Local $L_{(2)}$-Cohomology of Shimura Varieties

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Zusammenfassung

Auf der Baily-Borel-Kompaktifizierung einer Shimura-Varietät gibt es zwei kanonische Komplexe von Garben: Den Schnittkohomologiekomplex mittlerer Perversität und den $L_{(2)}$-Komplex. Nach der von Looijenga und Saper und Stern bewiesenen Zucker-Vermutung sind diese Komplexe quasi-isomorph.

Ausgehend von Frankes Beweis der Borel-Vermutung konstruieren wir eine Spektralsequenz, die den Halm der Kohmologiegarbe des $L_{(2)}$-Komplexes an einem Punkt berechnet. Durch eine einfache Anwendung der Vogan-Zuckerman-Klassifikation unitärer Darstellungen mit Kohomologie und kombinatorischer Argumente, wie sie von Borel, Casselman, Saper und Stern entwickelt wurden, zeigen wir die Verschwindungsaussagen, die den Schnittkohomologiekomplex charakterisiert. Auf diese Weise erhalten wir einen neuen Beweis er Zuckerschen Vermutung und darüber hinaus eine analytische Beschreibung der Einschränkung des Schnittkohomologiekomplexes auf ein Randstratum.

Summary

On the Baily-Borel-Compactification of a Shimura-Variety there are two canonical complexes of sheaves: The intersection cohomology complex of middle perversity and the $L_{(2)}$-complex. Zucker's conjecture as proved by Looijenga and Saper and Stern states that they are quasi-isomorphic.

Using Frankes proof of the Borel-Conjecture, we construct a spectral sequence that computes the stalk cohomology of the $L_{(2)}$-complexes at a point. By an easy application of the Vogan-Zuckerman classification of irreducible unitary representations with non-trivial cohomology and the application of combinatorial arguments developped by Borel, Casselman, Saper und Stern, we prove the vanishing assertions, that characterizes the intersection complex. In this way we obtain a new proof of Zucker's conjecture as well as an analytic description of the restriction of the intersection complex to a boundary stratum.

Contents

0 Introduction

Let $\mathbb{A}$ denote the adèle ring of $\mathbb{Q}$ and $\mathbb{A}_f$ its finite part. Let $\mathcal{G}$ be a connected reductive group defined over $\mathbb{Q}$. Let $\mathcal{A}_\mathcal{G}$ be the maximal $\mathbb{Q}$-split torus in the center of $\mathcal{G}$ and let $A_\mathcal{G} = \mathcal{A}_\mathcal{G}(\mathbb{R})^+$ the identity component of its real points. We assume that $\mathcal{A}_\mathcal{G}$ coincides with the maximal central $\mathbb{R}$-split torus. Let K_∞ be a maximal compact subgroup of $\mathcal{G}(\mathbb{R})$ and $K \subseteq K_\infty$ an open subgroup. Let

$$\mathcal{X} = \mathcal{G}(\mathbb{R})/A_\mathcal{G}K \tag{0.1}$$

be the symmetric space attached to these data. The group $\mathcal{G}(\mathbb{Q})$ is embedded diagonally in $\mathcal{G}(\mathbb{A})$ and as such acts on the space

$$\mathcal{X} \times \mathcal{G}(\mathbb{A}_f) = \mathcal{G}(\mathbb{A})/A_\mathcal{G}K$$

from the left. Let R denote the $\mathcal{G}(\mathbb{Q})$-relation on $\mathcal{X} \times \mathcal{G}(\mathbb{A}_f)$ and let $\overline{R}$ be its closure. One is interested in the quotient space

$$\mathrm{Sh} = \overline{R}\backslash\mathcal{X} \times \mathcal{G}(\mathbb{A}_f), \tag{0.2}$$

the "Shimura-Variety" attached to $\mathcal{G}$ and the particular choice of K. Classical examples are $\mathcal{G} = \mathbb{G}_m/\mathbb{Q}$ and $\mathcal{G} = \mathrm{Gl}_2/\mathbb{Q}$. In the first case $A_\mathcal{G} = \mathbb{R}^{>0}$ and with $K = \{1\}$ one can identify $\mathcal{X}$ with $\{\pm 1\}$. Hence (0.2) equals

$$\mathbb{Q}^\times\backslash\{\pm 1\} \times \mathbb{A}_f^\times = \mathbb{Q}^\times\backslash\mathbb{A}^\times/\mathbb{R}^{>0}$$

which is simply the idèle class group of $\mathbb{Q}$ divided by $\mathbb{R}^{>0}$. In the second case $A_\mathcal{G} = \mathbb{R}^{>0}$ and with $K = \mathrm{SO}(2)$ one can identify $\mathcal{X}$ with $\mathbb{C}\backslash\mathbb{R}$. Hence (0.2) becomes

$$\mathrm{Gl}_2(\mathbb{Q})\backslash(\mathbb{C}\backslash\mathbb{R}) \times \mathrm{Gl}_2(\mathbb{A}_f).$$

The group $\mathcal{G}(\mathbb{A}_f)$ acts on (0.2) from the right. For arithmetical applications it is important to keep track of this action.

If $\mathcal{X}$ is a hermitian symmetric space (0.2) has a natural $\mathcal{G}(\mathbb{A}_f)$-equivariant compactification

$$j\colon\ \mathrm{Sh} \subseteq \mathrm{Sh}^*, \tag{0.3}$$

its Satake-Baily-Borel-compactification, which has a natural $\mathcal{G}(\mathbb{A}_f)$-equivariant stratification

$$\mathrm{Sh}^* = \bigsqcup_{\mathcal{O}\in\mathfrak{P}^*} \partial_\mathcal{O}\,\mathrm{Sh}^*, \tag{0.4}$$

where $\mathfrak{P}^*$ is a certain set of standard rational parabolic subgroups $\mathcal{O}$ of $\mathcal{G}$. The strata $\partial_\mathcal{O}\,\mathrm{Sh}^*$, $\mathcal{O} \in \mathfrak{P}^*$, are called rational boundary components. The space Sh appears on the right hand side of (0.4) as $\partial_\mathcal{G}\,\mathrm{Sh}^*$.

Let E be finite dimensional algebraic representation of $\mathcal{G}(\mathbb{C})$ and let $\mathbb{E}$ be the associated automorphic local system on Sh. It carries a natural $\mathcal{G}(\mathbb{A}_f)$-action compatible with the action on Sh making it a $\mathcal{G}(\mathbb{A}_f)$-equivariant sheaf. Since the $j\colon \mathrm{Sh} \hookrightarrow \mathrm{Sh}^*$ is canonical, the intersection cohomology sheaf $\mathscr{I}^\bullet(\mathbb{E})$ as an object of the derived category of $\mathcal{G}(\mathbb{A}_f)$-equivariant sheaves on Sh^* is interesting. Our aim is to find an expression of $\mathscr{I}^\bullet(\mathbb{E})$ in analytical terms.

The possibility of doing this was conjectured by Zucker in [Zuc83]. In the rank one case Zuckers conjecture was proofed by Borel, see [Bor87]. Borel and Casselman proofed it for groups of rank 2, see [BC85]. The first proofs were given independently and by completely different methods in [SS90] and [Loo88]. Later there were proofs by [LR91] and more recently in [Sap05]. Among other things we add a new proof this list.

Let us give an overview over this thesis.

In the first section we sketch the construction of Satake-compactifications of $\mathrm{Sh} \subseteq \mathrm{Sh}^*$ which form a more general class of compactifications than the Satake-Baily-Borel compactification. This is done mainly by straightforward passage to the limit over compact open subgroups of $\mathcal{G}(\mathbb{A}_f)$ using known results. We recall Zucker's quotient map from the reductive Borel-Serre compactification to a Satake compactification and use it to describe convenient neighborhood basis for points in Sh^*.

In the second section we introduce, using the language of $(\mathfrak{g}, K)$-modules, certain complexes

$$\mathscr{A}^\bullet_{(2)}(\mathbb{E}) \tag{0.5}$$

of $\mathcal{G}(\mathbb{A}_f)$-equivariant sheaves on Sh^*. Up to a simple twist (0.5) is $\mathcal{G}(\mathbb{A}_f)$-equivariantly isomorphic to a sub-complex of the direct image of deRham complex of smooth $\mathbb{E}$-valued differential forms on Sh. For technical reasons we introduce logarithmic modifications $\mathscr{A}^\bullet_{(2)\pm\log}(\mathbb{E})$ of (0.5) where the weight condition are slightly relaxed in the $(+\log)$-case or strengthend in the $(-\log)$-case. Let us denote by $\mathscr{A}^\bullet_{(2)+?}(\mathbb{E})$, $? \in \{+\log, -\log, 0\}$, any one of these complexes. We prove a local regularization result to be used later.

As a next step we study the restriction

$$\mathscr{A}^\bullet_{(2)+?}(\mathbb{E})|_{\partial_{\mathcal{O}} \mathrm{Sh}^*} \tag{0.6}$$

to a fixed rational boundary component $\partial_{\mathcal{O}} \mathrm{Sh}^*$. The first step is to express (0.6) by means of an induction procedure. The second step then is to show that we may pass to $\mathcal{N}_{\mathcal{O}}$-invariants using Hodge theory. As a result we get:

0.7 Theorem: *The restriction of $\mathscr{A}^\bullet_{(2)+?}(\mathbb{E})$ to $\partial_{\mathcal{O}} \mathrm{Sh}^*$ is a possibly infinite dimensional automorphic local system.*

In the proof Theorem 0.7 we have to make a certain assumption (*). It is fullfilled for example if K is maximal or even stronger if we assume $\mathcal{G}$ to be semi-simple and simply connected. As long as one is interested only in Zucker's conjecture this may be seem to be a weak assumption but in view of possible applications to Shimura varieties in the sense of Deligne it would be desirable to proceed without (*). We assume it from now on.

As a consequence of Theorem 0.7 we obtain the following generalization of a Theorem of Nair.

0.8 Theorem: *If* Sh^* *is an equal-rank Satake-compactification there are natural quasi-isomorphisms*

$$\mathscr{A}^\bullet_{(2)-\log}(\mathbb{E}) \hookrightarrow \mathscr{A}^\bullet_{(2)}(\mathbb{E}) \hookrightarrow \mathscr{A}^\bullet_{(2)+\log}(\mathbb{E}).$$

of $\mathcal{G}(\mathbb{A}_f)$*-equivariant sheaves on* Sh^*.

The complexes $\mathscr{A}^\bullet_{(2)+?}(\mathbb{E})$ are our candidates for the intersection cohomology sheaf $\mathscr{I}^\bullet(\mathbb{E})$.

The Levi component $\mathcal{L}$ of $\mathcal{O}$ contains a certain normal $\mathbb{Q}$-subgroup $\mathcal{L}_l$, its "link" or "linear" factor. There is also a "hermitian" factor $\widetilde{\mathcal{L}}_h \subseteq \mathcal{L}$ such that $\mathcal{L}$ is the almost direct product of $\widetilde{\mathcal{L}}_h$ and $\mathcal{L}_l$. For a parbolic subgroup $\mathcal{R} \subseteq \mathcal{L}_l$ we set $\widetilde{\mathcal{R}} = \mathcal{R}\widetilde{\mathcal{L}}_h\mathcal{N}_{\mathcal{O}} \subseteq \mathcal{G}$. We assume for the simplicity of this introduction that $\mathcal{L}_l(\mathbb{R})$ is the full centralizer of $\partial_{\mathcal{O}}\mathcal{X}^*$ inside $\mathcal{L}(\mathbb{R})$.

Passing to functions invariant under $\mathcal{A}_{\mathcal{O}}(\mathbb{R})^+$, where $\mathcal{A}_{\mathcal{O}} \subseteq \mathcal{L}_{\mathcal{O}}$ is the maximal central $\mathbb{Q}$-split torus in the Levi component of $\mathcal{O}$ we reduce the computation of the stalk of the local systems of Theorem 0.7 to weighted cohomology of the link factor. Applying the main results of [Fra98] we obtain.

0.9 Theorem: *Let* $E_{\mathcal{G},\Lambda}$ *denote the representation of* $\mathcal{G}(\mathbb{C})$ *with highest weight* $\Lambda \in \overline{\check{\mathfrak{h}}^+}$, *where* $\mathfrak{h} \subseteq \mathfrak{g}$ *is some Cartan subalgebra of* $\mathfrak{g}$ *containing* $\mathfrak{a}_{\mathcal{O}}$. *Let* $s \in \partial_{\mathcal{O}}\,\mathrm{Sh}^*$ *and* $\mathfrak{O}_s = s.\mathcal{L}_l(\mathbb{A}_f)$ *be its right orbit under* $\mathcal{L}_l(\mathbb{A}_f)$. *Then there is a spectral sequence of* $\mathcal{L}_l(\mathbb{A}_f)$*-modules converging to*

$$H^{p+q}(\mathscr{A}^\bullet_{(2)+\log}(\mathbb{E}_{\mathcal{G},\Lambda})(\mathfrak{O}_s)). \tag{0.10}$$

Its $E_1^{p,q}$*-term is*

$$\bigoplus_{\{\mathcal{P}\}} \bigoplus_{k=0}^{\mathrm{rk}(\{\mathcal{P}\})} \bigoplus_{w} \operatorname*{colim}_{t \in M^{k,T,p}_{\mathscr{I}_w,\{\mathcal{P}\},\tau_w,+}} \mathrm{Ind}^{\mathcal{L}_l(\mathbb{A}_f)}_{\mathcal{R}_t(\mathbb{A}_f)} \left(H^{p+q-l(w)}_{(\mathfrak{m}_{\mathcal{R}_t}, K\cap\mathcal{R}_t(\mathbb{R}))}(V(u_t) \otimes E_{\mathcal{L},w(\Lambda+\rho_{\mathfrak{h}})-\rho_{\mathfrak{h}}}) \otimes \mathbb{C}_{-\lambda_t-\rho_{\mathcal{R}_t}} \right) \tag{0.11}$$

where $w \in W(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$ *is an element satisfying*

(1) $w^{-1}\alpha$ *is positive for all positive roots* α *appearing in* $\mathfrak{l}_{\widetilde{\mathcal{R}}_t,\mathbb{C}}$,

(2)

$$\lambda_t = -w(\Lambda + \rho_{\mathfrak{h}})|_{\check{\mathfrak{a}}^{\mathcal{O}}_{\widetilde{\mathcal{R}}_t}}$$

(3)
$$\Re(w(\Lambda+\rho_{\mathfrak{h}})|_{\mathfrak{a}_{\widetilde{\mathcal{R}}_t}}) \in \overline{\check{\mathfrak{a}}_{\widetilde{\mathcal{R}}_t}^{\mathcal{O}+}} + \check{\mathfrak{a}}_{\mathcal{O}}^{\mathcal{G}}$$

(4)
$$\lambda_t = -w(\Lambda+\rho_{\mathfrak{h}})|_{\check{\mathfrak{a}}^{\mathcal{O}}_{\widetilde{\mathcal{R}}_t}} \text{ and } \Re(w(\Lambda+\rho_{\mathfrak{h}})|_{\mathfrak{a}_{\mathcal{O}}}) \in \overline{{}^+\check{\mathfrak{a}}_0^{\mathcal{O}}}.$$

For a proof of Zucker's Conjecture it is not nessessary to understand all the notation in the statement of Theorem 0.9 but let us at least try to explain some of it here. The first sum is over associate classes of parabolics in $\mathcal{L}_l$ whose rank is denoted by $\mathrm{rk}(\{\mathcal{P}\})$. The third sum is explained in the Theorem. The colimit is over a certain finite groupoid $M^{k,T,p}_{\mathscr{J}_w,\{\mathcal{P}\},\tau_w,+}$ of certain triples $t=(\mathcal{R}_t,\lambda_t,\ldots)$ whose morphisms encode the various functional equations satisfied by the Eisenstein series used in the construction of the spectral sequence in [Fra98]. The entry $\mathcal{R}_t$ is a parabolic $\mathbb{Q}$-subgroup of $\mathcal{L}_l$ satisfying a certain conditions depending on k and $\{\mathcal{P}\}$. The space $V(u_t)$ is a space of cusp forms on the Levi component of $\mathcal{R}_t$. The group $\widetilde{\mathcal{R}}_t$ denotes the unique parabolic subgroup of $\mathcal{O}$ containing the full hermitian factor of $\mathcal{L}$ as explained above. The second entry λ_t in t denotes the point at which a certain Eisenstein series is to be evaluated. Since certain infinitesimal characters must match in order that the to give a non-trivial cohomology class we get condition (2). The real part of λ_t has to lie in the closure of the positive Weyl chamber of the link group. This condition gives condition (3). The first condition comes out of Kostant's Theorem on $\mathfrak{n}$-cohomology applied to $\mathfrak{n}_{\widetilde{\mathcal{R}}_t}$-invariants. Finally the fourth condition comes from a growth condition on certain functions.

In any case, the direct sums and the colimit are finite. From a well-known finiteness result for the space of cusp forms we infer that $H^\bullet(\mathscr{A}_{(2)+\log}(\mathbb{E})(\mathfrak{O}_s))$ is an admissible $\mathcal{L}_l(\mathbb{A}_f)$-module and consequently that the automorphic local system of Theorem 0.7 is constructible. Finiteness also allows us, using the previously mentioned regularization result, to show that $\mathscr{A}^\bullet_{(2)+\log}(\mathbb{E})$ is Verdier dual to $\mathscr{A}^\bullet_{(2)-\log}(\mathbb{E}^\vee)$.

To prove Zucker's Conjecture it remains to show that the $E_1^{p,q}$-term vanishes provided

$$p+q+\mathrm{prk}_{\mathbb{Q}}(\widetilde{\mathcal{R}}_t) \leq \frac{1}{2}\,\mathrm{codim}_{\mathbb{R}}(\partial_{\mathcal{O}}\,\mathrm{Sh}^* \subseteq \mathrm{Sh}^*). \tag{0.12}$$

In fact this estimate is somewhat better than needed. It implies in particular that only cuspidal Eisenstein classes contribute in the critical dimension. Since the E_1-term vanishes iff

$$H^{p+q-l(w)}_{(\mathfrak{m}_{\mathcal{R}_t},K\cap\mathcal{R}_t(\mathbb{R}))}(V(u_t)\otimes E_{\mathcal{L},w(\Lambda+\rho_{\mathfrak{h}})-\rho_{\mathfrak{h}}}) = \{0\}$$

it suffices to prove the following abstract vanishing result.

0.13 Theorem: *Let $\mathcal{R} \subset \mathcal{L}_l$ be rational standard parabolic subgroup. Let $F \subseteq H^p_{\mathfrak{n}_{\tilde{\mathcal{R}}}}(E)$ be an irreducible $\mathcal{L}_{\tilde{\mathcal{R}}}(\mathbb{C})$ submodule in degree p. Assume that the lowest weight of F is of the form $\tilde{\lambda} + \rho_{\mathfrak{h}}$ with*

$$\lambda = \tilde{\lambda}|_{\mathfrak{a}_{\tilde{\mathcal{R}}}} \in (-\overline{{}^{+}\check{\mathfrak{a}}_{\tilde{\mathcal{R}}}}) \cap (\overline{\check{\mathfrak{a}}^{\mathcal{O}+}_{\tilde{\mathcal{R}}}} + \check{\mathfrak{a}}_{\mathcal{O}}) \tag{0.14}$$

and that there exists an irreducible unitary $(\mathfrak{m}_{\mathcal{R}}, K_{\mathcal{R}})$-module V such that

$$H^q_{(\mathfrak{m}_{\mathcal{R}}, K_{\mathcal{R}})}(V \otimes F) \neq \{0\}.$$

If Sh^* *is the Satake-Baily-Borel-compactification of a hermitian locally symmetric space then (0.12) holds.*

The prove uses the Vogan-Zuckerman classification of irreducible unitary representation with cohomology and combinatorial results of Borel, Casselman and Saper and Stern.

The result is that

$$\mathscr{I}^\bullet(\mathbb{E}) \cong \mathscr{A}^\bullet_{(2)+?}(\mathbb{E})$$

and consequently we may use Theorem 2.69 to obtain information about the restriction $\mathscr{I}^\bullet(\mathbb{E})|_{\partial_{\mathcal{O}} \mathrm{Sh}^*}$. Of course the spectral sequence need not converge at E_1. It does however if the highest weight of E is regular. If it does not converge one may still use it to compute Euler characteristics or virtual characters.

Let us recall some standard notation. Let $\mathcal{P}_0$ be a minimal $\mathbb{Q}$-parabolic subgroup of $\mathcal{G}$ and $\mathcal{A}_0$ be the maximal $\mathbb{Q}$-split torus in the center of a Levi component of $\mathcal{P}_0$. Let $X^*(\mathcal{P}_0)$ be the group of rational characters of $\mathcal{P}_0$ and $X_*(\mathcal{A}_0)$ the group of rational cocharacters of $\mathcal{A}_0$. Set

$$\check{\mathfrak{a}}_0 = X^*(\mathcal{P}_0) \otimes_{\mathbb{Z}} \mathbb{R}$$

and

$$\mathfrak{a}_0 = X_*(\mathcal{A}_0) \otimes_{\mathbb{Z}} \mathbb{R}.$$

A rational parabolic is called standard if it contains $\mathcal{P}_0$. Similarly define for a pair $(\mathcal{P}, \mathcal{A}_{\mathcal{P}})$ of a standard rational parabolic $\mathcal{P} \supset \mathcal{P}_0$ and a maximal $\mathbb{Q}$-split torus $\mathcal{A}_{\mathcal{P}} \supseteq \mathcal{A}_0$ in the center of some Levi-component of $\mathcal{P}$ vector spaces $\check{\mathfrak{a}}_{\mathcal{P}} = X^*(\mathcal{P}) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathfrak{a}_{\mathcal{P}} = X_*(\mathcal{A}_{\mathcal{P}}) \otimes_{\mathbb{Z}} \mathbb{R}$. Restriction of characters from $\mathcal{P}$ to $\mathcal{A}_{\mathcal{P}}$ is one-to-one and identifies $X^*(\mathcal{P})$ with a subgroup of finite index in $X^*(\mathcal{A}_{\mathcal{P}})$. We identify $X^*(\mathcal{P}) \otimes_{\mathbb{Z}} \mathbb{R}$ with $X^*(\mathcal{A}_{\mathcal{P}}) \otimes_{\mathbb{Z}} \mathbb{R}$ via this map. The natural pairing $X^*(\mathcal{A}_0) \otimes_{\mathbb{Z}} X_*(\mathcal{A}_0) \to \mathbb{Z}$ gives a canonical isomorphism of $\check{\mathfrak{a}}_0$ with the dual of $\mathfrak{a}_0$. In particular $\mathfrak{a}_0$ is up to canonical isomorphism independent of the chosen Levi decomposition. The same applies to $\mathfrak{a}_{\mathcal{P}}$ for any standard parabolic $\mathcal{P} \supset \mathcal{P}_0$. Let us denote $\langle\ ,\ \rangle$ the canonical pairing between $\check{\mathfrak{a}}_{\mathcal{P}}$ and $\mathfrak{a}_{\mathcal{P}}$. Restriction of characters from $\mathcal{P}$ to $\mathcal{P}_0$ induces a natural injection $\check{\mathfrak{a}}_{\mathcal{P}} \hookrightarrow \check{\mathfrak{a}}_0$ that is split by the dual of the natural inclusion $\mathfrak{a}_{\mathcal{P}} \hookrightarrow \mathfrak{a}_0$ given by the inclusion $\mathcal{A}_{\mathcal{P}} \subset \mathcal{A}_0$. We get a natural direct sum decomposition

$$\check{\mathfrak{a}}_0 = \check{\mathfrak{a}}_{\mathcal{P}} \oplus \check{\mathfrak{a}}_0^{\mathcal{P}}.$$

Similarly we have

$$\mathfrak{a}_0 = \mathfrak{a}_{\mathcal{P}} \oplus \mathfrak{a}_0^{\mathcal{P}}.$$

These are orthogonal with respect to $\langle\ ,\ \rangle$. In this way we identify $\mathfrak{a}_{\mathcal{P}}$ etc. as subspaces of $\mathfrak{a}_0$ and similarly for $\check{\mathfrak{a}}_{\mathcal{P}}$. For another standard pair $(\mathcal{Q}, \mathcal{A}_{\mathcal{Q}})$ we let $\mathfrak{a}_{\mathcal{P}}^{\mathcal{Q}}$ be the intersection of $\mathfrak{a}_{\mathcal{P}}$ and $\mathfrak{a}_0^{\mathcal{Q}}$ in $\mathfrak{a}_0$. Let $\mathfrak{g}$ be the Lie algebra of $\mathcal{G}(\mathbb{R})$. Denote by $\Phi_0 \subset X^*(\mathcal{A}_0) \subset \check{\mathfrak{a}}_0$ the system of roots in $\mathfrak{g}$. The pair $(\Phi_0, \check{\mathfrak{a}}_0^{\mathcal{G}})$ is a root system. Let $\check{\Phi}_0 \subset \mathfrak{a}_0$ be the dual root system. For $\alpha \in \Phi_0$ there is a unique dual root $\check{\alpha} \in \check{\Phi}_0$. Let $\Delta_0 \subseteq \Phi_0$ be the set of simple roots. For to rational parabolics $\mathcal{R} \supseteq \mathcal{P} \supseteq \mathcal{P}_0$ let $\Delta_{\mathcal{P}}^{\mathcal{R}}$ be the set of those simple roots $\alpha \in \Delta_0$ which occur in the Lie algebra of the radical of $\mathcal{P}$ but not in the Lie-algebra of the radical of $\mathcal{R}$. For $\alpha \in \Delta_{\mathcal{P}}^{\mathcal{R}}$ we denote by $\check{\alpha}$ the corresponding coroot and by $\{\varpi_\alpha^{\mathcal{R}}\}_{\alpha \in \Delta_{\mathcal{P}}^{\mathcal{R}}}$ the base of $\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{R}}$ dual to the base of $\mathfrak{a}_{\mathcal{P}}^{\mathcal{R}}$ given by $\{\check{\alpha}\}_{\alpha \in \Delta_{\mathcal{P}}^{\mathcal{R}}}$. Dually define $\check{\varpi}_\alpha^{\mathcal{R}} \in \mathfrak{a}_{\mathcal{P}}^{\mathcal{R}}$. Write ϖ_α for $\varpi_\alpha^{\mathcal{G}}$. The positive open Weyl chamber $\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{R}+}$ is the open cone spanned by the $\varpi_\alpha^{\mathcal{R}}$ for $\alpha \in \Delta_{\mathcal{P}}^{\mathcal{R}}$ Denote by ${}^+\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{R}} \subset \check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{R}}$ the open cone spanned by all $\alpha \in \Delta_{\mathcal{P}}^{\mathcal{R}}$. Define $\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{R}+}$ and ${}^+\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{R}} \subset \check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{R}}$ dually. The Weyl $W_{\mathcal{P}}^{\mathcal{R}}$ is the group of automorphisms of $\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{R}}$ generated by the reflections

$$s_\alpha(\lambda) = \lambda - 2\frac{\langle \lambda, \check{\alpha}\rangle}{\langle \alpha, \check{\alpha}\rangle}\alpha.$$

We let it operate trivially on the orthogonal complement of $\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{R}}$ in $\check{\mathfrak{a}}_0$. It operates in a natural way on $\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{R}}$. We simply write W for $W_{\mathcal{P}_0}^{\mathcal{G}}$.

Recall the definition of the standard height function. Let V be the set of places of $\mathbb{Q}$ and let $\mathbb{A}$ denote the its adèle group. Let $\mathcal{P} \subseteq \mathcal{G}$ be a standard $\mathbb{Q}$-rational parabolic. For $v \in V$ and $p \in \mathcal{P}(\mathbb{Q}_v)$ we define $H_{\mathcal{P},v}(p) \in \mathfrak{a}_{\mathcal{P}}$ by

$$\exp(\langle \chi, H_{\mathcal{P},v}(p)\rangle) = |\chi(p)|_v \tag{0.15}$$

for every rational character χ of $\mathcal{P}$. This is well defined since $\check{\mathfrak{a}}_{\mathcal{P}}$ is spanned by the rational characters of $\mathcal{P}$. It is a group homomorphism and factorizes over $\mathcal{P}(\mathbb{Q})\mathcal{N}_{\mathcal{P}}(\mathbb{A})\backslash\mathcal{P}(\mathbb{A})$. If K_v is a good maximal compact subgroup of $\mathcal{G}(\mathbb{Q}_v)$ $H_{\mathcal{P},v}$ can be extended be means of the Iwasawa decomposition $\mathcal{G}(\mathbb{Q}_v) = \mathcal{P}(\mathbb{Q}_v)K_v$ by setting

$$H_{\mathcal{P},v}(g_v) = H_{\mathcal{P},v}(p_v)$$

where $g_v = p_v k_v$. This is well defined since $H_{\mathcal{P},v}(\mathcal{P}(\mathbb{Q}_v) \cap K_v)$ is a compact subgroup of $\mathfrak{a}_{\mathcal{P}}$ and hence trivial. Since $\mathfrak{a}_{\mathcal{P}}$ is abelian it depends only on the choice of the conjugacy class of K_v and hence is independent of any choice for the infinite place. Let $\mathbb{K}_f = K_2 \times K_3 \times K_5 \times \ldots \subset \mathcal{G}(\mathbb{A}_f)$ be a fixed good maximal compact subgroup of $\mathcal{G}$. Define $H_{\mathcal{P}}$ on $\mathcal{G}(\mathbb{A})$ by

$$H_{\mathcal{P}}(g) = \sum_{v \in V} H_{\mathcal{P},v}(g_v).$$

Its dependence on the choice of $\mathbb{K} = K_\infty \mathbb{K}_f$ will be suppressed for simplicity. By definition and the product formula it factorizes over

$$\mathcal{P}(\mathbb{Q})\mathcal{N}_{\mathcal{P}}(\mathbb{A})\backslash\mathcal{G}(\mathbb{A})/K_\infty\mathbb{K}_f.$$

The function $H_{\mathcal{P}}$ can be computed from the action of $\mathcal{G}(\mathbb{A})$ on a suitably chosen set of finite dimensional $\mathbb{K}$-spherical representations, see [Fra98], proof of Theorem 1.

The universal enveloping algebra of $\mathfrak{g}$ is a $\mathbb{C}$-algebra and denoted by $\mathfrak{U}(\mathfrak{g}_\mathbb{C})$ or simply $\mathfrak{U}(\mathfrak{g})$.

If it exists, we denote by $\lim_{\mathcal{D}}$ and $\mathrm{colim}_{\mathcal{D}}$ the categorial limit and colimit over a small category $\mathcal{D}$. Filtered (co-)limits, inductive limits and projective limits are denoted by the same symbols.

If S is a topological space and $\mathfrak{B}$ a basis for its topology, we use the notion of a $\mathfrak{B}$-(pre-)sheaf as in [Gro60], §3.2.

The algebraic dual space of a vector space V will be denoted $V^\vee$. If V is a topological vector space we write V' for its topological dual.

If E and F are two locally convex spaces we denote by $E\widehat{\otimes}_\pi F$ respectively $E\widehat{\otimes}_\epsilon F$ Grothendieck's completed π- respectively ϵ-tensor products. If one of the factors is nuclear we write $\widehat{\otimes} = \widehat{\otimes}_\pi = \widehat{\otimes}_\epsilon$ instead.

1 Satake Compactifications

We introduce Satake compactifications of symmetric spaces via spherical representations as in [Cas97, §6]. The equivalence with Satakes original construction is discussed in [Sap04]. See [BJ06] for examples. Our discussion of the adelic Satake compactifications is similar to expositions in [Fra], [Oss07] and [Roh96] for the case of the (reductive) Borel-Serre compactification. Let $\mathcal{G}$, $\mathcal{A}_\mathcal{G}$, $A_\mathcal{G}$, K_∞, K, $\mathcal{X}$, R, $\overline{R}$ and Sh be as in the introduction.

1.1 Compactification of the symmetric Space

Let ${}_\mathbb{R}\mathcal{P}_0$ be a minimal real parabolic subgroup with unipotent radical N_0. We may assume that ${}_\mathbb{R}\mathcal{L}_0(\mathbb{R}) = {}_\mathbb{R}\mathcal{P}_0(\mathbb{R}) \cap \theta({}_\mathbb{R}\mathcal{P}_0(\mathbb{R}))$, where θ is the Cartan involution associated to K_∞ is, defined over $\mathbb{R}$. Let ${}_\mathbb{R}\mathcal{A}_0$ be the maximal $\mathbb{R}$-split torus in the center of ${}_\mathbb{R}\mathcal{L}_0$.

Let ${}_\mathbb{R}\mathfrak{a}_0$ be the Lie algebra of ${}_\mathbb{R}\mathcal{A}_0(\mathbb{R})$ and ${}_\mathbb{R}\mathfrak{a}_0^{\mathcal{G}+}$ the positive Weyl chamber in it. Let $W({}_\mathbb{R}\mathfrak{a}_0, \mathfrak{g})$ be the Weyl group generated by reflections about $\alpha \in {}_\mathbb{R}\Delta_0$, the set of simple restricted roots.

A finite dimensional irreducible representation (π, V) of $\mathcal{G}(\mathbb{R})$ is called spherical if

$$V^K = \{v \in V | \pi(k)v = v \text{ for all } k \in K\} \subseteq V$$

is non-trivial. A spherical vector is a non-zero element of V^K. Up to scalars there is at most one spherical vector. Let $\mathfrak{h} = \mathfrak{t}_0 \oplus {}_\mathbb{R}\mathfrak{a}_0$ be a fundamental θ-stable Cartan subalgebra of $\mathfrak{g}$ and introduce a compatible ordering on the set of roots of $\mathfrak{g}_\mathbb{C}$ with respect to $\mathfrak{h}_\mathbb{C}$, e.g. by choosing a Borel subalgebra of $\mathfrak{g}_\mathbb{C}$ contained in ${}_\mathbb{R}\mathfrak{p}_{0,\mathbb{C}} = \mathrm{Lie}({}_\mathbb{R}\mathcal{P}_0(\mathbb{R}))_\mathbb{C}$. Let $\chi_0 \in \mathfrak{h}_\mathbb{C}^\vee$ be the highest weight of π. The Theorem of Cartan-Helgason, [Kna02], Theorem 8.49, characterizes spherical representations by the vanishing of $\chi_0|_\mathfrak{t}$ and a certain parity condition on the restricted highest weight $\lambda_0 = \chi_0|_{{}_\mathbb{R}\mathfrak{a}_0}$. It implies that there is up to $\mathbb{R}$-equivalence a unique real structure on V such that π is defined over $\mathbb{R}$. Fix such a real structure and write $V = V(\mathbb{R})$ for the set of real points of V. Let

$$V = V_{\lambda_0} \oplus \bigoplus_{\lambda \neq \lambda_0} V_\lambda$$

be the restricted weight decomposition of V with respect to the action of ${}_\mathbb{R}\mathfrak{a}_0$. Let v be a spherical vector and write

$$v = \sum_\lambda v_\lambda$$

accordingly. The Iwasawa decomposition for $\mathcal{G}(\mathbb{R})$ implies $v_{w\lambda_0} \in V_{w\lambda_0} - \{0\}$ for all $w \in W(\mathfrak{a}_0, \mathfrak{g})$ since by [Kna02], Proposition 7.32, w has representative in K. A triple (π, V, v) with (π, V) a spherical representation and a spherical vector $v \in V^K$ is called a spherical triple.

Set $x_{\mathcal{G}} = A_{\mathcal{G}}K \in \mathcal{X}$. Let

$$\mathcal{X}_+ \subseteq \mathcal{X}$$

be a connected component of $\mathcal{X}$. Since K_∞ meets all connected components of $\mathcal{G}(\mathbb{R})$ there is a class $kK \in K_\infty/K$ such that $\tilde{x}_{\mathcal{G}} = kx_{\mathcal{G}} \in \mathcal{X}_+$ and the map

$$\begin{aligned} \mathcal{G}(\mathbb{R})^+K/A_{\mathcal{G}}K &\to \mathcal{X}_+ \\ g &\mapsto kgk^{-1}\tilde{x}_{\mathcal{G}} \end{aligned}$$

is a diffeomorphism. A spherical triple (π, V, v) yields a smooth map

$$\begin{array}{rcl} \iota\colon \mathcal{X}_+ & \to & \mathbb{P}(V) \\ kgk^{-1}\tilde{x}_{\mathcal{G}} & \mapsto & [\pi(g)v]. \end{array}$$

It maps $\tilde{x}_{\mathcal{G}} \in \mathcal{X}_+$ to the line generated by v and is independent of the particular choice of v. Call a spherical triple (π, V, v) admissible if π is non-trivial on every non-compact $\mathbb{R}$-simple factor of $\mathcal{G}$. If (π, V, v) is admissible, ι is one-to-one. In this case the closure of $\iota(\mathcal{X}_+) \subset \mathbb{P}(V)$ is called the Satake compactification of $\mathcal{X}_+ \subseteq \mathcal{X}$. Let us denote it by $\overline{\mathcal{X}}_+$. Let $\overline{\mathcal{X}}$ be the disjoint union of $\overline{\mathcal{X}}_+$ where $\mathcal{X}_+$ runs through $\pi_0(\mathcal{X})$. The natural action of $\mathcal{G}(\mathbb{R})$ on $\overline{\mathcal{X}}$ is continuous and leaves $\mathcal{X}$ fixed.

The strategy for analyzing the connected components $\overline{\mathcal{X}}_+$ is to use affine coordinates of $\mathbb{P}(V)$ corresponding to the hyperplane

$$v_{\lambda_0} + \bigoplus_{\lambda \neq \lambda_0} V_\lambda \subset V - \{0\}$$

in conjunction with the Cartan decomposition of $K {}_{\mathbb{R}}\mathcal{A}_0(\mathbb{R})^+ K$ of $\mathcal{G}(\mathbb{R})^+K$. Here ${}_{\mathbb{R}}\mathcal{A}_0(\mathbb{R})^+$ is the image of the closure of the positive Weyl chamber in ${}_{\mathbb{R}}\mathfrak{a}_0$ under the exponential mapping. For example the action of $a \in {}_{\mathbb{R}}\mathcal{A}_0(\mathbb{R})^+$ is given by

$$a.x_{\mathcal{G}} = [\pi(a)v] = v_0 + \sum_{\lambda \neq \lambda_0} \prod_{\alpha \in {}_{\mathbb{R}}\Delta_0} e^{-\langle \lambda_0 - \lambda, \check{\varpi}_\alpha \rangle \langle \alpha, \log(a) \rangle} v_\lambda$$

where log denotes the inverse of $\exp\colon {}_{\mathbb{R}}\mathfrak{a}_0 \to {}_{\mathbb{R}}\mathcal{A}_0(\mathbb{R})^+$. We recall that $(\check{\varpi}_\alpha)_{\alpha \in {}_{\mathbb{R}}\Delta_0}$ is the basis of ${}_{\mathbb{R}}\mathfrak{a}_0$ dual to the basis ${}_{\mathbb{R}}\Delta_0$ of ${}_{\mathbb{R}}\check{\mathfrak{a}}_0$. The behavior of $a.x_{\mathcal{G}}$ as a varies is governed by the factors

$$e^{-\langle \lambda_0 - \lambda, \check{\varpi}_\alpha \rangle \langle \alpha, \log(a) \rangle}.$$

There are uniquely determined non-negative integers m_α such that

$$\lambda_0 - \lambda = \sum_{\alpha \in {}_{\mathbb{R}}\Delta_0} m_\alpha \alpha.$$

Define the support of λ to be the set

$$\operatorname{supp}(\lambda) = \{\alpha \in {}_{\mathbb{R}}\Delta_0 |\ m_\alpha > 0\} = \{\alpha \in {}_{\mathbb{R}}\Delta_0 |\ m_\alpha \neq 0\} \subseteq {}_{\mathbb{R}}\Delta_0.$$

Set

$$\delta = \{\alpha \in {}_{\mathbb{R}}\Delta_0 | \; s_\alpha \lambda_0 \neq \lambda_0\}.$$

A subset $\kappa \subseteq {}_{\mathbb{R}}\Delta_0$ is said to be δ-connected if $\kappa \cup \delta$ is connected as a subset of the Coxeter graph. Equivalently $\kappa \cup \{\lambda_0\}$ is connected as a subset of the inner product space ${}_{\mathbb{R}}\check{\mathfrak{a}}_0$.

The following proposition gives a rough description of the possible restricted weights of V that turns out to be sufficient for the study of Satake compactifications.

1.1 Proposition (Satake)**:** *Let λ be a restricted weight of ${}_{\mathbb{R}}\mathfrak{a}_0$ on V. Then* $\mathrm{supp}(\lambda) \subseteq {}_{\mathbb{R}}\Delta_0$ *is δ-connected. Conversely every δ-connected subset of ${}_{\mathbb{R}}\Delta_0$ is the support of some weight of V.*

A subset $\theta \subseteq {}_{\mathbb{R}}\Delta_0$ is called δ-saturated if it cannot be enlarged without enlarging its δ-connected component $\kappa(\theta)$. Under the action of $\mathcal{G}(\mathbb{R})$ the space $\overline{\mathcal{X}}$ decomposes into a disjoint union of $\mathcal{G}(\mathbb{R})$ orbits one for each δ-saturated subset of ${}_{\mathbb{R}}\Delta_0$. The orbit $\mathcal{X}$ corresponds to ${}_{\mathbb{R}}\Delta_0$.

A real parabolic subgroup $\mathcal{O} \subseteq \mathcal{G}$ is called δ-saturated (resp. δ-connected) if it is conjugate to a standard parabolic $\mathcal{O}'$ such that ${}_{\mathbb{R}}\Delta_0 - {}_{\mathbb{R}}\Delta_{\mathcal{O}'}$ is a δ-saturated (resp. δ-connected). For any real parabolic subgroup $\mathcal{O}$ let $\mathcal{O}^\kappa \subseteq \mathcal{O}$ be the unique parabolic such that the associated subset of ${}_{\mathbb{R}}\Delta_0$ is the δ-connected component of the set associated to $\mathcal{O}$. In particular $\mathcal{O}$ is δ-connected iff $\mathcal{O}^\kappa = \mathcal{O}$.

Let $\mathcal{O}$ be a δ-connected parabolic subgroup, $\mathcal{L}_{\mathcal{O}}(\mathbb{R}) = \mathcal{O}(\mathbb{R}) \cap \theta\mathcal{O}(\mathbb{R})$ the unique θ-stable Levi-component of $\mathcal{O}$ and $\mathcal{A}_{\mathcal{O}} \subseteq \mathcal{L}_{\mathcal{O}}$ the maximal $\mathbb{R}$-split torus in $\mathcal{L}_{\mathcal{O}}$. Let X be an element of the positive Weyl chamber $\mathfrak{a}_{\mathcal{O}}^{\mathcal{G}+}$. Then

$$v_{\mathcal{O}} = \lim_{n\to\infty} e^{-n\langle \lambda_0, X\rangle} \pi(e^{nX}) v \in V$$

exists and defines a point $[v_{\mathcal{O}}] \in \overline{\mathcal{X}}$. The boundary component

$$\partial_{\mathcal{O}}\overline{\mathcal{X}} \subseteq \overline{\mathcal{X}}$$

is the orbit of $[v_{\mathcal{O}}]$ under the action of $\mathcal{O}(\mathbb{R})$. It is easy to see that $\partial_{\mathcal{O}}\overline{\mathcal{X}}$ is contained in the set $\mathcal{N}_{\mathcal{O}}(\mathbb{R})$-fixed points in $\overline{\mathcal{X}}$. Hence the action of $\mathcal{O}(\mathbb{R})$ on $\partial_{\mathcal{O}}\overline{\mathcal{X}}$ induces an action of $\mathcal{L}_{\mathcal{O}}$. An element $g \in \mathcal{G}(\mathbb{R})$ induces a bijection

$$g \colon \partial_{\mathcal{O}}\overline{\mathcal{X}} \to \partial_{g\mathcal{O}g^{-1}}\overline{\mathcal{X}}.$$

The orbit decomposition of $\overline{\mathcal{X}}$ is refined by the decomposition into so called (real) boundary components It turns out that the union

$$\overline{\mathcal{X}} = \bigsqcup_{\mathcal{O}} \partial_{\mathcal{O}}\overline{\mathcal{X}}$$

where $\mathcal{O}$ runs through the set of all δ-saturated parabolics in $\mathcal{G}$ is disjoint.

It follows that the stabilizer of $\partial_{\mathcal{O}}\overline{\mathcal{X}} \subseteq \overline{\mathcal{X}}$ in $\mathcal{G}(\mathbb{R})$ is $\mathcal{O}(\mathbb{R})$. The actions of $\mathcal{N}_{\mathcal{O}}$ and $\mathcal{A}_{\mathcal{O}}$ on $\partial_{\mathcal{O}}\overline{\mathcal{X}}$ are trivial and $\mathcal{M}_{\mathcal{O}}$ acts transitively on $\partial_{\mathcal{O}}\overline{\mathcal{X}}$. The closure of $\partial_{\mathcal{O}}\overline{\mathcal{X}}$ is the union of all $\partial_{\mathcal{O}_1}\overline{\mathcal{X}}$ such that $\mathcal{O}_1^{\kappa} \subseteq \mathcal{O}^{\kappa}$, i.e.

$$\overline{\partial_{\mathcal{O}}\overline{\mathcal{X}}} = \bigsqcup_{\substack{\mathcal{O}_1 \\ \mathcal{O}_1^{\kappa} \subseteq \mathcal{O}^{\kappa}}} D_{\mathcal{O}_1}.$$

It agrees with the set of fixed points of $\mathcal{N}_{\mathcal{O}}(\mathbb{R})$ on $\overline{\mathcal{X}}$.

As it turns out the construction is hereditary. More precisely set $V_{\mathcal{O}} = V^{\mathcal{N}_{\mathcal{O}}}$ and let $\pi_{\mathcal{O}}$ be the representation of $\mathcal{L}_{\mathcal{O}} = \mathcal{O}/\mathcal{N}_{\mathcal{O}}$ on $V_{\mathcal{O}}$. Then $v_{\mathcal{O}} \in V_{\mathcal{O}}^{K_{\mathcal{O}}}$ and $\partial_{\mathcal{O}}\overline{\mathcal{X}}$ is the image of $\mathcal{L}_{\mathcal{O}}(\mathbb{R})/K_{\mathcal{O}}$ in an affine subspace of $\mathbb{P}(V)$ isomorphic to $\mathbb{P}(V_{\mathcal{O}})$. In general the representation of $\mathcal{L}_{\mathcal{O}}(\mathbb{R})$ on $V_{\mathcal{O}}$ will be trivial on some simple non-compact real factor of $\mathcal{L}_{\mathcal{O}}(\mathbb{R})$ and $\partial_{\mathcal{O}}\overline{\mathcal{X}}$ will only be the symmetric space of a suitable quotient of $\mathcal{L}_{\mathcal{O}}$.

1.2 Compactification of the locally symmetric Space

A real boundary component $\partial_{\mathcal{O}}\overline{\mathcal{X}} \subseteq \overline{\mathcal{X}}$ is called geometrically rational if $\mathcal{O}$ is defined over $\mathbb{Q}$ and if there is a normal $\mathbb{Q}$-subgroup $\mathcal{L}_{\mathcal{O},l} \subseteq \mathcal{L}_{\mathcal{O}}$ such that $\mathcal{L}_{\mathcal{O},l}(\mathbb{R})$ is normal and cocompact in

$$\mathcal{L}_{\mathcal{O}}(\mathbb{R})_l := \operatorname{Ker}(\mathcal{L}_{\mathcal{O}}(\mathbb{R}) \to \mathbb{P}\,\mathrm{Gl}(V_{\mathcal{O}}))$$

This determines $\mathcal{L}_{\mathcal{O},l}$ up to $\mathbb{Q}$-simple $\mathbb{R}$-anisotropic factors.

The Satake compactification $\overline{\mathcal{X}}$ is called geometrically rational if the closure of every classical Siegel $\mathfrak{S} \subset \mathcal{X}$ as defined in [Bor69] meets only geometrically rational boundary components. The definition guarantees that for every boundary component $\partial_{\mathcal{O}}\overline{\mathcal{X}}$ met by $\overline{\mathfrak{S}}$ the image of $\mathcal{O}(\mathbb{R}) \cap \Gamma$ in $\operatorname{Aut}(\partial_{\mathcal{O}}\mathcal{X}^*)$ is arithmetic. In [Cas97] Casselman gives a criterion for a spherical representation to be geometrically rational involving only the $\mathbb{Q}$-index of $\mathcal{G}$. It is a result of Baily and Borel, [BB66], Lemma 4.5, that the class of Satake compactifications corresponding to the Baily-Borel compactification in the hermitian case is geometrically rational. This result was extended to cover most equal-rank Satake compactifications by Saper in [Sap04]. A Satake compactification is equal-rank if the real boundary components $\partial_{\mathcal{O}}\overline{\mathcal{X}}$ are equal-rank symmetric spaces.

Assume geometric rationality from now on. Boundary components $\partial_{\mathcal{O}}\overline{\mathcal{X}}$ met by the closure of Siegel sets are called rational and are written as $\partial_{\mathcal{O}}\mathcal{X}^*$. For every $\partial_{\mathcal{O}}\mathcal{X}^*$ make some fixed choice of $\mathcal{L}_{\mathcal{O},l} \subset \mathcal{L}_{\mathcal{O}}$. Then $\mathcal{L}_{\mathcal{O},h} = \mathcal{L}_{\mathcal{O}}/\mathcal{L}_{\mathcal{O},l}$ is a semi-simple adjoint $\mathbb{Q}$-group.

Let $\mathfrak{P}$ be the set of rational parabolic subgroups of $\mathcal{G}$ and let $\mathfrak{P}^*$ be the subset of those that appear as normalizers of rational boundary components. Let

$$\mathcal{X}^* = \bigsqcup_{\mathcal{O} \in \mathfrak{P}^*} \partial_{\mathcal{O}}\mathcal{X}^*$$

be the disjoint union of all rational boundary components. Consider this for the moment equipped with the topology of the disjoint union. In [Cas97] it is proven that $\mathfrak{P}^*$ consists precisely of the $\mathcal{G}(\mathbb{Q})$-conjugates of rational standard parabolic subgroups that are δ-saturated real parabolic subgroups.

It is well known that for an arithmetic subgroup $\Gamma \subset \mathcal{G}(\mathbb{Q})$ one may choose a classical Siegel domain $\mathfrak{S} \subseteq \mathcal{X}$ and elements $\gamma_1, \ldots, \gamma_n \in \mathcal{G}(\mathbb{Q})$ such that

$$\mathfrak{F} = \bigcup_{i=1}^{n} \gamma_i \mathfrak{S},$$

is a fundamental domain for Γ. In particular $\Gamma\mathfrak{F} = \mathcal{X}$.

1.2 Theorem (Satake)**:** *Fix some arithmetic subgroup $\Gamma \subset \mathcal{G}(\mathbb{Q})$ and let $\mathfrak{F}$ be a fundamental domain for Γ as above. Then there exists a unique topology on $\mathcal{X}^*$ such that*

(1) *The topology induced on the closure $\mathfrak{F}^*$ of $\mathfrak{F}$ in $\mathcal{X}^*$ is the usual topology on $\mathfrak{F}^*$ as a subset of $\mathbb{P}(V)$.*

(2) *The group Γ acts as a group of homeomorphisms on $\mathcal{X}^*$.*

(3) *If x and x' are in different Γ-orbits then there exist neighborhoods $U \ni x$ and $U' \ni x'$ such that $\Gamma U \cap U' = \emptyset$.*

(4) *If $x \in \mathcal{X}^*$ is a point and $\Gamma_x \subset \Gamma$ is its isotropy group then there is a Γ_x invariant neighborhood $U \ni x$ such that $\gamma U \cap U \neq \emptyset$ implies $\gamma \in \Gamma_x$ for all $\gamma \in \Gamma$.*

This topology on $\mathcal{X}^*$, the Satake topology, is Hausdorff and does not depend on the particular arithmetic subgroup Γ and $\mathfrak{F}$ used in its definition. The quotient space $\Gamma\backslash\mathcal{X}^*$ is compact and Hausdorff.

By definition the space $\Gamma\backslash\mathcal{X}^*$ is the Satake compactification of $\Gamma\backslash\mathcal{X}$ with respect to the representation (π, V, v). Recall that $\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A}_f)/\mathbb{K}^f$ is finite by [PR94], Theorem 5.1. Let us define the Satake compactification of

$$\mathrm{Sh}(\mathbb{K}^f) = \mathcal{G}(\mathbb{Q})\backslash(\mathcal{X} \times \mathcal{G}(\mathbb{A}_f))/\mathbb{K}^f.$$

The inclusion $\mathcal{G}(\mathbb{R}) \subset \mathcal{G}(\mathbb{A})$ yields a natural homeomorphism

$$\coprod_{[h]\in\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A}_f)/\mathbb{K}^f} \mathcal{G}(\mathbb{Q}) \cap h\mathbb{K}^f h^{-1}\backslash\mathcal{X} \xrightarrow{\cong} \mathrm{Sh}(\mathbb{K}^f)$$
$$(\mathcal{G}(\mathbb{Q}) \cap h\mathbb{K}^f h^{-1})gA_{\mathcal{G}}K \mapsto \mathcal{G}(\mathbb{Q})A_{\mathcal{G}}(g,h)K\mathbb{K}^f.$$

Identifying $\mathrm{Sh}(\mathbb{K}^f)$ with the right hand side of (1.2) we define the Satake compactification of $\mathrm{Sh}(\mathbb{K}^f)$ as

$$\mathrm{Sh}(\mathbb{K}^f)^* := \coprod_{[h]\in\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A}_f)/\mathbb{K}^f} \mathcal{G}(\mathbb{Q}) \cap h\mathbb{K}^f h^{-1}\backslash\mathcal{X}^*.$$

It is a Theorem of Zucker, [Zuc83], that $\mathrm{Sh}(\mathbb{K}^f)^*$ is a quotient of the reductive Borel-Serre compactification. Let us briefly recall how the latter is constructed. Let $\mathcal{O} \in \mathfrak{P}$ and $q_{\mathcal{O}} \colon \mathcal{O} \twoheadrightarrow \mathcal{L}_{\mathcal{O}}$ be the Levi quotient. Set $Z_{\mathcal{O}} = q_{\mathcal{O}}^{-1}(A_{\mathcal{O}}) \supset \mathcal{N}_{\mathcal{O}}$ and let

$$\partial_{\mathcal{O}} \mathcal{X}^{\wedge} = \mathcal{X}/Z_{\mathcal{O}}$$

be the quotient space. For $\mathcal{O} \subseteq \mathcal{P}$ we have $Z_{\mathcal{P}} \subseteq Z_{\mathcal{O}}$ and there is a canonical quotient map

$$\widehat{\pi}_{\mathcal{P},\mathcal{O}} \colon \partial_{\mathcal{P}} \mathcal{X}^{\wedge} \twoheadrightarrow \partial_{\mathcal{O}} \mathcal{X}^{\wedge}.$$

Let

$$\mathcal{X}^{\wedge} = \bigsqcup_{\mathcal{O} \in \mathfrak{P}} \partial_{\mathcal{O}} \mathcal{X}^{\wedge}$$

be disjoint union (of topological spaces). For $\mathcal{O} \in \mathfrak{P}$ let

$$\mathcal{X}^{\wedge}(\mathcal{O}) = \bigsqcup_{\mathcal{O} \subseteq \mathcal{P} \in \mathfrak{P}} \partial_{\mathcal{P}} \mathcal{X}^{\wedge} \subseteq \mathcal{X}^{\wedge}$$

and set

$$\widehat{\pi}_{\mathcal{O}} = \sqcup_{\mathcal{O} \subseteq \mathcal{P} \in \mathfrak{P}} \widehat{\pi}_{\mathcal{P},\mathcal{O}} \colon \mathcal{X}^{\wedge}(\mathcal{O}) \twoheadrightarrow \partial_{\mathcal{O}} \mathcal{X}^{\wedge}.$$

This is a continuous map. One has $\widehat{\pi}_{\mathcal{O}} = \widehat{\pi}_{\mathcal{P},\mathcal{O}} \widehat{\pi}_{\mathcal{O}}$ and $\mathcal{X}^{\wedge}(\mathcal{O}) \supseteq \mathcal{X}^{\wedge}(\mathcal{P})$ for $\mathcal{O} \subseteq \mathcal{P}$. For $x \in \mathcal{X}^{\wedge}$ let $\mathcal{O}(x)$ be the unique rational parabolic such that $x \in \partial_{\mathcal{O}(x)} \mathcal{X}^{\wedge}$. There is a unique extension of the $\mathcal{G}(\mathbb{Q})$-action on $\mathcal{X}$ to an action on $\mathcal{X}^{\wedge}$ with the property that

$$\widehat{\pi}_{\gamma \mathcal{O} \gamma^{-1}}(\gamma x) = \gamma \widehat{\pi}_{\mathcal{O}}(x)$$

for all $\mathcal{O} \in \mathfrak{P}$ and $x \in \mathcal{X}$. Indeed, write $x \in \partial_{\mathcal{O}} \mathcal{X}^{\wedge}$ as $x = \widehat{\pi}_{\mathcal{O}}(y)$ for some $y \in \mathcal{X}$ and set for any $\gamma \in \mathcal{G}(\mathbb{Q})$ set $\gamma x := \widehat{\pi}_{\gamma \mathcal{O} \gamma^{-1}}(\gamma y) = \widehat{\pi}_{\mathcal{G}, \gamma \mathcal{O} \gamma^{-1}}(\gamma y)$. This is well defined since $\gamma Z_{\mathcal{O}} \gamma^{-1} = Z_{\gamma \mathcal{O} \gamma^{-1}}$ and is clearly the unique action with the required properties. Let

$$\overline{\mathbb{R}} = \mathbb{R} \cup \infty.$$

and define a function

$$d_{\mathcal{O}} \colon \mathcal{X}^{\wedge}(\mathcal{O}) \to \overline{\mathfrak{a}}_{\mathcal{O}} = \prod_{\alpha \in \Delta_{\mathcal{O}}} \overline{\mathbb{R}} \varpi_{\alpha}$$

as follows:

$$d_{\mathcal{O}}(x) = H_{\mathcal{O},\infty}(o) + \sum_{\alpha \in \Delta_{\mathcal{O}(x)}} \infty \cdot \varpi_{\alpha}$$

where $o \in \mathcal{O}$ is such that $\pi_{\mathcal{G},\mathcal{O}(x)}(o.X_{\mathcal{G}}) = x$. The function $d_{\mathcal{O}}$ is well defined since $H_{\mathcal{O},\infty}$ vanishes on $\mathcal{O} \cap K$. Let $\mathcal{O} \in \mathfrak{P}$, $V \subseteq \partial_{\mathcal{O}} \mathcal{X}^{\wedge}$ an open set and $T > D$ be a real number. Set

$$U^{\wedge}(\mathcal{O}, T, V) = \{x \in \mathcal{X}^{\wedge}(\mathcal{O}) | \langle \alpha, d_{\mathcal{O}}(x) \rangle > T,\ \alpha \in \Delta_{\mathcal{O}},\ \widehat{\pi}_{\mathcal{O}}(x) \in V\}$$

It is clear that any element of $\mathcal{X}^\wedge$ is contained in some $U^\wedge(\mathcal{O}, T, V)$. If

$$x \in U^\wedge(\mathcal{O}_1, T_1, V_1) \cap U^\wedge(\mathcal{O}_2, T_2, V_2)$$

and $x \in \partial_{\mathcal{O}}\mathcal{X}^\wedge$ let T be the maximum of T_1 and T_2 and

$$V = \partial_{\mathcal{O}}\mathcal{X}^\wedge \cap U^\wedge(\mathcal{O}_1, T_1, V_1) \cap U^\wedge(\mathcal{O}_2, T_2, V_2).$$

Then V is open and

$$x \in U^\wedge(\mathcal{O}, T, V) \subseteq U^\wedge(\mathcal{O}_1, T_1, V_1) \cap U^\wedge(\mathcal{O}_2, T_2, V_2).$$

Define a topology on $\mathcal{X}^\wedge$ by letting $U \subseteq \mathcal{X}^\wedge$ be open iff for any $x \in U$ there are $\mathcal{O}$, T, V as above such that $x \in U^\wedge(\mathcal{O}, T, V) \subseteq U$.

1.3 Lemma: Let $\mathcal{O} \in \mathfrak{P}$. Then

(1) The sets $\mathcal{X}$ and $\mathcal{X}^\wedge(\mathcal{O})$ are open in $\mathcal{X}^\wedge$ and the complement $\partial\mathcal{X}^\wedge$ of $\mathcal{X}$ is closed.

(2) The set $U^\wedge(\mathcal{O}, T, V)$ is open.

(3) The action of $\mathcal{G}(\mathbb{Q})$ on $\mathcal{X}^\wedge$ is continuous.

(4) The action of $\mathcal{O}$ on $\partial_{\mathcal{O}}\mathcal{X}^\wedge$ is continuous.

(5) $\mathcal{X}^\wedge$ is Hausdorff and contains $\mathcal{X}$ as an open dense subset.

The proof is easy.

For $\mathbb{K}^f \subset \mathcal{G}(\mathbb{A}_f)$ the reductive Borel-Serre compactification of $\mathrm{Sh}(\mathbb{K}^f)$ is the quotient space

$$\mathrm{Sh}(\mathbb{K}^f)^\wedge = \mathcal{G}(\mathbb{Q})\backslash\mathcal{X}^\wedge \times \mathcal{G}(\mathbb{A}_f)/\mathbb{K}^f.$$

1.4 Theorem (Borel, Serre, Zucker)**:** *The space*

$$\mathrm{Sh}(\mathbb{K}^f)^\wedge = \mathcal{G}(\mathbb{Q})\backslash\mathcal{X}^\wedge \times \mathcal{G}(\mathbb{A}_f)/\mathbb{K}^f.$$

is a compact Hausdorff space containing $\mathrm{Sh}(\mathbb{K}^f)$ *as an open dense subspace.*

To treat the various Satake compactifications and the reductive Borel-Serre compactification on the same footing let ? stand for either $*$ or $\wedge$. Clearly $\mathrm{Sh}(\mathbb{K}^f) = \mathrm{Sh}/\mathbb{K}^f$ since the $\mathcal{G}(\mathbb{Q})$ left action commutes with the $\mathcal{G}(\mathbb{A}_f)$ right action. Given compact open subgroups $\mathbb{K}_1^f \subset \mathbb{K}_2^f$ there is a canonical quotient map

$$\mathrm{Sh}(\mathbb{K}_1^f)^? \twoheadrightarrow \mathrm{Sh}(\mathbb{K}_2^f)^?$$

extending a similar map $\mathrm{Sh}(\mathbb{K}_1^f) \twoheadrightarrow \mathrm{Sh}(\mathbb{K}_2^f)$. Hence $(\mathrm{Sh}(\mathbb{K}^f)^?)_{\mathbb{K}^f}$ is a projective system of compact Hausdorff spaces. Its projective limit

$$\lim_{\mathbb{K}^f} \mathrm{Sh}(\mathbb{K}^f)^?,$$

is a compact Hausdorff space receiving a canonical map from Sh. We will see that this gives the desired compactification $\mathrm{Sh}^?$ of Sh. We need another description. Let $R^? \subset (\mathcal{X}^? \times \mathcal{G}(\mathbb{A}_f))^2$ be the graph of the $\mathcal{G}(\mathbb{Q})$-relation on $\mathcal{X}^? \times \mathcal{G}(\mathbb{A}_f)$ and let $\overline{R}^?$ be its closure. Let

$$\mathrm{Sh}^? = \overline{R}^? \backslash (\mathcal{X}^? \times \mathcal{G}(\mathbb{A}_f))$$

be the quotient space of $\mathcal{X}^? \times \mathcal{G}(\mathbb{A}_f)$ by $\overline{R}^?$.

The following Theorem is probably well-known. It extends Proposition 2.1.10 in [Del79] to Satake compactifications.

1.5 Theorem: *There is a canonical homeomorphism*

$$\mathrm{Sh}^? \xrightarrow{\cong} \lim_{\mathbb{K}^f} \mathrm{Sh}(\mathbb{K}^f)^?.$$

The space $\mathrm{Sh}^?$ *is a compact Hausdorff space. The closure* $\overline{R}^?$ *of the* $\mathcal{G}(\mathbb{Q})$ *relation on* $\mathcal{X}^? \times \mathcal{G}(\mathbb{A}_f)$ *may be described as follows: Let* $(x, g), (x', g') \in \mathcal{X}^? \times \mathcal{G}(\mathbb{A}_f)$ *be two points. Denote by* $\mathfrak{Z}_x^? = \{\gamma \in \mathcal{G}(\mathbb{Q}) | \gamma x = x\}$ *the stabilizer of* x *and by* $\mathbb{Z}_x^? \subset \mathcal{G}(\mathbb{A}_f)$ *its closure in* $\mathcal{G}(\mathbb{A}_f)$*. Then* $(x, g)\overline{R}^?(x', g')$ *if and only if there are elements* $\gamma \in \mathcal{G}(\mathbb{Q})$ *and* $m \in \mathbb{Z}_x^?$ *such that* $(x', g') = (\gamma x, \gamma m x)$*.*

Theorem 1.5 is probably well-known. It extends Proposition 2.1.10 in [Del79] to all Satake-compactifications and the reductive Borel-Serre compactification.

Since we mostly consider a fixed Satake compactification we write $\mathbb{Z}_x$ instead $\mathbb{Z}_x^*$ for simplicity.

Proof: The proof is straightforward. Assume $? = *$ and let us proceed in several steps.

(Step 1) The space Sh^* is compact. It suffices to show that there is a compact set $\mathbb{F} \subset \mathcal{X}^* \times \mathcal{G}(\mathbb{A}_f)$ such that $\mathcal{G}(\mathbb{Q})\mathbb{F} = \mathcal{X}^* \times \mathcal{G}(\mathbb{A}_f)$. Since $R^* \subseteq \overline{R}^*$ this implies

$$\overline{R}^*[\mathbb{F}] = \mathcal{X}^* \times \mathcal{G}(\mathbb{A}_f)$$

where $\overline{R}[\mathbb{F}]$ denotes the $\overline{R}^*$-saturation of $\mathbb{F}$. Fix a compact open subgroup $\mathbb{K}^f \subset \mathcal{G}(\mathbb{A}_f)$. Let $g_i \in \mathcal{G}(\mathbb{A}_f)$, $1 \le i \le n$, be a set of representatives for $\mathcal{G}(\mathbb{Q}) \backslash \mathcal{G}(\mathbb{A}_f) / \mathbb{K}^f$. Let

$$\mathfrak{F}_i^* \subset \mathcal{X}^*$$

be a compact fundamental domain for the action of

$$\Gamma_i = \mathcal{G}(\mathbb{Q}) \cap g_i \mathbb{K}^f g_i^{-1}$$

on $\mathcal{X}^*$ as in Theorem 1.2. Set

$$\mathbb{F} = \bigcup_i \mathfrak{F}_i^* \times g_i \mathbb{K}^f.$$

Let $(y, h) \in \mathcal{X}^* \times \mathcal{G}(\mathbb{A}_f)$ be some point. There are an index $1 \leq i \leq n$ and elements $\delta \in \mathcal{G}(\mathbb{Q})$ and $k \in \mathbb{K}^f$ such that $h = \delta g_i k$. Furthermore because $\mathfrak{F}_i^*$ is a fundamental domain for Γ_i there are $\gamma \in \Gamma_i$ and $x \in \mathfrak{F}_i$ such that $y = \delta\gamma x$. Hence

$$(y, h) = (\delta\gamma x, h) = \delta(\gamma x, g_i k) = \delta\gamma(x, g_i g_i^{-1}\gamma^{-1} g_i k) \in \mathcal{G}(\mathbb{Q})\mathbb{F}$$

since $g_i^{-1}\gamma^{-1} g_i k \in \mathbb{K}^f$.

(Step 2) The space $\lim\ \mathrm{Sh}(\mathbb{K}^f)^*$ is Hausdorff since its diagonal can be written as

$$\Delta_{\lim\ \mathrm{Sh}(\mathbb{K}^f)^*} = \bigcap_{\mathbb{K}^f} p_{\mathbb{K}^f}^{-1}(\Delta_{\mathrm{Sh}(\mathbb{K}^f)^*}).$$

(Step 3) Existence of the map f. Let $\mathbb{K}^f \subset \mathcal{G}(\mathbb{A}_f)$ be some compact open subgroup. Then there is a canonical quotient map

$$f_{\mathbb{K}^f}\colon \mathrm{Sh}^* \twoheadrightarrow \mathrm{Sh}^* / \mathbb{K}^f =: \mathrm{Sh}^*(\mathbb{K}^f) = \mathrm{Sh}(\mathbb{K}^f)^*$$

since in the presence of the $\mathbb{K}^f$-relation the difference between R and its closure $\overline{R}$ disappears, i.e.

$$R \circ R_{\mathbb{K}^f} = \overline{R}^* \circ R_{\mathbb{K}^f}$$

where $R_{\mathbb{K}^f}$ denotes the $\mathbb{K}^f$-relation and "$\circ$" denotes the composition of relations. For $\mathbb{K}_1^f \subseteq \mathbb{K}_2^f$ these maps satisfy $f_{\mathbb{K}_2^f} = \pi_{\mathbb{K}_2^f}^{\mathbb{K}_1^f} \circ f_{\mathbb{K}_1^f}$. Hence a continuous map

$$f = \lim\ f_{\mathbb{K}^f}\colon \mathrm{Sh}^* \to \lim\ \mathrm{Sh}^*(\mathbb{K}^f).$$

(Step 4) The map f is onto. This follows immediately from the following well-known fact from general topology, see [Roh96], Lemma 1.8.

1.6 Lemma: Let $(\mathcal{X}_\alpha, \pi_\beta^\alpha)$ and $(Y_\alpha, \pi_\beta^\rho)$ be two projective systems of topological spaces with $\mathcal{X}_\alpha$ compact and Y_α Hausdorff for every index α. Let $\mathcal{X} = \lim\ \mathcal{X}_\alpha$ and $Y = \lim\ Y_\alpha$. Let $f_\alpha\colon \mathcal{X}_\alpha \to Y_\alpha$ be a compatible system of continuous maps such that every f_α is onto with non-empty compact fibres. Then the limit map $f = \lim\ f_\alpha$ is continuous with non-empty compact fibres.

(Step 5) The map f is injective and the relation $\overline{R}$ is given as in the theorem. Indeed let $\mathbb{K}_n^f \subset \mathcal{G}(\mathbb{A}_f)$, $n \in \mathbb{N}$, form a basis of neighborhoods of the unit element. Let s,$s' \in \mathrm{Sh}^*$ be two points such that $f_{\mathbb{K}_n^f}(s) = f_{\mathbb{K}_n^f}(s')$ for every $n \in \mathbb{N}$. In terms of representatives (x, g),$(x', g') \in \mathcal{X}^* \times \mathcal{G}(\mathbb{A}_f)$ of s and s' this means that there is a sequence of $\gamma_n \in \mathcal{G}(\mathbb{Q})$ and $k_n \in \mathbb{K}_n^f$ such that

$$(x', g') = (\gamma_n x, \gamma_n g k_n)$$

for every n. Rewrite this as

$$\gamma_1(x, g'') = (x', g') = (\gamma_n x, \gamma_n g k_n) = \gamma_1(\mu_n x, \mu_n k_n' g)$$

with $g'' = \gamma_1^{-1}g'$, $\mu_n = \gamma_1^{-1}\gamma_n$ and $k'_n = gk_ng^{-1} \in g\mathbb{K}^f_n g^{-1}$. From this we see that $\mu_n \in \mathbb{Z}_x$ and $\mu_n k'_n = g''g^{-1}$. Since the sets $g\mathbb{K}^f_n g^{-1}$ form a neighborhood base of the identity, the series of the $\mu_n \in \mathbb{Z}_x$ converges to $g''g^{-1}$. Now set

$$m = \lim_n \mu_n = g''g^{-1} \in \mathbb{Z}_x.$$

Then

$$(x', g') = \gamma_1(x, mg)$$

and this implies $(x', g')\overline{R}^*(x, g)$ or $s = s'$ and f is injective as claimed. On the other hand we already know f to be onto and hence the closure of the $\mathcal{G}(\mathbb{Q})$-relation cannot be bigger than claimed.

(Step 6) The map f is a homeomorphism. By the preceding steps there is a canonical continuous bijection

$$f\colon \operatorname{Sh}^* \to \lim\ \operatorname{Sh}^*(\mathbb{K}^f),$$

the space Sh^* is compact and $\lim\ \operatorname{Sh}^*(\mathbb{K}^f)$ is Hausdorff. Hence f is an homeomorphism. □

Note that by the description of $\overline{R}^?$ the right action of $\mathcal{G}(\mathbb{A}_f)$ on $\mathcal{X}^? \times \mathcal{G}(\mathbb{A}_f)$ induces a right action of $\mathcal{G}(\mathbb{A}_f)$ on $\operatorname{Sh}^?$. For every standard parabolic subgroup $\mathcal{O} \in \mathfrak{P}^?$ we set

$$\partial_{\mathcal{O}} \operatorname{Sh}^? = \overline{R}^? \backslash \overline{R}^?(\partial_{\mathcal{O}}\mathcal{X}^? \times \mathcal{G}(\mathbb{A}_f)) = \overline{R}^? \backslash (\partial_{\mathcal{O}}\mathcal{X}^? \times \mathcal{G}(\mathbb{A}_f)).$$

1.7 Corollary: (1) *The set $\mathfrak{B}^?$ of open subsets of $\operatorname{Sh}^?$ that are invariant from the right under some compact open subgroup of $\mathcal{G}(\mathbb{A}_f)$ is a basis for the topology of $\operatorname{Sh}^?$.*

(2) *If the point $s \in \operatorname{Sh}^?$ is represented by $(x, g) \in \mathcal{X}^? \times \mathcal{G}(\mathbb{A}_f)$ then its stabilizer $\mathbb{Z}_s$ is the closed subgroup $g^{-1}\mathbb{Z}_x g \subseteq \mathcal{G}(\mathbb{A}_f)$.*

(3) *We have*

$$\operatorname{Sh}^? = \bigsqcup_{\mathcal{O}} \partial_{\mathcal{O}} \operatorname{Sh}^?$$

or in other words Sh^* is the disjoint union of the spaces $\partial_{\mathcal{O}} \operatorname{Sh}^*$ where $\mathcal{O} \in \mathfrak{P}^*$ runs through the set of standard δ-saturated rational parabolic subgroups.

(4) *The canonical map $\operatorname{Sh} \to \operatorname{Sh}^?$ identifies Sh with the open dense subset $\partial_{\mathcal{G}} \operatorname{Sh}^? \subseteq \operatorname{Sh}^?$. In particular the set $\mathfrak{B} = \mathfrak{B}^* \cap \operatorname{Sh}$ of open sets in Sh invariant from the right under some compact open subgroup is a basis for the topology of Sh.*

(5) *The quotient mapping $p\colon \mathcal{X}^? \times \mathcal{G}(\mathbb{A}_f) \twoheadrightarrow \operatorname{Sh}^?$ is an open mapping. Similarly its restriction*

$$p|_{\partial_{\mathcal{O}}\mathcal{X}^? \times \mathcal{G}(\mathbb{A}_f)}\colon \partial_{\mathcal{O}}\mathcal{X}^? \times \mathcal{G}(\mathbb{A}_f) \twoheadrightarrow \partial_{\mathcal{O}} \operatorname{Sh}^?$$

is open.

Proof: (1) results from the description of $\mathrm{Sh}^?$ as projective limit and (2) from the explicit description of the equivalence relation in the theorem. (3) is implied by the similar assertion about $\mathcal{X}^?$ and the fact that every rational parabolic subgroup is conjugate by some $\gamma \in \mathcal{G}(\mathbb{Q})$ to a standard parabolic subgroup. (4) follows from the existence of arbitrarily small neighborhoods invariant under some compact open subgroup $\mathbb{K}^f$ and the analogous fact for the classical Satake compactifications which is an immediate consequence of Theorem 1.2. (5) is a consequence of the description of $\overline{R}^?$ in Theorem 1.5. Indeed, let $V \times \Omega$ be a basic open subset. Then its $\overline{R}^?$-saturation is

$$\bigcup_{x \in V} \mathcal{G}(\mathbb{Q})(\{1\} \times \mathbb{Z}_x^?)(V \times \Omega).$$

which is visibly open. □

1.3 Zucker's Quotient Map

Recall that in [Zuc83], 3.7-3.10, Zucker constructs a continuous $\mathcal{G}(\mathbb{Q})$-equivariant surjection $p\colon \mathcal{X}^\wedge \twoheadrightarrow \mathcal{X}^*$ extending the identity mapping $\mathcal{X} \to \mathcal{X}$. Furthermore p is compatible with the stratifications, i.e.

$$p^{-1}(\partial_{\mathcal{O}}\mathcal{X}^*) = \bigcup_{\mathcal{P} \subseteq \mathcal{O}} \partial_{\mathcal{P}}\mathcal{X}^\wedge$$

where $\mathcal{P}$ runs through all elements of $\mathfrak{P}$ contained in $\mathcal{O}$ whose δ-saturation is $\mathcal{O}$ and the diagrams

$$\begin{array}{ccc} \partial_{\mathcal{P}}\mathcal{X}^\wedge & & \\ \Big\downarrow{\scriptstyle \widehat{\pi}_{\mathcal{P},\mathcal{Q}}} & \searrow^{p} & \\ & & \partial_{\mathcal{O}}\mathcal{X}^* \\ & \nearrow_{p} & \\ \partial_{\mathcal{Q}}\mathcal{X}^\wedge & & \end{array}$$

commute for all rational parabolics $\mathfrak{P}^* \ni \mathcal{O} \supset \mathcal{P} \supset \mathcal{Q}$ whose δ-saturation is $\mathcal{O}$.

1.8 Theorem: *There is a unique quotient map* $q\colon \mathrm{Sh}^\wedge \twoheadrightarrow \mathrm{Sh}^*$ *extending the identity map on* Sh.

Proof: There can be at most one such map since Sh is dense in $\mathrm{Sh}^\wedge$ and Sh^* is Hausdorff. For the existence note that $p\colon \mathcal{X}^\wedge \twoheadrightarrow \mathcal{X}^*$ induces a natural continuous $\mathcal{G}(\mathbb{Q})$-equivariant surjection $p' = p \times 1\colon \mathcal{X}^\wedge \times \mathcal{G}(\mathbb{A}_f) \twoheadrightarrow \mathcal{X}^* \times \mathcal{G}(\mathbb{A}_f)$. Equivariance implies that $R^\wedge \subseteq (p' \times p')^{-1}(R^*)$ and consequently

$$\overline{R}^\wedge \subseteq \overline{(p' \times p')^{-1}(R^*)} \subseteq (p' \times p')^{-1}(\overline{R}^*).$$

Hence p gives rise to a natural continuous surjection $q\colon \mathrm{Sh}^\wedge \twoheadrightarrow \mathrm{Sh}^*$ which is automatically a quotient map since $\mathrm{Sh}^\wedge$ and Sh^* are compact Hausdorff spaces. $\square$

For an open set $V \subseteq \partial_{\mathcal{O}}\mathcal{X}^*$ set

$$U^*(\mathcal{O}, T, V) = \text{ image of } U^\wedge(\mathcal{O}, T, (p|_{\partial_{\mathcal{O}}\mathcal{X}^\wedge \twoheadrightarrow \partial_{\mathcal{O}}\mathcal{X}^*})^{-1}(V)) \text{ in } \mathcal{X}^*$$

where $p\colon \mathcal{X}^\wedge \twoheadrightarrow \mathcal{X}^*$ is the map constructed by Zucker. From the construction of p it follows that the image of $U^\wedge(\mathcal{P}, T, V)$ in $\mathcal{X}^*$ is $U^*(\mathcal{O}, T, \widehat{\pi}_{\mathcal{O},\mathcal{P}}^{-1}(V))$ where $\mathcal{O} \supset \mathcal{P}$ is the δ-saturation of $\mathcal{P}$ and $\widehat{\pi}_{\mathcal{O},\mathcal{P}}\colon \partial_{\mathcal{O}}\mathcal{X}^\wedge \twoheadrightarrow \partial_{\mathcal{P}}\mathcal{X}^\wedge$ is the projection map. It is easily checked that these sets satisfy the conditions to be a neighborhood base for a uniquely determined topology on $\mathcal{X}^*$. This topology will in general be different from the Satake topology on $\mathcal{X}^*$. So let us write $\widetilde{\mathcal{X}}^*$ for it.

1.9 Proposition: *The product topology on* $\widetilde{\mathcal{X}}^* \times \mathcal{G}(\mathbb{A}_f)$ *induces the Satake topology on the quotient space* Sh^*.

PROOF: In the commutative diagram

$$\begin{array}{ccc}
\mathcal{X}^\wedge \times \mathcal{G}(\mathbb{A}_f) & \xrightarrow{\quad p' \quad} & \mathcal{X}^* \times \mathcal{G}(\mathbb{A}_f) \\
\Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \rho} \\
\mathrm{Sh}^\wedge & \xrightarrow[\quad q \quad]{} & \mathrm{Sh}^*
\end{array}$$

all maps are continuous and onto. The maps π, q, ρ are quotient maps. Let $s \in \partial_{\mathcal{O}}\,\mathrm{Sh}^*$ be represented by $(x, g) \in \partial_{\mathcal{O}}\mathcal{X}^* \times \mathcal{G}(\mathbb{A}_f)$. Let $s \in U \subseteq \mathrm{Sh}^*$ be some subset containing s. We have to show that $\rho^{-1}(U)$ is a neighborhood of (x, g) in $\mathcal{X}^* \times \mathcal{G}(\mathbb{A}_f)$ iff an only if it contains some set of the form $U^*(\mathcal{O}, T, V) \times \Omega$ with $x \in V \subset \partial_O\mathcal{X}^*$, $T > 0$ and Ω chosen appropriately.

To see this assume that every point (x, g) of $\rho^{-1}(U)$ contains a set of the form $U^*(\mathcal{O}, T, V) \times \Omega$. Then any point in $(\rho \circ p')^{-1}(U)$ contains a set $U^\wedge(\mathcal{O}, \widetilde{V}, T) \times \Omega$ where $\widetilde{V} \subseteq \partial_{\mathcal{O}}\mathcal{X}^\wedge$ is the preimage of V under the quotient mapping $p|_{\partial_{\mathcal{O}}\mathcal{X}^\wedge}\colon \partial_{\mathcal{O}}\mathcal{X}^\wedge \twoheadrightarrow \partial_{\mathcal{O}}\mathcal{X}^*$. It follows $(\rho \circ p')^{-1}(U)$ is open. Since it equals $(q \circ \pi)^{-1}(U)$ and $q \circ \pi$ is a quotient mapping U is also open in Sh^*.

Conversely assume that U is a neighborhood of $s \in \partial_{\mathcal{O}}\,\mathrm{Sh}^*$. Then $(q \circ \pi)^{-1}(U)$ is a neighborhood of $(q \circ \pi)^{-1}(\{s\})$. Hence $(q \circ \pi)^{-1}(\{s\})$ can be covered by sets $U^\wedge(\mathcal{P}_i, V_i, T_i) \times \Omega_i \subseteq (q \circ \pi)^{-1}(U)$ for i in some indexing set I where $\mathcal{O}^\kappa \subseteq \mathcal{P}_i \subseteq \mathcal{O}$, $V_i \subseteq \partial_{\mathcal{P}_i}\mathcal{X}^\wedge$ open, $T_i > 0$ and $\Omega_i \subseteq \mathcal{G}(\mathbb{A}_f)$ is an open. Since $q^{-1}(\{s\}) \subseteq \mathrm{Sh}^\wedge$ is compact there are finitely indices $i = 1, \dots, N \in I$ such that the images of $U^\wedge(\mathcal{P}_i, V_i, T_i) \times \Omega_i$ cover $q^{-1}(\{s\})$. Since the quotient maps $\partial_{\mathcal{P}_i}\mathcal{X}^\wedge \times \mathcal{G}(\mathbb{A}_f) \twoheadrightarrow \partial_{\mathcal{P}_i}\,\mathrm{Sh}^\wedge$ are open we may replace V_i, $i = 1, \dots, N$ by the larger open set

$$\widetilde{V}_i = \mathrm{pr}_{\partial_{\mathcal{P}_i}\mathcal{X}^\wedge \times \mathcal{G}(\mathbb{A}_f) \twoheadrightarrow \partial_{\mathcal{P}_i} Sh^\wedge}\, \overline{R}^\wedge[V_i \times \Omega_i]) \subseteq \partial_{\mathcal{P}_i}\mathcal{X}^\wedge.$$

Having done this the sets $U^\wedge(\mathcal{P}_i, \widetilde{V}_i, T_i) \times \Omega_i$ also cover $(q \circ \pi)^{-1}(\{s\})$ and are contained in $(q \circ \pi)^{-1}(U)$. Replacing $\mathcal{P}_i$ by $\mathcal{O}$ and $\tilde{V}_i$ by its preimage under the map $\widehat{\pi}_{\mathcal{O},\mathcal{P}_i} \colon \partial_{\mathcal{O}}\mathcal{X}^\wedge \twoheadrightarrow \partial_{\mathcal{P}_i}\mathcal{X}^\wedge$ we see that

$$(q \circ \pi)^{-1}(\{s\}) \subseteq \bigcup_{i=1}^{N} U^\wedge(\mathcal{O}, \widehat{\pi}^{-1}_{\mathcal{O},\mathcal{P}_i}(\widetilde{V}_i), T_i) \times \Omega_i \subseteq (q \circ \pi)^{-1}(U)$$

Now setting $V = \bigcap_{i=1}^{N} \widehat{\pi}^{-1}_{\mathcal{O},\mathcal{P}_i}(\widetilde{V}_i)$, $T = \max_{i=1}^{N}\{T_i\}$ and $\Omega = \bigcap_{i=1}^{N} \Omega_i$ we find that

$$(q \circ \pi)^{-1}(\{s\}) \subseteq U^\wedge(\mathcal{O}, V, T) \times \Omega \subseteq (q \circ \pi)^{-1}(U)$$

or

$$\sigma^{-1}(\{s\}) \subseteq U^*(\mathcal{O}, V, T) \times \Omega \subseteq \sigma^{-1}(U).$$

Hence $\sigma^{-1}(U)$ is open in $\widetilde{\mathcal{X}}^* \times \mathcal{G}(\mathbb{A}_f)$. □

1.10 Corollary: (1) *Let $x \in \partial_{\mathcal{O}}\mathcal{X}^*$, $x' \in \partial_{\mathcal{O}'}\mathcal{X}^*$, $g \in \mathcal{G}(\mathbb{A}_f)$ and let $\mathbb{K}^f \subset \mathcal{G}(\mathbb{A}_f)$ be an open compact subgroup. Then there exist neighborhoods $x \in V \subseteq \partial_{\mathcal{O}}\mathcal{X}^*$ and $x' \in V' \subseteq \partial_{\mathcal{O}'}\mathcal{X}^*$ and a real number T such that for all $\gamma \in \mathcal{G}(\mathbb{Q})$*

$$\gamma(U^*(\mathcal{O}, V, T) \times g\mathbb{K}^f) \cap (U^*(\mathcal{O}', V', T) \times g\mathbb{K}^f) \neq \emptyset$$

implies $\gamma\mathcal{O}'\gamma^{-1} = \mathcal{O}$. In particular $\gamma \in \mathcal{O}(\mathbb{Q}) \cap g\mathbb{K}^f g^{-1}$ if $\mathcal{O} = \mathcal{O}'$.

(1) *If $x \in \partial_{\mathcal{O}}\mathcal{X}^*$ and $\mathbb{K}^f \subset \mathcal{G}(\mathbb{A}_f)$ is compact then there exists a neighborhood $x \in V \subseteq \partial_{\mathcal{O}}\mathcal{X}^*$ and a real number T such $\gamma \in \mathcal{G}(\mathbb{Q})$ and*

$$\gamma(U^*(\mathcal{O}, T, V) \times g\mathbb{K}^f) \cap (U^*(\mathcal{O}, T, V) \times g\mathbb{K}^f) \neq \emptyset$$

implies $\gamma \in Z_l(\mathbb{Q}) \cap g\mathbb{K}^f g^{-1}$ where Z_l is the preimage of $\mathcal{L}_l$ in $\mathcal{O}$.

Proof: (1) By the preceding Proposition it suffices to produce some neighborhoods $U \in \mathcal{X}^*$ and $U' \in \mathcal{X}^*$ of x and x' such that $\gamma(U \times \mathbb{K}^f) \cap (U' \times \mathbb{K}^f) \neq \emptyset$ implies $\gamma\mathcal{O}'\gamma^{-1} = \mathcal{O}$. Let $\Gamma = \mathcal{G}(\mathbb{Q}) \cap \mathbb{K}^f$. If x and x' are in the same Γ-orbit we are done. Otherwise there exist by Theorem 1.2(3) neighborhoods U' and U' such that $\Gamma U \cap U' = \emptyset$ and this implies

$$\gamma(U \times \mathbb{K}^f) \cap (U' \times \mathbb{K}^f) = \emptyset$$

for all $\gamma \in \mathcal{G}(\mathbb{Q})$.

(2) is proved analogously using Theorem 1.2(4). □

2 Weighted L_2-Cohomology

2.1 Basic Sheaves and Modules

Let (E, σ) be an irreducible finite dimensional algebraic representation of $\mathcal{G}(\mathbb{C})$ in a complex vector space E. By irreducibility $\mathcal{A}_{\mathcal{G}}$ acts on E by a character ζ_E which is nessecarily rational since $\mathcal{A}_{\mathcal{G}}$ is $\mathbb{Q}$-split. Let

$$p\colon \mathcal{G}(\mathbb{A})/A_{\mathcal{G}}K \to \mathrm{Sh}$$

be the projection. For an open set $U \subseteq \mathrm{Sh}$ let

$$\mathbb{E}(U) = \{s\colon p^{-1}(U) \to E \text{ locally constant} | s(\gamma g) = \sigma(\gamma)s(g)\} \tag{2.1}$$

where the transformation rule is assumed to hold for all $\gamma \in \mathcal{G}(\mathbb{Q})$. This defines a sheaf $\mathbb{E}$ on Sh - the automorphic local system associated to E.

2.2 Proposition ([Oss07])**:** *Let $H \subseteq \mathcal{G}(\mathbb{A}_f)$ be a closed subgroup, S be a locally compact space on which H acts continuously on right and F a sheaf of complex vector spaces on S. Then the following assertions are equivalent:*

(1) *The group H acts continuously on the left on the étale space $|F| \xrightarrow{\pi} S$ such that for all $s \in S$, $h \in H$ and $f \in F_s$*

$$\pi(hf) = sh^{-1}.$$

(2) *For $h, h_1 \in H$ there is an isomorphism of sheaves*

$$T(h)\colon F \to h_*F$$

such that $T(hh_1) = T(h)T(h_1)$ and for all open sets $V \subset U \subseteq S$ with $\overline{V} \subset U$ compact and all sections $s \in F(U)$ there is an open compact subgroup $\mathbb{K}$ of H with $\overline{V}\mathbb{K} \subseteq U$ and $T(k)(s|_V) = s|_{Vk^{-1}}$ for all $k \in \mathbb{K}$.

If F satisfies one of these conditions then it said to be an H-equivariant sheaf on S.

One easily checks that

$$T(h)\colon \mathbb{E}(U) \to (h_*\mathbb{E})(U) = \mathbb{E}(Uh^{-1})$$

mapping $s(\cdot)$ to $s(\cdot h)$ makes $\mathbb{E}$ a $\mathcal{G}(\mathbb{A}_f)$-equivariant sheaf. Let $\zeta \in \check{\mathfrak{a}}_{\mathcal{G}}$, let $U \subseteq \mathrm{Sh}$ be open and $\mathbb{C}(\zeta)(U)$ the space of locally constant $\mathbb{C}$-valued functions on U. With the $\mathcal{G}(\mathbb{A}_f)$-action given by

$$\begin{aligned} T(h)\colon \mathbb{C}(\zeta)(U) &\to \mathbb{C}(\zeta)(Uh^{-1}) \\ f &\mapsto e^{\langle \zeta, H_{\mathcal{G}}(h)\rangle} f(\cdot h) \end{aligned}$$

$\mathbb{C}(\zeta)$ becomes a $\mathcal{G}(\mathbb{A}_f)$-equivariant sheaf on Sh. If F is any $\mathcal{G}(\mathbb{A}_f)$-equivariant sheaf on Sh let

$$F(\zeta) = F \otimes \mathbb{C}(\zeta) \tag{2.3}$$

be the twisted sheaf.

Recall that a $\mathfrak{U}(\mathfrak{g})$-module V together with a K-action is called $(\mathfrak{g}, K)$-module if

(a) K acts locally finite, i.e. every $v \in V$ lies in a finite dimensional K-invariant subspace on which K acts smoothly and

(b) the actions of $\mathfrak{g}$ and K are compatible, i.e. the action of $\mathfrak{k} \subseteq \mathfrak{g}$ on V is the differentiation of the K-action on V and $\mathrm{Ad}(k)(D)(k.v) = k(D.v)$ for all $D \in \mathfrak{U}(\mathfrak{g})$ and all $k \in K$.

A morphism of $(\mathfrak{g}, K)$-modules is linear map that commutes with the actions of $\mathfrak{U}(\mathfrak{g})$ and K.

For a $(\mathfrak{g}, K)$-module V one defines the a complex of vector space

$$(C^\bullet_{(\mathfrak{g},K)}(V), d)$$

by setting

$$C^p_{(\mathfrak{g},K)}(V) = \mathrm{Hom}_K(\Lambda^p(\mathfrak{g}/\mathfrak{k}), V)$$

and for $\eta \in C^p_{(\mathfrak{g},K)}(V)$ and

$$\begin{aligned} d\eta(X_0, \ldots, X_p) := &\sum_{i=0}^{p} (-1)^i X_i(\eta(X_0, \ldots, \widehat{X_i}, \ldots, X_p)) + \\ &\sum_{i<j} (-1)^{i+j} \eta([X_i, X_j], X_0, \ldots \widehat{X_i}, \ldots, \widehat{X_j} \ldots, X_p). \end{aligned}$$

Here the $X_i \in \mathfrak{g}$ are lifts of $X_i + \mathfrak{k}$. The differential d is well-defined since $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$, we consider only K-equivariant linear mappings η and the differentiation of the K-action on V agrees with the $\mathfrak{k}$-action. The equation $d^2 = 0$ follows from the Jacobi identity for $\mathfrak{g}$ and $(XY - YX)v = [X, Y]v$ for all $v \in V$ and $X, Y \in \mathfrak{g}$. The formation of $(C^\bullet_{(\mathfrak{g},K)}(V), d)$ is functorial in V and commutes with direct limits of $(\mathfrak{g}, K)$-modules. The $(\mathfrak{g}, K)$-cohomology of V is the cohomology $H^\bullet_{(\mathfrak{g},K)}(V)$ of the complex $(C^\bullet_{(\mathfrak{g},K)}(V), d)$. It commutes with direct limits as well. The zeroth $(\mathfrak{g}, K)$-cohomology space is

$$V^{(\mathfrak{g},K)} = \{v \in V | Xv = 0,\ kv = v \text{ for all } X \in \mathfrak{g},\ k \in K\}.$$

We shall also need for a $\mathfrak{g}$-module V the associated Chevalley-Eilenberg complex

$$C^\bullet_{\mathfrak{g}}(V) = C^\bullet_{(\mathfrak{g},\{e\})}(V).$$

Its cohomology is denoted by $H^\bullet_{\mathfrak{g}}(V)$.

For applications it is preferable to work with a slightly smaller complex than the full $(\mathfrak{g}, K)$-complex to be defined as follows. Set

$$\mathfrak{m}_{\mathcal{G}} = \mathfrak{g}^{\mathrm{der}} + \mathfrak{k}.$$

where $\mathfrak{g}^{\mathrm{der}} = [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$ is the derived algebra of $\mathfrak{g}$. This subalgebra of $\mathfrak{g}$ stable under $\mathrm{Ad}(K)$ and equals the orthogonal complement of $\mathfrak{a}_0 = {}_{\mathbb{R}}\mathfrak{a}_0$ in $\mathfrak{g}$. Now define

$$C^\bullet_{(\mathfrak{m}_\mathcal{G}, K)}(V) = \mathrm{Hom}_K(\Lambda^\bullet(\mathfrak{m}_\mathcal{G}/\mathfrak{k}), V) \tag{2.4}$$

in the same way as before.

The category of $(\mathfrak{g}, K)$-modules is complete. The kernel of a morphism is the kernel of the underlying linear map. The product of a family of $(\mathfrak{g}, K)$-modules is the space of K-finite elements in the usual product of vector spaces. The existence of arbitrary limits allows one to define sheaves of $(\mathfrak{g}, K)$-modules on a topological space in the usual way.

For $U \in \mathfrak{B}$ let

$$S(U)$$

be the space of $\mathbb{K}$-finite smooth functions on the preimage in $\mathcal{G}(\mathbb{Q})A_\mathcal{G}\backslash\mathcal{G}(\mathbb{A})$ of U. With the usual actions of $\mathfrak{g}$ and K by (infinitesimal) right translations $S(\cdot)$ is a $\mathfrak{B}$-presheaf of $(\mathfrak{g}, K)$-modules on Sh. Right translation by $h \in \mathcal{G}(\mathbb{A}_f)$ defines an isomorphism

$$T(h)\colon S(U) \to (h_*S)(U) = S(Uh^{-1}).$$

It is easy to see that the sheaf associated to S is $\mathcal{G}(\mathbb{A}_f)$-equivariant. By the Poincaré Lemma, the map

$$\begin{aligned} \mathbb{E}(U) \to C^0_{(\mathfrak{m}_\mathcal{G}, K)}(S(U) \otimes E) &\to C^\bullet_{(\mathfrak{m}_\mathcal{G}, K)}(S(U) \otimes E) \\ s \mapsto (g \mapsto \sigma(g_\infty)^{-1} s(g)) \end{aligned}$$

induces an $\mathcal{G}(\mathbb{A}_f)$-equivariant quasi-isomorphism $\mathbb{E} \to C^\bullet_{(\mathfrak{m}_\mathcal{G}, K)}(S \otimes E)$.

An admissible weight function is a smooth function

$$\rho\colon \mathcal{G}(\mathbb{Q})A_\mathcal{G}\backslash\mathcal{G}(\mathbb{A})/\mathbb{K}_f \to (0, \infty)$$

with the property that for every $D \in \mathfrak{U}(\mathfrak{g})$ there exists a constant $c_D > 0$ such that

$$|(D\rho)(g)| < c_D \rho(g) \tag{2.5}$$

holds for all $g \in \mathcal{G}(\mathbb{A})$. For such a weight function, there exists a neighborhood U of the identity in $\mathcal{G}(\mathbb{A})$ such that

$$\frac{1}{2}\rho(gh) < \rho(g) < 2\rho(gh) \tag{2.6}$$

holds for $g \in \mathcal{G}(\mathbb{A})$ and $h \in U$. Two admissible weight functions ρ and ρ' are called equivalent, written $\rho \sim \rho'$, if there are positive constants c, C such that $c\rho' \le \rho \le C\rho'$. By (2.6) all admissible weight functions are equivalent if Sh is compact. Otherwise there are many weight functions by Proposition 2.7 below. For real numbers D we define Siegel-like sets

$$\mathfrak{S}(D) = \{g \in \mathcal{G}(\mathbb{A}) | \langle \alpha, H_0(g) \rangle > D \text{ for all } \alpha \in \Delta_0\}.$$

It follows from reduction theory that we may find some $D << 0$ such that $\mathcal{G}(\mathbb{Q})\mathfrak{S}(D) = \mathcal{G}(\mathbb{A})$. We will often assume some D with this property chosen and fixed. For any other standard parabolic subgroup $\mathcal{P} \supset \mathcal{P}_0$ and and any real number $T > D$ we set

$$\mathfrak{S}(\mathcal{P}, D, T) = \{g \in \mathfrak{S}(D) | \langle \alpha, H_{\mathcal{P}}(g) \rangle > D \text{ for all } \alpha \in \Delta_0 - \Delta_0^{\mathcal{P}}\}.$$

2.7 Proposition ([Fra98], Proposition 1)**:** *For every $\lambda \in \check{\mathfrak{a}}_0^{\mathcal{G}}$ there exists up to equivalence a unique admissible weight function ρ_λ whose restriction to $\mathfrak{S}(D)$ is equivalent to* $\exp(\langle \lambda, H_0(.) \rangle)$*. There is up to equivalence only one admissible weight function satisfying this condition. In addition ρ_λ may be assumed to satisfy the following condition: If D has been fixed as above, then there exists a real number T such that*

$$\rho_\lambda(ng) = \rho_\lambda(g)$$

where $\mathcal{P} = M\mathcal{A}\mathcal{N}$ is a standard parabolic subgroup, $n \in \mathcal{N}(\mathbb{A})$, and $g \in \mathfrak{S}(\mathcal{P}, D, T)$.

As a warning, we note that ρ_0 denotes the half sum of positive roots as well as the weight function corresponding to $\lambda = 0$.

For $U \in \mathfrak{B}$ and ρ an admissible weight function let

$$S_\rho(U) \subseteq S(U) \tag{2.8}$$

be the subspace of functions $f \in S(U)$ such that for any left invariant differential operator $D \in \mathfrak{U}(\mathfrak{g})$ there is a constant $C_{D,f}$ depending only on D and f such that

$$|(Df)(g)| \leq C_{D,f} \cdot \rho(g)^{-1} \rho_{\rho_0}(g) \tag{2.9}$$

for all g in the preimage $W \subseteq \mathcal{G}(\mathbb{Q})A_{\mathcal{G}} \backslash \mathcal{G}(\mathbb{A})$ of U. For a finite set $\mathbb{S} \subset \widehat{\mathbb{K}}$ of $\mathbb{K}$-types let $S_\rho(U)_{\mathbb{S}}$ be the subspace of functions whose translates under $\mathbb{K}$ span a finite dimensional representation of $\mathbb{K}$ that decomposes into a finite sum of irreducible representations lying in $\mathbb{S}$. The space $S_\rho(U)_{\mathbb{S}}$ is a Fréchet space with seminorms

$$p_{\rho,D,U}(f) = \sup_{g \in W} |(Df)(g)| \rho(g) \rho_{\rho_0}(g)^{-1}.$$

Give

$$S_\rho(U) = \operatorname*{colim}_{\mathbb{S} \subset \widehat{\mathbb{K}}} S_\rho(U)_{\mathbb{S}}$$

the direct limit locally convex topology.

Fix some $\lambda \in \check{\mathfrak{a}}_0^{\mathcal{G}+}$, let ρ_λ be as in Proposition 2.7 with $\rho_\lambda \geq 2$ and set

$$w_n = (\log \rho_\lambda(g))^n$$

for every $n \in \mathbb{Z}$. The functions w_n are admissible weight functions. Using the Leibniz rule one easily sees that the product of any admissible weight function ρ with w_n is again admissible. Set

$$S_{\rho-\log}(U) = \lim_n S_{\rho w_n}(U) \quad \text{(intersection)}$$

and

$$S_{\rho+\log}(U) = \operatorname*{colim}_n S_{\rho w_n}(U) \quad \text{(union)}$$

and equip them with the limit respectively colimit locally convex topologies.

Let $j\colon \mathrm{Sh} \hookrightarrow \mathrm{Sh}^*$ be the inclusion map. Let here and in the following ? denote an element of $\{-\log, 0, +\log\}$. Let

$$\mathscr{S}_{\rho+?} = \mathrm{Sheaf}(j_* S_{\rho+?})$$

be the sheaf on Sh^* associated to the presheaf $j_* S_{\rho+?}$. The sheaves $\mathscr{S}_{\rho+?}$ are $\mathcal{G}(\mathbb{A}_f)$-equivariant sheaves of locally convex $(\mathfrak{g}, K)$-modules. Since Sh^* is compact

$$\mathscr{S}_{\rho+?}(\mathrm{Sh}^*) = S_{\rho+?}(\mathrm{Sh}). \tag{2.10}$$

2.2 Borel's Regularization Result

For any open set $U \subset \mathrm{Sh}^*$ let us endow

$$\Gamma_c(U, \mathscr{S}_{\rho^{-1}-?}) = \Gamma_c(U, j_* S_{\rho^{-1}-?}) = \operatorname*{colim}_{L \subset U} \Gamma_L(U, j_* S_{\rho^{-1}-?}) \tag{2.11}$$

the direct limit locally convex topology where of course $L \subseteq U$ runs through the directed set of compact subsets of U.

Let $\mathscr{S}'_{\rho+?}$ be the sheaf associated to the $\mathfrak{B}^*$-presheaf

$$\mathfrak{B}^* \ni U \mapsto \operatorname*{colim}_{\mathbb{S} \subseteq \widehat{\mathbb{K}}} (\Gamma_c(U, \mathscr{S}_{\rho^{-1}-?})')_{\mathbb{S}}.$$

Integration against the fixed quotient measure dg on $\mathcal{G}(\mathbb{Q})A_{\mathcal{G}}\backslash\mathcal{G}(\mathbb{A})$ yields an inclusion

$$\mathscr{S}_{\rho+?} \subset \mathscr{S}'_{\rho+?}. \tag{2.12}$$

We want to show that it induces a quasi-isomorphism

$$C^\bullet_{(\mathfrak{m}_{\mathcal{G}}, K)}(\mathscr{S}_{\rho+?} \otimes E) \cong C^\bullet_{(\mathfrak{m}_{\mathcal{G}}, K)}(\mathscr{S}'_{\rho+?} \otimes E)$$

with coefficients in any finite dimensional representation E of $\mathcal{G}(\mathbb{C})$. To this end we need a suitably localized version of Theorem 2.5 in [Bor83]. This was known in some form to Borel but never published, see [Bor83], introduction, and [BC85], §3.

Recall that a Polish space is a separable complete metrisable topological space. A Suslin space is a locally convex space such that there exists a Polish locally convex space mapping onto it. Separable Fréchet spaces are Suslin and the class of Suslin space is stable under closed subspaces, arbitrary limits, inductive limits and the formation of the strong dual. A locally convex space is called quasi-complete if all its closed bounded sets are complete.

Fix some norm $|\cdot|$ on $\mathfrak{g}$. Let

$$S_0' \xrightarrow{\varphi_0} S_1' \xrightarrow{\varphi_1} S_2' \xrightarrow{\varphi_2} \cdots$$

be an inductive system of quasi-complete nuclear Suslin spaces with K-action such that every element of S_i' lies in some finite dimensional K-invariant subspace. Assume that there exists a descending sequence U_i of connected open neighborhoods of $K \subseteq \mathcal{G}(\mathbb{R})$ such that $KU_iK = U_i$. Assume that for every $g \in U_i$ there is a linear continuous map

$$\pi_i(g)\colon S_i' \to S_{i+1}'$$

satisfying

a: For all $k \in K$ and $f \in S_i'$

$$\varphi_i(kf) = \pi_i(k)f.$$

b: For all $g \in U_i$, $h \in U_{i+1}$ with $hg \in U_i$ and all $f \in S_i'$

$$\varphi_{i+1}(\pi_i(hg)f) = \pi_{i+1}(h)\pi_i(g)f.$$

c: For $X \in \mathfrak{g}$ and $f \in S_i'$ the limit

$$\pi_i(X)f := \frac{d}{dt}\pi_i(\exp(tX))f\Big|_{t=0} \in S_{i+1}',$$

exists and for every continuous seminorm μ on S_i' there exists a continuous semi norm ν on S_i' and $\epsilon > 0$ such that

$$\mu\big(\exp(X)_i f - f - X_i f\big) \leq |X|\nu(f)$$

for all $f \in S_i'$ and $|X| < \epsilon$.

¿¿From these conditions it follows easily that

$$\varphi_{i+1}\pi_i(X)m = \pi_{i+1}(X)\varphi_i(m)$$

and $\mathfrak{g}$ acts on

$$S' = \operatorname*{colim}_i S'_i$$

which gets the structure of a $(\mathfrak{g}, K)$-module in this way. Let $\alpha \in C_c^\infty(U_i)$ be $\mathrm{Ad}(K)$-finite. Then the integral[1]

$$\pi_i(\alpha) f = \int_{U_i} \alpha(g) \pi_i(g) f \, dg$$

preserves K-finiteness and defines a linear continuous map from S'_i to S'_{i+1}. Let $(S_i)_{i=0}^\infty \subseteq (S'_i)_{i=0}^\infty$ be an inductive subsystem consisting of quasi-complete nuclear Suslin spaces with K-action such that the embedding $S_i \subseteq S'_i$ is continuous and $\pi_i(U_i) S_i \subseteq S_{i+1}$. Assume that condition **c** holds for the system $(S_i)_{i=0}^\infty$ and set $S = \operatorname{colim}_i S_i$. For $g \in U_i$ define an operator

$$\theta_{i,g} \colon \mathrm{Hom}(\Lambda^p \mathfrak{g}, S'_i) \to \mathrm{Hom}(\Lambda^p \mathfrak{g}, S'_{i+1})$$

by

$$(\theta_{i,g} \eta)(X_1, \ldots, X_p) = \pi_i(g) \eta(\mathrm{Ad}(g)^{-1} X_1, \ldots, \mathrm{Ad}(g)^{-1} X_p) \in S'_{i+1}.$$

Similarly for $X \in \mathfrak{g}$ define

$$\theta_{X,i} \colon \mathrm{Hom}(\Lambda^p \mathfrak{g}, S'_i) \to \mathrm{Hom}(\Lambda^p \mathfrak{g}, S'_{i+1})$$

by

$$(\theta_{X,i} \eta)(X_1, \ldots, X_p) = \pi_i(X) \eta_i(X_1, \ldots, X_p) - \sum_k \varphi_i(\eta(X_1, \ldots, [X, X_k], \ldots, X_p)).$$

Define differentials

$$d_i \colon \mathrm{Hom}(\Lambda^p \mathfrak{g}, S'_i) \to \mathrm{Hom}(\Lambda^{p+1} \mathfrak{g}, S'_{i+1})$$

by

$$(d_i \eta)(X_0, \ldots, X_p) = \sum_k \pi_i(X_k) \eta_i(X_0, \ldots, \widehat{X_k}, \ldots, X_p) + \sum_{k<l} (-1)^{k+l} \varphi_i(\eta([X_k, X_l], X_0, \ldots, \widehat{X_k}, \ldots, \widehat{X_l}, \ldots, X_p)).$$

[1] The integral is to be understood in the sense of Pettis, see Definition 3.26 in [Rud91]. In the situations we consider its existence follows in some cases from [Rud91] Theorem 3.27. More generally one can invoke [Sch73] to see that the locally convex spaces we consider are in fact all Suslin and [Tho75] where Pettis-integrals of functions with values in complete Suslin spaces are considered.

If one defines operators

$$i_X\colon \operatorname{Hom}(\Lambda^p\mathfrak{g}, S_i') \to \operatorname{Hom}(\Lambda^{p-1}\mathfrak{g}, S_i')$$

by

$$(i_X\eta)(X_1,\ldots,X_{p-1}) = \eta(X, X_1,\ldots,X_{p-1})$$

then

$$\theta_{i,X}\eta = i_X d_i\eta + d_i i_X\eta \tag{2.13}$$

holds in $\operatorname{Hom}(\Lambda^p\mathfrak{g}, S_i')$.

Assume that the exponential map is one-to-one on a neighborhood of the the support of α and let $\ln\colon \operatorname{supp}(\alpha) \to \mathfrak{g}$ denote its inverse. Define operators

$$E_{i,\alpha}\colon \operatorname{Hom}(\Lambda^p\mathfrak{g}, S_i') \to \operatorname{Hom}(\Lambda^{p+1}\mathfrak{g}, S_{i+1}')$$

by

$$E_{i,\alpha}\eta = \int_{U_i}\alpha(g)\int_0^1 \theta_{i,e^{t\ln(g)}} i_{\ln(g)}\eta\, dt dg. \tag{2.14}$$

If $\alpha \in C_c^\infty(U_{i+1})$ is $\operatorname{Ad}(K)$-finite with

$$\int_{U_i}\alpha(g)dg = 1$$

equation (2.13) implies that

$$\varphi_{i+1}(\theta_{i,\alpha}\eta - \varphi_i\eta) = d_{i+1}E_{i,\alpha}\eta + E_{i+1,\alpha}d_i\eta. \tag{2.15}$$

The prove of equation 2.15 uses Fubini's Theorem and the fundamental Theorem of calculus. In our situation (complete nuclear Suslin spaces) every weakly Pettis integrable function is automatically automatically Lebesgue-Bochner integrable by Theorem 7 and its Corollary in [Tho75]. The fundamental Theorem of Calculus follows from the definition of the Pettis integral, the Hahn-Banach Theorem and the complex valued case for continously differentiable functions from elementary calculus.

2.16 Proposition: *Let E be a finite dimensional representation of $\mathcal{G}(\mathbb{C})$. Assume that $\pi_i(\alpha)S_i' \subseteq S_{i+1}$ for all $\operatorname{Ad}(K)$-finite $\alpha \in C_c^\infty(U_i)$,*

$$E_{i,\alpha}C_{\mathfrak{g}}^\bullet(S_i'\otimes E) \subseteq C_{\mathfrak{g}}^\bullet(S_{i+1}'\otimes E)$$

and similarly for S_i. Then the inclusion $S \subseteq S'$ induces an isomorphism in $(\mathfrak{g}, K)$-cohomology with coefficients in E.

Proof: We claim that the inclusion $S \subseteq S'$ induces an isomorphism

$$H^\bullet_{\mathfrak{g}}(S \otimes E) \cong H^\bullet_{\mathfrak{g}}(S' \otimes E). \tag{2.17}$$

The inductive systems $S'_i = S'_i \otimes E$ and $S'_i = S_i \otimes E$ satisfy our assumptions from above. Hence we assume E to be trivial. Let

$$\eta \in C^\bullet_{\mathfrak{g}}(S')$$

satisfy $d\eta = 0$. Since $\mathrm{Hom}(\Lambda^\bullet \mathfrak{g}, \cdot)$ commutes with direct limits we may assume that η is the image of

$$\eta_i \in C^\bullet_{\mathfrak{g}}(S'_i)$$

satisfying $d\eta_i = 0$. Formula (2.15) implies

$$\varphi_{i+1}(\theta_{i,\alpha}\eta_i - \varphi_i(\eta_i)) = d_{i+1}E_{i.\alpha}\eta_i$$

for a suitable $\alpha \in C^\infty_c(U_i)$. Since by assumption $\theta_{i,\alpha}\eta_i \in C^\bullet_{\mathfrak{g}}(S_{i+1})$ this implies that (2.17) is onto. Similarly let

$$\eta \in C^\bullet_{\mathfrak{g}}(S')$$

such that there exists $\mu \in C^\bullet_{\mathfrak{g}}(S')$ with $d\mu = \eta$ There exist representatives $\mu_i \in C^\bullet_{\mathfrak{g}}(S'_i)$ and $\eta_{i+1} \in C^\bullet_{\mathfrak{g}}(S'_{i+1})$ such that $d_i\mu_i = \eta_{i+1}$. Applying (2.15) to μ_i one finds

$$\varphi_{i+2}\varphi_{i+1}\eta_{i+1} = d_{i+2}(\varphi_{i+1}\theta_{i,\alpha}\mu_i - E_{i+1,\alpha}\eta_{i+1}) \in C^\bullet_{\mathfrak{g}}(S_{i+3})$$

for a suitable $\alpha \in C^\infty_c(U_{i+1})$. This finishes the proof of (2.17). The argument given by Borel in the proof of part (ii) of Theorem 2.5 in [Bor83] now carries over to our situation without change. □

Let $V_i \in \mathfrak{B}^*$, $i \in \mathbb{N}_0$, be a neighborhood base of $s \in \mathrm{Sh}^*$ such that $\overline{V_{i+1}} \subset V_i$. Let W_i be the preimage of $V_i \cap \mathrm{Sh}$ in $\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})$. Define locally convex spaces with K-action by

$$S_{\rho,i} = j_* S_\rho(V_i)$$

and

$$S'_{\rho,i} = \operatorname*{colim}_{\mathbb{S}\subset\widehat{\mathbb{K}}} (\Gamma_c(V_i, j_* S_{\rho^{-1}})')_{\mathbb{S}}$$

where the last space carries the topology of the $\mathbb{K}$-finite elements in the strong topological dual of $\Gamma_c(V_i, j_* S_\rho)$ as in (2.11). Restriction makes $(S_{\rho,i})_{i=0}^\infty$ an inductive continuously embedded subsystem of $(S'_{\rho,i})_{i=0}^\infty$. On both systems K acts by right translations.

By our assumption on V_i there is a neighborhood U_i of K in $\mathcal{G}(\mathbb{R})$ such that $KU_iK = U_i$ and $W_{i+1}U_iU_i \subseteq W_i$. Replacing U_i by $\bigcap_{j=0}^i U_i$ we may assume that the U_i form a descending sequence. Let $f \in S'_{\rho,i}$ and $g \in U_i$. Define $\pi_i(g)f$ by the composition

$$C_c^\infty(W_i)' \xrightarrow{R_g} C_c^\infty(W_i g^{-1})' \xrightarrow{\cdot|_{W_{i+1}}} C_c^\infty(W_{i+1})'.$$

In this way S'_{ρ_i} form an inductive system of which $S_{\rho,i}$ is a continuously embedded subsystem. It has the properties **a.-c.** as described before the statement of Proposition 2.16. Set $S_{\rho,s} = \operatorname{colim}_i S_{\rho,i}$ and $S'_{\rho,s} = \operatorname{colim}_i S'_{\rho,i}$.

2.18 Proposition: *The inclusion $(S_{\rho,i})_{i=0}^\infty \subseteq (S'_{\rho,i})_{i=0}^\infty$ satisfies the conditions of Proposition 2.16. It defines isomorphisms*

$$H^\bullet_{(\mathfrak{g},K)}(S_{\rho,s} \otimes E) \cong H^\bullet_{(\mathfrak{g},K)}(S'_{\rho,s} \otimes E)$$

and

$$H^\bullet_{(\mathfrak{m}_\mathcal{G},K)}(S_{\rho,s} \otimes E) \cong H^\bullet_{(\mathfrak{m}_\mathcal{G},K)}(S'_{\rho,s} \otimes E)$$

with coefficients in every finite dimensional representation E of $\mathcal{G}(\mathbb{C})$.

Proof: It follows easily from (2.14) that $E_{i,\alpha} S_{\rho,i} \subseteq S_{\rho,i+2}$ and $E_{i,\alpha} S'_{\rho,i} \subseteq S'_{\rho,i+2}$ provided $\alpha \in C_c^\infty(U_{i+1})$.

It remains to show that $\pi_i(\alpha) S'_{\rho,i} \in S_{\rho,i+1}$ for all $\operatorname{Ad}(K)$-finite $\alpha \in C_c^\infty(U_i)$. Let $f \in S'_{\rho,i+1}$. There exist finitely many differential operators $D_j \in \mathfrak{U}(\mathfrak{g})$ such that

$$|f(\psi)| \leq \max_j \|D_j \psi | L_{2,\rho^{-1}}(W_i)\|$$

for all $\psi \in \Gamma_K(V_i, j_* S_{\rho^{-1}})$ where $V_{i+1} \subseteq K \subseteq V_i$ is compact. Since ρ is admissible, there exists $A > 0$ such that $\rho(g)^{-1} \leq A \rho(gh)^{-1}$ for all $h \in \operatorname{supp}(\alpha)$. For $\beta \in C^\infty(U_i)$ with $\operatorname{supp}(\beta) \subseteq \operatorname{supp}(\alpha)$ and every $\phi \in C_c^\infty(W_{i+1})$

$$\begin{aligned}
\|\pi_i(\check{\beta})\phi | L_{2,\rho^{-1}}(W_i)\|^2 &= \int_{W_i} \left| \int_{\mathcal{G}(\mathbb{R})} \phi(gh)\beta(h^{-1}) dh \right|^2 \rho(g)^{-2} dg \\
&\leq A^2 \|\beta | L_1\| \int_{\mathcal{G}(\mathbb{R})} \int_{W_i} |\phi(gh)|^2 \rho(gh)^{-2} d(gh) |\beta(h^{-1})| dh \\
&\leq A^2 \|\beta | L_1\|^2 \|\phi | L_{2,\rho^{-1}}(W_{i+1})\|^2.
\end{aligned}$$

This implies

$$\begin{aligned}
|D\pi_i(\alpha) f(\phi)| &\leq \max_j \|D_j \pi_i(\check{\alpha}) D^* \phi | L_{2,\rho^{-1}}(W_i)\| \\
&\leq \max_j \|\pi_i(((D_j)_r D\alpha)^\vee)\phi | L_{2,\rho^{-1}}(W_i)\| \\
&\leq \left(A \max_j \|(D_j)_r D\alpha | L_1(U_i)\| \right) \|\phi | L_{2,\rho^{-1}}(W_{i+1})\|
\end{aligned}$$

for any $\phi \in C_c^\infty(W_{i+1})$. It follows that $D\pi_i(\alpha)f(\phi)$ extends uniquely to a continuous linear functional on $L_{2,\rho^{-1}}(W_{i+1})$. By the Riesz representation theorem there exists $f_{\alpha,D} \in L_{2,\rho}$ such that

$$D\pi_i(\alpha)f(\phi) = \int_{W_{i+1}} f_D(g)\phi(g)\, dg$$

for all $\phi \in C_c^\infty(W_{i+1})$. For $D = 1 \in \mathfrak{U}(\mathfrak{g})$ the function $f_{\alpha,1} \in L_{2,\rho}$ is in fact representable by a $\mathbb{K}$-finite smooth function in $S_{\rho,i+1}$. This shows that $\pi_i(\alpha)f \in S_{\rho,i+1}$ as claimed.

□

2.19 Theorem: *The inclusion (2.12) induces a quasi-isomorphism*

$$C^\bullet_{(\mathfrak{m}_\mathcal{G},K)}(\mathscr{S}_{\rho+?} \otimes E) \to C^\bullet_{(\mathfrak{m}_\mathcal{G},K)}(\mathscr{S}'_{\rho+?} \otimes E)$$

for every finite-dimensional representation E.

Proof: Since $(\mathfrak{m}_\mathcal{G}, K)$-cohomology commutes with direct limits, the question is local. For $? = 0$

$$\mathscr{S}_{\rho,s} = \operatorname*{colim}_i S_{\rho,i}$$

and

$$\mathscr{S}'_{\rho,s} = \operatorname*{colim}_i S'_{\rho,i}$$

and the assertion follows from Proposition 2.18. In case that $? = \pm\log$ express the stalks as

$$\mathscr{S}_{\rho-\log,s} = \operatorname*{colim}_i \lim_n S_{\rho w_n,i}$$

and

$$\mathscr{S}_{\rho+\log,s} = \operatorname*{colim}_i \operatorname*{colim}_n S_{\rho w_n,i}$$

and similarly for the primed spaces. The assertion follows now from the passage to the (co-)limit. □

2.20 Proposition (Zucker)**:** *Every open covering of* Sh^* *(or* $\mathrm{Sh}^\wedge$*) has a subordinate smooth partition of unity with bounded differentials, i.e. for any function φ of this partition of unity and any differential operator $D \in \mathfrak{U}(\mathfrak{g})$ there is some constant C_D such that $|D\varphi(s)| \leq C_D$ for all $s \in$* Sh.

Proof: Any open covering of $\mathrm{Sh}^?$ may refined by a finite $\mathfrak{B}^?$-covering $(U_i)_{i=1}^n$. Let $\mathbb{K}_i^f \subseteq \mathcal{G}(\mathbb{A}_f)$ be an open compact subgroup fixing U_i. If we set $\mathbb{K}^f = \bigcap \mathbb{K}_i^f$ then the open sets $(U_i/\mathbb{K}^f)_{i=1}^n$ cover $\mathrm{Sh}^?(\mathbb{K}^f)$. In this situation it is proved in [Zuc83], proof of Proposition 4.4, that there is a partition of unity with bounded differentials. Pulling it back to $\mathrm{Sh}^?$ we get the result in the adelic setting. □

2.21 Corollary: *The sheaves $\mathscr{S}_{\rho+?}$, $\mathscr{S}'_{\rho+?}$ are fine. The complexes*

$$C^{\bullet}_{(\mathfrak{m}_{\mathcal{G}},K)}(\mathscr{S}_{\rho+?}\otimes E)$$

and

$$C^{\bullet}_{(\mathfrak{m}_{\mathcal{G}},K)}(\mathscr{S}_{\rho+?}\otimes E)$$

consist of fine sheaves.

Proof: The sheaves $\mathscr{S}_{\rho+?}$ and $\mathscr{S}'_{\rho+?}$ are are modules over the sheaf of continuous functions on Sh^* restricting to smooth functions on Sh whose differentials with respect to any $D\in\mathfrak{U}(\mathfrak{g})$ extend to continuous functions of Sh^*. By Proposition 2.20 this is a soft sheaf of rings with unity and the result follows from standard facts from sheaf theory. □

2.22 Corollary: *The map (2.12) induce isomorphisms*

$$H^{\bullet}_{(\mathfrak{m}_{\mathcal{G}},K)}(S_{\rho+?}(\mathrm{Sh})\otimes E)\cong H^{\bullet}_{(\mathfrak{m}_{\mathcal{G}},K)}(\mathscr{S}'_{\rho+?}(\mathrm{Sh}^*)\otimes E).$$

for every finite dimensional representation E.

Proof: This follows from equation (2.10), Theorem 2.19, Corollary 2.21 and an application of the hypercohomology spectral sequence to the cone

$$\mathrm{Cone}(H^{\bullet}_{(\mathfrak{m}_{\mathcal{G}},K)}(\mathscr{S}_{\rho+?}))\otimes E\to C^{\bullet}_{(\mathfrak{m}_{\mathcal{G}},K)}(\mathscr{S}'_{\rho+?}\otimes E)).$$

□

2.3 Restriction to the Boundary

Let $\partial_{\mathcal{O}}\mathrm{Sh}^*\subseteq\mathrm{Sh}^*$ a rational boundary component. Since $\mathcal{O}$ will be fixed from now on we will omit the index $\mathcal{O}$ if possible, i.e. we will write $\mathcal{L}$ instead of $\mathcal{L}_{\mathcal{O}}$, $\mathcal{L}_l$ instead of $\mathcal{L}_{\mathcal{O},l}$ and so forth. Set $K_{\mathcal{O}}=K\cap\mathcal{O}(\mathbb{R})$. Let $\mathfrak{B}_{\mathcal{O}}$ be the set of open subsets of

$$\lim_{\mathbb{K}^f}\ \mathcal{O}(\mathbb{Q})\backslash\mathcal{X}\times\mathcal{G}(\mathbb{A}_f)/\mathbb{K}^f$$

that are invariant under some compact open subgroup of $\mathcal{G}(\mathbb{A}_f)$. For $U\in\mathfrak{B}_{\mathcal{O}}$ let $\widetilde{U}$ be its preimage in $\mathcal{O}(\mathbb{Q})A_{\mathcal{G}}\backslash\mathcal{O}(\mathbb{R})\times\mathcal{G}(\mathbb{A}_f)$. Denote by

$$S_{\rho,\mathcal{O}}(U)$$

be the space of smooth $K_{\mathcal{O}}\mathbb{K}_f$-finite functions f on $\widetilde{U}$ such that for every $D\in\mathfrak{U}(\mathfrak{o})$ there exists a constant $C_{D,f}\geq 0$ such that

$$|(Df)(g)|\leq C_{D,f}\rho_{\rho_0}(g)\rho(g)^{-1}. \tag{2.23}$$

Set

$$S_{\rho+\log,\mathcal{O}}(U)=\operatorname*{colim}_{n\in\mathbb{Z}}S_{\rho w_n,\mathcal{O}}(U)$$

and

$$S_{\rho-\log,\mathcal{O}}(U) = \lim_{n\in\mathbb{Z}} S_{\rho w_n,\mathcal{O}}(U).$$

It is easy to see that $U \mapsto S_{\rho+?,\mathcal{O}}(U)$ is a $\mathfrak{B}_{\mathcal{O}}$-presheaf of $(\mathfrak{o}, K_{\mathcal{O}})$-modules with respect to the natural actions of $\mathfrak{o}$ and $K_{\mathcal{O}}$. Set

$$\mathrm{Sh}^*(\mathcal{O}) = (\lim_{\mathbb{K}^f}\ \mathcal{O}(\mathbb{Q})\backslash\mathcal{X}\times\mathcal{G}(\mathbb{A}_f)/\mathbb{K}^f) \sqcup \partial_{\mathcal{O}}\,\mathrm{Sh}^*$$

with the topology defined by glueing along the natural projection

$$\lim_{\mathbb{K}^f}\ \mathcal{O}(\mathbb{Q})\backslash\mathcal{X}\times\mathcal{G}(\mathbb{A}_f)/\mathbb{K}^f \twoheadrightarrow \partial_{\mathcal{O}}\,\mathrm{Sh}^*.$$

Let

$$j'\colon \lim_{\mathbb{K}^f}\ \mathcal{O}(\mathbb{Q})\backslash\mathcal{X}\times\mathcal{G}(\mathbb{A}_f)/\mathbb{K}^f \hookrightarrow \mathrm{Sh}^*(\mathcal{O})$$

denote the inclusion and let

$$\mathscr{S}_{\rho+?,\mathcal{O}}$$

be the the sheaf of $(\mathfrak{o}, K_{\mathcal{O}})$-modules on $\mathrm{Sh}^*(\mathcal{O})$ associated to the presheaf $j'_*S_{\rho\pm\log,\mathcal{O}}$. It is a $\mathcal{G}(\mathbb{A}_f)$-equivariant sheaf of $(\mathfrak{o}, K_{\mathcal{O}})$-modules on $\mathrm{Sh}^*(\mathcal{O})$. Let

$$\kappa\colon \mathrm{Sh}^*(\mathcal{O}) \to \mathrm{Sh}^*$$

denote the natural mapping and let

$$\kappa'\colon \lim_{\mathbb{K}^f}\ \mathcal{O}(\mathbb{Q})\backslash\mathcal{X}\times\mathcal{G}(\mathbb{A}_f)/\mathbb{K}^f \to \mathrm{Sh}$$

be its restriction. Since κ commutes with the $\mathcal{G}(\mathbb{A}_f)$-action

$$\kappa_*(\mathscr{S}_{\rho+?,\mathcal{O}})$$

is a $\mathcal{G}(\mathbb{A}_f)$-equivariant sheaf of $(\mathfrak{o}, K_{\mathcal{O}})$-modules. There is a canonical homomorphism

$$S_{\rho+?} \to \kappa'_*S_{\rho+?,\mathcal{O}}$$

of presheaves of $(\mathfrak{o}, K_{\mathcal{O}})$-modules on Sh induced by restricting functions on $\mathcal{G}(\mathbb{Q})A_{\mathcal{G}}\backslash\mathcal{G}(\mathbb{A})$ to functions on $\mathcal{O}(\mathbb{Q})A_{\mathcal{G}}\backslash\mathcal{O}(\mathbb{R})\times\mathcal{G}(\mathbb{A}_f)$. Hence a canonical homomorphism

$$j_*S_{\rho\pm\log} \to j_*\kappa'_*S_{\rho\pm\log,\mathcal{O}} = \kappa_*j'_*S_{\rho\pm\log,\mathcal{O}} \to \kappa_*\mathscr{S}_{\rho\pm\log,\mathcal{O}}$$

which by the universal property of the associated sheaf and adjunction gives a canonical homomorphism

$$\kappa^*\mathscr{S}_{\rho+?} \to \mathscr{S}_{\rho+?,\mathcal{O}} \tag{2.24}$$

of $\mathcal{G}(\mathbb{A}_f)$-equivariant sheaves of $(\mathfrak{o}, K_{\mathcal{O}})$-modules on $\mathrm{Sh}^*(\mathcal{O})$.

Recall that the restriction functor from the category of $(\mathfrak{g}, K)$-modules to the category of $(\mathfrak{o}, K_{\mathcal{O}})$-modules has a right adjoint functor

$$\mathrm{Ind}_{(\mathfrak{o},K_{\mathcal{O}})}^{(\mathfrak{g},K)} \colon \langle(\mathfrak{o}, K_{\mathcal{O}}) - \mathrm{Mod}\rangle \Rightarrow \langle(\mathfrak{g}, K) - \mathrm{Mod}\rangle.$$

To ease notation let us write Ind for this functor in the following. It can be described as follows: For a $(\mathfrak{o}, K_{\mathcal{O}})$-module W let $\mathrm{Ind}(W)$ be the space of functions $f \colon K \to W$ satisfying

$$k_1 f(k) = f(k_1 k) \tag{2.25}$$

for all $k_1 \in K_{\mathcal{O}}$ and whose right translates

$$(kf)(\cdot) = f(\cdot\, k)$$

for $k \in K$ span a finite dimensional subspace on which K acts smoothly. Define an action of $X \in \mathfrak{g}$ on $\mathrm{Ind}(W)$ by

$$(Xf)(k) = X_{\mathfrak{o}}(f(k)) + \frac{d}{dt} f(k e^{tX_{\mathfrak{k}}})\Big|_{t=0}$$

where $X = \mathrm{Ad}(k)^{-1} X_{\mathfrak{o}} + X_{\mathfrak{k}}$ with $X_{\mathfrak{o}} \in \mathfrak{o}$ and $X_{\mathfrak{k}} \in \mathfrak{k}$ is some decomposition. The action is well-defined by (2.25) and together with right translation by elements of K turns $\mathrm{Ind}(W)$ into a $(\mathfrak{g}, K)$-module. Evaluation at the unit element is a homomorphism

$$\begin{array}{rcl} \epsilon_W \colon \mathrm{Ind}(W) & \to & W \\ f & \mapsto & f(e) \end{array}$$

of $(\mathfrak{o}, K_{\mathcal{O}})$-modules. One easily checks that for every $(\mathfrak{g}, K)$-module V

$$\begin{array}{rcl} \mathrm{Hom}_{(\mathfrak{g},K)}(V, \mathrm{Ind}(W)) & \leftrightharpoons & \mathrm{Hom}_{(\mathfrak{o},K_{\mathcal{O}})}(V, W) \\ \varphi & \mapsto & \psi_\phi = \epsilon_w \circ \varphi \\ \psi_\varphi(v)(k) = \psi(k^{-1}v) & \leftarrow & \psi \end{array}$$

are natural inverse bijections.

It follows from the finiteness condition in its definition that Ind commutes with direct limits. Being right adjoint it is clear that Ind commutes with arbitrary limits. In particular Ind transforms sheaves of $(\mathfrak{o}, K_{\mathcal{O}})$-modules to sheaves of $(\mathfrak{g}, K)$-modules and (2.24) induces a canonical homomorphism

$$\kappa^* \mathscr{S}_{\rho+?} \to \mathrm{Ind}\, \mathscr{S}_{\rho+?,\mathcal{O}}$$

of $\mathcal{G}(\mathbb{A}_f)$-equivariant sheaves of $(\mathfrak{g}, K)$-modules on $\mathrm{Sh}^*(\mathcal{O})$. Induction commutes with passage to the stalk.

2.26 Proposition: *The canonical homomorphism*

$$\kappa^* \mathscr{S}_{\rho+?} \to \operatorname{Ind}_{(\mathfrak{o},K_{\mathcal{O}})}^{(\mathfrak{g},K)}(\mathscr{S}_{\rho+?,\mathcal{O}})$$

restricts to an isomorphism

$$\mathscr{S}_{\rho+?}|_{\partial_{\mathcal{O}} \operatorname{Sh}^*} \cong \operatorname{Ind}_{(\mathfrak{o},K_{\mathcal{O}})}^{(\mathfrak{g},K)}(\mathscr{S}_{\rho+?,\mathcal{O}})|_{\partial_{\mathcal{O}} \operatorname{Sh}^*} \tag{2.27}$$

of $\mathcal{G}(\mathbb{A}_f)$*-equivariant sheaves of* $(\mathfrak{g}, K)$*-modules over the boundary stratum* $\partial_{\mathcal{O}} \operatorname{Sh}^*$*. In particular for every* $s \in \partial_{\mathcal{O}} \operatorname{Sh}^*$ *there is a canonical isomorphism*

$$\mathscr{S}_{\rho+?,s} \cong \operatorname{Ind}_{(\mathfrak{o},K_{\mathcal{O}})}^{(\mathfrak{g},K)}(\mathscr{S}_{\rho+?,\mathcal{O},s}). \tag{2.28}$$

of $(\mathfrak{g}, K, \mathbb{Z}_s)$*-modules.*

Proof: Let

$$\epsilon \colon \mathscr{S}_{\rho,s} \to \mathscr{S}_{\rho,\mathcal{O},s}$$

be the map obtained from the canonical homomorphism

$$\kappa^* \mathscr{S}_\rho \to \mathscr{S}_{\rho,\mathcal{O}}$$

by passage to the stalk at s. It is a homomorphism of $(\mathfrak{o}, K_{\mathcal{O}}, \mathbb{Z}_s)$-modules. To prove the proposition for $? = 0$ it suffices to show that for every $(\mathfrak{g}, K)$-module V the natural map

$$\begin{array}{rcl} \operatorname{Hom}_{(\mathfrak{g},K)}(V, \mathscr{S}_{\rho,s}) & \leftrightarrow & \operatorname{Hom}_{(\mathfrak{o},K_{\mathcal{O}})}(V, \mathscr{S}_{\rho,\mathcal{O},s}) \\ \varphi & \mapsto & \psi_\varphi(v) = \epsilon \circ \varphi \\ \varphi_\psi(v)(g,h) = \psi(kv)(o,h) & \leftarrow & \psi \end{array}$$

where $g = ok \in \mathcal{O}(\mathbb{R})K = \mathcal{G}(\mathbb{R})$ is a bijection. More precisely φ_ψ is defined as follows: Let $g = ok$ be an Iwasawa decomposition of g and v in V. Let $f_{\psi,v,k}$ be an $\mathbb{K}^f$-invariant representative of $\psi(kv)$. Let (x, g) represent s, $U \subset \partial_{\mathcal{O}}\mathcal{X}^*$ a bounded neighborhood of x and T a large real number. Let

$$W^*(\mathcal{O}, T, U) \subseteq \mathcal{O} \tag{2.29}$$

be the preimage of $U^*(\mathcal{O}, T, U) \cap X$. We may assume $f_{\psi,v,k}$ to be defined on

$$\mathcal{O}(\mathbb{Q})A_{\mathcal{G}} \backslash \mathcal{O}(\mathbb{Q}) W^*(\mathcal{O}, T, U) \times g\mathbb{K}^f$$

By choosing T larger, V and $\mathbb{K}^f$ smaller we may assume by Corollary 1.10 that the natural map

$$\mathcal{O}(\mathbb{Q})A_{\mathcal{G}} \backslash \mathcal{O}(\mathbb{Q})(W^*(\mathcal{O}, T, U)K \times g\mathbb{K}^f)/\mathbb{K}^f \to \\ \mathcal{G}(\mathbb{Q})A_{\mathcal{G}} \backslash \mathcal{G}(\mathbb{Q})(U^*(\mathcal{O}, T, U) \cap \mathcal{X} \times g\mathbb{K}^f)/\mathbb{K}^f$$

is one-to-one. Here we view $U^*(\mathcal{O},T,U)\cap\mathcal{X}$ as a subset of $\mathcal{G}(\mathbb{R})$. Now let $\varphi_\psi(v)$ be the the image in $\mathscr{S}_{\rho+?,s}$ of the function

$$\begin{array}{rcl}\mathcal{G}(\mathbb{Q})A_{\mathcal{G}}(U^*(\mathcal{O},T,U)\cap\mathcal{X}\times g_f\mathbb{K}^f) & \to & \mathbb{C}\\ \gamma(g,g_fk_f) & \mapsto & f_{\psi,v,k}(\delta o,\delta g_fk_f)\end{array}$$

where we choose $\delta\in\mathcal{O}(\mathbb{Q})$, $o\in\mathcal{O}(\mathbb{R})$, $k_f\in\mathbb{K}^f$ and $k\in K$ such that

$$\delta(ok,g_fl_f)=\gamma(g,g_fk_f).$$

The germ $\varphi_\psi(v)$ satisfies the estimates in (2.9) because it satisfies (2.23) and U is bounded. This is a well-defined function whose germ depends only on ψ and v. One checks that $v\mapsto\varphi_\psi(v)$ is a homomorphism of $(\mathfrak{g},K)$-modules and that $\psi_{\varphi_\psi}=\psi$ and $\varphi_{\psi_\varphi}=\varphi$. The cases $?=\pm\log$ follow by passage to the limit. $\square$

The inclusion $\mathfrak{o}\cap\mathfrak{m}_{\mathcal{G}}\subseteq\mathfrak{m}_{\mathcal{G}}$ induces natural isomorphisms

$$\Lambda^\bullet\mathfrak{o}/\mathfrak{k}_{\mathcal{O}}\cong\Lambda^\bullet\mathfrak{g}/\mathfrak{k}$$

as well as

$$\Lambda^\bullet(\mathfrak{o}\cap\mathfrak{m}_{\mathcal{G}})/\mathfrak{k}_{\mathcal{O}}\cong\Lambda^\bullet\mathfrak{m}_{\mathcal{G}}/\mathfrak{k}.$$

One can show that for every finite-dimensional representation E of $\mathcal{G}(\mathbb{C})$ and every $(\mathfrak{o},K_{\mathcal{O}})$-module W the map $\epsilon_W\colon\operatorname{Ind}(W)\to W$ induces natural isomorphisms

$$C^\bullet_{(\mathfrak{g},K)}(\operatorname{Ind}^{(\mathfrak{g},K)}_{(\mathfrak{o},K_{\mathcal{O}})}(W)\otimes E)\cong C^\bullet_{(\mathfrak{o},K_{\mathcal{O}})}(W\otimes E)$$

and

$$C^\bullet_{(\mathfrak{m}_{\mathcal{G}},K)}(\operatorname{Ind}^{(\mathfrak{g},K)}_{(\mathfrak{o},K_{\mathcal{O}})}(W)\otimes E)\cong C^\bullet_{(\mathfrak{o}\cap\mathfrak{m}_{\mathcal{G}},K_{\mathcal{O}})}(W\otimes E)$$

of cochain complexes. Passage to cohomology yields natural (Frobenius reciprocity) isomorphisms

$$H^\bullet_{(\mathfrak{g},K)}(\operatorname{Ind}^{(\mathfrak{g},K)}_{(\mathfrak{o},K_{\mathcal{O}})}(W)\otimes E)\cong H^\bullet_{(\mathfrak{o},K_{\mathcal{O}})}(W\otimes E) \tag{2.30}$$

and

$$H^\bullet_{(\mathfrak{m}_{\mathcal{G}},K)}(\operatorname{Ind}^{(\mathfrak{g},K)}_{(\mathfrak{o},K_{\mathcal{O}})}(W)\otimes E)\cong H^\bullet_{(\mathfrak{m}_{\mathcal{G}}\cap\mathfrak{o},K_{\mathcal{O}})}(W\otimes E).$$

For a sheaf $\mathscr{V}$ of $(\mathfrak{g},K)$-modules on a topological space let us denote by

$$\mathscr{H}^\bullet_{(\mathfrak{m}_{\mathcal{G}},K)}(\mathscr{V})$$

the cohomology sheaves of the complex $C^\bullet_{(\mathfrak{m}_{\mathcal{G}},K)}(\mathscr{V})$.

2.31 Corollary: *There is a canonical Frobenius reciprocity morphism*

$$\mathscr{H}^\bullet_{(\mathfrak{m}_{\mathcal{G}},K)}(\kappa^*\mathscr{S}_{\rho+?}\otimes E)\to\mathscr{H}^\bullet_{(\mathfrak{o}\cap\mathfrak{m}_{\mathcal{G}},K_{\mathcal{O}})}(\mathscr{S}_{\rho+?,\mathcal{O}}\otimes E)$$

of $\mathcal{G}(\mathbb{A}_f)$*-equivariant sheaves on* $\operatorname{Sh}^*(\mathcal{O})$ *that restricts to a canonical isomorphism*

$$\mathscr{H}^\bullet_{(\mathfrak{m}_{\mathcal{G}},K)}(\mathscr{S}_{\rho+?}|_{\partial_{\mathcal{O}}\operatorname{Sh}^*}\otimes E)\cong\mathscr{H}^\bullet_{(\mathfrak{o}\cap\mathfrak{m}_{\mathcal{G}},K_{\mathcal{O}})}(\mathscr{S}_{\rho+?,\mathcal{O}}|_{\partial_{\mathcal{O}}\operatorname{Sh}^*}\otimes E)$$

on $\partial_{\mathcal{O}}\operatorname{Sh}^*$.

2.4 Hodge Theory in N-Direction

Let $U \subseteq \partial_{\mathcal{O}} \operatorname{Sh}^*$ be open and $f \in S_{\rho+?,\mathcal{O}}(U)$. Then for every $X \in \mathfrak{n}$ the function Xf is again an element of $S_{\rho+?,\mathcal{O}}(U)$ and we get a natural action of $\mathfrak{n}$ on the sheaf $\mathscr{S}_{\rho+?,\mathcal{O}}$. Let

$$\mathscr{S}^{\mathfrak{n}}_{\rho+?,\mathcal{O}} \subseteq \mathscr{S}_{\rho+?,\mathcal{O}}$$

be the subsheaf of $\mathfrak{n}$-invariant sections.

2.32 Proposition: *For every finite dimensional representation F of $\mathcal{O}$ the inclusion*

$$\mathscr{S}^{\mathfrak{n}}_{\rho+?,\mathcal{O}} \subset \mathscr{S}_{\rho+?,\mathcal{O}} \tag{2.33}$$

induces a quasi-isomorphism

$$C^{\bullet}_{(\mathfrak{o}\cap\mathfrak{m}_{\mathcal{G}},K_{\mathcal{O}})}(\mathscr{S}^{\mathfrak{n}}_{\rho+?,\mathcal{O}}|_{\partial_{\mathcal{O}}\operatorname{Sh}^*} \otimes F) \subset C^{\bullet}_{(\mathfrak{o}\cap\mathfrak{m}_{\mathcal{G}},K_{\mathcal{O}})}(\mathscr{S}_{\rho+?,\mathcal{O}}|_{\partial_{\mathcal{O}}\operatorname{Sh}^*} \otimes F)$$

of complexes of $\mathcal{G}(\mathbb{A}_f)$-equivariant sheaves of vector spaces on $\partial_{\mathcal{O}}\operatorname{Sh}^*$, i.e. the induced map on the cohomology sheaves

$$\mathscr{H}^{p}_{(\mathfrak{o}\cap\mathfrak{m}_{\mathcal{G}},K_{\mathcal{O}})}(\mathscr{S}^{\mathfrak{n}}_{\rho+?,\mathcal{O}}|_{\partial_{\mathcal{O}}\operatorname{Sh}^*} \otimes F) \cong \mathscr{H}^{p}_{(\mathfrak{o}\cap\mathfrak{m}_{\mathcal{G}},K_{\mathcal{O}})}(\mathscr{S}_{\rho+?,\mathcal{O}}|_{\partial_{\mathcal{O}}\operatorname{Sh}^*} \otimes F)$$

is an $\mathcal{G}(\mathbb{A}_f)$-equivariant isomorphism.

Proof: It is clear that the inclusion (2.33) induces a $\mathcal{G}(\mathbb{A}_f)$-equivariant homomorphism of the cohomology sheaves. This reduces us to a local problem. Let $s \in \partial_{\mathcal{O}}\operatorname{Sh}^*$ be a point. Let

$$K = \mathscr{S}_{\rho+?,\mathcal{O},s}/\mathscr{S}^{\mathfrak{n}}_{\rho+?,\mathcal{O},s} = (\mathscr{S}_{\rho+?,\mathcal{O}}/\mathscr{S}^{\mathfrak{n}}_{\rho+?,\mathcal{O}})_s$$

be the quotient $(\mathfrak{o}, K_{\mathcal{O}})$-module. We have to show that $H^{\bullet}_{(\mathfrak{o}\cap\mathfrak{m}_{\mathcal{G}},K_{\mathcal{O}})}(K\otimes F) = 0$. Looking at the E_2-term of the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p_{(\mathfrak{m}_{\mathcal{L}},K_{\mathcal{O}})}(H^q_{\mathfrak{n}}(K\otimes V)) \Rightarrow H^{p+q}_{(\mathfrak{o}\cap\mathfrak{m}_{\mathcal{G}},K_{\mathcal{O}})}(K\otimes F)$$

one sees that it suffices to show that $H^{\bullet}_{\mathfrak{n}}(K\otimes F)$ vanishes. Since $\mathfrak{n}$ is nilpotent

$$F^{\mathfrak{n}} \neq 0.$$

The long exact sequence in $\mathfrak{n}$-cohomology associated to

$$0 \to F^{\mathfrak{n}} \to F \to F/F^{\mathfrak{n}} \to 0$$

and induction on the dimension of F shows that we may assume F to be the trivial one dimensional representation of $\mathfrak{n}$. Assume this from now on. Let

$$\{1\} = N_0 \subset N_1 \subset N_2 \subset \ldots \subset N_l = N$$

be the ascending central series for N, i.e. N_i is the preimage in N of the center of N/N_{i-1}. All N_i are connected algebraic subgroups of N defined over $\mathbb{Q}$. If we set $\mathfrak{n}_i = \mathrm{Lie}(N_i)$ then

$$\{0\} = \mathfrak{n}_0 \subset \mathfrak{n}_1 \subset \mathfrak{n}_2 \subset \ldots \subset \mathfrak{n}_l = \mathfrak{n}$$

is an $\mathrm{Ad}(N)$-stable filtration of $\mathfrak{n}$. Then

$$\mathscr{S}^{\mathfrak{n}}_{\rho+?,\mathcal{O},s} \subseteq \mathscr{S}^{\mathfrak{n}_{l-1}}_{\rho+?,\mathcal{O},s} \subseteq \ldots \subseteq \mathscr{S}^{\mathfrak{n}_1}_{\rho+?,\mathcal{O},s} \subseteq \mathscr{S}_{\rho+?,\mathcal{O},s}$$

is an ascending sequence of N-modules. Hence it suffices to show that the inclusions

$$\mathscr{S}^{\mathfrak{n}_{i+1}}_{\rho+?,\mathcal{O},s} \subseteq \mathscr{S}^{\mathfrak{n}_i}_{\rho+?,\mathcal{O},s}$$

induce an isomorphism in $\mathfrak{n}$-cohomology. Since $\mathfrak{n}_i$ acts trivially on both sides using the Hochschild-Serre spectral sequence again for the normal subalgebra $\mathfrak{n}_{i+1}/\mathfrak{n}_i \subset \mathfrak{n}/\mathfrak{n}_i$ it suffices to show that it induces an isomorphism on $\mathfrak{n}_{i+1}/\mathfrak{n}_i$-cohomology. Since $\mathfrak{n}_{i+1}/\mathfrak{n}_i$-cohomology commutes with direct limits it suffices to show that the inclusion

$$S_{\rho+?,\mathcal{O}}(\mathcal{O}(\mathbb{Q})A_{\mathcal{G}}\backslash\mathcal{O}(\mathbb{Q})W^*(\mathcal{O},T,V) \times g\mathbb{K}^f/K_{\mathcal{O}})^{\mathbb{K}^f,\mathfrak{n}_{i+1}} \subseteq$$
$$S_{\rho+?,\mathcal{O}}(\mathcal{O}(\mathbb{Q})A_{\mathcal{G}}\backslash\mathcal{O}(\mathbb{Q})W^*(\mathcal{O},T,V) \times g\mathbb{K}^f/K_{\mathcal{O}})^{\mathbb{K}^f,\mathfrak{n}_i}$$

induces an isomorphism on $\mathfrak{n}_{i+1}/\mathfrak{n}_i$-cohomology for every compact open subgroup $\mathbb{K}^f \subseteq \mathbb{K}_f$. This inclusion is split in an $\mathfrak{n}_{i+1}/\mathfrak{n}_i$-equivariant way by

$$\sigma_?\colon f \mapsto \int_{N_{i+1}/(g\mathbb{K}^f g^{-1}\cap N_{i+1}(\mathbb{Q}))N_i} f(n.)dn.$$

Hence it suffices to show that the $\mathfrak{n}_{i+1}/\mathfrak{n}_i$-cohomology of the module

$$\mathscr{K}_? = \mathrm{Ker}(\sigma_?)$$

vanishes. Let $\nu_1,\ldots,\nu_n$ be an orthonormal basis of $\mathfrak{n}_{i+1}/\mathfrak{n}_i$ with respect to some fixed $K_{\mathcal{O}}$-invariant inner product and set $\Delta = -\sum \nu_i^2$. Then Δ induces the zero endomorphism on the $\mathfrak{n}_{i+1}/\mathfrak{n}_i$-cohomology of every $\mathfrak{n}_{i+1}/\mathfrak{n}_i$-module since Δ, having zero constant term, acts by zero on the trivial $\mathfrak{n}_{i+1}/\mathfrak{n}_i$-module $\mathbb{C}$. Hence to show that the $\mathfrak{n}_{i+1}/\mathfrak{n}_i$-cohomology of $\mathscr{K}_?$ vanishes it suffices to show that Δ is an automorphism of $\mathscr{K}_?$. Let $\mathscr{L} \supset \mathscr{K}_?$ be the space of smooth $K_{\mathcal{O}}$-finite and $\mathbb{K}^f$-invariant functions on

$$\mathcal{O}(\mathbb{Q})A_{\mathcal{G}}\backslash\mathcal{O}(\mathbb{Q})W^*(\mathcal{O},T,V) \times g\mathbb{K}^f$$

such that

$$\int_{N_{i+1}/(g\mathbb{K}^f g^{-1}\cap N_{i+1}(\mathbb{Q}))N_i} f(n.)dn = 0.$$

We claim that Δ is invertible on $\mathscr{L}$ and that $\Delta^{-1}(\mathscr{K}_?) \subseteq \mathscr{K}_?$. Indeed we will write down the inverse Δ^{-1} explicitly using Fourier series as follows. Let $\Gamma \subset G = \mathfrak{n}_{i+1}/\mathfrak{n}_i$ be the preimage of $(g\mathbb{K}^f g^{-1} \cap N_{i+1}(\mathbb{Q}))N_i/N_i$ under the exponential mapping $\exp\colon G \xrightarrow{\cong} N_{i+1}/N_i$. The function

$$(\nu, o) \mapsto f(\exp(\nu)o, g)$$

is a Γ-invariant, $K_{\mathcal{O}}$-finite and smooth function on $G \times W^*(\mathcal{O}, T, V)$. Its Fourier expansion is

$$f(\exp(\nu)o, g) = \sum_{\chi \in \Gamma'} c_\chi(f, o) \exp(i\langle \chi, \nu \rangle)$$

where

$$c_\chi(f, o) = \int_{\Gamma \backslash G} f(\exp(\nu)o, g) \exp(-i\langle \chi, \nu \rangle) d\nu$$

is the χ's Fourier coefficient and the sum runs over all elements of the lattice

$$\Gamma' = \{\chi \in G' | \chi(\Gamma) \subset 2\pi\mathbb{Z}\}.$$

We want to compute the coefficient $c_\chi(\Delta f, o)$. For the vector field ν_i we get

$$\begin{aligned} c_\chi(\nu_i f, o) &= \int_{\Gamma \backslash G} (\nu_i f)(\exp(\nu)o, g) \exp(-i\langle \chi, \nu \rangle) d\nu \\ &= \frac{d}{dt} \int_{\Gamma \backslash G} f(\exp(\nu + t \operatorname{Ad}(o)\nu_i)o, g) \exp(-i\langle \chi, \nu \rangle) d\nu \Big|_{t=0} \\ &= \frac{d}{dt} \exp(it\langle \chi, \operatorname{Ad}(o)\nu_i \rangle) \Big|_{t=0} c_\chi(f, o) \\ &= i\langle \operatorname{Ad}(o)'\chi, \nu_i \rangle \cdot c_\chi(f, o). \end{aligned}$$

Since the ν_i are an orthonormal basis of G we find

$$c_\chi(\Delta f, o) = \sum_i \langle \operatorname{Ad}(o)'\chi, \nu_i \rangle^2 c_\chi(f, o) = \| \operatorname{Ad}(o)'\chi | G' \|^2 c_\chi(f, o).$$

Hence if all zeroth Fourier coefficients $c_0(f, o)$, $o \in W^*(\mathcal{O}, T, V)$, vanish we may set

$$\begin{aligned} c_\chi(\Delta^{-1} f, o) &= \frac{c_\chi(f, o)}{\| \operatorname{Ad}(o)'\chi | G' \|^2} \text{ for all } \chi \neq 0 \\ c_0(\Delta^{-1} f, o) &= 0. \end{aligned} \tag{2.34}$$

This defines a function $\Delta^{-1} f \in \mathscr{L}$ since a function on a torus is smooth precisely if its Fourier coefficients are rapidly decaying and the euclidian product on G was chosen in a $K_{\mathcal{O}}$-invariant way so that Δ and Δ^{-1} commute with the action of $K_{\mathcal{O}}$.

For $k \in \mathbb{Z}$ let $\mathscr{L}_k \subset \mathscr{L}$ be the space of functions $f \in \mathscr{L}$ that satisfy (2.23) for k and all $A \in \mathfrak{U}(\mathfrak{n}_{i+1})$. We claim that Δ^{-1} preserves $\mathscr{L}_k$. Indeed it follows from (2.34) that

$$c_\chi(f, o) = \| \operatorname{Ad}(o)'\chi|G'\|^{-2l} c_\chi(\Delta^l f, o)$$

for all $l \in \mathbb{N}$ and $\chi \neq 0$. If we assume V to be bounded then by the definition of $W^*(\mathcal{O}, T, V)$ there is a constant C such that

$$\frac{1}{\| \operatorname{Ad}(o)'\chi|G'\|} \leq \frac{C}{\|\chi|G'\|}.$$

Recall that we may assume after enlarging T if necessary that the weight functions ρ_τ, ρ_{ρ_0} and w_k are N-invariant from the left. Assuming this we estimate for every l such that $2l + 2 > \dim(G)$

$$\begin{aligned}
|\Delta^{-1} f(o, g)| &\leq \sum_{\chi \neq 0} |c_\chi(\Delta^{-1} f, o)| \\
&= \left(\sum_{\chi \neq 0} \frac{1}{\| \operatorname{Ad}(o)'\chi|G'\|^{2l+2}} \right) |c_\chi(\Delta^l f, o)| \\
&\leq \left(\sum_{\chi \neq 0} \frac{C^{2l+2}}{\|\chi|G'\|^{2l+2}} \right) \max_{n \in N} |\Delta^l f(no)| \\
&\leq (C^{2l+2} \cdot C_{\Delta^l, f}) \cdot \max_{n \in N} \rho_\tau(no)^{-1} \rho_{\rho_0}(no) w_k(no) \\
&\leq (C^{2l+2} \cdot C_{\Delta^l, f} \cdot C') \cdot \rho_\tau(o)^{-1} \rho_{\rho_0}(o) w_k(o)
\end{aligned}$$

where in the last inequality we have used that $f \in \mathscr{L}_k$. Hence $\Delta^{-1}(\mathscr{L}_k) \subseteq \mathscr{L}_k$.

Let $\mathscr{L}_{k,l}$ be the subspace of $f \in \mathscr{L}_k$ such that $Af \in \mathscr{L}_k$ for all $A \in \mathfrak{U}^{\leq l}(\mathfrak{o})$. Because $[\mathfrak{o}, \mathfrak{n}_{i+1}] \subseteq \mathfrak{n}_{i+1}$ and hence $\mathfrak{U}^{\leq l}(\mathfrak{o})\mathfrak{U}(\mathfrak{n}_{i+1}) = \mathfrak{U}^{\leq l}(\mathfrak{o})\mathfrak{U}(\mathfrak{n}_{i+1})$ it follows that $\mathscr{L}_{k,l}$ is $\mathfrak{U}(\mathfrak{n}_{i+1})$-invariant since $\mathscr{L}_k$ is $\mathfrak{U}(\mathfrak{n}_{i+1})$-invariant by definition. We will show by induction on l that $\mathscr{L}_{k,l}$ is invariant under Δ. For $l = 0$ this is true since $\mathscr{L}_{k,0} = \mathscr{L}_k$. Let $l > 0$ and $A \in \mathfrak{U}^{\leq l+1}(\mathfrak{o})$. We may assume that $A = A'X$ with $X \in \mathfrak{o}$ and $A' \in \mathfrak{U}^{\leq l}(\mathfrak{o})$. For $f \in \mathscr{L}_{k,l+1}$ we get using the induction hypotheses, $[\Delta, X] \in \mathfrak{U}(\mathfrak{n})$ and the $\mathfrak{U}(\mathfrak{n})$-invariance of $\mathscr{L}_{k,l+1}$ that

$$\begin{aligned}
A\Delta^{-1} f &= (A'\Delta^{-1})(\Delta X)\Delta^{-1} f \\
&= (A'\Delta^{-1})(X\Delta + [\Delta, X])\Delta^{-1} f \\
&= A'\Delta^{-1}(Xf) + A'(\Delta^{-1}[\Delta, X]\Delta^{-1}) f \in \mathscr{L}_k
\end{aligned}$$

Hence $\Delta^{-1} f \in \mathscr{L}_{k,l+1}$ as claimed.

It follows that the space $\mathscr{L}_{k,\infty} = \bigcap_{l=0}^{\infty} \mathscr{L}_{k,l}$ is Δ^{-1}-stable. But now we are done since

$$\mathscr{K}_0 = \mathscr{L}_{0,\infty},$$

$$\mathscr{K}_{+\log} = \bigcup_{k\in\mathbb{Z}} \mathscr{L}_{k,\infty}$$

and

$$\mathscr{K}_{-\log} = \bigcap_{k\in\mathbb{Z}} \mathscr{L}_{k,\infty}.$$

Hence the trivial endomorphism induced by Δ on $H^\bullet_{(\mathfrak{o},K_\mathcal{O})}(\mathscr{K}_?)$ is invertible. $\square$

Similar arguments may be found in [BL84, §1.2], [Lan73, §2], [Oss07] and [Zuc83, §4].

2.35 Corollary: *There is a canonical isomorphism of $\mathcal{G}(\mathbb{A}_f)$-equivariant sheaves*

$$\mathrm{Tot}^\bullet \mathscr{H}^\bullet_{(\mathfrak{l},K_\mathcal{O})}(\mathscr{S}^\mathfrak{n}_{\rho+?,\mathcal{O}}|_{\partial_\mathcal{O}\,\mathrm{Sh}^*} \otimes H^\bullet_\mathfrak{n}(E)) \cong \mathscr{H}^\bullet_{(\mathfrak{o},K_\mathcal{O})}(\mathscr{S}^\mathfrak{n}_{\rho+?,\mathcal{O}}|_{\partial_\mathcal{O}\,\mathrm{Sh}^*} \otimes E)$$

where $\mathrm{Tot}^\bullet$ *denotes the total complex.*

PROOF: For any subset $V \subseteq \partial_\mathcal{O}\,\mathrm{Sh}^*$ there is a natural spectral sequence with E_2-term

$$\begin{aligned} E_2^{p,q} &= H^p_{(\mathfrak{l},K_\mathcal{O})}(H^q_\mathfrak{n}(\mathscr{S}^\mathfrak{n}_{\rho+?,\mathcal{O}}|_{\partial_\mathcal{O}\,\mathrm{Sh}^*}(V) \otimes E)) \\ &= H^p_{(\mathfrak{l},K_\mathcal{O})}(\mathscr{S}^\mathfrak{n}_{\rho+?,\mathcal{O}}|_{\partial_\mathcal{O}\,\mathrm{Sh}^*}(V) \otimes H^q_\mathfrak{n}(E)) \end{aligned}$$

converging to

$$H^{p+q}_{(\mathfrak{o},K_\mathcal{O})}(\mathscr{S}^\mathfrak{n}_{\rho+?,\mathcal{O}}|_{\partial_\mathcal{O}\,\mathrm{Sh}^*}(V) \otimes E).$$

It follows from Kostant's Theorem on $\mathfrak{n}$-cohomology and its proof that there is a canonical $\mathcal{L}(\mathbb{C})$-equivariant splitting $H^\bullet_\mathfrak{n}(E) \hookrightarrow Z^\bullet_\mathfrak{n}(E)$ in the derived category of $\mathcal{L}(\mathbb{C})$-modules. Furthermore any irreducible representation F of $\mathcal{L}(\mathbb{C})$ appears at most once in $H^\bullet_\mathfrak{n}(E)$. Hence the spectral sequence degenerates to a canonical splitting:

$$\bigoplus_{p+q=n} H^p_{(\mathfrak{l},K_\mathcal{O})}(\mathscr{S}^\mathfrak{n}_{\rho+?,\mathcal{O}}|_{\partial_\mathcal{O}\,\mathrm{Sh}^*}(V) \otimes H^q_\mathfrak{n}(E)) \cong H^n_{(\mathfrak{o},K_\mathcal{O})}(\mathscr{S}^\mathfrak{n}_{\rho+?,\mathcal{O}}|_{\partial_\mathcal{O}\,\mathrm{Sh}^*}(V) \otimes E)$$

that is compatible with restriction to smaller open sets. $\square$

2.5 Automorphic Local Systems

Let $\widetilde{\mathcal{L}}_h \subseteq \mathcal{L}$ be a semi-simple $\mathbb{Q}$-subgroup such that that the projection $\mathcal{L} \to \mathcal{L}_h$ restricts to an isogeny $\widetilde{\mathcal{L}}_h \to \mathcal{L}_h$. Since $\mathcal{L}_h$ is semi-simple we may choose $\widetilde{\mathcal{L}}_h$ to be the image of the universal covering of $\mathcal{L}_h$ under the morphism associated to some splitting of $\mathfrak{l} \to \mathfrak{l}_h$ in $\mathcal{L}$. Fix $\widetilde{\mathcal{L}}_h$ and let $\widetilde{\mathfrak{l}}_h \subseteq \mathfrak{l}$ be its Lie algebra. In general $\widetilde{\mathcal{L}}_h(\mathbb{R}) \to \mathcal{L}_h(\mathbb{R})$ will be neither one-to-one nor onto. Let

$$\partial_O \widetilde{\mathcal{X}}^* \subseteq \partial_O \mathcal{X}^*$$

be some $\widetilde{\mathcal{L}}_h(\mathbb{R})$-orbit and

$$\mathcal{L}(\mathbb{R})^1 \subseteq \mathcal{L}(\mathbb{R})$$

be the subgroup normalizing $\partial_O \widetilde{\mathcal{X}}^*$. Let $\mathcal{L}(\mathbb{R})_l \subseteq \mathcal{L}(\mathbb{R})$ the normal subgroup acting trivially on $\partial_O \mathcal{X}^*$. Because $\mathcal{L}(\mathbb{R})$ acts transitively on $\partial_{\mathcal{O}} \mathcal{X}^*$ one has

$$\mathcal{L}(\mathbb{R})^1 = \mathcal{L}(\mathbb{R})_l \widetilde{\mathcal{L}}_h(\mathbb{R}). \tag{2.36}$$

Set

$$\mathcal{L}(\mathbb{Q})^1 = \mathcal{L}(\mathbb{R})^1 \cap \mathcal{L}(\mathbb{Q})$$

we have

$$\mathcal{L}_l(\mathbb{Q}) \subseteq \mathcal{L}(\mathbb{Q})_l.$$

Similarly define $\mathcal{O}(L)^1$ and $\mathcal{O}(L)_l$ for any $\mathbb{Q}$-algebra L. Since there exist sections $\mathcal{L} \to \mathcal{O}$ defined over $\mathbb{Q}$, the images of $\mathcal{O}(L)^1$ and $\mathcal{O}(L)_l$ in $\mathcal{L}(L)$ agree with $\mathcal{L}(L)^1$ and $\mathcal{L}(L)_l$ for $L|\mathbb{Q}$. Let $\overline{\mathcal{L}(\mathbb{Q})_l}$ be the closure of $\mathcal{L}(\mathbb{Q})_l$ in $\mathcal{L}(\mathbb{A}_f)$. It is the image of $\mathbb{Z}_s$ under the quotient $p\colon \mathcal{O}(\mathbb{A}_f) \to \mathcal{L}(\mathbb{A}_f)$.

Let $\mathcal{K} \subseteq K_{\mathcal{O}} \widetilde{\mathcal{L}}_h(\mathbb{C})$ be a maximal compact subgroup and set $\widetilde{K}_l = \mathcal{L}(\mathbb{R})_l \cap \mathcal{K}$ and $K_l = \mathcal{L}_l(\mathbb{R}) \cap \mathcal{K}$.

We assume from now on that matters can be arranged in a way that

$$\mathcal{L}(\mathbb{R})_l \cap \mathcal{K} = \mathcal{L}(\mathbb{R})_l \cap K_{\mathcal{O}}. \tag{*}$$

This is for example always the case, if we start with the full maximal compact subgroup or, even stronger, if we assume $\mathcal{G}$ to be semi-simple and simply connected. Set

$$\widetilde{K}_l = \mathcal{L}(\mathbb{R})_l \cap \mathcal{K} = \mathcal{L}(\mathbb{R})_l \cap \mathcal{K}$$

and

$$K_l = \mathcal{L}_l(\mathbb{R}) \cap \mathcal{K}.$$

Set

$$\widetilde{\mathbb{K}}_l^f = \overline{\mathcal{L}(\mathbb{Q})_l} \cap p(\mathbb{K}^f \cap \mathcal{O}(\mathbb{A}_f))$$

where $p\colon \mathcal{O}(\mathbb{A}_f) \to \mathcal{L}(\mathbb{A}_f)$ is the projection and $\mathbb{K}^f \subseteq \mathcal{G}(\mathbb{A}_f)$ is any compact open subgroup and let $\widetilde{\mathbb{K}}_l = \widetilde{K}_l(\widetilde{\mathbb{K}_f})_l$. The groups $\mathbb{K}_l^f$ form a neighborhood basis of the identity in $\overline{\mathcal{L}(\mathbb{Q})_l}$ consisting of compact but not necessarily open subgroups. Similarly set $\mathbb{K}_l^f = \overline{\mathcal{L}_l(\mathbb{Q})} \cap \widetilde{\mathbb{K}}_l^f$. For an open $\widetilde{\mathbb{K}}_l^f$-invariant set

$$W \subseteq \mathcal{L}(\mathbb{Q})_l A_{\mathcal{G}} \backslash (\mathcal{L}(\mathbb{R})_l / \widetilde{K}_l) \times \overline{\mathcal{L}(\mathbb{Q})_l}$$

let

$$S_{l,\rho}(W)$$

be the space of $\widetilde{\mathbb{K}}_l$-finite smooth functions on the preimage in $\mathcal{L}(\mathbb{Q})_l A_{\mathcal{G}} \backslash \mathcal{L}(\mathbb{R})_l \times \overline{\mathcal{L}(\mathbb{Q})_l}$ of W satisfying (2.23). Let

$$W^*(\mathcal{O}, T)_l \subseteq \mathcal{L}(\mathbb{R})_l$$

be the image of $W^*(\mathcal{O},T,V)\cap \mathcal{O}(\mathbb{R})_l$ in $\mathcal{L}(\mathbb{R})$ and similarly

$$W_l^*(\mathcal{O},T) = W^*(\mathcal{O},T)_l \cap \mathcal{L}_l(\mathbb{R}).$$

It is independent of $V \subseteq \partial_{\mathcal{O}}\mathcal{X}^*$. The direct limit

$$S_{l,\rho,s} = \operatorname*{colim}_{T,\mathbb{K}^f} \; S_{l,\rho}(W^*(\mathcal{O},T)_l \times \widetilde{\mathbb{K}}_l^f)^{\widetilde{\mathbb{K}}_l^f} \tag{2.37}$$

is a $(\widetilde{\mathfrak{l}}_l, \widetilde{K}_l, \overline{\mathcal{L}(\mathbb{Q})}_l)$-module where an element of $\overline{\mathcal{L}(\mathbb{Q})}_l$ acts by the inverse of a rational approximation on the left and $\widetilde{\mathfrak{l}}_l$ denotes the Lie algebra of $\mathcal{L}(\mathbb{R})_l$.

Set

$$\Lambda = \mathcal{L}(\mathbb{Q})^1\overline{\mathcal{L}(\mathbb{Q})}_l.$$

and let $\mathrm{Rep}_{\mathcal{O},\mathbb{C}}$ denote the category of all complex representations V of Λ with the following properties:

(1) The restriction to $\overline{\mathcal{L}(\mathbb{Q})}_l$ is admissible in the sense that the spaces of $\widetilde{\mathbb{K}}_l^f$-invariants, for every $\mathbb{K}^f \subseteq \mathcal{G}(\mathbb{A}_f)$ compact and open, are finite dimensional and exhaust V.

(2) For every $\mathbb{K}^f \subseteq \mathcal{G}(\mathbb{A}_f)$ the action of $\widetilde{\mathcal{L}}_h(\mathbb{Q})$ on the finite dimensional space $V^{\widetilde{\mathbb{K}}^f}$ is algebraic.

The category $\mathrm{Rep}_{\mathcal{O},\mathbb{C}}$ is a semi-simple abelian category as is easily seen.

2.38 Lemma: There is a canonical differential graded action of Λ on

$$C^\bullet_{(\widetilde{\mathfrak{l}}_l,\widetilde{K}_l)}(S_{l,\rho,s}\otimes H^\bullet_{\mathfrak{n}}(E)).$$

making it an object of $\mathrm{Rep}_{\mathcal{O},\mathbb{C}}$. The restriction of the Λ-action to $\overline{\mathcal{L}(\mathbb{Q})}_l$ agrees with the one induced from right translations on $S_{l,\rho,s}$.

Proof: Let $\widetilde{S}_{l,\rho,s}$ be the direct limit of the space of smooth $\widetilde{K}_l\widetilde{\mathcal{L}}_h(\mathbb{C})$-finite functions defined on open subsets

$$W^*(\mathcal{O},T)_l\widetilde{\mathcal{L}}_h(\mathbb{C}) \subseteq \mathcal{L}(\mathbb{Q})_l C\backslash \mathcal{L}(\mathbb{R})_l\widetilde{\mathcal{L}}_h(\mathbb{C}) \times \overline{\mathcal{L}(\mathbb{Q})}_l$$

such that for every $k\in\mathcal{K}$ the function $l\mapsto f(lk)$ of $l\in\mathcal{L}(\mathbb{R})_l$ belongs to $S_{l,\rho}(W^*(\mathcal{O},T)_l\times\overline{\mathcal{L}(\mathbb{Q})}_l)$. Note that elements of $\widetilde{S}_{l,\rho,s}$ are algebraic in the direction of $\widetilde{\mathcal{L}}_h(\mathbb{C})$. The restriction map

$$\widetilde{S}_{l,\rho,s} \to S_{l,\rho,s} \tag{2.39}$$

is $(\widetilde{\mathfrak{l}}_l,\widetilde{K}_l)$-equivariant and defines an isomorphism

$$\widetilde{S}_{l,\rho,s} \cong \mathrm{Ind}^{(\mathfrak{l},\mathcal{K})}_{(\widetilde{\mathfrak{l}}_l,\widetilde{K}_l)}(S_{l,\rho,s}).$$

As in the discussion before Corollary 2.31 (2.39) induces an isomorphism

$$C^\bullet_{(\mathfrak{l},\mathcal{K})}(\widetilde{S}_{l,\rho,s}\otimes H^\bullet_{\mathfrak{n}}(E)) \cong C^\bullet_{(\widetilde{\mathfrak{l}}_l,\widetilde{K}_l)}(S_{l,\rho,s}\otimes H^\bullet_{\mathfrak{n}}(E)) \tag{2.40}$$

By (2.36) and since $\mathcal{L}(\mathbb{R})^1$ normalizes $\mathcal{L}(\mathbb{R})_l$ the inclusion $\mathcal{L}(\mathbb{R})_l \subseteq \mathcal{L}(\mathbb{R})^1$ induces a natural homeomorphism

(2.41)

$$\begin{aligned}\mathcal{L}(\mathbb{Q})_l A_{\mathcal{G}} \backslash \mathcal{L}(\mathbb{R})_l \widetilde{\mathcal{L}}_h(\mathbb{C}) \times \overline{\mathcal{L}(\mathbb{Q})}_l &\cong \mathcal{L}(\mathbb{Q})^1 A_{\mathcal{G}} \backslash \mathcal{L}(\mathbb{R})^1 \widetilde{\mathcal{L}}_h(\mathbb{C}) \times (\overline{\mathcal{L}(\mathbb{Q})}_l \mathcal{L}(\mathbb{Q})^1)\\ [(l, l_l)] &\mapsto [(l, l_l)]\\ [(\gamma^{-1} l, \gamma^{-1} l_l \gamma)] &\leftarrow [(l, l_l \gamma)].\end{aligned}$$

for every $l \in \mathcal{L}(\mathbb{R})_l \widetilde{\mathcal{L}}_h(\mathbb{C}) = \mathcal{L}(\mathbb{R})^1 \widetilde{\mathcal{L}}_h(\mathbb{C})$, $l_l \in \overline{\mathcal{L}(\mathbb{Q})}_l$ and $\gamma \in \mathcal{L}(\mathbb{Q})^1$. The action of Λ on (2.41) by right multiplication can be transported to a Λ-action on $\widetilde{S}_{l,\rho,s}$ and yields the desired Λ-action via the identification in (2.40). The action is obviously admissible when restricted to $\overline{\mathcal{L}(\mathbb{Q})}_l$. Since E is algebraic, the action of $\mathcal{O}$ on $H^\bullet_n(E)$ is algebraic as well. Since $\mathcal{K} \cap \widetilde{\mathcal{L}_h}(\mathbb{C})$ is Zariski-dense in $\widetilde{\mathcal{L}_h}(\mathbb{C})$ and the formation of the Chevalley-Eilenberg complex involves the passage to $\mathcal{K}$-invariants the action of $\widetilde{\mathcal{L}}_h(\mathbb{Q})$ is algebraic. □

Let $\partial_{\mathcal{O}} \widetilde{\mathrm{Sh}}^*$ be the image of $\partial_{\mathcal{O}} \widetilde{\mathcal{X}}^* \times \mathcal{G}(\mathbb{A}_f)$ in $\partial_{\mathcal{O}} \mathrm{Sh}^*$ and let

$$p \colon \partial_{\mathcal{O}} \widetilde{\mathcal{X}}^* \times \mathcal{G}(\mathbb{A}_f) \to \partial_{\mathcal{O}} \widetilde{\mathrm{Sh}}^*$$

denote the projection. For $V \in \mathrm{Rep}_{\mathcal{O},\mathbb{C}}$ define the associated automorphic local system $\mathbb{A}V$ on $\partial_{\mathcal{O}} \widetilde{\mathcal{X}}^*$ as follows: For an open set $U \subseteq \partial_{\mathcal{O}} \widetilde{\mathrm{Sh}}^*$ let

$$\mathbb{A}V(U) = \{ s \colon p^{-1}(U) \to V \text{ locally constant} |\ s(\gamma(x,g)) = \gamma s(x,g) \}$$

where the transformation rule is assumed to hold for all $\gamma \in \mathcal{L}(\mathbb{Q})^1 \subseteq \Lambda$. The association $V \mapsto \mathbb{A}V$ defines a functor

$$\mathbb{A} \colon \mathrm{Rep}_{\mathcal{O},\mathbb{C}} \to \mathcal{G}(\mathbb{A}_f)\text{-equivariant sheaves } \partial_{\mathcal{O}} \mathrm{Sh}^* .$$

Recall that a sheaf on Sh^* is called weakly constructible if its restriction to each boundary component is a locally constant sheaf of possibly infinite dimensional vector spaces.

2.42 Theorem: (1) *The inclusion*

(2.43)

$$C^\bullet_{(\mathfrak{m}_{\mathcal{L}_l}, K_l)}((\mathscr{S}^{\mathfrak{n}}_{\rho,\mathcal{O}}|_{\partial_{\mathcal{O}} \widetilde{\mathrm{Sh}}^*} \otimes H^\bullet_{\mathfrak{n}}(E))^{\widetilde{\mathfrak{l}}_h}) \hookrightarrow C^\bullet_{(\mathfrak{m}_{\mathcal{L}}, K_{\mathcal{O}})}((\mathscr{S}^{\mathfrak{n}}_{\rho,\mathcal{O}}|_{\partial_{\mathcal{O}} \widetilde{\mathrm{Sh}}^*} \otimes H^\bullet_{\mathfrak{n}}(E)))$$

is a quasi-isomorphism.

(1)*Restricting differential forms to the link defines a canonical $\mathcal{G}(\mathbb{A}_f)$-equivariant isomorphism of complexes of sheaves on $\partial_{\mathcal{O}} \widetilde{\mathrm{Sh}}^*$*

$$C^\bullet_{(\mathfrak{m}_{\mathcal{L}}, K_l)}((\mathscr{S}^{\mathfrak{n}}_{\rho,\mathcal{O}}|_{\partial_{\mathcal{O}} \widetilde{\mathrm{Sh}}^*} \otimes H^\bullet_{\mathfrak{n}}(E))^{\widetilde{\mathfrak{l}}_h}) \to \mathbb{A}V^\bullet \tag{2.44}$$

where $\mathbb{A}V^\bullet$ is the differential graded automorphic local system associated to the Λ-action on

$$V^\bullet = C^\bullet_{(\mathfrak{l}_l, K_l)}(S_{l,\rho,s} \otimes H^\bullet_{\mathfrak{n}}(E))$$

constructed in Lemma 2.38. More precisely, the map sends an element f on the LHS that is defined over a suitable neighborhood of a boundary point to the class of the function defined on a suitable subset of $\partial_{\mathcal{O}}\widetilde{\mathfrak{X}}^* \times \mathcal{G}(\mathbb{A}_f)$ that maps (x, g) to the function

$$(l_l, l_f) \mapsto \sigma(l) f(l_l l, l_f g; \mathrm{Ad}(l)^{-1} X) \in H^\bullet_{\mathfrak{n}}(E)$$

where (l_l, l_f) is contained in a suitable subset of $\mathcal{L}(\mathbb{R})_l \times \overline{\mathcal{L}(\mathbb{Q})}_l$,

$$X \in \Lambda^\bullet \mathfrak{l}_{l,\mathbb{C}}/\mathfrak{k}_{l,\mathbb{C}} = \Lambda^\bullet \mathfrak{l}_{\mathbb{C}}/\operatorname{Lie}(\mathcal{K})_{\mathbb{C}}$$

is a polyvector, $l \in \mathcal{L}(\mathbb{R})^1$ is a preimage of $x \in \partial_{\mathcal{O}}\widetilde{\mathfrak{X}}^*$ and σ denotes the action of $\mathcal{L}(\mathbb{C})$ on $H^\bullet_{\mathfrak{n}}(E)$. In particular for $s \in \partial_{\mathcal{O}}\widetilde{\mathrm{Sh}}^*$ there is a canonical isomorphism

$$C^\bullet_{(\mathfrak{m}_{\mathcal{L}} \cap \widetilde{\mathfrak{l}}_l, \widetilde{K}_l)}((\mathscr{S}^{\mathfrak{n}}_{\rho,\mathcal{O},s} \otimes H^\bullet_{\mathfrak{n}}(E)))^{\widetilde{\mathfrak{l}}_h}) \cong C^\bullet_{(\mathfrak{m}_{\mathcal{L}} \cap \widetilde{\mathfrak{l}}_l, \widetilde{K}_l)}(S_{l,\rho,s} \otimes H^\bullet_{\mathfrak{n}}(E)) \tag{2.45}$$

of differential graded $\mathbb{Z}_s$-modules.

Proof: The first point follows from the Poincaré Lemma. To prove the second point it suffices to construct the map (2.44) prior to sheafification of the $\mathfrak{B}^*$-presheaf $j_* S_{\rho,\mathcal{O}}$ on neighborhoods of the form $U = [U^*(\mathcal{O}, T, V) \times g\mathbb{K}^f] \in \mathfrak{B}^*$ provided it is sufficiently natural. Then it will be compatible with restrictions and extend to the associated sheaf.

The map sends

$$f \in C^q_{(\mathfrak{m}_{\mathcal{L}} \cap \widetilde{\mathfrak{l}}_l, \widetilde{K}_l)}\left((S_{\rho,\mathcal{O}}(U)^{\mathfrak{n},\mathbb{K}^f} \otimes H^\bullet_{\mathfrak{n}}(E))^{\widetilde{\mathfrak{l}}_h}\right)$$

to a function

$$s_f \colon p^{-1}(U) \to V = C^\bullet_{(\mathfrak{m}_{\mathcal{L}} \cap \widetilde{\mathfrak{l}}_l, \widetilde{K}_l)}(S_{l,\rho,s} \otimes H^\bullet_{\mathfrak{n}}(E))$$

defined as follows. Let $(x, g) \in \partial_{\mathcal{O}}\widetilde{X}^*$, $[(l_\infty, l_f)] \in \mathcal{L}(\mathbb{Q})_l A_{\mathcal{G}} \backslash \mathcal{L}(\mathbb{R})_l \times \overline{\mathcal{L}(\mathbb{Q})}_l$. Let $(l, g) \in \mathcal{L}(\mathbb{R})^1 \times \mathcal{G}(\mathbb{A}_f)$ represent (x, g) and set

$$(s_f(x, g))(l, l_f; X) = \sigma(l) f(l_l l, l_f g; \mathrm{Ad}(l)^{-1} X) \in H^\bullet_{\mathfrak{n}}(E).$$

We have to check that it is well-defined. It does not depend on the choices of l and l_l since f is $\mathfrak{n}$- as well as $K_{\mathcal{O}}$-invariant. It does not depend on the choice of l_f since f is right $\mathbb{K}^f$-invariant. It does not depend on the choice of (l, l_f) representing its class $[(l_\infty, l_f)]$ because f is $\mathcal{O}(\mathbb{Q})$-invariant on the left. The function s_f is locally constant and $\widetilde{\mathbb{K}}^f_l$-invariant on $\partial_{\mathcal{O}}\widetilde{\mathfrak{X}}^* \times \mathcal{G}(\mathbb{A}_f)$ since

f is $\widetilde{\mathfrak{l}}_h$ as well as $\mathbb{K}^f$-invariant. It is $\widetilde{K}_l$-invariant because f is $K_{\mathcal{O}}$-invariant. It remains to show that

$$(1) \qquad (\gamma s_f)(x,g) = s_f(\gamma(x,g))$$

for all $\gamma \in \mathcal{L}(\mathbb{Q})^1$. Let $\widetilde{s}_f$ be the image of s_f under (2.40) considered as a function on the right hand side of (2.41). Then (1) is equivalent to

$$(\widetilde{s}_f(x,g))(l_l l_h, l_f\gamma; X) = s_f(\gamma(x,g)(l_l l_h, l_f; X)$$

for all $o_l \in \mathcal{L}(\mathbb{R})_l$, $l_h \in \widetilde{\mathcal{L}}_h(\mathbb{C})$, $l_f \in \Lambda$, $X \in \mathfrak{l}_{\mathbb{C}}$ and $\gamma \in \mathcal{L}(\mathbb{Q})^1$. By continuity we may assume that $l_f \in \mathcal{L}(\mathbb{Q})^1$ and replacing γ by $l_f\gamma$ we may assume that $l_f = e$. Choose a decomposition $\gamma^{-1} = m_l m_h \in \mathcal{L}(\mathbb{R})_l\widetilde{\mathcal{L}}_h(\mathbb{C})$. Since $\widetilde{\mathcal{L}}_h(\mathbb{C})$ is connected and $\mathrm{Lie}(\mathcal{K})_{\mathbb{C}}$ contains $\widetilde{\mathfrak{l}}_{h,\mathbb{C}}$, $\mathcal{K}$-invariance implies $\widetilde{\mathcal{L}}_h(\mathbb{C})$-invariance. By (2.36) there is a decomposition $\gamma^{-1} = m_l m_h \in \mathcal{L}(\mathbb{R})_l\widetilde{\mathcal{L}}_h(\mathbb{R})$. We calculate:

$$\begin{aligned}
&\widetilde{s}_f(x,g)(l_l l_h,\gamma;X) = \\
&\quad(\widetilde{s}_f(x,g))((\gamma^{-1}l_l\gamma)m_l m_h l_h, e; X) = \\
&\quad\sigma(m_h l_h)^{-1}\widetilde{s}_f(x,g)(\gamma^{-1}l_l\gamma m_l, e; \mathrm{Ad}(m_h l_h)X) = \\
&\quad\sigma(m_h l_h)^{-1}s_f(x,g)(\gamma^{-1}l_l\gamma m_l, e; \mathrm{Ad}(m_h l_h)X) = \\
&\quad\sigma(m_h l_h)^{-1}\sigma(m_l^{-1}l)f(\gamma^{-1}l_l\gamma m_l(m_l^{-1}l), g; \mathrm{Ad}(m_l^{-1}l)^{-1}\,\mathrm{Ad}(m_h l_h)X) = \\
&\quad\sigma(l_h)^{-1}\sigma(\gamma l)f(\gamma^{-1}l_l(\gamma l), g; \mathrm{Ad}(\gamma l)^{-1}\,\mathrm{Ad}(l_h)X) = \\
&\quad\sigma(l_h)^{-1}\sigma(\gamma l)f(l_l(\gamma l), (\gamma g); \mathrm{Ad}(\gamma l)^{-1}\,\mathrm{Ad}(l_h)X) = \\
&\quad\sigma(l_h)^{-1}s_f(\gamma(x,g))(l_l, e; \mathrm{Ad}(l_h)X) = \widetilde{s}_f(\gamma(x,g))(l_l l_h, e; X).
\end{aligned}$$

Hence the map (2.44) is well-defined. Since both sides of (2.44) are local systems it suffices to show that (2.44) induces an isomorphism of the stalks. This is easy to see. □

Theorem 2.42 describes the restriction of

$$C^\bullet_{(\mathfrak{m}_{\mathcal{G}}\,\mathfrak{l},K_{\mathcal{O}})}(\mathscr{S}^{\mathfrak{n}}_{\rho,\mathcal{O}}|_{\partial_{\mathcal{O}}\widetilde{\mathrm{Sh}}^*} \otimes H^\bullet_{\mathfrak{n}}(E)))$$

to $\partial_{\mathcal{O}}\widetilde{\mathrm{Sh}}^*$. In general we may find finitely many $\widetilde{\mathcal{L}}_h(\mathbb{R})$-orbits

$$\partial_{\mathcal{O}}\widetilde{\mathcal{X}}^*_1, \ldots, \partial_{\mathcal{O}}\widetilde{\mathcal{X}}^*_n \subseteq \partial_{\mathcal{O}}\mathcal{X}^*$$

such that the images

$$\partial_{\mathcal{O}}\widetilde{\mathrm{Sh}}^*_i = [\partial_{\mathcal{O}}\widetilde{X}^*_1 \times \mathcal{G}(\mathbb{A}_f)] \subseteq \partial_{\mathcal{O}}\,\mathrm{Sh}^*$$

are disjoint and

$$\partial_{\mathcal{O}}\,\mathrm{Sh}^* = \bigcup_{i=1}^n \partial_{\mathcal{O}}\widetilde{\mathrm{Sh}}^*_i.$$

is a $\mathcal{G}(\mathbb{A}_f)$-stable decomposition of the boundary component. Hence Theorem 2.42 describes the situation completely.

2.6 Logarithmic Modifications in the Equal Rank Case

From results of [BC83] and [Fra98] Nair deduced the following

2.46 Theorem ([Nai99], Theorem 4.1)**:** *If* Sh *is an equal-rank locally symmetric space then the natural inclusions*

$$C^\bullet_{(\mathfrak{m}_\mathcal{G},K)}(S_{(2)-\log}(\mathrm{Sh})\otimes E) \subset C^\bullet_{(\mathfrak{m}_\mathcal{G},K)}(S_{(2)}(\mathrm{Sh})\otimes E) \subset C^\bullet_{(\mathfrak{m}_\mathcal{G},K)}(S_{(2)+\log}(\mathrm{Sh})\otimes E)$$

induce isomorphisms in cohomology.

We need the following generalization of this:

2.47 Theorem: *If* Sh^* *is an equal-rank Satake compactification then the natural inclusions of complexes of sheaves*

$$C^\bullet_{(\mathfrak{m}_\mathcal{G},K)}(\mathscr{S}_{(2)-\log}\otimes E) \subseteq C^\bullet_{(\mathfrak{m}_\mathcal{G},K)}(\mathscr{S}_{(2)}\otimes E) \subseteq C^\bullet_{(\mathfrak{m}_\mathcal{G},K)}(\mathscr{S}_{(2)+\log}\otimes E)$$

are quasi-isomorphisms.

Proof: Set $\mathscr{A}^\bullet_{(2)+?} = C^\bullet_{(\mathfrak{m},K)}(\mathscr{S}_{(2)+?}\otimes E)$. Let

$$\mathscr{K}^\bullet = \mathrm{Cone}(\mathscr{A}^\bullet_{(2)} \hookrightarrow \mathscr{A}^\bullet_{(2)+\log})$$

be the cone of the second inclusion. We have to show that its cohomology sheaves vanish. Assume the contrary and let $\partial_\mathcal{O}\mathrm{Sh}^*$ be inclusion maximal with the property that

$$\mathscr{H}^\bullet(\mathscr{K}^\bullet)|_{\partial_\mathcal{O}\mathrm{Sh}^*} \neq \{0\}.$$

Let

$$\mathfrak{P}^*_{\mathcal{O},\mathscr{K}^\bullet} \subset \mathfrak{P}^* - \{\mathcal{O}\}$$

be the set of all standard parabolics $\mathcal{P}\in\mathfrak{P}^*$, $\mathcal{P}\neq\mathcal{O}$, such that there is some $\mathcal{Q}\in\mathfrak{P}^*$ with

$$\mathscr{H}^\bullet(\mathscr{K}^\bullet)|_{\partial_\mathcal{Q}\mathrm{Sh}^*} \neq 0$$

and

$$\partial_\mathcal{P}\mathrm{Sh}^* \subseteq \overline{\partial_\mathcal{Q}\mathrm{Sh}^*}.$$

Set

$$\mathrm{Sh}^*_{\mathcal{O},\mathscr{K}^\bullet} = \mathrm{Sh}^* - \bigcup_{\mathcal{P}\in\mathfrak{P}^*_{\mathcal{O},\mathscr{K}^\bullet}} \partial_\mathcal{P}\mathrm{Sh}^*$$

This is an open subset of Sh^* containing $\partial_\mathcal{O}\mathrm{Sh}^*$ as a closed subset. Let

$$j\colon\ \mathrm{Sh}^*_{\mathcal{O},\mathscr{K}^\bullet} \hookrightarrow \mathrm{Sh}^* \hookleftarrow \partial_\mathcal{O}\mathrm{Sh}^*: i$$

denote the inclusions. By our assumption on $\mathcal{O}$ and the definition of $\mathfrak{P}^*_{\mathcal{O},\mathscr{K}^\bullet}\subset \mathfrak{P}^* - \{\mathcal{O}\}$ the natural homomorphism

$$j^*\mathscr{K}^\bullet \to i_*i^*j^*\mathscr{K}^\bullet$$

is a quasi-isomorphism. Since the formation of the cone commutes with exact functors there is a natural isomorphism

$$i^* j^* \mathscr{K}^\bullet = \mathrm{Cone}(\mathscr{A}^\bullet_{(2)}|_{\partial_{\mathcal{O}} \mathrm{Sh}^*} \to \mathscr{A}^\bullet_{(2)+\log}|_{\partial_{\mathcal{O}} \mathrm{Sh}^*})$$

By Theorem 2.42 $\mathscr{A}^\bullet_{(2)}|_{\partial_{\mathcal{O}} \mathrm{Sh}^*}$ and $\mathscr{A}^\bullet_{(2)+\log}|_{\partial_{\mathcal{O}} \mathrm{Sh}^*}$ are quasi-isomorphic to a graded automorphic local systems on $\partial_{\mathcal{O}} \mathrm{Sh}^*$. Since the category of automorphic local systems is semi-simple there is a graded automorphic local system $\mathbb{A}V^\bullet$ and a quasi-isomorphism

$$\mathbb{A}V^\bullet \to i^* j^* \mathscr{K}^\bullet.$$

By definition of $\mathrm{Sh}^*_{\mathcal{O},\mathscr{K}^\bullet}$ the cohomology sheaves of $j^*\mathscr{K}^\bullet$ are supported on $\partial_{\mathcal{O}} \mathrm{Sh}^*$ and it follows the the induced map

$$i_* \mathbb{A}V^\bullet \to i_* i^* j^* \mathscr{K}^\bullet.$$

is a quasi-isomorphism as well. There results a commutative diagram

$$\begin{array}{ccc}
\mathbb{H}^\bullet_c(\mathrm{Sh}^*_{\mathcal{O},\mathscr{K}^\bullet}, j^*\mathscr{K}^\bullet) & \longrightarrow & \mathbb{H}^\bullet(\mathrm{Sh}^*_{\mathcal{O},\mathscr{K}^\bullet}, j^*\mathscr{K}^\bullet) \\
\downarrow & & \downarrow \\
\mathbb{H}^\bullet_c(\mathrm{Sh}^*_{\mathcal{O},\mathscr{K}^\bullet}, i_* i^* j^*\mathscr{K}^\bullet) & \longrightarrow & \mathbb{H}^\bullet(\mathrm{Sh}^*_{\mathcal{O},\mathscr{K}^\bullet}, i_* i^* j^*\mathscr{K}^\bullet) \\
\uparrow & & \uparrow \\
\mathbb{H}^\bullet_c(\mathrm{Sh}^*_{\mathcal{O},\mathscr{K}^\bullet}, i_*\mathbb{A}V^\bullet) & \longrightarrow & \mathbb{H}^\bullet(\mathrm{Sh}^*_{\mathcal{O},\mathscr{K}^\bullet}, i_*\mathbb{A}V^\bullet) \\
\uparrow & & \uparrow \\
H^\bullet_c(\partial_{\mathcal{O}} \mathrm{Sh}^*, \mathbb{A}V^\bullet) & \longrightarrow & H^\bullet(\partial_{\mathcal{O}} \mathrm{Sh}^*, \mathbb{A}V^\bullet) \\
\downarrow & \nearrow & \\
H^\bullet_{\mathrm{cusp}}(\partial_{\mathcal{O}} \mathrm{Sh}^*, \mathbb{A}V^\bullet) & &
\end{array}$$

of graded vector spaces where the vertical arrows are isomorphisms and the horizontal arrows are induced by the natural map from hypercohomology with compact supports to ordinary hypercohomology. We have used the well-known fact that there is a natural isomorphism from cohomology with compact support to cuspidal cohomology with values in an automorphic local system. It is a result of Borel that the diagonal arrow is a split inclusion. On the other hand the top horizontal arrow naturally factors over

$$\begin{array}{ccccc}
 & & \mathbb{H}^\bullet(\mathrm{Sh}^*, \mathscr{K}^\bullet) & & \\
 & \nearrow & & \searrow & \\
\mathbb{H}^\bullet_c(\mathrm{Sh}^*_{\mathcal{O},\mathscr{K}^\bullet}, j^*\mathscr{K}^\bullet) & & \longrightarrow & & \mathbb{H}^\bullet(\mathrm{Sh}^*_{\mathcal{O},\mathscr{K}^\bullet}, j^*\mathscr{K}^\bullet)
\end{array}$$

where the diagonal arrows are extension by zero and restriction respectively. Now there is a converging spectral sequence

$$E_2^{pq} = H^p(H^q(\mathrm{Sh}^*, \mathscr{K}^\bullet), d_{\mathscr{K}^\bullet}) \Rightarrow \mathbb{H}^{p+q}(\mathrm{Sh}^*, \mathscr{K}^\bullet).$$

Since by Proposition 2.21 the sheaves $\mathscr{K}^p$ are fine we have $H^q(\mathrm{Sh}^*, \mathscr{K}^\bullet) = 0$ for $q > 0$. The terms $E_2^{\bullet,0}$ vanishes by Theorem 2.46. Hence $\mathbb{H}^{p+q}(\mathrm{Sh}^*, \mathscr{K}^\bullet)$ vanishes and therefore

$$H^\bullet_{\mathrm{cusp}}(\partial_{\mathcal{O}} \mathrm{Sh}^*, \mathbb{A}V^\bullet) = 0.$$

But by a Theorem of Clozel [Clo86] any non-zero automorphic local system on the equal-rank boundary stratum $\partial_{\mathcal{O}} \mathrm{Sh}^*$ has non-zero cohomology. Since cohomology with compact support commutes with direct limits by [God58], Theorem 4.12.1, we may apply this in our situation and it follows that

$$\mathscr{H}^\bullet(\mathscr{K}^\bullet)|_{\partial_{\mathcal{O}} \mathrm{Sh}^*} \cong \mathbb{A}V^\bullet = 0$$

contrary to our assumption on $\mathcal{O} \in \mathscr{P}^*$. This proves that $\mathscr{A}^\bullet_{(2)} \to \mathscr{A}^\bullet_{(2)+\log}$ is a quasi-isomorphism. The remaining case is proved in the same way. □

2.7 Reduction to Weighted Cohomology of the Link

Let $\overline{\mathcal{L}_l(\mathbb{Q})}$ be the closure of $\mathcal{L}_l(\mathbb{Q})$ in $\mathcal{L}_l(\mathbb{A}_f)$. Consider the inclusion

$$\mathcal{L}_l(\mathbb{Q})A_{\mathcal{G}}\backslash\mathcal{L}_l(\mathbb{R}) \times \overline{\mathcal{L}_l(\mathbb{Q})} \subseteq \mathcal{L}(\mathbb{Q})_l A_{\mathcal{G}}\backslash\mathcal{L}(\mathbb{R})_l \times \overline{\mathcal{L}(\mathbb{Q})_l}.$$

Let U be $\widetilde{K}_l$- and $\overline{\mathcal{L}(\mathbb{Q})_l}$-invariant open subset of the RHS and U_1 its intersection with the left hand side. Then U_1 is K_l- and $\overline{\mathcal{L}_l(\mathbb{Q})}$-invariant and the restriction of functions defines a $(\mathfrak{l}_l, K_l, \overline{\mathcal{L}_l(\mathbb{Q})_l})$-equivariant map

$$S_{l,\rho+?}(W) \to S_{\rho+?}(W_1). \tag{2.48}$$

Recall the smooth induction functor for example from [Car79], §1.8. We claim that (2.48) it induces a canonical isomorphism

$$S_{l,\rho+?}(W) \cong \mathrm{Ind}_{(\mathfrak{l}_l,K_l,\overline{\mathcal{L}_l(\mathbb{Q})})}^{(\widetilde{\mathfrak{l}}_l,\widetilde{K}_l,\overline{\mathcal{L}(\mathbb{Q})_l})}(S_{\rho+?}(W_1)) := \mathrm{Ind}_{\overline{\mathcal{L}_l(\mathbb{Q})}}^{\overline{\mathcal{L}(\mathbb{Q})_l}} \mathrm{Ind}_{(\mathfrak{l}_l,K_l)}^{(\widetilde{\mathfrak{l}}_l,\widetilde{K}_l)}(S_{\rho+?}(W_1)) \tag{2.49}$$

of $(\widetilde{\mathfrak{l}}_l, \widetilde{K}_l, \overline{\mathcal{L}_l(\mathbb{Q})_l})$-modules. Indeed, by our choice of $\mathcal{L}_l$ the quotient

$$\mathcal{L}(\mathbb{R})_l/\mathcal{L}_l(\mathbb{R})$$

is a compact group and hence $\mathcal{L}(\mathbb{R})_l = \mathcal{L}_l(\mathbb{R})\widetilde{K}_l$. This enables us to define the inverse map in (2.49). By Frobenius reciprocity (2.49) induces an isomorphism

$$C^\bullet_{(\mathfrak{m}_{\mathcal{L}}\cap\widetilde{\mathfrak{l}}_l,\widetilde{K}_l)}(S_{l,\rho+?}(W) \otimes H^\bullet_{\mathfrak{n}}(E)) \cong \mathrm{Ind}_{\overline{\mathcal{L}_l(\mathbb{Q})}}^{\overline{\mathcal{L}(\mathbb{Q})_l}} C^\bullet_{(\mathfrak{m}_{\mathcal{L}_l},K_l)}(S_{l,\rho+?}(W_1) \otimes H^\bullet_{\mathfrak{n}}(E)). \tag{2.50}$$

Let $F \subseteq H^\bullet_{\mathfrak{n}}(E)$ be an irreducible $\mathcal{L}(\mathbb{C})$-submodule. Let $\lambda_{F,\mathcal{O}} - \rho_{\mathcal{O}} \in \check{\mathfrak{a}}$ be the character by which $\mathfrak{a}$ acts on F. Define for a weight $\tau_l \in \check{\mathfrak{a}}_0^{\mathcal{L}_l}$ the $(\mathfrak{m}_l, K_l, \mathcal{L}_l(\mathbb{A}_f))$-module

$$S_{\rho_{-\tau_l}+?}(\mathcal{L}_{\mathcal{O},l}(\mathbb{Q})\mathcal{A}_{\mathcal{O}}(\mathbb{R})^+ \backslash \mathcal{L}_{\mathcal{O},l}(\mathbb{A}_f)) \tag{2.51}$$

as the space of global section defined after (2.8) with $\mathcal{G}$ replaced by $\mathcal{L}_l$. More precisely for $? = 0$ (2.51) is the space of smooth $\mathbb{K}_l = K_l\mathbb{K}_l^f$-finite functions on $\mathcal{L}_l(\mathbb{Q})\mathcal{A}_{\mathcal{O}}(\mathbb{R})^+ \backslash \mathcal{L}_l(\mathbb{A}_f)$ such that

$$|Df(l)| \ll \rho_{\rho_0^{\mathcal{L}_l}}(l)\rho_{-\tau_l}(l)^{-1}.$$

for all $D \in \mathfrak{U}(\mathfrak{l}_l)$. Here $\rho_0^{\mathcal{L}_l}$ is the half sum of positive roots of $\mathfrak{a}_0^{\mathcal{L}_l}$ in the Lie algebra of the unipotent radical of $\mathcal{P}_{0,l} := \mathcal{L}_l \cap (\mathcal{P}_0/\mathcal{N}_{\mathcal{O}})$ and the weight functions are produced by Proposition 2.7. For $? = \pm\log$ modify in the same way as after (2.8).

2.52 Proposition: *Let $F \subseteq H^\bullet_{\mathfrak{n}}(E)$ be an irreducible $\mathcal{L}(\mathbb{C})$-submodule and $\lambda_{F,\mathcal{O}} - \rho_{\mathcal{O}} \in \check{\mathfrak{a}}_{\mathcal{O}}$ the rational character by which $\mathcal{A}_{\mathcal{O}}$ acts on F. Assume that $\rho = \rho_{-\tau}$ with $\tau \in \mathfrak{a}_0^{\mathcal{G}}$ as in Proposition 2.7 and let $\rho_{-\tau_F}$ be the one associated to*

$$\tau_F = -\sum_{\alpha \in \Delta_0 - \Delta_0^{\mathcal{O}}} \langle \lambda_{F,\mathcal{O}} + \tau, \check{\varpi}_\alpha \rangle \alpha|_{\mathfrak{a}_{0,l}^{\mathcal{O}}} \in \check{\mathfrak{a}}_{0,l}^{\mathcal{O}}. \tag{2.53}$$

on $\mathcal{L}_l(\mathbb{A})$. Then

$$H^p_{\mathfrak{a}}(S_{l,\rho_{-\tau}+\log}(\mathcal{L}_l(\mathbb{Q})A_{\mathcal{G}} \backslash \mathcal{L}_l(\mathbb{Q})W^*(\mathcal{O},T,V) \cap \mathcal{L}_l(\mathbb{R}) \times \overline{\mathcal{L}_l(\mathbb{Q})}) \otimes F)$$

is $(\mathfrak{l}_l, K_l, \overline{\mathcal{L}_l(\mathbb{Q})})$-equivariantly isomorphic to

$$S_{\rho_{-\tau_F}+\log}(\mathcal{L}_l(\mathbb{Q})\mathcal{A}_{\mathcal{O}}(\mathbb{R})^+ \backslash \mathcal{L}_l(\mathbb{R}) \times \overline{\mathcal{L}_l(\mathbb{Q})}) \otimes F$$

if $p = 0$ and $(\lambda_{F,\mathcal{O}} + \tau)|_{\mathfrak{a}_{\mathcal{O}}} \in {}^+\check{\mathfrak{a}}_{\mathcal{O}}$ and zero otherwise. Similarly

$$H^p_{\mathfrak{a}}(S_{l,\rho-\log}(\mathcal{L}_l(\mathbb{Q})A_{\mathcal{G}} \backslash \mathcal{L}_l(\mathbb{Q})W^*(\mathcal{O},T,V) \cap \mathcal{L}_l(\mathbb{R}) \times \overline{\mathcal{L}_l(\mathbb{Q})}) \otimes F)$$

is $(\mathfrak{l}_l, K_l, \overline{\mathcal{L}_l(\mathbb{Q})})$-equivariantly isomorphic to

$$S_{\rho_{-\tau_F}-\log}(\mathcal{L}_l(\mathbb{Q})\mathcal{A}_{\mathcal{O}}(\mathbb{R})^+ \backslash \mathcal{L}_l(\mathbb{R}) \times \overline{\mathcal{L}_l(\mathbb{Q})}) \otimes F$$

if $p = 0$ and $(\lambda_{F,\mathcal{O}} + \tau)|_{\mathfrak{a}_{\mathcal{O}}} \in {}^+\check{\mathfrak{a}}_{\mathcal{O}}$ and zero otherwise.

Proof: We treat only the $(+\log)$-case the $(-\log)$-case being similar (and not needed in the following). Let us treat the case $p = 0$ first. Let

$$f \in H^0_{\mathfrak{a}}(S_{l,\rho_{-\tau}}([W_l^*(\mathcal{O},T) \times \mathbb{K}_l^f])^{\mathbb{K}_l^f} \otimes F)$$

and assume that $f(l)$, $l \in W_l^*(\mathcal{O},T)$, is non-zero. The sets $W_l^*(\mathcal{O},T) \subseteq \mathcal{L}_l(\mathbb{R})$ were defined on page 51. The set $W_l^*(\mathcal{O},T)$ is stable under multiplication by elements of the form e^X with $X \in \mathfrak{a}_{\mathcal{O}}^+$. Since f is $\mathfrak{a}_{\mathcal{O}}$-invariant

$$f(l,k) = e^{\langle \lambda_{F,\mathcal{O}} - \rho_{\mathcal{O}}, X\rangle} f(le^X, k)$$

for all $X \in \mathfrak{a}_{\mathcal{O}}^+$. Hence there exists $n \in \mathbb{Z}$ such that

$$\begin{aligned} 0 \neq |f(l,k)| &= e^{t\langle \lambda_{F,\mathcal{O}} - \rho_{\mathcal{O}}, X\rangle} |f(le^{tX}, k)| \\ &\ll e^{t\langle \lambda_{F,\mathcal{O}} - \rho_{\mathcal{O}}, X\rangle} \rho_{-\tau}(le^{tX})^{-1} \rho_{\rho_0}(le^{tX}, e) w_n(le^{tX}, e) \\ &\ll e^{t\langle \lambda_{F,\mathcal{O}} - \rho_{\mathcal{O}} + \rho_0 + \tau, X\rangle} w_n(le^{tX}) = e^{t\langle \lambda_{F,\mathcal{O}} + \tau, X\rangle} w_n(le^{tX}, e). \end{aligned} \tag{1}$$

for $X \in \mathfrak{a}_{\mathcal{O}}^+$ and all $t \geq 0$. For $t \to +\infty$ $w_n(le^{tX})$ growth at most polynomially in t. Then (1) forces $\langle \lambda_{F,\mathcal{O}} + \tau, X\rangle \leq 0$. Since X was arbitrary $\lambda_{F,\mathcal{O}} + \tau \in {}^+\check{\mathfrak{a}}_{\mathcal{O}}$ as claimed in the $(+\log)$-case. In the $(-\log)$-case $\langle \lambda_{F,\mathcal{O}} + \tau, X\rangle = 0$ is excluded since $w_n(le^{tX})$ tends to zero for $t \to +\infty$ if $n < 0$. This proves the vanishing assertion in degree $p = 0$.

For $\alpha \in \Delta_0 - \Delta_0^{\mathcal{O}}$ the function

$$\begin{aligned} \mathcal{L}_l(\mathbb{R}) &\to \mathbb{R} \\ l &\mapsto \langle \alpha, H_0(\widetilde{l})\rangle, \end{aligned}$$

where $\tilde{l} \in \mathcal{O}(\mathbb{R})$ is a preimage of l, is a well-defined smooth function. Hence

$$\begin{aligned} \phi\colon \mathcal{L}_l(\mathbb{R}) &\to W_l^*(\mathcal{O},T) \\ l &\mapsto l \exp\left(\sum_{\alpha \in \Delta_0 - \Delta_0^{\mathcal{O}}} (T + 1 - \langle \alpha, H_0(\tilde{l})\rangle) \check{\varpi}_\alpha \right) \end{aligned}$$

is also well-defined and smooth. The function

$$\begin{aligned} \mathcal{L}_l(\mathbb{Q}) \mathcal{A}_{\mathcal{O}}(\mathbb{R})^+ \backslash \mathcal{L}_l(\mathbb{R}) \times \overline{\mathcal{L}_l(\mathbb{Q})} &\mapsto F \\ \tilde{f}\colon (l, \gamma k) &\mapsto e^{\langle \lambda_{F,\mathcal{O}} - \rho_{\mathcal{O}}, H_0(\gamma^{-1}\tilde{l})\rangle} f(\phi(\gamma^{-1} l), e) \end{aligned}$$

is well-defined and smooth. Let us compute the weight $\tau_F \in \check{\mathfrak{a}}_{0,l}^{\mathcal{O}}$ we have to choose in order to get the desired isomorphism in degree zero. Set $X = H_0(\tilde{l})$ and compute

$$\begin{aligned} &\langle \lambda_{F,\mathcal{O}} - \rho_{\mathcal{O}} + \rho_0 + \tau, X - \sum_{\alpha \in \Delta_0 - \Delta_0^{\mathcal{O}}} (T + 1 - \langle \alpha, X\rangle) \check{\varpi}_\alpha \rangle \\ &= \langle \rho_{o,l} + \lambda_{F,\mathcal{O}} + \tau, \sum_{\alpha \in \Delta_0 - \Delta_0^{\mathcal{O}}} \langle \alpha, X\rangle \check{\varpi}_\alpha \rangle + C_T \\ &= \langle \rho_{o,l} + \sum_{\alpha \in \Delta_0 - \Delta_0^{\mathcal{O}}} \langle \lambda_{F,\mathcal{O}} + \tau, \check{\varpi}_\alpha \rangle \alpha|_{\mathfrak{a}_{0,l}^{\mathcal{O}}}, X\rangle + C_T \end{aligned}$$

with some constant C_T. Since the constant C_T changes the estimates in question only by a non-zero constant it follows that

$$\tau_F = -\sum_{\alpha\in\Delta_0-\Delta_0^{\mathcal{O}}} \langle \lambda_{F,\mathcal{O}}+\tau, \breve{\varpi}_\alpha\rangle \alpha|_{\mathfrak{a}_{0,l}^{\mathcal{O}}}.$$

does the job.

To prove the vanishing in higher degrees let $\breve{\varpi}_1,\ldots,\breve{\varpi}_r$ be the basis of $\mathfrak{a}_{\mathcal{O}}$ dual to the basis $\alpha_1,\ldots,\alpha_r\in\Delta_0-\Delta_0^{\mathcal{O}}$. Using the Hochschild Serre spectral sequence inductively for the filtration

$$\{0\}\subseteq\langle\breve{\varpi}_r\rangle\subset\langle\breve{\varpi}_{r-1},\breve{\varpi}_r\rangle\subset\ldots\subset\langle\breve{\varpi}_1,\ldots,\breve{\varpi}_r\rangle=\mathfrak{a}$$

of $\mathfrak{a}$ we see that it suffices to prove that the endomorphism induced by D_j, $1\le j\le r$, on

$$H^0_{\mathbb{R}\breve{\varpi}_{j+1}}(H^0_{\mathbb{R}\breve{\varpi}_{j+2}}(\ldots H^0_{\mathbb{R}\breve{\varpi}_r}(S_{l,\rho_\tau}([W_l^*(\mathcal{O},T)\times\mathbb{K}_l^f])^{\mathbb{K}_l^f}\otimes F)\ldots))=\bigcap_{i>j}\mathrm{Ker}(\frac{\partial}{\partial\breve{\varpi}_i})$$

is an epimorphism. Let us treat $(-\log)$-case first. We have to distinguish two cases:

(CASE A): $\langle\tau-\lambda_{F,\mathcal{O}},\breve{\varpi}_j\rangle\le 0$: In this case we may define a right inverse to the action of $\breve{\varpi}_j$ on $\bigcap_{i>j}\mathrm{Ker}(\frac{\partial}{\partial\breve{\varpi}_i})$ by setting

$$(R_jf)(p):=-\int_0^\infty f(p\exp(t\breve{\varpi}_j))dt.$$

This is well-defined since the function we integrate decays faster the any polynomial if $t\to\infty$. For the same reason we may differentiate under the integral sign. It follows that R_j leaves $\bigcap_{i>j}\mathrm{Ker}(\frac{\partial}{\partial\breve{\varpi}_i})$ stable. We compute

$$\begin{aligned}(\breve{\varpi}_jR_jf)(p)&=-\int_0^\infty \frac{d}{ds}f(p\exp((s+t)\breve{\varpi}_j))\Big|_{s=0}dt\\&=-\int_0^\infty\frac{df(p\exp(.\breve{\varpi}_j))}{dt}dt\\&=-(f(\infty)-f(p))\\&=f(p)\end{aligned}$$

and R_j is a right inverse to $\breve{\varpi}_j$.

(CASE B): $\langle\tau-\lambda_{F,\mathcal{O}},\breve{\varpi}_j\rangle>0$: In this case we define R_j by the formula

$$(R_jf)(p):=\int_0^{\langle\alpha_j,H_{\mathcal{O},\infty}(p)\rangle-T} f(p\exp(-t\breve{\varpi}_j))dt.$$

and find compute again

$$
\begin{aligned}
(\breve{\varpi}_j R_j f)(p) &= \frac{d}{ds} \int_0^{\langle \alpha_j, H_{\mathcal{O},\infty}(p)\rangle + s - T} f(p \exp((s-t)\breve{\varpi}_j)) dt \bigg|_{s=0} \\
&= \lim_{s \to 0} \frac{1}{s} \int_{\langle \alpha_j, H_{\mathcal{O},\infty}(p)\rangle - T}^{\langle \alpha_j, H_{\mathcal{O},\infty}(p)\rangle + s - T} f(p \exp((s-t)\breve{\varpi}_j)) dt + \\
&\quad \frac{d}{ds} \int_0^{\langle \alpha_j, H_{\mathcal{O},\infty}(p)\rangle - T} f(p \exp((s-t)\breve{\varpi}_j)) dt \bigg|_{s=0} \\
&= f(p \exp((T - \langle \alpha_j, H_{\mathcal{O},\infty}(p)\rangle)\breve{\varpi}_j)) + \\
&\quad (f(p) - f(p \exp((T - \langle \alpha_j, H_{\mathcal{O},\infty}(p)\rangle)\breve{\varpi}_j))) \\
&= f(p).
\end{aligned}
$$

In the $(+\log)$ case the definition of R_j is the same besides that we have to distinguish the cases are A : $\langle \tau - \lambda_{F,\mathcal{O}}, \breve{\varpi}_j \rangle < 0$ and B: $\langle \tau - \lambda_{F,\mathcal{O}}, \breve{\varpi}_j \rangle \geq 0$. $\square$

For any $\tau_l \in \breve{\mathfrak{a}}_0^{\mathcal{L}_l}$ the canonical restriction map

$$
S_{\rho - \tau_l + \log}(\mathcal{L}_l(\mathbb{Q})\mathcal{A}_{\mathcal{O}}(\mathbb{R})^+ \backslash \mathcal{L}_l(\mathbb{A})) \to S_{\rho - \tau_l + \log}(\mathcal{L}_l(\mathbb{Q})\mathcal{A}_{\mathcal{O}}(\mathbb{R})^+ \backslash \mathcal{L}_l(\mathbb{R}) \times \overline{\mathcal{L}_l(\mathbb{Q})})
$$

induces an isomorphism

$$
\begin{aligned}
(2.54) \quad & S_{\rho - \tau_l + \log}(\mathcal{L}_l(\mathbb{Q})\mathcal{A}_{\mathcal{O}}(\mathbb{R})^+ \backslash \mathcal{L}_l(\mathbb{A})) \cong \\
& \qquad \operatorname{Ind}_{\overline{\mathcal{L}_l(\mathbb{Q})}}^{\mathcal{L}_l(\mathbb{A}_f)} S_{\rho - \tau_l + \log}(\mathcal{L}_l(\mathbb{Q})\mathcal{A}_{\mathcal{O}}(\mathbb{R})^+ \backslash \mathcal{L}_l(\mathbb{R}) \times \overline{\mathcal{L}_l(\mathbb{Q})}).
\end{aligned}
$$

On the level of $\mathcal{G}(\mathbb{A}_f)$-equivariant sheaves induction corresponds to sections over an $\mathcal{L}_l(\mathbb{A}_f)$-orbit. For $s = [(x,g)] \in \partial_{\mathcal{O}} \operatorname{Sh}^*$ let

$$
\mathfrak{O}_s = s.g^{-1}\mathcal{L}_l(\mathbb{A}_f)g
$$

be the orbit of s under right action of $g^{-1}\mathcal{L}_l(\mathbb{A}_f)g$. By reduction theory the quotient $\overline{\mathcal{L}_l(\mathbb{Q})} \backslash \mathcal{L}_l(\mathbb{A}_f)$ is compact, the orbit map

$$
\begin{aligned}
\overline{\mathcal{L}(\mathbb{Q})}_l \backslash \overline{\mathcal{L}(\mathbb{Q})}_l \mathcal{L}_l(\mathbb{A}_f) &\to \mathfrak{O}_s \\
l &\mapsto s.(g^{-1}lg)
\end{aligned}
$$

is a homeomorphism and $\mathfrak{O}_s$ is compact. It is in fact just a profinite set. Recall that for a sheaf $\mathscr{F}$ on $\partial_{\mathcal{O}} \operatorname{Sh}^*$ one defines its sections over $\mathfrak{O}_s$ as the direct limit

$$
\mathscr{F}(\mathfrak{O}_s) = \operatorname*{colim}_{\mathfrak{O}_s \subseteq U} \mathscr{F}(U)
$$

where U runs through the set of neighborhoods of $\mathfrak{O}_s$. If $\mathscr{F}$ is a $\mathcal{G}(\mathbb{A}_f)$-equivariant sheaf then $\mathscr{F}(\mathfrak{O}_s)$ is an admissible $\mathcal{L}_l(\mathbb{A}_f)$-module because $\mathfrak{O}_s$

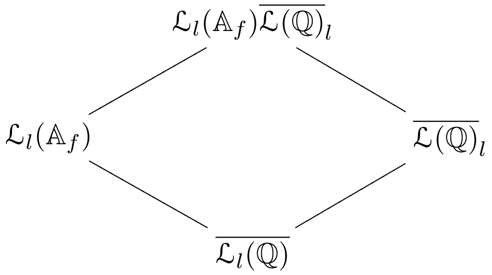
is compact. It is not difficult to see that the restriction of section over $\mathfrak{O}_s$ to $\{s\}$

$$S^{\mathfrak{n}}_{\rho+?}(\mathfrak{O}_s) \to S^{\mathfrak{n}}_{\rho+?,s}$$

is $\overline{\mathcal{L}(\mathbb{Q})}_l$-equivariant and induces a canonical isomorphism
(2.55)

$$H^{\bullet}_{(\mathfrak{l},K_{\mathcal{O}})}(S^{\mathfrak{n}}_{\rho+?}(\mathfrak{O}_s)\otimes H^{\bullet}_{\mathfrak{n}}(E)) \cong \operatorname{Ind}^{\overline{\mathcal{L}(\mathbb{Q})}_l\mathcal{L}_l(\mathbb{A}_f)}_{\overline{\mathcal{L}(\mathbb{Q})}_l} H^{\bullet}_{(\mathfrak{l},K_{\mathcal{O}})}(S^{\mathfrak{n}}_{\rho+?,s}\otimes H^{\bullet}_{\mathfrak{n}}(E)).$$

Induction in the stages along the diagram



and collecting our results obtained so far immediately gives.

2.56 Theorem: *Let $\mathfrak{O}_s \subseteq \partial_{\mathcal{O}}\operatorname{Sh}^*$ be the $\mathcal{L}_l(\mathbb{A}_f)$-orbit of a point $s\in\partial_{\mathcal{O}}\operatorname{Sh}^*$. Let $\tau\in\check{\mathfrak{a}}^{\mathcal{G}}_0$. For irreducible $\mathcal{L}(\mathbb{C})$-submodule $F\subseteq H^l_{\mathfrak{n}}(E)$ let $\lambda_{F,\mathcal{O}}-\rho_{\mathcal{O}}\in\check{\mathfrak{a}}_{\mathcal{O}}$ be the character of $\mathcal{A}$ on F. Let $\rho_{-\tau_F}$ be the weight function associated to*

$$\tau_F = -\sum_{\alpha\in\Delta_0-\Delta_0^{\mathcal{O}}}\langle\lambda_{F,\mathcal{O}}+\tau,\check{\varpi}_\alpha\rangle\alpha|_{\mathfrak{a}^{\mathcal{O}}_{0,l}}\in\check{\mathfrak{a}}^{\mathcal{O}}_{0,l}.$$

Then there is a canonical $\mathcal{L}_l(\mathbb{A}_f)\overline{\mathcal{L}(\mathbb{Q})}_l$-equivariant isomorphism

$$\text{(2.57)}\quad \overline{H}^{\bullet}_{(\mathfrak{m}_{\mathcal{G}},K)}(S_{\rho_{-\tau}\pm\log}(\mathfrak{O}_s)\otimes E)\cong$$
$$\bigoplus_F \operatorname{Ind}^{\mathcal{L}_l(\mathbb{A}_f)\overline{\mathcal{L}(\mathbb{Q})}_l}_{\mathcal{L}_l(\mathbb{A}_f)} H^{\bullet}_{(\mathfrak{m}_{\mathcal{L}_l},K_l)}(S_{\rho_{-\tau_F}\pm\log}(\mathcal{L}_l(\mathbb{Q})\mathcal{A}_{\mathcal{O}}(\mathbb{R})^+\backslash\mathcal{L}_l(\mathbb{A}_f))\otimes F)[-l]$$

where F runs through all irreducible submodules of $H^l_{\mathfrak{n}}(E)$ such that $(\lambda_{F,\mathcal{O}}+\tau)|_{\mathfrak{a}_{\mathcal{O}}}\in\overline{{}^+\check{\mathfrak{a}}_{\mathcal{O}}}$ in the $(+\log)$-case and $(\lambda_{F,\mathcal{O}}+\tau)|_{\mathfrak{a}_{\mathcal{O}}}\in{}^+\check{\mathfrak{a}}_{\mathcal{O}}$ in the $(-\log)$-case respectively.

2.58 Corollary: *The local cohomology group at $s\in\partial_{\mathcal{O}}\operatorname{Sh}^*$*

$$\overline{H}^p_{(\mathfrak{g},K)}(S_{\rho_{-\tau}\pm\log,s}\otimes E)$$

vanishes (is an admissible $\overline{\mathcal{L}(\mathbb{Q})}_l$-module) precisely if

$$H^{p-l}_{(\mathfrak{m}_{\mathcal{L}_l},K_l)}(S_{\rho_{-\tau_F}\pm\log}(\mathcal{L}_{\mathcal{O},l}(\mathbb{Q})\mathcal{A}_{\mathcal{O}}(\mathbb{R})^+\backslash\mathcal{L}_{\mathcal{O},l}(\mathbb{A}_f))\otimes F)$$

vanishes (is an admissible $\mathcal{L}_l(\mathbb{A}_f)$-module) for all $F\subseteq H^l_{\mathfrak{n}}(E)$ satisfying the conditions in Theorem 2.56.

PROOF: This follows from the fact that a module smoothly induced from an admissible non-zero representation is non-zero. This is easy to see (with the notation of [Car79] §1.8): For $K \subseteq G$ compact open and small enough there is a non-zero vector $v \in V^{H \cap K}$. Now define $f\colon G \to V$ by $f(g) = hv$ for $g = hk \in HK$ and zero otherwise. Then $f \in \mathrm{Ind}_H^G(V)$ and $f(e) = v \neq 0$. Similarly a $\mathcal{L}_l(\mathbb{A}_f)\overline{\mathcal{L}(\mathbb{Q})}_l$-equivariant sheaf on $\mathfrak{O}_s$ is zero if its global section vanish since this is true for sheaves on a finite Hausdorff set. □

2.8 The Eisenstein Spectral Sequence for the Link

To compute the right hand side of (2.57) we apply the main results of [Fra98] to the group $\mathcal{L}_l$ and a fixed irreducible $\mathcal{L}(\mathbb{C})$-submodule $F \subseteq H^\bullet_{\mathfrak{n}}(E)$. The fact that the restriction of F to $\mathcal{L}_l$ is a multiple of an irreducible representation of $\mathcal{L}_l(\mathbb{C})$ doesn't cause any trouble. Let us briefly recall some notation. For unexplained notation we refer to [Fra98].

Let $\mathcal{P}$ be a standard $\mathbb{Q}$-parabolic subgroup of $\mathcal{L}_l$, $\{\mathcal{P}\}$ its associated class and $k \geq 0$ a non-negative integer. Let $U^k_{\{\mathcal{P}\}}$ be the set of triples $(\mathcal{R}, \widetilde{\Lambda}, \chi)$ with the following properties:

(1) $\mathcal{R} = \mathcal{M}_{\mathcal{R}}\mathcal{A}_{\mathcal{R}}\mathcal{N}_{\mathcal{R}}$ is a standard parabolic subgroup containing an element of $\mathcal{P}$ and $\mathrm{rk}(\{\mathcal{P}\}) = \mathrm{rk}(\mathcal{R}) + k$.

(2) $\widetilde{\Lambda}\colon \mathcal{A}_{\mathcal{R}}(\mathbb{Q})\mathcal{A}_{\mathcal{R}}(\mathbb{R})^+\backslash\mathcal{A}_{\mathcal{R}}(\mathbb{A}) \to U(1)$ is a continuous character.

(3) $\chi\colon \mathfrak{Z}(\mathfrak{m}_{\mathcal{R},\mathbb{C}}) \to \mathbb{C}$ is a unitary character of the center of the universal enveloping algebra of $\mathfrak{m}_{\mathcal{R}}$. Recall that χ is called unitary if

$$\chi(D^*) = \overline{\chi(D)}$$

for all $D \in \mathfrak{Z}(\mathfrak{m}_{\mathcal{R},\mathbb{C}})$. Here D^* denote the adjoint operator, i.e.

$$(cX_1 \cdots X_k)^* = (-1)^k \cdot \bar{c} \cdot \overline{X}_k \cdots \overline{X}_1$$

where the bar denotes complex conjugation with respect to $\mathbb{R} \subset \mathbb{C}$ or $\mathfrak{m}_{\mathcal{R}} \subseteq \mathfrak{m}_{\mathcal{R},\mathbb{C}}$ respectively.

We associate to any such triple $u = (\mathcal{R}, \widetilde{\Lambda}, \chi)$ the space $V(u)$ of square integrable $\mathbb{K} \cap \mathcal{R}(\mathbb{A})$-finite functions

$$\mathcal{R}(\mathbb{Q})\mathcal{N}_{\mathcal{R}}(\mathbb{A})\mathcal{A}_{\mathcal{R}}(\mathbb{R})^+\backslash\mathcal{R}(\mathbb{A}) \to \mathbb{C}$$

with the following properties:

(i) For every rational standard parabolic $\mathcal{Q} \subseteq \mathcal{R}$ not belonging to $\{\mathcal{P}\}$, the constant term

$$f_{\mathcal{N}_{\mathcal{Q}}}(.) = \int_{\mathcal{N}_{\mathcal{Q}}(\mathbb{Q})\backslash\mathcal{N}_{\mathcal{Q}}(\mathbb{A})} f(n.)dn$$

of f along $\mathcal{N}_{\mathcal{Q}}$ is orthogonal to the space of cusp forms on $\mathcal{M}_{\mathcal{Q}}(\mathbb{Q})\backslash\mathcal{M}_{\mathcal{Q}}(\mathbb{A})$

(ii) $f(ag) = \Lambda(a)f(g)$ for all $a \in \mathcal{A}_{\mathcal{R}}(\mathbb{A})$.

(iii) f lies in the χ-eigenspace of $\mathfrak{Z}(\mathfrak{m}_{\mathcal{R},\mathbb{C}})$.

2.59 Lemma: The subspace of functions in $V(u)$ having some fixed $\mathbb{K}\cap\mathcal{R}(\mathbb{A})$-type is a finite dimensional.

Proof: The space of $\mathbb{K}^f \cap \mathcal{R}(\mathbb{A}_f)$-invariant functions in $V(u)$ for some fixed u and $\mathbb{K}^f$ decomposes into a finite direct sum of irreducible unitary $(\mathfrak{m}_{\mathcal{R}}, K_{\mathcal{R}})$-modules. Indeed any function $f \in V(u)$ is by (iii), K-finiteness and elliptic regularity is automatically real analytic. Hence f generates some irreducible unitary $(\mathfrak{m}_{\mathcal{R}}, K_{\mathcal{R}})$-submodule of

$$L_{2,\text{discrete}}(\mathcal{L}_{\mathcal{R}}(\mathbb{Q})\mathcal{A}_{\mathcal{R}}(\mathbb{R})^+\backslash\mathcal{L}_{\mathcal{R}}(\mathbb{A}))^{\mathbb{K}^f\cap\mathcal{R}(\mathbb{A}_f)}$$

by [Wal88], Proposition 1.6.6. But the latter is known to decompose into a direct sum of irreducible unitary $(\mathfrak{m}_{\mathcal{R}}, K_{\mathcal{R}})$-modules with finite multiplicities. By [Wal88], Theorem 5.5.6, there are only finitely many irreducible unitary $(\mathfrak{m}_{\mathcal{R}}, K_{\mathcal{R}})$-modules having infinitesimal character χ and the claim follows. □

Let $W(u)$ be the space of $\mathbb{K}\cap\mathcal{L}_l(\mathbb{A})$-finite functions

$$f\colon \mathcal{R}(\mathbb{Q})\mathcal{N}_{\mathcal{R}}(\mathbb{A})\mathcal{A}_{\mathcal{R}}(\mathbb{R})^+\backslash\mathcal{O}(\mathbb{A}) \to \mathbb{C}$$

such that for every $k \in \mathbb{K}\cap\mathcal{L}_l(\mathbb{A})$, the function

$$\begin{aligned}\mathcal{R}(\mathbb{Q})\mathcal{A}_{\mathcal{R}}(\mathbb{R})^+\backslash\mathcal{R}(\mathbb{A}) &\to \mathbb{C}\\ r &\mapsto f(rk)\end{aligned}$$

lies in $V(u)$.

Let $\mathscr{J}_F \subseteq \mathfrak{Z}(\mathfrak{m}_{\mathcal{L}_l})$ be the annihilator of $F^\vee$ in the center of the universal enveloping algebra of $\mathfrak{m}_{\mathcal{L}_l}$. It is an ideal of finite codimension. Let $\mathfrak{h} \supseteq \mathfrak{a}_0^{\mathcal{G}}$ be a Cartan subalgebra of $\mathfrak{g}$.

Let $M^k_{\mathscr{J}_F,\{\mathcal{P}\}}$ the set of triples $t = (\mathcal{R}, \Lambda, \chi)$ such that (1) holds and

(2)' $\Lambda\colon \mathcal{A}_{\mathcal{R}}(\mathbb{Q})\mathcal{A}_{\mathcal{L}_l}(\mathbb{R})^+\backslash\mathcal{A}_{\mathcal{R}}(\mathbb{A}) \to \mathbb{C}^\times$ is a continuous character. If $\lambda_t \in (\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{L}_l})_{\mathbb{C}}$ is the differential of the archimedean component of Λ. We assume that its real part $\Re(\lambda_t)$ is contained in $\overline{\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{L}_l+}}$ and that

$$\lambda_t \in \operatorname{supp}_{u_t} \mathscr{J}_F = \{\mu \in (\check{\mathfrak{a}}_{\mathcal{R}}^{\mathcal{L}_l})_{\mathbb{C}} | \gamma(\mathscr{J}_F)(\mu + \chi_t) = \{0\}\}$$

where

$$\gamma\colon \mathfrak{Z}(\mathfrak{m}_{\mathcal{L}_l,\mathbb{C}}) \to S(\mathfrak{h}\cap\mathfrak{m}_{\mathcal{L}_l,\mathbb{C}})^{W(\mathfrak{m}_{\mathcal{L}_l,\mathbb{C}},\mathfrak{h}\cap\mathfrak{m}_{\mathcal{L}_l,\mathbb{C}})}$$

is Harish-Chandra's isomorphism as in [Wal88], Theorem 3.2.2.

(3)' If $\widetilde{\Lambda}(a) = \Lambda(a)e^{-\langle\lambda_t, H(a)\rangle}$ then $u_t = (\mathcal{R}, \widetilde{\Lambda}, \chi) \in U^k_{\{\mathcal{P}\}}$.

For $\tau_F \in \overline{\check{\mathfrak{a}}_0^{\mathcal{L}_l+}}$ let $M^k_{\mathscr{J}_F,\{\mathcal{P}\},\tau_F,+} \subseteq M^k_{\mathscr{J}_F,\{\mathcal{P}\}}$ be the subset consisting of those triples for which

$$\Re(\lambda_t) \in \tau_F - \overline{{}^+\check{\mathfrak{a}}_0^{\mathcal{L}_l}}. \tag{2.60}$$

For $t = (\mathcal{R}, \Lambda, \chi) \in M^k_{\mathscr{J}_F,\{\mathcal{P}\},\tau_F,+}$ let $M(t)$ be the $(\mathfrak{l}_l, K_l, \mathcal{L}_l(\mathbb{A}_f))$-modules

$$M(t) = W(u_t) \otimes D_t = \mathrm{Ind}_{\mathcal{R}(\mathbb{A})}^{\mathcal{L}_l(\mathbb{A})} V(u_t) \otimes D_t$$

where D_t is the symmetric algebra $S((\mathfrak{a}_{\mathcal{R}}^{\mathcal{L}_l})_{\mathbb{C}})$. These spaces can be given the structure of $(\mathfrak{l}_l, K_l, \mathcal{L}_l(\mathbb{A}_f))$-modules.

Let

$$\mathfrak{Fin}_{\mathscr{J}_F}(S_{\rho-\tau_F+\log}(\mathcal{L}_l(\mathbb{Q})\mathcal{A}(\mathbb{R})^+\backslash\mathcal{L}_l(\mathbb{A})))_{\{\mathcal{P}\}}, \tag{2.61}$$

the subspace of functions in $S_{\rho-\tau_F\pm\log}(\mathcal{L}_l(\mathbb{Q})\mathcal{A}(\mathbb{R})^+\backslash\mathcal{L}_l(\mathbb{A})))_{\{\mathcal{P}\}}$ that are annihilated by some power of $\mathscr{J}_F$. Note that (2.61) is a space of automorphic forms on $\mathcal{L}_l(\mathbb{A})$. By considering the constant term of f along standard parabolics $\mathcal{P} \subseteq \mathcal{L}_l$ Franke defines a filtration of finite length on the spaces in (2.61). It depends on the choice of a certain integer valued functions $T \colon \overline{\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{L}_l+}} \to \mathbb{Z}$. The filtration steps are denoted by

$$\mathfrak{Fin}_{\mathscr{J}_F}(S_{\rho-\tau_F+\log}(\mathcal{L}_l(\mathbb{Q})\mathcal{A}(\mathbb{R})^+\backslash\mathcal{L}_l(\mathbb{A})))^{T,i}_{\{\mathcal{P}\}} \tag{2.62}$$

where $i \in \mathbb{Z}$. For $i \in \mathbb{Z}$ let

$$M^{k,T,i}_{\mathscr{J}_F,\{\mathcal{P}\},\tau_F,+}$$

be the set of those elements of $M^k_{\mathscr{J}_F,\{\mathcal{P}\},\tau_F,+}$ such that $T(\Re(\lambda_t)_+) = i$ where we let for $\mu \in \check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{L}_l}$ denote μ_+ the unique point in the closed convex set $\overline{\check{\mathfrak{a}}_{\mathcal{P}}^{\mathcal{L}_l+}}$ with minimal distance to μ. The set $M^{k,T,i}_{\mathscr{J}_F,\{\mathcal{P}\},\tau_F,+}$ can be turned into a groupoid with certain Weyl sets as morphism sets in such a way that the Eisenstein transform factorizes over its colimit encoding the various functional equations satisfied by Eisenstein series.

2.63 Theorem ([Fra98], Theorem 14)**:** *If $\tau_F \in \overline{\check{\mathfrak{a}}_0^{\mathcal{L}_l+}}$ there is an isomorphism*

$$\begin{aligned} \bigoplus_{k=0}^{\mathrm{rk}(\{\mathcal{P}\})} \operatorname*{colim}_{M^{k,T,i}_{\mathscr{J}_F,\{\mathcal{P}\},\tau_F,+}} M(t) &\cong \\ &\cong \mathrm{Gr}^i\left(\mathfrak{Fin}_{\mathscr{J}_F}(S_{\rho-\tau_F+\log}(\mathcal{L}_l(\mathbb{Q})\mathcal{A}_{\mathcal{L}_l}(\mathbb{R})^+\backslash\mathcal{L}_l(\mathbb{A})))^{T,\bullet}_{\{\mathcal{P}\}}\right) \end{aligned} \tag{2.64}$$

of $(\mathfrak{l}_l, K_l, \mathcal{L}_l(\mathbb{A}_f))$-modules.

The existence of the filtration implies that there is a convergent spectral sequence associated to the filtered $(\mathfrak{l}_l, K_l, \mathcal{L}_l(\mathbb{A}_f))$-module (2.62) with E_1 term

$$\begin{aligned} E_1^{p,q} = \bigoplus_{k=0}^{\mathrm{rk}(\{\mathcal{P}\})} \operatorname*{colim}_{M^{k,T,i}_{\mathscr{J}_F,\{\mathcal{P}\},\tau_F,+}} H^{p+q}_{(\mathfrak{m}_l,K_l)}(M(t) \otimes F) \\ \Rightarrow H^{p+q}_{(\mathfrak{m}_l,K_l)}(\mathfrak{Fin}_{\mathscr{J}_F}(S_{\rho-\tau_F+\log}(\mathcal{L}_l(\mathbb{Q})\mathcal{A}_{\mathcal{L}_l}(\mathbb{R})^+\backslash\mathcal{L}_l(\mathbb{A})))_{\{\mathcal{P}\}} \otimes F). \end{aligned} \tag{2.65}$$

Under the assumptions $\tau_F \in \overline{\check{\mathfrak{a}}_0^{\mathcal{L}_l+}}$ as in [Fra98],Theorem 2.63, the higher right derived functors of $\mathfrak{Fin}_{\mathscr{J}_F}$ vanish on the modules $S_{\rho-\tau_F \pm \log}(\mathcal{L}_l(\mathbb{Q})\mathcal{A}_{\mathcal{L}_l}(\mathbb{R})^+\backslash\mathcal{L}_l(\mathbb{A}))$ by Theorem 16 in the same paper. In this case there is by ibid., Theorem 7(4), a canonical isomorphism

$$
(2.66)\quad H^{p+q}_{(\mathfrak{m}_l,K_l)}(\mathfrak{Fin}_{\mathscr{J}_F}(S_{\rho-\tau_F\pm\log}(\mathcal{L}_l(\mathbb{Q})\mathcal{A}_{\mathcal{L}_l}(\mathbb{R})^+\backslash\mathcal{L}_l(\mathbb{A})))_{\{\mathcal{P}\}}\otimes F)\cong H^{p+q}_{(\mathfrak{m}_l,K_l)}(S_{\rho-\tau_F\pm\log}(\mathcal{L}_l(\mathbb{Q})\mathcal{A}_{\mathcal{L}_l}(\mathbb{R})^+\backslash\mathcal{L}_l(\mathbb{A})))_{\{\mathcal{P}\}}\otimes F).
$$

Now consider τ_F with F satisfying $\lambda_{F,\mathcal{O}}+\tau\in\overline{{}^+\check{\mathfrak{a}}_{\mathcal{O}}^{\mathcal{G}}}$ as in Theorem 2.52. Combining (2.66), (2.65) and summing over the associated classes of parabolic subgroups $\{\mathcal{P}\}$ we see that (2.65) computes the summand corrsesponding to $F\subset H^{\bullet}_{\mathfrak{n}_{\mathcal{O}}}(E)$ in (2.57) provided that $\tau_F\in\check{\mathfrak{a}}_0^{\mathcal{L}_l}$ lies in $\overline{\check{\mathfrak{a}}_0^{\mathcal{L}_l+}}$. Before we state the Theorem let us prove this and let us review Kostant's Theorem. Clearly $\langle\alpha,\beta\rangle\le 0$ for all $\alpha\in\Delta_0-\Delta_0^{\mathcal{O}}$ and $\beta\in\Delta_0^{\mathcal{O}}$ and consequently

$$
\langle\tau_F,\beta\rangle=-\sum_{\alpha\in\Delta_0-\Delta_0^{\mathcal{O}}}\langle\lambda_{F,\mathcal{O}}+\tau,\check{\varpi}_\alpha\rangle\langle\alpha,\beta\rangle\ge 0
$$

for all $\beta\in\Delta_0^{\mathcal{O}}$ since we only consider F's with $\lambda_{F,\mathcal{O}}+\tau\in\overline{{}^+\check{\mathfrak{a}}_{\mathcal{O}}^{\mathcal{G}}}$.

For a parabolic $\mathbb{Q}$-subgroup $\mathcal{R}\subseteq\mathcal{L}_l$ let $\widetilde{\mathcal{R}}\subseteq\mathcal{O}$ be the unique parabolic $\mathbb{Q}$-subgroup of $\mathcal{O}$ with $\widetilde{\mathcal{R}}/\mathcal{N}_{\mathcal{O}}=\mathcal{R}\widetilde{\mathcal{L}}_h$. There is a canonical isomorphism

$$
(2.67)\qquad H^r_{\mathfrak{n}_{\widetilde{\mathcal{R}}}}(E)=\bigoplus_{p+q=r}H^p_{\mathfrak{n}_{\widetilde{\mathcal{R}}}/\mathfrak{n}_{\mathcal{O}}}(H^r_{\mathfrak{n}_{\mathcal{O}}}(E)),
$$

for any parabolic $\mathbb{Q}$-subgroup $\mathcal{R}\subseteq\mathcal{L}_l$, see [Sch94], §4.10. Write $E_{\mathcal{G},\Lambda}$ of the representation of $\mathcal{G}(\mathbb{C})$ with highest weight $\Lambda\in\overline{\check{\mathfrak{h}}^+}$. Let us assume that $E=E_{\mathcal{G},\Lambda}$ has highest weight $\Lambda\in\overline{\check{\mathfrak{h}}^+}$. By Kostant's theorem on $\mathfrak{n}$-cohomology, [Wal88] Theorem 9.6.2, the LHS of (2.67) can be computed as a sum:

$$
(2.68)\qquad H^r_{\mathfrak{n}_{\widetilde{\mathcal{R}}}}(E)=\bigoplus_w E_{\mathcal{L}_{\widetilde{R}},w(\Lambda+\rho_{\mathfrak{h}})+\rho_{\mathfrak{h}}}
$$

where w runs over all elements of length $l(w)=r$ in the Weyl group $W(\mathfrak{h}_{\mathbb{C}},\mathfrak{g}_{\mathbb{C}})$ of $\mathfrak{h}\subseteq\mathfrak{g}$ with the property that $w^{-1}\alpha>0$ for all positive roots of $\mathfrak{h}$ in $\mathfrak{l}_{\widetilde{\mathcal{R}},\mathbb{C}}$. Let us write τ_w instead of τ_F if $F=E_{\mathcal{L}_{\widetilde{R}},w(\Lambda+\rho_{\mathfrak{h}})+\rho_{\mathfrak{h}}}$ and similarly $\mathscr{J}_w$ instead of $\mathscr{J}_F$.

2.69 Theorem: *Assume that $\tau\in\overline{\check{\mathfrak{a}}_0^{\mathcal{G}+}}$ and let $\Lambda\in\overline{\check{\mathfrak{h}}^+}$ be the highest weight of E. With the notations as in Theorem 2.56 there is a spectral sequence of $\mathcal{L}_l(\mathbb{A}_f)\overline{\mathcal{L}(\mathbb{Q})}_l$-modules converging to*

$$
H^{p+q}_{(\mathfrak{m}_{\mathcal{G}},K)}(\mathscr{S}_{\rho-\tau+\log}(\mathfrak{O}_s)\otimes E).
$$

Its E_1-term is

$$\bigoplus_{\{\mathcal{P}\}} \bigoplus_{k=0}^{\mathrm{rk}(\{\mathcal{P}\})} \bigoplus_{w} \operatorname*{colim}_{t \in M^{k,T,i}_{\mathscr{J}_w,\{\mathcal{P}\},\tau_w,+}} \tag{2.70}$$
$$\mathrm{Ind}_{\mathcal{R}_t(\mathbb{A}_f)}^{\mathcal{L}_l(\mathbb{A}_f)\overline{\mathcal{L}(\mathbb{Q})}_l} \left(H^{p+q-l(w)}_{(\mathfrak{m}_{\mathcal{R}_t}, K\cap\mathcal{R}_t(\mathbb{R}))}(V(u_t) \otimes E_{\mathcal{L},w(\Lambda+\rho_{\mathfrak{h}})-\rho_{\mathfrak{h}}}) \otimes \mathbb{C}_{-\lambda_t-\rho_{\mathcal{R}_t}} \right)$$

where $w \in W(\mathfrak{h}_{\mathbb{C}}, \mathfrak{m}_{\mathcal{G},\mathbb{C}})$ is an element satisfying

(1) $w^{-1}\alpha$ *is positive for all positive roots α appearing in $\mathfrak{l}_{\tilde{\mathcal{R}}_t,\mathbb{C}}$,*

(2)
$$\Re(w(\Lambda+\rho_{\mathfrak{h}})|_{\mathfrak{a}_{\tilde{\mathcal{R}}}} \in \overline{\check{\mathfrak{a}}^{\mathcal{O}+}_{\tilde{\mathcal{R}}_t}} + \check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{O}}$$

(3)
$$\lambda_t = -w(\Lambda+\rho_{\mathfrak{h}})|_{\check{\mathfrak{a}}^{\mathcal{O}}_{\tilde{\mathcal{R}}}} \text{ and } \Re(w(\Lambda+\rho_{\mathfrak{h}})|_{\mathfrak{a}_{\mathcal{O}}}) \in \overline{{}^+\check{\mathfrak{a}}^{\mathcal{O}}_0}.$$

2.71 Remark: If Λ is regular, the spectral sequence degenerates at the E_1-term by [Fra98], Theorem 19. By the same Theorem, the objects of the groupoids $M^{k,T,i}_{\mathscr{J},\{\mathcal{P}\},\tau,\pm}$ don't have non-trivial automorphisms. Hence the colimit becomes isomorphic to a direct sum once representatives for the isomorphism classes of $M^{k,T,i}_{\mathscr{J},\{\mathcal{P}\},\tau,\pm}$ are chosen.

3 Theorem of Loojenga, Saper and Stern

3.1 Statement of Zucker's "Conjecture"

Let us briefly recall the definition of middle perversity intersection cohomology of the stratified space Sh^*. Let

$$\mathscr{I}^\bullet(\mathbb{E})$$

be a bounded complex of $\mathcal{G}(\mathbb{A}_f)$-equivariant sheaves on Sh^* with the following properties:

(1) The cohomology sheaves $\mathscr{H}^p(\mathscr{I}^\bullet(\mathbb{E}))$ vanish for $p < 0$.

(2) There is a $\mathcal{G}(\mathbb{A}_f)$-equivariant quasi-isomorphism of $\mathbb{E}$ with $\mathscr{I}^\bullet(\mathbb{E})|_{\mathrm{Sh}}$.

(3) The cohomology sheaves $\mathscr{H}^\bullet(\mathscr{J}^\bullet(\mathbb{E}))$ are weakly constructible with respect to the stratification of Sh^* by its boundary components, i.e.

$$\mathscr{H}^\bullet(\mathscr{I}^\bullet(\mathbb{E}))|_{\partial_{\mathcal{O}}\mathrm{Sh}^*}$$

is a graded $\mathcal{G}(\mathbb{A}_f)$-equivariant locally constant sheaf of vector spaces on $\partial_{\mathcal{O}}\mathrm{Sh}^*$.

(4) The stalk of the locally constant $\mathcal{G}(\mathbb{A}_f)$-equivariant sheaves in (3) at a point $s \in \partial_{\mathcal{O}}\mathrm{Sh}^*$ are admissible modules under the stabilizer $\mathbb{Z}_s$.

(5) If

$$\mathscr{D}_{\mathrm{Sh}^*}(\mathscr{I}^\bullet(\mathbb{E}))^\bullet$$

denotes the Verdier dual complex of $\mathscr{I}^\bullet(\mathbb{E})$ there is a $\mathcal{G}(\mathbb{A}_f)$-equivariant quasi-isomorphism

$$\mathscr{D}_{\mathrm{Sh}^*}(\mathscr{I}^\bullet(\mathbb{E}))^\bullet \cong \mathscr{I}^\bullet(\mathbb{E}^\vee).$$

(6) If $s \in \partial_{\mathcal{O}}\mathrm{Sh}^*$ then

$$\mathscr{H}^p(\mathscr{I}^\bullet(\mathbb{E}))_s = 0$$

for $p \geq \frac{1}{2}\operatorname{codim}_{\mathbb{R}}(\partial_{\mathcal{O}}\mathrm{Sh}^*)$.

It is known that the properties (1)-(6) characterize $\mathscr{I}^\bullet(\mathbb{E})$ uniquely up to quasi-isomorphism. It follows that the intersection cohomology with coefficients in $\mathbb{E}$

$$IH^\bullet(\mathrm{Sh}^*, \mathbb{E}) := \mathbb{H}^\bullet(\mathrm{Sh}^*, \mathscr{I}^\bullet(\mathbb{E}))$$

is well-defined.

(7) If $\mathscr{I}^\bullet(\mathbb{E})$ consists of fine sheaves, the hypercohomology spectral sequence degenerates to an isomorphism

$$IH^\bullet(\mathrm{Sh}^*, \mathbb{E}) \cong H^\bullet(\mathscr{I}^\bullet(\mathbb{E})(\mathrm{Sh}^*)).$$

Set

$$\mathscr{A}^\bullet_{(2)}(\mathbb{E}) := C^\bullet_{(\mathfrak{m}_{\mathcal{G}}, K)}(\mathscr{S}_{\rho_0} \otimes E)(\zeta_E).$$

3.1 Theorem: *There is a quasi-isomorphism*

$$\mathscr{I}^\bullet(\mathbb{E}) \cong \mathscr{A}^\bullet_{(2)}(\mathbb{E}). \tag{3.2}$$

In particular

$$IH^\bullet(\mathrm{Sh}^*, \mathbb{E}) \cong H^\bullet_{(\mathfrak{m}_\mathcal{G}, K)}(S_{\rho_0}(\mathrm{Sh}) \otimes E)(\zeta_E). \tag{3.3}$$

as $\mathcal{G}(\mathbb{A}_f)$*-modules.*

Theorem 3.1 is our version of the Theorem of Loojenga and Saper-Stern the of which will take up the rest of this section. If the properties (1)-(7) are known (3.3) follows from (2.10). Hence it suffices to check properties (1)-(7) for $\mathscr{A}^\bullet_{(2)}(\mathbb{E})$.

3.2 First Part of the Proof - Verdier Duality

Proof (of 3.1, properties (1)-(4) and (7)):

(1) is obvious for $\mathscr{A}^\bullet_{(2)}(\mathbb{E})$.

(2) follows from the twisted Poincaré Lemma and the obvious isomorphism

$$\mathscr{A}^\bullet_{(2)}(\mathbb{E})|_{\mathrm{Sh}} \cong \mathscr{A}^\bullet(\mathbb{E})|_{\mathrm{Sh}}.$$

(3) is immediate from Theorem 2.42

(4) follows from Theorem 2.69 and Lemma 2.59.

(7) is Proposition 2.20. □

To prove (5) let us more generally prove

3.4 Theorem: *Let* $\mathbb{E}^\vee$ *be the automorphic local system associated to the dual representation of* E*. There is an isomorphism*

$$\mathscr{D}_{\mathrm{Sh}^*}(C^\bullet_{(\mathfrak{m}_\mathcal{G}, K)}(\mathscr{S}_{\rho \pm \log} \otimes E))^\bullet \cong C^\bullet_{(\mathfrak{m}_\mathcal{G}, K)}(\mathscr{S}_{\rho^{-1} \mp \log} \otimes E^\vee)$$

in the derived category of $\mathcal{G}(\mathbb{A}_f)$*-equivariant sheaves on* Sh^**. Here* $\mathscr{D}_{\mathrm{Sh}^*}$ *denotes the Verdier duality functor.*

3.5 Lemma: Let $D^\bullet$ be a bounded complex of ultrabornological spaces each possessing a web of type $\mathscr{C}$, i.e. such that the closed graph theorem of De Wilde holds (see [Obe82], Folgerung 6.1.4). Assume that cohomology spaces are finite dimensional. Then the natural inclusion

$$D^{\bullet,\prime} \subseteq D^{\bullet,\vee}$$

of the topological dual into the algebraic dual is a quasi-isomorphism.

Proof: We prove this by induction on the length n of the interval in which $C^\bullet$ is non-zero. If $n = 0$ there is nothing to prove. Let $n > 0$. After translation we may assume that $C^p = 0$ for $p < 0$. The sequence

$$0 \to H^0(C^\bullet)[0] \to C^\bullet \to C^\bullet / H^0(C^\bullet)[0] \to 0$$

is topologically exact since $H^0(C^\bullet)$ is finite dimensional. The algebraic and topological dualizing functors are exact. The exactness of the latter follows from the Hahn-Banach Theorem. Since

$$H^0(C^\bullet)$$

is finite dimensional by assumption the canonical inclusion

$$H^0(C^\bullet)' \subseteq H^0(C^\bullet)^\vee$$

is an isomorphism. Hence we may replace $C^\bullet$ by $C^\bullet/H^0(C^\bullet)$ and assume that $H^0(C^\bullet) = 0$. Since $\mathrm{Im}(d^0)$ is of finite codimension in $\mathrm{Ker}(d^1)$ it is a closed subspace of C^1. Hence $\mathrm{Im}(d^0)$ possesses a web of type $\mathscr{C}$. By De Wildes closed graph theorem the canonical map

$$d^0 \colon C^0 \to \mathrm{Im}(d^0) \subseteq C^1$$

is an isomorphism and identifies C^0 with a closed subspace of C^1. If we let $D^\bullet$ be the complex

$$C^0 \overset{\mathrm{id}_{C^0}}{\to} C^0 \to 0 \to 0 \to \dots$$

there is a sequence

$$0 \to D^\bullet \overset{(\mathrm{id}_{C^0}, d^0)}{\to} C^\bullet \to \tau_{\geq 1} C^\bullet \to 0$$

of locally convex spaces. We have seen that it is topologically exact. By induction the assertion of the Lemma is true for $\tau_{\geq 1}C^\bullet$. It is trivially true for $D^\bullet$. Functoriality of the long exact sequence shows the Lemma for $C^\bullet$. $\square$

Proof (of 3.4): It suffices to show

$$\mathscr{D}_{\mathrm{Sh}^*} C^\bullet_{(\mathfrak{m}_\mathcal{G}, K)}(\mathscr{S}_{\rho \pm \log} \otimes E) \cong C^\bullet_{(\mathfrak{m}_\mathcal{G}, K)}(\mathscr{S}_{\rho^{-1} \mp \log} \otimes E^\vee).$$

Since $C^\bullet_{(\mathfrak{m}_\mathcal{G}, K)}(\mathscr{S}^\bullet_{\rho \pm \log} \otimes E)$ is a complex of fine sheaves the Verdier dual complex is

$$\mathscr{D}_{\mathrm{Sh}^*} C^\bullet_{(\mathfrak{m}_\mathcal{G}, K)}(\mathscr{S}_{\rho \pm \log} \otimes E) = C^{\dim_\mathbb{R} \mathrm{Sh}^* - \bullet}_{(\mathfrak{m}_\mathcal{G}, K)}(\Gamma_c(U, \mathscr{S}_{\rho \pm \log}) \otimes E))^\vee.$$

By definition of the locally convex topology $\Gamma_c(U, \mathscr{S}_{\rho \pm \log})$ is an ultrabornological space. It possesses a web of type $\mathscr{C}$ in the sense of De Wilde by [Obe82], Satz 6.2.7. Consequently the Lemma applies to

$$C^\bullet = C^\bullet_{(\mathfrak{m}_\mathcal{G}, K)}(\Gamma_c(U, \mathscr{S}_{\rho \pm \log}) \otimes E)$$

provided that the cohomology spaces are finite dimensional. We have seen that the cohomology sheaves $\mathscr{H}^\bullet(\mathscr{A}_{(2)+?}(\mathbb{E}))$ are constructible on any Satake compactification. Hence every point of Sh^* has a neighborhood base consisting of open sets $U \in \mathfrak{B}^*$ such that the cohomology groups of $C^\bullet$ are finite dimensional (after passage to a $\mathbb{K}^f$-invariant subspace). It follows from Lemma 3.5 that

$$C^{\dim_{\mathbb{R}} \mathrm{Sh}^* - \bullet}_{(\mathfrak{m}_{\mathcal{G}},K)}(\Gamma_c(U, \mathscr{S}_{\rho\pm\log}) \otimes E))' \subset C^{\dim_{\mathbb{R}} \mathrm{Sh}^* - \bullet}_{(\mathfrak{m}_{\mathcal{G}},K)}(\Gamma_c(U, \mathscr{S}_{\rho\pm\log}) \otimes E))^\vee$$

is a quasi-isomorphism for such U. Using the perfect pairing

$$\Lambda^p(\mathfrak{m}_{\mathcal{G},\mathbb{C}}/\mathfrak{k}_{\mathbb{C}})^\vee \otimes \Lambda^{\dim_{\mathbb{R}} \mathrm{Sh}^* - p}(\mathfrak{m}_{\mathcal{G},\mathbb{C}}/\mathfrak{k}_{\mathbb{C}})^\vee \to \Lambda^{\dim_{\mathbb{R}} \mathrm{Sh}^*}(\mathfrak{m}_{\mathcal{G},\mathbb{C}}/\mathfrak{k}_{\mathbb{C}})^\vee \cong \mathbb{C}$$

there is an isomorphism

$$C^{\dim_{\mathbb{R}} \mathrm{Sh}^* - \bullet}_{(\mathfrak{m}_{\mathcal{G}},K)}(\Gamma_c(U, \mathscr{S}_{\rho\pm\log}) \otimes E))' \cong C^{\bullet}_{(\mathfrak{m}_{\mathcal{G}},K)}(\Gamma_c(U, \mathscr{S}_{\rho\pm\log})' \otimes E^\vee))$$

By Theorem 2.19 the inclusion

$$C^{\bullet}_{(\mathfrak{m}_{\mathcal{G}},K)}(\mathscr{S}_{\rho^{-1}\mp\log} \otimes E^\vee)) \subset C^{\bullet}_{(\mathfrak{m}_{\mathcal{G}},K)}(\Gamma_c(U, \mathscr{S}_{\rho\pm\log})' \otimes E^\vee))$$

is a quasi-isomorphism. □

3.6 Corollary: *If* Sh* *is an equal-rank Satake compactification there is a quasi-isomorphism*

$$\mathscr{D}_{\mathrm{Sh}^*}\mathscr{A}^\bullet_{(2)}(\mathbb{E}) \cong \mathscr{A}^\bullet_{(2)}(\mathbb{E}^\vee).$$

Proof: This is immediate from Theorem 3.4 and Theorem 2.47. □

3.3 Second Part of the Proof - Estimates

Choose a fundamental θ-stable Cartan subalgebra $\mathfrak{b} \subseteq \mathfrak{m}_{\widetilde{\mathcal{R}}}$ and extend it to a θ-stable Cartan subalgebra $\mathfrak{h} = \mathfrak{a}_{\widetilde{\mathcal{R}}} \oplus \mathfrak{b}$ of $\mathfrak{g}$. Fix a θ-stable positive system $\Phi(\mathfrak{m}_{\widetilde{\mathcal{R}},\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})^+$ of roots. Set

$$\Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})^+ = \Phi(\mathfrak{m}_{\widetilde{\mathcal{R}},\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})^+ \cup \Phi(\mathfrak{n}_{\widetilde{\mathcal{R}},\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}).$$

The aim of this section is to prove the following:

3.7 Theorem: *Let $F \subseteq H^p_{\mathfrak{n}_{\widetilde{\mathcal{R}}}}(E)$ be an irreducible $\mathcal{L}_{\widetilde{\mathcal{R}}}(\mathbb{C})$ submodule in degree p. Assume that the lowest weight of F is of the form $\tilde{\lambda} + \rho_{\mathfrak{h}}$ with*

$$\lambda = \tilde{\lambda}|_{\mathfrak{a}_{\widetilde{\mathcal{R}}}} \in (-\overline{{}^+\check{\mathfrak{a}}_{\widetilde{\mathcal{R}}}}) \cap (\overline{\check{\mathfrak{a}}^{\mathcal{O}+}_{\widetilde{\mathcal{R}}}} + \check{\mathfrak{a}}_{\mathcal{O}}) \tag{3.8}$$

and that there exists an irreducible unitary $(\mathfrak{m}_{\mathcal{R}}, K_{\mathcal{R}})$-module V such that

$$H^q_{(\mathfrak{m}_{\mathcal{R}},K_{\mathcal{R}})}(V \otimes F) \neq \{0\}.$$

If Sh* *is the Satake-Baily-Borel-compactification of a hermitian locally symmetric space then*

$$p + q + \mathrm{prk}_{\mathbb{Q}}(\widetilde{\mathcal{R}}) \leq \frac{1}{2}\operatorname{codim}_{\mathbb{R}}(\partial_{\mathcal{O}}\mathrm{Sh}^* \subseteq \mathrm{Sh}^*). \tag{3.9}$$

Property (7) is an immediate consequence of Theorem 3.7 since it shows that the E_1-term of the spectral sequence in Theorem 2.69 vanishes if the total degree is large enough.

A similar Proposition is proved in [SS90], Proposition 11.1, for λ in the cone

$$\{\mu | \langle \mu, \check{\alpha} \rangle \leq 0 \text{ for all } \alpha \in \Phi(\mathfrak{a}_{\tilde{\mathcal{R}}}, \mathfrak{n}_{\mathcal{O}})\} \tag{3.10}$$

and the proposition will be proved if we can show that the cone described by (3.8) is contained in the one described by (3.10).

Recall that $l(F)$ equals the number of positive roots $\alpha \in \Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ such that $w^{-1}\alpha$ is negative. Since $\Lambda + \rho_{\mathfrak{h}}$ is regular and dominant the condition $w^{-1}\check{\alpha} < 0$ is equivalent to

$$\langle \tilde{\lambda}, \check{\alpha} \rangle = -\langle \Lambda + \rho_{\mathfrak{h}}, w^{-1}\check{\alpha} \rangle > 0.$$

Recall that in Kostant's theorem only Weyl group elements w are considered with the property that $w^{-1}\alpha$ is positive for any positive root appearing in $\mathfrak{l}_{\tilde{\mathcal{R}},\mathbb{C}}$. Using this fact we may compute $l(F)$ as

$$l(F) = \text{ number of roots } \alpha \in \Phi(\mathfrak{n}_{\tilde{\mathcal{R}},\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \text{ such that } \langle \alpha, \tilde{\lambda} \rangle > 0 \tag{3.11}$$

or in other words the number of root hyperplanes separating $\tilde{\lambda}$ from the negative Weyl chamber in $\mathfrak{h}$. We also write $l(\tilde{\lambda})$ for $l(F)$.

The Cartan subalgebra $\mathfrak{h}$ decomposes under θ as

$$\mathfrak{b} = \mathfrak{b}_{\mathfrak{s}} \oplus \mathfrak{b}_{\mathfrak{k}}$$

where $\mathfrak{b}_{\mathfrak{s}} = \mathfrak{b} \cap \mathfrak{s}$ and $\mathfrak{b}_{\mathfrak{k}} = \mathfrak{b} \cap \mathfrak{k}$. Note that $\mathfrak{b}_{\mathfrak{s}} \cap \mathfrak{m}_{\tilde{\mathcal{R}},h} = 0$ since $\mathfrak{b}$ is fundamental and $\mathfrak{m}_{\tilde{\mathcal{R}},h}$ equal-rank by assumption.

Recall that a θ-stable parabolic subalgebra of $\mathfrak{m}_{\mathcal{R}}$ is a parabolic subalgebra $\mathfrak{q} \subseteq \mathfrak{m}_{\mathcal{R},\mathbb{C}}$ such that

- $\theta\mathfrak{q} = \mathfrak{q}$ and
- $\mathfrak{l}_{\mathfrak{q},\mathbb{C}} = \mathfrak{q} \cap \overline{\mathfrak{q}}$ is a Levi subalgebra of $\mathfrak{q}$.

It is clear that $\mathfrak{l}_{\mathfrak{q},\mathbb{C}}$ is defined over $\mathbb{R}$. Write $\mathfrak{l}_{\mathfrak{q}} \subseteq \mathfrak{m}_{\mathcal{R}}$ for the set of its reals points. The Levi-group of $\mathfrak{q}$ is defined to be

$$L_{\mathfrak{q}} \subseteq \{g \in \mathcal{M}_{\mathcal{R}}(\mathbb{R}) | \operatorname{Ad}(g)(\mathfrak{q}) \subseteq \mathfrak{q}\} \subseteq \mathcal{M}_{\mathcal{R}}(\mathbb{R}).$$

Note that θ-stable parabolic subalgebras of $\mathfrak{m}_{\mathcal{R}}$ are only defined over $\mathbb{C}$ and that not every parabolic subalgebra of $\mathfrak{m}_{\mathcal{R},\mathbb{C}}$ that satisfies $\theta\mathfrak{q} = \mathfrak{q}$ is θ-stable. After conjugation by an element of $K_{\mathcal{R}}$ we may assume that $\mathfrak{b} \subset \mathfrak{q}$ and $\mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}}$ is a standard parabolic subalgebra of $\mathfrak{k}_{\mathbb{C}} \cap \mathfrak{o}_{\mathbb{C}}$ with respect to some of positive compact roots.

Conversely, any θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{m}_{\mathcal{R}}$ such that $\mathfrak{q}\cap\mathfrak{k}_{\mathbb{C}} \subseteq \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{o}_{\mathbb{C}}$ is standard is of the form

$$\mathfrak{q}(\nu) = \mathfrak{b}_{\mathbb{C}} \oplus \bigoplus_{\langle \nu,\check{\alpha}\rangle \geq 0} \mathfrak{m}_{\mathcal{R},\mathbb{C},\alpha}$$

where $\nu \in i\check{\mathfrak{b}}_{\mathfrak{k}}$ is some dominant character and α is positive with respect to the standard Borel subalgebra of $\mathfrak{m}_{\mathcal{R},\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}$. In this case

$$L(\nu) = L_{\mathfrak{q}(\nu)} \subseteq \{g \in \mathcal{M}_{\mathcal{R}}(\mathbb{R}) |\, \mathrm{Ad}(g)(\nu) \subseteq \nu\}.$$

Associated to $\mathfrak{q}(\nu)$ and some $\mu \in i\check{\mathfrak{b}}$ inducing a unitary character of $L_{\mathfrak{q}}$ one associates an irreducible unitary $(\mathfrak{m}_{\mathcal{R}}, K_{\mathcal{R}})$-module $A_{\mathfrak{q}}(\mu)$, c.f. [KV95]. These modules interpolate between the discrete series representations (if existent) corresponding to $A_{\mathfrak{q}}(\mu)$ for $\mathfrak{q}$ a θ-stable Borel subalgebra and the trivial representation $A_{\mathfrak{m}_{\mathcal{R},\mathbb{C}}}(0)$. If

$$H^{\bullet}_{(\mathfrak{m}_{\mathcal{R}},K_{\mathcal{R}})}(V \otimes F) \neq \{0\}$$

then there exists by [Wal88], Theorem 9.7.1., a θ-stable parabolic subalgebra of the form $\mathfrak{q}(\nu)$ of $\mathfrak{m}_{\mathcal{R}}$ such that $F/\mathfrak{n}_{\mathfrak{q}}F =: \mathbb{C}_{\mu}$ is a unitary character μ of $L(\nu)$ and such that V is unitarily equivalent to $A_{\mathfrak{q}(\nu)}(\mu)$. Hence we may assume $V = A_{\mathfrak{q}(\nu)}(\mu)$ for our purpose. By [Wal88], Theorem 9.6.6, the $(\mathfrak{m}_{\mathcal{R}}, K_{\mathcal{R}})$-cohomology of the module $A_{\mathfrak{q}}(\mu)$ can be non-zero only in degrees less or equal than

$$m(F,V) = \frac{1}{2}(\dim_{\mathbb{R}}(\mathfrak{m}_{\mathcal{R}} \cap \mathfrak{s}) + \dim_{\mathbb{R}}(\mathfrak{l}_{\mathfrak{q}} \cap \mathfrak{s})). \tag{3.12}$$

This number is independent of the nilradical of $\mathfrak{q}$ and consequently we may assume $\mathfrak{q}$ to be standard if we are only interested in the vanishing assertion. Since $F/\mathfrak{n}(\nu)F$ is one-dimensional $\mu = \tilde{\lambda} + \rho_{\mathfrak{h}}$ is the lowest weight of F. By [BC83], §1, $(\tilde{\lambda} + \rho_{\mathfrak{h}})|_{\mathfrak{b}} \in i\check{\mathfrak{b}}_{\mathfrak{k}}$ and in particular

$$\theta((\tilde{\lambda} + \rho_{\mathfrak{h}})|_{\mathfrak{b}}) = (\tilde{\lambda} + \rho_{\mathfrak{h}})|_{\mathfrak{b}} \tag{3.13}$$

where we write θ for the induced involution on the dual $\check{\mathfrak{b}}$ of the θ-stable Cartan subalgebra.

3.14 Lemma: Let $\lambda \in \check{\mathfrak{a}}_{\tilde{\mathcal{R}}}$ then

$$\lambda \in (-\overline{{}^{+}\check{\mathfrak{a}}^{\mathcal{G}}_{\tilde{\mathcal{R}}}}) \cap (\overline{\check{\mathfrak{a}}^{\mathcal{O}+}_{\tilde{\mathcal{R}}}} + \check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{O}})$$

implies that

$$\langle \lambda, \check{\alpha}\rangle \leq 0$$

for all roots $\alpha \in \Phi(\mathfrak{n}_{\mathcal{O}}, \mathfrak{a}_{\mathcal{R}})$.

PROOF: The $\mathbb{Q}$-root system $\Phi(\mathfrak{g}, \mathfrak{a}_0)$ is of type C_l or BC_l where l is the $\mathbb{Q}$-rank of $\mathcal{G}$. It follows that the restricted root system $\Phi(\mathfrak{g}, \mathfrak{a}_{\widetilde{\mathcal{R}}})$ is of type C_r or BC_r where r is the $\mathbb{Q}$-parabolic rank of $\widetilde{\mathcal{R}}$. Let $\beta_1, \ldots, \beta_r \in \check{\mathfrak{a}}_{\widetilde{\mathcal{R}}}$ such that

$$\Phi(\mathfrak{n}^{\mathcal{O}}_{\widetilde{\mathcal{R}}}, \mathfrak{a}_{\widetilde{\mathcal{R}}}) = \{\frac{1}{2}(\beta_i - \beta_j) | i < j\}$$

and

$$\Phi(\mathfrak{n}_{\mathcal{O}}, \mathfrak{a}_{\widetilde{\mathcal{R}}}) = \begin{cases} \{\frac{1}{2}(\beta_i + \beta_j) | i \leq j\} & , \quad C_r \\ \{\frac{1}{2}(\beta_i + \beta_j) | i \leq j\} \cup \{\frac{1}{2}\beta_i\} & , \quad BC_r \end{cases}$$

in the case BC_r. This description follows from [BB66], Proposition 2.9 and its proof. The condition $\lambda \in (-\overline{{}^{+}\check{\mathfrak{a}}_{\widetilde{\mathcal{R}}}})$ is equivalent to

$$\langle \lambda, \check{\beta}_1 \rangle, \ \langle \lambda, \check{\beta}_1 + \check{\beta}_2 \rangle, \ldots, \ \langle \lambda, \check{\beta}_1 + \cdots + \check{\beta}_r \rangle \leq 0$$

while the condition $\lambda \in \overline{\check{\mathfrak{a}}^{\mathcal{O}+}_{\widetilde{\mathcal{R}}}} + \check{\mathfrak{a}}^{\mathcal{G}}_{\mathcal{O}}$ is equivalent to

$$\langle \lambda, \check{\beta}_i \rangle \leq \langle \lambda, \check{\beta}_j \rangle \text{ for every } i > j.$$

It follows that

$$\langle \lambda, \check{\beta}_i \rangle \leq \frac{1}{i}(\langle \lambda, \check{\beta}_1 \rangle + \cdots + \langle \lambda, \check{\beta}_i \rangle) = \frac{1}{i}\langle \lambda, \check{\beta}_1 + \cdots + \check{\beta}_i \rangle \leq 0$$

for $i = 1, \ldots, r$. This implies the Lemma since any $\alpha \in \Phi(\mathfrak{n}_{\mathcal{O}}, \mathfrak{a}_{\widetilde{\mathcal{R}}})$ is contained in the positive cone spanned by the β_i. $\square$.

PROOF (of 3.7): As explained above, the proposition will follow from [SS90], Proposition 11.1, in view of Lemma 3.14. It remains to see how. As a first step let us rewrite the RHS of (3.9). The decomposition

$$\mathfrak{g} = \mathfrak{l}_{\widetilde{\mathcal{R}}} \oplus \widetilde{\mathfrak{l}}_h \oplus \mathfrak{n}_{\widetilde{\mathcal{R}}} \oplus \theta\mathfrak{n}_{\widetilde{\mathcal{R}}}$$

implies

$$\text{(3.15)} \qquad \begin{aligned} \operatorname{codim}_{\mathbb{R}}(\partial_{\mathcal{O}} \operatorname{Sh}^* \subseteq \operatorname{Sh}^*) &= \dim(\mathfrak{g} \cap \mathfrak{s}) - \dim(\widetilde{\mathfrak{l}}_h \cap \mathfrak{s}) \\ &= \dim(\mathfrak{l}_{\widetilde{\mathcal{R}}} \cap \mathfrak{s}) + \dim(\mathfrak{n}_{\widetilde{\mathcal{R}}}). \end{aligned}$$

As in [SS90] let $\Phi^+_{\mathfrak{s}} \subseteq \Phi(\mathfrak{m}_{\mathcal{R},\mathbb{C}}, \mathfrak{h}_{l,\mathbb{C}})$, $\mathfrak{h}_{l,\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{m}_{\mathcal{R},\mathbb{C}}$, be a subset containing all positive imaginary roots and exactly one root from each pair $\{\alpha, \theta\alpha\}$ of positive complex roots. If

$$A^- = \Phi^+_{\mathfrak{s}} \cap \Phi(\mathfrak{l}_{\mathfrak{q}}, \mathfrak{h}_{l,\mathbb{C}})$$

then

$$\langle \tilde{\lambda} + \rho_{\mathfrak{h}}, \check{\alpha} \rangle = 0$$

for all $\alpha \in A^-$. Lemma 3.14 guarantees condition (i) of Proposition 11.1 in [SS90] with

$$\beta = -\tilde{\lambda} - \rho_{\mathfrak{h}} + \frac{1}{2} \sum_{\alpha \in \Phi(\mathfrak{n}_{\widetilde{\mathcal{R}},\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})} \alpha$$

since

$$\rho_{\mathfrak{h}}|_{\mathfrak{a}_{\widetilde{\mathcal{R}}}} = \rho_{\widetilde{\mathcal{R}}} = \frac{1}{2} \sum_{\alpha \in \Phi(\mathfrak{n}_{\widetilde{\mathcal{R}},\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})} \alpha|_{\mathfrak{a}_{\widetilde{\mathcal{R}}}}.$$

Condition (ii) of the same Proposition 11.1 is satisfied since

$$\frac{1}{2} \sum_{\alpha \in \Phi(\mathfrak{n}_{\widetilde{\mathcal{R}},\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})} \alpha|_{\mathfrak{b}_{\mathfrak{s}}} = 0$$

and (3.13). Hence we may apply that Proposition and get

$$l(F) + |A^-| + \frac{1}{2} \dim(\mathfrak{h} \cap \mathfrak{s}) \leq \frac{1}{2} \dim \mathfrak{n}_{\widetilde{\mathcal{R}}}$$

or

$$l(F) + \frac{1}{2}(\dim \mathfrak{m}_{\mathcal{R}} \cap \mathfrak{s} + 2|A^-| + \dim(\mathfrak{b} \cap \mathfrak{s})) + \frac{1}{2} \dim \mathfrak{a}_{\widetilde{\mathcal{R}}} \leq \frac{1}{2}(\dim \mathfrak{m}_{\mathcal{R}} \cap \mathfrak{s} + \dim \mathfrak{n}_{\widetilde{\mathcal{R}}}).$$

Now using

$$\dim(\mathfrak{l}_{\mathfrak{q}} \cap \mathfrak{s}) = 2|A^-| + \dim(\mathfrak{b} \cap \mathfrak{s})$$

and adding $\frac{1}{2} \dim \mathfrak{a}_{\widetilde{\mathcal{R}}}$ on both sides we get

$$l(F) + m(F, V) + \dim \mathfrak{a}_{\widetilde{\mathcal{R}}} \leq \frac{1}{2}(\dim \mathfrak{l}_{\widetilde{\mathcal{R}}} \cap \mathfrak{s} + \dim \mathfrak{n}_{\widetilde{\mathcal{R}}}) = \frac{1}{2} \operatorname{codim}_{\mathbb{R}}(\partial_{\mathcal{O}} \operatorname{Sh}^* \subseteq \operatorname{Sh}^*)$$

by (3.12) and (3.15). □

Let us remark that it is not difficult to check (3.9) case by case. The reason is that the "extra space" $(\operatorname{prk}_{\mathbb{Q}}(\widetilde{\mathcal{R}}) - 1)$ in (3.9) allows one to get rid of the $\mathbb{Q}$-structure. More precisily it is sufficient to show an estimate similar to (3.9) for real parabolics and real boundary components of $\overline{\mathcal{X}}$. But there are only a few irreducible hermitian bounded symmetric domains. In the general equal-rank case the list expands and the calculations become a little more involved. The author has proved (3.9) all classical hermitian symmetric domains and some equal rank cases in this way.

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