A E\left[\int_{t_n}^{t} e^{-rs}\left(\pi_c\left(X_s, I_s\right) - r\right) ds \,\middle|\, \mathcal{F}_{t_n}\right]\right] \leq 0\,. \tag{8.16}$$

Now we have $\lim_{s\nearrow t} X_s = X_t$ a.s. and $\lim_{s\nearrow t} I_s = I_{t-}$ a.s. It follows that

$$Z_n := \frac{1}{t - t_n}\int_{t_n}^{t} e^{-rs}\left(\pi_c\left(X_s, I_s\right) - r\right) ds \to e^{-rt}\left(\pi_c\left(X_t, I_{t-}\right) - r\right) =: Z \qquad \text{a.s.}$$

The sequence $(Z_n)_{n\in\mathbb{N}}$ is bounded below by $-r$ and bounded above by $\sup_{s\leq\hat{T}} \pi_c\left(X_s, I_0\right)$ which is integrable by the assumption. Hence, convergence also takes place in L^1.

By a martingale convergence argument (see, e.g., Loéve (1978), Chapter *IX*, Complement 10, p. 75), we have

$$\lim_{n\to\infty} E\left[Z_n \,|\, \mathcal{F}_{t_n}\right] = E\left[Z \,|\, \mathcal{F}_{t-}\right] = Z \quad a.s. \text{ and } L^1,$$

where the last equality follows from the assumption that there is no fixed information surprise at t. By Theorem 8.4.5 (applied to the stopping time $\tau = t1_A + \hat{T}1_{A^c}$), we have

$$\pi_c\left(X_t, I_{t-}\right) - r > \pi_c\left(X_t, I_t\right) - r \geq 0 \qquad \text{on } A\,.$$

As $Z_n = \frac{1}{t-t_n}\int_{t_n}^{t} e^{-rs}\left(\pi_c\left(X_s, I_s\right) - r\right) ds$ is bounded below by $-r$, we can apply Fatou's Lemma, Equation (8.16) and the assumption that $\mathbb{P}(A) > 0$ to obtain

$$\begin{aligned}
0 &< E\left[1_A\, e^{-rt}\left(\pi_c\left(X_t, I_{t-}\right) - r\right)\right] \\
&= E\left[1_A \lim_{n\to\infty} Z_n\right] \\
&\leq \lim_{n\to\infty}\inf E\left[1_A Z_n\right] \\
&= \lim_{n\to\infty}\inf\; E\left[1_A\left(Z_n - E\left[Z_n \,|\, \mathcal{F}_{t_n}\right]\right)\right] + \lim_{n\to\infty}\inf\; E\left[1_A E\left[Z_n \,|\, \mathcal{F}_{t_n}\right]\right] \\
&\leq 0\,.
\end{aligned} \tag{8.17}$$

Note that the first limit in (8.17) is zero because both $(Z_n)_{n\in\mathbb{N}}$ and $\left(E\left[Z_n \,|\, \mathcal{F}_{t_n}\right]\right)_{n\in\mathbb{N}}$ converge in L^1 to Z. The second summand is always nonnegative. This is a contradiction.

■

As one special case, the irreversible investment under certainty studied in Arrow (1968) possesses complete information set during the whole investment plan. As a result, lumpy investment takes place only at the initial time:

Corollary 8.4.8. *In Arrows model, jumps occur only at time* 0.

In addition, it is worthwhile to point out that the model has no fixed surprises if X is a diffusion.

Corollary 8.4.9. *If* $(\mathcal{F}_s)_{s\in[0,\hat{T}]}$ *is the augmented filtration of a Wiener process* W *and* X *a diffusion given by*

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t\,,$$

then for all $t > 0$*, we have* $\mathbb{P}[\Delta I_t > 0] = 0$ *for the optimal investment plan* I*, provided* (8.15) *holds true.*

Similarly, the model has no fixed surprises for Lévy processes.

Corollary 8.4.10. *If* $(\mathcal{F}_s)_{s\in[0,\hat{T}]}$ *is the augmented filtration of a Lévy process* X*,then for all* $t > 0$*, we have* $\mathbb{P}[\Delta I_t > 0] = 0$ *for the optimal investment plan* I*, provided* (8.15) *holds true.*

8.5 Comparative Statics

An advantage of our approach to irreversible investment is that it easily leads to general monotone comparative statics. We are going to illustrate it in this section with two comparative statics results.

First, it is shown that the base capacity is monotonically increasing in shocks X whenever the operating profit function has increasing differences in shocks and capacity. A $\mathcal{C}^2$–function is supermodular or equivalently exhibits increasing differences (see Topkis (1978)) if and only if the function π satisfies

$$\frac{\partial^2}{\partial x \partial c}\pi(x,c) \geq 0\,.$$

The general theory by Topkis (1978) and Milgrom and Shannon (1994) suggests that this property is necessary to have monotone comparative statics. As there is no general theory of stochastic dominance for stochastic processes, we order the set of all stochastic processes by the partial order of being greater or equal almost surely everywhere.

Theorem 8.5.1. *Let* X *and* Y *be two progressively measurable stochastic processes with* $X_t \geq Y_t$ *for all* $t \in [0,\hat{T}]$ *almost surely. Denote the optimal base capacity under* X *(resp.* Y*) by* l^X *(resp.* l^Y*). Assume that the operating profit function is supermodular. Then the base capacity is monotonically increasing in* X*, i.e.* $l^X_t \geq l^Y_t$ *for all* $t \in [0,\hat{T}]$ *a.s.*

Proof: As the base capacity level is the essential infimum of all candidates L_t^τ in (8.9), it is enough to show that the $L_t^{X,\tau}$ corresponding to the exogenous stock X is larger than that to Y, or equivalently, $L_t^{X,\tau} > L_t^{Y,\tau}$.

$L_t^{\cdot,\tau}$ is by definition the unique solution of the first–order condition (8.9). Thus, we get an equality of the two conditions subject to different shocks X and Y as follows:

$$\begin{aligned} 0 &= E\left[\int_t^\tau e^{-(r+\delta)s}\left[\pi_c\left(X_s, L_t^{X,\tau}e^{-\delta(s-t)}\right) - (r+\delta)\right] ds \,\middle|\, \mathcal{F}_t\right] \\ &= E\left[\int_t^\tau e^{-(r+\delta)s}\left[\pi_c\left(Y_s, L_t^{Y,\tau}e^{-\delta(s-t)}\right) - (r+\delta)\right] ds \,\middle|\, \mathcal{F}_t\right]. \end{aligned} \tag{8.18}$$

In particular, we have for the set $A = \{L_t^{X,\tau} < L_t^{Y,\tau}\} \in \mathcal{F}_t$

$$0 = E\left[1_A \int_t^\tau e^{-(r+\delta)s}\left[\pi_c\left(X_s, L_t^{X,\tau}e^{-\delta(s-t)}\right) - \pi_c\left(Y_s, L_t^{Y,\tau}e^{-\delta(s-t)}\right)\right] ds \,\middle|\, \mathcal{F}_t\right].$$

On the set A, the integrand is strictly positive, because π is supermodular and $\pi_{cc} < 0$. Hence, the expectation can only be zero, if A has measure 0. Then, $L_t^{X,\tau} \geq L_t^{Y,\tau}$ a.s. follows. ■

Remark 8.5.2. *An alternative proof via Topkis (1978) is also possible. Moreover, Theorem* 10 *in Milgrom and Shannon (1994) suggests that supermodularity is necessary for this type of monotone comparative statics.*

Remark 8.5.3. *Having considered a monotone shift in the shock, one would also like to ask what happens if the exogenous shock process becomes more risky. Unfortunately, there is no general theory of second order stochastic dominance for stochastic processes. A natural definition inspired by the well–known fact of second order stochastic dominance would be an unanimity principle for expected utility maximizers. Fix a certain discount rate ρ and consider arbitrary risk–averse expected utility maximizers that live from time t to some stopping time $\tau > t$. If all these rational agents would rather consume Y than X, then we call X riskier than Y. Formally, let X and Y be two progressively measurable stochastic processes. X is riskier than Y if*

$$E\left[\int_t^\tau e^{-\rho s}u(X_s)ds \,\middle|\, \mathcal{F}_t\right] \leq E\left[\int_t^\tau e^{-\rho s}u(Y_s)ds \,\middle|\, \mathcal{F}_t\right]$$

for all monotone and increasing functions $u : \mathbb{R} \to \mathbb{R}$ and all times $\tau > t$. However, one can show that such a definition boils down to the condition that X dominating Y in an almost sure sense which is exactly the monotonicity condition considered in Theorem 8.5.1.

We conclude this section by establishing a plausible result that the firm size is decreasing in the user cost of capital.

Theorem 8.5.4. *The base capacity is decreasing in the user cost of capital $r + \delta$.*

The complete proof is provided in Appendix D.3.

8.6 Solutions for Lévy Shocks & Cobb–Douglas Functions

Generally, numerical methods have to be adopted to identify solutions. Nevertheless, a closed–form solution can be obtained for an infinitely–lived firm ($\hat{T} = \infty$) when the multiplicative economic shock is characterized by an exponential Lévy process and the firm is endowed with the operating profit function of the form

$$\pi(X_t, C_t) = \frac{1}{1-\alpha} X_t^{\alpha} C_t^{1-\alpha}, \quad 0 < \alpha < 1. \tag{8.19}$$

This construction is consistent with a competitive firm who produces at decreasing returns to scale or with a monopolist firm facing with a constant elasticity demand function and constant returns to scale production (as shown by Abel and Eberly (1996) and Morellec (2001)). Clearly, this function is concave with the first derivative $\pi_C = X_t^{\alpha} C_t^{-\alpha}$. In particular, the economic shock X_t is modelled by

$$X_t = x_0 e^{Y_t},$$

where x_0 is the initial value at $t = 0$ and Y_t is a Lévy process with zero initial value. Moreover, the interest and discount rate are assumed to be constant over time.

Computation of the Base Capacity As introduced in Section 8.3, the irreversible investment decision problem is solved by calculating the first–order condition and solving the achieved backward equation (8.14). Here, it is reduced to

$$E\left[\int_{\tau}^{\infty} e^{-(r+\delta)s} X_s^{\alpha} \left(e^{-\delta s} \sup_{\tau \le u \le s} l_u e^{\delta u}\right)^{-\alpha} ds \,\middle|\, \mathcal{F}_{\tau}\right] = e^{-(r+\delta)\tau}, \tag{8.20}$$

which can be explicitly solved by means of the strong Markov property and time homogeneity of Lévy processes, as given in the following theorem.

Theorem 8.6.1. *When the production function is of form* (8.19) *and the exogenous economic shock is characterized by an exponential Lévy process, the base capacity is identified as* $l_t = \kappa X_t$ *with*

$$\kappa \;=\; \left(\frac{1}{r+\delta} E\left[e^{\alpha \underline{Z}_{\tau(r+\delta)}}\right]\right)^{\frac{1}{\alpha}}, \tag{8.21}$$

where $Z_t = Y_t + \delta t$, $\underline{Z}_t$ *is defined as* $\underline{Z}_t = \inf_{0 \le u \le t} Z_u$ *and* $\tau(r+\delta)$ *is an independent exponential distributed time with parameter* $r + \delta$.

Proof: The backward equation for Lévy processes has been solved in Bank and Riedel (2001b). In order to keep the thesis self-contained, we repeat the proof here.

Make the ansatz $l_u = \kappa X_u$ for a constant κ to be determined. Then the left–hand side of Equation (8.20) can be reduced into

$$\begin{aligned}
& E\left[\int_\tau^\infty e^{-(r+\delta)s} X_s^\alpha \left(e^{-\delta s}\sup_{\tau\le u\le s} l_u e^{\delta u}\right)^{-\alpha} ds \,\middle|\, \mathcal{F}_\tau\right] \\
= \;& E\left[\int_\tau^\infty e^{-(r+\delta)s} X_s^\alpha \inf_{\tau\le u\le s} (\kappa X_u)^{-\alpha} e^{-\alpha\delta(u-s)} ds \,\middle|\, \mathcal{F}_\tau\right] \\
= \;& \kappa^{-\alpha} E\left[\int_\tau^\infty e^{-(r+\delta)s} \inf_{\tau\le u\le s} \left(\frac{x_0 e^{Y_s}}{x_0 e^{Y_u}}\right)^\alpha e^{-\alpha\delta(u-s)} ds \,\middle|\, \mathcal{F}_\tau\right] \\
= \;& \kappa^{-\alpha} E\left[\int_\tau^\infty e^{-(r+\delta)s} \inf_{\tau\le u\le s} e^{\alpha[(Y_s-Y_\tau)+\delta(s-\tau)-(Y_u-Y_\tau)-\delta(u-\tau)]} ds \,\middle|\, \mathcal{F}_\tau\right] \\
= \;& \kappa^{-\alpha} E\left[\int_0^\infty e^{-(r+\delta)(t+\tau)} \inf_{0\le u\le t} e^{\alpha[(Y_t+\delta t)-(Y_u+\delta u)]} dt\right],
\end{aligned}$$

where the last two step follows from the strong Markov property and the independence of the increments of the past for the Lévy process Y. Denote $Z_t = Y_t + \delta t$ which is clearly also a Lévy process. Then the backward equation (8.20) is solved if we set

$$\kappa = \left(E\left[\int_0^\infty e^{-(r+\delta)t} \inf_{0\le u\le t} e^{\alpha(Z_t-Z_u)} dt\right]\right)^{\frac{1}{\alpha}}.$$

Let $\overline{Z}_t = \sup_{0\le u\le t} Z_u$ and $\underline{Z}_t = \inf_{0\le u\le t} Z_u$. The expression of κ can be simplified as

$$\begin{aligned}
\kappa &= \left(E\left[\int_0^\infty e^{-(r+\delta)t} \inf_{0\le u\le t} e^{\alpha(Z_t-Z_u)} dt\right]\right)^{\frac{1}{\alpha}} \\
&= \left(E\left[\int_0^\infty e^{-(r+\delta)t} e^{\alpha(Z_t-\overline{Z}_t)} dt\right]\right)^{\frac{1}{\alpha}} \\
&= \left(\frac{1}{r+\delta} E\left[e^{\alpha(Z_{\tau(r+\delta)}-\overline{Z}_{\tau(r+\delta)})}\right]\right)^{\frac{1}{\alpha}} \\
&= \left(\frac{1}{r+\delta} E\left[e^{\alpha \underline{Z}_{\tau(r+\delta)}}\right]\right)^{\frac{1}{\alpha}},
\end{aligned}$$

where $\tau(r+\delta)$ is an independent exponential distributed time with parameter $r+\delta$ and the last equality is achieved by the duality theorem that $Z_t - \overline{Z}_t$ has the same distribution as $\underline{Z}_t$ (see Bertoin (1996), Chapter *VI*, Proposition 3, p. 158). ■

Remark 8.6.2. *According to the optimal investment policy, it is always maintained that* $C_t \ge l_t$ *at all time* $t \in [0, \hat{T}]$. *With the derived solution* $l_t = \kappa X_t$, *one can easily obtain*

$$\pi_c = X_t^\alpha C_t^{-\alpha} \le X_t^\alpha (\kappa X_t)^{-\alpha} = \kappa^{-\alpha},$$

where $\kappa^{-\alpha} = (r+\delta)/E\left[e^{\alpha \underline{Z}_{\tau(r+\delta)}}\right]$. *Obviously, the expectation term is valued only in* $(0,1]$. *It follows thus that the marginal operating profit under the optimal investment plan is always kept below the user cost of capital times a markup factor.*

The threshold κ can be computed in closed form by some properties of Lévy processes as in the following theorem.

Theorem 8.6.3. *$E\left[e^{\alpha \underline{Z}_{\tau(r+\delta)}}\right]$ in Equation (8.21) is specified as*

- $\frac{(r+\delta)\,(\Phi(r+\delta)-\alpha)}{\Phi(r+\delta)\,(r+\delta-\Psi(\alpha))}$ *for Lévy processes $(Z_t)_{t\in[0,\infty)}$ with only negative jumps.*
- $\Psi^-_{(r+\delta)}(\alpha)$*, the right Wiener–Hopf factor of $(Z_t)_{t\in[0,\,\infty)}$, which is readily recognized for BMs and Lévy processes of exponential type.*

Computation of the Firm's Overall Profit and Well–Posedness of the Problem
The preceding theorem obtains a solution of the stochastic backward equation for all Lévy processes. Taking it as a candidate for the optimal policy, we then have to check for optimality that it gives an admissible investment and that the resulted firm's value is finite. In infinite horizon models, this usually requires a constraint on the interest rate and on the growth rate of X. Here, only one condition is already sufficient as stated in the theorem below.

Theorem 8.6.4. *Assume that $r+\delta > \Psi(1)$ where $\Psi(1)$ is the Lévy–Laplace exponent of Z defined by $\Psi(1) = \log E[e^{Z_1}]$. Then the base capacity policy that keeps the capacity just above the base capacity $l_t = \kappa X_t$ is optimal. The overall profit of the firm is given by*

$$\Pi(I^*) = \kappa x_0 \frac{\alpha}{1-\alpha} E\left[e^{\overline{Z}_{\tau(r+\delta)}}\right].$$

Proof: Recall that

$$C_t^{l,\delta} = e^{-\delta t}\sup_{s\le t} l_s e^{\delta s} = x_0 \kappa e^{-\delta t}\sup_{s\le t} e^{Z_s}\,.$$

$I^{l,\delta}$ is admissible if and only if $E\left[\int_0^\infty e^{-rt} dI_t^{l,\delta}\right] < \infty$. Expanding it by $dI_t^{l,\delta} = dC_t^{l,\delta} + \delta C_t^{l,\delta}dt$ and taking integration by parts yields

$$\begin{aligned} E\left[\int_0^\infty e^{-rt} dI_t^{l,\delta}\right] &= E\left[\int_0^\infty e^{-rt}\left(dC_t^{l,\delta} + \delta C_t^{l,\delta}dt\right)\right] \\ &= E\left[e^{-r\infty}C_\infty^{l,\delta}\right] + E\left[\int_0^\infty e^{-rt}(r+\delta)C_t^{l,\delta}dt\right] \\ &= x_0\kappa E\left[e^{-(r+\delta)\infty}\sup_{s\le\infty} e^{Z_s}\right] + x_0\kappa E\left[e^{\overline{Z}_{\tau(r+\delta)}}\right] \\ &< \infty\,. \end{aligned}$$

Hence, the admissibility is guaranteed if

$$E\left[e^{\overline{Z}_{\tau(r+\delta)}}\right] < \infty\,. \tag{8.22}$$

It is a sufficient condition since it implies $E\left[e^{-(r+\delta)\infty}\sup_{s\le\infty} e^{Z_s}\right]=0$ a.s.

The Wiener–Hopf factorization tells that (8.22) holds true if and only if

$$E\int_0^\infty e^{-(r+\delta)s+Z_s}ds=\int_0^\infty e^{[\Psi(1)-(r+\delta)]s}ds<\infty$$

and hence

$$r+\delta>\Psi(1)\,, \tag{8.23}$$

where $\Psi(1)$ is the Lévy–Laplace exponent of Z defined by $E[e^{Z_t}]=e^{t\Psi(1)}$.

In this case, $I^{l,\delta}$ is the optimal investment plan with expected discounted cost at

$$E\left[\int_0^\infty e^{-rt}dI_t^{l,\delta}\right]=x_0\kappa E\left[e^{\overline{Z}_{\tau(r+\delta)}}\right]\,.$$

Meanwhile, the optimal investment policy generates the overall profit

$$\begin{aligned}\Pi(I^*) &= E\left[\int_0^\infty e^{-rt}\left(\frac{1}{1-\alpha}X_t^\alpha C_t^{*\,1-\alpha}dt-dI_t^*\right)\right]\\ &= \frac{x_0\kappa^{1-\alpha}}{(1-\alpha)(r+\delta)}E\left[e^{\alpha Z_{\tau(r+\delta)}+(1-\alpha)\overline{Z}_{\tau(r+\delta)}}\right]-\kappa x_0E\left[e^{\overline{Z}_{\tau(r+\delta)}}\right]\\ &= \frac{x_0\kappa^{1-\alpha}}{(1-\alpha)(r+\delta)}E\left[e^{\alpha(Z_{\tau(r+\delta)}-\overline{Z}_{\tau(r+\delta)})}\right]E\left[e^{\overline{Z}_{\tau(r+\delta)}}\right]-\kappa x_0E\left[e^{\overline{Z}_{\tau(r+\delta)}}\right],\end{aligned}$$

where the last equality is obtained since $\overline{Z}_t$ and $Z_t-\overline{Z}_t$ are independent by Theorem $VI.5(i)$ in Bertoin (1996).

It can be further simplified due to $\kappa^{-\alpha}=(r+\delta)\left(E\left[e^{\underline{Z}_{\tau(r+\delta)})}\right]\right)^{-1}$ and duality theorem

$$\begin{aligned}\Pi(I^*) &= \kappa\frac{x_0}{1-\alpha}E\left[e^{\overline{Z}_{\tau(r+\delta)}}\right]-\kappa x_0E\left[e^{\overline{Z}_{\tau(r+\delta)}}\right]\\ &= \kappa x_0\frac{\alpha}{1-\alpha}E\left[e^{\overline{Z}_{\tau(r+\delta)}}\right]\,.\end{aligned}$$

It is worth to note that (8.23) is also necessary to achieve the well–posedness of our profit maximization problem. ■

Remark 8.6.5. *For GBMs, the irreversible investment problem is well–posed whenever* $r>\mu+\frac{1}{2}\sigma^2$ *where* μ *and* σ *are the constant drift and volatility of* X*. This basically coincides those results in Pindyck (1988) and Bertola (1998). Boyarchenko (2004) derives the result for exponential Lévy processes under the additional restriction that the capacity remains bounded. This assumption is not required in our work.*

Specific Examples In order to well illustrate this method and the derived *base capacity policy*, two examples are provided based on the specific model setup as follows:

Example 8.6.6. *As mostly often assumed in the literature, X is a GBM, that is,*

$$Y_t = \sigma W_t \,,$$

where W_t is the standard Wiener process and the constant volatility $\sigma = 0.20$. Additionally, the production parameter is given as $\alpha = 0.80$. The constant interest and discount rate are $r = 8\%$, $\delta = 2\%$, respectively.

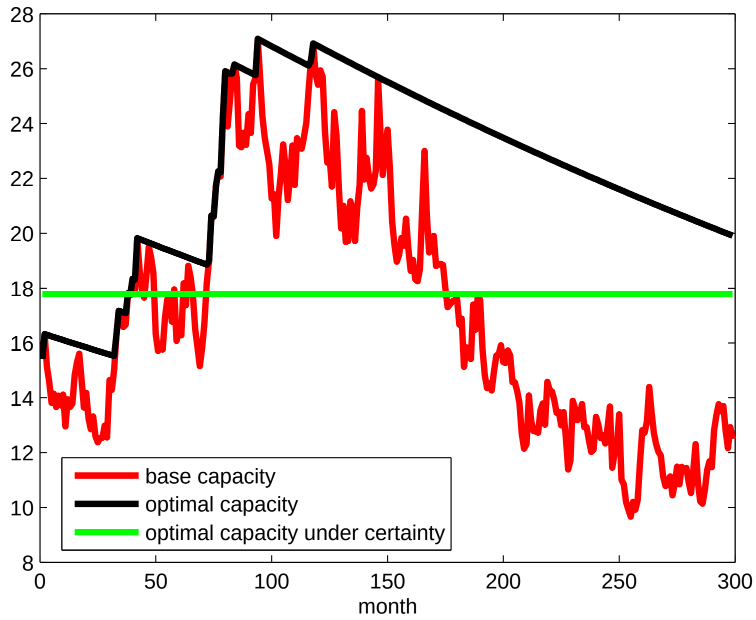

Figure 8.1: Optimal Capacity Level under Certainty and Uncertainty with Geometric Brownian Motion Modelled Shocks

As shown in Figure 8.1, the base capacity evolves according to a GBM with a continuous path but in nowhere differentiable fashion. Investment is undertaken if and only if the current capacity is discounted or becomes lower than the base capacity. In any case, the optimal capacity is maintained to be equal or higher than the base capacity, although sometimes the firm would like to disinvest, which is impossible due to irreversibility of the investment. Consequently, the investment plan in this case only consists of singular investment and no investment. Jump in investment appears only at the initial time. Moreover, the initial jump is below the optimal capacity level under certainty that equals the marginal operating profit with the user cost of capital. Clearly, it coincides with Theorem (8.4.5) *that irreversibility leads to underinvestment.*

Example 8.6.7. *The next example models the economic shock by a Compound Poisson process*

$$Y_t = \mu t + \sum_{n=1}^{N_t} J_n \,,$$

where the drift term is constant with $\mu = 0.05$, $(N_t)_{t\geq 0}$ *is a Poisson process of intensity* $\lambda = 0.05$ *and* $J = (J_n)_{n\in\mathbb{N}}$ *a sequence of independent identically distributed random variables with density*

$$f(j) = \begin{cases} pc^+e^{-c^+j} & j \geq 0\,, \\ (1-p)c^-e^{c^-j} & j < 0\,. \end{cases}$$

with $c^+ = 0.10$, $c^- = 0.45$ *and* $p = 0.70$. *Under this assumption, the economic shock at time* t *has in all* N_t *possible upward and downward jumps which occur with probability* 70% *and* 30%*, respectively. Each positive/negative jump is exponentially distributed with parameter* c^+/c^-. *Keep all the other model parameters constant as given in Example* (8.6.7)*. In this way,* κ *and hence the base capacity can be identified and plotted in Figure* 8.2*.*

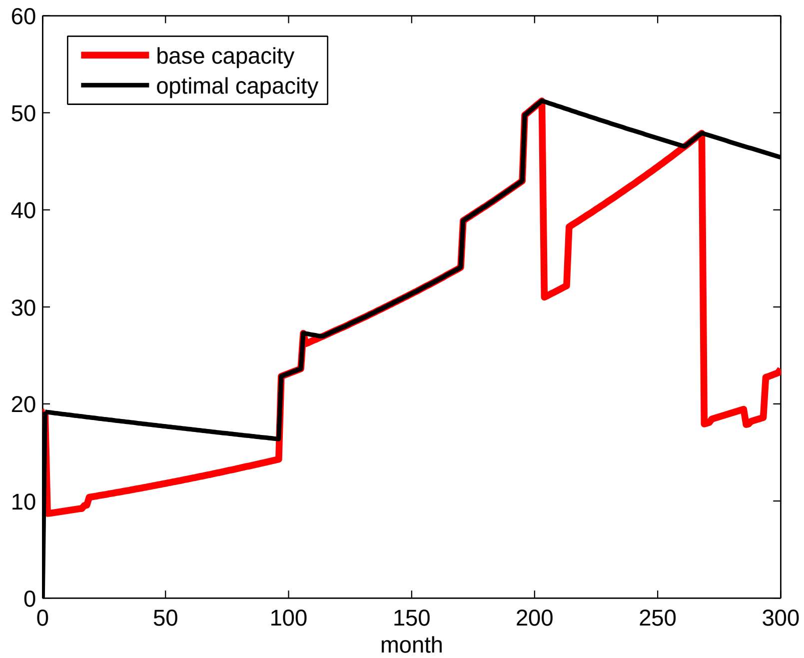

Figure 8.2: Optimal Capacity Level under Uncertainty with Compound Poisson Process Modelled Shocks

Obviously in this case where there is no BM term, there exist lump sum and also smooth investment, but no singular investment. Consequently, the whole investment plan can be easily divided into free and blocked intervals after jumps. Meanwhile as clearly observed, all the jumps in the optimal investment occur only at information surprises, i.e., when X *jumps upward.*

8.7 Conclusion

This chapter studies sequential irreversible investment decision problems under uncertainty. The same problem is solved in Pindyck (1988) by the standard real options approach. The dynamic capacity choice problem is treated as a sequence of optimal stopping problems. Instead of focusing on how much to invest at each time, he starts from when the infinitesimal stock of capital should be invested. This is exactly the starting point of our method, which is based on Bank and Riedel (2001b) and first applied in this work to the real options theory, to concern the marginal effect of investment at any given time. Similarly, Bertola (1998) solves the maximization problem (8.2) by identifying the optimality condition in the sense of marginal effect. On this basis, different techniques are applied to achieve the the optimal threshold investment level. Pindyck (1988) obtains the optimal trigger level of the investment by solving Hamilton–Jacobi–Bellman equations. Sticking to the marginal effect, Bertola (1998) identifies the marginal profit from investment and solves its stochastic differential equation after assuming that there is a control barrier on the marginal profit. While, the method of our work considers the marginal investment problem as a singular control problem and characterizes the optimal investment policy by constructing and tracking a base capacity and solving our key stochastic backward equation.

This method is advantageous mainly in the following four aspects. First, it applies well to a general model which is free of any distributional and parametric assumptions. General existence and uniqueness theorem is derived for both finite and infinite horizons, which is to our knowledge the first result in the literature. Second, this method incorporates an economic interpretation in the derivation, enabling one to better understand the irreversible investment problem. More importantly, it allows for a general qualitative characterization of the optimal investment. Generally, the investment plan can be characterized by three different phenomena: smooth continuous investment, lump sum investment and singular investment. The marginal operating profit is equal to the user cost of capital only in free intervals where smooth investment occurs at positive rates. While in blocked intervals during which there is no investment, the equality of the marginal profit and the user cost of capital is maintained only in expectation on average over time. After time zero, lump sum investment is only possible with fixed surprises. Singular investment takes place in a nowhere differentiable fashion whenever the uncertainty is (partly) modelled by diffusions. In addition, this method gives some monotone comparative statics results: When the operating profit function is supermodular, the base capacity increases monotonically with the exogenous shock; and the firm size always declines with the user cost of capital. Finally, explicit solutions is obtained for an infinitely–lived firm where he is endowed with the operating profit function of Cobb–Douglas type and the multiplicative economic shock is modelled by an exponential Lévy process. In this context, the base capacity is identified as the exogenous shock multiplied by a factor κ, which recovers the well–known result in the literature.

Chapter 9

Incomplete Market Consideration — Utility Maximization[1]

9.1 Introduction

The management objective of real option models up to now is maximizing the expected profit of the project. This model is standard and can be easily validated in a complete financial market: The cash flow of the investment can be spanned by those products traded in the market such that the value of the project is exactly the EPV of those cash flows under the unique risk neutral measure. As shown in the previous two chapters, the stochastic representation method is surely not restricted to those assumptions of market completeness and risk neutrality[2]. However, these two assumptions are too demanding and highly irrelevant to the practice. One one hand, the market is incomplete due to many already acknowledged factors, e.g., transaction costs. Moreover, real assets on which real options are contingent are typically not available in the (financial) market. In most cases, e.g., an investment for R&D and new products, it is almost impossible to replicate those cash flows from the project by the available financial products, which directly violated with the complete market assumption. On the other hand, investors hold in general different attitudes to risk and hence have different preferences for an investment. Consequently, the motivation of this chapter is to incorporate subjective risk preferences in the irreversible investment valuation. To this end, the standard decision problem is combined with utility functions which are usually used in economics to define and measure risk preferences.

The limitation of complete market assumption is first addressed in McDonald and Siegel (1986). They divide risks from investment into unsystematic (i.e., diversifiable) and systematic risks. The decision maker is risk neutral in this case only to unsystematic risks but averse towards the remaining unhedgeable risks. The risk neutrality issue is also mentioned in Dixit and Pindyck (1994) but without real treatment mainly due to the infeasibility and complicated computation that would be involved in the dynamic pro-

[1] This chapter is based on Su (2006).

[2] The dynamic programming method does not require the strict assumptions either.

gramming method. Nevertheless, they state that a utility function can be defined to account for market incompleteness and to determine the correct discount factor ρ. The idea is then developed by Hugonnier and Morellec (2005) for a GBM and a power utility. This utility maximization model is extended in Henderson (2005), Miao and Wang (2005), Henderson and Hobson (2002) and Hugonnier and Morellec (2006) by introducing a correlated asset which is traded in the market to account for market incompleteness. Therefore, their works focus more on the wealth allocation in riskless and risky assets. Moreover, they are all somewhat restricted to a GBM and a specific utility function such as an exponential or a power utility.

In this chapter, we focus mainly on the risk aversion impact on the investment problem and hence follow the model construction of Hugonnier and Morellec (2005). To be more specific, the framework involves a risk averse manager facing with a one–shot irreversible investment under uncertainty. Particularly, an infinite horizon is assumed in order to facilitate comparisons to the classical models of Dixit and Pindyck (1994) (discussed in Chapter 7). The resulting utility–based decision problem is then solved again by using our new stochastic representation method. Similar to the value–maximization decision rule, a *shadow utility process* is obtained such that the investment is initiated whenever the net utility from the investment becomes non–negative. The *shadow utility process* measures the real utility gain after compensating the manager's decision of giving up the right to wait.

In contrast with other approaches, our method works well for a general classs of increasing and concave (risk–averse) utility function and all semi–martingale processes, provided that the objective optimization function is bounded to be finite. More importantly, it allows the feasibility of analytical solution form for an exponential Lévy process modelled uncertainty and a power utility. The power utility function is a well–known specification characterizing the manager's risk preferences with a constant relative risk aversion (CRRA). In this way, the standard real option model with risk neutrality is one special case of the utility–based model when the risk aversion coefficient tends to zero.

Our results reveal that risk aversion has both qualitative and quantitative impact on the threshold value and hence the optimal investment time. Qualitatively, the firm due to risk aversion has a relatively high incentive to delay the investment, which in turn leads to a higher threshold compared to risk neutrality. This slowing down effect of risk aversion on the investment is not restricted to GBMs but valid for all exponential Lévy processes. Quantitatively, we study through an example the combined effect of utility consideration and jumps on the trigger value. It is demonstrated that under risk aversion and negative jumps, the critical price can be higher or lower than that under the GBM modification. It is argued in the paper that it is not counter–intuitive. Given a certain estimated variance, there are two opposite effects of jumps on the threshold value: on one hand the trigger value rises as a response to possible negative jumps; on the other hand the trigger value declines with the jump coefficients due to the decrease in diffusion uncertainty. The first increasing effect is nevertheless greatly magnified by risk aversion.

The remainder of this chapter is structured in the following way: The investment problem faced by a decision maker who is risk–averse is considered in Section 9.2 in the utility optimization framework and the utility–based decision rule is then derived by means of the stochastic representation method. Section 9.3 presents an explicit characterization of the solution for the case of a power utility function and an exponential Lévy process. The implications shown by the result, especially the combination effect of jumps and risk aversion on the threshold investment value, are discussed in Section 9.4 with the help of a numerical example. Section 9.5 finally summarizes this chapter by a concluding remark. Technical details are presented in Appendix E.

9.2 Utility–Based Decision Rule

To show the impact of risk aversion on the investment policy, the simplest generalization of Dixit and Pindyck (1994) model is used here where the firm has an opportunity to invest but faces an incomplete market. As done typically in the literature, subjective risk preferences are characterized by a utility function. The resulted utility maximization problem is then treated by the stochastic representation method.

9.2.1 Utility–Based Irreversible Investment Decision Problem

Consider again the model in Chapter 7. The firm has to make a decision when to undertake a single irreversible investment. Nevertheless, the manager or decision maker is risk averse. Therefore, we define especially an expected utility function to represent the risk aversion

$$p \longmapsto E\left[\int_0^\infty e^{-\rho t} U(p_t) dt\right],$$

where the utility function $U(\cdot)$ is supposed to be increasing, concave and continuously differentiable. In addition to the investment opportunity, the firm has wealth I that can be invested either in the risky project or in risk–free assets with the interest rate r. Furthermore, the manager's time horizon is supposed to be infinite. In this case, the firm maximizes the expected utility of the investment

$$F = \max_{\tau \in \mathcal{T}([0,\infty))} E\left[e^{-\rho\tau}\left(\int_\tau^\infty e^{-\rho(t-\tau)}(U(P_t) - U(rI))dt\right)^+\right], \tag{9.1}$$

where $U(rI)$ is the utility lost which could be otherwise gained by investing risk–free cash flow stream rI.

To guarantee the well–posedness of the problem, the expected utility of the future revenue from investment has to be finite, namely, $E\left[\int_0^\infty e^{-\rho t} U(P_t) dt\right] < \infty$. Given the model construction, we now turn to the derivation of the decision rule as well as the value of the option to invest.

9.2.2 Investment Policy and Project Value

One advantage of assuming market completeness and risk neutrality is the convenience of characterizing the optimal investment strategy. The introduction of the utility function makes the partial differential equation in a complicated form which greatly hinders the generalization of the classical model. Hugonnier and Morellec (2005) solves only the problem based on GBM by taking the results of Karatzas and Shreve (1999) on the expected utility optimization. As the first contribution of this chapter, our treatment with the stochastic representation method improves the literature available by generalizing the result to any semi–martingale processes. Formally, the decision rule of the utility–based irreversible investment is figured out by the following theorem.

Theorem 9.2.1. *A risk averse investor who has to decide on the investment timing in the problem* (9.1) *will undertake the investment at time*

$$\tau^* = \inf \left\{ t \geq 0 \,\middle|\, \xi_t^{um} \geq \frac{U(rI)}{\rho} \right\} ,$$

where ξ_t^{um} *is the solution to the representation of the form*

$$E\left[\int_t^\infty e^{-\rho s} U(P_s)\, ds \middle| \mathcal{F}_t\right] = E\left[\int_t^\infty \rho e^{-\rho s} \sup_{t\leq v\leq s} \xi_v^{um}\, ds \middle| \mathcal{F}_t\right] . \tag{9.2}$$

Proof: This result can be easily obtained by following the same argument in the proof of Theorem 7.2.1. The investment at time τ^* is shown to be always positive and in addition arrive the maximum utility by comparing with any earlier and later investments. The two regularity conditions required for applicability of the method is also maintained since $U(\cdot)$ is concave and continuously differentiable. Notably, $E\left[\int_0^\infty e^{-\rho t} U(P_t) dt\right] < \infty$ is the only condition necessary for both well–posedness of the problem and also the method. ■

In this utility framework, ξ_t^{um} can be naturally interpreted as the *shadow utility* that the firm attains from the investment. By investing the firm gives up the option to wait and hence the possible utility gain while postponing the project. The obtained *shadow utility* measures the direct or real utility gain from the project after deducting the utility lost. In this way, investment occurs only when the subjective valuation of the project amounts high enough to cover the full investment cost, namely the investment cost which can be otherwise invested in risk–free assets and the opportunity cost of abandoning the real option.

The theorem provides the utility–maximizing investment rule for any increasing and concave utility function. To derive a specific investment decision and examine the impact of risk aversion on the decision, the model has to be further specified. In the following, we consider the case of a CRRA utility function and uncertainty modelled by an exponential Lévy process.

9.3 Explicit Solutions for CRRA Utility Functions

A power utility function

$$U(x) = \frac{1}{1-\alpha} x^{1-\alpha} \tag{9.3}$$

is the typical modification exhibiting constant relative risk aversion, where the parameter $\alpha > 0$, $\alpha \neq 1$ is the coefficient of relative risk aversion. The higher it is, the more risk averse the firm is. In particular, it is risk neutral when $\alpha = 0$, which is exactly the original profit–maximization problem. Further assume that the price of the produced good follows an exponential Lévy process as given in (7.8). Under this construction, a simple application of the result above yields the following theorem.

Theorem 9.3.1. *Suppose that the decision maker takes the power utility* (9.3) *and faces uncertainty modelled by an exponential Lévy process. Whenever the exponential growth rate of the utility is bounded from above by the discount factor, i.e.,* $\rho > \Psi(1-\alpha)$*, the utility–based irreversible investment model is well defined and the shadow utility process* ξ_t^{um} *is obtained as*

$$\xi_t^{um} = \frac{\theta}{1-\alpha} P_t^{1-\alpha} ,$$

where $\theta = \frac{1}{\rho} E\left[e^{(1-\alpha)\underline{Y}_{\tau(\rho)}}\right]$. *The expected utility under the optimal investment rule is given by*

$$F = \frac{E\left[e^{-\rho\tau^* + (1-\alpha)Y_{\tau^*}}\right]}{\rho} E\left[U(P_{\tau(\rho)})\right] - E\left[e^{-\rho\tau^*}\right] \frac{U(rI)}{\rho} ,$$

where $\tau(\rho)$ *is an exponential distributed time with parameter* ρ.

Proof: First, the utility maximization problem is well–posed if and only if

$$E\Big[\int_0^\infty e^{-\rho s} U(P_s) ds\Big] \;<\; \infty ,$$

or equivalently

$$E\Big[\int_0^\infty e^{-\rho s + (1-\alpha)Y_s} ds\Big] \;<\; \infty .$$

Using Fubini's theorem and Lévy–Laplace exponent $\Psi(z)$ yields

$$\begin{aligned} E\Big[\int_0^\infty e^{-\rho s + (1-\alpha)Y_s} ds\Big] &= \int_0^\infty e^{-\rho s} E\Big[e^{(1-\alpha)Y_s}\Big] ds \\ &= \int_0^\infty e^{-\rho s} e^{s\Psi(1-\alpha)} ds . \end{aligned}$$

Clearly, $\rho > \Psi(1-\alpha)$ is the necessary condition for the well–posedness of the problem.

To find the solution to the representation form (9.2), we construct $\xi_t^{um} = \frac{\theta}{1-\alpha} P_t^{1-\alpha}$. Substituting this and $P_t = P_0 e_t^Y$ into the right hand side of the representation form yields

$$\begin{aligned}
& E\left[\int_{\tau^*}^{\infty} \rho e^{-\rho s} \sup_{\tau^* \le v \le s} \xi_v^{um}\, ds \Big| \mathcal{F}_{\tau^*}\right] \\
= \; & E\left[\int_{\tau^*}^{\infty} \rho e^{-\rho s} \sup_{\tau^* \le v \le s} \frac{\theta}{1-\alpha} P_v^{1-\alpha}\, ds \Big| \mathcal{F}_{\tau^*}\right] \\
= \; & E\left[\int_{\tau^*}^{\infty} \rho e^{-\rho s} \sup_{\tau^* \le v \le s} e^{(1-\alpha)Y_v}\, ds \Big| \mathcal{F}_{\tau^*}\right] \frac{\theta}{1-\alpha} P_0^{1-\alpha} \\
= \; & E\left[\int_{0}^{\infty} \rho e^{-\rho s} \sup_{0 \le v \le s} e^{(1-\alpha)Y_v}\, ds\right] \frac{\theta}{1-\alpha} P_0^{1-\alpha} e^{-\rho\tau^* + (1-\alpha)Y_{\tau^*}},
\end{aligned}$$

where the last equality is obtained by applying the strong Markov property of Lévy processes. Simultaneously, the left hand side can be also reduced into

$$\begin{aligned}
& E\left[\int_{\tau^*}^{\infty} e^{-\rho s} U(P_s)\, ds \Big| \mathcal{F}_{\tau^*}\right] \\
= \; & E\left[\int_{\tau^*}^{\infty} e^{-\rho s} \frac{1}{1-\alpha} P_s^{1-\alpha}\, ds \Big| \mathcal{F}_{\tau^*}\right] \\
= \; & \frac{P_0^{1-\alpha}}{\rho(1-\alpha)} e^{-\rho\tau^* + (1-\alpha)Y_{\tau^*}} E\left[\int_0^{\infty} \rho e^{-\rho s} e^{(1-\alpha)Y_s}\, ds\right] \\
= \; & \frac{P_0^{1-\alpha}}{\rho(1-\alpha)} e^{-\rho\tau^* + (1-\alpha)Y_{\tau^*}} E\left[e^{(1-\alpha)Y_{\tau(\rho)}}\right],
\end{aligned}$$

where $\tau(\rho)$ is an exponential distributed time with parameter ρ. Then, equating these two parts and using the Wiener–Hopf identity, we have

$$\theta \;=\; \frac{1}{\rho} E\left[e^{(1-\alpha)\underline{Y}_{\tau(\rho)}}\right],$$

where $\underline{Y}_t$ is as previously defined the running infimum process of the Lévy process Y. With the value of θ, the expected utility of the investment becomes easy after some computation as below.

$$\begin{aligned}
F \;=\; & E\left[\int_{\tau^*}^{\infty} e^{-\rho t} U(P_t) dt\right] - E\left[\int_{\tau^*}^{\infty} e^{-\rho t} U(rI) dt\right] \\
= \; & E\left[\int_{\tau^*}^{\infty} \rho e^{-\rho s} \sup_{\tau^* \le v \le s} \xi_v^{um}\, ds \Big| \mathcal{F}_{\tau^*}\right] - \frac{U(rI)}{\rho} E\left[e^{-\rho\tau^*}\right] \\
= \; & E\left[\int_{\tau^*}^{\infty} \rho e^{-\rho s} \sup_{\tau^* \le v \le s} \frac{\theta}{1-\alpha} (P_0 e^{Y_v})^{1-\alpha}\, ds \Big| \mathcal{F}_{\tau^*}\right] - \frac{U(rI)}{\rho} E\left[e^{-\rho\tau^*}\right] \\
= \; & \frac{\theta}{1-\alpha} P_0^{1-\alpha} E\left[e^{-\rho\tau^* + (1-\alpha)Y_{\tau^*}} \int_0^{\infty} \rho e^{-\rho s} e^{(1-\alpha)\overline{Y}_s}\, ds \Big| \mathcal{F}_{\tau^*}\right] - \frac{U(rI)}{\rho} E\left[e^{-\rho\tau^*}\right] \\
= \; & \frac{\theta}{1-\alpha} P_0^{1-\alpha} E\left[\int_0^{\infty} \rho e^{-\rho s} e^{(1-\alpha)\overline{Y}_s}\, ds\right] E\left[e^{-\rho\tau^* + (1-\alpha)Y_{\tau^*}}\right] - \frac{U(rI)}{\rho} E\left[e^{-\rho\tau^*}\right] \\
= \; & \frac{E\left[P_0^{1-\alpha} e^{(1-\alpha)Y_{\tau(\rho)}}/(1-\alpha)\right]}{\rho} E\left[e^{-\rho\tau^* + (1-\alpha)Y_{\tau^*}}\right] - \frac{U(rI)}{\rho} E\left[e^{-\rho\tau^*}\right],
\end{aligned}$$

where the third step is achieved again due to the strong Markov property of Lévy processes and the expectation term can be furthermore written into two expectations since the integral part is fully independent with the information at time τ^*. The final result is arrived after substituting θ and applying Wiener–Hopf factorization. ■

The above theorem gives the optimal investment strategy and the project value. In the framework of utility, the value from investment can be understood as the compensation required according to the decision maker's subjective criterion by giving up the right to wait. This is also shown clearly in the final result which can be decomposed into the expected utility and the discount factor from the optimal investment time. In the next section, we concentrate on the threshold, providing both qualitative and quantitative analysis of the risk aversion effect on the trigger value.

9.4 Effect of Risk Aversion on Investment Decision

The utility framework constructed here yields several implications regarding risk aversions. These results are shown in two categories as follows.

Risk Aversion Slows Down Investment. The critical expected utility from the project for issuing the investment is identified as

$$E\left[\int_{\tau^*}^{\infty} e^{-\rho t} U(P_t)\, dt \middle| \mathcal{F}_{\tau^*}\right] \geq \theta \cdot \frac{U(rI)}{\rho},$$

where $\theta = E\left[e^{(1-\alpha)\overline{Y}_{\tau(\rho)}}\right]$. At first glance, it gives an image that the investment rule is almost the same as that of the profit–maximization problem but in terms of utility.

Despite the similar form, utility–maximization highlights the risk attitude of investors and its effect on the decision, compared to the profit–maximization problem. The first influence of maximizing the utility is the occurrence of investment even when the project revenue (before netting of the cost) creates a negative utility. In case of slight risk aversion ($0 \leq \alpha < 1$), the threshold utility is always positive and the multiplicative factor θ always larger than 1. While, the expected utility turns out to be negative, when investors are more risk averse with $\alpha > 1$. The net utility is nevertheless positive as $\theta < 1$. Intuitively, investment gives higher utility than merely holding the money I, although the cash flow from the project is too risky according to their subjective judgement. As a result, the firm in both cases would like to invest if and only if the utility from the project covers both the utility of the investment cost and the lost utility due to the delay in investment.

Furthermore as argued by Hugonnier and Morellec (2005), in the GBM model, the critical investment level is quite high relative to the profit–maximization threshold and it increases monotonically with the relative risk aversion coefficient α. The intuition behind is clear: The decision maker is risk averse and would prefer the project with less risk. Hence, when facing uncertainty in the future revenue, he has a strong incentive to delay the investment. This is also true when extending the model to more general processes, e.g., a Lévy process with possible unexpected shocks.

Theorem 9.4.1 (Comparative statics analysis of the risk aversion). *The threshold value is obtained as* $P^* = \eta^{-\frac{1}{1-\alpha}} rI$ *with* $\eta = E\left[e^{(1-\alpha)\underline{Y}_{\tau(\rho)}}\right]$ *for any exponential Lévy process utility maximization model. It is monotonically increasing in the risk aversion coefficient* α.

PROOF: The proof is provided in Appendix E. ■

Combination Effect of Risk Aversion and Jumps on Decision One may ask the question in this model construction how jumps affect the threshold value combined with risk aversion? In order to answer this question, we provide a specific example where a firm has to make an investment decision for a project. Assume that the output log–price has mean and variance equal to $m_1 = -0.03$ and $m_2 = 0.04$, respectively. In addition, the discount rate is supposed to be $\rho = 15\%$[3]. Clearly, the decision rule is dependent on the model he chooses: the estimated mean and variance have to be fitted to the model and hence may result in different trigger values. Suppose that the firm mainly focuses on 4 investment scenarios:

I. The project value is Gaussian distributed; and profit is the correct measure.

II. The project value is Gaussian distributed; managers are nevertheless risk averse and choose to check the utility that the project creates.

III. The project value is specified by a jump–diffusion process with negative jumps since e.g., more competitors may come into the market in the future and the price is greatly influenced by another product's price etc.; and profit is the correct measure.

IV. Project value is specified by a jump–diffusion process; and managers are also risk averse.

If the project value is lognormal distributed, the stochastic process of P_t is simply a GBM as described in Equation (6.1) and the drift and volatility are completely determined by $\mu = m_1$ and $\sigma^2 = m_2$. The jump–diffusion process with negative jumps is characterized as a special case of (7.11) with $p = 0$ (i.e., the jump follows a compound Poisson process whose jump size is exponential distributed.). In this case, the its mean and variance are obtained as

$$m_1 = \Psi'(0) \qquad \text{and} \qquad m_2 = \Psi''(0)\,.$$

It helps to uniquely determine the drift and volatility term by the values of m_1, m_2, c and λ.

[3]A relatively low expectation of the log–price and a high discount factor are assumed in order to draw a picture for a certain interval of α which is large enough for illustration. Such a seemingly unreal assumption is required to make all the chosen parameters economically sensible.

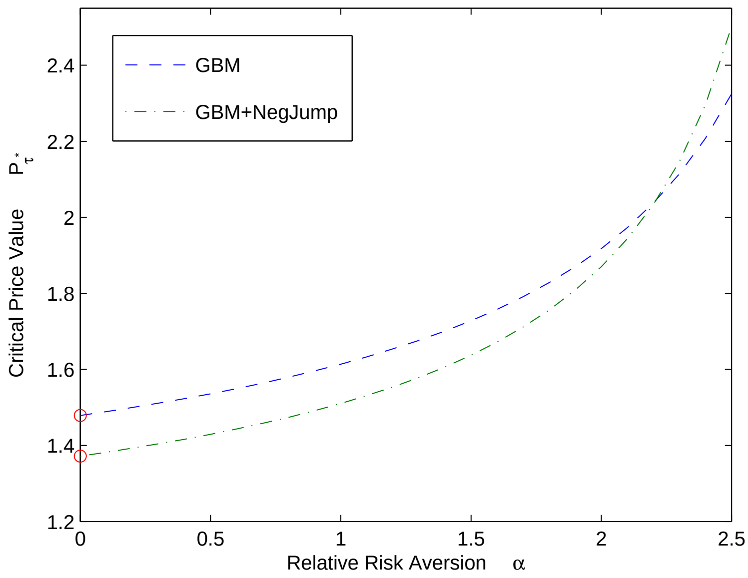

Figure 9.1: Investment Thresholds of Four Models with Parameters $m_1 = -0.03$, $m_2 = 0.04$, $c = 7.5$ and $\lambda = 1.0$ (for Negative Jump): Two Red Circles and Two Lines Describe Four Models with Different Risk Attitudes and Underlying Processes.

Figure 9.1 gives the critical price value for the four different scenarios. Profit maximization is one special case of utility–based model. Thus, the investment thresholds for Scenario *I* and *III* are the two red circles corresponding to $\alpha = 0$. By contrast, the investment threshold obtained from a simple NPV criterion, it is 1. It constitutes the most well–known result from the real options theory: irreversibility and time flexibility drive investors to wait until much larger thresholds. In presence of risk aversion, the trigger values of Scenario *II* and *IV* become even larger and as expected are heavily dependent on the relative risk aversion coefficient α. It is shown in the figure that P_{τ^*} in both cases increases monotonically with α, confirming the result of Theorem 9.4.1. In other words, risk aversion increases firms' initiative to postpone the investment. Consequently, the profit–maximization model in general gives a *wrong* investment decision when the investor is indeed risk averse. Moreover, with the same mean and variance, it is the Non–Gaussian model whose investment trigger value is more affected by α.

It is well acknowledged that in the profit maximizing model a GBM gives a higher threshold value than a jump–diffusion process with negative jumps. It is in fact one essential argument in the literature, recommending the introduction of jumps. However, this property is not maintained when we include additionally utility in the model. A jump–diffusion process gives a lower critical value only when α is small. With a large α, utility consideration may give a higher value than that for a GBM. We find this fact in many numerical analysis even for a jump–diffusion process with positive jumps. See for instance Figure 9.2.

To get more insights on the impact of jumps, we plot in Figure 9.3 investment trigger

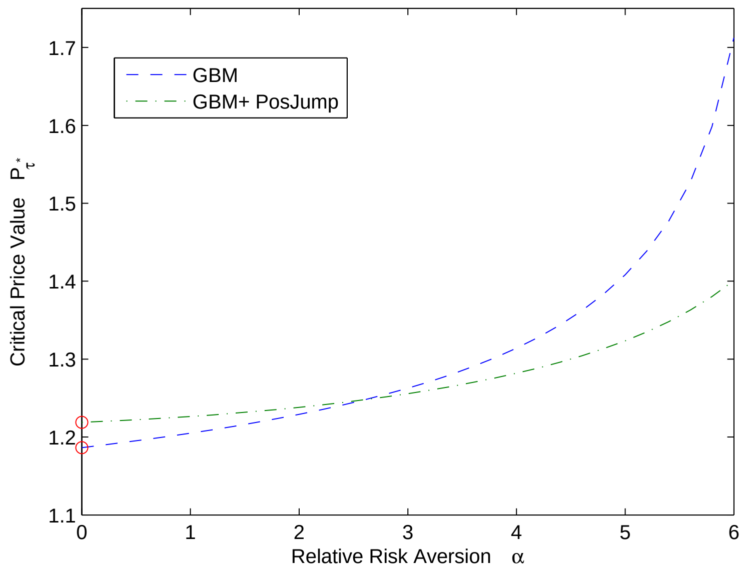

Figure 9.2: Investment Thresholds of Four Models with Parameters $m_1 = 0.07$, $m_2 = 0.04$, $c = 8.5$ and $\lambda = 0.5$ (for Positive Jump): Two Red Circles and Two Lines Describe Four Models with Different Risk Attitudes and Underlying Processes.

values for a family of jump–diffusion processes with various jump coefficients and two different relative risk aversion coefficients of $\alpha = 0$ and $\alpha = 1.75$. In the profit–maximization model ($\alpha = 0$), P_{τ^*} always decreases with λ and c. That is, given certain fixed estimates on the log–price's mean and variance, the risk neutral decision maker is less hesitant to invest when he expects either higher negative jumps or negative jumps with a larger probability. It is because the addition of jump terms decreases the diffusion uncertainty which is in effect the biggest "loss of information" and hence involves the highest uncertainty in comparison to other processes with the same instantaneous volatility. To clarify it, the BM is standard normal distributed and could be obtained as the limit of the average sum of a large number of i.i.d. random variables with finite variance. The larger the number is, the closer their distributions are and thus the more the information is missing! In this sense, the BM modification follows the principle of maximum entropy or minimum information while remaining consistent with the given knowledge — the estimated variance (see Boyarchenko and Levendorskiǐ (2004b) and also the detailed mathematical argument in Bouchard and Potters (2000)). Consequently, the decision maker is better informed of the future profits of the investment or equivalently faces less uncertainty by increasing the relevant parameters of negative jumps. It in turn decreases the threshold value.

However, an increase in jumps has an ambiguous effect on the trigger value when considering utility maximization or risk aversion. As observed in the right plot of Figure 9.3 for $\alpha = 1.75$, P_{τ^*} rises slightly with λ when $c = 5$ but decreases when $c = 7.5$. To our knowledge, there are two opposite effects of jumps on the trigger value for the case of a fixed estimate on variance: It decreases on one hand the diffusion volatility of the

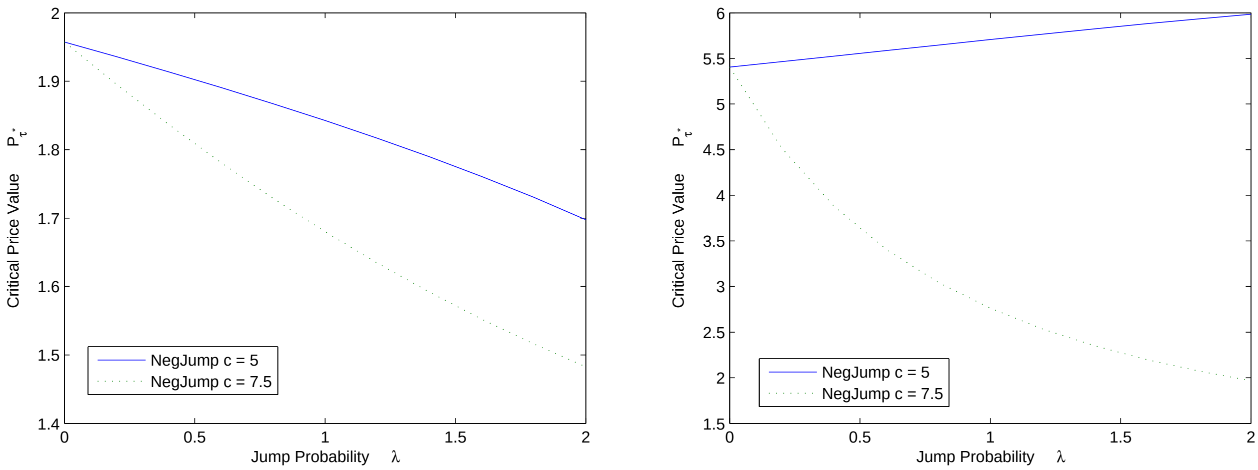

Figure 9.3: Investment Trigger Value vs. Jump Coefficients
for $\alpha = 0$ (left) and $\alpha = 1.75$ (right) with Estimates $m_1 = -0.06$ and $m_2 = 0.16$

project which in turn lowers the threshold value; on the other hand (negative) jumps increase the trigger value as a response to the expectation of gloomy economic conditions. In case of risk aversion, the second effect of jumps is greatly exaggerated, as risk averse investors would prefer to a larger extent waiting in order to avoid a great loss. Therefore, the critical price value turns out to be much higher than that in the profit–maximization model although with the same jump coefficients. Moreover, P_{τ^*} even increases with λ when $c = 5$. The declining relationship is recovered only when the jump term is high enough (as in the case of $c = 7.5$) to overcome the second effect additionally driven by the risk aversion.

9.5 Conclusion

This chapter formulates and analyzes an utility–based model for irreversible investment problems. This model recognizes the market incompleteness and most importantly the risk aversion of investors. By doing so, this chapter extends the classical real options model which is heavily dependent on the assumption of a complete market and risk neutrality. The objective problem is solved by means of our new stochastic representation method. The optimal investment time can be easily identified as the first time when the *shadow utility* process rises high enough to compensate the utility lost of putting the investment cost on risky investment instead of risk–free assets.

Although the investment decision rule is in structure similar to the profit–maximization model with the only difference of being measured in utility, the exercise threshold exhibit a great dependence with respect to uncertainty and risk aversion. Our conclusions show that the standard real options model leads to over–investment. Incorporating risk aversion increases the investment trigger value and hence a recommendation of later–investment. This result is intuitive and understandable since decision makers have more incentive to wait for avoiding possible risks. In addition, we also examine the combination impact of unexpected jumps and risk aversion on the threshold value with the help of a numerical example. Unlike the observation in the profit maximization model that the inclusion of negative jumps lowers the trigger value, the threshold in presence of risk aversion can be also higher than that under the modification of GBM. This is because risk aversion has two opposite influence on the critical value given a certain estimate of variance. It hints that model specification of the underlying uncertainty is greatly important for the optimal investment strategy.

Various extensions of this utility–based model can be done. For instance, the partial spanning concept can be introduced in the model as done in Henderson (2005). By considering one correlated asset traded in the financial market, part of risks of investment can be then hedged. It gives then a even more generalized model for the case of an incomplete market. Meanwhile, other qualitative features may be found for the decision rule if using other utility functions, although complicated calculation or numerical method may be required.

Chapter 10

Conclusion

The second part of this thesis deals with the irreversible investment problem under uncertainty. In contrast with the standard real options theory, all the related problems are solved by applying an alternative stochastic representation method.

In order to clearly elucidate the method and the resulted decision rule, the standard one–shot investment model is first considered. After identifying the maximization problem of the expected present value from the investment, we rewrite the expected revenue of the project in form of an expected present value of a running supremum process. This supremum process is defined as *shadow revenue* of the investment which is especially significant in this method, signifying the optimal investment time. On this basis, we derive our new *Shadow NPV* rule. To put it in a simple word, the project is launched at the time when the *shadow revenue* of the investment becomes equal to or higher than the investment cost. The *shadow revenue* here means the economic value of the project by deducting the entire investment cost. Therefore, it measures the willingness of the decision maker to pay the investment cost and meanwhile to give up the option to wait. Clearly, it is different from the NPV which is the money value of the project without taking into account the opportunity cost incurred when postponing the investment. In this sense, our method improves the conventional NPV rule by specifying the proper NPV. Moreover, this method is closely associated with the expected present value from the investment which the concern of practitioners lies in. It enables then easy understanding and possibly wider application of the real options theory into the reality.

This method is advantageous than other approaches available in that it fits for any semi–martingale processes even with stochastic interest rate. Meanwhile, our method makes it feasible to obtain a closed–form solution for an infinite–lived investment with an exponential Lévy process or a time–inhomogeneous diffusion process modelled uncertainty. The threshold values in the case of exponential Lévy processes are in the simple form as in the benchmark GBM model. Therefore, our investment policy generalizes the standard real options theory and meanwhile provides additional intuitions even within in the GBM model.

Considering that capacity can be built up gradually, this dissertation extends the standard model with the possibility of capacity expansion and develops a general theory to sequential irreversible investments when the firm faces uncertain economic situations. We construct for this purpose a generalized model where a value–maximizing firm has a concave operating profit function and is subject to a multiplicative economic shock modelled by a semimartingale. As one of the main contributions of this dissertation, the existence and uniqueness theorem is first derived for both finite and infinite horizon cases. Then, the optimal investment rule is derived in a very detailed and intuitive way by the stochastic representation method.

Here, the sequential irreversible investment problem is treated as a sequence of singular control problems. As the starting point, the necessary and sufficient optimality conditions are first derived by investigating the relationship of the marginal profit and cost of investment. In principal, the marginal profit by installing any infinitesimal unit of capital has to be lower than or equal to the marginal cost. Investment occurs "at the margin" if and only if the capacity is depreciated or the investment cost declines such that the marginal profit becomes equal to the marginal cost. Then, the marginal investment problem is further solved by constructing and characterizing a *base capacity*, the optimal capacity that the firm should hold. In this way, the optimal investment policy is defined as: if the current capacity is lower than or equal to the base capacity, investment is undertaken at once to increase the capacity to the base capacity; otherwise just keep the current capacity. The first means to characterize the base capacity is through solving optimal stopping problems. In order to gain the maximal flexibility for future decisions, the base capacity is found in a "cautious way" to be the lowest capacity level that makes the optimality condition binding. Alternatively by another more tractable approach, the optimal capacity is characterized via a stochastic backward equation. Due to irreversibility, the base capacity can not be exactly matched at all times. More feasibly, we are going to track it as closely as possible. Since the capacity is always maintained over or at the base capacity, the feasible capacity is recognized in form of the running maximum of the base capacity. Then after combining it with the optimality condition, the base capacity is finally calculated by solving a stochastic backward equation, which is shown to have always one unique solution.

Based on the optimal investment policy, a thorough analysis is carried out with illustrative examples for qualitative characterization and comparative statics of the irreversible investment. Generally, the optimal investment plan can be characterized by three different phenomena: smooth continuous investment, lump sum investment and singular investment. The smooth investment occurs in an absolutely continuous way at a positive rate during free intervals. Through such intervals, the marginal profit is equal to the user cost of capital, as in the case of reversible investment. However, the equality is maintained only in expectation on average over time in blocked intervals which start with an investment in lumps and continue with no further investment. Lump sum investment consists of all the jumps in investment. They are demonstrated to take place only with fixed surprises which means an information discontinuity at a given date. Nevertheless, the capacity never jumps to an excess capacity with respect to the operating profit. The remaining investment phenomenon is defined as singular investment. It is singular with respect to an infinitesimal time, that is, the set of the investment occurrence time is of Lebesgue measure zero. Hence, it occurs continuously but in a nowhere differentiable fashion. Generally, singular investment exists whenever the uncertainty is (partly) modelled by BMs. Moreover, this method leads to some general comparative statics results. First, the base capacity is shown to be monotonically increasing in the exogenous shock when the operating profit function is supermodular or equivalently has increasing differences in capacity and exogenous economic shocks. Besides, a natural result is obtained that the firm size always declines with the user cost of capital.

Generally, numerical methods have to be used to identify the base capacity according to the algorithm given in the thesis. Nevertheless, closed–form solutions of the optimal investment policy are possible for some special cases. To emphasize this feature, a specific model is constructed on the basis of Pindyck (1988) and Abel and Eberly (1996). In the model, an infinitely–lived firm is endowed with the operating profit function of Cobb–Douglas type. Specifically, the multiplicative economic shock is modelled by an exponential Lévy process with possible rare and unexpected jumps. Under this construction, the base capacity is explicitly solved by means of strong Markov property and time homogeneity of Lévy processes. Particularly, the base capacity is characterized simply by the exogenous economic shock multiplied by a constant factor expressed in terms of expectation. In this way, the marginal profit under the optimal investment plan is always kept below the user cost of capital times a markup factor.

As another extension of the standard real options theory, we are concerned with the market incompleteness and risk aversion. To this end, a utility–based model is discussed. Our proposed method handles the problem perfectly. The utility maximization problem is similarly reduced into a representation problem but in terms of the running supremum of the *shadow utility process*. A *shadow utility* decision rule is finally determined: the investment decision is committed if and only if the *shadow* utility gain from the investment rises up to fully cover the utility loss due to the investment. In the utility framework, it is even more intuitively understandable. The *shadow utility* records the willingness of investors to undertake the project netting of the subjective value for both giving up the option to invest and paying the investment cost into risky assets. Otherwise, he can wait for a better information while gaining utility from investing the wealth into risk–free savings.

In particular, when the firm is risk averse with CRRA and faces an exponential Lévy process modelled uncertainty, the investment rule is obtained in an analytically tractable form. Consistent with economic intuition, trigger values tend to increase with risk aversion because the risk averse decision maker has more incentive to delay the investment. Hence, investment is slowed down in the presence of risk aversion. In this sense, the standard real options decision rule may be wrong by assuming a risk preference free framework. With fixed estimates of the mean and variance of the underlying log–price processes, the threshold value under the Gaussian modification can be lower or higher than that under the jump–diffusion process modification with negative jumps, heavily depending on the relative risk aversion coefficient and jump component parameters. We argue that this is not a counter–intuitive result: Given the estimated variance, the increasing effect of negative jumps on the trigger value is greatly magnified by the risk aversion even if the jump term simultaneously lowers the trigger value by means of decreasing diffusion uncertainty.

Before concluding this thesis, we would like to give a remark on the discrepancies between real options and financial options. In our mind, they, although share the similar payoff structure, differ in several prospects which are summarized mainly in the following four points: First, real options are on physical assets and mostly have no clean environment for evaluation as financial options trading in somewhat independent financial markets. They are subject both to macro–economical (supplies, demands and so on) and micro–economical (like strategic competition) conditions. Second, real options generally have longer time to expiration than financial options. Perpetual horizon is sometimes not imaginary but practical, e.g., for the case of land. Third, real options have more peculiar risks of each individual project. Indeed, it is essential for investment opportunity valuation. Most importantly, analysis of these two options have different objectives: Pricing and especially hedging are usually two basic concerns of financial options; while for real options, the decision rule (the trigger value determination) is much more important.

Although the mathematical treatment is identical to that for financial options, the real options theory is not a mere adaptation of those financial options approaches. In fact, a real option problem can be viewed as a problem of dynamic optimization under uncertainty of a real asset given some possible options. The real options theory provides a framework to model the available managerial flexibility dynamically with uncertainty. This is the essence of the real options concept. As Amram and Kulatilaka (1999) say, *real options "is a way of thinking"*. Therefore, more work and extension have to be done in order to better model real options, fully incorporating their properties and facilitating its application in practice. With this objective, this dissertation takes our first step and will move further on the way.

Appendix A

Proof of Chapter 3

A.1 Proof of Theorem 3.2.1

Proof: First, the Lagrange function for the minimization problem is formed

$$
\begin{aligned}
\mathcal{L} &= \sum_{i=1}^{N} \omega_i e^{-rT} E^{\mathcal{Q}} \left[(S_i(T) - k_i)^+ \right] + \lambda \left(\sum_{i=1}^{N} \omega_i k_i - K \right) \\
&= \sum_{i=1}^{N} \omega_i e^{-rT} \int_{max(k_i,0)}^{\infty} (x_i - k_i)\, f_i(x_i) dx_i + \lambda \left(\sum_{i=1}^{N} \omega_i k_i - K \right)
\end{aligned}
$$

where $f_i(x_i)$ is the lognormal density function under the risk–neutral martingale measure $\mathcal{Q}$ for the stock S_i. A necessary and sufficient condition for the sequence k_i to minimize the Lagrange function is found through the first–order conditions (FOCs):

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial k_i} &= -\omega_i e^{-rT} \frac{\partial max\,(k_i, 0)}{\partial k_i} \left\{ max\,(k_i, 0) - k_i \right\} f_i \left[max\,(k_i, 0) \right] \\
&\quad -\omega_i e^{-rT} \int_{max(k_i,0)}^{\infty} f_i(x_i) dx_i + \lambda \omega_i \\
&= 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= \sum_{i=1}^{N} \omega_i k_i - K = 0\,.
\end{aligned}
$$

These conditions can be further simplified to

$$
\frac{\partial \mathcal{L}}{\partial k_i} = -\omega_i \left(e^{-rT} \int_{max(k_i,0)}^{\infty} f_i(x_i) dx_i - \lambda \right) = 0 \qquad \forall\, i = 1, \cdots, N \tag{A.1}
$$

$$
\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_{i=1}^{N} \omega_i k_i - K = 0 \tag{A.2}
$$

since the term of the first condition $\omega_i e^{-rT} \frac{\partial max(k_i,0)}{\partial k_i} \left\{ max\,(k_i, 0) - k_i \right\} f_i \left[max\,(k_i, 0) \right]$ is always equal to zero no matter which value $max\,(k_i, 0)$ is going to take.

With these conditions, one can first prove that $k_i \in [0, K]$ $\forall$ $i = 1, \cdots, N$ is always satisfied. Assume any specific $\bar{i}$ we have $k_{\bar{i}} < 0$. This implies that

$$\frac{\partial \mathcal{L}}{\partial k_i}|_{k_i = k_{\bar{i}}} = -\omega_i \left(e^{-rT} - \lambda \right) = 0 \,.$$

In this case, FOC (A.1) can be reduced to

$$\int_{max(k_i,0)}^{\infty} f_i(x_i) dx_i = 1 \qquad \forall \, i = 1, \cdots, N \,,$$

which implies the result that $k_i \leq 0$ $\forall$ $i = 1, \cdots, N$. This contradicts however the second condition (A.2). Therefore, k_i's are always positive and lie in the interval $[0, K]$.

Then, given $k_i \in [0, K]$, $\forall \, i = 1, \cdots, N$, FOC (A.1) can be stated as

$$\Phi(d_2(S_i, k_i)) = \Phi(d_2(S_j, k_j)) \qquad \forall \quad i, j$$

where $d_2(S_i, k_i) = \frac{\ln\left(\frac{S_i(0)}{k_i}\right) + (r - q_i - \frac{1}{2}\sigma_i^2)T}{\sigma_i \sqrt{T}}$ as defined in the BS formula, and Φ again denotes the standard normal cumulative distribution function.

Furthermore, $\Phi(x)$ is bijective, Condition (A.1) can be reduced to

$$d_2(S_i, k_i) = d_2(S_j, k_j) \quad \forall \, i = 1, \cdots, N.$$

Then k_i can be all expressed in k_1 as

$$k_i = S_i \left(\frac{k_1}{S_1} \right)^{\frac{\sigma_i}{\sigma_1}} \exp\left\{ T \left[\left(1 - \frac{\sigma_i}{\sigma_1} \right) \left(r + \frac{1}{2}\sigma_1 \sigma_i \right) + \left(\frac{\sigma_i}{\sigma_1} q_1 - q_i \right) \right] \right\} \tag{A.3}$$

In summary, the optimal k_i's are all positive and determined by solving the system of equations

$$\begin{aligned} k_i &= S_i \left(\frac{k_1}{S_1} \right)^{\frac{\sigma_i}{\sigma_1}} \exp\left\{ T \left[\left(1 - \frac{\sigma_i}{\sigma_1} \right) \left(r + \frac{1}{2}\sigma_1 \sigma_i \right) + \left(\frac{\sigma_i}{\sigma_1} q_1 - q_i \right) \right] \right\} \\ \sum_{i=1}^{N} \omega_i k_i &= K \,. \end{aligned}$$

The existing problem is whether there is always a solution and whether the solution is unique. This is shown in the following way:

First, k_i is a strictly increasing function of k_1 since the first derivative of k_i with respect to k_1

$$k_i' = \frac{S_i \sigma_i}{S_1 \sigma_1} \left(\frac{k_1}{S_1} \right)^{\frac{\sigma_i}{\sigma_1} - 1} \exp\left\{ T \left[\left(1 - \frac{\sigma_i}{\sigma_1} \right) \left(r + \frac{1}{2}\sigma_1 \sigma_i \right) + \left(\frac{\sigma_i}{\sigma_1} q_1 - q_i \right) \right] \right\}$$

is always larger than zero.

Then the sum of the k_i's as a function of k_1 given by

$$g(k_1) = \sum_{i=1}^{N} k_i = \sum_{i=1}^{N} S_i \left(\frac{k_1}{S_1}\right)^{\frac{\sigma_i}{\sigma_1}} \exp\left\{T\left[\left(1-\frac{\sigma_i}{\sigma_1}\right)\left(r+\frac{1}{2}\sigma_1\sigma_i\right)+\left(\frac{\sigma_i}{\sigma_1}q_1-q_i\right)\right]\right\}$$

is also continuous and increasing in k_1, which could be proven again by checking its first derivative. Moreover,

$$g(k_1 = 0) = 0\,,$$

and

$$\begin{aligned} g(k_1 = K) &= \sum_{1=1}^{n} S_i \left(\frac{K}{S_1}\right)^{\frac{\sigma_i}{\sigma_1}} \exp\left\{T\left[\left(1-\frac{\sigma_i}{\sigma_1}\right)\left(r+\frac{1}{2}\sigma_1\sigma_i\right)+\left(\frac{\sigma_i}{\sigma_1}q_1-q_i\right)\right]\right\} \\ &= K + S_2 \left(\frac{K}{S_1}\right)^{\frac{\sigma_2}{\sigma_1}} \exp\left\{T\left[\left(1-\frac{\sigma_2}{\sigma_1}\right)\left(r+\frac{1}{2}\sigma_1\sigma_2\right)+\left(\frac{\sigma_2}{\sigma_1}q_1-q_2\right)\right]\right\} + \cdots \\ &\geq K\,. \end{aligned}$$

As a consequence, there is always a unique solution $k_i \in [0, K]$. ■

A.2 Derivation of the Basket Covariance Matrix

This appendix derives the covariance matrix of the original underlying basket, $\sum_{i=1}^{N} \omega_i S_i(t)$ ($t \in [0,T]$), which can be then decomposed by means of PCA to select those dominant assets for hedging. As assumed in Equation (1.1), all the underlying assets in the basket are correlated GBMs with constant drift $r - q_i$ and volatility σ_i. Then expressed in a vector, the weighted stock price $\omega\mathcal{S}$ has N elements in form of

$$\omega_i S_i(t) = \omega_i S_i(0) e^{(r-q_i-\frac{1}{2}\sigma_i^2)t+\sigma_i W_i(t)}$$

for all $i = 1, \cdots, N$.

The expected value of each weighted asset under the risk neutral measure $\mathcal{Q}$ is calculated based on the knowledge that the Wiener process is normal distributed, i.e., $W_i(t) \sim \mathcal{N}(0,t)$

$$\begin{aligned} E^{\mathcal{Q}}[\omega_i S_i(t)] &= \int_{-\infty}^{\infty} \omega_i S_i(0) e^{(r-q_i-\frac{1}{2}\sigma_i^2)t+\sigma_i x} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \\ &= \omega_i S_i(0) e^{(r-q_i-\frac{1}{2}\sigma_i^2)t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(x^2-2t\sigma_i x)} dx \\ &= \omega_i S_i(0) e^{(r-q_i-\frac{1}{2}\sigma_i^2)t+\frac{1}{2}\sigma_i^2 t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(x-t\sigma_i)^2} dx \\ &= \omega_i S_i(0) e^{(r-q_i)t}\,. \end{aligned}$$

Analogously, we obtain $\forall\ i = 1, \cdots, N$

$$\begin{aligned}
E^{\mathcal{Q}}\left[(\omega_i S_i(t))^2\right] &= E^{\mathcal{Q}}\left[\omega_i^2 S_i^2(0) e^{2(r-q_i-\frac{1}{2}\sigma_i^2)t+2\sigma_i W_i(t)}\right] \\
&= \int_{-\infty}^{\infty} \omega_i^2 S_i^2(0) e^{(2r-2q_i-\sigma_i^2)t+2\sigma_i x} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \\
&= \omega_i^2 S_i^2(0) e^{(2r-2q_i-\sigma_i^2)t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(x^2-4t\sigma_i x)} dx \\
&= \omega_i^2 S_i^2(0) e^{(2r-2q_i-\sigma_i^2)t+2\sigma_i^2 t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2t}(x-2t\sigma_i)^2} dx \\
&= \omega_i^2 S_i^2(0) e^{2(r-q_i)t+\sigma_i^2 t}
\end{aligned}$$

and for $i \neq j$ and $i, j = 1, \cdots, N$

$$\begin{aligned}
E^{\mathcal{Q}}\left[\omega_i S_i(t) \omega_j S_j(t)\right] &= E^{\mathcal{Q}}\left[\omega_i S_i(0) \omega_j S_j(0) e^{(r-q_i-\frac{1}{2}\sigma_i^2)t+\sigma_i W_i(t)} e^{(r-q_j-\frac{1}{2}\sigma_j^2)t+\sigma_j W_j(t)}\right] \\
&= \omega_i S_i(0) \omega_j S_j(0) e^{(2r-q_i-q_j-\frac{1}{2}\sigma_i^2-\frac{1}{2}\sigma_j^2)t} E^{\mathcal{Q}}\left[e^{\sigma_i W_i(t)+\sigma_j W_j(t)}\right] \\
&= \omega_i S_i(0) \omega_j S_j(0) e^{(2r-q_i-q_j-\frac{1}{2}\sigma_i^2-\frac{1}{2}\sigma_j^2)t} e^{\frac{1}{2}(\sigma_i^2+\sigma_j^2)t+\sigma_i\sigma_j\rho_{ij}t} \\
&= \omega_i S_i(0) \omega_j S_j(0) e^{(2r-q_i-q_j)t+\sigma_i\sigma_j\rho_{ij}t} .
\end{aligned}$$

In this way, the covariance matrix of the entire basket is achieved with N variances on the diagonal

$$\begin{aligned}
Var^{\mathcal{Q}}\left[\omega_i S_i(t)\right] &= E^{\mathcal{Q}}\left[(\omega_i S_i(t))^2\right] - \left(E^{\mathcal{Q}}\left[\omega_i S_i(t)\right]\right)^2 \\
&= \omega_i^2 S_i^2(0) e^{2(r-q_i)t+\sigma_i^2 t} - \left(\omega_i S_i(0) e^{(r-q_i)t}\right)^2 \\
&= \omega_i^2 S_i^2(0) e^{2(r-q_i)t}\left(e^{\sigma_i^2 t} - 1\right)
\end{aligned}$$

and other off–diagonal elements

$$\begin{aligned}
Cov^{\mathcal{Q}}[\omega_i S_i(t), \omega_j S_j(t)] &= E^{\mathcal{Q}}[\omega_i S_i(t)\omega_j S_j(t)] - E^{\mathcal{Q}}[\omega_i S_i(t)]E^{\mathcal{Q}}[\omega_j S_j(t)] \\
&= \omega_i S_i(0)\omega_j S_j(0) e^{(2r-q_i-q_j)t+\sigma_i\sigma_j\rho_{ij}t} - \omega_i S_i(0)\omega_j S_j(0) e^{(2r-q_i-q_j)t} \\
&= \omega_i S_i(0)\omega_j S_j(0) e^{(2r-q_i-q_j)t}\left(e^{\sigma_i\sigma_j\rho_{ij}t} - 1\right) .
\end{aligned}$$

Appendix B

Preliminaries on Lévy Processes

The purpose of this appendix is to give a brief overview of Lévy processes and some well-known mathematical properties which are often used in the second part of this thesis. The corresponding proofs and detailed account can be found in (almost) each literature on Lévy processes (cf. Bertoin (1996) and Sato (1999)).

B.1 Lévy Process: Definition and Concept

Definition B.1.1 (Lévy Process). *A real valued stochastic process $Y = (Y_t)_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ is a Lévy process if and only if it possesses the following properties:*

(a) Y is a càdlàg, adapted process, i.e., the paths of Y are right continuous with left limits $\mathbf{P}$-a.s.

(b) $\mathbf{P}[Y_0 = 0] = 1$ $\mathbf{P}$-a.s.

(c) For $0 \leq s < t$, the random increments $Y_t - Y_s$ are independent of the σ-field $\mathcal{F}_s$.

(d) For $0 \leq s < t$, $Y_t - Y_s$ has the same distribution as Y_{t-s}.

Such a process is sometimes also called a process with stationary independent increments and usually viewed as a mixture of independent processes of a Gaussian and a pure jump process. Hence, the GBM is a specific form when the jump component is absent, i.e., its sample paths are continuous. The full extent to which we can characterize Lévy processes is made via their Lévy–Laplace exponent (the Laplace transform of their law or their moment generating function):

Definition B.1.2 (Lévy–Laplace Exponent). *The moment generating function of a Lévy process is represented in the form*

$$E[e^{zY_t}] = e^{t\Psi(z)},$$

where the function $\Psi(z)$ is referred to as the Lévy–Laplace exponent of Y.

Note that the law of a Lévy process is uniquely determined by its Lévy–Laplace exponent. It is because that the Lévy–Laplace exponent characterizes uniquely all distributions of Y at each fixed time t. Moreover, due to the property of stationary independent increments it is trivial to have $\Psi(z) = \Psi(1)$, the Lévy–Laplace Exponent of Y_1. On this basis. the *characteristic equation* of the Lévy process Y is identified as $\Psi(z) = \rho$ for any $\rho > 0$.

B.2 Some Examples of Lévy Processes

Lévy Process is a big family of processes with several categories. Here, we name only those examples of Lévy processes which are actively used in the main text. For each example, we specify the associated Lévy–Laplace exponent and the characteristic equation.

B.2.1 GBM

GBM is one of the simplest Lévy processes. Especially, it is the only Lévy process with continuous paths, as a Lévy process pins down to a GBM when the jump component is absent. Suppose a GBM with constant drift μ and volatility σ:

$$Y_t = \mu t + \sigma W_t, \tag{B.1}$$

where $(W_t)_{t\geq 0}$ is the standard Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$.

Given this construction, one can find that

$$\begin{aligned} E[e^{zY_t}] &= \int_{\mathbb{R}} e^{z(\mu t+\sigma y)} \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}} dy \\ &= e^{(\mu z + \frac{1}{2}\sigma^2 z^2)t}, \end{aligned}$$

which implies that the Lévy–Laplace exponent of a GBM is obtained as $\Psi(z) = \frac{1}{2}\sigma^2 z^2 + \mu z$.

Then, its characteristic equation can be easily identified in form of

$$\frac{1}{2}\sigma^2 z^2 + \mu z - \rho = 0\,,$$

which is a quadratic equation giving one positive and negative root, β^+ and β^-, respectively.

B.2.2 Mixed Jump–Diffusion Processes

The next example is a mixed jump–diffusion process which consists of a GBM and a pure Poisson process

$$Y_t = \mu t + \sigma W_t - \eta N_t, \tag{B.2}$$

where $(N_t)_{t\geq 0}$ is a Poisson process with intensity λ and $\eta > 0$ is a constant. In this way, it is a Lévy process with negative jumps only.

With the Poisson distribution, we know that

$$\begin{aligned} E[e^{z(-\eta N_t)}] &= \sum_{n\geq 0} e^{-\eta z n} e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\ &= e^{\lambda t \left(e^{-\eta z} - 1\right)}, \end{aligned}$$

Then, the Lévy–Laplace exponent of the defined mixed jump–diffusion process can be found to be $\Psi(z) = \frac{1}{2}\sigma^2 z^2 + \mu z + \lambda\left(e^{-\eta z} - 1\right)$ and the characteristic equation

$$\frac{1}{2}\sigma^2 z^2 + \mu z + \lambda\left(e^{-\eta z} - 1\right) = \rho.$$

Clearly, this is complicated to solve due to the existence of one exponential term. Nevertheless, one can easily tell that this equation has one positive and one negative root by examining the figures of two functions $f_1 = \frac{1}{2}\sigma^2 z^2 + \mu z - (\lambda + \rho)$ and $f_2 = -\lambda e^{-\eta z}$.

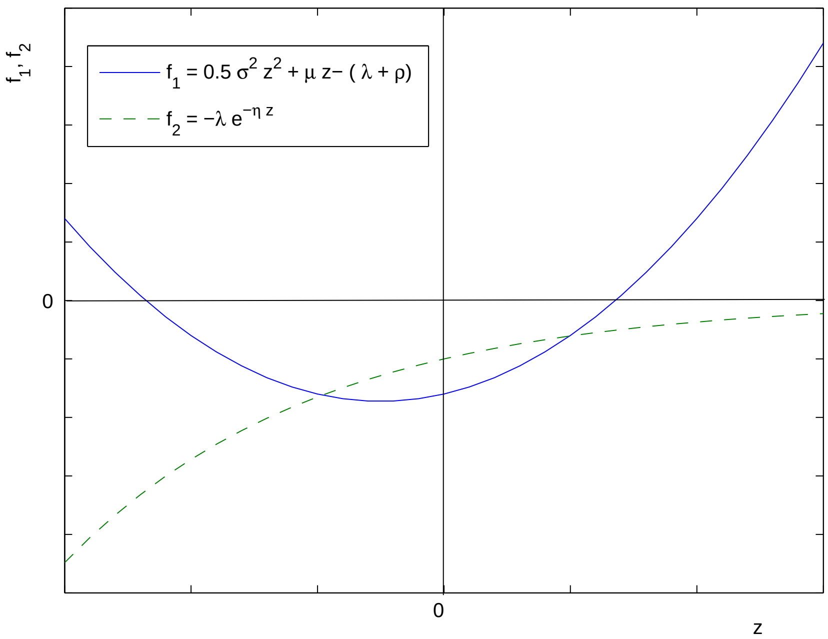

Figure B.1: Graphical Proof by Drawing the Plots of Two Functions

As shown in Figure $B.1$, f_1 is an upward parabola with the vertex $(-\frac{\mu}{\sigma^2}, -(\lambda + \rho + \frac{\mu^2}{2\sigma^2}))$. Meanwhile, it intersects with the vertical axis at the point $(0, -(\lambda + \rho))$. The function f_2 is an increasing concave function over the entire range $(-\infty, \infty)$, passing through the point $(0, -\lambda)$. All these factors guarantee that these two functions have altogether two roots at both sides of the origin.

B.2.3 GBM Combined with a Compound Poisson Process

As shown in the name, this process is a combination of a GBM and a jump component characterized by a compound Poisson process. Formally, it is a process of the form

$$Y_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} J_k, \tag{B.3}$$

where $(N_t)_{t\geq 0}$ is again a Poisson process of intensity λ and jump sizes $J = (J_k)_{k\in\mathbb{N}}$ are represented by a sequence of independent identically distributed random variables. The inclusion of compound Poisson processes obviously complicates the calculation of the Lévy–Laplace exponent. Following the result of the above two examples, we focus only on the jump component

$$\begin{aligned} E[e^{z\sum_{k=1}^{N_t} J_k}] &= \sum_{n\geq 0} E[e^{z\sum_{k=1}^{n} J_k} e^{-\lambda t}]\frac{(\lambda t)^n}{n!} \\ &= \sum_{n\geq 0}\left(\int_{\mathbb{R}} e^{zj}F(dj)\right)^n e^{-\lambda t}\frac{(\lambda t)^n}{n!} \\ &= e^{-\lambda t(\int_{\mathbb{R}} e^{zj}F(dj)-1)}, \end{aligned}$$

where $F(j)$ is the associated cumulative distribution function of J.

This process is widely used in finance to model asset (stock, bond, currency etc.) prices. One well–known example is the double exponential jump–diffusion process where J has a double exponential distribution (e.g. Kou (2004)). That is, the density of J is given by

$$f(j) = \begin{cases} pc^+e^{-c^+j} & j \geq 0, \\ (1-p)c^-e^{c^-j} & j < 0. \end{cases}$$

where the parameters $c^{\pm} > 0$ and $0 \leq p \leq 1$. Under this assumption, the project value at time t has in all N_t possible upward and downward jumps which occur with probability p and $1-p$, respectively. Each positive/negative jump is exponentially distributed with the parameter c^+/c^-. Such construction with an exponential distribution allows some results in analytical form (e.g. the property of first hitting times), which is indeed advantageous in application. For instance, its Lévy–Laplace exponent can be determined after further calculation as

$$\Psi(z) = \mu z + \frac{1}{2}\sigma^2 z^2 + \lambda p\frac{z}{c^+ - z} - \lambda(1-p)\frac{z}{c^- + z}\,.$$

Accordingly, the the characteristic equation $\Psi(z) = \rho$ follows

$$\mu z + \frac{1}{2}\sigma^2 z^2 + \lambda p\frac{z}{c^+ - z} - \lambda(1-p)\frac{z}{c^- + z} = \rho\,.$$

This equation has always two positive roots $\beta^+_{1/2}$ and two negative roots $\beta^-_{1/2}$ since it is essentially a quadratic function of form $ax^4 + bx^3 + cx^2 + dx + e = 0$ after rearranging and then can be solved analytically[1]. Those who are interested in the technique originally due to Ferrari are referred to Boyer and Merzbach (1991) and Borwein and Erélyi (1995).

B.3 Fluctuation Theorem

One significant fluctuation theory is the Wiener–Hopf factorization which is indeed the main technique used in this dissertation. Define $\overline{Y}_t = \sup_{0\leq s\leq t} Y_t$ and $\underline{Y}_t = \inf_{0\leq s\leq t} Y_t$ as the running supremum and infimum of the corresponding Lévy process Y_t. The Wiener–Hopf factorization describes exactly the relationship of these two processes and the Lévy–Laplace exponent.

Theorem B.3.1. *Let τ be an exponential random time with parameter ρ, independent of Y, and denoted as $\tau(\rho)$. Then we have the Wiener–Hopf factorization*

$$\frac{\rho}{\rho - \Psi(z)} = \Psi_\rho^+(z)\Psi_\rho^-(z),$$

where $\Psi_\rho^+(z) = E[e^{z\overline{Y}_{\tau(\rho)}}]$ and $\Psi_\rho^-(z) = E[e^{z\underline{Y}_{\tau(\rho)}}]$ represent the so–called Wiener–Hopf left and right factor, respectively.

Proof: First note that

$$\begin{aligned} E[e^{zY_{\tau(\rho)}}] &= \int_0^\infty \rho E[e^{zY_t}]e^{-\rho t}dt \\ &= \rho \int_0^\infty e^{-t\Psi(z)}e^{-\rho t}dt \\ &= \frac{\rho}{\rho - \Psi(z)}. \end{aligned}$$

Using the excursion theory (or simply Theorem $VI.5(i)$ in Bertoin (1996)), the random variables $\overline{Y}_t$ and $\overline{Y}_t - Y_t$ can be proven to be independent and hence

$$\begin{aligned} E[e^{zY_{\tau(\rho)}}] &= E[e^{z\overline{Y}_{\tau(\rho)}}e^{-z(\overline{Y}_{\tau(\rho)}-Y_{\tau(\rho)})}] \\ &= E[e^{z\overline{Y}_{\tau(\rho)}}]E[e^{-z(\overline{Y}_{\tau(\rho)}-Y_{\tau(\rho)})}] \\ &= E[e^{z\overline{Y}_{\tau(\rho)}}]E[e^{z\underline{Y}_{\tau(\rho)}}], \end{aligned}$$

where the third equality follows by duality, i.e., when Y is time reversed over a fixed time interval, it has the same law as $-Y$. In this way, taking all these equality into consideration yields the identity in the theorem. ■

This factorization is unique (Bertoin (1996)). Hence, it is possible to obtain analytical forms of the two Wiener–Hopf factors. For instance, for the case of a GBM, they are

[1]The sign of the roots are determined or guaranteed additionally by the regularity condition in order to achieve the economic sensibility.

related to the two roots of the characteristic equation $\Psi(z) = \rho$. Assume β^+ and β^- be the positive and negative roots of the characteristic equation. Then the two factors are given by

$$\Psi^+_\rho(z) = \frac{\beta^+}{\beta^+ - z} \qquad \text{and} \qquad \Psi^-_\rho(z) = \frac{\beta^-}{\beta^- - z}.$$

For general Lévy processes, the characterization of the two factors is not that trivial. However, it is still possible to identify them by means of residue calculus. For instance, Boyarchenko and Levendorskiĭ (2002a) provide the general solution form to regular Lévy processes of exponential type.

Example B.3.2. *Take the specification in Section B.2.3, namely the GBM combined with a compound Poisson process with a double exponential distributed jump size, as an example. It is one regular exponential Lévy process. Its two Wiener–Hopf factors are found to be closely related to the roots $\beta^\pm_{1/2}$ of the characteristic equation $\Psi(z) = \rho$*

$$\Psi^+_\rho(z) = \frac{\beta^+_1}{\beta^+_1 - z} \frac{\beta^+_2}{\beta^+_2 - z} \frac{c^+ - z}{c^+}$$

and

$$\Psi^-_\rho(z) = \frac{\beta^-_1}{\beta^-_1 - z} \frac{\beta^-_2}{\beta^+_2 - z} \frac{c^- - z}{c^-}.$$

Among other important formulae in the fluctuation theory, the Pecherskii–Rogozin identity deals with the first hitting time of Lévy processes. Define $\tau_y = \inf\{t > 0 \mid Y_t \geq y\}$ the first passage time of the level y. For general Lévy processes, their sample paths may be not continuous at level y due to the possible upward jumps. Equivalently, there maybe an overshoot above the level with the value of $Y_{\tau_y} - y$. The Pecherskii–Rogozin identity expresses the double Laplace transform of the joint distribution of the first passage time and the overshoot as stated in the following theorem[2]:

Theorem B.3.3. *For every triple (α, β, ρ), where $\alpha > 0$, $\beta \geq 0$ and $\rho > 0$,*

$$\int_0^\infty e^{-\rho y} E[e^{-\alpha\tau_y - \beta(Y_{\tau_y} - y)}]dy = \frac{1}{\rho - \beta}\left(1 - \frac{\Psi^+_\alpha(-\rho)}{\Psi^+_\alpha(-\beta)}\right).$$

B.4 Spectrally Negative Lévy Processes

Now we turn to the case in which Lévy processes have no positive jumps[3]. Formally, it is called a spectrally negative Lévy process. In this case, the corresponding characteristic

[2]For more detail and mathematical proof on the identity, refer to Bertoin (1996) and Alili and Kyprianou (2005).

[3]Note that by considering the dual process, similar results can be derived for the case when Y has only positive jumps.

equation $\Psi(z) = \rho$ has only one unique positive root β^+. The Wiener–Hopf factors are given by

$$\Psi_\rho^+(z) = \frac{\beta^+}{\beta^+ - z} \qquad \text{and} \qquad \Psi_\rho^-(z) = \frac{\rho}{\rho - \Psi(z)} \frac{\beta^+ - z}{\beta^+} .$$

In addition, a well–behaved distribution can be specified for the supremum $\overline{Y}_{\tau(\rho)}$ evaluated at an independent exponentially distributed time τ with parameter ρ. More precisely, $\overline{Y}_{\tau(\rho)}$ has an exponential distribution with parameter β^+ as shown by Bertoin (1996) in Chapter VII.

Another essential property of a spectrally negative Lévy process is that its paths have no upward discontinuity, hence zero overshoots over a specific level, due to the lack of upward jumps. As a result, the above Pecherskii–Rogozin identity can be simplified correspondingly. Applying the martingale theory as done in Bertoin (1996), the Laplace transform of the hitting time is obtained as

$$E[e^{-\alpha\tau_y} 1_{[\tau_y < \infty]}] = e^{-y\beta^+} , \tag{B.4}$$

where β^+ denotes the unique positive root of the characteristic equation $\Psi(z) = \alpha$.

Appendix C

Proof of Chapter 7

C.1 Proof of $e^{-\rho\tau}\pi_\tau \geq e^{-\rho\tau}\xi_\tau^{pm}$ for $\tau \in [0, \hat{T}]$.

Proof:

$$\begin{aligned} e^{-\rho\tau}\pi_\tau &= E\left[\int_\tau^{\hat{T}} \rho e^{-\rho t} \sup_{\tau \leq v < t} \xi_v^{pm}\, dt + e^{-\rho\hat{T}} \sup_{\tau \leq v \leq \hat{T}} \xi_v^{pm} \middle| \mathcal{F}_\tau\right] \\ &\geq E\left[\int_\tau^{\hat{T}} \rho e^{-\rho t} \xi_\tau^{pm}\, dt + e^{-\rho\hat{T}} \xi_\tau^{pm} \middle| \mathcal{F}_\tau\right] \\ &= e^{-\rho\tau}\xi_\tau^{pm}. \end{aligned}$$

■

C.2 Proof of Theorem 7.3.2

Proof: (i) The solution for κ: By the definition of the Wiener–Hopf factorization, it is easy to observe that the expectation form in κ is indeed the left Wiener–Hopf factor $\Psi_\rho^+(1)$ of the Lévy process Y_t. Thus, κ is obtained as

$$\kappa = (\rho - \Psi(1))E\left[e^{\overline{Y}_{\tau(\rho)}}\right] = (\rho - \Psi(1))\Psi_\rho^+(1).$$

The left Wiener–Hopf factor is readily recognized for the defined Lévy process Y_t. For a Lévy process with no positive jumps, it is more simple to solve by using the well-known exponential distribution of the supremum process $\overline{Y}_{\tau(\rho)}$ (Bertoin (1996), Chapter VII). More precisely, the running supremum at an exponentially distributed time with parameter ρ has an exponential distribution with parameter β^+, the unique positive root of the characteristic equation. In this way,

$$\begin{aligned}\kappa &= (\rho-\Psi(1))E\left[e^{\overline{Y}_{\tau(\rho)}}\right]\\ &= (\rho-\Psi(1))\int_0^\infty \beta^+ e^y e^{-\beta^+ y}dy\\ &= (\rho-\Psi(1))\left[-\frac{\beta^+}{\beta^+-1}e^{-(\beta^+-1)y}\Big|_0^\infty\right]\\ &= (\rho-\Psi(1))\frac{\beta^+}{\beta^+-1}\,.\end{aligned}$$

(*ii*) The value of the option to invest: The option value is easy for the case of Lévy processes with no positive jumps. In this case, there is no upward discontinuity due to the lack of upward jumps, hence zero overshoots over the critical level. Therefore, the project expected value at the optimal investment time is exactly $P_{\tau^*}=\kappa I$. Substituting $I=\frac{P_{\tau^*}}{\kappa}$ and the trigger level of the Lévy process $y^*=Y_{\tau^*}=\ln\frac{\kappa I}{P_0}$ yields the real option value

$$\begin{aligned}F &= E\Big[e^{-\rho\tau^*}(\pi_{\tau^*}-I)^+\Big]\\ &= E\left[e^{-\rho\tau^*}\left(\frac{P_{\tau^*}}{\rho-\Psi(1)}-\frac{P_{\tau^*}}{\kappa}\right)\right]\\ &= E\left[e^{-\rho\tau^*}\right]\left(E\left[e^{\overline{Y}_{\tau(\rho)}}\right]-1\right)\frac{P_{\tau^*}}{\kappa}\,.\end{aligned}$$

Then according to the Laplace transform of the hitting time $E[e^{-\rho\tau_y}1_{[\tau_y<\infty]}]=e^{-y\beta^+}$ where $\tau_y=\inf\{t\geq 0\,|\,Y_t\geq y\}$ (Bertoin (1996)), this can be further reduced to

$$\begin{aligned}F &= e^{-y^*\beta^+}\left(E\left[e^{\overline{Y}_{\tau(\rho)}}\right]-1\right)\frac{P_{\tau^*}}{\kappa}\\ &= \left(\frac{\kappa I}{P_0}\right)^{-\beta^+}\left(E\left[e^{\overline{Y}_{\tau(\rho)}}\right]-1\right)I\\ &= \left(E\left[e^{\overline{Y}_{\tau(\rho)}}\right]-1\right)\left(\frac{P_0}{\kappa}\right)^{\beta^+}I^{1-\beta^+}\,.\end{aligned}$$

However, such a nice form is not possible for a general Lévy process since the continuity at the level y^* is not guaranteed any more. Thus, a possible overshoot has to be considered in this context. As

$$\begin{aligned}F &= E\left[e^{-\rho\tau^*}\pi_{\tau^*}\right]-E\left[e^{-\rho\tau^*}I\right]\\ &= I\left[E\Big[e^{-\rho\tau^*}\frac{P_{\tau^*}}{I\,(\rho-\Psi(1))}\Big]-E\Big[e^{-\rho\tau^*}\Big]\right]\\ &= I\Big[E\Big[e^{-\rho\tau^*}\frac{\kappa P_{\tau^*}}{P^*\,(\rho-\Psi(1))}\Big]-E\Big[e^{-\rho\tau^*}\Big]\Big]\\ &= I\Big[E\Big[e^{-\rho\tau^*}E\Big[e^{\overline{Y}_{\tau(\rho)}}\Big]e^{Y_{\tau^*}-y^*}\Big]-E\Big[e^{-\rho\tau^*}\Big]\Big]\\ &= I\Big[E\Big[e^{\overline{Y}_{\tau(\rho)}}\Big]E\Big[e^{-\rho\tau^*+(Y_{\tau^*}-y^*)}\Big]-E\Big[e^{-\rho\tau^*}\Big]\Big]\end{aligned}$$

with $P^* = P_0 e^{y^*} = \kappa I$, the option value is known by calculating the two expectations. Based on the Pecherskii-Rogozin identity, their Laplace transforms are obtained in form of

$$\int_0^\infty e^{-qy} E\Big[e^{-\rho\tau^* + (Y_{\tau^*} - y)}\Big] dy = \frac{1}{q+1}\left(1 - \frac{\Psi_\rho^+(-q)}{\Psi_\rho^+(1)}\right)$$

and

$$\int_0^\infty e^{-qy} E\left[e^{-\rho\tau^*}\right] dy = \frac{1 - \Psi_\rho^+(-q)}{q}\,.$$

Thus, simple analytical formulae are only possible for some specific cases. In general, numerical methods have to be used to get the final solution.

■

C.3 Derivation of Explicit Solutions to CIR Processes

Bank and Föllmer (2003) claries how to solve the backward equation of a time–homogeneous diffusion process, of which CIR is one special case. To make the proof complete, we repeat their argument and specify it for CIR processes.

The project expected operating profit π_t can be easily demonstrated to be a CIR process with the initial value π_0. On this basis, assume that the solution ξ^{pm} is of the form $\xi_t^{pm} = \kappa(\pi_t)$, where κ is a strictly increasing continuous function on $[0, \infty)$ with $\kappa(0) = 0$ and $\kappa(+\infty) = +\infty$. Then the representation problem (7.5) can be reduced to

$$\pi_0 = E\left[\int_0^\infty \rho e^{-\rho t} \sup_{0\le u\le t} \kappa(\pi_u)\, dt \Big| \pi_0\right], \qquad \forall \quad \pi_0 \in [0, +\infty)$$

by means of strong Markov property.

Defining $\tau(\rho)$ to be an independent exponentially distributed random time with parameter ρ yields then

$$\pi_0 = E\left[\sup_{0\le u\le \tau(\rho)} \kappa(\pi_u) \Big| \pi_0\right], \qquad \forall \quad \pi_0 \in [0, +\infty)\,. \tag{C.1}$$

Since κ is an increasing function, the right–hand side of $(C.1)$ can be rewritten as

$$\begin{aligned} E\left[\sup_{0\le u\le \tau(\rho)} \kappa(\pi_u) \Big| \pi_0\right] &= E\left[\kappa\,(\sup_{0\le u\le \tau(\rho)} \pi_u) \Big| \pi_0\right] \\ &= \int_0^\infty \mathbb{P}_{\pi_0}\left[\kappa\,(\sup_{0\le u\le \tau(\rho)} \pi_u) > y\right] dy \end{aligned} \tag{C.2}$$

where $\mathbb{P}_{\pi_0}[.]$ denotes the conditional probability. The probability can be further calculated as

$$\begin{aligned}
\mathbb{P}_{\pi_0}\Big[\kappa(\sup_{0\le u\le \tau(\rho)} \pi_u) > y\Big] &= \mathbb{P}_{\pi_0}\Big[\sup_{0\le u\le \tau(\rho)} \pi_u > \kappa^{-1}(y)\Big] \\
&= \begin{cases} 1 & \text{if } \pi_0 \ge \kappa^{-1}(y) \text{ or } \kappa(\pi_0) \ge y\,, \\ \mathbb{P}_{\pi_0}\Big[\tau_{\kappa^{-1}(y)} < \tau(\rho)\Big] & \text{otherwise.} \end{cases}
\end{aligned}$$

and by using Fubini's theorem

$$\begin{aligned}
\mathbb{P}_{\pi_0}[\tau_{\kappa^{-1}(y)} < \tau(\rho)] &= \int_0^\infty \rho e^{-\rho t}\left\{\int_0^t \mathbb{P}_{\pi_0}[\tau_{\kappa^{-1}(y)} \in ds]\right\} dt \\
&= \int_0^\infty \left\{\int_t^\infty \rho e^{-\rho t} dt\right\} \mathbb{P}_{\pi_0}[\tau_{\kappa^{-1}(y)} \in ds] \\
&= \int_0^{+\infty} e^{-\rho t}\mathbb{P}_{\pi_0}[\tau_{\kappa^{-1}(y)} \in ds] \\
&= E\Big[e^{-\rho\tau_{\kappa^{-1}(y)}}\Big|\pi_0\Big]\,.
\end{aligned}$$

Clearly, $E\Big[e^{-\rho\tau_{\kappa^{-1}(y)}}\Big|\pi_0\Big]$ is the Laplace transform of the first passage time

$$\tau_{\kappa^{-1}(y)} = \inf\{t \ge 0 |\ \pi_t = \kappa^{-1}(y)\}\,.$$

The Laplace transform for the CIR process π_t is obtained according to Karlin and Taylor (1981), p. 245

$$E[e^{-\rho\tau_y}\ |\pi_0 = v_0] = \frac{{}_1F_1(\frac{\rho}{\gamma}, \frac{2\gamma\mu}{\sigma^2}; \frac{2\gamma}{\sigma^2}v_0)}{{}_1F_1(\frac{\rho}{\gamma}, \frac{2\gamma\mu}{\sigma^2}; \frac{2\gamma}{\sigma^2}y)}\,, \tag{C.3}$$

where ${}_1F_1(a, b; x)$ stands for the confluent hypergeometric function. Then denote $\varphi_\rho(y) = {}_1F_1(\frac{\rho}{\gamma}, \frac{2\gamma\mu}{\sigma^2}; \frac{2\gamma}{\sigma^2}y)$ a continuous and strictly increasing convex function, we have

$$\mathbb{P}_{\pi_0}[\tau_{\kappa^{-1}(y)} < \tau(\rho)] = \frac{\varphi_\rho(\pi_0)}{\varphi_\rho(\kappa^{-1}(y))}\,.$$

Substituting this and $(C.2)$ into $(C.1)$ yields

$$\pi_0 = \int_0^{\kappa(\pi_0)} dy + \int_{\kappa(\pi_0)}^\infty \frac{\varphi_\rho(\pi_0)}{\varphi_\rho(\kappa^{-1}(y))} dy = \kappa(\pi_0) + \varphi_\rho(\pi_0)\int_{\pi_0}^\infty \frac{d\kappa(z)}{\varphi_\rho(z)}\,,$$

where the last equality is obtained by replacing $y = \kappa(z)$.

Putting this in differential form results in

$$d\pi_0 = d\kappa(\pi_0) + d\varphi_\rho(\pi_0)\int_{\pi_0}^{+\infty} \frac{d\kappa(z)}{\varphi_\rho(z)} - d\kappa(\pi_0),$$

or equivalently

$$\frac{1}{\varphi'_\rho(\pi_0)} = \int_{\pi_0}^{+\infty} \frac{d\kappa(z)}{\varphi_\rho(z)} \, .$$

Differentiating it again yields

$$d\frac{1}{\varphi'_\rho(\pi_0)} = -\frac{d\kappa(\pi_0)}{\varphi_\rho(\pi_0)} \, .$$

This gives

$$\kappa(\pi_0) = -\int_0^{\pi_0} \varphi_\rho(y) d\frac{1}{\varphi'_\rho(y)} = \pi_0 - \frac{\varphi_\rho(\pi_0)}{\varphi'_\rho(\pi_0)} \qquad \forall \quad \pi_0 \in [0, +\infty) \, ,$$

where we use the partial integration in the last step and $\lim_{y\to 0}\frac{\varphi_\rho(y)}{\varphi'_\rho(y)} = 0$ due to the convexity of φ_ρ .

As $\varphi_\rho(y)$ is a continuous and strictly increasing convex function, the obtained function $\kappa(v)$ is indeed strictly increasing and continuous. Consequently, it validates the assumption for the function and for the procedure given above. In this way, the trigger project value is then expressed as

$$\pi_{\tau^*} = I + \frac{\varphi_\rho(\pi_{\tau^*})}{\varphi'_\rho(\pi_{\tau^*})} \, .$$

The option to invest is then valued as

$$\begin{aligned} E\Big[e^{-\rho\tau^*}(\pi_{\tau^*} - I)^+ \Big| \mathcal{F}_0\Big] &= E\Big[e^{-\rho\tau^*}\Big|\pi_0\Big](\pi_{\tau^*} - I) \\ &= \frac{\varphi_\rho(\pi_0)}{\varphi_\rho(\pi_{\tau^*})}\frac{\varphi_\rho(\pi_{\tau^*})}{\varphi'_\rho(\pi_{\tau^*})} \\ &= \frac{\varphi_\rho(\pi_0)}{\varphi'_\rho(\pi_{\tau^*})} \, . \end{aligned}$$

Remark C.3.1. *The Laplace transform of the first passage time for a CIR process is not trivial to calculate. However thanks to Karlin and Taylor (1981) (p. 245), it can be obtained by solving a differential equation combined with a boundary condition.*

Define $u(\pi) = 1/E\big[e^{-\rho\tau_\pi}\big|\pi_0\big]$, $0 < \pi_0 < \pi < \infty$. *Then,* $u(\pi)$ *satisfies the differential equation*

$$\frac{1}{2}\sigma^2\pi u''(\pi) + \gamma(\mu - \pi)u'(\pi) \;\; = \;\; \rho u(\pi) \qquad \text{(C.4)}$$

with the boundary condition

$$u(\pi_0) = 1. \tag{C.5}$$

Setting $V = \frac{\sigma^2 y}{2\gamma}$ *and* $u(\pi) = f(y)$, *Equation* (C.4) *becomes then*

$$yf''(y) + (\frac{2\gamma\mu}{\sigma^2} - y)f'(y) - \frac{\rho}{\gamma}f(y) = 0\,.$$

Recall that the Kummer ordinary differential equation (see Abramowitz and Stegun (1969), Chapter 13*)*

$$yf''(y) + (\eta - y)f'(y) - \alpha f(y) = 0,$$

where $\alpha, \eta > 0$ *are two constants, has two independent solutions expressed in terms of the confluent hypergeometric function: One is* ${}_1F_1(\alpha, \eta; y)$[1]*, positive and strictly increasing for* $y \in [0, \infty)$ *with* ${}_1F_1(\alpha, \eta; 0) = 1$ *and* $\lim_{y\to\infty} {}_1F_1(\alpha, \eta; y) = \infty$*; while the other is positive but strictly decreasing for* $y \in [0, \infty)$ *when* $\alpha > 0$, $\eta > 1$ *and especially has the limit of* ∞ *when* y *approaches to* 0*.*

Take $\lim_{\pi\to 0} u(\pi) = 0$ *into the consideration. The second solution should be neglected, hence the solution of Equation* (C.4) *is obtained as*

$$u(\pi) = A{}_1F_1(\frac{\rho}{\gamma}, \frac{2\gamma\mu}{\sigma^2}; \frac{2\gamma}{\sigma^2}\pi)\,.$$

*Finally, combining it with the boundary condition (C.*5*) yields*

$$A = 1/{}_1F_1(\frac{\rho}{\gamma}, \frac{2\gamma\mu}{\sigma^2}; \frac{2\gamma}{\sigma^2}\pi_0).$$

It hence gives the Laplace transform of the first passage time of the CIR process

$$E[e^{-\rho\tau_y}\,|\pi_0] = \frac{{}_1F_1(\frac{\rho}{\gamma}, \frac{2\gamma\mu}{\sigma^2}; \frac{2\gamma}{\sigma^2}\pi_0)}{{}_1F_1(\frac{\rho}{\gamma}, \frac{2\gamma\mu}{\sigma^2}; \frac{2\gamma}{\sigma^2}y)}\,.$$

As required in the derivation of $\kappa(y)$, $\varphi_\rho(y) = {}_1F_1(\frac{\rho}{\gamma}, \frac{2\gamma\mu}{\sigma^2}; \frac{2\gamma}{\sigma^2}y)$ *should be a strictly increasing convex function. To show it, one property of the confluent hypergeometric function is greatly helpful, which is* ${}_1F_1'(\alpha, \eta; x) = \frac{\alpha}{\eta}{}_1F_1'(\alpha+1, \eta+1; x)$ *as stated in Abramowitz and Stegun (1969). In this way, one can easily verify the convexity of* $\varphi_\rho(y)$ *by showing* $\varphi_\rho'(y) > 0$ *and* $\varphi_\rho''(y) > 0$ *as well.*

[1]The confluent hypergeometric function is defined in the literature as ${}_1F_1(\alpha, \eta; y) = \sum_{m=0}^{\infty} \frac{1}{m!}\frac{(\alpha)_m}{(\eta)_m}$ where $(\alpha)_0 = 1$ and $(\alpha)_m = \alpha(\alpha+1)\cdots(\alpha+m-1)$.

Remark C.3.2. *Note that the solution provided here applies to all diffusion processes*

$$dP_t = \mu(P_t)dt + \sigma(P_t)dW_t,$$

where $\mu(P_t)$ and $\sigma(P_t)$ denote the state–dependent drift and volatility, since it is a Markov process and always satisfies the strong Markov property. However for general diffusion processes, the Laplace transform of the first passage time may be not given in an explicit form. It is nevertheless still possible to find some function φ_ρ by means of the strong Markov property as

$$E[e^{-\rho\tau_z}|Y_t = x] = E[e^{-\rho\tau_y}|Y_t = x]E[e^{-\rho\tau_z}|Y_t = y],$$

for any $0 < x < y < z, \rho > 0$ and $\tau_y = \inf\{t \geq 0, |\pi_t = y\}$ the first passage times. The Laplace transforms of these first passage times can then be expressed as

$$E[e^{-\rho\tau_y}|Y_t = x] = \frac{\varphi_\rho(x)}{\varphi_\rho(y)}$$

for some continuous and strictly increasing function $\varphi_\rho(y) : (0, \infty) \to (0, \infty)$ (cf. Itô and McKean (1965), p. 130). Then, the solution form provided above is valid if and only if $\varphi_\rho(y)$ is strictly convex and continuously differentiable.

Appendix D

Proof of Chapter 8

D.1 Proof of the Existence and Uniqueness Theorem

D.1.1 The Finite Horizon

Theorem D.1.1. *Under Assumption* 8.2.4*, there always exists a unique optimal investment plan* I^**.*

PROOF: For simplicity, assume in the proof that the interest and depreciation rate r and δ are positive constants. The argument goes through also in the case of bounded, nonnegative processes with the corresponding and obvious changes, which is easily done without any difficulties but in terms of clumsier formula.

First, uniqueness follows directly from strict concavity and the fact that capacity is linear in investment. Hence, it is not necessary to be more addressed here. The existence proof is not that trivial and consists of three steps. First, Assumption 8.2.4 (i) is shown to guarantee the finiteness of $\Pi(I)$. Step 2 demonstrates that one can restrict attention to those investment plans I which lead to the capacities that satisfy $E\big[C^I_{\hat{T}}\big] \leq K$, where the constant K is as defined in Assumption 8.2.4 (ii). In the third step, a suitable variant of Komlos' Theorem (Komlós (1967), see also Balder (1990) and Kabanov (1999)) is applied to obtain a sequence of investment plans (I^n) that converges in the Cesaro sense almost surely to some investment plan I^*. Concavity of the profit functional ensures the optimality of I^*.

Step 1. From Equation (8.1), one can write $dI_t = dC^I_t + \delta C^I_t dt$. This yields

$$\int_0^{\hat{T}} e^{-rt} dI_t = \int_0^{\hat{T}} e^{-rt} dC^I_t + \int_0^{\hat{T}} \delta e^{-rt} C^I_t dt\,.$$

Integration by parts gives

$$\int_0^{\hat{T}} e^{-rt} dC^I_t = e^{-r\hat{T}} C^I_{\hat{T}} + \int_0^{\hat{T}} r e^{-rt} C^I_t dt\,,$$

and hence

$$\int_0^{\hat{T}} e^{-rt} dI_t = e^{-r\hat{T}} C^I_{\hat{T}} + \int_0^{\hat{T}} (r+\delta) e^{-rt} C^I_t dt\,.$$

It follows then

$$\begin{aligned}\int_0^{\hat{T}} e^{-rt}\Big(\pi\left(X_t, C^I_t\right)\,dt - dI_t\Big) &\le \int_0^{\hat{T}} e^{-rt}\Big(\pi\left(X_t, C^I_t\right) - (r+\delta)C^I_t\Big)\,dt \\ &\le \int_0^{\hat{T}} e^{-rt}\pi^*(X_t, r, \delta)\,dt\,.\end{aligned}$$

This implies consequently

$$\Pi(I) \le E\int_0^{\hat{T}} e^{-rt}\pi^*(X(t), r, \delta)dt < \infty\,,$$

and the problem has always a finite value $v^\star = \sup\limits_I \Pi(I) < \infty$.

Step 2. In this step, an investment plan $\hat{I}$ with the corresponding capacity $\hat{C}$ is constructed such that it gives an upper bound for all reasonable plans in the sense that it is not worthwhile to have a higher capacity than $\hat{C}$. The basic idea is that it does not make sense to have a capacity higher than that one would have in the reversible case, c^*. A complication arises from the fact that $c^*(X_s, r, \delta)$ will generally be a process of unbounded variation and thus may not be a feasible capacity.

The trick here is to construct the investment plan that leads to a capacity $\hat{C} \ge c^*$ in a minimal way. Set

$$\hat{C}_t = e^{-\delta t}\sup_{s\le t}\left(c^*_s e^{\delta s}\right)\,, \tag{D.1}$$

where the notation is slightly abused by writing

$$c^*_s = c^*\left(X_s, r, \delta\right)\,.$$

Because of Assumption 8.2.4 and $\delta \ge 0$, $\hat{C}_{\hat{T}}$ is integrable as

$$E\Big[\hat{C}_{\hat{T}}\Big] = E\Big[\sup_{s\le\hat{T}} c^*_s e^{-\delta(\hat{T}-s)}\Big] \le E\Big[\sup_{s\le\hat{T}} c^*_s\Big] < \infty\,,$$

where $E\Big[\sup\limits_{s\le\hat{T}} c^*_s\Big]$ is obviously equal to K specified in Assumption 8.2.4 with the deterministic r and δ.

The investment plan

$$\hat{I}_t = \hat{C}_t + \int_0^t \delta\hat{C}_s ds \tag{D.2}$$

is the feasible plan that leads to the capacity $\hat{C}$.

The claim to be demonstrated is that one can restrict attention to plans I with capacity $C^I \leq \hat{C}$. Let I be given and write $C = C^I$ for shorter notation. Define $\bar{C}_t = \min\left\{C_t, \hat{C}_t\right\}$ and $\bar{A}_t = e^{\delta t}\bar{C}_t$. Note that $(A_t)_{t\in[0,\hat{T}]}$ is also nondecreasing as $(C_t)_{t\in[0,\hat{T}]}$. The corresponding investment plan with capacity $C^{\bar{I}} = \bar{C}$ is denoted as $\bar{I}_t = \int_0^t e^{\delta s} d\bar{A}_s$.

Under this construction, the claim is valid if $\bar{I}$ is shown to be at least as good as I. Integration by parts yields

$$\begin{aligned}
\Pi(\bar{I}) - \Pi(I) &= E\int_0^{\hat{T}} e^{-rt}\left(\pi\left(X_t, \bar{C}_t\right) - (r+\delta)\bar{C}_t\right) dt - Ee^{-r\hat{T}}\bar{C}_{\hat{T}} \\
&\quad - E\int_0^{\hat{T}} e^{-rt}\left(\pi\left(X_t, C_t\right) - (r+\delta)C_t\right) dt + Ee^{-r\hat{T}}C_{\hat{T}} \\
&= E\int_0^{\hat{T}} e^{-rt}\left[\left(\pi\left(X_t, \bar{C}_t\right) - (r+\delta)\bar{C}_t\right) - \left(\pi\left(X_t, C_t\right) - (r+\delta)C_t\right)\right] dt \\
&\quad - Ee^{-r\hat{T}}\left(\bar{C}_{\hat{T}} - C_{\hat{T}}\right) .
\end{aligned}$$

The last term is nonnegative because $\bar{C} \leq C$. The integrand in the first term is either zero when $\bar{C} = C$; or nonnegative when $\bar{C} < C$. In the second case of $\bar{C} < C$, it is clear that $C_t > \bar{C}_t \geq c_t^*$. As C_t is located at the right of the maximum c^* and the function $c \mapsto \pi(x, c) - (r+\delta)c$ is concave, one can find out that

$$\pi\left(X_t, \bar{C}_t\right) - (r+\delta)\bar{C}_t > \pi\left(X_t, C_t\right) - (r+\delta)C_t \, .$$

These arguments altogether lead to $\Pi(\bar{I}) \geq \Pi(I)$ as desired.

Step 3. By the proceeding step, the auxiliary problem

$$\sup_{I:\, C^I \leq \hat{C}} \Pi(I) = v^*$$

has the same value as the original problem. Choose an optimal sequence (I^n) for this auxiliary problem. Its value at time $\hat{T}$ has the following property:

$$I^n_{\hat{T}} = C^{I^n}_{\hat{T}} + \delta \int_0^{\hat{T}} C^{I^n}_s ds \leq (1+\delta\hat{T})C^{I^n}_{\hat{T}} \leq (1+\delta\hat{T})\hat{C}_{\hat{T}}.$$

This suggests that

$$\sup_n E\left[I^n_{\hat{T}}\right] < \infty \, .$$

With this condition, Kilos Theorem (in the variant of Kabanov (1999)) can be thus applied here: Assume without loss of generality that (I^n) converges in the Cesaro sense almost surely to some I^*, that is,

$$J_t^n \triangleq \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} I_t^k = I_t^* \qquad a.s.$$

Through linearity, the corresponding capacities C^k converge also in the Cesaro sense almost surely. Moreover, the concavity of the profit function in capacity yields the final result that

$$\Pi\big(I^*\big) \geq \lim_{n\to\infty} \sup \Pi\left(J^n\right) = v^*\,.$$

Therefore, I^* is the optimal investment plan that maximizes the firm's net profit. ■

D.1.2 Existence for the Infinite Horizon Case

Of course, the naive generalization of Assumption 8.2.4 with

$$E\big[\sup_{t<\infty} c_t^*\big] < \infty$$

is sufficient (by repeating the proof above for the finite horizon case). However, it is too strong in the infinite horizon case because the overall maximum of the process will be in many contexts infinity. Indeed, the following weaker version of Assumption 8.2.4 is sufficient to guarantee the existence of the optimal sequential investment plan with the infinite horizon.

Assumption D.1.2. *(i)* $E\big[\int_0^\infty e^{-\int_0^t r_s ds} \pi^*\left(X_t, r_t, \delta_t\right) dt\big] < \infty \;\forall\, t \in [0, \hat{T}]$.

(ii) $K \triangleq E\big[\int_0^\infty e^{-\int_0^t r_s ds} d\hat{I}_t\big] < \infty$ *for* $\hat{I}$ *as given by* (D.2).

We thus assume only that the running supremum of the optimal policy under reversibility c^* is integrable for all finite times. This is enough to construct the candidate $\hat{I}$ as in $(D.2)$. In addition, we have to impose the condition that his policy is admissible in order to have an admissible upper bound. From there on, the proof proceeds as in the finite horizon case. Thus, we omit the details here.

Theorem D.1.3. *Under Assumption D.1.2, there always exists one unique optimal investment plan I^* for the infinite–horizon sequential irreversible investment problem.*

D.2 Proof of Theorem 8.3.5

PROOF: Bank and El Karoui (2004) perform a detailed analysis of the adjusted first–order equation (8.14). In particular, they show that the base capacity is the unique progressively measurable process that solves (8.14) (Theorem 1 and 3 therein). Given that the base capacity solves (8.14), we check now the first–order conditions (8.6) and

(8.7). Let I^* denote the investment plan that finances $C^{l,\delta}$. From (8.8), the gradient at time t is given by

$$G_t'(I^*) = e^{\delta t} E\left[\int_t^{\hat{T}} e^{-(r+\delta)s}\pi_c\left(X_s, C_s^{l,\delta}\right) ds \,\middle|\, \mathcal{F}_t\right].$$

As $C^{l,\delta}$ tracks the level l, the marginal profit of investment can be written as

$$G_t'(I^*) = e^{\delta t} E\left[\int_t^{\hat{T}} e^{-(r+\delta)s}\pi_c\left(X_s, e^{-\delta s}\sup_{u\le s} l_s e^{\delta s}\right) ds \,\middle|\, \mathcal{F}_t\right].$$

Trivially, we have $\sup_{u\le s} l_s e^{\delta s} \ge \sup_{t\le u\le s} l_s e^{\delta s}$ and as the marginal profit is decreasing in c, it follows with the help of the backward equation that

$$G_t'(I^*) \le e^{\delta t} E\left[\int_t^{\hat{T}} e^{-(r+\delta)s}\pi_c\left(X_s, e^{-\delta s}\sup_{t\le u\le s} l_s e^{\delta s}\right) ds \,\middle|\, \mathcal{F}_t\right] = e^{-rt}.$$

This proves (8.6). When $dI_t^* > 0$, the process $(l_s e^{\delta s})_{s\in[0,\hat{T}]}$ reaches a new running maximum at time t, that is,

$$l_t e^{\delta t} > l_u e^{\delta u} \quad \text{for all } u < t\,.$$

In this case, we have

$$\sup_{u\le s} l_s e^{\delta s} = \sup_{t\le u\le s} l_s e^{\delta s},$$

which leads to

$$G_t'(I^*) = e^{\delta t} E\left[\int_t^{\hat{T}} e^{-(r+\delta)s}\pi_c\left(X_s, e^{-\delta s}\sup_{t\le u\le s}\left(l_s e^{\delta s}\right)\right) ds \,\middle|\, \mathcal{F}_t\right] = e^{-rt}$$

and (8.7) is also satisfied by I^*. ■

D.3 Proof of Theorem 8.5.4

Proof: On the basis of (8.10), it is sufficient to check the relationship of the user cost of capital and L_t^τ. Fix $t < \tau$. For simplicity, we drop the subscript t, τ for the candidates L_t^τ in the proof and reparameterize the user cost of capital as $a = r + \delta$. In this way, the candidates L^a solve the equation

$$E\left[\int_t^{\tau} e^{-as}\left[\pi_c\left(X_s, L^a e^{-\delta(s-t)}\right) - a\right] ds \,\middle|\, \mathcal{F}_t\right] = 0\,.$$

By multiplying e^{at}, we have

$$E\left[\int_t^{\tau} e^{-a(s-t)}\left[\pi_c\left(X_s, L^a e^{-\delta(s-t)}\right) - a\right] ds \,\middle|\, \mathcal{F}_t\right] = 0\,.$$

Now let $a > b$ and L^a and L^b be the candidates of these two user cost of capital, respectively. We aim to show that $L^a \leq L^b$ almost surely.

Start with the contrary case: set $A = \{L^a > L^b\}$ and assume $\mathbb{P}(A) > 0$. As $A \in \mathcal{F}_t$, we have

$$\begin{aligned} 0 &= E\left[\int_t^\tau e^{-a(s-t)}\left[\pi_c\left(X_s, L^a e^{-\delta(s-t)}\right) - a\right] ds\, 1_A \,\middle|\, \mathcal{F}_t\right] \\ &\quad - E\left[\int_t^\tau e^{-b(s-t)}\left[\pi_c\left(X_s, L^b e^{-\delta(s-t)}\right) - b\right] ds\, 1_A \,\middle|\, \mathcal{F}_t\right]. \end{aligned}$$

on the set A, $\pi_c\left(X_s, L^a e^{-\delta(s-t)}\right) \leq \pi_c\left(X_s, L^b e^{-\delta(s-t)}\right)$. Moreover, $e^{-a(s-t)} < e^{-b(s-t)}$ for all $s > t$. Thus, we obtain

$$\begin{aligned} 0 &= E\left[\int_t^\tau e^{-a(s-t)}\left[\pi_c\left(X_s, L^a e^{-\delta(s-t)}\right) - a\right] ds\, 1_A \,\middle|\, \mathcal{F}_t\right] \\ &\quad - E\left[\int_t^\tau e^{-b(s-t)}\left[\pi_c\left(X_s, L^b e^{-\delta(s-t)}\right) - b\right] ds\, 1_A \,\middle|\, \mathcal{F}_t\right] \\ &= E\left[\int_t^\tau e^{-a(s-t)}\pi_c\left(X_s, L^a e^{-\delta(s-t)}\right) ds\, 1_A \,\middle|\, \mathcal{F}_t\right] - E\left[\int_t^\tau e^{-b(s-t)}\pi_c\left(X_s, L^b e^{-\delta(s-t)}\right) ds\, 1_A \,\middle|\, \mathcal{F}_t\right] \\ &\quad + E\left[\int_t^\tau \left(be^{-b(s-t)} - ae^{-a(s-t)}\right) ds\, 1_A \,\middle|\, \mathcal{F}_t\right] \\ &\leq E\left[\int_t^\tau \left(be^{-b(s-t)} - ae^{-a(s-t)}\right) ds\, 1_A \,\middle|\, \mathcal{F}_t\right] \\ &= E\left[\left(e^{-b(\tau-t)} - e^{-a(\tau-t)}\right) 1_A \,\middle|\, \mathcal{F}_t\right] \\ &< 0. \end{aligned}$$

This contradiction shoes that $\mathbb{P}(A) = 0$ as desired.

■

Appendix E

Proof of Chapter 9

E.1 Proof of Theorem 9.4.1

PROOF: Denote the critical price of the output P^*. Based on Theorem 9.2.1 and 9.3.1, it should be identified as

$$\frac{\theta}{1-\alpha}(P^*)^{1-\alpha} = \frac{U(rI)}{\rho}.$$

Further simplification yields then

$$P^* = \left(\frac{1}{\rho\theta}\right)^{\frac{1}{1-\alpha}} rI = \eta^{-\frac{1}{1-\alpha}} rI\,,$$

where $\eta = \rho\theta = E\left[e^{(1-\alpha)\underline{Y}_{\tau(\rho)}}\right]$.

To get the comparative statics result with respect to the relative risk aversion coefficient, we need to calculate $\frac{\partial P^*}{\partial \alpha}$:

$$\begin{aligned}
\frac{\partial P^*(\alpha)}{\partial \alpha} &= \eta^{-\frac{1}{1-\alpha}}\left[\left(-\frac{1}{1-\alpha}\right)' \ln\eta + \left(-\frac{1}{1-\alpha}\right)\frac{\eta'(\alpha)}{\eta}\right] rI \\
&= \eta^{-\frac{1}{1-\alpha}}\left[-\frac{1}{(1-\alpha)^2} \ln\eta + \left(-\frac{1}{1-\alpha}\right)\frac{\eta'(\alpha)}{\eta}\right] rI \\
&= -\frac{1}{1-\alpha}\eta^{-\frac{1}{1-\alpha}} \underbrace{\left[\frac{1}{1-\alpha}\ln\eta + \frac{\eta'(\alpha)}{\eta}\right]}_{:=D} rI\,, \qquad \text{(E.1)}
\end{aligned}$$

where $\eta'(\alpha) = \frac{\partial}{\partial\alpha}E\left[e^{(1-\alpha)\underline{Y}_{\tau(\rho)}}\right] = E\left[-\underline{Y}_{\tau(\rho)}e^{(1-\alpha)\underline{Y}_{\tau(\rho)}}\right]$ by using the Fubini's theorem and we define especially the term in bracket as D for convenience. Note that $\underline{Y}_{\tau(\rho)}$ is always negative as $Y_0 = 0$. Therefore, we have always $\eta > 0$ and $\eta' > 0$. Again by applying the Fubini's theorem and assuming that the distribution of $\underline{Y}_{\tau(\rho)}$ is $F(\underline{Y}_{\tau(\rho)})$ in the interval $(-\infty, 0]$, we find first that

$$\begin{aligned}\frac{1}{1-\alpha}\ln\eta &= \frac{1}{1-\alpha}\ln E\left[e^{(1-\alpha)\underline{Y}_{\tau(\rho)}}\right]\\ &= \frac{1}{1-\alpha}\ln\int_{-\infty}^{0} e^{(1-\alpha)\underline{Y}_{\tau(\rho)}}dF(\underline{Y}_{\tau(\rho)})\\ &= \int_{-\infty}^{0}\frac{1}{1-\alpha}\ln e^{(1-\alpha)\underline{Y}_{\tau(\rho)}}dF(\underline{Y}_{\tau(\rho)})\\ &= \int_{-\infty}^{0}\underline{Y}_{\tau(\rho)}dF(\underline{Y}_{\tau(\rho)})\\ &= E\left[\underline{Y}_{\tau(\rho)}\right]\ .\end{aligned}$$

In this way, we have D further reduced as

$$\begin{aligned}D &= E\left[\underline{Y}_{\tau(\rho)}\right] + \frac{E\left[-\underline{Y}_{\tau(\rho)}e^{(1-\alpha)\underline{Y}_{\tau(\rho)}}\right]}{E\left[e^{(1-\alpha)\underline{Y}_{\tau(\rho)}}\right]}\\ &= \frac{E\left[-\underline{Y}_{\tau(\rho)}e^{(1-\alpha)\underline{Y}_{\tau(\rho)}}\right] - E\left[-\underline{Y}_{\tau(\rho)}\right]E\left[e^{(1-\alpha)\underline{Y}_{\tau(\rho)}}\right]}{E\left[e^{(1-\alpha)\underline{Y}_{\tau(\rho)}}\right]}\\ &= \frac{Cov[-\underline{Y}_{\tau(\rho)}, e^{(1-\alpha)\underline{Y}_{\tau(\rho)}}]}{E\left[e^{(1-\alpha)\underline{Y}_{\tau(\rho)}}\right]}\ .\end{aligned}$$

Clearly, the sign of the covariance, namely D, is heavily dependent on α. For $\underline{Y}_{\tau(\rho)} \in (-\infty, 0]$ always, one can easily find that D turns out to be negative for $0 < \alpha < 1$ and positive for $\alpha > 1$. Taking it back to Equation $(E.1)$ gives then the final result that $\frac{\partial P^*}{\partial\alpha}$ is always positive for any value of $\alpha \in [0, 1)$ and $(1, \infty)$. ■

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