Essays on Dual Risk Measures and the Asymptotic Term Structure

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Klaas Schulze
aus Kaiserslautern

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Dekan:	Prof. Dr. Christian Hillgruber
Erstreferent:	Prof. Dr. Frank Riedel
Zweitreferent:	JProf. Dr. Eva Lütkebohmert-Holtz

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Für meine Eltern

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Contents

Introduction

This dissertation covers two distinct topics in the field of financial economics: risk measurement and the term structure of interest rates.

The phenomenon of *risk* plays a ubiquitous role in finance and insurance as well as in economics, since it is involved in nearly all financial and economic activities. Risk is a key aspect in diverse situations, as the formation of investment decisions and the operation of financial markets. It is the daily occupation of several professions, as rating agents, financial regulators, and portfolio and fund managers. Without it, the profession of investment banking would reduce to simple accounting, as stated by Machina & Rothschild (2008), and the insurance industry would cease to exist.

Due to its central role, risk and its various aspects are frequently addressed in economic literature. The idea of risk entered economic theory with the dissertation of Knight (1921) and his famous distinction between risk and uncertainty. Von Neumann & Morgenstern (1944) accomplished the formal incorporation of risk by their axiomatization of the expected utility hypothesis. Remarkably, this hypothesis already appears in Bernoulli (1738). By considering the expected utility instead of the expected return of a random payoff and introducing the idea of diminishing marginal utility Bernoulli (1738) solved the well-known St. Petersburg paradox, which was posed by his cousin Nicholas Bernoulli in 1713.

The primary step in coping with risk is its *measurement*. To this end actuarial and financial literature has proposed a variety of measures of risk over the last decades, for an overview confer e.g. Bühlmann (1970) and Föllmer & Schied (2004). This research in quantifying the risk of a random payoff lead from simple moment-considerations, as the Sharpe ratio, to the commonly-used Value-at-Risk and more sophisticated measures, as e.g. expected shortfall, utility-based or spectral risk measures. Since some of these measures expose unfavorable properties, e.g. Value-at-Risk penalizes diversification, Artzner et al. (1999) introduce axioms of coherence in order to ensure consistent risk measures. As pointed out by Denuit et al. (2006) broad classes of risk measures base on the actuarial principle of an equivalent utility premium: they are obtained by deriving the indifference price, that is the price for which a decision-maker is indifferent between accepting the financial position by paying this price or not entering the position. The indifference price is thus the minimal certain side payment or capital requirement, which makes a position acceptable.

The primary idea in the first part of this thesis is not to ask for the *price* in this context, but for the level of *risk aversion* of the investor, which leaves him indifferent. The rational behind this idea is obvious: a financial position, which is exposed to high risk, is rejected by the majority of risk-averse agents and an indifferent agent has to be relatively risk-affine compared to the majority. And vice versa, for a low-risk position an agent has to be relatively risk-averse in order to be indifferent. Consequently the indifference risk aversion level provides inference to the risk of the position. I call this idea risk measurement via the indifference risk aversion principle.

In order to apply this principle I consider the elementary setting of agents with exponential utility in the expected utility paradigm of decision-making. Following the indifference risk aversion principle I ask which agent of this family is exactly indifferent. Thereby I define the *indifference measure*, which maps a position to the positive constant Arrow-Pratt coefficient of the agent who is indifferent to the position in question. This consideration follows the pathbreaking work of Aumann & Serrano (2008). They introduce a risk measure, they call the index of *riskiness*, which is defined as the reciprocal of the indifference measure.

In order to ensure the *existence* of riskiness Aumann & Serrano (2008) provide three conditions on the random payoff. Their third condition is although sufficient, but not necessary for existence and excludes some central distributions of risk. In order to include these distribution I relax their third condition. The main result in Chapter 1 derives three conditions which are equivalent to the existence of riskiness. Appealingly, a consideration of economic acceptance behavior shows that these conditions on the distribution of a random payoff (namely the possibility of losses, a positive expected value, and no heavy negative tail) are the relevant case for a non-trivial decision under risk.

Having clarified for which financial positions the index of riskiness, and hence the indifference measure, exist, the next elementary question is their *computation*. Whereas Aumann & Serrano (2008) calculate the riskiness of trivial cases and positions, which are normally distributed, it remains defined implicitly for all other positions. Calculating riskiness explicitly is demanding, since the implicit definition involves an improper integral, which can be difficult to solve and need not exist. Chapter 1 observes a connection to the theory of Laplace transforms and reduces the problem of explicit calculation of riskiness to the easier problem of inverting two-sided Laplace transforms. Moreover, I provide closed-form solutions of several distributions, including the Exponential, Poisson, and Gamma distribution.

I believe that Chapter 1 contributes to both, theory and practice of riskiness and the indifference measure. On the theoretical side it derives equivalent conditions for existence. Naturally, existence is the basis for the proceeding research on riskiness and the indifference measure. Moreover, relaxing the third condition of Aumann &

Serrano (2008) is economically relevant as it incorporates several continuous distributions of risk, which were excluded before, e.g. the Exponential, Gamma, Laplace and Logistic distribution. On the practical side, explicit calculations are necessary for any application of riskiness and the indifference measure. They allow the index of riskiness to quantify the risk involved in a position. Deriving explicit numbers gives riskiness a meaning beside its appealing axiomatic properties.

These computations show that the indifference measure maps positions, which are exposed to high risk, to low parameters and vice versa. This is in line with the intuition of the indifference risk aversion principle. By convention, a typical risk measure associates high-risk positions with high numbers and low-risk position with low numbers. This suggests to *reverse* the indifference measure in order to receive a conventional risk measure. An outstanding example for this reversion is riskiness, which is the reciprocal of the indifference measure. To verify this intuition, that a reversed indifference measure is a sensible risk measure, I consider the axiom of duality.

This axiom is also introduced in the seminal paper of Aumann & Serrano (2008). The *duality* axiom asserts, roughly speaking, that less risk-averse agents accept riskier gambles. More precisely, it poses a condition on risk measures: A risk measure is dual, if every agent accepts all positions, which are less risky (by means of the risk measure in question) than a position, which is already accepted by a more risk-averse agent. There are two reasons for this behavior: first, the agent in question is relatively more risk-affine, and second, the gambles in question are relatively less risky. This makes duality a natural assumption, which can be seen as a requirement on risk measures. Aumann & Serrano (2008) characterize riskiness uniquely by duality and the axiom of positive homogeneity. Since positive homogeneity is of minor importance (also in their opinion), Chapter 2 relaxes it and considers the whole class of dual risk measures. This class is of relevance as it contains all risk measures satisfying the natural axiom of duality.

As it turns out, the crucial key to this class of risk measures is the indifference measure, which induces an ordering on the set of positions. The main insight of Chapter 2 is that all dual risk measures can be *characterized* by the indifference measure. The primary theorem of Chapter 2 yields, that any dual risk measure reverses the ordering induced by the indifference measure. A consequence is a representation theorem which decomposes dual risk measures into the indifference measure and a reversing function. Further I derive that the opposite implication also holds true: any measure reversing the ordering of the indifference measure is necessarily dual. Hence Chapter 2 provide an easy method to construct dual risk measures by composing the indifference measure with an arbitrary reversing function. Overall, I present a handy equivalent condition for duality.

This characterization supports the risk measurement via the indifference risk aversion principle. It verifies the intuition, that reversed indifference measures are sensible risk measures, as they satisfy the natural duality axiom. Moreover, it shows that every dual risk measure necessarily bases on the indifference measure. Thus the indifference measure, introduced and analyzed in Chapter 1 and derived by the indifference risk aversion principle, is crucial for the class of dual risk measures, as shown in Chapter 2. In addition, it is very appealing that the indifference measure, that is derived by exponential utility solely, ensures duality, which in turn ensures consistency with a notion of comparative risk aversion for general utility functions beyond exponential utility. This highlights the outstanding role of the indifference measure and exponential utility among concave utility functions.

As mentioned above, the *importance* of the classical equivalent utility principle based on the indifference price is unquestioned in finance, insurance, and economics. Similarly the equivalent utility principle based on indifference risk aversion can add to the accuracy of risk measurement. This new principle illuminates a different aspect of the broad phenomenon of risk. It is conceptually appealing, since the link between risk and attitude towards risk is natural. In the context of risk measurement (opposed to risk valuation) it is more direct than the relation between risk and an indifference price. Moreover, it is attractive that no fixed level of risk aversion has to be assumed and that a position itself is measured without changing it by adding a side payment. These appealing properties underline the potential of the principle and in particular dual risk measures. Of course, there are many important aspects in risk measurement, and the indifference risk aversion principle is at most one of them. In the light of the statement "risk is what risk-averters hate" by Machina & Rothschild (2008) it is a promising one.

To summarize I briefly list the *contribution* of Chapter 1 and Chapter 2 to the theory of risk measurement: First, a new principle for risk measurement is introduced, which is conceptually appealing and has the potential to add to the accuracy of risk measurement. Second, the abstract principle is made concrete by considering the indifference measure. The indifference measure constitutes a new type of risk measure which differs conceptually from all classical risk measures except for the index of riskiness. Its existence, and thereby the existence of riskiness, is clarified by relating it to acceptance behavior and it is computed explicitly by linking it to the theory of Laplace transforms. Third, I introduce the class of dual risk measures and characterize it by a simple equivalent condition. This equivalence provides a representation theorem and a construction method for dual risk measures. It further proves the intuition, given by the indifference risk aversion principle, that reversed indifference measures are meaningful risk measures.

The second topic of this thesis is distinct from the first topic as it addresses the *term structure of interest rates*. Hereby I focus on a long-term perspective. The long-term behavior of the yield curve is essential for the valuation of long-term interest rate sensitive products. These products include fixed-income securities, insurance and annuity contracts, and perpetuities. For pricing and hedging of these instruments finance practitioners require a term structure for 100 years or more, whereas in most markets only 30 years are observable. Thus they rely on theoretical models, which extrapolate the evolution of the term structure beyond observable maturities. For the resulting limiting term structure Chapter 3 derives two results: under no arbitrage long zero-bond yields and long forward rates are monotonically increasing and equal to their minimal future value. Both results are inspired the seminal work of Dybvig et al. (1996). They are fairly general as they require only the minimal setting of buy-and-hold trading of Chapter 3 and barely impose restrictions on bond prices. My results contribute to the theory of interest rates theory as they apply to virtually all arbitrage-free term structure models and impose restrictions on their long-term behavior by excluding various behavior of limiting yield curves. These implications serve as caution for modelers that not every specification is consistent with the considered notions of arbitrage. Specifically, setting up the asymptotic yield or forward rate as a diffusion process or a process with systematic jumps necessarily imposes arbitrage opportunities.

Every chapter in this thesis forms a self-explaining unit.

Chapter 1

Existence and Computation of the Aumann-Serrano Index of Riskiness

Aumann & Serrano (2008) introduce the index of riskiness to quantify the risk of a financial position. We clarify for which positions this index of riskiness exists by considering the acceptance behavior of CARA-agents. Furthermore, we derive numerical and closed-form solutions for the riskiness of several distributions of risk. These existence and computation results base on a connection of riskiness to the theory of Laplace transforms.

1.1 Introduction

Aumann & Serrano (2008) introduce the economic index of riskiness, which associates the risk of a gamble or financial position to a real number. The index is essentially characterized by an axiom, which asserts that every agent a fortiori accepts a position, which is less risky (in term of the risk measure in question) than a position, which is accepted by another agent, who is uniformly more risk-averse compared to the first agent. Aumann & Serrano (2008) derive several appealing theoretical properties of the index, including continuity, positive homogeneity and consistency with first and second order stochastic dominance.

In this chapter we clarify, for which financial positions the index of riskiness exists, and we derive explicit calculations of riskiness.

In their highly inspiring paper Aumann & Serrano (2008) consider the class of positions with positive mean and potential negative outcomes. For infinite positions, these two conditions do not imply the existence of riskiness, as we show by considering a position, which is lognormally distributed. Therefore Aumann & Serrano (2008) add a third condition, which is sufficient, but not necessary for existence. It excludes several central distributions of risk, whose riskiness exists. This suggests to relax this third condition and raises the question for the set of equivalent conditions of existence. We answer this question by our main result, which states that the index of riskiness of a position exists, if and only if, the position satisfies the following three conditions: (*i*) it has negative outcomes, (*ii*) it has a positive mean, and (*iii*) it does not have a heavy negative tail. [1] Therefore we identify the class of positions, whose riskiness exists. Relaxing the third condition of Aumann & Serrano (2008) to condition (*iii*) is important as it incorporates several continuous distributions of risk, as the Exponential, Gamma, Laplace, and Logistic distribution.

We illustrate this existence result by the following economic intuition and show that conditions (*i*), (*ii*) and (*iii*) are mild assumptions, as they ensure that the decision problem is non-trivial. As Aumann & Serrano (2008) show, riskiness can be interpreted as the reciprocal of the risk aversion parameter of an agent with CARA-utility, who is indifferent between accepting the position or not. Thus the index of riskiness exists, if and only if, there exists a positive number α, such that

[1] We state these conditions formally by denoting a random variable describing the position by X, the measure of a probability space by $\mathcal{P}$ and the expectation under this measure by E. Condition (*i*) is given by $\mathcal{P}(X < 0) > 0$, condition (*ii*) by $\mathrm{E}X > 0$ and condition (*iii*) by the existence of a positive number α with $\mathrm{E}\exp(-\alpha X) < \infty$. The third condition in Aumann & Serrano (2008) is given by $\mathrm{E}\exp(-\alpha X) < \infty$ for all $\alpha > 0$.

an agent of CARA-utility with parameter α has a zero expected utility by accepting the position.[2] However, such an indifferent agent does not necessarily exist. This happens in the following two cases: First, if all CARA-agents with a positive parameter accept the position. This case, called Case A, holds exactly for positions which have only positive outcomes. Such positions involve no risk at all and thus Case A is not relevant for risk measures, as the decision is trivial. Case A is excluded by condition (*i*). Second, if all CARA-agents reject the position. This case, called Case R, holds exactly for positions with negative mean or heavy negative tail, as we show. A position with negative mean is rejected by all risk-neutral or risk-averse agents in a expected utility framework. Thus the decision is obvious and there is no reason to consult a risk measure. Such positions are excluded by condition (*ii*). A position with a heavy negative tail involves risk, which is not bounded. In spite of a potentially positive mean all risk-averse CARA-agents decline such a position, regardless how close to risk-neutrality they are. These positions are excluded by condition (*iii*).

In the remaining case under conditions (*i*), (*ii*) and (*iii*), there necessarily exists a CARA-agent accepting the position and another CARA-agent rejecting the position. Our result shows that this yields also the existence of an indifferent CARA-agent (called Case I), and thus the index of riskiness. Intuitively this follows by the continuity of expected utility in parameter α.[3] Overall we show that riskiness exists exactly for positions which involve risk and have a positive expectation and no heavy tail in the meantime, which is the relevant case for decision-making under risk.

Having clarified when the index of riskiness exists, the next elementary question is its computation. Aumann & Serrano (2008) calculate the riskiness of trivial cases and positions which are normally distributed. For all other positions the in-

[2]By referring to CARA-agents we can abstract from wealth effects, since the decisions of an CARA-agent are the same under all wealth levels.

[3]However, the proof is not direct, since the CARA-expected utility does not necessarily exist for all $\alpha > 0$.

dex of riskiness is defined implicitly. Calculating riskiness explicitly is demanding, since the implicit definition involves an improper integral, which can be difficult to solve and need not exist. In this chapter we derive a method, which in principle computes explicitly the riskiness of arbitrary positions. This method is based on our observation that the expected utility of a CARA-agent essentially equals the bilateral Laplace transform of the position in question. Hence the theory of Laplace transforms is the suitable technique on the mathematical level to consider existence and computation of riskiness. This connection to Laplace transforms reduces the problem of calculating riskiness to the easier problem of inverting Laplace transforms, which is still a demanding task. Using this method, we provide numerical calculations of the riskiness of arbitrary positions, which are described by common distributions of risk. Moreover, we provide closed-form solutions of several distributions, including the exponential, Poisson, and Gamma distribution. These solutions involve the non-elementary Lambert W-function.

We believe that this chapter contributes to both, theory and practice of the index of riskiness. On the theoretical side it clarifies, for which positions it actually exists by deriving equivalent conditions. Existence is the basis for the proceeding research on riskiness. Moreover, relaxing the third condition of Aumann & Serrano (2008) to condition (*iii*) is economically relevant as it incorporates several continuous distributions of risk, which were excluded before. These distributions include the following classes: Exponential, Gamma, Laplace, and Logistic distribution. On the practical side, explicit calculations are necessary for any application of riskiness. They allow the index of riskiness to quantify the risk involved in a position. Unquestionable the measurement of risk is central in economic, financial, and actuarial sciences: Coping with risk plays a pervasive role, since the phenomenon of risk is a key aspect in diverse situations, as the formation of investment decisions and the operation of financial markets. Deriving explicit numbers gives riskiness a meaning beside its axiomatic properties and allows for comparisons to classical measures of risk.

This chapter is organized as follows: The following section asks more formally for the existence of riskiness and provides the connection to Laplace transforms. This question is answered in Section 3 by relating three essential cases of CARA-acceptance behavior to the properties of respective positions. In Section 4 riskiness is calculated explicitly by deriving numerical and closed-form solutions. The Appendix provides details on several integral transforms and the Lambert W-function.

1.2 The Problem

We use standard notation. The triple $(\Omega, \mathcal{F}, \mathcal{P})$ denotes a probability space and E denotes the expectation operator with respect to $\mathcal{P}$. A *financial position* or *gamble* is described by a random variable $X : \Omega \to \mathbb{R}$. $X(\omega)$ is the discounted net worth of the position at the end of a given period under scenario ω. By $\mathfrak{X}$ we denote the set of financial positions. For full generality we set $\mathfrak{X} := \mathfrak{L}^0(\Omega, \mathcal{F}, \mathcal{P}) - \ker(\| \cdot \|_p)$, i.e. the space of Borel-measurable functions on $(\Omega, \mathcal{F})$ minus the kernel of the seminorm $\| \cdot \|_p$. Thus we consider all real-valued random variables except for random variables, which equal zero almost surely. We exclude the trivial case of vanishing positions for the convenience that some inequalities below hold strictly. This is no restriction in our context of decision-making, as vanishing positions do not affect expected utility. Since the set $\mathfrak{X}$ also contains positions with infinite mean we refer to Delbaen (2002) and Delbaen (2009) for details on risk measures for non-integrable random variables. We state the main problem of this chapter.

Problem 1. *For any position $X \in \mathfrak{X}$ find a positive real number $\alpha^*(X)$ with*

$$Ee^{-\alpha^*(X)\,X} = 1. \tag{1.1}$$

For a position X we call a positive solution of equation (1.1) the *positive root* $\alpha^*(X)$. The index of riskiness R, introduced in Aumann & Serrano (2008), is defined as the reciprocal of the positive root

$$R(X) := \frac{1}{\alpha^*(X)}.$$

Following Aumann & Serrano (2008) the positive root $\alpha^*(X)$ can be interpreted as the parameter of a CARA-agent, who is indifferent between accepting the position X or not. An agent is called CARA with parameter α, if she has a constant Arrow-Pratt coefficient of absolute risk aversion (CARA), which is defined by

$$r_u(x) := \frac{-u''(x)}{u'(x)},$$

confer Arrow (1965), Arrow (1971) and Pratt (1964). Any CARA-agent has a utility function, which is up to additive and multiplicative constants of the form

$$\bar{u}_\alpha(x) := \frac{1}{\alpha}(1 - e^{-\alpha x})$$

with $\alpha \neq 0$. Such a CARA-agent is indifferent, if her expected utility equals zero, i.e.

$$\mathrm{E}\, \bar{u}_\alpha(X) = 0,$$

which is equivalent to equation (1.1). Thus the existence of an indifferent CARA-agent with positive parameter α^* is equivalent to the existence of a positive root α^*.

The main insight to solve Problem 1 is the observation that the left hand side of equation (1.1) is the two-sided Laplace transform of X. The two-sided Laplace transform $\mathfrak{L}_f : \mathbb{R} \to \mathbb{R}$ of a function f is given by the integral

$$\mathfrak{L}_f(\alpha) := \int_{\mathbb{R}} e^{-\alpha x} f(x)\, dx.$$

For an extensive introduction to Laplace transformation we refer to Feller (1966) and Doetsch (1974). By denoting the continuous density function of X with f_X we rewrite the expectation of equation (1) and receive the Laplace transform of f_X, (or of X in shorter notation to avoid double indexing):

$$\mathrm{E}\, e^{-\alpha X} = \int_{\mathbb{R}} e^{-\alpha x} f_X(x)\, dx = \mathfrak{L}_X(\alpha).$$

Hence we can solve Problem 1 by finding a positive point α^* of the Laplace transform, where it equals one

$$\mathfrak{L}_X(\alpha^*) = 1. \tag{1.2}$$

This reduces Problem 1 to the problem of inverting the Laplace transform of X, since α^* can be, if it exists, expressed more explicitly by

$$\alpha^*(X) = \mathfrak{L}_X^{-1}(1),$$

where $\mathfrak{L}^{-1}$ denotes the inverse function of $\mathfrak{L}_X(\alpha)$ in argument α mapping into the positive numbers. Note that we do not refer to the common inverse of Laplace transforms in argument X yielding the density f_X. Since the theory on Laplace transforms is highly sophisticated and Laplace transforms are derived for virtually all distributions, this solves Problem 1 in principle.

For convenience we focus on a continuous notation here. Strictly speaking, the relation of $\mathrm{E}\, e^{-\alpha X}$ to the Laplace transform holds only, if the position X has a continuous density function. Nonetheless our results hold analogously for discrete positions: The properties of the Laplace transform we use below, i.e. essentially strict convexity and the moment-relation, are similarly fulfilled by $\mathrm{E}\, e^{-\alpha X}$ as a function in argument α, regardless if X is continuous or discrete. For a unifying approach we refer to Appendix A.1, in which we relate to the moment-generating function $\mathfrak{M}(X, \alpha)$ by

$$\mathrm{E}\, e^{-\alpha X} = \mathfrak{M}(X, -\alpha).$$

However, for the sake of brevity we focus on the relation to the Laplace transform in this chapter. This appears more illustrating, since the positive root is the positive point of Laplace transform equalling one. Since there are some more integral transforms, which help to solve Problem 1, we consider these transforms in Appendix A.1.

The economic interpretation with acceptance behavior of CARA-agents is given by the following relation between the Laplace transform and the expected CARA-utility for $\alpha > 0$

$$\mathfrak{L}_X(\alpha) = 1 - \alpha\,\mathrm{E}\,\bar{u}_\alpha(X), \quad \mathrm{E}\,\bar{u}_\alpha(X) = \frac{1}{\alpha} - \frac{1}{\alpha}\mathfrak{L}_X(\alpha), \tag{1.3}$$

and, in particular

$$\mathfrak{L}_X(\alpha) = 1 \quad \Leftrightarrow \quad \mathrm{E}\,\bar{u}_\alpha(X) = 0.$$

1.3 Existence of Riskiness

In this section we derive for which positions the index of riskiness exists by discussing three cases of acceptance behavior. We start by showing, which positions are accepted by all CARA-agents.

Lemma 2. *All CARA-agents* ***a****ccept a position* $X \in \mathfrak{X}$*, i.e.* $E\bar{u}_\alpha(X) > 0$ *for all* $\alpha > 0$ *and called Case A, if and only if the position* X *satisfies*

(i^c) *no losses are possible, i.e.* $X \geq 0$.

Proof. We start assuming condition (i^c), which is the complement of condition (i) of the introduction. As the trivial position $X \equiv 0$ is excluded, there is a positive outcome, i.e. $\mathcal{P}(X > 0) > 0$. We consider the Laplace transform for $\alpha > 0$

$$\begin{aligned}\mathfrak{L}_X(\alpha) \;&=\; \int_{\mathbb{R}_0^+} e^{-\alpha x} f_X(x)\,dx \;=\; \mathcal{P}(X=0) + \int_{\mathbb{R}^+} e^{-\alpha x} f_X(x)\,dx \\ &<\; \mathcal{P}(X=0) + \int_{\mathbb{R}^+} f_X(x)\,dx \;=\; \mathcal{P}(X=0) + \mathcal{P}(X>0) \;=\; 1,\end{aligned}$$

where the strict inequality follows by $e^{-\alpha x} < 1$ for all $\alpha > 0$ and $x > 0$. Consequently by equation (1.3) it holds for all $\alpha > 0$

$$\mathrm{E}\,\bar{u}_\alpha(X) > 0,$$

which yields Case A. A typical Laplace transform in Case A is plotted in Figure 1.1. The other direction of the lemma we show by contraposition. We assume,

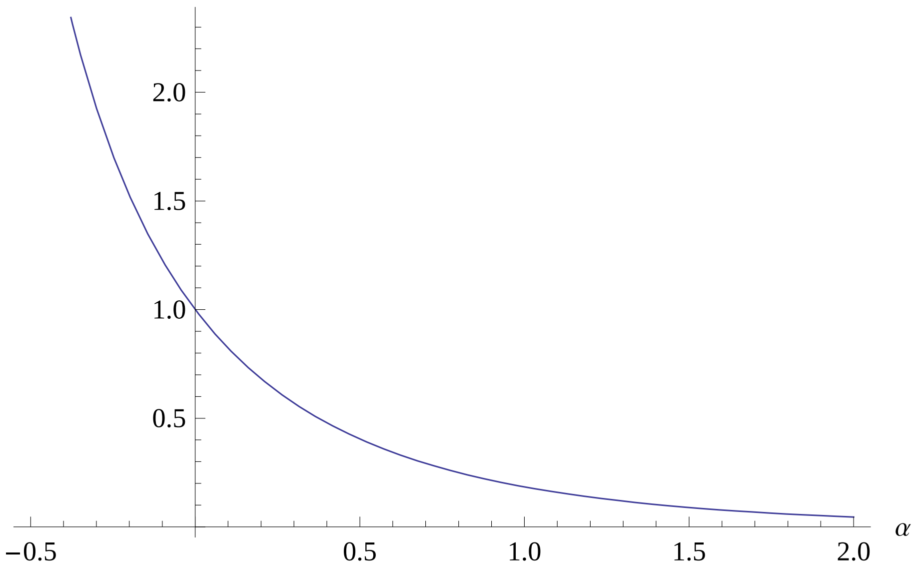


Figure 1.1: Laplace transform $\mathfrak{L}_X(\alpha)$ in Case A.

that losses are possible, i.e. $\mathcal{P}(X < 0) > 0$. Then it holds by definition of the Laplace transform

$$\lim_{\alpha\to\infty} \frac{1}{\alpha}\,\mathfrak{L}_X(\alpha) = \infty.$$

This yields $\lim_{\alpha\to\infty} \mathrm{E}\,\bar{u}_\alpha(X) = -\infty$ by equation (1.3). Thus for sufficiently large α there exists a CARA-agent rejecting the position, which contradicts Case A and completes the proof. □

In the next case we derive the positions, which are rejected by all CARA-agents.

Lemma 3. *All CARA-agents* ***r****eject a position* $X \in \mathfrak{X}$*, i.e.* $E\bar{u}_\alpha(X) < 0$ *for all* $\alpha > 0$ *and called Case R, if and only if the position* X *satisfies*

(ii^c) *X has a non-positive mean, i.e.* $EX \leq 0$*, or*

(iii^c) *X has a heavy negative tail, i.e.* $Ee^{-\alpha X} = \infty$ *for all* $\alpha > 0$.

Proof. We start by assuming condition (ii^c), which yields

$$\mathfrak{L}_X'(0) = -\mathrm{E}X \geq 0,$$

since the k-th derivative of the Laplace transform at zero equals minus the k-th moment of X. The Laplace transform is strictly convex on the interval, where it exists. To see this we consider the second derivative, which equals by the continuity of the exponential

$$\mathfrak{L}_X''(\alpha) = \int_{\mathbb{R}} x^2 e^{-\alpha x} f_X(x)\, dx \; > \; 0.$$

This integral is strictly positive for all $X \in \mathfrak{X}$, since the integrand is strictly positive for all $x \neq 0$ and positions which equal zero almost surely are excluded. The strict convexity and the nonnegative slope in 0 yields for $\alpha > 0$

$$\mathfrak{L}_X(\alpha) > \mathfrak{L}_X(0) = 1.$$

This is illustrated in Figure 1.2 and by equation (1.3) this is equivalent to

$$\mathrm{E}\, \bar{u}_\alpha(X) < 0,$$

for $\alpha > 0$, which yields Case R. This is natural, since even a risk-neutral agent rejects this position. Assuming condition (iii^c) it holds by definition

$$\mathfrak{L}_X(\alpha) = \infty$$

for $\alpha > 0$, which yields by equation (1.3)

$$\mathrm{E}\, \bar{u}_\alpha(X) = -\infty,$$

for $\alpha > 0$, and hence Case R. The other implication of the lemma we show again by contraposition. By assuming condition (iii), the complement of condition (iii^c), there exists some $\alpha > 0$ with $\mathfrak{L}_X(\alpha) < \infty$. Thus the Laplace transform exists on the interval $[\,0, b\,)$ for some $b > 0$. By the moment-relation

$$\mathfrak{L}_X'(0) = -\mathrm{E}X$$

this yields $\mathrm{E}X < \infty$. By condition (ii), the complement of (ii^c), the Laplace transform is further decreasing in 0, and hence there exists some $\alpha > 0$ with

$$\mathfrak{L}_X(\alpha) < 1,$$

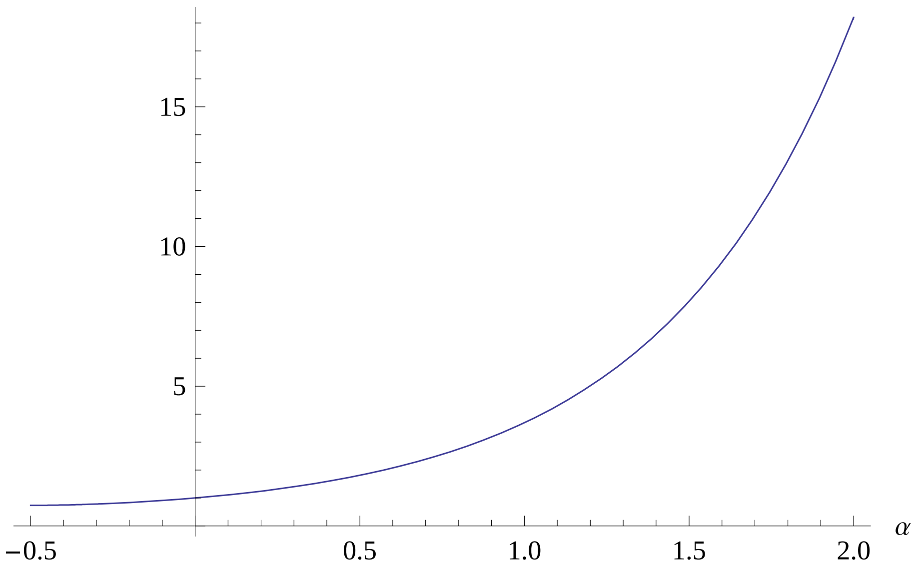


Figure 1.2: Laplace transform $\mathfrak{L}_X(\alpha)$ in Case R.

which yields $\mathrm{E}\, u_\alpha(X) > 0$ by equation (1.3). This contradicts Case R and closes the proof. □

The following theorem provides that the index of riskiness exists for all positions of the final case.

Theorem 4 (Existence of Riskiness)**.** *There exists a CARA-agent, who is **i**ndifferent to a position $X \in \mathfrak{X}$, i.e. there exists some $\alpha > 0$ with $E\bar{u}_\alpha(X) = 0$ and called Case I, if and only if the position X features the following three conditions*

- (i) *losses are possible, i.e. $\mathcal{P}(X < 0) > 0$, and*
- (ii) *X has a positive mean, i.e. $EX > 0$, and*
- (iii) *X has no heavy negative tail, i.e. $Ee^{-\alpha X} < \infty$ for some $\alpha > 0$.*

Proof. From Case I it follows by contraposition, that the position satisfies conditions (i), (ii) and (iii): If one of the conditions is violated, Case A or Case R holds, what is shown in Lemmata 2 and 3, and contradicts Case I. The other implication of the

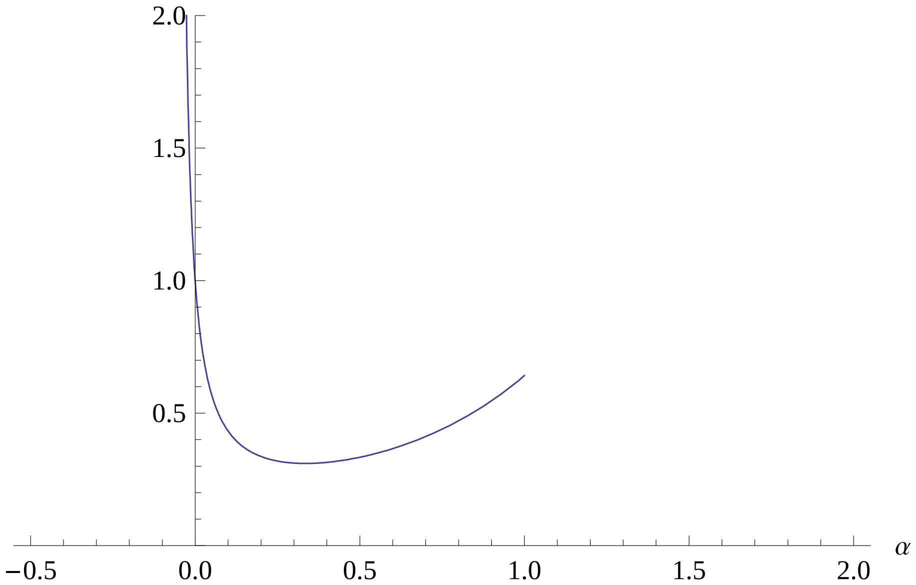


Figure 1.3: Laplace transform $\mathfrak{L}_X(\alpha)$ with no indifferent CARA-agent.

theorem is not so direct using Lemmata 2 and 3, since it is not directly clear that Case I is the complement of Case A or R. Although this complementarity is intuitive it does not hold, if there only exist CARA-agents accepting or rejecting the position and not an indifferent one. This case is illustrated in Figure 1.3, in which agents with parameter $\alpha < 1$ accept the position. Agents with parameter $\alpha \geq 1$ reject it, since the Laplace transform is unbounded and their expected utility is minus infinity. The continuity of expected utility does not directly help out, since it fails to exist for the rejecting agents. To see that this cannot happen and we provide the following proof for the other implication, which shows that the Laplace transform grows unboundedly, when approaching the minimal point, where it is infinite. Thus Case I actually is the complement of Case A or R.

We start assuming conditions (i), (ii) and (iii) are satisfied and consider the minimal value b, such that the Laplace transform is infinite

$$b(X) := \inf_{\alpha \geq 0} \{\mathfrak{L}_X(\alpha) = \infty\},$$

and set $b(X) := \infty$, if the Laplace transform is finite on $\mathbb{R}^+$. By condition (iii), there exists a positive α with $\mathfrak{L}_X(\alpha) < \infty$ and we have $b(X) > 0$. By definition of b and its strict convexity the Laplace transform is finite on the interval $[\,0, b\,)$. Notice that this yields again that the mean of X is finite. We consider a monotone sequence $(b_n)_{n\in\mathbb{N}}$ with $b_n \nearrow b$. It remains to show that

$$\lim_{n\to\infty} \mathfrak{L}_X(b_n) = \infty. \tag{1.4}$$

Then the Laplace transform is exploding on the right-hand side of the interval $[\,0, b\,)$, where it exists. On the left-hand side of this interval it is smaller than one by condition (ii): $\mathfrak{L}'_X(0) = -\mathrm{E}X < 0$. In consequence it equals one at some point α^* with $0 < \alpha^* < b$ by the continuity of the Laplace transform on the interval $[\,0, b\,)$. This is illustrated in Figure 1.4 and yields a positive root and thus Case I.

We close the proof by showing the remaining equation (1.4). In case of $b(X) = \infty$, equation (1.4) follows, as it holds by definition of the Laplace transform under condition (i)

$$\lim_{\alpha\to\infty} \mathfrak{L}_X(\alpha) = \infty.$$

In case of $b(X) < \infty$, we consider the following trivial decomposition

$$\mathfrak{L}_X(b) = \mathfrak{L}^-_X(b) + \mathfrak{L}^+_X(b),$$

with

$$\mathfrak{L}^-_X(b) := \int_{\mathbb{R}^-} e^{-bx} f_X(x)\,dx, \quad \mathfrak{L}^+_X(b) := \int_{\mathbb{R}^+_0} e^{-bx} f_X(x)\,dx.$$

Since the integral on the positive axis is finite for $b > 0$ by

$$\mathfrak{L}^+_X(b) \le \int_{\mathbb{R}^+_0} 1 f_X(x)\,dx \le 1,$$

we have $\mathfrak{L}^-_X(b) = \infty$ by definition of b. We define the following sequence of integrands

$$f_n(x) := e^{-b_n x}\,\mathbf{1}_{\{x<0\}}, \quad n \in \mathbb{N}.$$

So we have $0 \le f_1 \le f_2 \le ...$ and

$$\lim_{n\to\infty} f_n = e^{-bx}\,\mathbf{1}_{\{x<0\}}.$$

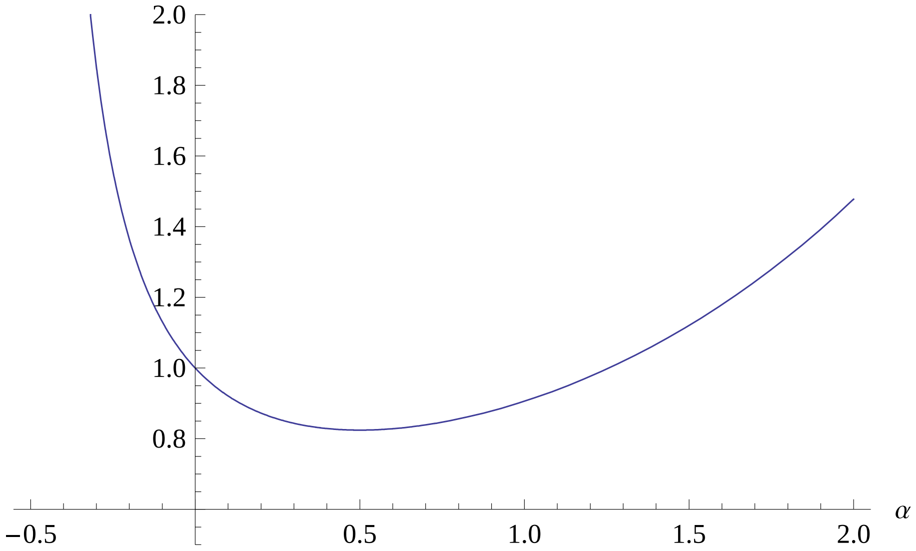


Figure 1.4: Laplace transform $\mathfrak{L}_X(\alpha)$ in Case I.

Thus the assumptions of the Beppo Levi's theorem of monotone convergence are satisfied and limit and integration can be interchanged

$$\lim_{n\to\infty} \int f_n\, f_X\, dx = \int \lim_{n\to\infty} f_n\, f_X\, dx,$$

which is by definition of f_n

$$\lim_{n\to\infty} \mathfrak{L}_X^-(b_n) = \mathfrak{L}_X^-(b) = \infty,$$

and establishes equation (1.4) by the nonnegativity of $\lim_{n\to\infty} \mathfrak{L}_X^+(b_n)$. □

Having clarified the existence of the index of riskiness, we address its uniqueness.

Lemma 5 (Uniqueness of Riskiness)**.** *If there exists a positive root of equation (1.2), it is unique.*

Proof. The existence of a positive root yields by Theorem 4 that conditions (i), (ii) and (iii) are met. Positive roots can only exists in the interval $(\,0, b\,)$ with again

$$b := \inf_{\alpha\geq 0}\, \{\mathfrak{L}_X(\alpha) = \infty\}.$$

Notice that it holds

$$\mathfrak{L}_X(0) = \int_{\mathbb{R}} f_X(x)\,dx = 1 \tag{1.5}$$

trivially for all density functions f_X and $\mathfrak{L}'_X(0) < 0$ by condition (ii). This yields $\mathfrak{L}'_X(\alpha^*) > 0$ for the minimal positive root α^* by the strict convexity of the Laplace transform on $(\,0, b\,)$. Again by convexity there thus cannot exist a second positive root in $(\,0, b\,)$. □

We have clarified for which gambles a positive root and thus riskiness exist by identifying the three essential Cases A, R and I of acceptance behavior. Hence we are in position to extent the index of riskiness from Case I, where it exists, to the set of all gambles in $\mathfrak{X}$. To do so we extend the positive root from Case I to a measure on $\mathfrak{X}$, we call indifference measure.

Definition 6 (Indifference Measure)**.** *The* indifference measure $\overline{\alpha} : \mathfrak{X} \to \mathbb{R}_0^+ \cup \{\infty\}$ *is defined by*

$$\overline{\alpha}(X) := \begin{cases} \alpha^*(X) & \textit{in Case I,} \\ \infty & \textit{in Case A,} \\ 0 & \textit{in Case R.} \end{cases}$$

The extension to Cases A and R is chosen in a natural way: In Case A, a position is accepted by CARA-agents with arbitrary big parameter. The parameter of a potential indifferent agent is hence unbounded and set to infinity. In Case R, all CARA-agents reject the position regardless how small their positive parameter is. Thus a potential indifferent agent must have a non-positive parameter and it is set to zero. By the reciprocity of riskiness and the positive root we follow Aumann & Serrano (2008) in extending riskiness.

Definition 7 (Extended Riskiness)**.** Extended Riskiness $\overline{R} : \mathfrak{X} \to \mathbb{R}_0^+ \cup \{\infty\}$ *is* *defined by*

$$\overline{R}(X) := \begin{cases} \frac{1}{\alpha^*(X)} & \text{in Case I,} \\ 0 & \text{in Case A,} \\ \infty & \text{in Case R.} \end{cases}$$

In Case A no losses are possible. This risk-free situation is reflected by a riskiness of zero. Whereas in Case R no risk-averse agent accepts the position, and hence riskiness is infinite.

In the remainder of this section we consider an example of a position, which satisfies condition (i) and (ii) and violates condition (iii). This is of interest in this context, since it illustrates that condition (i) and (ii) are not sufficient for the existence of riskiness. This position of Case R is accepted by a risk neutral agent, but rejected by all risk-averse CARA-agents, regardless how close to risk-neutrality they are. This example is inspired by Example 3.11 of Romano & Siegel (1986). The position is lognormally distributed and has a heavy negative tail.

Example 8. *Let X have standard normal distribution. We consider the random variable $Y := 2 - e^X$. Hence Y is up to additive and multiplicative constants a standard lognormal distribution. To show that the Laplace transform of Y explodes on the positive real axis we consider for $\alpha > 0$*

$$\begin{aligned} \mathfrak{L}_Y(\alpha) &= \mathfrak{L}_{2-e^X}(\alpha) = E\exp(\alpha\exp(X) - 2\alpha) \\ &= (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} \exp\big(\alpha\exp(x) - \frac{x^2}{2} - 2\alpha\big)\, dx \\ &\geq (2\pi)^{-\frac{1}{2}} \int_0^\infty \exp\big(\alpha(1 + x + \frac{x^2}{2} + \frac{x^3}{6}) - \frac{x^2}{2} - 2\alpha\big)\, dx \\ &= \infty. \end{aligned}$$

The inequality follows by omitting integration on the negative axis and by reducing the exponential of x to a third-degree polynomial. Since this polynomial has the positive coefficient α, the integrand tends to infinity for $x \to \infty$. Therefore *the Laplace transform is infinite for any positive α and condition (iii) is violated. Nonetheless condition (i) and (ii) are satisfied, since it holds*

$$\mathcal{P}(Y < 0) = \mathcal{P}(X > \log(2)) > 0 \quad \textit{and} \quad EY = 2 - e^{\frac{1}{2}} > 0.$$

1.4 Calculations of Riskiness

In this section we derive explicit solutions for the riskiness of positions whose distributions take common forms. In the first subsection we compute the riskiness for standard distributions numerically, whereas in the remainder of the section we present closed-form solutions for the riskiness for some particular distributions.

1.4.1 Numerical Solutions

We compute the index of riskiness exemplary for the following non-heavy tailed distributions of risk: Laplace, Logarithmic, Logistic, Uniform and Binomial distribution. Theorem 4 provides for which parameters of the distributions riskiness exists. For the two-parameter distributions we thus consider pairs of parameter which ensure that the distributions satisfy conditions (i) and (ii). For the one-parameter Logarithmic distribution, its support is the positive real line for all parameters. Hence no losses are possible. To meet condition (i) of Theorem 4 nonetheless, we shift the distribution to the left, i.e. considering the position $X - s$ for $s \geq 0$. The economic interpretation of s is the deterministic price to enter the position X. We also shift the Binomial distribution, since its support is a subset of the nonnegative numbers.

The connection to Laplace transforms, presented in section 2, reduces these calculations of riskiness to the invertion of Laplace transforms. We compute positives roots of equation (1.2) and thus positive roots of Problem 1 using the Newton method. Regarding the shift of the Logarithmic and Binomial distribution it holds for the Laplace transform all $\alpha \in \mathbb{R}$

$$\mathfrak{L}_{X-s}(\alpha) = e^{s\alpha}\,\mathfrak{L}_X(\alpha).$$

In Table 1.1 we present exemplary some computations for different parameters. For each distribution except for the Binomial distribution we consider three pairs of parameters. The first pairs are standard parameters. The second pairs are chosen in such a way that the distribution has a lower mean compared with the first pairs. This yields an increase of risk for the positions. According to the axiom of monotonicity the computed riskiness is higher. The third pairs of parameters result that the distributions have a higher volatility compared with distributions of the second pairs. Since this again increases the risk of the positions, also computed riskiness increases. For the Binomial distribution we consider four triples of parameters. The first triple is standard. The second triples leads to a higher shift and thus to a lower mean compared to the first triple. This increases risk and riskiness is higher. The third triple leads to a lower success probability and once more increases risk and riskiness. The forth triple reduces the number of trials and substantially increases risk and riskiness.

1.4.2 Closed-form Solutions

In remainder of the section we derive closed-form solutions for several distributions, namely the Exponential, Normal, Poisson, and Gamma distribution. To find the positive root of equation (1.1) we are slightly more generally in the remainder of the section by including also non-positive solutions, i.e. some $\alpha^* \in \mathbb{R}$ with

$$\mathfrak{L}_X(\alpha^*) = 1. \tag{1.6}$$

We call the solution $\alpha^* = 0$ *trivial* solution, since it holds $\mathfrak{L}_X(\alpha)(0) = 1$ for all $X \in \mathfrak{X}$ by equation (1.5). A solution $\alpha^* \neq 0$ is called *non-trivial* solution.

Table 1.1: Positive Roots and Riskiness of Common Distributions.

Distribution of X	$\alpha^*(X)$	R(X)
Laplace(1 , 1)	0.71456	1.39947
Laplace(0.5 , 1)	0.44796	2.23233
Laplace(0.5 , 1.5)	0.21091	4.74144
Logarithmic(0.9) - 1.5	1.70229	0.58745
Logarithmic(0.9) - 2	0.63128	1.58408
Logarithmic(0.8) - 2	0.25185	3.97058
Logistic(1 , 1)	0.54281	1.84225
Logistic(0.5 , 1)	0.29502	3.38957
Logistic(0.5 , 1.5)	0.13330	7.50207
Uniform[-1 , 4]	2.54265	0.39329
Uniform[-1 , 2]	1.54078	0.64902
Uniform[-2 , 4]	0.77039	1.29805
Binomial(5 , 0.5) - 1	3.28128	0.30475
Binomial(5 , 0.5) - 1.5	1.80107	0.55522
Binomial(5 , 0.4) - 1.5	0.91123	1.09741
Binomial(4 , 0.4) - 1.5	0.21165	4.72469

1.4.3 Exponential Distribution

Let X be exponentially distributed with parameter $\lambda > 0$. Thus the density function is of the following form:

$$f_\lambda(x) := \lambda\, e^{-\lambda x}\, \mathbf{1}_{\{x \geq 0\}}.$$

Since the support of X is the positive real line, we shift the position by a number s to the left in order to meet condition (i). The economic interpretation for such a position is natural: The agent buys an investment, which has a constant hazard rate λ. She benefits from the stochastic lifetime of the investment, e.g. a light bulb, car or manufacturing facility. For this benefit she pays the fixed price s. The Laplace transform of $X - s$ is given by

$$\mathfrak{L}_{X-s}(\alpha) = e^{\alpha s} \frac{\lambda}{\lambda + \alpha}.$$

Lemma 9. *Let $X \sim$ Exp(λ). The general solution for the set of roots of equation (1.6) is given by*

$$\alpha^*(X - s) = -\lambda - \frac{1}{s} \mathfrak{W}(-\lambda s\, e^{-\lambda s}), \tag{1.7}$$

where $\mathfrak{W}$ denotes the Lambert W-function for real arguments.

Proof. We consider the Laplace transform of α^* and set $z := -\lambda s\, e^{-\lambda s}$ for convenience

$$\mathfrak{L}_{X-s}(\alpha^*) = e^{\alpha^* s} \frac{\lambda}{\lambda + \alpha^*} = e^{-\lambda s - \mathfrak{W}(z)} \frac{-\lambda s}{\mathfrak{W}(z)} = \frac{z}{\mathfrak{W}(z)\, e^{\mathfrak{W}(z)}} = 1.$$

The last equality follows for all values of $\mathfrak{W}(z)$ as the denominator equals the nominator by $\mathfrak{W}(z)\, e^{\mathfrak{W}(z)} = z$, see equation (4.1) in Appendix B.1. Thus the Laplace transform satisfies equation (1.6). □

For details on the Lambert W-function we refer to Appendix B.1. Since this set of roots may only contain the trivial solution or also a non-trivial solution, which can be positive or negative, we discuss this in more detail. This discussion illuminates the relation between four aspects: the choice of parameters λ and s, the conditions of

Theorem 4, the shape of the Laplace transform and the number of roots of equation (1.6). It is illustrating to see how these aspects coincide in four different cases, which we distinguish by the choice of parameters, namely the choice of price s.

- We start with considering $s \leq 0$. In this case the price is negative and the position $X - s$ has only positive outcomes. Since no losses are possible, the position belongs to Case A and Lemma 2 yields that there is no positive root. The Laplace transform is shaped as in Figure 1.1. Thus the trivial root is the only one. How is this reflected in the solution (1.7)? By the negativity of s, it is $-\lambda s\, e^{-\lambda s} > 0$. Referring to the discussion of the Lambert W-function in Appendix B.1, this yields $\mathfrak{W}(-\lambda s\, e^{-\lambda s})$ has one value, namely the principal branch $\mathfrak{W}_0(-\lambda s\, e^{-\lambda s}) = -\lambda s$. This yields the trivial root $\alpha^*_{X-s} = 0$, which is unique.

- We assume $0 < s < \frac{1}{\lambda}$. By this positive price the position $X - s$ has potential negative outcomes, but still a positive mean. Since the exponential distribution does not have a heavy negative tail, the position belongs to Case I and Theorem 4 yields the existence of a positive root besides the trivial root. A typical Laplace transform is plotted in Figure 1.4. How are these two roots given by solution (1.7)? By choice of the price in this case it holds $e^{-1} > \lambda s\, e^{-\lambda s} > 0$. The first inequality follows by the relation $-1 > \log(\lambda s) - \lambda s$, which holds for all $\lambda s > 0$ and $0 < \lambda s < 1$. In consequence the Lambert W-function at point $-\lambda s\, e^{-\lambda s}$ is two-valued. The principal branch $\mathfrak{W}_0(-\lambda s\, e^{-\lambda s}) = -\lambda s$ yields the trivial root by $-\lambda s > -1$. For the second value of the alternate branch it holds $\mathfrak{W}_{-1}(-\lambda s\, e^{-\lambda s}) < -1$. This yields a positive root α^*_{X-s}.

- Let $s = \frac{1}{\lambda}$. The price is chosen, that the position has zero-mean. In this case the Laplace transform has slope 0 at point 0 and by strict convexity it is strictly greater than 1 in all other points, what is illustrated in Figure 1.5. This yields that the trivial root is unique, and so the position belongs to Case

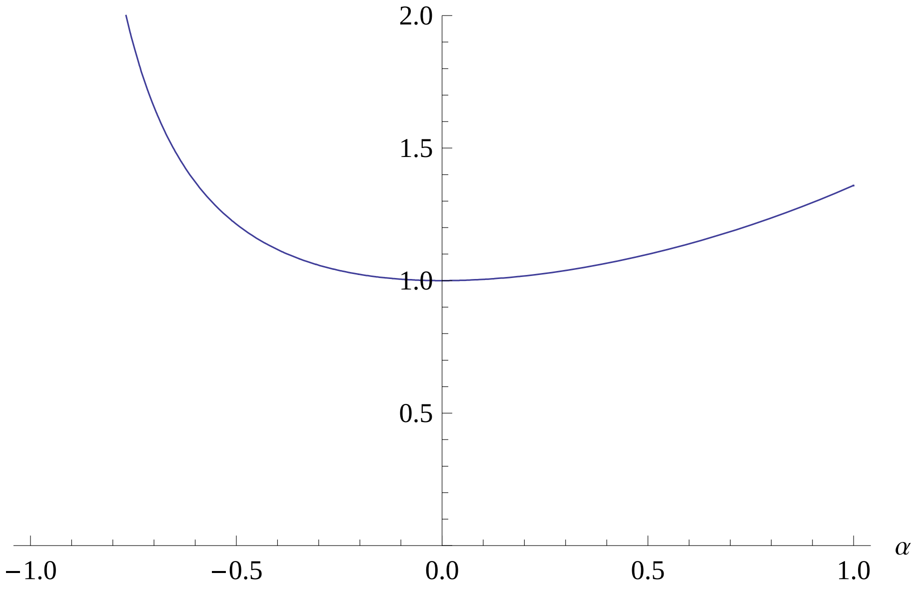


Figure 1.5: Laplace transform $\mathfrak{L}_X(\alpha)$ of an Exponential Distribution with Zero-Mean.

R. This is reflected by solution (1.7), since $\mathfrak{W}(-e^{-1})$ has one value by the principal branch $\mathfrak{W}_0(-e^{-1}) = -1$, yielding the unique trivial root.

- The last possible case is $\frac{1}{\lambda} < s$. The position has a negative mean and positive outcomes are possible. The corresponding Laplace transform has positive slope at point 0 and increases to infinity for $\alpha \to \infty$ and $\alpha \to -\infty$. Thus there is a negative root beside the trivial root, what is illustrated in Figure 1.6, and the position belongs to Case R. This is reflected in the solution (1.7): By $-\lambda s < -1$, $-\lambda s$ is the value of the alternate branch, i.e. $\mathfrak{W}_{-1}(-\lambda s\, e^{-\lambda s}) = -\lambda s$. This yields the trivial root $\alpha^*_{X-s} = 0$. For the second value of the principal branch it holds $\mathfrak{W}_0(-\lambda s\, e^{-\lambda s}) > -1$. This yields a negative root α^*_{X-s} by equation (1.7).

Summing up, riskiness exists if and only if $0 < s < \frac{1}{\lambda}$ holds, i.e. Case I, and is given by the reciprocal of

$$\alpha^*(X - s) = -\lambda - \frac{1}{s}\mathfrak{W}_{-1}(-\lambda s\, e^{-\lambda s}),$$

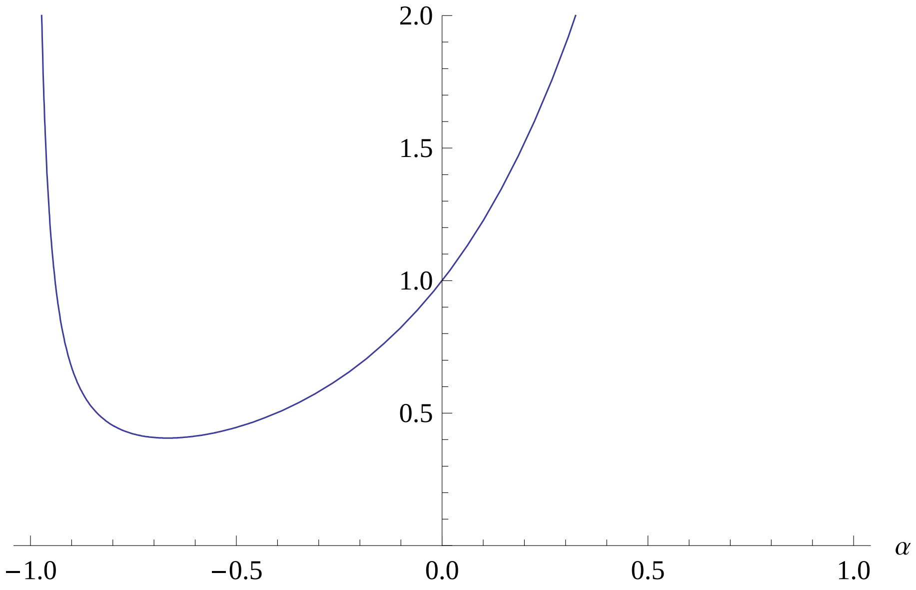


Figure 1.6: Laplace transform $\mathfrak{L}_X(\alpha)$ of an Exponential Distribution with Negative Root.

which is unique and positive. For e.g. $\lambda = 0.5$ and $s = 1$, it is $\alpha^* = 1.2564$ and hence a riskiness of $R = 0.7959$. The Laplace transform for these parameters is plotted in Figure 1.4.

1.4.4 Normal Distribution

Let X be normal distributed with mean $\mu \in \mathbb{R}$ and standard deviation $\sigma \in \mathbb{R}$, i.e.

$$f_{\mu,\sigma}(x) := \frac{1}{\sigma\sqrt{2\pi}} \exp\big(-\frac{(x-\mu)^2}{2\sigma^2}\big).$$

The Laplace transform of X is given by

$$\mathfrak{L}_X(\alpha) = \exp(-\mu\alpha + \frac{\alpha^2\sigma^2}{2}).$$

The non-trivial root of equation (1.6) is given by

$$\alpha^*(X) = \frac{2\mu}{\sigma^2},$$

for all $\mu, \sigma \in \mathbb{R}^{-\{0\}}$, which is derived in Aumann & Serrano (2008) and also direct by

$$\mathfrak{L}_X(\alpha^*) = \exp(-\frac{2\mu^2}{\sigma^2} + \frac{2\mu^2}{\sigma^2}) = 1.$$

Riskiness is given by $R(X) = \frac{\sigma^2}{2\mu}$ and exists, if and only if, it holds $\mu > 0$, i.e. Case I.

1.4.5 Poisson Distribution

Let X be Poisson distributed with parameter $\lambda > 0$, i.e.

$$f_\lambda(k) := \frac{\lambda^k\, e^{-\lambda}}{k!}.$$

Since the support of X is the set $\mathbb{N}_0$, we again shift by price s in order to satisfy condition (i). The Laplace transform of $X - s$ is given by

$$\mathfrak{L}_{X-s}(\alpha) = \exp(\lambda e^{-\alpha} - \lambda + \alpha s).$$

Lemma 10. *Let $X \sim \mathcal{P}(\lambda)$. The set of roots of equation (1.6) is given by*

$$\alpha^*(X - s) = \frac{\lambda}{s} + \mathfrak{W}(-\frac{\lambda}{s}\, e^{-\frac{\lambda}{s}}).$$

Proof. We insert these roots into the Laplace transform. For convenience we set $z := -\frac{\lambda}{s}\, e^{-\frac{\lambda}{s}}$ and use $\mathfrak{W}(z)\, e^{\mathfrak{W}(z)} = z$:

$$\begin{aligned}
\mathfrak{L}_{X-s}(\alpha^*) &= \exp\left(\lambda e^{-\alpha^*} - \lambda + \alpha^* s\right) \\
&= \exp\left(\lambda e^{-\frac{\lambda}{s} - \mathfrak{W}(z)} + s\,\mathfrak{W}(z)\right) \\
&= \exp\left(s\, e^{-\mathfrak{W}(z)}(-z + \mathfrak{W}(z) e^{\mathfrak{W}(z)})\right) \\
&= \exp\left(s\, e^{-\mathfrak{W}(z)}\, 0\right) \\
&= 1.
\end{aligned}$$

□

Riskiness exists if and only if it holds $0 < s < \lambda$, i.e. Case I, and is given by the reciprocal of

$$\alpha^*(X - s) = \frac{\lambda}{s} + \mathfrak{W}_0(-\frac{\lambda}{s}\, e^{-\frac{\lambda}{s}}).$$

For e.g. $\lambda = 1$ and $s = 0.5$, it is $\alpha^* = 1.5936$ and hence a riskiness of $R = 0.6275$.

1.4.6 Gamma Distribution

Let X be Gamma distributed with parameter $a, b > 0$, i.e.

$$f_{a,b}(x) := x^{a-1}\, \frac{e^{-x/b}}{b^a\, \Gamma(a)}.$$

The Gamma distribution is a generalization of the exponential distribution, since for $a \in \mathbb{N}$ it has the same distribution as the sum of a random variables, which are exponentially distributed with parameter $1/b$. The Laplace transform of $X - s$ exists on $(-1/b, \infty)$ and is given by

$$\mathfrak{L}_{X-s}(\alpha) = e^{\alpha s}\, (1 + b\, \alpha)^{-a}.$$

Lemma 11. *Let $X \sim \Gamma(a, b)$. The set of roots of equation (1.6) is given by*

$$\alpha^*(X - s) = -\frac{1}{b} - \frac{a}{s}\mathfrak{W}(-\frac{s}{ab}\, e^{-\frac{s}{ab}}).$$

Proof. We consider the Laplace transform and set $z := -\frac{s}{ab}\, e^{-\frac{s}{ab}}$

$$\begin{aligned}
\mathfrak{L}_{X-s}(\alpha^*) &= e^{\alpha^* s}\, (1 + b\, \alpha^*)^{-a} \\
&= \frac{\exp(-\frac{s}{b} - a\, \mathfrak{W}(z))}{\big(-\frac{a\, b}{s}\mathfrak{W}(z)\big)^a} \\
&= \frac{(-\frac{a\, b}{s})^{-a}\, e^{-\frac{s}{b}}}{\mathfrak{W}(z)^a\, e^{a\, \mathfrak{W}(z)}} \\
&= \frac{z^a}{\big(\mathfrak{W}(z)\, e^{\mathfrak{W}(z)}\big)^a} \\
&= 1.
\end{aligned}$$

□

Riskiness exists if and only if it holds $0 < s < ab$, i.e. Case I, and is given by the reciprocal of

$$\alpha^*(X - s) = -\frac{1}{b} - \frac{a}{s}\mathfrak{W}_{-1}(-\frac{s}{ab}\, e^{-\frac{s}{ab}}).$$

For e.g. $a = 2, b = 1$ and $s = 1$, it is $\alpha^* = 2.5129$ and hence a riskiness of $R = 0.3980$.

Chapter 2

A Characterization of Dual Risk Measures via the Indifference Measure

Decisions involving risk depend on two distinct aspects: (i) the risk of the position and (ii) the investor's attitude towards risk. The literature captures the first aspect by risk measures and the second by risk aversion. We connect both concepts by considering a new principle for risk measurement, which refers to the indifference risk aversion level. Applying this principle in an elementary setting provides the indifference measure. This measure is shown to be a meaningful risk measure, as it characterizes all so-called dual risk measures, which respect comparative risk aversion.

2.1 Introduction

Whenever facing a position with uncertain outcomes people are exposed to risk. These positions appear in diverse situations and include e.g. any stock, portfolio, credit, gamble or lottery. Conceptually, whether or not an individual enters such a position depends, following Diamond & Stiglitz (1974), on two distinct aspects:

(i) the objective attributes of the position, in particular how risky it is, and

(ii) the subjective attitude towards risk of the investor, in particular how risk-averse she is.

For both aspects the literature provides sophisticated theories. Concerning the first, it suggests numerous *risk measures*, which quantify the risk of a financial position. Famous examples are the commonly used Value-at-Risk, Expected Shortfall, spectral risk measures, and risk measures based on moments and expected utility. Since some of these examples expose unfavorable properties (e.g. Value-at-Risk penalizes diversification), Artzner et al. (1999) and Föllmer & Schied (2002) introduce axioms of coherence to ensure consistent risk measures. Concerning the second aspect, the classic contribution of Arrow (1965), Arrow (1971) and Pratt (1964) measures the attitude towards risk by defining a coefficient of *risk aversion* based on the subjective utility function.

This chapter aims to connect both concepts. For this purpose we will introduce the indifference risk aversion principle and the class of dual risk measures, which respect comparative risk aversion.

Quantifying the risk of a random payoff for aspect (i) usually bases on the actuarial principle of an equivalent utility premium: As pointed out by Denuit et al. (2006) broad classes of common risk measures are obtained by deriving the price, such that a decision-maker is indifferent between accepting the financial position and paying the price or not entering the position. This indifference price is thus the minimal certain side payment or capital requirement, which makes a position acceptable.

Our primary idea in this context is not to ask for the *price*, but for the level of *risk aversion* of an investor, which leaves him indifferent. The rationale behind this idea is obvious: a financial position, which is exposed to high risk, is rejected by the majority of risk-averse agents and an indifferent agent has to be relatively risk-affine compared to the majority. And vice versa, for a low-risk position an agent has to be relatively risk-averse in order to be indifferent. Consequently the indifference risk aversion level provides inference to the risk of the position. We call this plausible idea risk measurement via the *indifference risk aversion principle*. This principle connects the two aspects (i) and (ii) of decision-making in a natural way.

The importance of the classical equivalent utility principle based on the indifference price is unquestioned in finance, insurance, and economics. Similarly the equivalent utility principle based on indifference risk aversion can add to the accuracy of risk measurement. This new principle illuminates a different aspect of the broad phenomenon of risk. It is conceptually appealing, since the link between risk and attitude towards risk is natural. In the context of risk measurement, opposed to risk valuation, it is more direct than the relation between risk and an indifference price. Moreover, it is attractive that no fixed level of risk aversion has to be assumed and that a position itself is measured without changing it by adding a side payment.

In order to apply this principle in a concrete setting we consider the family of risk-averse CARA-agents in the expected utility framework. Inspired by the indifference risk aversion principle we ask which agent of this family is exactly indifferent. Thereby we define the *indifference measure*, which maps a position to the positive constant Arrow-Pratt coefficient of the agent, who is indifferent to the position in question. By convention, a typical risk measure associates a position, which is exposed to high risk, with a high value. Since the indifference measure expresses high risk with a low parameter and low risk with a high parameter, it is natural to reverse it in order to receive a risk measure. In order to verify this intuition, that a reversed indifference measure is a sensible risk measure, we consider the axiom of duality.

This axiom is introduced in the pathbreaking paper of Aumann & Serrano (2008). The duality axiom asserts, roughly speaking, that less risk-averse agents accept riskier gambles. More precisely, it poses a condition on risk measures: A risk measure is dual, if every agent accepts all positions, which are less risky (by means of the risk measure in question) than a position, which is already accepted by a more risk-averse agent. There are two reasons for this behavior: first, the agent in question is relatively more risk-affine and second, the gambles in question are relatively less risky. This makes duality a natural assumption, which could be seen as a requirement on risk measures. Aumann & Serrano (2008) define a risk measure, called the index of riskiness, and show that it equals the reciprocal of the indifference measure. Further they characterize the index of riskiness by two axioms: duality and positive homogeneity. Since positive homogeneity is of minor importance (also in their opinion), we relax it and consider the whole class of dual risk measures. This class is of relevance as it contains all risk measures satisfying the natural axiom of duality.

As it turns out, the crucial key to this class of risk measures is the indifference measure, which induces an ordering on the set of positions. The main insight of this chapter is that all dual risk measures can be characterized by the indifference measure. Our primary theorem yields, that any dual risk measure reverses the ordering induced by the indifference measure. A consequence is a representation theorem which decomposes dual risk measures into the indifference measure and a reversing function. It turns out that the opposite implication also holds true: any measure reversing the ordering of the indifference measure is necessarily dual. Hence we derive a method to construct dual risk measures by composition of the indifference measure with an arbitrary reversing function. Overall, we provide a handy equivalent condition for duality.

This characterization supports the risk measurement via the indifference risk aversion principle. It verifies the intuition that reversed indifference measures are sensible risk measures, as they satisfy the natural duality axiom. Furthermore, it shows that every dual risk measure necessarily bases on the indifference measure, which is derived by the indifference risk aversion principle. It is in addition very appealing that the indifference measure, that is derived by exponential utility solely, ensures duality, which in turn ensures consistency with a notion of comparative risk aversion for general concave utility functions beyond exponential utility. This highlights the outstanding role of the indifference measure and exponential utility among concave utility functions.

Summing up, we believe that this chapter contributes to various aspects of decision-making under risk. First, it introduces a new principle for risk measurement, which is conceptually appealing and has the potential to add to the accuracy of risk measurement. Further this indifference risk aversion principle achieves to connect the theories of the two essential aspects (risk measurement and risk aversion) of individual decision-making. Second, the abstract principle is made concrete by considering the indifference measure. The indifference measure constitutes a new type of risk measure, which differs conceptually from all classical risk measures except for the index of riskiness. Third, the class of dual risk measures is introduced and characterized by a simple equivalent condition. This equivalence provides a representation theorem and a construction method for dual risk measures. It further proves the intuition, given by the principle, that the indifference measure is a meaningful risk measure.

The chapter is organized as follows: the following section provides the indifference measure, whereas Section 3 introduces the axiom of duality. The primary result is derived in Section 4, which provides the characterization of dual risk measures via the indifference measure.

2.2 Notation and the Indifference Measure

We use the standard notation from Chapter 1 and repeat it in brief. The triple $(\Omega, \mathcal{F}, \mathcal{P})$ denotes a probability space and E denotes the expectation operator with respect to $\mathcal{P}$. A *financial position* or *gamble* is described by a random variable $X : \Omega \to \mathbb{R}$. $X(\omega)$ is the discounted net worth of the position at the end of a given period under scenario ω. By $\mathfrak{X}$ we denote the set of financial positions. For full generality we set $\mathfrak{X} := \mathfrak{L}^0(\Omega, \mathcal{F}, \mathcal{P}) - \ker(\|\cdot\|_p)$, i.e. the space of Borel-measurable functions on $(\Omega, \mathcal{F})$ minus the kernel of the seminorm $\|\cdot\|_p$.

We consider a standard paradigm of decision-making, the expected utility framework. Let $u : \mathbb{R} \to \mathbb{R}$ denote a Bernoulli utility function for money, which is strictly monotonic, strictly concave, and in $\mathcal{C}^2(\mathbb{R})$. We identify an agent i by her utility function u_i and denote by $w_i \in \mathbb{R}$ agent i's wealth level. We say *i accepts X at w*, if accepting the position at wealth w increases agent i's expected utility compared to rejecting it, i.e.

$$\mathrm{E}u_i(X + w) > u_i(w).$$

Analogously, *i is indifferent to X at w*, if it holds $\mathrm{E}u_i(X + w) = u_i(w)$, and *$i$ rejects at w*, if it holds $\mathrm{E}u_i(X + w) < u_i(w)$.

For a utility function u the Arrow-Pratt coefficient of absolute risk aversion is defined by

$$r_u(x) := \frac{-u''(x)}{u'(x)}.$$

Any utility function with constant Arrow-Pratt coefficient of absolute risk aversion, called CARA, is up to additive and multiplicative constants of the form

$$\bar{u}_\alpha(x) := \tfrac{1}{\alpha}(1 - e^{-\alpha x}).$$

The corresponding agent is called CARA with parameter α. Since we only consider risk-averse agents, we concentrate on $\alpha > 0$. Such an agent is indifferent at wealth w, if it holds

$$\mathrm{E}\,\tfrac{1}{\alpha}\big(1 - e^{-\alpha(X+w)}\big) = \tfrac{1}{\alpha}\big(1 - e^{-\alpha w}\big),$$

which is equivalent to

$$\mathrm{E}\,e^{-\alpha X} = 1.$$

for all wealth levels $w \in \mathbb{R}$. Thus, the decisions of CARA-agents are the same under all wealth levels. By considering CARA-agents we abstract from wealth effects.

We are in position to apply the indifference risk aversion principal for risk measurement. For a given position we ask, which agent of the CARA-family is indifferent to the position. Thus we map a position $X \in \mathfrak{X}$ to the positive parameter of risk aversion of an indifferent CARA-agent.

Definition 12 (Indifference Measure)**.** *The* indifference measure $\overline{\alpha} : \mathfrak{X} \rightarrow \mathbb{R}_0^+ \cup \{\infty\}$ *is defined by*

$$\overline{\alpha}(X) := \begin{cases} \alpha^* & \text{the positive root of } Ee^{-\alpha^* X} = 1, \text{ if it exists,} \\ \infty & \text{if } Ee^{-\alpha X} < 1 \text{ holds for all } \alpha > 0, \\ 0 & \text{if } Ee^{-\alpha X} > 1 \text{ holds for all } \alpha > 0. \end{cases}$$

This definition yields a well-defined mapping, as derived in Chapter 1. The existence of a positive root is analyzed in detail and the three cases of the definition are related to properties of the position X. Moreover, Chapter 1 yields numerical and closed-form solutions for the positive root. Finally, we define a risk measure, which quantifies the risk of a position.

Definition 13 (Risk Measure)**.** *A mapping* $\rho : \mathfrak{X} \rightarrow \mathbb{R} \cup \{\infty, -\infty\}$ *is called a* risk measure*.*

Hence the indifference measure is formally a risk measure. The computations of Chapter 1 show that it maps high-risk positions to low parameters and vice versa. This is in line with the intuition of the indifference risk aversion principle. However, by convention a typical risk measure associates high-risk positions with high numbers and low-risk position with low numbers: For a given risk measure ρ, we say a position X is *riskier* than a position Y, if it holds $\rho(X) > \rho(Y)$. This suggests to reverse the indifference measure in order to receive a conventional risk measure. To verify the intuition, that a reversed indifference measure is a sensible risk measure, we consider the axiom of duality.

2.3 The Axiom of Duality

In this section we consider the axiom of duality and a notion of comparative risk aversion, which are introduced by Aumann & Serrano (2008).

Definition 14 (Uniform Comparative Risk Aversion)**.** *Agent* i is as least as risk-averse as j *(denoted by $i \trianglerighteq j$), if for all positions $X \in \mathfrak{X}$ and wealth levels $w_i, w_j \in \mathbb{R}$ holds*

If agent i accepts X at w_i, then agent j accepts X at w_j.

Agent i is more risk-averse than j *(denoted by $i \triangleright j$), if it is $i \trianglerighteq j$ and not $j \trianglerighteq i$; (or equivalently, if for all $X \in \mathfrak{X}$ and $w_i, w_j \in \mathbb{R}$ holds: if agent i accepts X at w_i, then agent j accepts X at w_j, and there additionally exist $Y \in \mathfrak{X}$ and $w_i', w_j' \in \mathbb{R}$ such that i rejects Y at w_i' and j accepts Y at w_j').*

This notion of uniform comparative risk aversion induces a partial ordering of the agents by their risk aversion and is essential for the definition of duality. An agent j is, roughly speaking, more "risk-affine" than agent i, if she accepts at all wealth levels all positions, which are accepted by the more risk-averse agent i at some wealth level. Thus this notion compares "uniformly" in wealth. For a detailed discussion of this notion and its relation to the classic and "local" notion of comparative risk aversion we refer to Aumann & Serrano (2008). Aumann & Serrano (2008) provide several arguments supporting the uniform notion in comparison to the local notion. Most convincing is the fact that it appears in the antecedence of the axiom, which in turn makes assuming the axiom a weaker condition.

Definition 15 (Duality)**.** *A risk measure ρ is* dual*, if for all agents i, j, positions $X, Y \in \mathfrak{X}$ and wealth levels $w_i, w_j \in \mathbb{R}$ it holds*

$$\text{If } i \triangleright j,\ i \text{ accepts } X \text{ at } w_i, \text{ and } \rho(X) > \rho(Y), \text{ then } j \text{ accepts } Y \text{ at } w_j. \tag{2.1}$$

In words, a risk measure is dual, if every agent accepts any position, which is less risky (by means of ρ) than a position, which is accepted by a more risk-averse agent.

In other words, duality of a risk measure ρ states that, if the more risk-averse of two agents accepts the riskier (by means of ρ) of two positions, than more than ever the less risk-averse agent accepts the less risky position. There are two reasons for this behavior: first, the second agent is more risk-affine, and second, the second position is less risky. In the light of these two reasons assuming duality appears natural. By definition a dual risk measure respects comparative risk aversion.

2.4 Representation Theorem

We present the main theorem, which states that any dual risk measure reverses the ordering, which is induced by the indifference measure $\overline{\alpha}$ on the set of positions $\mathfrak{X}$. In addition, the other implication of the theorem yields that an arbitrary risk measure reversing the ordering is necessarily dual and thus provides a criterium for duality.

Theorem 16 (Characterization of Dual Risk Measures)**.** *A risk measure ρ is dual if and only if for all positions $X, Y \in \mathfrak{X}$ it holds*

$$\rho(X) > \rho(Y) \;\rightarrow\; \overline{\alpha}(X) \leq \overline{\alpha}(Y). \tag{2.2}$$

Proof. If-Direction: Suppose equation (2.2) holds. To show that ρ is dual we fix any agents i, j with $i \rhd j$ and any two positions $X, Y \in \mathfrak{X}$ with $\rho(X) > \rho(Y)$. We have to show that j accepts Y at w_j, if i accepts X at w_i.

We start the proof with some preparations. Without loss of generality we normalize both utility functions to $u_i(0) = u_j(0) = 0$ and $u_i'(0) = u_j'(0) = 1$. Also without loss of generality we set $w_i = w_j = 0$, since for e.g. $w_i \neq 0$ we consider the agent $\tilde{i}$ with

$$\tilde{u}_i(x) := \frac{u_i(x + w_i) - u_i(w_i)}{u_i'(w_i)},$$

who accepts a position at wealth 0 if and only if i accepts this position at w_i.

Further we consider the following three statements about the Arrow-Pratt coefficients r_i, r_j of agents i, j, which we will use below:

$$\text{It holds } 0 < r_i(x) < \infty \text{ for all agents } i \text{ and } x \in \mathbb{R}. \tag{2.3}$$

$$\text{If it holds } r_i(x) \leq r_j(x) \text{ for all } x \text{, then it holds } u_i(x) \geq u_j(x) \text{ for all } x. \tag{2.4}$$

$$\text{It is } i \trianglerighteq j \text{ if and only if it holds } r_i(x_i) \geq r_j(x_j) \text{ for all } x_i, x_j. \tag{2.5}$$

Statement (2.3) follows by the assumption of twice differentiability, strict monotonicity and strict concavity on utility functions. The proofs of statements (2.4) and (2.5) are given in Aumann & Serrano (2008), Corollary 3 and Proposition 4.1.2, and are not repeated here. We set

$$\begin{aligned} \underline{r} &:= \inf_{x \in \mathbb{R}} r_i(x), \quad \text{and} \\ \overline{r} &:= \sup_{x \in \mathbb{R}} r_j(x). \end{aligned}$$

By $i \triangleright j$ and statement (2.5) it follows the inequality $\underline{r} \geq \overline{r}$, and together with statement (2.3) the inequalities

$$\infty > \underline{r} \geq \overline{r} > 0.$$

Having finished the preparations we assume that agent i accepts X at 0, i.e.

$$\mathrm{E} u_i(X) > 0,$$

in order to show that agent j accepts Y at 0. By $r_i(x) \geq \underline{r}$ for all x and statement (2.4) we have for all x

$$u_i(x) \leq \bar{u}_{\underline{r}}(x) := \frac{1}{\underline{r}}(1 - e^{\underline{r}x}),$$

and hence

$$\mathrm{E}\bar{u}_{\underline{r}}(X) > 0,$$

which means that a CARA-agent with parameter $\underline{r}$ accepts X. This gives us $\overline{\alpha}(X) > 0$ by definition of the indifference measure. Note that it holds

$$\underline{r} < \overline{\alpha}(X),$$

since otherwise the inequality $\underline{r} \geq \overline{\alpha}(X)$ yields $\overline{\alpha}(X) < \infty$ and further by statement (2.4) the inequality $\bar{u}_{\underline{r}} \leq \bar{u}_{\overline{\alpha}(X)}$, which in turn yields by the defining equation $\mathrm{E}\bar{u}_{\overline{\alpha}(X)}(X) = 0$ the contradiction

$$\mathrm{E}\bar{u}_{\underline{r}}(X) \leq 0.$$

We use equation (2.2), which yields the inequality $\overline{\alpha}(X) \leq \overline{\alpha}(Y)$ by the given assumption $\rho(X) > \rho(Y)$, and we sum up the upper inequalities by

$$0 < \overline{r} \leq \underline{r} < \overline{\alpha}(X) \leq \overline{\alpha}(Y).$$

By definition of CARA-utility $\bar{u}$ it holds the following statement

$$\text{If it is } \alpha < \beta \text{, then it holds } \bar{u}_{\alpha}(x) > \bar{u}_{\beta}(x) \text{ for all } x \neq 0. \tag{2.6}$$

Thus we have the ordering

$$\bar{u}_{\overline{r}} > \bar{u}_{\overline{\alpha}(Y)} \quad \text{on} \quad \mathbb{R}^{-\{0\}}$$

by $\overline{r} < \overline{\alpha}(Y)$ and assuming $\overline{\alpha}(Y) < \infty$. This ordering finally yields by the defining equation $\mathrm{E}\bar{u}_{\overline{\alpha}(Y)}(Y) = 0$

$$\mathrm{E}\bar{u}_{\overline{r}}(Y) > 0,$$

for all $Y \in \mathfrak{X}$. In case of $\overline{\alpha}(Y) = \infty$ this follows directly by definition of the indifference measure. The relation $r_j(x) \leq \overline{r}$ for all x and statement (2.4) close the proof that ρ is dual by yielding that agent j accepts Y, i.e.

$$\mathrm{E}u_j(Y) > 0.$$

Only If-Direction: The converse direction we show by contraposition: If equation (2.2) is violated, then ρ is not dual. Hence we have to show that there exist some agents i, j, some positions $X, Y \in \mathfrak{X}$, and some wealth levels w_j, w_j, which satisfy the assumptions of equation (2.1), and that agent j does not accept Y at w_j.

Without loss of generality we again normalize both utility functions and set $w_i = w_j = 0$. By the violation of equation (2.2), there exist some positions $X, Y \in \mathfrak{X}$ with $\rho(X) > \rho(Y)$ and $\overline{\alpha}(X) > \overline{\alpha}(Y)$. For the proof we choose these two positions and two CARA-agents i, j, which are defined by their parameters

$$r_i = \min\Big(\frac{\overline{\alpha}(X) + \overline{\alpha}(Y)}{2}\ ,\ \overline{\alpha}(Y) + 1\Big),$$
$$r_j = \min\Big(\frac{\overline{\alpha}(X) + 3\,\overline{\alpha}(Y)}{4}\ ,\ \overline{\alpha}(Y) + \frac{1}{2}\Big).$$

This choice yields the following ordering

$$0 \leq \overline{\alpha}(Y) < r_j < r_i < \overline{\alpha}(X) \leq \infty,$$

for arbitrary $\overline{\alpha}(X), \overline{\alpha}(Y) \in \mathbb{R}_0^+ \cup \{\infty\}$. Notice that by $\overline{\alpha}(X) > \overline{\alpha}(Y)$ it holds $\overline{\alpha}(X) > 0$ and $\overline{\alpha}(Y) < \infty$. By the upper ordering the antecedence of equation (2.1) is satisfied: The ordering

$$i \rhd j$$

follows by $r_i > r_j$, statement (2.5) and its contraposition. The second assumption in the antecedence of equation (2.1)

$$\mathrm{E}\bar{u}_{r_i}(X) > 0$$

follows trivially by definition of the indifference measure in case of $\overline{\alpha}(X) = \infty$. In case of $\overline{\alpha}(X) < \infty$ we have $\mathrm{E}\bar{u}_{\overline{\alpha}(X)}(X) = 0$. Then the second assumption follows by $r_i < \overline{\alpha}(X)$ and statement (2.6) for all $X \in \mathfrak{X}$. The third assumption in the antecedence,

$$\rho(X) > \rho(Y),$$

holds trivially by choice of X and Y. What remains to be shown is that agent j does not accept Y at 0. In case of $\overline{\alpha}(Y) = 0$ this follows trivially as j is a CARA-agent. In case of $\overline{\alpha}(Y) > 0$ we have $\mathrm{E}\bar{u}_{\overline{\alpha}(Y)}(Y) = 0$. Then it follows

$$\mathrm{E}\bar{u}_{r_j}(Y) < 0$$

by $r_j > \overline{\alpha}(Y)$ and statement (2.6), which completes the proof. □

For formal clarity we restate the theorem in a set-theoretic notation.

Definition 17. *The* strict ordering *on $\mathfrak{X}$ induced by a risk measure ρ is defined by*

$$\prec_\rho := \{(X, Y) \in \mathfrak{X} \times \mathfrak{X} \mid \rho(X) < \rho(Y)\}.$$

The (set-theoretic) kernel *of a risk measure ρ is defined by*

$$\ker_\rho := \{(X, Y) \in \mathfrak{X} \times \mathfrak{X} \mid \rho(X) = \rho(Y)\}.$$

Corollary 18 (Characterization in Set-Theoretic Notation)**.** *A risk measure ρ is dual if and only if it holds*

$$\prec_\rho \; \subset \; \{\prec_{-\overline{\alpha}} \cup \ker_{\overline{\alpha}}\}.$$

The next theorem yields that essentially any dual risk measure can be decomposed into the indifference measure and a reversing function. We call a function $\phi : \mathbb{R}_0^+ \cup \{\infty\} \to \mathbb{R} \cup \{\infty, -\infty\}$ *reversing*, if it holds

$$\phi(x) \leq \phi(y)$$

for all $x, y \in \mathbb{R}_0^+ \cup \{\infty\}$ with $x > y$. In particular, any function with a non-positive derivative is reversing. Moreover, the theorem provides a method to obtain dual risk measures by composing the indifference measure with an arbitrary reversing function.

Theorem 19 (Representation Theorem)**.** *If a risk measure ρ is dual and it holds $\ker_{\overline{\alpha}} \subset \ker_\rho$, then it has the following representation in terms of the indifference measure*

$$\rho = \phi \circ \overline{\alpha},$$

where ϕ is a reversing function. On the other hand, for any reversing function ϕ, the risk measure ρ, defined by the composition

$$\rho := \phi \circ \overline{\alpha},$$

is dual.

Proof. In order to proof the second statement of the theorem we fix a reversing function ϕ. To show duality of the composition $\rho = \phi \circ \overline{\alpha}$ it is by Theorem 16 sufficient to show the contraposition of equation (2.2), i.e.

$$\overline{\alpha}(X) > \overline{\alpha}(Y) \quad \rightarrow \quad \rho(X) \leq \rho(Y)$$

for all $X, Y \in \mathfrak{X}$. This implication is trivial, since for X, Y in $\mathfrak{X}$ with $\overline{\alpha}(X) > \overline{\alpha}(Y)$ it holds

$$\rho(X) = \phi(\overline{\alpha}(X)) \leq \phi(\overline{\alpha}(Y)) = \rho(Y)$$

as the function ϕ is reversing.

In order to proof the first statement of the theorem we fix a risk measure ρ with $ker_{\overline{\alpha}} \subset ker_{\rho}$. To construct a reversing function ϕ we require some notation. The kernel of the indifference measure defines an equivalence relation by

$$X \sim Y \quad :\Leftrightarrow \quad \overline{\alpha}(x) = \overline{\alpha}(Y).$$

The quotient set of $\mathfrak{X}$ modulo equivalence in terms of the indifference measure

$$\mathfrak{X} / \sim \; := \{ [X]_{\sim} \mid X \in \mathfrak{X} \}$$

consists of the equivalence classes $[X]_{\sim} := \{Y \in \mathfrak{X} \,|\, Y \sim X\}$ and is also called coimage of the indifference measure. There is a natural isomorphism between the image and the coimage of a function, which maps an image to the equivalence class of the inverse images of this image. Thus the following function is well-defined

$$\begin{array}{rccl} \alpha^{-1}: & \mathbb{R}_0^+ \cup \{\infty\} & \rightarrow & \mathfrak{X} / \sim \\ & x & \mapsto & \{ X \in \mathfrak{X} \mid \overline{\alpha}(X) = x \}. \end{array}$$

Intuitively the function α^{-1} maps an Arrow-Pratt coefficient to the set of positions to which an CARA-agent of this coefficient is indifferent. It is essentially an inverse of the indifference measure, as it holds

$$\begin{array}{rccl} \alpha^{-1} \circ \overline{\alpha}: & \mathfrak{X} & \rightarrow & \mathfrak{X} / \sim \\ & X & \mapsto & [X]_{\sim}. \end{array}$$

Next we consider the function

$$\begin{aligned} \rho^{\sim}: \quad \mathfrak{X}/\sim \; &\to \; \mathbb{R}\cup\{\infty,-\infty\} \\ [X]_{\sim} \; &\mapsto \; \rho(X), \end{aligned}$$

which intuitively maps all positions with the same indifference measure to their ρ-measure. Thus for the well-definition of this function, it must hold, that two positions of the same indifference measure have the same ρ-measure, or more formally

$$X, Y \in [X]_{\sim} \;\; \to \;\; \rho(X) = \rho(Y)$$

for all $X, Y \in \mathfrak{X}$. This is ensured by the assumption $ker_{\overline{\alpha}} \subset ker_{\rho}$, which is equivalent to

$$\overline{\alpha}(X) = \overline{\alpha}(Y) \;\; \to \;\; \rho(X) = \rho(Y)$$

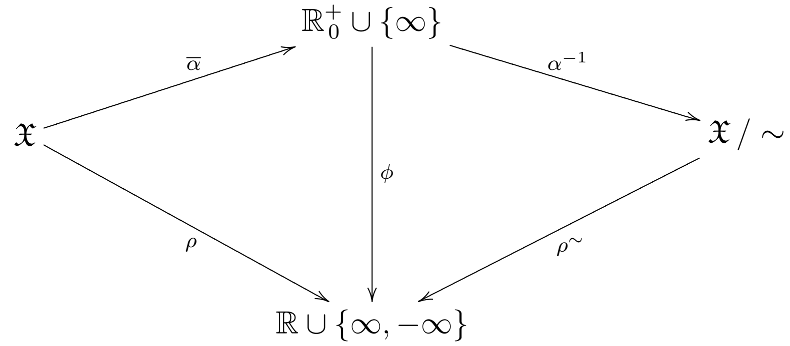
for all $X, Y \in \mathfrak{X}$. We refer to a remark on the weakness of this assumption below the proof. We are in position to construct the reversing function by

$$\phi \; := \rho^{\sim} \circ \alpha^{-1}.$$

This provides the representation $\rho = \phi \circ \overline{\alpha}$, since it holds by the above definitions

$$\phi \circ \overline{\alpha} \; = \; \rho^{\sim} \circ \alpha^{-1} \circ \overline{\alpha} \; = \; \rho.$$

The relation of all considered functions is illustrated in the following diagram:



In order to close the proof of the first statement, it remains to show that the function ϕ is reversing. To show this we fix $x, y \in \mathbb{R}_0^+ \cup \{\infty\}$ with $x > y$ and any two representatives X, Y with

$$X \in \alpha^{-1}(x),$$
$$Y \in \alpha^{-1}(y).$$

By definition of the equivalence classes it holds

$$\overline{\alpha}(X) = x > y = \overline{\alpha}(Y).$$

Since the measure ρ is dual by assumption, Theorem 16 thus provides by the contraposition of equation (2.2)

$$\rho(X) \leq \rho(Y).$$

By construction of the reversing function this gives us

$$\begin{aligned}\phi(x) = \rho^{\sim}(\alpha^{-1}(x)) &= \rho^{\sim}([X]_{\sim}) = \rho(X) \\ &\leq \rho(Y) = \rho^{\sim}([Y]_{\sim}) = \rho^{\sim}(\alpha^{-1}(y)) = \phi(y),\end{aligned}$$

yielding that the function ϕ is reversing. □

Finally we remark that the assumption $ker_{\overline{\alpha}} \subset ker_{\rho}$ is natural in the context of representing dual risk measures by the indifference measure. If it is violated, there exist two positions X, Y with $\overline{\alpha}(X) = \overline{\alpha}(Y)$ and $\rho(X) \neq \rho(Y)$. In this case there trivially cannot exist a composition $\rho = \phi \circ \overline{\alpha}$. Hence, a dual risk measure, which violates $ker_{\overline{\alpha}} \subset ker_{\rho}$, has some freedom within the equivalence classes $[X]_{\sim}$. However, even in this case the indifference measure determines the structure of a dual risk measure ρ: For all positions X, Y with $\overline{\alpha}(X) \neq \overline{\alpha}(Y)$ it either holds $\overline{\alpha}(X) < \overline{\alpha}(Y)$ or $\overline{\alpha}(X) > \overline{\alpha}(Y)$. Then the relation $\rho(X) \geq \rho(Y)$ or $\rho(X) \leq \rho(Y)$ is given by the contraposition of equation (2.2). Consequently, on the quotient set $X / \sim$ the structure of a dual risk measure is given as its ordering of the equivalence classes is determined by the indifference measure. Thus, roughly speaking, as far as the indifference measure can tell (i.e. modulo its equivalence relation), it determines the structure of arbitrary dual risk measures.

Chapter 3

Asymptotic Maturity Behavior of the Term Structure

Pricing and hedging of long-term interest rate sensitive products require to extrapolate the term structure beyond observable maturities. For the resulting limiting term structure we show two results: under no arbitrage long zero-bond yields and long forward rates (i) are monotonically increasing and (ii) equal their minimal future value. Both results constrain the asymptotic maturity behavior of stochastic yield curves. They are fairly general and require only buy-and-hold strategies. Hence our framework embeds various arbitrage-free term structure models and imposes restrictions on their specification.

3.1 Introduction

Bond markets play a prominent role among international financial markets. It is popular to set up bond market models on the term structure of interest rates. Its stochastic modeling is a central topic in mathematical finance. The recent literature has advanced in modeling the evolution of the term structure dynamically by integrating asset-pricing theory. Traditionally, many models in this approach concentrate on the short-term behavior. However, the focus on the long rate is also of great interest: The long-term behavior is essential for the valuation of long-term interest rate sensitive products. These products include fixed-income securities, insurance and annuity contracts, and perpetuities. For pricing and hedging of these instruments finance practitioners require a term structure for 100 years or more, whereas in most markets only 30 years are observable. Hence models are required, which capture the evolution of the yield curve beyond limited observable maturities. To derive joint properties of such models, we examine the limiting term structure in general and show the following two results: Under no arbitrage in a frictionless bond market with infinitely increasing maturities, long zero-bond yields and long forward rates satisfy two properties:

(i) *Asymptotic Monotonicity:* Both rates are monotonically increasing in time,

(ii) *Asymptotic Minimality:* Both rates equal their minimal future value.

This chapter derives both results in a general framework for term structure models.

The first result of asymptotic monotonicity states that both asymptotic rates cannot fall over time and hence excludes that tomorrow's long rate is less than today's long rate. Both rates still may increase with positive probability, but they cannot increase almost surely. This is denied by the second result of asymptotic minimality as it excludes systematic jumps of the long rates. Consequently both results cause that asymptotic maturity behavior of the term structure is not arbitrary. They reduce potential realizations of stochastic yields curves by excluding a multitude of asymptotic behavior under no arbitrage.

Both results are fairly general, since we derive them under weak assumptions. To show properties of the long rates, we assume their existence. The only further assumption is to postulate no arbitrage in a bond market with infinitely increasing maturities. For this purpose we provide a definition of arbitrage in bond markets, which distinguishes between the economically sound notions of bounded and vanishing risk. Instead of addressing a certain term structure model, we rather present a framework for term structure models, which refers to a frictionless bond market.

A model in our framework is given by a family of bond prices for any maturity. We do not have to impose a structure on these bond prices, except for adaption and boundedness, which both appear natural. Furthermore, our results require only a minimal setting of elementary buy-and-hold trading, which gives them a concrete and applicable character. Nonetheless our results also apply in more sophisticated settings, as they e.g. extend to continuous trading, which is of vast theoretical importance. As a result of this generality in bond prices and trading, our framework embeds virtually all existing arbitrage-free term structure models. In consequence the results and their implications apply to all these models.

Summing up, asymptotic monotonicity and minimality are important, as they exclude various behavior of limiting yield curves. Since we derive them in a general framework, they impose severe restrictions on the long-term behavior of virtually all term structure models. These theoretical implications serve as a benchmark for modelers specifying an arbitrage-free term structure. Specifically, setting up the asymptotic yield or forward rate as a diffusion process or a process with systematic jumps necessarily imposes arbitrage opportunities.

The actual asymptotic behavior of 16 well-known term structure models is analyzed in detail by Yao (1999b). All these models satisfy asymptotic monotonicity, although they are not necessarily arbitrage-free. In the models of Dothan (1978) and Heath et al. (1992) the result applies under existence of the long rates without further parameter specification. On first sight the result seems to be violated in the model by Brennan & Schwartz. (1979). The long rate is specified exogenously and decreases under certain parameter choice, but it refers to a consol bond, instead of a zero-coupon bond. In some term structure models, including Vasicek (1977), Cox et al. (1985) and Chen (1996), long bond yields are constant over time. This specific behavior suggests to consider the generalization of our two results, stating that long yields are constant over time. However, it is impossible to derive this generalization, which in consequence closes our results. This can be seen by considering the discrete binomial model by Ho & Lee (1986). This model is an example of an arbitrage-free bond market with infinitely increasing maturities. Its long yield rises with each upward shock, which appears with positive Bernoulli-probability at all discrete time points. Considering this model is furthermore illustrative for both results, since they are not violated, although the bond yield underlies downward shocks, which have a permanent effect. For this discussion we refer to Dybvig et al. (1996).

The two results of this chapter base on Dybvig et al. (1996). They came up with the genuine idea of showing the results by a no arbitrage-argument. Since the topic is also addressed by several other authors, we compare the different approaches and comment on our respective generalizations in Section 3.5. Furthermore the notion of arbitrage, we introduce to derive the results, is compared to closely-related classical notions. We discuss under which conditions the results can be transferred to these notions.

This chapter is organized as follows: The following Section provides the general framework for term structure models and the definition of arbitrage. In this setting Section 3 presents the proof of asymptotic monotonicity by explicitly constructing an arbitrage strategy. Using a related proof, asymptotic minimality is established in Section 4. The related literature and related notions of arbitrage are discussed in Section 5. Section 6 concludes.

3.2 Modeling the Bond Market and Arbitrage

This section presents the formal setting of the bond market and introduces the notion of arbitrage in the limit. The bond market is defined by a family of zero-coupon bond prices, confer also Musiela & Rutkowski (1997). A *zero-coupon bond* is a financial security that pays one unit of cash to its holder at a fixed later date T, called *maturity*. We assume these bonds to be default-free.

Definition 20 (Bond Market)**.** *A* Bond Market *consists of a filtered probability* *space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ *and an adapted uniformly bounded process* $B(\cdot, T)$ *for every* $T \geq 0$.

This definition is very general, since the only two restrictions for bond prices, adaption and uniform boundedness, are not very demanding.[1] Bond prices are just assumed to be given, so they can be observed in some real market or generated by some procedure. A model in our general approach is thus given by a family of almost arbitrary bond prices. We refer to price processes, which are denoted in a continuous-time setting. This is no essential restriction, since our results can be derived analogously in a discrete-time setting. The reason to consider the continuous-time version, is that our framework will include both, models with discrete and continuous trading, as we state below. More demanding is requiring the existence of bond price for arbitrarily large T. Notice that we do not need the existence of the limiting process $\lim_{T\to\infty} B(\cdot, T)$, which in practice equals zero. As common we assume the filtration to satisfy the usual conditions.

For a given bond we consider the constant yield from holding it over the time interval $[t, T]$. This yield-to-maturity is called *bond yield* or *zero-coupon rate*, denoted by $z(t, T)$ and defined via

$$B(t, T) = \exp\left(-z(t, T)(T - t)\right). \tag{3.1}$$

The *instantaneous forward rate* is the interest rate fixed at time t for lending over the infinitesimal interval $[T, T + dt]$. It is denoted by $f(t, T)$ and connected to bond

[1] The boundedness is a t-uniform almost sure boundedness and formally given by the existence of a bound $K \in \mathbb{R}$ with $-K < B(t,T) < K$ almost surely for all $t \geq 0$. This restriction is not severe: zero-coupon bond prices are under the absence of arbitrage and negative interests rates always nonnegative and less or equal to the principal of one unit of cash. Thus the principal is a uniform bound. Also in practice such bonds prices lie between zero and one. Assuming adaption to the filtration is also natural.

prices and bond yields by

$$B(t,T) = \exp\left(-\int_t^T f(t,u)du\right),$$
$$z(t,T) = \frac{1}{T-t}\int_t^T f(t,u)du. \tag{3.2}$$

The concept of continuously compounded forward rates is a mathematical idealization, whereas bond prices and bond yields are observable in practice. Nevertheless all three concepts are, from a theoretical point of view, equivalent in defining the so-called *term structure of interest rates* or *yield curve* at time t, which relates maturity T to the bond yield $z(t,T)$. For our asymptotic view of the term structure we define the following almost sure limits:

$$\text{the } \textit{long bond yield} \quad z_L(t) := \lim_{T\to\infty} z(t,T), \tag{3.3}$$
$$\text{and the } \textit{long forward rate} \quad f_L(t) := \lim_{T\to\infty} f(t,T),$$

which do not necessarily exist. If these limits exists, i.e. the sequence of long rates converges for growing maturities, then this convergence is assumed to imply that these limits are almost surely bounded. This can be supported by the analogy to the deterministic case: A converging sequence of numbers implies that its limit is finite. An argument for the existence is given by Yao (1999b) and empirically by Malkiel (1966), who show that the yield curve levels out for growing maturity.

We now consider how bonds can be traded in our market. Here we can take three simplifying assumptions: (i) an investor trades at a finite number of dates in the continuous-time setting, (ii) an investor trades a finite number of bonds out of the infinite number of bonds available, and (iii) an investor has a finite trading horizon. This elementary setting of buy-and-hold trading of finite portfolios in finite time is a minimal requirement to construct an arbitrage strategy in order to prove our results by contradiction. Nonetheless this arbitrage also works in more complicated settings, since in continuous trading models with infinite portfolios and horizons it is naturally also possible to trade buy-and-hold of finite portfolios in finite time. Thus our results also apply to more complicated settings, as we require the minimal setting, but we also allow for more complicated trading in the meantime. In consequence, our framework embeds also continuous-time trading models with infinite portfolios and horizons. For a rigorous generalization to infinite portfolios, we refer to Björk et al. (1997). Concerning continuous-time trading there is a large literature, which culminates in Delbaen & Schachermayer (1994). It shows that stochastic integration with respect to general integrands is a powerful and sophisticated tool to model continuous trading. Technically speaking, the fact that these stochastic integrals equal Riemann-sums on the set of simple integrands, which represent buy-and-hold strategies, yields that our results extent to continuous-time trading.

The fact that our arbitrage strategies only require elementary buy-and-hold trading has three advantages. First, as mentioned above our framework includes both, models with discrete trading and models with continuous trading. Second, although continuous trading is of immense theoretical importance, in practice it is only an ideal approximation: the only possible way of trading is buy-and-hold. This gives our arbitrage strategies a more applicable character, since they do not rely on unfeasible continuous trading. Third, we can include a broader class of bond prices. In continuous trade settings semimartingales proved to be tailor-made in modeling bond prices. Delbaen & Schachermayer (1994) show that their famous concept of no-arbitrage implies that bond prices are semi-martingales. In contrast to this we may also include non-semimartingales in our setting for the following two reasons: First, there no technical problems: elementary Riemann-sums are, opposed to stochastic integrals, well-defined for non-semimartingales. Second, it can also make sense: non-semimartingales do not necessarily impose arbitrage possibilities in continuous time settings, if one permits continuous trade, as recently shown by Jarrow et al. (2008). Allowing for non-semimartingales, for example the fractional Brownian motion, is significant, since non-semimartingales appear more regularly in the empirical literature estimating price processes, see Lo (1991) and references therein.

Formally we express buy-and-hold trading of finite, say k, portfolios at a finite number m of dates in the following standard way. Let $\mathfrak{T} = (T^1, ..., T^k)$ be a vector of maturities, which identify the corresponding bonds. Moreover, an investor can invest into the numéraire B, in which the bonds are expressed.[2] By $B(\cdot, \mathfrak{T}) = \big(B(\cdot),\ B(\cdot, T^1), ...,\ B(\cdot, T^k)\big)$ we denote the $k+1$-dimensional process of the numéraire and the traded bonds. A *bond trading strategy* is a pair $(\Phi, \mathfrak{T})$, where $\Phi = (\Phi^0, ..., \Phi^k)$ is a $k+1$-dimensional simple integrand with bounded support. A *simple integrand* with bounded support is a sum of the form

$$\Phi = \sum_{l=1}^{m} f_l \,\cdot\, \mathbf{1}_{(\tau_{l-1}, \tau_l]},$$

where $0 = \tau_0 \leq \tau_1 \leq ... \leq \tau_m$ are finite stopping times and the random variables $f_l = (f_l^0, ..., f_l^k)$ are $\mathcal{F}_{\tau_{l-1}}$-measurable for $l = 1, ..., m$. At each date τ_{l-1} the agent buys f_l^0 units of the numéraire and f_l^i units of the bond $B(\cdot, T^i)$ and holds them until time τ_l. The measurability requirements exclude insider trade and clairvoyance. The finite time horizon is ensured, since τ_m is a finite stopping time. The *value* of a trading strategy $(\Phi, \mathfrak{T})$ at date u is given by the following Riemann-sums

$$\begin{aligned}\mathbf{V}(\Phi, \mathfrak{T})(u) &= \sum_{l=1}^{m} \big\langle f_l\,,\, B(u \wedge \tau_l, \mathfrak{T}) - B(u \wedge \tau_{l-1}, \mathfrak{T})\big\rangle \\ &= \sum_{l=1}^{m} \sum_{i=0}^{k} f_l^i \,\big(B(u \wedge \tau_l, T^i) - B(u \wedge \tau_{l-1}, T^i)\big),\end{aligned}$$

and accumulates gains and losses up to time u. Since the numéraire is a traded asset and its price process, priced by itself, equals one at all times, it serves as a cash box, which finances buys and sells. Hence we do not have to check, if bond trading strategies are self-financing. Notice that our results do not depend on the choice of numéraire.

[2]To consider discounted bond prices, we can change the numéraire e.g. to the money market account, given by $B(t) := \exp(\int_0^t f(u,u)du)$, or to an account rolling over certain bonds, e.g. $B(t) := \frac{B(t,[t]+1)}{\prod_{n=1}^{[t]+1} B(n-1,n)}$.

The last concept we formalize is the concept of arbitrage. Intuitively, an arbitrage is described by a risk-free strategy with a chance of "making something out of nothing". As we are interested in the behavior of limiting yields, we need to define an arbitrage trading bonds, which approximate the asymptotic bond. This is done by considering a sequence of bond trading strategies, in which all maturities increase infinitely.

Definition 21 (Arbitrage in the Limit). *A sequence of bond trading strategies* $(\Phi_j, \mathfrak{T}_j)_{j\in\mathbb{N}}$ *with* $\lim_{j\to\infty} T_j^i = \infty$ *for all* i *is called a* simple arbitrage in the limit with bounded risk (SALBR)*, if its value process satisfies the following conditions:*

$$\text{(a)}\ \boldsymbol{V}(\Phi_j, \mathfrak{T}_j)(u)^- < K_j \quad a.s.\ for\ all \quad j \in \mathbb{N},\ u \geq 0,$$

$$\text{(b)}\ \lim_{j\to\infty} \boldsymbol{V}(\Phi_j, \mathfrak{T}_j)(\tau_m) \geq 0 \quad a.s.,$$

$$\text{(c)}\ \mathbf{P}\left(\lim_{j\to\infty} \boldsymbol{V}(\Phi_j, \mathfrak{T}_j)(\tau_m) = \infty\right) > 0,$$

where $K_j \in \mathbb{R}$ *and* $\boldsymbol{V}(\Phi, \mathfrak{T})^- := \max\big(0\,, -\boldsymbol{V}(\Phi, \mathfrak{T})\big)$ *denotes the negative part. The sequence is called a* simple arbitrage in the limit with vanishing risk (SALVR)*, if it additionally satisfies*

$$\text{(d)}\ \lim_{j\to\infty} \boldsymbol{V}(\Phi_j, \mathfrak{T}_j)(u)^- = 0 \quad a.s.\ uniformly\ in\ u.$$

All limits in the definition are almost sure limits. These limits are allowed to take infinite values with positive probability such that condition (c) is satisfiable. Note that the maturities of traded bonds explode and not the trading time. The absence of arbitrage in the limit with bounded or vanishing risk in the bond market via simple integrands is denoted by "*no simple arbitrage in the limit with bounded risk*" (NSALBR) or "*no simple arbitrage in the limit with vanishing risk*" (NSALVR), respectively. Since condition (d) is additional, (NSALBR) implies (NSALVR). These two notions are inspired by classical definitions of arbitrage, as introduced in formal notation by Delbaen & Schachermayer (1994) and Kabanov & Kramkov (1994). For a detailed comparison of these notions we refer to the last section.

In the remainder of the section we discuss the intuition of the conditions given in the definition. $\mathbf{V}(\Phi, \mathfrak{T})(\tau_m)$ describes the final value or payoff from trading the strategy $(\Phi, \mathfrak{T})$, since it holds by construction $\mathbf{V}(\Phi, \mathfrak{T})(u) = \mathbf{V}(\Phi, \mathfrak{T})(\tau_m)$ for all $u \in [\tau_m, \infty)$. Condition (b) asserts that this final payoff from trading long bonds is risk-free, as it is asymptotically nonnegative. Condition (c) ensures that there is a chance to gain an arbitrary big payoff, as the final value grows to infinity with positive probability. Conditions (a) and (d) refer, besides to the final value, also to the whole trading interval. Condition (a) states that the value of each strategy is bounded from below, and thus ensures that risk is bounded uniformly over the time. This condition is often called "admissibility of the strategy" in the literature and is required to exclude trivial arbitrage opportunities by doubling strategies. Condition (d) asserts that risk vanishes, as it ensures that the negative parts of the value process converge to zero uniformly over all dates.

3.3 Asymptotic Monotonicity

In this section we prove that long bond yields and long forward rates can never fall over time in an arbitrage-free setting. Given the bond market from the previous section the following holds:

Theorem 22 (Asymptotic Monotonicity)**.** *If $z_L(s)$ and $z_L(t)$ exist almost surely for $s < t$, then under (NSALVR) it holds*

$$z_L(s) \leq z_L(t) \quad a.s.$$

Before proving the result we give some intuition: The theorem states that tomorrow's infinite bond yield can never be less than today's infinite bond yield. If this statement was wrong, the long bond yield would fall with positive probability. Then we would buy a bond with high long yield today. We sell this bond tomorrow, when it has a lower long yield with positive probability. By relation (3.1) this bond is more expensive in the limit. To capture the asymptotic long yield we consider a sequence of such trades, indexed by the maturity of the bond. We buy a precise number of shares, such that today's costs tend to zero for limiting maturity. In case the long bond yield falls, tomorrow's bond is more expensive and we have a profit with positive probability. This strategy yields a simple arbitrage in the limit with vanishing risk contradicting (NSALVR). Note that the gain is realized in finite time t, only the maturity of traded bonds tends to infinity.

Proof. We prove the almost sure-result of asymptotic monotonicity by contradiction: Assuming the long yield falls with positive probability, we will construct a simple arbitrage in the limit with vanishing risk, which contradicts the assumption of (NSALVR).

The first step in constructing this arbitrage strategy is to find a $\mathcal{F}_s$- measurable random variable y, which is smaller than $z_L(s)$ and larger than $z_L(t)$ with positive probability. Therefore we consider the essential infimum of $z_L(t)$ conditioned on $z_L(s)$. We denote it by $\operatorname{ess\,inf}\big(z_L(t)\,\big|\,z_L(s)\big)$ and it satisfies by definition

$$\operatorname*{ess\,inf}_{\omega\in A}\big(z_L(t)\,\big|\,z_L(s)\big)(\omega)=\operatorname*{ess\,inf}_{\omega\in A} z_L(t)(\omega)\quad \forall A\in\mathcal{D},$$

where $\mathcal{D}$ is the σ-algebra generated by $z_L(s)$. For an extensive discussion of conditional essential suprema we refer to Barron et al. (2003). This reference provides also the existence of $\operatorname{ess\,inf}\big(z_L(t)\,\big|\,z_L(s)\big)$ and its $\mathcal{D}$-measurability. Intuitively it is the largest $\mathcal{D}$-measurable random variable, which is almost surely smaller or equal to $z_L(t)$. We define the following set, which plays a crucial role in the proof:

$$\Omega_1 := \{z_L(s) > \operatorname{ess\,inf}\big(z_L(t)\,\big|\,z_L(s)\big)\}.$$

The set Ω_1 consists of the states, in which the long yield can potentially fall, as seen from date s, i.e. the states where the long yield at date s is almost surely higher than the lowest possible long yield at date t, which can occur after the considered state at date s. Clearly Ω_1 also contains the states, in which the long yield actually falls, i.e. $\{z_L(s) > z_L(t)\} \subset \Omega_1$. By assuming the contrary of the almost sure-result of asymptotic monotonicity, i.e.

$$\mathbf{P}(z_L(s) > z_L(t)) > 0,$$

it thus follows $\mathbf{P}(\Omega_1) > 0$. By the adaptedness of the bond yields we have $\mathcal{D} \subset \mathcal{F}_s$. This yields trivially $\operatorname{ess\,inf}\big(z_L(t)\,\big|\,z_L(s)\big) \in \mathcal{F}_s$, and hence $\Omega_1 \in \mathcal{F}_s$. Now we can define the random variable y for the first step by

$$y := \frac{1}{2}\, z_L(s) + \frac{1}{2} \operatorname{ess\,inf}\big(z_L(t)\,\big|\,z_L(s)\big).$$

By definition y is $\mathcal{F}_s$-measurable and it holds

$$z_L(s) > y > \operatorname{ess\,inf}\big(z_L(t)\,\big|\,z_L(s)\big) \quad \text{on} \quad \Omega_1, \tag{3.4}$$

which means that, if the long yield at date s can potentially fall, it is larger than y. The convergence of the long yields, which is given by assumption, implies that the limits $z_L(s)$ and $z_L(t)$ are almost surely bounded, as stated after equation (3.3). Since $\operatorname{ess\,inf}\big(z_L(t)\,\big|\,z_L(s)\big)$ is smaller or equal to $z_L(t)$ and bounded from below by the lower bound of $z_L(t)$, it is also almost surely bounded and so is y by construction. To finalize the first step it remains to show, that y is larger than $z_L(t)$ with positive probability. Therefore we assume the contrary, i.e. $y \le z_L(t)$ almost surely. Proposition 2.6.b in Barron et al. (2003) then yields $y \le \operatorname{ess\,inf}\big(z_L(t)\,\big|\,z_L(s)\big)$ almost surely. This expresses the intuition that the conditional essential infimum is the largest $\mathcal{D}$-measurable random variable, which is almost surely smaller or equal to $z_L(t)$, and thus larger than y. Moreover it contradicts the last inequality of (3.4) in the light of $\mathbf{P}(\Omega_1) > 0$, and we obtain

$$\mathbf{P}(y > z_L(t)) > 0. \tag{3.5}$$

Having completed the first step we are in position to construct the arbitrage in the limit strategy. Therefore we consider the following sequence of bond trading strategies $(\Phi_j, \mathfrak{T}_j)_{j\in\mathbb{N}}$, given by

$$\Phi_j^0(u) := 0, \quad \Phi_j^1(u) := \exp(y(j-s))\ \mathbf{1}_{\Omega_1}\,\mathbf{1}_{(s,t]}(u), \quad \mathfrak{T}_j := (j),$$

for all $j \in \mathbb{N}$. Each strategy of the sequence is a trivial buy-and-hold trade: at date s we buy $\exp(y(j-s))$ units of the bond with maturity j and sell them at date t. We only trade on the set Ω_1. Note that on the complement of Ω_1 asymptotic monotonicity holds trivially. We first show that each strategy is predictable, a property which is questioned in Hubalek et al. (2002). Since the only buying date is date s, it suffices to show that $\Phi(s+)$ is $\mathcal{F}_s$-measurable. This is the case since y and Ω_1 are $\mathcal{F}_s$-measurable, as we already pointed out. It remains to show that the sequence constitutes an arbitrage in the limit with vanishing risk and we consider the value of the strategy

$$\mathbf{V}(\Phi_j, \mathfrak{T}_j)(u) = \exp(y(j-s))\left(B(u\wedge t, j) - B(u \wedge s, j)\right)\mathbf{1}_{\Omega_1}.$$

By the above mentioned almost sure boundedness of y the number of shares $\exp(y(j-s))$ is almost surely bounded for any given j. Since bounded prices are almost sure u-uniformly bounded by definition, the value process $\mathbf{V}(\Phi_j, \mathfrak{T}_j)(u)$ is almost sure u-uniformly bounded for every j by construction. Thus risk is bounded and condition (a) of Definition 21 is satisfied. The value process allows for the trivial representation by the difference $\mathbf{V}(u) = M(u) - S(u)$, with minuend M and subtrahend S, given by

$$\begin{aligned}
M_j^\Phi(u) &:= \exp(y(j-s))\, B(u,j)\,\mathbf{1}_{\Omega_1}\,\mathbf{1}_{(s,t]}(u) + \exp(y(j-s))\, B(t,j)\,\mathbf{1}_{\Omega_1}\,\mathbf{1}_{(t,\infty)}(u)\\
S_j^\Phi(u) &:= \exp(y(j-s))\, B(s,j)\,\mathbf{1}_{\Omega_1}\,\mathbf{1}_{(s,\infty)}(u)\\
&= \exp(y(j-s))\ \exp(-z(s,j)(j-s))\,\mathbf{1}_{\Omega_1}\,\mathbf{1}_{(s,\infty)}(u)\\
&= \exp\left((y - z(s,j))(j-s)\right)\mathbf{1}_{\Omega_1}\,\mathbf{1}_{(s,\infty)}(u) \qquad (3.6)
\end{aligned}$$

The term (3.6) is derived by relation (3.1). It converges almost surely to zero for $j \to \infty$, since $z_L(s)$ converges almost surely and $z_L(s) > y$ holds on Ω_1 by equation (3.4). Hence the subtrahend S converges to zero uniformly in u. By the nonnegativity of the minuend M it follows $\mathbf{V}^- \leq S$. Thus risk is vanishing and condition (d) of definition 21 is satisfied. The zero-convergence of the subtrahend implies in particular for the final date t

$$\begin{aligned}\lim_{j\to\infty} \mathbf{V}(\Phi_j, \mathfrak{T}_j)(t) = \lim_{j\to\infty} M_j(t) = \lim_{j\to\infty} \exp(y(j-s))\, B(t,j)\, \mathbf{1}_{\Omega_1} \\ = \lim_{j\to\infty} \exp(y(j-s))\, \exp(-z(t,j)(j-t))\, \mathbf{1}_{\Omega_1}.\end{aligned}$$

By the nonnegativity of the last term condition (b) of Definition 21 is met. The almost sure existence of the limiting final payoff is ensured by the convergence of $z_L(t)$. Moreover this last term grows unboundedly for $j \to \infty$ on the set $\{y > z_L(t)\}$. Since this set is a subset of Ω_1 and since it has a positive probability by equation (3.5), condition (c) of Definition 21 is satisfied. Consequently the sequence $(\Phi_j, \mathfrak{T}_j)_{j\in\mathbb{N}}$ constitutes a simple arbitrage in the limit with vanishing risk, which contradicts the assumption of (NSALVR) and closes the proof. □

As shown in Dybvig et al. (1996), Theorem 1 for a discrete time setting or easily derived by relation (3.2), $f_L(t)$ equals $z_L(t)$ almost surely, if $f_L(t)$ exists almost surely. As a result, the following corollary states that the infinite forward rate cannot fall over time.

Corollary 23 (Asymptotic Monotonicity). *If $f_L(s)$ and $f_L(t)$ exist almost surely for $s < t$, then under (NSALVR) it holds*

$$f_L(s) \leq f_L(t) \quad a.s.$$

3.4 Asymptotic Minimality

In the previous section we have seen that the long bond yield cannot fall over time. It certainly may rise, what can be seen e.g. by considering the discrete binomial model by Ho & Lee (1986), whose long yield rises with positive Bernoulli-probability at each discrete time point. But may it rise almost surely? This question is denied by the following result: the long bond yield always equals its minimal future value. Thus today's long bond yield cannot be bounded away from the support of possible future long yields and so jumps with probability one are excluded.

Theorem 24 (Asymptotic Minimality)**.** *If $z_L(s)$ and $z_L(t)$ exist almost surely for $s < t$, then under (NSALBR) it holds*

$$z_L(s) = \operatorname{ess\,inf}\big(z_L(t) \,\big|\, z_L(s)\big) \quad a.s.$$

Before proving the result we again give some intuition: If the theorem was wrong, tomorrow's long bond yield would rise with probability one due to asymptotic monotonicity. Then we could short-sell a bond today and liquidate it tomorrow, when it is cheaper in the limit, since the long yield is higher. We consider a sequence of such trades, which is indexed by maturity. We sell a precise number of shares, such that we are paid today to enter the position and have asymptotically no costs tomorrow. This arbitrage opportunity is realized in finite time and contradicts the assumption.

Proof. We start by observing a direct corollary of asymptotic monotonicity. Under the given assumptions it holds

$$z_L(s) \leq \operatorname{ess\,inf}\big(z_L(t) \,\big|\, z_L(s)\big) \quad \text{a.s.}$$

To show the remaining inequality, we assume its contrary, i.e. $\mathbf{P}(\Omega_2) > 0$, with

$$\Omega_2 = \big\{z_L(s) < \operatorname{ess\,inf}\big(z_L(t) \,\big|\, z_L(s)\big)\big\}.$$

The set Ω_2 consists of the sates, in which the long rate at date s is strictly less than the lowest possible long rate at date t as seen from date s. The first step in constructing the arbitrage strategy is to find a $\mathcal{F}_s$-measurable random variable, which is larger than $z_L(s)$ and smaller than $z_L(t)$ with positive probability. As we will see, the random variable y from the previous proof of Theorem 22 also does this job. Since the reasoning here is similar to the previous proof, we provide a condensed version. By definition we have $\operatorname{ess\,inf}\big(z_L(t)\,\big|\,z_L(s)\big) \in \mathcal{F}_s$, and hence $\Omega_2 \in \mathcal{F}_s$. We recall the random variable y for the first step:

$$y = \frac{1}{2}\,z_L(s) + \frac{1}{2}\operatorname{ess\,inf}\big(z_L(t)\,\big|\,z_L(s)\big).$$

By definition y is $\mathcal{F}_s$-measurable and we have the following ordering on Ω_2

$$z_L(s) < y < \operatorname{ess\,inf}\big(z_L(t)\,\big|\,z_L(s)\big) \leq z_L(t). \tag{3.7}$$

Having completed the first step we construct the arbitrage strategy. Therefore we consider the following sequence of bond trading strategies $(\Psi_j, \mathfrak{T}_j)_{j\in\mathbb{N}}$, given by

$$\Psi_j^0(u) := 0, \quad \Psi_j^1(u) := -\exp(y(j-s))\;\mathbf{1}_{\Omega_2}\,\mathbf{1}_{(s,t]}(u), \quad \mathfrak{T}_j := (j),$$

for all $j \in \mathbb{N}$. Each strategy of the sequence is a trivial buy-and-hold trade: at date s we sell $\exp(y(j-s))$ units of the bond with maturity j to rebuy them at date t. We only trade on the set Ω_2, as on the complement of Ω_2 asymptotic minimality holds trivially. Each strategy is predictable, since y and Ω_2 are $\mathcal{F}_s$-measurable. To see that this sequence constitutes a simple arbitrage in the limit with bounded risk, we consider the value process, which equals for all j

$$\mathbf{V}(\Psi_j, \mathfrak{T}_j)(u) = -\exp(y(j-s))\,\big(B(u\wedge t, j) - B(u\wedge s, j)\big)\,\mathbf{1}_{\Omega_2}.$$

The number of shares $\exp(y(j-s))$ is bounded for any given j, since y is bounded by equation (3.7) and the convergence of the bond yields. As bond-prices are u-uniformly bounded by definition, the value process is u-uniformly bounded for every j and thus satisfies condition (a) of Definition 21. To close the proof we consider the final value at date t

$$\begin{aligned}\mathbf{V}(\Psi_j, \mathfrak{T}_j)(t) &= -\exp(y(j-s))\,\big(B(t,j) - B(s,j)\big)\,\mathbf{1}_{\Omega_2}\\ &= \exp(y(j-s))\,\exp(-z(s,j)(j-s))\,\mathbf{1}_{\Omega_2}\\ &\quad -\exp(y(j-s))\,\exp(-z(t,j)(j-t))\,\mathbf{1}_{\Omega_2} \qquad (3.8)\\ &=: M_j^{\Psi}(t) - S_j^{\Psi}(t).\end{aligned}$$

Due to the inequality $y < z_L(t)$ in (3.7) the subtrahend $S_j^{\Psi}(t)$ converges almost surely to zero on the set Ω_2. By the nonnegativity of the minuend $M_j^{\Psi}(t)$ condition (b) of Definition 21 is met. Moreover, the minuend $M_j^{\Psi}(t)$ explodes on Ω_2 by the inequality $z_L(s) < y$ in (3.7). As we have $\mathbf{P}(\Omega_2) > 0$ by assumption, condition (c) of Definition 21 is also satisfied. Consequently $(\Psi_j, \mathfrak{T}_j)_{j\in\mathbb{N}}$ is a simple arbitrage in the limit with bounded risk, which completes the proof. □

The result of asymptotic minimality can again be expanded to long forward rates.

Corollary 25 (Asymptotic Minimality). *If $f_L(s)$ and $f_L(t)$ exist almost surely for $s < t$, then under (NSALBR) it holds*

$$f_L(s) = \operatorname{ess\,inf}\big(f_L(t) \,\big|\, f_L(s)\big) \quad a.s.$$

3.5 The Literature

In the first part of this section we compare the literature on asymptotic monotonicity to our approach. In the second part of the section we refer to asymptotic minimality and clarify an apparent contradiction to existing literature. In the third part we compare our notions of arbitrage to closely-related classical notions and analyze, if our results can be transferred to these notions.

3.5.1 Asymptotic Monotonicity

Since several authors worked on the topic of asymptotic monotonicity, we concentrate on the papers, which are closely related to ours. The most important contribution to asymptotic monotonicity is by Dybvig et al. (1996). They came up with the genuine idea of showing this result by a no arbitrage-argument and initiated the proceeding literature. Since the emphasis in their paper is on intuition, we aim to make their original idea more rigorous in this chapter. For this purpose, we provide a stringent formal setting, in which all objects are defined in mathematical terms.

Compared to Dybvig et al. (1996), this setting is extended in several aspects. We stress out two aspects: First by using a continuous-time setting we also allow for continuous trading, which is of vast theoretical importance. Second, whereas in Dybvig et al. (1996) all dates, except for date t, are deterministic, our setting is a filtered probability space, in which all dates are stochastically modeled.

MacCollugh (2000) comments on Dybvig et al. (1996) and states that the proof is defective, since it includes an error. This critic is valid, but it only refers to the proof in the body of Dybvig et al. (1996). The proof in the Appendix of Dybvig et al. (1996) is not affected, since the problem stems from the set $\{y = \operatorname{ess\,inf}\big(z_L(t) \,\big|\, z_L(s)\big)\}$, where the invested amount equals the essential infimum, and y is chosen to be strictly greater than the infimum. Neither our proof is affected for the same reason.

Yao (1999a) derives asymptotic monotonicity rigorously under additional assumptions in a jump-diffusion context.

There is another approach to prove asymptotic monotonicity: Hubalek et al. (2002) provide a stringent proof by assuming the existence of an equivalent martingale measure. In contrast to this we approach arbitrage in a less abstract way by a positive definition. We see two advantages in this approach: First by introducing the concrete notion of arbitrage, we can construct the arbitrage strategy explicitly and tell an arbitrageur to buy how many of which bonds. Hence our proof is more illustrative. Second we do not require the fundamental theorem of asset pricing and, thus we are able to leave the common semimartingale-setting. As a result, whereas in the setting of Hubalek et al. (2002) bond prices are modeled as semimartingales, our setting may also include non-semimartingales. Without affecting the validity of their elegant and conveniently brief proof, Hubalek et al. (2002) criticize Dybvig et al. (1996) erroneously in proposing that the strategy is anticipative. They state that in consequence one has to assume implicitly in Dybvig et al. (1996) that the long bond yield at time t, denoted by $z_L(t)$, is $\mathcal{F}_s$-measurable for $s < t$. But this assumption is not necessary, since the strategy in Dybvig et al. (1996) is not anticipating: The strategy does not depend on $z_L(t)$, but on its unconditioned essential infimum. The essential infimum is in turn a property of the distribution, which does not depend on the realization of $z_L(t)$. In the setting of Dybvig et al. (1996), with stochastic modeling only at date t, this infimum is just a number and its measurability poses no problem. In our general setting it neither poses a problem. By considering the essential infimum conditioned on $z_L(s)$, which is hence known at date s, we can construct an arbitrage strategy, which is not anticipating.

3.5.2 Asymptotic Minimality

Dybvig et al. (1996) also address the second result of asymptotic minimality. They derive it in spaces with a finite number of states. On the other hand, they provide a counter-example for infinite spaces. In this section we briefly clarify the resulting contradiction to our Theorem 24 and we show that it is only ostensible. The reason for the contradiction does not lie in the different setting of the bond market mentioned above. It is solely located in the definition of arbitrage. In Dybvig et al. (1996) this definition is presented in an intuitive way. The crucial difference to our notions of arbitrage is the claim for ω-uniform convergence. The counter-example (see Example 2 in Dybvig et al. (1996)) presents a long yield, which is almost surely, but not ω-uniformly almost surely converging. In consequence the corresponding arbitrage strategies are neither ω-uniformly converging and asymptotic minimality cannot be proved under the no-arbitrage notion of Dybvig et al. (1996) in general spaces. However, in finite spaces almost sure convergence implies uniform convergence and asymptotic minimality is shown in Dybvig et al. (1996) for finite spaces.[3]

Opposed to this, asymptotic minimality is established in general spaces under

[3]Assuming that the bond yield is ω-uniformly almost surely converging at date s and t, the counter example is excluded and asymptotic minimality can be derived for general spaces under the no-arbitrage notion of Dybvig et al. (1996) analogously to the finite space proof.

(NSALBR) in section 4. We show that there is arbitrage in the (NSALBR)-sense in the counter-example. Therefore we consider the almost sure limits of the bond yield in the counter-example, which equal in continuous-time notion

$$z_L(s) = \min(r_2, r_1 - \log(1-p)), \qquad z_L(t) = r_2,$$

where $r_1, r_2 > 0$ and $p \in (0,1)$. If it holds $r_2 \leq r_1 - \log(1-p)$, asymptotic minimality holds trivially. If it holds $r_2 > r_1 - \log(1-p)$, the long yield is strictly growing between dates s and t. Then by the choice of $y = \frac{1}{2}(r_1 + r_2 - \log(1-p))$ equation (3.7) holds true and the strategies $(\Psi_j, \mathfrak{T}_j)_{j\in\mathbb{N}}$ constitute a simple arbitrage with bounded risk, as derived in the proof of Theorem 24.

3.5.3 Related Notions of Arbitrage

There are several notions of arbitrage in the literature. We compare our Definition 21 to those definitions of arbitrage, which are closely related. Moreover we discuss under which conditions asymptotic monotonicity and minimality hold true under these classical notions.

We start with the notions of Delbaen & Schachermayer (1994). They refer to a stock price process and the integrator is fixed throughout the sequence. In our notion the integrator is a bond price and it varies within the sequence to approximate the limiting bond. Nonetheless we transfer the two notions of Delbaen & Schachermayer (1994) and their convergence criteria to a bond market and interpret it for bond prices with limiting maturity in our notation.

Definition 26 (No free lunch with bounded or vanishing risk)**.** *A sequence of bond trading strategies* $(\Phi_j, \mathfrak{T}_j)_{j\in\mathbb{N}}$ *with* $\lim_{j\to\infty} T_j^i = \infty$ *for all* i *is called a* free lunch with bounded risk (FLBR), *if its value process satisfies the following conditions:*

$$\text{(a)}\quad \boldsymbol{V}(\Phi_j, \mathfrak{T}_j)(u)^- < K_j \quad \textit{a.s. for all} \quad j \in \mathbb{N},\ u \geq 0$$

$$\text{(b)}\quad \lim_{j\to\infty} \boldsymbol{V}(\Phi_j, \mathfrak{T}_j)(\tau_m) \geq 0 \quad \textit{a.s.},$$

$$(\text{c}^{\text{DS}})\ \mathbf{P}\left(\lim_{j\to\infty} \boldsymbol{V}(\Phi_j, \mathfrak{T}_j)(\tau_m) > 0\right) > 0,$$

where $K_j \in \mathbb{R}$. *The sequence is called a* free lunch with vanishing risk (FLVR)*, if it additionally satisfies*

$$(\text{d}^{\text{DS}})\ \lim_{j\to\infty} \boldsymbol{V}(\Phi_j, \mathfrak{T}_j)(\tau_m)^- = 0 \quad \textit{a.s. uniformly in } \omega.$$

Notice that condition (a) is the admissibility condition on strategies in Delbaen & Schachermayer (1994). Together with condition (b) it yields that risk of the final value is j-uniformly bounded, which is central for the intuition of (FLBR).[4] The interpretation of (NFLBR) in the upper definition shows that there is essentially no difference to (NSALBR). Since condition (c^{DS}) is slightly weaker than condition (c), asymptotic monotonicity and asymptotic minimality hold true under (NFLBR).

The crucial difference between the vanishing risk notions, is that (FLVR), except for the admissibility condition (a), only considers the final value of the strategy, whereas (SALVR) also imposes restrictions on the value of all dates, as risk is vanishing uniformly over the trading period. In this sense (NSALVR) is a weaker assumption as (NFLVR). On the other hand, condition (d^{DS}) of (FLVR) calls additionally for almost sure ω-uniform convergence, compared to (SALVR). Thus we consider in the following lemma, when ω-uniform convergence of risk is ensured.

Lemma 27 (Uniform Convergence in ω)**.** *If $z_L(s)$ and $z_L(t)$ exist as almost sure ω-uniform limits, then the negative parts of the arbitrage strategies $(\Phi_j, \mathfrak{T}_j)_{j\in\mathbb{N}}$ and*

[4]More formally: By (b) the almost sure lower bounds K_j on the final value $\mathbf{V}(\Phi_j, \mathfrak{T}_j)(\tau_m)$ converge to zero and so there exists a $J \in \mathbb{N}$ with $K_j < 1$ for all $j > J$. By (a) we have $\mathbf{V}(\Phi_j, \mathfrak{T}_j)(\tau_m)^- < K_j \in \mathbb{R}$ for all $j \in \mathbb{N}$. In consequence we have $\mathbf{V}(\Phi_j, \mathfrak{T}_j)(\tau_m)^- < K$ almost surely with $K := \max(1, L)$ and $L := \max_{j<J} K_j < \infty$.

$(\Psi_j, \mathfrak{T}_j)_{j\in\mathbb{N}}$ *of the previous proofs satisfy:*

$$\lim_{j\to\infty} \boldsymbol{V}(\Phi_j, \mathfrak{T}_j)(u)^- = 0 \quad \textit{a.s. uniformly in } \omega \textit{ and in } u \geq 0,$$

$$\lim_{j\to\infty} \boldsymbol{V}(\Psi_j, \mathfrak{T}_j)(t)^- = 0 \quad \textit{a.s. uniformly in } \omega.$$

Proof. Notice that by $\mathbf{V}^- \leq S$ the negative parts converge almost sure ω-uniformly, if the respective subtrahends of the decomposition given in the proofs converge almost sure ω-uniformly. The limit of the subtrahend of $S_j^\Phi(u)$ depends crucially on the limit $z_L(s)$ solely: If $z(s,j)$ converges almost sure ω-uniformly, so it does $S_j^\Phi(u)$, confer equation (3.6). Moreover this convergence is uniform in u. Analogously, if $z(t,j)$ converges almost sure ω-uniformly, so it does $S_j^\Psi(t)$, confer equation (3.8). However, $S_j^\Psi(u)$ for arbitrary $u \geq 0$ depends on the convergence behavior of $z(u,j)$ with $u \in (s,t]$, and thus its risk is not necessarily vanishing. □

By this lemma arbitrage strategies $(\Phi_j, \mathfrak{T}_j)_{j\in\mathbb{N}}$ and $(\Psi_j, \mathfrak{T}_j)_{j\in\mathbb{N}}$ satisfy condition (d^{DS}), since risk of the final value is uniformly converging to zero. In consequence Asymptotic Monotonicity and Asymptotic Minimality hold true under (NFLVR) for ω-uniformly converging yields.

Finally we address the notion of *asymptotic arbitrage*, which is introduced by Kabanov & Kramkov (1994). The idea of asymptotic arbitrage already appears in the arbitrage pricing theory, which is introduced by Ross (1976) and extended e.g. by Huberman (1982). Kabanov & Kramkov (1994) and Kabanov & Kramkov (1998) define a large financial market by a sequence of market models. To transfer our bond market of Definition 20 to this setting, we consider the sequence of market models, which consists of copies of our probability space $B_j = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ for all $j \in \mathbb{N}$. The price process of the j-th market S_j is given by the bond price $B(\cdot, j)$. So we are able to rewrite asymptotic arbitrage in our notation.

Definition 28 (Asymptotic Arbitrage)**.** *A sequence of bond trading strategies* $(\Phi_j, \mathfrak{T}_j)_{j\in\mathbb{N}}$ *with* $\mathfrak{T}_j = (j)$ *is called a* asymptotic arbitrage of the first kind (AA1)*, if its value* *process satisfies the following conditions:*

$$\text{(a)} \quad \boldsymbol{V}(\Phi_j, \mathfrak{T}_j)(u)^- < K_j \quad a.s.\ \textit{for all} \quad j \in \mathbb{N},\ u \geq 0,$$

$$(\mathrm{b}^{\mathrm{KK}}) \ \lim_{j\to\infty} K_j = 0,$$

$$(\mathrm{c}^{\mathrm{KK}}) \ \lim_{j\to\infty} \mathbf{P}\left(\boldsymbol{V}(\Phi_j, \mathfrak{T}_j)(\tau_m) > 1\right) > 0,$$

where $K_j \in \mathbb{R}$.

Condition (a) is the admissibility condition on strategies. Condition (b^{KK}) asserts that risk is vanishing: There are lower bounds on an asymptotic arbitrage, which are uniform over Ω and time and converge to zero. Thus (AA1) is a stronger notion compared to (SALBR). It it also stronger than (SALVR) in the sense that it claims additionally for ω-uniform convergence of vanishing risk. Strictly speaking it is weaker than (SALVR) in the sense that the almost sure existence of the limiting payoff is not asserted and e.g. an oscillating value is permitted. Finally condition (c^{KK}) is essentially the same as condition (c). Since Lemma 27 ensures that the risk of the arbitrage strategies $(\Phi_j, \mathfrak{T}_j)_{j\in\mathbb{N}}$ is vanishing uniformly over Ω and time, asymptotic monotonicity holds true under (NAA1) for ω-uniformly converging yields $z_L(s)$ and $z_L(t)$. However, asymptotic minimality does not necessarily hold true under (NAA1).

3.6 Conclusion

In order to analyze the long-term behavior of the term structure we consider families of almost arbitrary bond prices with infinitely increasing maturities. In this general framework for term structure models we derive the Dybvig-Ingersoll-Ross result of non-falling long bond yields. Introducing a notion of arbitrage we prove this result by constructing an arbitrage strategy explicitly. This strategy requires only a minimal setting of buy-and-hold trading and is not anticipating, as proposed in the literature. Furthermore we extend the second Dybvig-Ingersoll-Ross result to general spaces: Long bond yields and forward rates equal their minimal future value. Both results impose restrictions on arbitrage-free term structure models, since they exclude a multitude of asymptotic maturity behavior. These severe implications serve as caution for modelers that not every specification is consistent with the considered notions of arbitrage. Specifically, setting up a long yield, which decreases with positive probability or increases almost surely, imposes arbitrage opportunities.

Chapter 4

Appendices

A.1 Moment-Generating Function and Integral Transforms

The central question of Chapter 1, the existence and computation of the index of riskiness, is focused in Problem 1. In this section we consider the relation of Problem 1 to the moment-generating function and several integral transforms. These connections reduce Problem 1 to the simpler problem of inverting integral transforms.

The well-known *moment-generating function* is the expectation $\mathrm{E}e^{\alpha X}$ and provides a unifying approach to solve Problem 1, regardless if X is continuous or discrete. If the position $X \in \mathfrak{X}$ has a continuous density function f_X, the moment-generating function is defined by

$$\mathfrak{M}(X, \alpha) := \int_{-\infty}^{\infty} e^{\alpha x} f_X(x)\, dx,$$

whereas for discrete positions $X \in \mathfrak{X}$ it is given by

$$\mathfrak{M}(X, \alpha) := \sum_{x \in \mathrm{Im}(X)} e^{\alpha x}\, \mathcal{P}(X = x).$$

In both cases we have the relation

$$\mathrm{E}e^{-\alpha X} = \mathfrak{M}(X, -\alpha).$$

In consequence a positive root of Problem 1 is given by α^* with

$$\mathfrak{M}(X, -\alpha^*) = 1,$$

and can be expressed more explicitly by

$$\alpha^*(X) = -\mathfrak{M}^{-1}(X, 1),$$

where $\mathfrak{M}^{-1}$ is the inverse of the moment-generating function in argument α mapping into the negative numbers. The moment-generating function receives its name from the moment-relation

$$\frac{\delta^n\,\mathfrak{M}(X, 0)}{\delta\alpha^n} = \mathrm{E}X^n.$$

Moreover, for $X \neq 0$ almost surely the moment-generating function is strictly convex in α, where it exists. In the continuous case this follows by the continuity of the exponential:

$$\frac{\delta^2\,\mathfrak{M}(X, \alpha)}{\delta\alpha^2} = \int_{-\infty}^{\infty} x^2 e^{\alpha x}\, f_X(x)\, dx > 0.$$

In the discrete case this follows directly by

$$\frac{\delta^2\,\mathfrak{M}(X, \alpha)}{\delta\alpha^2} = \sum_{x \,\in\, \mathrm{Im}(X)} x^2 e^{\alpha x}\, \mathcal{P}(X = x) > 0.$$

By these two properties the proofs in chapter 1 holds analogously for the moment-generating function instead of the Laplace transform. In consequence our results of chapter 1 hold for all positions $X \in \mathfrak{X}$, regardless if X is continuous or discrete.

In the following we assume that the density function f_X is continuous. As derived in chapter 1 Problem 1 is closely related to the bilateral Laplace transform by

$$\mathrm{E}e^{-\alpha X} = \mathfrak{L}_X(\alpha).$$

Moreover, there are further similar relations to integral transforms. First, we consider the *unilateral* Laplace transform, given by

$$\mathfrak{L}^1(f(x), \alpha) := \int_0^{\infty} e^{-\alpha x} f(x)\, dx.$$

Problem 1 is related to the unilateral Laplace transform by

$$\mathrm{E}e^{-\alpha X} = \mathfrak{L}^1(f_X(x), \alpha) + \mathfrak{L}^1(f_X(-x), -\alpha).$$

Second, the connection to the *characteristic function*, and the continuous *Fourier transform*, defined by

$$\begin{aligned} \phi_X(\alpha) &:= \int_{-\infty}^{\infty} e^{-i\alpha x} f_X(x)\, dx, \\ \mathfrak{F}(f, \alpha) &:= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\alpha x} f(x)\, dx, \end{aligned}$$

is given by

$$\begin{aligned} \mathrm{E}e^{-\alpha X} &= \phi_X(-i\alpha), \\ \mathrm{E}e^{-\alpha X} &= \sqrt{2\pi}\, \mathfrak{F}(f_X, -i\alpha). \end{aligned}$$

While inverting the characteristic function and the Fourier transform we have to restrict to non-complex solutions. Third, for the *Mellin transform*, given by

$$M(f, \alpha) := \int_0^{\infty} x^{\alpha-1} f(x)\, dx,$$

it holds

$$\mathrm{E}e^{-\alpha X} = M(f_X(-\log x), \alpha).$$

Inverting one of the considered integral transforms helps to solve Problem 1, since the theory on these various integral transforms is highly sophisticated.

B.1 The Lambert W-Function

We consider the Lambert W-function, which is also called *product log* or *Omega function*. It is the inverse function of

$$f(x) = x\, e^x = z,$$

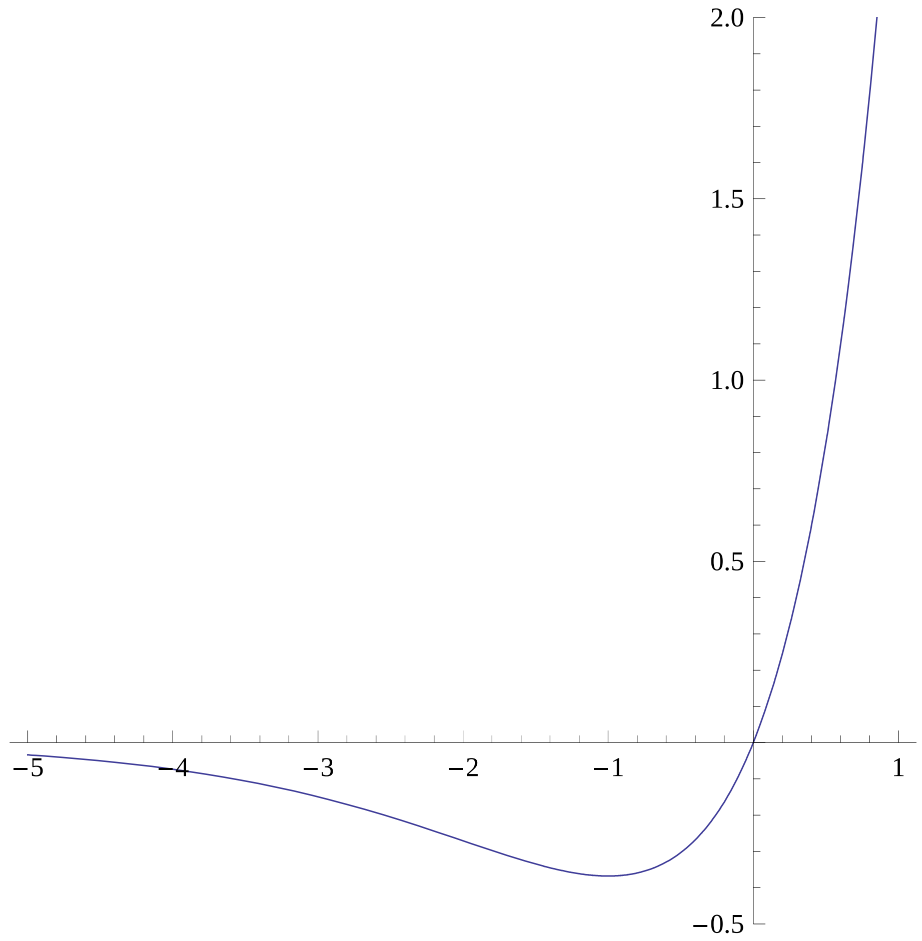


Figure 4.1: The Function $f(x) = x\,e^x$.

which is plotted in Figure 4.1. We denote the *Lambert W-function* by $\mathfrak{W}$ and it satisfies for the above arguments x and z

$$\mathfrak{W}(z) = x.$$

This provides the relation we frequently use

$$\mathfrak{W}(z)\,e^{\mathfrak{W}(z)} = z. \tag{4.1}$$

We consider the Lambert W-function for real arguments x and z and do not refer to complex solutions for the inversion of a complex function f. Since the real function $f : \mathbb{R} \to \mathbb{R}$ is not injective on $(-\infty, 0]$, the real Lambert W-function is multivalued.

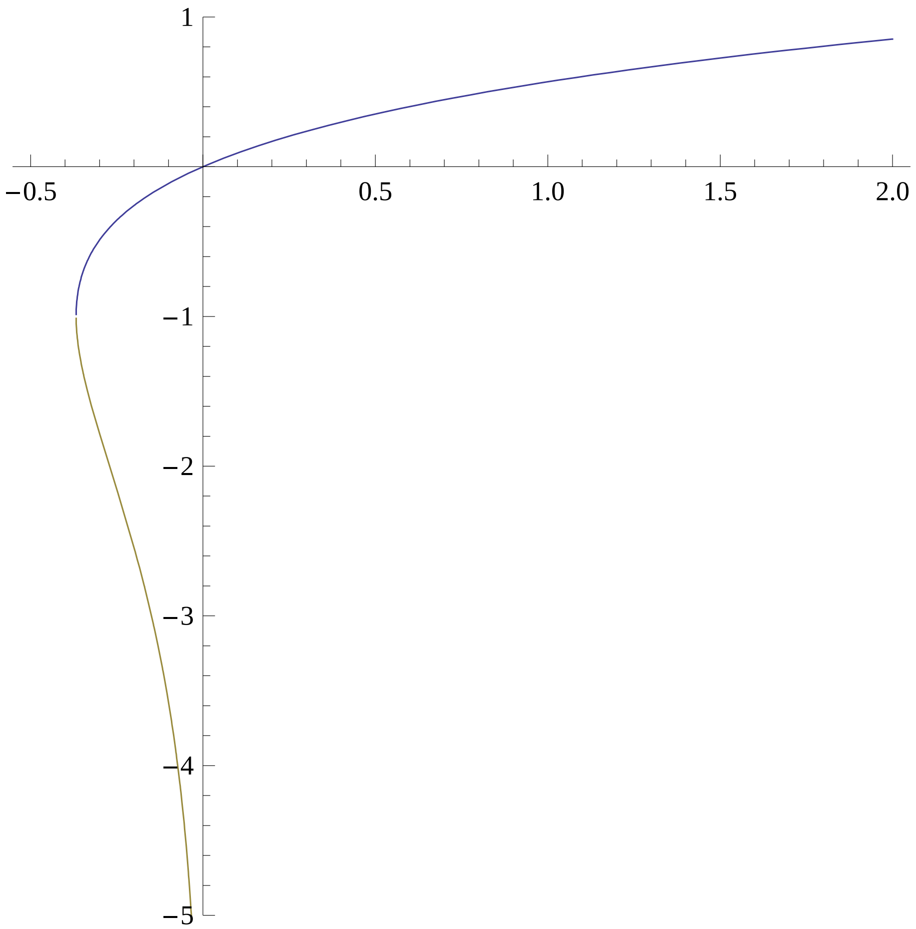


Figure 4.2: The Two Branches of the Real Lambert W-Function.

It consists of two single-valued branches: the principal branch $\mathfrak{W}_0 : [-\frac{1}{e}, \infty) \to [-1, \infty)$ and the alternate branch $\mathfrak{W}_{-1} : (-\frac{1}{e}, 1) \to (-\infty, -1)$. These two branches are plotted in Figure 4.2. The Lambert W-function $\mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is two-valued on the interval $(-\frac{1}{e}, 0)$. The number of values depends on argument z and is formally given by

$$\mathfrak{W}(z) = \begin{cases} (\emptyset, \emptyset) & \text{for} \quad z < -\frac{1}{e}, \\ (\mathfrak{W}_0, \mathfrak{W}_1) & \text{for} \quad -\frac{1}{e} < z < 0, \\ (\mathfrak{W}_0, \emptyset) & \text{for} \quad z = -\frac{1}{e} \quad \text{or} \quad z \geq 0. \end{cases}$$

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