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1. Introduction

In this thesis we construct an additive category whose objects are embedded graphs (or in particular knots) in the 3-sphere and where morphisms are formal linear combinations of 3-manifolds. Our definition of correspondences relies on the Alexander branched covering theorem [**1**], which shows that all compact oriented 3-manifolds can be realized as branched coverings of the 3-sphere, with branched locus an embedded (not necessarily connected) graph. The way in which a given 3-manifold is realized as a branched cover is highly not unique. It is precisely this lack of uniqueness that makes it possible to regard 3-manifolds as correspondences. In fact, we show that, by considering a 3-manifold **M** realized in two different ways as a covering of the 3-sphere as defining a correspondence between the branch loci of the two covering maps, we obtain a well defined associative composition of correspondences given by the fibered product.
An equivalence relation between correspondences given by 4-dimensional cobordisms is introduced to conveniently reduce the size of the spaces of morphisms. We construct a 2-category where morphisms are coverings as above and 2-morphisms are cobordisms of branched coverings. We discuss how to pass from embedded graphs to embedded links using the relation of *b*-homotopy on branched coverings, which is a special case of the cobordism relation.
We associate to the set of correspondences with composition a convolution algebra and we describe natural time evolutions induced by the multiplicity of the covering maps. We prove that, when considering correspondences modulo the equivalence relation of cobordism, this time evolution is generated by a Hamiltonian with discrete spectrum and finite multiplicity of the eigenvalues.
Similarly, in the case of the 2-category, we construct an algebra of functions of cobordisms, with two product structures corresponding to the vertical and horizontal composition of 2-morphisms. We consider a time evolution on this algebra, which is compatible with the vertical composition of 2-morphism given by gluing of cobordisms, that corresponds to the Euclidean version of Hartle–Hawking gravity. This has the effect of weighting each cobordism according to the corresponding Einstein–Hilbert action.
We also show that evolutions compatible with the vertical composition of 2-morphisms can be obtained from the linearized version of the gluing formulae for gauge theoretic moduli spaces on 4-manifolds. The linearization is given by an index theorem and this suggests that time evolutions compatible with both the vertical and horizontal compositions may be found by considering an index pairing for the bivariant Chern character on KK-theory classes associated to the geometric correspondences. We outline the argument for such a construction. Our category constructed using 3-manifolds as morphisms is motivated by the problem of developing a suitable notion of *spectral correspondences* in noncommutative geometry, outlined in the last chapter of the book [**17**]. The spectral correspondences described in [**17**] will be the product of a finite noncommutative geometry by a "manifold part".
The latter is a smooth compact oriented 3-manifold that can be seen as a correspondence in the sense described in the present paper. We discuss the problem of extending the construction presented here to the case of products of manifolds by finite noncommutative spaces in the last section of the first chapter.

Chapter two begins with a discussion of how to pass from the case where the branch loci of the coverings are embedded multi-connected graph to more special case where these loci are links and knots. This is achieved using the "Alexander trick" and the equivalence relation of *b*-homotopy of branched covering. Passing to knots and links allows us to make use in our context of some invariants and known constructions for knots and links and investigate analogs for embedded graphs. An interesting homology theory for knots and links that we consider here is the one introduced by

Khovanov in [**43**]. We recall the basic definition and properties of Khovanov homology and we give some explicit examples of how it is computed for very simple cases such as the *Hopf link*. We also recall, at the beginning of Chapter 2, the construction of the cobordism group for links and for knots and their relation. We then consider the question of constructing a similar cobordism group for embedded graphs in the 3-sphere. We show that this can actually be done in two different ways, both of which reduce to the same notion for links. The first one comes from the description of the cobordisms for links in terms of sequences of two basic operations, called "fusion" and "fission", which in terms of cobordisms correspond to the basic cobordisms obtained by attaching or removing a 1-handle. We define analogous operations of fusion and fission for embedded graphs and we introduce an equivalence relation of cobordism by iterated application of these two operations. The second possible definition of cobordism of embedded graphs is the one that we already used in Chapter 1 in section 7 as part of the definition of cobordisms of branched coverings, as the induced cobordism of the branched loci in the 3-sphere realized by an embedded surface (meaning here 2-complex) in $S^3 \times [0,1]$ with boundary the union of the given graphs. While for links, where cobordisms are realized by smooth surfaces, these can always be decomposed into a sequence of handle attachments, hence into a sequence of fusions and fissions, in the case of graphs not all cobordisms realized by 2-complexes can be decomposed as fusions and fissions, hence the two notions are no longer equivalent. We then return to homology again and discuss the question of extending Khovanov homology from links to embedded graphs. We propose two possible approaches to this purpose and we explain completely only one of them, while only sketching the other. The first idea is to try and combine the Khovanov complex, which is based on resolving in different ways crossings in a planar diagram, with the complex for the *graph homology*, which is not sensitive to the graph being embedded, but it has a good control over the combinatorial complexity of edges and vertices. We only sketch in one very simple example how one can try to combine these two differentials. We then take on a different approach. This is based on a result of Kauffman that constructs a topological invariant of embedded graphs in the 3-sphere by associating to such a graph a family of links and knots obtained using some local replacements at each vertex in the graph. He showed that it is a topological invariant by showing that the resulting knot and link types in the family thus constructed are invariant under a set of Reidemeister moves for embedded graphs that determine the ambient isotopy class of the embedded graphs. We build on this idea and simply define the Khovanov homology of an embedded graph to be the sum of the Khovanov homologies of all the links and knots in the Kauffman invariant associated to this graph. Since this family of links and knots is a topologically invariant, so is the Khovanov homology of embedded graphs defined in this manner. We close Chapter two by giving an example of computation of Khovanov homology for an embedded graph using this definition.

The appendix collects some known preliminary notions and background material that is needed elsewhere in the text.

Graphs Category and Three-manifolds as correspondences

1. Three-manifolds as correspondences

For the moment, we only work in the PL (piecewise linear) category, with proper PL maps. This is no serious restriction as, in the case of 3-dimensional and 4-dimensional manifolds, there is no obstruction in passing from the PL to the smooth category. When we refer to embedded graphs in $\mathbf{S^3}$, we mean PL embeddings of 1-complexes in $\mathbf{S^3}$ with no order zero or order one vertices.
Let $\mathbf{M^3}$ and $\mathbf{N^3}$ be smooth compact oriented 3-manifolds without boundary. A branched covering $p : \mathbf{M^3} \to \mathbf{N^3}$ is a continuous surjective map with the property that there exists a 1-dimensional subcomplex E in $\mathbf{N^3}$ such that on the complement of E the map

$$p : \mathbf{M^3} \smallsetminus p^{-1}(E) \to \mathbf{N^3} \smallsetminus E \tag{1.1}$$

is an actual (smooth) covering space. The manifold $\mathbf{M^3}$ is called the covering manifold, $\mathbf{N^3}$ the base, and E is called the branching set or branch locus.

1.1. 3-manifolds and branched covers.

We begin by recalling the following well known results that will be useful in the rest of our work (see [**54**]).

THEOREM 1.1. *(Alexander branched covering theorem): Suppose* $\mathbf{M^3}$ *is a compact oriented 3-dimensional manifold without boundary. Then there exists a branched covering* $p : \mathbf{M^3} \to \mathbf{S^3}$ *with branch locus an embedded (not necessarily connected) graph.*

In particular, this includes the special cases where the branch loci are *knots* or *links*.

In the case where the branch locus is a graph we in general only assume that the multiplicities, *i.e.* the number of points in the fiber $p^{-1}(x)$, is constant along 1-simplices (edges) of the graph, with compatibilities at the vertices, meaning that if two edges e_1 and e_2 of a graph G meet at a vertex v and m_1 and m_2 are the multiplicities of the covering over these vertices, then the multiplicity m over the vertex v divides both m_i, that is, multiplicities of adjacent edges have a common divisor. However, to simplify some of the arguments that follow, we will often make a stronger assumption on the coverings, which is to require that the multiplicities are constant on connected components of the graph.

Notice that Theorem 1.1 does not impose any condition on the order of the covering. In fact, it is known (see [**54**]) that one can strengthen the Alexander branched covering theorem to the following form.

THEOREM 1.2. *(Hilden-Montesinos Theorem): For any compact oriented 3-manifold* $\mathbf{M^3}$ *without boundary, there exists a 3-fold covering* $p : \mathbf{M^3} \to \mathbf{S^3}$ *of the 3-sphere branched along a knot* K.

DEFINITION 1.3. In the above, let m be the order of the the covering map (1.1) that is, $\#p^{-1}(x) = m$ for $x \in \mathbf{N^3} \smallsetminus E$. Suppose that the branch locus is an embedded graph of components $E = G_1 \cup \cdots \cup G_n$ and assume for simplicity that $\#p^{-1}(x) = n_i$ for all $x \in G_i \subset E$, with $G_i \cap G_j = \emptyset$, for $i \neq j$ and $1 \leq n_i < m$. We denote the integers n_i the multiplicities of the components of the branch set. The branching indices of the components G_i are positive integers b_{ij} for $j = 1 \dots n_i$ satisfying

$$\sum_{j=1}^{n_i} b_{ij} = m. \tag{1.2}$$

In other words, the integer b_{ij} counts how many components of the covering (1.1) come together at a point in $p^{-1}(G_i)$.

The data listed in Definition 1.3 above are not completely arbitrary. In fact, it is well known [**24**] that a branched covering $p : \mathbf{M} \to \mathbf{S^3}$ is uniquely determined by the restriction to the complement of the branch locus $L \subset \mathbf{S^3}$, which is a covering space of order m

$$p : \mathbf{M} \smallsetminus p^{-1}(E) \to \mathbf{S^3} \smallsetminus E. \tag{1.3}$$

This gives an equivalent description of branched coverings in terms of representations of the fundamental group of the complement of the branch locus [**25**]. We recall it here below as it will be useful in the following.

LEMMA 1.4. *Assigning a branched cover $p : \mathbf{M} \to \mathbf{S^3}$ of order m branched along a graph E is the same as assigning a representation*

$$\sigma_E : \pi_1(\mathbf{S^3} \smallsetminus E) \to S_m, \tag{1.4}$$

where S_m denotes the group of permutations of m elements.

PROOF. It suffices in fact to specify the representation up inner automorphisms of the group S_m. Thus, we do not have to worry about the choice of a base point for the fundamental group. By the observation above (see [**24**]), for a codimension two branch locus, there is a *unique* way of extending a covering (1.3) to a branched cover $p : \mathbf{M} \to \mathbf{S^3}$, so that the remaining data (multiplicities and branch indices over the points of the branch locus) are uniquely determined by assigning the datum (1.3). □

EXAMPLE 1.5. *(Cyclic branched coverings):* We represent $\mathbf{S^3}$ as $\mathbb{R}^3 \cup \{\infty\}$. Let l be a straight line chosen in $\mathbb{R}^3$. Consider the quotient map $p : \mathbb{R}^3 \to \mathbb{R}^3/(\mathbb{Z}/n\mathbb{Z})$ that identifies the points of $\mathbb{R}^3$ obtained from each other by a rotation by an angle of $\frac{2\pi}{n}$ about the axis l. Upon identifying $\mathbb{R}^3 \simeq \mathbb{R}^3/(\mathbb{Z}/n\mathbb{Z})$, this extends to a map $p : \mathbf{S^3} \to \mathbf{S^3}$ which is an n-fold covering branched along the unknot $l \cup \{\infty\}$ and with multiplicity one over the branch locus.

These cyclic branched coverings are useful to construct other more complicated branched coverings by performing surgeries along framed links (see [**54**]).

1.2. Correspondences and morphisms. The main idea we present in this section is to define morphisms $\phi : G \to G'$ between graphs as formal finite linear combinations

$$\phi = \sum_i a_i \mathbf{M}_i \tag{1.5}$$

with $a_i \in \mathbb{Q}$ and $\mathbf{M}_i$ compact oriented smooth 3-manifolds with submersions

$$\pi_i : \mathbf{M}_i \to \mathbf{S^3}$$

and

$$\pi_i' : \mathbf{M}_i \to \mathbf{S^3}$$

that are branched covers, respectively branched along G and G'. We use the notation

$$G \subset \mathbf{S^3} \stackrel{\pi_G}{\longleftarrow} \mathbf{M} \stackrel{\pi_{G'}}{\longrightarrow} \mathbf{S^3} \supset G' \tag{1.6}$$

for a 3-manifold that is realized in two ways as a covering of $\mathbf{S^3}$, branched along the graph G or G'. This definition makes sense, since the way in which a given 3-manifold $\mathbf{M}$ is realized as a branched cover of $\mathbf{S^3}$ branched along a knot is not unique.

EXAMPLE 1.6. *(Poincaré homology sphere):* Let $\mathbf{P}$ denote the Poncaré homology sphere. This smooth compact oriented 3-manifold is a 5-fold cover of $\mathbf{S^3}$ branched along the *trefoil knot* (that is, the $(2,3)$ torus knot), or a 3-fold cover of $\mathbf{S^3}$ branched along the $(2,5)$ torus knot, or also a 2-fold cover of $\mathbf{S^3}$ branched along the $(3,5)$ torus knot. For details see [**54**], [**44**].

We can extend the definition above to the case where the manifolds $\mathbf{M}$ are smooth and compact (without boundary) but not necessarily connected. In this case, if $\mathbf{M} = \mathbf{M}_1 \cup \cdots \cup \mathbf{M}_n$, with $\mathbf{M}_i$ connected we identify the morphism

$$\phi = \sum_i \mathbf{M}_i$$

with the morphism defined by $\mathbf{M}$. This corresponds to introducing a first simple equivalence relation on morphisms.

DEFINITION 1.7. Let $\mathbf{M}$ be a disjoint union of two smooth compact connected 3-manifolds without boundary $\mathbf{M} = \mathbf{M}_1 \cup \mathbf{M}_2$, with compatible covering maps $\pi_G = (\pi_{G,1}, \pi_{G,2})$ and $\pi_{G'} = (\pi_{G',1}, \pi_{G',2})$. Then we set

$$\phi_{\mathbf{M}} = \phi_{\mathbf{M}_1} + \phi_{\mathbf{M}_2}. \tag{1.7}$$

where we let $\phi_{\mathbf{M}} : G \to G'$ denote the morphism defined by a manifold $\mathbf{M}$ as in (1.6).

1.3. The set of geometric correspondences. We define the set of geometric correspondences $Hom(G, G')$ between two embedded graphs G and G' in the following way.

DEFINITION 1.8. Given two embedded graphs G and G' in $\mathbf{S^3}$, let $Hom(G, G')$ denote the set of 3-manifolds $\mathbf{M}$ that can be represented as branched covers as in (1.6), for some graphs E and E', respectively containing G and G' as subgraphs. We also assume that, for all G the set $Hom(G, G)$ also contains the sphere $\mathbf{S^3}$ as trivial (unbranched) covering.

We explain in §2 below why here we need to allow for larger graphs E and E' instead of just assuming the branch loci to be the given G and G' as we suggested earlier in (1.5). We explain in §2.3 below why we include the unbranched covering in $Hom(G, G)$.

To avoid logical complications in dealing with the "set" of all 3-manifolds, we describe the $Hom(G, G')$ in terms of the following set of representation theoretic data. As we have seen in Lemma 1.4 above (see [**24**]), a branched covering $p : \mathbf{M} \to \mathbf{S^3}$ is uniquely determined by the restriction to the complement of the branch locus $E \subset \mathbf{S^3}$. This gives an equivalent description of branched coverings in terms of representations of the fundamental group of the complement of the branch locus [**25**]. The representation is determined up to inner automorphisms, hence there is no dependence on the choice of a base point for the fundamental group in (1.3).

Thus, in terms of these representations, the spaces of morphisms $Hom(G, G')$ are identified with the set of data

$$\mathcal{R}_{G,G'} \subset \bigcup_{n,m,G\subset E,G'\subset E'} \mathrm{Hom}(\pi_1(\mathbf{S^3} \setminus E), S_n) \times \mathrm{Hom}(\pi_1(\mathbf{S^3} \setminus E'), S_m), \tag{1.8}$$

where the E, E' are embedded graphs, $n, m \in \mathbb{N}$, and where the subset $\mathcal{R}_{G,G'}$ is determined by the condition that the pair of representations (σ_1, σ_2) define the same 3-manifold.

1.4. Covering moves and correspondences. To get some more feeling for the type of correspondences we are dealing with, we recall here a result on covering moves which, from our point of view, describes when a given 3-manifold $\mathbf{M}$ is a correspondence between two graphs G and G'. Suppose given a compact oriented smooth 3-manifold $\mathbf{M}$ and a map π_L realizing this 3-manifold as a covering of $\mathbf{S^3}$ branched along a link (or a knot) L. By the stronger form of the Hilden-Montesinos theorem, we can assume that it is a 3-fold cover. It is known that such a covering can be represented by a colored link (see for instance [**52**]). Notice that the same manifold has many different representations as a colored link, as the following statement illustrates.

THEOREM 1.9. *(Equivalence Theorem,* [**52**]*) Two colored link diagrams represent the same manifold if and only if they can be related (up to colored Reidemeister moves) by a finite sequence of moves of the four types described in* [**52**].

In this theorem we see that the manifold is a covering, branched over another link, obtained by simple moves called colored moves applied to the first link. Thus, one can see that it is quite easy to provide examples of different links that realize the same 3-manifold as branched cover of $\mathbf{S^3}$, with the given link as branch locus. As a consequence of this result we obtain the following statement.

LEMMA 1.10. *Let* $\mathbf{M}$ *be a compact 3-manifold that is realized as a branched cover of* $\mathbf{S^3}$*, branched along a knot* K*. Then the manifold* $\mathbf{M}$ *belongs to* $Hom(K,K')$*, for all knots* K' *that are obtained from* K *by the covering moves of* [**52**].

2. Composition of correspondences

We now explain why in Definition 1.8 we need to assume that the covering maps are branched on graphs containing the given graphs G and G'. This has to do with having a well defined composition of morphisms.

In fact, if we only require the branch loci to be exactly G and G', our preliminary definition of morphisms as elements of the form (1.5) runs immediately into a problem with the composition law. In fact, it is natural to define the composition of geometric correspondences of the form (1.5) to be given by the fibered product, as in [**18**].

DEFINITION 2.1. Suppose given

$$G \subset \mathbf{S^3} \overset{\pi_G}{\longleftarrow} \mathbf{M} \overset{\pi_{G'}}{\longrightarrow} \mathbf{S^3} \supset G' \quad \text{and} \quad G' \subset \mathbf{S^3} \overset{\tilde{\pi}_{G'}}{\longleftarrow} \tilde{\mathbf{M}} \overset{\tilde{\pi}_{G''}}{\longrightarrow} \mathbf{S^3} \supset G''. \tag{2.1}$$

One defines the composition $\mathbf{M} \circ \tilde{\mathbf{M}}$ as

$$\mathbf{M} \circ \tilde{\mathbf{M}} := \mathbf{M} \times_{G'} \tilde{\mathbf{M}}, \tag{2.2}$$

where the fibered product $\mathbf{M} \times_{G'} \tilde{\mathbf{M}}$ is defined as

$$\mathbf{M} \times_{G'} \tilde{\mathbf{M}} := \{(x,x') \in \mathbf{M} \times \tilde{\mathbf{M}} | \pi_{G'}(x) = \tilde{\pi}_{G'}(x')\}. \tag{2.3}$$

The composition $\mathbf{M} \circ \tilde{\mathbf{M}}$ defined in this way satisfies the following property.

PROPOSITION 2.2. *Assume that the maps of* (2.1) *have the following multiplicities. The map* π_G *is of order* m *for* $x \in \mathbf{S^3} \smallsetminus G$ *and of order* n *for* $x \in G$*; the map* $\pi_{G'}$ *is of order* m' *for* $x \in \mathbf{S^3} \smallsetminus G'$ *and* n' *for* $x \in G'$*; the map* $\tilde{\pi}_{G'}$ *is of order* $\tilde{m}'$ *for* $x \in \mathbf{S^3} \smallsetminus G'$ *and of order* $\tilde{n}'$ *for* $x \in G'$*; the map* $\tilde{\pi}_{G''}$ *is of order* $\tilde{m}''$ *for* $x \in \mathbf{S^3} \smallsetminus G''$ *and* $\tilde{n}''$ *for* $x \in G''$*. For simplicity assume that*

$$G \cap \pi_G(\pi_{G'}^{-1}(G')) = \emptyset \quad \text{and} \quad G'' \cap \tilde{\pi}_{G''}(\tilde{\pi}_{G'}^{-1}(G')) = \emptyset. \tag{2.4}$$

Then the fibered product $\hat{\mathbf{M}} = \mathbf{M} \times_{G'} \tilde{\mathbf{M}}$ *is a smooth 3-manifold with submersions*

$$E \subset \mathbf{S^3} \overset{\hat{\pi}_E}{\longleftarrow} \hat{\mathbf{M}} \overset{\hat{\pi}_{E''}}{\longrightarrow} \mathbf{S^3} \supset E''. \tag{2.5}$$

where

$$E = G \cup \pi_G(\pi_{G'}^{-1}(G')) \tag{2.6}$$

$$E'' = G'' \cup \tilde{\pi}_G(\tilde{\pi}_{G'}^{-1}(G')) \tag{2.7}$$

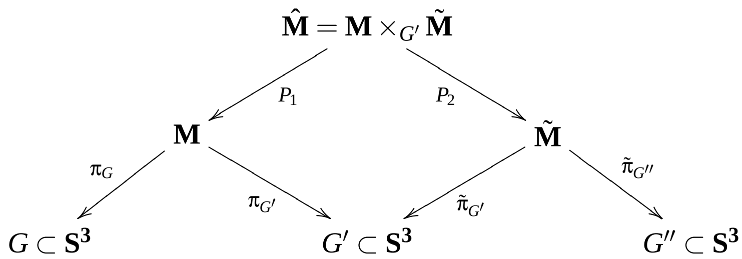
The fibers of the map $\hat{\pi}_E$ *have cardinality*

$$\#\hat{\pi}_E^{-1}(x) = \begin{cases} m\tilde{m}' & x \in \mathbf{S^3} \smallsetminus (G \cup \pi_G(\pi_{G'}^{-1}(G')) \\ \tilde{m}'n & x \in G \\ m\tilde{n}' & x \in \pi_G(\pi_{G'}^{-1}(G') \end{cases} \tag{2.8}$$

Similarly, the fibers of the map $\hat{\pi}_{E''}$ *have cardinality*

$$\#\hat{\pi}_E^{-1}(x) = \begin{cases} \tilde{m}''m' & x \in \mathbf{S^3} \smallsetminus (\tilde{\pi}_{G''}(\tilde{\pi}_{G'}^{-1}(G')) \cup G'') \\ \tilde{m}''n' & x \in \tilde{\pi}_{G''}(\tilde{\pi}_{G'}^{-1}(G')) \\ m'\tilde{n}'' & x \in G'' \end{cases} \tag{2.9}$$

PROOF. Consider the diagram



The fibered product $\hat{\mathbf{M}}$ is by definition a subset of the product $\mathbf{M} \times \tilde{\mathbf{M}}$ defined as the preimage $\hat{\mathbf{M}} = (\pi_{G'} \times \pi_{G''})^{-1}(\Delta(\mathbf{S^3}))$, where $\Delta(\mathbf{S^3})$ is the diagonal embedding of $\mathbf{S^3}$ in $\mathbf{S^3} \times \mathbf{S^3}$. This defines a smooth 3-dimensional submanifold of $\mathbf{M} \times \tilde{\mathbf{M}}$. In general $\hat{\mathbf{M}}$ needs not be connected. The restriction to $\hat{\mathbf{M}} \subset \mathbf{M} \times \tilde{\mathbf{M}}$ of the projections $P_1 : \mathbf{M} \times \tilde{\mathbf{M}} \to \mathbf{M}$ and $P_2 : \mathbf{M} \times \tilde{\mathbf{M}} \to \tilde{\mathbf{M}}$ defines projections

$$\mathbf{M} \stackrel{P_1}{\leftarrow} \hat{\mathbf{M}} \stackrel{P_2}{\rightarrow} \tilde{\mathbf{M}}. \tag{2.10}$$

We first show that these maps are branched covers, respectively branched along $\pi_{G'}^{-1}(G') \subset \mathbf{M}$ and $\tilde{\pi}_{G'}^{-1}(G') \subset \tilde{\mathbf{M}}$. For a point $x \in \mathbf{M}$ the preimage $P_1^{-1}(x) \subset \hat{\mathbf{M}}$ consists of

$$P_1^{-1}(x) = \{y \in \tilde{\mathbf{M}} \,|\, \tilde{\pi}_{G'}(y) = \pi_{G'}(x)\}.$$

There are two cases: if the point $s = \pi_{G'}(x) \in \mathbf{S^3}$ lies in the complement of the graph G' then $\#\tilde{\pi}_{G'}^{-1}(s) = \tilde{m}'$, while if $s = \pi_{G'}(x) \in G'$ then $\#\tilde{\pi}_{K'}^{-1}(s) = \tilde{n}' < \tilde{m}'$. We see from this that the map $P_1 : \hat{\mathbf{M}} \to \mathbf{M}$ is a branched cover of order $\tilde{m}'$, with branch locus the set of points $\{x \in \mathbf{M} \,|\, \pi_{G'}(x) \in G'\} = \pi_{G'}^{-1}(G')$. A similar argument for the fibers

$$P_2^{-1}(y) = \{x \in \mathbf{M} \,|\, \pi_{G'}(x) = \tilde{\pi}_{G'}(y)\}$$

shows that the map $P_2 : \hat{\mathbf{M}} \to \tilde{\mathbf{M}}$ is a branched cover of order m' branched along the set $\tilde{\pi}_{G'}^{-1}(G')$. Now we consider the composite maps

$$\hat{\pi}_G = \pi_G \circ P_1 : \hat{\mathbf{M}} \to \mathbf{S^3} \quad \text{and} \quad \hat{\pi}_{G''} = \tilde{\pi}_{G''} \circ P_2 : \hat{\mathbf{M}} \to \mathbf{S^3}.$$

We show that these maps are also branched covers, with the order and multiplicities as specified in (2.8) and (2.9). Consider the preimages $\hat{\pi}_G^{-1}(s)$ for $s \in \mathbf{S^3}$. For a point $s \in \mathbf{S^3} \smallsetminus (G \cup \pi_G(\pi_{G'}^{-1}(G')))$ we have $\#\hat{\pi}_G^{-1}(s) = \#\pi_G^{-1}(s) \cdot \#P_1^{-1}(x)$, for $x \in \mathbf{M} \smallsetminus \pi_{G'}^{-1}(G')$. This gives

$$\#\hat{\pi}_G^{-1}(s) = mm', \quad \forall s \in \mathbf{S^3} \smallsetminus (G \cup \pi_G(\pi_{G'}^{-1}(G'))).$$

If we consider instead a point $s \in G$, by assumption that $G \cap \pi_G(\pi_{G'}^{-1}(G')) = \emptyset$ we know that the point $x \in \pi_G^{-1}(s)$ are in $\mathbf{M} \smallsetminus \pi_{G'}^{-1}(G')$, hence we get

$$\#\hat{\pi}_G^{-1}(s) = nm', \quad \forall s \in G \subset \mathbf{S^3}$$

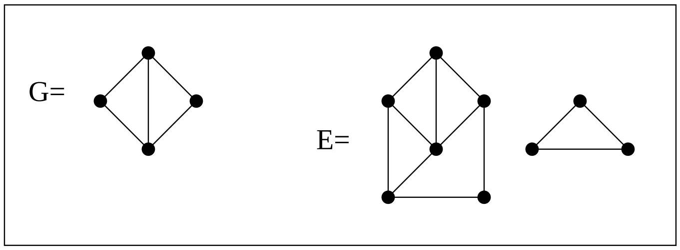


FIGURE 1

Finally, by the same reasoning we obtain

$$\#\hat{\pi}_G^{-1}(s) = mn', \quad \forall s \in \pi_G(\pi_{G'}^{-1}(G'))$$

This gives the result of (2.8) The case of the composite map $\hat{\pi}_{G''} = \tilde{\pi}_{G''} \circ P_2$ is analyzed in the same way and it yields the multiplicities of (2.9). □

REMARK 2.3. The assumption (2.4) need not hold in general, where one typically has

$$G \cap \pi_G(\pi_{G'}^{-1}(G')) \neq \emptyset \quad \text{or} \quad G'' \cap \tilde{\pi}_{G''}(\tilde{\pi}_{G'}^{-1}(G')) \neq \emptyset. \tag{2.11}$$

One still obtains that E and E'' are embedded graphs, and the counting of the multiplicities and branched indices will be more involved but the argument remains essentially analogous to the one given in Proposition 2.2.

This shows that, for the composition $\hat{\mathbf{M}} = \mathbf{M} \circ \tilde{\mathbf{M}}$, the maps $\hat{\pi}_G$ and $\hat{\pi}_{G''}$ are no longer coverings branched along G and G''. In fact, the branch loci are now larger graphs

$$E = G \cup \pi_G(\pi_{G'}^{-1}(G')) \quad \text{and} \quad E'' = G'' \cup \tilde{\pi}_{G''}(\tilde{\pi}_{G'}^{-1}(G')) \tag{2.12}$$

and the multiplicities are different on different parts of the graph. Thus, in order to have a well defined composition law, we need to enlarge the class of morphisms from our initial proposal (1.5) to include what we obtained as the result of the composition of morphisms in the class (1.5).

DEFINITION 2.4. A morphism $\phi : G \to G'$ is a finite linear combination $\sum_i a_i \mathbf{M}_i$, with coefficients $a_i \in \mathbb{Q}$ and where the $\mathbf{M}_i$ are smooth compact oriented 3-manifolds with branched covering maps

$$G \subset E_i \subset \mathbf{S^3} \xleftarrow{\pi_{E_i}} \mathbf{M}_i \xrightarrow{\pi_{E_i'}} \mathbf{S^3} \supset E_i' \supset G', \tag{2.13}$$

where E_i are embedded graphs in $\mathbf{S^3}$ where $E_i = G \cup G_{i,1} \cup \cdots G_{i,g_i}$, and $E_i' = G' \cup G'_{i,1} \cup \cdots G'_{i,g_i'}$.

The graph E_i contain G $(G \subset E_i)$ but not necessarily as a connected component, see for example Figure 1.

Notice that we need to assume in the definition above that the graphs E_i and E_i' are not necessarily the same for different $\mathbf{M}_i$, though they all contain G (respectively G'). This is because in the argument of Proposition 2.2 we see that the graphs $E = G \cup \pi_G \pi_{G'}^{-1}(G')$ and $E'' = G'' \cup \tilde{\pi}_{G''} \tilde{\pi}_{G'}^{-1}(G')$ along which the composite morphism $\hat{\mathbf{M}}$ is ramified do not depend only on the subgraphs G, G' and G'' but also on the projection maps π_G and $\pi_{G'}$ (respectively $\tilde{\pi}_{G'}$ and $\tilde{\pi}_{G''}$), hence on the manifolds $\mathbf{M}$ and $\tilde{\mathbf{M}}$. This means that, when we consider the composition of morphisms according to Definition 2.4, we do so according to the following definition.

DEFINITION 2.5. Let $\mathbf{M}$ and $\tilde{\mathbf{M}}$ be smooth compact oriented 3-manifolds with branched covering maps

$$E \subset \mathbf{S^3} \xleftarrow{\pi_E} \mathbf{M} \xrightarrow{\pi_{E_1'}} \mathbf{S^3} \supset E_1' \quad \text{and} \quad E_2' \subset \mathbf{S^3} \xleftarrow{\tilde{\pi}_{E_2'}} \tilde{\mathbf{M}} \xrightarrow{\tilde{\pi}_{E''}} \mathbf{S^3} \supset E'', \tag{2.14}$$

with graphs E, E_1', E_2' and E'' with $G \subset E$, $G' \subset E_1'$ and $G' \subset E_2'$ and $G'' \subset E''$. The composition $\mathbf{M} \circ \tilde{\mathbf{M}}$ is given by the fibered product

$$\mathbf{M} \circ \tilde{\mathbf{M}} := \mathbf{M} \times_{G'} \tilde{\mathbf{M}}, \tag{2.15}$$

with

$$\mathbf{M} \times_{G'} \tilde{\mathbf{M}} := \{(x,y) \in \mathbf{M} \times \tilde{\mathbf{M}} \,|\, \pi_{E_1'}(x) = \tilde{\pi}_{E_2'}(y)\}. \tag{2.16}$$

The result of Proposition 2.2 adapts to this case to show the following result.

LEMMA 2.6. *Let $G' \subset E_1$ and $G' \subset E_2$ be two graphs in $\mathbf{S^3}$. Consider branched coverings*

$$G \subset S^3 \overset{\pi_G}{\leftarrow} \mathbf{M} \overset{\pi_1}{\to} S^3 \supset E_1 \quad E_2 \subset S^3 \overset{\pi_2}{\leftarrow} \tilde{\mathbf{M}} \overset{\pi_{G''}}{\to} G''.$$

The composition $\hat{\mathbf{M}} = \mathbf{M} \times_{G'} \tilde{\mathbf{M}}$ is a branched cover

$$G \cup \pi_G \pi_1^{-1}(E_2) \subset S^3 \overset{\hat{\pi}_G}{\leftarrow} \hat{\mathbf{M}} \overset{\hat{\pi}_{G''}}{\to} S^3 \supset G'' \cup \pi_{G''} \pi_2^{-1}(E_1).$$

PROOF. Consider first the projections $P_1 : \mathbf{M} \times_{G'} \tilde{\mathbf{M}} \to \mathbf{M}$ and $P_2 : \mathbf{M} \times_{G'} \tilde{\mathbf{M}} \to \tilde{\mathbf{M}}$. They are branched covers, respectively branched over $\pi_1^{-1}(E_2)$ and $\pi_2^{-1}(E_1)$. In fact, we have

$$P_1^{-1}(x) = \{(x,y) \in \mathbf{M} \times \tilde{\mathbf{M}} \,|\, \pi_1(x) = \pi_2(y)\} = \{y \in \tilde{M} \,|\, \pi_2(y) = \pi_1(x)\}.$$

Thus, the map P_1 is branched over the points $x \in \mathbf{M}$ such that $\pi_1(x)$ lies in the branch locus of the map π_2, that is, the points $\{x \in \pi_1^{-1}(E_2)\}$. Similarly, the branch locus of the map P_2 is the set of points $y \in \pi_2^{-1}(E_1) \subset \tilde{\mathbf{M}}$. Thus, the composite map $\hat{\pi}_E = \pi_G \circ P_1 : \hat{\mathbf{M}} \to \mathbf{S^3}$ is branched over the graph $E = G \cup \pi_G \pi_1^{-1}(E_2)$ and the map $\hat{\pi}_{E''} = \pi_{G''} \circ P_2 : \hat{\mathbf{M}} \to \mathbf{S^3}$ is branched over the graph $E = G'' \cup \pi_{G''} \pi_2^{-1}(E_1)$. □

COROLLARY 2.7. *Let*

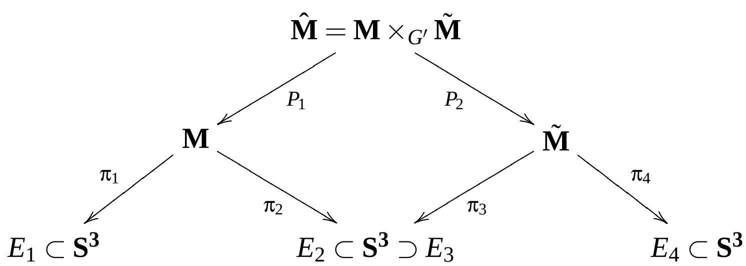
$$G \subset E_1 \subset \mathbf{S^3} \overset{\pi_1}{\leftarrow} \mathbf{M} \overset{\pi_2}{\to} \mathbf{S^3} \supset E_2 \supset G'$$

and

$$G' \subset E_3 \subset \mathbf{S^3} \overset{\pi_3}{\leftarrow} \tilde{\mathbf{M}} \overset{\pi_4}{\to} \mathbf{S^3} \supset E_4 \supset G''$$

be morphisms from G to G' and from G' to G'', respectively, in the sense of Definition 2.4. Then the composition $\hat{\mathbf{M}} = \mathbf{M} \circ \tilde{\mathbf{M}} = \mathbf{M} \times_{G'} \tilde{\mathbf{M}}$ of (2.15), (2.16) *is also a morphism from G to G'' in the sense of Definition 2.4.*

PROOF. The composition is given by the diagram



As in (2.10), the restriction to $\hat{\mathbf{M}} \subset \mathbf{M} \times \tilde{\mathbf{M}}$ of the projections $P_1 : \mathbf{M} \times \tilde{\mathbf{M}} \to \mathbf{M}$ and $P_2 : \mathbf{M} \times \tilde{\mathbf{M}} \to \tilde{\mathbf{M}}$ defines projections $P_1 : \hat{\mathbf{M}} \to \mathbf{M}$ and $P_2 : \hat{\mathbf{M}} \to \tilde{\mathbf{M}}$. Lemma 2.6 shows that they are branched covers, respectively branched along $\pi_2^{-1}(E_3) \subset \mathbf{M}$ and $\pi_3^{-1}(E_2) \subset \tilde{\mathbf{M}}$, so that the resulting maps $\hat{\pi}_1 = \pi_1 \circ P_1$ and $\hat{\pi}_2 = \pi_4 \circ P_2$ from $\hat{\mathbf{M}}$ to $\mathbf{S^3}$ are branched covers, branched along $E_1 \cup \pi_1 \pi_2^{-1}(E_3)$ and $E_4 \cup \pi_4 \pi_3^{-1}(E_2)$, respectively. Since $G \subset E_1 \cup \pi_1 \pi_2^{-1}(E_3)$ and $G'' \subset E_4 \cup \pi_4 \pi_3^{-1}(E_2)$, we obtain that

$$G \subset E_1 \cup \pi_1 \pi_2^{-1}(E_3) \subset \mathbf{S^3} \overset{\hat{\pi}_1}{\leftarrow} \hat{\mathbf{M}} \overset{\hat{\pi}_2}{\to} \mathbf{S^3} \supset E_4 \cup \pi_4 \pi_3^{-1}(E_2) \supset G''$$

is a morphism in $Hom(G,G'')$ in the sense of Definition 2.4. □

We then describe explicitly the multiplicities of the covering maps $\hat{\pi}_i : \hat{\mathbf{M}} \to \mathbf{S^3}$. To simplify the computation we work under the assumption that the multiplicities are constant on connected components of the graph and not just on the individual simplices (up to homotopy it is always possible to reduce to this case).

LEMMA 2.8. *Let* $\mathbf{M}$ *and* $\tilde{\mathbf{M}}$ *be as in Corollary 2.7 above. Assume that the graphs* E_i*, for* $i = 1,\dots,4$ *, have components*

$$E_i = G_{i0} \cup G_{i1} \cup \cdots \cup G_{ig_i}, \tag{2.17}$$

with $G_{10} = G$*,* $G_{20} = G' = G_{30}$ *and* $G_{40} = G''$ *and* $g_i \geq 0$ *for* $i = 1,\dots,4$ *is the number of the components of the graph* G_i*. Also assume that the maps* π_i *have multiplicities*

$$\#\pi_i^{-1}(x) = \begin{cases} m_i & x \in \mathbf{S^3} \smallsetminus E_i \\ n_{ij} & x \in E_{ij} \quad j = 1,\dots,g_i. \end{cases} \tag{2.18}$$

Then the composite maps

$$E_1 \cup \pi_1\pi_2^{-1}(E_3) \subset \mathbf{S^3} \overset{\hat{\pi}_1}{\leftarrow} \hat{\mathbf{M}} \overset{\hat{\pi}_2}{\to} \mathbf{S^3} \supset E_4 \cup \pi_4\pi_3^{-1}(E_2)$$

have multiplicities

$$\#\hat{\pi}_1^{-1}(x) = \begin{cases} m_1 m_3 & x \in \mathbf{S^3} \smallsetminus \hat{E}_1 \\ m_1 n_{3j} & x \in \pi_1\pi_2^{-1}(G_{3j}) \;\; j = 0,\dots,g_3 \\ n_{1j} m_3 & x \in G_{1j} \;\; j = 0,\dots,g_1 \end{cases} \tag{2.19}$$

$$\#\hat{\pi}_2^{-1}(x) = \begin{cases} m_2 m_4 & x \in \mathbf{S^3} \smallsetminus \hat{E}_2 \\ n_{2j} m_4 & x \in \pi_4\pi_3^{-1}(G_{2j}) \;\; j = 0,\dots,g_2 \\ m_2 n_{4j} & x \in G_{4j} \;\; j = 0,\dots,g_4, \end{cases} \tag{2.20}$$

where $\hat{E}_1 = E_1 \cup \pi_1\pi_2^{-1}(E_3)$ *and* $\hat{E}_4 = E_4 \cup \pi_4\pi_3^{-1}(E_2)$*.*

PROOF. The argument is analogous to Proposition 2.2. For a point $x \in \mathbf{M}$ the preimage $P_1^{-1}(x) \subset \hat{\mathbf{M}}$ consists of

$$P_1^{-1}(x) = \{y \in \tilde{\mathbf{M}} \,|\, \pi_3(y) = \pi_2(x)\}.$$

Thus, the map P_1 has multiplicities

$$\#P_1^{-1}(x) = \begin{cases} m_3 & x \in \mathbf{M} \smallsetminus \pi_2^{-1}(E_3) \\ n_{3j} & x \in \pi_2^{-1}(G_{3j}) \;\; j = 0,\dots,g_3. \end{cases} \tag{2.21}$$

Thus there are then three cases for the map $\hat{\pi}_1 = \pi_1 \circ P_1$: if the point $\pi_1(x) = s \in \mathbf{S^3}$ lies in the complement of both E_1 and $\pi_1\pi_2^{-1}(E_3)$ then $\#\hat{\pi}_1^{-1}(s) = m_1 m_3$. If the point $\pi_1(x) = s$ is in a component G_{1j} then $\#\hat{\pi}_1^{-1}(s) = n_{1j} m_3$. Finally, if $\pi_1(x) = s$ is in $\pi_1\pi_2^{-1}(G_{3j})$, for one of the components G_{3j} of E_3, then $\#\hat{\pi}_1^{-1}(s) = m_1 n_{3j}$. This gives the multiplicities of (2.19). The case of the composite map $\hat{\pi}_2 = \pi_4 \circ P_2$ is analogous. We first notice that the multiplicities for the map P_2 are $\#P_2^{-1}(x) = m_2$ for $x \in \tilde{m} \smallsetminus \pi_3^{-1}(E_2)$ and $\#P_2^{-1}(x) = n_{2j}$ for $x \in \pi_3^{-1}(G_{2j})$, for $j = 0,\dots,g_2$. Then arguing as before we see that there are three possible cases, as before, for the multiplicities for $\hat{\pi}_2$. If the point $\pi_4(x) = s \in \mathbf{S^3}$ is in the complement of both E_4 and $\pi_4\pi_3^{-1}(E_2)$, then $\#\hat{\pi}_2^{-1}(s) = m_2 m_4$. If the point $\pi_1(x) = s$ is in a component G_{4j}, then $\#\hat{\pi}_2^{-1}(s) = n_{4j} m_2$. Finally, if $\pi_1(x) = s$ is in $\pi_4\pi_3^{-1}(G_{2j})$, then $\#\hat{\pi}_2^{-1}(s) = n_{2j} m_4$. This gives the multiplicities of (2.20). □

The general case where the multiplicities change on different simplices within the same connected component can be treated similarly only the formulae become more involved. The argument one then uses to derive explicit formulae for the branching indices is also analogous.

2.1. Example of correspondences and composition. We give simple example of composition of morphisms by fibered product.

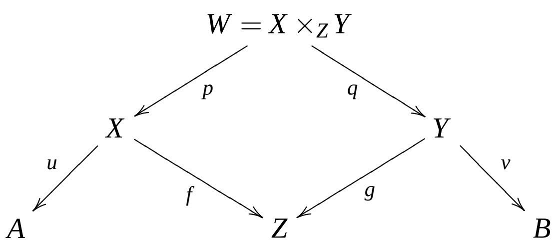
EXAMPLE 2.9. In this example one can use the cyclic branched covering maps we mentioned before in Example 1.5. Consider the fibered product $\mathbf{M}_m \circ \mathbf{M}_n$ of

$$O \subset \mathbf{S^3} \leftarrow \mathbf{M}_n \rightarrow \mathbf{S^3} \supset O \quad O \subset \mathbf{S^3} \leftarrow \mathbf{M}_m \rightarrow \mathbf{S^3} \supset O, \tag{2.22}$$

where $\mathbf{M}_n$ and $\mathbf{M}_m$ are, respectively, the n-fold and m-fold branched cyclic coverings, branched over the trivial knot O. Then the composition $\mathbf{M}_m \circ \mathbf{M}_n$ is the cyclic branched cover $\mathbf{M}_{mn}$, which is a morphism between two unknots.

2.2. Associativity of composition. We now prove that the composition of morphisms defined in the previous section is associative. We begin by stating a very simple lemma that will be useful in the proof.

LEMMA 2.10. *Consider a commutative diagram*



where all the maps are submersions and $p(x,y) = x$, $q(x,y) = y$. Then, for any $b \in B$ one has

$$upq^{-1}v^{-1}(b) = uf^{-1}gv^{-1}(b) \subset A.$$

PROOF. Let $V \subset Y$ be the set $V = v^{-1}(b)$. Its preimage under q is the set

$$\{(x,y) \in X \times Y \,|\, y \in V,\ g(y) = f(x)\} = \{(x,y) \in X \times Y \mid f(x) = g(y) \in g(V)\}.$$

Thus, the image $pq^{-1}(V) = \{x \in X \mid f(x) \in g(V)\} = f^{-1}g(V)$. This implies $upq^{-1}(V) = uf^{-1}g(V)$, hence the statement follows. □

We now compare the compositions $\mathbf{M}_1 \circ (\mathbf{M}_2 \circ \mathbf{M}_3)$ and $(\mathbf{M}_1 \circ \mathbf{M}_2) \circ \mathbf{M}_3$ of morphisms $\mathbf{M}_i \in Hom(G_i, G_{i+1})$.

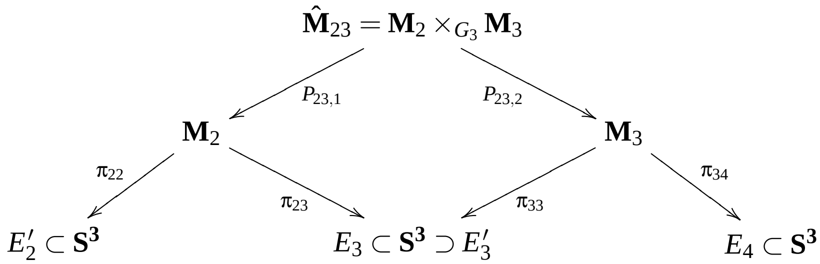
PROPOSITION 2.11. *Suppose given branched covers*

$$\begin{aligned} E_1 &\subset S^3 \overset{\pi_{11}}{\leftarrow} \mathbf{M}_1 \overset{\pi_{12}}{\rightarrow} S^3 \supset E_2 \\ E_2' &\subset S^3 \overset{\pi_{22}}{\leftarrow} \mathbf{M}_2 \overset{\pi_{23}}{\rightarrow} S^3 \supset E_3 \\ E_3' &\subset S^3 \overset{\pi_{33}}{\leftarrow} \mathbf{M}_3 \overset{\pi_{34}}{\rightarrow} S^3 \supset E_4, \end{aligned} \tag{2.23}$$

where E_1 is a graph containing the subgraph G_1, E_2 and E_2' are graphs containing the subgraph G_2, E_3 and E_3' are graphs containing a given subgraph $G_3$and E_4 is a graph containing the subgraph G_4. The composition is associative

$$\mathbf{M}_1 \circ (\mathbf{M}_2 \circ \mathbf{M}_3) = (\mathbf{M}_1 \circ \mathbf{M}_2) \circ \mathbf{M}_3. \tag{2.24}$$

PROOF. Consider first the composition $\hat{\mathbf{M}}_{23} := \mathbf{M}_2 \circ \mathbf{M}_3 = \mathbf{M}_2 \times_{G_3} \mathbf{M}_3$. It is given by the diagram



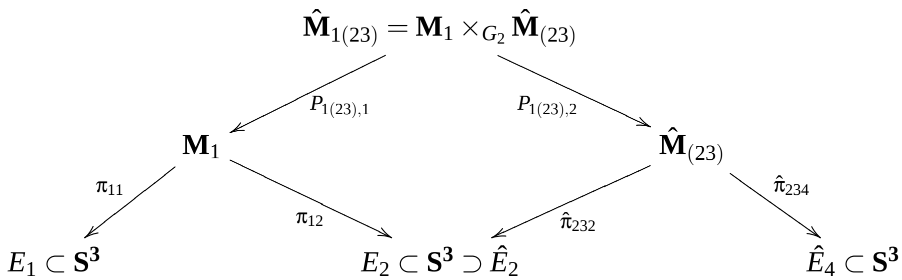
with $\hat{\pi}_{232} = \pi_{22} \circ P_{23,1}$ and $\hat{\pi}_{234} = \pi_{34} \circ P_{23,2}$. By Lemma 2.6, $\hat{\mathbf{M}}_{23}$ is a branched cover

$$\hat{E}_2 \subset S^3 \stackrel{\hat{\pi}_{232}}{\leftarrow} \hat{\mathbf{M}}_{23} \stackrel{\hat{\pi}_{234}}{\rightarrow} S^3 \supset \hat{E}_4,$$

with

$$\hat{E}_2 = E_2' \cup \pi_{22}\pi_{23}^{-1}(E_3') \quad \hat{E}_4 = E_4 \cup \pi_{34}\pi_{33}^{-1}(E_3). \tag{2.25}$$

Then the composition $\hat{\mathbf{M}}_{1(23)} := \mathbf{M}_1 \circ \hat{\mathbf{M}}_{23} = \mathbf{M}_1 \circ (\mathbf{M}_2 \circ \mathbf{M}_3)$ is given by the diagram



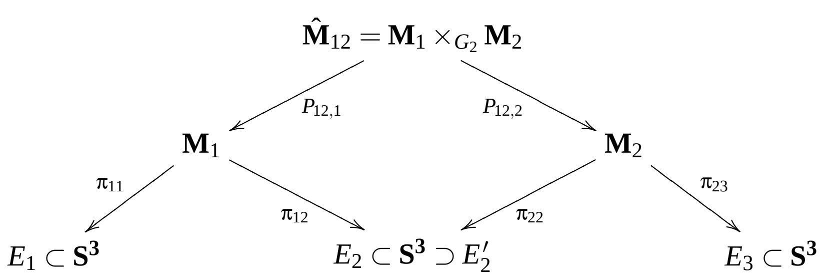
where $\hat{E}_2 = E_2' \cup \pi_{22}(\pi_{23}^{-1}(E_3'))$ and $\hat{E}_4 = E_4 \cup \pi_{34}(\pi_{33}^{-1}(E_3))$. We use the notation $\hat{\pi}_{J_1} := \pi_{11} \circ P_{1(23),1}$ and $\hat{\pi}_{J_4} := \hat{\pi}_{234} \circ P_{1(23),2}$. By Lemma 2.6, $\hat{\mathbf{M}}_{1(23)}$ is a covering

$$J_1 \subset S^3 \stackrel{\hat{\pi}_{J_1}}{\leftarrow} \hat{\mathbf{M}}_{1(23)} \stackrel{\hat{\pi}_{J_4}}{\rightarrow} S^3 \supset J_4,$$

with branch locus the graphs

$$J_1 = E_1 \cup \pi_{11}\pi_{12}^{-1}(\hat{E}_2) \quad J_4 = \hat{E}_4 \cup \hat{\pi}_{234}\hat{\pi}_{232}^{-1}(E_2). \tag{2.26}$$

Consider now the composition $\hat{\mathbf{M}}_{12} := \mathbf{M}_1 \circ \mathbf{M}_2$. It is given by the diagram



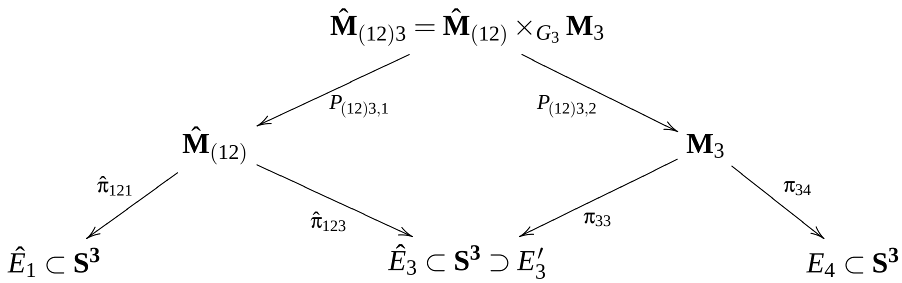
with $\hat{\pi}_{121} = \pi_{11} \circ P_{12,1}$ and $\hat{\pi}_{123} = \pi_{23} \circ P_{12,2}$. By Lemma 2.6 above, this is a branched cover

$$\hat{E}_1 \subset S^3 \stackrel{\hat{\pi}_{121}}{\leftarrow} \hat{\mathbf{M}}_{12} \stackrel{\hat{\pi}_{123}}{\rightarrow} S^3 \supset \hat{E}_3$$

where the graphs $\hat{E}_1$ and $\hat{E}_2$ are given by

$$\hat{E}_1 = E_1 \cup \pi_{11}\pi_{12}^{-1}(E_2') \quad \hat{E}_3 = E_3 \cup \pi_{23}\pi_{22}^{-1}(E_2). \tag{2.27}$$

Then the composition $\hat{\mathbf{M}}_{(12)3} := \hat{\mathbf{M}}_{12} \circ \mathbf{M}_3 = (\mathbf{M}_1 \circ \mathbf{M}_2) \circ \mathbf{M}_3$ is given by the diagram



with $\hat{E}_1 = E_1 \cup \pi_{11}(\pi_{12}^{-1}(E_2'))$ and $\hat{E}_3 = E_3 \cup \pi_{24}(\pi_{23}^{-1}(E_2))$. We have $\hat{\pi}_{I_1} := \hat{\pi}_{121} \circ P_{(12)3,1}$ and $\hat{\pi}_{I_4} := \pi_{34} \circ P_{(12)3,2}$. Again by Lemma 2.6 this is a branched covering

$$I_1 \subset S^3 \overset{\hat{\pi}_{I_1}}{\leftarrow} \hat{\mathbf{M}}_{(12)3} \overset{\hat{\pi}_{I_4}}{\to} S^3 \supset I_4,$$

with branch locus the graphs

$$I_1 = \hat{E}_1 \cup \hat{\pi}_{121}\hat{\pi}_{123}^{-1}(E_3') \quad I_4 = E_4 \cup \pi_{34}\pi_{33}^{-1}(\hat{E}_3). \tag{2.28}$$

Thus, we need to compare the branch locus

$$E_1 \cup \pi_{11}\pi_{12}^{-1}(\hat{E}_2) = E_1 \cup \pi_{11}\pi_{12}^{-1}(E_2') \cup \pi_{11}\pi_{12}^{-1}\pi_{22}\pi_{23}^{-1}(E_3')$$

with

$$\hat{E}_1 \cup \hat{\pi}_{121}\hat{\pi}_{123}^{-1}(E_3') = E_1 \cup \pi_{11}\pi_{12}^{-1}(E_2') \cup \hat{\pi}_{121}\hat{\pi}_{123}^{-1}(E_3').$$

Using Lemma 2.10, we now see that

$$\hat{\pi}_{121}\hat{\pi}_{123}^{-1}(E_3') = \pi_{11}\pi_{12}^{-1}\pi_{22}\pi_{23}^{-1}(E_3'),$$

so that the branch loci $J_1 = I_1$ agree. Similarly, we now compare the branch locus

$$\hat{E}_4 \cup \hat{\pi}_{234}\hat{\pi}_{232}^{-1}(E_2) = E_4 \cup \pi_{34}\pi_{33}^{-1}(E_3) \cup \hat{\pi}_{234}\hat{\pi}_{232}^{-1}(E_2)$$

with

$$E_4 \cup \pi_{34}\pi_{33}^{-1}(\hat{E}_3) = E_4 \cup \pi_{34}\pi_{33}^{-1}(E_3) \cup \pi_{34}\pi_{33}^{-1}\pi_{23}\pi_{22}^{-1}(E_2).$$

Again using Lemma 2.10, we see that

$$\hat{\pi}_{234}\hat{\pi}_{232}^{-1}(E_2) = \pi_{34}\pi_{33}^{-1}\pi_{23}\pi_{22}^{-1}(E_2)$$

so that the branch loci $J_4 = I_4$ also coincide. It remains to check that the multiplicities also agree. As before, to simplify the computation let us assume that $E_i = G_{i0} \cup \cdots \cup G_{ig_i}$ where the G_{ij} are subgraphs with $g_i \geq 0$ the number of the components of the graph G_i, and with $G_{10} = G_1$, $G_{20} = G_{20}' = G_2$, $G_{30} = G_{30}' = G_3$ and $G_{40} = G_4$. We also need to fix some notation for the multiplicities of each branched covering map. We assume that the maps π_{ii} and π_{ii+1}, $i = 1,2,3$, of (2.23) have multiplicities

$$\#\pi_{11}^{-1}(x) = \begin{cases} m_{11} & x \in \mathbf{S^3} \smallsetminus E_1 \\ n_{11,j} & x \in G_{1j} \; j = 0,\dots,g_1 \end{cases}$$
$$\#\pi_{12}^{-1}(x) = \begin{cases} m_{12} & x \in \mathbf{S^3} \smallsetminus E_2 \\ n_{12,j} & x \in G_{2j} \; j = 0,\dots,g_2 \end{cases}$$
$$\#\pi_{22}^{-1}(x) = \begin{cases} m_{22} & x \in \mathbf{S^3} \smallsetminus E_2' \\ n_{22,j} & x \in G_{2j}' \; j = 0,\dots,g_2' \end{cases} \tag{2.29}$$
$$\#\pi_{23}^{-1}(x) = \begin{cases} m_{23} & x \in \mathbf{S^3} \smallsetminus E_3 \\ n_{23,j} & x \in G_{3j} \; j = 0,\dots,g_3 \end{cases}$$
$$\#\pi_{33}^{-1}(x) = \begin{cases} m_{33} & x \in \mathbf{S^3} \smallsetminus E_3' \\ n_{33,j} & x \in G_{3j}' \; j = 0,\dots,g_3' \end{cases}$$
$$\#\pi_{34}^{-1}(x) = \begin{cases} m_{34} & x \in \mathbf{S^3} \smallsetminus E_4 \\ n_{34,j} & x \in G_{4j} \; j = 0,\dots,g_4 \end{cases}$$

By Lemma 2.6 we then know that the multiplicities of the composite maps are of the form

$$\#\hat{\pi}_{121}^{-1}(s) = \begin{cases} m_{11}m_{22} & s \in \mathbf{S^3} \smallsetminus \hat{E}_1 \\ n_{11,j}m_{22} & s \in G_{1j} \; j = 0,\dots,g_1 \\ m_{11}n_{22,j} & s \in \pi_{11}\pi_{12}^{-1}(G_{2j}') \; j = 1,\dots,g_2' \end{cases} \tag{2.30}$$

$$\#\hat{\pi}_{123}^{-1}(s) = \begin{cases} m_{12}m_{23} & s \in \mathbf{S^3} \smallsetminus \hat{E}_3 \\ m_{12}n_{23,j} & s \in G_{3j} \; j = 0,\dots,g_3 \\ n_{12,j}m_{23} & s \in \pi_{23}\pi_{22}^{-1}(G_{2j}) \; j = 0,\dots,g_2 \end{cases} \tag{2.31}$$

$$\#\hat{\pi}_{232}^{-1}(s) = \begin{cases} m_{22}m_{33} & s \in \mathbf{S^3} \smallsetminus \hat{E}_2 \\ n_{22,j}m_{33} & s \in G_{2j}' \; j = 0,\dots,g_2' \\ m_{22}n_{33,j} & s \in \pi_{22}\pi_{23}^{-1}(G_{3j}') \; j = 0,\dots,g_3' \end{cases} \tag{2.32}$$

$$\#\hat{\pi}_{234}^{-1}(s) = \begin{cases} m_{23}m_{34} & s \in \mathbf{S^3} \smallsetminus \hat{E}_4 \\ m_{23}n_{34,j} & s \in G_{4j} \; j = 0,\dots,g_4 \\ n_{23,j}m_{34} & s \in \pi_{34}\pi_{33}^{-1}(G_{3j}) \; j = 0,\dots,g_3 \end{cases} \tag{2.33}$$

Now we check that the composition is associative by checking that the multiplicities also agree, as we saw for the branched loci. We will begin with $\hat{\mathbf{M}}_{(12)3} = \hat{\mathbf{M}}_{(12)} \circ \mathbf{M}_3$. The projection $P_{(12)3,1} : \hat{\mathbf{M}}_{(12)3} \to \hat{\mathbf{M}}_{(12)}$ is a branched cover branched along $\hat{\pi}_{123}^{-1}(E_3') \subset \hat{\mathbf{M}}_{(12)}$ with multiplicities

$$\#P_{(12)3,1}^{-1}(x) = \begin{cases} m_{33} & x \in \hat{\mathbf{M}}_{(12)} \smallsetminus \pi_{123}^{-1}(E_3') \\ n_{33,j} & x \in \pi_{123}^{-1}(G_{3j}') \; j = 0,\dots,g_3' \end{cases} \tag{2.34}$$

Similarly, the projection $P_{(12)3,2}:\hat{\mathbf{M}}_{(12)3}\to\mathbf{M}_3$ has multiplicities

$$\#P^{-1}_{(12)3,2}(x)=\begin{cases} m_{12}m_{23} & x\in\mathbf{M}_3\smallsetminus\pi_{33}^{-1}(\hat{E}_3)\\ m_{12}n_{23,j} & x\in\pi_{33}^{-1}(G_{3j})\ \ j=0,\dots,g_3\\ n_{12,j}m_{23} & x\in\pi_{33}^{-1}(\pi_{23}\pi_{22}^{-1}(G_{2j}))\ \ j=0,\dots,g_2\end{cases}\tag{2.35}$$

Now consider the composite maps $\hat{\pi}_{I_1}=\hat{\pi}_{121}\circ P_{(12)3,1}$ and $\hat{\pi}_{I_4}=\pi_{34}\circ P_{(12)3,2}$. These are branched as described above with multiplicities

$$\#\hat{\pi}_{I_1}^{-1}(x)=\begin{cases} m_{11}m_{22}m_{33} & x\in\mathbf{S^3}\smallsetminus I_1\\ n_{11,j}m_{22}m_{33} & x\in G_{1j}\ \ j=0,\dots,g_1\\ m_{11}n_{22,j}m_{33} & x\in\pi_{11}\pi_{12}^{-1}(G'_{2j})\ \ j=0,\dots,g'_2\\ m_{11}m_{22}n_{33,j} & x\in\hat{\pi}_{121}\hat{\pi}_{123}^{-1}(G'_{3j})\ \ j=0,\dots,g'_3.\end{cases}\tag{2.36}$$

Similarly, for the map $\hat{\pi}_{I_4}=\pi_{34}\circ P_{(12)3,2}$ we obtain the multiplicities

$$\#\hat{\pi}_{I_4}^{-1}(x)=\begin{cases} m_{12}m_{23}m_{34} & x\in\mathbf{S^3}\smallsetminus I_4\\ m_{12}m_{23}n_{34,j} & x\in G_{4j}\ \ j=0,\dots,g_4\\ m_{12}n_{23,j}m_{34} & x\in\pi_{34}\pi_{33}^{-1}(G_{3j})\ \ j=0,\dots,g_3\\ n_{12,j}m_{23}m_{34} & x\in\pi_{34}\pi_{33}^{-1}\pi_{23}\pi_{22}^{-1}(G_{2j})\ \ j=0,\dots,g_2.\end{cases}\tag{2.37}$$

We now compare the multiplicities of the maps $\hat{\pi}_{I_1}$ and $\hat{\pi}_{I_2}$ to those obtained from the other composition. Namely, we consider the maps $\hat{\pi}_{J_1}=\pi_{11}\circ P_{1(23),1}$ and $\hat{\pi}_{J_2}=\hat{\pi}_{234}\circ P_{1(23),2}$. The map $P_{1(23),1}:\hat{\mathbf{M}}_{1(23)}\to\mathbf{M}_1$ is a branched cover, branched over $\pi_{12}^{-1}(\hat{E}_2)$, with $\hat{E}_2=E'_2\cup\pi_{22}(\pi_{23}^{-1}(E'_3))$. It has multiplicities as follows.

$$\#P^{-1}_{1(23),1}(x)=\begin{cases} m_{22}m_{33} & x\in\mathbf{M}_1\smallsetminus\pi_{12}^{-1}(\hat{E}_2)\\ n_{22,j}m_{33} & \pi_{12}^{-1}(G'_{2j})\ \ j=0,\dots,g'_2\\ m_{22}n_{33,j} & \pi_{12}^{-1}(\pi_{22}\pi_{23}^{-1}(G'_{3j}))\ \ j=0,\dots,g'_3\end{cases}\tag{2.38}$$

By the same argument, the projection map $P_{1(23),2}$ is a branched cover of $\hat{\mathbf{M}}_{(23)}$, branched over over $\hat{\pi}_{232}^{-1}(E_2)$ with multiplicities as follows

$$\#P^{-1}_{1(23),2}(x)=\begin{cases} m_{12} & x\in\mathbf{S^3}\smallsetminus\hat{\pi}_{232}^{-1}(E_2)\\ n_{12,j} & x\in\hat{\pi}_{232}^{-1}(G_{2j})\ \ j=0,\dots,g_2.\end{cases}\tag{2.39}$$

This gives for the composition $\hat{\pi}_{J_1}=\pi_{11}\circ P_{1(23),1}$ the multiplicities

$$\#\hat{\pi}_{J_1}^{-1}(x)=\begin{cases} m_{11}m_{22}m_{33} & x\in\mathbf{S^3}\smallsetminus J_1\\ n_{11,j}m_{22}m_{33} & x\in G_{1j}\ \ j=0,\dots,g_1\\ m_{11}n_{22,j}m_{33} & x\in\pi_{11}\pi_{12}^{-1}(G'_{2j})\ \ j=0,\dots,g'_2\\ m_{11}m_{22}n_{33,j} & x\in\pi_{11}\pi_{12}^{-1}\pi_{22}\pi_{23}^{-1}(G'_{3j})\ \ j=0,\dots,g'_3\end{cases}\tag{2.40}$$

Similarly we have

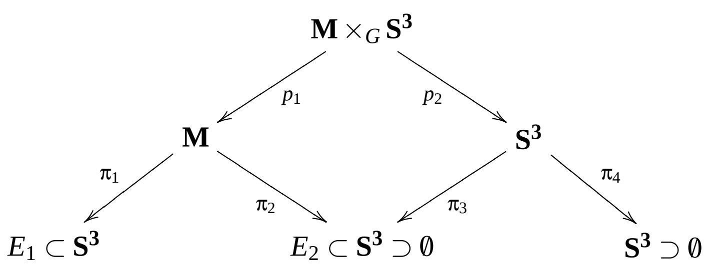
$$\#\hat{\pi}_{J_4}^{-1}(x) = \begin{cases} m_{12}m_{23}m_{34} & x \in \mathbf{S^3} \smallsetminus J_1 \\ m_{12}m_{23}n_{34,j} & x \in G_{4j} \;\; j = 0,\dots,g_4 \\ m_{12}n_{23,j}m_{34} & x \in \pi_{34}\pi_{33}^{-1}(G_{3j}) \;\; j = 0,\dots,g_3 \\ n_{12,j}m_{23}m_{34} & x \in \hat{\pi}_{234}\hat{\pi}_{232}^{-1}(G_{2j}) \;\; j = 0,\dots,g_2. \end{cases} \tag{2.41}$$

We can then see by direct comparison that the multiplicities of the maps $\hat{\pi}_{I_1}$ and $\hat{\pi}_{J_1}$ agree and so do the multiplicities of the maps $\hat{\pi}_{I_4}$ and $\hat{\pi}_{J_4}$. A similar argument can be used to compare the branching indices and show that they also match. This completes the proof that the composition is associative. □

2.3. Trivial covering and composition. Now we consider the question of the existence of an identity element for composition, *i.e.* whether there exists a 3-manifold $\mathbf{U}$, which is an element of $Hom(G',G')$ for any given embedded graph G' and with the property that, for all $\mathbf{M} \in Hom(G,G')$, the compositions $\mathbf{U} \circ \mathbf{M} = \mathbf{M}$ and $\mathbf{M} \circ \mathbf{U} = \mathbf{M}$. To this purpose, it is convenient to allow, in addition to the morphisms in $Hom(G,G)$ given by branched covers $G \subset E \subset \mathbf{S^3} \leftarrow \mathbf{M} \to \mathbf{S^3} \supset E' \supset G$ also an additional morphism representing the *unbranched case*, as we did in our definition of morphisms. Since the 3-sphere $\mathbf{S^3}$ has trivial fundamental group, we know that an unbranched covering can only be the trivial one $\mathbf{S^3} \to \mathbf{S^3}$ given by the identity (multiplicity one everywhere). We assume that the trivial covering $id : \mathbf{S^3} \to \mathbf{S^3}$ belongs to $Hom(G,G)$ for all G. We then have the following proposition.

PROPOSITION 2.12. *The trivial covering* $id : \mathbf{S^3} \to \mathbf{S^3}$ *is the identity element* $\mathbf{U}$ *for composition.*

PROOF. consider the diagram



where E_1 and E_2 are two graphs that contain G and G' respectively and are the branching locus of π_1 and π_2, respectively. The maps $\pi_3 = \pi_4 = id$ are the identity map of the trivial covering $id : \mathbf{S^3} \to \mathbf{S^3}$. The notation $\emptyset \subset \mathbf{S^3}$ means that this is an unbranched cover (empty graph). The fibered product satisfies

$$\mathbf{M} \times_{G'} \mathbf{S^3} = \{(m,s) \in \mathbf{M} \times \mathbf{S^3} \,|\, \pi_2(m) = s\} = \bigcup_{s \in \mathbf{S^3}} \pi_2^{-1}(s) = \mathbf{M}.$$

So the projection map p_1 is just the identity map $id : \mathbf{M} \to \mathbf{M}$, with the composite map $\hat{\pi}_G = \pi_1 \circ p_1 = \pi_1$. The projection map $p_2 : \mathbf{M} \times_{G'} \mathbf{S^3} \to \mathbf{S^3}$ that sends $(m,s) \mapsto s$ for $m \in \pi_2^{-1}(s)$ is just the map $p_2 = \pi_2$, and so is $\hat{\pi}_G = \pi_4 \circ p_2 = \pi_2$. Thus, we see that $\mathbf{M} \times_G \mathbf{S^3} = \mathbf{M}$ with $\pi_G = \pi_1$ and $\pi_{G'} = \pi_2$. This shows that $\mathbf{M} \circ \mathbf{U} = \mathbf{M}$. The argument for the composition $\mathbf{U} \circ \mathbf{M}$ is analogous. □

3. Representations and compositions of correspondences

We reinterpret the composition of correspondences described in §2 above from the point of view of representations of fundamental groups, using the characterization of branched coverings as in Lemma 1.4 above. The results of this section are not needed for the rest of our work, but we include them here for completeness.

We first discuss some facts about covering spaces. Let $p : X \to Z$ be a covering space of order m. If $\tilde{X}$ denotes the universal cover of X, and $G = \pi_1(Z,x)$ the fundamental group, then we have $X = \tilde{X}/N_1$ for N_1 a normal subgroup of G, with G/N_1 the group of deck transformations of the covering $X \to Z$, with $\#G/N_1 = m$. The covering $p : X \to Z$ is uniquely specified by assigning a representation $\sigma : \pi_1(Z) \to S_m$, determined up to inner automorphisms of S_m. Suppose now that $p : X \to Z$ is the composite of two covering maps $p = p_2 \circ p_1$,

$$X \stackrel{p_1}{\to} Y \stackrel{p_2}{\to} Z.$$

The covering $Y = \tilde{X}/N_2$ of Z is similarly obtained from a normal subgroup N_2 of G, so that its group of deck transformations is G/N_2, with $\#G/N_2 = n_2$. As above the covering is determined by a representation $\sigma_2 : \pi_1(Z) \to S_{n_2}$. Notice that this factors through the quotient G/N_2. Similarly, the space X, viewed as a covering of Y is determined by a representation $\sigma_1 : \pi_1(Y) = N_2 \to S_{n_1}$, where $n_1 = \#N_2/N_1$ and N_1 is a normal subgroup of N_2 and $H = N_2/N_1$ is the group of deck transformations of the covering $p_1 : X \to Y$. We have $m = n_1 n_2$. The representations σ, σ_1 and σ_2 are related in the following way.

LEMMA 3.1. *Let G and N_2 be as above. Then the representation $\sigma : G \to S_m$ is given by*

$$x_{i,j} \mapsto x_{\sigma(\gamma)(i,j)} = x_{\sigma_1(h_j)(i),\sigma_2(g)(j)}, \tag{3.1}$$

where $g = \gamma \mod N_2$ and $h_j \in N_2 \simeq \pi_1(Y,\tilde{x}_j)$ is determined by an identification $\tilde{\gamma}_j = h_j g$ of the homotopy classes of paths $\mathcal{P}_{\tilde{x}_j, g\tilde{x}_j} \simeq \pi_1(Y,\tilde{x}_j)g$, where $\tilde{\gamma}_j$ is the lift of the path $\gamma \in \pi_1(Z,x)$ to the covering Y starting at the point $\tilde{x}_j \in p_2^{-1}(x)$.

PROOF. Consider a path $\gamma \in \pi_1(Z,x)$ and a chosen point $\tilde{x} \in p_2^{-1}(x) \subset Y$. We denote by $\tilde{\gamma}$ the unique path lifting γ starting at $\tilde{\gamma}(0) = \tilde{x}$. It has $\tilde{\gamma}(1) = g\tilde{x} \in p_2^{-1}(x)$, where $g \in G/N_2$ is the corresponding deck transformation with $g = \gamma \mod N_2$. We can identify the set of homotopy classes $\mathcal{P}_{\tilde{x},g\tilde{x}}$ with the set

$$\pi_1(Y,\tilde{x})g := \{\tilde{\gamma} \circ \gamma' \,|\, \gamma' \in \pi_1(Y,\tilde{x})\},$$

with $g = \tilde{\gamma} \mod N_2$. For $g = g_1 g_2 \in G/N_2$, we obtain

$$\pi_1(Y,\tilde{x})g = \{\tilde{\gamma}_2 \circ \tilde{\gamma}_1 \circ \gamma' \,|\, \gamma' \in \pi_1(Y,\tilde{x})\} = \{\tilde{\gamma}_2 \circ \tilde{\gamma}_1 \circ \tilde{\gamma}_1^{-1} \circ \gamma'' \circ \tilde{\gamma}_1 \,|\, \gamma'' \in \pi_1(Y, g_1\tilde{x})\} \tag{3.2}$$

with $g_i = \tilde{\gamma}_i \mod N_2$. Let us then look more precisely at the representation $\sigma_1 : \pi_1(Y) \to S_{n_1}$ describing the covering $p_1 : X \to Y$. Suppose given elements $h \in N_2 \subset G$, and $\gamma h \gamma^{-1} \in N_2$, for some $\gamma \in G$. Then, we identify $N_2 = \pi_1(Y,\tilde{x})$ for a choice of a base point $\tilde{x} \in p_2^{-1}(x) \subset Y$. The lift of the path γ to the covering Y in general will not be close but will send the initial point $\tilde{x}$ to the point $g\tilde{x}$ where $g = \gamma \mod N_2$ the class in G/N_2 acting as the group of deck transformations. Thus, the path $\gamma h \gamma^{-1}$ in $G = \pi_1(Z,x)$ defines an element in $\pi_1(Y, g\tilde{x})$, for the new base point. Thus, when we consider the representation $\sigma_1 : \pi_1(Y,\tilde{x}) \to S_{n_1}$ and we identify it with a representation $\sigma_1 : N_2 \to S_{n_1}$, we should more precisely regard this as a pair $(\sigma_1, \tilde{x})$ of a representation of N_2 and a choice of a base point that gives the identification $N_2 \simeq \pi_1(Y,\tilde{x})$. Then the action of an element $\gamma \in G$ by conjugation on $h \in N_2$ produces an element $\gamma h \gamma^{-1} \in N_2$, as well as a deck transformation $g = \gamma \mod N_2$ that changes the base point $\tilde{x}$ to $g\tilde{x}$. The representation $\sigma_1 : \pi_1(Y,\tilde{x}) \to S_{n_1}$ is not invariant under this action, because the base point is not preserved, but the set of pairs $(\sigma_1, \tilde{x})$ is and it is acted upon by G as

$$Ad_\gamma : (\sigma_1, \tilde{x}) \mapsto (\sigma_1 \circ Ad_\gamma, g\tilde{x}), \tag{3.3}$$

for $\sigma_1 \circ Ad_\gamma : N_2 \to S_{n_1}$ given by $\sigma_1 \circ Ad_\gamma(h) = \sigma_1(\gamma h \gamma^{-1})$ and $g \in G/N_2$ given by $g = \gamma \mod N_2$. Equivalently, we think of the pairs $(\sigma_1, \tilde{x})$ as a representation $\sigma_1 : N_2 \to (S_{n_1})^{n_2}$, where $n_2 = \# p_2^{-1}(x)$, that maps

$$\sigma_1(h) = (\tilde{s} h \tilde{s}^{-1})_{s \in G/H}, \tag{3.4}$$

where the $\tilde{s}$ are a chosen lift of the $s \in G/H$. We can write (3.4) equivalently as $(\sigma_1(h_s))_{s\in G/H}$, or again equivalently as $(\sigma_1(h_j))_{j=1\dots,n_2} \in (S_{n_1})^{n_2}$ as in (3.1). The action (3.3) becomes of the form

$$Ad_\gamma : \sigma_1(h) \mapsto \sigma_g \sigma_1(h) \sigma_g^{-1}, \tag{3.5}$$

where $g = \gamma \mod N_2$ and σ_g is the permutation in S_{n_2} that sends the point $s \in G/H$ to $sg \in G/H$. This shows that the representation σ_1 satisfies

$$\sigma_1(\gamma h \gamma^{-1}) = \sigma_g \sigma_1(h) \sigma_g^{-1}, \quad \text{with} \quad g = \gamma \mod N_2, \tag{3.6}$$

for permutations $\sigma_g \in S_{n_2}$ as above. The expression (3.1) defines an element in S_m for $m = n_1 n_2$. To see that it is a representation of G it suffices to show compatibility with the product. For $\gamma = \gamma_1\gamma_2$ we have

$$\sigma_{\gamma_1\gamma_2} = \sigma_1(h)\sigma_2(g),$$

where by (3.6) and (3.2)

$$\sigma_1(h) = \sigma_1(h_1)\sigma_{g_1}\sigma_1(h_2)\sigma_{g_1}^{-1}$$

By construction the matrices $\sigma_g \in S_{n_2}$ are the elements $\sigma_2(g)$ of the representation $\sigma_2 : \pi_1(Z) \to S_{n_2}$ describing the covering $p_1 : X \to Y$. □

We now describe the composition of correspondences of the form (2.1) in terms of representations of the fundamental groups of the complement of the branch loci. Suppose given, as before, two 3-manifolds $\mathbf{M}$ and $\tilde{\mathbf{M}}$ with branched covering maps as in (2.1),

$$G \subset \mathbf{S^3} \xleftarrow{\pi_G} \mathbf{M} \xrightarrow{\pi_{G'}} \mathbf{S^3} \supset E_1 \supset G' \quad \text{and} \quad G' \subset E_2 \subset \mathbf{S^3} \xleftarrow{\tilde{\pi}_{G'}} \tilde{\mathbf{M}} \xrightarrow{\tilde{\pi}_{G''}} \mathbf{S^3} \supset G'',$$

These correspond to the data of representations

$$\begin{array}{ll} \sigma_G : \pi_1(\mathbf{S^3} \smallsetminus G) \to S_m & \sigma_{E_1} : \pi_1(\mathbf{S^3} \smallsetminus E_1) \to S_{m'} \\ \tilde{\sigma}_{E_2} : \pi_1(\mathbf{S^3} \smallsetminus E_2) \to S_{\tilde{m}'} & \tilde{\sigma}_{G''} : \pi_1(\mathbf{S^3} \smallsetminus G'') \to S_{\tilde{m}''}, \end{array} \tag{3.7}$$

where E_1 and E_2 are two graphs containing the subgraph G'

PROPOSITION 3.2. *The composition $\hat{\mathbf{M}} = \mathbf{M} \times_{G'} \tilde{\mathbf{M}}$ is the branched covering*

$$E = G \cup \pi_G \pi_{G'}^{-1}(E_2) \subset \mathbf{S^3} \xleftarrow{\hat{\pi}_E} \hat{\mathbf{M}} \xrightarrow{\hat{\pi}_{E''}} \mathbf{S^3} \supset E'' = G'' \cup \tilde{\pi}_{G''} \tilde{\pi}_{G'}^{-1}(E_1), \tag{3.8}$$

with $\hat{\pi}_E = \pi_G \circ P_1$ and $\hat{\pi}_{E''} = \tilde{\pi}_{G''} \circ P_2$. This corresponds to the representations

$$\hat{\sigma}_E : \pi_1(\mathbf{S^3} \smallsetminus E) \to S_{m\tilde{m}'} \quad \text{and} \quad \hat{\sigma}_{E''} : \pi_1(\mathbf{S^3} \smallsetminus E'') \to S_{\tilde{m}''m'} \tag{3.9}$$

given by

$$\hat{\sigma}_E(\gamma) = \tilde{\sigma}_{E_2}(\pi_{G'}(\underline{\gamma}))\sigma_G(\iota_G(\gamma)) \quad \text{and} \quad \hat{\sigma}_{E''}(\gamma) = \sigma_{E_1}(\tilde{\pi}_{G'}(\underline{\gamma}))\tilde{\sigma}_{G''}(\iota_{G''}(\gamma)). \tag{3.10}$$

Here $\iota_G : \pi_1(\mathbf{S^3} \smallsetminus E) \to \pi_1(\mathbf{S^3} \smallsetminus G)$ and $\iota_{G''} : \pi_1(\mathbf{S^3} \smallsetminus E'') \to \pi_1(\mathbf{S^3} \smallsetminus G'')$ are the group homomorphisms induced by inclusion. The elements $\underline{\gamma}$ denote the collection of lifts of $\iota_G(\gamma)$ to paths in $\mathbf{M}$ (or of $\iota_{G''}(\gamma)$ to $\tilde{\mathbf{M}}$, respectively), depending on the choice of a point in the fiber of the covering $\pi_G : \mathbf{M} \to \mathbf{S^3}$ (respectively, $\tilde{\pi}_{G''} : \tilde{\mathbf{M}} \to \mathbf{S^3}$).

PROOF. Let $\mathbf{G}_E = \pi_1(\mathbf{S^3} \smallsetminus E, s)$ and $\mathbf{G}_G = \pi_1(\mathbf{S^3} \smallsetminus G, s)$. Since $\hat{\pi}_E$ is a branched covering map of order $m\tilde{m}'$, branched along $E = G \cup \pi_G \pi_{G'}^{-1}(E_2)$, then by Lemma 1.4 the covering is determined by the datum of a representation

$$\hat{\sigma}_E : \pi_1(\mathbf{S^3} \smallsetminus E) \to S_{m\tilde{m}'}.$$

The covering can be described in terms of a normal subgroup $N_E = (\hat{\pi}_E)_* \pi_1(\hat{\mathbf{M}} \smallsetminus \hat{\pi}_E^{-1}(E))$ of $\mathbf{G}_E$ with $\mathbf{G}_E/N_E$ the group of deck transformations with $\#\mathbf{G}_E/N_E = m\tilde{m}'$. On the other hand, $\hat{\pi}_E$ is a composition of two covering maps $\hat{\pi}_E = \pi_G \circ P_1$. Thus, we can use the result of Lemma 3.1 above to describe it in terms of the representations associated to π_G and P_1. The covering π_G corresponds to a normal subgroup $N_G = (\pi_G)_* \pi_1(\mathbf{M} \smallsetminus \pi^{-1}(G))$ of $\mathbf{G}_G$ such that $\mathbf{G}_G/N_G$ is the group of deck transformations, of order $\#\mathbf{G}_G/N_G = m$. The covering π_G is determined by a representation $\sigma_G : \pi_1(\mathbf{S^3} \smallsetminus G) \to S_m$. In the same way, the covering map P_1 is branched along the set $\pi_{G'}^{-1}(E_2) \subset \mathbf{M}$, hence the covering is specified by a representation

$$\sigma_{P_1} : \pi_1(\mathbf{M} \smallsetminus \pi_{G'}^{-1}(E_2)) \to S_{\tilde{m}'}.$$

In terms of normal subgroups, this covering corresponds to a subgroup $N_{E_2} \subset \pi_1(\mathbf{M} \smallsetminus \pi_{G'}^{-1}(E_2))$. The quotient $H = \pi_1(\mathbf{M} \smallsetminus \pi_{G'}^{-1}(E_2))/N_{E_2}$ gives the group of deck transformations of the covering with $\#H = \tilde{m}'$. Consider the group homomorphism

$$(\pi_{G'})_* : \pi_1(\mathbf{M} \smallsetminus \pi_{G'}^{-1}(E_2)) \to \pi_1(\mathbf{S^3} \smallsetminus E_2) \tag{3.11}$$

induced by the covering map $\pi_{G'} : \mathbf{M} \to \mathbf{S^3}$, branched along E_1. This induces a map of representations

$$\mathrm{Hom}(\pi_1(\mathbf{S^3} \smallsetminus E_2), S_{\tilde{m}'}) \to \mathrm{Hom}(\pi_1(\mathbf{M} \smallsetminus \pi_{G'}^{-1}(E_2)), S_{\tilde{m}'})$$

given by composition $\sigma \mapsto \sigma \circ (\pi_{G'})_*$. Let $\tilde{\sigma}_{E_2} : \pi_1(\mathbf{S^3} \smallsetminus E_2) \to S_{\tilde{m}'}$ be the representation that describes the covering

$$\tilde{\mathbf{M}} \xrightarrow{\tilde{\pi}_{G'}} \mathbf{S^3} \supset E_2 \supset G'.$$

Claim: The representation σ_{P_1} satisfies $\sigma_{P_1} = \tilde{\sigma}_{E_2} \circ (\pi_{G'})_*$.

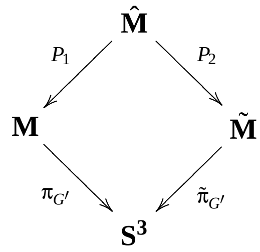
PROOF. For a chosen base point $x \in \mathbf{M} \smallsetminus \pi_{G'}^{-1}(E_2)$, let $\gamma \in \pi_1(\mathbf{M} \smallsetminus \pi_{G'}^{-1}(E_2), x)$. Let then $\hat{\gamma}$ be a lifting of the path γ to $\hat{\mathbf{M}}$, which starts at a chosen point $(x, y_1) \in \hat{\mathbf{M}}$, with $y_1 \in P^{-1}(x)$ and $\pi_{G'}(x) = \tilde{\pi}_{G'}(y)$. We denote by (x, y_2) the endpoint of $\hat{\gamma}$. This is another point in the same fiber, that is, with $y_2 \in P^{-1}(x)$ and $(x, y_2) = \sigma_{P_1}(\gamma)(x, y_1)$. The point (x, y_2) is uniquely determined by (x, y_1) and the homotopy class of γ. By definition, the permutation $\tilde{\sigma}_{E_2}(\gamma) \in S_{\tilde{m}'}$ is the permutation

$$\sigma_{P_1}(\gamma) : (x, y_1) \mapsto (x, y_2). \tag{3.12}$$

On the other hand the image $(\pi_{G'})_*(\gamma)$ under the group homomorphism (3.11) determines an element in $\pi_1(\mathbf{S^3} \smallsetminus E_2, \pi_{G'}(x))$. Let us denote this element by γ', with $\pi_{G'}(x)$ the base point. Then for any given point $\tilde{y}_1 \in \tilde{\mathbf{M}}$ such that $\pi_{G'}(x) = \tilde{\pi}_{G'}(\tilde{y}_1)$, there exists a unique lift $\tilde{\gamma}'$ of γ', which starts at $\tilde{y}_1 \in \tilde{\pi}_{G'}^{-1}(\pi_{G'}(x))$. We denote by $\tilde{y}_2$ the endpoint of this path. This is also a point in the fiber $\tilde{\pi}_{G'}^{-1}(\pi_{G'}(x))$ and it is uniquely determined by $\tilde{y}_1$ and γ'. The permutation $\tilde{\sigma}_{E_2}(\gamma') \in S_{\tilde{m}'}$ is given by

$$\tilde{\sigma}_{E_2}(\gamma') : \tilde{y}_1 \mapsto \tilde{y}_2. \tag{3.13}$$

Notice that, since $\pi_{G'}(x) = \tilde{\pi}_{G'}(\tilde{y}_1)$, we have $(x, \tilde{y}_1) \in \hat{\mathbf{M}}$. Thus, as above, we can consider the lift $\hat{\gamma}$ of γ to $\hat{\mathbf{M}}$ that starts at this point $(x, \tilde{y}_1) \in P_1^{-1}(x)$. We want to show that the endpoint of this path is $(x, \tilde{y}_2) \in \hat{\mathbf{M}}$ with $\tilde{y}_2$ the endpoint of the path $\tilde{\gamma}'$, as above. This will imply, by (3.12) and (3.13) that the permutations $\sigma_{P_1}(\gamma)$ and $\tilde{\sigma}_{E_2}(\gamma')$ are the same. Now, since the diagram



is commutative, we have

$$\tilde{\pi}_{G'}(P_2(\hat{\gamma})) = \pi_{G'}(P_1(\hat{\gamma})) = \pi_{G'}(\gamma) = \gamma'. \tag{3.14}$$

This mean that $P_2(\hat{\gamma})$ is a lifting path of γ', which starts at $(x,\tilde{y}_1)$. By the uniqueness of the lifting for a chosen initial point, we have $P_2(\hat{\gamma}) = \tilde{\gamma}'$, so that both paths end at the same point $(x,\tilde{y}_2)$. This implies that $\sigma_{P_1}(\gamma) = \tilde{\sigma}_{E_2}(\gamma') \in S_{\tilde{m}'}$, which proves the claim. We now apply the result of Lemma 3.1.

Consider a path $\gamma \in \mathbf{G}_E$. Under the restriction map

$$\iota_G : \pi_1(\mathbf{S^3} \smallsetminus (G \cup \pi_G \pi_{G'}^{-1}(E_2))) \to \pi_1(\mathbf{S^3} \smallsetminus G)$$

induced by the inclusion, we can identify γ with an element $\iota_G(\gamma) = \gamma \in \mathbf{G}_G$, hence we can apply to it the representation σ_G to obtain an element $\sigma_G(\iota_G(\gamma)) \in S_m$. For a chosen base point $x \in \pi_G^{-1}(s) \subset \mathbf{M} \setminus \pi_G^{-1}(G) \cup \pi_{G'}^{-1}(E_2)$, there exist $gx \in \pi_G^{-1}(s)$ such that the unique lifting $\overline{\gamma}$ of γ with starting point x ends at the point gx. The deck transformation g is the element of $\mathbf{G}_G/N_G$ satisfying $g = \gamma \bmod N_G$. Thus, in the same way as before, we can parameterize the set of lifts of elements in $\pi_1(\mathbf{S^3} \smallsetminus (G \cup \pi_G \pi_{G'}^{-1}(E_2)), s)$ with the set

$$\cup_{g \in G_G/N_G} \pi_1(\mathbf{M} \smallsetminus (\pi_G^{-1}(G) \cup \pi_G^{-1}(E_2)), x) g.$$

Again we have a group homomorphism

$$\iota_{E_2} : \pi_1(\mathbf{M} \smallsetminus (\pi_G^{-1}(G) \cup \pi_G^{-1}(E_2)), x) \to \pi_1(\mathbf{M} \smallsetminus \pi_G^{-1}(E_2)), x)$$

induced by the inclusion. Thus, we can apply the representation $\sigma_{P_1} : \pi_1(\mathbf{M} \smallsetminus \pi_{G'}^{-1}(E_2)) \to S_{\tilde{m}'}$ to an element $\iota_{E_2}(\tilde{\gamma})$, for $\tilde{\gamma}$ in $\pi_1(\mathbf{M} \smallsetminus (\pi_G^{-1}(G) \cup \pi_G^{-1}(E_2)), x)$, such that $\tilde{\gamma} g$ describes a lift of γ to $\mathbf{M}$ as above. The change of base point $x \mapsto gx$ corresponds to an action $\alpha \mapsto \gamma\alpha\gamma^{-1}$ on the normal subgroup

$$(\pi_G)_* \pi_1(\mathbf{M} \smallsetminus (\pi_G^{-1}(G) \cup \pi_G^{-1}(E_2)), x) \subset \pi_1(\mathbf{S^3} \smallsetminus (G \cup \pi_G \pi_{G'}^{-1}(E_2)), s).$$

As in the proof of Lemma 3.1, we can shift the pairs $(\sigma_{P_1} \circ \iota_{E_2}, x)$ to $(\sigma_{P_1} \circ \iota_{E_2} \circ Ad_\gamma, gx)$ with the action

$$\sigma_{P_1} \circ \iota_{E_2} \circ Ad_\gamma : \pi_1(\mathbf{M} \smallsetminus (\pi_G^{-1}(G) \cup \pi_{G'}^{-1}(E_2))) \to S_{\tilde{m}'}$$

given by

$$\sigma_{P_1} \circ \iota_{E_2} \circ Ad_\gamma(\alpha) = \sigma_{P_1} \circ \iota_{E_2}(\gamma\alpha\gamma^{-1})$$

for g the image of γ in the quotient of $\pi_1(\mathbf{S^3} \smallsetminus (G \cup \pi_G \pi_{G'}^{-1}(E_2)), s)$ by the normal subgroup $N = (\pi_G)_* \pi_1(\mathbf{M} \smallsetminus (\pi_G^{-1}(G) \cup \pi_G^{-1}(E_2)), x)$. Since our covering map π_G is of order m, then the representations (σ_{P_1}, x) define an m-vector of representations, or equivalently a single map

$$\sigma_{P_1} : \pi_1(\mathbf{M} \smallsetminus \pi_{G'}^{-1}(E_2)) \to (S_{\tilde{m}'})^m.$$

We write this equivalently as in Lemma 3.1 in the form $(\sigma_{P_1}(\alpha_j))_{j=1\dots,m} \in (S_{\tilde{m}'})^m$. We then have $\sigma_{P_1}(\gamma\alpha\gamma^{-1}) = \sigma_g \sigma_{P_1}(\alpha) \sigma_g^{-1}$ with $g = \gamma \bmod N$, where σ_g is the permutation in S_m determined by the deck transformation g, so that we get $\hat{\sigma}_E(\gamma) = \sigma_{P_1}(\alpha_j) \sigma_G(\gamma)$. We then apply the result of the Claim above, and replace $\sigma P_1(\alpha_j) = \tilde{\sigma}_{G'} \circ (\pi_{G'})_*(\alpha_j)$ and this complete the proof of the statement. $\square$

4. Semigroupoids and additive categories

A semigroupoid can be thought of as a generalized semigroup in which only certain multiplications are possible.

A semigroupoid on a set $\mathcal{S}$ is a set $\mathcal{G}$ together with the following pair of maps (s,r)

$$\mathcal{G} \underset{r}{\overset{s}{\rightrightarrows}} \mathcal{S}$$

s is called the source while r is called the range. To each element $\alpha \in \mathcal{G}$ we assigns an arrow from $s(\alpha) = x$ to $r(\alpha) = y$ in $\mathcal{S}$

$$s(\alpha) = x \xrightarrow{\alpha} r(\alpha) = y$$

Define the set of composable pairs

$$\mathcal{G}^2 = \{(\alpha,\beta) \in \mathcal{G} \times \mathcal{G} \,|\, s(\alpha) = r(\beta)\}$$

with a product $m : \mathcal{G}^2 \longrightarrow \mathcal{G}$ defined by

$$m(\alpha,\beta) = \alpha\beta = \alpha \circ \beta.$$

Now if $\beta : s(\beta) = x \longrightarrow r(\beta) = y = s(\alpha)$ And $\alpha : s(\alpha) = y \longrightarrow z = r(\alpha)$ Then

$$\alpha\beta : x = s(\beta) = s(\alpha\beta) \longrightarrow z = r(\alpha) = r(\alpha\beta)$$

as in the diagram

$$x \xrightarrow{\beta} y \xrightarrow{\alpha} z, \qquad x \xrightarrow{\alpha\beta = \alpha\circ\beta} z$$

The multiplication m is an associative *i.e.* $\alpha(\beta\delta) = (\alpha\beta)\delta$.

An embedding $\gamma : \mathcal{S} \longrightarrow \mathcal{G}$ is called a unit section if it satisfies

$$\gamma(r(\alpha))\alpha = \alpha = \alpha\gamma(s(\alpha)), \quad \forall \alpha \in \mathcal{G}.$$

Notice that it is not necessary in general that all the $\gamma(r(\alpha)) = \gamma(s(\alpha)) = \gamma$, but if they are all equal then $\mathcal{G}$ is a semigroup and γ is the unit of the semigroup.

A semigroupoid is the same thing as a small category which is a category in which both objects and $Hom(,)$ are actually sets. We denote by $\mathcal{U}(\mathcal{G})$ the set of units of $\mathcal{G}$. A semigroupoid is regular if, for all $\alpha \in \mathcal{G}$ there exist units γ and γ' such that $\gamma\alpha$ and $\alpha\gamma'$ are defined. Such units, if they exist, are unique (for each α). To each unit $\gamma \in \mathcal{U}(\mathcal{G})$ in a regular semigroupoid one associates a subsemigroupoid $\mathcal{G}_\gamma = \{\alpha \in \mathcal{G} \,|\, \gamma(s(\alpha)) = \gamma\}$.

A semigroupoid (*cf.* [**37**]) gives rise to a hierarchy of sets

$$\mathcal{G}^0 = \gamma(\mathcal{S}) \simeq \mathcal{S}$$

$$\mathcal{G}^1 = \mathcal{G}$$

$$\mathcal{G}^2 = \{(\alpha,\beta) \in \mathcal{G} \times \mathcal{G} \,|\, s(\alpha) = r(\beta)\}$$

$$\mathcal{G}^3 = \{(\alpha,\beta,\gamma) \in \mathcal{G} \times \mathcal{G} \times \mathcal{G} \,|\, s(\gamma) = r(\beta), s(\beta) = r(\alpha)\}$$

and so on, by considering successive compositions of morphisms.

We can reformulate the results on embedded graphs and 3-manifolds obtained in the previous section in terms of semigroupoids in the following way.

LEMMA 4.1. *The set of compact oriented 3-manifolds forms a regular semigroupoid, whose set of units is identified with the set of embedded graphs.*

PROOF. We let $\mathcal{G}$ be the collection of data $\alpha = (\mathbf{M}, G, G')$ with $\mathbf{M}$ a closed oriented 3-manifold with branched covering maps to $\mathbf{S^3}$ of the form (1.6). We define a composition rule as in Definition 2.5, given by the fibered product. In the multi-connected case, for

$$M = M_1 \cup M_2 \cup \cdots \cup M_k \tag{4.1}$$

with (M_i, G, G') as in (1.6) with M_i connected, we extend the composition $M \circ \tilde{M}$ to mean

$$M \circ \tilde{M} = M_1 \circ \tilde{M} \cup M_2 \circ \tilde{M} \cup \cdots \cup M_k \circ \tilde{M}, \tag{4.2}$$

and similarly for $\tilde{M}$ multi-connected. It is necessary to include the multi-connected case since the fibered product of connected manifolds may consist of different connected components. We impose the condition that the composition of $\alpha = (\mathbf{M}_1, G_1, G_1')$ and $\beta = (\mathbf{M}_2, G_2, G_2')$ is only defined when the $G_1' = G_2$. By Lemma 2.12, we know that, for each $\alpha = (\mathbf{M}, G, G') \in \mathcal{G}$ the source and range are given by the trivial coverings $\gamma = \mathbb{U}_G = (\mathbb{U}, G, G)$ and $\gamma' = \mathbb{U}_{G'} = (\mathbb{U}, G', G')$. That is, we can identify them with $s(\alpha) = G$ and $r(\alpha) = G'$. Thus, the set of units $\mathcal{U}(\mathcal{G})$ is the set of embedded graphs in $\mathbf{S^3}$. □

For a given embedded graph G, the subsemigroupoid $\mathcal{G}_G$ is given by the set of all 3-manifolds that are coverings of $\mathbf{S^3}$ branched along embedded graphs E containing G as a subgraph.

Given a semigroupoid $\mathcal{G}$, and a commutative ring R, one can define an associated semigroupoid ring $R[\mathcal{G}]$, whose elements are finitely supported functions $f : \mathcal{G} \to R$, with the associative product

$$(f_1 * f_2)(\alpha) = \sum_{\alpha_1, \alpha_2 \in \mathcal{G} : \alpha_1 \alpha_2 = \alpha} f_1(\alpha_1) f_2(\alpha_2). \tag{4.3}$$

Elements of $R[\mathcal{G}]$ can be equivalently described as finite R-combinations of elements in $\mathcal{G}$, namely $f = \sum_{\alpha \in \mathcal{G}} a_\alpha \delta_\alpha$, where $a_\alpha = 0$ for all but finitely many $\alpha \in \mathcal{G}$ and $\delta_\alpha(\beta) = \delta_{\alpha,\beta}$, the Kronecker delta. The following statement is a semigroupoid version of the representations of groupoid algebras generalizing the regular representation of group rings.

LEMMA 4.2. *Suppose given a unit $\gamma \in \mathcal{U}(\mathcal{G})$. Let $\mathcal{H}_\gamma$ denote the R-module of finitely supported functions $\xi : \mathcal{G}_\gamma \to R$. The action*

$$\rho_\gamma(f)(\xi)(\alpha) = \sum_{\alpha_1 \in \mathcal{G}, \alpha_2 \in \mathcal{G}_\gamma : \alpha = \alpha_1 \alpha_2} f(\alpha_1) \xi(\alpha_2), \tag{4.4}$$

for $f \in R[\mathcal{G}]$ and $\xi \in \mathcal{H}_\gamma$, defines a representation of $R[\mathcal{G}]$ on $\mathcal{H}_\gamma$.

PROOF. We have

$$\rho_\gamma(f_1 * f_2)(\xi)(\alpha) = \sum (f_1 * f_2)(\alpha_1) \xi(\alpha_2)$$
$$= \sum_{\beta_1 \beta_2 = \alpha_1 \in \mathcal{G}} \sum_{\alpha_1 \alpha_2 = \alpha} f_1(\beta_1) f_2(\beta_2) \xi(\alpha_2) = \sum_{\beta_1 \beta = \alpha} f_1(\beta_1) \rho_\gamma(f_2)(\xi)(\beta),$$

hence $\rho_\gamma(f_1 * f_2) = \rho_\gamma(f_1) \rho_\gamma(f_2)$. Since for elements of a semi-groupoid the range satisfies $s(\alpha\beta) = s(\beta)$, the action is well defined on $\mathcal{H}_\gamma$. □

In the next section we see that in fact the difference in the representation (4.4) between the semigroupoid and the groupoid case manifests itself in the compatibility with the involutive structure.

A semigroupoid is just an equivalent formulation of a small category, so the result above simply states that embedded graphs form a small category with the sets $Hom(G, G')$ as morphisms. Passing from the semigroupoid $\mathcal{G}$ to $R[\mathcal{G}]$ corresponds to passing from a small category to its additive envelope, as follows.

5. Categories of graphs and correspondences

In the previous discussion on correspondences we introduced a category of graphs and correspondences, see Lemma 4.1 above. We will later refine them by introducing suitable equivalence relations on the correspondences. Here we first describe the additive envelope of the small category of Lemma 4.1.

DEFINITION 5.1. We let $\mathcal{K}$ denote the category whose objects $Obj(\mathcal{K})$ are graphs $G \subset \mathbf{S^3}$ and whose morphisms $\phi \in Hom(G,G')$ are $\mathbb{Q}$-linear combinations $\sum_i a_i \mathbf{M}_i$ of 3-manifold $\mathbf{M}_i$ with submersions π_E and $\pi_{E'}$ to $\mathbf{S^3}$ as in Definition 2.4, including the trivial (unbranched) covering in all the $Hom(G,G)$ as in Proposition 2.12.

LEMMA 5.2. *The category $\mathcal{K}$ is a small pre-additive category.*

PROOF. Notice that $Obj(\mathcal{K})$ is a set, since tamely embedded graphs in $\mathbf{S^3}$ can be identified with linearly embedded graphs in $\mathbf{S^3}$ and that 3-manifolds are here described by representation theoretic data $\pi_1(M \smallsetminus E') \longrightarrow S_m$ that also form a set, so that $\mathcal{K}$ is a small category. We have seen that the trivial unbranched covering is the identity for composition. This shows that, for each object $G \in Obj(\mathcal{K})$, there is an identity morphism $id_G \in Hom(G,G)$. We have also proved that associativity of composition holds. Thus, $\mathcal{K}$ is a category.

DEFINITION 5.3. A pre-additive category $\mathcal{C}$ is a category such that, for any $\mathcal{O},\mathcal{O}' \in Obj(\mathcal{C})$ the set of morphisms $Hom(\mathcal{O},\mathcal{O}')$ is an abelian group and the composition of maps is a bilinear operation, that is, for $\mathcal{O},\mathcal{O}',\mathcal{O}'' \in Obj(\mathcal{C})$ the composition

$$\circ : Hom(\mathcal{O},\mathcal{O}') \otimes Hom(\mathcal{O}',\mathcal{O}'') \longrightarrow Hom(\mathcal{O},\mathcal{O}'')$$

is a bilinear homomorphism.

In our case, the set of morphisms $Hom(G,G')$ is an abelian group with the addition of coefficients. In fact, we can write morphisms $\phi = \sum_i a_i \mathbf{M}_i$ equivalently as $\phi = \sum_{\mathbf{M}} a_{\mathbf{M}} \mathbf{M}$, where the sum ranges over the set of all 3-manifolds that are branched covers

$$G \subset E \subset \mathbf{S^3} \stackrel{\pi_E}{\longleftarrow} \mathbf{M} \stackrel{\pi_{E'}}{\longrightarrow} \mathbf{S^3} \supset E' \supset G'$$

and all but finitely many of the coefficients $a_{\mathbf{M}}$ are zero. Then, for $\phi = \sum a_{\mathbf{M}} \mathbf{M}$ and $\phi' = \sum b_{\mathbf{M}} \mathbf{M}$, we have $\phi + \phi' = \sum_{\mathbf{M}} (a_{\mathbf{M}} + b_{\mathbf{M}}) \mathbf{M}$. The composition rule given by the fibered product of 3-manifolds extends to linear combinations by

$$\phi' \circ \phi = (\sum_i a_i \mathbf{M}_i) \circ (\sum_j b_j \mathbf{M}_j) = \sum_{i,j} a_i b_j \mathbf{M}_i \circ \mathbf{M}_j.$$

This gives a bilinear homomorphism

$$Hom(G,G') \otimes Hom(G',G'') \to Hom(G,G'').$$

This shows that $\mathcal{K}$ is a pre-additive category. □

DEFINITION 5.4. Suppose given a pre-additive category $\mathcal{C}$. Then the additive category $Mat(\mathcal{C})$ is defined as follows (*cf.* [**3**]).

(1) The objects in $Obj(Mat(\mathcal{C}))$ are formal direct sums $\bigoplus_{i=1}^n \mathcal{O}_i$ of objects $\mathcal{O}_i \in Obj(\mathcal{C})$, where we allow for the direct sum to be possibly empty.
(2) If $F : \mathcal{O}' \to \mathcal{O}$ is a morphism in $Mat(\mathcal{C})$ with objects $\mathcal{O} = \bigoplus_{i=1}^m \mathcal{O}_i$ and $\mathcal{O}' = \bigoplus_{j=1}^n \mathcal{O}_j$ then $F = F_{ij}$ is a $m \times n$ matrix of morphisms $F_{ij} : \mathcal{O}'_j \to \mathcal{O}_i$ in $\mathcal{C}$. The abelian group structure on $Hom_{Mat(\mathcal{C})}(\mathcal{O}',\mathcal{O})$ is given by matrix addition and the abelian group structure of $Hom_{\mathcal{C}}(\mathcal{O}'_j,\mathcal{O}_i)$.

(3) The composition of morphisms in $Mat(\mathcal{C})$ is defined by the rule of matrix multiplication and the composition of morphisms in $\mathcal{C}$.

Then $Mat(\mathcal{C})$ is called the *additive closure* of $\mathcal{C}$. For more details see for instance [**3**].

In the following, for simplicity of notation, we continue to use the notation $\mathcal{K}$ for the additive closure of the category $\mathcal{K}$ of Definition 5.1. Notice that we could equally choose to work with $\mathbb{Z}$-linear combinations instead of $\mathbb{Q}$-linear combinations in the definition of morphisms, since for a pre-additive category one requires that composition is $\mathbb{Z}$-bilinear.

6. Convolution algebra and time evolution

Consider as above the semigroupoid ring (algebra) $\mathbb{C}[\mathcal{G}]$ of complex valued functions with finite support on $\mathcal{G}$, with the associative convolution product (4.3),

$$(f_1 * f_2)(\mathbf{M}) = \sum_{\mathbf{M}_1,\mathbf{M}_2\in\mathcal{G}:\mathbf{M}_1\circ\mathbf{M}_2=\mathbf{M}} f_1(\mathbf{M}_1)\, f_2(\mathbf{M}_2). \tag{6.1}$$

We define an involution on the semigroupoid $\mathcal{G}$ by setting

$$Hom(G,G') \ni \alpha = (\mathbf{M},G,G') \mapsto \alpha^\vee = (\mathbf{M},G',G) \in Hom(G',G), \tag{6.2}$$

where, if α corresponds to the 3-manifold $\mathbf{M}$ with branched covering maps

$$G \subset E \subset \mathbf{S^3} \stackrel{\pi_G}{\leftarrow} \mathbf{M} \stackrel{\pi_{G'}}{\rightarrow} \mathbf{S^3} \supset E' \supset G'$$

then $\alpha^\vee$ corresponds to the same 3-manifold with maps

$$G' \subset E' \subset \mathbf{S^3} \stackrel{\pi_{G'}}{\leftarrow} \mathbf{M} \stackrel{\pi_G}{\rightarrow} \mathbf{S^3} \supset E \supset G$$

taken in the opposite order. In the following, for simplicity of notation, we write $\mathbf{M}^\vee$ instead of $\alpha^\vee = (\mathbf{M},G',G)$.

LEMMA 6.1. *The algebra $\mathbb{C}[\mathcal{G}]$ is an involutive algebra with the involution*

$$f^\vee(\mathbf{M}) = \overline{f(\mathbf{M}^\vee)}. \tag{6.3}$$

PROOF. We clearly have $(af_1+bf_2)^\vee = \bar{a}f_1^\vee + \bar{b}f_2^\vee$ and $(f^\vee)^\vee = f$. We also have

$$(f_1 * f_2)^\vee(\mathbf{M}) = \sum_{\mathbf{M}^\vee=\mathbf{M}_1^\vee\circ\mathbf{M}_2^\vee} \overline{f_1}(\mathbf{M}_1^\vee)\overline{f_2}(\mathbf{M}_2^\vee) = \sum_{\mathbf{M}=\mathbf{M}_2\circ\mathbf{M}_1} f_2^\vee(\mathbf{M}_2)\, f_1^\vee(\mathbf{M}_1)$$

so that $(f_1 * f_2)^\vee = f_2^\vee * f_1^\vee$ □

6.1. Time evolution. Given an algebra $\mathcal{A}$ over $\mathbb{C}$, a time evolution is a 1-parameter family of automorphisms $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$. There is a natural time evolution on the algebra $\mathbb{C}[\mathcal{G}]$ obtained as follows.

LEMMA 6.2. *Suppose given a function $f \in \mathbb{C}[\mathcal{G}]$. Consider the action defined by*

$$\sigma_t(f)(\mathbf{M}) := \left(\frac{n}{m}\right)^{it} f(\mathbf{M}), \tag{6.4}$$

where $\mathbf{M}$ a covering as in (1.6), *with the covering maps π_G and $\pi_{G'}$ respectively of generic multiplicity n and m. This defines a time evolution on $\mathbb{C}[\mathcal{G}]$.*

PROOF. Clearly $\sigma_{t+s} = \sigma_t \circ \sigma_s$. We check that $\sigma_t(f_1 * f_2) = \sigma_t(f_1) * \sigma_t(f_2)$. By (6.1), we have

$$\sigma_t(f_1 * f_2)(\mathbf{M}) = \left(\frac{n}{m}\right)^{it} (f_1 * f_2)(\mathbf{M})$$

$$= \sum_{\mathbf{M}_1,\mathbf{M}_2 \in \mathcal{G}:\mathbf{M}_1\circ\mathbf{M}_2=\mathbf{M}} \left(\frac{n_1}{m_1}\right)^{it} f_1(\mathbf{M}_1) \left(\frac{n_2}{m_2}\right)^{it} f_2(\mathbf{M}_2) = (\sigma_t(f_1) * \sigma_t(f_2))(\mathbf{M}),$$

where n_i, m_i are the generic multiplicities of the covering maps for $\mathbf{M}_i$, with $i = 1,2$. In fact, we know by Lemma 2.6 that $n = n_1 n_2$ and $m = m_1 m_2$. The time evolution is compatible with the involution (6.3), since we have

$$\sigma_t(f^\vee)(\mathbf{M}) = \left(\frac{n}{m}\right)^{it} f^\vee(\mathbf{M}) = \left(\frac{n}{m}\right)^{it} \overline{f(\mathbf{M}^\vee)} = \overline{\left(\frac{m}{n}\right)^{it} f(\mathbf{M}^\vee)} = \overline{\sigma_t(f)(\mathbf{M}^\vee)} = (\sigma_t(f))^\vee(\mathbf{M}).$$

□

Similarly, we define the left and right time evolutions on $\mathcal{A}$ by setting

$$\sigma_t^L(f)(M) := n^{it} f(M), \quad \sigma_t^R(f)(M) := m^{it} f(M), \tag{6.5}$$

where n and m are the multiplicities of the two covering maps as above. The same argument of Lemma 6.2 shows that the $\sigma_t^{L,R}$ are time evolutions. One sees by construction that they commute, *i.e.* that $[\sigma_t^L, \sigma_t^R] = 0$. The time evolution (6.4) is the composite

$$\sigma_t = \sigma_t^L \sigma_{-t}^R. \tag{6.6}$$

The involution exchanges the two time evolutions by

$$\sigma_t^L(f^\vee) = (\sigma_{-t}^R(f))^\vee. \tag{6.7}$$

6.2. Creation and annihilation operators. Given an embedded graph $G \subset \mathbf{S^3}$, consider, as above, the set $\mathcal{G}_G$ of all 3-manifolds that are branched covers of $\mathbf{S^3}$ branched along an embedded graph $E \supset G$. On the vector space $\mathcal{H}_G$ of finitely supported complex valued functions on $\mathcal{G}_G$ we have a representation of $\mathbb{C}[\mathcal{G}]$ as in Lemma 4.2, defined by

$$(\rho_G(f)\xi)(\mathbf{M}) = \sum_{\mathbf{M}_1 \in \mathcal{G}, \mathbf{M}_2 \in \mathcal{G}_G : \mathbf{M}_1 \circ \mathbf{M}_2 = \mathbf{M}} f(\mathbf{M}_1)\xi(\mathbf{M}_2). \tag{6.8}$$

It is natural to consider on the space $\mathcal{H}_G$ the inner product

$$\langle \xi, \xi' \rangle = \sum_{\mathbf{M} \in \mathcal{G}_G} \overline{\xi(\mathbf{M})} \xi'(\mathbf{M}). \tag{6.9}$$

Notice however that, unlike the usual case of groupoids, the involution (6.3) given by the transposition of the correspondence does not agree with the adjoint in the inner product (6.9), namely $\rho_\gamma(f)^* \neq \rho_\gamma(f^\vee)$.

The reason behind this incompatibility is that semigroupoids behave like semigroup algebras implemented by isometries rather than like group algebras implemented by unitaries. The model case for an adjoint and involutive structure that is compatible with the representation (6.8) and the pairing (6.9) is therefore given by the algebra of creation and annihilation operators. (See the appendix for more information on the general properties of creation and annihilation operators.)

We need the following preliminary result.

LEMMA 6.3. *Suppose given elements* $\alpha = (\mathbf{M}, G, G')$ *and* $\alpha_1 = (\mathbf{M}_1, G_1, G_1')$ *in* $\mathcal{G}$. *If there exists an element* $\alpha_2 = (\mathbf{M}_2, G_2, G_2')$ *in* $\mathcal{G}(G_2, G_2')$ *such that* $\alpha = \alpha_1 \circ \alpha_2 \in \mathcal{G}$, *then* α_2 *is unique.*

PROOF. We have $\mathbf{M} = \mathbf{M}_1 \circ \mathbf{M}_2$. We denote by $E \supset G$, $E' \supset G'$ and $E_1 \supset G_1$ and $E_1' \supset G_1'$ the embedded graphs that are the branching loci of the covering maps π_G, $\pi_{G'}$ and π_{G_1}, $\pi_{G_1'}$ of $\mathbf{M}$ and $\mathbf{M}_1$, respectively. By construction we know that for the composition $\alpha_1 \circ \alpha_2$ to be defined in $\mathcal{G}$ we need to have $G_1' = G_2$. Moreover, by Lemma 2.6 we know that $E = E_1 \cup \pi_{G_1}\pi_{G_1'}^{-1}(E_2)$ and $E' = E_2' \cup \pi_{G_2'}\pi_{G_2}^{-1}(E_1')$, where E_2 and E_2' are the branch loci of the two covering maps of $\mathbf{M}_2$. The manifold $\mathbf{M}_2$ and the branched covering maps π_{G_2} and $\pi_{G_2'}$ can be reconstructed by determining the multiplicities, branch indices, and branch loci E_2, E_2'. The n-fold branched covering $\pi_G : \mathbf{M} \to S^3 \supset E \supset G$ is equivalently described by a representation of the fundamental group $\pi_1(S^3 \smallsetminus E) \to S_n$. Similarly, the n_1-fold branched covering $\pi_{G_1} : \mathbf{M}_1 \to S^3 \supset E_1 \supset G_1$ is specified by a representation $\pi_1(S^3 \smallsetminus E_1) \to S_{n_1}$. Given these data, we obtain the branched covering $P_1 : \mathbf{M} \to \mathbf{M}_1$ such that $\pi_G = \pi_{G_1} \circ P_1$ in the following way. The restrictions $\pi_G : \mathbf{M} \smallsetminus \pi_G^{-1}(E) \to S^3 \smallsetminus E$ and $\pi_{G_1} : \mathbf{M}_1 \smallsetminus \pi_{G_1}^{-1}(E) \to S^3 \smallsetminus E$ are ordinary coverings, and we obtain from these the covering $P_1 : \mathbf{M} \smallsetminus \pi_G^{-1}(E) \to \mathbf{M}_1 \smallsetminus \pi_{G_1}^{-1}(E)$. Since this is defined on the complement of a set of codimension two, it extends uniquely to a branched covering $P_1 : \mathbf{M} \to \mathbf{M}_1$. The image under $\pi_{G_1'}$ of the branch locus of P_1 and the multiplicities and branch indices of P_1 then determine uniquely the manifold $\mathbf{M}_2$ as a branched covering $\pi_{G_2} : \mathbf{M}_2 \to S^3 \supset E_2$. Having determined the branched covering π_{G_2} we have the covering maps realizing $\mathbf{M}$ as the fibered product of $\mathbf{M}_1$ and $\mathbf{M}_2$, hence we also have the branched covering map $P_2 : \mathbf{M} \to \mathbf{M}_2$. The knowledge of the branch loci, multiplicities and branch indices of $\pi_{G'}$ and P_2 then allows us to identify the part of the branch locus E' that constitutes E_2' and the multiplicities and branch indices of the map $\pi_{G_2'}$. This completely determines also the second covering map $\pi_{G_2'} : \mathbf{M}_2 \to S^3 \supset E_2'$. ☐

We denote in the following by the same notation $\mathcal{H}_G$ the Hilbert space completion of the vector space $\mathcal{H}_G$ of finitely supported complex valued functions on $\mathcal{G}_G$ in the inner product (6.9). We denote by $\delta_{\mathbf{M}}$ the standard orthonormal basis consisting of functions $\delta_{\mathbf{M}}(\mathbf{M}') = \delta_{\mathbf{M},\mathbf{M}'}$, with $\delta_{\mathbf{M},\mathbf{M}'}$ the Kronecker delta.
Given an element $\mathbf{M} \in \mathcal{G}$, we define an associated bounded linear operator $A_{\mathbf{M}}$ on $\mathcal{H}_G$ of the form

$$(A_{\mathbf{M}}\xi)(\mathbf{M}') = \begin{cases} \xi(\mathbf{M}'') & \text{if } \mathbf{M}' = \mathbf{M} \circ \mathbf{M}'' \\ 0 & \text{otherwise.} \end{cases} \tag{6.10}$$

Notice that (6.10) is well defined because of Lemma 6.3.

LEMMA 6.4. *The adjoint of the operator* (6.10) *in the inner product* (6.9) *is given by the operator*

$$(A^*_{\mathbf{M}}\xi)(\mathbf{M}') = \begin{cases} \xi(\mathbf{M} \circ \mathbf{M}') & \textit{if the composition is defined} \\ 0 & \textit{otherwise.} \end{cases} \tag{6.11}$$

PROOF. We have

$$\langle \xi, A_{\mathbf{M}}\zeta \rangle = \sum_{\mathbf{M}' = \mathbf{M} \circ \mathbf{M}''} \overline{\xi(\mathbf{M}')}\zeta(\mathbf{M}'') = \sum_{\mathbf{M}''} \overline{\xi(\mathbf{M} \circ \mathbf{M}'')}\zeta(\mathbf{M}'') = \langle A^*_{\mathbf{M}}\xi, \zeta \rangle.$$

☐

We regard the operators $A_{\mathbf{M}}$ and $A^*_{\mathbf{M}}$ as the annihilation and creation operators on $\mathcal{H}_G$ associated to the manifold $\mathbf{M}$. They satisfy the following relations.

LEMMA 6.5. *The products $A^*_{\mathbf{M}}A_{\mathbf{M}} = P_{\mathbf{M}}$ and $A_{\mathbf{M}}A^*_{\mathbf{M}} = Q_{\mathbf{M}}$ are given, respectively, by the projection $P_{\mathbf{M}}$ onto the subspace of $\mathcal{H}_G$ given by the range of composition by $\mathbf{M}$, and the projection $Q_{\mathbf{M}}$ onto the subspace of $\mathcal{H}_G$ spanned by the $\mathbf{M}'$ with $s(\mathbf{M}') = r(\mathbf{M})$.*

PROOF. This follows directly from (6.10) and (6.11). ☐

The following result shows the relation between the algebra $\mathbb{C}[\mathcal{G}]$ and the algebra of creation and annihilation operators $A_{\mathbf{M}}, A^*_{\mathbf{M}}$.

LEMMA 6.6. *The algebra of linear operators on $\mathcal{H}_G$ generated by the $A_{\mathbf{M}}$ is the image $\rho_G(\mathbb{C}[\mathcal{G}])$ of $\mathbb{C}[\mathcal{G}]$ under the representation ρ_G of* (6.8).

PROOF. Every function $f \in \mathbb{C}[\mathcal{G}]$ is by construction a finite linear combination $f = \sum_{\mathbf{M}} a_{\mathbf{M}} \delta_{\mathbf{M}}$, with $a_{\mathbf{M}} \in R$. Under the representation ρ_G we have

$$(\rho_G(\delta_{\mathbf{M}})\xi)(\mathbf{M}') = \sum_{\mathbf{M}'=\mathbf{M}_1\circ\mathbf{M}_2} \delta_{\mathbf{M}}(\mathbf{M}_1)\xi(\mathbf{M}_2) = (A_{\mathbf{M}}\xi)(\mathbf{M}'). \tag{6.12}$$

□

This shows that, when working with the representations ρ_G the correct way to obtain an involutive structure is by extending the algebra generated by the $A_{\mathbf{M}}$ to include the $A^*_{\mathbf{M}}$, instead of using the involution (6.3) of $\mathbb{C}[\mathcal{G}]$.

6.3. Hamiltonian. Given a representation $\rho : \mathcal{A} \to \mathrm{End}(\mathcal{H})$ of an algebra $\mathcal{A}$ with a time evolution σ, one says that the time evolution, in the representation ρ, is generated by a Hamiltonian H if for all $t \in \mathbb{R}$ one has

$$\rho(\sigma_t(f)) = e^{itH}\rho(f)e^{-itH}, \tag{6.13}$$

for an operator $H \in \mathrm{End}(\mathcal{H})$.

LEMMA 6.7. *The time evolutions σ^L_t and σ^R_t of* (6.5) *and $\sigma_t = \sigma^L_t \sigma^R_{-t}$ of* (6.4) *extend to time evolutions of the involutive algebra generated by the operators $A_{\mathbf{M}}$ and $A^*_{\mathbf{M}}$ by*

$$\begin{array}{ll} \sigma^L_t(A_M) = n^{it}A_M & \sigma^L_t(A^*_M) = n^{-it}A^*_M \\ \sigma^R_t(A_M) = m^{it}A_M & \sigma^R_t(A^*_M) = m^{-it}A^*_M \\ \sigma_t(A_M) = \left(\frac{n}{m}\right)^{it} A_M & \sigma_t(A^*_M) = \left(\frac{n}{m}\right)^{-it} A^*_M. \end{array} \tag{6.14}$$

PROOF. The result follows directly from (6.12) and the condition $\sigma_t(T^*) = (\sigma_t(T))^*$. □

We then have immediately the following result.

LEMMA 6.8. *Consider the unbounded linear operators $H^L_{G'}$ and $H^R_{G'}$ on the space $\mathcal{H}_{G'}$ defined by*

$$(H^L_{G'}\xi)(\mathbf{M}) = \log(n)\,\xi(\mathbf{M}), \quad (H^R_{G'}\xi)(\mathbf{M}) = \log(m)\,\xi(\mathbf{M}) \tag{6.15}$$

for $\mathbf{M}$ a geometric correspondence of the form

$$G \subset E \subset \mathbf{S^3} \xleftarrow{\pi_G} \mathbf{M} \xrightarrow{\pi_{G'}} \mathbf{S^3} \supset E' \supset G'$$

with π_G and $\pi_{G'}$ branched coverings of order n and m, respectively. Then $H^L_{G'}$ and $H^R_{G'}$ are, respectively, Hamiltonians for the time evolutions σ^L_t and σ^R_t in the representation $\rho_{G'}$ of (6.8).

PROOF. It is immediate to check that

$$\rho_{G'}(\sigma^L_t(f)) = e^{-itH^L}\rho_{G'}(f)e^{itH^L} \quad \text{and} \quad \rho_{G'}(\sigma^R_t(f)) = e^{-itH^R}\rho_{G'}(f)e^{itH^R},$$

for $f \in \mathbb{C}[\mathcal{G}]$. In fact, it suffices to use the explicit form of the time evolutions on the creation and annihilation operators given in Lemma 6.7 above to see that they are implemented by the Hamiltonians $H^L_{G'}$ and $H^R_{G'}$. □

An obvious problem with this time evolution is the fact that the Hamiltonian typically can have infinite multiplicities of the eigenvalues. For example, by the strong form of the Hilden-Montesinos theorem [**54**] and the existence of universal knots [**33**], there exist knots K such that all compact oriented 3-manifolds can be obtained as a 3-fold branched cover of $\mathbf{S^3}$, branched along K. For this reason it is useful to consider time evolutions on a convolution algebra of geometric correspondences that takes into account the equivalence given by 4-dimensional cobordisms. We turn to this in §7 and §8 below.

7. Equivalence of correspondences

It is quite clear that, in our first definition of the category $\mathcal{K}$ of knots with correspondences given by branched covers of the 3-sphere, we typically have spaces of morphisms that are "too large" to deal with effectively. The following result illustrates one of the problems we encounter.

LEMMA 7.1. *There are choices of embedded graphs G, G' for which $Hom(G, G')$ is the $\mathbb{Q}$-vector space spanned by all compact oriented connected 3-manifolds.*

PROOF. To find such example it is suffices to restrict to the case where G and G' are knots. The result is an immediate consequence of the existence of *universal knots* (see the appendix and also [**33**], [**35**]). A knot G is universal if all compact oriented connected 3-manifolds can be obtained as branched covers of $\mathbf{S^3}$ branched along the same knot G. It suffices to choose G and G' to be universal knots to obtain the stated result. □

Thus, it is clear that it is necessary to impose a suitable equivalence relation $\sim$ on correspondences and redefine our category as the category $\mathcal{K}^{\sim}$ where objects are graphs and the morphisms are $\mathbb{Q}$-linear combinations $\phi = \sum_i a_i [M_i]$ of equivalence classes of branched covers with the properties described above. This will allow us to work with smaller spaces of morphisms. It is well known that, whenever one defines morphisms via correspondence, be it cycles in the product as in the case of motives or submersions as in the case of geometric correspondence, the most delicate step is always deciding up to what equivalence relation correspondences should be considered. In fact, as the case of motives clearly show (*cf.* [**40**]) the properties of the category change drastically when one changes the equivalence relation on correspondences. In the case of 3-manifolds with the structure of branched covers, there is a natural notion of equivalence, which is given by cobordisms of branched covers.

7.1. Cobordisms of branched covers. Hilden and Little (*cf.* [**37**]) gave us a suitable notion of equivalence relation of branched coverings obtained by using cobordisms. Namely, suppose given two compact oriented 3-manifolds $\mathbf{M}_1$ and $\mathbf{M}_2$ that are branched covers of $\mathbf{S^3}$, with covering maps $\pi_1 : \mathbf{M}_1 \to \mathbf{S^3}$ and $\pi_2 : \mathbf{M}_2 \to \mathbf{S^3}$, respectively branched along 1-dimensional simplicial complex E_1 and E_2. A cobordism of branched coverings is a 4-dimensional manifold W with boundary $\partial W = \mathbf{M}_1 \cup -\mathbf{M}_2$ (where the minus sign denotes the change of orientation), endowed with a submersion $q : W \to \mathbf{S^3} \times [0,1]$, with $\mathbf{M}_1 = q^{-1}(\mathbf{S^3} \times \{0\})$ and $\mathbf{M}_2 = q^{-1}(\mathbf{S^3} \times \{1\})$ and $q|_{\mathbf{M}_1} = \pi_1$ and $q|_{\mathbf{M}_2} = \pi_2$. One also requires that the map q is a covering map branched along a surface $S \subset S^3 \times [0,1]$ such that $\partial S = E_1 \cup -E_2$, with $E_1 = S \cap (\mathbf{S^3} \times \{0\})$ and $E_2 = S \cap (\mathbf{S^3} \times \{1\})$. Since in the case of both 3-manifolds and 4-manifolds there is no substantial difference in working in the *PL* or smooth categories, we keep formulating everything in the *PL* setting. We adapt easily this notion to the case of our correspondences. We simply need to modify the definition above to take into account the fact that our correspondences have two (not just one) covering maps to $\mathbf{S^3}$, so that the cobordisms have to be chosen accordingly.

DEFINITION 7.2. Suppose given two morphisms $\mathbf{M}_1$ and $\mathbf{M}_2$ in $Hom(G,G')$, of the form

$$G \subset E_1 \subset \mathbf{S^3} \stackrel{\pi_{G,1}}{\longleftarrow} \mathbf{M}_1 \stackrel{\pi_{G',1}}{\longrightarrow} \mathbf{S^3} \supset E_1' \supset G'$$

$$G \subset E_2 \subset \mathbf{S^3} \stackrel{\pi_{G,2}}{\longleftarrow} \mathbf{M}_2 \stackrel{\pi_{G',2}}{\longrightarrow} \mathbf{S^3} \supset E_2' \supset G'.$$

Then a cobordism between $\mathbf{M}_1$ and $\mathbf{M}_2$ is a 4-dimensional manifold W with boundary $\partial W = \mathbf{M}_1 \cup -\mathbf{M}_2$, endowed with two branched covering maps

$$S \subset \mathbf{S^3} \times [0,1] \stackrel{q}{\longleftarrow} W \stackrel{q'}{\longrightarrow} \mathbf{S^3} \times [0,1] \supset S', \tag{7.1}$$

branched along surfaces $S, S' \subset \mathbf{S^3} \times [0,1]$. The maps q and q' have the properties that $\mathbf{M}_1 = q^{-1}(\mathbf{S^3} \times \{0\}) = q'^{-1}(\mathbf{S^3} \times \{0\})$ and $\mathbf{M}_2 = q^{-1}(\mathbf{S^3} \times \{1\}) = q'^{-1}(\mathbf{S^3} \times \{1\})$, with $q|_{\mathbf{M}_1} = \pi_{G,1}$, $q'|_{\mathbf{M}_1} = \pi_{G',1}$, $q|_{\mathbf{M}_2} = \pi_{G,2}$ and $q'|_{\mathbf{M}_2} = \pi_{G',2}$. The surfaces S and S' have boundary $\partial S = E_1 \cup -E_2$ and $\partial S' = E_1' \cup -E_2'$, with $E_1 = S \cap (\mathbf{S^3} \times \{0\})$, $E_2 = S \cap (\mathbf{S^3} \times \{1\})$, $E_1' = S' \cap (\mathbf{S^3} \times \{0\})$, and $E_2' = S' \cap (\mathbf{S^3} \times \{1\})$.

Here By "surface" we mean a 2-dimensional simplicial complex that is PL-embedded in $\mathbf{S^3} \times [0,1]$, with boundary $\partial S \subset \mathbf{S^3} \times \{0,1\}$ given by 1-dimensional simplicial complexes, *i.e.* embedded graphs.

LEMMA 7.3. *We set $\mathbf{M}_1 \sim \mathbf{M}_2$ if there exists a cobordims W as in Definition 7.2. This is an equivalence relation.*

PROOF. (1) **Reflexivity.** Consider $\mathbf{M}$ in $Hom(G,G')$ specified by a diagram

$$E_1 \subset \mathbf{S^3} \stackrel{\pi_1}{\longleftarrow} \mathbf{M} \stackrel{\pi_2}{\longrightarrow} \mathbf{S^3} \supset E_1'.$$

We can choose $W = \mathbf{M} \times [0,1]$ as a cobordism of $\mathbf{M}$ with itself. This has $\partial W = \mathbf{M} \cup -\mathbf{M}$, with covering maps

$$E_1 \times [0,1] \subset \mathbf{S^3} \times [0,1] \stackrel{q_1}{\longleftarrow} W = \mathbf{M} \times [0,1] \stackrel{q_2}{\longrightarrow} \mathbf{S^3} \times [0,1] \supset E_1' \times [0,1]$$

branched along the surfaces $S = E_1 \times [0,1]$ and $S' = E_1' \times [0,1]$ in $\mathbf{S^3} \times [0,1]$. These have $\partial S = E_1 \cup -E_1'$ and $\partial S' = E_1 \cup -E_1'$, as needed. The covering maps q_1 and q_2 have the properties that

$$\mathbf{M} = q_1^{-1}(\mathbf{S^3} \times \{0\}) = q_2^{-1}(\mathbf{S^3} \times \{0\}) = q_1^{-1}(\mathbf{S^3} \times \{1\}) = q_2^{-1}(\mathbf{S^3} \times \{1\}).$$

Thus, this satisfies all the properties of Definition 7.2 above.

(2) **Symmetry.** Given $\mathbf{M}_1 \sim \mathbf{M}_2$, there exist a cobordism W satisfying the properties of Definition 7.2. Now consider $\overline{W}$, which is the same manifold W, with the opposite orientation. This is also a cobordism between $\mathbf{M}_2$ and $\mathbf{M}_1$, that is, it has boundary $\partial \overline{W} = \overline{\partial W} = \overline{(\mathbf{M}_1 \cup -\mathbf{M}_2)} = -\mathbf{M}_1 \cup \mathbf{M}_2$. It is also endowed with two branched covering maps

$$\overline{S} \subset \mathbf{S^3} \times \overline{[0,1]} \stackrel{\overline{q}}{\longleftarrow} \overline{W} \stackrel{\overline{q}'}{\longrightarrow} \mathbf{S^3} \times \overline{[0,1]} \supset \overline{S'}, \tag{7.2}$$

branched along the surfaces $(\overline{S}, \overline{S'}) \subset \mathbf{S^3} \times \overline{[0,1]}$, where here again $\overline{S}$, $\overline{S'}$, $\overline{q}$, $\overline{q'}$ and $\overline{[0,1]}$ are the same as S, S', q, q' and $[0,1]$, but taken with the opposite orientation. The maps $\overline{q}$ and $\overline{q'}$ have the property that

$$\mathbf{M}_1 = q^{-1}(\mathbf{S^3} \times \{0\}) = \overline{q}^{-1}(\mathbf{S^3} \times \{1\}) = q'^{-1}(\mathbf{S^3} \times \{0\}) = \overline{q'}^{-1}(\mathbf{S^3} \times \{1\}),$$

$$\mathbf{M}_2 = q^{-1}(\mathbf{S^3} \times \{1\}) = \overline{q}^{-1}(\mathbf{S^3} \times \{0\}) = q'^{-1}(\mathbf{S^3} \times \{1\}) = \overline{q'}^{-1}(\mathbf{S^3} \times \{0\}),$$

with $\overline{q}|_{\mathbf{M}_1} = \pi_{G,1}$, $\overline{q'}|_{\mathbf{M}_1} = \pi_{G',1}$, $\overline{q}|_{\mathbf{M}_2} = \pi_{G,2}$ and $\overline{q'}|_{\mathbf{M}_2} = \pi_{G',2}$. The surfaces $\overline{S}$ and $\overline{S'}$ have boundary $\partial \overline{S} = \overline{(\partial S)} = \overline{(E_1 \cup -E_2)} = -E_1 \cup E_2$ and $\partial \overline{S} = \overline{(\partial S)} = \overline{(E_1 \cup -E_2)} = -E_1 \cup E_2$, with $E_1 = \overline{S} \cap (\mathbf{S^3} \times \{0\})$, $E_2 = \overline{S} \cap (\mathbf{S^3} \times \{1\})$, $E_1' = \overline{S'} \cap (\mathbf{S^3} \times \{0\})$, and $E_2' = \overline{S'} \cap (\mathbf{S^3} \times \{1\})$. Thus, this shows that $\mathbf{M}_2 \sim \mathbf{M}_1$.

(3) **Transitivity.** Assume that $\mathbf{M}_1 \sim \mathbf{M}_2$ and $\mathbf{M}_2 \sim \mathbf{M}_3$. We want to show that $\mathbf{M}_1 \sim \mathbf{M}_3$. Since $\mathbf{M}_1 \sim \mathbf{M}_2$, there exists a cobordism W_1 as in Definition 7.2 with a diagram

$$S_1 \subset \mathbf{S^3} \times [0,1] \xleftarrow{q_1} W_1 \xrightarrow{q_1'} \mathbf{S^3} \times [0,1] \supset S_1', \tag{7.3}$$

and $\partial S_1 = E_1 \cup -E_2$, $\partial S_1' = E_1' \cup -E_2'$. Similarly, since $\mathbf{M}_2 \sim \mathbf{M}_3$, there exist a cobordism W_2, which also satisfies the properties of Definition 7.2, with covering maps

$$S_2 \subset \mathbf{S^3} \times [0,1] \xleftarrow{q_2} W_2 \xrightarrow{q_2'} \mathbf{S^3} \times [0,1] \supset S_2', \tag{7.4}$$

where $\partial S_2 = E_2 \cup -E_3$ and $\partial S_2' = E_2' \cup -E_3'$. Now we use the "collar neighborhood" property. Consider the sets

$$U_1 = q_1^{-1}(S^3 \times [1-\varepsilon, 1]), \quad U_1' = (q_1')^{-1}(S^3 \times [1-\varepsilon, 1]),$$
$$U_2 = q_2^{-1}(S^3 \times [0, \varepsilon]), \qquad U_2' = (q_2')^{-1}(S^3 \times [0, \varepsilon]).$$

For a sufficiently small $\varepsilon > 0$ these have the property that there exist homeomorphisms

$$\phi_1 : U_1 \to \mathbf{M}_2 \times [1-\varepsilon, 1], \quad \phi_1' : U_1' \to \mathbf{M}_2 \times [1-\varepsilon, 1],$$
$$\phi_2 : U_2 \to \mathbf{M}_2 \times [0, \varepsilon], \qquad \phi_2 : U_2' \to \mathbf{M}_2 \times [0, \varepsilon].$$

Here we can replace homeomorphisms by PL-homeomorphism of diffeomorphism if we work in the PL or smooth category. Moreover, under this identification, we also have, for $i = 1, 2$, identifications

$$f_i := \psi_i q_i \phi_i^{-1} = \pi_{G,i} \times id, \;\; g_i := \psi_i' q_i' (\phi_i')^{-1} = \pi_{G',i} \times id, \tag{7.5}$$

where the $\psi_i : \mathbf{S^3} \times [1-\varepsilon, 1] \to \mathbf{S^3} \times [1-\varepsilon, 1]$ and $\psi_i' : \mathbf{S^3} \times [0, \varepsilon] \to \mathbf{S^3} \times [0, \varepsilon]$ are homeomorphisms with the property that

$$\psi_1(S_1 \cap (\mathbf{S^3} \times [1-\varepsilon, 1])) = E_2 \times [1-\varepsilon, 1]$$
$$\psi_1'(S_1' \cap (\mathbf{S^3} \times [1-\varepsilon, 1])) = E_2' \times [1-\varepsilon, 1]$$
$$\psi_2(S_2 \cap (\mathbf{S^3} \times [0, \varepsilon])) = E_2 \times [0, \varepsilon]$$
$$\psi_2'(S_2' \cap (\mathbf{S^3} \times [0, \varepsilon])) = E_2' \times [0, \varepsilon].$$

Thus, f_1 is branched along $E_2 \times [1-\varepsilon, 1]$, f_2 is branched along $E_2' \times [1-\varepsilon, 1]$, g_1 is branched along $E_2 \times [0, \varepsilon]$ and g_2 is branched along $E_2' \times [0, \varepsilon]$. Now fix a homeomorphism $h : [1-\varepsilon, 1] \to [0, \varepsilon]$ and define

$$W = W_1 \cup_{\mathbf{M}_2} W_2 = W_1 \cup W_2 / \sim,$$

which is the quotient of the disjoint union $W_1 \cup W_2$ by the equivalence relation generated by requiring that $w_1 \sim w_2$ whenever $w_1 \in U_1 \cap U_1'$ and $w_2 \in U_2 \cap U_2'$ with $h\phi_1(w_1) = \phi_2(w_2)$ and $h\phi_1'(w_1) = \phi_2'(w_2)$. We can assume in the following, possibly after passing to a smaller $\varepsilon > 0$, that $U_1 = U_1'$ and $U_2 = U_2'$, so we just use the notation U_1 and U_2 for both the ϕ and ϕ' maps. We then need to check that $W = W_1 \cup_{\mathbf{M}_2} W_2$ defined as above satisfies all the properties of Definition 7.2. First, we check that W is a 4-dimensional manifold with boundary $\partial W = \mathbf{M}_1 \cup \mathbf{M}_3$, endowed with two branched covering maps

$$\hat{S} \subset \mathbf{S^3} \times [0,1] \xleftarrow{\Pi_1} W \xrightarrow{\Pi_2} \mathbf{S^3} \times [0,1] \supset \hat{S}'. \tag{7.6}$$

Here we use an identification $\mathbf{S^3} \times [0,1] \simeq \mathbf{S^3} \times \mathcal{I}$, where $\mathcal{I}$ is the interval obtained by identifying two copies of the interval $[0,1]$ by gluing $[1-\varepsilon,1]$ and $[0,\varepsilon]$,

$$\mathcal{I} = [0,1] \cup_{h:[1-\varepsilon,1]\to[0,\varepsilon]} [0,1].$$

This means that we identify $s_1 \sim s_2$, for $s_1 \in \mathbf{S^3} \times [1-\varepsilon,1]$ and $s_2 \in \mathbf{S^3} \times [0,\varepsilon]$ whenever $h(\psi_1(s_1)) = \psi_2(s_2)$ and $h(\psi_1'(s_1)) = \psi_2'(s_2)$. In order to define the functions Π_1 and Π_2 of (7.6), we first need the following fact. If $w_1 \sim w_2$, with $w_1 \in U_1$ and $w_2 \in U_2$, then $q_1(w_1) \sim q_2(w_2)$. In fact, suppose that $w_1 \sim w_2$. This means that $h\phi_1(w_1) = \phi_2(w_2)$ and $h\phi_1'(w_1) = \phi_2'(w_2)$. Such w_1 and w_2 have images $q_1(w_1) \in \mathbf{S^3} \times [1-\varepsilon,1]$ and $q_2(w_2) \in \mathbf{S^3} \times [0,\varepsilon]$. We apply the maps ψ_i and obtain $h\psi_1(q_1(w_1)) = (\pi_{G,1} \times h)(w_1) = \psi_2(q_2(w_2))$, which means that $q_1(w_1) \sim q_2(w_2)$. The same argument shows that, conversely, if $q_1(w_1) \sim q_2(w_2)$ then $w_1 \sim w_2$. Thus, we can define the functions Π_1 and Π_2 of (7.6) by setting

$$\begin{aligned} \Pi_1(w) &= \begin{cases} q_1(w) & w \in W_1 \\ q_2(w) & w \in W_2 \end{cases} \\ \Pi_2(w) &= \begin{cases} q_1'(w) & w \in W_1 \\ q_2'(w) & w \in W_2 \end{cases} \end{aligned} \tag{7.7}$$

This gives well defined maps on the quotient $W = W_1 \cup_{\mathbf{M}_2} W_2$ of the above equivalence relation. By construction, these two maps Π_1 and Π_2 are branched, respectively, along surfaces $\hat{S}, \hat{S}' \subset \mathbf{S^3} \times [0,1]$, where

$$\hat{S} = S_1 \cup_{L_2} S_2 = S_1 \cup S_2 / \sim,$$

which is again the quotient of the disjoint union $S_1 \cup S_2$ by the equivalence relation $s_1 \sim s_2$ when $s_1 \in S_1 \cap (\mathbf{S^3} \times [1-\varepsilon,1])$ and $s_2 \in S_2 \cap (\mathbf{S^3} \times [0,\varepsilon])$ with $h(\psi_1(s_1)) = \psi_2(s_2)$, *i.e.* the identification obtained by gluing the two surfaces along the common boundary components given by the link E_2. The surface $\hat{S}'$ is obtained in the same way. Moreover, the maps Π_1 and Π_2 have the properties that

$$\mathbf{M}_1 = q_1^{-1}(\mathbf{S^3} \times \{0\}) = \Pi_1^{-1}(\mathbf{S^3} \times \{0\}) = q_1'^{-1}(\mathbf{S^3} \times \{0\}) = \Pi_2^{-1}(\mathbf{S^3} \times \{0\})$$

and

$$\mathbf{M}_3 = q_2^{-1}(\mathbf{S^3} \times \{1\}) = \Pi_1^{-1}(\mathbf{S^3} \times \{1\}) = q_2'^{-1}(\mathbf{S^3} \times \{1\}) = \Pi_1^{-1}(\mathbf{S^3} \times \{1\}).$$

The surfaces $\hat{S}$ and $\hat{S}'$ have boundary $\partial\hat{S} = E_1 \cup -E_3$ and $\partial\hat{S}' = E_1' \cup -E_3'$, with $E_1 = \hat{S} \cap (\mathbf{S^3} \times \{0\})$, $E_3 = \hat{S} \cap (\mathbf{S^3} \times \{1\})$, $E_1' = \hat{S}' \cap (\mathbf{S^3} \times \{0\})$, and $E_3' = \hat{S}' \cap (\mathbf{S^3} \times \{1\})$.

□

PROPOSITION 7.4. *Let $\mathbf{M}_1 \sim \mathbf{M}_2$ in $Hom(G,G')$ and $\mathbf{M}_1' \sim \mathbf{M}_2'$ in $Hom(G',G'')$. Then the compositions satisfy*

$$\mathbf{M}_1' \circ \mathbf{M}_1 \sim \mathbf{M}_2' \circ \mathbf{M}_2.$$

PROOF. Suppose given $\mathbf{M}_1$ and $\mathbf{M}_2 \in Hom(G,G')$ and $\mathbf{M}_1'$ and $\mathbf{M}_2' \in Hom(G',G'')$ with

$$\begin{aligned} &G \subset E_{11} \subset S^3 \stackrel{\pi_{11}}{\leftarrow} \mathbf{M}_1 \stackrel{\pi_{12}}{\rightarrow} S^3 \supset E_{12} \supset G' \\ &G \subset E_{21} \subset S^3 \stackrel{\pi_{21}}{\leftarrow} \mathbf{M}_2 \stackrel{\pi_{22}}{\rightarrow} S^3 \supset E_{22} \supset G' \\ &G' \subset E_{11}' \subset S^3 \stackrel{\pi_{11}'}{\leftarrow} \mathbf{M}_1' \stackrel{\pi_{12}'}{\rightarrow} S^3 \supset E_{12}' \supset G'' \\ &G' \subset E_{21}' \subset S^3 \stackrel{\pi_{21}'}{\leftarrow} \mathbf{M}_2' \stackrel{\pi_{22}'}{\rightarrow} S^3 \supset E_{22}' \supset G''. \end{aligned} \tag{7.8}$$

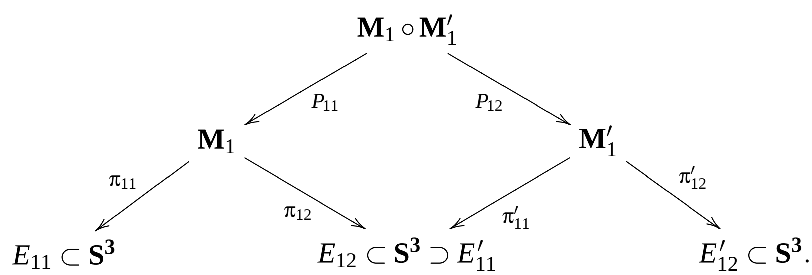
Since $\mathbf{M}_1 \sim \mathbf{M}_2$, there exist a cobordism W_1 such that $\partial W_1 = \mathbf{M}_1 \cup -\mathbf{M}_2$, endowed with two branched covering maps

$$S_{11} \subset \mathbf{S^3} \times [0,1] \xleftarrow{q_1} W_1 \xrightarrow{q_1'} \mathbf{S^3} \times [0,1] \supset S_{12}, \tag{7.9}$$

branched along surfaces $S_{11}, S_{12} \subset \mathbf{S^3} \times [0,1]$. These surfaces have boundary $\partial S_{11} = E_{11} \cup -E_{21}$ and $\partial S_{12} = E_{12} \cup -E_{22}$, and the branched covering maps satisfy $q_1|_{\mathbf{M}_1} = \pi_{11}$, $q_1'|_{\mathbf{M}_1} = \pi_{12}$, $q_1|_{\mathbf{M}_2} = \pi_{21}$ and $q_1'|_{\mathbf{M}_2} = \pi_{22}$, with the properties of Definition 7.2. In the same way, $\mathbf{M}_1' \sim \mathbf{M}_2'$ mean that there exists a cobordism W_2 with $\partial W_2 = \mathbf{M}_1' \cup -\mathbf{M}_2'$, with branched covering maps

$$S_{21} \subset \mathbf{S^3} \times [0,1] \xleftarrow{q_2} W_2 \xrightarrow{q_2'} \mathbf{S^3} \times [0,1] \supset S_{22}, \tag{7.10}$$

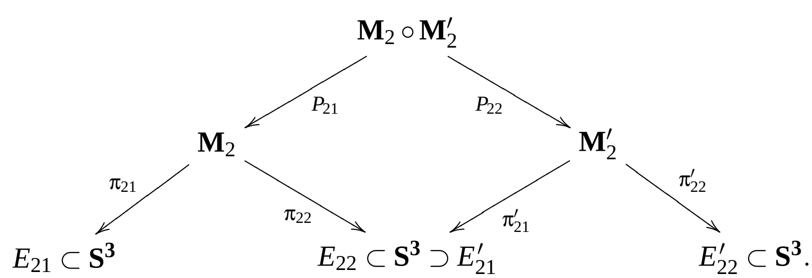
branched along surfaces $S_{21}, S_{22} \subset \mathbf{S^3} \times [0,1]$. These surfaces have boundary $\partial S_{21} = E_{11}' \cup -E_{21}'$ and $\partial S_{22} = E_{12}' \cup -E_{22}'$, and the maps satisfy $q_2|_{\mathbf{M}_1'} = \pi_{11}'$, $q_2'|_{\mathbf{M}_1'} = \pi_{12}'$, $q_2|_{\mathbf{M}_2'} = \pi_{21}'$ and $q_2'|_{\mathbf{M}_2'} = \pi_{22}'$, with the properties of Definition 7.2. The composition $\mathbf{M}_1 \circ \mathbf{M}_1'$ corresponds to the diagram



Corollary 2.7 shows that the composite maps $\hat{\pi}_1 = \pi_{11} \circ P_{11}$ and $\hat{\pi}_2 = \pi_{12}' \circ P_{12}$ are branched coverings

$$I_1 = (E_{11} \cup \pi_{11}\pi_{12}^{-1}(E_{11}')) \subset \mathbf{S^3} \stackrel{\hat{\pi}_1}{\leftarrow} \mathbf{M}_1 \circ \mathbf{M}_1' \stackrel{\hat{\pi}_2}{\rightarrow} \mathbf{S^3} \supset I_2 = (E_{12}' \cup \pi_{12}'\pi_{11}'^{-1}(E_{12})).$$

Thus, $\mathbf{M}_1 \circ \mathbf{M}_1'$ is a morphism in $Hom(G, G'')$. Similarly, for the composition $\mathbf{M}_2 \circ \mathbf{M}_2'$ we consider the diagram



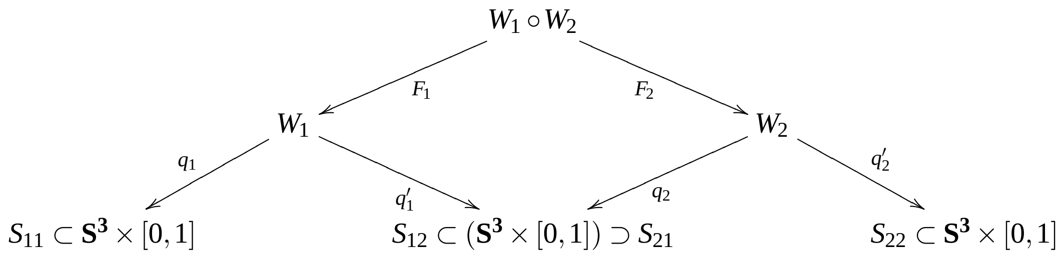
Again by Corollary 2.7 we know that the maps $\hat{\pi}_1' = \pi_{21} \circ P_{21}$ and $\hat{\pi}_2' = \pi_{22}' \circ P_{22}$ are branched coverings with

$$I_3 = (E_{21} \cup \pi_{21}\pi_{22}^{-1}(E_{21}')) \subset \mathbf{S^3} \stackrel{\hat{\pi}_1'}{\leftarrow} \mathbf{M}_2 \circ \mathbf{M}_2' \stackrel{\hat{\pi}_2'}{\rightarrow} \mathbf{S^3} \supset I_4 = (E_{22}' \cup \pi_{22}'\pi_{21}'^{-1}(E_{22})),$$

hence $\mathbf{M}_2 \circ \mathbf{M}_2'$ is also a morphism in $Hom(G, G'')$. Now, in order to show that $\mathbf{M}_1' \circ \mathbf{M}_1 \sim \mathbf{M}_2' \circ \mathbf{M}_2$, we define a new 4-dimensional manifold given by the fibered product

$$W_1 \circ W_2 := \{(x,y) \in W_1 \times W_2 | q_1'(x) = q_2(y)\}. \tag{7.11}$$

This has branched covering maps obtained as in the diagram below,



The maps $T_1 = q_1 \circ F_1$ and $T_2 = q'_2 \circ F_2$ are branched along the surfaces $\hat{S}_1, \hat{S}_2$ with $\hat{S}_1 = S_{11} \cup q_1(q_1'^{-1}(S_{21}))$ and $\hat{S}_2 = S_{22} \cup q'_2(q_1^{-1}(S_{12}))$, *i.e.* we have a branched covering

$$\hat{S}_1 \subset \mathbf{S^3} \times [0,1] \stackrel{T_1}{\leftarrow} W_1 \circ W_2 \stackrel{T_2}{\rightarrow} \mathbf{S^3} \times [0,1] \supset \hat{S}_2.$$

We claim that $W_1 \circ W_2$ is a cobordism between $\mathbf{M}_1 \circ \mathbf{M}'_1$ and $\mathbf{M}_2 \circ \mathbf{M}'_2$ and that it satisfies all the properties of Definition 7.2. To show this, we first prove that the boundary is given by

$$\partial(W_1 \circ W_2) = \partial W_1 \circ \partial W_2 = (\mathbf{M}_1 \cup -\mathbf{M}_2) \circ (\mathbf{M}'_1 \cup -\mathbf{M}'_2) = (\mathbf{M}_1 \circ \mathbf{M}'_1) \cup -(\mathbf{M}_2 \circ \mathbf{M}'_2).$$

First we want to prove that $\partial(W_1 \circ W_2) = \partial W_1 \circ \partial W_2$.
By the definition of $W_1 \circ W_2$ we know that it is a submanifold $W_1 \circ W_2 \subset W_1 \times W_2$, defined by imposing the condition $q'_1(w_1) = q_2(w_2)$ on pairs $(w_1, w_2) \in W_1 \times W_2$, hence

$$\partial(W_1 \circ W_2) \subset \partial(W_1 \times W_2) = \partial W_1 \times W_2 \cup W_1 \times \partial W_2.$$

In fact, we have $\partial(W_1 \circ W_2) = (W_1 \circ W_2) \cap \partial(W_1 \times W_2)$ Let $(w_1, w_2) \in \partial(W_1 \circ W_2) \subset W_1 \circ W_2$. Suppose that $(w_1, w_2) \in \partial W_1 \times W_2$. Then, since $w_1 \in \partial W_1$, it has image $q'_1(w_1) \in \mathbf{S^3} \times \{0\}$ or in $\mathbf{S^3} \times \{1\}$. Say $q'_1(w_1) \in \mathbf{S^3} \times \{0\}$ (the other case is analogous). The condition $q'_1(w_1) = q_2(w_2)$ then implies that $q_2(w_2) \in \mathbf{S^3} \times \{0\}$, which means that $w_2 \in q_2^{-1}(\mathbf{S^3} \times \{0\}) \in \partial W_2$. This shows that an element $(w_1, w_2) \in \partial(W_1 \circ W_2)$ satisfies $(w_1, w_2) \in \partial W_1 \times \partial W_2$, hence that $\partial(W_1 \circ W_2) \subset \partial W_1 \times \partial W_2$.
Conversely, an element $(w_1, w_2) \in \partial W_1 \times \partial W_2$, with the property that $q'_1(w_1) = q_2(w_2)$ is in $W_1 \circ W_2 \cap \partial(W_1 \times W_2) = \partial(W_1 \circ W_2)$. This completes the proof that

$$\partial(W_1 \circ W_2) = \partial W_1 \circ \partial W_2.$$

Next we prove that $(\mathbf{M}_1 \cup -\mathbf{M}_2) \circ (\mathbf{M}'_1 \cup -\mathbf{M}'_2) = (\mathbf{M}_1 \circ \mathbf{M}'_1) \cup -(\mathbf{M}_2 \circ \mathbf{M}'_2)$. This follows from the following simple general fact. Suppose given disjoint unions $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$, and $Z = Z_1 \cup Z_2$, with submersions $f_i : X_1 \to Z_i$ and $g_i : Y_i \to Z_i$. Let $f : X \to Z$ and $g : Y \to Z$ be defined by $f(x) = f_i(x)$ for $x \in X_i$ and $g(y) = g_i(y)$ for $y \in Y_i$, for $i = 1,2$. Then the fibered product satisfies

$$X \times_Z Y = (X_1 \times_{Z_1} Y_1) \cup (X_2 \times_{Z_2} Y_2).$$

In fact, one has

$$\begin{aligned} X \times_Z Y = & \ \{(x,y) \in X \times Y \mid f(x) = g(y)\} \\ = & \ \{(x,y) \in X_1 \times Y_1 \mid f_1(x) = g_1(y)\} \cup \{(x,y) \in X_2 \times Y_2 \mid f_2(x) = g_2(y)\}. \end{aligned}$$

The result then follows by applying this general fact to $X_i = \mathbf{M}_i$, $Y_i = \mathbf{M}'_i$, $Z_1 = \mathbf{S^3} \times \{0\}$ and $Z_2 = \mathbf{S^3} \times \{1\}$. Moreover, we have

$$\begin{aligned}
\partial \hat{S_1} = \;& \partial(S_{11} \cup q_1(q_1'^{-1}(S_{21}))) = \partial S_{11} \cup \partial q_1(q_1'^{-1}(S_{21})) \\
= \;& E_{11} \cup -E_{21} \cup q_1(q_1'^{-1}(E'_{11} \cup -E'_{21})) = E_{11} \cup -E_{21} \cup q_1(q_1'^{-1}(E'_{11})) \cup q_1(q_1'^{-1}(E'_{21})) \\
= \;& E_{11} \cup q_1(q_1'^{-1}(E'_{11})) \cup (-E_{21} \cup q_1(q_1'^{-1}(E'_{21}))) \\
= \;& E_{11} \cup \pi_{11}(\pi_{12}^{-1}(E'_{11})) \cup (-E_{21} \cup \pi_{21}(\pi_{22}^{-1}(E'_{21}))) \\
= \;& I_1 \cup -I_3.
\end{aligned}$$

By the same calculation one can get $\partial \hat{S_2} = I_2 \cup -I_4$. Finally, we need to show that

$$T_1^{-1}(\mathbf{S^3} \times \{0\}) = \mathbf{M}_1 \circ \mathbf{M}'_1 = T_2^{-1}(\mathbf{S^3} \times \{0\})$$
$$T_1^{-1}(\mathbf{S^3} \times \{1\}) = \mathbf{M}_2 \circ \mathbf{M}'_2 = T_2^{-1}(\mathbf{S^3} \times \{1\}).$$

We just consider the case of $T_1^{-1}(\mathbf{S^3} \times \{0\})$, as the argument for the other cases is analogous. We have

$$\begin{aligned}
T_1^{-1}(\mathbf{S^3} \times \{0\}) &= F_1^{-1} q_1^{-1}(\mathbf{S^3} \times \{0\}) \\
&= \{(x,y) \in W_1 \times W_2 : q'_1(x) = q_2(y), q_1(x) \in \mathbf{S^3} \times \{0\}\} \\
&= \{(x,y) \in q_1^{-1}(\mathbf{S^3} \times \{0\}) \times W_2 : q'_1(x) = q_2(y)\}, \qquad (7.12)
\end{aligned}$$

while we have

$$\begin{aligned}
\mathbf{M}_1 \circ \mathbf{M}'_1 &= q_1^{-1}(\mathbf{S^3} \times 0) \circ q_2^{-1}(\mathbf{S^3} \times 0) \\
&= \{(x,y) \in q_1^{-1}(\mathbf{S^3} \times \{0\}) \times q_2^{-1}(\mathbf{S^3} \times \{0\}) : q'_1(x) = q_2(y)\}. \qquad (7.13)
\end{aligned}$$

In comparing (7.12) and (7.13), we see that, in order to show that $T_1^{-1}(\mathbf{S^3} \times \{0\}) = q_1^{-1}(\mathbf{S^3} \times 0) \circ q_2^{-1}(\mathbf{S^3} \times 0)$ it suffices to show that points $(x,y) \in F_1^{-1} q_1^{-1}(\mathbf{S^3} \times \{0\})$ necessarily have also $y \in q_2^{-1}(\mathbf{S^3} \times 0)$ not just in W_2. This follows from the condition $q'_1(x) = q_2(y)$. In fact, given $(x,y) \in F_1^{-1} q_1^{-1}(\mathbf{S^3} \times \{0\})$ then $q'_1(x) = q_2(y)$, but $q'_1(q_1^{-1}(\mathbf{S^3} \times \{0\})) \subset \mathbf{S^3} \times \{0\}$, hence $q_2(y) \in \mathbf{S^3} \times \{0\}$, which implies $y \in q_2^{-1}(\mathbf{S^3} \times \{0\})$. This shows that the two sets of (7.12) and (7.13) are equal.
A similar argument can be used to show that $\mathbf{M}_1 \circ \mathbf{M}'_1 = T_2^{-1}(\mathbf{S^3} \times \{0\})$ and that $\mathbf{M}_2 \circ \mathbf{M}'_2 = T_1^{-1}(\mathbf{S^3} \times \{1\}) = T_2^{-1}(\mathbf{S^3} \times \{1\})$. □

LEMMA 7.5. *Let G and G' be embedded graphs in $\mathbf{S^3}$ and let $Hom(G,G')$ be the set of geometric correspondences as in equation 1.6. Let*

$$Hom(G,G',\sim) := Hom(G,G')/\sim \qquad (7.14)$$

denote the quotient of $Hom(G,G')$ by the equivalence relation of cobordism of Definition 7.2. There is an induced associative composition

$$\circ : Hom(G,G',\sim) \times Hom(G',G'',\sim) \to Hom(G,G'',\sim). \qquad (7.15)$$

As in §4 above, given a commutative ring R we define $Hom_{R,\sim}(G,G')$ to be the free R-module generated by $Hom(G,G',\sim)$, that is, the set of finite R-combinations $\phi = \sum_{[\mathbf{M}]} a_{[\mathbf{M}]} [\mathbf{M}]$, with $[\mathbf{M}] \in Hom(G,G',\sim)$ and $a_{[\mathbf{M}]} \in R$ with $a_{[\mathbf{M}]} = 0$ for all but finitely many $[\mathbf{M}]$. We write $Hom_{\sim}(G,G')$ for $Hom_{\mathbb{Z},\sim}(G,G')$. We then construct a category $\mathcal{K}_{R,\sim}$ of embedded graphs and correspondences in the following way.

DEFINITION 7.6. The category $\mathcal{K}_{R,\sim}$ has objects the embedded graphs G in $\mathbf{S^3}$ and morphisms the $Hom_{R,\sim}(G,G')$

After passing to $Mat(\mathcal{K}_{R,\sim})$ one obtains an additive category of embedded graphs and correspondences, which one still denotes $\mathcal{K}_{R,\sim}$.

7.2. Time evolutions and equivalence. We return now to the time evolutions (6.5) and (6.4) on the convolution algebra $\mathbb{C}[\mathcal{G}]$. After passing to equivalence classes by the relation of cobordism, we can consider the semigroupoid $\bar{\mathcal{G}}$ which is given by the data $\alpha = ([\mathbf{M}], G, G')$, where $[\mathbf{M}]$ denotes the equivalence class of $\mathbf{M}$ under the equivalence relation of branched cover cobordism. Lemma 7.3 shows that the composition in the semigroupoid $\mathcal{G}$ induces a well defined composition law in $\bar{\mathcal{G}}$. We can then consider the algebra $\mathbb{C}[\bar{\mathcal{G}}]$ with the convolution product as in (6.1),

$$(f_1 * f_2)([\mathbf{M}]) = \sum_{[\mathbf{M}_1],[\mathbf{M}_2]\in\bar{\mathcal{G}}:[\mathbf{M}_1]\circ[\mathbf{M}_2]=[\mathbf{M}]} f_1([\mathbf{M}_1])\, f_2([\mathbf{M}_2]). \tag{7.16}$$

The involution $f \mapsto f^\vee$ is also compatible with the equivalence relation, as it extends to the involution on the cobordisms W that interchanges the two branched covering maps.

LEMMA 7.7. *The time evolutions* (6.5) *and* (6.4) *descend to well defined time evolutions on the algebra* $\mathbb{C}[\bar{\mathcal{G}}]$.

PROOF. The result follows from the fact that the generic multiplicity of a branched covering is invariant under branched cover cobordisms. Thus, we have an induced time evolution of the form

$$\sigma_t^L(f)[\mathbf{M}] := n^{it} f[\mathbf{M}], \quad \sigma_t^R(f)[\mathbf{M}] := m^{it} f[\mathbf{M}], \quad \sigma_t(f)[\mathbf{M}] := \left(\frac{n}{m}\right)^{it} f[\mathbf{M}], \tag{7.17}$$

where each representative in the class $[\mathbf{M}]$ has branched covering maps with multiplicities

$$G \subset E \subset \mathbf{S^3} \stackrel{n:1}{\leftarrow} \mathbf{M} \stackrel{m:1}{\rightarrow} \mathbf{S^3} \supset E' \supset G'.$$

We see that the time evolution is compatible with the involution as in Lemma 6.2. ☐

7.3. Representations and Hamiltonian. Similarly, we can again consider representations of $\mathbb{C}[\bar{\mathcal{G}}]$ as in (6.8)

$$(\rho(f)\xi)[\mathbf{M}] = \sum_{[\mathbf{M}_1]\in\bar{\mathcal{G}},[\mathbf{M}_2]\in\bar{\mathcal{G}}_G:[\mathbf{M}_1]\circ[\mathbf{M}_2]=[\mathbf{M}]} f[\mathbf{M}_1]\xi[\mathbf{M}_2]. \tag{7.18}$$

As in the previous case, we define on the space $\bar{\mathcal{H}}_G$ of finitely supported functions $\xi : \bar{\mathcal{G}}_G \to \mathbb{C}$ the inner product

$$\langle \xi, \xi' \rangle = \sum_{[\mathbf{M}]} \overline{\xi[\mathbf{M}]}\xi'[\mathbf{M}]. \tag{7.19}$$

Once again we see that, in this representation, the adjoint does not correspond to the involution $f^\vee$ but it is instead given by the involution in the algebra of creation and annihilation operators

$$(A_{[\mathbf{M}]}\xi)[\mathbf{M}'] = \begin{cases} \xi[\mathbf{M}''] & \text{if } [\mathbf{M}'] = [\mathbf{M}] \circ [\mathbf{M}''] \\ 0 & \text{otherwise} \end{cases} \tag{7.20}$$

$$(A^*_{[\mathbf{M}]}\xi)[\mathbf{M}'] = \begin{cases} \xi[\mathbf{M} \circ \mathbf{M}'] & \text{if the composition is possible} \\ 0 & \text{otherwise.} \end{cases} \tag{7.21}$$

Again we have $\rho_G(\delta_{[\mathbf{M}]}) = A_{[\mathbf{M}]}$ so that the algebra generated by the $A_{[\mathbf{M}]}$ is the same as the image of $\mathbb{C}[\bar{\mathcal{G}}]$ in the representation ρ_G and the algebra of the creation and annihilation operators $A_{[\mathbf{M}]}$ and $A^*_{[\mathbf{M}]}$ is the involutive algebra in $\mathcal{B}(\bar{\mathcal{H}}_G)$ generated by $\mathbb{C}[\bar{\mathcal{G}}]$. In fact, the same argument we used before shows that $A^*_{[\mathbf{M}]}$ defined as in (7.21) is the adjoint of $A_{[\mathbf{M}]}$ in the inner product (7.19).

We then have the following result. We state it for the time evolution σ_t^R, while the case of σ_t^L is analogous.

THEOREM 7.8. *The Hamiltonian* $H = H_G^R$ *generating the time evolution* σ_t^R *in the representation* (7.18) *has discrete spectrum*

$$\mathrm{Spec}(H) = \{\log(n)\}_{n\in\mathbb{N}},$$

with finite multiplicities

$$1 \le N_n \le \#\pi_3(B_n), \tag{7.22}$$

where B_n *is the classifying space for branched coverings of order* n*.*

PROOF. It was proven in [**11**] that the n-fold branched covering spaces of a manifold $\mathbf{M}$, up to cobordism of branched coverings, are parameterized by the homotopy classes

$$B_n(\mathbf{M}) = [\mathbf{M}, B_n], \tag{7.23}$$

where the B_n are classifying spaces. In particular, cobordism equivalence classes of n-fold branched coverings of the 3-sphere are classified by the homotopy group

$$B_n(\mathbf{S^3}) = \pi_3(B_n). \tag{7.24}$$

The rational homotopy type of the classifying spaces B_n is computed in [**11**] in terms of the fibration

$$K(\pi, j-1) \to \bigvee^{t-1} \Sigma K(\pi, j-1) \to \bigvee^{t} K(\pi, j), \tag{7.25}$$

which holds for any abelian group π and any positive integers $t, j \ge 2$, with Σ denoting the suspension. For the B_n one finds

$$B_n \otimes \mathbb{Q} = \bigvee^{p(n)} K(\mathbb{Q}, 4) \tag{7.26}$$

with the fibration

$$S^3 \otimes \mathbb{Q} \to \bigvee^{p(n)-1} S^4 \times \mathbb{Q} \to B_n \otimes \mathbb{Q}, \tag{7.27}$$

where $p(n)$ is the number of partitions of n. The rational homotopy groups of B_n are computed from the exact sequence of the fibration (7.27) (see [**11**]) and are of the form $\pi_n(B_k) \otimes \mathbb{Q} = \mathbb{Q}^D$ with

$$D = \begin{cases} p(n) & k = 4 \\ Q(\frac{k-1}{3}, p(n)-1) & k = 1,4,10 \mod 12, \text{ with } k \neq 1,4 \\ Q(\frac{k-1}{3}, p(n)-1) + Q(\frac{k-1}{6}, p(n)-1) & k \equiv 7 \mod 12 \\ 0 & \text{otherwise} \end{cases} \tag{7.28}$$

where

$$Q(a,b) = \frac{1}{a}\sum_{d|a} \mu(d) b^{a/d}$$

with $\mu(d)$ the Möbius function. The result (7.28) then implies that the homotopy groups $\pi_3(B_n)$ satisfy $\pi_3(B_n) \otimes \mathbb{Q} = 0$. Moreover, in [**11**] the classifying spaces B_n are constructed explicitly by fitting together the classifying space $BO(2)$, that carries the information on the branch locus, with the classifying space BS_k, for S_k the group of permutations of k elements. For example, in the case of normalized simple coverings of [**7**], the classifying space is a mapping cylinder $BO(2) \cup_{BD_k} BS_k$, with D_k the dihedral group, over the maps induced by the inclusion $D_k \hookrightarrow O(2)$ as the subgroup leaving the set of k-th roots of unity globally invariant, and $D_k \to S_k$ giving the permutation action on the k-th roots of unity. In the case of [**11**] that we consider here, where more general branched coverings are considered, the explicit form of B_k in terms of $BO(2)$ and BS_k is more complicated, as it also involves a union over partitions of k, which accounts for the different choices of branching indices, of data of disk bundles associated to each partition.

The skeleta of the classifying space have finitely generated homology in each degree, *i.e.* they are spaces of finite type, and simply connected in the case of [**11**]. By a result of Serre it is known that, for simply connected spaces of finite type, the homotopy groups are also finitely generated (*cf.* also §0.a of [**29**]). The condition $\pi_3(B_n)\otimes\mathbb{Q}=0$ then implies that the groups $\pi_3(B_n)$ are finite for all n. By the same argument used in Lemma 6.8, the Hamiltonian generating the time evolution in the representation (7.18) is of the form

$$(H\xi)[\mathbf{M}] = \log(n)\,\xi[\mathbf{M}], \tag{7.29}$$

where $\mathbf{M}$ is a branched cover of $\mathbf{S^3}$ of order n branched along $E\supset G$, for the given embedded graph G specifying the representation. Thus, the multiplicity of the eigenvalue $\log(n)$ is the number of cobordism classes $[\mathbf{M}]$ branched along an embedded graph containing G as a subgraph. This number $N_n = N_n(G)$ is bounded by $1\le N_n(G)\le \#\pi_3(B_n)$. □

The result can be improved by considering, instead of the Brand classifying spaces B_n of branched coverings, the more refined Tejada classifying spaces $B_n(\ell)$ introduced in [**59**], [**8**]. In fact, the homotopy group $\pi_3(B_n)$ considered above parameterizes branched cobordism classes of branched coverings where the branch loci are embedded manifolds of codimension two. Since in each cobordism class there are representatives with such branch loci (*cf.* the discussion in Section 2 in Chapter 2 below) we can work with B_n and obtain the coarse estimate above. However, in our construction we are considering branch loci that are, more generally, embedded graphs and not just links. Similarly, our cobordisms are branched over 2-complexes, not just embedded surfaces. In this case, the appropriate classifying spaces are the generalizations $B_n(\ell)$ of [**59**], [**8**]. These are such that $B_n(2)=B_n$ and $B_n(\ell)$, for $\ell>2$, allows for branched coverings and cobordisms where the branch locus has strata of some codimension $2\le r\le\ell$. We have then the following more refined result.

COROLLARY 7.9. *The multiplicity $N_n(G)$ of the eigenvalue $\log(n)$ of the Hamiltonian H_G satisfies the estimate*

$$1\le N_n(G)\le \#\pi_3(B_n(4)). \tag{7.30}$$

PROOF. In our construction, we are considering branched coverings of the 3-sphere with branch locus an embedded graph $E\supset G$, up to branched cover cobordism, where the cobordisms are branched over a 2-complex. Thus, the branch locus E has strata of codimension two and three and the branch locus for the cobordism has strata of codimension two, three, and four. Thus, we can consider, instead of the classifying space B_n, the more refined $B_n(4)$. The results of [**8**] show that $\pi_3(B_n)\cong\pi_3(B_n(3))$, while there is a surjection $\pi_3(B_n(3))\to\pi_3(B_n(4))$, so that we have $\#\pi_3(B_n(4))\le\#\pi_3(B_n)$. Thus, the same argument of Theorem 7.8 above, using cobordisms with stratified branch loci, gives the finer estimate (7.30) for the multiplicities. □

We can then consider the partition function for the Hamiltonian of the time evolution (7.17). To stress the fact that we work in the representation $\rho=\rho_G$ associated to the subsemigroupoid $\mathcal{G}_G$ for a given graph G, we write $H=H(G)$. We then have

$$Z_G(\beta)=\mathrm{Tr}(e^{-\beta H(G)})=\sum_n \exp(-\beta\log(n))\,N_n(G). \tag{7.31}$$

Thus, the question of whether the summability condition $\mathrm{Tr}(e^{-\beta H})<\infty$ holds depends on an estimate of the asymptotic growth of the cardinalities $\#\pi_3(B_n)$ for large $n\to\infty$, by the estimate

$$\zeta(\beta)=\sum_n n^{-\beta}\le Z_G(\beta)\le\sum_n \#\pi_3(B_n)\,n^{-\beta}. \tag{7.32}$$

This corresponds to the question of studying a generating function for the numbers $\#\pi_3(B_n)$. We will not pursue this in the present text, but we hope to return to it in future work.

Notice that there is evidence in the results of [**8**] in favor of some strong constraints on the growth of the numbers $\#\pi_3(B_n)$ (hence of the $\#\pi_3(B_n(4))$), based on the periodicities along certain arithmetic progressions of the localizations at primes. In fact, it is proved in [**7**] that, at least for the classifying spaces BR_n of normalized simple branched coverings, in the stable range $n > 4$ and for any given prime p, the localizations $\pi_3(BR_n)_{(p)}$ satisfy the periodicity

$$\pi_3(BR_n)_{(p)} = \pi_3(BR_{n+2^{a+i+1}p^{b+j}})_{(p)},$$

for $n = 2^a p^b m$ with $(2,m) = (p,m) = 1$. The number $2^i p^j$ is determined by homotopy theoretic data as described in Proposition 11 of [**7**]. Thus, one can consider associated zeta functions

$$Z_p(\beta) = \sum_n \#\pi_3(BR_n)_{(p)} n^{-\beta}. \tag{7.33}$$

If a finite summability $\mathrm{Tr}(e^{-\beta H}) < \infty$ holds for sufficiently large $\beta >> 0$, then one can recover invariants of embedded graphs as zero temperature KMS functionals, by considering functionals of the Gibbs form

$$\varphi_{G,\beta}(f) = \frac{\mathrm{Tr}(\rho_G(f)e^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}, \tag{7.34}$$

where, for instance, f is taken to be an invariant of embedded graphs in 3-manifolds and $f(\mathbf{M}) := f(\pi_G^{-1}(G))$, for $\pi_G : \mathbf{M} \to \mathbf{S^3}$ the branched covering map. In this case, in the zero temperature limit, *i.e.* for $\beta \to \infty$, the weak limits of states of the form (7.34) would give back the invariant of embedded graphs in $\mathbf{S^3}$ in the form

$$\lim_{\beta\to\infty} \varphi_{K,\beta}(f) = f(\mathbb{U}_G).$$

Notice that, to the purpose of studying KMS states for the algebra with time evolution, the convergence of the partition function $Z_G(\beta)$ is not needed, as KMS states need not necessarily be of the Gibbs form (7.34), *cf.* [**27**]. However, it is still useful to consider the question of the convergence of the partition function $Z_G(\beta)$, since Gibbs states of the form (7.34) may have applications to constructing interesting zeta functions for embedded graphs $G \subset S^3$.
For instance, suppose given an invariant F of cobordism classes of embedded graphs in S^3. Cobordism for embedded graphs can be defined, for connected graphs, as in [**57**], and in the multi-connected case using the same basic relation (attaching a 1-handle) as in the case of links, as in [**36**]. An example of such an invariant can be obtained, for instance, by considering the collection of links $T(G)$ constructed in [**42**] as an invariant of an embedded graph G and define a total linking number of $T(G)$ by adding the total linking numbers of all the links in the collection. Given such an invariant F, one can then consider, for a set of representatives of the classes $[M] \in \pi_3(B_n)$, the values $F(\pi_{G'}\pi_G^{-1}(G))$ and form the series

$$\sum_n \sum_{[M]\in\pi_3(B_n)} F(\pi_{G'}\pi_G^{-1}(G))\, n^{-\beta}, \tag{7.35}$$

where the inner sum is over the classes $[M] \in \pi_3(B_n)$ such that M is a branched cover of S^3 branched along a graph $E \supset G$. Similarly, one can form variations of this same concept based on the zeta functions (7.33). When the function F on the set of the $\{\pi_{G'}\pi_G^{-1}(G)\}$ is either bounded or of some growth $\sim n^\alpha$, then the convergence of $Z_G(\beta)$ (or of the $Z_p(\beta)$ of (7.33)) would ensure the convergence of (7.35). Obviously such zeta functions are very complicated objects, even for very simple graphs G and it would be difficult to compute them explicitly, but it would be interesting to see whether some variant of this idea might have relevance in the context of spin networks, spin foams, and dynamical triangulations.

Finally, notice that, while the Hamiltonian H of the time evolution σ_t^L has finite multiplicities in the spectrum after passing to the quotient by the equivalence relation of cobordism (similarly for

σ_t^R), the infinitesimal generator for the time evolution $\sigma_t = \sigma_t^L \sigma_{-t}^R$ still has infinite multiplicities. In fact, the time evolution (6.4) is generated by an unbounded operator D that acts on a densely defined domain in $\mathcal{H}_G$ by

$$D\,\delta_{\mathbf{M}} = \log\left(\frac{n}{m}\right)\,\delta_{\mathbf{M}}, \tag{7.36}$$

with n and m the multiplicities of the two covering maps, as above. This operator is not a good physical Hamiltonian since is does not have a lower bound on the spectrum. It has the following property.

LEMMA 7.10. *The operator D of* (7.36) *has bounded commutators $[D,a]$ with the elements of the involutive algebra generated (algebraically) by the $A_{[\mathbf{M}]}$ and $A^*_{[\mathbf{M}]}$.*

PROOF. It suffices to check that the commutators $[D, A_{[\mathbf{M}]}]$ and $[D, A^*_{[\mathbf{M}]}]$ are bounded. We have

$$[D, A^*_{[\mathbf{M}]}]\delta_{[\mathbf{M}']} = \left(\log\left(\frac{nn'}{mm'}\right) - \log\left(\frac{n'}{m'}\right)\right)\delta_{[\mathbf{M}\circ\mathbf{M}']} = \log\left(\frac{n}{m}\right)\delta_{[\mathbf{M}\circ\mathbf{M}']}.$$

The case of $[D, A_{[\mathbf{M}]}]$ is analogous. □

Notice, however, that D fails to be a Dirac operator in the sense of spectral triples, because of the infinite multiplicities of the eigenvalues.

8. Convolution algebras and 2-semigroupoids

In noncommutative geometry, it is customary to replace the operation of taking the quotient by an equivalence relation by forming a suitable convolution algebra of functions over the graph of the equivalence relation. This corresponds to replacing an equivalence relation by the corresponding groupoid and taking the convolution algebra of the groupoid, *cf.* [**14**]. In our setting, we can proceed in a similar way and, instead of taking the quotient by the equivalence relation of cobordism of branched cover, as we did above, keep the cobordisms explicitly and work with a 2-category.

LEMMA 8.1. *The data of embedded graphs in the 3-sphere, 3-dimensional geometric correspondences, and 4-dimensional branched cover cobordisms form a 2-category $\mathcal{G}^2$.*

PROOF. We already know that embedded graphs and geometric correspondences form semigroupoid with associative composition of morphisms given by the fibered product of geometric correspondences. Suppose given geometric correspondences $\mathbf{M}_1$ $\mathbf{M}_2$ and $\mathbf{M}_3$ in $Hom(G,G')$, and suppose given cobordisms W_1 and W_2 with $\partial W_1 = \mathbf{M}_1 \cup -\mathbf{M}_2$ and $\partial W_2 = \mathbf{M}_2 \cup -\mathbf{M}_3$. As we have seen in Lemma 7.3, for the transitive property of the equivalence relation, the gluing of cobordisms $W_1 \cup_{\mathbf{M}_2} W_2$ gives a cobordism between $\mathbf{M}_1$ and $\mathbf{M}_3$ and defines in this way a composition of 2-morphisms that has the right properties for being the vertical composition in the 2-category. Similarly, suppose given correspondences $\mathbf{M}_1, \tilde{\mathbf{M}}_1 \in Hom(G,G')$, and $\mathbf{M}_2, \tilde{\mathbf{M}}_2 \in Hom(G',G'')$, with cobordisms W_1 and W_2 with $\partial W_1 = \mathbf{M}_1 \cup -\tilde{\mathbf{M}}_1$ and $\partial W_2 = \mathbf{M}_2 \cup -\tilde{\mathbf{M}}_2$. Again by the argument of Lemma 7.3, we know that the fibered product $W_1 \circ W_2$ defines a cobordism between the compositions $\mathbf{M}_1 \circ \mathbf{M}_2$ and $\tilde{\mathbf{M}}_1 \circ \tilde{\mathbf{M}}_2$. This gives the horizontal composition of 2-morphisms. By the results of Lemma 7.3 and an argument like that of Proposition 2.11, one sees that both the vertical and horizontal compositions of 2-morphisms are associative. □

In the following, we denote the compositions of 2-morphisms by the notation

$$\text{horizontal (fibered product): } W_1 \circ W_2 \qquad \text{vertical (gluing): } W_1 \bullet W_2. \tag{8.1}$$

We obtain a convolution algebra associated to the 2-semigroupoid described above. Consider the space of complex valued functions with finite support

$$f : \mathcal{U} \to \mathbb{C} \tag{8.2}$$

on the set

$$\mathcal{U} = \cup_{\mathbf{M}_1,\mathbf{M}_2 \in \mathcal{G}} \mathcal{U}_{(\mathbf{M}_1,\mathbf{M}_2)},$$

of branched cover cobordisms

$$\mathcal{U}_{(\mathbf{M}_1,\mathbf{M}_2)} = \{W \,|\, \mathbf{M}_1 \stackrel{W}{\sim} \mathbf{M}_2\}, \tag{8.3}$$

with

$$S \subset \mathbf{S^3} \times I \stackrel{q}{\leftarrow} W \stackrel{q'}{\to} \mathbf{S^3} \times I \supset S',$$

where $\sim$ denotes the equivalence relation given by branched cover cobordisms with $\partial W = \mathbf{M}_1 \cup -\mathbf{M}_2$, compatibly with the branched cover structures as in §7 above.
As in the case of the sets $Hom(G,G')$ of geometric correspondences discussed in §2, the collection $\mathcal{U}_{(\mathbf{M}_1,\mathbf{M}_2)}$ of cobordisms is a set because it can be identified with a set of branched covering data of a representation theoretic nature. In fact, as a PL manifold, one such cobordism W can be specified by assigning a representation

$$\sigma_W : \pi_1((\mathbf{S^3} \times I) \smallsetminus S) \to S_n, \tag{8.4}$$

which determines a covering space on the complement of the branch locus S. This space of functions (8.2) can be made into an algebra $\mathcal{A}(\mathcal{G}^2)$ with the associative convolution product of the form

$$(f_1 \bullet f_2)(W) = \sum_{W = W_1 \bullet W_2} f_1(W_1) f_2(W_2), \tag{8.5}$$

which corresponds to the vertical composition of 2-morphisms, namely the one given by the gluing of cobordisms. Similarly, one also has on $\mathcal{A}(\mathcal{G}^2)$ an associative product which corresponds to the horizontal composition of 2-morphisms, given by the fibered product of cobordisms, of the form

$$(f_1 \circ f_2)(W) = \sum_{W = W_1 \circ W_2} f_1(W_1) f_2(W_2). \tag{8.6}$$

We also have an involution compatible with both the horizontal and vertical product structure. In fact, consider the two involutions on the cobordisms W

$$W \mapsto \bar{W} = -W, \quad W \mapsto W^\vee, \tag{8.7}$$

where the first is the orientation reversal, so that if $\partial W = \mathbf{M}_1 \cup -\mathbf{M}_2$ then $\partial \bar{W} = \mathbf{M}_2 \cup -\mathbf{M}_1$, while the second extends the involution $\mathbf{M} \mapsto \mathbf{M}^\vee$ and exchanges the two branch covering maps, that is, if W has covering maps

$$S \subset \mathbf{S^3} \times I \stackrel{q}{\leftarrow} W \stackrel{q'}{\to} \mathbf{S^3} \times I \supset S'$$

then $W^\vee$ denotes the same 4-manifold but with covering maps

$$S' \subset \mathbf{S^3} \times I \stackrel{q'}{\leftarrow} W \stackrel{q}{\to} \mathbf{S^3} \times I \supset S.$$

We define an involution on the algebra $\mathcal{A}(\mathcal{G}^2)$ by setting

$$f^\dagger(W) = \overline{f}(\bar{W}^\vee) \tag{8.8}$$

LEMMA 8.2. *The involution $f \mapsto f^\dagger$ makes $\mathcal{A}(\mathcal{G}^2)$ into an involutive algebra with respect to both the vertical and the horizontal product.*

PROOF. We have $(f^\dagger)^\dagger = f$ since the two involutions $W \mapsto \bar{W}$ and $W \mapsto W^\vee$ commute. We also have $(af_1 + bf_2)^\dagger = \bar{a} f_1^\dagger + \bar{b} f_2^\dagger$. For the two product structures, we have

$$\begin{aligned} \bar{W} &= \bar{W}_1 \circ \bar{W}_2 && \text{for } W = W_1 \circ W_2 \\ W^\vee &= W_1^\vee \bullet W_2^\vee && \text{for } W = W_1 \bullet W_2 \end{aligned}$$

which gives

$$(f_1 \circ f_2)^\dagger(W) = \sum_{\bar{W}^\vee = \bar{W}_1^\vee \circ \bar{W}_2^\vee} \overline{f_1}(\bar{W}_1^\vee)\overline{f_2}(\bar{W}_2^\vee) = (f_2^\dagger \circ f_1^\dagger)(W)$$

$$(f_1 \bullet f_2)^\dagger(W) = \sum_{\bar{W}^\vee = \bar{W}_1^\vee \bullet \bar{W}_2^\vee} \overline{f_1}(\bar{W}_1^\vee)\overline{f_2}(\bar{W}_2^\vee) = (f_2^\dagger \bullet f_1^\dagger)(W).$$

□

9. Vertical and horizontal time evolutions

We say that σ_t is a *vertical time evolution* on $\mathcal{A}(\mathcal{G}^2)$ if it is a 1-parameter group of automorphisms of $\mathcal{A}(\mathcal{G}^2)$ with respect to the product structure given by the vertical composition of 2-morphisms as in (8.5), namely

$$\sigma_t(f_1 \bullet f_2) = \sigma_t(f_1) \bullet \sigma_t(f_2).$$

Similarly, a *horizontal time evolution* on $\mathcal{A}(\mathcal{G}^2)$ satisfies

$$\sigma_t(f_1 \circ f_2) = \sigma_t(f_1) \circ \sigma_t(f_2).$$

We give some simple examples of one type or the other first and then we move on to more subtle examples.

LEMMA 9.1. *The time evolution by order of the coverings defined in* (6.4) *extends to a horizontal time evolution on* $\mathcal{A}(\mathcal{G}^2)$.

PROOF. This clearly follows by taking the order of the cobordisms as branched coverings of $\mathbf{S^3} \times I$. It is not a time evolution with respect to the vertical composition. □

LEMMA 9.2. *Any numerical invariant that satisfies an inclusion-exclusion principle*

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B) \tag{9.1}$$

defines a vertical time evolution by

$$\sigma_t(f)(W) = \exp(it(\chi(W) - \chi(\mathbf{M}_2)))\, f(W), \tag{9.2}$$

for $\partial W = \mathbf{M}_1 \cup -\mathbf{M}_2$.

PROOF. This also follows immediately by direct verification, since

$$\sigma_t(f_1 \bullet f_2)(W) = e^{it(\chi(W) - \chi(\mathbf{M}_2))} \sum_{W = W_1 \cup_{\mathbf{M}} W_2} f_1(W_1) f_2(W_2)$$

$$= e^{it(\chi(W_1) + \chi(W_2) - \chi(\mathbf{M}) - \chi(\mathbf{M}_2))} \sum_{W = W_1 \cup_{\mathbf{M}} W_2} f_1(W_1) f_2(W_2)$$

$$= \sum_{W = W_1 \cup_{\mathbf{M}} W_2} e^{it(\chi(W_1) - \chi(\mathbf{M}))} f_1(W_1) e^{it(\chi(W_2) - \chi(\mathbf{M}_2))} f_2(W_2) = (\sigma_t(f_1) \bullet \sigma_t(f_2))(W).$$

□

In particular, the following are two simple examples of this type of time evolution.

EXAMPLE 9.3. Setting $\chi(W)$ to be the Euler characteristic gives a time evolution as in (9.2). Since the 4-dimensional volume of the boundary 3-manifold $\mathbf{M}$ is zero, also setting $\chi(W) = Vol(W)$ gives a time evolution.

A more elaborate example of this type is given in §11 below.

10. Vertical time evolution: Hartle–Hawking gravity

We describe here a first non-trivial example of a vertical time evolution, which is related to the Hartle–Hawking formalism of Euclidean quantum gravity [**28**]. The classical Euclidean action for gravity on a 4-manifold W with boundary is of the form

$$S(W,g) = -\frac{1}{16\pi}\int_W R\,dv - \frac{1}{8\pi}\int_{\partial W} K\,dv, \tag{10.1}$$

where R is the scalar curvature and K is the trace of the II fundamental form. In the Hartle–Hawking approach to quantum gravity, the transition amplitude between two 3-dimensional geometries $\mathbf{M}_1$ and $\mathbf{M}_2$, endowed with Riemannian structures $g_{\mathbf{M}_1}$ and $g_{\mathbf{M}_2}$ is given by

$$\langle(\mathbf{M}_1,g_1),(\mathbf{M}_2,g_2)\rangle = \int e^{iS(g)}D[g], \tag{10.2}$$

in the Lorentzian signature, where the formal functional integration on the right hand side involves also a summation over topologies, meaning a sum over all cobordisms W with $\partial W = \mathbf{M}_1 \cup -\mathbf{M}_2$. In the Euclidean setting the probability amplitude $e^{iS(g)}$ is replaced by $e^{-S(g)}$, with $S(g)$ the Euclidean action (10.1). We have suppressed the dependence of the probability amplitude on a quantization parameter $\hbar$. This suggests setting

$$\sigma_t(f)(W,g) := e^{itS(W,g)}f(W,g), \tag{10.3}$$

with $S(W,g)$ as in (10.1). For (10.3) to define a vertical time evolution, *i.e.* for it to satisfy the compatibility $\sigma_t(f_1 \bullet f_2) = \sigma_t(f_1)\bullet\sigma_t(f_2)$ with the vertical composition, we need to impose conditions on the metrics g on W so that the gluing of the Riemannian data near the boundary is possible when composing cobordisms $W_1 \bullet W_2 = W_1 \cup_{\mathbf{M}} W_2$ by gluing them along a common boundary $\mathbf{M}$.
For instance, one can assume cylindrical metrics near the boundary, though this is does not correspond to the physically interesting case of more general space-like hypersurfaces. Also, one needs to restrict here to cobordisms that are smooth manifolds, or to allow for weaker forms of the Riemannian structure adapted to PL manifolds. Then, formally, one obtains states for this vertical time evolution that can be expressed in the form of a functional integration as

$$\varphi_\beta(f) = \frac{\int f(W,g)e^{-\beta S(g)}D[g]}{\int e^{-\beta S(g)}D[g]}. \tag{10.4}$$

We give in the next section a more mathematically rigorous example of vertical time evolution.

11. Vertical time evolution: gauge moduli and index theory

Consider again the vertical composition $W_1 \bullet W_2 = W_1 \cup_{\mathbf{M}_2} W_2$ given by gluing two cobordisms along their common boundary. In order to construct interesting time evolutions on the corresponding convolution algebra, we consider the spectral theory of Dirac type operators on these 4-dimensional manifolds with boundary, *cf.* [**6**].
Consider first the simpler case where X is a closed connected 4-manifold and $\mathbf{M}$ is a hypersurface that partitions $X \setminus M$ in two components $X = X_1 \cup_M X_2$ with boundary $\partial X_1 = M = -\partial X_2$. We assume that X is endowed with a cylindrical metric on a collar neighborhood $M \times [-1,1]$ of the hypersurface

M. Let $\mathcal{D}$ be an elliptic differential operator on X of Dirac type. We take it to be the Dirac operator assuming that X is a spin 4-manifold. The restriction $\mathcal{D}|_{M\times[-1,1]}$ has the form

$$\mathcal{D}|_{M\times[-1,1]} = c(\frac{\partial}{\partial s} + \mathcal{B}),$$

where c denotes Clifford multiplication by ds and $\mathcal{B}$ is the self-adjoint tangential Dirac operator on M. We let $P_{\geq}$ denote the spectral Atiyah–Patodi–Singer boundary conditions, *i.e.* the projection onto the subspace of the Hilbert space of square integrable spinors $L^2(M,\mathcal{S}^+|M)$ spanned by the eigenvectors of $\mathcal{B}$ with non-negative eigenvalues. Here $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$ is the spinor bundle on X, with $\mathcal{D}^+ : C^\infty(X,\mathcal{S}^+) \to C^\infty(X,\mathcal{S}^-)$. The projection $P_{\leq}$ is defined similarly. Let $\mathcal{D}_i$ denote the Dirac operator on X_i with APS boundary conditions, that is,

$$\mathcal{D}_1^+ : C^\infty(X_1,\mathcal{S}^+,P_{\leq}) \to C^\infty(X_1,\mathcal{S}^-), \quad \mathcal{D}_2^+ : C^\infty(X_1,\mathcal{S}^+,P_{\geq}) \to C^\infty(X_2,\mathcal{S}^-),$$

where

$$C^\infty(X_1,\mathcal{S}^+,P_{\leq}) = \{\psi \in C^\infty(X_1,\mathcal{S}^+) \,|\, P_{\leq}(\psi|_M) = 0\},$$
$$C^\infty(X_2,\mathcal{S}^+,P_{\geq}) = \{\psi \in C^\infty(X_2,\mathcal{S}^+) \,|\, P_{\geq}(\psi|_M) = 0\}.$$

The index of the Dirac operator $\mathcal{D}$ is computed by the Atiyah–Singer index theorem and is given by a local formula, while the index of $\mathcal{D}_i$ is given by the Atiyah–Patodi–Singer index theorem and consists of a local formula, together with a correction given by an eta invariant of the boundary manifold M. Moreover, one has the following splitting formula for the index (*cf.* [**6**], p.77)

$$\mathrm{Ind}(\mathcal{D}) = \mathrm{Ind}(\mathcal{D}_1) + \mathrm{Ind}(\mathcal{D}_2) - \dim \mathrm{Ker}(\mathcal{B}). \tag{11.1}$$

In the case of 4-manifolds $W = W_1 \cup_M W_2$, where $\partial W = M_1 \cup -M_3$, $\partial W_1 = M_1 \cup -M_2$, and $\partial W_2 = M_2 \cup -M_3$, one can modify the above setting by imposing APS boundary conditions at both ends of the cobordims. Namely, we assume that W is a smooth manifold with boundary endowed with a Riemannian metric with cylindrical ends $M_1 \times [0,1]$ and $M_3 \times [-1,0]$, as well as a cylindrical metric on a collar neighborhood $M \times [-1,1]$. Thus, the operator $\mathcal{D}$ will be the Dirac operator with APS boundary conditions $P_{\geq}$ and $P_{\leq}$ at M_1 and M_3, and similarly for the operators $\mathcal{D}_1$ and $\mathcal{D}_2$. We also denote by $\mathcal{B}$, $\mathcal{B}_1$ and $\mathcal{B}_2$ the tangential Dirac operators on M, M_1 and M_2, respectively. We then obtain a time evolution on the algebra $\mathcal{A}(\mathcal{G}^2)$ with the product (8.5) associated to the splitting of the index, in the following way.

LEMMA 11.1. *Let $W = W_1 \cup_M W_2$ be a composition of 4-dimensional cobordisms with metrics as above, and with $\mathcal{D}$, $\mathcal{D}_i$ the corresponding Dirac operators with APS boundary conditions. We let*

$$\delta(W) := \mathrm{Ind}(\mathcal{D}) - \dim \mathrm{Ker}(\mathcal{B}_2). \tag{11.2}$$

Then setting

$$\sigma_t(f)(W) = \exp(it\delta(W))\; f(W) \tag{11.3}$$

defines a time evolution on $\mathcal{A}(\mathcal{G}^2)$ with the product (8.5) *of vertical composition.*

PROOF. Using the splitting formula (11.1) for the index one sees immediately that

$$\sigma_t(f_1 \bullet f_2)(W) = \sum_{W=W_1\bullet W_2} e^{it\delta(W)} f_1(W_1) f_2(W_2)$$
$$= \sum_{W=W_1\bullet W_2} e^{it(\mathrm{Ind}\mathcal{D}_1 + \mathrm{Ind}\mathcal{D}_2 - \dim\mathrm{Ker}\mathcal{B} - \dim\mathrm{Ker}\mathcal{B}_2)} f_1(W_1) f_2(W_2)$$
$$= \sum_{W=W_1\bullet W_2} e^{it\delta(W_1)} f_1(W_1) e^{it\delta(W_2)} f_2(W_2) = \sigma_t(f_1) \bullet \sigma_t(f_2)(W).$$

□

The type of spectral problem described above arises typically in the context of invariants of 4-dimensional geometries that behave well under gluing. A typical such setting is given by the topological quantum field theories, as outlined in [**2**], where to every 3-dimensional manifolds one assigns functorially a vector space and to every cobordism between 3-manifolds a linear map between the vector spaces. In the case of Yang–Mills gauge theory, the gluing theory for moduli spaces of anti-self-dual $SO(3)-$connections on smooth 4-manifolds (see [**58**] for an overview) shows that if $\mathbf{M}$ is a compact oriented smooth 3-manifold that separates a compact smooth 4-manifold X in two connected pieces

$$X = X_+ \cup_{\mathbf{M}} X_- \tag{11.4}$$

glued along the common boundary $\mathbf{M} = \partial X_+ = -\partial X_-$, then the moduli space $\mathcal{M}(X)$ of gauge equivalence classes of framed anti-self-dual $SO(3)$-connections on X decomposes as a fibered product

$$\mathcal{M}(X) = \mathcal{M}(X_+) \times_{\mathcal{M}(\mathbf{M})} \mathcal{M}(X_-), \tag{11.5}$$

where $\mathcal{M}(X_\pm)$ are moduli spaces of anti-self-dual $SO(3)$-connections on the 4-manifolds with boundary and $\mathcal{M}(\mathbf{M})$ is a moduli space of gauge classes of flat connections on the 3-manifold $\mathbf{M}$. The fibered product is over the restriction maps induced by the inclusion of $\mathbf{M}$ in $X_\pm$. In particular, at the linearized level, the virtual dimensions of the moduli spaces satisfy

$$\dim \mathcal{M}(X) = \dim \mathcal{M}(X_+) + \dim \mathcal{M}(X_-) - \dim \mathcal{M}(\mathbf{M}). \tag{11.6}$$

In Donaldson–Floer theory the virtual dimension of the moduli space for the 3-manifold is zero, the deformation complex being given by a self-adjoint elliptic operator, however we allow here for the possibility that $\mathcal{M}(\mathbf{M})$ might be of positive dimension. We then obtain a time evolution on the algebra $\mathcal{A}(\mathcal{G}^2)$ with the product (8.5) associated to the instanton moduli spaces in the following way.

LEMMA 11.2. *Let W be a branched cover cobordism with $\partial W = \mathbf{M}_1 \cup -\mathbf{M}_2$. Let $\mathcal{M}(W)$ denote the moduli space of gauge equivalence classes of framed anti-self-dual $SO(3)$-connections on W. Let $\mathcal{M}(\mathbf{M}_i)$ be the moduli space of gauge equivalence classes of flat framed connections on $\mathbf{M}_i$. We set*

$$\delta(W) = \dim \mathcal{M}(W) - \dim \mathcal{M}(\mathbf{M}_2). \tag{11.7}$$

Then setting

$$\sigma_t(f)(W) = \exp(it\delta(W))\, f(W) \tag{11.8}$$

defines a time evolution on $\mathcal{A}(\mathcal{G}^2)$ with the product (8.5) *of vertical composition.*

PROOF. We assume in this discussion that the moduli spaces satisfy the gluing theorem so that

$$\mathcal{M}(W) = \mathcal{M}(W_1) \times_{\mathcal{M}(\mathbf{M})} \mathcal{M}(W_2) \tag{11.9}$$

for $W = W_1 \cup_{\mathbf{M}} W_2$ with $\partial W_1 = \mathbf{M}_1 \cup -\mathbf{M}$ and $\partial W_2 = \mathbf{M} \cup -\mathbf{M}_2$. Strictly speaking, the result (11.5) holds for a compact 4-manifold X, while here we are dealing with a 4-manifold W with boundary. The same technique used in analyzing the moduli spaces $\mathcal{M}(X_\pm)$ in (11.5) can be used to treat $\mathcal{M}(W)$. A detailed discussion of the gluing theory that yields (11.9) is beyond the scope of this short paper. Assuming (11.9) we see immediately that

$$\sigma_t(f_1 \bullet f_2)(W) = \sum_{W = W_1 \bullet W_2} e^{it\delta(W)} f_1(W_1) f_2(W_2)$$

$$= \sum_{W = W_1 \bullet W_2} e^{it(\dim \mathcal{M}(W_1) + \dim \mathcal{M}(W_2) - \dim \mathcal{M}(\mathbf{M}) - \dim \mathcal{M}(\mathbf{M}_2))} f_1(W_1) f_2(W_2)$$

$$= \sum_{W = W_1 \bullet W_2} e^{it\delta(W_1)} f_1(W_1) e^{it\delta(W_2)} f_2(W_2) = \sigma_t(f_1) \bullet \sigma_t(f_2)(W).$$

□

One can define similar time evolutions using other moduli spaces on 4-manifolds that satisfy suitable gluing formulae, such as the Seiberg–Witten moduli spaces, with the gluing theory discussed in [**12**]. Notice that we are only using a very coarse invariant extracted from the moduli spaces, namely the (virtual) dimension. This only depends on the linearized theory. Typically, the virtual dimension is computed via an index theorem $\delta(W) = \mathrm{Ind}\,\mathcal{D}_W$ where $\mathcal{D}_W : \Omega^{odd} \to \Omega^{ev}$, for an elliptic complex

$$\Omega^0 \stackrel{\mathcal{D}_0}{\to} \Omega^1 \stackrel{\mathcal{D}_1}{\to} \Omega^2$$

where the elliptic operators $\mathcal{D}_1$ and $\mathcal{D}_0$ correspond, respectively, to the linearization of the nonlinear elliptic equations and to the infinitesimal gauge action. Thus, the fact that (11.8) becomes a direct consequence of the additivity of the index

$$\mathrm{Ind}\,\mathcal{D}_W = \mathrm{Ind}\,\mathcal{D}_{W_1} + \mathrm{Ind}\,\mathcal{D}_{W_2}. \tag{11.10}$$

12. Horizontal time evolution: bivariant Chern character

The time evolution of Lemma 11.2, however, does not detect the structure of W as a branched cover of $\mathbf{S^3} \times I$ branched along an embedded surface $S \subset \mathbf{S^3} \times I$. Thus, there is no reason why a time evolution defined in this way should also be compatible with the other product given by the horizontal composition of 2-morphisms. The interpretation of the time evolution (11.8) in terms of the additivity of the index (11.10), however, suggests a possible way to define other time evolutions, also related to properties of an index, which would be compatible with the horizontal composition. Although we are working here in the commutative context, in view of the extension to noncommutative spectral correspondences outlined in the next section, we give here a formulation using the language of KK-theory and cyclic cohomology that carries over naturally to the noncommutative cases. In noncommutative geometry, one thinks of the index theorem as a pairing of K-theory and K-homology, or equivalently as the pairing $\langle ch_n(e), ch_n(x)\rangle$ of Connes–Chern characters

$$ch_n : K_i(\mathcal{A}) \to HC_{2n+i}(\mathcal{A}) \quad \text{and} \quad ch_n : K^i(\mathcal{A}) \to HC^{2n+i}(\mathcal{A}), \tag{12.1}$$

under the natural pairing of cyclic homology and cohomology, *cf.* [**14**]. Recall that cyclic (co)homology has a natural description in terms of the derived functors Ext and Tor in the abelian category of cyclic modules (*cf.* [**15**]), namely

$$HC^n(\mathcal{A}) = \mathrm{Ext}^n_\Lambda(\mathcal{A}^\natural, \mathbb{C}^\natural) \quad \text{and}\; HC_n(\mathcal{A}) = \mathrm{Tor}^\Lambda_n(\mathbb{C}^\natural, \mathcal{A}^\natural), \tag{12.2}$$

where Λ denotes the cyclic category and $\mathcal{A}^\natural$ is the cyclic module associated to an associative algebra $\mathcal{A}$. It was shown in [**50**] that the characters (12.1) extend to a bivariant Connes–Chern character

$$ch_n : KK^i(\mathcal{A}, \mathcal{B}) \to \mathrm{Ext}^{2n+i}_\Lambda(\mathcal{A}^\natural, \mathcal{B}^\natural) \tag{12.3}$$

defined on KK-theory, which sends the Kasparov products

$$\circ : KK^i(\mathcal{A}, \mathcal{C}) \times KK^j(\mathcal{C}, \mathcal{B}) \to KK^{i+j}(\mathcal{A}, \mathcal{B})$$

to the Yoneda products,

$$\mathrm{Ext}^{2n+i}_\Lambda(\mathcal{A}^\natural, \mathcal{C}^\natural) \times \mathrm{Ext}^{2m+j}_\Lambda(\mathcal{C}^\natural, \mathcal{B}^\natural) \to \mathrm{Ext}^{2(n+m)+i+j}(\mathcal{A}^\natural, \mathcal{B}^\natural), \tag{12.4}$$

with the natural cap product pairings

$$\mathrm{Tor}^\Lambda_m(\mathbb{C}^\natural, \mathcal{A}^\natural) \otimes \mathrm{Ext}^n_\Lambda(\mathcal{A}^\natural, \mathcal{B}^\natural) \to \mathrm{Tor}^\Lambda_{m-n}(\mathbb{C}^\natural, \mathcal{B}^\natural) \tag{12.5}$$

corresponding to an index theorem

$$\psi = ch(x)\phi, \quad \text{with} \quad \phi(e \circ x) = \psi(e). \tag{12.6}$$

requires a modification of both KK-theory and cyclic cohomology. Such a general form of the bivariant Connes–Chern character is given in [**20**]. The construction of [**18**] of geometric correspondences realizing KK-theory classes shows that, given manifolds X_1 and X_2, classes in $KK(X_1,X_2)$ are realized by geoemtric data (Z,E) of a manifold Z with submersions $X_1 \leftarrow Z \to X_2$ and a vector bundle E on Z. The Kasparov product $x \circ y \in KK(X_1,X_3)$, for $x = kk(Z,E) \in KK(X_1,X_2)$ and $y = kk(Z',E') \in KK(X_2,X_3)$, is given by the fibered product $x \circ y = kk(Z \circ Z', E \circ E')$, where

$$Z \circ Z' = Z \times_{X_2} Z' \quad \text{and} \quad E \circ E' = \pi_1^* E \times \pi_2^* E'.$$

To avoid momentarily the complication caused by working with manifolds with boundary, we consider the simpler situation where W is a 4-manifold endowed with branched covering maps to a compact 4-manifold X (for instance $\mathbf{S^3} \times S^1$ or S^4) instead of $\mathbf{S^3} \times [0,1]$,

$$S \subset X \xleftarrow{q} W \xrightarrow{q'} X \supset S' \tag{12.7}$$

branched along surfaces S and S' in X. We can then think of an elliptic operator $\mathcal{D}_W$ on a 4-manifold W, which has branched covering maps as in (12.7), as defining an unbounded Kasparov bimodule, *i.e.* as defining a KK-class $[\mathcal{D}_W] \in KK(X,X)$. We can think of this class as being realized by a geometric correspondence in the sense of [**18**]

$$[\mathcal{D}_W] = kk(W, E_W),$$

with the property that, for the horizontal composition $W = W_1 \circ W_2 = W_1 \times_X W_2$ we have

$$[\mathcal{D}_{W_1}] \circ [\mathcal{D}_{W_2}] = kk(W_1, E_{W_1}) \circ kk(W_2, E_{W_2}) = kk(W, E_W) = [\mathcal{D}_W].$$

The bivariant Chern character maps these classes to elements in the Yoneda algebra

$$ch_n([\mathcal{D}_W]) \in \mathcal{Y} := \oplus_j \mathrm{Ext}^{2n+j}(\mathcal{A}^\natural, \mathcal{A}^\natural) \tag{12.8}$$

$$ch_n([\mathcal{D}_{W_1}]) ch_m([\mathcal{D}_{W_2}]) = ch_{n+m}([\mathcal{D}_{W_1}] \circ [\mathcal{D}_{W_2}]).$$

Let $\chi : \mathcal{Y} \to \mathbb{C}$ be a character of the Yoneda algebra. Then by composing $\chi \circ ch$ we obtain

$$\chi ch([\mathcal{D}_{W_1}] \circ [\mathcal{D}_{W_2}]) = \chi ch([\mathcal{D}_{W_1}]) \chi ch([\mathcal{D}_{W_2}]) \in \mathbb{C}.$$

This can be used to define a time evolution for the horizontal product of the form

$$\sigma_t(f)(W) = |\chi ch([\mathcal{D}_W])|^{it} f(W)$$

13. Noncommutative spaces and spectral correspondences

We return now briefly to the problem of spectral correspondences of [**17**], mentioned in the introduction. Recall that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consists of the data of a unital involutive algebra $\mathcal{A}$, a representation $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ as bounded operators on a Hilbert space $\mathcal{H}$ and a self-adjoint operator D on $\mathcal{H}$ with compact resolvent, (A hermitian linear operator L in a Hilbert space $\mathcal{H}$ is said to have a compact resolvent if there is a complex number $\lambda \in \rho(L)$ for which the resolvent $R(\lambda, L) = (L - \lambda I)^{-1}$is compact) such that $[D, \rho(a)]$ is a bounded operator for all $a \in \mathcal{A}$. We extend this notion to a correspondence in the following way, following [**17**].

DEFINITION 13.1. A spectral correspondence is a set of data $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{H}, D)$, where $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital involutive algebras, with representations $\rho_i : \mathcal{A}_i \to \mathcal{B}(\mathcal{H})$, $i = 1, 2$, as bounded operators on a Hilbert space $\mathcal{H}$, such that

$$[\rho_1(a_1), \rho_2(a_2)] = 0, \quad \forall a_1 \in \mathcal{A}_1,\ \forall a_2 \in \mathcal{A}_2, \tag{13.1}$$

and with a self-adjoint operator D with compact resolvent, such that

$$[[D, \rho_1(a_1)], \rho_2(a_2)] = 0, \quad \forall a_1 \in \mathcal{A}_1,\ \forall a_2 \in \mathcal{A}_2, \tag{13.2}$$

and such that $[D, \rho_1(a_1)]$ and $[D, \rho_2(a_2)]$ are bounded operators for all $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$. A spectral correspondence is even if there exists an operator γ on $\mathcal{H}$ with $\gamma^2 = 1$ and such that D anticommutes with γ and $[\gamma, \rho_i(a_i)] = 0$ for all $a_i \in \mathcal{A}_i$, $i = 1, 2$. A spectral correspondence is odd if it is not even.

One might relax the condition of compact resolvent on the operator D if one wants to allow more degenerate types of operators in the correspondences, including possibly $D \equiv 0$, as seems desirable in view of the considerations of [**17**]. For our purposes here, we consider this more restrictive definition. Notice also that the condition (13.2) also implies $[[D, \rho_2(a_2)], \rho_1(a_1)] = 0$ because of (13.1). A more refined notion of spectral correspondences as morphisms between spectral triples, in a setting for families, is being developed by B. Mesland, [**48**]. We first show that our geometric correspondences define commutative spectral correspondences and then we give a noncommutative example based on taking products with finite geometries as in [**17**].

LEMMA 13.2. *Suppose given a compact connected oriented smooth 3-manifold with two branched covering maps* $\mathbf{S^3} \xleftarrow{\pi_1} \mathbf{M} \xrightarrow{\pi_2} \mathbf{S^3}$. *Given a choice of a Riemannian metric and a spin structure on* $\mathbf{M}$, *this defines a spectral correspondence for* $\mathcal{A}_1 = \mathcal{A}_2 = C^\infty(\mathbf{S^3})$.

PROOF. We consider the Hilbert space $\mathcal{H} = L^2(\mathbf{M}, S)$, where S is the spinor bundle on $\mathbf{M}$ for the chosen spin structure. Let $\partial\!\!\!/_{\mathbf{M}}$ be the corresponding Dirac operator. The covering maps π_i, for $i = 1, 2$, determine representations $\rho_i : C^\infty(\mathbf{S^3}) \to \mathcal{B}(\mathcal{H})$, by $\rho_i(f) = c(f \circ \pi_i)$, where c denotes the usual action of $C^\infty(\mathbf{M})$ on $\mathcal{H}$ by Clifford multiplication on spinors. Then we have $[\partial\!\!\!/_{\mathbf{M}}, \rho_i(f)] = c(d(f \circ \pi_i))$, which is a bounded operator on $\mathcal{H}$. All the commutativity conditions are satisfied in this case. □

Let $\mathcal{A}$ and $\mathcal{B}$ be finite dimensional unital (noncommutative) involutive algebras. Let V be a finite dimensional vector space with commuting actions of $\mathcal{A}$ and $\mathcal{B}$. Let $T \in \mathrm{End}(V)$ be a linear map such that $[[T, a], b] = 0$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Then we obtain noncommutative spectral correspondences of the type described in the last section of [**17**] in the following way.

LEMMA 13.3. *The cup product* $S_{\mathbf{M}} \cup S_F$ *of* $S_{\mathbf{M}} = (C^\infty(\mathbf{S^3}), C^\infty(\mathbf{S^3}), L^2(\mathbf{M}, S), \partial\!\!\!/_{\mathbf{M}})$ *and* $S_F = (A, B, V, T)$ *defines a noncommutative spectral correspondence for the algebras* $C^\infty(\mathbf{S^3}) \otimes \mathcal{A}$ *and* $C^\infty(\mathbf{S^3}) \otimes \mathcal{B}$.

PROOF. We simply adapt the usual notion of cup product for spectral triples to the case of correspondences. If the correspondence (A, B, V, T) is even, with grading γ, then we consider the Hilbert space $\mathcal{H} = L^2(\mathbf{M}, S) \otimes V$ and the operator $D = T \otimes 1 + \gamma \otimes \partial\!\!\!/_{\mathbf{M}}$. Then the usual argument for cup products of spectral triples show that $(C^\infty(\mathbf{S^3}) \otimes A, C^\infty(\mathbf{S^3}) \otimes B, \mathcal{H}, D)$ is an odd spectral correspondence. Similarly, if (A, B, V, T) is odd, then take $\mathcal{H} = L^2(\mathbf{M}, S) \otimes V \oplus L^2(\mathbf{M}, S) \otimes V$, with the diagonal actions of $C^\infty(\mathbf{S^3}) \otimes \mathcal{A}$ and $C^\infty(\mathbf{S^3}) \otimes \mathcal{B}$. Consider then the operator

$$D = \begin{pmatrix} 0 & \delta^* \\ \delta & 0 \end{pmatrix},$$

for $\delta = T \otimes 1 + i \otimes \partial\!\!\!/_{\mathbf{M}}$. Then, by the same standard argument that holds for spectral triples, the data $(C^\infty(\mathbf{S^3}) \otimes A, C^\infty(\mathbf{S^3}) \otimes B, \mathcal{H}, D)$ form an even spectral correspondence with respect to the $\mathbb{Z}/2\mathbb{Z}$

grading
$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
In either case, we denote the resulting correspondence $(C^\infty(\mathbf{S^3}) \otimes A, C^\infty(\mathbf{S^3}) \otimes B, \mathcal{H}, D)$ as the cup product $S_{\mathbf{M}} \cup S_F$. □

We can then form a convolution algebra on the space of correspondences, using the equivalence relation given by cobordism of branched covering spaces of §7 above, as in §8 above. This requires extending the equivalence relation defined by cobordisms of branched coverings to the case of the product by a finite geometry. We propose the following construction. The existence of a cobordism W of branched coverings between two geometric correspondences $\mathbf{M}_1$ and $\mathbf{M}_2$ in $Hom(G,G')$ implies the existence of a spectral correspondence *with boundary* of the form
$$S_W = (C^\infty(\mathbf{M}_1), C^\infty(\mathbf{M}_2), L^2(W,S), \partial\!\!\!/_W).$$
We will not discuss here the setting of spectral triples with boundary. A satisfactory theory was recently developed by Chamseddine and Connes (*cf.* [**13**]). We only recall here briefly the following notions, from [**16**]. A spectral triple with boundary $(\mathcal{A}, \mathcal{H}, D)$ is *boundary even* if there is a $\mathbb{Z}/2\mathbb{Z}$-grading γ on $\mathcal{H}$ such that $[a,\gamma]=0$ for all $a \in \mathcal{A}$ and $Dom(D) \cap \gamma Dom(D)$ is dense in $\mathcal{H}$. The boundary algebra $\partial\mathcal{A}$ is the quotient $\mathcal{A}/(J \cap J^*)$ by the two-sided ideal $J = \{a \in \mathcal{A} \,|\, aDom(D) \subset \gamma Dom(D)\}$. The boundary Hilbert space $\partial\mathcal{H}$ is the closure in $\mathcal{H}$ of $D^{-1}KerD_0^*$, where D_0 is the symmetric operator obtained by restricting D to $Dom(D) \cap \gamma Dom(D)$. The boundary algebra acts on the boundary Hilbert space by $a - D^{-2}[D^2, a]$. The boundary Dirac operator ∂D is defined on $D^{-1}KerD_0^*$ and satisfies $\langle \xi, \partial D \eta \rangle = \langle \xi, D\eta \rangle$ for $\xi \in \partial\mathcal{H}$ and $\eta \in D^{-1}KerD_0^*$. It has bounded commutators with $\partial\mathcal{A}$. One can extend from spectral triples to correspondences, by having two commuting representations of $\mathcal{A}_1$ and $\mathcal{A}_2$ on $\mathcal{H}$ with the properties above and such that the resulting boundary data $(\partial\mathcal{A}_1, \partial\mathcal{A}_2, \partial\mathcal{H}, \partial D)$ define a spectral correspondence. If one wants to extend to the product geometries the condition of cobordism of geometric correspondences, it seems that one is inevitably faced with the problem of defining spectral triples with corners. In fact, if S_W and S_F are both spectral triples with boundary, then their cup product $S_W \cup S_F$ would no longer give rise to a spectral triple with boundary but to one with corners. At present there isn't a well defined theory of spectral triples with corners. However, we can still propose a way of dealing with products of cobordisms by finite noncommutative geometries, which remains within the theory of spectral triples with boundary. To this purpose, we assume that the finite part S_F is an ordinary spectral triple, while only the cobordims part is a spectral triple with boundary. We then relate the cup product $S_W \cup S_F$ to the spectral correspondences $S_{\mathbf{M}_i} \cup S_{F_i}$ via the boundary ∂S_W and bimodules relating the S_{F_i} to S_F. More precisely, we consider the following data. Suppose given $\mathbf{M}_i \in Hom(G,G')$, $i=1,2$ as above and finite spectral correspondences $S_{F_i} = (A_i, B_i, V_i, T_i)$. Then we say that the cup products $S_{\mathbf{M}_i} \cup S_{F_i}$ are related by a spectral cobordism if the following conditions hold. The geometric correspondences are equivalent $\mathbf{M}_1 \circ \mathbf{M}_2$ via a cobordism W. There exist finite dimensional (noncommutative) algebras R_i, $i=1,2$ together with R_i–A_i bimodules E_i and B_i–R_i bimodules F_i, with connections. There exists a finite spectral correspondence $S_F = (R_1, R_2, V_F, D_F)$ such that $S_W \cup S_F = (\mathcal{A}, \mathcal{H}, \mathcal{D})$ is a spectral triple with boundary in the sense of Chamseddine–Connes with
$$\partial\mathcal{A} = \oplus_{i=1,2} C^\infty(\mathbf{M}_i) \otimes R_i$$
$$\partial\mathcal{H} = \oplus_{i=1,2} L^2(\mathbf{M}_i, S) \oplus (E_1 \otimes_{A_1} V_i \otimes_{B_i} F_i)$$
and $\partial\mathcal{D}$ gives the cup product of the Dirac operators $\partial_{\mathbf{M}_i}$ with the T_i, with the latter twisted by the connections on E_i and F_i.

We do not give more details here. In fact, in order to use this notion to extend the equivalence relation of cobordims of branched coverings and the 2-category we considered in §8 above to the noncommutative case, one needs a gluing theory for spectral triples with boundary that makes it possible to define the horizontal and vertical compositions of 2-morphisms as in the case of $W_1 \circ W_2$ and $W_1 \bullet W_2$. The analysis necessary to develop such gluing results is beyond the scope our study and the problem will be considered in future work .

2

Knots, Khovanov Homology

1. Introduction

In the previous chapter we presented a construction of a 2-category whose objects are embedded graphs in the 3-sphere, whose 1-morphisms are 3-manifolds realized in different ways as branched coverings of the 3-sphere with the given embedded graphs as branch loci, and with 2-morphisms given by 4-dimensional branched cover cobordisms. We also studied time evolutions on algebras obtained from this 2-category. We would then like to see if one can obtain suitable cohomology theories that can define functors for our category to some category of vector spaces. To be precise, since we are working with a 2-category, we should expect to land in some 2-category of 2-vector spaces, for instance in the sense introduced by Kapranov and Voevodsky [**41**], or in a more explicit form in a 2-category of 2-matrices as constructed by Elgueta in [**21**]. In this chapter we only make some preliminary steps in this direction, leaving a more detailed investigation to future work. We begin this chapter by a remark on the results of the previous chapter, in §2 below, which shows that one can pass from the case where the branch loci of the coverings are embedded graphs to the more restrictive case of links using the equivalence relation of *b*-homotopy of branched coverings. This result suggests that we may be able to seek a suitable cohomology theory for our purposes by a suitable extension of known cohomological constructions for knots and links in the 3-sphere. Following this idea, we then recall the definition and main properties of Khovanov homology for knots and links. We also recall the notion of cobordism groups for knots and links and their relation. The following part of this chapter first deals with extending the notion of cobordism from links to embedded graphs. We find that this can be done in two possible ways, which respectively extend two notions of cobordisms that are known to be equivalent in the case of links but are no longer equivalent for graphs. We then discuss how to extend Khovanov homology from links to embedded graphs. The first idea is to combine the Khovanov complex with the complex of graph homology, where the Khovanov complex accounts for the crossings and the complexity of the embedding in the 3-sphere and the graph homology accounts for the combinatorial complexity of the graph. We only show in an example how one can associate to each term of the graph homology complex for a planar diagram of an embedded graph in the 3-sphere a cubical complex that resolves the crossings in each of the graphs involved in the graph homology complex. However, instead of continuing along this line of thought to construct rigorously a double complex, we show that one can bypass several difficulties and achieve a satisfactory notion of Khovanov homology for embedded graphs through a different procedure, which is based on a result of Kauffman. This result is a construction of a topological invariant of embedded graphs, which is given by a finite collection of knots and links, obtained by performing certain cutting operations at the vertices of the graph. Thus, this first step incorporates the combinatorial complexity of the graph, in a way similar to what graph homology does in the approach described above, while the topology of the embedding is retained by the links and knots in the resulting family. Since the family of links is itself a topological invariant, any further invariant computed out of them will also be. Thus, one can then proceed to define a Khovanov homology for an embedded graph as being the sum of the Khovanov homologies of all the knots and links in the Kauffman invariant of the graph. This is well defined and a topological invariant.

2. From graphs to knots

The Alexander branched covering theorem is greatly refined by the Hilden–Montesinos theorem, which ensures that all compact oriented 3-manifolds can be realized as branched covers of the 3-sphere, branched along a knot or a link (see [**31**], [**49**], *cf.* also [**54**]). One can see how to pass from a branch locus that is a multi-connected graph to one that is a link or a knot in the following way, [**10**]. One says that two branched coverings $\pi_0 : \mathbf{M} \to \mathbf{S^3}$ and $\pi_1 : \mathbf{M} \to \mathbf{S^3}$ are *b*-homotopic if there exists a homotopy $H_t : \mathbf{M} \to \mathbf{S^3}$ with $H_0 = \pi_0$, $H_1 = \pi_1$ and H_t a branched covering, for all $t \in [0,1]$, with branch locus an embedded graph $G_t \subset \mathbf{S^3}$. The "Alexander trick" shows that two branched coverings of the 3-ball $D^3 \to D^3$ that agree on the boundary $S^2 = \partial D^3$ are *b*-homotopic. Using this trick, one can pass, by a *b*-homotopy, from an arbitrary branched covering to one that is *simple*, namely where all the fibers consist of at least $n-1$ points, n being the order of the covering. Simple coverings are *generic*. The same argument shows ([**10**], Corollary 6.6) that any branched covering $\mathbf{M} \to \mathbf{S^3}$ is *b*-homotopic to one where the branch set is a link. We restrict to the case where the embedded graphs G and G' are knots K and K' and we consider geometric correspondences $Hom(K,K')$ modulo the equivalence relation of *b*-homotopy. Namely, we say that two geometric correspondences $\mathbf{M}_1, \mathbf{M}_2 \in Hom(K,K')$ are *b*-homotopic if there exist two homotopies Θ_t, Θ'_t relating the branched covering maps

$$\mathbf{S^3} \stackrel{\pi_{K,i}}{\longleftarrow} \mathbf{M} \stackrel{\pi_{K',i}}{\longrightarrow} \mathbf{S^3}.$$

Since we have the freedom to modify correspondences by *b*-homotopies, we can as well assume that the branch loci are links. Thus, we are considering geometric correspondences of the form

$$K \subset L \subset \mathbf{S^3} \stackrel{\pi_K}{\longleftarrow} \mathbf{M} \stackrel{\pi_{K'}}{\longrightarrow} \mathbf{S^3} \supset L' \supset K', \tag{2.1}$$

where the branch loci are links L and L', containing the knots K and K', respectively. Notice also that, if we are allowed to modify the coverings by *b*-homotopy, we can arrange so that, in the composition $\mathbf{M}_1 \circ \mathbf{M}_2$, the branch loci $L \cup \pi_k \pi_1^{-1}(L'_2)$ and $L'' \cup \pi_{K''} \pi_2^{-1}(L'_1)$ are links in $\mathbf{S^3}$. We denote by $[\mathbf{M}]_b$ the equivalence class of a geometric correspondence under the equivalence relation of *b*-homotopy. The equivalence relation of *b*-homotopy is a particular case of the relation of branched cover cobordism that we considered above. In fact, the homotopy Θ_t can be realized by a branched covering map $\Theta : \mathbf{M} \times [0,1] \to \mathbf{S^3} \times [0,1]$, branched along a 2-complex $S = \cup_{t \in [0,1]} G_t$ in $\mathbf{S^3} \times [0,1]$. Thus, by the same argument used to prove the compatibility of the composition of geometric correspondences with the equivalence relation of cobordism, we obtain the compatibility of composition

$$[\mathbf{M}_1]_b \circ [\mathbf{M}_2]_b = [\mathbf{M}_1 \circ \mathbf{M}_2]_b. \tag{2.2}$$

The *b*-homotopy is realized by the cobordism $(\mathbf{M}_1 \circ \mathbf{M}_2) \times [0,1]$ with the branched covering maps $\hat{\Theta} = \Theta \circ P_1$ and $\hat{\Theta}' = \Theta' \circ P_2$. While the knots K and K' are fixed in the construction of $Hom(K,K')$, the other components of the links L and L', when we consider the correspondences up to *b*-homotopy, are only determined up to knot cobordism with trivial linking numbers (*i.e.* as classes in the knot cobordism subgroup of the link cobordism group, see [**36**]). To make the role of the link components more symmetric, it is then more natural in this setting to consider a category where the objects are cobordism classes of knots $[K]$, $[K']$ and where the morphisms are given by the *b*-homotopy classes of geometric correspondences $Hom([K],[K'])_b$. The time evolution considered in the previous chapter still makes sense on the corresponding semigroupoid ring, since the order of the branched cover is well defined on the *b*-equivalence class and multiplicative under composition of morphisms.

3. Khovanov Homology

In the following we recall a homology theory for knots and links embedded in the 3-sphere. We discuss later in this chapter how to extend it to the case of embedded graphs.

3.1. Khovanov Homology for links. In recent years, many papers have appeared that discuss properties of Khovanov Homology theory, which was introduced in [**43**]. For each link $L \in \mathbf{S^3}$, Khovanov constructed a bi-graded chain complex associated with the diagram D for this link L and applied homology to get a group $Kh^{i,j}(L)$, whose Euler characteristic is the normalised Jones polynomial.

$$\sum_{i,j}(-1)^i q^j dim(Kh^{i,j}(L)) = J(L)$$

He also proved that, given two diagrams D and D' for the same link, the corresponding chain complexes are chain equivalent, hence, their homology groups are isomorphic. Thus, Khovanov homology is a link invariant.

3.2. The Link Cube. Let L be a link with n crossings. At any small neighborhood of a crossing we can replace the crossing by a pair of parallel arcs and this operation is called a resolution. There are two types of these resolutions called 0-resolution (Horizontal resolution) and 1-resolution (Vertical resolution) as illustrated in figure (1).

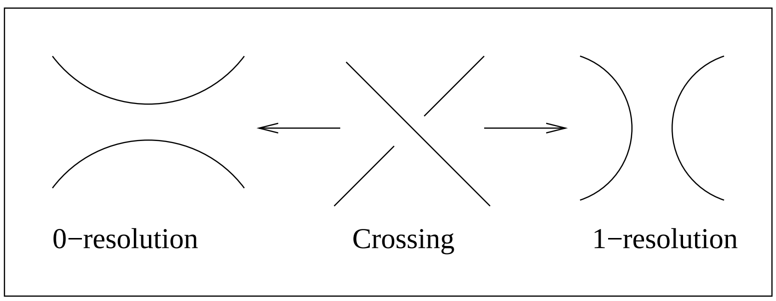


FIGURE 1. 0 and 1- resolutions to each crossing

We can construct a n-dimensional cube by applying the 0 and 1-resolutions n times to each crossing to get 2^n pictures called smoothings (which are one dimensional manifolds) S_α. Each of these can be indexed by a word α of n zeros and ones, *i.e.* $\alpha \in \{0,1\}^n$. Let ξ be an edge of the cube between two smoothings S_{α_1} and S_{α_2}, where S_{α_1} and S_{α_2} are identical smoothings except for a small neighborhood around the crossing that changes from 0 to 1-resolution. To each edge ξ we can assign a cobordism Σ_ξ (orientable surface whose boundary is the union of the circles in the smoothing at either end)

$$\Sigma_\xi : S_{\alpha_1} \longrightarrow S_{\alpha_2}$$

This Σ_ξ is a product cobordism except in the neighborhood of the crossing, where it is the obvious saddle cobordism between the 0 and 1-resolutions. Khovanov constructed a complex by applying a $1+1$-dimensional TQFT (Topological Quantum Field Theory) which is a monoidal functor (see the appendix 4), by replacing each vertex S_α by a graded vector space V_α and each edge (cobordism) Σ_ξ by a linear map d_ξ, and we set the group $CKh(D)$ to be the direct sum of the graded vector spaces for all the vertices and the differential on the summand $CKh(D)$ is a sum of the maps d_ξ for all edges ξ such that Tail(ξ)= α *i.e.*

$$d^i(v) = \sum_{\xi} sign(-1) d_\xi(v) \tag{3.1}$$

where $v \in V_\alpha \subseteq CKh(D)$ and $sign(-1)$ is chosen such that $d^2 = 0$.
An element of $CKh^{i,j}(D)$ is said to have homological grading i and q-grading j where

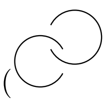
$$i = |\alpha| - n_- \tag{3.2}$$

$$j = deg(v) + i + n_- + n_+ \tag{3.3}$$

for all $v \in V_\alpha \subseteq CKh^{i,j}(D)$, $|\alpha|$ is the number of 1's in α, and n_-, n_+ represent the number of negative and positive crossings respectively in the diagram D.

DEFINITION 3.1. Let $V = \bigoplus_m V_m$ be a graded vector space with homogenous components V_m. Then the graded dimension of V is defined to be $qdimV = \sum_m q^m dim(V_m)$, which is a Laurant polynomial in variable q.

DEFINITION 3.2. For a graded vector space V and an integer n we can define a new graded vector space $V\{n\}$ (called a shifted version of V) by $V\{n\}_m = V_{m-n}$

Here we give an example of computing the homology of the Hopf link $CKh^{*,*}($  $)$

EXAMPLE 3.3. [**61**] Consider the n-cube diagram of the Hopf link figure (2) ,

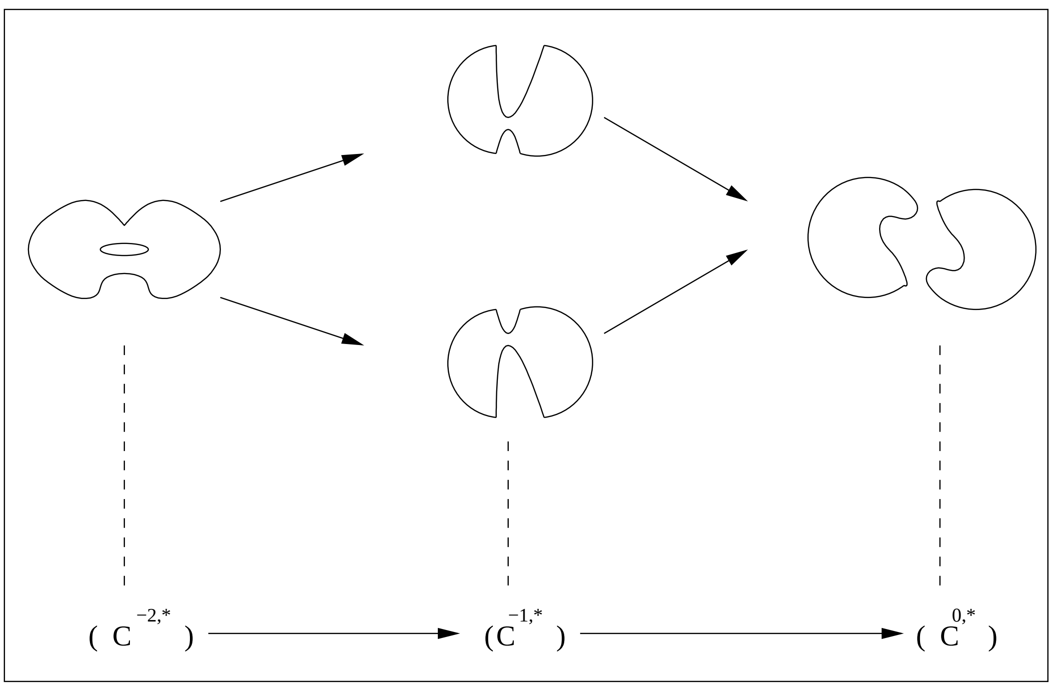


FIGURE 2. n-cube diagram of the Hopf link

We associate to each smoothing (vertex) in the n-cube a graded vector space as follows: Put $V =$

$\mathbb{Q}\{1,x\}$ ($\mathbb{Q}$-vector space with $1,x$ basis elements). Grade the two basis elements by $deg(1) = 1$ and $deg(x) = -1$. Associate to each vertex α a graded vector space $V_\alpha = V^{k_\alpha}\{|\alpha| + n_+ - 2n_-\}$ where $|\alpha|$ is the number of $1's$ in α and k_α is the number of circles os smoothings in the vertex α. Set

$$CKh^{i,*} = \bigoplus_{\alpha\in\{0,1\}^n} V_\alpha$$

In this particular case, the n-cube is given in figure (3).

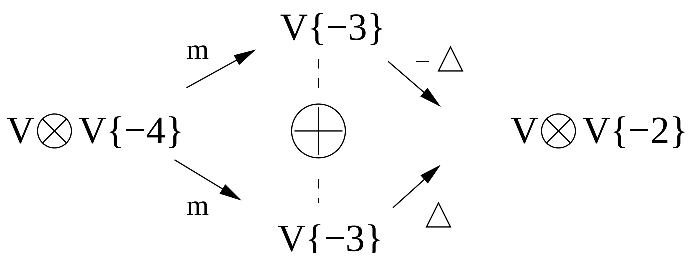


FIGURE 3

The linear map $d_\xi : V_\alpha \longrightarrow V_{\alpha'}$ is either a fusion map given by multiplication $m : V \otimes V \longrightarrow V$ or the splitting map given by co-multiplication $\triangle : V \longrightarrow V \otimes V$, corresponding geometrically to the contribution of a disk where the crossing change happens, and the identity map outside the disk. The multiplication map m is defined by

$$m(1\otimes 1) = 1, \quad m(1\otimes x) = m(x\otimes 1) = x, \quad m(x\otimes x) = 0$$

and the co-product $\triangle$ by

$$\triangle(1) = 1\otimes x + x\otimes 1, \quad \triangle(x) = x\otimes x.$$

Now following [**61**], by using the equation (3.1), we see that the homology can be given in a table as follows.

j \ i	-2	-1	0
0			$\mathbb{Q}$
-1			
-2			$\mathbb{Q}$
-3			
-4	$\mathbb{Q}$		
-5			
-6	$\mathbb{Q}$		

3.3. Properties. [**61**],[**46**] Here we give some properties of Khovanov homology.

PROPOSITION 3.4. *(1) If D' is a diagram obtained from D by the application of a Reidemeister move then the complexes $(CKh^{*,*}(D))$ and $(CKh^{*,*}(D'))$ are homotopy equivalent.*

(2) For an oriented link L with diagram D, the graded Euler characteristic satisfies

$$\sum(-1)^i qdim(Kh^{i,*}(L)) = J(L) \tag{3.4}$$

where $J(L)$ is the normalised Jones Polynomials for a link L and

$$\sum(-1)^i qdim(Kh^{i,*}(D)) = \sum(-1)^i qdim(CKh^{i,*}(D))$$

(3) *Let L_{odd} and L_{even} be two links with odd and even number of components then $Kh^{*,even}(L_{odd}) = 0$ and $Kh^{*,odd}(L_{even}) = 0$*

(4) *For two oriented link diagrams D and D', the chain complex of the disjoint union $D \sqcup D'$ is given by*

$$CKh(D \sqcup D') = CKh(D) \otimes CKh(D'). \tag{3.5}$$

(5) *For two oriented link L and L', the Khovanov homology of the disjoint union $L \sqcup L'$ satisfies*

$$Kh(L \sqcup L') = Kh(L) \otimes Kh(L').$$

(6) *Let D be an oriented link diagram of a link L with mirror image D^m diagram of the mirror link L^m. Then the chain complex $CKh(D^m)$ is isomorphic to the dual of $CKh(D)$ and*

$$Kh(L) \cong Kh(L^m)$$

3.4. Links Cobordisms. Recall that in section (7.2) we defined branched cover cobordism as a cobordism W with boundary $M_1 \cup -M_2$ with two branched covering maps.

$$S \subset \mathbf{S^3} \times [0,1] \xleftarrow{q} W \xrightarrow{q'} \mathbf{S^3} \times [0,1] \supset S', \tag{3.6}$$

branched along compact oriented surfaces S and $S' \subset \mathbf{S^3} \times [0,1]$ with boundary $\partial S = L_0 \cup -L_1$ and $\partial S' = L'_1 \cup -L'_2$. Here we assume that the L_i and L'_i are links. We discuss later in this chapter what happens in the case of embedded graphs. In this section we want to recall how one constructs a linear map between the homologies of the boundary links by following Khovanov [**43**]. The first idea is, we can decompose S into elementary subcobordisms S_t for finitely many $t \in [0,1]$ with

$$S_t = S \cap \mathbf{S^3} \times [0,t]$$

and

$$\partial S_t = L_{t-1} \cup -L_t$$

where L_{t-1} and L_t are one dimensional manifolds, not necessary links. Using a small isotopy we can obtain that they are links for some $t \in [0,1]$. Here we assume that S is a smooth embedded surface. A smooth embedded surface S can be represented by a one parameter family $D_t, t \in [0,1]$ of planar diagrams of oriented links L_t for finitely many $t \in [0,1]$ and this representation is called a *movie M*. Between any two consecutive clips of a movie the diagrams will differ by one of the "Reidemeister moves" or "Morse moves". The Reidemeister moves are in figure (1) in the appendix or the first moves in figure (17) and the Morse moves are given in figure (4).

These two types of moves will be called *local moves*. This means that between any two consecutive diagrams there is a local move either of Reidemeister or of Morse type. The necessary condition is that the projection diagram D_0 in the first clip in M should be the a projection of the link L_0 and the projection diagram in the D_1 in the last clip of the movie M should be the projection of the link L_1 (boundary of S). Notice that the orientation of S induces an orientation on all intersection links L_t. To show that, let v be a tangent vector to L_t. Then orient v in the positive direction if (v,w) gives the orientation of S where w is the tangent vector to S in the direction of increasing of t. Khovanov constructed a chain map between complexes of two consecutive diagrams that changed by a local move, hence a homomorphisms between their homologies. The composition of these chain maps defines a homomorphism between the homology groups of the diagrams of the boundary links.

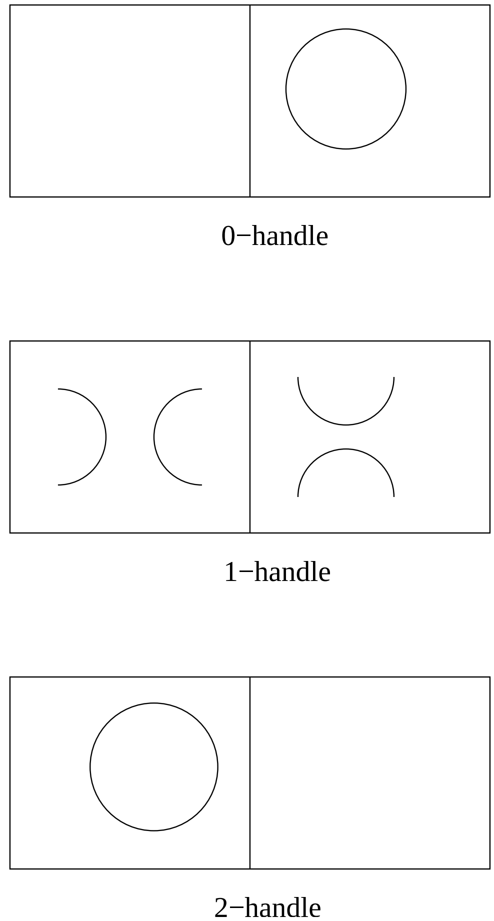


FIGURE 4. Morse moves

3.5. Constructing the homology map from local moves. We recall here more explicitly how one obtains the maps associated to the two types of local moves described above.

(1) Reidemeister moves: The idea of constructing the homology map is getting a homotopy equivalence between the chain complexes of consecutive clips see proposition 3.4 no.1. Let $D_{t_{r-1}}, D_{t_r}$ be two consecutive diagrams in two consecutive clips in the movie M which differ by one of the Reidemeister moves of the first type or the second type with D_{t_r} is the one with more crossings, then by **[43]** the chain complex of the diagram D_t can be split into direct sum of chain complex CKh_{t_r} and contractible chain complex C proposition 3.4 no.4

$$CKh(D_{t_r}) = CKh_{t_r} \oplus C \tag{3.7}$$

Then we can get a homomorphism which respects the filtration, see the appendix (2),

$$\Xi : CKh_{t_r} \longrightarrow CKh(D_{t_{r-1}})$$

to show $CKh(D_{t_{r-1}})$ is equivalent to $CKh(D_{t_r})$ it is enough to see that the composition of Ξ with the projection map onto the first summand 3.7 is chain equivalent to the composition of the inclusion i and Ξ^{-1}.

For the third move then $CKh(D_{t_{r-1}})$ and $CKh(D_{t_r})$ can be split both into a chain complex and a contractible chain complexes C_1 and C_2 as above 3.7,

$$CKh(D_{t_{r-1}}) = CKh_{t_{r-1}} \oplus C_1 \tag{3.8}$$

and

$$CKh(D_{t_r}) = CKh_{t_r} \oplus C_2 \tag{3.9}$$

and by the same way we can get as above an isomorphism map

$$\Xi : CKh_{t_{r-1}} \longrightarrow CKh_{t_r}$$

then we have equivalent chain complexes

$$CKh(D_{t_r}) = CKh_{t_r} \oplus C_2 \xrightarrow{Pr} CKh_{t_r} \xrightarrow{\Xi} CKh_{t_{r-1}} \xrightarrow{i} CKh(D_{t_{r-1}}) = CKh_{t_{r-1}} \oplus C_1 = CKh(D_{t_{r-1}})$$

Now we get a homotopy equivalence of chain complexes of D_{t_r} in the movie M that differ by one of the Reidemeister move from the previous diagram which induce a map in homology $\Upsilon : Kh^{*,*}(D_{t_{r-1}}) \longrightarrow Kh^{*,*}(D_{t_r})$ with respects the filtration on $Kh^{*,*}$.

(2) Morse moves: Let D_0 and D_1 be two diagrams that differ by one of the Morse moves. For 0 or 2-handle there is a simple closed curve that one will add or remove from the consecutive diagrams. Then $CKh^{*,*}(D_{t-r} \bigsqcup \bigcirc) = CKh^{*,*}(D_{t_r}) \otimes V$ (where $V = \mathbb{Q}\{1, x\}$ is a vector space with two basis elements [**61**]). we can define for 0-handle a map [**43**]

$$\phi_t = I_d \otimes i : CKh^{*,*}(D_{t_{r-1}}) \longrightarrow CKh^{*,*+1}(D_{t_r}) = CKh^{*,*}(D_{t_{r-1}} \otimes V)$$

The increasing in q-grading 3.3 by 1 comes from the fact that V has identity with q-grading equal 1 [**61**], and $i : Q \longrightarrow V$ is the unit of frobenius algebra [**61**]. The same operation works for 2-handle with the map $\phi_t = I_d \otimes \varepsilon$, where ε is the co-unit map $\varepsilon : V \longrightarrow Q$. Then we can define the homology map

$$\Upsilon_{S_t} : Kh^{*,*}(D_{t_{r-1}}) \longrightarrow Kh^{*,*+1}(D_{t_r})$$

For the 1-handle move the map $CKh^{*,*}(D_{t_{r-1}}) \longrightarrow CKh^{*,*-1}(D_{t_r})$ is constructed by [**43**], [**61**] and by applying homology we can get

$$\Upsilon_{S_t} : Kh^{*,*}(D_{t_{r-1}}) \longrightarrow Kh^{*,*-1}(D_{t_r})$$

Let $\Upsilon_S : Kh^{*,*}(D_0) \longrightarrow Kh^{*,*+\chi(S)}(D_1)$ be a composition of consecutive maps $\Upsilon_{S_t}, t \in [0,1]$. One can see that the q-grading is changed by adding the Euler characteristic of cobordism S and this change comes from Morse moves since the q-grading does not change by using Reidemeister moves, and by the sum over all these changes we get $\chi(S)$, see [**43**].

4. Knots and Links Cobordism Groups

A notion of knot cobordism group and link cobordism group can be given by using cobordism classes of knots and links to form a group [**23**],[**36**]. The link cobordism group splits into the direct sum of the knot cobordism group and an infinite cyclic group which represents the linking number, which is invariant under link cobordism [**36**]. In this part we will give a survey about both knot and link cobordism groups. In a later part of this chapter we will show that the same idea can be adapted to construct a graph cobordism group by using the definition of cobordisms between graphs given in section 7.1 definition(7.2).

4.1. Knot cobordism group. We recall the concept of cobordism between knots introduced in [**23**]. Two knots K_1 and K_2 are called knot cobordic if there is a locally flat cylinder S in $\mathbf{S^3} \times [0,1]$ with boundary $\partial S = K_1 \cup -K_2$ where $K_1 \subset \mathbf{S^3} \times \{0\}$ and $K_2 \subset \mathbf{S^3} \times \{1\}$. We then write $K_1 \sim K_2$. The critical points in the cylinder are assumed be minima (birth), maxima (death), and saddle points. In the birth point at some t_0 there is a sudden appearance of a point. The point becomes an unknotted circle in the level immediately above t_0. At a maxima or death point, a circle collapses to a point and disappearance from higher levels.

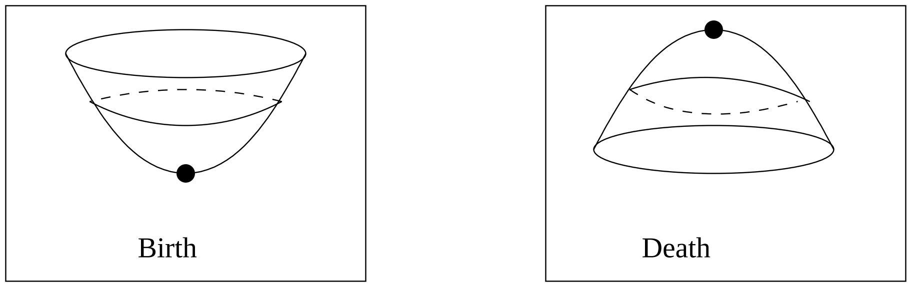


FIGURE 5. Death and Birth Points

For the saddle point, two curves touch and rejoin as illustrated in figure 6

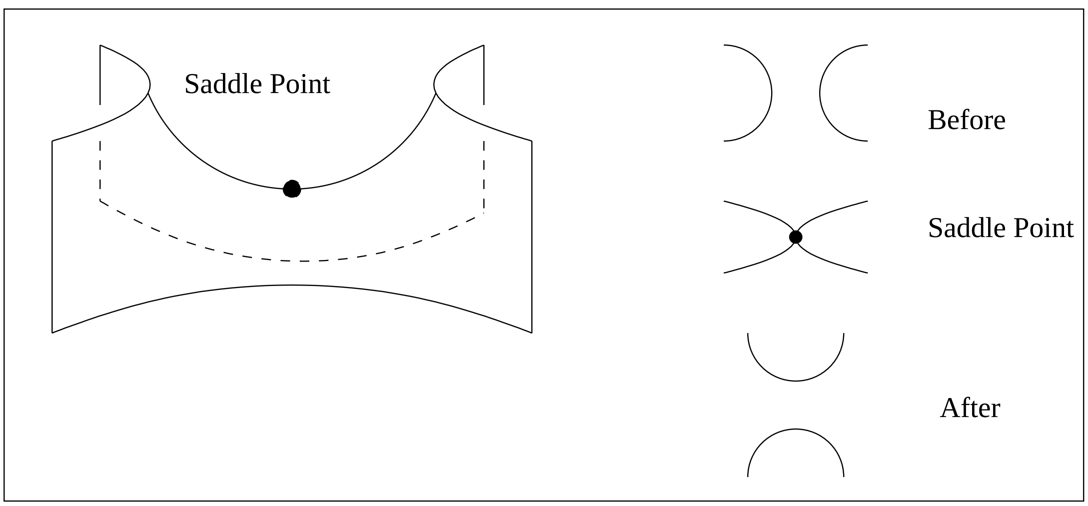


FIGURE 6. Saddle Point

These saddle points are of two types: *negative* if with increasing t the number of the cross sections decreases and *positive* if the number increases.
A transformation [**23**] from one cross section to another is called negative hyperbolic transformation if there is only one saddle point between the two cross sections and if the number of components decreases. We can define analogously a positive hyperbolic transformation.

DEFINITION 4.1. [**36**] We say that two knots K_1 and K_2 are related by an elementary cobordism if the knot K_2 is obtained by $r-1$ negative hyperbolic transformations from a split link consisting of K_1 together with $r-1$ circles.

What we mean by split link is a link with n components $(K_i, i = 1....n)$ in $\mathbf{S^3}$ such that there are mutually disjoint n 3-cells $(D_i, i = 1....n)$ containing $K_i, i = 1, 2..., n$

LEMMA 4.2. [**36**] *Two knots are called knot cobordic if and only if they are related by a sequence of elementary cobordisms*

It is well known that the oriented knots form a commutative semigroup under the operation of composition #. Given two knots K_1 and K_2, we can obtain a new knot by removing a small arc from each knot and then connecting the four endpoints by two new arcs. The resulting knot is called the composition of the two knots K_1 and K_2 and is denoted by $K_1\#K_2$.
Notice that, if we take the composition of a knot K with the unknot $\bigcirc$ then the result is again K.

LEMMA 4.3. *The set of oriented knots with the connecting operation # forms a semigroup with identity* $\bigcirc$

Fox and Milnor [**23**] showed that composition of knots induces a composition on knot cobordism classes $[K]\#[K']$. This gives an abelian group G_K with $[\bigcirc]$ as identity and the negative is $-[K] = [-K]$, where the $-K$ denotes the reflected inverse of K.

THEOREM 4.4. *The knot cobordism classes with the connected sum operation # form an abelian group, called the knot cobordism group and denoted by* G_K.

4.2. Link cobordism group. [**36**] For links, the conjunction operation & between two links gives a commutative semigroup. $L_1\&L_2$ is a link represented by the union of the two links $l_1 \cup l_2$ where l_1 represents L_1 and l_2 represents L_2 with mutually disjoint 3-cells D_1 and D_2 contain l_1 and l_2 respectively. Here "represents" means that we are working with ambient isotopy classes L_i of links (also called *link types*) and the l_i are chosen representatives of these classes. In the following we loosely refer to the classes L_i also simply as links, with the ambient isotopy equivalence implicitly understood. The zero of this semigroup is the link consisting of just the empty link. The link cobordism group is constructed [**36**] using the conjunction operation and the cobordism classes. We recall below the definition of cobordism of links.
Let L be a link in $\mathbf{S^3}$ containing r-components $k_1, ..., k_r$ with a split sublink $L_s = k_1 \cup k_2 \cup .... \cup k_t$, $t \leq r$ of L. Define a knot $\hat{\mathbf{K}}$ to be $k_1 + k_2 + ... + k_t + \partial B_{t+1} + \partial B_{t+2} + ... + \partial B_r$ where $\{B_{t+1}, B_{t+2}, B_{t+3} ...., B_r\}$ are disjoint bands in $\mathbf{S^3}$ spanning L_s [**36**]. The operation $+$ means additions in the homology sense. Put $L_1 = L_s \cup k_{t+1} \cup k_{t+2} .... \cup k_r$ and $L_2 = \hat{\mathbf{K}} \cup k_{t+1} \cup k_{t+2} .... \cup k_r$. Now, the operation of replacing L_1 by L_2 is called **fusion** and L_2 by L_1 is called **fission**.

DEFINITION 4.5. [**36**] Two links will be called link cobordic if one can be obtained from the other by a sequence of fusions and fissions. This equivalence relation is denoted by $\simeq$. $[L]$ denotes the link cobordism class of L.

THEOREM 4.6. [**36**] *The link cobordism classes with the conjunction operation form an abelian group, called the link cobordism group and denoted by* G_L.

PROOF. For two cobordism classes $[L_1]$ and $[L_2]$ the multiplication between them is well defined and given by

$$[L_1]\&[L_2] = [L_1\&L_2]$$

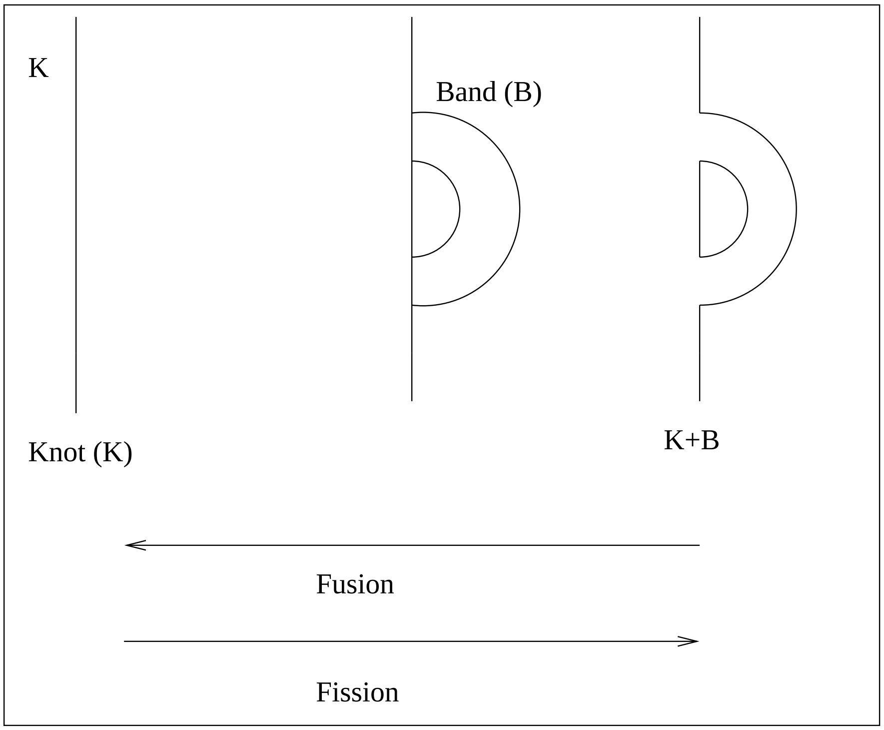


FIGURE 7. Band

The zero of this operation is the class [◯] which is the trivial link of a countable number of components. The negative of $[L]$ is $-[L] = [-L]$, where $-L$ denoted the reflected inverse of L. □

LEMMA 4.7. *For any link L, a conjunction $L\& -L$ is link cobordic to zero.*

To study the relation between the knot cobordism group G_K and link cobordism group G_L define a natural mapping $f : G_K \longrightarrow G_L$ which assigns to each knot cobordism class $[k]$ the corresponding link cobordism class $[L]$ where L is the knot k viewed as a one-component link. We claim that f is a homomorphism. f is well defined from the following lemma

LEMMA 4.8. [**36**] *Two knots are link cobordic if and only if they are knot cobordic.*

Now, $K_1\#K_2$ is a fusion of $K_1\&K_2$ then $K_1\#K_2$ is cobordic to $K_1\&K_2$, therefore f is a homomorphism. Again by using the lemma 4.8, if a knot is link cobordic to zero then it is also knot cobordic to zero, and hence $ker(f)$ consists of just ◯.

LEMMA 4.9. *f is an isomorphism of G_K onto a subgroup of G_L.*

THEOREM 4.10. [**36**] *$f(G_K)$ is a direct summand of G_L and it is a subgroup of G_L whose elements have total linking number zero. The other summand is isomorphic to the additive group of integers.*

5. Graphs and cobordisms

The rest of this chapter will be dedicated to extending some of the notions recalled above for knots and links, to the case of embedded graphs in the 3-sphere. In this section we describe how to obtain a cobordism group for graphs, in two possible ways and the relation between them and with the cobordism group of links.

5.1. Some basic facts about graphs. We recall here some basic facts about graphs. A Graph G is an ordered triple $(V(G), E(G), \phi_G)$ consisting of a nonempty set $V(G)$ of vertices (zero-dimensional), a set $E(G)$, disjoint from $V(G)$, of edges or loops or lines (one-dimensional), and an incidence function ϕ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. If e is an edge and u and v are vertices such that $\phi_G(e) = uv$, then e is said to join u and v. The vertices u and v are called the ends of e. Each vertex is indicated by a point, and each edge by a line joining the points which represent its ends.

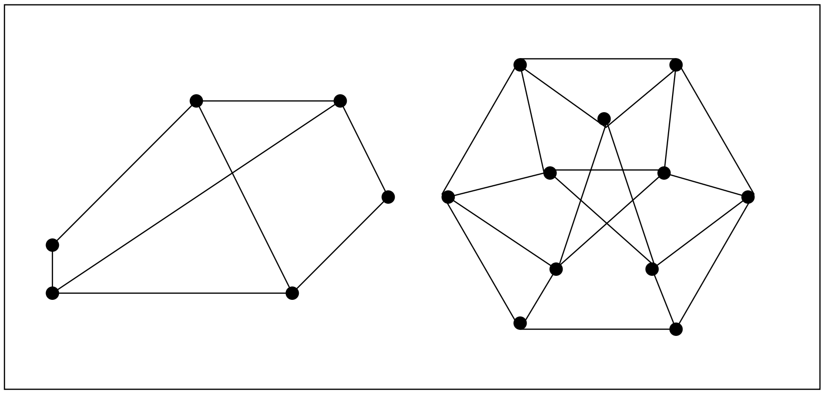


FIGURE 8

Most of the definitions and concepts in graph theory are suggested by the graphical representation. The ends of an edge are said to be incident with the edge, and vice versa. Two vertices which are incident with a common edge are adjacent, as are two edges which are incident with a common vertex. An edge with identical ends is called a loop, and an edge with distinct ends a link.
A graph is finite if both its vertex set and edge set are finite, we call a graph with just one vertex trivial and all other graphs nontrivial. A graph is simple if it has no loops and no two of its edges join the same pair of vertices. We use the symbols $v(G)$ (sometimes $|G|$) and $\varepsilon(G)$ (sometimes$||G||$) to denote the number of vertices and edges in the graph G.
Two graphs G and H are identical (written G = H) if $V(G) = V(H)$ and $E(G) = E(H)$, and $\phi_G = \phi_H$. If two graphs are identical then they can clearly be represented by identical diagram. Two graphs G and H are said to be isomorphic $G \simeq H$ if there are bijections $\theta : V(G) \longrightarrow V(H)$ and $\psi : E(G) \longrightarrow E(H)$ such that $\phi_G(e) = uv$ if and only if $\phi_h(\psi(e)) = \theta(u)\theta(v)$; such pair (θ, ψ) of mapping is called isomorphism between G and H.
A class of graphs that is closed under isomorphism is called a graph property. A simple graph in which each pair of distinct vertices is joined by an edge is called a complete graph. Up to isomorphism there is just one complete graph on n vertices and denoted K_n or K^n. K_3 is called a triangle. For an arbitrary edge $e \in E(G)$ we can define $G - e$ to be the graph G with deleted edge e, and by G/e the graph obtained by contacting edge e *i.e.* by identifying the vertices incident to e and deleting e. A graph is said to be *digraph* if each graph edge is replaced by a directed graph edge *i.e.* graph whose edge have direction and are called *arcs*. A complete graph in which each edge is bidirected (symmetric pair of directed edges) is called a complete directed graph. The inverse of a directed graph G is a graph $-G$ with the same number of vertices but reverse direction of the edges

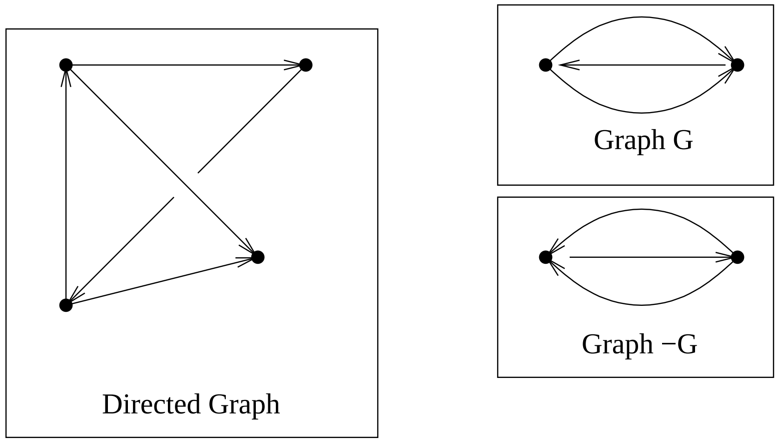


FIGURE 9. Directed Graphs

5.1.1. *subgraphs.* A graph H is a subgraph of G written $(H \subseteq G)$ if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and ϕ_H is the restriction of ϕ_G to $E(H)$. When $H \subseteq G$ but $(H \neq G)$, we write $H \subset G$ and call proper subgraph of G. A spanning subgraph of G is a subgraph H with $V(H) = V(G)$. The union $G_1 \cup G_2$ of G_1 and G_2 is the subgraph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. If G_1 and G_2 are disjoint sometimes denote their union by $G_1 + G_2$. The intersection $G_1 \cap G_2$ of G_1 and G_2 is defined similarly, but in this case G_1 and G_2 must have at least one vertex in common.

5.2. Graph cobordism group. In this section we construct cobordism groups for embedded graphs by extending the notions of cobordisms used in the case of links. In definition (7.2) we have already introduced a concept of cobordism between graphs. We recall here the definition of cobordisms of graphs that we used in the previous chapter.

DEFINITION 5.1. Two graphs E_1 and E_2 are called cobordic if there is a surface S have the boundary $\partial S = E_1 \cup -E_2$ with $E_1 = S \cap (\mathbf{S^3} \times \{0\})$, $E_2 = S \cap (\mathbf{S^3} \times \{1\})$ and we set $E_1 \sim E_2$. Here by "surfaces" we mean 2-dimensional simplicial complexes that are PL-embedded in $\mathbf{S^3} \times [0,1]$. $[E]$ denotes the cobordism class of the graph E.

By using the graph cobordism classes and the conjunction operation &, we can induce a graph cobordism group. $E_1 \& E_2$ is a graph represented by the union of the two graphs $E_1 \cup E_2$ with mutually disjoint 3-cells D_1 and D_2 containing (representatives of) E_1 and E_2, respectively. Here again we do not distinguish in the notation between the ambient isotopy classes of embedded graphs (graph types) and a choice of representatives.

LEMMA 5.2. *The graph cobordism classes in the sense of Definition 5.1 with the conjunction operation form an abelian group called the graph cobordism group and denoted by G_E.*

PROOF. For two cobordism classes $[E_1]$ and $[E_2]$ the operation between them is given by

$$[E_1] \& [E_2] = [E_1 \& E_2].$$

This operation is well defined. To show that : Suppose $E_1 \sim F_1$, for two graphs E_1 and F_1. Then there exists a surface S_1 with boundary $\partial S_1 = E_1 \cup -F_1$. Suppose also, $E_2 \sim F_2$, for two graphs E_2 and F_2. Then there exists a surface S_2 with boundary $\partial S_2 = E_2 \cup -F_2$. We want to show that $E_1 \& E_2 \sim F_1 \& F_2$, *i.e.* we want to find a surface S with boundary $\partial S = (E_1 \& E_2) \cup -(F_1 \& F_2)$.

Define the cobordism S to be $S_1 \& S_2$ where $S_1 \& S_2$ represents $S_1 \cup S_2$ with mutually disjoint 4-cells $D_1 \times [0,1]$ and $D_2 \times [0,1]$, containing S_1 and S_2 respectively with $D_1 \times \{0\}$ containing E_1, $D_2 \times \{0\}$ containing F_1, $D_1 \times \{1\}$ containing E_2 and $D_2 \times \{1\}$ containing F_2. The boundary of S is given by,

$$\partial S = \partial(S_1 \& S_2) = \partial S_1 \& \partial S_2 = \partial S_1 \cup \partial S_2 = (E_1 \cup -F_1) \cup (E_2 \cup -F_2) = (E_1 \& E_2) \cup -(F_1 \& F_2)$$

Then the operation is well defined. The zero of this operation is the class $[\bigcirc]$ which is the trivial graph of a countable number of components. The negative of $[E]$ is $-[E] = [-E]$, where $-E$ denoted the reflected inverse of E. □

5.3. Fusion and fission for embedded graphs. We now describe a special kind of cobordisms between embedded graphs, namely the basic cobordisms that correspond to attaching a 1-handle and that give rise to the analog in the context of graphs of the operations of fusion and fission described already in the case of links. Let E be a graph containing n-components with a split subgraph $E_s = G_1 \cup G_2 \cup G_3 ... \cup G_t$. We can define a new graph $\hat{E}$ to be $G_1 + G_2 + G_3 ... + G_t + \partial B_{t+1} + \partial B_{t+2} + ... + \partial B_n$ where $\{B_{t+1}, B_{t+2}, B_{t+3} ...., B_n\}$ are disjoint bands in $\mathbf{S^3}$ spanning E_s. The operation $+$ means addition in the homology sense. Put $E_1 = E_s \cup G_{t+1} \cup G_{t+2} .... \cup G_n$ and $E_2 = \hat{E} + G_{t+1} + G_{t+2} .... + G_n$. Now, The operation of replacing E_1 by E_2 is called **fusion** and E_2 by E_1 is called **fission**.

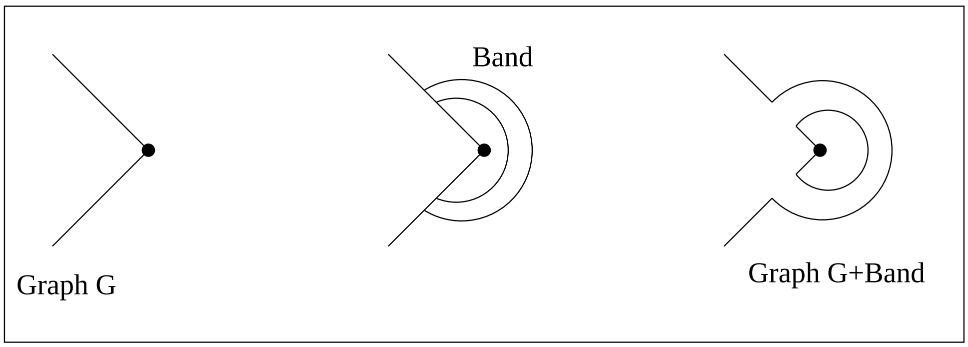


FIGURE 10

Notice that, in order to make sure that all resulting graphs will still have at least one vertex, one needs to assume that the 1-handle is attached in such a way that there is at least an intermediate vertex in between the two segments where the 1-handle is attached, as the figure above illustrates.

REMARK 5.3. Unlike the case of links, a fusion and fission for graphs does not necessarily change the number of components. For example see the figure below.

We can use the operations of fusion and fission described above to give another possible definition of cobordism of embedded graphs.

DEFINITION 5.4. Two graphs will be called graph cobordic if one can be obtained from the other by a sequence of fusions and fissions. We denote this equivalence relation by $\simeq$, and by $[E]$ the graph cobordism class of E.

Thus we have two corresponding definitions for the graph cobordism group. One can check from the definition of fusion and fission that they gives the existence of a cobordism (surface) between two graphs E_1 and E_2.

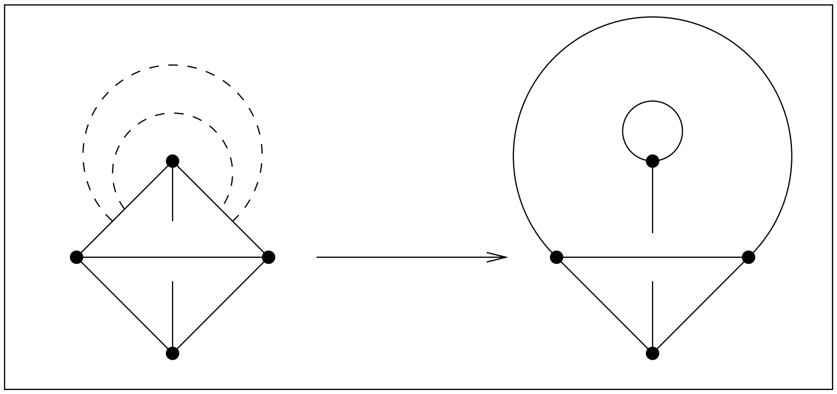


FIGURE 11

LEMMA 5.5. *Two graphs E_1 and E_2 that are cobordant in the sense of Definition 5.4 are also cobordant in the sense of Definition 5.1. The converse, however, is not necessary true.*

PROOF. As we have seen, a fusion/fission operation is equivalent to adding or removing a band to a graph and this implies the existence of a saddle cobordism given by the attached 1-handle, as illustrated in figure (6). By combining this cobordism with the identity cobordism in the region outside where the 1-handle is attached, one obtains a PL-cobordism between E_1 and E_2. This shows that cobordims in the sense of Definition 5.4 implies cobordism in the sense of Definition 5.1. The reason why the converse need not be true is that, unlike what happens with the cobordisms given by embedded smooth surfaces used in the case of links, the cobordisms of graphs given by PL-embedded 2-complexes are not always decomposable as a finite set of fundamental saddle cobordims given by a 1-handle. Thus, having a PL-cobordism (surface in the sense of a 2-complex) between two embedded graphs E_1 and E_2 does not necessarily imply the existence of a finite sequence of fusions and fissions. □

LEMMA 5.6. *The graph cobordism classes in the sense of Definition 5.4 with the conjunction operation form an abelian group called the graph cobordism group and denoted by G_F.*

PROOF. The proof is the same as the proof on lemma 5.2 since fusion and fission are a special case of cobordisms. □

The result of Lemma 5.5 shows that there are different equivalence classes $[E_1] \neq [E_2]$ in G_F that are identified $[E_1] = [E_2]$ in G_E. Thus, the number of cobordism classes when using Definition 5.1 is smaller that the number of classes by the fusion/fission method of Definition 5.4.

6. Homology theories for embedded graphs

In this part we will present a method to extend Khovanov homology from links to embedded graphs $G \subset \mathbf{S}^3$. Our construction is obtained by using Khovanov homology for links, applied to a family of knots and links associated to an embedded graph. This family is obtained by a result of Kauffman [**42**] as a fundamental topological invariant of embedded graphs obtained by associating to an embedded graph G in three-space a family of knots and links constructed by some operations of cutting graphs at vertices. Before we give this construction, we motivate the problem of extending Khovanov homology to embedded graphs by recalling another known construction of a homology theory, *graph homology*, which is defined for abstract graphs and captures the combinatorial complexity of the graph. The homology theory we seek to construct will combine aspects of Khovanov and graph homology, in as it captures information both on the embedding, as in Khovanov homology, and on the combinatorics of the graph, as in graph homology.

6.1. Graph homology. We recall here the construction and some basic properties of graph homology. As we discuss below, graph homology can be regarded as a categorification of the chromatic polynomial of a graph, in the same way as Khovanov homology gives a categorification of the Jones polynomial of a link. A construction is given for a graded homology theory for graphs whose graded Euler characteristic is the *Chromatic Polynomial* of the graph [**30**]. Laure Helm-Guizon and Yongwu Rong used the same technique to get a graded chain complex. Their construction depends on the edges in the vertices of the cube $\{0,1\}^n$ whose elements are connected subgraphs of the graph G. In this subsection we recall the construction of Laure Helm-Guizon and Yongwu Rong.

6.1.1. *Chromatic Polynomial.* let G be a graph with set of vertices $V(G)$ and set of edges $E(G)$. For a positive integer t, let $\{1,2,...,t\}$ be the set of t-colors. A coloring of G is an assignment of a t-color to each vertex of G such that vertices that are connected by an edge in G always have different colors. Let $P_G(t)$ be the number on t-coloring of G *i.e.* is the number of vertex colorings of G with t colors (in a vertex coloring two vertices are colored differently whenever they are connected by an edge e), then $P_G(t)$ satisfies the Deletion-Contraction relation

$$P_G(t) = P_{G-e}(t) + P_{G/e}(t)$$

In addition to that $P_{K_n}(t) = t^n$ where K_n is the graph with n vertices and n edges. $P_G(t)$ is called *Chromatic Polynomial*. Another description can be give to $P_G(t)$, let $s \subset E(G)$, define G_s to be the graph whose vertex set is the same vertex set of G with edge set s. Put $k(s)$ the number of connected components of G_s. Then we have

$$P_G(t) = \sum_{s \subseteq E(G)} (-1)^{|s|} t^{k(s)}$$

6.1.2. *Constructing n-cube for a Graph.* First we want to give an introduction to the type of algebra that we will use it in our work later.

DEFINITION 6.1. [**30**] Let $\mathcal{V} = \oplus_i V_i$ be a graded $\mathbb{Z}$-module where $\{V_i\}$ denotes the set of homogenous elements with degree i, and the graded dimension of $\mathcal{V}$ is the power series

$$qdim\mathcal{V} = \sum_i q^i dim_{\mathbb{Q}}(V_i \otimes \mathbb{Q})$$

.

We can define the tensor product and directed sum for the graded $\mathbb{Z}$-module as follows:

THEOREM 6.2. [**30**] *Let $\mathcal{V}$ and $\mathcal{W}$ be a graded $\mathbb{Z}$-modules, then $\mathcal{V} \otimes \mathcal{W}$ and $\mathcal{V} \oplus \mathcal{W}$ are both graded $\mathbb{Z}$-module with*

(1) $qdim(\mathcal{V} \oplus \mathcal{W}) = qdim(\mathcal{V}) + qdim(\mathcal{W})$
(2) $qdim(\mathcal{V} \otimes \mathcal{W}) = qdim(\mathcal{V}) \cdot qdim(\mathcal{W})$

Let G be a graph with edge set $E(G)$ and $n = |E(G)|$ represents the cardinality of $E(G)$. We need first to order the edges in $E(G)$ and denote the edges by $\{e_1, e_2, ..., e_n\}$. Consider the n-dimensional cube $\{0,1\}^n$ [**30**],(see the figure (12)). Each vertex can be indexed by a word $\alpha \in \{0,1\}^n$. This vertex α corresponded to a subset $s = s_\alpha$ of $E(G)$. This is the set of edges of G that are incident to the chosen vertex. Then $e_i \in s_\alpha$ if and only if $\alpha_i = 1$. Define $|\alpha| = \sum \alpha_i$ (height of α) to be the number of $1's$ in α or equivalently the number of edges in s_α. We associate to each vertex α in the cube $\{0,1\}^n$, a graded vector space V_α as follows [**30**]. Let V_α be a graded free $\mathbb{Z}$-module with 1 and x

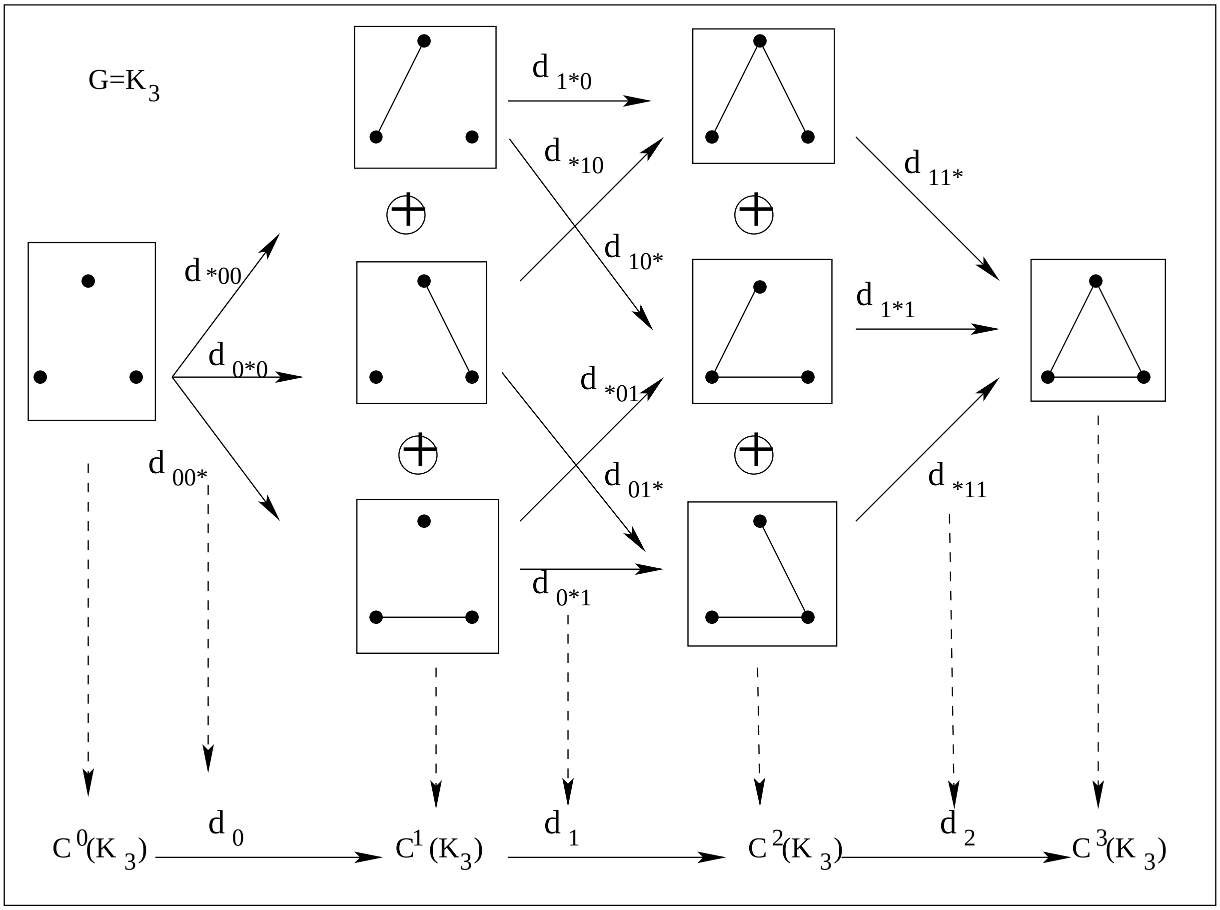


FIGURE 12

basis elements with degree 0 and 1 respectively, then $V_\alpha = \mathbb{Z} \oplus \mathbb{Z}x$ with $qdim(V_\alpha) = 1+q$ and hence, $qdim(V_\alpha^{\otimes k}) = (1+q)^k$.

Consider G_{s_α}, the graph with vertex set $V(G)$ and edge set s_α. Replace each component of G_{s_α} by a copy of V_α and take the tensor product over all components.

Define the graded vector space $\mathcal{V}_\alpha = V_\alpha^{\otimes k}$ where k is the number of the components of G_{s_α}. Set the vector space $\mathcal{V}$ to be the direct sum of the graded vector space for all the vertices. The differential map d^i, defined by using the edges of the cube $\{0,1\}^n$. We can label each edge of $\{0,1\}^n$ by a sequence of $\{0,1,\star\}^n$ with exactly one $\star$. The tail of the edge labeled by $\star = 0$ and the head by $\star = 1$. To define the differential we need first to define *Per-edge* maps between the vertices of the cube $\{0,1\}^n$. These maps is defined to be a linear maps such that every square in the cube $\{0,1\}^n$ is commutative. Define the *per-edge* map $d_\xi : \mathcal{V}_{\alpha_1} \longrightarrow \mathcal{V}_{\alpha_2}$ for the edge ξ with tail α_1 and head α_2 as follows: Take $\mathcal{V}_{\alpha_i} = V^{\otimes k_i}$ for $i = 1,2$ with k_i is the number of the connected components of $G_{s_{\alpha_i}}$. Let e be the edge and $s_{\alpha_2} = s_{\alpha_1} \cup \{e\}$, then there are two possible cases. First one (easy case): d_ξ will be the identity map if the edge e joins a component r of $G_{s_{\alpha_1}}$ to itself. Then $k_1 = k_2$ with a natural correspondence between the components of $G_{s_{\alpha_1}}$ and $G_{s_{\alpha_2}}$. Second one: if e joins two different components of $G_{s_{\alpha_1}}$, say r_1 and r_2, then $k_2 = k_1 - 1$ and the components of $G_{s_{\alpha_2}}$ are $r_1 \cup r_2 \cup \{e\} \cup \ldots . \cup r_{k_1}$. Define d_ξ to be the identity map on the tensor factor coming from $r_3, r_4, \ldots, r_{k_1}$. Also define d_ξ on the remaining tensor factor to be the multiplication map $V_\alpha \otimes V_\alpha \longrightarrow V_\alpha$ sending $x \otimes y$ to xy. The differential $d^i : \mathcal{V}^i \longrightarrow \mathcal{V}^{i+1}$ in given by

$$d^i = \sum_{|\xi|=i} \text{sign}(\xi) d_\xi$$

Where $\text{sign}(\xi)$ is chosen so that $d^2 = 0$.

THEOREM 6.3. [**61**],[**30**] *The following properties hold for graph homology.*

- *The graded Euler characteristic for the graph homology given by*
$$\sum_{i,j} (-1)^i q^j dim(Kh^{i,j}(G)) = P_G(t)$$
where $P_G(t)$ is the chromatic polynomial
- *In graph homology a short exact sequence*
$$0 \longrightarrow CKh^{i-1,j}(G/e) \longrightarrow CKh^{i,j}(G) \longrightarrow CKh^{i,j}(G-e) \longrightarrow 0$$
can be constructed by using the deletion-contraction relation for a given edge $e \in G$. This gives a long exact sequence
$$\cdots \longrightarrow Kh^{i-1,j}(G/e) \longrightarrow Kh^{i,j}(G) \longrightarrow Kh^{i,j}(G-e) \longrightarrow \cdots$$

6.2. Graph homology and Khovanov homology. A first idea of how to obtain a homology theory that extends Khovanov homology for embedded graph is to combine the chain complex that computes Khovanov homology, constructed using the basic cobordisms near each crossing of a planar diagram, and the chain complex of graph homology which is based on removing edges from the graph.

Since we are going to concentrate later on a different approach to constructing a Khovanov homology for embedded graphs, we only give here a simple example illustrating how to associate to each level in the graph homology complex a corresponding cubical complex as in Khovanov homology with differentials between these induced by the graph homology differentials, but we do not pursue the details of this construction further at present.

The aim of the approach we sketch briefly here would be to obtain a double complex that combines the graph homology complex and a version of the Khovanov complex. We recall briefly the notion of a double complex.

DEFINITION 6.4. Let $\mathcal{C}$ be an additive category. A double complex $(C^{*,*}(A), dx)$ in $\mathcal{C}$ is the data of $(C^{i,j}, d'^{i,j}x, d''^{i,j}x), (i,j) \in \mathbb{Z} \times \mathbb{Z}$, where $C^{i,j} \in \mathcal{C}$ and the differentials $d'^{i,j}x : C^{i,j} \longrightarrow C^{i+1,j}$ (vertical differential) and $d''^{i,j}x : C^{i,j} \longrightarrow C^{i,j+1}$ (horizontal differential) satisfy:

$$d'^2 x = d''^2 x = 0, d' \circ d'' = d'' \circ d'$$

in the commutative diagram

$$\begin{array}{ccccc}
C^{i,j} & \xrightarrow{d''} & C^{i,j+1} & \xrightarrow{d''} & C^{i,j+2} \\
\downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d'} \\
C^{i+1,j} & \xrightarrow{d''} & C^{i+1,j+1} & \xrightarrow{d''} & C^{i+1,j+2} \\
\downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d'} & & \downarrow{\scriptstyle d'} \\
C^{i+2,j} & \xrightarrow{d''} & C^{i+2,j+1} & \xrightarrow{d''} & C^{i+2,j+2}
\end{array}$$

We now look at a simple example of an embedded graph with a small number of vertices and of crossings, to illustrate how one can try to combine Khovanov and graph homology.

For a graph G with n crossings, one can follow the same idea of Khovanov and construct an associated n-cube by applying the 0 and 1-resolutions illustrated in figure (13)

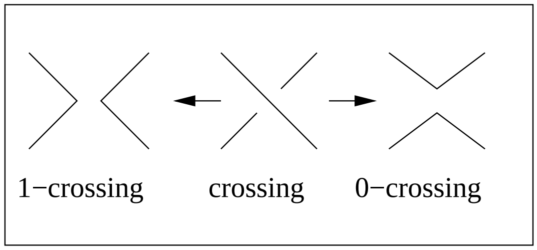


FIGURE 13. 0 and 1- resolutions to each crossing in a Graph G

To each vertex α in the n-cube we can associate a graded $\mathbb{Z}$-module $\mathcal{M}_\alpha$ and sum over all columns to get a complex C.

$$C = \bigoplus_{\alpha\ n-cube} \mathcal{M}_\alpha$$

And to each edge in the cube we associate a differential d. A differential D on the summand C is the sum of the maps d for all edges, such that $d^2 = 0$. Consider the diagram in figure (14)

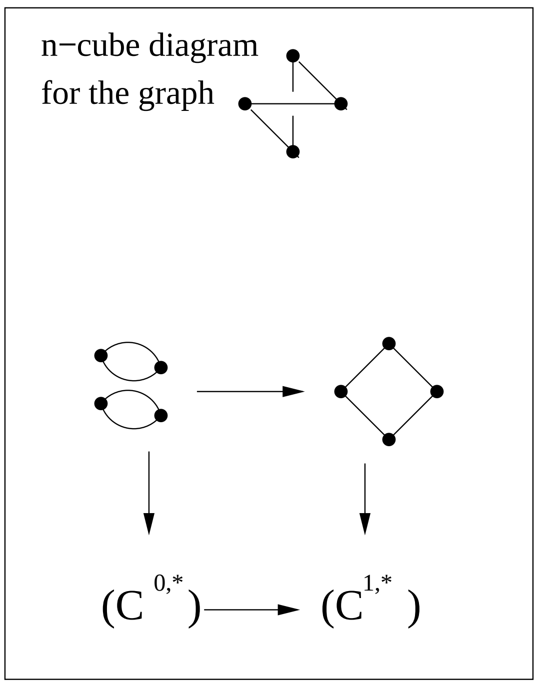


FIGURE 14

One can then try to combine the n-cube complex obtained in this way with the complex computing graph homology as described in the previous section. Consider the example of the planar diagram of an embedded graph as illustrated in figure (15).

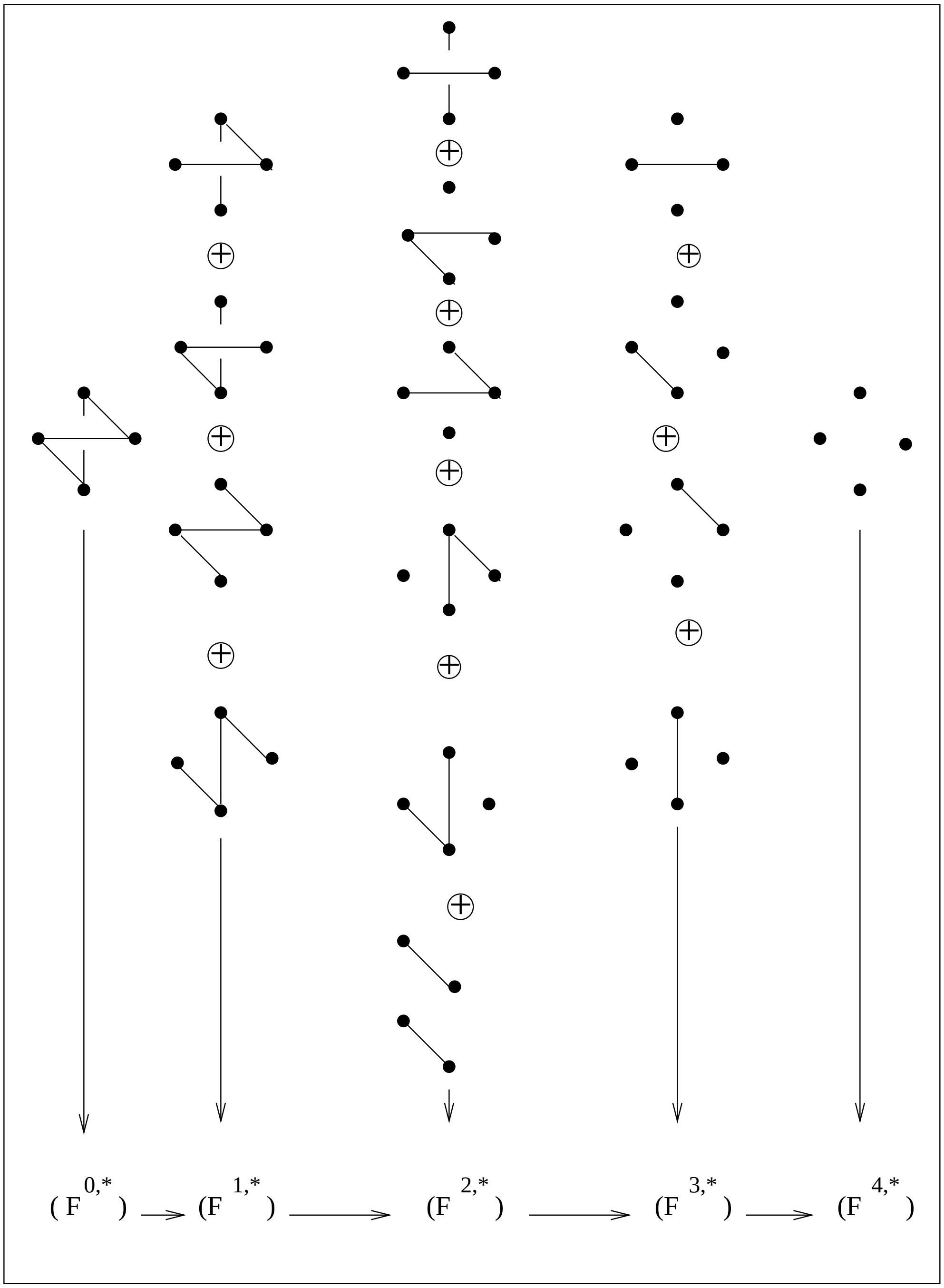


FIGURE 15

Figure (14) shows the graph together with the associated two terms cubical complex obtained by resolving the crossing in the two different ways, while Figure (15) shows the graph homology complex for the same graph. Consider then the diagram in figure (16) This shows how to associate to each term in the first two steps of the graph homology complex a corresponding cubical complex.

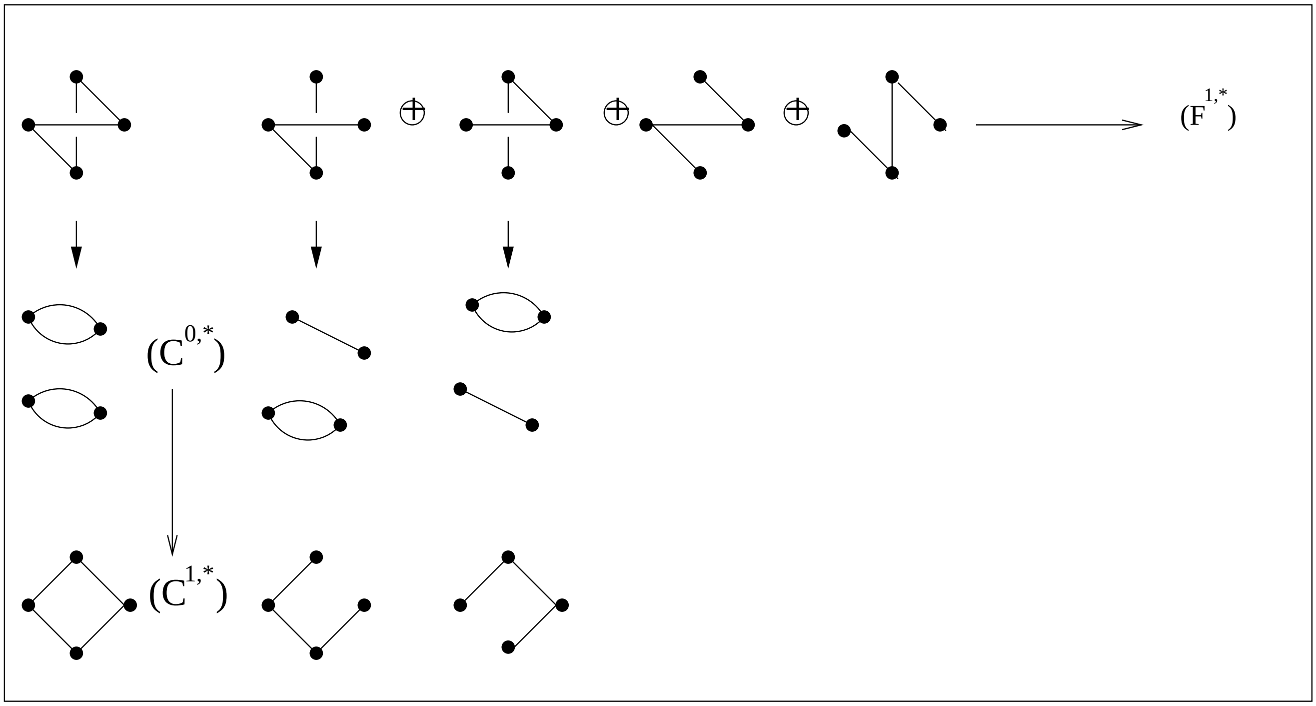


FIGURE 16

Instead of continuing in more generality this approach, we show in the next section a more direct and simpler approach to constructing a Khovanov homology of embedded graphs. The approach we present below will have the advantage that the proof of topological invariance will immediately follow from Kauffman's result and will not require checking that the graph Reidemeister moves induce chain homotopies of the complexes involved.

To this purpose, we first review a useful result of Kauffman in the next paragraphs.

6.3. Kauffman's invariant of Graphs. We give now a survey of the Kauffman theory and show how to associate to an embedded graph in $\mathbf{S^3}$ a family of knots and links. We then use these results to give our definition of Khovanov homology for embedded graphs. In [**42**] Kauffman introduced a method for producing topological invariants of graphs embedded in $\mathbf{S^3}$. The idea is to associate a collection of knots and links to a graph G so that this family is an invariant under the expanded Reidemeister moves defined by Kauffman and reported here in figure (17).

He defined in his work an ambient isotopy for non-rigid (topological) vertices. (Physically, the rigid vertex concept corresponds to a network of rigid disks each with (four) flexible tubes or strings emanating from it.) Kauffman proved that piecewise linear ambient isotopies of embedded graphs in $\mathbf{S^3}$ correspond to a sequence of generalized Reidemeister moves for planar diagrams of the embedded graphs.

THEOREM 6.5. [**42**] *Piecewise linear (PL) ambient isotopy of embedded graphs is generated by the moves of figure (17), that is, if two embedded graphs are ambient isotopic, then any two diagrams of them are related by a finite sequence of the moves of figure (17).*

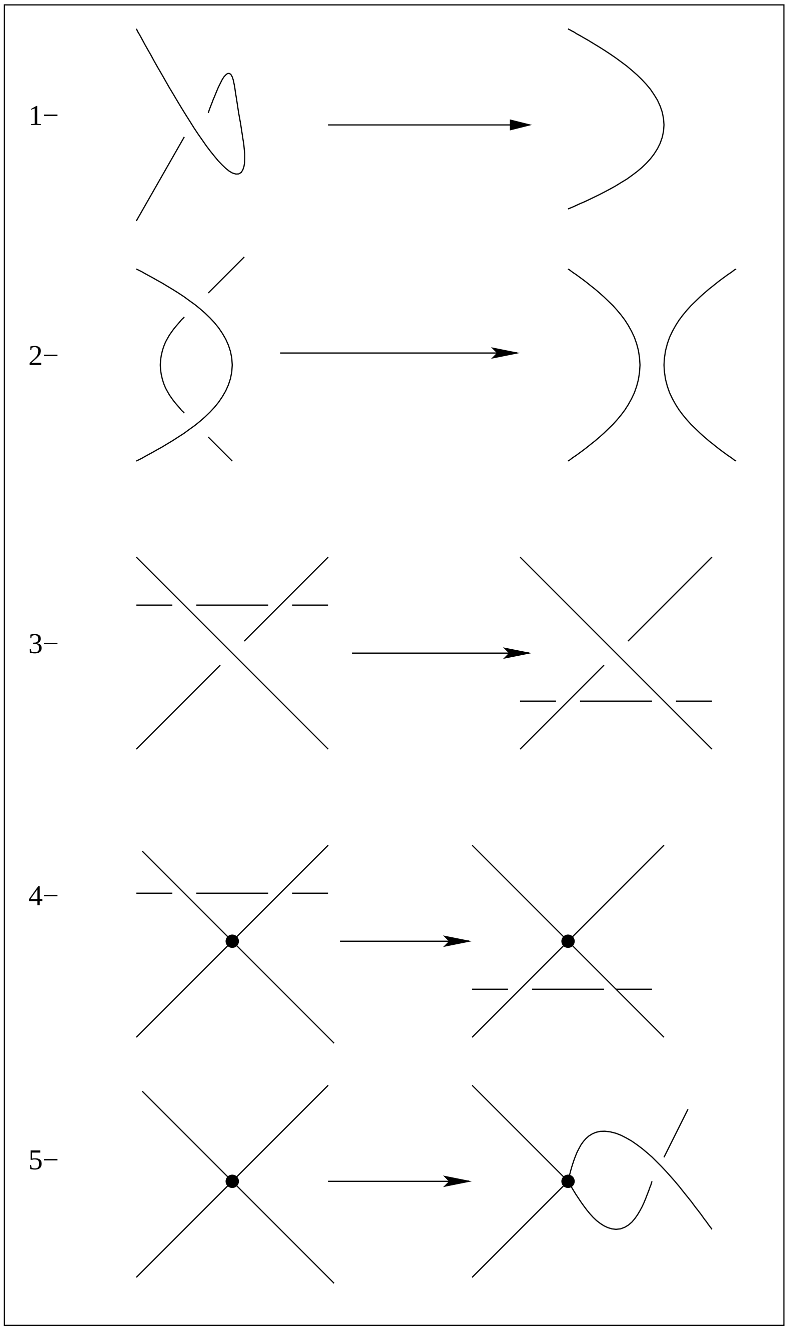


FIGURE 17. Generalized Reidemeister moves by Kauffman

Let G be a graph embedded in $\mathbf{S^3}$. The procedure described by Kauffman of how to associate to G a family of knots and links prescribes that we should make a local replacement as in figure 18 to each vertex in G. Such a replacement at a vertex v connects two edges and isolates all other edges at that vertex, leaving them as free ends. Let $r(G,v)$ denote the link formed by the closed curves formed by this process at a vertex v. One retains the link $r(G,v)$, while eliminating all the remaining unknotted

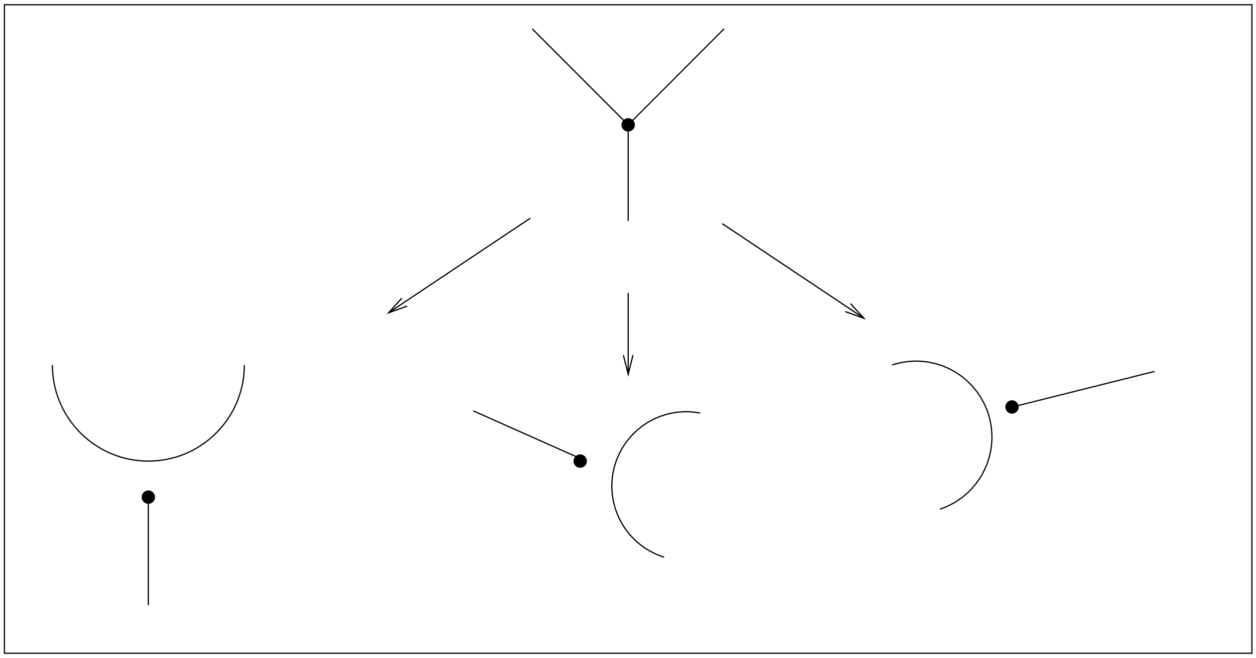


FIGURE 18. local replacement to a vertex in the graph G

arcs. Define then $T(G)$ to be the family of the links $r(G,v)$ for all possible replacement choices,

$$T(G) = \cup_{v \in V(G)} r(G,v).$$

For example see figure (19).

THEOREM 6.6. [**42**] *Let G be any graph embedded in* $\mathbf{S^3}$*, and presented diagrammatically. Then the family of knots and links* $T(G)$*, taken up to ambient isotopy, is a topological invariant of G.*

For example, in the figure (19) the graph G_2 is not ambient isotopic to the graph G_1, since $T(G_2)$ contains a non-trivial link.

6.4. Definition of Khovanov homology for embedded graphs. Now we are ready to speak about a new concept of Khovanov homology for embedded graphs by using Khovanov homology for the links (knots) and Kauffman theory of associate a family of links to an embedded graph G, as described above.

DEFINITION 6.7. Let G be an embedded graph with $T(G) = \{L_1, L_2, ..., L_n\}$ the family of links associated to G by the Kauffman procedure. Let $Kh(L_i)$ be the usual Khovanov homology of the link L_i in this family. Then the Khovanov homology for the embedded graph G is given by

$$Kh(G) = Kh(L_1) \oplus Kh(L_2) \oplus .... \oplus Kh(L_n)$$

Its graded Euler characteristic is the sum of the graded Euler characteristics of the Khovanov homology of each link, *i.e.* the sum of the Jones polynomials,

$$\sum_{i,j,k} (-1)^i q^j dim(Kh^{i,j}(L_k)) = \sum_k J(L_k). \tag{6.1}$$

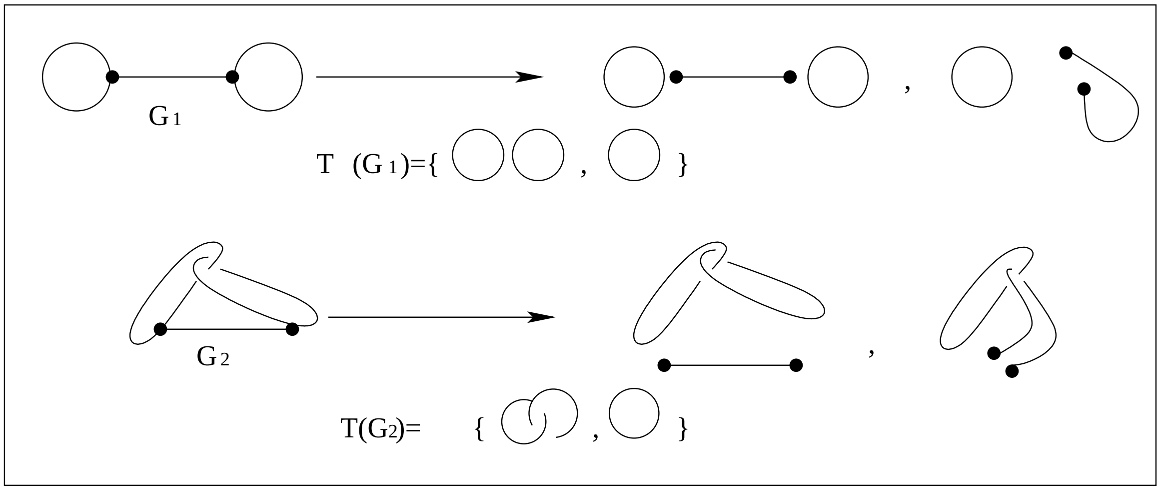


FIGURE 19

We show some simple explicit examples.

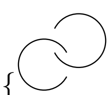
EXAMPLE 6.8. In figure (19) $T(G_1) = \{\bigcirc\bigcirc, \bigcirc\}$ then for $Kh(\bigcirc) = \mathbb{Q}$

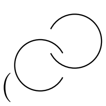
$$Kh(G_1) = Kh(\bigcirc\bigcirc) \oplus Kh(\bigcirc)$$

Now, from proposition 3.5 no.5

$$Kh(G_1) = Kh(\bigcirc) \otimes Kh(\bigcirc) \oplus Kh(\bigcirc)$$

$$Kh(G_1) = \mathbb{Q} \otimes \mathbb{Q} \oplus \mathbb{Q} = \mathbb{Q} \oplus \mathbb{Q}$$

$T(G_2) = \{$  $, \bigcirc\}$ then

$Kh(G_2) = Kh($  $) \oplus Kh(\bigcirc)$

$Kh(G_2) =$

j \ i	-2	-1	0
0			$\mathbb{Q} \oplus \mathbb{Q}$
-1			
-2			$\mathbb{Q}$
-3			
-4	$\mathbb{Q}$		
-5			
-6	$\mathbb{Q}$		

7. Questions and Future Work

We sketch briefly an outline of ongoing work where the construction presented in this paper is applied to other constructions related to noncommutative geometry and knot invariants.

7.1. Categorification and homology invariants. We have constructed a category of knots and links, or more generally of embedded graphs, where it is possible to use homological algebra to construct complexes and cohomological invariants. The process of categorifications in knot theory, applied to a different category of knots, has already proved very successful in deriving new knot invariants such as Khovanov homology. We have begun investigating in this second chapter how to associate cohomologies to the objects in our category. We need to understand how to combine these with the rest of the categorical structure described in the first chapter, to obtain a functor from our 2-category to a 2-category of 2-vector spaces. We also plan to study filtrations, long exact sequences, and spectral sequences for Khovanov homology of embedded graphs.

7.2. Time evolutions and moduli spaces. We have constructed vertical time evolutions from virtual dimensions of moduli spaces. It would be more interesting to construct time evolutions on the algebra of correspondences, in such a way that the actual gauge theoretic invariants obtained by integrating certain differential forms over the moduli spaces can be recovered as low temperature equilibrium states. The formal path integral formulations of gauge theoretic invariants of 4-manifolds suggests that something of this sort may be possible, by analogy to the case we described of Hartle–Hawking gravity. In the case of the horizontal time evolution, it would be interesting to see if that can also be related to gauge theoretic invariants. The closest model available would be the gauge theory on embedded surfaces developed in [**45**].

7.3. Noncommutative spaces and dynamical systems. Another way to construct noncommutative spaces out of the geometric correspondences considered here is via the subshifts of finite type constructed in [**56**] out of the representations $\sigma : \pi_1(\mathbf{S^3} \setminus L) \to S_m$ describing branched coverings. A subshift of finite type naturally determines a noncommutative space in the form of associated Cuntz–Krieger algebras. The covering moves (or colored Reidemeister moves) of [**52**] will determine correspondences between these noncommutative spaces.

APPENDIX A

1. Branched Covering

1.1. Branched Coverings of Manifolds. We work here in the PL category with piecewise linear maps. Manifolds have piecewise linear local charts and all maps $\phi : \mathbf{M}^m \to \mathbf{N}^m$ between m-manifolds are assumed to be PL and proper (*i.e.* the preimage of a compact set is compact). A PL, proper, finite-to-one and open map $\phi : \mathbf{M}^m \longrightarrow \mathbf{N}^m$ between manifolds is called a branched covering. The singular set is the set of points of $\mathbf{M}^m$ at which ϕ fails to be a local homeomorphism. It is a subpolyhedron of $\mathbf{M}^m$ of codimension 2. The branched set, or branch locus, of ϕ (denoted by B_ϕ) is the image of the singular set of the branched covering ϕ in $\mathbf{N}^m$. The fibers of ϕ are finite sets $\phi^{-1}(y)$, for all $y \in \mathbf{N}^m$. The degree $\deg\phi$ is the maximum cardinality of a fiber.

EXAMPLE 1.1. Let $D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$ and let $P : D^2 \longrightarrow D^2$ be the map given by the formula $P(z) = z^n$. Then P is an n-fold branched covering with unique branched point $z = 0$.

Two branched coverings $\phi_0, \phi_1 : \mathbf{M}^m \longrightarrow \mathbf{N}^m$ are equivalent if there exist homeomorphisms $f : \mathbf{M}^m \longrightarrow \mathbf{M}^m$ and $g : \mathbf{N}^m \longrightarrow \mathbf{N}^m$ such that $g\phi_0 = \phi_1 f$. They are b-homotopic if there is a homotopy $\theta_t : \mathbf{M}^m \longrightarrow \mathbf{N}^m$, for $0 \leq t \leq 1$, such that $\theta_0 = \phi_0$, $\theta_1 = \phi_1$, and such that each θ_t is a branched covering. We recall briefly the "Alexander trick" for branched covering. Any branched covering $\phi : \mathbf{D^3} \longrightarrow \mathbf{D^3}$ is b-homotopic to the cone $C(\phi|\partial\mathbf{D^3})$, hence two such branched coverings that agree on $\partial\mathbf{D^3}$ are b-homotopic.
Suppose that $\phi : \mathbf{M}^3 \longrightarrow \mathbf{N}^3$ is branched covering and that $\mathbf{N}^3$ is orientable. The orientation of $\mathbf{N}^3$ determines an orientation of $\mathbf{M}^3$ such that ϕ is orientation- preserving. To show this, triangulate $\mathbf{M}^3$ and $\mathbf{N}^3$ so that ϕ is simplicial and orient the m-simplices of $\mathbf{M}^3$ so that ϕ is orientation-preserving on each simplex. A branched covering $\phi : \mathbf{M}^3 \longrightarrow \mathbf{N}^3$ which preserves specified orientations of $\mathbf{M}^3$ and $\mathbf{N}^3$ is called an oriented branched covering . A branched covering $\phi : \mathbf{M}^3 \longrightarrow \mathbf{N}^3$ of degree $n \geq 2$ is *simple* provided that it is of local degree 2 and, for each $x \in \mathbf{N}^3$, the fiber $\phi^{-1}(x)$ over x consists of at least $n-1$ points (hence it contains at most one singular point).

PROPOSITION 1.2. [**5**] *Let $\phi : \mathbf{M}^3 \longrightarrow \mathbf{N}^3$ be a simple branched covering of degree n between compact manifolds, and let $\xi : \mathbf{M}^3 \longrightarrow \mathbf{N}^3$ be any other branched covering. If ξ is close enough to ϕ in the compact-open topology, then ξ is also simple.*

LEMMA 1.3. [**5**] *Any Branched covering $\phi : \mathbf{D}^2 \longrightarrow \mathbf{D}^2$ is b-homotopic rel$\partial\mathbf{D}^2$ to a simple branched covering.*

THEOREM 1.4. [**5**] *Let M^3 be a connected orientable 3-manifold with connected boundary and let $\phi : \partial M^3 \longrightarrow S^2$ be a simple branched covering of degree $n \geq 3$. Then there is a simple branched covering $\varphi : M^3 \longrightarrow D^3$ which extends ϕ.*

In [**38**] A. Hurwitz introduced a way of associating data to every branched covering $\phi : M^n \longrightarrow N^n$ of degree n which are called the *Hurwitz system* for ϕ. This is defined in the following way: a branched covering $\phi_0 : M^3 \setminus \phi^{-1}(B_\phi) \longrightarrow N^3 \setminus B_\phi$ can be determined by a representation

$$\sigma(\phi) : \pi_1(N^m \setminus B_\phi) \to S_m, \tag{1.1}$$

where S_m is the symmetric group. One then has the following result.

THEOREM 1.5. *(Hurwitz Existence Theorem)*
For any finite set $B \subset N^m$ and representation $\sigma : \pi_1(N^m \setminus B) \to S_n$, there is a degree n branched covering $\phi : M^m \longrightarrow N^m$, where M^m is not necessary connected, with $B_\phi \subset B$ and $\sigma(\phi) \subset \sigma$.

One also has the following result that generalizes Lemma 1.3 above.

THEOREM 1.6. [**5**] *Any branched covering $\phi : \mathbf{M}^3 \longrightarrow \mathbf{N}^3$ of degree n is b-homotopic to a simple branched covering.*

This can be used to show (see [**5**]) that one can, up to homotopy, always reduce to the case where the branch locus is a manifold, which in the case of branched coverings of 3-manifolds means a 1-manifold in the 3-sphere, that is, a link.

COROLLARY 1.7. *Any branched covering $\phi : \mathbf{M}^3 \longrightarrow \mathbf{N}^3$ is branched homotopic to one with branched set a* 1*-manifold.*

In the first chapter, in order to have well defined compositions of morphisms, we did not want to consider coverings up to homotopy, so we had to keep also branch loci that are embedded graphs and not just links.

Another result from the general theory of branched coverings of 3-manifolds that we used extensively in our work is the fact that all 3-manifolds are branched coverings of the 3-sphere. We report here a simple argument that shows why this is the case. It assumes the fact that all 3-manifolds admit a Heegaard splitting.

THEOREM 1.8. *Let $\mathbf{M}^3$ be a closed orientable 3-manifold, and $n \geq 3$ an integer. Then there exists a simple branched covering $\phi : \mathbf{M}^3 \longrightarrow \mathbf{S^3}$ of degree n.*

PROOF. Let $\mathbf{M}^3 = H_- \cup H_+$ be a Heegaard decomposition where H_- and H_+ are handlebodies identified along their boundary. Let $\mathbf{S^3} = \mathbf{D^3}_- \cup \mathbf{D^3}_+$ where $\mathbf{D^3}_-$ and $\mathbf{D^3}_+$ are the upper and lower hemispheres. By the Hurwitz Existence theorem 1.5 there is a simple branched covering $\xi : \partial H_- \longrightarrow \partial \mathbf{D^3}_-$. By the extension theorem 1.4 ξ extends to a simple branched covering $\phi_- : H_- \longrightarrow \mathbf{D^3}_-$ and to a simple branched covering $\phi_+ : H_+ \longrightarrow \mathbf{D^3}_+$ of degree n. Just set $\phi = \phi_- \cup \phi_+$ □

The fact that all compact PL 3-manifolds admit a Heegaard splitting is also easy to check. In fact, take a triangulation of the 3-manifold. A small tubular neighborhood of the 1-skeleton of the triangulation gives a handlebody H_- and the complement of this tubular neighborhood can also be seen to be a handlebody H_+ of the same genus. Their common boundary is the genus g surface along which the gluing of the two handlebody happens.

2. Filtration

A finite length filtration of a chain complex C is a sequence of subcomplexes

$$0 = C_k \subset C_{k-1} \subset C_{k-2} \subset \ldots\ldots \subset C_0 = C$$

A map $f : C \longrightarrow C'$ between two filtered chain complexes is said to respect the filtration if $f(C_i) \subset C'_i$. A map f is a filtered map of degree k if $f(C_i) \subset C'_{i+k}$. By defining a filtration $\{C_i\}$ on a chain complex C, one can induce another filtration $\{F_i\}$ on $H_*(C)$ defined as follows : a class $[x] \in H_*(C)$ is in F_i if and only if it has a representative which is an element of C_i. Notice that if $f : C \longrightarrow C'$ is a filtred chain map of degree k, then it is easy to see that the induced map $f_* : H_*(C) \longrightarrow H_*(C')$ is also filtered of degree k. A finite length filtration $\{C_i\}$ on C induces a spectral sequence which converges to the associated graded group of the induced filtration $\{F_i\}$. The associated grading of a filtration is defined as follows : an element $x \in C$ has grading i if and only if $x \in C_i$ and $x \notin C_{i+1}$. The associated graded group is the quotient group C_i / C_{i+1}.

3. Knot and link

A *link* or a *knot* in $\mathbf{S^3}$ is a smooth embedding of a disjoint family of circles (link) or a single circle (knot), *i.e.* it is a collection of disjoint smooth simple closed curves, which is a 1-dimensional closed submanifold of $\mathbf{S^3}$.

DEFINITION 3.1. Two links L_1 and L_2 are said to be equivalent if there is a homeomorphism of $\mathbf{S^3}$ taking one to the other. Two links L_1 and L_2 are said to be ambient isotopic if there exists a continuous family of homeomorphisms ϕ_t of $\mathbf{S^3}$ beginning from the identity $\phi_0 = id$ and ending with a homeomorphism ϕ_1 with $L_1 = \phi_1(L_2)$. The ambient isotopy class of a link is called the link type.

A link (knot) is said to be trivial if it is equivalent to a circles (circle). The relation between equivalence and ambient isotopy is the following. Given a choice of the orientation on the 3-sphere $\mathbf{S^3}$, if the homeomorphism of $\mathbf{S^3}$ that gives the equivalence between L_1 and L_2 is orientation preserving, then there is a continuous family of homeomorphisms of $\mathbf{S^3}$ beginning from the identity and ending with a homeomorphism taking L_1 to L_2 which is an ambient isotopy. Thus, two links L_1 and L_2 are ambient isotopic if there exists an orientation preserving homeomorphisms ϕ of $\mathbf{S^3}$ with $L_1 = \phi(L_2)$.

A knot is said to be tame if it is isotopic to a polygonal knot. Non-tame knot exist and are called wild. The set of tame knot types is countable. A knot is called smooth if it is a smooth submanifold of $\mathbf{S^3}$.

Let K be a tame knot type. One can project K onto a plane in such a way that the image is a nodal curve. By drawing the nodal points as crossings that remember the 3-dimensional positions of the two crossing strands of the knot, one obtains a picture called a *knot diagram D* of K. One can define the link diagrams in the same way.

DEFINITION 3.2. The minimal crossing number $c(K)$ of a given knot type K is the minimum number of crossings among all the planar diagrams representing K.

DEFINITION 3.3. A knot invariant is a mathematical object associated with each knot, in such a way that the object attributed to two ambient isotopic knots is the same (or isomorphic in the appropriate category).

For example, the crossing number is a knot invariant. Let K be a knot type in $\mathbf{S^3}$. We can reflect its image through a plane to get a knot K^m called the mirror image of K. If K ambient isotopic to its mirror image K^m then K is called achiral and if they are not ambient isotopic then the knot is called chiral. For example, the Figure-8 knot is achiral.

DEFINITION 3.4. A knot diagram D is called alternating if, when we proceed along the nodal curve, we pass alternately over, under, over and so on, at each crossing.

The usual planar diagram for the trefoil knot is alternating.

The following result is a well known and very useful characterization of ambient isotopy of knots and links in terms of their planar diagrams.

PROPOSITION 3.5. *Two diagrams represent the same link or knot type if and only if we can get one from the other by finite sequence of Reidemeister moves as in figure (1).*

Let K be an embedded knot in $\mathbf{S^3}$. We define the knot complement as the complement of the knot in $\mathbf{S^3}$ *i.e.* the topological space $\mathbf{S^3} - K$. Let K and K' be two ambient isotopic knots in $\mathbf{S^3}$, and let $\phi : \mathbf{S^3} \longrightarrow \mathbf{S^3}$ be an orientation preserving homeomorphism of $\mathbf{S^3}$ with $\phi(K) = K'$. The restriction $\phi|(\mathbf{S^3} - K) \longrightarrow (\mathbf{S^3} - K)$ is also a homeomorphism. Thus, two ambient isotopic knots have homeomorphic knot complements.

A knot or a link called reducible (composite knot) if it can be expressed as the *connected sum* of two non-trivial knots or links. Recall that, if we have two knots K_1 and K_2, then the connected sum

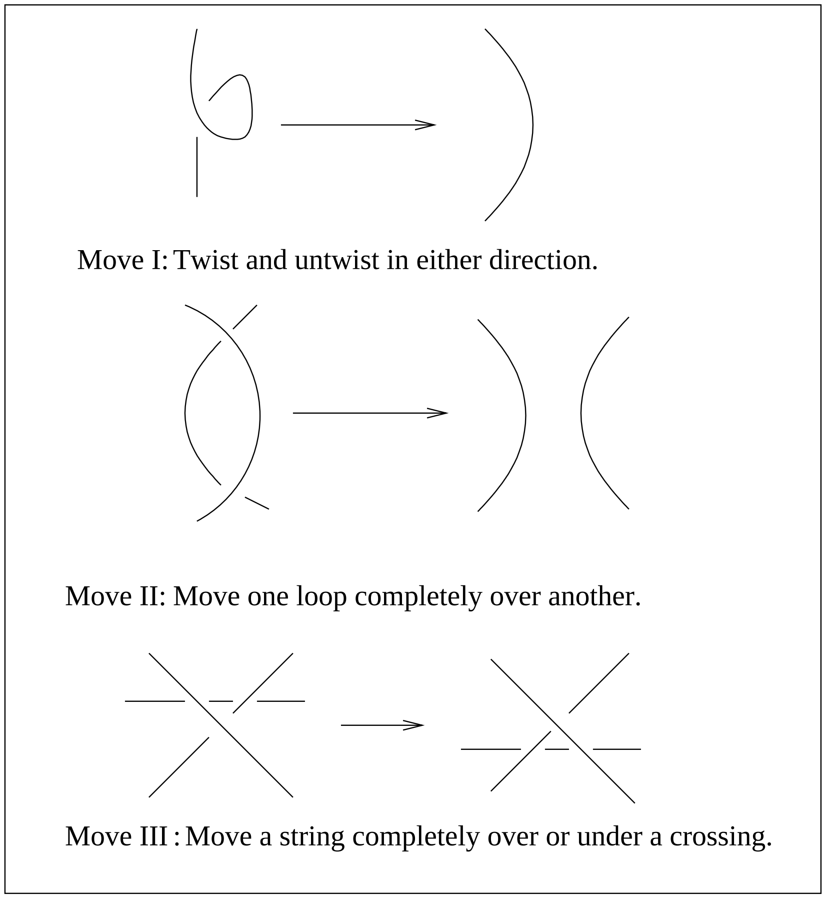


FIGURE 1. Reidemeister move

of K_1 and K_2, denoted by $K_1\#K_2$, is formed in the following way. Take a knot projection of K_1 and K_2, and put them next to each other. Select a small *arc* on each of the two knots K_1 and K_2. Delete a segment of arc from each, and connect the endpoints by adding two new arcs each connecting an endpoint on one of the two knots to an endpoint on the other. A knot is composite if it is a connected sum of two non-trivial knots. The knots K_1 and K_2 are *factor knots* of $K_1\#K_2$. The decomposition of knots into *prime* factors is unique up to the order of each summand in the connected sum (like the unique prime factorization of natural numbers). For example, the trefoil knot is a prime knot.

However, unlike the case of prime numbers, here there are two choices of how to connect the endpoints of the arcs in performing a connected sum. These yield the same result whenever one of the knots is invertible.

THEOREM 3.6. *The composition $K_1\#K_2$ is unique if and only If one of the two knots K_1 or K_2 is invertible (i.e. it can be deformed by an ambient isotopy into the same knot with the reverse orientation).*

3.1. Universal Knot. In 1982 the concept of *universal link* for 3-manifolds was introduced by W.Thurston [**60**]. He gave an example of a six components universal link. A link (or a knot) U_L is said to be universal if every closed orientable 3-manifold can be realized as a branched covering of $\mathbf{S^3}$ in such a way that the branched set is U_L.

Hilden, Lozano, and Montesinos constructed the first example of a universal knot in [**34**]. They also gave a description of 2 and 4-components universal links. Their result is based on the use of the following result.

THEOREM 3.7. [**34**] *Let L be a link in $\mathbf{S^3}$ with $n+1$-components. Then there is a link L' in $\mathbf{S^3}$ with $2n+4$-components and a map $p:\mathbf{S^3}\longrightarrow\mathbf{S^3}$ such that*

(1) p is a $2n+5$ to 1 branched covering map, branched along a knot k.
(2) $p^{-1}(k)=L'$ and $L\subset L'$

The first example of a universal knot first obtained by Hilden, Lonzano, and Montesinos is very complicated, but simpler examples where constructed later. Not all knots are universal knots. For example, the question of whether the Figure-8 knot is universal remained open for some time and was eventually proved by Hilden, Lozano, and Montesinos in [**33**].

4. Topological Quantum Field Theory

A topological quantum field theory (or topological field theory or TQFT) is a quantum field theory which computes topological invariants. In physics, topological quantum field theories are the low energy effective theories of topologically ordered states, such as fractional quantum Hall states, string-net condensed states, and other strongly correlated quantum liquid states. In 1988 [**2**] Atiyah gave a description of topological QFT with axioms. The basic idea is that a TQFT is a functor from a certain category of cobordisms to the category of vector spaces.

DEFINITION 4.1. [**2**] In dimension d, TQFT is a monoidal functor $Z:Cob(d+1)\longrightarrow Vect$, where $Cob(d+1)$ is the category whose objects are closed, oriented d-manifolds M without boundary. The cobordism morphism $W:M\longrightarrow M'$ is a smooth, oriented, compact $d+1$-dimensional manifold with boundary $\partial W=M\sqcup -M'$. Two cobordisms W_1 and W_2 are equivalent if there is an orientation-preserving diffeomorphism $f:W_1\longrightarrow W_2$. *Vect* is a symmetric monoidal category of finite dimensional complex vector space where morphisms are linear maps $L:V_1\longrightarrow V_2$ with dual $L^*:V_2\longrightarrow V_1$.

This functor satisfies the following axioms [**2**]:

(1) For a cobordism W with boundary $\partial W=M_1\sqcup -M_2$, then $Z(W)=Z(M_1)\longrightarrow Z(M_2)$ is a homomorphism, *i.e.* a linear map of vector spaces.
(2) Z is *involutory*, that is, $Z(-M)=Z(M)^*$, where $-M$ denotes M with the opposite orientation and $Z(M)^*=\mathrm{Hom}(Z(M),\mathbb{C})$ is the dual vector space.
(3) Z is *multiplicative*, that is, $Z(M_1\sqcup M_2)=Z(M_1)\otimes Z(M_2)$.
(4) Z is *Associative*: for composite cobordisms (gluing) $W=W_1\cup_{M_2}W_2$ with $\partial W_1=M_1\sqcup -M_2$ and $\partial W_2=M_2\sqcup -M_3$ then $Z(W)=Z(W_1)\circ Z(W_2)\in Hom(Z(M_1),Z(M_3))$.
(5) $Z(\emptyset)=\mathbb{C}$.
(6) $Z(M\times I)$ is the identity endomorphism of $Z(M)$.

REMARKS 4.2.
- The identity endomorphism of $Z(M)$ in (6) and the functoriality of Z imply homotopy invariance.
- Let W be a closed (d+1)-dimensional manifold (with empty boundary). Then by (5) the vector $Z(W)$ is just a complex number. This means that a TQFT assigns a numerical invariant to each closed (d+1)-manifolds.
- Let $W=W_1\bigsqcup_{M_2}W_2$ as in (3) with $M_1=M_3=\emptyset$. Then one can cut W along a d-manifold M_2 and one obtains
$$Z(W)=\langle Z(W_1),Z(W_2)\rangle$$
where $\langle\cdot,\cdot\rangle$ denotes the pairing of the vector space $Z(M_2)$ with its dual $Z(M_2)^*=Z(-M_2)$.

Here one can give a physical explanation of the meaning to these axioms. In dimension 3 we can suppose that M is a physical space with an imaginary time $M \times I$ and a Hilbert space $Z(M)$ of the quantum theory associated to the Hamiltonian H with evolution operator e^{itH} (where t is the coordinate on the interval I). In axiom (6) the Hamiltonian H vanishes. Thus having a topological QFT implies that there is no real dynamics taking place along the cylinder $M \times I$. Notice that, for a manifold W with $\partial W = \overline{M}_1 \cup M_2$, there can still be an interesting propagation from M_1 to M_2 and this reflects the nontrivial topology of W.

Topological quantum field theories had many important applications in modern geometry, among these the work of Gromov [**26**] on pseudo-holomorphic curves in symplectic geometry. In TQFT (and in particular for example in [**26**]) a vector $Z(W)$ in the Hilbert space $Z(M)$ is called a *vacuum state* if $\partial W = M$ and for a closed manifold W the number $Z(W)$ is the *vacuum-vacuum* expectation value. In analogy with the statical mechanics it is also called the partition function.

5. 2-Category

In category theory, a 2-category is a small category $\mathcal{C}_2$ with "morphisms between morphisms". 2-categories are the first case of higher order categories and they are constructed as follows:

- $\mathcal{C}_2$ is defined as a small category enriched over Cat which is defined as a category of small categories and functors. Here we mean by enriched category a category whose $Hom - Sets$ are replaced by objects from some other category. More precisely, a 2-category consists of the following data.
- A class of objects $(A, B, ...) \in Cat$ called 0-cells.
- For all 0-cells A and B, we can define a set $\mathcal{C}_2(A,B)$ which is defined as a $Hom_{\mathcal{C}_2}(A,B)$ of objects $f : A \longrightarrow B$ which are called 1-cells.
- A morphism $\alpha : f_1 \longrightarrow f_2$ for any two morphisms f_1 and f_2 of $\mathcal{C}_2$. These 2-morphisms are called 2-cells.
- The 2-categorical compositions of 2-morphisms is denoted as $\bullet$ and is called *vertical composition*.
- For all objects A, B and C, there is a functor

  $$\circ : \mathcal{C}_2(A,B) \times \mathcal{C}_2(B,C) \longrightarrow \mathcal{C}_2(A,C)$$

  called *horizontal composition*, which is associative and admits the identity 2-cells I_A as identities.
- For any object A there is a functor from the terminal category (with one object and one arrow) to $\mathcal{C}_2(A,A)$.

The notion of 2-category differs from the more general notion of a bicategory in that composition of 1-morphisms is required to be strictly associative, whereas in a bicategory it needs only be associative up to a 2-isomorphism.

There are three different ways to obtain a category from a 2-category, all of which we use in Chapter 1. They are summarized as follows.

- *Forgetting 2-morphisms*: one is left with the category consisting of the objects and 1-morphisms of the 2-category.
- *Forgetting objects*: one obtains a category whose objects are the 1-morphisms of the 2-category and whose morphisms are the 2-morphisms of the 2-category.
- *Equivalence relation*: one uses the 2-morphisms to define an equivalence relation on the set of 1-morphisms and obtains in this way a category whose objects are the same as the objects of the 2-category and whose morphisms are the equivalence classes of 1-morphisms of the two category modulo the equivalence relation generated by the 2-morphisms.

6. Group Rings

A group ring is a ring $R[G]$ constructed from a ring R and a group G. As an R-module, the ring $R[G]$ is the free module over R generated by the elements of G, that is, the elements of the group ring are finite linear combinations of elements of G with coefficients in R,

$$R[G] = \{\sum_{g \in G} \alpha_g g | \alpha_g \in R\}$$

with all but finitely many of the α_g being 0.

The R-module $R[G]$ is a ring with addition of formal linear combinations

$$(\sum_{g \in G, a_g \in R} a_g g) + (\sum_{g \in G, b_g \in R} b_g g) = \sum_{g \in G} (a_g + b_g) g \tag{6.1}$$

and multiplication defined by the group operation in G extended by linearity and distributivity, and the requirement that elements of R commute with elements of G,

$$(\sum_{g \in G, a_g \in R} a_g g)(\sum_{h \in G, b_h \in R} b_h h) = \sum_{g,h \in G} (a_g b_h) gh. \tag{6.2}$$

If R has a unit element, then this is the unique bilinear multiplication for which $(1g)(1h) = (1gh)$. In this case, G can be identified with the elements $1g$ of $R[G]$. The identity element of G is the multiplicative unit in the ring $R[G]$. If R is commutative, then $R[G]$ is an associative algebra over R. If $R = F$ is a field, then $F[G]$ is an algebra, called the group algebra.

We have the following equivalent descriptions of the group ring

DEFINITION 6.1. [**51**] Let G be a group and R a ring. Define the set $R[G]$ to be one of the following equivalent statements:

- The set of all formal R-linear combinations of elements of G.
- The set of all functions $f : G \longrightarrow R$ with $f(g) = 0$ for all but finitely many $g \in G$.
- The free R-module with basis G.

The ring structure is given as above by (6.1) and (6.2).

If R and G are both commutative,*i.e.* R is a commutative ring and G is an abelian group, then $R[G]$ is commutative. If H is a subgroup of G, then $R[H]$ is a subring of $R[G]$. Similarly, if S is a subring of R, then $S[G]$ is a subring of $R[G]$.

6.1. Group algebra over a finite group. [51]

We recall briefly the example of group algebras for finite groups. These occur naturally in the theory of group representations of finite groups. As we have seen above, when R is a field F the group algebra $F[G]$ is a vector space over F, with a canonical basis e_g given by the elements $g \in G$ and with elements given by formal sums

$$v = \sum_{g \in G} x_g e_g$$

As we saw in general for group rings, the algebra structure is defined by the multiplication in the group,

$$e_g . e_h = e_{gh}$$

Thinking of the free vector space as F-valued functions on G, the algebra multiplication can be written equivalently as convolution of functions.

The group algebra is an algebra over itself; under the correspondence of representations over R and $R[G]$ modules, it is the regular representation of the group. Written as a representation, it is the representation $g \mapsto \rho_g$ with the action given by $\rho(g).e_h = egh$, or

$$\rho(g).r = \sum_{h\in G} k_h \rho(g).e_h = \sum_{h\in G} k_h.e_{gh}$$

For a finite group, the dimension of the vector space $F[G]$ is equal to the number of elements in the group. The field F is commonly taken to be the complex numbers $\mathbb{C}$ or the reals $\mathbb{R}$. The group algebra $\mathbb{C}[G]$ of a finite group over the complex numbers is a semisimple ring. This result, Maschke's theorem, allows us to understand $C[G]$ as a finite product of matrix rings with entries in $\mathbb{C}$.

6.2. Groupoids, semigroups, semigroupoids and their rings. In the first chapter of our work we introduced algebras that are generalizations of group rings. They are generalizations in two different senses. First of all one can pass from groups to groupoids and define the groupoid ring $R[\mathcal{G}]$. In a different direction one has generalizations where one replaces the group by a semigroup and have the corresponding semigroup ring $R[S]$. In our case, we work with a generalization of both of these concepts which is a semigroupoid $\mathcal{S}$ and the corresponding ring $R[\mathcal{S}]$. We recall here these different notions and stress the way in which they differ from one another and from the original notion of group ring recalled above.

6.3. Groupoid Ring. A groupoid $\mathfrak{G}$ is a small category in which each morphism is an isomorphism. Thus $\mathfrak{G}$ has a set of morphisms, which we call elements of $\mathfrak{G}$, a set Y of objects together with range (target) and source functions $r,s : \mathfrak{G} \longrightarrow Y$ such that, for $g_1, g_2 \in \mathfrak{G}$ with $r(g_1) = s(g_2)$, then the product or composite $g_2 g_1 = g_2 \circ g_1$ exists, with $s(g_2 g_1) = s(g_1)$ and $r(g_2 g_1) = r(g_2)$. The composition is associative. For $\gamma : Y \longrightarrow \mathfrak{G}$ and for an element $x \in Y$ the element $\gamma(x)$ is denoted by 1_x and it acts as the identity, and each element g has an inverse g^{-1} such that $s(g^{-1}) = r(g)$ and $r(g^{-1}) = s(g)$, with $g^{-1}g = \gamma(s(g))$ and $gg^{-1} = \gamma(r(g))$.

In a groupoid $\mathfrak{G}$, for y_1, y_2 we define the set $\mathfrak{G}(y_1, y_2)$ of all morphisms with initial point y_1 and final point y_2. We say that $\mathfrak{G}$ is transitive if, for all $y_1, y_2 \in Y$ the set $\mathfrak{G}(y_1, y_2)$ is non-empty. For $y \in Y$ we denote the set $\{g \in \mathfrak{G} : s(g) = y\}$ by $\mathfrak{G}_y$. Let $\mathfrak{G}$ be a groupoid. The transitive component of $x \in Y$, denoted by $C(\mathfrak{G})_y$, is the full subgroupoid of $\mathfrak{G}$ on those objects $x \in Y$ such that $\mathfrak{G}(y,x)$ is non-empty.

DEFINITION 6.2. Let $\mathfrak{G}$ be a groupoid and R a ring or a field. The groupoid ring (or groupoid algebra in the field case) $R[\mathfrak{G}]$ consists of all finite formal sums of the form $\sum_i^n r_i g_i$ where $r_i \in R$ and $g_i \in \mathfrak{G}$, which satisfy the following conditions.

(1) If $\sum_{i=1}^n r_i g_i = \sum_{i=1}^n s_i g_i$ then $r_i = s_i$, for $i = 1, 2, \dots, n$.

(2) $(\sum_{i=1}^n r_i g_i) + (\sum_{i=1}^n s_i g_i) = \sum_{i=1}^n (r_i + s_i) g_i$.

(3) $(\sum_{i=1}^n r_i g_i)(\sum_{i=1}^n s_i g_i) = (\sum_{i=1}^n k_i t_i)$ where $k_i = \sum_{i,j}^n r_i s_j$ and $t_i = g_i g_j$.

(4) $r_i g_i = g_i r_i$ for all $r_i \in R$ and $g_i \in \mathfrak{G}$.

(5) $r \sum_{i=1}^n r_i g_i = \sum_{i=1}^n r r_i g_i$, for $r, r_i \in R$.

Notice that since $1 \in R$ and $g_i \in \mathfrak{G}$, we have $\mathfrak{G} = 1.\mathfrak{G} \subset R[\mathfrak{G}]$ and $R \subset \mathfrak{G}$ if and only if $\mathfrak{G}$ has identity, otherwise $R \nsubseteq \mathfrak{G}$.

6.4. Semigroup Ring. We now similarly recall the notion of semigroup ring, which is another generalization of the concept of group ring recalled in §6 above.

The construction of the semigroup ring is not far from what we said before for the group ring. We try to illustrate the concept of semigroup from another perspective. Let $\mathcal{S}$ be a semigroup and let R

be a commutative ring. We define the semigroup ring $R[\mathcal{S}]$ to be the set of functions $f:\mathcal{S}\longrightarrow R$ that send all but finitely elements of $\mathcal{S}$ to zero,

$$f(s)=\sum_{m\in\mathcal{S}} a_m\delta_m(s),$$

where $a_m\in R$ and $\delta_m(s)=\delta_{m,s}$ is the Kronecker delta function, and all but finitely many of the coefficients are $a_m=0$. Clearly the set of such functions has the structure of an R-module if is a ring, or a vector space if R is a field. From the product in the semigroup $\mathcal{S}$ we can also define a product on the semigroup ring $R[\mathcal{S}]$ as follows. Let (x,y) be a pair of elements of $\mathcal{S}$ with $xy=s\in\mathcal{S}$ then we set

$$(f*g)(s)=\sum_{xy=s} f(x)g(y). \tag{6.3}$$

This is analogous to the way one defines the product in the group ring. In fact, it takes the product of all non-zero components of f and g and collects the resulting terms whose indices multiply to the same element of the semigroup. With this additive and multiplicative structure, one can check that, as in the case of groups, the set $R[\mathcal{S}]$ is in fact a ring (or an algebra if R is a field).

In this text we have assumed the convention that semigroups have a unit. However, the definition above makes sense also for the case where one does not require $\mathcal{S}$ to have a unit. In some text the semigroup ring of a semigroup with unit is called a *monoid ring*. It is then a unital ring with a unit given by the identity (unit) of the semigroup. If $\mathcal{S}$ is a group we recover the same definition of *group ring* discussed in §6 above. If in any of these cases we start with a commutative semigroup we get a commutative ring.

Notice that if $\mathcal{S}$ is a group, for the group ring $R[\mathcal{S}]$, since $xy=s$ only if $y=x^{-1}s$, we can rewrite the product formula (6.3) in the equivalent form

$$(f*g)(s)=\sum_{x\in\mathcal{S}} f(x)g(x^{-1}s).$$

This way of multiplying two functions on a group is called *convolution* product.

The notion of semigroupoid and semigroupoid ring is described in detail in Chapter 1. It is similar to the groupoid case, in as the compositions are only defined when the range of the first element agrees with the source of the second, and it is also similar to the semigroup case, in the sense that not all elements have an inverse. The notion of semigroupoid ring or algebra that we consider there is still a natural generalization of the notion of groupoid ring, as the ones we recalled in this appendix.

7. Creation and annihilation operators

The unitaries $U_k: f\mapsto (U_k f)(n)=f(n+k)$, for $k\in\mathbb{Z}$, acting on $\ell^2(\mathbb{Z})$, induce isometries

$$(S_n f)(k)=\begin{cases} f(k+n) & k+n\geq 0\\ 0 & k+n<0,\end{cases} \tag{7.1}$$

acting on the Hilbert space $\ell^2(\mathbb{N}\cup\{0\})=\ell^2(\mathbb{Z}/V)$ with $V=\{\pm 1\}$.

LEMMA 7.1. *The operators S_n of* (7.1)*, for $n\in\mathbb{Z}$, satisfy the relations $S_n^*=S_{-n}$ and*

$$S_n^*S_n=P_n \quad \text{and} \quad S_nS_n^*=P_{-n} \tag{7.2}$$

with P_n the projection $P_n f(k)=f(k)\chi_{[n,\infty)}(k)$, which is the identity for $n<0$. The operators S_n also satisfy the relations

$$S_nS_m=P_{-n}S_{n+m}. \tag{7.3}$$

PROOF. First notice that the S_k satisfy

$$(S_n^* f)(k) = \begin{cases} f(k-n) & k-n \geq 0 \\ 0 & k-n < 0. \end{cases} \tag{7.4}$$

In fact, we have

$$\langle S_n f, \psi \rangle = \sum_{k \in \mathbb{N} \cup \{0\}} f(k+n) \chi_{[0,\infty)}(k+n) \psi(k)$$

$$= \sum_{u \in \mathbb{Z}} f(u) \chi_{[0,\infty)}(u) \chi_{[0,\infty)}(u-n) \psi(u-n) = \sum_{u \in \mathbb{N} \cup \{0\}} f(u) \chi_{[0,\infty)}(u-n) \psi(u-n).$$

Thus, we have $S_n^* = S_{-n}$. we then have

$$S_n^* S_n f(u) = \chi_{[n,\infty)}(k) f(k) = P_n f(k)$$

and

$$S_n S_n^* f(k) = \chi_{[-n,\infty)}(k) f(k) = P_{-n} f(k).$$

This is in fact a particular case of the following relations. The relation $U_n U_m = U_{n+m}$ satisfied by the unitaries acting on $\ell^2(\mathbb{Z})$ descends to the relation (7.3) between the isometries S_n acting on $\ell^2(\mathbb{N} \cup \{0\})$. In fact, we have

$$S_n P_m f(k) = \chi_{[-n,\infty)}(k) \chi_{[m-n,\infty)}(k) f(k+n) = P_{m-n} S_n f(k)$$

where

$$P_m S_n f(k) = \chi_{[-n,\infty)}(k) \chi_{[m,\infty)}(k)\, f(k+n).$$

Thus, we see that $S_n S_m = P_{-n} S_{n+m}$, since

$$(S_n S_m f)(k) = \chi_{[-n,\infty)}(k) \chi_{[-(n+m),\infty)}(k) f(k+m+n) = (P_{-n} S_{n+m}\, f)(u).$$

□

Thus, we see that, even in the case of a commutative group like $\mathbb{Z}$, where the algebra of the U_m is commutative, we obtain a noncommutative algebra of isometries S_m,

$$S_m S_n = P_{-m} S_{n+m} \neq P_{-n} S_{n+m} = S_n S_m.$$

Notice however that, if n and m are both positive, then $P_{-m} = 1 = P_{-n}$ so that $S_n S_m = S_m S_n = S_{n+m}$. Notice also that the fact that the algebra generated by the isometries S_n is associative follows from the fact that the projections P_n commute among themselves, as they are given by multiplication operators by the characteristic functions $\chi_{[n,\infty)}$. In fact, we have

$$S_n(S_m S_k) = S_n P_{-m} S_{m+k} = P_{-m-n} S_n S_{m+k} = P_{-m-n} P_{-n} S_{n+m+k}$$

$$(S_n S_m) S_k = P_{-n} S_{n+m} S_k = P_{-n} P_{-(n+m)} S_{n+m+k},$$

with $P_{-(n+m)} P_{-n} = P_{-n} P_{-(n+m)}$. For $n > 0$, we also have $S_{-1}^n = P_1 S_{-2} S_{-1}^{n-1} = P_1 \cdots P_{n-1} S_{-n} = P_{n-1} S_{-n} = S_{-n}$, since $P_{n-1} S_{-n} f(k) = \chi_{[n-1,\infty)}(k) \chi_{[n,\infty)}(k) f(k-n) = \chi_{[n,\infty)}(k) f(k-n)$. Similarly, $S_1^n = P_{-1} S_2 S_1^{n-1} = P_{-1} \cdots P_{-n+1} S_n = S_n$ since $P_{-n+k} = 1$, for $k = 0, \ldots, n-1$.

Clearly, the algebra of the S_n we described here is generated by a single isometry S_{-1}, which is the isometry that describes the "phase" part of the creation operator in quantum mechanics, see [**22**]. In fact, recall that the creation and annihilation operators $a^\dagger$ and a, with $a^* = a^\dagger$, act on $\ell^2(\mathbb{N} \cup \{0\})$ by

$$a^\dagger e_n = \sqrt{n+1}\, e_{n+1} \quad \text{and} \quad a\, e_n = \sqrt{n}\, e_{n-1}, \tag{7.5}$$

with the commutation relation $[a^\dagger, a] = 1$. It is well known that, while the operators $a^\dagger$ and a do not have a polar decomposition in terms of a unitary and a self-adjoint operator, they have a decomposition in terms of an isometry and a self-adjoint operator in the form

$$a^\dagger = N^{1/2} S_{-1} \quad \text{and} \quad a = S_1 N^{1/2}, \tag{7.6}$$

where $N e_n = n e_n$ is the grading operator on $\ell^2(\mathbb{N} \cup \{0\})$ and S_1 and S_{-1} are the isometries described above, $S_1 e_n = e_{n-1}$ and $S_{-1} e_n = e_{n+1}$. Notice that the grading operator N acting on $\ell^2(\mathbb{N} \cup \{0\})$ defines a 1^+-summable self-adjoint operator with compact resolvent and with the property that the commutators with the operators S_n are bounded. Namely, one has the commutation relation

$$[N, S_n] = -n S_n. \tag{7.7}$$

The Hamiltonian associated to the creation and annihilation operators is of the form (see [**22**])

$$H = a^\dagger a, \quad \text{with} \quad \mathrm{Spec}(H) = \mathbb{N} \cup \{0\}. \tag{7.8}$$

The corresponding partition function at inverse temperature $\beta > 0$ is of the form

$$Z(\beta) = \mathrm{Tr}(e^{-\beta H}) = \sum_{n \geq 0} \exp(-\beta n) = (1 - \exp(-\beta))^{-1}. \tag{7.9}$$

8. A quick introduction to Dirac operators

8.1. Clifford Algebra. Let $V \simeq \mathbb{R}^n$ be a vector space with non degenerate symmetric bilinear form g. Over a field of characteristic different than 2, such a bilinear form can always be determined by the corresponding *quadric form* q defined as $q(v) = g(v, v)$, by setting $2g(u, v) = q(u, v) - q(u) - q(v)$.

DEFINITION 8.1. The Clifford Algebra $CL(V, g)$ is an algebra over $\mathbb{R}$ generated by the vectors $v \in V$, subject to the relation $uv + vu = 2g(u, v)$ for all $u, v \in V$.

8.1.1. *concepts in Riemannian geometry.* Let M be a compact smooth n-dimensional manifold without boundary. Define a Riemannian metric on M to be a symmetric bilinear form

$$g : \mathfrak{F}(M) \times \mathfrak{F}(M) \longrightarrow C(M),$$

where $\mathfrak{F}(M) = \Gamma(M, T_{\mathbb{C}} M)$ is the space of continuous vector fields on M, and $C(M)$ is the commutative C^*-algebra of continuous functions on M. Then g satisfies the following properties.

(1) $g(X, Y)$ is a real function if X, Y are real vector fields.
(2) g is $C(M)$-bilinear *i.e.* $g(fX, Y) = g(X, fY) = fg(X, Y)$ for all $f \in C(M)$. In this condition g is given by a continuous family of symmetric bilinear map $g_x : T_x M \times T_x M \longrightarrow R$, where g_x is positive definite.
(3) $g(X, X) \geq 0$ for X real, with $g(X, X) = 0$ only if $X = 0$ in $\mathfrak{F}(M)$.

8.1.2. *Dirac Operator.* Let (M, g) be a smooth compact Riemannian m-manifold without boundary with a Clifford algebra bundle $CL(M)$. A Clifford module is a module over $CL(M)$. Any Clifford module Λ that is finitely generated and projective is of the form $\Lambda = \Gamma(M, E)$ for $E \longrightarrow M$ a complex vector bundle. For $E \longrightarrow M$ a smooth complex vector bundle of Clifford modules $\Lambda = \Gamma(M, E)$, we can define the Clifford multiplication which is a bundle map $c : CL(M) \longrightarrow Hom(E, E)$ which is given fiberwise by maps $c : CL(T_x M, g_x) \longrightarrow Hom_{\mathbb{C}}(E_x, E_x)$.

Any choice of a smooth connection

$$\nabla : C^\infty(M, E) \longrightarrow C^\infty(M, T^* M \otimes E)$$

defines an operator of *Dirac type* by setting $\mathcal{D} = c \circ \nabla$. We use here the identification of tangent and cotangent bundle $TM \cong T^* M$ induced by the Riemannian metric.

Consider a small open chart domain $U \subset M$, where the cotangent bundle is trivial, *i.e.* $T^*M\,|_U \simeq U \times \mathbb{R}^n$. For any local coordinate $(x_1, x_2, ..., x_n)$ over a chart domain U, the local coordinate of the cotangent bundle $T^*M\,|_U$ are $(x, \xi) = (x_1, x_2, ..., x_n, \xi_1, \xi_2, ..., \xi_n)$ where $\xi \in T_x^*M$. A differential operator acting on smooth local sections $f \in \Gamma(U, E)$ is an operator P of the form

$$P = \sum_{|\alpha| \leq d} a_\alpha(x) D^\alpha$$

with $a_\alpha \in \Gamma(U, EndE)$, and where $D^\alpha = D_1^{\alpha_1}.D_2^{\alpha_2}.D_3^{\alpha_3}...D_n^{\alpha_n}$. and $D_j = -i\frac{\partial}{\partial x_j}$, with a positive integer d representing the order of P. Let $E \longrightarrow M$ be a vector bundle of rank r. By Fourier transform, we can write for $f \in C^\infty(U, \mathbb{R}^n)$:

$$\begin{aligned} Pf(x) &= (2\pi)^n \int_{\mathbb{R}^n} \exp^{ix\xi} p(x,\xi) \hat{f}(\xi) d^n\xi \\ &= (2\pi)^n \int \int_{\mathbb{R}^{2n}} \exp^{i(x-y)\xi} p(x,\xi) f(\xi) d^n y d^n \xi, \end{aligned} \tag{8.1}$$

where $p(x,\xi)$ is a polynomial of order d in the ξ-variable, called the *complete symbol* of the operator P. Then we can isolate the homogeneous part

$$p(x,\xi) = \sum_{j=0}^{d} p_{d-j}(x,\xi)$$

where $p_{d-j}(x, t\xi) = t^{d-j} p_{d-j}(x,\xi)$ for $t > 0$.

DEFINITION 8.2. An element p is called a *classical symbol* if we can find a sequence of terms $p_d(x,\xi), p_{d-1}(x,\xi), ...$ with

$$p(x,\xi) \sim \sum_{j>0}^{\infty} p_{d-j}(x,\xi)$$

such that $p_{d-j}(x, t\xi) = t^{d-j} p_{d-j}(x,\xi)$ for $t > 0$.

DEFINITION 8.3. A classical *pseudo-differential operator* of order d over the chart domain $U \subset \mathbb{R}^n$ is an operator P defined by 8.1, for which $p(x,\xi)$ is a classical symbol, whose leading term $p_d(x,\xi)$ does not vanish. This leading term is called the *principal symbol* of P, and we also denote it by $\sigma(P)(x,\xi) = p_d(x,\xi)$.

For an operator of Dirac type, which is a first-order differential operator on $\Gamma(M,E)$, we get $\sigma(\mathcal{D}) \in \Gamma(T^*M, \pi^*(EndE))$ and for the property of $p(x,\xi) = c(dx^j)(\xi_j - i\omega_j(x))$ we get

$$\sigma(\mathcal{D})(x,\xi) = c(\xi_j dx^j) = c(\xi)$$

and

$$\sigma(\mathcal{D}^2)(x,\xi) = (\sigma(\mathcal{D})(x,\xi))^2 = (c(\xi))^2 = g(\xi,\xi)$$

Notice that $\sigma(\mathcal{D}^2)$ only vanishes when $\xi = 0$, that is, on the zero section of T^*M.

DEFINITION 8.4. Let P a classical pseudo-differential operator, then P is called *elliptic* if $\sigma(P)(x,\xi)$ is invertible when $\xi \neq 0$.

An operators $\mathcal{D}$ of Dirac type is elliptic and so is its square $\mathcal{D}^2$. On a compact manifold without boundary this implies that it is Fredholm (has finite dimensional kernel and cokernel), hence its index $Ind(\mathcal{D}) = \dim Ker(\mathcal{D}) - \dim Coker(\mathcal{D})$ is well defined. The Atiyah-Singer index theorem gives a local formula, in terms of integration of a differential form, for the index. On a manifold with boundary, the Fredholm property depends on the choice of boundary conditions. With the Atiyah-Patodi-Singer boundary conditions one still has a Fredholm operator and an index formula, now with an additional term that is an eta invariant for the operator restricted to the boundary manifold.

9. Concepts of Cyclic Cohomology

Cyclic cohomology of non-commutative algebras is playing in non-commutative geometry a similar role to that of de Rham cohomology in differential topology [**14**]. The first appearance of the Cyclic cohomology was in the cohomology theory for algebras. The cyclic cohomology $HC^*(\mathcal{A})$ of an algebra $\mathcal{A}$ over $\mathbb{R}$ or $\mathbb{C}$ is the cochain complex $\{C^*_\lambda(\mathcal{A}), b\}$, where $C^*_\lambda(\mathcal{A}), n \geq 0$ consists of the $(n+1)$-linear forms ϑ on $\mathcal{A}$ satisfying the cyclicity condition[**19**]

$$\vartheta(a^0, a^1, \ldots a^n) = (-1)^n \vartheta(a^1, a^2, \ldots, a^0) \tag{9.1}$$

where $a^0, a^1, \ldots, a^n \in \mathcal{A}$ and the coboundary operator is given by

$$(b\vartheta)(a^0, a^1, \ldots, a^n) = \sum_{j=0}^{n} (-1)^j (a^0, \ldots, a^j a^{j+1}, \ldots, a^{n+1}) + (-1)^{n+1} \vartheta(a^{n+1} a^0, \ldots, a^n)$$

$C^*_\lambda(\mathcal{A})$ then consists of all continuous $(n+1)$-linear forms on $\mathcal{A}$ satisfying 9.1. Cyclic cohomology provides numerical invariants of K-theory classes as follows. For an even integer n, given an n-dimensional cyclic cocycle ϑ on $\mathcal{A}$, then the scalar

$$\vartheta \otimes Tr(E, E, \ldots, E) \tag{9.2}$$

is invariant [**19**] under homotopy, for an idempotent

$$E^2 = E \in M_N(\mathcal{A}) = \mathcal{A} \otimes M_N(\mathbb{C})$$

This gives the pairing $\langle [\vartheta], [E] \rangle$ between cyclic homology and K-theory. For a manifold M let $\mathcal{A} = C^\infty(M)$ with

$$\vartheta(f^0, f^1, \ldots, f^n) = \langle \Omega, f^0 df^1 \wedge df^2 \wedge \ldots \wedge df^n \rangle$$

where $f^1, f^2, \ldots, f^n \in \mathcal{A}$ and Ω is a closed n-dimensional de Rham form on M. Then the invariant 9.2 up to normalization is equal to $\langle \Omega, ch^*(\tau) \rangle$ where $ch^*(\tau)$ denotes the Chern character of the rank N vector bundle τ on M whose fiber at $x \in M$ is the range of $E(x) \in M_N(\mathbb{C})$. To any algebra $\mathcal{A}$ one can associate a module $\mathcal{A}^\natural$ over the cyclic category by assigning to each integer $n \geq 0$ the vector space $C^n(\mathcal{A})$ of $(n+1)$-linear forms $\vartheta(a^0, a^1, \ldots, a^n)$ on $\mathcal{A}$ and to the generating morphisms the operators $\delta_i : C^{n-1} \longrightarrow C^n$ and $\sigma_i : C^{n+1} \longrightarrow C^n$ defined above. One thus [**19**], obtains the desired interpretation of the cyclic cohomology group of $\mathcal{K}$-algebra $\mathcal{A}$ over a ground ring $\mathcal{K}$ in terms of derived functors over the cyclic category

$$HC^n(\mathcal{A}) \simeq Ext^n_\Lambda(\mathcal{K}^\natural, \mathcal{A}^\natural)$$

and

$$HC_n(\mathcal{A}) \simeq Tor^\Lambda_n(\mathcal{A}^\natural, \mathcal{K}^\natural)$$

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