CPPI Strategies in Discrete Time

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CPPI Strategies in Discrete Time

Abstract

In general, the purpose of portfolio insurance strategies is to limit the downside risk of risky portfolios. The constant proportion portfolio insurance (CPPI) is a prominent example of a portfolio insurance strategy. Based on a dynamic trading rule, the CPPI provides payoffs greater than some minimum wealth level at some specified time horizon. The great advantage of the CPPI is its particularly simple trading rule, which basically only requires the knowledge of the current portfolio value and thus makes the CPPI applicable to any kind of risky portfolio. Under the assumption of a complete financial market where trading takes place in continuous time, it is well known that the payoffs provided by the CPPI are greater than a pre-specified minimum wealth level with certainty. In this thesis we are concerned with various sources of market incompleteness. One source of market incompleteness are trading restrictions. Restricting the possibility of making changes to the portfolio to a fixed set of trading dates allows for payoffs below the minimum wealth level. The associated risk is called gap risk. The assumption of a fixed set of trading dates is well suited for the derivation of various risk-measures related to gap risk. Analyzing the gap risk is important with respect to the effectiveness of the CPPI if trading in continuous time is not possible. One natural reason for the assumption of trading restrictions are transaction costs. However, in the presence of transaction costs the frequency of monitoring the portfolio is generally larger than the willingness to rebalance the portfolio. With respect to transaction costs it is reasonable only to rebalance the portfolio upon relevant changes in the portfolio value or the underlying assets. This rationale leads to the notion of triggered trading dates. It turns out that triggered trading dates are also better suited with respect to analyzing modifications of the CPPI. The basic CPPI exhibits at least three structural problems. First, it requires the assumption of unlimited borrowing which can be explicitly modelled with the introduction of a borrowing constraint. Second, in the case of a good performance of the portfolio, it is well possible that the minimum wealth level becomes insignificant in comparison to the portfolio value. This can be modelled by increasing the minimum wealth level upon good performances of the portfolio. Third, the exposure to the underlying risky assets can become arbitrarily small such that portfolio may basically only consist of riskless assets. Explicitly defining a minimum on the exposure to the risky assets provides another modification. All modifications can be analyzed in a setup with triggered trading dates. While the use of triggered trading dates allows for the modelling of transaction costs also for the modifications of the CPPI, choosing small triggers allows for approximations of the continuous-time case for which analytic expressions for the modifications are not known in the literature so far either.

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Introduction

Financial strategies designed to limit downside risk and at the same time to profit from rising markets are summarized in the class of portfolio insurance strategies. Among others, Leland and Rubinstein (1976), Grossman and Vila (1989) as well as Basak (1995) define a portfolio insurance strategy as a trading strategy which guarantees a minimum level of wealth at a specified time horizon.[1] This definition has to be understood as a minimum requirement for a portfolio insurance strategy. Surely, strategies that require stronger conditions such as permanently keeping the portfolio value above some minimum level as in El Karoui, Jeanblanc, and Lacoste (2005) or keeping the portfolio value permanently above some stochastic minimum level as for example in Grossman and Zhou (1993) and Cvitanić and Karatzas (1995) are included in the definition of portfolio insurance strategies.

The optimality of an investment strategy depends on the risk profile of the investor. If the risk profile is given in the form of an utility function, in order to determine the optimal rule, one has to solve for the strategy which maximizes the expected utility. Approaches that model portfolio insurers as utility maximizers where the maximization problem includes an additional constraint for keeping the portfolio value above some certain (not necessarily constant or deterministic) level can be found for example in Cox and Huang (1989), Brennan and Schwartz (1989), Grossman and Vila (1989), Grossman and Zhou (1993, 1996), Basak (1995), Cvitanic and Karatzas (1995, 1999), Browne (1999), Tepla (2000, 2001). In a fairly general framework, El Karoui, Jeanblanc, and Lacoste (2005) show that the solution to the maximization problem, when the portfolio is to be kept above a certain constant level (permanently or only at some specified time), is given by the unconstrained solution with an additional put option written on the unconstrained solution. Unconstrained solution is to be understood in the sense of an optimal choice about the assets to invest in if the portfolio insurance constraint is ignored. The put

[1] An alternative definition can be found in Leland (1980) and Brennan and Schwartz (1989) who refer to the term portfolio insurance with respect to any strategy whose payoff at some specified time horizon is a convex function of the payoff of some reference portfolio.

option on this optimal choice then accounts for the satisfaction of the portfolio insurance constraint. This is what is often called *three fund separation*. It goes back to Cox and Huang (1989) and can be viewed as a generalization of the famous *two fund separation* by Markowitz (1952). Portfolio insurance strategies that employ options to satisfy the portfolio insurance condition are called option-based portfolio insurance (OBPI). The appropriate options do not need to be traded assets. It is well known that in a complete market model any claim is attainable such that options can be replicated by a self-financing dynamic trading strategy. However, there are various sources of market frictions such as borrowing constraints, short selling restrictions and transaction costs that cause a market to become incomplete. A market incompleteness may result in the relevant option of an OBPI not being attainable in the sense that it may not be possible to be replicated with a self-financing strategy. Consequently it is well possible that an optimal strategy in a complete market ceases to be optimal if a source of market incompleteness is introduced. Furthermore, solving the optimization problem in an incomplete market is usually quite complicated or even impossible. Another problem is posed by model risk. This is generated by the possible inconsistency between the unknown true model and the model the risk manager relies on in order to determine the optimal strategy. That is, one has to use some (educated) assumptions about the data-generating processes. However, strategies which are based on an optimality criterion with respect to some assumed model, fail to be optimal if the true model deviates from the assumed one.

In absence of an optimal solution due to the presence of market frictions, an alternative approach is to define a stylized trading rule that, if followed by the portfolio manager, satisfies the constraint of a portfolio insurance strategy. One of the most prominent of such strategies is the constant proportion portfolio insurance (CPPI). The CPPI was introduced by Black and Jones (1987) and Perold (1986). While the CPPI can be found to emerge as a special case of the maximization with HARA utility functions which goes back as far as Merton (1971), Black and Perold (1992) show the CPPI to be utility maximizing with respect to a piecewise HARA utility function if the only source of incompleteness is due to borrowing constraints. The properties of CPPI strategies in continuous time have been widely studied in the literature. Apart from the above mentioned Bookstaber and Langsam (2000) focus on the comparison of different portfolio insurance strategies with respect to path dependency. Black and Rouhani (1989) and Bertrand and Prigent (2002a) compare the properties of the CPPI and the OBPI. Several modifications of the CPPI are compared in a Monte Carlo study in Boulier and Kanniganti (1995). While Black and Perold (1992) further develop the properties of the CPPI strategy in continuous time based on a standard lognormal model, they also introduce discrete-time trading based on triggered trading dates and show how to include borrowing constraints and transaction costs. The analysis of the CPPI has also been conducted under alternative model assumptions, such as a stochastic volatility model and jump processes in Bertrand and Prigent (2003) and Bertrand and Prigent (2002b).

While it is an appealing feature, that the CPPI can be found to be utility maximizing under certain conditions, this is not the main intention with a stylized trading rule. The great advantage of the CPPI lies in an extremely simple trading rule and its flexibility. Consider, for example, a fund manager that has to keep a portfolio consisting of many different assets above a certain level. In the option based approach, the manager is required to either buy or replicate put options to insure the portfolio. While put options on the single assets in the portfolio might be available on the market, usually an appropriate option on the whole portfolio will not be. Also, insuring the portfolio with options on all single assets is likely to be too expensive, such that the manager might have no choice but to make assumptions about parameters such as the drift and volatility of the portfolio and determine a suitable replication strategy for the appropriate option. Surely, the success of the strategy hinges critically on the assumptions, such that these must frequently be checked and the strategy adapted if necessary. Furthermore, the manager might want to change the composition of the portfolio from time to time. Altering the composition of the portfolio will usually also lead to an altered appropriate option to insure the portfolio and therefore also to a different trading strategy. In contrast to this, in order to insure the portfolio with a CPPI strategy, basically all information needed is the current portfolio value. Surely, any projections about the future performance of a CPPI strategy critically hinge on the model assumptions as well. However, the strategy itself does not. It is this great simplicity, that has let the CPPI to become a frequently applied strategy among practitioners and caused the market to produce a large number of CPPI based products.

Clearly, since the CPPI is based on a stylized trading rule, it can only be optimal in the utility maximizing sense with respect to certain conditions on the market environment. Nevertheless, as an optimal strategy mostly is not available, it is important to investigate the performance of the CPPI with respect to satisfying the portfolio insurance condition as well as with respect to different performance measures under the consideration of various sources of market incompleteness. To a large extent, this is the focus of this work. In particular, the main source of market incompleteness will be caused by trading restrictions. Although the standard lognormal model along the lines of Black and Scholes (1973) is used for the underlying assets, trading will be restricted to discrete time. In chapter 1, which is strongly based on Balder, Brandl, and Mahayni (2009), it is assumed, that trading can only take place at a discrete set of fixed trading dates. While, if trading in continuous time is permitted, it can be shown, that the CPPI strategy always yields a portfolio value at a pre-specified future time that is greater than some minimum level, thus always satisfies the portfolio insurance constraint, this is not true any more if trading is restricted to a discrete set of fixed trading dates. Under discrete-time trading the potential losses in underlying assets and hence also in the portfolio value may be so large from one trading date to another, such that with the lowered portfolio value it is not possible any more to meet the portfolio insurance constraint with certainty, even if this was still possible at the previous trading date. This is what is commonly labelled gap risk or overnight risk. An accumulation of the gap risks yields the probability of the discrete CPPI not satisfying the portfolio insurance constraint, i.e. the shortfall probability. The shortfall probability is one of several risk-measures that are employed in order to investigate the effectiveness of the CPPI in discrete time with respect to keeping the portfolio insurance constraint, another risk-measure is the expected shortfall. However, also other properties of the discrete CPPI such as the moments and sensitivities with respect to the model parameters are provided. It is also shown, that the discrete CPPI converges to the continuous time version as the number of permitted trading dates turns to infinity.

In chapter 2 a different kind of discrete-time trading is employed. In contrast to the discrete set of fixed trading dates in chapter 1, in principle trading is permitted at any time. The fact that trading is permitted at any time does not mean that trading in continuous time is possible. A natural reason for the introduction of such a trading restriction are transaction costs. Based on the methodology of Black and Perold (1992), it is assumed that trading takes place upon changes in the underlying assets, i.e. the trading dates are assumed to be triggered. While choosing a certain number of fixed trading dates in chapter 2 can be viewed as a strategic decision, so can choosing the right triggers here. It turns out that triggered trading dates result in appealing properties of a discrete version of the CPPI based on these trading dates. For example, as a direct consequence of the construction, it also turns out that with the so-discretized CPPI the portfolio insurance constraint can be satisfied with certainty which means there is no gap risk. In addition, it is possible to find an analytic expression for the distribution of the discrete CPPI. There are several structural problems of the CPPI. One of these structural problems is the requirement of the assumption of unlimited borrowing. While first an analytical expression for the requirement of certain borrowing levels is derived, it is shown later, that the introduction of a borrowing limit changes the properties of the

CPPI considerably. The CPPI with borrowing constraints will be called a capped CPPI and important properties such as the distribution are derived and compared. It is also possible to introduce transaction costs without borrowing constraints or in addition to a borrowing limit.

The analysis in chapter 3 is also based on the methodology of triggered trading dates. However, while in the first two chapters the focus is on trading restrictions, the focus here is on modifying the CPPI. While a portfolio insurance that guarantees a minimum level of wealth at a specified future time might be appropriate for a short time horizon, for a long time horizon it is well possible for the portfolio value to increase to a level that makes the portfolio insurance insignificant in comparison. A modification of the CPPI that increases the level of portfolio insurance as the portfolio increases is proposed. The modification is quite similar to a TIPP strategy as proposed by Estep and Kritzman (1988). Although Grossman and Zhou (1993) and Cvitanić and Karatzas (1995) prove the optimality of the strategy with respect to a CRRA utility function, the properties of the strategy have hardly been analyzed. After the investigation of this modification another structural problem, the cash-lock, which is common to CPPI structures in general is tackled. The term cash-lock refers to a situation where the portfolio is completely invested into the riskless asset. While such a situation in a strict sense can only occur as a result of a fixed set of trading dates as in chapter 1 or as a result of jumps in the underlying assets, in a wider sense in can be used to describe situations where the investment into risky assets is very small. It turns out that the modified CPPI increases this problem. The problem is tackled by modifying the CPPI further such as to require a minimum fraction of the portfolio value to be invested into the risky assets. However, while such a condition clearly solves the cash-lock problem, it opens up the possibility of a violation of the portfolio insurance condition again.

Chapter 1

The Discrete CPPI with Fixed Trading Dates

A CPPI investor specifies two parameters, a constant *multiplier* and a minimum level of wealth at some future time, the *guarantee*. The present value of the guarantee is called the *floor*. Then the *exposure*, i.e. the amount which is invested in a risky asset, is determined by the product of the multiplier and the excess of the portfolio value over the floor. The excess of the portfolio value over the floor is called the *cushion* such that the exposure equals the product of the multiplier and the cushion. The remaining part, i.e. the difference of the portfolio value and the asset exposure is invested in a riskless asset. This implies that the strategy is self-financing. Self-financing means that funds are neither taken from nor added to the portfolio. The procedure is best explained on the basis of an example. Suppose, the portfolio value equals 1000, the multiplier is chosen to be equal to 6 and the portfolio insurance condition requires the portfolio value to be larger than 900 in one year, which reflects the guarantee. Assuming for simplicity that the riskfree interest rate equals 0%, then the floor is equal to the guarantee. Consequently, the cushion is 100 and the exposure is 600. If now the portfolio value decreases to a value of 950 due to a bad performance of the risky asset, the cushion drops to 50 and consequently the exposure drops to 300. Vice versa, if the portfolio value increases due to a good performance of the risky asset, the exposure increases as well. Hence, the CPPI is a pro-cyclical strategy. If the risky asset keeps decreasing, the exposure of the CPPI will approach zero at the same time such that the guarantee can still be met with the investment in the riskless asset. If the price process of the risky asset does not permit jumps, the continuous-time application of the CPPI ensures that the portfolio value does not fall below the floor. The strategy outperforms the prescribed floor unless there is a sudden drop in market prices such that the manager is not able to rebalance the portfolio adequately.

In this chapter, we assume that trading is restricted to a given set of fixed trading dates. Surely, from one trading date to another the risky asset can drop so much such as to yield a portfolio value below the floor and thus violate the portfolio insurance condition. We propose a discrete-time version of a simple CPPI strategy which satisfies three conditions. The strategy is self-financing, the asset exposure is non–negative and the value process converges. Assuming that the underlying price process is given by a geometric Brownian motion, trading restrictions in the sense of discrete-time trading are sufficient to model the possibility of a floor violation. The advantage of a model setup along the lines of Black and Scholes (1973) is that risk measures, such as the shortfall probability and the expected shortfall which are implied by the discrete-time CPPI method can be given in closed form. Once the risk measures are determined, the gap risk can be priced easily. However, the main focus is not the pricing. Instead, the relevant risk measures are used to discuss criteria which must be satisfied such that the CPPI strategy is still effective if applied in discrete time.[1] For example, it turns out that for a small number of rehedges, the shortfall probability, i.e. the probability that the strategy falls below the floor at the terminal date, may as well first increase in the trading frequency before it decreases. However, after a critical number of rehedges, the shortfall probability is always decreasing in the number of rehedges. The change in monotonicity can be interpreted in terms of a minimal number of rehedges which is necessary such that a portfolio protection can be achieved by applying the CPPI technique in discrete time. Obviously, the critical number of rehedges depends on the model parameters.

The outline of the chapter is as follows. Section 1.1 gives the model setup and reviews the structure and the properties of continuous-time CPPI strategies. A discrete-time version of a CPPI strategy where the asset exposure is restricted to be non-negative is defined in section 1.2. The properties of the discrete-time version are derived in analogy to the continuous-time version. The assumption that the asset price increments are independent and identically distributed yields a closed-form solution for the shortfall probability and the expected shortfall. The calculations are given in section 1.3 which also includes a sensitivity analysis of the risk measures with respect to model and strategy parameters. Section 1.4 illustrates the results and discusses criteria which ensure that the discrete-

[1] It is worth mentioning that while arbitrage free pricing is based on the expectation under the martingale measure, the risk measures must be determined with respect to the *real world measure*.

time strategy is effective, i.e. the portfolio protection is still valid in discrete time. In section 1.5 it will be shown, that the discrete-time version of the CPPI converges to the continuous-time version as the trading restrictions vanish. Section 1.6 concludes the paper.

1.1 Model Setup and the simple CPPI in continuous time

All stochastic processes are defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T^*]}, P)$ which satisfies the usual hypotheses. We consider two investment possibilities: a risky asset S and a riskless bond B which grows with constant interest rate r, i.e. $dB_t = B_t r\, dt$ where $B_0 = b$. The evolution of the risky asset S, a stock or benchmark index, is given by a geometric Brownian motion, i.e.

$$d\,S_t = S_t\,(\mu\, dt + \sigma\, dW_t)\,, \quad S_0 = s, \tag{1.1}$$

where $W = (W_t)_{0\le t\le T}$ denotes a standard Brownian motion with respect to the *real world* measure P and μ, σ are constants with $\mu > r \ge 0$ and $\sigma > 0$. A continuous–time investment strategy or saving plan for the interval $[0, T]$ can be represented by a predictable process $(\alpha_t)_{0\le t\le T}$ where α_t denotes the fraction of the portfolio value at time t which is invested in the risky asset S. If there are no additional borrowing restrictions, we can, w.l.o.g., restrict ourselves to strategies which are self–financing, i.e. strategies where money is neither injected nor withdrawn during the trading period $]0, T[$. Thus, the amount which is invested at date t in the riskless bond B is given in terms of the fraction $1-\alpha_t$. $V = (V_t)_{0\le t\le T}$ denotes the portfolio value process which is associated with the strategy α, i.e. V_t is the solution of

$$dV_t(\alpha) = V_t\left(\alpha_t\frac{dS_t}{S_t} + (1-\alpha_t)\frac{dB_t}{B_t}\right), \text{ where } V_0 = x. \tag{1.2}$$

Notice that there are alternative possibilities for portfolio insurance. Let T denote the terminal trading date. For example, one might think of T as the retirement day. The minimal wealth which must be obtained is denoted by G. The guaranteed amount is assumed to be less than the terminal value of a pure bond investment, i.e. we assume $G < e^{rT}V_0$. Besides a pure bond investment, a trivial possibility is given by a static trading strategy where at the initial time $t = 0$ the present value of the guarantee, i.e. Ge^{-rT} is invested in the bond B and the remaining part, i.e. the *surplus* $V_0 - e^{-rT}G$, is invested in the risky asset S. Thus, although $\alpha_t = \frac{(V_0 - e^{-rT}G)}{V_t}\frac{S_t}{S_0}$ is stochastic, the strategy is static in the sense that there are no rebalancing decisions involved during the interval $]0, T]$. Abstracting from stochastic interest rates, the above strategy honors the guarantee G independent of the stochastic process generating the asset prices. Another example of portfolio insurance is given by a stop–loss–strategy which is represented by a portfolio fraction $\alpha_t = 1_{\{V_t > e^{-r(T-t)}G\}}$. Here, everything is invested in the asset until the surplus (or cushion) $V_t - e^{-r(T-t)}G$ is exhausted. This means that the strategy is effective with respect to the guarantee if continuous–time monitoring (trading) is possible and the asset price process does not permit jumps. Together, the above strategies can be used to explain the basic idea of the constant proportion portfolio insurance. A combination of continuous–time monitoring and keeping the cushion under control yields the CPPI approach.

However, in a complete market there is a second possibility, the option based portfolio insurance approach. The completeness implies that there is a self–financing and duplicating strategy in S and B for any claim with payoff $h(S_T)$ at T. Notice that for $h(S_T) = \lambda\left(S_T + \left[\frac{G}{\lambda} - S_T\right]^+\right) = G + \lambda\left[S_T - \frac{G}{\lambda}\right]^+$ and $\lambda > 0$ it holds $h(S_T) \geq G$. Buying λ assets and λ put–options with strike $\frac{G}{\lambda}$ enables a portfolio insurance, too.[2] If the associated options are not traded, they must be synthesized by a hedging strategy in S and B. If the concept of perfect hedging is impeded by market incompleteness, the OBPI and the CPPI can both violate the purpose of portfolio insurance. In terms of model risk, i.e. the problem that one does not know which process can describe the true data generating process adequately, the OBPI approach causes more problems than the CPPI technique. The composition of the CPPI strategy is model independent. In contrast to this, it is necessary to incorporate a volatility guess in order to implement the OBPI approach with synthetic options. Thus, there is an additional error introduced by using the wrong hedging model.

In the following, we concentrate on the CPPI approach. It is worth mentioning that even without an utility based justification, the CPPI is an important strategy in practice.[3] We fix the notation and review the basic form and properties of continuous–time CPPI strategies. Recall that the basic idea of the CPPI approach is to invest the amount

[2]Or buying λ call options with strike $\frac{G}{\lambda}$ and a riskless investment of Ge^{-rT}.

[3]Besides the importance of CPPI strategies in the context of hedge funds, the CPPI technique has recently been extended to the credit derivatives market, c.f. Fletcher (2005). ABN Amro created the first credit CPPI product in April 2004. It is called Rente Booster.

of portfolio value which is above the present value of the guarantee in the risky asset S. Normally, the symbol F is used to denote the *floor*. The floor is defined by $F_t := e^{-r(T-t)}G$ and thus denotes the present value of the guarantee G. This is equivalent to

$$dF_t = F_t r\, dt \quad \text{with } F_0 = e^{-rT}G.$$

The surplus is called *cushion* and denoted by C, i.e. $C_t := V_t - F_t$. If the cushion is monitored in continuous time, it is even possible to invest a multiple of the cushion in the risky asset. Let m denote the multiplier, then the fraction α of a CPPI strategy is given by[4]

$$\alpha_t := \frac{mC_t}{V_t}.$$

Notice that there are various modifications of the CPPI, some of which will be considered in chapters 2 and 3. For this reason, we call a continuous-time CPPI strategy which satisfies the above form *simple*. Notice that a simple CPPI strategy is given in terms of the guarantee G and the multiplier $m \geq 1$. In addition to the protection feature, this ensures that the value of the CPPI strategy is convex in the asset price[5], at least in a continuous-time setup with continuous asset paths. Throughout this chapter, the guarantee is given exogenously, i.e. it is the minimal value of wealth which is needed at T. We review some basic properties of the continuous-time CPPI technique. First, consider the cushion process $(C_t^{cont})_{0\leq t\leq T}$. We use the notation C^{cont} for the cushion process in continuous time and likewise V^{cont} for the value process in continuous time in order to distinguish from several discrete-time cushion and value processes yet to be introduced.

Lemma 1.1.1

If the asset price dynamic is lognormal, i.e. if it satisfies equation (1.1), the cushion process $(C_t^{cont})_{0\leq t\leq T}$ of a simple CPPI is lognormal, too. In particular, it holds

$$dC_t^{cont} = C_t^{cont}\left((r + m(\mu - r)\, dt + \sigma m\, dW_t\right).$$

[4]For simplicity, we abstract from borrowing constraints in this chapter. Borrowing constraints are discussed in chapter 2. In the current framework, they could be modelled by $\alpha_t = \frac{\min\{m(V_t-F_t), pV_t\}}{V_t}$ with $p \geq 0$.

[5]Note that this property ensures that the CPPI is also a portfolio insurance strategy with respect to the definition of Leland (1980) and Brennan and Schwartz (1989).

PROOF: Notice that $C_t^{cont} = V_t^{cont} - F_t$ implies

$$\begin{aligned} dC_t^{cont} &= d\left(V_t^{cont} - F_t\right) \\ &= V_t^{cont}\left(\frac{mC_t^{cont}}{V_t^{cont}}\frac{dS_t}{S_t} + \left(1 - \frac{mC_t^{cont}}{V_t^{cont}}\right)\frac{dB_t}{B_t}\right) - F_t\frac{dB_t}{B_t} \\ &= C_t^{cont}\left(m\frac{dS_t}{S_t} - (m-1)r\,dt\right). \end{aligned}$$

The rest of the proof follows with equation (1.1). □

Proposition 1.1.2

The t–value of the a simple CPPI with parameter m and G is

$$V_t^{cont} = Ge^{-r(T-t)} + \frac{V_0 - Ge^{-rT}}{S_0^m}\exp\left\{\left(r - m\left(r - \frac{1}{2}\sigma^2\right) - m^2\frac{\sigma^2}{2}\right)t\right\}S_t^m.$$

PROOF: Notice that the assertion can also be rewritten as

$$V_t^{cont} = F_t + \frac{C_0}{S_0^m}\exp\left\{\left(r - m\left(r - \frac{1}{2}\sigma^2\right) - m^2\frac{\sigma^2}{2}\right)t\right\}S_t^m.$$

The proof of this equation is well-known, c.f. for example Bertrand and Prigent (2002a). Together with

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}$$

it follows that

$$C_t^{cont} = C_0 e^{\left(r + m(\mu - r) - \frac{1}{2}m^2\sigma^2\right)t + \sigma m W_t} \tag{1.3}$$

which matches the result of lemma 1.1.1. □

Proposition 1.1.2 illustrates the basic property of a simple CPPI. The t-value of the strategy consists of the present value of the guarantee G, i.e. the floor at t, and a non-negative part which is proportional to $\left(\frac{S_t}{S_0}\right)^m$. Thus, the value process of a simple CPPI strategy is path independent.[6] The payoff above the guarantee is linear for $m = 1$ and it is strictly convex for $m > 1$. In financial terms, the payoff of a CPPI strategy with $m > 1$ can be interpreted as a power claim. The portfolio protection is efficient with probability one, i.e. the terminal value of the strategy is higher than the guarantee with probability one. Notice that the lognormality of the asset price process implies the lognormality of the cushion process. Therefore, it is immediately clear that the strategy does not fall below the floor in all scenarios where the asset price dynamic is lognormal. Clearly, the assumption of lognormality is not necessary. In general, the CPPI in continuous time

[6] Notice that this is not true if one deviates from the concept of a simple CPPI.

Expectation and Standard Deviation of a simple CPPI

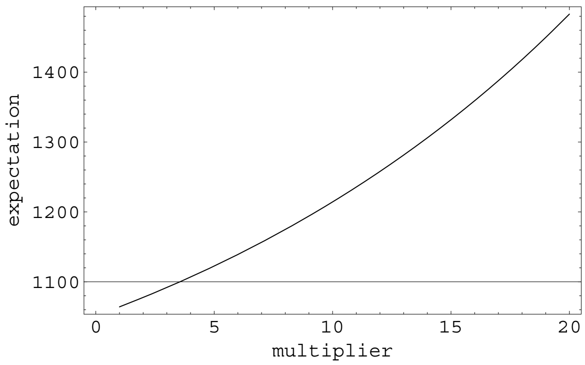

Figure 1.1: Expected terminal value of a simple CPPI with $V_0 = 1000$, $G = 800$, $T = 1$ and varying m for $\sigma = 0.1$, $\mu = 0.1$ and $r = 0.05$.

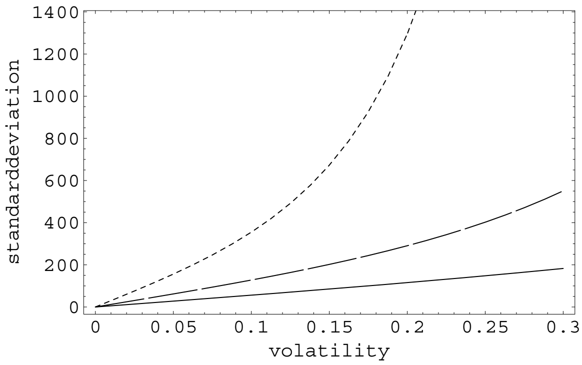

Figure 1.2: Standard deviation of the terminal value of a simple CPPI with $V_0 = 1000$, $G = 800$, $T = 1$ and varying σ for $\mu = 0.1$, $r = 0.05$ and $m = 2$ ($m = 4$, $m = 8$ respectively).

will satisfy the portfolio insurance condition whenever the sample-paths of the underlying risky asset are assumed to be continuous.

The expected value and the variance of a simple CPPI are easily calculated as follows.

Lemma 1.1.3

$$\begin{aligned} E\left[V_t^{cont}\right] &= F_t + \left(V_0 - Ge^{-rT}\right)\exp\left\{\left(r + m(\mu - r)\right)t\right\} \\ Var\left[V_t^{cont}\right] &= \left(V_0 - Ge^{-rT}\right)^2 \exp\left\{2\left(r + m(\mu - r)\right)t\right\}\ \left(\exp\left\{m^2\sigma^2 t\right\} - 1\right). \end{aligned}$$

Proof: With proposition 1.1.2 it follows

$$\begin{aligned} E\left[\ln\left(V_t^{cont} - F_t\right)\right] &= \ln C_0 + \left(r + m(\mu - r) - \frac{1}{2}m^2\sigma^2\right)t \\ Var\left[\ln\left(V_t^{cont} - F_t\right)\right] &= \sigma^2 m^2 t \end{aligned}$$

while it is well-known that for $X \sim N\left(\mu_X, \sigma_X\right)$ we have

$$E\left[e^X\right] = e^{\mu_X + \frac{1}{2}\sigma_X^2},\ \ Var\left[e^X\right] = e^{2\mu_X} e^{\sigma_X^2}\left(e^{\sigma_X^2} - 1\right).$$

□

It is worth mentioning that the expected terminal value of a simple CPPI strategy is independent of the volatility σ. In contrast, the standard deviation increases exponentially in the volatility of the asset S, c.f. figures 1.1 and 1.2. Intuitively, this property explains that the effectiveness of a CPPI strategy with respect to various sources of market incompleteness does not only depend on the asset price drift but even more importantly on the volatility of the underlying asset. In particular, this is the case for large values of the multiplier.

1.2 Trading restrictions

We assume now that trading is restricted to a discrete set of dates and define a discrete-time version of the simple CPPI strategy satisfying the following three conditions. Firstly, the value process of the discrete-time version converges in distribution to the value process of the simple continuous-time CPPI strategy. Secondly, the discrete-time version is a self-financing strategy. This means, that after the initial investment $V_0 = x$, there is no in- or outflow of funds. Thirdly, the strategy does not allow for a negative asset exposure. Notice that the first condition implies that the cushion process of the discrete-time version converges to a lognormal process in distribution. However, the cushion process with respect to a discrete-time set of trading dates may also be negative. Therefore, to avoid a negative asset exposure, this must be captured by the definition of the discrete–time version.

Let $\mathcal{T}^n$ denote a sequence of equidistant refinements of the interval $[0, T]$, i.e.

$$\mathcal{T}^n = \{t_0 = 0 < t_1 < \cdots < t_{n-1} < t_n = T\},$$

where $t_{k+1} - t_k = \frac{T}{n}$ for $k = 0, \cdots, n-1$. The restriction that trading is only possible immediately after $t_k \in \mathcal{T}^n$ implies that the number of shares held in the risky asset is constant on the intervals $]t_i, t_{i+1}]$ for $i = 0, \ldots, n-1$. However, the fractions of wealth which are invested in the assets change as asset prices fluctuate. Thus, it is necessary to consider the number of shares held in the risky asset η and the number of bonds β, i.e. the tupel $\phi = (\eta, \beta)$. With respect to the continuous-time simple CPPI strategies, it holds

$$\begin{aligned} \eta_t &= \frac{\alpha_t V_t^{cont}}{S_t} = \frac{m C_t^{cont}}{S_t}, \\ \beta_t &= \frac{(1-\alpha_t) V_t^{cont}}{B_t} = \frac{V_t^{cont} - m C_t^{cont}}{B_t}. \end{aligned}$$

The following argumentation illustrates that a time-discretized strategy $\phi^{\mathcal{T}^n}$ which is defined by

$$\phi_t^{\mathcal{T}^n} := \phi_{t_k} \text{ for } t \in]t_k, t_{k+1}],\ k = 0, \ldots, n-1$$

is in general not self-financing. The value process $V^{\mathcal{T}^n} := V(\phi; \mathcal{T}^n)$ which is associated with the discrete-time version of ϕ, i.e. with $\phi^{\mathcal{T}^n}$, is defined by $V_0^{\mathcal{T}^n} := V_0$ and

$$\begin{aligned} V_t(\phi; \mathcal{T}^n) &:= \eta_{t_k} S_t + \beta_{t_k} B_t \quad \text{for } t \in]t_k, t_{k+1}] \\ &= V_t(\phi) - (\eta_t - \eta_{t_k}) S_t - (\beta_t - \beta_{t_k}) B_t \quad \text{for } t \in]t_k, t_{k+1}], \end{aligned}$$

where

$$V_t(\phi) := \eta_t S_t + \beta_t B_t.$$

If ϕ is self-financing, this is not necessarily true for $\phi^{\mathcal{T}^n}$. Notice that $\phi^{\mathcal{T}^n}$ is self-financing iff

$$\begin{aligned} \eta_{t_k} S_{t_{k+1}} + \beta_{t_k} B_{t_{k+1}} &= \eta_{t_{k+1}} S_{t_{k+1}} + \beta_{t_{k+1}} B_{t_{k+1}} \quad \text{for all } k = 0, \ldots, n-1 \\ \Longleftrightarrow V_{t_{k+1}}(\phi; \mathcal{T}^n) &= V_{t_{k+1}}(\phi) \quad \text{for all } k = 0, \ldots, n-1. \end{aligned}$$

Obviously, this is only true in the limit, i.e. for $n \to \infty$. It is worth mentioning that it is not even clear whether the above time-discretized version is mean-self-financing with respect to the *real world measure*, c.f. for example Mahayni (2003). In order to specify a meaningful discrete-time version of a simple CPPI strategy, it is necessary to admit only self-financing strategies. This is equal to the condition that

$$\beta_t^{\mathcal{T}^n} = \frac{1}{B_{t_k}} \left(V_{t_k}^{\mathcal{T}^n} - \eta_t^{\mathcal{T}^n} S_{t_k} \right) \quad \text{for } t \in]t_k, t_{k+1}] \tag{1.4}$$

which is reflected in the following definition.

Definition 1.2.1 (Discrete–time CPPI)
A strategy $\phi^{\mathcal{T}^n} = \left(\eta^{\mathcal{T}^n}, \beta^{\mathcal{T}^n} \right)$ *where for* $t \in]t_k, t_{k+1}]$ *and* $k = 0, \ldots, n-1$

$$\begin{aligned} \eta_t^{\mathcal{T}^n} &:= \max \left\{ \frac{m(V_{t_k}^{\mathcal{T}^n} - F_{t_k})}{S_{t_k}}, 0 \right\} \\ \beta_t^{\mathcal{T}^n} &:= \frac{1}{B_{t_k}} \left(V_{t_k}^{\mathcal{T}^n} - \eta_t^{\mathcal{T}^n} S_{t_k} \right) \end{aligned}$$

is called simple discrete-time CPPI.

Recall that constant proportion portfolio insurance means that the fraction of wealth α which is invested in the risky asset is given proportionally to the difference of the portfolio value and the floor, i.e. the cushion. Note that this basic trading rule of the CPPI is immanent in definition 1.2.1. In addition, we do not allow for short positions in the risky asset, i.e. the asset exposure is bounded below by zero. This is achieved by considering the positive part of the cushion in definition 1.2.1. Also, the self-financing condition from equation 1.4 is reflected. In order to distinguish from the discretization with triggered trading dates, which is introduced in chapter 2, from now on we will denote the value process and the cushion process of a simple discrete-time CPPI with respect to a *fixed* set of trading dates as defined in definition 1.2.1 with V^{fi} and C^{fi}, respectively. The cushion process C^{fi} is defined by $C_t^{fi} := V_t^{fi} - F_t$.

Proposition 1.2.2 (Discrete–time cushion process)
Let $t_s := \min\left\{t_k \in \mathcal{T}^n | C_{t_k}^{fi} \leq 0\right\}$ *denote the first trading date at which the portfolio value process of the simple discrete-time CPPI is not strictly above the floor. Further set* $t_s = \infty$ *if the minimum is not attained. Then it holds*

$$C_{t_{k+1}}^{fi} = C_0 e^{r(t_{k+1} - \min\{t_s, t_{k+1}\})} \prod_{i=1}^{\min\{s,k+1\}} \left(m \frac{S_{t_i}}{S_{t_{i-1}}} - (m-1) e^{r\frac{T}{n}} \right).$$

PROOF: Notice that

$$\begin{aligned} V_{t_{k+1}}^{fi} &= \max\left\{\frac{mC_{t_k}^{fi}}{S_{t_k}}, 0\right\} S_{t_{k+1}} + \left(V_{t_k}^{fi} - \max\left\{\frac{mC_{t_k}^{fi}}{S_{t_k}}, 0\right\} S_{t_k}\right) \frac{B_{t_{k+1}}}{B_{t_k}} \\ &= \begin{cases} F_{t_k}\frac{B_{t_{k+1}}}{B_{t_k}} + C_{t_k}^{fi}\left(m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)\frac{B_{t_{k+1}}}{B_{t_k}}\right) & \text{for } C_{t_k}^{fi} > 0 \\ V_{t_k}^{fi}\frac{B_{t_{k+1}}}{B_{t_k}} & \text{for } C_{t_k}^{fi} \leq 0. \end{cases} \end{aligned}$$

Together with $F_{t_k}\frac{B_{t_{k+1}}}{B_{t_k}} = F_{t_{k+1}}$ it follows

$$C_{t_{k+1}}^{fi} = \begin{cases} C_{t_k}^{fi}\left(m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{r\frac{T}{n}}\right) & \text{for } C_{t_k}^{fi} > 0 \\ C_{t_k}^{fi} e^{r\frac{T}{n}} & \text{for } C_{t_k}^{fi} \leq 0, \end{cases}$$

for all $k = 0, \ldots, n-1$, from which the assertion becomes apparent. □

Notice that the value process V^{fi} converges in distribution to the value process V^{cont} if the trading restrictions vanish, i.e. if $n \to \infty$. The proof of the convergence statement is based on the convergence of the corresponding expectation and variance. Therefore, it is postponed to section 1.5 where the moments are known.

1.3 Risk Measures of Discrete–Time CPPI

Recall that the basic idea of a CPPI strategy is portfolio protection. Heuristically, the usage of these strategies is explained by an investor who wants to participate in bullish markets but does not want the terminal value of the strategy to end up below a guaranteed amount G. Thus, the investor is completely risk averse for values below the floor (or guarantee). As motivated in the previous sections, as soon as a source of market incompleteness is considered, i.e. a restriction on the set of trading dates, the concept of a perfect portfolio protection is impeded, in particular for dynamic strategies. With the exception of static portfolio insurance strategies, there is a positive probability that the terminal value is below the guaranteed amount. In particular, this is true for CPPI and OBPI strategies which include a synthetic put. The use of such *constrained* strategies or strategies which include a gap risk can be explained as follows. On the one hand, one might think of an investor who accepts, because of market incompleteness, a strategy which gives the guaranteed amount with a certain success probability. On the other hand, one might think of retail products which are based on the CPPI method and are thus also hedged by a CPPI strategy. Normally, the buyer of such a product gets the guaranteed amount even in the case that the strategy fails to provide it. Here, the issuer takes the gap risk and considers this in his product pricing. In both cases, the risk profile of the CPPI is of great interest. It is necessary to compute risk measures which allow a characterization if the constrained CPPI is still effective in terms of portfolio insurance.

In the following, we take the view of an investor who uses the CPPI as a savings plan with portfolio protection. A CPPI strategy contradicts the original idea of the portfolio insurance if it results in a very high gap risk, i.e. if the shortfall probability and the expected shortfall are prohibitively high. The investor has to decide whether this additional risk is not too high in terms of a portfolio insurance. In addition to the expected final value and its standard deviation, we consider the shortfall probability and the expected shortfall given default as the risk measures which determine the effectiveness of the discrete–time CPPI strategy.[7] The shortfall probability is the probability that the final value of the discrete–time CPPI strategy is less or equal to the guaranteed amount G. Intuitively, one can also define a local shortfall probability (given that no prior shortfall happened before). Additionally, we use the expected shortfall given default to describe the amount which is lost if a shortfall occurs.

[7]Notice that the shortfall probability is not a coherent risk measure, i.e. it is not sub–additive. In contrast, the expected shortfall given default is a coherent risk measure. We remain within the class of stylized strategies, i.e. the CPPI strategies. Thus, it is in fact not a problem even if the effectiveness of the strategies is analyzed by using a risk measure which is not sub–additive. For details on coherent risk measures we refer to the work of Artzner, Delbaen, Eber, and Heath (1999).

Definition 1.3.1 (Risk measures)

$$\begin{aligned} P^{SF} &:= P\left(V_T^{fi} \leq G\right) = P\left(V_T^{fi} \leq F_T\right) && \textit{shortfall probability} \\ P^{LSF}_{t_i,t_{i+1}} &:= P\left(V_{t_{i+1}}^{fi} \leq F_{t_{i+1}} | V_{t_i}^{fi} > F_{t_i}\right) && \textit{local shortfall probability} \\ ES &:= E\left[G - V_T^{fi} | V_T^{fi} \leq G\right] && \textit{expected shortfall given default.} \end{aligned}$$

It turns out that, in contrast to a discrete-time option based strategy with a synthetic put, the calculation of the shortfall probability implied by a CPPI strategy is very simple. This is easily explained if one observes that the shortfall event is equivalent to the event that the stopping time which is defined in proposition 1.2.2 is prior to the terminal date. It is convenient to consider the following lemma.

Lemma 1.3.2

Let $A_k := \left\{\frac{S_{t_k}}{S_{t_{k-1}}} > \frac{m-1}{m} e^{r\frac{T}{n}}\right\}$ *for* $k = 1, \ldots, n$*, then it holds*

$$\{t_s > t_i\} = \bigcap_{j=1}^{i} A_j \text{ and } \{t_s = t_i\} = A_i^C \cap \left(\bigcap_{j=1}^{i-1} A_j\right) \quad \text{for } i = 1, \ldots, n.$$

Proof: According to the proof of proposition 1.2.2 it holds

$$C_{t_{k+1}}^{fi} = \begin{cases} C_{t_k}^{fi}\left(m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{r\frac{T}{n}}\right) & \text{for } C_{t_k}^{fi} > 0 \\ C_{t_k}^{fi} e^{r\frac{T}{n}} & \text{for } C_{t_k}^{fi} \leq 0. \end{cases}$$

The rest of the proof follows immediately with the definition of the stopping time t_s and

$$m\frac{S_{t_{k+1}}}{S_{t_k}} - (m-1)e^{r\frac{T}{n}} > 0 \Longleftrightarrow \frac{S_{t_{k+1}}}{S_{t_k}} > \frac{m-1}{m} e^{r\frac{T}{n}}.$$

□

Lemma 1.3.3

The local shortfall probability is independent of t_i *and* t_{i+1}*, i.e.*

$$P^{LSF}_{t_i,t_{i+1}} = P^{LSF} = \mathcal{N}(-d_2) \tag{1.5}$$

$$\textit{where } d_2 := \frac{\ln \frac{m}{m-1} + (\mu - r)\frac{T}{n} - \frac{1}{2}\sigma^2\frac{T}{n}}{\sigma\sqrt{\frac{T}{n}}}. \tag{1.6}$$

Proof: Notice that in view of lemma 1.3.2

$$P^{\text{LSF}}_{t_i,t_{i+1}} = P\left(V_{t_{i+1}}^{fi} \leq F_{t_{i+1}} | V_{t_i}^{fi} > F_{t_i}\right) = P\left(t_s = t_{i+1} | t_s > t_i\right) = P\left(A_1^C\right),$$

where the last equality follows from the assumption that the asset price increments are independent and identically distributed (iid). □

Shortfall probability

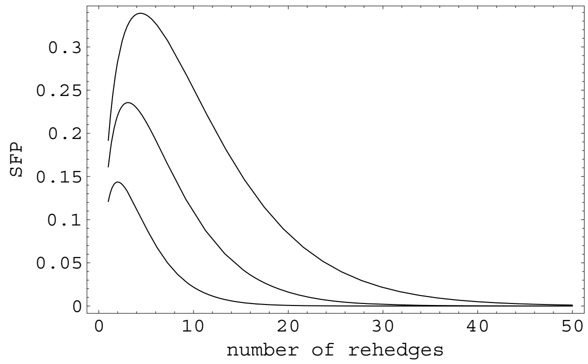

Figure 1.3: Shortfall probability dependent on the number of rehedges. The parameters are $V_0 = 1000$, $G_T = 1000$, $m = 12$ (15 and 18 respectively), $\mu = 0.085$, $r = 0.05$ and $\sigma = 0.1$.

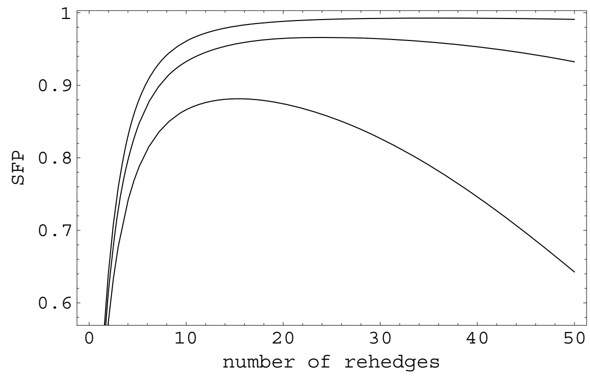

Figure 1.4: Shortfall probability dependent on the number of rehedges. The parameters are $V_0 = 1000$, $G_T = 1000$, $m = 12$ (15 and 18 respectively), $\mu = 0.085$, $r = 0.05$ and $\sigma = 0.3$.

Proposition 1.3.4
The shortfall probability P^{SF} is given in terms of the local shortfall probability P^{LSF}, i.e.

$$P^{SF} = 1 - \left(1 - P^{LSF}\right)^n .$$

Proof: The above lemma is a direct consequence of lemmas 1.3.2, 1.3.3 and the independence of asset price increments, i.e.

$$P^{\text{SF}} = 1 - P(t_s = \infty) \;\; = \;\; 1 - \left(1 - P^{\text{LSF}}\right)^n .$$

□

It will be shown later, c.f. lemma 1.5.2, that the shortfall probability converges to zero as continuous-time trading is approached, i.e. $\lim_{n\to\infty} P^{\text{SF}} = 0$. At first glance, it might be tempting to think that the shortfall probability is monotonically decreasing in the hedging frequency, i.e. the number of rehedges n. In general, this is only true after a sufficiently large n is reached. The effect that the shortfall probability is increasing for small n is more pronounced for high volatilities and high multipliers, c.f. figure 1.3 and figure 1.4.[8] Let n^* denote the number of rehedges such that the shortfall probability is increasing in n for $n \leq n^*$ and decreasing for $n \geq n^*$. The critical level n^* is to be interpreted as a minimal number of rehedges which is necessary such that the CPPI method is effective for $m > 1$

[8]It is straightforward to show that the shortfall probability is monotonically increasing in m and σ.

in discrete time. Consider for example a guaranteed amount G given by $G = e^{rT}\frac{m-1}{m}V_0$ such that $\alpha_0 = 1$, i.e. the initial exposure in the risky asset coincides with the initial portfolio value. If in addition n is chosen to be one, i.e. there is no rehedge until T, the discrete-time CPPI strategy coincides with a pure asset investment. Obviously, the CPPI method can not be effective for $n = 1$, i.e. a pure asset investment is not in the spirit of the CPPI method. Thus, it is intuitively clear that a minimal number of rehedges becomes necessary such that the CPPI method applies if trading is restricted to discrete time. The critical level n^* and its implications are further discussed in section 1.4 where the effectiveness of the discrete-time CPPI method is studied in detail.

If a shortfall is possible, one should also consider the amount of the shortfall or a risk measure which describes the amount of the shortfall. One possibility is given by the expected shortfall ES which is introduced in definition 1.3.1. It turns out that in order to determine the expected shortfall, it is convenient to decompose the expected terminal payoff into two parts. One part is given by the expected terminal value if a shortfall occurs and the other by the expectation on the set where the terminal value is above the guarantee.

Proposition 1.3.5 (Expected terminal value)
It holds

$$\begin{aligned} E\left[V_T^{fi}\right] &= G + \left(V_0 - Ge^{-rT}\right)\left[E_1^n + E_2 \frac{e^{rT} - E_1^n}{e^{r\frac{T}{n}} - E_1}\right] \\ \textit{where} \quad E_1 &:= me^{\mu\frac{T}{n}}\mathcal{N}(d_1) - e^{r\frac{T}{n}}(m-1)\mathcal{N}(d_2) \\ E_2 &:= me^{\mu\frac{T}{n}}\mathcal{N}(-d_1) - e^{r\frac{T}{n}}(m-1)\mathcal{N}(-d_2), \end{aligned}$$

d_2 is the same as in lemma 1.3.3 and $d_1 := d_2 + \sigma\sqrt{\frac{T}{n}}$.

Proof: First notice that

$$E\left[V_T^{fi}\right] = E\left[V_T^{fi}\, 1_{\{t_s=\infty\}}\right] + E\left[V_T^{fi}\, 1_{\{t_s\le t_n\}}\right] \tag{1.7}$$

where the first expectation on the right hand side can be written as

$$\begin{aligned} E\left[V_T^{fi}\, 1_{\{t_s=\infty\}}\right] &= E\left[F_T \prod_{i=1}^{n} 1_{A_i}\right] + E\left[C_T^{fi} \prod_{i=1}^{n} 1_{A_i}\right] \\ &= G\left(1 - P^{\mathrm{SF}}\right) + E\left[C_T^{fi} \prod_{i=1}^{n} 1_{A_i}\right] \end{aligned} \tag{1.8}$$

with the help of lemma 1.3.2.

We now show that

$$E\left[C_{t_i}^{fi}\prod_{j=1}^{i}1_{A_j}\right]=C_0\left(E_1\right)^i,\ \forall\ i=1,\ldots,n. \tag{1.9}$$

The following calculations are based on proposition 1.2.2, lemma 1.3.2 and the assumption that the asset price increments are independent and identically distributed (iid).

$$\begin{aligned}
E\left[C_{t_i}^{fi}\prod_{j=1}^{i}1_{A_j}\right] &= C_0E\left[\prod_{j=1}^{i}\left(m\frac{S_{t_j}}{S_{t_{j-1}}}-(m-1)e^{r\frac{T}{n}}\right)1_{A_j}\right]\\
&= C_0\prod_{j=1}^{i}E\left[\left(m\frac{S_{t_j}}{S_{t_{j-1}}}-(m-1)e^{r\frac{T}{n}}\right)1_{A_j}\right]\\
&= C_0\left(E\left[\left(m\frac{S_{t_1}}{S_0}-(m-1)e^{r\frac{T}{n}}\right)1_{A_1}\right]\right)^i
\end{aligned}$$

Notice that the last expectation matches the definition of E_1, i.e. it holds

$$E_1=\frac{m}{S_0}e^{\mu\frac{T}{n}}E\left[e^{-\mu\frac{T}{n}}\left(S_{t_1}-\frac{m-1}{m}S_0e^{r\frac{T}{n}}\right)^+\right]=me^{\mu\frac{T}{n}}\mathcal{N}\left(d_1\right)-e^{r\frac{T}{n}}(m-1)\mathcal{N}\left(d_2\right)$$

such that equation (1.9) is proven.

For the second expectation on the righthand side of equation (1.7), observe that

$$E\left[V_T^{fi}\ 1_{\{t_s\le t_n\}}\right] = \sum_{i=1}^{n}E\left[V_T^{fi}\ 1_{\{t_s=t_i\}}\right]. \tag{1.10}$$

We now show that

$$E\left[V_T^{fi}\ 1_{\{t_s=t_i\}}\right]=G\ P^{\mathrm{LSF}}(1-P^{\mathrm{LSF}})^i+e^{r(T-t_i)}C_0E_2E_1^{i-1},\ \forall i=1,\ldots,n \tag{1.11}$$

From lemma 1.3.2 we know

$$\begin{aligned}
E\left[V_T^{fi}\ 1_{\{t_s=t_i\}}\right] &= E\left[e^{r(T-t_i)}V_{t_i}^{fi}\ 1_{\left\{V_{t_i}^{fi}\le F_{t_i}\right\}}\prod_{j=1}^{i-1}1_{\left\{V_{t_j}^{fi}>F_{t_j}\right\}}\right]\\
&= e^{r(T-t_i)}E\left[V_{t_i}^{fi}\ 1_{A_i^C}\prod_{j=1}^{i-1}1_{A_j}\right].
\end{aligned}$$

Notice from proposition 1.2.2 that if there is no shortfall until t_{i-1}, it holds

$$C_{t_i}^{fi} = C_{t_{i-1}}^{fi}\left(m\frac{S_{t_i}}{S_{t_{i-1}}}-(m-1)e^{r\frac{T}{n}}\right)$$

such that

$$
\begin{aligned}
&E\left[V_T^{fi}\, 1_{\{t_s=t_i\}}\right] \\
&= G\, E\left[1_{\{t_s=t_i\}}\right] + E\left[C_T^{fi} 1_{\{t_s=t_i\}}\right] \\
&= G\, E\left[1_{A_i^C} \prod_{j=1}^{i-1} 1_{A_j}\right] + e^{r(T-t_i)} E\left[C_{t_{i-1}}^{fi}\left(m\frac{S_{t_i}}{S_{t_{i-1}}} - (m-1)e^{r\frac{T}{n}}\right) 1_{A_i^C} \prod_{j=1}^{i-1} 1_{A_j}\right].
\end{aligned}
$$

With equation (1.9) and the assumption that the asset price increments are iid, it follows

$$
\begin{aligned}
E\left[V_T^{fi}\, 1_{\{t_s=t_i\}}\right] &= G\, P^{\text{LSF}}\left(1-P^{\text{LSF}}\right)^{i-1} \\
&\quad + e^{r(T-t_i)} C_0 E_1^{i-1} E\left[\left(m\frac{S_{t_1}}{S_0} - (m-1)e^{r\frac{T}{n}}\right)\, 1_{A_1^C}\right]
\end{aligned}
$$

where it is straightforward to check that the expectation satisfies the definition of E_2. This proves equation (1.11). Now a combination of equations (1.10) and (1.11) yields

$$
\begin{aligned}
E\left[V_T^{fi}\, 1_{\{t_s\le t_n\}}\right] &= \sum_{i=1}^{n}\left(G\, P^{\text{LSF}}(1-P^{\text{LSF}})^{i-1} + e^{r(T-t_i)} C_0 E_2 E_1^{i-1}\right) \\
&= G\, P^{\text{SF}} + C_0 E_2 \frac{e^{rT} - E_1^n}{e^{r\frac{T}{n}} - E_1} \qquad (1.12)
\end{aligned}
$$

such that together with equations (1.7), (1.8) and (1.9) the proposition is proven. □

The calculation of the expected shortfall ES is now straightforward.[9]

Corollary 1.3.6 (Expected Shortfall)
The expected shortfall ES which is defined as in definition 1.3.1 is given by

$$
ES = -\frac{C_0 E_2 \frac{e^{rT} - E_1^n}{e^{r\frac{T}{n}} - E_1}}{P^{SF}}.
$$

PROOF: According to the definition, it holds

$$
\text{ES} = E\left[G - V_T^{fi} | t_s < \infty\right] = G - \frac{E\left[V_T^{fi}\, 1_{\{t_s\le t_n\}}\right]}{P^{SF}}.
$$

The proof is completed by inserting equation (1.12). □

---

[9]The same is true for the price of the associated gap risk, i.e. the price of an option where the payoff at T is given by $(G - V_T^{fi})^+$. Notice that by standard financial theory, the t_0–price is given by the expected value of the discounted payoff under the martingale measure. However, the risk measures which are considered here must be given with respect to the *real world measure*.

Proposition 1.3.7 (Variance of final value)
With d_1 and d_2 as defined above, it holds

$$Var\left[V_T^{fi}\right] = (V_0 - Ge^{-rT})^2 \left[\widetilde{E}_1^n + \widetilde{E}_2 \frac{e^{2rT} - \widetilde{E}_1^n}{e^{2r\frac{T}{n}} - \widetilde{E}_1}\right] - (E[V_T^{fi}] - G)^2$$

where

$$\begin{aligned}
\widetilde{E}_1 &:= m^2 e^{(2\mu+\sigma^2)\frac{T}{n}} \mathcal{N}(d_3) - 2m(m-1)e^{(\mu+r)\frac{T}{n}} \mathcal{N}(d_1) + (m-1)^2 e^{2r\frac{T}{n}} \mathcal{N}(d_2), \\
\widetilde{E}_2 &:= m^2 e^{(2\mu+\sigma^2)\frac{T}{n}} \mathcal{N}(-d_3) - 2m(m-1)e^{(\mu+r)\frac{T}{n}} \mathcal{N}(-d_1) + (m-1)^2 e^{2r\frac{T}{n}} \mathcal{N}(-d_2)
\end{aligned}$$

and

$$d_3 := \frac{\ln\frac{m}{m-1} + (\mu - r)\frac{T}{n} + \frac{3}{2}\sigma^2\frac{T}{n}}{\sigma\sqrt{\frac{T}{n}}}.$$

PROOF: Notice that

$$Var\left[V_T^{fi}\right] = Var\left[C_T^{fi}\right] = E\left[\left(C_T^{fi}\right)^2\right] - \left(E\left[V_T^{fi}\right] - G\right)^2 \tag{1.13}$$

where

$$E\left[\left(C_T^{fi}\right)^2\right] = E\left[\left(C_T^{fi}\right)^2 1_{\{t_s=\infty\}}\right] + \sum_{i=1}^{n} E\left[\left(C_T^{fi}\right)^2 1_{\{t_s=t_i\}}\right]. \tag{1.14}$$

Analogously to the proof of proposition 1.3.5 it is straightforward to show for all $i = 1, \ldots, n$

$$E\left[\left(C_{t_i}^{fi}\right)^2 \prod_{j=1}^{i} 1_{A_j}\right] = C_0^2 \left(E\left[\left(m\frac{S_{t_1}}{S_0} - (m-1)e^{r\frac{T}{n}}\right)^2 1_{A_1}\right]\right)^i = C_0^2 \left(\widetilde{E}_1\right)^i \tag{1.15}$$

which gives an expression for the first term on the right hand side of equation (1.14). For the second term on the right hand side of equation (1.14) note first that

$$E[(C_T^{fi})^2 1_{\{t_s=t_i\}}] = e^{2r(T-t_i)} E\left[(C_{t_i}^{fi})^2 1_{\{t_s=t_i\}}\right]$$

where

$$\begin{aligned}
E\left[(C_{t_i}^{fi})^2 1_{\{t_s=t_i\}}\right] &= E\left[(C_{t_i}^{fi})^2 1_{A_i^C} \prod_{j=1}^{i-1} 1_{A_j}\right] \\
&= E\left[\left(C_{t_{i-1}}^{fi}\right)^2 \left(m\frac{S_{t_i}}{S_{t_{i-1}}} - (m-1)e^{r\frac{T}{n}}\right)^2 1_{A_i^C} \prod_{j=1}^{i-1} 1_{A_j}\right] \\
&= E\left[\left(m\frac{S_{t_1}}{S_0} - (m-1)e^{r\frac{T}{n}}\right)^2 1_{A_1^C}\right] C_0^2 \widetilde{E}_1^{i-1} \\
&= C_0^2 \widetilde{E}_2 \widetilde{E}_1^{i-1}
\end{aligned}$$

Sensitivity of risk measures

Risk measures	Strategy parameter		Model parameter	
	G	m	μ	σ
Mean	↓	↑	↑	↑
Stdv.	↓	↑	↑	↑
P^{SF}	–	↑	↓	↑
ESF	↓	↑	↑	↑

Table 1.1: Sensitivity analysis of risk measures. We use the symbol ↑ for monotonically increasing and ↓ for monotonically decreasing.

with the help of equation (1.15). Therefore the sum in equation (1.14) is given by

$$\sum_{i=1}^{n} E\left[(C_T^{fi})^2 1_{\{t_s=t_i\}}\right] = C_0^2 \widetilde{E}_2 \sum_{i=1}^{n} e^{2r(T-t_i)} \widetilde{E}_1^{i-1} = C_0^2 \widetilde{E}_2 \frac{e^{2rT} - \widetilde{E}_1^n}{e^{2r\frac{T}{n}} - \widetilde{E}_1}$$

which combined with equations (1.13), (1.14), and (1.15) yields the assertion in the proposition. □

Before we study the effectiveness of the time–discretized CPPI in detail, we briefly perform a sensitivity analysis of the risk measures. In order to avoid a lengthy discussion of all possible sensitivities, we summarize the main results in table 1.1. The corresponding proofs are straightforward. Notice that the shortfall probability is independent of G, c.f. proposition 1.3.4. Partial differentiation immediately yields that the shortfall probability is increasing in σ and m but decreasing in μ. In contrast, the sensitivity analysis of the other risk measures is tedious. For example, the monotonicity of the expected terminal value, i.e. $E[V_T^{fi}]$, in σ is shown at the end of the section in lemma 1.3.8. Likewise, similar arguments to the ones presented here can be used to show that the expected terminal payoff is also increasing in μ and m. Monotonicity in G and V_0 is immanent. With respect to the standard deviation, it is intuitively clear that the volatility σ has a positive effect on the standard deviation, so does m. It is worth mentioning that both the shortfall probability and the expected shortfall are increasing in m and σ. This implies that a discrete-time CPPI is not effective in discrete time if either the standard deviation is too large in comparison to the multiplier or vice versa.

We end the section by proving the sensitivity of the expected terminal value with respect to the volatility.

Lemma 1.3.8

The expected terminal value of the simple discrete CPPI is increasing in the volatility σ, i.e.

$$\frac{\partial\, E\left[V_T^{fi}\right]}{\partial\, \sigma} > 0$$

Proof: With proposition 1.3.5 and the definition of E_2 it follows that

$$E[V_T^{fi}] \quad = \quad V_0 e^{rT} + (V_0 - F_0) m \left(e^{(\mu-r)\frac{T}{n}} - 1\right) \frac{e^{rT} - E_1^n}{1 - E_1 e^{-r\frac{T}{n}}}.$$

It is straightforward to show that $E_1 > e^{r\frac{T}{n}}$. For $\mu > r$, the expected terminal value of the discrete CPPI strategy is always larger than the investment in the riskless asset. This is quite intuitive. Now, consider the derivative with respect to σ, i.e.

$$\frac{\partial\, E[V_T^{fi}]}{\partial\, \sigma} \quad = \quad m\,(V_0 - F_0)\left(e^{(\mu-r)\frac{T}{n}} - 1\right) \frac{-n\, E_1^{n-1} \frac{\partial\, E_1}{\partial\, \sigma}\left(1 - E_1 e^{-r\frac{T}{n}}\right) - \left(E_1^n - e^{rT}\right)\frac{\partial\, E_1}{\partial\, \sigma} e^{-r\frac{T}{n}}}{\left(1 - E_1 e^{-r\frac{T}{n}}\right)^2}.$$

For $\mu > r$, the leading factors are positive. Besides, we have $\frac{\partial\, E_1}{\partial\, \sigma} > 0$, the proof of which is omitted here. In particular, analogous calculations as for the determination of the vega of a call-option price in a Black/Scholes–type model are needed. Finally, it is to show that

$$-n\, E_1^{n-1}\left(1 - E_1 e^{-r\frac{T}{n}}\right) - \left(E_1^n - e^{rT}\right) e^{-r\frac{T}{n}} \geq 0.$$

An application of Bernoulli's inequality gives

$$\begin{aligned}
& n\, E_1^{n-1}\left(E_1 e^{-r\frac{T}{n}} - 1\right) - e^{-r\frac{T}{n}}\left(E_1^n - e^{rT}\right) \\
&= e^{-r\frac{T}{n}}\left(n\, E_1^{n-1}\left(E_1 - e^{r\frac{T}{n}}\right) - E_1^n + E_1^n\left(1 + \frac{e^{r\frac{T}{n}} - E_1}{E_1}\right)^n\right) \\
&\geq e^{-r\frac{T}{n}}\left(n\, E_1^{n-1}\left(E_1 - e^{r\frac{T}{n}}\right) - E_1^n + E_1^n\left(1 + n\,\frac{e^{r\frac{T}{n}} - E_1}{E_1}\right)\right) = 0.
\end{aligned}$$

Notice that due to $E_1 > e^{r\frac{T}{n}}$, the above inequality is also strict. □

1.4 Effectiveness of the discrete-time CPPI method

As shown above, the effectiveness of the discrete-time CPPI method depends on the strategy parameters, i.e. the multiplier m, the number of rehedges n and the guarantee G, as well as the model parameters μ and σ. The most important influences are caused by the multiplier m and the volatility σ. Therefore, all examples are considered for varying multipliers and volatilities. If not mentioned otherwise, we consider a model scenario where $\mu = 0.085$, $\sigma = 0.1$ (0.2 or 0.3, respectively) and $r = 0.05$. The maturity time of the CPPI strategy is equal to one year ($T = 1$), the initial investment coincides with the guarantee, i.e. $V_0 = G = 1000$. Thus, the goal of the strategies under consideration is to ensure 100% of the initial capital. This is in accordance to guaranteed fund management.[10] For the multiplier m we consider the values 12, 15 and 18. Here, the initial asset exposure $m\left(V_0 - e^{-rT}G\right)$ is 585.247 for $m = 12$, 731.559 for $m = 15$ and 877.870 for $m = 18$ such that the relative initial asset investment varies between 0.585 and 0.88. A high multiplier is convenient in order to emphasize all effects and to highlight the effect of a small change in volatility.

First, we consider the question whether the discrete-time CPPI method gives a good approximation of the continuous-time CPPI for a finite number of rehedges n. Recall that the value process of the discrete–time CPPI converges to the value process of the continuous–time CPPI in distribution, c.f. proposition 1.5.1. Since the cushion process of the continuous–time CPPI is lognormal, the payoff distribution of the continuous–time CPPI is described by its mean and its standard deviation. These numbers are summarized in table 1.2. In addition, table 1.2 summarizes the moments and risk measures for various numbers of rehedges n.

Now consider the shortfall probability. Observe, that in the case where $\sigma = 0.1$, a monthly CPPI–strategy ($n = 12$) with a multiplier $m = 12$ implies a shortfall probability of only 0.01. In contrast, a volatility of $\sigma = 0.2$ gives a shortfall probability of more than 0.5. Thus, the monthly CPPI strategy ensures a significant protection level for $\sigma = 0.1$ while the concept of portfolio insurance is already impeded for $\sigma = 0.2$. Here (for $\sigma = 0.2$), even a weekly rehedging, i.e. $n = 48$ is not enough to achieve a shortfall probability of less than 0.05. This illustrates that the effectiveness of the discrete–time CPPI method is very sensitive to the volatility of the asset price process. Besides, the higher the multiplier, the more pronounced the effect is. For example, notice that the shortfall probability for a CPPI–strategy with $n = 24$ and $m = 18$ is 0.049 for $\sigma = 0.1$ but 0.86 for $\sigma = 0.2$.

Recall that the shortfall probability is not necessarily monotonically decreasing in the number of rehedges. A very large shortfall probability implies that the number of rehedges

[10] It is worth mentioning that the probability that the CPPI portfolio value is higher than the OBPI value increases in the percentage of the insured initial investment, c.f. Bertrand and Prigent (2003). Recall that $V_T^{\mathrm{OBPI}} = G + [S_T - G]^+$. Thus, the above effect is intuitively explained by observing that the probability of exercising the embedded call option is decreasing in the strike.

Moments and risk measures of the CPPI

n	m	Mean	Stdv.	SFP	ESF
12	12	1077.53 (1080.23)	125.04 (703.03)	**0.0115 (0.5430)**	5.463 (25.933)
24	12	1077.77 (1078.60)	132.01 (948.79)	0.0002 (0.3195)	2.981 (12.296)
48	12	1077.90 (1077.98)	135.88 (1133.36)	0.0000 (0.0580)	1.574 (5.802)
96	12	1077.97 (1077.97)	137.92 (1249.06)	0.0000 (0.0009)	0.000 (3.037)
∞	12	1078.03 (1078.03)	140.04 (1387.90)		
n	m	Mean	Stdv.	SFP	ESF
12	15	1085.94 (1074.28)	206.30 (1874.59)	0.0767 (0.7592)	8.901 (57.01)
24	15	1086.22 (1090.92)	226.81 (3361.17)	0.0069 (0.6610)	4.836 (27.86)
48	15	1086.44 (1087.43)	238.86 (4936.18)	0.0000 (0.3258)	2.597 (11.03)
96	15	1086.56 (1086.60)	245.46 (6130.89)	0.0000 (0.0333)	1.364 (5.02)
∞	15	1086.67 (1086.67)	252.51 (7801.45)		
n	m	Mean	Stdv.	SFP	ESF
12	18	1095.70 (1120.63)	339.07 (4924.65)	0.2094 (0.8691)	13.911 (118.32)
24	18	1095.65 (1111.58)	396.37 (12759.40)	**0.0494 (0.8593)**	7.296 (64.66)
48	18	1095.90 (1101.08)	432.75 (25691.30)	0.0015 (0.6767)	3.908 (23.70)
96	18	1096.08 (1096.68)	453.66 (39053.60)	0.0000 (0.2131)	2.067 (8.30)
∞	18	1096.27 (1096.27)	476.83 (62763.30)		

Table 1.2: The time horizon is $T = 1$ year and the guarantee G is equal to the initial investment $V_0 = 1000$. The model parameters are given by $\mu = 0.085$, $r = 0.05$ and $\sigma = 0.1$ ($\sigma = 0.2$ respectively). The case $n = \infty$ represents the continuous-time CPPI.

is still too low to achieve an effective portfolio protection. For example, one might think of the extreme case that $n = 1$, i.e. the case where the portfolio is held constantly on the trading period $[0, T]$. Obviously, a portfolio protection can only be achieved if only the surplus is invested in the risky asset. One can argue that the CPPI method is not effective if the number of rehedges n is still in a region where the shortfall probability is increasing in n. Thus, it is convenient to determine the minimal number n^* such that an increase in the number of portfolio rebalancing dates is able to reduce the shortfall probability. For different combinations of σ and m, the critical number n^* is illustrated in table 1.3.[11] However, n^* can only be used as a number which is at least necessary to achieve an effective portfolio insurance.

[11] Compare also the remarks in the last section referring to figure 1.3 and figure 1.4.

Minimal number of rehedges

m	σ	n^*	m	σ	n^*	m	σ	n^*
12	0.1	2.00	15	0.1	3.08	18	0.1	4.40
12	0.2	7.00	15	0.2	11.09	18	0.2	16.11
12	0.3	15.35	15	0.3	24.44	18	0.3	35.64

Table 1.3: Minimal number n^* of rehedges such that the shortfall probability is decreasing in n.

One solution to ensure the effectiveness of the discrete-time CPPI method is given by the possibility to determine the contract parameters such that the probability of falling below the guarantee is bounded from above by a confidence level γ, for example $\gamma = 0.99$ (or $\gamma = 0.95$). This can be explained by an investor who is aware of market incompleteness and accepts a small shortfall probability with respect to the guarantee. Again, we consider the same model scenario where $T = 1$, $\mu = 0.085$, $r = 0.05$, $V_0 = G = 1000$ and distinguish between $\sigma = 0.1$ and $\sigma = 0.2$. For illustration, we determine (n, m)–tupels which give a shortfall probability of 0.01 and 0.05. The resulting values as well as the corresponding other risk measures are given in table 1.4. For example, observe that in the case of $\sigma = 0.1$, the CPPI method with monthly rehedging and a multiplier of 11.84 ensures that the capital is maintained with a probability of 0.99. At the same time the expected payoff and the variance of the payoff are similar in magnitude to the ones obtained by a direct investment in S, i.e. for the expectation compare 1077 to 1088 and for the standard deviation compare 121.75 to 109.14.[12] Therefore, in the case where $\sigma = 0.1$, even a monthly rehedging is enough to give a high success probability if the multiplier

[12]A direct investment of V_0 in the asset S gives for $\sigma = 0.1$ ($\sigma = 0.2$ respectively)

$$\begin{aligned} E\left[V_0 \frac{S_T}{S_0}\right] &= V_0 e^{\mu T} = 1088.72 \;\; (1088.72) \\ \sqrt{\operatorname{Var}\left[V_0 \frac{S_T}{S_0}\right]} &= V_0 \sqrt{e^{(2\mu+\sigma^2)T} - e^{2\mu T}} = 109.144 \;\; (219.939) \\ P\left(V_0 \frac{S_T}{S_0} \le G\right) &= 0.212 \;\; (0.373). \end{aligned}$$

Risk profile for discrete-time CPPI strategies with a shortfall probability of 0.01 (0.05).

$\sigma = 0.1$				
n	m	Mean	Stdv.	ES
12	11.843 (14.124)	1077.118 (1083.377)	121.752 (178.420)	5.313 (7.770)
24	15.446 (18.024)	1087.558 (1095.730)	246.087 (398.225)	5.157 (7.319)
36	18.146 (20.956)	1096.273 (1106.154)	432.362 (774.426)	5.149 (7.217)
48	20.386 (23.389)	1104.150 (1115.646)	717.129 (1419.070)	5.186 (7.219)
60	22.336 (25.507)	1111.528 (1124.588)	1152.310 (2511.390)	5.243 (7.267)

$\sigma = 0.2$				
n	m	Mean	Stdv.	ES
12	6.065 (7.152)	1063.302 (1065.747)	107.138 (150.350)	4.478 (6.432)
24	7.879 (9.128)	1067.464 (1070.485)	204.334 (316.650)	4.275 (5.931)
36	9.234 (10.605)	1070.748 (1074.241)	345.136 (591.266)	4.190 (5.720)
48	10.358 (11.829)	1073.591 (1077.500)	554.966 (1048.690)	4.145 (5.605)
60	11.335 (12.893)	1076.156 (1080.449)	868.650 (1804.760)	4.121 (5.535)

Table 1.4: For a given discretization in terms of n, the multiplier is determined such that the implied shortfall probability is 0.01 (0.05 respectively).

is chosen appropriately.[13] However, in case of a volatility scenario where $\sigma = 0.2$, the multiplier is to be chosen much more conservatively. Finally, it is worth mentioning that it is sufficient to control the shortfall probability if one also wants to control the expected shortfall which is unarguably a more convincing risk measure. In the above example, keeping the shortfall probability on a 0.01 level is approximately the same as keeping the expected shortfall at a level of 5.2.

[13] Again, it is worth mentioning that although a multiplier of approximately 12 seems to be fairly large, it is to be interpreted in combination with the low volatility. In particular, a multiplier of $m = 11.843$ implies that for a guarantee $G = V_0 = 1000$ the initial amount invested in S is given by

$$\alpha V_0 = m(V_0 - F_0) = 11.843(1000 - e^{-0.05}1000) = 577.59.$$

1.5 Convergence

Based on the results of section 1.3, the purpose of this section is to show that the simple discrete CPPI converges to the simple continuous CPPI as the trading restrictions vanish. In particular, the following proposition will be shown.

Proposition 1.5.1 (Convergence)
For $n \to \infty$, the value process V^{fi} converges to the value process V^{cont} in distribution, i.e. $V^{fi} \xrightarrow{\mathcal{L}} V^{cont}$. In particular, it holds

$$\begin{aligned}\lim_{n\to\infty} E\left[V_T^{fi}\right] &= G + (V_0 - Ge^{-rT})\exp\{(r + m(\mu - r))\,T\}\\ \lim_{n\to\infty} Var\left[V_T^{fi}\right] &= \left(V_0 - Ge^{-rT}\right)^2 \exp\{2\,(r + m(\mu - r))\,T\}\ \left(\exp\left\{m^2\sigma^2 T\right\} - 1\right).\end{aligned}$$

First, we consider the convergence of the shortfall probability, the expected terminal value and the variance of the terminal value.

Lemma 1.5.2
The shortfall probability converges to zero if the trading restrictions vanish, i.e.

$$\lim_{n\to\infty} P^{SF} = 0$$

Proof: Let $f \in \mathcal{C}^1(\mathbb{R})$ such that $\lim_{x\to\infty} f(x) = 1$. With

$$\lim_{x\to\infty} (f(x))^x = \lim_{x\to\infty} \left(1 + \frac{x(f(x)-1)}{x}\right)^x = e^{\lim_{x\to\infty} x(f(x)-1)}$$

together with an application of L'Hôpital's rule, i.e.

$$\lim_{x\to\infty} x(f(x)-1) = \lim_{x\to\infty} \frac{f(x)-1}{x^{-1}} = \lim_{x\to\infty} -x^2 \frac{\partial f}{\partial x}(x),$$

it follows $\lim_{x\to\infty} (f(x))^x = e^{\lim_{x\to\infty} -x^2 \frac{\partial f}{\partial x}(x)}$ if $\lim_{x\to\infty} -x^2 \frac{\partial f}{\partial x}(x)$ exists. Thus,

$$\begin{aligned}\lim_{n\to\infty} P^{SF} &= \lim_{n\to\infty} 1 - (1 - P^{LSF})^n = 1 - \lim_{n\to\infty} \mathcal{N}(d_2)^n\\ &= 1 - e^{\lim_{n\to\infty} -n^2 \frac{\partial \mathcal{N}(d_2)}{\partial n}} = 1 - e^{\lim_{n\to\infty} -n^2 \mathcal{N}'(d_2) \frac{\partial d_2}{\partial n}}.\end{aligned}$$

The rest of the proof follows immediately with the definition of d_2, c.f. proposition 1.3.5, and

$$\lim_{n\to\infty} e^{-n} n^k = 0 \ \text{ for all } \ k \in \mathbb{N}.$$

□

Lemma 1.5.3
The expected value of the discrete–time CPPI converges to the expected value of a simple CPPI if the trading restrictions vanish, i.e.

$$\lim_{n\to\infty} E\left[V_T^{fi}\right] = G + (V_0 - Ge^{-rT})e^{(r+m(\mu-r))T}$$

PROOF: According to proposition 1.3.5 it holds

$$E[V_T^{fi}] \;=\; G + (V_0 - Ge^{-rT})E_1^n + (V_0 - Ge^{-rT})e^{-r\frac{T}{n}}E_2\frac{e^{rT} - E_1^n}{1 - E_1 e^{-r\frac{T}{n}}}. \tag{1.16}$$

First, we consider the limit of E_1^n. Using the definition of E_1, c.f. proposition 1.3.5, it is straightforward to show that $\lim\limits_{n\to\infty} E_1 = 1$. According to the proof of lemma 1.5.2 it holds

$$\lim_{n\to\infty} E_1^n = e^{\lim\limits_{n\to\infty} -n^2\frac{\partial E_1}{\partial n}}$$

where

$$\begin{aligned} -n^2\frac{\partial E_1}{\partial n} \;&=\; m\mu T e^{\mu\frac{T}{n}}\mathcal{N}(d_1) - (m-1)rTe^{r\frac{T}{n}}\mathcal{N}(d_2) \\ &\quad - \; m e^{\mu\frac{T}{n}} n^2 \mathcal{N}'(d_1)\frac{\partial d_1}{\partial n} + (m-1)e^{r\frac{T}{n}} n^2 \mathcal{N}'(d_2)\frac{\partial d_2}{\partial n}. \end{aligned}$$

Notice that the last two terms on the right-hand side vanish for $n \to \infty$. Besides, with the definitions of d_1 and d_2, c.f. proposition 1.3.5, it immediately follows

$$\lim_{n\to\infty} \mathcal{N}(d_1) = 1, \quad \lim_{n\to\infty} \mathcal{N}(d_2) = 1$$

such that

$$\lim_{n\to\infty} (V_0 - Ge^{-rT})E_1^n \;=\; (V_0 - Ge^{-rT})e^{m\mu T - (m-1)rT}.$$

Thus, it is still to show that the last term of the right hand side of equation (1.16) converges to zero. Inserting E_2 according to its definition, c.f. proposition 1.3.5, the relevant term is

$$(V_0 - Ge^{-rT})(e^{rT} - E_1^n)\left(1 + \frac{m\left(e^{(\mu-r)\frac{T}{n}} - 1\right)}{1 - E_1 e^{-r\frac{T}{n}}}\right).$$

Notice that $\lim\limits_{n\to\infty} m\left(e^{(\mu-r)\frac{T}{n}} - 1\right) = 0$ and $\lim\limits_{n\to\infty} 1 - E_1 e^{-r\frac{T}{n}} = 0$. With the rule of L'Hôpital and similar arguments as above it follows

$$\begin{aligned} \lim_{n\to\infty}\frac{m\left(e^{(\mu-r)\frac{T}{n}} - 1\right)}{1 - E_1 e^{-r\frac{T}{n}}} \;&=\; \lim_{n\to\infty}\left(\frac{-\frac{m(\mu-r)T}{n^2}e^{(\mu-r)\frac{T}{n}}}{-\frac{E_1 rTe^{-r\frac{T}{n}}}{n^2} - e^{-r\frac{T}{n}}\frac{\partial E_1}{\partial n}}\right) \\ &=\; \frac{m(\mu-r)T}{rT - (m\mu T - (m-1)rT)} = -1. \end{aligned}$$

□

Lemma 1.5.4

The variance of the discrete–time CPPI converges to the variance of a simple CPPI if the trading restrictions vanish, i.e.

$$\lim_{n\to\infty} Var\left[V_T^{fi}\right] = (V_0 - Ge^{-rT})^2 e^{2(r+m(\mu-r))T}(e^{m^2\sigma^2 T} - 1)$$

Proof: Recall that according to proposition 1.3.7 it holds

$$Var[V_T^{fi}] = (V_0 - Ge^{-rT})^2 \widetilde{E}_1^n + (V_0 - Ge^{-rT})^2 e^{-2r\frac{T}{n}} \widetilde{E}_2 \frac{e^{2rT} - \widetilde{E}_1^n}{1 - e^{-2r\frac{T}{n}}\widetilde{E}_1} - (E[V_T^{fi}] - G)^2.$$

Analogously to the proof of lemma 1.5.3, it can be shown that $\widetilde{E}_2 \frac{e^{2rT}-\widetilde{E}_1^n}{1-e^{-2r\frac{T}{n}}\widetilde{E}_1} \stackrel{n\to\infty}{\longrightarrow} 0$. For the convergence of $\widetilde{E}_1^n$ we use again that $\lim_{n\to\infty} \widetilde{E}_1^n = e^{\lim_{n\to\infty} -n^2 \frac{\partial \widetilde{E}_1}{\partial n}}$. Notice that

$$\begin{aligned} -n^2 \frac{\partial \widetilde{E}_1}{\partial n} &= m^2 e^{(2\mu+\sigma^2)\frac{T}{n}}(2\mu+\sigma^2)T\mathcal{N}(d_3) - 2m(m-1)e^{(\mu+r)\frac{T}{n}}(\mu+r)T\mathcal{N}(d_1) \\ &\quad + (m-1)^2 e^{2r\frac{T}{n}} 2rT\mathcal{N}(d_2) - m^2 e^{(2\mu+\sigma^2)\frac{T}{n}} n^2 \mathcal{N}'(d_3)\frac{\partial d_3}{\partial n} \\ &\quad + 2m(m-1)e^{(\mu+r)\frac{T}{n}} n^2 \mathcal{N}'(d_1)\frac{\partial d_1}{\partial n} - (m-1)^2 e^{2r\frac{T}{n}} n^2 \mathcal{N}'(d_2)\frac{\partial d_2}{\partial n}. \end{aligned}$$

Similar arguments to the ones given in the proofs of lemma 1.5.3 and lemma 1.5.2 imply

$$\lim_{n\to\infty} \widetilde{E}_1^n = e^{m^2(2\mu+\sigma^2)T - 2m(m-1)(\mu+r)T + (m-1)^2 2rT} = e^{2(r+m(\mu-r))T} e^{m^2\sigma^2 T}.$$

Finally, lemma 1.5.3 immediately gives

$$\lim_{n\to\infty} (E\left[V_T^{fi}\right] - G)^2 = (V_0 - Ge^{-rT})^2 e^{2(r+m(\mu-r))T}.$$

□

In order to prove proposition 1.5.1 it is remains to show that, for $n \to \infty$, the limiting distribution of $\ln(V_T^{fi})$ is a normal distribution . Let

$$\zeta_n := \sum_{i=1}^n \underbrace{\ln\left(m\frac{S_{t_i}}{S_{t_{i+1}}} - (m-1)e^{r\frac{T}{n}}\right)}_{=:\xi_{i,n}}.$$

In view of lemma 1.5.2, i.e. $\lim_{n\to\infty} P^{SF} = 0$, it is sufficient to show that the limiting distribution of ζ_n is a normal distribution. Applying the results for rowwise independent arrays of Gnedenko and Kolmogorov (1954), c.f. in particular theorem 1 in Ch.5 §26, it remains to show that

$$\sum_{i=1}^n P(|\xi_{i,n}| > \epsilon) \stackrel{n\to\infty}{\longrightarrow} 0 \quad \text{for all } \epsilon > 0.$$

Using the independency, one only needs to show that

$$nP(|\xi_{1,n}| > \epsilon) \overset{n\to\infty}{\longrightarrow} 0 \quad \text{for all } \epsilon > 0.$$

This proof is straightforward, i.e. it is given by arguments which are similar to the ones used in the proofs of the above lemmas.

1.6 Conclusion

The introduction of market incompleteness and model risk impedes the concept of dynamic portfolio insurance, i.e. the technique of constant proportion portfolio insurance. The introduction of tradings restrictions is one possibility to model a gap risk in the sense that a CPPI strategy can not be adjusted adequately. Measuring the risk that the value of a CPPI strategy is less than the floor (or guaranteed amount) is of practical importance for at least two reasons. On the one hand, CPPI strategies are common in hedge funds and retail products. Often, a CPPI strategy is pre–specified in the term sheet of hedge funds. In addition, it is combined with a guarantee for the investor. Thus, an additional option is written. The option is exercised if the value of the CPPI strategy is below the floor. On the other hand, CPPI strategies can be used to protect return guarantees which are embedded in unit-linked life insurance contracts. The terminal date T is interpreted as the time of retirement and the guarantee is interpreted as the amount which is at least needed by the insured. The assumption that the insurer wants to back up the guarantee by a simple and discrete-time investment strategy highlights some advantages in favor of the CPPI method. Firstly, it is computationally very simple and it can easily be applied in discrete time. Secondly, the composition of a CPPI strategy is independent of the model assumption of the investor or insurer who might use a misspecified model. Thirdly, the riskiness in terms of commonly used risk measures which is induced by trading restrictions can be given in closed form. In particular, this is also true for the price of an additional option which is normally included in CPPI-based products.

The analysis of the risk measures of a discrete-time CPPI strategy poses various problems which are to be considered. Basically, it is necessary to check the associated risk measures and to determine whether the strategy is still effective in terms of portfolio protection. For example, the protection feature is violated if the shortfall probability of the CPPI strategy under consideration exceeds the shortfall probability of a pure asset investment. Formally, the last one can be interpreted as a static CPPI. Intuitively, this explains the result that the shortfall probability of a discrete–time CPPI is only decreasing in the hedge frequency after a sufficiently high number of rehedges. Below this critical number, the shortfall probability increases such that additional adjustments of the strategy yield a shortfall probability which is even higher than the one of a pure asset investment. This effect is even more pronounced for high asset price volatilities and high multipliers. Thus, if one restricts the set of admissible strategies to those strategies which satisfy a confidence level of protection, the choice of the CPPI–multiplier is naturally restricted. A similar reasoning is applied to other risk measures such as the expected shortfall.

Chapter 2

The Discrete CPPI with Triggered Trading Dates

In this chapter we are also concerned with the question of how the CPPI can be performed in discrete time. In the previous chapter a discretization of the CPPI strategy based on a fixed set of trading dates was presented. The focus was on the discussion of the default risk that emerged as a consequence of the discretization. Default risk was understood in the sense that the payoff of the discrete version of the CPPI will not be greater than some given guarantee with probability one. However, since the CPPI strategy in continuous time does not incorporate default risk, the pure existence of default risk in the discrete version has to be viewed as a major drawback. Although as a consequence of convergence the default risk can be made arbitrarily small by choosing smaller distances between the trading dates, it is unpleasant that the discrete-time version of the CPPI loses the most important feature of the continuous-time CPPI, portfolio protection with probability one. A second drawback is that the default risk between any two trading dates is constant, independent of the size of the cushion. This makes things even worse, because it is clearly an unfavorable feature that no matter how far the portfolio is above the guarantee, the probability of falling below the guarantee from one trading date to the next is always the same.

Here we present a different kind of discretization. Instead of taking fixed trading dates, the portfolio is adjusted at triggered trading dates based on the performance of the portfolio. By triggered trading dates we mean that trading only takes place whenever the underlying risky asset or alternatively the cushion process has gained or lost a certain proportion, for example 10%. Note that triggered trading dates emerge naturally as a consequence of transaction costs. In the presence of transaction costs the frequency of monitoring the portfolio and rebalancing the portfolio is usually not identical since the willingness to rebalance is reduced due to the induced costs. In particular, while the frequency of monitoring may be very large in order to reduce the gap risk, one would like to only trade upon relevant changes in the portfolio value, the cushion value or the underlying assets such as to reduce the costs of rebalancing. This can be achieved with triggered trading dates.

In this chapter we use a model where the frequency of monitoring is infinite, i.e. continuous monitoring, but trading takes place in discrete time dependent on changes in the cushion value or the underlying assets. If trading takes place upon changes in the value of the cushion, it is quite intuitive, that the cushion process cannot fall below zero because it can always lose only a predetermined proportion of its previous value from one trading date to another. Since the cushion is the difference between the portfolio value and the amount theoretically needed to be rolled over in the riskless asset such as to yield the guarantee at maturity, the cushion can never become negative if from one trading date to the next only a fixed proportion of the previous cushion value can be lost. As a result we get a discrete-time version of the CPPI that keeps the portfolio protection feature with probability one.

However, this very appealing result comes at a cost. From a formal point of view, it is clear, that for this strategy to be able to work, we must have a model in which the underlying assets (the risky asset as well as the riskless asset if stochastic) have continuous sample paths and we must assume a market that allows for the instantaneous execution of trading orders. From a practical point of view, this strategy is slightly more difficult to perform than the strategy with fixed trading dates since it requires continuous monitoring of the portfolio. The acceptance of continuous monitoring will be the price for the riddance of default risk.

This strategy was first investigated by Black and Perold (1992). They show that the payoff of the discrete CPPI with triggered trading dates only depends on the number of trading dates (which can possibly be arbitrarily large) and the terminal values of the risky and the riskless asset. In general, this is a very appealing result in itself and they also show that the inclusion of transaction costs does not change the basic structure of this result. However, in order to be able to deduce quantitative results like the moments or the distribution of the terminal payoff from their formula, the joined distribution of the number of trading dates and the terminal values of the underlying assets would be needed which is not known so far.

This chapter consists of three topics. The first topic is the presentation and discussion of the simple CPPI with a discretization based on triggered trading dates. In particular, analytical expressions in terms of Laplace transforms for the case of a standard Black/Scholes type market are provided, such that the only numerics required will be the inversion of a Laplace transform. Note that with a Monte-Carlo simulation it is very hard to find reliable results, since the simulation of the portfolio value is equivalent to simulating the m^{th} moment of the underlying risky asset[1] and therefore the simulation errors are hard to control in particular for high values of m. In contrast to this, the Laplace transform can be calculated very fast and with high precision[2]. The main results include the distribution and the moments of the terminal value, the distribution of the number of trading dates and the distribution of the maximum amount of borrowing required. Also convergence to the continuous-time strategy is shown as the trading restrictions vanish. The second topic is dedicated to a structural problem of the simple CPPI, the requirement of unlimited borrowing. An attempt to limit the borrowing leads to the introduction of a modification of the CPPI, the *capped CPPI*. We are able to present analytical expressions based on Laplace transforms also for the capped CPPI. Finally, the influence of transaction costs on both, the simple and the capped CPPI, is discussed. Note, that we use a basic Black/Scholes type setup and will make frequent use of the independent and identical increments property of Brownian motion. The use of the independent and identical increments property combined with the assumption of continuous sample paths basically limit our results to a log-normal model. Principally, both assumptions can be relaxed but analytical expressions do not seem possible any more if they are relaxed. Additionally, relaxing the continuous sample paths condition will allow for the possibility of default risk.

Note, that the discretization with triggered trading dates is advantageous from a theoretical point of view not only because of the riddance of default risk. It opens the door for an analytical consideration of various modifications of the CPPI strategy, the first of which is the capped CPPI. In the next chapter this discretization will be used as a vehicle to consider several other modifications. While it already seems very hard to find an analytical expression for the distribution of the payoff of the simple CPPI if a fixed set of trading dates is used as in the previous chapter, it is even harder to find any analytical

[1] See e.g. Bertrand and Prigent (2002a)

[2] See Abate and Valkó (2004) for further information on the numerical inversion of the Laplace transform and an appropriate algorithm. We use their *Mathematica*-package for our calculations.

expressions for modifications. The combination of the use of triggered trading dates and Laplace transforms translates the continuous-time problem into a random-walk problem.

From an applied point of view, the discretization with triggered trading dates can also be seen as an approximation to the following strategy. Suppose, the CPPI strategy is to be performed in discrete time with fixed trading dates where the trading dates are very frequent, for example daily (which is currently quit common) or even several times a day. As changes made to the portfolio are always subject to transaction costs, it might be decided not to change the portfolio on every single trading date but only if the changes to be made are of significant size. This strategy would require a hybrid model between fixed and triggered trading dates, but the discretization with triggered trading dates will yield a good approximation.

In particular, the chapter is organized as follows. The basic model and definitions will be introduced in section 2.1. The focus is on the simple CPPI in section 2.2. The capped CPPI will be introduced and discussed in section 2.3 and transaction costs are considered in section 2.4. Finally, in section 2.5, the strategies will be discussed with respect to long maturity times. A conclusion of the chapter is given in section 2.6.

2.1 Basic Model and Definitions

As in the previous chapter, we assume a Black-Scholes type market with two investment opportunities, a risky asset and a bond with values S_t and B_t at time t, respectively. The risky asset is assumed to follow a geometric Brownian motion, i.e. $dS_t = S_t(\mu dt + \sigma dW_t)$. The bond is assumed to be riskless and grows at a constant rate r, i.e. $dB_t = B_t r dt$. In the previous chapter we assumed a given set of fixed trading dates $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ at which trading took place while between two trading dates the strategy was a buy and hold strategy. Here we do not assume fixed trading dates but we assume an increasing potentially infinite sequence of random variables $0 = \tau_0 < \tau_1 < \tau_2 < \ldots$ at which trading takes place. However, we also assume here that no changes to the portfolio are made between the trading dates. In order to distinguish the discretization with triggered trading dates from the discretization with fixed trading dates we use the notation C_t^{tr} and V_t^{tr} for the cushion process and the value process at some time t respectively. As the floor is identical to the floor in the discretization with fixed trading dates, we keep the notation F_t. It is clear, that if the trading rule of the simple CPPI is followed on each trading date τ_i, i.e. invest the amount $mC_{\tau_i}^{tr}$ in the risky asset and the rest, $V_{\tau_i}^{tr} - mC_{\tau_i}^{tr}$, in the riskless asset, for $t \in (\tau_i, \tau_{i+1}]$ we find

$$C_t^{tr} = C_{\tau_i}^{tr} e^{r(t-\tau_i)} \left(m \frac{S_t e^{-r(t-\tau_i)}}{S_{\tau_i}} - m + 1 \right) \tag{2.1}$$

similarly to proposition 1.2.2 in the previous chapter, since this is just a consequence of the fact that no changes are made to the portfolio between any two trading dates. However, while for fixed trading dates there is always the probability of a default, i.e. a negative cushion, this possibility can be avoided by defining the trading dates appropriately. Suppose the trading dates are defined by

$$\begin{aligned} \tau_0(\omega) &:= 0, \\ \tau_n(\omega) &:= \min \left\{ \inf_{t > \tau_{n-1}} e^{-r(t-\tau_{n-1})} C_t^{tr} = k_u C_{\tau_{n-1}}^{tr}, \inf_{t > \tau_{n-1}} e^{-r(t-\tau_{n-1})} C_t^{tr} = k_d C_{\tau_{n-1}}^{tr} \right\}, \end{aligned} \tag{2.2}$$

where $k_u > 1$ and $k_d \in (0, 1)$ are some constants, the triggers. This recursive definition of the trading dates means that trading takes place whenever the discounted cushion process has gained the fraction $k_u - 1$ or lost the fraction or $1 - k_d$ relative to the value of the discounted cushion process at the previous trading date. In the following we will refer to a fractional change of $k_u - 1$ as an up-move and to a fractional change of $k_d - 1$ as a down-move. Note that the number of trading dates is not bounded from above a priori. From equation (2.2) it is obvious, that at each trading date the discounted cushion process has either multiplied with k_u relative to its value at the previous trading date as a result of an up-move or multiplied with k_d as a result of a down-move. Since both, k_u and k_d are positive constants, the discounted cushion process can never become negative by construction. The value of the discounted cushion process can become arbitrarily small due to frequent down-moves and hence frequent multiplication with k_d, but it will remain positive. However, if the discounted cushion process is positive at all times, clearly the cushion process is also positive at all times and therefore no default can occur by construction. The only thing we have to make sure is that no problem similar to the paradox of Achilles and the turtle can occur.[3]

[3]The paradox of Achilles and the turtle is as follows. Achilles and a turtle are having a race to find out who is the fastest runner. Achilles is comfortable to easily win the race and therefore they agree on the turtle getting a head start of some distance. At some time after the race has started, Achilles will reach the starting point of the turtle but the turtle will have moved forward as well. By the time, Achilles reaches the point where the turtle was when he had reached the starting point of the turtle, the turtle will have moved forward again and so on. The consequence is, that Achilles will never be able to overtake the turtle. He can shrink the lead of the turtle to an infinitesimal distance if he indeed is faster than the turtle, but he can never overtake.

To this end, notice that a combination of equations (2.1) and (2.2) yields

$$\begin{aligned} m\frac{S_t e^{-r(t-\tau_i)}}{S_{\tau_i}} - m + 1 &= k_{u,d} \\ \Leftrightarrow \quad \frac{S_t e^{-r(t-\tau_i)}}{S_{\tau_i}} &= 1 + \frac{k_{u,d}-1}{m} \end{aligned} \tag{2.3}$$

such that the trading dates could be equivalently defined by fractional changes of $\frac{k_u-1}{m}$ or $\frac{k_d-1}{m}$ in the discounted risky asset instead of fractional changes of $k_u - 1$ or $k_d - 1$ in the discounted cushion process. While this relation holds in more general model setups and is also mentioned in Black and Perold (1992), in our setup it is also equivalent to changes in the Brownian motion driving the risky asset. From the dynamics of the risky asset we know

$$S_t = S_{\tau_i} e^{(\mu - \frac{1}{2}\sigma^2)(t-\tau_i) + \sigma(W_t - W_{\tau_i})}$$

which yields

$$\begin{aligned} & \frac{S_t e^{-r(t-\tau_i)}}{S_{\tau_i}} = 1 + \frac{k_{u,d}-1}{m} \\ \Leftrightarrow \quad & e^{(\mu - r - \frac{1}{2}\sigma^2)(t-\tau_i) + \sigma(W_t - W_{\tau_i})} = 1 + \frac{k_{u,d}-1}{m} \\ \Leftrightarrow \quad & \frac{\mu - r - \frac{1}{2}\sigma^2}{\sigma}(t-\tau_i) + W_t - W_{\tau_i} = \frac{1}{\sigma}\log\left(1 + \frac{k_{u,d}-1}{m}\right) \end{aligned}$$

which relates the definition of the trading dates to changes in a Brownian motion with drift. In particular, the trading dates could equivalently be defined by trading whenever the Brownian motion with drift δ, $W_t^\delta := \delta t + W_t$, has lost the quantity

$$a(k_d) := \frac{1}{\sigma}\log\left(1 - \frac{1-k_d}{m}\right) \tag{2.4}$$

or gained the quantity

$$b(k_u) := \frac{1}{\sigma}\log\left(1 + \frac{k_u-1}{m}\right) \tag{2.5}$$

where

$$\delta := \frac{\mu - r - \frac{1}{2}\sigma^2}{\sigma}. \tag{2.6}$$

Note that $a(k_d) < 0$ and $b(k_u) > 0$. For most parts of the chapter we will shorten the notation to $a = a(k_d)$ and $b = b(k_u)$. The situation is as schematically depicted in figure 2.1. For the first trading date to occur, the Brownian motion with drift must hit either of the barriers a and b. If the barrier a is hit, the discounted cushion process of the simple CPPI has multiplied with k_d and likewise if the barrier b is hit, the discounted cushion process has multiplied with k_u. From that point onwards, for the second trading date to occur, the Brownian must again hit the same barriers a or b and so on until neither of the barriers is hit any more up to maturity time T. In the remainder of the chapter we will need information about the trading dates, i.e. their distribution. However, we know now that the distribution of the time between any two successive trading dates $\tau_{i+1} - \tau_i$ is given by the time the Brownian motion with drift needs to hit the barriers a or b. Due to the independent and identical increments property of Brownian motion, it is clear that $\tau_{i+1} - \tau_i$ must be independently identically distributed. Coming back to the problem of Achilles and the turtle, the independent and identical distribution of the difference between any two trading dates is sufficient to conclude that such a problem can not occur in our setup. It is now crucial to find expressions for the hitting time densities. Fortunately the double barrier problem has been solved already such that from Hall (1997) we can take the following lemma.

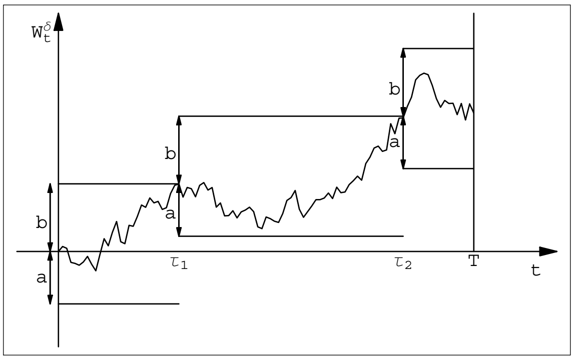

Figure 2.1: Sample path of Brownian motion with drift hitting exactly twice the upper barrier before T.

Lemma 2.1.1 (Hitting time densities)

Let $(W_t^\delta)_{t\geq 0}$ a Brownian motion with drift δ started from 0 and $a < 0 < b$ two constants. Let further $t_0 > 0$ and define $g_1(t|\gamma,\delta) := \frac{\gamma}{\sqrt{2\pi t^3}} e^{-\frac{1}{2}(\frac{\gamma^2}{t}+\delta^2 t)}$ and $g_2(t|\gamma,\delta) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{1}{2}(\frac{\gamma^2}{t}+\delta^2 t)}$.[4]

a) The defective probability density function $p_u(t|a,b,\delta)$, $t \in [0,t_0]$ for the time of hitting the upper barrier b before the lower barrier a and before time t_0 is given by

$$p_u(t|a,b,\delta) = e^{b\delta}\left(\sum_{j=0}^{\infty} g_1(t|2j(b-a)+b,\delta) - \sum_{j=0}^{\infty} g_1(t|2j(b-a)+b-2a,\delta)\right)$$

b) The defective probability density function $p_d(t|a,b,\delta)$, $t \in [0,t_0]$ for the time of hitting the lower barrier a before the upper barrier b is given by

$$p_d(t|a,b,\delta) = e^{a\delta}\left(\sum_{j=0}^{\infty} g_1(t|2j(b-a)-a,\delta) - \sum_{j=0}^{\infty} g_1(t|2j(b-a)+2b-a,\delta)\right)$$

c) The joint probability density function $p^{t_0}(z|a,b,\delta)$ of $W_{t_0}^\delta$ and hitting none of the two barriers a and b up to time t_0 is given by

$$\begin{aligned} p^{t_0}(z|a,b,\delta) = e^{\delta z}\Bigg(& g_2(t_0|z,\delta) + \sum_{j=1}^{\infty} g_2(t_0|z-2j(b-a),\delta) + \sum_{j=1}^{\infty} g_2(t_0|z+2j(b-a),\delta) \\ & - \sum_{j=1}^{\infty} g_2(t_0|z+2j(b-a)-2b,\delta) - \sum_{j=1}^{\infty} g_2(t_0|z-2j(b-a)-2a,\delta)\Bigg) \end{aligned}$$

From our discussion above, it is clear that with a, b, δ as in (2.4), (2.5), (2.6), respectively, for all i, $\int_0^{T-\tau_i} p_u(t|a,b,\delta)dt$ is the probability (conditioned on the knowledge of time τ_i) of the discounted cushion process increasing by the factor k_u, $\int_0^{T-\tau_i} p_d(t|a,b,\delta)dt$ is the probability of the discounted cushion process decreasing by the factor k_d and $\int_a^b p^{T-\tau_i}(z|a,b,\delta)dz$ is the probability of the discounted cushion process staying within the bounds $(k_d C_{\tau_i}^{tr} e^{-r\tau_i}, k_u C_{\tau_i}^{tr} e^{-r\tau_i})$. Since our results are all based on Laplace transforms we will need the following proposition. The Laplace transform of some function $f(t)$ at the point s with respect to the variable t will be denoted by $\mathcal{L}_{t,s}\{f(t)\}$ and likewise the inverse Laplace transform of some function $f(s)$ with respect to the variable s will be denoted by $\mathcal{L}_{s,t}^{-1}\{f(t)\}$.[5]

[4]Note, that $e^{\gamma\delta}g_1$ and $\delta e^{\gamma\delta}g_2$ are the densities of the inverse gaussian and the reciprocal inverse gaussian distribution respectively. See for example Barndorff-Nielsen and Koudou (1998).

[5]See also section A.2 in the appendix for a brief introduction to Laplace transforms.

Proposition 2.1.2 (Laplace transforms of the hitting time densities)
The Laplace transforms of the densities $p_u(t|a,b,\delta)$, $p_d(t|a,b,\delta)$ and $p^t(z|a,b,\delta)$ with respect to t are given by

$$\text{a)}\quad u(s|a,b,\delta) := \mathcal{L}_{t,s}\left\{p_u(t|a,b,\delta)\right\} = e^{b\delta-b\sqrt{2s+\delta^2}}\frac{1-e^{2a\sqrt{2s+\delta^2}}}{1-e^{-2(b-a)\sqrt{2s+\delta^2}}}$$

$$\text{b)}\quad d(s|a,b,\delta) := \mathcal{L}_{t,s}\left\{p_d(t|a,b,\delta)\right\} = e^{a\delta+a\sqrt{2s+\delta^2}}\frac{1-e^{-2b\sqrt{2s+\delta^2}}}{1-e^{-2(b-a)\sqrt{2s+\delta^2}}}$$

$$\text{c)}\quad \rho(s,z|a,b,\delta) := \mathcal{L}_{t,s}\left\{p^t(z|a,b,\delta)\right\} = \begin{cases} d(s)\frac{e^{-a\delta-a\sqrt{2s+\delta^2}}}{\sqrt{2s+\delta^2}}\left(e^{\delta z+z\sqrt{2s+\delta^2}} - e^{\delta z-z\sqrt{2s+\delta^2}+2a\sqrt{2s+\delta^2}}\right) & , z\le 0 \\ u(s)\frac{e^{-b\delta+b\sqrt{2s+\delta^2}}}{\sqrt{2s+\delta^2}}\left(e^{\delta z-z\sqrt{2s+\delta^2}} - e^{\delta z+z\sqrt{2s+\delta^2}-2b\sqrt{2s+\delta^2}}\right) & , z>0. \end{cases}$$

PROOF: With a hint to footnote (4) it is well known that the Laplace transform of g_1 is given by $\mathcal{L}_{t,s}\left\{g_1(t|\gamma,\delta)\right\} = e^{-\gamma\sqrt{2s+\delta^2}}$, $\quad \gamma > 0$. With this we get

$$\begin{aligned}
&\mathcal{L}_{t,s}\left\{p_u(t|a,b,\delta)\right\} \\
&= e^{b\delta}\left(\sum_{j=0}^{\infty}\mathcal{L}_{t,s}\left\{g_1(t|2j(b-a)+b,\delta)\right\} - \sum_{j=0}^{\infty}\mathcal{L}_{t,s}\left\{g_1(t|2j(b-a)+b-2a,\delta)\right\}\right) \\
&= e^{b\delta}\left(\sum_{j=0}^{\infty}e^{-(2j(b-a)+b)\sqrt{2s+\delta^2}} - \sum_{j=0}^{\infty}e^{-(2j(b-a)+b-2a)\sqrt{2s+\delta^2}}\right)
\end{aligned}$$

and an application of the summation formula for a geometric series yields

$$\mathcal{L}_{t,s}\left\{p_u(t|a,b,\delta)\right\} = e^{b\delta-b\sqrt{2s+\delta^2}}\left(\frac{1}{1-e^{-2(b-a)\sqrt{2s+\delta^2}}} - \frac{e^{2a\sqrt{2s+\delta^2}}}{1-e^{-2(b-a)\sqrt{2s+\delta^2}}}\right)$$

proofing a). The proof of b) is completely analogous. For c) note that with respect to footnote (4) we know $\mathcal{L}_{t,s}\left\{g_2(t|\gamma,\delta)\right\} = \frac{e^{-|\gamma|\sqrt{2s+\delta^2}}}{\sqrt{2s+\delta^2}}$. Therefore

$$\begin{aligned}
&\mathcal{L}_{t,s}\left\{p^t(z|a,b,\delta)\right\} \\
&= e^{\delta z}\left(\frac{e^{-|z|\sqrt{2s+\delta^2}}}{\sqrt{2s+\delta^2}} + \sum_{j=1}^{\infty}\frac{e^{(z-2j(b-a))\sqrt{2s+\delta^2}}}{\sqrt{2s+\delta^2}} + \sum_{j=1}^{\infty}\frac{e^{-(z+2j(b-a))\sqrt{2s+\delta^2}}}{\sqrt{2s+\delta^2}}\right. \\
&\qquad \left. - \sum_{j=1}^{\infty}\frac{e^{-(z+2j(b-a)-2b)\sqrt{2s+\delta^2}}}{\sqrt{2s+\delta^2}} - \sum_{j=1}^{\infty}\frac{e^{(z-2j(b-a)-2a)\sqrt{2s+\delta^2}}}{\sqrt{2s+\delta^2}}\right) \\
&= \frac{e^{\delta z}}{\sqrt{2s+\delta^2}(1-e^{-2(b-a)\sqrt{2s+\delta^2}})}\left(e^{-|z|\sqrt{2s+\delta^2}}(1-e^{-2(b-a)\sqrt{2s+\delta^2}}) + e^{z\sqrt{2s+\delta^2}-2(b-a)\sqrt{2s+\delta^2}}\right. \\
&\qquad \left. + e^{-z\sqrt{2s+\delta^2}-2(b-a)\sqrt{2s+\delta^2}} - e^{-z\sqrt{2s+\delta^2}+2a\sqrt{2s+\delta^2}} - e^{z\sqrt{2s+\delta^2}-2b\sqrt{2s+\delta^2}}\right)
\end{aligned}$$

Considering the cases $z \le 0$ and $z > 0$ directly yields the result. □

For most parts of the chapter we will use the shorter notation $u(s) = u(s|a, b, \delta)$, $d(s) = d(s|a, b, \delta)$ and $\rho(s, z) = \rho(s, z|a, b, \delta)$.

2.2 The Simple Discrete CPPI With Triggered Trading Dates

We start this section by giving the distribution of the number of trading dates. The number of trading dates is important in particular for a comparison with the discrete CPPI with fixed trading dates as in the previous chapter as well as with respect to section 2.4 where we will consider transaction costs.

Proposition 2.2.1 (Distribution of the number of trading dates)
Let $N(\omega) := \sup\{n \in \mathbb{N}_0 | \tau_n \leq T\}$ be the number of trading dates. Then the distribution of N is given by:

a) For $n \in \mathbb{N}_0$: $\quad P(N = n) = \mathcal{L}^{-1}_{s,T}\left\{\frac{1-u(s)-d(s)}{s}(u(s)+d(s))^n\right\}$

b) For $n \in \mathbb{N}_0$: $\quad P(N \leq n) = \mathcal{L}^{-1}_{s,T}\left\{\frac{1-(u(s)+d(s))^{n+1}}{s}\right\}$

and the expected number of trading dates is given by

c) $E[N] = \mathcal{L}^{-1}_{s,T}\left\{\frac{1}{s}\frac{u(s)+d(s)}{1-u(s)-d(s)}\right\}$

Proof: For $n \geq 1$, let $f_{\tau_n}(t)$ denote the probability density function of τ_n. Note that $f_{\tau_1}(t) = p_u(t|a, b, \delta) + p_d(t|a, b, \delta)$ and further $f_{\tau_n} = \underbrace{f_{\tau_1} * \cdots * f_{\tau_1}}_{n \text{ times}}$ where $*$ denotes the convolution. Therefore

$$\begin{aligned} P(N(\omega) = n) &= P(\tau_n \leq T \wedge \tau_{n+1} > T) \\ &= \int_0^T f_{\tau_n}(t) P(\tau_{n+1} > T | \tau_n = t) dt \\ &= \left(f_{\tau_n}(\cdot) * P(\tau_1 > \cdot)\right)(T) \\ &= \mathcal{L}^{-1}_{s,T}\left\{(u(s)+d(s))^n \int_a^b \rho(s, z) dz\right\} \end{aligned}$$

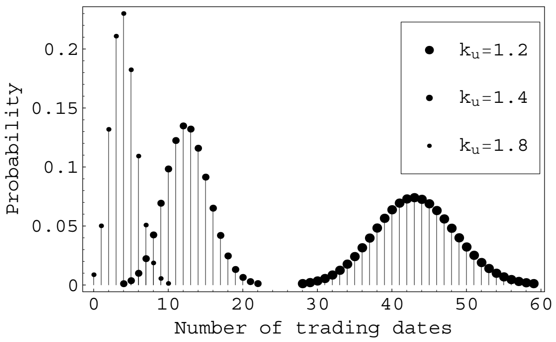

Figure 2.2: Point probabilities for different values of k_u and $k_d = k_u^{-1}$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.3$, $T = 1$.

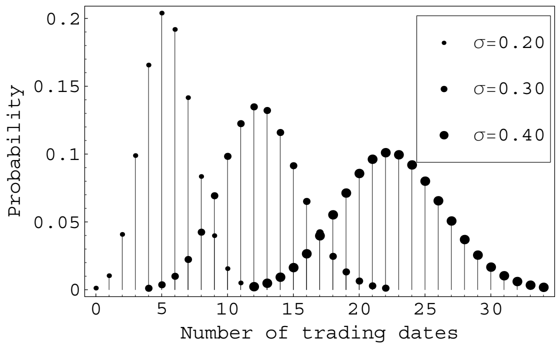

Figure 2.3: Point probabilities for different values of the volatility σ and $k_u = 1.4$, $k_d = k_u^{-1}$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $T = 1$.

follows from the convolution property of the Laplace transform[6]. Direct calculation shows that $\int_a^b \rho(s,z)dx = \frac{1-u(s)-d(s)}{s}$. The proof can be found in the appendix, proposition A.3.1. This proofs part a) for $n \geq 1$ and the case $n = 0$ is trivial. Part b) follows from a) by summing over n. As for part c), note that

$$E[N] = \sum_{n=0}^{\infty} nP(N = n)$$

and hence the result follows from part a) and

$$\sum_{n=0}^{\infty} nx^n = x \sum_{n=0}^{\infty} \frac{\partial x^n}{\partial x} = x \frac{\partial \sum_{n=0}^{\infty} x^n}{\partial x} = \frac{x}{(1-x)^2}$$

□

It is important to notice that the distribution of the number of trading dates only depends on $u(s)$ and $d(s)$, i.e. the Laplace transform of the hitting time densities. This means in particular, that the number of trading dates is independent of the guarantee $G = F_T$. This is no surprise, since the trading dates are defined on relative changes in the cushion and the guarantee only influences the size of the cushion at time $t = 0$.[7]

In figure 2.2 and 2.3 we have depicted the distribution of the number of trading dates for different values of k_u and σ respectively. It is striking at first glance that the distributions

[6]See proposition A.2.5,b) in the appendix.

[7]Also, Proposition 2.2.1 only requires the i.i.d. increments property and not the continuous sample paths property such that the very same formula holds if the risky asset is modelled by a Lévy-process. Of course $u(s)$ and $d(s)$ have to be adjusted in that case.

look very symmetric, although there are possibly arbitrarily many trading dates and clearly no less than zero. Also, it can be seen, that there is approximately a quadratic relation between the number of trading dates and the volatility. Doubling the volatility means quadrupling the number of trading dates. Intuitively we can do comparative statics on the basis of equations (2.4), (2.5) and (2.6). The drift δ of the Brownian motion has a minor influence on the number of trading dates and therefore also μ and r do not influence the number of trading dates much. However, the size of the barriers a and b is crucial. Doubling either of these barriers in size approximately results in half as many trading dates. Therefore, since in view of equations (2.4) and (2.5) doubling the volatility σ means cutting the barriers a and b half in size, it is intuitively clear, that the number of trading dates must quadruple which is confirmed by figure 2.3. Also, doubling the multiplier m approximately means quadrupling the number of trading dates. Using a simple first-order Taylor approximation for the exponential function, $e^x \approx 1 + x$, we find $a \approx -\frac{1-k_d}{\sigma m}$ and $b \approx \frac{k_u-1}{\sigma m}$ from which this effect is apparent. While likewise the number of trading dates can be found approximately proportional with respect to $k_u - 1$ and $1 - k_d$ if only either $k_u - 1$ or $1 - k_d$ is changed, it is more difficult to determine the influence of k_u if the symmetric case $k_d = \frac{1}{k_u}$ is considered as in figure 2.2. Clearly, the number of trading dates is less than quadratic in $k_u - 1$ for the symmetric case. Finally, it is obvious that the number of trading dates must be approximately proportional with respect to maturity time T.

Unfortunately, since the formulas in proposition 2.2.1 involve an inverse Laplace transform it is not possible to directly do any comparative statics on the basis of these formulas. However, as the inverse Laplace transform can be expressed as an integral[8], the formulas can easily be derived by exchanging the inverse Laplace transform and the differentiation. For example we can write

$$\frac{\partial}{\partial \sigma} P(N \leq n) = \mathcal{L}^{-1}_{s,T} \left\{ -\frac{1}{s} \frac{\partial}{\partial \sigma} (u(s) + d(s))^{n+1} \right\}$$

and likewise for the other parameters if a precise comparative statics is required.

[8] See appendix A.2.

Proposition 2.2.2 (Moments of the simple CPPI with triggered trading dates)

Let $k_u > 1$, $k_d \in (0,1)$. The j-th moment of the terminal value of the cushion is given by

$$E\left[(C_T^{tr})^j\right] = C_0^j e^{jrT} \mathcal{L}_{s,T}^{-1} \left\{ \frac{\int_a^b (me^{\sigma z} - m + 1)^j \rho(s,z) dz}{1 - k_u^j u(s) - k_d^j d(s)} \right\}$$

where the integral is given explicitly in proposition A.3.1.

Proof: Let $\Omega^{i,n} := \{\omega \in \Omega | \quad \tau_n \leq T \wedge C_{\tau_n}^{tr} = C_0 e^{r\tau_n} k_u^i k_d^{n-i}\}, i \in \{0, ..., n\}, n \in \mathbb{N}_0$ be the event that the discounted cushion process performs i up-moves and $n-i$ down-moves. Note that $(\Omega^{i,n} \cap \{\tau_{n+1} > T\})_{i \in \{0,...,n\}, n \in \mathbb{N}_0}$ is a partitioning of Ω. Let further $f_{i,n}(t)$ denote the probability density function for the first time at which the event $\Omega^{i,n}$ occurs. Since there are $\binom{n}{i}$ different ways of the discounted cushion process performing i up-moves and $n-i$ down-moves, we have

$$f_{i,n} = \binom{n}{i} \underbrace{p_u(\cdot|a,b,\delta) * \cdots * p_u(\cdot|a,b,\delta)}_{i \text{ times}} * \underbrace{p_d(\cdot|a,b,\delta) * \cdots * p_d(\cdot|a,b,\delta)}_{n-i \text{ times}}$$

and hence

$$\mathcal{L}_{t,s}\{f_{i,n}(t)\} = \binom{n}{i} u(s)^i d(s)^{n-i}$$

follows from the convolution property of the Laplace transform and proposition 2.1.2. From equations (2.1) and (2.2) we know that $(C_T^{tr})^j = C_0^j e^{jrT} k_u^{ji} k_d^{j(n-i)} (me^{\sigma(W_T^\delta - W_{\tau_n}^\delta)} - m + 1)^j$ on the set $\Omega^{i,n} \cap \{\tau_{n+1} > T\}$. Further we can find

$$\begin{aligned}
& E\left[(me^{\sigma(W_T^\delta - W_{\tau_n}^\delta)} - m + 1)^j 1_{\Omega^{i,n}} 1_{\{\tau_{n+1} > T\}}\right] \\
&= \int_0^T f_{i,n}(t) E\left[(me^{\sigma(W_T^\delta - W_{\tau_n}^\delta)} - m + 1)^j 1_{\{\tau_{n+1} > T\}} | \tau_n = t\right] dt \\
&= \int_0^T f_{i,n}(t) \int_a^b (me^{\sigma z} - m + 1)^j p^{T-t}(z|a,b,\delta) dz dt \\
&= \int_a^b (me^{\sigma z} - m + 1)^j \left(f_{i,n}(\cdot) * p^{(\cdot)}(z|a,b,\delta)\right)(T) dz \\
&= \mathcal{L}_{s,T}^{-1} \left\{ \binom{n}{i} u(s)^i d(s)^{n-i} \int_a^b (me^{\sigma z} - m + 1)^j \rho(s,z) dz \right\}.
\end{aligned}$$

m	$E[N]$	k_u	Mean	Stdv.
12	12	1.4080 (1.9768)	1076.87 (1074.28)	126.08 (626.14)
12	24	1.2756 (1.6245)	1077.42 (1075.97)	132.46 (872.32)
12	48	1.1885 (1.4117)	1077.72 (1076.94)	136.07 (1076.37)
12	96	1.1301 (1.2768)	1077.87 (1077.48)	138.01 (1214.38)
18	12	1.6654 (2.7373)	1091.69 (1082.85)	332.24 (2999.90)
18	24	1.4389 (2.0579)	1093.80 (1088.25)	389.85 (8763.30)
18	48	1.2950 (1.6730)	1094.98 (1091.82)	428.42 (19614.13)
18	96	1.2011 (1.4414)	1095.61 (1093.92)	451.17 (33035.68)

Table 2.1: Moments of the discrete CPPI with triggered trading dates. The parameters are $T = 1$, $V_0 = 1000$, $G = 1000$, $\mu = 0.085$, $r = 0.05$ and $\sigma = 0.1$ ($\sigma = 0.2$ respectively).

Therefore we get

$$
\begin{aligned}
E\left[(C_T^{tr})^j\right] &= \sum_{n=0}^{\infty}\sum_{i=0}^{n} E\left[(C_T^{tr})^j 1_{\Omega^{i,n}} 1_{\{\tau_{n+1}>T\}}\right] \\
&= C_0^j e^{jrT} \mathcal{L}_{s,T}^{-1}\left\{\int_a^b (me^{\sigma z}-m+1)^j \rho(s,z)dz \sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i} k_u^{ij} u(s)^i k_d^{(n-i)j} d(s)^{n-i}\right\} \\
&= C_0^j e^{jrT} \mathcal{L}_{s,T}^{-1}\left\{\frac{\int_a^b (me^{\sigma z}-m+1)^j \rho(s,z)dz}{1-k_u^j u(s) - k_d^j d(s)}\right\}
\end{aligned}
$$

using the binomial formula and the summation formula for the geometric series. □

From the moments of the cushion, we can easily deduce the moments of the terminal value of the portfolio since $V_T^{tr} = G + C_T^{tr}$. In particular, the expected value and the variance are immediately given by

$$
\begin{aligned}
E\left[V_T^{tr}\right] &= G + E\left[C_T^{tr}\right] \\
Var\left[V_T^{tr}\right] &= Var\left[C_T^{tr}\right] = E\left[(C_T^{tr})^2\right] - E\left[C_T^{tr}\right]^2.
\end{aligned}
$$

Table 2.1 shows the expected terminal value and the standard deviation of the terminal value for different values of m and different values of the triggers k_u and k_d. We have chosen $k_d = \frac{1}{k_u}$ and k_u such that the expected number of trading dates are 12, 24, 48 and 96. Remember, that it is quite easy to translate the values of k_u into changes in the discounted risky asset with the help of equation (2.3). For example, in order to have an expected number of 96 trading dates per year with $m = 18$ and $\sigma = 0.20$, trading must take place whenever the discounted risky asset has gained 2.45% or lost 1.7%. The other parameter values are chosen such as to equal the choice of table 1.2 for the simple discrete-time CPPI with fixed trading dates and the simple continuous-time CPPI. It is important to notice that the moments of the discrete CPPI with fixed trading dates are mostly closer to the moments of the continuous CPPI than the moments of the discrete CPPI with triggered trading dates if the expected number of triggered trading dates equals the number of fixed trading dates. In particular in view of the very high shortfall probabilities of the CPPI with fixed trading dates for the case $m = 18$ and $\sigma = 0.20$ this is quite remarkable.

The lower standard deviation of the CPPI with triggered trading dates in comparison with the CPPI with fixed trading dates can surely partly be explained with the missing possibility of a shortfall. However, since the expected terminal value is also mostly lower and considering the magnitude of the standard deviation, the lower standard deviation must also or mainly stem from the large payoffs. The reason here seems to be the choice of the triggers, in particular $k_d = \frac{1}{k_u}$. This choice favors conservative adaptations of the portfolio since the lower barrier $1 - k_d$ is smaller than the upper barrier $k_u - 1$. If $1 - k_d$ was to be chosen such as to equal $k_u - 1$, i.e. $k_d = 2 - k_u$, then last line in table 2.1 would read

m	$E[N]$	k_u	Mean	Stdv.
18	96	1.1836 (1.3671)	1096.24 (1096.35)	469.94 (54750.79)

giving a higher expectation and standard deviation of the terminal value than for the CPPI with fixed trading dates.

Let us now turn to the distribution of the terminal value of the CPPI. It will simplify things considerably if $k_d = \frac{1}{k_u}$ and we will make this assumption from now on. Let N denote the number of trading dates before time T, then τ_N is the last trading date before time T. Further, let n denote the number of net up-moves, i.e. the number of up-moves minus the number of down-moves at time τ_N and therefore also at time T. Then it immediately follows from the definition of the trading dates, i.e. equation (2.2), that

$$C^{tr}_{\tau_N} = C_0 e^{r\tau_N} k_u^n.$$

Also, since τ_N is the last trading date before time T, it follows that

$$C_T^{tr} \in (C_0 e^{rT} k_u^{n-1}, C_0 e^{rT} k_u^{n+1})$$

since otherwise there would be another up- or down-move. Now, suppose some $x \in (G, \infty)$ and choose n_x such that $C_0 e^{rT} k_u^{n_x} \leq x - G < C_0 e^{rT} k_u^{n_x+1}$, then it is apparent that whenever the number of net up-moves up to time T is less than n_x, $n < n_x$, the condition $V_T^{tr} \leq x$ or equivalently $C_T^{tr} \leq x - G$ must be satisfied, while for all $n > n_x + 1$ the condition $C_T^{tr} \leq x - G$ can not be satisfied. For the cases $n = n_x$ and $n = n_x + 1$, it depends upon the behavior of C_t^{tr} in the interval $(\tau_N, T]$ whether the condition $C_T^{tr} \leq x - G$ is satisfied or not. It is this simple idea, that lies beneath our expression for the distribution of the terminal value of the CPPI.

Proposition 2.2.3 (Distribution of the simple CPPI)
Let $k_u > 1$, $k_d = k_u^{-1}$. Further let [9],

$$\begin{aligned} n_x &:= \left\lfloor \frac{\log \frac{x-G}{C_0 e^{rT}}}{\log k_u} \right\rfloor \\ y_1(x) &:= \frac{1}{\sigma} \log \left(\frac{x-G}{m C_0 e^{rT} k_u^{n_x}} + \frac{m-1}{m} \right) \\ y_2(x) &:= \frac{1}{\sigma} \log \left(\frac{x-G}{m C_0 e^{rT} k_u^{n_x+1}} + \frac{m-1}{m} \right) \end{aligned}$$

for $x \in (G, \infty)$. Then the distribution of the terminal value of the simple CPPI with triggered trading dates, i.e. the probability $P(V_T^{tr} \leq x)$, is given by:

$$\mathcal{L}_{s,T}^{-1} \left\{ \frac{1 - u(s) - d(s)}{s} Q(n_x - 1, s) + q(n_x, s) \int_a^{y_1(x)} \rho(s,z) dz + q(n_x + 1, s) \int_a^{y_2(x)} \rho(s,z) dz \right\}$$

where $q(k, s) = q(k|u(s), d(s))$, $Q(k, s) = Q(k|u(s), d(s))$ for all k as in lemma A.1.2 and the integrals as in proposition A.3.2.[10]

Proof: Note that n_x is the solution to

$$\max \left\{ n \in \mathbb{Z} | G + C_0 e^{rT} k_u^n \leq x \right\}$$

and therefore the condition $V_T^{tr} \leq x$ is satisfied for all $n < n_x$, independent of the behavior of C_t^{tr} in the interval $(\tau_N, T]$ (remember that n denotes the number of net up-moves at

[9]For some $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the largest integer less or equal to x. Likewise, we will later use the notation $\lceil x \rceil$ to denote the smallest integer larger or equal to x.

[10]We will mostly use the shorter notation $q(k, s)$ and $Q(k, s)$ from now on.

time τ_N and τ_N is the last trading date before maturity time T). Therefore

$$P(V_T^{tr} \leq x,\ n = k) = P(n = k)$$

for all $k \in \mathbb{Z},\ k < n_x$ is immediate. Similarly to the proof of proposition 2.2.2 let us now define

$$\Omega^{k,i} := \left\{ \omega \in \Omega | \quad \tau_{|k|+2i} \leq T \wedge C^{tr}_{\tau_{|k|+2i}} = C_0 e^{r\tau_{|k|+2i}} k_u^k \right\},\ i \in \mathbb{N}_0,\ k \in \mathbb{Z}$$

and $f_{k,i}(t)$ denote the probability density function for the first time at which the event $\Omega^{k,i}$ occurs. The Laplace transform of $f_{k,i}$ is then given by

$$\mathcal{L}_{t,s}\left\{ f_{k,i}(t) \right\} = \begin{cases} \binom{|k|+2i}{i} u(s)^i d(s)^{|k|+i} & ,\ k < 0 \\ \binom{|k|+2i}{i} u(s)^{|k|+i} d(s)^i & ,\ k \geq 0. \end{cases}$$

Therefore we get

$$\begin{aligned}
P(n = k) &= \sum_{i=0}^{\infty} P(\Omega^{k,i}, \tau_{|k|+2i+1} > T) \\
&= \sum_{i=0}^{\infty} \int_0^T f_{k,i}(t) P(\tau_{|k|+2i+1} > T |\ \tau_{|k|+2i} = t) dt \\
&= \sum_{i=0}^{\infty} \int_0^T f_{k,i}(t) \int_a^b p^{T-t}(z|a,b,\delta) dz dt \\
&= \sum_{i=0}^{\infty} \int_a^b \left(f_{k,i}(\cdot) * p^{(\cdot)}(z|a,b,\delta) \right)(T) dz \\
&= \begin{cases} \mathcal{L}^{-1}_{s,T} \left\{ \sum\limits_{i=0}^{\infty} \binom{|k|+2i}{i} u(s)^i d(s)^{|k|+i} \int\limits_a^b \rho(s,z) dz \right\} & ,\ k < 0 \\ \mathcal{L}^{-1}_{s,T} \left\{ \sum\limits_{i=0}^{\infty} \binom{|k|+2i}{i} u(s)^{|k|+i} d(s)^i \int\limits_a^b \rho(s,z) dz \right\} & ,\ k \geq 0 \end{cases} \\
&= \mathcal{L}^{-1}_{s,T} \left\{ q(k,s) \int_a^b \rho(s,z) dz \right\}
\end{aligned}$$

where the last equality follows from a glimpse at lemma A.1.2. Note the analogy to random walks which occurs as a consequence of the convolution property of Laplace transforms that turns convolutions into products.

Let us now turn to the case $n = n_x$. We know that

$$V_T^{tr} = G + C^{tr}_{\tau_N} \frac{C_T^{tr}}{C^{tr}_{\tau_N}} = G + C_0 e^{rT} k_u^{n_x} \left(m e^{\sigma(W_T^\delta - W_{\tau_N}^\delta)} - m + 1 \right)$$

and therefore, in this case,

$$V_T^{tr} \leq x \;\Leftrightarrow\; a < W_T^\delta - W_{\tau_N}^\delta \leq \frac{1}{\sigma}\log\left(\frac{x-G}{mC_0e^{rT}k_u^{n_x}} + \frac{m-1}{m}\right) = y_1(x)$$

from which

$$P(V_T^{tr} \leq x,\; n = n_x) = \mathcal{L}_{s,T}^{-1}\left\{q(n_x, s)\int\limits_a^{y_1(x)} \rho(s,z)dz\right\}$$

can be concluded analogously to the cases $n < n_x$. Finally, for the case $n = n_x + 1$, we find

$$P(V_T^{tr} \leq x,\; n = n_x + 1) = \mathcal{L}_{s,T}^{-1}\left\{q(n_x + 1, s)\int\limits_a^{y_2(x)} \rho(s,z)dz\right\}$$

analogously to the case $n = n_x$. The assertion is now a direct consequence of

$$P(V_T^{tr} \leq x) = \sum_{k=-\infty}^{n_x+1} P(V_T^{tr} \leq x,\; n = k)$$

since $\int\limits_a^b \rho(s,z)dz = \frac{1-u(s)-d(s)}{s}$ is already known. □

Corollary 2.2.4 (Density of the simple CPPI)
In the notation of proposition 2.2.3, the probability density function of the terminal value of the simple CPPI with triggered trading dates, $p_{V_T^{tr}}(x)$, is given by:

$$p_{V_T^{tr}}(x) \;=\; \mathcal{L}_{s,T}^{-1}\left\{q(n_x, s)\rho(s, y_1(x))\frac{\partial\, y_1}{\partial\, x} + q(n_x + 1, s)\rho(s, y_2(x))\frac{\partial\, y_2}{\partial\, x}\right\}$$

where

$$\frac{\partial\, y_1}{\partial\, x} = \frac{1}{\sigma(x-G) + \sigma(m-1)C_0e^{rT}k_u^{n_x}}$$

$$\frac{\partial\, y_2}{\partial\, x} = \frac{1}{\sigma(x-G) + \sigma(m-1)C_0e^{rT}k_u^{n_x+1}}$$

Proof: The formulas can be immediately verified from theorem 2.2.3 by differentiation.
□

In figure 2.4 we have plotted different densities of the terminal value of the simple CPPI. The triggers k_u are chosen such as to yield 3, 6, 12, 24 expected transactions per year. The choice of the other parameters deviates from our usual choice in chapter 1. The reason for this is that the density of the simple CPPI (continuous as well as discrete) becomes

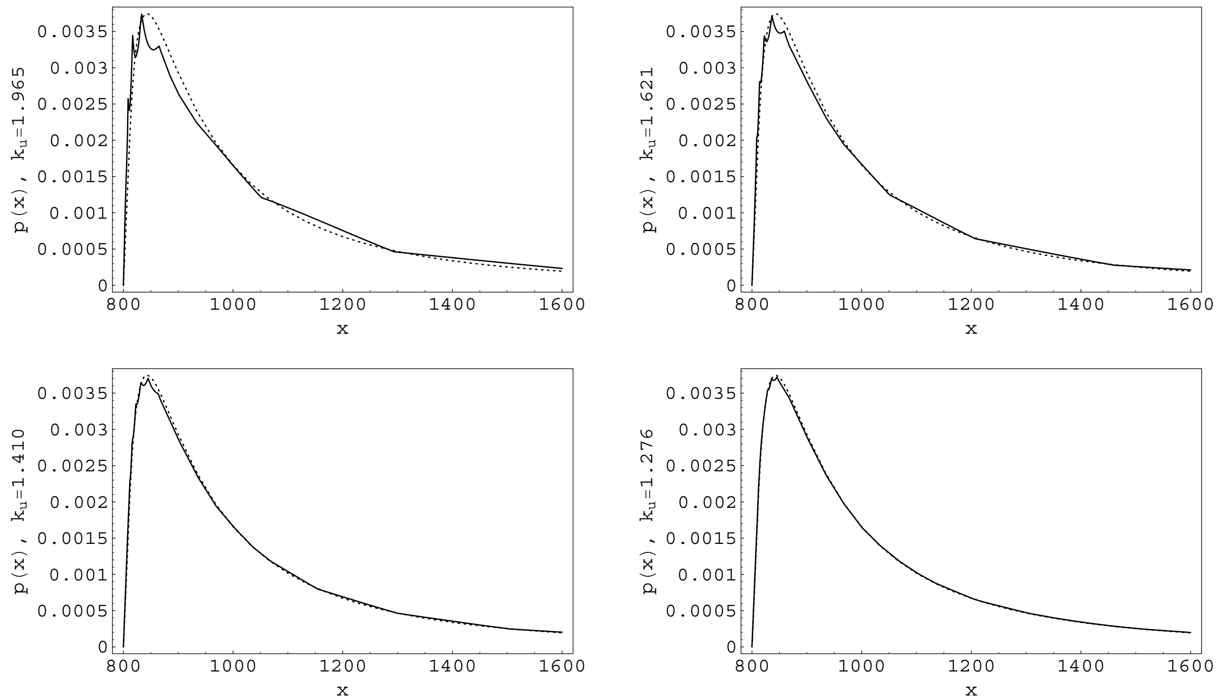

Figure 2.4: Densities of the terminal value of the discrete CPPI for $G = 800$, $V_0 = 1000$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.30$, $T = 1$, and the k_u chosen to give approximately 3, 6, 12, 24 expected trading dates. The dotted line is the continuous CPPI.

very spiky close to the guarantee for high values of m which makes it impossible to see differences between densities of the discrete and the continuous version. The volatility has been set to a high value of $\sigma = 0.30$ to pronounce the differences. We will use these parameters as a standard throughout this chapter. It is not difficult to notice the spikes in figure 2.4. They are placed at the values $x = C_0 e^{rT} k_u^n$ for $n \in \mathbb{Z}$ and thus refer to the trading dates, where changes in the portfolio are made. It is not surprising that the density is not differentiable in these points. However, it is quite remarkable, how few transactions are required to resemble the density of the continuous CPPI. This is even more surprising if one keeps in mind, that for the same parameters the discretization with fixed trading dates yields default risks of 11.15%, 4.52%, 0.44% and 0.003% respectively.

In order to further investigate how good or bad the discrete version of the CPPI resembles the continuous version, we consider the terminal value of the simple continuous-time CPPI conditioned on the terminal value of the simple CPPI with triggered trading dates taking some fixed value. In particular, we can show

Proposition 2.2.5 (Conditional Distribution)
Using the same notation as in proposition 2.2.3 and the additional notation

$$\begin{aligned} n_{1,u,x}(j) &:= \max\{n_x, 0\} + j \\ n_{1,d,x}(j) &:= -\min\{n_x, 0\} + j \\ n_{2,u,x}(j) &:= \max\{n_x + 1, 0\} + j \\ n_{2,d,x}(j) &:= -\min\{n_x + 1, 0\} + j \end{aligned}$$

for $j \in \mathbb{N}_0$*, it holds:*

a) *Given that the terminal value of the discrete simple CPPI equals* x*,* $V_T^{tr} = x$*, the terminal value of the continuous simple CPPI,* V_T^{cont}*, can only take values from the discrete set*

$$\bigcup_{j=0}^{\infty} v_{1,j}(x) \cup \bigcup_{j=0}^{\infty} v_{2,j}(x),$$

where

$$\begin{aligned} v_{1,j}(x) &:= G + C_0 e^{(r-\frac{1}{2}m(m-1)\sigma^2)T + n_{1,d,x}(j)m\sigma a + n_{1,u,x}(j)m\sigma b + my_1(x)} \\ v_{2,j}(x) &:= G + C_0 e^{(r-\frac{1}{2}m(m-1)\sigma^2)T + n_{2,d,x}(j)m\sigma a + n_{2,u,x}(j)m\sigma b + my_2(x)} \end{aligned}$$

b) *The distribution of the terminal value of the simple continuous-time CPPI conditional on* $V_T^{tr} = x$ *is a discrete distribution and for* $i \in \{1, 2\}$ *and for all* $j \in \mathbb{N}_0$*:*

$$P(V_T^{cont} = v_{i,j}(x) | V_T^{tr} = x) = \frac{\mathcal{L}_{s,T}^{-1}\left\{\binom{n_{i,d,x}(j)+n_{i,u,x}(j)}{n_{i,u,x}(j)} u(s)^{n_{i,u,x}(j)} d(s)^{n_{i,d,x}(j)} \rho(s, y_i(x)) \frac{\partial y_i}{\partial x}\right\}}{p_{V_T^{tr}}(x)}$$

Proof: For part a) note that there are two possibilities for the terminal value of the discrete CPPI, V_T^{tr}, to take the value x. First, the discounted (discrete) cushion process has performed exactly net n_x up-moves at the last trading date τ_N and $W_T^\delta - W_{\tau_N}^\delta = y_1(x)$. Second, the discounted (discrete) cushion process has performed exactly $n_x + 1$ net up-moves and $W_T^\delta - W_{\tau_N}^\delta = y_2(x)$. Therefore we know that $W_{\tau_N}^\delta$ can take any of the values $n_{i,d,x}(j)a + n_{i,u,x}(j)b$ for $i \in \{1, 2\}$,$j \in \mathbb{N}_0$, dependent on the exact number of up- and down-moves. Since we know from lemma 1.1.1 that

$$\begin{aligned} C_t^{cont} &= C_0 e^{(r+m(\mu-r)-\frac{1}{2}m^2\sigma^2)t+\sigma m W_t} \\ &= C_0 e^{(r-\frac{1}{2}m(m-1)\sigma^2)t+\sigma m W_t^\delta}, \end{aligned}$$

we find that V_T^{cont} can take the values

$$\begin{aligned} V_T^{cont} &= G + C_{\tau_N}^{cont} \frac{C_T^{cont}}{C_{\tau_N}^{cont}} \\ &= G + C_0 e^{(r-\frac{1}{2}m(m-1)\sigma^2)\tau_N + n_{i,d,x}(j)m\sigma a + n_{i,u,x}(j)m\sigma b} e^{(r-\frac{1}{2}m(m-1)\sigma^2)(T-\tau_N)+m\sigma y_i(x)} \\ &= G + C_0 e^{(r-\frac{1}{2}m(m-1)\sigma^2)T + n_{i,d,x}(j)m\sigma a + n_{i,u,x}(j)m\sigma b + m\sigma y_i(x)} \end{aligned}$$

for $i \in \{1,2\}$,$j \in \mathbb{N}_0$.

For part b) is suffices to notice that

$$\begin{aligned} P(V_T^{cont} = v_{i,j}(x) | V_T^{tr} = x) &= \frac{P(V_T^{cont} = v_{i,j}(x),\ V_T^{tr} \in dx)}{P(V_T^{tr} \in dx)} \\ &= \frac{P(V_T^{tr} \in dx,\ \tau_{n_{i,u,x}(j)+n_{i,d,x}(j)} \leq T < \tau_{n_{i,u,x}(j)+n_{i,d,x}(j)+1})}{P(V_T^{tr} \in dx)}. \end{aligned}$$

□

Before we discuss the implications of the conditional distribution of proposition 2.2.5, we establish L^2 convergence for $k_u \to 1$ of the terminal values of the simple discrete-time CPPI with triggered trading dates and the simple continuous-time CPPI as an application.

Proposition 2.2.6 (Convergence)
The terminal value of the simple discrete-time CPPI with triggered trading dates converges to the terminal value of the simple continuous-time CPPI in L^2 as $k_u \to 1$:

$$\lim_{k_u \to 1} E\left[(V_T^{cont} - V_T^{tr})^2\right] = 0.$$

PROOF: First note that

$$E\left[(V_T^{cont} - V_T^{tr})^2\right] = E\left[(C_T^{cont} - C_T^{tr})^2\right] = E\left[(C_T^{cont})^2\right] - 2E\left[C_T^{cont} C_T^{tr}\right] + E\left[(C_T^{tr})^2\right]$$

where

$$E\left[(C_T^{cont})^2\right] = C_0^2 e^{(2r+2m(\mu-r)+m^2\sigma^2)T}$$

is known from lemma 1.1.3. Further we know from proposition 2.2.2 that

$$E\left[(C_T^{tr})^2\right] = C_0^2 e^{2rT} \mathcal{L}_{s,T}^{-1} \left\{ \frac{\sum_{i=0}^{2} \binom{2}{i} m^i (1-m)^{2-i} \frac{1-e^{i\sigma b}u(s) - e^{i\sigma a}d(s)}{s - i(\mu-r) - i(i-1)\frac{\sigma^2}{2}}}{1 - k_u^2 u(s) - k_d^2 d(s)} \right\}$$

and find

$$\lim_{k_u \to 1} \frac{1 - e^{i\sigma b}u(s) - e^{i\sigma a}d(s)}{1 - k_u^2 u(s) - k_d^2 d(s)} = \frac{s - i(\mu - r) - i(i-1)\frac{\sigma^2}{2}}{s - 2m(\mu - r) - m^2\sigma^2}$$

with the rule of L'Hospital. Therefore

$$\begin{aligned}
\lim_{k_u \to 1} E\left[(C_T^{tr})^2\right] &= C_0^2 e^{2rT} \mathcal{L}_{s,T}^{-1} \left\{ \frac{\sum_{i=0}^{2} \binom{2}{i} m^i (1-m)^{2-i}}{s - 2m(\mu - r) - m^2\sigma^2} \right\} \\
&= C_0^2 e^{2rT} \mathcal{L}_{s,T}^{-1} \left\{ \frac{1}{s - 2m(\mu - r) - m^2\sigma^2} \right\} \\
&= E\left[(C_T^{cont})^2\right]
\end{aligned}$$

follows with the help of lemma A.2.5,d). Using proposition 2.2.5 we further find

$$\begin{aligned}
E\left[C_T^{cont} C_T^{tr}\right] &= \int_0^\infty x E\left[C_T^{cont} | C_T^{tr} = x\right] p_{C_T^{tr}}(x) dx \\
&= C_0^2 e^{(2r - \frac{1}{2}m(m-1)\sigma^2)T} \mathcal{L}_{s,T}^{-1} \left\{ \frac{\int_a^b e^{m\sigma y}(me^{\sigma y} - m + 1)dy}{1 - k_u e^{m\sigma b} u(s) - k_d e^{m\sigma a} d(s)} \right\}.
\end{aligned}$$

Writing

$$\int_a^b e^{m\sigma y}(me^{\sigma y} - m + 1)dy = \sum_{i=0}^{1} m^i (1-m)^{1-i} \frac{1 - e^{(m+i)\sigma b} - e^{(m+i)\sigma a}}{s - (m+i)(\mu - r) - (m+i)(m+i-1)\frac{\sigma^2}{2}}$$

we can apply the rule of L'Hospital again to give

$$\lim_{k_u \to 1} \frac{1 - e^{(m+i)\sigma b} - e^{(m+i)\sigma a}}{1 - k_u e^{m\sigma b}u(s) - k_d e^{m\sigma a} d(s)} = \frac{s - (m+i)(\mu - r) - (m+i)(m+i-1)\frac{\sigma^2}{2}}{s - 2m(\mu - r) - \frac{3}{2}m^2\sigma^2 + \frac{1}{2}m\sigma^2}$$

from which

$$\begin{aligned}
\lim_{k_u \to 1} E\left[C_T^{cont} C_T^{tr}\right] &= C_0^2 e^{(2r - \frac{1}{2}m(m-1)\sigma^2)T} \mathcal{L}_{s,T}^{-1} \left\{ \frac{1}{s - 2m(\mu - r) - \frac{3}{2}m^2\sigma^2 + \frac{1}{2}m\sigma^2} \right\} \\
&= C_0^2 e^{(2r - \frac{1}{2}m(m-1)\sigma^2)T} e^{(2m(\mu - r) + \frac{3}{2}m^2\sigma^2 - \frac{1}{2}m\sigma^2)T} \\
&= E\left[(C_T^{cont})^2\right]
\end{aligned}$$

follows again with lemma A.2.5,d). □

Figure 2.5 shows the probabilities for the terminal value of the continuous CPPI to take different values under the condition that the terminal value of the discrete CPPI equals

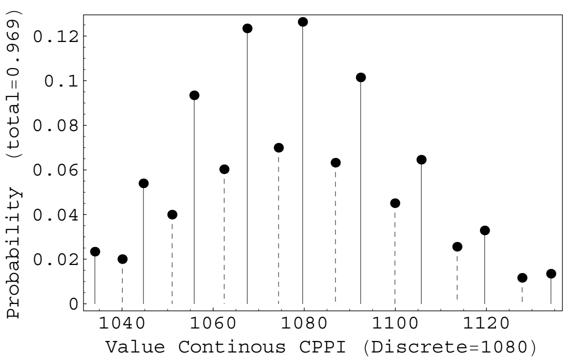

Figure 2.5: Probabilities $P(V_T^{cont}|\ V_T^{tr} = 1080)$ and $G = 800$, $V_0 = 1000$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.30$, $T = 1$, $k_u = 1.276$.

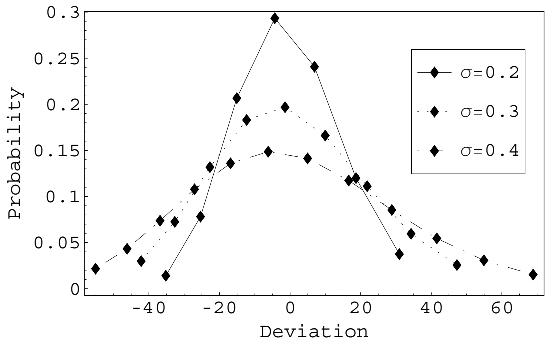

Figure 2.6: Probabilities $P(V_T^{cont} - V_T^{tr}|\ V_T^{tr} = 1051)$ for different values of the volatility σ and $G = 800$, $V_0 = 1000$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $T = 1$, $k_u = 1.276$.

1080. All probabilities greater than 1% are depicted and the values of the continuous CPPI vary from 1035 to 1135 which seems to be a very large range. All parameters have the same values as in figure 2.4, picture 4. In particular with this picture in mind one might have expected a better result. Observe that the conditioned distribution consists of two parts. The two parts consist of the values $v_{1,j}(x)$ and $v_{2,j}(x)$, respectively. However, conditioning on the portfolio values $V_T^{tr} = G + C_0e^{rT}k_u^n$ for $n \in \mathbb{Z}$, which relates to the values at the trading dates, will result in the collapse of one part. Observe also that the distribution is skewed to the right and for increasing values of V_T^{tr} it will even be more so. This has already been observed by Black and Perold (1992). They noticed that reversals, i.e. an up-move followed by a down-move or vice versa, increase the continuous CPPI relative to the discrete CPPI. This result they call "volatility cost". Since the number of reversals is bounded from below by zero but unbounded from above, it is clear that the conditioned distribution must be skewed to the right. Conditioned on the performance of the discrete CPPI, the continuous CPPI takes its minimum value if there is no reversal, i.e. if the number of net up-moves equals the number of up-moves.

In figure 2.6 we have chosen V_T^{tr} appropriately such that the conditional distribution collapses to only one part, in particular $V_T^{tr} = V_0e^{rT}$ which equals the performance of the riskless asset, and depicted three distributions for different volatilities. All point probabilities that belong to the same value of the volatility are connected to make the picture clearer. A very important observation is that the variance of the conditional distributions increases as the volatility of the risky asset increases. This is important because the discrete CPPI possesses a certain "self-regulation" property. The term self-regulation is to be understood on the following chain of arguments. It is intuitively clear, that an increase in the volatility of the risky asset will increase the deviation between the discrete and continuous CPPI if k_u is adjusted such as to keep the number of expected trading dates constant. However, an increase in the volatility will result in an increase in the number of trading dates if k_u is kept constant. This exploits the convergence and thus reduces the deviation to the continuous CPPI. This "self-regulation" property puts the question about the relevance of the volatility for the deviation between the continuous CPPI and the discrete CPPI as one might think that the increase in the number of trading dates could make up for the increased volatility. We learn from figure 2.6 that this is clearly not the case. The (conditional) variance of the deviation between discrete and continuous CPPI still increases in the volatility, irrespective of the larger number of trading dates.

The simple CPPI requires the assumption of unlimited borrowing. In particular this means that for any given borrowing level, there is a positive probability of the simple CPPI requiring even more borrowing if the trading rule is to be followed. We can not expect our discrete version to change this basic fact, as the strategy converges to the continuous one for a large number of trading dates. However, it is interesting to investigate how exactly the borrowing requirement changes in the discrete version. An expression for the probability of the discrete CPPI requiring more borrowing than some borrowing level $Z \in \mathbb{R}_0^+$ is given in the following proposition.

Proposition 2.2.7 (Borrowing requirement)
Let $Z \in \mathbb{R}_0^+$ and $\bar{n} := \left\lceil \frac{\log \frac{F_0+Z}{(m-1)C_0}}{\log k_u} \right\rceil - 1$. Then the probability for the simple discrete-time CPPI with triggered trading dates to require at least a discounted amount of Z to be invested into the risky asset in addition to the current portfolio value V_t^{tr} at some point before maturity time T is given by:

$$P(\ mC_t^{tr} \geq V_t^{tr} + Ze^{rt} \text{ for some } t \in [0,T]\ ) = \mathcal{L}_{s,T}^{-1} \left\{ \frac{h(\bar{n}+1, s)}{s} \right\}$$

where $h(k,s) = h(k|u(s), d(s))$ as in lemma A.1.1.

Proof: The condition can be rewritten in the following way:

$$mC_t^{tr} \geq V_t^{tr} + Ze^{rt} \quad \Leftrightarrow \quad (m-1)C_t^{tr} \geq (F_0 + Z)e^{rt}.$$

Since rebalancing takes place only at trading dates, additional capital will also only be required at trading dates. However, at trading dates the cushion process takes the form $C_\tau^{tr} = C_0 k_u^n e^{r\tau}$, where n denotes the net up-moves at time τ. Therefore the first time when the condition is satisfied is determined by the net up-moves, i.e.

$$\min \{n \in \mathbb{Z} | (m-1)C_0 k_u^n \geq F_0 + Z\}$$

and it is obvious that $\bar{n}+1$ is the solution to this minimization problem. From lemma A.1.1 we know that the Laplace transform of the probability density function of the first time the discounted cushion process performs net $\bar{n}+1$ up-moves is given by $h(\bar{n}+1|u(s), d(s))$. Therefore with lemma A.2.5 part c) it is clear that the Laplace transform of the appropriate probability is given by $\frac{h(\bar{n}+1,s)}{s}$ which proves the assertion. □

Figure 2.7 depicts the probabilities of the continuous and the discrete CPPI to require more borrowing than the amount Ze^{rt} at least at some point in time, $t \in [0, T]$, depending on the borrowing level Z. The curve with the jumps stems from the discrete CPPI. It is apparent from the figure that the probability for the requirement of borrowing is reduced compared to the continuous CPPI for virtually any borrowing level. The jumps in the curve stem from the fact, that the discrete CPPI can only require borrowing at trading dates and not in between. However, it is also apparent from figure 2.7 that borrowing is a critical issue in general for both, the discrete and the continuous CPPI.

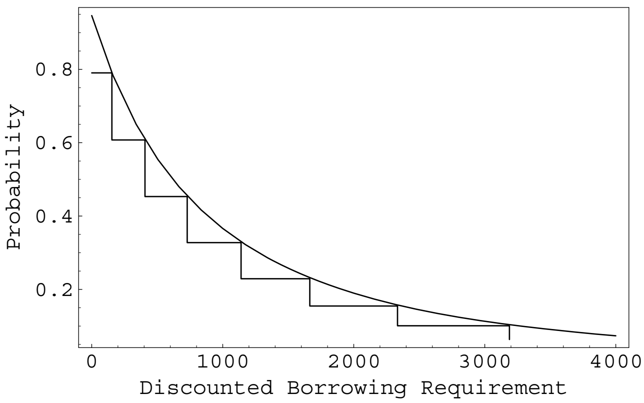

Figure 2.7: Borrowing requirement of the discrete and continuous CPPI with parameters $G = 800$, $V_0 = 1000$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.3$, $T = 1$, $k_u = 1.276$.

2.3 Limited Borrowing - The Capped CPPI

In this section we deal with the problem of limited borrowing. As we know from section 2.2, the simple CPPI requires the assumption of unlimited borrowing both in discrete time as in continuous time. If the trading rule is to be followed strictly there is a positive probability for the CPPI to require more borrowing than any given borrowing level at some point in time. In practice, this is a major drawback. It is clear that the trading rule cannot be followed strictly in all cases in practice since unlimited borrowing does not exist. But consequently, estimations about the outcome of the strategy based on the formulas of the simple CPPI, be it the discrete or continuous version, must be flawed. Indeed, we know from section 2.2 and proposition 2.2.7 in particular that the amount of borrowing the simple CPPI requires, is directly linked to the size of the cushion and therefore as well to the portfolio value. Hence, introducing a borrowing limit will only affect the "good" paths and lead to a significant change in particular in the expected terminal value. Estimations based on the formulas for the simple CPPI will overestimate the expected terminal value of the strategy. In the following, we assume a borrowing limit $Z \in \mathbb{R}_0^+$ and Z will denote the maximum borrowing allowed in discounted terms, i.e. at time t the total borrowing will be restricted to Ze^{rt}. In particular, $Z = 0$ refers to the case of no borrowing such that the maximum exposure will always be equal to the current portfolio value. The resulting strategy will be referred to as *capped CPPI*. The value process and the cushion process of the capped CPPI at some time t will be denoted by V_t^{Cap} and C_t^{Cap} respectively. On the basis of our discrete time model we will only be concerned with borrowing limits in discrete time. For borrowing limits in continuous time see Balder (2007). The introduction of a borrowing limit immediately leads to two different cases. Already at time $t = 0$ it is possible for the capped CPPI to require more borrowing than the borrowing limit permits. The condition for this is $mC_0 \geq V_0 + Z$. In this case, already at time $t = 0$ the borrowing constraint is binding and our trading rule changes such that the amount $V_0 + Z$ (in contrast to mC_0 in the unrestricted case) is invested in the risky asset and the amount Z is borrowed. The situation where the borrowing limit is binding will be referred to as a situation of *full exposure*. Analogously $mC_0 < V_0 + Z$ refers to the case where the borrowing constraint is not binding at time $t = 0$ and the investment in the risky asset is mC_0 like in the case of the simple CPPI whereas the investment in the riskless asset is $V_0 - mC_0$.

We proceed to determine the distribution of the capped CPPI and start by investigating the somewhat simpler case $mC_0 \geq V_0 + Z$. Since the portfolio value V_0 and in addition the borrowed amount Z is invested in the risky asset, the portfolio value evolves according to

$$V_t^{Cap} = (V_0 + Z)\frac{S_t}{S_0} - Ze^{rt} \tag{2.7}$$

and the first trading date is now defined by the first time the condition

$$mC_t^{Cap} = V_t^{Cap} + Ze^{rt} \tag{2.8}$$

holds, as this is the first time, the investment into the risky asset is determined by the trading rule of the CPPI and not the borrowing constraint. Together with equation (2.7)

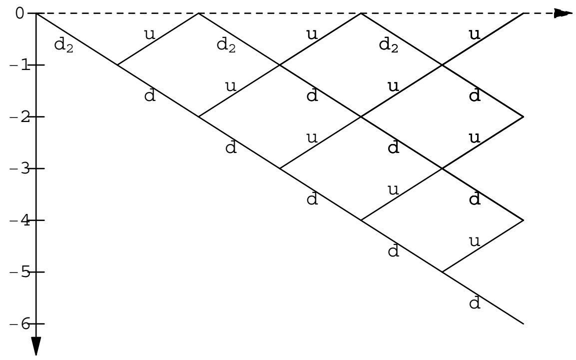

Figure 2.8: Binomial tree with maximum level zero and changing probabilities at the maximum level.

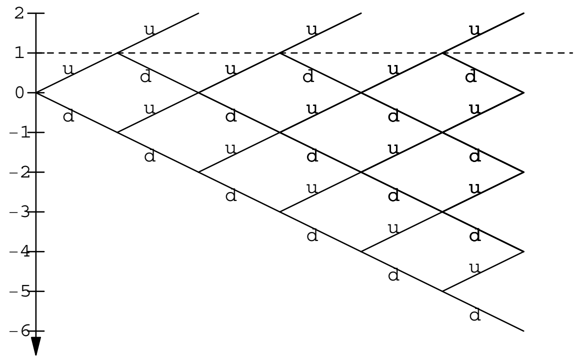

Figure 2.9: Binomial tree with maximum level greater zero and unaltered probabilities at the maximum level.

and $V_t^{Cap} = F_t + C_t^{Cap}$, this can be rewritten as $\frac{S_t e^{-rt}}{S_0} = \frac{m(F_0+Z)}{(m-1)(V_0+Z)}$ or equivalently

$$W_t^\delta = \frac{1}{\sigma}\log\left(\frac{m(F_0+Z)}{(m-1)(V_0+Z)}\right) =: a' \qquad (2.9)$$

and the time defined by equation (2.8) or equivalently equation (2.9) therefore is τ_1. Note that τ_1 is a dummy trading date as the exposure to the risky asset is still maximal and no trading takes place. However, it is the first time, the exposure to the risky asset is determined by the trading rule of the CPPI and not the borrowing limit. In contrast to conditions (2.4) and (2.5), condition (2.9) only reflects a one-sided barrier on the Brownian motion. The first trading date occurs, if the Brownian motion (W_t^δ) decreases to a'. It is easy to verify, that for the case where there is only one barrier, analogous expressions to the ones given in proposition 2.1.2 are given by

$$\begin{aligned} d(s|a,\infty,\delta) &:= \lim_{b\to\infty} d(s|a,b,\delta) = e^{a\delta + a\sqrt{2s+\delta^2}} \\ \rho(s,z|a,\infty,\delta) &:= \lim_{b\to\infty} \rho(s,z|a,b,\delta) = \begin{cases} \frac{e^{\delta z + z\sqrt{2s+\delta^2}} - e^{\delta z - z\sqrt{2s+\delta^2} + 2a\sqrt{2s+\delta^2}}}{\sqrt{2s+\delta^2}} & , \; z \le 0 \\ \frac{e^{\delta z - z\sqrt{2s+\delta^2}}\left(1 - e^{2a\sqrt{2s+\delta^2}}\right)}{\sqrt{2s+\delta^2}} & , \; z > 0 \end{cases} \end{aligned}$$

for some $a < 0$ and some $\delta \in \mathbb{R}$ and thus the Laplace transform of the probability density function of the time for the first trading date to occur is given by $d(s|a',\infty,\delta)$ with δ as in equation (2.6). The second trading date will be the first time, the portfolio is rebalanced and it can only occur as the result of a down-move since the strategy is still at full exposure. The situation at τ_1 is as depicted as a binomial tree in figure 2.8 where the start of the tree refers to time τ_1. The tree starts at level 0 which refers to the case of full exposure. Since the exposure is at its maximum, there can only be a down-move. With respect to our discretization this down move occurs if the discounted cushion process decreases by the fraction $1 - k_d$ which is equivalent to condition (2.4). Therefore, the Laplace transform for the time of this down-move to occur is given by $d(s|a(k_d), \infty, \delta)$. After this down-move the strategy is at level -1 from where both, up- and down-moves, are possible. Hence we set the maximum number of up-moves to 0 and count the net up-moves as $n = 0, -1, -2, \ldots$.

Proposition 2.3.1 (Distribution of the capped CPPI, case $mC_0 \geq V_0 + Z$)
Let $Z \in \mathbb{R}_0^+$ the maximum amount of borrowing allowed, $C := \frac{F_0+Z}{m-1}$ and a' as in equation (2.9) and $d'(s) = d(s|a', \infty, \delta)$. Further let

$$\begin{aligned} n'_x &:= \min\left\{ \left\lfloor \frac{\log \frac{x-G}{Ce^{rT}}}{\log k_u} \right\rfloor, 0 \right\} \\ y'_1(x) &:= \frac{1}{\sigma} \log\left(\frac{x-G}{mCe^{rT}k_u^{n'_x}} + \frac{m-1}{m} \right) \\ y'_2(x) &:= \frac{1}{\sigma} \log\left(\frac{x-G}{mCe^{rT}k_u^{n'_x+1}} + \frac{m-1}{m} \right) \\ y'_3(x) &:= \frac{1}{\sigma} \log \frac{xe^{-rT} + Z}{V_0 + Z} \end{aligned}$$

for all $x \in (G, \infty)$. Then the distribution of the terminal value of the capped CPPI, i.e. the probability $P(V_T^{Cap} \leq x)$, is given by:

$$P(V_T^{Cap} \leq x) = \mathcal{L}^{-1}_{s,T}\left\{ d'(s) \frac{1-u(s)-d(s)}{s} Q_0(n'_x - 1, s) \right\}$$
$$+ \begin{cases} \mathcal{L}^{-1}_{s,T}\left\{ d'(s) \left(q_0(n'_x, s) \int_a^{y'_1(x)} \rho(s,z)dz + q_0(n'_x+1, s) \int_a^{y'_2(x)} \rho(s,z)dz \right) \right\} & , n'_x < -1 \\ \mathcal{L}^{-1}_{s,T}\left\{ d'(s) \left(q_0(n'_x, s) \int_a^{y'_1(x)} \rho(s,z)dz + q_0(n'_x+1, s) \int_a^{y'_2(x)} \rho(s,z|a,\infty,\delta)dz \right) \right\} & , n'_x = -1 \\ \mathcal{L}^{-1}_{s,T}\left\{ d'(s) q_0(n'_x, s) \int_a^{y'_1(x)} \rho(s,z|a,\infty,\delta)dz + \int_{a'}^{y'_3(x)} \rho(s,z|a',\infty,\delta)dz \right\} & , n'_x = 0 \end{cases}$$

with $q_0(k,s) = q_0(k|u(s), d(s), d(s|a,\infty,\delta))$ and $Q_0(k,s) = Q_0(k|u(s), d(s), d(s|a,\infty,\delta))$ as in lemma A.1.4. Expressions for the integrals are given in propositions A.3.2 and A.3.4.

PROOF: If $\tau_1 < T$, we know that the cushion of the capped CPPI at time τ_N is given by $C_{\tau_N}^{Cap} = C_{\tau_1}^{Cap} e^{r(\tau_N - \tau_1)} k_u^n$, where n denotes the number of net up-moves at τ_N. From equation (2.8) it is apparent that $C_{\tau_1}^{Cap} = e^{r\tau_1} C$ and hence $C_{\tau_N}^{Cap} = Ce^{r\tau_N} k_u^n$. Furthermore it is known from equation (2.9) that $\tau_1 < T$ is equivalent to the Brownian motion with drift hitting a' and the appropriate Laplace transform is $d'(s)$. Once τ_1

has occurred, the three cases $n \leq n'_x - 1$, $n = n'_x$ and $n = n'_x + 1$ can be considered analogously to the proof of proposition 2.2.3. Note that n'_x is the solution to $\max\left\{n \in \{0, -1, \dots\} \mid G + Ce^{rT}k_u^n \leq x\right\}$. Hence, for $n \leq n'_x - 1$, we know $V_T^{Cap} < G + Ce^{rT}k_u^{n'_x} \leq x$ regardless of how the cushion develops in the time interval $(\tau_N, T]$. Similarly to proposition 2.2.3 the probability of all cases $n \leq n'_x - 1$ without any restriction to the further development of the cushion is given by

$$\mathcal{L}_{s,T}^{-1}\left\{d'(s)\int_a^b \rho(s,z)dz \sum_{k=-\infty}^{n'_x-1} q_0(k,s)\right\}$$

where the differences are that the net up-moves are counted by $q_0(k,s)$[11] instead of $q(k,s)$ and that $d'(s)$ has to be added as a factor accounting for trading date τ_1.

The situation is more complicated for the cases $n = n'_x$ and $n = n'_x + 1$. Due to the fact, that the maximum number of net up-moves is 0, three more cases have to be considered: $n'_x < -1$, $n'_x = -1$, $n'_x = 0$. Let us start with the case $n = n'_x$ and $n'_x < -1$. In this case the portfolio value at time T is given by

$$V_T^{Cap} = G + Ce^{rT}k_u^{n'_x}\left(me^{\sigma(W_T^\delta - W_{\tau_N}^\delta)} - m + 1\right)$$

and hence $V_T^{Cap} \leq x \Leftrightarrow a < W_T^\delta - W_{\tau_N}^\delta \leq y'_1(x)$. We immediately conclude that the probability of $V_T^{Cap} \leq x$ in this case is given by

$$\mathcal{L}_{s,T}^{-1}\left\{d'(s)q_0(n'_x)\int_a^{y'_1(x)} \rho(s,z)dz\right\}.$$

For the cases $n = n'_x + 1$, $n'_x < -1$ and $n = n'_x$, $n'_x = -1$ the probability can be calculated analogously to yield

$$\mathcal{L}_{s,T}^{-1}\left\{d'(s)q_0(n'_x + 1)\int_a^{y'_2(x)} \rho(s,z)dz\right\}$$

and

$$\mathcal{L}_{s,T}^{-1}\left\{d'(s)q_0(n'_x)\int_a^{y'_1(x)} \rho(s,z)dz\right\}$$

respectively. However, the situation is different in the case $n = n'_x + 1$ and $n'_x = -1$. Since $n = n'_x + 1 = 0$ the exposure is at its maximum (where there can only be a down-move and

[11]See lemma A.1.4 where the corresponding random walk problem is solved.

the probability of a down-move as well as the probability of not having further trading dates are altered) and hence $\rho(s, z|a, \infty, \delta)$ must be used instead of $\rho(s, z)$ to yield

$$\mathcal{L}_{s,T}^{-1}\left\{ d'(s) q_0(n'_x + 1) \int_a^{y'_2(x)} \rho(s, z|a, \infty, \delta) dz \right\}$$

and likewise the case $n = n'_x$ and $n'_x = 0$ gives

$$\mathcal{L}_{s,T}^{-1}\left\{ d'(s) q_0(n'_x) \int_a^{y'_1(x)} \rho(s, z|a, \infty, \delta) dz \right\}.$$

Clearly, the case $n = n'_x + 1$, $n'_x = 0$ can not happen, since $n \leq 0$. However, so far, the additional assumption for all cases was $\tau_1 < T$, made at the beginning of the proof. It is possible that there is never a first trading date τ_1 up to maturity time T, i.e. $\tau_1 \geq T$. In this case, the exposure will always be at its maximum, i.e. the capped CPPI is nothing more than a pure investment into the risky asset (leveraged if $Z > 0$). The portfolio value at time T is then given by

$$V_T^{Cap} = (V_0 + Z)\frac{S_T}{S_0} - Ze^{rT} = (V_0 + Z)e^{rT}e^{\sigma W_T^\delta} - Ze^{rT} \tag{2.10}$$

and $V_T^{Cap} \leq x \Leftrightarrow a' < W_T^\delta \leq y'_3(x)$. Therefore the probability in this case is given by

$$\mathcal{L}_{s,T}^{-1}\left\{ \int_{a'}^{y'_3(x)} \rho(s, z|a', \infty, \delta) dz \right\}$$

completing the proof. □

Now consider the case $mC_0 < V_0 + Z$ where the borrowing is not binding at time $t = 0$. Since the up-moves determine the size of the cushion and therefore also the investment in the risky asset and the required borrowing, it is clear that the introduction of a borrowing limit induces a maximum number of net up-moves which we will denote by $\bar{n}$. Allowing for $\bar{n} + 1$ up-moves and then changing the portfolio according to the trading rule of the CPPI, i.e. invest m times the cushion into the risky asset, would violate the borrowing constraint. Therefore the situation is as depicted in figure 2.9 such that the number of net up-moves can take the values $\bar{n}, \bar{n} - 1, \bar{n} - 2, \ldots$. From proposition 2.2.7 it is known that $\bar{n}$ is given by

$$\bar{n} = \left\lceil \frac{\log \frac{F_0 + Z}{(m-1)C_0}}{\log k_u} \right\rceil - 1. \tag{2.11}$$

As the trading rule of the CPPI can not be followed after net $\bar{n}+1$ up-moves due to the borrowing constraint, it would be the simplest way to define the strategy such that whenever a situation of net $\bar{n}$ up-moves occurs, only a down-move is possible and no changes are made to the portfolio if the (discounted) cushion process keeps increasing. However, with this simple rule, full exposure would only be possible in special cases. For being fully invested at some trading date τ, the condition $mC_\tau^{Cap} = V_\tau^{Cap} + Ze^{r\tau}$ must hold and we get the condition

$$(m-1)C_0 k_u^{\bar{n}} = F_0 + Z \tag{2.12}$$

using $C_\tau^{Cap} = C_0 e^{r\tau} k_u^{\bar{n}}$ and $C_0 = V_0 - F_0$. It is obvious that condition (2.12) can only hold for discrete values of k_u. In order to make full exposure possible for the non-suited values of k_u, we proceed in the following way. Suppose the cushion has performed net $\bar{n}+1$ up-moves at some time τ. Then the current portfolio value is given by

$$V_\tau^{Cap} = F_\tau + C_0 e^{r\tau} k_u^{\bar{n}+1} \tag{2.13}$$

and the amount $V_\tau^{Cap} + Ze^{r\tau}$ is invested into the risky asset since $mC_\tau^{Cap} \geq V_\tau^{Cap} + Ze^{r\tau}$ and thus the borrowing limit is binding. The situation is then similar to the case $mC_0 \geq V_0 + Z$ where the borrowing limit is binding already at time $t = 0$. Analogously to that case the portfolio evolves according to

$$V_t^{Cap} = (V_\tau^{Cap} + Ze^{r\tau})\frac{S_t}{S_\tau} - Ze^{rt} \tag{2.14}$$

and we define the next trading date by

$$mC_t^{Cap} = V_t^{Cap} + Ze^{rt} \tag{2.15}$$

analogously to condition (2.8). Equations (2.13), (2.14) and (2.15) together yield

$$W_t^\delta - W_\tau^\delta = \frac{1}{\sigma}\log\left(\frac{m(F_0+Z)}{(m-1)(F_0 + C_0 k_u^{\bar{n}+1} + Z)}\right) =: a'' \tag{2.16}$$

and so the appropriate Laplace transforms are given by $d(s|a'', \infty, \delta)$ and $\rho(s, z|a'', \infty, \delta)$. It is important to notice that while the situation at the beginning is as depicted in figure 2.9, the situation after $\bar{n}+1$ up-moves is as depicted in figure 2.8.

Proposition 2.3.2 (Distribution of the capped CPPI, case $mC_0 < V_0 + Z$)
In the notation of propositions 2.2.3, 2.3.1 and additionally a'' as in equation (2.16), $d''(s) = d(s|a'', \infty, \delta)$ and

$$y_3''(x) := \frac{1}{\sigma} \log \frac{xe^{-rT} + Z}{F_0 + C_0 k_u^{\bar{n}+1} + Z}$$

for all $x \in (G, \infty)$, the distribution of the terminal value of the capped CPPI, $P(V_T^{Cap} \leq x)$, is given by:

$$P(V_T^{Cap} \leq x) = P_1(x) + P_2(x)$$

where

$$P_1(x) = \mathcal{L}_{s,T}^{-1}\left\{\frac{1-u(s)-d(s)}{s} Q_{\bar{n}}(\min\{n_x, \bar{n}\} - 1, s)\right\}$$
$$+ \begin{cases} \mathcal{L}_{s,T}^{-1}\left\{ q_{\bar{n}}(n_x, s) \int\limits_a^{y_1(x)} \rho(s,z)dz + q_{\bar{n}}(n_x+1, s) \int\limits_a^{y_2(x)} \rho(s,z)dz \right\} & , n_x < \bar{n} \\ \mathcal{L}_{s,T}^{-1}\left\{ q_{\bar{n}}(n_x, s) \int\limits_a^{y_1(x)} \rho(s,z)dz \right\} & , n_x = \bar{n} \\ \mathcal{L}_{s,T}^{-1}\left\{ q_{\bar{n}}(\bar{n}, s) \frac{1-u(s)-d(s)}{s} \right\} & , n_x > \bar{n} \end{cases}$$

and

$$P_2(x) = \mathcal{L}_{s,T}^{-1}\left\{h(\bar{n}+1, s)d''(s)\frac{1-u(s)-d(s)}{s} Q_0(n_x' - 1, s)\right\}$$
$$+ \begin{cases} \mathcal{L}_{s,T}^{-1}\Bigg\{ h(\bar{n}+1,s)d''(s) \Bigg(q_0(n_x', s) \int\limits_a^{y_1'(x)} \rho(s,z)dz \\ \qquad + q_0(n_x'+1, s) \int\limits_a^{y_2'(x)} \rho(s,z)dz \Bigg) \Bigg\} & , n_x' < -1 \\ \mathcal{L}_{s,T}^{-1}\Bigg\{ h(\bar{n}+1,s)d''(s) \Bigg(q_0(n_x', s) \int\limits_a^{y_1'(x)} \rho(s,z)dz \\ \qquad + q_0(n_x'+1, s) \int\limits_a^{y_2'(x)} \rho(s,z|a,\infty,\delta)dz \Bigg) \Bigg\} & , n_x' = -1 \\ \mathcal{L}_{s,T}^{-1}\Bigg\{ h(\bar{n}+1,s)d''(s) q_0(n_x', s) \int\limits_a^{y_1'(x)} \rho(s,z|a,\infty,\delta)dz \\ \qquad + h(\bar{n}+1,s) \int\limits_{a''}^{y_3''(x)} \rho(s,z|a'',\infty,\delta)dz \Bigg\} & , n_x' = 0 \end{cases}$$

Notice that $q_{\bar{n}}(k,s) = q_{\bar{n}}(k|u(s), d(s))$ and $Q_{\bar{n}}(k,s) = Q_{\bar{n}}(k|u(s), d(s))$ as in lemma A.1.3, while $q_0(k,s) = q_0(k|u(s), d(s), d(s|a,\infty,\delta))$ and $Q_0(k,s) = Q_0(k|u(s), d(s), d(s|a,\infty,\delta))$ as in lemma A.1.4. Expressions for the integrals are given in propositions A.3.2 and A.3.4.

PROOF: We determine the probability $P(V_T^{Cap} \leq x)$ as the sum of two cases. The first case gives the joint probability of $V_T^{Cap} \leq x$ and the cap never becoming relevant (i.e. net $\bar{n}+1$ up-moves never occur up to time T). The second case gives the joint probability of $V_T^{Cap} \leq x$ and the cap becoming active at some point in time before T. The two cases refer to $P_1(x)$ and $P_2(x)$ respectively. We start with the first case. Since the cap is never to become active, the capped CPPI behaves exactly like the simple CPPI, but the probability for the net up-moves must be calculated according to lemma A.1.3. It is therefore apparent that for $n_x < \bar{n}$, the expression for $P_1(x)$ must equal the expression in proposition 2.2.3 with $q(k,s)$ exchanged by $q_{\bar{n}}(k,s)$. For $n_x = \bar{n}$, the term $q_{\bar{n}}(n_x+1,s)\int_a^{y_2(x)} \rho(s,z)dz$ must vanish, since net $n_x + 1 = \bar{n}+1$ up-moves would violate the assumption that the net up-moves do not surpass $\bar{n}$. For $n_x > \bar{n}$ it is apparent from the fact that there can maximally be net $\bar{n}$ up-moves and from the definition of n_x that

$$V_T^{Cap} < G + C_0 e^{rT} k_u^{\bar{n}+1} \leq G + C_0 e^{rT} k_u^{n_x} \leq x$$

such that $V_T^{Cap} \leq x$ is always satisfied. Hence, here the probability $P(V_T^{Cap} \leq x)$ must be given by

$$\mathcal{L}_{s,T}^{-1}\left\{\frac{1-u(s)-d(s)}{s}\sum_{k=-\infty}^{\bar{n}} q_{\bar{n}}(k,s)\right\} = \mathcal{L}_{s,T}^{-1}\left\{\frac{1-u(s)-d(s)}{s} Q_{\bar{n}}(\bar{n},s)\right\}.$$

Let us now turn to the second case where net $\bar{n}+1$ up-moves do occur up to time T. Once the cushion process has performed $\bar{n}+1$ up-moves, the situation is equivalent to the situation where $mC_0 \geq V_0 + Z$. Hence the expression for $P_2(x)$ is very similar to the expression for $P(V_T^{Cap} \leq x)$ in proposition 2.3.1 and we restrict ourselves here to explain the differences. While $d'(s)$ was needed in proposition 2.3.1 to leave the borrowing limit, here $d'(s)$ must be replaced by the product of $h(\bar{n}+1,s)$ and $d''(s)$ for first reaching and then leaving the borrowing limit. After $\bar{n}+1$ up-moves (which we suppose to occur at time τ) the portfolio evolves according to equation (2.14) and thus $V_T^{Cap} = (V_\tau^{Cap} + Ze^{r\tau})\frac{S_T}{S_\tau} - Ze^{rT}$ if the borrowing limit keeps being binding until maturity time T. Hence, with equation (2.13)

$$\begin{aligned}
V_T^{Cap} \leq x \quad &\Leftrightarrow \quad (F_\tau + C_0 e^{r\tau} k_u^{\bar{n}+1} + Ze^{r\tau})\frac{S_T}{S_\tau} \leq x + Ze^{rT} \\
&\Leftrightarrow \quad (F_0 + C_0 k_u^{\bar{n}+1} + Z)e^{\sigma(W_T^\delta - W_\tau^\delta)} \leq x + Ze^{rT} \\
&\Leftrightarrow \quad W_T^\delta - W_\tau^\delta \leq y_3''(x)
\end{aligned}$$

and it becomes obvious that

$$\int_{a'}^{y_3'(x)} \rho(s,z|a',\infty,\delta)dz$$

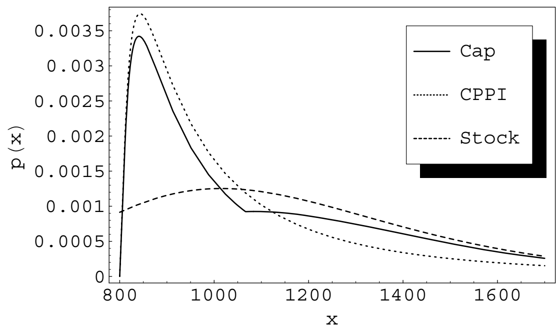

Figure 2.10: Densities of the terminal values of the risky asset, the capped and uncapped CPPI. The parameters are $V_0 = 1000$, $G = 800$, $Z = 0$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.30$, $T = 1$, $k_u = 1.01$.

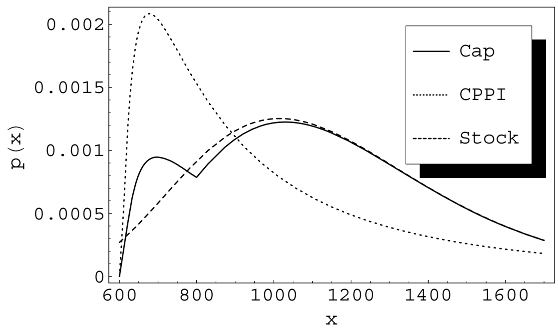

Figure 2.11: Densities of the terminal values of the risky asset, the capped and uncapped CPPI. The parameters are $V_0 = 1000$, $G = 600$, $Z = 0$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.30$, $T = 1$, $k_u = 1.01$.

from proposition 2.3.1 must be replaced by

$$h(\bar{n}+1, s) \int_{a''}^{y_3''(x)} \rho(s, z|a'', \infty, \delta) dz$$

here. □

The probability density function of the terminal value of the capped CPPI can immediately be deduced from propositions 2.3.1 and 2.3.2 by differentiation analogously to corollary 2.2.4. Figures 2.10 and 2.11 depict the density functions for the terminal value of the CPPI, the capped CPPI and a pure investment into the risky asset. While for figure 2.10 we have used our usual choice of parameters, in figure 2.11 the guarantee is reduced such that the initial exposure, mC_0, is considerably larger than the initial portfolio value V_0 which makes the borrowing constraint binding already at time $t = 0$. The trigger k_u has been chosen extremely small, such as to refrain from the effects of the discretization and resemble the continuous-time case instead. A large k_u leads to spikes in the density function similar to figure 2.4.

The first thing to notice is that the capped CPPI is bimodal. Unsurprisingly this is a direct consequence of the change in the strategy whenever the borrowing limit is binding. The position of the break in the density can be deduced from the condition $mC_T^{Cap} = V_T^{Cap} + Ze^{rT}$, which is the condition for the borrowing limit being binding at maturity T. Solving for V_T^{Cap} gives

$$V_T^{Cap} = \frac{mG + Ze^{rT}}{m-1} \tag{2.17}$$

for the position of the break. What can also be seen from the figures is that the right tail of the capped CPPI and the risky asset seem to be similar while the right tail of the simple CPPI is significantly different. Indeed, since the borrowing of the simple CPPI is directly linked to the portfolio value, high payoffs of the capped CPPI can only be achieved when the cap is binding. However, if the cap is binding, the capped CPPI is only a pure investment in the risky asset (leveraged if $Z > 0$) such that the behavior of the capped CPPI for large values of the risky asset must be identical to the behavior of the risky asset itself. The distribution of the simple CPPI on the other hand has a much fatter right tail than the risky asset. These fat tails of the simple CPPI reflect the possibility to create exorbitant gains. It is exactly this theoretical possibility to create large gains that accounts for the very high expectation and standard deviation of the terminal value of the simple CPPI and that must be paid with a large probability for the terminal value to end up close to the guarantee. The capped CPPI effectively solves this problem.

On the other hand, the left tail of the capped CPPI is very similar to that of the simple CPPI while the left tail of the risky asset must be considerably different as there is no portfolio protection for a pure investment in the risky asset. The left tail behavior of the capped CPPI is unsurprising, as the capped CPPI is identical to the simple CPPI as long as the cap is not binding which is the case for low portfolio values.

Generally it can be said that in comparison to the simple CPPI, the capped CPPI shifts probability mass from the right tail towards the middle while in comparison to a pure investment in the risky asset the capped CPPI shifts probability mass from the left tail towards the middle.

Proposition 2.3.3 (Moments of the cushion of the capped CPPI)
In the notation of proposition 2.3.2, the j-th moment of the cushion of the capped CPPI is given by
Case $mC_0 \geq V_0 + Z$:

$$\begin{aligned} E\left[(C_T^{Cap})^j\right] &= \left(Ce^{rT}\right)^j \mathcal{L}_{s,T}^{-1}\left\{ d'(s)\tilde{Q}_0(-1,s)\int_a^b (me^{\sigma z}-m+1)^j \rho(s,z)dz\right\} \\ &+ \left(Ce^{rT}\right)^j \mathcal{L}_{s,T}^{-1}\left\{ d'(s)\tilde{q}_0(0,s)\int_a^\infty (me^{\sigma z}-m+1)^j \rho(s,z|a,\infty,\delta)dz\right\} \\ &+ e^{jrT}\mathcal{L}_{s,T}^{-1}\left\{ \int_{a'}^\infty \left((V_0+Z)e^{\sigma z}-(Z+F_0)\right)^j \rho(s,z|a',\infty,\delta)dz\right\} \end{aligned}$$

Case $mC_0 < V_0 + Z$:

$$E\left[(C_T^{Cap})^j\right] =$$
$$\left(C_0e^{rT}\right)^j \mathcal{L}_{s,T}^{-1}\left\{\tilde{Q}_{\bar{n}}(\bar{n},s)\int_a^b(me^{\sigma x}-m+1)^j\rho(s,z)dz\right\}$$
$$+\left(Ce^{rT}\right)^j \mathcal{L}_{s,T}^{-1}\left\{h(\bar{n}+1,s)d''(s)\tilde{Q}_0(-1,s)\int_a^b(me^{\sigma z}-m+1)^j\rho(s,z)dz\right\}$$
$$+\left(Ce^{rT}\right)^j \mathcal{L}_{s,T}^{-1}\left\{h(\bar{n}+1,s)d''(s)\tilde{q}_0(0,s)\int_a^\infty(me^{\sigma z}-m+1)^j\rho(s,z|a,\infty,\delta)dz\right\}$$
$$+e^{jrT}\mathcal{L}_{s,T}^{-1}\left\{h(\bar{n}+1,s)\int_{a''}^\infty\left((F_0+C_0k_u^{\bar{n}+1}+Z)e^{\sigma z}-(Z+F_0)\right)^j\rho(s,z|a'',\infty,\delta)dz\right\}$$

where
$\tilde{q}_0(k,s)=q_0(k|k_u^ju(s),k_d^jd(s),k_d^jd(s|a,\infty,\delta))$, $\tilde{Q}_0(k,s)=Q_0(k|k_u^ju(s),k_d^jd(s),k_d^jd(s|a,\infty,\delta))$ as in lemma A.1.4, $\tilde{q}_{\bar{n}}(k,s)=q_{\bar{n}}(k,s|k_u^ju(s),k_d^jd(s))$, $\tilde{Q}_{\bar{n}}(k,s)=Q_{\bar{n}}(k|k_u^ju(s),k_d^jd(s))$ as in lemma A.1.3 and the integrals as in propositions A.3.1 and A.3.3.

Proof: As the proof is very similar to the proofs of propositions 2.3.1 and 2.3.2, we only show the case $mC_0 \geq V_0 + Z$ where the strategy starts with full exposure. Recall from the beginning of the section that the first trading date τ_1 is defined by the first time the trading rule of the CPPI requires full exposure and the Laplace transform of the probability density function of τ_1 is given by $d'(s)$. Further recall from the proof of proposition 2.3.1 that $C_{\tau_N}^{Cap}=Ce^{r\tau_N}k_u^n$ where n denotes the number of net up-moves at the last trading date before maturity, τ_N. Therefore, for $k\leq -1$, we have

$$E\left[(C_T^{Cap})^j 1_{\{n=k\}}1_{\{\tau_1<T\}}\right]$$
$$=E\left[\left(Ce^{rT}k_u^k\left(me^{\sigma(W_T^\delta-W_{\tau_N}^\delta)}-m+1\right)\right)^j 1_{\{n=k\}}1_{\{\tau_1<T\}}\right]$$
$$=\left(Ce^{rT}\right)^j\mathcal{L}_{s,T}^{-1}\left\{k_u^{jk}d'(s)q_0(k|u(s),d(s),d(s|a,\infty,\delta))\int_a^b\left(me^{\sigma z}-m+1\right)^j\rho(s,z)dz\right\}$$

as a consequence of lemma A.1.4. Now carefully notice that

$$k_u^{jk}q_0(k|u(s),d(s),d(s|a,\infty,\delta))=q_0(k|k_u^ju(s),k_d^jd(s),k_d^jd(s|a,\infty,\delta))=\tilde{q}_0(k,s)$$

follows from the definition of the function q in lemma A.1.4 and $k_d=\frac{1}{k_u}$. Likewise, for

m	$E[N]$	k_u	Mean	Stdv.
12	12	1.4080 (1.9768)	1072.57 (1066.58)	77.60 (**133.97**)
12	24	1.2756 (1.6245)	1072.71 (1066.75)	78.11 (135.30)
12	48	1.1885 (1.4117)	1072.78 (1066.84)	78.37 (136.04)
12	96	1.1301 (1.2768)	1072.82 (1066.88)	78.50 (**136.39**)

Table 2.2: Moments of the capped CPPI. The parameters are $T = 1$, $V_0 = 1000$, $G = 1000$, $\mu = 0.085$, $r = 0.05$, $Z = 0$ and $\sigma = 0.1$ ($\sigma = 0.2$ respectively).

$k = 0$, we find

$$\begin{aligned} & E\left[(C_T^{Cap})^j 1_{\{n=0\}} 1_{\{\tau_1 < T\}}\right] \\ & = \left(Ce^{rT}\right)^j \mathcal{L}_{s,T}^{-1}\left\{ d'(s)\tilde{q}_0(0,s) \int_a^\infty (me^{\sigma z} - m + 1)^j \rho(s, z|a, \infty, \delta) dz \right\}. \end{aligned}$$

Finally, if the borrowing constraint is always binding, it follows with equation (2.10)

$$\begin{aligned} E\left[(C_T^{Cap})^j 1_{\{\tau_1 \geq T\}}\right] &= E\left[(V_T^{Cap} - G)^j 1_{\{\tau_1 \geq T\}}\right] \\ &= e^{jrT} E\left[\left((V_0 + Z)e^{\sigma W_T^\delta} - Z - F_0\right)^j 1_{\{\tau_1 \geq T\}}\right] \\ &= e^{jrT} \mathcal{L}_{s,T}^{-1}\left\{ \int_{a'}^\infty ((V_0 + Z)e^{\sigma z} - (Z + F_0))^j \rho(s, z|a', \infty, \delta) dz \right\} \end{aligned}$$

and hence the assertion of the proposition follows with

$$E\left[(C_T^{Cap})^j\right] = \sum_{k=-\infty}^{0} E\left[(C_T^{Cap})^j 1_{\{n=k\}} 1_{\{\tau_1 < T\}}\right] + E\left[(C_T^{Cap})^j 1_{\{\tau_1 \geq T\}}\right]$$

□

Table 2.2 shows the moments of the capped CPPI when no borrowing is permitted. The parameters are chosen such as to match the parameters in table 2.1. For a better comparison the values for the trigger k_u have been chosen identical to the values in table 2.1. Hence, the column $E[N]$ refers to the expected number of trading dates for the simple CPPI. The expected number of trading dates for the capped CPPI will clearly be lower in comparison, as one of the effects of the capped CPPI is that no trading takes place while the cap is binding. The chosen parameter constellation implies an initial exposure of 585.25 such that the borrowing limit is not binding at time $t = 0$. It is remarkable, how little the values for the capped CPPI resemble those for the simple CPPI. A comparison of the values of the mean confirm what was said at the beginning of the section, the cap affects only the "good" paths, such that the simple CPPI overestimates the expected terminal value. The same argument holds for the considerably reduced values of the standard deviation of the capped CPPI. However, it is remarkable how little the values for the standard deviation vary with respect to the number of trading dates. For $\sigma = 20\%$ the values standard deviation only vary between 133.97 and 136.39. While figure 2.4 already suggested that very few trading dates are necessary to resemble the continuous simple CPPI well, the values for the standard deviation in table 2.2 suggest that this impression is even more valid if borrowing limits are introduced. Note that, compared with table 2.1, we have omitted to display the values for $m = 18$. The reason is that the initial exposure in this case equals 877.87 which is already close to the borrowing limit such that the effects of the case $m = 12$ are only highlighted but no new information can be drawn.

We will now proceed to discuss the influence of the strategy parameters on the capped CPPI. In order to sharpen the intuition about the behavior of the strategy it seems to be well suited to look at extreme values of the parameters. An overview is given in (2.18).

$$\begin{aligned} Z \to \infty \quad &\longrightarrow \quad \text{Simple Discrete} - \text{Time CPPI} \\ Z = 0,\ G \to 0 \quad &\longrightarrow \quad \text{Risky Asset} \\ Z = 0,\ m \to \infty \quad &\longrightarrow \quad \text{Stop} - \text{Loss} \\ Z = 0,\ mC_0 \geq V_0,\ k_u \to \infty \quad &\longrightarrow \quad \text{Stop} - \text{Loss} \end{aligned} \tag{2.18}$$

Let us first consider the borrowing limit Z. We know that the condition for the cap to be active is given by $mC_t^{Cap} \geq V_t^{Cap} + Ze^{rt}$. Therefore it is intuitively clear that the probability of the cap to be active is decreasing in the borrowing limit Z and converges to 0 for $Z \to \infty$, such that in the limit case the cap will never be active. Hence, as Z turns to infinity, the capped CPPI will converge to the simple discrete-time CPPI with all other parameters identical. Now suppose the case $G \to 0$. If $G = 0$, i.e. there is no guarantee, the cushion is always equal to the portfolio value, since $C_t^{Cap} = V_t^{Cap} - F_t$ and $F_t = Ge^{-r(T-t)}$. As a consequence, the condition $mC_t^{Cap} \geq V_t^{Cap} + Ze^{rt}$ is always satisfied if borrowing is not allowed since it can be rewritten as $(m-1)V_t^{Cap} \geq 0$. Therefore the cap is always active and the capped CPPI collapses to a pure investment into the risky asset. It can also be observed in figures 2.10 and 2.11 that for the smaller value of G, the density of the capped CPPI resembles the risky asset much better. Note, that this is not the case for $Z > 0$. In this case the condition $mC_t^{Cap} \geq V_t^{Cap} + Ze^{rt}$ can only be rewritten to $(m-1)V_t^{Cap} \geq Ze^{rt}$ such that there is a positive probability of the cap not being active. However, looking at this condition, it is clear that for $G = 0$ the borrowing limit Z must be large in order for the capped CPPI to differ significantly from a buy and hold strategy where $V_0 + Z$ is invested into the risky asset and the amount Z borrowed at time $t = 0$. Let us now turn to the case $m \to \infty$. As m becomes large, the cushion must become very small to violate the condition for the cap being active. Therefore, it is intuitive that in the limit case, the cap must always be active unless the cushion becomes zero, in which case the portfolio value at maturity equals the guarantee. This is exactly what a stop-loss strategy does. A formal proof of the convergence in m can be found in Black and Perold (1992). Finally we consider the case $k_u \to \infty$. When k_u turns to infinity, at the same time $k_d = \frac{1}{k_u}$ turns to 0. It is obvious, that in this case, there will be no trading at all unless the cushion approaches zero. Therefore, if $mC_0 \geq V_0$ and no borrowing is permitted, the capped CPPI converges to the stop-loss strategy analogously to the case $m \to \infty$. If borrowing is permitted and $mC_0 \geq V_0$, the capped CPPI still converges to a stop-loss strategy but a leveraged one in the sense that more than the initial portfolio value V_0 will be invested in the risky asset at time $t = 0$ and this investment will be held until the cushion approaches zero. In the case $mC_0 < V_0$, the capped CPPI turns to a mixture of a riskless investment and a stop-loss strategy. Generally it can be said that the non-leveraged capped CPPI is a hybrid between a pure investment into the risky asset (or a stop-loss strategy) and the simple CPPI as its distribution possesses the tail of the risky asset while close to the guarantee the capped CPPI shows the smooth stop-loss feature of the simple CPPI.

2.4 Transaction costs

In this section we investigate how transaction costs change our results from the previous sections. For simplicity we assume that the cost of a transaction is given by some fraction of the transaction size, i.e. we are only concerned with proportional transaction costs. Note that the introduction of a fixed component in the transaction costs can result in the cushion becoming negative and thus lead to default risk if the cushion size is very small. For economic reasons, in such cases, one would resort to changing the strategy and omit the transactions if the cushion size is very small. This problem can be avoided if only proportional transaction costs are considered.

For the definition of the transaction costs we follow Black and Perold (1992). Denote the proportional factor of the transaction costs by β such that e.g. $\beta = 1\%$ means that for any transaction, 1% of the transaction size is lost in value. Further denote by C_{τ_i} and $C_{\tau_i +}$

the cushion before and after the transaction, respectively, such that C_{t+} is the cushion process net of transaction costs. We do not use the specific notation C^{tr} and C^{Cap} as the general procedure of the implementation of transaction costs holds for both strategies. The trading rule of our strategy will be to invest the quantity mC_{τ_i+} into the risky asset at time τ_i such that the transaction costs are immediately implemented in the strategy. At time τ_{i-1} the investment in the risky asset is $mC_{\tau_{i-1}+}$ according to the trading rule. At time τ_i this investment will have evolved to $mC_{\tau_{i-1}+}\frac{S_{\tau_i}}{S_{\tau_{i-1}}}$ and the trading rule will require to invest the amount mC_{τ_i} into the risky asset. Therefore the transaction costs will be $\beta m\left|C_{\tau_i+} - C_{\tau_{i-1}+}\frac{S_{\tau_i}}{S_{\tau_{i-1}}}\right|$. We can now find the cushion net of transaction costs at time τ_i through the equation

$$C_{\tau_i+} = C_{\tau_i}\beta m\left|C_{\tau_i+} - C_{\tau_{i-1}+}\frac{S_{\tau_i}}{S_{\tau_{i-1}}}\right| \tag{2.19}$$

and from the definition of the trading dates in (2.2) we know

$$C_{\tau_i} = C_{\tau_{i-1}+}e^{r(\tau_i-\tau_{i-1})}k_u \tag{2.20}$$

and

$$C_{\tau_i} = C_{\tau_{i-1}+}e^{r(\tau_i-\tau_{i-1})}k_d \tag{2.21}$$

for and up- and down-move respectively. Furthermore, a combination of equations (2.1) and (2.20) yields

$$\frac{S_{\tau_i}}{S_{\tau_{i-1}}} = \frac{k_u+m-1}{m}e^{r(\tau_i-\tau_{i-1})} \tag{2.22}$$

in case of an up-move while a combination of equations (2.1) and (2.21) yields

$$\frac{S_{\tau_i}}{S_{\tau_{i-1}}} = \frac{k_d+m-1}{m}e^{r(\tau_i-\tau_{i-1})} \tag{2.23}$$

in case of a down-move. Combining equations (2.19), (2.20) and (2.22) now gives

$$\begin{aligned} C_{\tau_i+} &= C_{\tau_{i-1}+}e^{r(\tau_i-\tau_{i-1})}k_u - \beta m\left(C_{\tau_i+} - C_{\tau_{i-1}+}\frac{k_u+m-1}{m}e^{r(\tau_i-\tau_{i-1})}\right) \\ \Leftrightarrow \quad C_{\tau_i+} &= C_{\tau_{i-1}+}e^{r(\tau_i-\tau_{i-1})}\underbrace{\left(1+\frac{1+\beta}{1+\beta m}(k_u-1)\right)}_{=:\hat{k}_u} \end{aligned} \tag{2.24}$$

in case of an up-move and likewise equations (2.19), (2.21) and (2.23) yield

$$\begin{aligned} C_{\tau_i+} &= C_{\tau_{i-1}+}e^{r(\tau_i-\tau_{i-1})}k_d + \beta m\left(C_{\tau_i+} - C_{\tau_{i-1}+}\frac{k_d+m-1}{m}e^{r(\tau_i-\tau_{i-1})}\right) \\ \Leftrightarrow \quad C_{\tau_i+} &= C_{\tau_{i-1}+}e^{r(\tau_i-\tau_{i-1})}\underbrace{\left(1-\frac{1-\beta}{1-\beta m}(1-k_d)\right)}_{=:\hat{k}_d} \end{aligned} \tag{2.25}$$

in case of a down-move. It is easy to see that generally $\hat{k}_u < k_u$ and $\hat{k}_d < k_d$ for $\beta > 0$ such that a comparison with (2.2) leads to the insight that the cushion process net of transaction costs is gernerally smaller than the cushion process without transaction costs, as it should be. Note, that the cushion process is not supposed to become negative. Therefore from (2.25) we find a lower bound for k_d to be given by

$$\hat{k}_d \geq 0 \quad \Leftrightarrow \quad k_d \geq 1 - \frac{1-\beta m}{1-\beta}.$$

For $0 < k_d < 1 - \frac{1-\beta m}{1-\beta}$ the cushion process without transaction costs will still always be positive but the transaction costs will cause the cushion to become negative on the first down-move such that the amount G can not be guaranteed any more.

It is important to keep in mind, that while the probability of an up-move or down-move of the cushion remains unchanged in the presence of transaction costs, i.e. still hinges on the triggers k_u and k_d, the discounted cushion only multiplies with $\hat{k}_u < k_u$ in case of an up-move and $\hat{k}_d < k_d$ in case of a down-move. It is fairly easy to include transaction costs in the propositions of section 2.2. Basically, replacing k_u by $\hat{k}_u$ and k_d by $\hat{k}_d$ wherever they don't refer to a probability is all there is to do. For example, in proposition 2.2.3, k_u and k_d must be replaced in the expressions for n_x, y_1, y_2 while they must not be replaced in the expressions for a, b and hence also the expressions for u, d, ρ and q remain unchanged. However, the condition $k_d = \frac{1}{k_u}$ must be changed to $\hat{k}_d = \frac{1}{\hat{k}_u}$. Generally, apart from the trivial case $m = 1$, if $k_d = k_u^{-1}$ holds, $\hat{k}_d = \hat{k}_u^{-1}$ will not hold. This condition can be satisfied by first calculating $\hat{k}_u$ as defined in equation (2.24), then putting $\hat{k}_d = \frac{1}{\hat{k}_u}$ and finally calculating k_d from $\hat{k}_d$, using the definition of $\hat{k}_d$ in equation (2.25). It is slightly more difficult to include transaction costs in the propositions of section 2.3, in particular for the case $mC_0 < V_0$. The reason for this is the change in the trading rule at the first time the cap becomes binding. Since the cap is binding at that time, the investment into the risky asset is less than it would be without the cap and therefore the transaction costs are also less. In section 2.3 we have omitted to present the density of the terminal value of the capped CPPI. As an example of how to implement transaction costs, we will now give this density in the presence of transaction costs. The density without transaction costs can be deduced by setting $\beta = 0$.

Corollary 2.4.1 (Density of the capped CPPI, case $mC_0 < V_0 + Z$)
Let $\beta \geq 0$ the factor for the transaction costs, $k_u > 0$, $\hat{k}_u$ as in eq. (2.24), $\hat{k}_d = \frac{1}{\hat{k}_u}$, $k_d = 1 - \frac{1-\beta}{1-\beta m}(1 - \hat{k}_d)$ and $Z \in \mathbb{R}_+$ the maximum amount of borrowing allowed. Further let $\bar{n}$ as in equation (2.11) and n_x, $y_1(x)$, $y_2(x)$ as in proposition 2.2.3 and C, n'_x, $y'_1(x)$, $y'_2(x)$ as in proposition 2.3.1 with k_u and k_d exchanged by $\hat{k}_u$ and $\hat{k}_d$. Additionally let $V :=$

$\frac{F_0+C_0\hat{k}_u^{\bar{n}}k_u+Z+\beta C_0\hat{k}_u^{\bar{n}}(k_u+m-1)}{1+\beta}$, $a'' := \frac{1}{\sigma}\log\frac{mC}{V}$, $y_3''(x) := \frac{1}{\sigma}\log\frac{xe^{-rT}+Z}{V}$ *for all* $x \in [G,\infty)$. *Then the density of the terminal value of the capped CPPI in the presence of transaction costs is given by:*

$$p_{V_T^{Cap}}(x) = p_1(x) + p_2(x)$$

where

$$p_1(x) = \begin{cases} \mathcal{L}_{s,T}^{-1}\left\{q_{\bar{n}}(n_x,s)\rho(s,y_1(x))\frac{\partial y_1}{\partial x} + q_{\bar{n}}(n_x+1,s)\rho(s,y_2(x))\frac{\partial y_2}{\partial x}\right\} & , \ n_x < \bar{n} \\ \mathcal{L}_{s,T}^{-1}\left\{q_{\bar{n}}(n_x,s)\rho(s,y_1(x))\frac{\partial y_1}{\partial x}\right\} & , \ n_x = \bar{n} \\ 0 & , \ n_x > \bar{n} \end{cases}$$

and

$$p_2(x) = \begin{cases} \mathcal{L}_{s,T}^{-1}\Big\{h(\bar{n}+1,s)d''(s)\Big(q_0(n_x',s)\rho(s,y_1'(x))\frac{\partial y_1'}{\partial x} \\ \qquad\qquad +q_0(n_x'+1,s)\rho(s,y_2'(x))\frac{\partial y_2'}{\partial x}\Big)\Big\} & , \ n_x' < -1 \\ \mathcal{L}_{s,T}^{-1}\Big\{h(\bar{n}+1,s)d''(s)\Big(q_0(n_x',s)\rho(s,y_1'(x))\frac{\partial y_1'}{\partial x} \\ \qquad\qquad +q_0(n_x'+1,s)\rho(s,y_2'(x)|a,\infty,\delta)\frac{\partial y_2'}{\partial x}\Big)\Big\} & , \ n_x' = -1 \\ \mathcal{L}_{s,T}^{-1}\Big\{h(\bar{n}+1,s)d''(s)q_0(n_x',s)\rho(s,y_1'(x)|a,\infty,\delta)\frac{\partial y_1'}{\partial x} \\ \qquad +h(\bar{n}+1,s)\rho(s,y_3''(x)|a'',\infty,\delta)\frac{\partial y_3''}{\partial x}\Big\} & , \ n_x' = 0 \end{cases}$$

PROOF: The expression for $p_1(x)$ follows immediately from proposition 2.3.2 by differentiation. For $p_2(x)$ we have to take into account that after net $\bar{n}+1$ up-moves (which we suppose to happen at time τ) the trading rule changes such that instead of $mC_{\tau+}^{Cap}$ the amount $V_{\tau+}^{Cap} + Ze^{r\tau}$ is invested into the risky asset. Since $mC_\tau^{Cap} \geq V_\tau^{Cap} + Ze^{r\tau}$, the transaction costs for the changes made to the portfolio will therefore be lower. Denote the trading date before τ with τ', then the amount invested into the risky asset at time τ' was $mC_{\tau'+}^{Cap}$ and it follows from equation (2.3) that this amount has evolved to

$$\begin{aligned} mC_{\tau'+}^{Cap}\frac{S_\tau}{S_{\tau'}} &= C_{\tau'+}^{Cap}e^{r(\tau-\tau')}(k_u+m-1) \\ &= C_0e^{r\tau}\hat{k}_u^{\bar{n}}(k_u+m-1) \end{aligned}$$

up to time τ. It follows that

$$Ve^{r\tau} = V_\tau^{Cap} + Ze^{r\tau} - \beta\left(Ve^{r\tau} - C_0e^{r\tau}\hat{k}_u^{\bar{n}}(k_u+m-1)\right)$$

determines the amount $Ve^{r\tau}$ to be invested into the risky asset at time τ. This equation can be solved for V to yield

$$V = \frac{V_\tau e^{-r\tau} + Z + \beta C_0\hat{k}_u^{\bar{n}}(k_u+m-1)}{1+\beta}$$

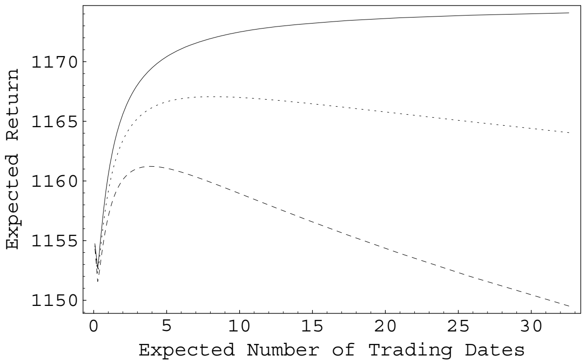

Figure 2.12: Expected terminal value of the simple CPPI for different values of the transaction costs. The parameters are $V_0 = 1000$, $G = 800$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.20$, $T = 1$, $Z = 0$. From top to bottom the curves are for $\beta = 0$, $\beta = 0.2\%$ and $\beta = 0.5\%$.

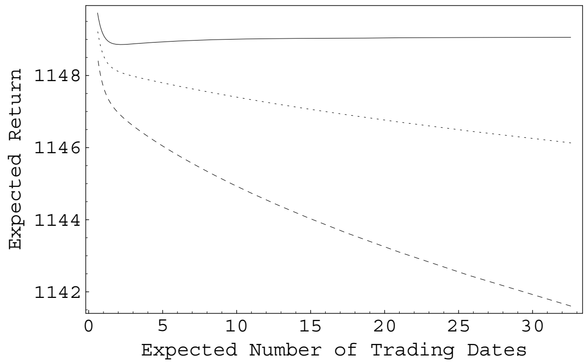

Figure 2.13: Expected terminal value of the capped CPPI for different values of the transaction costs. The parameters are $V_0 = 1000$, $G = 800$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.20$, $T = 1$, $Z = 0$. From top to bottom the curves are for $\beta = 0$, $\beta = 0.2\%$ and $\beta = 0.5\%$.

from which it follows immediately that V matches the definition in the corollary if it is taken into account that

$$V_\tau^{Cap} = e^{r\tau}\left(F_0 + C_0 \hat{k}_u^{\bar{n}} k_u\right)$$

is the portfolio value at time τ before the transaction. Now, analogously to the proof of proposition 2.3.2, the expressions for a'' and y_3'' can be found and the expression for $p_2(x)$ then follows from proposition 2.3.2 by differentiation. □

Figure 2.12 shows the expected terminal value of the simple CPPI depending on the expected number of trading dates with and without transaction costs. While it can be shown that the expected terminal value of the simple CPPI in continuous time is greater than the expected terminal value of the simple CPPI in discrete time, i.e. the global maximum of the expected terminal value is attained for $E[N] \to \infty$ or equivalently $k_u \to 1$, this is not true any more if transaction costs are considered. In the presence of transaction costs, the expected terminal value exhibits a local maximum in the number of trading dates. While it is rather intuitive that a large number of trading dates causes the expected terminal value to decrease, it is surprising how few trading dates are sufficient to produce this effect. Figure 2.12 shows that the maximum of the expected terminal value is approximately at 7.5 expected trading dates for $\beta = 0.2\%$ and at 3.5 expected trading dates (per year!) for $\beta = 0.5\%$. Note also that it is possible for the expected terminal value to have a local minimum for large k_u (or equivalently for very few expected trading dates). A similar effect occurs for $m \to \infty$. Black and Perold (1992) show that the expected terminal value of the simple CPPI has a maximum for very large m before

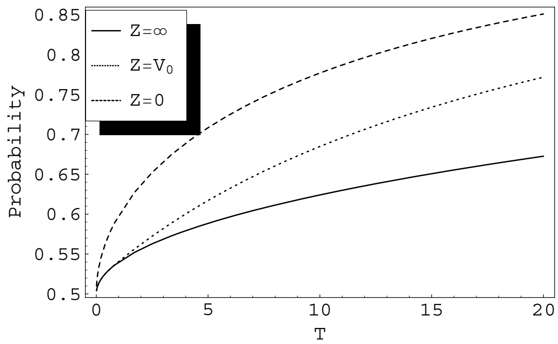

Figure 2.14: Probability of the CPPI and the capped CPPI performing better than the riskless asset. The parameters are $V_0 = 1000$, $F_0 = 750$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.20$, $k_u = 1.01$, $Z = \infty,\ V_0,\ 0$.

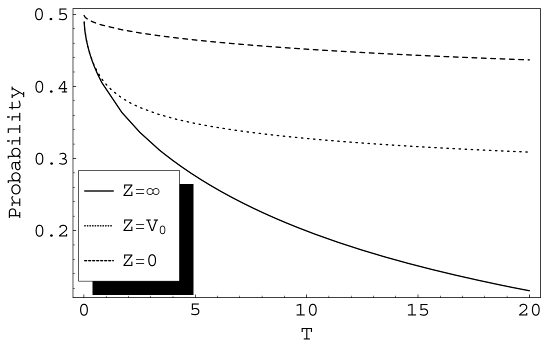

Figure 2.15: Probability of the CPPI and the capped CPPI performing better than the riskless asset. The parameters are $V_0 = 1000$, $F_0 = 750$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.30$, $k_u = 1.01$, $Z = \infty,\ V_0,\ 0$.

converging to the expected terminal value of a stop-loss strategy. Similarly, for large k_u the simple CPPI converges to a stop-loss strategy. With the help of the expected terminal value of a stop-loss strategy it can be shown, that in spite of the local minimum for large k_u, the global maximum can not be at $k_u \to \infty$.

For the capped CPPI, the situation is different. The global maximum without transaction costs can be attained for $E[N] \to \infty$ with the same effects as for the simple CPPI. However, figure 2.13 shows, that the global maximum of the expected terminal value can also be attained for $k_u \to \infty$ which reflects the case of a stop-loss strategy. This effect occurs if the initial exposure is close to or even greater than the maximum exposure. For initial exposures well below the maximum exposure, the situation will be as for the simple CPPI in figure 2.12.

2.5 Long Maturities

So far we have omitted to discuss the influence of the maturity time T on the behavior of both, the simple and the capped CPPI. As in section 2.3 it will be instructive to discuss the influence of the maturity time in terms of the extreme case $T \to \infty$. However, there is a good reason for the omittance. The simple as well as the capped CPPI were introduced as strategies that aim to guarantee a certain fixed amount G at maturity time T. With a fixed guarantee G, the initial floor is given by $F_0 = Ge^{-rT}$ and the initial exposure equals $m(V_0 - F_0) = m(V_0 - Ge^{-rT})$. For large maturities, it is obvious that the initial floor turns to zero while the initial exposure turns to mV_0. Hence, it is trivial that the guarantee becomes irrelevant in the long run.

In order to meaningfully investigate the influence of the maturity time, we therefore resort to keeping the initial floor F_0 constant which in turn makes the guarantee a function of the maturity time, i.e. $G = F_T = F_0 e^{rT}$. Notice, that this is a change in the interpretation of the strategies. In contrast to guaranteeing a fixed amount at maturity, keeping F_0 constant can be interpreted as guaranteeing a minimum rate of return of r on some fraction, $\frac{F_0}{V_0}$, of the initial wealth. For a given maturity T, both interpretations are obviously equivalent.

Throughout this section we will mainly focus on the probability of the simple and the capped CPPI performing better than the riskless asset. Figures 2.14 and 2.15 show this probability as a function of the maturity time, i.e. $P(V_T > V_0 e^{rT})$. The three cases refer to no borrowing, a maximum borrowing of the initial portfolio value and unlimited borrowing which reflects the case of the simple CPPI. We have basically used our standard parameter set with $\sigma = 20\%$ in figure 2.14 and $\sigma = 30\%$ in figure 2.15. The trigger k_u was chosen very small such as to reflect an approximation of the continuous-time case. The initial floor was chosen such as to yield an initial exposure equal to V_0. Without borrowing (the case $Z = 0$) this means that the capped CPPI is exactly at full exposure at time $t = 0$. The first thing to notice is that with respect to the probability of beating the riskless asset the capped CPPI performs worse in both figures if borrowing is permitted. Also it can be seen, that in figure 2.14 all probabilities are increasing in the maturity time, while in figure 2.15 all probabilities are decreasing. For the simple CPPI, the probability of beating the riskless asset may converge to 0 or to 1 as the maturity time T turns to infinity dependent on the exact parameter constellation. This behavior stems from the dynamics of the simple CPPI. We know from lemma 1.1.1 that the simple CPPI in continuous time follows a geometric Brownian motion. In particular we know that the terminal value of the simple CPPI in continuous time is given by

$$\begin{aligned} V_T^{cont} &= G + C_0 e^{(r+m(\mu-r)-\frac{1}{2}m^2\sigma^2)T+m\sigma W_T} \\ &= V_0 e^{rT} + C_0 e^{rT}\left(e^{(m(\mu-r)-\frac{1}{2}m^2\sigma^2)T+m\sigma W_T} - 1\right) \end{aligned}$$

and it is obvious that the probability of beating the riskless asset hinges on the drift term $m(\mu - r) - \frac{1}{2}m^2\sigma^2$. More specifically it is the sign of the term

$$\mu - r - \frac{1}{2}m\sigma^2 \tag{2.26}$$

that determines whether the probability of the simple CPPI beating the riskless asset approaches 1 or 0 in the long run. If the sign is positive, the probability will approach

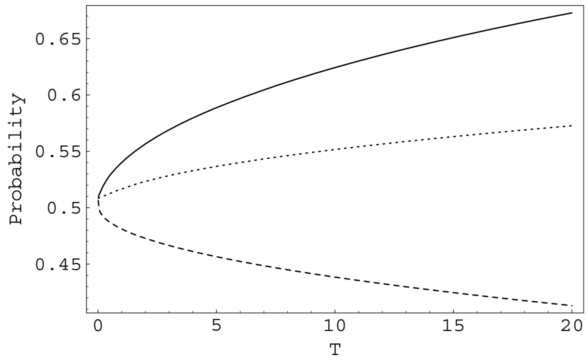

Figure 2.16: Probability of the simple CPPI performing better than the riskless asset for different values of the transaction costs $\beta = 0$, $\beta = 0.2\%$, $\beta = 0.5\%$. The parameters are $V_0 = 1000$, $F_0 = 750$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.20$, $k_u = 1.085$.

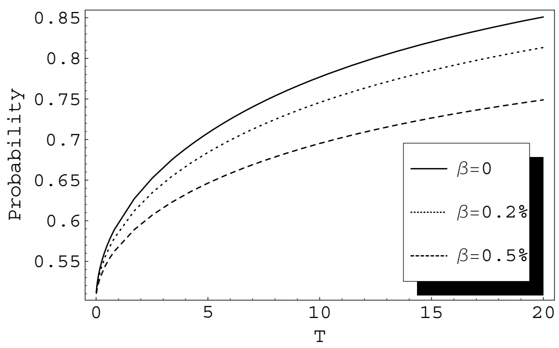

Figure 2.17: Probability of the capped CPPI performing better than the riskless asset for different values of the transaction costs. The parameters are $V_0 = 1000$, $F_0 = 750$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.20$, $k_u = 1.085$ and $Z = 0$.

one and if the sign is negative, the probability will approach zero. The sign of the term (2.26) critically depends on the multiplier m and the volatility σ of the risky asset and is very stringent as can be seen in figure 2.18. If the parameters are chosen to be $\mu = 0.15$, $r = 0.05$ and $\sigma = 0.20$, the multiplier can maximally be set to $m = 5$ if the probability of the simple CPPI beating the riskless asset is not supposed to converge to zero. This maximum on the multiplier is interesting in particular with respect to the fact that the expected terminal value of the simple CPPI is increasing in the multiplier. Therefore, for large m and a large maturity time T, the probability of the simple CPPI performing better than the riskless asset will be close to zero, while the expected payoff will be huge at the same time. This is a feature that is rather suited for a lottery than for a meaningful portfolio insurance strategy.

Things are not as simple for the capped CPPI. In figure 2.14 the probability of outperforming the riskless asset approaches 1 in the long run for both capped CPPI strategies, with and without borrowing, such that the only difference is in the speed of convergence. However, in figure 2.15, for neither of the strategies the probability of beating the riskless asset will approach 0 in the long run. Indeed, if no borrowing is permitted and if the initial floor is set to $F_0 = \frac{m-1}{m} V_0$, which yields an initial exposure of V_0, it is possible to show

$$\lim_{T\to\infty} \lim_{k_u \to 1} P(V_T^{Cap} > V_0 e^{rT}) = \begin{cases} 1 - \frac{\left[\mu - r - \frac{1}{2} m\sigma^2\right]^-}{(m-1)\frac{\sigma^2}{2}} & , \ \mu - r - \frac{\sigma^2}{2} > 0 \\ 0 & , \ \mu - r - \frac{\sigma^2}{2} \leq 0 \end{cases} \tag{2.27}$$

with the rule of L'Hôpital and lemma A.2.5,e).[12] From equation (2.27) it is apparent that the probability of the capped CPPI beating the riskless asset depends on the sign of the same term as for the simple CPPI but in addition on the sign of the term $\mu - r - \frac{\sigma^2}{2}$. The terminal value of a pure investment into the risky asset is given by

$$\begin{aligned} V_T &= V_0 e^{(\mu - \frac{\sigma^2}{2})T + \sigma W_T} \\ &= V_0 e^{rT} e^{\sigma W_T^\delta} \end{aligned}$$

such that the sign of $\delta = \frac{\mu - r - \frac{\sigma^2}{2}}{\sigma}$ determines whether the probability of a pure investment into the risky asset outperforming the riskless asset turns to 0 or 1 in the long run. In other words, the long run probability of the capped CPPI beating the riskless asset is dependent on both, the long run probability of the simple CPPI and the risky asset beating the riskless asset. Note that equation (2.27) only holds for the special case $F_0 = \frac{m-1}{m} V_0$ and $Z = 0$. For $F_0 > \frac{m-1}{m} V_0$ the probability will be lower while for $F_0 < \frac{m-1}{m} V_0$ the probability will be larger. Similarly, the probability is decreasing in the borrowing limit. Also, equation (2.27) does only hold strictly for the continuous-time limit case $k_u \to 1$, however the distortions for reasonable values $k_u > 1$ are minute.

Figures 2.16 and 2.17 show the probability of the simple and the capped CPPI beating the riskless asset for different values of the proportional transaction costs. The discretization parameter has been set to $k_u = 1.085$ which approximately gives 96 expected trading dates per year for the simple CPPI. While the probability for the simple CPPI is still increasing for the small transaction costs $\beta = 0.2\%$, it can be seen that the probability already decreases for $\beta = 0.5\%$. In fact, there are still only two cases for the simple CPPI, either the probability turns to 0 or to 1 in the long run. In comparison, the probability for the capped CPPI is still increasing even for the larger transaction costs $\beta = 0.5\%$. However, while the probability still turns to 1 for the smaller transaction costs $\beta = 0.2\%$, for $\beta = 0.5\%$ the probability only turns to around 89% for $T \to \infty$. Notice that while the discretization parameter k_u has only marginal effects on the long term probability of beating the riskless asset, the effect becomes significant when transaction costs are considered. For example, choosing $k_u = 1.26$, which yields about 12 trading dates per year, causes both, the probability of the simple and the capped CPPI for the larger transaction costs $\beta = 0.5\%$ to approach 1.

Figure 2.19 shows the expected yield of the capped CPPI, as defined by $\frac{1}{T} \log E\left[\frac{V_T^{Cap}}{V_0}\right]$,[13] as a function of T. The parameters are identical to the ones in figure 2.17, with the

[12] The notation $[X]^-$ denotes the negative part of X, i.e. $[X]^- := [-X]^+$.

[13] Note that this is rather the yield of the expectation than the expected yield, as the expected yield is

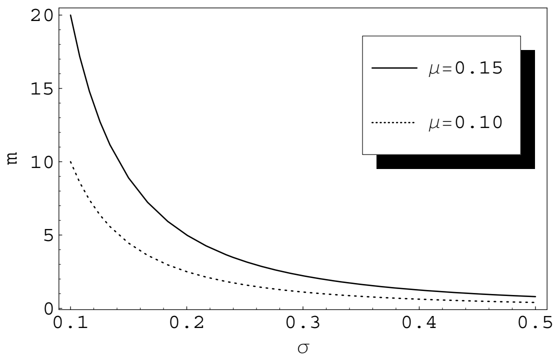

Figure 2.18: Combinations of m and σ such as to yield $\mu - r - \frac{m}{2}\sigma^2 = 0$ for different μ. The riskfree interest rate is set to $r = 0.05$.

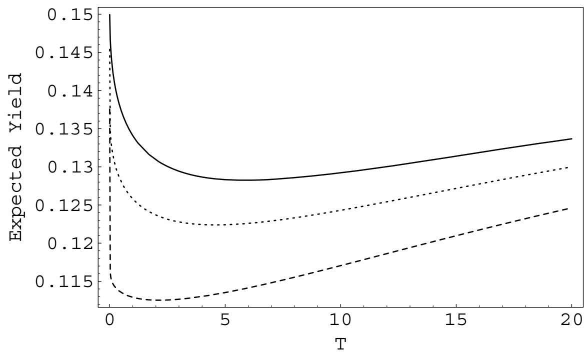

Figure 2.19: Expected yield of the capped CPPI for different values of the transaction costs. The parameters are $V_0 = 1000$, $F_0 = 750$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.20$, $k_u = 1.085$ and $Z = 0$. From top to bottom the curves are for $\beta = 0,\ 0.2\%,\ 0.5\%$.

exception of a higher volatility $\sigma = 30\%$. The higher volatility is chosen such as to produce the case $\mu - r - \frac{1}{2}m\sigma^2 < 0$. It is clear that the expected yield must approach 15% for $T \to 0$ as this is the drift of the risky asset and the initial exposure is 100%. It can be seen that the expected yield is increasing in the long run. Indeed, it can be shown that

$$\lim_{T\to\infty} \lim_{k_u \to 1} \frac{1}{T} \log \frac{E\left[V_T^{Cap}\right]}{V_0} = \mu$$

independent of the other parameters. Although this result might seem appealing, there is a major drawback. The reason for the large expected yield is that in the long run, the capped CPPI outgrows the guarantee. Once the cap is binding, the higher drift of the risky asset compared with the riskless asset results in a tendency of the capped CPPI never reach a situation again where the cap is not binding. Exactly those paths, that keep the cap permanently binding, are responsible for the high expected yield and make the guarantee increasingly irrelevant over time. This feature of the capped CPPI makes it a strategy that is rather suited for short than for long maturities.

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usually defined by $\frac{1}{T}E\left[\log \frac{V_T^{Cap}}{V_0}\right]$. The difference is that the yield of the expectation is not risk-adjusted. For example, for the risky asset it holds

$$\frac{1}{T}\log E\left[\frac{S_T}{S_0}\right] = \mu \qquad \text{while} \qquad \frac{1}{T}E\left[\log \frac{S_T}{S_0}\right] = \mu - \frac{\sigma^2}{2}.$$

We use the yield of the expectation as a tool to meaningfully investigate the expected terminal value for large maturity times T.

2.6 Conclusion

The discretization with triggered trading dates of the CPPI strategy solves the problem of continuous trading at cost of continuous monitoring while maintaining the feature that the terminal value be above some pre-specified guaranteed amount almost surely. It is this feature, that distinguishes a discretization with triggered trading dates from one with fixed trading dates. It is shown that as the triggers become small, the discrete CPPI converges to the continuous CPPI. Also the discrete CPPI possesses a certain self-regulation property with respect to the volatility. If the triggers are kept constant, an increase of the volatility of the risky asset will lead to a higher expected number of trading dates, thus exploiting the convergence. However, an increase of the volatility will lead to a larger difference between the discrete and the continuous CPPI and the self-regulation property can only dampen but not compensate this effect. In particular with respect to hedging this is an important insight. The introduction of transaction costs results in the expected payoff of the discrete CPPI not being monotonously increasing in the expected number of trading dates any more. Moreover the expected payoff will have a maximum, such that there is an optimal number of expected trading dates. In continuous time, the simple CPPI strategy requires the assumption of unlimited borrowing. Although also the discrete version of the simple CPPI requires unlimited borrowing, the borrowing requirement is reduced.

The introduction of a borrowing limit changes the properties the CPPI significantly. In particular the moments of second order or higher change dramatically if a borrowing limit is assumed. While for the simple CPPI the exposure is directly linked to the portfolio value, the introduction of a borrowing limit is equivalent to the introduction of a maximum exposure ratio. As a consequence the capped CPPI shows two different tail behaviors. For large portfolio values the CPPI behaves like a pure investment in the risky asset, while for low portfolio values, the capped CPPI behaves like the simple CPPI. Dependent on the parameters the density of the payoff of the capped CPPI can be bimodal.

Chapter 3

Floor Adjustments on CPPI

The previous chapters were mainly dedicated to discretizing the simple CPPI strategy. In addition, the structural problem of unlimited borrowing was solved by introducing a capped version of the CPPI. For the simple and the capped CPPI, the floor develops over time like an investment in the riskless asset and the goal is to keep the portfolio value above the floor at any point in time. Therefore, given a fixed maturity, a fixed amount can be guaranteed. For short maturities this seems to be a reasonable target. However, for long maturities, due to the higher drift of the risky asset, the portfolio value is likely to outgrow the floor, such that the floor and therefore the guaranteed amount at maturity can easily become insignificant in size in comparison to the portfolio value. Surely, once the portfolio value has outgrown the floor, it would be undesired to fall back to the floor in a market crash. Therefore for long maturities, it might be a desirable feature to increase the floor farther upwards in bullish markets such as to protect the gains to date. The Time Invariant Portfolio Protection (TIPP) strategy, which was first formulated by Estep and Kritzman (1988), is designed to do exactly this. The basic idea of the TIPP is to increase the current floor together with the portfolio value. In particular, the current floor in the TIPP will be a certain fraction of the maximum of all past portfolio values. That means, whenever the portfolio value reaches a new all-time high, the new floor will be set to a certain fraction γ of the current portfolio value. In contrast to the CPPI, where the floor increases at the riskfree interest rate, the guarantee in the TIPP solely hinges on the maximum of the portfolio values and remains constant otherwise. Apart from the differences in the definition of the floor, the TIPP trading rule is the same as the trading rule of the CPPI, at each point in time a multiple m of the current cushion, i.e. the difference between the current portfolio value and the current floor, is invested into the risky asset and the rest into the riskless asset. It is apparent that in the TIPP strategy the current floor will always be significant in size compared with the current portfolio value as the ratio between floor and portfolio value can never fall below γ. Currently there are several investment funds on the market which perform TIPP strategies, for example Zurich Financial Services, who were the first to offer such a structure with their "Protected Profits Fund" in 2003 or Barclay Capital's "Prosper". While for short maturities these companies usually perform capped CPPI strategies with a fixed guarantee at maturity, Prosper as well as the Protected Profits Fund are offered as open ended products, which is consistent with our discussion above.

In this chapter we are concerned with a strategy very similar to the TIPP. While in the TIPP the current floor remains constant unless the portfolio value reaches a new maximum, we increase the floor by the riskfree interest rate as for the CPPI. However, we also adopt the feature of the TIPP and tie the current floor to the maximum of the portfolio value. In particular, in the strategy to be considered, the current floor will be set to a certain fraction γ of the portfolio value whenever this fraction is greater than the current floor and the strategy will be referred to as *CPPI with floor adjustment*. Grossman and Zhou (1993) as well as Cvitanić and Karatzas (1995) show that if the floor is defined as described for the CPPI with floor adjustment, then, with respect to the expected long-term growth rate of a utility of constant relative risk aversion as well as with respect to the expected long term growth rate of logarithmic utility, it is optimal to use the trading rule of the CPPI.

While the CPPI with floor adjustment is better suited as a long term portfolio insurance strategy in a certain sense as its construction prohibits the portfolio insurance condition, i.e. the floor, to be outgrown by the portfolio value, it turns out to be more susceptible with respect to a structural problem of the CPPI, the cash-lock. The cushion and hence the exposure to the risky asset can become arbitrarily small for the CPPI with floor adjustment as well as for the simple and the capped CPPI such that the investor effectively might end up holding a bond. This problem is referred to as *cash-lock* and it is a problem that all CPPI structures share. As a reaction to the cash-lock problem, CPPI products are often offered with a minimum exposure ratio. This means, that CPPI strategies are modified such that the exposure to the risky asset is never less than a certain fraction of the current portfolio value and it results in a CPPI strategy that switches to a constant mix strategy for low cushion values and switches back when the cushion value increases again. While a minimum exposure ratio solves the cash-lock problem by construction, it contradicts the idea of a portfolio insurance at the same time. The introduction of a minimum exposure ratio causes default risk even in a continuous-time setup without market frictions if no additional provision is taken. We investigate the effects of the minimum exposure ratio in particular with respect to the drawback of the default risk caused and the gains of an increased expected yield. Finally we consider the case where put-options are employed to cover the default risk caused by the introduction of a minimum exposure ratio. It turns out that a provision against the default risk is quite expensive.

After introducing the CPPI with floor adjustment in section 3.1 and a modification in section 3.2 we investigate and compare the cash-lock for the CPPI with floor adjustment with the simple and the capped CPPI in section 3.3. A minimum exposure ratio is introduced in section 3.4. Section 3.5 is dedicated to the investigation of the costs of hedging against the default risk caused by the introduction of a minimum exposure ratio.

3.1 The CPPI with Floor Adjustment

We use the same model setup as in the previous chapter, i.e. changes in the portfolio take place whenever the yield of the discounted cushion process is equal to $k_u - 1$ or $k_d - 1$ whichever occurs first. For simplicity we assume $k_d = \frac{1}{k_u}$ throughout this chapter. Suppose an initial investment of V_0. The floor at time $t = 0$ is set to $F_0 = \gamma V_0$ where $\gamma \in (0, 1)$ is a constant and accordingly the initial cushion is equal to $C_0 = (1 - \gamma)V_0$. As for the simple and the capped CPPI, the floor increases at the riskfree rate r such that $F_t = e^{rt} F_0$ at some time t. From equation (2.2) we know that if at some trading date τ the portfolio has performed net n up-moves, the portfolio value of the simple CPPI is given by

$$V_0 e^{r\tau}(\gamma + (1 - \gamma)k_u^n). \tag{3.1}$$

However, here we want to adapt the rule that if the fraction gamma of the current portfolio value is greater than the current floor, i.e. $\gamma V_t > F_t$, the floor is adjusted and set to $F_t = \gamma V_t$. Note that this is equivalent to saying that the floor is adjusted whenever the discounted portfolio value has reached a new maximum. Indeed, it can immediately be seen that $\gamma V_t > F_t$ is equivalent to $V_t e^{-rt} > \frac{F_0}{\gamma} = V_0$ such that the first adjustment to the floor is made when the discounted portfolio value surpasses V_0. Since $\gamma + (1 - \gamma)k_u > 1$, it is obvious from equation (3.1) that the discounted portfolio value reaches a new maximum in our discrete-time setup when the discounted cushion process performs net one up-move. In order to distinguish the CPPI with floor adjustment from the other strategies, at some time t the portfolio value, the cushion and the floor will be denoted by V_t^{FA}, C_t^{FA} and

F_t^{FA} respectively. Suppose that the level of one net up-move is reached the first time at τ. Then the portfolio value is given by $V_\tau^{FA} = V_0 e^{r\tau} c$ where

$$c = c(\gamma, k_u) := \gamma + (1-\gamma)k_u \tag{3.2}$$

and the floor adjustment rule sets the new floor to $F_\tau^{FA} = \gamma V_\tau^{FA} = \gamma V_0 e^{r\tau} c$ and thereby the cushion to $C_\tau^{FA} = (1-\gamma)V_0 e^{r\tau} c$. From time τ onwards the situation resembles the situation at the beginning. The time τ of the floor adjustment can be interpreted as a reset to the CPPI such that a new CPPI is started with floor F_τ^{FA}. Therefore the floor at time $t > \tau$ is given by $F_t^{FA} = F_\tau^{FA} e^{r(t-\tau)}$ and when the discounted cushion process performs net one up-move (suppose at time τ'), the floor is adapted to $F_{\tau'}^{FA} = \gamma V_{\tau'}^{FA}$ and the CPPI strategy is restarted again and so on. From equation (3.1) it is immediate that at time τ' the portfolio value is given by $V_\tau^{FA} e^{r(\tau'-\tau)} c = V_0 e^{r\tau'} c^2$ such that between two floor adjustments the strategy is a simple CPPI strategy while on every floor adjustment, the portfolio multiplies with the constant c and the strategy is restarted. It is important to notice that γ can not be chosen independently of the multiplier m with respect to borrowing. Since the cushion at the time of a floor adjustment equals $C_\tau^{FA} = (1-\gamma)V_\tau^{FA}$, the investment into the risky asset equals $m(1-\gamma)V_\tau^{FA}$ according to the trading rule of the CPPI. Therefore, in contrast to the capped CPPI where a maximum exposure ratio was explicitly modelled, the CPPI with floor adjustment automatically incorporates a maximum exposure ratio given by $m(1-\gamma)$. Any borrowing limit can be modelled with a suitable choice of the parameters m and γ. In particular, choosing

$$\gamma = \frac{m-1}{m} \tag{3.3}$$

yields a maximum exposure ratio of 100%, thereby ruling out borrowing while allowing for the possibility of the portfolio being completely invested into the risky asset. For $\gamma < \frac{m-1}{m}$ the strategy will possibly require borrowing and for $\gamma > \frac{m-1}{m}$ the maximum exposure ratio will be below 100% such that some fraction of the portfolio value is always invested into the riskless asset. As a first result we calculate the moments of the strategy.

Proposition 3.1.1 (Moments of the CPPI with floor adjustment)
Let $\gamma \in (0,1)$. Then the moments of the CPPI with floor adjustment, $E\left[(V_T^{FA})^j\right]$, are given by

$$V_0^j e^{jrT} \sum_{i=0}^{j} \binom{j}{i} \gamma^{j-i} (1-\gamma)^i \mathcal{L}_{s,T}^{-1} \left\{ \frac{1-k_u^i h(1,s)}{1-c^j h(1,s)} \frac{\int_a^b (me^{\sigma z} - m + 1)^i \rho(s,z) dz}{1 - k_u^i u(s) - k_d^i d(s)} \right\}$$

where h as in lemma A.1.1, the constant c as in equation (3.2) and the integral as in A.3.1.[1]

Proof: Suppose the number of floor adjustments equals $k \in \mathbb{N}_0$. Then the terminal value of the CPPI with floor adjustment is given by $V_T^{FA} = V_0 e^{rT} c^k (\gamma + (1-\gamma) k_u^l (m e^{\sigma(W_T^\delta - W_{\tau_N}^\delta)} - m + 1))$ where $l = 0, -1, -2, ...$ denotes the number of net up-moves between the time of the last floor adjustment and time T. Hence, the j-th power of the terminal value is given by

$$\begin{aligned}(V_T^{FA})^j &= V_0^j e^{jrT} c^{kj} \left(\gamma + (1-\gamma) k_u^l (m e^{\sigma(W_T^\delta - W_{\tau_N}^\delta)} - m + 1)\right)^j \\ &= V_0^j e^{jrT} c^{kj} \sum_{i=0}^{j} \binom{j}{i} \gamma^{j-i} (1-\gamma)^i k_u^{il} (m e^{\sigma(W_T^\delta - W_{\tau_N}^\delta)} - m + 1))^i\end{aligned}$$

We know that the Laplace transform for the first time of net k up-moves is given by $h(k,s)$. Further we know that the Laplace transform for the last hit before time T yielding net l up-moves while never surpassing the level 0 is given by $q_0(l,s)$ and the Laplace transform for having no further trading date is given by $\rho(s,z)$. Therefore the j-th moment of the terminal value of the CPPI with floor adjustment, conditioned on having k floor adjustments and the last trading date before time T being at level l, is given by

$$V_0^j e^{jrT} \sum_{i=0}^{j} \binom{j}{i} \gamma^{j-i} (1-\gamma)^i \mathcal{L}_{s,T}^{-1} \left\{ h(k,s) c^{kj} q_0(l,s) k_u^{il} \int_a^b (m e^{\sigma z} - m + 1)^i \rho(s,z) dz \right\}.$$

Since $k_u^{il} q_0(l,s) = q_0(l | k_u^i u(s), k_d^i d(s))$ and $h(k,s) = h(1,s)^k$, summation over all $k \in \mathbb{N}_0$ and $l \in \mathbb{Z} \setminus \mathbb{N}$ immediately yields

$$V_0^j e^{jrT} \sum_{i=0}^{j} \binom{j}{i} \gamma^{j-i} (1-\gamma)^i \mathcal{L}_{s,T}^{-1} \left\{ \frac{Q_0(0 | k_u^i u(s), k_d^i d(s)) \int_a^b (m e^{\sigma z} - m + 1)^i \rho(s,z) dz}{1 - c^j h(1,s)} \right\}$$

using the definition of Q_0. From lemma A.1.3 we know that

$$Q_0(0 | k_u^i u(s), k_d^i d(s)) = \frac{q_0(0 | k_u^i u(s), k_d^i d(s))}{1 - h(-1 | k_u^i u(s), k_d^i d(s))} = \frac{1 - k_u^i h(1,s)}{1 - k_u^i u(s) - k_d^i d(s)}$$

where the second equality follows immediately from direct calculation using the identities (A.4). □

[1] As in the previous chapter, we will frequently use short notations such as $h(k,s) = h(k|u(s), d(s))$, $q(k,s) = q(k|u(s), d(s))$, $q_{\bar{n}}(k,s) = q_{\bar{n}}(k|u(s), d(s))$, $Q_{\bar{n}}(k,s) = Q_{\bar{n}}(k|u(s), d(s))$ and so on.

Before we give the moments of the continuous-time version, notice the similarity between the formula for the moments of the CPPI with floor adjustment in proposition 3.1.1 and the formula for the moments of the cushion of the simple CPPI in proposition 2.2.2.

Corollary 3.1.2 (Moments, continuous-time case)
The j-th moment of the continuous-time version of the CPPI with floor adjustment is given by

$$2V_0^j e^{jrT} \sum_{i=0}^{j} \frac{\binom{j}{i}\gamma^{j-i}(1-\gamma)^i}{\theta_{1,i}+\theta_{2,j}} \left(\theta_{1,i} e^{(\theta_{1,i}^2-\tilde{\delta}^2)\frac{T}{2}} \mathcal{N}(-\theta_{1,i}\sqrt{T}) + \theta_{2,j} e^{(\theta_{2,j}^2-\tilde{\delta}^2)\frac{T}{2}} \mathcal{N}(\theta_{2,j}\sqrt{T})\right)$$

where

$$\tilde{\delta} = \frac{\mu - r - \frac{1}{2}m\sigma^2}{\sigma}, \quad \theta_{1,i} = \tilde{\delta} + im\sigma, \quad \theta_{2,j} = \tilde{\delta} + jm\sigma(1-\gamma)$$

Proof: In view of proposition 3.1.1 we first need to determine the limit case $k_u \to 1$. Notice that

$$\lim_{k_u \to 1} \frac{1 - k_u^i h(1,s)}{1 - c^j h(1,s)} = \frac{\sqrt{2s + \tilde{\delta}^2} - \tilde{\delta} - im\sigma}{\sqrt{2s + \tilde{\delta}^2} - \tilde{\delta} - jm\sigma(1-\gamma)}$$

follows after lengthy calculations as an application of the rule of L'Hospital. Similarly to the proof of proposition 2.2.6 also

$$\lim_{k_u \to 1} \frac{\int_a^b (me^{\sigma z} - m + 1)^i \rho(s,z) dz}{1 - k_u^i u(s) - k_d^i d(s)} = \frac{1}{s - im(\mu - r) - i(i-1)m^2\frac{\sigma^2}{2}}$$

follows with the rule of L'Hospital. Now notice that

$$\frac{1}{s - im(\mu - r) - i(i-1)m^2\frac{\sigma^2}{2}} = \frac{2}{(\sqrt{2s+\tilde{\delta}^2} - \theta_{1,i})(\sqrt{2s+\tilde{\delta}^2} + \tilde{\delta} + im\sigma)}$$

and hence the moments of the continuous-time version of the CPPI with floor adjustment are given by

$$V_0^j e^{jrT} \sum_{i=0}^{j} \binom{j}{i} \gamma^{j-i}(1-\gamma)^i \mathcal{L}_{s,T}^{-1} \left\{ \frac{2}{\left(\sqrt{2s+\tilde{\delta}^2} - \tilde{\delta} - jm\sigma(1-\gamma)\right)\left(\sqrt{2s+\tilde{\delta}^2} + \tilde{\delta} + im\sigma\right)} \right\}$$

Due to the particularly simple structure, the involved Laplace transforms can be inverted analytically, such as to yield the assertion in the corollary. □

Notice that the expression given in corollary 3.1.2 is closed-form. In particular, it does not contain any Laplace transform since the relevant Laplace transforms could be inverted

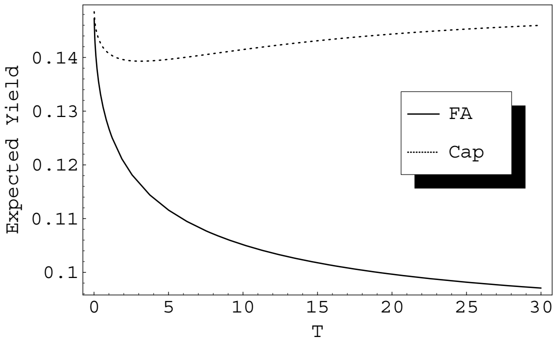

Figure 3.1: Expected yield of the capped CPPI and the CPPI with floor adjustment as a function of the maturity time T. The parameters are $k_u = 1.01$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.20$, $\gamma = \frac{m-1}{m} = 0.75$ and $F_0 = \gamma V_0$ for the capped CPPI.

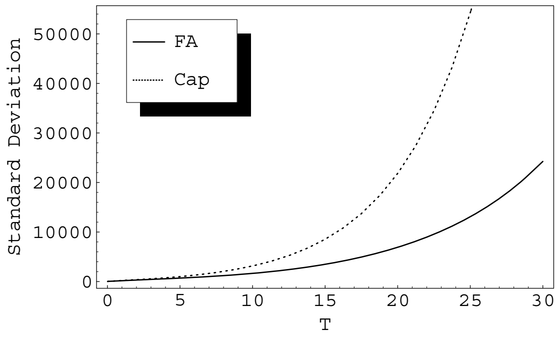

Figure 3.2: Standard deviation of the capped CPPI and the CPPI with floor adjustment as a function of the maturity time T. The parameters are $k_u = 1.01$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.20$, $\gamma = \frac{m-1}{m} = 0.75$ and $F_0 = \gamma V_0$ for the capped CPPI.

explicitly. Remember that it is far from usual that the inverse of a Laplace transform can be calculated analytically[2]. All Laplace transforms given in chapter 2 must be calculated numerically. However, one might wonder whether it is possible to find a closed-form solution for the continuous-time version of the capped CPPI, i.e. the limit case $k_u \to 1$. Unfortunately, although it is possible to find the Laplace transform for the limit case, its analytical inversion does not seem possible for the capped CPPI.

Figure 3.1 depicts the expected yield, defined by $\frac{1}{T} \log \frac{E[V_T]}{V_0}$, for the capped CPPI and the CPPI with floor adjustment as a function of the maturity time T. The dotted line is the capped CPPI. While the capped CPPI seems to converge to the drift, μ, of the risky asset, the CPPI with floor adjustment seems to be monotonously decreasing to a value considerably below μ. Indeed, we know that

$$\lim_{T\to\infty} \lim_{k_u\to 1} \frac{1}{T} \log \frac{E\left[V_T^{Cap}\right]}{V_0} = \mu \tag{3.4}$$

for the capped CPPI from the previous chapter and, with the help of corollary 3.1.2 and proposition A.2.5,e), it can be shown that

$$\lim_{T\to\infty} \lim_{k_u\to 1} \frac{1}{T} \log \frac{E\left[V_T^{FA}\right]}{V_0} = r + m(1-\gamma)[\mu - r - m\gamma\frac{\sigma^2}{2}]^+$$

for the CPPI with floor adjustment and in particular

$$\lim_{T\to\infty} \lim_{k_u\to 1} \frac{1}{T} \log \frac{E\left[V_T^{FA}\right]}{V_0} = r + [\mu - r - (m-1)\frac{\sigma^2}{2}]^+ \tag{3.5}$$

[2]See Polyanin and Manzhirov (1998) for a good reference on correspondence tables of Laplace transforms.

if γ is chosen as in equation (3.3) such as to yield a maximum exposure ratio of 100%. For the parameter values in figure 3.1 this gives a rate of convergence of 9% for the CPPI with floor adjustment. The capped CPPI converging to μ is a confirmation of what was already mentioned at beginning of the chapter. In the long run, the higher drift of the risky asset results in a tendency of the capped CPPI to reach the maximum exposure ratio of 100% (without leverage) and remain at that level unless a massive decrease in the risky asset occurs and hence the behavior of the capped CPPI becomes similar to a pure investment in the risky asset. For the CPPI with floor adjustment this is not true. Let us choose γ as in equation (3.3) such that the strategy starts with a 100% investment in the risky asset at time $t = 0$ and the instantaneous yield at the beginning consequently equals μ. From the point of full exposure, if the portfolio value increases, the floor is adjusted and the exposure ratio remains at its maximum. However, if the portfolio value decreases, funds are shifted from the risky asset towards the riskless asset irrespective of how often the floor has been adjusted before. This is the crucial difference to the capped CPPI. If the portfolio value in the capped CPPI is large, a moderate decrease in the risky asset will not change the exposure ratio. In the CPPI with floor adjustment, no matter what the portfolio value is, decreases in the risky asset will pull the exposure ratio below 100%. It is therefore intuitive, that the expected yield of the CPPI with floor adjustment must decrease in the maturity time if the initial exposure ratio is 100%. A closer look at the term $r + [\mu - r - (m-1)\frac{\sigma^2}{2}]^+$ yields the insight that for a high volatility of the risky asset or a high multiplier, the expected yield of the CPPI with floor adjustment very easily converges to the risk-free interest rate r. For example one of the standard parameter constellations from chapter 2, $\mu = 0.15$, $\sigma = 0.30$, $m = 4$ already gives $r + [\mu - r - (m-1)\frac{\sigma^2}{2}]^+ = r$. This insight makes the choice of the underlying as well as the choice of the multiplier crucial decisions for the success of a CPPI with floor adjustment as a long maturity or even open ended product.

Figure 3.2 shows that the standard deviation of the CPPI with floor adjustment is considerably reduced compared to the one of the capped CPPI. Since in contrast to the capped CPPI the exposure ratio of the CPPI with floor adjustment is not likely to remain at 100%, a shift of funds to the riskless asset must result in a decrease of variance.

Proposition 3.1.3 (Distribution of the CPPI with floor adjustment)

Let $\bar{n}_x := \left\lfloor \frac{\log \frac{x}{\gamma V_0 e^{rT}}}{\log c} \right\rfloor$, $\underline{n}_x := \max\left\{ \left\lfloor \frac{\log \frac{x}{V_0 e^{rT}}}{\log c} \right\rfloor, 0 \right\}$, $n_x(i) := \left\lfloor \frac{\log \frac{x - \gamma c^i V_0 e^{rT}}{(1-\gamma) c^i V_0 e^{rT}}}{\log k_u} \right\rfloor$ *and*

$$\begin{aligned} y_1(i,x) &:= \frac{1}{\sigma}\log\left(\frac{x - c^i \gamma V_0 e^{rT}}{mc^i(1-\gamma)V_0 e^{rT} k_u^{n_x(i)}} + \frac{m-1}{m}\right) \\ y_2(i,x) &:= \frac{1}{\sigma}\log\left(\frac{x - c^i \gamma V_0 e^{rT}}{mc^i(1-\gamma)V_0 e^{rT} k_u^{n_x(i)+1}} + \frac{m-1}{m}\right) \end{aligned}$$

for all $x \in (\gamma V_0 e^{rT}, \infty)$*. Then the probability* $P(V_T^{FA} \leq x)$ *is given by*

$$\mathcal{L}^{-1}_{s,T}\left\{ \frac{1}{s}(1 - h(\underline{n}_x, s)) + \sum_{i=\underline{n}_x}^{\bar{n}_x} h(i,s)\left(Q_0(n_x(i)-1, s)\int_a^b \rho(s,z)dz \right.\right.$$
$$\left.\left. + q_0(n_x(i), s)\int_a^{y_1(i,x)} \rho(s,z)dz + q_0(n_x(i)+1, s)\int_a^{y_2(i,x)} \rho(s,z)dz \right)\right\}$$

with h*,* q_0*,* Q_0 *as in lemmas A.1.1, A.1.3 and the integrals as in proposition A.3.2.*

PROOF: Note that $\underline{n}_x$ is the solution to $\max\left\{i \in \mathbb{N}_0 | V_0 e^{rT} c^i \leq x\right\}$. Suppose the number of floor adjustments is i. Then it is clear that $V_T \leq V_0 e^{rT} c^i(\gamma + (1-\gamma)k_u) = V_0 e^{rT} c^{i+1}$ and therefore the condition $V_T \leq x$ always holds for $0 \leq i < \underline{n}_x$. The Laplace transform for the density of the first time of having $\underline{n}_x$ floor adjustments is given by $h(1,s)^{\underline{n}_x} = h(\underline{n}_x, s)$ and therefore, using proposition A.2.5,c), the probability for having $\underline{n}_x$ or more floor adjustments is given by $\mathcal{L}^{-1}_{s,T}\left\{\frac{h(\underline{n}_x,s)}{s}\right\}$. Consequently, using proposition A.2.5,d), the probability for having less than $\underline{n}_x$ floor adjustments equals $\mathcal{L}^{-1}_{s,T}\left\{\frac{1-h(\underline{n}_x,s)}{s}\right\}$ which accounts for the first term in the expression for $P(V_T \leq x)$. For the other terms, first note that $\bar{n}_x$ is the solution to $\max\left\{i \in \mathbb{N}_0 | \gamma V_0 e^{rT} c^i \leq x\right\}$. Since $\gamma V_0 e^{rT} c^i$ is the guarantee after i floor adjustments, we know that always $V_T > x$ for $i > \bar{n}_x$. However, for $i \in \{\underline{n}_x, \ldots, \bar{n}_x\}$, once the i-th floor adjustment has occurred, the situation is very similar to that of the capped CPPI in proposition 2.3.1. From that time on the maximally allowed number of net up-moves is zero because otherwise there would be a further floor adjustment. Note, that the definitions of $n_x(i)$, $y_1(i,x)$ and $y_2(i,x)$ match the definitions of proposition 2.3.1 with $G = \gamma V_0 e^{rT} c^i$ and $C = (1-\gamma)V_0 c^i$. Therefore,

$$\mathcal{L}^{-1}_{s,T}\left\{ h(i,s)\left(Q_0(n_x(i)-1, s)\int_a^b \rho(s,z)dz + q_0(n_x(i), s)\int_a^{y_1(i,x)} \rho(s,z)dz \right.\right.$$
$$\left.\left. + q_0(n_x(i)+1, s)\int_a^{y_2(i,x)} \rho(s,z)dz \right)\right\}$$

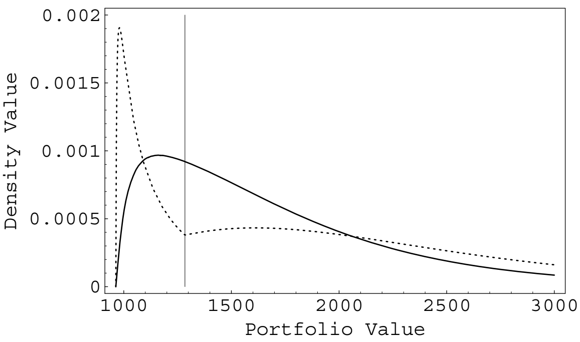

Figure 3.3: Densities of the capped CPPI and the CPPI with floor adjustment. The parameters are $k_u = 1.01$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.20$, $\gamma = \frac{m-1}{m} = 0.75$, $T = 5$ and $F_0 = \gamma V_0$ for the capped CPPI.

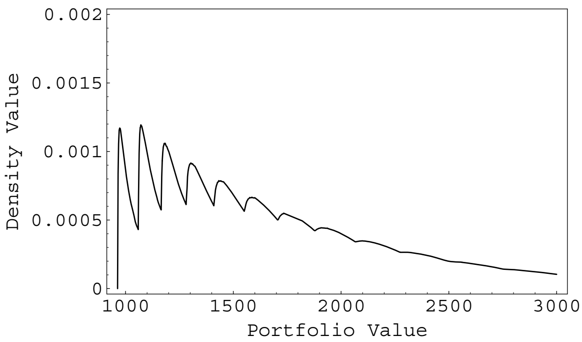

Figure 3.4: Density of the CPPI with floor adjustment. The parameters are $k_u = 1.40$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.20$, $\gamma = \frac{m-1}{m} = 0.75$, $T = 5$.

gives the joint probability of having i floor adjustments and $V_T \leq x$. Summing over all $i \in \{\underline{n}_x, \ldots, \bar{n}_x\}$ completes the proof. □

It is straightforward deduce the density of the CPPI with floor adjustment by means of differentiation.

Corollary 3.1.4 (Density of the CPPI with floor adjustment)
In the notation of proposition 3.1.3, the density of the terminal value of the CPPI with floor adjustment is given by

$$p_{V_T^{FA}}(x) = \mathcal{L}_{s,T}^{-1}\left\{\sum_{i=\underline{n}_x}^{\bar{n}_x} h(i,s)\left(q_0(n_x(i),s)\rho(s,y_1(i,x))\frac{\partial\, y_1(i,x)}{\partial\, x}\right.\right.$$
$$\left.\left.+q_0(n_x(i)+1,s)\rho(s,y_2(i,x))\frac{\partial\, y_2(i,x)}{\partial\, x}\right)\right\}$$

where

$$\frac{\partial\, y_1(i,x)}{\partial\, x} = \frac{1}{\sigma}\frac{1}{x - c^i V_0 e^{rT}\left(\gamma - (m-1)(1-\gamma)k_u^{n_x(i)}\right)}$$
$$\frac{\partial\, y_2(i,x)}{\partial\, x} = \frac{1}{\sigma}\frac{1}{x - c^i V_0 e^{rT}\left(\gamma - (m-1)(1-\gamma)k_u^{n_x(i)+1}\right)}$$

Note that the structure of the formula in corollary 3.1.4 is very similar to the structure of the density of the simple CPPI in corollary 2.2.4, the main difference being the sum. However, this difference is crucial with respect to the numerical complexity. While the sum is finite for any fixed $k_u > 1$, the upper summation limit $\bar{n}_x$ turns to infinity as $k_u \to 1$.

Figure 3.3 depicts the densities of the capped CPPI (dotted line) and the CPPI with floor adjustment. The discretization parameter k_u was chosen very small such as to approximate the continuous-time case. Figure 3.4 shows the density of the CPPI with floor adjustment for the same parameters but a larger discetization parameter. The occurrence of the spikes in the density for larger values of the discretization parameter is not very surprising. Since between any two floor adjustments, the CPPI with floor adjustment is identical to a simple CPPI, the spikes reflect the spike of the simple CPPI close to the guarantee. However, for small values of the discretization parameter, the CPPI with floor adjustment becomes uni-modal as can be observed in figure 3.3. This is in contrast to the capped CPPI where the one break in the density as given in equation (2.17) remains, independent of k_u. Generally it can be said that the CPPI with floor adjustment shifts probability mass from both edges towards the middle compared with the capped CPPI. The vertical line is at the point $V_T = V_0 e^{rT}$ and therefore symbolizes the payoff of the riskless asset. From equation (2.17) it is immediate that the placement of the break in the density of the capped CPPI is equal to the payoff of the riskless asset whenever there is no borrowing permitted and the initial exposure ratio is 100%. Figure 3.3 also suggests that the probability of beating the riskless asset is larger for the CPPI with floor adjustment than for the capped CPPI. Indeed, it is known from the previous chapter that

$$\lim_{T\to\infty}\lim_{k_u\to 1} P(V_T^{Cap} > V_0 e^{rT}) = \begin{cases} 1 - \frac{\left[\mu - r - \frac{1}{2}m\sigma^2\right]^-}{(m-1)\frac{\sigma^2}{2}} & , \ \mu - r - \frac{\sigma^2}{2} > 0 \\ 0 & , \ \mu - r - \frac{\sigma^2}{2} \leq 0 \end{cases} \tag{3.6}$$

is the long term probability for the capped CPPI beating the riskless asset while it can be shown (using propositions 3.1.3 and A.2.5,e)) that

$$\lim_{T\to\infty}\lim_{k_u\to 1} P(V_T^{FA} > V_0 e^{rT}) = \gamma^{\frac{\left[\mu - r - \frac{1}{2}m\sigma^2\right]^-}{m(1-\gamma)\frac{\sigma^2}{2}}}$$

and in particular

$$\lim_{T\to\infty}\lim_{k_u\to 1} P(V_T^{FA} > V_0 e^{rT}) = \left(\frac{m-1}{m}\right)^{\frac{\left[\mu - r - \frac{1}{2}m\sigma^2\right]^-}{\frac{\sigma^2}{2}}} \tag{3.7}$$

for γ as in equation (3.3) is the long term probability for the CPPI with floor adjustment beating the riskless asset. An application of Bernoulli's inequality to equation (3.7)

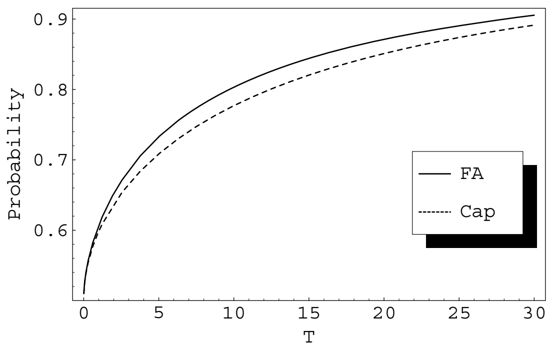

Figure 3.5: Probability of the CPPI with floor adjustment and the capped CPPI performing better than the riskless asset. The parameters are $V_0 = 1000$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.20$, $k_u = 1.01$, $\gamma = \frac{m-1}{m} = 0.75$ and $Z = 0$, $F_0 = \gamma V_0$ for the capped CPPI.

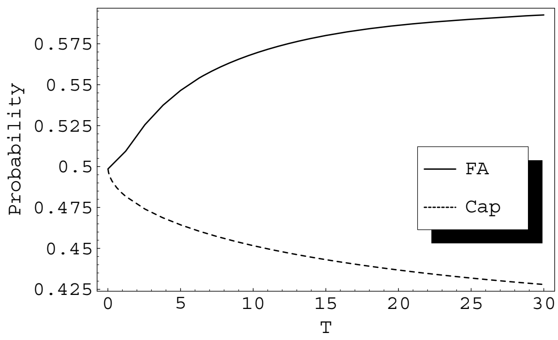

Figure 3.6: Probability of the CPPI with floor adjustment and the capped CPPI performing better than the riskless asset. The parameters are $V_0 = 1000$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.30$, $k_u = 1.01$, $\gamma = \frac{m-1}{m} = 0.75$ and $Z = 0$, $F_0 = \gamma V_0$ for the capped CPPI.

immediately yields

$$\left(\frac{m-1}{m}\right)^{\frac{\left[\mu-r-\frac{1}{2}m\sigma^2\right]^-}{\frac{\sigma^2}{2}}} \geq 1 - \frac{\left[\mu-r-\frac{1}{2}m\sigma^2\right]^-}{m\frac{\sigma^2}{2}} \geq 1 - \frac{\left[\mu-r-\frac{1}{2}m\sigma^2\right]^-}{(m-1)\frac{\sigma^2}{2}}$$

such that at least in the long run, the probability of beating the riskless asset is larger for the CPPI with floor adjustment.

Figures 3.5 and 3.6 depict the probability of the capped CPPI and the CPPI with floor adjustment outperforming the riskless asset as a function of the maturity time T. The parameter constellations only differ in the volatility. Using equations (3.6) and (3.7) we find that the probability of beating the riskless asset tends to one in the long run for both strategies in figure 3.5 while it tends to 40.74% for the capped CPPI and 59.96% for the CPPI with floor adjustment in figure 3.6. Figure 3.6 is particularly interesting with respect to the long term yield of the capped CPPI and the CPPI with floor adjustment, i.e. equations (3.4) and (3.5). While the probability of outperforming the riskless asset is only about 40% but the expected yield equals the drift, μ, of the risky asset for the capped CPPI, the probability of outperforming the riskless asset is 60% but the expected yield is equal to the riskfree rate r.

3.2 Increased initial floor levels

First, we generalize the strategy of the previous section section slightly. In the previous section the floor was adjusted and set to γV_τ^{FA} whenever $F_\tau^{FA} > \gamma V_\tau^{FA}$ on some trading date τ, which was seen to be equivalent to adjusting the floor whenever the maximum exposure ratio $m(1-\gamma)$ is surpassed. In this section we will use the same rule for floor adjustments. However, while in the previous section, the initial floor was set to $F_0 = \gamma V_0$, here we investigate the effects of increased initial floors, i.e. $F_0 \in [\gamma V_0, V_0)$. The choice $F_0 = \gamma V_0$ means that the strategy starts with the maximum exposure ratio at time $t = 0$. Combined with a choice of $\gamma = \frac{m-1}{m}$ as in equation (3.3), which is the condition for making an exposure ratio of 100% possible while ruling out borrowing, this means that the strategy starts with the whole portfolio invested into the risky asset. Choosing larger values for the initial floor results in the strategy starting with less than the maximum exposure ratio, in particular with a choice of $F_0 > \gamma V_0$ and $\gamma = \frac{m-1}{m}$ the strategy starts with an exposure ratio, which is less than 100% but the possibility of the exposure ratio rising up to 100% in the future remains. While in the previous section the initial exposure ratio was equal to the maximum exposure ratio and therefore floor adjustments were made after net one up-move, the situation is slightly different here. Since the initial exposure ratio is lower than the maximum exposure ratio, it will take more than net one up-move to surpass the maximum exposure ratio.

Suppose that the initial floor is set to $F_0 = \frac{\gamma V_0}{\gamma + (1-\gamma)k_d^{\hat{n}}}$ where $\hat{n} \in \mathbb{N}_0$. It is obvious that $\hat{n} = 0$ yields $F_0 = \gamma V_0$ and $\lim\limits_{\hat{n}\to\infty} \frac{\gamma V_0}{\gamma + (1-\gamma)k_d^{\hat{n}}} = V_0$ such that the value for F_0 is chosen from a discrete subset of the interval $(\gamma V_0, V_0)$. The restriction to this discrete subset is not necessary and is only done for simplicity. Suppose now that the cushion process performs net $\hat{n}$ up-moves up to time τ. Then the portfolio value at time τ is given by

$$\begin{aligned} V_\tau^{FA} &= F_0 e^{r\tau} + (V_0 - F_0) k_u^{\hat{n}} e^{r\tau} \\ &= \frac{V_0 e^{r\tau}}{\gamma + (1-\gamma)k_d^{\hat{n}}} \end{aligned}$$

and the exposure ratio is given by

$$\frac{m(V_\tau^{FA} - F_\tau^{FA})}{V_\tau^{FA}} = m(1-\gamma)$$

which is the maximum exposure ratio. Therefore, the first time the maximum exposure ratio is surpassed is after net $\hat{n}+1$ up-moves. Like in the last section, at this time the current floor is increased to γ times the current portfolio value such that the exposure ratio becomes equal to the maximum exposure ratio. From this time onwards the strategy is identical to the strategy in the previous section. Since after the first floor adjustment, which is made after $\hat{n}+1$ net up-moves, the exposure ratio is at its maximum, it requires only one net up-move to adjust the floor again. Formally, the only difference to the strategy in the previous section is, that the first floor adjustment requires net $\hat{n}+1$ up-moves. Calculating the moments and the distribution of the strategy is a straightforward adaption of propositions 3.1.1 and 3.1.3 and we restrict ourselves to giving the moments in the following corollary.

Corollary 3.2.1 (Moments, increased initial floor)
Let $\gamma \in [\frac{m-1}{m}, 1)$, $\hat{n} \in \mathbb{N}_0$ and $F_0 = \frac{\gamma V_0}{\gamma+(1-\gamma)k_d^{\hat{n}}}$. Then the j-th moment of the CPPI with floor adjustment and increased initial floor is given by

$$\left(\frac{V_0 e^{rT}}{\gamma+(1-\gamma)k_d^{\hat{n}}}\right)^j \sum_{i=0}^{j} \binom{j}{i} \gamma^{j-i}(1-\gamma)^i \cdot$$
$$\mathcal{L}^{-1}_{s,T}\left\{\frac{c^j h(\hat{n}+1,s)}{1-c^j h(1,s)} Q_0(0|k_u^i u(s), k_d^i d(s)) \int_a^b (me^{\sigma z}-m+1)^i \rho(s,z)dz\right.$$
$$\left. + k_d^{i\hat{n}} Q_{\hat{n}}(\hat{n}|k_u^i u(s), k_d^i d(s)) \int_a^b (me^{\sigma z}-m+1)^i \rho(s,z)dz\right\}$$

Figure 3.7 depicts the expected yield as a function of the time to maturity for different values of the initial floor. It can be seen that the expected yield is not always monotonous in the maturity time. While for low values of the initial floor it seems to be monotonously decreasing, it seems to be monotonously increasing for high values of the initial floor. For medium values of the initial floor, however, the expected yield has a maximum in the maturity time. Nevertheless, the size of the initial floor does not seem to have any impact on the long term performance of the strategy, but only influences the short term performance. Irrespective of the size of the initial floor the expected yield converges to the same value, which is equal to $r + [\mu - r - (m-1)\frac{\sigma^2}{2}]^+$ as is known from the previous section.

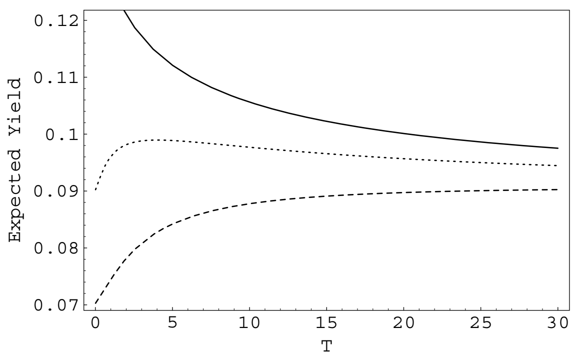

Figure 3.7: Expected yield of the CPPI with floor adjustment as a function of the maturity. The parameters are $k_u = 1.01$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\sigma = 0.20$, $\gamma = 0.75$. The three lines are (from top to bottom) for $\hat{n} = 0,\ 37,\ 62$ giving approximately $F_0 = 750,\ 900,\ 950$ respectively.

Let us now take a different view on the increased initial floors. Suppose the CPPI with floor adjustment is offered as an open ended fund product where customers can invest or withdraw their investment at any time. Without loss of generality we can assume a two customer case here. At time $t = 0$, a company launches an open ended CPPI product with floor adjustment and customer one invests the amount K_1 at time $t = 0$ and we assume $F_0 = \gamma V_0 = \gamma K_1$. At a later point in time $\tau > 0$, which we assume to be a trading date for simplicity, customer two wants to enter and invest the amount K_2. At time τ the wealth of customer one is equal to $K_1 c^i e^{r\tau}(\gamma + (1-\gamma)k_d^n)$ for some $i, n \in \mathbb{N}_0$. This means, that customer one has had i floor adjustments after which the portfolio decreased such that the next floor adjustment requires net $n+1$ up-moves. If the strategy is not to be changed by the investment of customer two, the first floor adjustment for customer two will also be after net $n+1$ up-moves. After the arrival of customer two, the (joined) portfolio value is given by

$$\begin{aligned} V_\tau &= K_1 c^i e^{r\tau}(\gamma + (1-\gamma)k_d^n) + K_2 \\ &= \left(K_1 c^i e^{r\tau} + \frac{K_2}{\gamma + (1-\gamma)k_d^n}\right)(\gamma + (1-\gamma)k_d^n) \end{aligned} \tag{3.8}$$

such that from equation (3.8) it is apparent that for customer two, entering a running CPPI with floor adjustment is equivalent to investing into a CPPI with floor adjustment and an increased initial floor. In view of this, the above mentioned irrelevance of the long term performance of the CPPI with floor adjustment of the initial floor is an ideal result. Irrespective of when a customer joins an open ended CPPI with floor adjustment and how the performance has been in the past, the long term performance will always be the same. Surely, the short term performance may be considerably different and heavily depends on the past performance as is obvious from figure 3.7.

3.3 The cash-lock problem

The term *cash-lock* refers to a situation where the portfolio value of the CPPI is lower than the floor at some point in time, i.e. the cushion is negative. In a cash-lock situation, the investment strategy is to fully invest the portfolio into the riskless asset and this is where the terminology stems from. However, if the price process of the risky asset is continuous, this situation can not occur in continuous time and neither in discrete time with triggered trading dates as in the previous chapter. A cash-lock can only occur if trading takes place in discrete time with fixed trading dates as in chapter 1 or if the price process of the risky asset is not continuous. Since in our setup a cash-lock can not occur, we resort to a slightly modified definition of cash-lock. In the following, we will call a situation ε-*cash-lock* if the exposure ratio of the strategy falls below some $\varepsilon > 0$. Although all versions of the CPPI considered in this and the previous chapter can fully recover from an ε-cash-lock situation, once the exposure ratio has become small, the expected time for a recovery is large such that a very small exposure ratio is already an unpleasant situation to be in.

Proposition 3.3.1 (ε-Cash-Lock)

Let $\varepsilon > 0$, $\gamma \in (0,1)$ and $\underline{n} := \left\lfloor \frac{\log \frac{\gamma\varepsilon}{(m-\varepsilon)(1-\gamma)}}{\log k_u} \right\rfloor$. Then, with $h(n,s) = h(n|u(s),d(s))$ as in lemma A.1.1 and $h_k(n,s) = h_k(n|u(s),d(s))$ as in lemma A.1.5, it holds

a) the probability of the exposure ratio of the simple CPPI with $F_0 = \gamma V_0$ falling below ε at some point in time before maturity time T is given by

$$\mathcal{L}^{-1}_{s,T}\left\{\frac{h(\underline{n},s)}{s}\right\}$$

b) the probability of the exposure ratio of the CPPI with floor adjustment falling below ε at some point in time before maturity time T is given by

$$\mathcal{L}^{-1}_{s,T}\left\{\frac{1}{s}\frac{h_0(\underline{n},s)}{1-h_{\underline{n}+1}(1,s)}\right\}$$

c) with $F_0 = \gamma V_0$, the probability of the exposure ratio of the capped CPPI falling below ε at some point in time before maturity time T is given by

$$\mathcal{L}^{-1}_{s,T}\left\{\frac{1}{s}h_{\bar{n}-1}(\underline{n},s) + \frac{1}{s}\frac{h_{\underline{n}+1}(\bar{n},s)d''(s)d(s|a,\infty,\delta)h_0(\underline{n}'+1,s)}{1-d(s|a,\infty,\delta)h_{\underline{n}'+2}(1,s)}\right\}$$

for the case $mC_0 < V_0 + Z$ and by

$$\mathcal{L}^{-1}_{s,T}\left\{\frac{1}{s}\frac{d'(s)d(s|a,\infty,\delta)h_0(\underline{n}'+1,s)}{1-d(s|a,\infty,\delta)h_{\underline{n}'+2}(1,s)}\right\}$$

for the case $mC_0 \geq V_0 + Z$, where $\bar{n}$,$d'(s)$ $d''(s)$ as in propositions 2.3.1, 2.3.2 and additionally $\underline{n}' := \left\lfloor \frac{\log\left(\frac{(m-1)F_0\varepsilon}{(m-\varepsilon)(F_0+Z)}\right)}{\log k_u} \right\rfloor$.

Proof: From equation (3.1) we know that at some trading date τ the value of the simple CPPI is given by $V_0 e^{r\tau}(\gamma + (1-\gamma)k_u^n)$ for some $n \in \mathbb{Z}$. Therefore the exposure ratio at τ is given by $\frac{m(1-\gamma)k_u^n}{\gamma+(1-\gamma)k_u^n}$ and it follows immediately that the exposure ratio is less than or equal to ε for all $n \leq \underline{n}$. We know that the Laplace transform of the density for the first time of having net $\underline{n}$ up-moves is given by $h(\underline{n}, s)$. Hence, with the help of proposition A.2.5,c) part a) of the proposition is immediate.

For part b) note that the value of the CPPI with floor adjustment is given by $V_0 e^{r\tau} c^i (\gamma + (1-\gamma)k_u^n)$ at some trading date τ for some $i \in \mathbb{N}_0$ and some $n \in \mathbb{Z} \setminus \mathbb{N}$. Hence, the exposure ratio is again given by $\frac{m(1-\gamma)k_u^n}{\gamma+(1-\gamma)k_u^n}$ and less than or equal to ε for all $n \leq \underline{n}$. The Laplace transform of the density for the first time of having i floor adjustments while not reaching the ε-cash-lock and then $\underline{n}$ net up-moves while not having a further floor adjustment is given by $(h_{\underline{n}+1}(1,s))^i h_0(\underline{n}, s)$. Since there can be possibly arbitrarily many floor adjustments, summing over $i \in \mathbb{N}_0$ yields the result.

For part c) note that $h_{\bar{n}}(\underline{n}, s)$ is the Laplace transform for reaching the ε-cash-lock before the cap becomes active, which explains the first summand in c) for the case $mC_0 < V_0 + Z$. Likewise $h_{\underline{n}+1}(\bar{n}+1, s)$ is the Laplace transform for reaching the cap before the ε-cash-lock. The term $d''(s)$ is the Laplace transform for going down to the situation where $mC_t^{Cap} = V_t^{Cap} + Ze^{rt}$, i.e. the situation where according to the trading rule of the CPPI the complete portfolio plus the maximum borrowing must be invested in the risky asset. From that point, it requires $\underline{n}'$ up-moves to reach the ε-cash-lock. The term $d(s|a,\infty,\delta)h_{\underline{n}'+2}(1,s)$ stands for going down one level and going up to the situation of full exposure again without reaching the ε-cash-lock before. The term $\frac{1}{1-d(s|a,\infty,\delta)h_{\underline{n}'+2}(1,s)}$ therefore accounts for the fact that there can be arbitrarily many switches between full exposure and less than full exposure. Finally, $d(s|a,\infty,\delta)h_0(\underline{n}'+1, s)$ stands for going down to less than full exposure and reaching the ε-cash-lock while not reaching full exposure again. Hence, the case $mC_0 < V_0 + Z$ becomes apparent and the case $mC_0 < V_0 + Z$ is analogous. □

Figures 3.8 and 3.9 show the probability of an ε-cash-lock occurring as a function of the maturity time for the simple CPPI, the capped CPPI and the CPPI with floor adjustment. It is clear, that the ε-cash-lock probability must be increasing in the maturity time for all strategies, since a longer maturity time increases the overall variance. It can be seen that the ε-cash-lock probability of the CPPI with floor adjustment converges to one in both figures, the ε-cash-lock probability of the capped CPPI converges to a value considerably below one in both figures and the ε-cash-lock probability of the simple CPPI converges to a value below one in figure 3.8 and to one in figure 3.9. These differences require some comment and we start with the capped CPPI. The capped CPPI has two basic barriers. One lower barrier for reaching the ε-cash-lock and one upper barrier for reaching the cap. There is always a positive probability of reaching the cap and this probability does not converge to zero if the time to maturity becomes large. However, once the cap is reached,

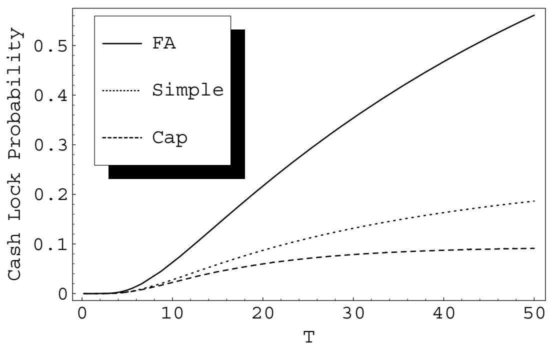

Figure 3.8: Probability of an ε-cash-lock of the simple CPPI, the capped CPPI and the CPPI with floor adjustment as a function of the maturity time. The parameters are $k_u = 1.01$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\gamma = 0.75$, $F_0 = \gamma V_0$, $\varepsilon = 0.01$ and $\sigma = 0.20$.

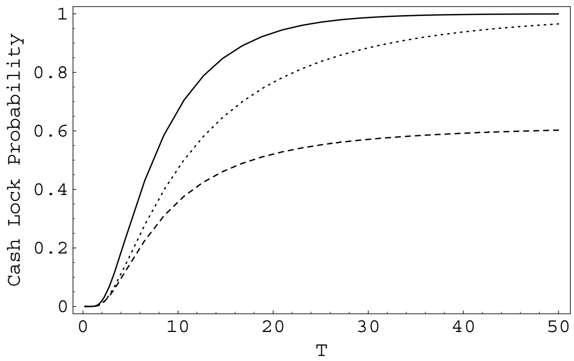

Figure 3.9: Probability of an ε-cash-lock of the simple CPPI, the capped CPPI and the CPPI with floor adjustment as a function of the maturity time. The parameters are $k_u = 1.01$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\gamma = 0.75$, $F_0 = \gamma V_0$, $\varepsilon = 0.01$ and $\sigma = 0.30$.

the strategy turns into a pure investment in the risky asset. We know that the value of the discounted risky asset at some time $t > \tau$ is given by $S_t = S_\tau e^{\sigma(W_t^\delta - W_\tau^\delta)}$. It is well known, that for Brownian motions with positive drifts, there is a positive probability of the Brownian motion never hitting lower barriers. Therefore, whenever the drift δ is positive and the strategy has reached full exposure, there is a positive probability of the strategy never reaching less than full exposure again.

For the simple CPPI the situation is somewhat different. From proposition 1.1.2 we know that the discounted cushion process of the continuous-time version is given by $e^{-rt}C_t^{cont} = C_0 e^{(m(\mu-r)-\frac{1}{2}m^2\sigma^2)t+\sigma m W_t}$. With the same argument about Brownian motions with drift, we find here that there is positive probability for the discounted cushion process never falling below some lower barrier, if $\mu - r - \frac{1}{2}m\sigma^2$ is positive. Since the exposure ratio is given by $\frac{me^{-rt}C_t^{cont}}{F_0+e^{-rt}C_t^{cont}}$ and directly dependent on the discounted cushion process, the same is true for the ε-cash-lock probability. However, for negative drifts the probability of an ε-cash-lock occurring in infinite time is indeed 1. The parameters in figures 3.8 and 3.9 are the same as in figures 3.5 and 3.6 and therefore also yield $\mu - r - \frac{1}{2}m\sigma^2 > 0$ and < 0 respectively. Surely, for our discrete version of the simple CPPI, this condition is only an approximation as we know that for $k_u \to \infty$ the simple CPPI converges to a stop-loss strategy. Nevertheless, for reasonable values of the discretization parameter k_u, the cash-lock probability of the discrete version of the simple CPPI is well explained by the cash-lock probability of the continuous time version.

For the CPPI with floor adjustment, note that after each floor adjustment, the fixed number of net $\underline{\mathrm{n}}$ up-moves (since $\underline{n}$ is negative these are actually down-moves) is required to reach the ε-cash-lock. Hence it is a simple consequence of the indefinitely increasing variance of the Brownian motion in infinite time that any fixed number of net down-moves is surpassed at some point in time. Therefore, independent of the parameters, the CPPI with floor adjustment will reach an ε-cash-lock situation with certainty within infinite time. Although, for a long-term or open ended strategy this is not a nice result, it is the trade-off for the increased portfolio insurance. Notice also that the probability of recovering from an ε-cash-lock situation is the same for all three strategies as their behavior is identical for low portfolio values close to the respective floor.

Nevertheless, as a consequence of the large ε-cash-lock probability it is an intuitive idea to further modify the CPPI with floor adjustment such as to introduce a minimum exposure ratio. Currently, CPPI products are often offered with a maximum as well as a minimum exposure ratio. While the maximum exposure ratio is a natural byproduct of the floor adjustments, the minimum exposure ratio must be modelled explicitly. This is what will be done in the next section.

3.4 The CPPI with Minimum Exposure Ratio

In this section, we provide the CPPI with floor adjustment with a minimum exposure ratio. The idea is to create a new strategy by performing a CPPI with floor adjustment whenever the exposure ratio according to the CPPI with floor adjustment is greater than some minimum exposure ratio $\lambda \in (0, 1)$ and investing exactly the fraction λ of the current portfolio value in the risky asset and the fraction $1 - \lambda$ in the riskfree asset whenever the exposure ratio would be less than λ according to the CPPI with floor adjustment. We will call this strategy *CPPI with minimum exposure ratio*. In continuous time, always investing exactly some fraction λ in the risky asset and the fraction $1 - \lambda$ is called a *constant mix* strategy. Therefore a situation where the minimum exposure ratio is binding will be referred to as the strategy being *in the constant mix part* as opposed to the strategy being *in the CPPI part*.

The first question in modelling the constant mix part of the strategy is how to define the trading dates. Up to now, the trading dates were defined upon changes in the discounted cushion process and this was equivalent to trading upon changes in the price process of the discounted risky asset. However, this equivalence does not hold anymore for the constant mix part of the strategy. It is impossible to define the trading dates of the constant mix part of the strategy based on changes in the discounted cushion process. Formally, this would mean that the portfolio is rebalanced according to the constant mix strategy whenever the discounted cushion process $e^{-rt}C_t = e^{-rt}(V_t - F_t)$ increases or decreases by certain fractions. The consequence of so-defined trading dates would be a problem equivalent to the paradox of Achilles and the turtle. Therefore it is clear, that the trading dates can not be defined upon changes of the cushion process, but they can be defined upon changes in the discounted price process of the risky asset. If the same fractions as for the CPPI part of the strategy were used, trading would take place whenever the discounted price process of the risky asset has gained the fraction $\frac{k_u-1}{m}$ or lost the fraction $\frac{1-k_d}{m}$. However, it is not clear, why the same fractions as for the CPPI part should be used. While being in the constant mix part, one might wish to trade at a higher or lower frequency. Although it is no problem to introduce different fractions for the constant mix part upon which the trading dates are defined, it is not our concern here to discretize the constant mix strategy. Therefore we resort to the simplest possibility and perform the constant mix part of the strategy in continuous time.

It might seem somewhat awkward to combine discrete time trading and continuous time trading and it is justified to ask what information can be drawn from such a model. Firstly, for k_u close to 1, the model will be a good approximation for the entirely continuous time pendant of the strategy and so far there does not exist a continuous time pendant in the literature. Secondly, one could argue that while a company performs the CPPI part of the strategy itself, whenever the constant mix part is reached, it invests into a constant mix product on the market and does not perform this part itself. For k_u considerably greater than 1, the differences between the results of our partially discrete time strategy and the entirely continuous-time pendant will clearly only stem from the CPPI part and this shall be the focus here. Thirdly, a constant mix strategy is not a portfolio insurance strategy. For a constant mix strategy there is always the possibility of the portfolio value decreasing to zero such that the portfolio can not be kept above a certain level. Hence, the CPPI with minimum exposure ratio is not a port-

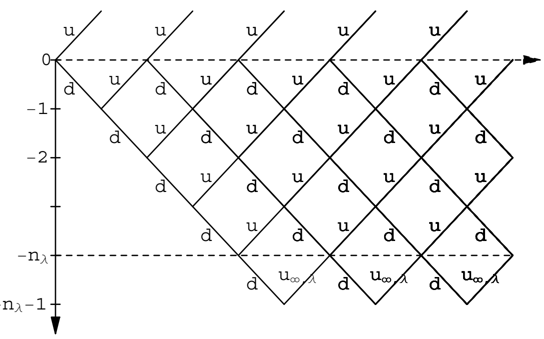

Figure 3.10: Binomial tree with both, a maximum and minimum barrier.

folio insurance strategy either. Reinstalling a portfolio protection feature for the CPPI with minimum exposure ratio becomes an important topic. The next section is dedicated to this topic and it entails considerable simplifications if the constant mix part of the strategy is performed in continuous time.

Let us now briefly recall the constant mix strategy in continuous time. Suppose, at some point in time τ, the amount V_τ is invested into a constant mix strategy with parameter λ. Then the dynamics of the portfolio value is given by

$$\begin{aligned} \frac{dV_t}{V_t} &= \lambda \frac{dS_t}{S_t} + (1-\lambda) r dt \\ &= (\lambda\mu + (1-\lambda)r)dt + \lambda\sigma dW_t \end{aligned} \tag{3.9}$$

for $t > \tau$ using the dynamics of the risky asset as defined in equation (1.1). It is apparent from equation (3.9) that the portfolio value of the constant mix strategy follows a geometric Brownian motion again if the dynamics of the price process of the risky asset follows a geometric Brownian motion. From equation (3.9) we therefore immediately conclude that

$$V_t = V_\tau e^{r(t-\tau)} e^{(\lambda(\mu-r)-\frac{1}{2}\lambda^2\sigma^2)(t-\tau)+\lambda\sigma(W_t-W_\tau)}$$

which can be rewritten as

$$V_t = V_\tau e^{r(t-\tau)} e^{\lambda\sigma(W_t^{\delta_\lambda} - W_\tau^{\delta_\lambda})} \tag{3.10}$$

where

$$\delta_\lambda := \frac{\mu - r - \frac{1}{2}\lambda\sigma^2}{\sigma} \tag{3.11}$$

and $W_t^{\delta_\lambda}$ denotes the Brownian motion with drift δ_λ.

In the following we will give a detailed description of the CPPI with minimum exposure ratio. At time $t = 0$ the strategy starts in the CPPI part such that a CPPI with floor adjustment is performed. From section 3.1 we know that at some trading date τ the portfolio value is given by $V_0 e^{r\tau} c^i(\gamma + k_d^n(1-\gamma))$ for some $i, n \in \mathbb{N}_0$. From section 3.3 it is known that the exposure ratio of the CPPI with floor adjustment is less than λ whenever $n > n_\lambda$ where

$$n_\lambda := \left\lfloor \frac{\log\left(\frac{\gamma\lambda}{(1-\gamma)(m-\lambda)}\right)}{\log k_d} \right\rfloor \tag{3.12}$$

and therefore the constant mix part of the strategy is reached, when the strategy has performed $n_\lambda+1$ down-moves since the most recent floor adjustment.[3] Hence, the portfolio value at the beginning of the constant mix part is given by

$$V_\tau^{ME} = V_0 e^{r\tau} c^i(\gamma + k_d^{n_\lambda+1}(1-\gamma)) \tag{3.13}$$

for some $i \in \mathbb{N}_0$, where V_t^{ME} denotes the value process of the CPPI with minimum exposure ratio at time t. Likewise C^{ME} and F^{ME} denote the cushion and the floor. After being in the constant mix part, the question is, when to enter the CPPI part again. The canonical choice would be the time, when the trading rule of the CPPI suggests to invest exactly the fraction λ of the portfolio into the risky asset. However, for simplicity, we define the time of reentering the CPPI part by

$$\tau' := \min_{t \geq \tau} V_t^{ME} = V_0 e^{rt} c^i(\gamma + k_d^{n_\lambda}(1-\gamma)) \tag{3.14}$$

such that from time τ' it requires exactly net n_λ up-moves to achieve full exposure again and exactly net $n_\lambda + 1$ up-moves for the next floor adjustment. In general, the exposure ratio a time τ' will be slightly higher than λ. It will be equal to λ if and only if $\frac{k_d^{n_\lambda}(1-\gamma)}{\gamma+k_d^{n_\lambda}(1-\gamma)} = \lambda$ which can only be the case for certain combinations of λ and k_d. We will now focus on the time needed to reenter the CPPI part of the strategy. A combination of equations (3.10), (3.13) and (3.14) yields

$$V_0 e^{rt} c^i(\gamma + k_d^{n_\lambda+1}(1-\gamma)) e^{\lambda\sigma(W_t^{\delta_\lambda} - W_\tau^{\delta_\lambda})} = V_0 e^{rt} c^i(\gamma + k_d^{n_\lambda}(1-\gamma))$$

or equivalently

$$W_t^{\delta_\lambda} - W_\tau^{\delta_\lambda} = \frac{1}{\lambda\sigma} \log\left(\frac{\gamma + k_d^{n_\lambda}(1-\gamma)}{\gamma + k_d^{n_\lambda+1}(1-\gamma)}\right) =: b_\lambda. \tag{3.15}$$

It is important to notice that equation (3.15) does not depend on the number of floor adjustments i such that the time to reach the CPPI part again, once being in the constant mix part, is independent of how often the floor has been adjusted before. From equation (3.15) it is immediate that the Laplace transform for the density of the time to reenter the CPPI part is given by

$$u_{\infty,\lambda}(s) := u(s| - \infty, b_\lambda, \delta_\lambda) := \lim_{a \to -\infty} u(s|a, b_\lambda, \delta_\lambda) = e^{b_\lambda \delta_\lambda - b_\lambda \sqrt{2s + \delta_\lambda^2}}$$

[3] Note that (3.12) is given in terms of down-moves which is in contrast to the formulas in the previous sections where the formulas were given in terms of up-moves. Although n_λ down-moves is equivalent to $-n_\lambda$ up-moves due to $k_d = \frac{1}{k_u}$, it is convenient to think in terms of down-moves in this section. The situation is as depicted in figure 3.10.

and the Laplace transform for the joint density of not reentering the CPPI part up to some point in time and the final value of the Brownian motion with drift δ_λ is given by

$$
\begin{aligned}
\rho_{\infty,\lambda}(s,z) &:= \rho_{\infty,\lambda}(s,z|-\infty,b_\lambda,\delta_\lambda)\\
&:= \lim_{a\to-\infty}\rho(s,z|a,b_\lambda,\delta_\lambda)\\
&= \begin{cases} \dfrac{\left(1-e^{-2b_\lambda\sqrt{2s+\delta_\lambda^2}}\right)e^{\delta_\lambda z+z\sqrt{2s+\delta_\lambda^2}}}{\sqrt{2s+\delta_\lambda^2}} & , \ z\le 0\\[2ex] \dfrac{e^{\delta_\lambda z-z\sqrt{2s+\delta_\lambda^2}}-e^{\delta_\lambda z+z\sqrt{2s+\delta_\lambda^2}-2b_\lambda\sqrt{2s+\delta_\lambda^2}}}{\sqrt{2s+\delta_\lambda^2}} & , \ z>0. \end{cases}
\end{aligned}
$$

We are now in a position to give the first result, the moments of the CPPI with minimum exposure ratio.

Proposition 3.4.1 (Moments of the CPPI with minimum exposure ratio)

Let $\gamma\in[\frac{m-1}{m},1)$, $\lambda\in(0,m(1-\gamma))$ and $F_0=\gamma V_0$. Then the j-th moment of the CPPI with minimum exposure ratio, $E\left[(V_T^{ME})^j\right]$, is given by

$$
\begin{aligned}
V_0^j e^{jrT}\mathcal{L}^{-1}_{s,T}\Bigg\{ & \frac{\sum_{i=0}^{j}\binom{j}{i}\gamma^{j-i}(1-\gamma)^i\xi_u(k_u^iu(s),k_d^id(s))\int_a^b(me^{\sigma z}-m+1)^i\rho(s,z)dz}{(1-\eta_{\infty,\lambda}(j,s))(1-c^jh_{-n_\lambda}(1,s))}\\
&+\frac{\eta_{\infty,\lambda}(j,s)\sum_{i=0}^{j}\binom{j}{i}\gamma^{j-i}(1-\gamma)^ik_d^{in_\lambda}\xi_d(k_u^iu(s),k_d^id(s))\int_a^b(me^{\sigma z}-m+1)^i\rho(s,z)dz}{(1-\eta_{\infty,\lambda}(j,s))c^jh_0(n_\lambda+1,s)}\\
&+\frac{\eta_{\infty,\lambda}(j,s)(\gamma+k_d^{n_\lambda+1}(1-\gamma))^j\int_{-\infty}^{b_\lambda}e^{j\lambda\sigma z}\rho_{\infty,\lambda}(s,z)dz}{(1-\eta_{\infty,\lambda}(j,s))c^jh_0(n_\lambda+1,s)u_{\infty,\lambda}(s)}\Bigg\}
\end{aligned}
$$

where

$$
\eta_{\infty,\lambda}(l,s):=\frac{c^lh_0(n_\lambda+1,s)h_0(-n_\lambda-1,s)u_{\infty,\lambda}(s)}{(1-c^lh_{-n_\lambda}(1,s))(1-h_{n_\lambda}(-1,s)u_{\infty,\lambda}(s))},\ l\in\mathbb{N}_0
$$

and

$$
\begin{aligned}
\xi_u(u,d) &:= \frac{1-h_{-n_\lambda}(1|u,d)-h_0(-n_\lambda-1|u,d)}{1-u-d}\\
\xi_d(u,d) &:= \frac{1-h_{n_\lambda}(-1|u,d)-h_0(n_\lambda+1|u,d)}{1-u-d}
\end{aligned}
$$

PROOF: We only proof the proposition for $j = 1$, the formula for arbitrary j is a straightforward generalization. First, we explain the term $\eta_{\infty,\lambda}(l, s)$. It can be understood as the strategy performing a certain cycle. The cycle consists of first having a possibly arbitrary number of floor adjustments without reaching the constant mix part of the strategy, then reach the constant mix part, switch possibly arbitrarily often between the constant mix part and the CPPI part without further floor adjustment before finally having another floor adjustment. Since the strategy starts with maximum exposure, one net up-move is sufficient for a floor adjustment. So, in the following we will refer to a situation with maximum exposure as the strategy being at level zero. Also we know that net $n_\lambda + 1$ down-moves are required to enter the constant mix part of the strategy and we will call this level $-n_\lambda - 1$. Therefore the Laplace transform for having a floor adjustment while not entering the constant mix part before is given by $h_{-n_\lambda}(1, s)$. We know that a floor adjustment means multiplying the portfolio with c and therefore c^j for the j-th moment. Suppose there are $i_1 \in \mathbb{N}_0$ floor adjustments, then we have $\left(c^j h_{-n_\lambda}(1, s)\right)^{i_1}$. The Laplace transform for reaching the constant mix part is given by $h_0(-n_\lambda - 1, s)$ and the Laplace transform for going back to the CPPI part is $u_{\infty,\lambda}(s)$. From that point, level $-n_\lambda$, the Laplace transform for entering the constant mix part while not having another floor adjustment before is given by $h_{n_\lambda}(-1, s)$. Therefore the term $\left(h_{n_\lambda}(-1, s)u_{\infty,\lambda}(s)\right)^{i_2}$ describes $i_2 \in \mathbb{N}_0$ switches between the CPPI part and the constant mix part. Adjusting the floor again after the last switch requires net $n_\lambda+1$ up-moves and the relevant Laplace transform for that is $h_0(n_\lambda + 1, s)$. In addition, a multiplication with c^j for the floor adjustment is needed again for the floor adjustment. Hence, in total we get

$$\left(ch_{-n_\lambda}(1, s)\right)^{i_1} h_0(-n_\lambda - 1, s)u_{\infty,\lambda}(s) \left(h_{n_\lambda}(-1, s)u_{\infty,\lambda}(s)\right)^{i_2} ch_0(n_\lambda + 1, s)$$

and summation over all possibilities $i_1,\ i_2 \in \mathbb{N}_0$ yields $\eta_{\infty,\lambda}(1, s)$. There can be an arbitrary number of such cycles and therefore the term

$$\sum_{i=0}^{\infty} \eta_{\infty,\lambda}(1, s)^i = \frac{1}{1 - \eta_{\infty,\lambda}(1, s)}$$

is found.

After a possibly arbitrary number of cycles, there are three cases. First, never reach the constant mix part again. Second, reach the constant mix part again, switch arbitrarily often between constant mix and CPPI without having another floor adjustment before eventually neither having another floor adjustment nor reaching the constant mix part again. Third, reach the constant mix part again, switch arbitrarily often between constant mix and CPPI without having another floor adjustment before never returning to the

CPPI part again. We will now calculate the expected value of the strategy for each of these cases separately.

For the first case, note that never reaching the constant mix part again means that there can still be arbitrarily many floor adjustments and the Laplace transform for that is given by $\frac{1}{1-ch_{-n_\lambda}(1,s)}$ including the multiplications with c for each floor adjustment. The value of the strategy at maturity T is given by

$$V_0 e^{rT} c^i (\gamma + (1-\gamma) k_d^n (m e^{\sigma(W_T^\delta - W_{\tau_N}^\delta)} - m + 1))$$

for some $i \in \mathbb{N}_0,\ n \in \{0, 1, \ldots, n_\lambda\}$ if the strategy ends in the CPPI part where τ_N is the last trading date before time T as usual. The c are accounted for implicitly with the probabilities of occurring floor adjustments as described above. After the last floor adjustment, the strategy is at level zero. It is apparent that $\xi_u(u(s), d(s))$ accounts for the probability of all paths that neither produce another floor adjustment nor reach the constant mix part. Hence, for the first case the expected guarantee is given by

$$V_0 e^{rT} \mathcal{L}_{s,T}^{-1} \left\{ \frac{\gamma \xi_u(u(s), d(s)) \int_a^b \rho(s,z)dz}{(1-\eta_{\infty,\lambda}(1,s))(1-ch_{-n_\lambda}(1,s))} \right\}. \tag{3.16}$$

Since the cushion multiplies with k_u for each up-move and k_d for each down move, we can implicity account for that by taking $\xi_u(k_u u(s), k_d d(s))$ and therefore the expected cushion is given by

$$V_0 e^{rT} \mathcal{L}_{s,T}^{-1} \left\{ \frac{(1-\gamma)\xi_u(k_u u(s), k_d d(s)) \int_a^b (m e^{\sigma z} - m + 1)\rho(s,z)dz}{(1-\eta_{\infty,\lambda}(1,s))(1-ch_{-n_\lambda}(1,s))} \right\}. \tag{3.17}$$

For the second case there can be arbitrarily many floor adjustments, hence $\frac{1}{1-ch_{-n_\lambda}(1,s)}$, before reaching the constant mix part with $h_0(-n_\lambda - 1, s)$, going back to the CPPI part with $u_{\infty,\lambda}(s)$ and switching arbitrarily often between the constant mix and the CPPI with $\frac{1}{1-h_{n_\lambda}(-1,s)u_{\infty,\lambda}(s)}$. Multiplication of the terms yields

$$\frac{h_0(-n_\lambda - 1, s)u_{\infty,\lambda}(s)}{(1-ch_{-n_\lambda}(1,s))(1-h_{n_\lambda}(-1,s)u_{\infty,\lambda}(s))} = \frac{\eta_{\infty,\lambda}(1,s)}{(1-\eta_{\infty,\lambda}(1,s))ch_0(n_\lambda+1,s)}.$$

The strategy then is at level $-n_\lambda$ and the probability of all paths that neither produce another floor adjustment nor reach the constant mix part is accounted for with $\xi_d(u(s), d(s))$.

With the same arguments as above, for the second case the expected guarantee is given by

$$V_0 e^{rT} \mathcal{L}^{-1}_{s,T} \left\{ \frac{\gamma \eta_{\infty,\lambda}(1,s) \xi_d(u(s), d(s)) \int_a^b \rho(s,z) dz}{(1-\eta_{\infty,\lambda}(1,s)) ch_0(n_\lambda + 1, s)} \right\} \tag{3.18}$$

and the expected cushion is given by

$$V_0 e^{rT} \mathcal{L}^{-1}_{s,T} \left\{ \frac{(1-\gamma) k_d^{n_\lambda} \eta_{\infty,\lambda}(1,s) \xi_d(k_u u(s), k_d d(s)) \int_a^b (m e^{\sigma z} - m + 1) \rho(s,z) dz}{(1-\eta_{\infty,\lambda}(1,s)) ch_0(n_\lambda + 1, s)} \right\}. \tag{3.19}$$

For the third case we analogously find the term

$$\frac{\eta_{\infty,\lambda}(1,s)}{(1-\eta_{\infty,\lambda}(1,s)) ch_0(n_\lambda + 1, s) u_{\infty,\lambda}(s)}$$

for all cases of the strategy reaching level $-n_\lambda - 1$. Since the strategy is not to enter the CPPI part again by assumption, the value of the portfolio at maturity is given by

$$V_0 e^{rT} c^i (\gamma + (1-\gamma) k_d^{n_\lambda + 1}) e^{\lambda \sigma (W_T^{\delta_\lambda} - W_{\tau_N}^{\delta_\lambda})}$$

and therefore the expected value at maturity must equal

$$V_0 e^{rT} \mathcal{L}^{-1}_{s,T} \left\{ \frac{\eta_{\infty,\lambda}(1,s)(\gamma + k_d^{n_\lambda+1}(1-\gamma)) \int_{-\infty}^{b_\lambda} e^{\lambda \sigma z} \rho_{\infty,\lambda}(s,z) dz}{(1-\eta_{\infty,\lambda}(1,s)) ch_0(n_\lambda + 1, s) u_{\infty,\lambda}(s)} \right\} \tag{3.20}$$

for the third case.

It can readily be seen, that the sum equations (3.16), (3.17), (3.18), (3.19) and (3.20) coincides with the formula in the proposition for $j = 1$. □

Note that while the formula given in proposition 3.4.1 is closed-form in terms of a Laplace transform, as all formulas presented so far, it does not seem possible to receive such an expression for the distribution of the CPPI with minimum exposure ratio. Although it is not difficult to find an expression containing an infinite sum of Laplace transforms (or equivalently a Laplace transform of an infinite sum), we therefore restrict ourselves to present some risk-measures.

Proposition 3.4.2 (Risk-Measures)

In the notation of proposition 3.4.1 and in addition

$$\begin{aligned} a_\lambda &:= \frac{1}{\lambda\sigma}\log\frac{\gamma}{\gamma+(1-\gamma)k_d^{n_\lambda+1}} \\ \eta_\lambda(s) &:= \frac{h_0(n_\lambda+1,s)h_0(-n_\lambda-1,s)u(s|a_\lambda,b_\lambda,\delta_\lambda)}{(1-h_{-n_\lambda}(1,s))(1-h_{n_\lambda}(-1,s)u(s|a_\lambda,b_\lambda,\delta_\lambda))},\ l\in\mathbb{N}_0, \end{aligned}$$

it holds:

a) The probability of the strategy being in default at maturity T, i.e. the shortfall probability $PSF = P(V_T^{ME} < F_T^{ME})$, is given by

$$\mathcal{L}_{s,T}^{-1}\left\{\frac{\eta_{\infty,\lambda}(0,s)}{(1-\eta_{\infty,\lambda}(0,s))h_0(n_\lambda+1,s)u_{\infty,\lambda}(s)}\int_{-\infty}^{a_\lambda}\rho_{\infty,\lambda}(s,z)dz\right\}$$

b) The probability of the strategy with maturity time T falling below the current floor at some time before T, i.e. $P\left(\exists t\in(0,T):\ V_t^{ME} < F_t^{ME}\right)$, is given by

$$\mathcal{L}_{s,T}^{-1}\left\{\frac{\eta_\lambda(s)}{(1-\eta_\lambda(s))h_0(n_\lambda+1,s)u(s|a_\lambda,b_\lambda,\delta_\lambda)}\frac{d(s|a_\lambda,b_\lambda,\delta_\lambda)}{s}\right\}$$

c) The expected shortfall at maturity, ESF, is determined through

$$ESF = E\left[F_T^{ME}-V_T^{ME}|V_T^{ME}<F_T^{ME}\right] = \frac{E\left[(F_T^{ME}-V_T^{ME})1_{\{V_T^{ME}<F_T^{ME}\}}\right]}{PSF}$$

where $E\left[(F_T^{ME}-V_T^{ME})1_{\{V_T^{ME}<F_T^{ME}\}}\right]$ is given by

$$V_0e^{rT}\mathcal{L}_{s,T}^{-1}\left\{\frac{\eta_{\infty,\lambda}(1,s)\left(\gamma\int_{-\infty}^{a_\lambda}\rho_{\infty,\lambda}(s,z)dz-(\gamma+(1-\gamma)k_d^{n_\lambda+1})\int_{-\infty}^{a_\lambda}e^{\lambda\sigma z}\rho_{\infty,\lambda}(s,z)dz\right)}{(1-\eta_{\infty,\lambda}(1,s))ch_0(n_\lambda+1,s)u_{\infty,\lambda}(s)}\right\}$$

Proof: We know that the value of the portfolio at maturity is given by $V_0e^{rT}c^i(\gamma+(1-\gamma)k_d^{n_\lambda+1})e^{\lambda\sigma(W_T^{\delta_\lambda}-W_{\tau_N}^{\delta_\lambda})}$ if the strategy is to end in the constant mix part. Since

$$V_0e^{rT}c^i(\gamma+(1-\gamma)k_d^{n_\lambda+1})e^{\lambda\sigma a_\lambda} = \gamma V_0e^{rT}c^i,$$

which equals the time T guarantee after i floor adjustments, part a) is immediately clear from the proof of proposition 3.4.1. It is also obvious from the proof of proposition 3.4.1

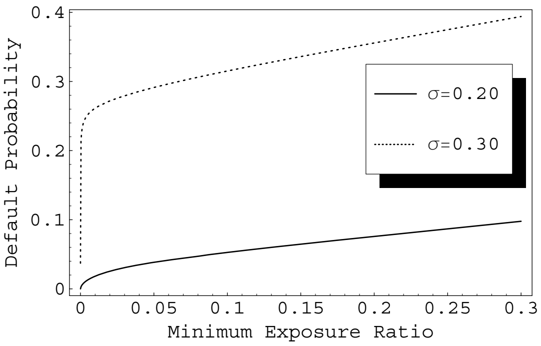

Figure 3.11: Probability of a default at maturity of the CPPI with minimum exposure ratio as a function of the minimum exposure ratio, λ, for different volatilities. The parameters are $k_u = 1.01$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\gamma = 0.75$ and $T = 30$.

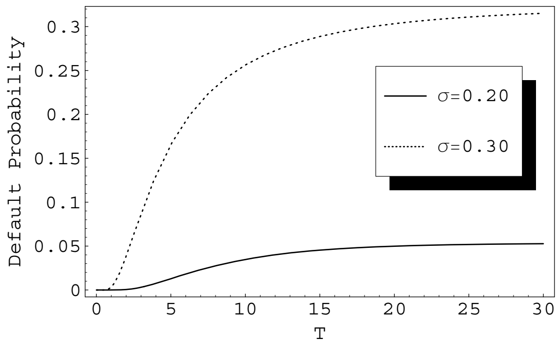

Figure 3.12: Probability of a default at maturity of the CPPI with minimum exposure ratio as a function of the maturity time T for different volatilities. The parameters are $k_u = 1.01$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\gamma = 0.75$ and $\lambda = 0.10$.

that

$$V_0 e^{rT} \mathcal{L}_{s,T}^{-1} \left\{ \frac{\gamma \eta_{\infty,\lambda}(1,s) \int_{-\infty}^{a_\lambda} \rho_{\infty,\lambda}(s,z) dz}{(1-\eta_{\infty,\lambda}(1,s)) ch_0(n_\lambda+1,s) u_{\infty,\lambda}(s)} \right\}$$

is the expected guarantee on the set of all paths that end in default and

$$V_0 e^{rT} \mathcal{L}_{s,T}^{-1} \left\{ \frac{\eta_{\infty,\lambda}(1,s)(\gamma+(1-\gamma)k_d^{n_\lambda+1}) \int_{-\infty}^{a_\lambda} e^{\lambda\sigma z} \rho_{\infty,\lambda}(s,z) dz}{(1-\eta_{\infty,\lambda}(1,s)) ch_0(n_\lambda+1,s) u_{\infty,\lambda}(s)} \right\}$$

is the expected portfolio value on the set of all paths that end in default. Hence, part c) is apparent.

For part b) note that $\eta_\lambda(s)$ is identical to $\eta_{\infty,\lambda}(0,s)$ only that $u_{\infty,\lambda}(s)$ has been exchanged for $u(s|a_\lambda, b_\lambda, \delta_\lambda)$, which is the Laplace transform for the strategy (once in the constant mix part) reaching the CPPI part again before the current portfolio value is below the current floor. Likewise, $d(s|a_\lambda, b_\lambda, \delta_\lambda)$ is the Laplace transform for the strategy, being in the constant mix part, reaching default while not going back to the CPPI part before. Hence, part b) follows. □

Figures 3.11 and 3.12 show the probability of the CPPI with minimum exposure ratio being in default at maturity as a function of the minimum exposure ratio λ and the maturity T respectively. The default probability is increasing in both variables. It is

CPPI with floor adjustment			
$\frac{E[N]}{T}$	k_u	Mean	Stdv.
12	1.2595 (1.4134)	8148.29 (4384.71)	8619.67 (5519.37)
24	1.1773 (1.2774)	7890.23 (4266.88)	8069.22 (4937.29)
48	1.1224 (1.1890)	7712.84 (4190.30)	7692.69 (4570.41)
96	1.0851 (1.1303)	7590.02 (4139.48)	7433.26 (4332.29)

CPPI with minimum exposure ratio					
$\frac{E[N]}{T}$	λ	Mean	Stdv.	SFP	ESF
12	10%	8433.07 (5377.21)	8618.26 (6497.90)	0.0467 (0.2951)	84.89 (159.84)
24	10%	8177.71 (5238.70)	8068.96 (5879.65)	0.0479 (0.2981)	84.15 (158.08)
48	10%	8000.68 (5146.05)	7692.52 (5482.67)	0.0485 (0.2995)	83.59 (156.82)
96	10%	7878.46 (5084.82)	7433.20 (5224.02)	0.0490 (0.3006)	83.21 (155.98)
12	30%	9535.64 (7275.82)	9089.27 (9196.78)	0.0890 (0.3651)	313.28 (599.16)
24	30%	9279.10 (7106.25)	8544.44 (8464.34)	0.0914 (0.3718)	310.28 (594.80)
48	30%	9095.30 (6983.79)	8163.74 (7965.56)	0.0927 (0.3751)	307.91 (590.82)
96	30%	8969.08 (6903.44)	7902.12 (7640.84)	0.0937 (0.3776)	306.27 (588.29)

Table 3.1: Moments and risk-measures of the CPPI with floor adjustment and the CPPI with minimum exposure ratio. The parameters are $T = 20$, $V_0 = 1000$, $m = 4$, $\gamma = \frac{m-1}{m} = 0.75$, $F_0 = \gamma V_0 = 750$, $\mu = 0.15$, $r = 0.05$ and $\sigma = 0.2$ ($\sigma = 0.3$ respectively).

clear, that the default probability must vanish as the minimum exposure ratio tends to zero, since in this case the CPPI with minimum exposure ratio converges to the CPPI with floor adjustment. However, in figure 3.11 it is interesting to notice, that in the case of $\sigma = 30\%$ already a minimum exposure ratio of 1% produces a default risk of approximately 26% for a maturity of 30 years. Also, the difference in the default probability for the two different volatilities is remarkable. The difference is even more remarkable in figure 3.12. While the default probability converges to around 32.2% as T turns to infinity for the case $\sigma = 30\%$ it only converges to approximately 5.3% for a volatility of $\sigma = 20\%$.

Table 3.1 compares the moments of the terminal values of the CPPI with floor adjustment and the CPPI with minimum exposure ratio for different volatilities, different values of the discretization parameter k_u and different values of the minimum exposure ratio λ. While the parameters agree with our usual set of parameters, the maturity time T has been chosen quite large to equal 20 years. This is consistent with our interest in the long term performance as well as suited to pronounce the effects of the minimum exposure ratio.

First notice that the discretization parameter k_u has been chosen such as to yield 12, 24, 48 and 96 expected trading dates per year (see the column $\frac{E[N]}{T}$). Although we have not presented the distribution of the trading dates in this chapter, it is straightforward to check that this distribution coincides with the distribution of the trading dates of the simple CPPI as given in proposition 2.2.1. For the CPPI with minimum exposure this is not true any more. The column $\frac{E[N]}{T}$ here merely is supposed to symbolize that the same discretization parameters as for the CPPI with floor adjustment have been chosen.

Notice that the values for the moments in table 3.1 are significantly decreasing in the number of expected trading dates. This is no coincidence. We know from section 2.3 that both, the simple and the capped CPPI, converge to a stop-loss strategy as k_u turns to infinity. This is also true for the CPPI with floor adjustment and the CPPI with minimum exposure ratio. The strategies start with full exposure at time $t = 0$ and for $k_u \to \infty$ the portfolio will never be rebalanced. While it is well-known that the stop-loss strategy has the same expected long term yield as the risky asset, i.e. the drift μ, it is known from section 3.1 that this is not true for the CPPI with floor adjustment. Also the long term yield of the CPPI with minimum exposure ratio will be lower than the drift of the risky asset. Therefore it is not surprising that the moments are decreasing in the number of expected trading dates. However, one must not be misled by the increase of the expected value for $k_u \to \infty$. Although a larger expected payoff seems appealing, no rebalancing of the portfolio also means, that there will be no floor adjustment, such that the portfolio protection is considerably lowered at the same time.

Now compare the values of the moments for the two different volatilities $\sigma = 20\%$ and $\sigma = 30\%$. Recall from equation (3.5) that the expected long term yield of the CPPI with floor adjustment equals the risk-free interest rate r for $\sigma = 30\%$. On this basis, the CPPI with floor adjustment can be ruled out as a reasonable long term strategy upon violation of the condition $\mu - r - (m-1)\frac{\sigma^2}{2} \geq 0$. Table 3.1 underpins this impression. The difference between the values for the two volatilities is remarkable.

A comparison between the moments of the CPPI with minimum exposure ratio and the moments of the CPPI with floor adjustment demonstrates once again the basic effect of the minimum exposure ratio. It provokes a trade-off between an increased expected payoff and the existence of a shortfall probability. For the smaller volatility, $\sigma = 20\%$, it is surprising that both, the gain in the expected payoff and the shortfall probability, are quite small for a minimum exposure ratio of $\lambda = 10\%$. For the minimum exposure ratio of $\lambda = 30\%$ the effect is more pronounced. Notice also that the moments vary considerably

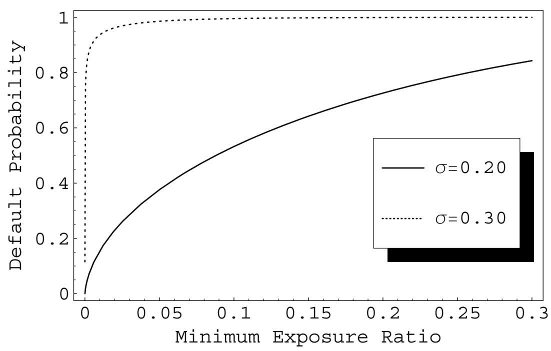

Figure 3.13: Probability of the CPPI with minimum exposure ratio becoming smaller than the floor at some time up to maturity as a function of the minimum exposure ratio for different volatilities. The parameters are $k_u = 1.01$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\gamma = 0.75$, $T = 30$ and $\sigma = 0.20,\ 0.30$.

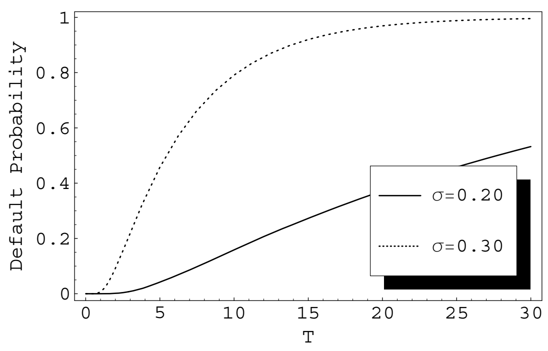

Figure 3.14: Probability of the CPPI with minimum exposure ratio becoming smaller than the floor at some time up to maturity as a function of the maturity time T for different volatilities. The parameters are $k_u = 1.01$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\gamma = 0.75$, $\lambda = 0.10$ and $\sigma = 0.20,\ 0.30$.

dependent on the discretization while the shortfall probability and the expected shortfall remain relatively constant in comparison. In the case of the larger volatility, $\sigma = 30\%$, the gain in the expected payoff compared to the CPPI with floor adjustment is remarkable, in particular for the larger minimum exposure ratio, but so is the increase of the shortfall probability.

The relevance of the risk-measures and acceptance of the CPPI with minimum exposure ratio very much depends on the contract specification between the issuer and the investor. For example, if the contract is such that the investor bears all default risk and pays the issuer only for performing the strategy, a default risk of around 5% (10%) for $\sigma = 20\%$ and $\lambda = 10\%$ ($\lambda = 30\%$) might well be acceptable for the avoidance of the cash-lock dependent on the investor's attitude towards risk. Keep in mind that the shortfall probability gives the probability of the portfolio value at maturity being lower than the floor. Due to the floor adjustments, this probability has little to do with the investor losing money compared to the initial investment. It is well possible that, compared to the initial investment, the terminal portfolio is quite large while below the floor at the same time such that the outcome is satisfactory irrespective of a shortfall. This situation occurs in particular if the portfolio value increases very much at the beginning and decreases later. The situation is much different if the issuer commits himself to guarantee at least the floor at maturity. In this case the risk-measures are crucial.

Figures 3.13 and 3.14 depict the probability of the CPPI with minimum exposure ratio becoming smaller than the floor at some time before maturity. Once the current portfolio value is below the current floor, default at maturity would be safe if all funds were switched to the riskless asset immediately. Hence, from this time onwards the strategy must perform better than the riskless asset to avoid default and so the situation could be called a *virtual default*. This probability converges to one, independent of the parameters, as the maturity T turns to infinity. The reason is the same as for the probability of an ε-cash-lock, the increasing variance of the risky asset for large maturities combined with the floor adjustment rule.

The importance of the virtual default probability also depends very much on the contract specifications. If the CPPI with minimum exposure ratio is offered as a fixed maturity product, then the virtual default probability is of minor interest, as there is no obligation before maturity time whatsoever. However, for example a surrender option that allows the investor to retrieve the maximum of the current portfolio value and the current floor at any time before maturity makes the virtual default probability relevant. The virtual default probability is the only one of relevance if the CPPI with minimum exposure ratio is offered as an open ended fund product where investors can come an go at any time. Our results show, that while the CPPI with minimum exposure ratio might be an acceptable long term investment strategy based on a fixed maturity, if offered as an open ended fund product the strategy must end in default sooner or later.

The default risk can be avoided by covering the potential losses caused by the minimum exposure ratio with the help of options. Such a hybrid between a CPPI and an OBPI is presented in the next section.

3.5 Hedging the CPPI with minimum exposure ratio

In section 3.4 it was shown that the introduction of a minimum exposure ratio causes significant default risk, such that hedging a CPPI with minimum exposure ratio becomes an important issue. First, we consider the CPPI with minimum exposure ratio with a fixed maturity T and determine the fair price at time $t = 0$ to cover potential losses that occur if the strategy does not end up above the floor F_T^{ME} at maturity, which we suppose is guaranteed by the issuer. More explicitly, the question will be what the price of the claim

$$\left(F_T^{ME} - V_T^{ME}\right)^+ \tag{3.21}$$

is. This claim is an European put-option written on the CPPI with minimum exposure ratio as underlying and the floor at maturity as strike. Note that if the i-th floor adjustment is made at some trading date τ, the portfolio value of the CPPI with minimum exposure ratio is given by $V_\tau^{ME} = V_0 e^{r\tau} c^i$, while the floor is given by $F_\tau^{ME} = \gamma V_0 e^{r\tau} c^i$. Therefore, the discounted claim $\left(e^{-rT} F_T^{ME} - e^{-rT} V_T^{ME}\right)^+$ can also be viewed as an infinitely increasing ladder put, based on the discounted value process of the CPPI with minimum exposure ratio, where the barriers are given by $V_0 c^i$ and the strikes are given by $\gamma V_0 c^i$ for $i \in \mathbb{N}_0$. From the fundamental theorem of asset pricing it is well-known that the price of the option in equation (3.21) is given by

$$E^{P^*}\left[e^{-rT}\left(F_T^{ME} - V_T^{ME}\right)^+\right]$$

where P^* is the equivalent martingale measure[4]. Note that pricing the option in equation (3.21) is quite similar to calculating the expected shortfall of the CPPI with minimum exposure ratio. Since the expected shortfall of the CPPI with minimum exposure ratio at maturity is given by

$$\frac{E\left[\left(F_T^{ME} - V_T^{ME}\right)^+\right]}{PSF},$$

formally the difference between the expectation for the expected shortfall and the price of the option is simply in discounting and taking expectations with respect to the martingale measure P^* instead of the physical measure P. Therefore, from proposition 3.4.2,c) we have the following corollary.

Corollary 3.5.1 (Price of a static hedge at time $t = 0$)
The fair price at time $t = 0$ for covering all potential losses of the CPPI with minimum exposure ratio, i.e. $E^{P^*}\left[e^{-rT}\left(F_T^{ME} - V_T^{ME}\right)^+\right]$, is given by

$$V_0 \mathcal{L}_{s,T}^{-1}\left\{\frac{\eta_{\infty,\lambda}(1,s)\left(\gamma \int\limits_{-\infty}^{a_\lambda} \rho_{\infty,\lambda}(s,z)dz - (\gamma + (1-\gamma)k_d^{n_\lambda+1}) \int\limits_{-\infty}^{a_\lambda} e^{\lambda\sigma z}\rho_{\infty,\lambda}(s,z)dz\right)}{(1-\eta_{\infty,\lambda}(1,s))ch_0(n_\lambda+1,s)u_{\infty,\lambda}(s)}\right\}$$

in the notation of proposition 3.4.2 with the additional assumption $\mu = r \Rightarrow \delta = -\frac{1}{2}\sigma$.

[4]See Harrison and Kreps (1979) and Harrison and Pliska (1981) for the notion of arbitrage free pricing.

Figure 3.15 shows the price of the option as a function of the maturity time T. It is noticeable that the price of the option is very large. For a volatility of 30% and a maturity of 30 years, the fair price of the option is around 17.5% of the initial investment. The reason for this large price are the floor adjustments. Since the strike of the put option equals the floor at maturity, each floor adjustment makes the option more valuable. As we know from the previous section, the floor at maturity is roughly equal to $\gamma e^{rT} \max_{t\in[0,T]} e^{-rt}V_t^{ME}$, therefore depends on the maximum of the value process of the CPPI with minimum exposure ratio and is very similar to a look-back option. It turns out that the price of the option is increasing in the maturity time but more importantly, the price is not bounded from above as the maturity time turns to infinity. Indeed, Duffie and Harrison (1993) show that the price of a perpetual look-back option can not be finite. Already in finite time the price of the option in equation (3.21) can be higher than the initial investment V_0. Hence, even if the simplest contract specification is considered, i.e. a fixed maturity time T and a guarantee from the issuer that is only related to the maturity time, the issuer might not be able to perform a static hedge against potential losses due to the guarantee by just buying (or synthesizing) the option in equation (3.21). A simple static hedge might thus not be a viable option.

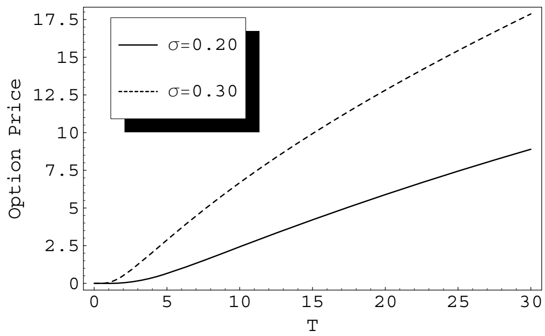

Figure 3.15: Price of the option at time $t = 0$ to cover all potential losses of the CPPI with minimum exposure ratio as a function of the maturity T and in percent of the initial investment V_0 for two different volatilities, $\sigma = 0.20$ and $\sigma = 0.30$. The other parameters are $k_u = 1.01$, $m = 4$, $\lambda = 0.20$, $r = 0.05$, $\gamma = 0.75$.

We therefore resort to considering claims of the form

$$\left(F_T^{ME} - V_T^{ME}\right)^+ 1_{\{F_T^{ME}=F_\tau^{ME}e^{r(T-\tau)}\}} \tag{3.22}$$

where τ is the time of some floor adjustment. The claim in equation (3.22) refers to a knock-out put option that starts at a time where a floor adjustment is made and covers all potential losses of the CPPI with minimum exposure ratio as long as there is no further floor adjustment, which is the knock-out condition. In the following, the idea is not to buy an option that covers all potential losses right at time $t = 0$ but to buy the knock-out option in equation (3.22) every time a floor adjustment is made. It is clear that if such a knock-out option is bought every time a floor adjustment is made, all potential losses of the CPPI are covered. From the point of view of time $t = 0$, the expected costs of this hedging strategy equal the price in corollary 3.5.1. However, we will not hedge the CPPI with minimum exposure ratio externally, but rather buy the knock-out options from the portfolio such as to produce a new self-financing strategy that features a minimum exposure ratio while keeping the cushion non-negativ. The resulting strategy can be viewed as a hybrid between an OBPI and a CPPI strategy and we will call it the *default protected CPPI with minimum exposure ratio.* We start by giving the price of the knock-out options.

Corollary 3.5.2 (Price of a hedge between two floor adjustments)
Suppose a CPPI with minimum exposure ratio that starts at time $t \in [0,T]$ with maturity time T and an initial investment of one unit, i.e. $V_t = 1$. Then the fair price at time t of the option to cover potential losses of the strategy if there is no floor adjustment, i.e. $\pi(t,T) := E^{P^*}\left[e^{-r(T-t)}\left(F_T^{ME} - V_T^{ME}\right)^+ 1_{\{F_T^{ME}=F_t^{ME}e^{r(T-t)}\}}|\mathcal{F}_t\right]$, is given by

$$\mathcal{L}^{-1}_{s,T-t}\left\{\frac{h_0(-(n_\lambda+1),s)\left(\gamma\int\limits_{-\infty}^{a_\lambda}\rho_{\infty,\lambda}(s,z)dz - (\gamma+(1-\gamma)k_d^{n_\lambda+1})\int\limits_{-\infty}^{a_\lambda}e^{\lambda\sigma z}\rho_{\infty,\lambda}(s,z)dz\right)}{1-h_{n_\lambda}(-1,s)u_{\infty,\lambda}(s)}\right\}$$

in the notation of proposition 3.4.2 with the additional assumption $\mu = r \Rightarrow \delta = -\frac{1}{2}\sigma$.

Suppose that at some time τ of a floor adjustment, the CPPI with minimum exposure ratio is to be hedged with a knock-out option. Then, according to corollary 3.5.2 the hedging costs at time τ are given by $V_\tau^{ME}\pi(\tau,T)$. However, if the option is to be bought from the portfolio, then the amount available for the CPPI with minimum exposure ratio decreases and hence $V_\tau^{ME}\pi(\tau,T)$ can not be the correct price any more, since this is the price if the amount invested into the CPPI with minimum exposure ratio equals V_τ^{ME}, which it does not any more after the option is acquired. It turns out that the price of the appropriate option is given by

$$\frac{V_\tau^{ME}}{1+\pi(\tau,T)}\pi(\tau,T) \tag{3.23}$$

since

$$V_\tau^{ME} - \frac{V_\tau^{ME}}{1+\pi(\tau,T)}\pi(\tau,T) = \frac{V_\tau^{ME}}{1+\pi(\tau,T)}$$

such that the amount invested into the CPPI with minimum exposure ratio after the option is bought equals $\frac{V_\tau^{ME}}{1+\pi(\tau,T)}$, which is consistent with the price in equation (3.23). Since the option knocks out on the next floor adjustment, this effectively means that the portfolio value decreases by the factor $\frac{1}{1+\pi(\tau,T)}$ on each floor adjustment. However, we also know, that the discounted portfolio value of the CPPI with minimum exposure ratio multiplies with the constant factor c on each floor adjustment. Therefore, the discounted value process of the default protected CPPI with minimum exposure ratio multiplies with the factor

$$\tilde{c}(\tau, T) := \frac{c}{1 + \pi(\tau, T)} \tag{3.24}$$

instead of c as for the CPPI with minimum exposure ratio. With the trivial inequality $\pi(\tau, T) \geq 0$ this means that the portfolio value of the default protected CPPI with minimum exposure ratio is generally lower than the portfolio value of the CPPI with minimum exposure ratio as long as the latter is larger than the floor.

Unfortunately, as a consequence of the dependence of $\tilde{c}(\tau, T)$ on the time to maturity $T-\tau$, it does not seem possible to find any analytical expressions for the moments or even the distribution of the default protected CPPI with minimum exposure ratio. Nevertheless we can find an expression for a lower bound of the expected terminal payoff. As a preparation the following proposition is needed.

Proposition 3.5.3 *The price $\pi(t, T)$ is increasing in the time to maturity $T - t$.*

PROOF: Since $\pi(t, T) = \pi(0, T - t)$, it is sufficient to show $\pi(0, T) \leq \pi(0, T')$ for $T' > T$. Let τ be the time of the first floor adjustment. Due to the fact that the CPPI with minimum exposure ratio is a self-financing strategy, the discounted value process $(V_t^{ME} e^{-rt})$ is a martingale under the martingale measure P^* and hence also the process $(\gamma - V_t^{ME} e^{-rt})$ is a martingale. Therefore also the stopped process $(\gamma - V_{t\wedge\tau}^{ME} e^{-r(t\wedge\tau)})$ is a martingale according to the Optional Sampling Theorem[5]. Hence we have

$$\begin{aligned}
\pi(0, T') &= E^{P^*}\left[e^{-rT'}\left(F_{T'}^{ME} - V_{T'}^{ME}\right)^+ 1_{\{F_{T'}^{ME} = F_0^{ME} e^{rT'}\}}\right] \\
&= E^{P^*}\left[\left(\gamma - V_{(T'\wedge\tau)}^{ME} e^{-r(T'\wedge\tau)}\right)^+\right] \\
&= E^{P^*}\left[E^{P^*}\left[\left(\gamma - V_{(T'\wedge\tau)}^{ME} e^{-r(T'\wedge\tau)}\right)^+ \middle| \mathcal{F}_T\right]\right] \\
&\geq E^{P^*}\left[E^{P^*}\left[\gamma - V_{(T'\wedge\tau)}^{ME} e^{-r(T'\wedge\tau)} \middle| \mathcal{F}_T\right]^+\right] \\
&= E^{P^*}\left[\left(\gamma - V_{(T\wedge\tau)}^{ME} e^{-r(T\wedge\tau)}\right)^+\right] \\
&= \pi(0, T)
\end{aligned}$$

[5]See for example Rogers and Williams (2000), p.159.

using Jensen's inequality. □

With the help of proposition 3.5.3 it is now obvious that $\tilde{c}(t,T) \geq \tilde{c}(0,T)$, such that a rough lower bound for the expected payoff of the default protected CPPI with minimum exposure ratio can be found by always using $\tilde{c}(0,T)$ instead of $\tilde{c}(t,T)$.

Corollary 3.5.4 (Expected payoff, lower bound)
A lower bound for the expected terminal payoff of the default protected CPPI with minimum exposure ratio is given by

$$\frac{V_0 e^{rT}}{1+\pi(0,T)} \mathcal{L}^{-1}_{s,T} \Bigg\{ \left(\frac{\xi_u(u(s),d(s))}{1-\tilde{c}(0,T)h_{-n_\lambda}(1,s)} + \frac{\eta_{\infty,\lambda}(s)\xi_d(u(s),d(s))}{\tilde{c}(0,T)h_0(n_\lambda+1,s)} \right) \frac{\gamma \int_a^b \rho(s,z)dz}{1-\eta_{\infty,\lambda}(s)}$$
$$+ \left(\frac{\xi_u(k_u u(s),k_d d(s))}{1-\tilde{c}(0,T)h_{-n_\lambda}(1,s)} + k_d^{n_\lambda} \frac{\eta_{\infty,\lambda}(s)\xi_d(k_u u(s),k_d d(s))}{\tilde{c}(0,T)h_0(n_\lambda+1,s)} \right) \frac{(1-\gamma)\int_a^b (me^{\sigma z}-m+1)\rho(s,z)dz}{1-\eta_{\infty,\lambda}(s)}$$
$$+ \frac{\eta_{\infty,\lambda}(s)\left(\gamma \int_{-\infty}^{a_\lambda} \rho_{\infty,\lambda}(s,z)dz + (\gamma + k_d^{n_\lambda+1}(1-\gamma)) \int_{a_\lambda}^{b_\lambda} e^{\lambda\sigma z}\rho_{\infty,\lambda}(s,z)dz\right)}{(1-\eta_{\infty,\lambda}(s))\tilde{c}(0,T)h_0(n_\lambda+1,s)u_{\infty,\lambda}(s)} \Bigg\}$$

where

$$\eta_{\infty,\lambda}(s) := \frac{\tilde{c}(0,T)h_0(n_\lambda+1,s)h_0(-n_\lambda-1,s)u_{\infty,\lambda}(s)}{(1-\tilde{c}(0,T)h_{-n_\lambda}(1,s))(1-h_{n_\lambda}(-1,s)u_{\infty,\lambda}(s))}, \quad l \in \mathbb{N}_0$$

and the other notation as in propositions 3.4.1 and 3.4.2.

Figures 3.16 and 3.17 show the expected yield per year of the CPPI with floor adjustment, the CPPI with minimum exposure ratio and the lower bound of the default protected CPPI with minimum exposure ratio as a function of the maturity time T as given by the formula $\frac{1}{T}\log\frac{E[V_T]}{V_0}$ for our standard parameter constellation and $\sigma = 20\%$ and $\sigma = 30\%$ respectively. The minimum exposure ratio was set to $\lambda = 10\%$. It is known from section 3.1 that the expected yield of the CPPI with floor adjustment converges to 9% for $\sigma = 20\%$ and to the riskfree interest rate of 5% for the large volatility $\sigma = 30\%$. Unsurprisingly, the expected yield of the CPPI with minimum exposure ratio is significantly larger in the long run, in particular for the larger volatility. This is a direct consequence of the constant mix part of the strategy. The expected yield of a pure constant mix strategy is given by $\lambda\mu + (1-\lambda)r$ which gives 6% for our parameters. It is intuitively clear that the expected

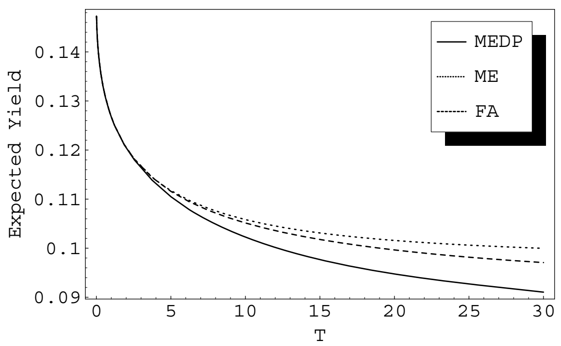

Figure 3.16: Expected yield of the CPPI with floor adjustment, the CPPI with minimum exposure ratio and the lower bound of the default protected CPPI with minimum exposure ratio as a function of the maturity time T. The parameters are $k_u = 1.01$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\gamma = 0.75$, $\lambda = 0.10$ and $\sigma = 0.20$.

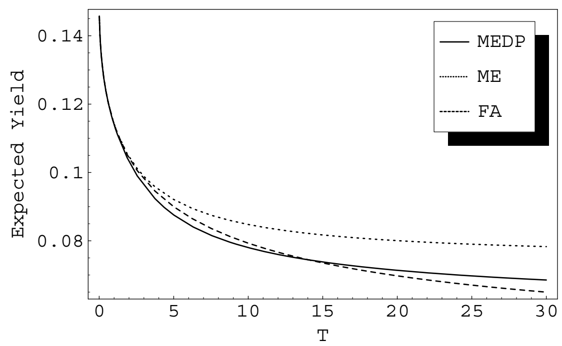

Figure 3.17: Expected yield of the CPPI with floor adjustment, the CPPI with minimum exposure ratio and the lower bound of the default protected CPPI with minimum exposure ratio as a function of the maturity time T. The parameters are $k_u = 1.01$, $m = 4$, $\mu = 0.15$, $r = 0.05$, $\gamma = 0.75$, $\lambda = 0.10$ and $\sigma = 0.30$.

yield of the CPPI with minimum exposure ratio will not be lower. Nevertheless, our numerical results suggest that for $T \to \infty$ the expected yield is considerably larger than 6% and converges to approximately 7.4%. For the default protected CPPI with minimum exposure ratio things look different. First consider figure 3.17. Although also the lower bound for the default protected CPPI with minimum exposure ratio will converge to a value larger than the riskfree interest rate and therefore perform better than the CPPI with floor adjustment in the long run, the default protection seems to be quite costly. In figure 3.16 the result is disastrous for the default protected CPPI with minimum exposure ratio. The high hedging costs seem to completely thwart the positive effects of the minimum exposure ratio, i.e. the avoidance of an ε-cash-lock. Although it must not be forgotten that the curve for the default protected CPPI with minimum exposure ratio only constitutes a lower bound which is very rough, this result is remarkable.

So far we have assumed a fixed maturity time. Suppose now that the default protected CPPI with minimum exposure ratio is offered as an open ended fund product. In this case the European type options from equation (3.22) must be replaced by perpetual American type options. It is well-known that the price of an European type option is always a lower bound for the price of an otherwise identical American type option. Since we know from proposition 3.5.3 that $\pi(0,T)$ is increasing in the maturity time T, it is apparent that

$$\pi(0,\infty) = \lim_{T\to\infty} \pi(0,T)$$

constitutes a lower bound for the price of the options to be bought if the default protected CPPI with minimum exposure ratio is offered as an open ended fund product. In addition to that, due to the infinite maturity, it is immediate that $\pi(t,\infty) = \pi(0,\infty)$ for all t. Hence, from the definition of $\tilde{c}$ in equation 3.24 we also find that $\tilde{c}(t,\infty) = \tilde{c}(0,\infty)$ for all t, such that $\tilde{c}$ is independent of time. Consequently, if $\tilde{c}(0,T)$ is replaced by $\tilde{c}(0,\infty)$ in corollary 3.5.4, then corollary 3.5.4 constitutes an *upper* bound for the expected payoff of the open ended strategy after T years. With respect to figures 3.16 and 3.17 it therefore must be said, that hedging the default risk induced by the minimum exposure ratio by consecutively buying knock-out options does not seem to be a viable strategy. Although, in principle, it is possible to hedge the minimum exposure ratio with knock-out options, the induced costs seem to be too large.

3.6 Conclusion

For long maturity times, the capital protected with conventional CPPI strategies such as the simple and the capped CPPI can become insignificantly small compared to the portfolio value. The CPPI with floor adjustment increases the level of protection by lifting the floor if the portfolio value increases and hence keeps the magnitude of the protected capital significant compared with the portfolio value. In particular, the portfolio value can never lose more than a given fraction of the maximum of the past portfolio values. While both, the expectation and variance of the terminal value, are lower in comparison with the capped CPPI, the probability of outperforming the riskless asset is increased. Hence, the CPPI with floor adjustment is better suited for conservative investors. Due to the increasing level of protection, the strategy is suited for both a long maturity and an open ended product. New investors joining at a later point in time will face a different short-term performance while receiving the same expected yield as earlier investors in the long run.

A well known problem that CPPI structures in general share is the cash-lock. The probability of a cash-lock is found to be increased for the CPPI with floor adjustment compared with the simple and capped CPPI and a direct consequence of the increased protection level. While it seems to be a natural idea to introduce a minimum exposure ratio such as to prevent an ε-cash-lock, it contradicts the idea of portfolio protection. If no further provision is taken, the portfolio value can not be kept above the floor and thus a minimum exposure ratio induces default risk. For an investor who is willing to accept a small default risk as a trade-off for avoiding an ε-cash-lock, a CPPI with minimum exposure ratio can be an adequate investment strategy. However, the strategy is can not be labelled a portfolio insurance strategy any more in the strict sense as it induces default risk.

Formally, the default risk induced by the introduction of a minimum exposure ratio can be covered using put-options written on the value process of the CPPI with minimum exposure ratio. However, it turns out, that the price of these options can be larger than the initial portfolio value for long maturities and is unbounded from above as the maturity time turns to infinity. The reason for this result is the combination of the minimum exposure ratio and the floor adjustments. Due to the floor adjustment, insuring the portfolio against default risk at the inception of the strategy means insuring potential gains of the strategy that are yet to be realized. For long maturities this is not possible. Although it is possible to successively insure the portfolio whenever gains have been realized, i.e. whenever floor adjustments are made, this procedure is very expensive. While the introduction of a minimum exposure ratio has positive effects on the expected payoff of the strategy, these positive effects seem to be made undone by the large costs of covering the default risk.

Appendix A

Mathematical Prerequisites

A.1 Some aspects about Random Walks

Consider a potentially infinitely repeated game where at each step the probability of winning one unit of money is u, the probability of loosing one unit of money is d and the probability of the game terminating immediately is ρ where $u + d + \rho = 1$. For $n \in \mathbb{N}_0$, let X_n denote the wealth of the player at the n-th step of the game and suppose $X_0 = 0$.[1]

Lemma A.1.1 *The probability of the player's wealth rising or falling to $k \in \mathbb{Z}$ at some step is given by*

$$P(\exists n \in \mathbb{N}_0 : \; X_n = k) = h(k|u,d)$$

where

$$h(k|u,d) := \begin{cases} \left(\frac{1+\sqrt{1-4ud}}{2d}\right)^k & , \; k < 0 \\ \left(\frac{1-\sqrt{1-4ud}}{2d}\right)^k & , \; k \geq 0. \end{cases}$$

Proof: The lemma will be proven together with the following lemma. □

Lemma A.1.2 *Let $N \in \mathbb{N}_0$ denote the number of steps after which the game terminates.*

a) The probability of the game terminating at wealth level $k \in \mathbb{Z}$ is given by

$$P(X_N = k) = \rho \, q(k|u,d)$$

[1] Most of the results presented in this section are well-known and can be found, at least for the special case $u + d = 1$, for example in Feller (1968) in the context of generating functions.

where

$$q(k|u,d) := \frac{h(k|u,d)}{\sqrt{1-4ud}}.$$

b) *The probability of the game terminating at or lower than wealth level $k \in \mathbb{Z}$ is given by*

$$P(X_N \leq k) = \rho Q(k|u,d)$$

where

$$Q(k|u,d) := \sum_{j=-\infty}^{k} q(j|u,d) = \begin{cases} \frac{q(k|u,d)}{1-h(-1|u,d)} & , \ k < 0 \\ \frac{1}{1-u-d} - \frac{q(k+1|u,d)}{1-h(1|u,d)} & , \ k \geq 0. \end{cases}$$

Proof: We start with part a). It is apparent that the probability of the game terminating at wealth level $k \in \mathbb{N}_0$ is given by

$$x_k := \begin{cases} \rho d^{-k} \sum_{n=0}^{\infty} \binom{2n-k}{n-k} (ud)^n & , k < 0 \\ \rho u^k \sum_{n=0}^{\infty} \binom{2n+k}{n+k} (ud)^n & , k \geq 0 \end{cases} \tag{A.1}$$

such that the proof of the lemma boils down to finding an expression for these sums. It is obvious that $x_{-k} = \frac{d^k}{u^k} x_k$ for $k \geq 0$ and therefore it is sufficient to consider the case $k \geq 0$. From equation (A.1) the difference equation

$$ux_k + dx_{k+2} = x_{k+1}$$

for the series (x_k) can easily be verified and we solve this difference equation with the standard method. The general solution to the difference equation is given by

$$x_k = A\lambda_1^k + B\lambda_2^k$$

where $\lambda_{1,2}$ are found from the characteristic equation

$$u + d\lambda^2 = \lambda \quad \Rightarrow \quad \lambda_{1,2} = \frac{1 \pm \sqrt{1-4ud}}{2d}$$

and A and B are some constants. From the general solution we know that the constants A and B must satisfy

$$x_0 = A + B, \quad x_1 = A\lambda_1 + B\lambda_2 \tag{A.2}$$

such that the problem has reduced to finding explicit expressions for x_0 and x_1. It can readily be checked that

$$\binom{2n}{n} (ud)^n = \binom{-\frac{1}{2}}{n} (-4ud)^n$$

and therefore

$$\begin{aligned} x_0 &= \rho \sum_{n=0}^{\infty} \binom{2n}{n} (ud)^n \\ &= \rho \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-4ud)^n \\ &= \frac{\rho}{\sqrt{1-4ud}} \end{aligned}$$

as a consequence of Newton's binomial formula[2]. From equation (A.1) $ux_{-1}+dx_1 = x_0-\rho$ can be verified and together with $x_{-1} = \frac{d}{u}x_1$ we find

$$x_1 = \frac{x_0-\rho}{2d} = \frac{\rho}{\sqrt{1-4ud}} \frac{1-\sqrt{1-4ud}}{2d}.$$

From condition (A.2) we now immediately deduce

$$A = 0, \quad B = \frac{\rho}{\sqrt{1-4ud}} = x_0$$

and therefore

$$x_k = \frac{\rho}{\sqrt{1-4ud}} \left(\frac{1-\sqrt{1-4ud}}{2d} \right)^k = \rho q(k|u,d)$$

for $k \geq 0$. For $k < 0$ we now get

$$\begin{aligned} x_k &= \frac{d^{-k}}{u^{-k}} \frac{\rho}{\sqrt{1-4ud}} \left(\frac{1-\sqrt{1-4ud}}{2d} \right)^{-k} \\ &= \frac{\rho}{\sqrt{1-4ud}} \left(\frac{1-\sqrt{1-4ud}}{2u} \right)^{-k} \\ &= \frac{\rho}{\sqrt{1-4ud}} \left(\frac{1+\sqrt{1-4ud}}{2d} \right)^{k} \\ &= \rho q(k|u,d) \end{aligned}$$

which completes the proof of part a) of lemma A.1.2. Also lemma A.1.1 now becomes apparent. Note, that x_k can also be written as $x_k = x_0 h(k|u,d)$ and can therefore be interpreted as first reaching a wealth level k with the probability $h(k|u,d)$, then terminate at that level with probability x_0.

For part b) we have to calculate the sums $\sum_{j=-\infty}^{k} q(k|u,d)$. For $k < 0$, the formula immediately follows from

$$\sum_{j=-\infty}^{k} q(j|u,d) = q(k|u,d) \sum_{j=-\infty}^{0} h(j|u,d) = q(k|u,d) \sum_{j=0}^{\infty} h(-1|u,d)^j$$

[2]See for example Feller (1968), p.51.

while for $k \geq 0$, note that

$$\sum_{j=-\infty}^{k} q(j|u,d) = \sum_{j=-\infty}^{\infty} q(j|u,d) - \sum_{j=k+1}^{\infty} q(j|u,d).$$

For the second sum on the right hand side of the equation we get

$$\sum_{j=k+1}^{\infty} q(j|u,d) = q(k+1|u,d) \sum_{j=0}^{\infty} h(k|u,d) = \frac{q(k+1|u,d)}{1-h(1|u,d)}$$

while for the first sum we find

$$\begin{aligned} \sum_{j=-\infty}^{\infty} q(j|u,d) &= q(0|u,d) \sum_{j=-\infty}^{0} h(j|u,d) + q(1|u,d) \sum_{j=0}^{\infty} h(j|u,d) \\ &= q(0|u,d) \left(\frac{1}{1-h(-1|u,d)} + \frac{h(1|u,d)}{1-h(1|u,d)} \right). \end{aligned}$$

Finally, the identity

$$\frac{1}{1-h(-1|u,d)} + \frac{h(1|u,d)}{1-h(1|u,d)} = \frac{q(0|u,d)^{-1}}{1-u-d}$$

yields the assertion. □

Lemma A.1.3

a) The joint probability of the game terminating at wealth level $k \in [\bar{n}, \bar{n}-1, \bar{n}-2, \ldots[$ and the wealth level never surpassing some maximum wealth level $\bar{n} \in \mathbb{N}_0$ is given by

$$P(X_N = k, \max_{n \in \{0,1,\ldots,N\}} X_n \leq \bar{n}) = \rho q_{\bar{n}}(k|u,d)$$

where

$$q_{\bar{n}}(k|u,d) := \begin{cases} q(k|u,d) \left(1 - (h(1|u,d)h(-1|u,d))^{\bar{n}+1}\right) & , \ k < 0 \\ q(k|u,d) \left(1 - (h(1|u,d)h(-1|u,d))^{\bar{n}+1-k}\right) & , \ k \geq 0. \end{cases}$$

b) The joint probability of the game terminating at or lower than wealth level $k \in [\bar{n}, \bar{n}-1, \bar{n}-2, \ldots[$ and the wealth level never surpassing some maximum wealth level $\bar{n} \in \mathbb{N}_0$ is given by

$$P(X_N \leq k, \max_{n \in \{0,1,\ldots,N\}} X_n \leq \bar{n}) = \rho Q_{\bar{n}}(k|u,d)$$

where

$$Q_{\bar{n}}(k|u,d) := \sum_{j=-\infty}^{k} q_{\bar{n}}(j|u,d) = \begin{cases} \frac{q_{\bar{n}}(k|u,d)}{1-h(-1|u,d)} & , \ k < 0 \\ \frac{q_{\bar{n}}(k|u,d)}{1-h(-1|u,d)} + \frac{1-h(k|u,d)}{1-u-d} & , \ k \geq 0. \end{cases}$$

PROOF: Throughout this proof we will use the simplified notation $q_{\bar{n}}(k) = q_{\bar{n}}(k|u,d)$, $q(k) = q(k|u,d)$ and $h(k) = h(k|u,d)$. Let $q_{\bar{n}}(k)$ denote the probability of the wealth level being k when the game terminates such that the probability of the game terminating at wealth level k is given by $\rho q_{\bar{n}}(k)$. First, consider the situation with a maximum wealth level of 0 and in particular the probability $q_0(0)$. Since we start with a wealth level 0 and the maximum wealth level is not to be surpassed, there are only two possibilities: The game terminates or the wealth level goes to -1. From level -1 the probability of reaching level 0 again sooner or later is given by $h(1|u,d)$. Once being at level 0 again, the situation is the same as at the start. Therefore the probability of terminating at level 0 is given by

$$\rho q_0(0) = \rho + \rho d h(1) q_0(0)$$

from which

$$q_0(0) = \frac{1}{1 - dh(1)}$$

can be deduced. By a similar chain of arguments we find that

$$\begin{aligned} & q_0(k) = d\left(q_0(k+1) + h(1)q_0(k)\right) \\ \Leftrightarrow \quad & q_0(k) = \frac{d}{1-dh(1)} q_0(k+1) = h(-1)q_0(k+1) \end{aligned}$$

and by recursion

$$q_0(k) = h(k)q_0(0). \tag{A.3}$$

Now, in the situation with an arbitrary maximum wealth level $\bar{n} \in \mathbb{N}_0$, it is apparent by a similar argument, that the probabilities are given by

$$q_{\bar{n}}(k) = \begin{cases} \sum_{j=0}^{\bar{n}} h(j) q_0(k-j) & k < 0 \\ \sum_{j=k}^{\bar{n}} h(j) q_0(k-j) & k \geq 0 \end{cases}$$

and with equation (A.3) we find for $k < 0$

$$\begin{aligned} q_{\bar{n}}(k) &= \sum_{j=0}^{\bar{n}} h(j) q_0(k-j) \\ &= q_0(0) h(k) \sum_{j=0}^{\bar{n}} (h(1)h(-1))^j \\ &= q_0(0) h(k) \frac{1-(h(1)h(-1))^{\bar{n}+1}}{1-h(1)h(-1)} \\ &= q(k)\left(1-(h(1)h(-1))^{\bar{n}+1}\right) \end{aligned}$$

and likewise for $k \geq 0$

$$q_{\bar{n}}(k) = q(k)\left(1-(h(1)h(-1))^{\bar{n}+1-k}\right)$$

which proofs part a) of the lemma. For part b) we only have to calculate the sums $\sum_{j=-\infty}^{k} q_{\bar{n}}(j)$. For $k < 0$, the formula immediately follows from

$$\sum_{j=-\infty}^{k} q_{\bar{n}}(j) = q_{\bar{n}}(k) \sum_{j=-\infty}^{0} h(j).$$

For $k \geq 0$ we have

$$\begin{aligned} \sum_{j=-\infty}^{k} q_{\bar{n}}(j) &= \sum_{j=-\infty}^{-1} q_{\bar{n}}(j) + \sum_{j=0}^{k} q_{\bar{n}}(j) \\ &= \frac{q_{\bar{n}}(-1)}{1-h(-1)} + \sum_{j=0}^{k} q(0) h(j) \left(1-(h(1)h(-1))^{\bar{n}+1-j}\right) \\ &= \frac{q_{\bar{n}}(-1)}{1-h(-1)} + q(0)\frac{1-h(k+1)}{1-h(1)} - q(0)\frac{1-h(-1)^{-(k+1)}}{1-h(-1)^{-1}} (h(1)h(-1))^{\bar{n}+1} \end{aligned}$$

and since $\frac{1}{1-h(-1)^{-1}} = -\frac{h(-1)}{1-h(-1)}$ we further get

$$\begin{aligned} &\sum_{j=-\infty}^{k} q_{\bar{n}}(j) \\ &= q(0)\left(\frac{h(-1)}{1-h(-1)} + \frac{1}{1-h(1)}\right) - q(0)\frac{h(1)}{1-h(1)} h(k) - q(0)\frac{h(k)(h(1)h(-1))^{\bar{n}+1-k}}{1-h(-1)} \\ &= q(0)\left(\frac{h(-1)}{1-h(-1)} + \frac{1}{1-h(1)}\right) - q(0)h(k)\left(\frac{h(1)}{1-h(1)} + \frac{1}{1-h(-1)}\right) + \frac{q_{\bar{n}}(k)}{1-h(-1)} \\ &= \frac{q_{\bar{n}}(k)}{1-h(-1)} + \frac{1-h(k)}{1-u-d} \end{aligned}$$

where the last equation follows from

$$\frac{h(-1)}{1-h(-1)} + \frac{1}{1-h(1)} = \frac{q(0)^{-1}}{1-u-d} \quad \text{and} \quad \frac{h(1)}{1-h(1)} + \frac{1}{1-h(-1)} = \frac{q(0)^{-1}}{1-u-d}. \tag{A.4}$$

□

Lemma A.1.4 *Suppose there is a maximum wealth level $\bar{n} \in \mathbb{N}_0$. Suppose further that whenever the wealth level equals $\bar{n}$, the probability of losing one unit of money equals d_2 and the probability of the game terminating is ρ_2 with $d_2 + \rho_2 = 1$, while whenever the wealth level is lower than $\bar{n}$, the probabilities of the game terminating, gaining one unit of money and loosing one unit of money are ρ, u and d, respectively, with $u + d + \rho = 1$.*

a) Then the probability of the game terminating at a wealth level $k \in [\bar{n}, \bar{n}-1, \bar{n}-2, \ldots[$ is given by

$$P(X_N = k) = \begin{cases} \rho_2 \; q_{\bar{n}}(k|u,d,d_2) & , \; k = \bar{n} \\ \rho \; q_{\bar{n}}(k|u,d,d_2) & , \; k < \bar{n} \end{cases}$$

where[3]

$$q_{\bar{n}}(k|u,d,d_2) := \begin{cases} \frac{1}{1-d_2 h(1|u,d)} h(\bar{n}|u,d) & , \; k = \bar{n} \\ q_{\bar{n}-1}(k|u,d) \; + \frac{d_2}{1-d_2 h(1|u,d)} h(\bar{n}|u,d) q_0(k+1-\bar{n}|u,d) & , \; k < \bar{n}. \end{cases}$$

b) The probability of the game terminating at or lower than a wealth level $k < \bar{n}$ is given by

$$P(X_N \le k) = \rho \; Q_{\bar{n}}(k|u,d,d_2)$$

where

$$Q_{\bar{n}}(k|u,d,d_2) := \sum_{j=-\infty}^{k} q_{\bar{n}}(j|u,d,d_2) = \begin{cases} \frac{q_{\bar{n}}(k|u,d,d_2)}{1-h(-1|u,d)} & , \; k < 0 \\ \frac{q_{\bar{n}}(k|u,d,d_2)}{1-h(-1|u,d)} + \frac{1-h(k|u,d)}{1-u-d} & , \; 0 \le k < \bar{n}. \end{cases}$$

Proof: For the proof we use the simplified notation $\tilde{q}_{\bar{n}}(k) = q_{\bar{n}}(k|u,d,d_2)$, $q_{\bar{n}}(k) = q_{\bar{n}}(k|u,d)$ and $h(k) = h(k|u,d)$. Consider first the situation with a maximum level $\bar{n} = 0$. Analogously to the proof of lemma A.1.3 we find

$$\rho_2 \tilde{q}_0(0) = \rho_2 + \rho_2 d_2 h(1) \tilde{q}_0(0) \quad \Rightarrow \quad \tilde{q}_0(0) = \frac{1}{1 - d_2 h(1)}$$

[3]For $\bar{n} = 0$, the term $q_{\bar{n}-1}(k|u,d)$ becomes meaningless and must be set equal to zero for the formula to hold.

and then for $k < 0$

$$\tilde{q}_0(k) = d_2\left(q_0(k+1) + h(1)\tilde{q}_0(k)\right) \quad \Rightarrow \quad \tilde{q}_0(k) = \frac{d_2}{1 - d_2 h(1)} q_0(k+1).$$

For the situation with an arbitrary maximum level $\bar{n} \in \mathbb{N}$ it is sufficient to notice that

$$\tilde{q}_{\bar{n}}(\bar{n}) = h(\bar{n})\tilde{q}_0(0)$$

and

$$\tilde{q}_{\bar{n}}(k) = q_{\bar{n}-1}(k) + h(\bar{n})\tilde{q}_0(k - \bar{n})$$

for $k < \bar{n}$. This proofs part a) of the lemma. Part b) is a immediate consequence of part a) and lemma A.1.3. □

Note, that the term $q_{\bar{n}-1}(k|u,d)$ in Lemma A.1.4 refers to Lemma A.1.3. Since $q_{\bar{n}}(k|u,d,d) = q_{\bar{n}}(k|u,d)$ for all k, Lemma A.1.4 is a generalization of Lemma A.1.3 and we use the same notation for both.

Lemma A.1.5 *Suppose a minimum (maximum) wealth level $k \in \mathbb{Z} \setminus \mathbb{N}$ ($k \in \mathbb{N}_0$) and a target wealth level $n \in \mathbb{N}_0$ ($n \in \mathbb{Z} \setminus \mathbb{N}$). Then the probability of reaching the target wealth level before falling below the minimum (rising above the maximum) wealth level is given by*

$$h_k(n|u,d) := \begin{cases} h(n|u,d)\frac{1-(h(1|u,d)h(-1|u,d))^{|k|+1}}{1-(h(1|u,d)h(-1|u,d))^{|n|+|k|+1}} & , \text{ for } k, n \in \mathbb{Z}, \ kn \leq 0 \\ 0 & , \text{ else} \end{cases}$$

Proof: Since the two cases are analogous, it is sufficient to consider a minimum wealth level $k \in \mathbb{Z} \setminus \mathbb{N}$ and a target wealth level $n \in \mathbb{N}_0$. For notational simplicity set $h(i) = h(i|u,d)$, $i \in \mathbb{Z}$ throughout this proof. From lemma A.1.1 it is known that $h(n)$ is the probability of ever reaching a wealth level n. However, since the wealth level can fall below k before reaching n, this probability is too large for the current situation and hence we subtract the probability $h(-(|k|+1))h(n+|k|+1)$ for falling below the minimum wealth level k and rising to the level n afterwards. Unfortunately, although $h(-(|k|+1))$ is the probability for falling below the minimum wealth level k, the wealth level might have risen to n before. Therefore the subtraction of $h(-(|k|+1))h(n+|k|+1)$ is too large and the probability $h(n)h(-(n+|k|+1))h(n+|k|+1)$ for rising to n, falling to k and rising to n again must be added. Surely this addition is too large again. Carrying on this procedure ad infinitum will give the probability in the assertion. Hence we find

$$\begin{aligned}
h_k(n|u,d) &= h(n) - h(-(|k|+1))h(n+|k|+1) \\
&\quad + h(n)h(-(n+|k|+1))h(n+|k|+1) \\
&\quad - h(-(|k|+1))h(n+|k|+1)h(-(n+|k|+1))h(n+|k|+1) + - \ldots \\
&= (h(n) - h(-(|k|+1))h(n+|k|+1)) \sum_{i=0}^{\infty} (h(-(n+|k|+1))h(n+|k|+1))^i \\
&= h(n) \frac{1-(h(1)h(-1))^{|k|+1}}{1-(h(1)h(-1))^{n+|k|+1}}
\end{aligned}$$

using the summation formula for the geometric series. □

A.2 Basics about Laplace Transforms

This section summarizes some important facts about Laplace transforms. All results are well-known and can be found for example in Davies (1985).

Definition A.2.1 (Laplace transform) *Let $f : [0,\infty) \mapsto \mathbb{R}$, then the Laplace transform of f is a function $\mathbb{C} \mapsto \mathbb{C}$ defined by*

$$\mathcal{L}_{t,s}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt.$$

Lemma A.2.2 (Existence) *Let $f : [0,\infty) \mapsto \mathbb{R}$ a piecewise continuous function and suppose $|f(t)| \leq Me^{\alpha t}$ for all $t \in [0,\infty)$ and some constants M and α. Then $\mathcal{L}_{t,s}\{f(t)\}$ is an analytic function for $\mathrm{Re}(s) > \alpha$.*

Lemma A.2.3 (Uniqueness) *Let f and g two functions with $\mathcal{L}_{t,s}\{f(t)\} = \mathcal{L}_{t,s}\{g(t)\}$ for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \alpha$ where alpha is some constant such that the Laplace transforms of both functions exist. Then $f(t) = g(t)$ for all $t \in [0,\infty)$ where $f(t)$ and $g(t)$ are continuous.*

Lemma A.2.4 (Inversion) *Suppose the Laplace transform of some function f is analytic for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \alpha$. Then*

$$f(t) = \frac{1}{2\pi\iota} \int_{\gamma-\iota\infty}^{\gamma+\iota\infty} e^{st} \mathcal{L}_{t,s}\{f(t)\}$$

for any $\gamma > \alpha$. The integral is known as the Bromwich-Integral.

Proposition A.2.5 (Properties) *Let f and g two functions such that both Laplace transforms, $\mathcal{L}_{t,s}\{f(t)\}$ and $\mathcal{L}_{t,s}\{g(t)\}$, exist for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \alpha$. Then it holds:*

a) Linearity: For any constants c_1 and c_2 the Laplace transform of $c_1 f(t) + c_2 g(t)$ is given by

$$\mathcal{L}_{t,s}\{c_1 f(t) + c_2 g(t)\} = c_1 \mathcal{L}_{t,s}\{f(t)\} + c_2 \mathcal{L}_{t,s}\{g(t)\}$$

b) Convolution: The Laplace transform of the convolution, $(f * g)(t)$, is given by

$$\mathcal{L}_{t,s}\{(f * g)(t)\} = \mathcal{L}_{t,s}\{f(t)\}\, \mathcal{L}_{t,s}\{g(t)\}$$

c) Integration:

$$\mathcal{L}_{t,s}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}\mathcal{L}_{t,s}\{f(t)\}$$

d) Special Case:

$$\mathcal{L}^{-1}_{s,t}\left\{\frac{1}{s-a}\right\} = e^{at}$$

e) Limit: If f is analytic on $\mathrm{Re}(s) > 0$, it holds

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0+} s \cdot \mathcal{L}_{t,s}\{f(t)\}$$

A.3 Some integrals

Proposition A.3.1 *Let a, b, $\delta \in \mathbb{R}$ with $a < 0$, $b > 0$ and $\rho(s, z|a, b, \delta)$ as in proposition 2.1.2. Let further A_1, A_2, $A_3 \in \mathbb{R}$ some constants. Then, for some $j \in \mathbb{N}_0$,*

$$\int_a^b (A_1 e^{zA_3} + A_2)^j \rho(s, z|a, b, \delta)dz = \sum_{i=0}^{j} \frac{\binom{j}{i} A_1^i A_2^{j-i} (1 - e^{ibA_3} u(s|a, b, \delta) - e^{iaA_3} d(s|a, b, \delta))}{s - i\delta A_3 - \frac{1}{2} i^2 A_3^2}$$

and the particular choice $\delta = \frac{\mu - r - \frac{1}{2}\sigma^2}{\sigma}$ and $A_3 = \sigma$ yields

$$\int_a^b (A_1 e^{\sigma z} + A_2)^j \rho(s, z|a, b, \delta)dz = \sum_{i=0}^{j} \frac{\binom{j}{i} A_1^i A_2^{j-i} (1 - e^{i\sigma b} u(s|a, b, \delta) - e^{i\sigma a} d(s|a, b, \delta))}{s - i(\mu - r) - i(i-1)\frac{\sigma^2}{2}}$$

while the choosing $j = 0$ gives

$$\int_a^b \rho(s, z|a, b, \delta)dz = \frac{1 - u(s|a, b, \delta) - d(s|a, b, \delta)}{s}$$

PROOF: First, an application of the binomial expansion yields

$$(A_1 e^{zA_3} + A_2)^j = \sum_{i=0}^{j} \binom{j}{i} A_1^i e^{ziA_3} A_2^{j-i}$$

such that only the integrals of the form $\int_a^b e^{ziA_3}\rho(s,z|a,b,\delta)dz$ need to be considered. Notice that

$$\int_a^0 e^{ziA_3}\rho(s,z|a,b,\delta)dz = d(s)\frac{e^{-a\delta - a\sqrt{2s+\delta^2}}}{\sqrt{2s+\delta^2}}\int_a^0 \left(e^{(\delta+iA_3+\sqrt{2s+\delta^2})z} - e^{(\delta+iA_3-\sqrt{2s+\delta^2})z+2a\sqrt{2s+\delta^2}}\right)dz$$

with

$$\begin{aligned}
&\int_a^0 \left(e^{(\delta+iA_3+\sqrt{2s+\delta^2})z} - e^{(\delta+iA_3-\sqrt{2s+\delta^2})z+2a\sqrt{2s+\delta^2}}\right)dz \\
&= \left.\frac{(\delta+iA_3-\sqrt{2s+\delta^2})e^{(\delta+iA_3+\sqrt{2s+\delta^2})z} - (\delta+iA_3+\sqrt{2s+\delta^2})e^{(\delta+iA_3-\sqrt{2s+\delta^2})z+2a\sqrt{2s+\delta^2}}}{(\delta+iA_3+\sqrt{2s+\delta^2})(\delta+iA_3-\sqrt{2s+\delta^2})}\right|_a^0 \\
&= \frac{1}{2}\frac{\sqrt{2s+\delta^2}(1+e^{2a\sqrt{2s+\delta^2}} - 2e^{iA_3a}e^{a\delta+a\sqrt{2s+\delta^2}}) - (\delta+iA_3)(1-e^{2a\sqrt{2s+\delta^2}})}{s - i\delta A_3 - \frac{1}{2}i^2A_3^2}
\end{aligned}$$

and likewise

$$\int_0^b e^{ziA_3}\rho(s,z|a,b,\delta)dz = u(s)\frac{e^{-b\delta + b\sqrt{2s+\delta^2}}}{\sqrt{2s+\delta^2}}\int_0^b \left(e^{(\delta+iA_3-\sqrt{2s+\delta^2})z} - e^{(\delta+iA_3+\sqrt{2s+\delta^2})z-2b\sqrt{2s+\delta^2}}\right)dz$$

with

$$\begin{aligned}
&\int_0^b \left(e^{(\delta+iA_3-\sqrt{2s+\delta^2})z} - e^{(\delta+iA_3+\sqrt{2s+\delta^2})z-2b\sqrt{2s+\delta^2}}\right)dz \\
&= \left.\frac{(\delta+iA_3+\sqrt{2s+\delta^2})e^{(\delta+iA_3-\sqrt{2s+\delta^2})z} - (\delta+iA_3-\sqrt{2s+\delta^2})e^{(\delta+iA_3+\sqrt{2s+\delta^2})z-2b\sqrt{2s+\delta^2}}}{(\delta+iA_3+\sqrt{2s+\delta^2})(\delta+iA_3-\sqrt{2s+\delta^2})}\right|_0^b \\
&= \frac{1}{2}\frac{\sqrt{2s+\delta^2}(1+e^{-2b\sqrt{2s+\delta^2}} - 2e^{iA_3b}e^{b\delta-b\sqrt{2s+\delta^2}}) + (\delta+iA_3)(1-e^{-2b\sqrt{2s+\delta^2}})}{s - i\delta A_3 - \frac{1}{2}i^2A_3^2}.
\end{aligned}$$

Summing up the two integrals $\int_a^0 e^{ziA_3}\rho(s,z|a,b,\delta)dz$ and $\int_0^b e^{ziA_3}\rho(s,z|a,b,\delta)dz$ now yields the assertion. □

Proposition A.3.2 *Let a, b, $\delta \in \mathbb{R}$ with $a < 0$, $b > 0$ and $u(s|a,b,\delta)$, $d(s|a,b,\delta)$, $\rho(s,z|a,b,\delta)$ as in proposition 2.1.2. Then the integral $\int_a^y \rho(s,z|a,b,\delta)dz$ is given by*

$$\begin{cases} -\frac{d(s|a,b,\delta)}{s} + d(s|a,b,\delta)\frac{(\delta+\sqrt{2s+\delta^2})e^{\delta y - y\sqrt{2s+\delta^2}+2a\sqrt{2s+\delta^2}} - (\delta-\sqrt{2s+\delta^2})e^{\delta y + y\sqrt{2s+\delta^2}}}{2s\sqrt{2s+\delta^2}e^{\delta a + a\sqrt{2s+\delta^2}}} & , \ y \leq 0 \\ \frac{1-d(s|a,b,\delta)}{s} - u(s|a,b,\delta)\frac{(\delta+\sqrt{2s+\delta^2})e^{\delta y - y\sqrt{2s+\delta^2}} - (\delta-\sqrt{2s+\delta^2})e^{\delta y + y\sqrt{2s+\delta^2}-2b\sqrt{2s+\delta^2}}}{2s\sqrt{2s+\delta^2}e^{\delta b - b\sqrt{2s+\delta^2}}} & , \ y > 0 \end{cases}$$

PROOF: Follows from direct calculation similar to the proof of proposition A.3.1. □

Proposition A.3.3 *Let a, $\delta \in \mathbb{R}$ with $a < 0$ and $d(s|a,\infty,\delta)$, $\rho(s,z|a,\infty,\delta)$ as in section 2.3. Let further A_1, A_2, $A_3 \in \mathbb{R}$ some constants. Then, for some $j \in \mathbb{N}_0$*

$$\int_a^\infty (A_1 e^{zA_3} + A_2)^j \rho(s,z|a,\infty,\delta)dz = \sum_{i=0}^{j} \frac{\binom{j}{i} A_1^i A_2^{j-i}(1 - e^{iaA_3} d(s|a,\infty,\delta))}{s - i\delta A_3 - \frac{1}{2}i^2 A_3^2}$$

and the particular choice $\delta = \frac{\mu - r - \frac{1}{2}\sigma^2}{\sigma}$ and $A_3 = \sigma$ yields

$$\int_a^\infty (A_1 e^{\sigma z} + A_2)^j \rho(s,z|a,\infty,\delta)dz = \sum_{i=0}^{j} \frac{\binom{j}{i} A_1^i A_2^{j-i}(1 - e^{i\sigma a} d(s|a,\infty,\delta))}{s - i(\mu - r) - i(i-1)\frac{\sigma^2}{2}}$$

PROOF: Follows immediately from proposition A.3.1 and $b \to \infty$. □

Proposition A.3.4 *Let a, $\delta \in \mathbb{R}$ with $a < 0$ and $d(s|a,\infty,\delta)$, $\rho(s,z|a,\infty,\delta)$ as in section 2.3. Then the integral $\int_a^y \rho(s,z|a,\infty,\delta)dz$ is given by*

$$\begin{cases} -\frac{d(s|a,\infty,\delta)}{s} + \frac{(\delta+\sqrt{2s+\delta^2})e^{\delta y - y\sqrt{2s+\delta^2}+2a\sqrt{2s+\delta^2}} - (\delta-\sqrt{2s+\delta^2})e^{\delta y + y\sqrt{2s+\delta^2}}}{2s\cdot\sqrt{2s+\delta^2}} & , \ y \leq 0 \\ \frac{1-d(s|a,\infty,\delta)}{s} - \frac{(1-e^{2a\sqrt{2s+\delta^2}})(\delta+\sqrt{2s+\delta^2})e^{\delta y - y\sqrt{2s+\delta^2}}}{2s\cdot\sqrt{2s+\delta^2}} & , \ y > 0 \end{cases}$$

PROOF: Follows immediately from proposition A.3.2 and $b \to \infty$. □

Proposition A.3.5 *Let b, δ, $A \in \mathbb{R}$ with $b > 0$ and $\rho_{\infty,\lambda}(s,z) = \lim_{a \to -\infty} \rho(s,z|a,b,\delta)$ as* *in section 3.4. Then the integral* $\int\limits_{-\infty}^{y} e^{Az}\rho_{\infty,\lambda}(s,z)dz$ *is given by*

$$\begin{cases} \frac{(1-e^{-2b\sqrt{2s+\delta^2}})e^{(\delta+\sqrt{2s+\delta^2}+A)y}}{\sqrt{2s+\delta^2}(\delta+\sqrt{2s+\delta^2}+A)} & , \; y \leq 0 \\ \\ \frac{1}{s-A\delta-\frac{A^2}{2}} + \frac{1}{\sqrt{2s+\delta^2}}\left(\frac{e^{(\delta-\sqrt{2s+\delta^2}+A)y}}{\delta-\sqrt{2s+\delta^2}+A} - \frac{e^{(\delta+\sqrt{2s+\delta^2}+A)y-2b\sqrt{2s+\delta^2}}}{\delta+\sqrt{2s+\delta^2}+A}\right) & , \; y > 0 \end{cases}$$

PROOF: Follows from direct calculation. □

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