Inverse spectral theory and relative determinants of elliptic operators on surfaces with cusps

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Summary

This thesis concerns relative determinants for Laplacians on surfaces with asymptotically cusps ends and the inverse spectral problem on surfaces with cusps. We consider (M, g), a surface with cusps, and a metric on the surface that is a conformal transformation of the initial metric $h = e^{2\varphi}g$.

In the first part we find conditions φ that make it possible to define the relative determinant of the pair (Δ_h, Δ_g). We prove Polyakov's formula for the relative determinant and study the extremal values of this determinant as a function of unit area metrics inside a conformal class. We prove that if the maximum exists it has to be attained at the metric of constant curvature. We discuss necessary conditions for the existence of a maximizer.

In the second part we restrict our attention to hyperbolic surfaces of fixed genus and a fixed number of cusps. We study the relative determinant as a function on the moduli space for this kind of surfaces and use the results in [19] to prove that it tends to zero at the boundary of the moduli space.

In the third part we return to general surfaces with cusps. We prove a splitting formula for the relative determinant and use it to prove compactness in the C^∞-topology of sets of isospectral metrics in a given conformal class. We assume that the conformal factors φ have support in a fixed compact set of M.

Contents

Introduction

In this thesis we study the relative (regularized) determinant of the Laplace operator on surfaces with cusps. Regularized determinants of elliptic operators play an important role in many fields of mathematics and mathematical physics. They were initially introduced by D. B. Ray and M. I. Singer in [41] in relation to R-torsion. Let A be a self-adjoint non-negative elliptic pseudo-differential operator of order m on a compact Riemannian manifold of dimension n. Then A has pure point spectrum consisting of a sequence of eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ of finite multiplicities. The regularized determinant of A is defined through the zeta function

$$\zeta(s) = \sum_{\lambda_j \neq 0} \lambda_j^{-s}, \quad \mathrm{Re}(s) > n/m.$$

After Seeley [38] it is well known that the zeta function admits a meromorphic extension to the complete complex plane that is regular at $s = 0$. Then the regularized determinant is defined as:

$$\det(A) = \exp\left(-\frac{d}{ds}\zeta(s)\bigg|_{s=0} \right).$$

The zeta function can also be expressed in terms of the heat semigroup associated to A:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty (\mathrm{Tr}(e^{-tA}) - \dim \mathrm{Ker}(A)) t^{s-1} dt.$$

This formula is only valid in the half-plane $\mathrm{Re}(s) > n/m$. In section 1.8 we explain in detail how to derive this formula when A is the Laplacian on a closed manifold.

The regularized determinant of the Laplacian on a compact Riemannian manifold is an important spectral invariant. For instance, in the 2-dimensional case, Osgood, Phillips and Sarnak (to whom we refer as OPS from now on) in [33] showed that the determinant, considered as a functional on the space of metrics, has very interesting extremal properties. They proved the following result: let M be a closed surface of genus p. Then in a given conformal class, among all metrics of unit area, there exists a unique metric of constant curvature at which the regularized determinant attains a maximum. They also proved a corresponding statement for compact surfaces with boundary and suitable conditions at the boundary.

This determinant can also be restricted to a function on the moduli space of hyperbolic metrics on a closed surface of genus p, $\mathcal{M}_p$:

$$\det \Delta_\cdot : \mathcal{M}_p \to \mathbb{R}, \ [\tau] \mapsto \det \Delta_\tau,$$

where τ is a metric of constant curvature and unit area. Sarnak in [37] made the very interesting conjecture that the function that assigns to an isometry class $\hat{g}$ its "height", given by $-\log \det \Delta_{\hat{g}}$, has a unique global minimum. If this conjecture is true, the global minimum could be taken as a "distinguished metric" on the surface.

The regularized determinant of the Laplacian can be used to study inverse spectral problems such as isospectral problems. Isospectral problems go back to 1960 when Leon Green asked if a Riemannian manifold was determined by its spectrum. The question was rephrased by Kac for planar domains in the very suggestive way: "Can one hear the shape of a drum?" see [20]. An important result is the well known existence of non-isometric manifolds that are isospectral, see [40] and the references therein. We also refer to [46] for a comprehensive survey of inverse spectral problems in geometry.

The isospectral problem on closed surfaces and simply connected planar domains was studied by OPS in [34]. In that paper the authors prove compactness of isospectral sets of isometry classes of metrics in the corresponding C^∞-topology. Two metrics g_1 and g_2 are called isospectral if the spectrum of the Laplacians Δ_{g_1} and Δ_{g_2} are the same including multiplicities. In particular the heat invariants a_j for $j \geq 0$ and the determinant $\det \Delta$ have the same values at g_1 and g_2. As the authors remark in the paper, the use of the regularized determinant of the Laplacian is essential in order to obtain compactness, since the heat invariants are not enough. On planar domains the problem has been studied by R. Melrose in [27] and OPS in [35], and for compact surfaces with boundary by Y. Kim in [22].

The isospectral problem also makes sense for certain non-compact manifolds. There scattering theory comes into play and we need to deal with inverse scattering theory. For example, on exterior planar domains the isospectral problem was studied by A. Hassell and S. Zelditch in [18]. There two exterior planar domains are called isophasal if they have the same scattering phase. Hassell and Zelditch prove that each class of isophasal exterior planar domains is sequentially compact in the C^∞-topology. In the proof they define a regularized determinant of the Laplacian that plays a fundamental role.

The goal of this thesis is to study how to extend the results of OPS in [33] and [34] to surfaces with cusps. As suggested by the work of Hassell and Zelditch, it is important to find a good definition of the determinant of the Laplace operator.

In the first part we study the definition of the determinant for surfaces with cusps and asymptotically cusps ends and its possible extremal values. Let (M, g) be a surface with cusps. This means that it is a smooth 2-dimensional Riemannian manifold of finite area such that outside a compact set the metric is hyperbolic. The hyperbolic ends are called cusps. The first thing to do is to define the determinant of the Laplacian Δ_g on M. It is well known that Δ_g has continuous spectrum therefore its zeta regularized determinant can not be defined. To solve this problem we use relative determinants. The relative determinant of a pair of non-negative self-adjoint operators (A, B) in a Hilbert space was introduced by W. Müller in [30]. If the operators (A, B) satisfy certain given conditions the relative determinant can be defined through a zeta function using the trace of $e^{-tA} - e^{-tB}$.

We start by fixing a class of metrics on M that are conformal to g and that satisfy suitable conditions. For any metric $h = e^{2\varphi} g$ in the conformal class $[g]$ we obtain a Laplacian which we denote by Δ_h. We will consider relative determinants of pairs (Δ_h, Δ_g) and also of pairs $(\Delta_h, \bar{\Delta}_{\beta,0})$, where $\bar{\Delta}_{\beta,0}$ is an operator over M that is associated to the cusps. We have to take into account the following technical detail: the operator Δ_h acting on $L^2(M, dA_g)$ is not self-adjoint. We therefore consider a unitary map $T : L^2(M, dA_g) \to L^2(M, dA_h)$ and the corresponding transformed operators. The map T then appears in all the corresponding statements and proofs.

For the relative determinant $\det(\Delta_h, \Delta_g)$ to make sense, the relative heat operator $T^{-1}e^{-t\Delta_h}T - e^{t\Delta_g}$ must be trace class for $t > 0$ and the trace must have suitable asymptotic expansions for large and small values of t. This is the case if the conformal factor φ has a specific decay at infinity. Chapters 2 and 3 are devoted to finding these decay conditions and to proving the properties required above. The first main result is:

Theorem 2.3 *Let $h = e^{2\varphi}g$ and let $i(z)$ be a function on M satisfying $i(z) = 1$ if $z \in M_0$, and $i(z) = y_j$ if $z = (x_j, y_j) \in Z_j$, for $j = 1, \ldots, m$. If $\varphi(z)$ and $\Delta_g\varphi(z)$ are $O(i(z)^{-1})$ as $i(z) \to \infty$, then $T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_g}$ is a trace class operator for all $t > 0$.*

Theorem 2.3 implies that the continuous spectra of the Laplacians Δ_h and Δ_g coincide. Since $\sigma_c(\Delta_g) = [1/4, \infty)$, it follows from [30, Lemma 2.2] that there exists a constant $\kappa > 0$ such that the relative trace has the following asymptotic expansion as $t \to \infty$:

$$\mathrm{Tr}(T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_g}) = O(e^{-\kappa t}). \tag{1}$$

The second main result is about the asymptotic expansion of the relative heat trace as $t \to 0$:

Theorem 3.4 *Let us use the notation of Theorem 1. If the functions $\varphi(z)$ and $\Delta_g\varphi(z)$ are $O(i(z)^{-32})$ as $i(z) \to \infty$, then there exists an expansion up to order two in t of* $\mathrm{Tr}(T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_g})$ *as $t \to 0$.*

In the proof of Theorem 3.4 we use mainly classical methods such as parametrices, Duhamel's principle, upper bounds of heat kernels and covering spaces.

We are now ready to define $\det(\Delta_h, \Delta_g)$ using the relative zeta function. By Theorem 3.4 and equation (1) we may define:

$$\zeta(s; \Delta_h, \Delta_g) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}(T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_g})t^{s-1}\, dt,$$

that converges in the a half plane $\mathrm{Re}(s) \geq 1$. This formula for $\zeta(s; \Delta_h, \Delta_g)$ is analogous to the formula in the compact case that expresses the zeta function in terms of the trace of the heat operator. The asymptotic expansions for the trace described above ensure that the relative zeta function has a meromorphic continuation to $\mathbb{C}$ that is analytic at $s = 0$. The relative determinant is then defined by:

$$\det(\Delta_h, \Delta_g) = \exp\left(-\frac{d}{ds}\zeta(s; \Delta_h, \Delta_g)\bigg|_{s=0} \right).$$

Although Theorem 3.4 allows us to define the relative determinant $\det(\Delta_h, \Delta_g)$; the result is not optimal since we would like to obtain a complete asymptotic expansion of the relative heat trace requiring lower decay. It seems possible to improve the statement using methods borrowed from Melroses' b-calculus but that will be part of another project.

In Chapter 4 we define the relative determinant for the pair $(\Delta_h, \Delta_{1,0})$. We study it as a functional on the space of metrics of a given fixed area inside the conformal class and look for its extremal values. The main result of Chapter 4 is a Polyakov-type formula for $\det(\Delta_h, \Delta_{1,0})$:

Theorem 4.5 Let (M, g) be a surface with cusps and let $h = e^{2\varphi}g$ be a conformal transformation of g with $\varphi(z)$ and $\Delta_g\varphi(z)$ being $O(i(z)^{-32})$ as $y = i(z) \to \infty$. For the corresponding relative determinants we have the following formula:

$$\log\det(\Delta_h, \Delta_{1,0}) = -\frac{1}{12\pi}\int_M |\nabla_g\varphi|^2\, dA_g - \frac{1}{6\pi}\int_M K_g\,\varphi\, dA_g + \log A_h + \log\det(\Delta_g, \Delta_{1,0}).$$

The proof of this formula follows the same lines as the proof in the compact case in [33]. The formula is the same as the one obtained by R. Lundelius in [26] for heights of pairs of admissible surfaces. Let us point out however that our methods are different from the ones in [26]. As in [33] and [26], we see that if there exists a maximum it is attained at the metric of constant curvature. The equation relating the curvature of the metrics g and $h = e^{2\varphi}g$ is $K_h = e^{-2\varphi}(\Delta_g\varphi + K_g)$. The study of this differential equation for φ together with the conditions of constant curvature in the cusps for g and constant curvature everywhere for h leads to a precise decay for the function φ at infinity. Unfortunately this decay is not included in the conditions required to define the determinant. Therefore the metric of constant curvature will not be in the conformal class that we consider unless we start with a metric of constant curvature.

In Chapter 5 we study the relative determinant as a function on the moduli space of hyperbolic surfaces with cusps. We work in $\mathcal{M}_{p,m}$, the moduli space of compact Riemann surfaces of genus p with m punctures and think of it as a space of complete hyperbolic metrics on a topological surface of genus p with m punctures. We define the free Laplacian as the Laplacian $\bar{\Delta}_{1,0}$ associated to the union of m cusps all starting at 1; notice that the Laplacian $\bar{\Delta}_{1,0}$ is chosen independently of $[g]$. Hence the relative determinant defines a function on the moduli space in the same way as in the compact case: $[g] \in \mathcal{M}_{p,m} \mapsto \det(\Delta_g, \bar{\Delta}_{1,0}) \in \mathbb{R}^+$, where $g \in [g]$ is hyperbolic. We start by analyzing the behavior of $\det(\Delta_g, \bar{\Delta}_{1,0})$ as $[g]$ approaches the boundary of $\mathcal{M}_{p,m}$. It is well known that each point of the boundary can be reached through a degenerating family of metrics. The degeneration arises from closed geodesics whose length converges to zero. Comparing the relative determinant with the determinant defined in [19] we prove the following theorem:

Theorem 5.4 Let $\mathcal{M}_{p,m}$ be the moduli space of hyperbolic surfaces with cusps. Consider the relative determinant $\det(\Delta_g, \bar{\Delta}_{1,0})$ as a function on $\mathcal{M}_{p,m}$. Then $\det(\Delta_g, \bar{\Delta}_{1,0})$ tends to zero if $[g]$ approaches $\overline{\mathcal{M}}_{m,p} \setminus \mathcal{M}_{p,m}$, the boundary of the moduli space.

This will imply that $\det(\Delta_g, \bar{\Delta}_{1,0})$ attains its maximum in the interior of the moduli space. The determinant as a function of the moduli space of hyperbolic surfaces of finite area was studied in [26] in terms of the heights of a degenerating family of hyperbolic surfaces and a fixed surface. There the author proves an asymptotic formula for the degeneration of the height. The regularized determinant as a function of the moduli space of metrics on surfaces with smooth boundary was studied by H. H. Khuri in [23] and by Y-H. Kim [22], obtaining different results under different conditions.

Finally, we study the isospectral problem inside a conformal class of the metric in a surface with cusps. In this setting, two metrics are isospectral if the resonances are the same for both metrics including multiplicities. For hyperbolic surfaces of finite area, W. Müller proved in [29] that the resonance set associated to the surface determines the surface up to finitely many possibilities. We restrict our attention to metrics inside a given conformal class. The main result of this part is the following theorem:

Theorem 7.5 *Let (M, g) be a surface with cusps, $K \subset M$ be compact and let $[g]_K = \{e^{2\varphi} g \mid \varphi \in C_c^\infty(M),\ \operatorname{supp}(\varphi) \subset K\}$ be the K-compactly supported conformal class of g. Then isospectral sets in $[g]_K$ are compact in the C^∞-topology.*

In the proof of Theorem 7.5 we use a splitting formula for the relative determinant. This formula relates $\det(\Delta_g, \Delta_{\beta,0})$ (with β big enough) to the determinant of the Dirichlet-to-Neumann operator acting on a submanifold of M homeomorphic to S^1. We prove this formula in Chapter 6. We finally discuss the possibility of generalizing Theorem 7.5 to conformal classes including metrics that have asymptotically cusp-ends.

In Chapter 1 we introduce notation and most of the background theory we use throughout the document. We include two appendices. Appendix A about Sobolev spaces and Appendix B with an explicit computation of the spectral shift function of the pair of operators $(\Delta_g, \Delta_{1,0})$.

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To my parents, Clara Inés and Camilo

Chapter 1

Background theory

In this chapter we introduce the notation, definitions and main results we use throughout this thesis. We start by defining the spaces over which we work and the operators in which we are interested. In the second section we state the main known results about spectral theory of surfaces with cusps. In the other sections we define conformal transformations of metrics, the injectivity radius of a Riemannian manifold, we state the Gauss-Bonnet formula for surfaces with cusps. We dedicate a section to summarizing some results we will use about heat kernels and their estimates. In the last section we give the definition of the regularized determinant of the Laplace operator on a closed manifold.

1.1 Notation and some definitions

A surface with cusps is a 2-dimensional Riemannian manifold that is complete, non-compact, has finite volume and is hyperbolic in the complement of a compact set. It admits a decomposition of the form

$$M = M_0 \cup Z_1 \cup \cdots \cup Z_m,$$

where M_0 is a compact surface with smooth boundary and for each $i = 1, ..., m$ we assume that

$$Z_i \cong [a_i, \infty) \times S^1, \quad g|_{Z_i} = y_i^{-2}(dy_i^2 + dx_i^2), \quad a_i > 0$$

The subsets Z_i are called cusps. Sometimes we denote Z_i by Z_{a_i} to indicate the "starting point" a_i. Instances of surfaces with cusps are quotients of the form $\Gamma(N)\backslash\mathbb{H}$, where $\mathbb{H}$ is the upper half plane and $\Gamma(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup, i.e. $\Gamma(N) = \{\gamma \in \mathrm{SL}_2(\mathbb{Z}) | \gamma \equiv \mathrm{Id} \pmod N\}$. These quotients play an important role in the theory of automorphic forms.

To any surface with cusps (M, g) we can associate a compact surface $\overline{M}$ such that (M, g) is diffeomorphic to the complement of m points in $\overline{M}$. Let p denote the genus of the compact surface $\overline{M}$; then the pair (p, m) is called the conformal type of M. In many of the proofs we set $m = 1$ and $a_1 = 1$, to simplify the equations.

For any oriented Riemannian manifold (M, g) the Laplace-Beltrami operator on functions is defined as $\Delta f = -\operatorname{div}\operatorname{grad} f$. It is equal to $\Delta = d^*d$. Note that we consider positive Laplacians. In local coordinates the Laplacian has the form

$$\Delta f = -\frac{1}{\sqrt{\det(g_{ij})}}\, \partial_j(\sqrt{\det(g_{ij})}\; g^{ij}\partial_i f),$$

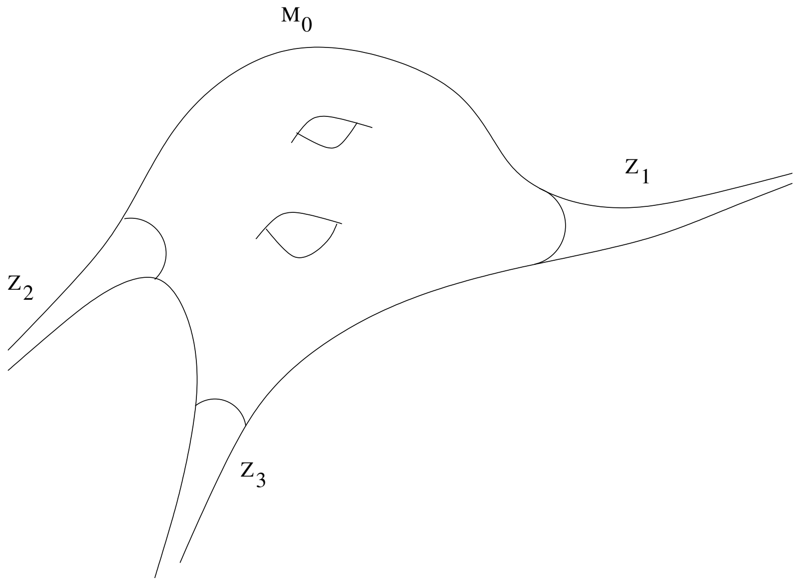

Figure 1.1: A surface with cusps

where $f \in C_c^\infty(M)$. On a cusp Z, the Laplacian is given by

$$\Delta_Z = -y^2 \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \right).$$

If (M, g) is complete, Δ has a unique closed extension that we denote by Δ_g. The gradient in coordinates is given by $\operatorname{grad} f = (g^{ij}\partial_j f)\partial_i$ and the Laplacian of the product of two functions f_1 and f_2 is given by

$$\Delta(f_1 f_2) = (\Delta f_1) f_2 + f_1(\Delta f_2) - 2\langle \operatorname{grad} f_1, \operatorname{grad} f_2 \rangle \tag{1.1}$$

Definition 1.1. *Let $a > 0$, let $\Delta_{a,0}$ denote the self-adjoint extension of the operator*

$$-y^2 \frac{\partial^2}{\partial y^2} : C_c^\infty((a, \infty)) \to L^2([a, \infty), y^{-2} dy)$$

obtained after imposing Dirichlet boundary conditions at $y = a$. The domain of $\Delta_{a,0}$ is then given by $\operatorname{Dom}(\Delta_{a,0}) = H_0^1([a, \infty)) \cap H^2([a, \infty))$, *where* $H_0^1([a, \infty)) = \{f \in H^1([a, \infty)) : f(a) = 0\}$.

Let $\bar{\Delta}_{a,0} = \oplus_{j=1}^m \Delta_{a_j,0}$ be defined as the direct sum of the self-adjoint operators operators $\Delta_{a_j,0}$ defined above. The operator $\bar{\Delta}_{a,0}$ acts on a subspace of $\oplus_{j=1}^m L^2([a_j, \infty), y_j^{-2} dy_j)$.

Now, let $a > 0$, let Z_a be endowed with the hyperbolic metric g and let $\Delta_{Z_a,D}$ be the self-adjoint extension of

$$-y^2 \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \right) : C_c^\infty((a, \infty) \times S^1) \to L^2(Z_a, dA_g)$$

obtained after imposing Dirichlet boundary conditions at $\{a\} \times S^1$. The operator $\Delta_{Z_a,D}$ can be decomposed as follows. Put

$$L_0^2(Z_a) = \{f \in L^2(Z_a, dA_g) | \int_{S^1} f(y, x) dx = 0 \text{ for a. e. } y \geq a\}. \tag{1.2}$$

The orthogonal complement of $L_0^2(Z_a)$ in $L^2(Z_a, dA_g)$ consists of functions that are independent of $x \in S^1$. Indeed, let $f \in L_0^2(Z_a)^\perp$, then $\int_{Z_a} f(y,x)\psi(y,x)dA_g(z) = 0$ for all $\psi \in L_0^2(Z_a)$, in particular for $\psi_n(x) = e^{-2\pi inx}$ with $n \neq 0$. This implies that in the Fourier decomposition of f, $f(y,x) = \sum_{n \in \mathbb{Z}_a} a_n(f,y)e^{2\pi inx}$, all the terms but the constant term are zero. Therefore $f(y,x) = a_0(f,y) = \int_{S^1} f(y,x)dx$, i.e. f is independent of x. The other inclusion is obvious. Then we can decompose $L^2(Z_a, dA_g)$ as the orthogonal direct sum

$$L^2(Z_a, dA_g) = L^2([a,\infty), y^{-2}dy) \oplus L_0^2(Z_a).$$

This decomposition is invariant under $\Delta_{Z_a,D}$ so in terms of this decomposition we can write $\Delta_{Z_a,D} = \Delta_{a,0} \oplus \Delta_{Z_a,1}$, where $\Delta_{Z_a,1}$ acts on $L_0^2(Z_a)$.

Remark 1.2. *The operator $\Delta_{Z_a,1}$ has compact resolvent; in particular it has only point spectrum, see Lemma 7.3 in [32]. In addition, the counting function for $\Delta_{Z_a,1}$, $N_{\Delta_{Z_a,1}}(\lambda) = \#\{\tilde{\lambda}_j \leq \lambda\}$, where $\{\tilde{\lambda}_j\}$ are the eigenvalues of $\Delta_{Z_a,1}$, satisfies $N_{\Delta_{Z_a,1}}(\lambda) \sim \frac{\lambda}{4\pi} A_g$. See [12, Thm.6]. This implies that the heat operator $e^{-t\Delta_{Z_a,1}}$ is trace class.*

1.2 Spectral theory of surfaces with cusps

For spectral theory for manifolds with cusps we refer to [28], [12], and the references therein. The results in [28] hold for any dimension. For surfaces in particular we refer to [29]. Here we only recall the main facts and definitions that we use in this document.

For a surface with cusps (M,g), the spectrum of the Laplacian $\sigma(\Delta_g)$ is the union of the point spectrum σ_p and the continuous spectrum σ_c. The point spectrum consist of a sequence of eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \dots$$

Each eigenvalue has finite multiplicity, and the counting function $N(\Lambda) = \#\{\lambda_j | \lambda_j \leq \Lambda^2\}$ for $\Lambda > 0$ satisfies $\limsup N(\Lambda)\Lambda^{-2} \leq A_g(4\pi)^{-1}$, where A_g denotes the area of (M,g). Depending on the metric, the set of eigenvalues may be infinite or not.

The continuous spectrum σ_c of Δ_g is the interval $[\frac{1}{4}, \infty)$ with multiplicity equal to the number of cusps of M. For a proof of this fact, see for example [28, p.206]. The spectral decomposition of the absolutely continuous part of Δ_g is described by the generalized eigenfunctions $E_j(z,s)$, for $j = 1, \dots, m$ with $z \in M$, $s \in \mathbb{C}$. To each cusp we can associate such generalized eigenfunctions, also called Eisenstein functions by analogy with the Eisenstein series for hyperbolic surfaces. They are closely related to the wave operators $W_\pm(\Delta_g, \bar{\Delta}_{a,0})$ and to the scattering matrix $S(\lambda)$. For details, see [28, sec.7].

Each Eisenstein function $E_j(z,s)$ is smooth as a function of $z \in M$ and is meromorphic as a function of $s \in \mathbb{C}$. It satisfies:

$$\Delta_g E_j(z,s) = s(1-s)E_j(z,s).$$

Its poles are contained in the union of the half-plane $\mathrm{Re}(s) < \frac{1}{2}$ and the interval $(\frac{1}{2}, 1]$. The restriction of $E_j(z,s)$ to the cusp Z_i satisfies

$$E_j((y_i,x_i),s) = \delta_{ji}y_i^s + C_{ji}(s)y_i^{1-s} + O(e^{-cy_i}), \quad \text{as } y_i \to \infty.$$

Let $C(s)$ be the $m \times m$ matrix $(C_{ji}(s))$. Then $C(s)$ is a meromorphic function of $s \in \mathbb{C}$. The scattering matrix $S(\lambda; \Delta_g, \bar{\Delta}_{a,0})$ given by the time-dependent approach to scattering theory is related to $C(s)$ by the equation $S(\frac{1}{4}+\lambda^2) = C(\frac{1}{2}+i\lambda)$, for $\lambda \in \mathbb{R}$. In this way $S(\lambda)$ has an extension to the double covering of $\mathbb{C}$ defined by $\lambda = s(1-s)$. This extension is meromorphic and we have that $S(s(1-s)) = C(s)$, see [30, p.342]. From now on in this setting we refer to the matrix-valued function $C(s)$ as the scattering matrix.

Let us summarize the main properties of the Eisenstein functions and the scattering matrix by recalling Theorem 7.24 in [28].

Theorem 1.3. *([28]) With the notation introduced above we have that the Eisenstein functions and the scattering matrix associated to the surface with cusps (M,g) satisfy the following properties:*

1. *(a) $C(s)$ is meromorphic on $\mathbb{C}$ with poles contained in the half-plane* $\mathrm{Re}(s) < \frac{1}{2}$ *and the interval $(\frac{1}{2}, 1]$.*
   (b) The poles $s_0 \in (\frac{1}{2}, 1]$ of $C(s)$ are simple.
   (c) $C(s)$ is holomorphic in a neighborhood of the line $\mathrm{Re}(s) = \frac{1}{2}$.
   (d) The matrix $C(s)$ is symmetric and satisfies the functional equation $C(s)C(1-s) = \mathrm{Id}$.

2. *For every $j = 1, \ldots m$,*
   (a) $E_j(z,s)$ is holomorphic in a neighborhood of the line $\mathrm{Re}(s) = \frac{1}{2}$.
   (b) The poles $s_0 \in (\frac{1}{2}, 1]$ of $E_j(z,s)$ are simple and if s_0 is a pole of E_j then s_0 is also a simple pole of $C_{jj}(s)$.
   (c) If s_0 is a pole of $E_j(z_0,s)$ of order n then s_0 is also a pole of $E_j(z,s)$ of order n for all $z \in M$, and n is the maximal order of the pole of $C_{ij}(s)$ at s_0, $i = 1, \ldots, m$.
   (d) The system of Eisenstein functions satisfies the functional equations:

$$E_i(z,s) = \sum_{j=1}^{m} C_{ij}(s) E_j(z, 1-s).$$

The scattering matrix also satisfies:

$$\overline{C(s)} = C(\bar{s}) \quad \text{and} \quad C(s)^* = C(\bar{s}).$$

Let $L^2_d(M, dA_g)$ be the subspace of $L^2(M, dA_g)$ spanned by the eigenfunctions of Δ_g, and let $\varphi_0, \varphi_1, \ldots$ be a basis of $L^2_d(M, dA_g)$ composed of normalized eigenfunctions of Δ_g. Then any $f \in C_0^\infty(M)$ has the following Fourier type expansion:

$$f(z) = \sum_k (\varphi_k, f)\varphi_k + \frac{1}{4\pi}\sum_{j=1}^{m}\int_{\mathbb{R}} E_j(z, \frac{1}{2}+i\lambda)\int_M E_j(w, \frac{1}{2}-i\lambda) f(w)\; dA_g(w)\; d\lambda.$$

A quantity of interest is the determinant of the scattering matrix which we denote by $\phi(s) = \det C(s)$. It satisfies the following equations:

$$\phi(s)\phi(1-s) = 1, \quad \overline{\phi(s)} = \phi(\bar{s}), \quad s \in \mathbb{C}\,.$$

The concept of resonance is very important in the spectral theory of surfaces with cusps. The definition of resonance is given in Chapter 7. The mathematical quantity that we use to define isospectral metrics here is the resonance set of the Laplacian. This set is defined as the union of the poles of the scattering matrix and the set $\{s_j | s_j(1-s_j)\text{is an eigenvalue}\}$.

1.3 Conformal transformations

In this section we explain how some geometrical quantities change under a conformal transformation of the metric, i.e., when we multiply a given metric by a function that is strictly positive in the complete manifold.

Definition 1.4. *A conformal transformation of the metric g on M is a metric h defined by $h = e^{2\varphi}g$ where $\varphi \in C^\infty(M)$.*

In this paper we consider conformal factors that are the exponential of smooth functions on M. These functions are often denoted by φ. Depending on the case they may have compact support or not. If the support is not compact we usually require some decay at infinity of the function as well as some of its derivatives. In what follows the metric h will always denote a conformal transformation of g.

Definition 1.5. *Two metrics g_1, g_2 are quasi-isometric if there exist constants $C_1, C_2 > 0$ such that*

$$C_1 g_1(z) \le g_2(z) \le C_2 g_1(z), \quad \textit{for all } z \in M,$$

in the sense of positive definite forms.

Quasi-isometric metrics have equivalent geodesic distances. The associated L^2-spaces coincide as sets, thought the inner product is not the same.

Remark 1.6. *If the function φ appearing in the conformal factor is bounded on M we have that the metrics g and $h = e^{2\varphi}g$ are quasi-isometric and the geodesic distances d_g and d_h are equivalent. If in addition the metric g is complete, so is the metric h.*

Let A_g denote the area of (M, g), dA_g the volume element, and $K_g(z)$ its Gaussian curvature. Let A_h, dA_h and K_h be the quantities corresponding to (M, h), for any conformal transformation h of g. Let Δ_h be the Laplacian associated to h. Then for the metrics g and h we have the following relations:

$$\begin{aligned} dA_h &= e^{2\varphi} dA_g \\ K_h &= e^{-2\varphi}(\Delta_g \varphi + K_g) \\ \Delta_h &= e^{-2\varphi}\Delta_g \end{aligned}$$

The domains of the Laplacians Δ_g and Δ_h lie in different Hilbert spaces. Thus sometimes it is necessary to consider a unitary map between the spaces $L^2(M, dA_g)$ and $L^2(M, dA_h)$. From the definition of the metrics and the transformation of the area element we have that the unitary map is:

$$T : L^2(M, dA_g) \to L^2(M, dA_h), \ f \mapsto e^{-\varphi} f. \tag{1.3}$$

For the transformed Laplacians we have the following expressions:

$$\begin{aligned} T^{-1}\Delta_h T f &= \Delta_h f + 2\langle \nabla_h f, \nabla_h \varphi \rangle_h - (\Delta_h \varphi + |\nabla_h \varphi|_h^2) f \\ &= e^{-2\varphi}\left(\Delta_g f + 2\langle \nabla_g f, \nabla_g \varphi \rangle_g - (\Delta_g \varphi + |\nabla_g \varphi|_g^2) f\right) \\ T\Delta_g T^{-1} f &= e^{2\varphi}\left(\Delta_h f - 2\langle \nabla_h \varphi, \nabla_h f \rangle_h + (\Delta_h \varphi - |\nabla_h \varphi|_h) f\right) \end{aligned} \tag{1.4}$$

To see this notice that $\Delta_h e^{-\varphi} = -e^{-\varphi}\Delta_h \varphi - e^{-\varphi}|\nabla_h \varphi|_h$. Then by equation (1.1)

$$\begin{aligned}
\Delta_h(e^{-\varphi} f) &= (\Delta_h e^{-\varphi}) f + e^{-\varphi}(\Delta_h f) - 2\langle \nabla_h e^{-\varphi}, \nabla_h f\rangle_h \\
&= (-e^{-\varphi}\Delta_h \varphi - e^{-\varphi}|\nabla_h \varphi|_h) f + e^{-\varphi}(\Delta_h f) + 2e^{-\varphi}\langle \nabla_h \varphi, \nabla_h f\rangle_h \\
T^{-1}\Delta_h T f &= \Delta_h f + 2\langle \nabla_h \varphi, \nabla_h f\rangle_h - (\Delta_h \varphi + |\nabla_h \varphi|_h) f
\end{aligned}$$

As for g we have $\Delta_g e^{\varphi} = e^{\varphi}\Delta_g \varphi - e^{\varphi}|\nabla_g \varphi|_g$. Then

$$\begin{aligned}
\Delta_g(e^{\varphi} f) &= (\Delta_g e^{\varphi}) f + e^{\varphi}(\Delta_g f) - 2\langle \nabla_g e^{\varphi}, \nabla_g f\rangle_g \\
&= (e^{\varphi}\Delta_g \varphi - e^{\varphi}|\nabla_g \varphi|_g) f + e^{\varphi}(\Delta_g f) - 2e^{\varphi}\langle \nabla_g \varphi, \nabla_g f\rangle_g \\
T\Delta_g T^{-1} f &= \Delta_g f - 2\langle \nabla_g \varphi, \nabla_g f\rangle_g + (\Delta_g \varphi - |\nabla_g \varphi|_g) f
\end{aligned}$$

Note that the operators $T^{-1}\Delta_h T$ and $T\Delta_g T^{-1}$ are self-adjoint in the corresponding transformed domain.

Let us first give a handwaving definition of what we mean by a surface with asymptotically cusps ends. The reason to do that is that we need flexibility in the conditions on the conformal factors:

A surface with asymptotically cusp ends is a surface (M, h) where the metric h is a conformal transformation of the metric on a surface with cusps (M, g) such that the conformal factor as well as some of its derivatives have a suitable decay in the cusps.

1.4 Injectivity radius

Let (M, g) be a Riemannian manifold. The injectivity radius at a point $z \in M$ is defined as the supremum of the radius of balls centered at $0 \in T_z M$ such that the exponential function exp_z is defined and injective in such balls. (This is equivalent to defining it as the minimal distance from the point z to its cut locus). Let us denote the injectivity radius at a point $z \in M$ by $\mathrm{inj}_g(z)$. The injectivity radius of M is the infimum of the injectivity radius at each point, i.e. $\mathrm{inj}_g(M) = \inf_{z\in M} \mathrm{inj}_g(z)$.

Let g and h be as above i.e. $h = e^{2\varphi} g$. We are particularly interested in the case where for $(y, x) \in Z$, $\varphi(y, x) = O(1/y)$ as $y \to \infty$. In this case, the metrics g and h are quasi-isometric and the geodesic distances are equivalent.

It is well known that on a surface with cusps (M, g) the injectivity radius is null. For an element in a cusp, $z = (y, x)$, we have that $\mathrm{inj}_g(z) \sim \frac{1}{y}$. If we assume in addition that $\Delta_g \varphi = O(1)$ as $y \to \infty$ then the surface (M, h) has bounded Gaussian curvature. This implies that there exist constants $c, c' > 0$ such that

$$\mathrm{inj}_h(z) \geq \min\{c\ \mathrm{inj}_g(z), c'\}, \quad z \in M,$$

see [32, Prop.2.1].

The injectivity radius is a very important quantity in geometry. Many generalizations of results in geometric analysis for compact manifolds hold for manifolds of bounded geometry, i.e. manifolds with bounded curvature and injectivity radius bounded away from zero. If the injectivity radius vanishes these results generally fail in their standard form. For example, the Sobolev embedding theorems and Rellich's lemma do not hold for surfaces with cusps.

1.5 Heat kernels and estimates

Heat semigroups and heat kernels are very useful and important tools when working with regularized determinants via zeta function regularization. The heat semigroup associated to a closed self-adjoint operator can be constructed using the spectral theorem. For the existence and uniqueness of the heat kernel on a complete open manifold with Ricci curvature bounded from below see [15]. For the main properties of heat kernels see [15] and [9]. On a complete Riemannian manifold of dimension n with Ricci curvature bounded from below, the heat kernel $K(x, y, t)$ is the smallest smooth positive fundamental solution of the heat equation on M, i.e. $K(x, y, t) \leq p(x, y, t)$ for every positive fundamental solution $p(x, y, t)$. The heat kernel is symmetric in the space variables and satisfies the conservation law $\int_M K(x, y, t)dV(y) = 1$, for all $t > 0$ and $x \in M$. Let (M, g) and $h = e^{2\varphi}g$ be as above, and let $e^{-t\Delta_h}$, $e^{-t\Delta_g}$, $e^{-t\Delta_{1,0}}$ denote the heat semigroups associated to the Laplacians Δ_h, Δ_g and $\Delta_{1,0}$, respectively. Since the Laplacians are positive, the heat equation is $\Delta + \partial_t = 0$. Let $K_h(z, z', t)$ and $K_g(z, z', t)$ denote the heat kernels corresponding to Δ_h and Δ_g respectively. We hope that this will not lead to confusions with the notation for the Gaussian curvatures and the heat kernels.

Like the Laplacians, the heat semigroups act on different spaces. The operator $e^{-t\Delta_h}$ may act on $L^2(M, dA_g)$, but it is not self-adjoint with respect to this inner product. To make $e^{-t\Delta_h}$ and $e^{-t\Delta_g}$ act on the same space and preserve self-adjointness we use the unitary map $T : L^2(M, dA_g) \to L^2(M, dA_h)$ defined by (1.3). The transformed operators $T^{-1}e^{-t\Delta_h}T$ and $Te^{-t\Delta_g}T^{-1}$ are self-adjoint on the corresponding space. Consider the integral kernel of the transformed operator $T^{-1}e^{-t\Delta_h}T : L^2(M, dA_g) \to L^2(M, dA_g)$. If $f \in L^2(M, dA_g)$, then

$$\begin{aligned} T^{-1}e^{-t\Delta_h}Tf(z) &= T^{-1}\left(\int_M K_h(z, z', t)e^{-\varphi(z')}f(z')\ dA_h(z')\right) \\ &= \int_M e^{\varphi(z)}K_h(z, z', t)e^{-\varphi(z')}f(z')e^{2\varphi(z')}\ dA_g(z') \\ &= \int_M e^{\varphi(z)}K_h(z, z', t)e^{\varphi(z')}f(z')\ dA_g(z'), \end{aligned}$$

thus $K_{T^{-1}e^{-t\Delta_h}T}(z, z', t) = e^{\varphi(z)}K_h(z, z', t)e^{\varphi(z')}$. This also follows from the general transformation law in the product space:

$$L^2(M \times M, dA_h \times dA_h) \to L^2(M \times M, dA_g \times dA_g), f(z, z') \mapsto f(z, z')e^{\varphi(z)}e^{\varphi(z')}$$

The kernel of $T^{-1}e^{-t\Delta_h}T$ restricted to the diagonal is $K_{T^{-1}e^{-t\Delta_h}T}(z, z, t) = K_h(z, z, t)e^{2\varphi(z)}$.

For the details of the construction of the heat kernel on a compact Riemannian manifold M and its asymptotic expansion for small t, we refer to [9], [6] and the references therein. If the manifold is closed, there exists a constant $c > 0$ such that for any fixed $0 < \tau < \infty$, the heat kernel satisfies the following bounds

$$K(x, y, t) \ll t^{-n/2}e^{-\frac{d(x,y)^2}{ct}}, \quad \text{for } t \leq \tau. \tag{1.5}$$

If the manifold has boundary, consider the closed self-adjoint extension of the Laplacian with Dirichlet boundary conditions. For the Dirichlet heat kernel, and given a compact set $K \subset M$ and a $T > 0$, there exists a positive constant c' such that

$$K_D(x, y, t) \leq c't^{-n/2}(e^{-\frac{d(x,y)^2}{ct}} + e^{\frac{d(y,\partial M)^2}{8t}}),$$

for $(x, y, t) \in K \times M \times (0, T]$, see [9, chapter VII].

1.5.1 The heat kernel on a surface with cusps

The heat kernel on a surface with cusps was constructed by W. Müller in [28]. Here we give a brief description of this construction and mention some of the main statements; for details see [28]. In this part we consider (M, g) with only one cusp, $M = M_0 \cup Z$ and $Z \cong [1, \infty) \times S^1$. The kernel of a parametrix for the heat operator is constructed by gluing together the heat kernel in the complete cusp $\mathbb{R}^+ \times S^1$ equipped with the hyperbolic metric and the heat kernel in a suitable compact manifold.

Let $\mathbb{R}^+ \times S^1$ be the complete cusp with the hyperbolic metric $y^{-2}(dy^2 + dx^2)$. Let Δ_1 be the unique self-adjoint extension of the Laplacian defined on $C_c^\infty(\mathbb{R}^+ \times S^1)$. It is unique because $\mathbb{R}^+ \times S^1$ with the hyperbolic metric is a complete Riemannian manifold. The notation for Δ_1 is arbitrary. Since the metric in the cusp Z_a, for any $a > 0$, is also the hyperbolic metric, the Laplacians $\Delta_{Z_a,D}$ and Δ_1 coincide when acting on $C_c^\infty((a, \infty) \times S^1)$.

The heat kernel for Δ_1 on $\mathbb{R}^+ \times S^1$ may be constructed using separation of variables and equals:

$$K_1((y, x), (y', x'), t) = \sum_{n \in \mathbb{Z}} \tilde{K}_n(y, y', t) e^{2\pi i n (x - x')},$$

where $\tilde{K}_n(y, y', t)$ is the heat kernel corresponding to the operator

$$D_n = -y^2 \left(\frac{\partial^2}{\partial y^2} - 4\pi^2 n^2 \right),$$

with domain containing $C_c^\infty(\mathbb{R}^+)$; see [28, page 218].

Let $M_2 = M_0 \cup ([1, 2] \times S^1)$, let W be a closed Riemannian manifold containing M_2 isometrically. Let Δ_W be the Laplacian on W, $e^{-t\Delta_W}$ be its corresponding heat operator and $K_2(z, z', t)$ be the fundamental solution of the heat equation on W restricted to $M_2 \times M_2 \times \mathbb{R}^+$.

We define the gluing functions ϕ_1, ϕ_2 and ψ_1, ψ_2 as follows: let $\varphi(b, c)$ denote any increasing C^∞ function of real variable u such that $\varphi = 0$ for $u \leq b$ and $\varphi = 1$ for $u \geq c$, and let

$$\begin{aligned} \phi_1 &= \varphi(1, 1 + \frac{1}{4}), & \psi_1 &= \varphi(1 + \frac{3}{8}, 1 + \frac{5}{8}) \\ \phi_2 &= 1 - \varphi(1 + \frac{3}{4}, 2), & \psi_2 &= 1 - \psi_1. \end{aligned}$$

We consider each of these functions as defined on the cylinder $Z = [1, \infty) \times S^1$ and extend them in the obvious way to M. Then we glue the two heat kernels together:

$$H(z, z', t) = \phi_1(z) K_1(z, z', t) \psi_1(z') + \phi_2(z) K_2(z, z', t) \psi_2(z').$$

The function $H(z, z', t)$ is the kernel of a parametrix for the heat operator associated to g.

Let $i(z)$ be the function given by:

$$i(z) = \begin{cases} 1, & \text{if } z \in M \setminus Z; \\ y, & \text{if } z \in Z \text{ and } z = (y, x). \end{cases} \tag{1.6}$$

The kernels K_1 and K_g satisfy the following estimates:

1. For $K_1(z,z',t)$, the heat kernel in the complete cusp, and for arbitrary $T>0$ there exist constants $C_1, c_2>0$ such that for $0<t<T$, $y,y'\geq 1$, and $k,l,m\in\mathbb{N}$ one has:

$$\left|\frac{\partial^k}{\partial t^k} d_z^l d_{z'}^m K_1(z,z',t)\right| \leq C_1(i(z)i(z'))^{\frac{1}{2}} t^{-1-k-l-m} \exp\left(-\frac{c_2 d_g^2(z,z')}{t}\right) \tag{1.7}$$

   where $d_g(z,z')$ is the Riemannian distance in (M,g), that is the hyperbolic distance in the cusp. The constants depend on T.

2. For $K_g(z,z',t)$, we have: for $T>0$ there exist $C_1,c_2>0$ such that

$$|K_g(z,z',t)| \leq C_1(i(z)i(z'))^{\frac{1}{2}} t^{-1} \exp\left(-\frac{c_2 d_g^2(z,z')}{t}\right) \tag{1.8}$$

   uniformly for $0<t<T$.

Note that the constants in the previous equations can be different but since we are dealing with inequalities we can choose the most convenient (the largest one in front of the inequalities and the smallest one in the exponents).

We also have that for every $l\in\mathbb{N}$ and $T>0$ there exist constants C_2, C_3 such that

$$|K_g(z,z',t)-H(z,z',t)| \leq C_2(i(z)i(z'))^{\frac{1}{2}} t^{l} \exp\left(-\frac{C_3 d_g^2(z,z')}{t}\right),$$

uniformly for $0<t<T$. The constants C_2, C_3 depend on $l\in\mathbb{N}$ and T. This follows from equations (4.9) and (4.10) in [28].

Let us now go back to the metric $h=e^{2\varphi}g$. We can extend it to a metric on the complete cusp $\widetilde{Z}=\mathbb{R}^+\times S^1$ in the following way: On $\widetilde{Z}$ we have the hyperbolic metric g_0. The metric $g|_Z=g_0$. We start by extending the function $\varphi|_Z$ to a smooth function $\widetilde{\varphi}$ on $\widetilde{Z}$ that vanishes in a small neighborhood of 0. Then on $(0,\infty)\times S^1$ we define h as $h:=e^{2\widetilde{\varphi}}g_0$. It is a complete metric and $h=g_0$ close to 0. In this way we can define the Laplacian on $(\widetilde{Z},h)$. Denote its unique self-adjoint extension by $\Delta_{1,h}$. Clearly $\Delta_{1,h}=e^{-2\widetilde{\varphi}}\Delta_1$. The heat kernel associated to $\Delta_{1,h}$ is denoted by $K_{1,h}(z,z',t)$, for $z,z'\in\widetilde{Z}$ and $t>0$.

1.5.2 Other heat kernels

In this subsection we introduce the other heat operators that we will use throughout this document.

For $a>1$ let $\Delta_{a,0}$ be the operator defined in section 1.1. The heat kernel $p_a(y,y',t)$ for $\Delta_{a,0}$ can be computed explicitly, see [8, sec.14.2] or [28, p.258]. It is given by

$$p_a(y,y',t) = \frac{e^{-t/4}}{\sqrt{4\pi t}}\,(yy')^{1/2}\left\{e^{-(\log(y/y'))^2/4t} - e^{-(\log(yy')-\log(a^2))^2/4t}\right\}, \tag{1.9}$$

for $y,y'>a$. This is easy to verify by direct computation. Also note that for $1\leq y\leq a$, $p_a(y,y',t)=0$.

The operator $e^{-t\Delta_{a,0}}$ acts on $L^2([a,\infty),y^{-2}dy)$. However, we can regard it as an operator acting on $L^2([1,\infty),y^{-2}dy)$. In order to do so, let us consider the following linear transformations: Let

$$J_{1,a}: L^2([a,\infty),y^{-2}dy)\to L^2([1,\infty),y^{-2}dy)$$

be the inclusion and

$$\rho_{a,1} : L^2([1,\infty), y^{-2}dy) \to L^2([a,\infty), y^{-2}dy)$$

be the restriction. If $f \in L^2([1,\infty), y^{-2}dy)$, then $e^{-t\Delta_{a,0}} f := J_{1,a} e^{-t\Delta_{a,0}} \rho_{a,1} f$.

Conversely, the operator $e^{-t\Delta_{1,0}}$ can be regarded as acting on $L^2([a,\infty), y^{-2}dy)$ in the following way: Let $f \in L^2([a,\infty), y^{-2}dy)$, $e^{-t\Delta_{1,0}} f := \rho_{a,1} e^{-t\Delta_{1,0}} J_{1,a} f$.

Now, let us assume that M can be decomposed as $M = M_0 \cup Z$ with $Z = [1,\infty) \times S^1$. Then we can make the operator $e^{-t\Delta_{a,0}}$ act on $L^2(M, dA_g)$ in the following way: let $\Pi_a : L^2(M, dA_g) \to L^2([a,\infty), y^{-2}dy)$, be the projection defined by $\Pi_a f(y) = \int_{S^1} f|_{Z_a}(y,x)dx$, where as before Z_a denotes the cusp $[a,\infty) \times S^1$, and let $J_a : L^2([a,\infty), y^{-2}dy) \to L^2(M, dA_g)$ be the inclusion. By an abuse of notation, for $f \in L^2(M, dA_g)$, we write $e^{-t\Delta_{a,0}} f := J_a e^{-t\Delta_{a,0}} \Pi_a f$. Explicitly,

$$e^{-t\Delta_{a,0}} f(z) = \int_a^\infty \int_{S^1} p_a(y,y',t)\; f|_{Z_a}(y',x')dx' \frac{dy'}{y'^2} \quad \text{for } z = (y,x) \in Z_a$$

and is zero otherwise. From the symmetry of $p_a(y,y',t)$, is clear that the operator $e^{-t\Delta_{a,0}}$ acting on $L^2(M, dA_g)$ is symmetric.

Recall the operator $\Delta_{Z,D}$ defined in section 1.1. The kernel of the operator $e^{-t\Delta_{Z,D}}$ is constructed by a classical method (see [9, chapter VII]) and it is given by:

$$K_{Z,D}((y,x),(y',x'),t) = K_1((y,x),(y',x'),t) + p_{1,D}((y,x),(y',x'),t) \tag{1.10}$$

where $y, y' \geq 1$, $x, x' \in S^1$, $t > 0$, and $p_{1,D}((y,x),(y',x'),t)$ is a function that decays exponentially as $t \to 0$ if (y,x) and (y',x') are away from the boundary. More precisely, for every $T > 0$ there exist constants $C, c > 0$ such that:

$$|p_{1,D}(z,z',t)| \leq Ct^{-1}(i(z)i(z'))^{1/2} e^{-\frac{c(d_g(z,\partial Z)+d_g(z',\partial Z))^2}{t}} \tag{1.11}$$

for all $z, z' \in Z$ and $0 < t < T$. For manifolds of dimension n, the power of t is $-n/2$.

For upper estimates of heat kernels on complete Riemannian manifolds, we refer to [11], Theorems 4, 6 and 7. There, the authors work with the heat kernel of a complete Riemannian manifold with sectional curvature bounded from above and from below by constants C and $-c$, with $C, c \geq 0$. They state that the derivatives of the heat kernel $K(x,y,t)$ are expected to satisfy similar inequalities as $K(x,y,t)$ itself satisfies, except for the powers of the time variable t which will be different and the constants will depend on the curvature of M and its covariant derivatives. We state here some of their results. Let M be a complete Riemannian manifold with bounded curvature. Then there exists a constant $c(n,k,T)$, with $n = \dim M$, k the lower bound of the curvature, and $T > 0$, such that for all $p, x \in M$, and for all $t \in [0,T]$, ([11, Thm 4])

$$K(p,x,t) \leq c(n,k,T)(\tilde{\delta}(p))^{-\alpha(n)} t^{-n/2} \exp\left(-\frac{\alpha(n)d^2(p,x)}{t}\right).$$

for some universal constant $\alpha(n) > 0$, where

$$\tilde{\delta}(p) = \min\left\{1, \delta(p), \frac{\pi}{12\sqrt{K}}\right\},$$

and $\delta(p)$ is the injectivity radius at p. For the derivatives of K one has, ([11, Thm 6])

$$|\nabla K|(x,p,t) \leq c(n,k,T)(\tilde{\delta}(p))^{-\alpha(n)} t^{-(n+1)/2} \exp\left(-\frac{\alpha(n)d^2(p,x)}{t}\right).$$

Finally, under the same hypothesis as above Theorem 7 in [11] states that there exists a constant $C(n, A_0, A_1, T)$ such that for all $p, x \in M$, and for all $t \in [0, T]$,

$$|\operatorname{hess} K|(p, x, t) \leq C(n, A_0, A_1, T)(\tilde{\delta}(p))^{-\alpha(n)} t^{-(n+2)/2} \exp\left(-\frac{\alpha(n) d^2(p, x)}{t}\right),$$

where A_0 is a bound for the curvature tensor and A_1 is a bound for its covariant derivative.

Now let us go back to the conformal transformation of the metric in a surface with cusps, $h = e^{2\varphi} g$. Let us assume that φ and $\Delta_g \varphi$ decay in the cusp as $O(1/y)$, as $y \to \infty$. Remember that in that case the metrics g and h are quasi-isometric, therefore the corresponding Riemannian distances are equivalent, i.e., there exist constants $C_1, C_2 > 0$ such that

$$C_1 d_g(z, z') \leq d_h(z, z') \leq C_2 d_g(z, z').$$

Transferring the results of [11] to our case we have that for the power of the injectivity radius $\alpha = 1/2$ and for $z = (y, x) \in Z$, $\delta(z) = \operatorname{inj}_g(z) \sim y^{-1} \sim \operatorname{inj}_h(z)$. Therefore the kernel of the heat operator for the metric h satisfies the same estimate as the one corresponding to g:

$$K_h(z, z', t) \ll (i(z)i(z'))^{\frac{1}{2}} t^{-1} \exp\left(-\frac{c_0 d_h^2(z, z')}{t}\right) \ll (i(z)i(z'))^{\frac{1}{2}} t^{-1} \exp\left(-\frac{c_1 d_g^2(z, z')}{t}\right) \quad (1.12)$$

uniformly for $0 < t < T$, where c_1 and c_2 are positive constants and the symbol $\ll$ means $\leq c$ times the expression, for some constant $c \geq 0$.

For the derivatives of the heat kernel K_*, where $*$ denotes the metric g or h, we obtain:

$$|\nabla K_*|(z, z', t) \leq c(n, k, T)\ (i(z)i(z'))^{1/2} t^{-3/2} \exp\left(-\frac{c_1 d_g^2(z, z')}{t}\right), \text{ and} \quad (1.13)$$

$$|\Delta_* K_*|(z, z', t) \leq C(n, A_0, A_1, T)\ (i(z)i(z'))^{1/2} t^{-2} \exp\left(-\frac{c_1 d_g^2(z, z')}{t}\right). \quad (1.14)$$

Now let $\Delta_{Z,h}$ be the self-adjoint extension of the operator

$$-e^{-2\varphi} y^2 (\partial_y^2 + \partial_x^2) : C_c^\infty(Z) \to L^2(Z, dA_h)$$

obtained after imposing Dirichlet boundary conditions at $\{1\} \times S^1$. Let $K_{Z,h}$ denote the kernel of the operator $e^{-t\Delta_{Z,h}}$. As in the case of the heat kernel associated to the operator $\Delta_{Z,D}$, equation (1.10), the kernel $K_{Z,h}$ is given by:

$$K_{Z,h}(z, z', t) = K_{1,h}(z, z', t) + p_{h,D}(z, z', t), \quad (1.15)$$

for $z, z' \in Z$ and $t > 0$ where the term $p_{h,D}(z, z', t)$ is determined by the boundary condition. By the same argument as above, we infer that $p_{h,D}(z, z', t)$ satisfies, up to some constants, the same estimate as the one given by equation (1.11).

1.6 Duhamel's Principle

There are several ways to state and use Duhamel's principle. In this section we refer to [9, VII.3]. Let Ω be a regular domain in a fixed Riemannian manifold M and let $\partial\Omega$ be its boundary. The boundary carries the outward unit normal vector field ν. A regular domain is a connected open subset which has compat closure and whose boundary is smooth.

Let $u, v \in C^2(\Omega \times (0,t))$ be such that they extend continuously to $\bar{\Omega}$, and their gradients extend to continuous vector fields on $\bar{\Omega}$. Let $[\alpha, \beta] \subseteq (0,t)$. Then

$$\begin{aligned}\int_\alpha^\beta d\tau \int_\Omega \{(Lu)(z, t-\tau)v(z,\tau) - u(z,t-\tau)(Lv)(z,\tau)\} dA(z) \\ = \int_\alpha^\beta d\tau \int_{\partial\Omega} \left\{ \frac{\partial u}{\partial \nu_x}(x, t-\tau)v(x,\tau) - u(x,t-\tau)\frac{\partial v}{\partial \nu_x}(x,\tau) \right\} dA(x) \\ + \int_\Omega u(z, t-\beta)v(z,\beta) - u(z,t-\alpha)v(z,\alpha) dA(z),\end{aligned} \quad (1.16)$$

where $L = \Delta - \frac{\partial}{\partial t}$. In the notation of [9] the Laplacian is negative. Since we work with positive Laplacians we need to arrange the signs.

Duhamel's principle can be applied in the non-compact setting under certain assumptions on the decay of the functions. This is the case of the heat kernels on surfaces with cusps and asymptotically cusp ends. Using equation (1.16) and the properties of the heat kernels, we obtain the equations that we will use throughout this thesis. One of then is the following:

$$K_h(z, z', t)e^{2\varphi(z')} - K_g(z, z', t) = \int_0^t d\tau \int_M K_h(z, w, s)e^{2\varphi(w)}(\Delta_g - \Delta_h)K_g(w, z', t-s) dA_g(w) ds$$

In terms of the operators, Duhamel's principle can be written as:

$$T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_g} = \int_0^t T^{-1}e^{-s\Delta_h}T(\Delta_g - T^{-1}\Delta_h T)e^{-(t-s)\Delta_g}\, ds.$$

1.7 Gauss-Bonnet formula

For a surface M with m cusps, the Euler characteristic is given by $\chi(M) = (2 - 2p - m)$, where p is the genus of the compact surface $\overline{M}$ defined in section 1.1. A Gauss-Bonnet formula is valid in this setting:

$$\int_M K_g dA_g = 2\pi\chi(M),$$

where K_g denotes the Gaussian curvature of the metric g. The same formula is valid for the metric $h = e^{2\varphi}g$ when φ and $\Delta_g\varphi$ have a suitable decay at infinity, since:

$$\int_M K_h\, dA_h = \int_M e^{-2\varphi}(\Delta_g\varphi + K_g)e^{2\varphi}\, dA_g = \int_M \Delta_g\varphi\, dA_g + \int_M K_g\, dA_g = \int_M K_g\, dA_g.$$

1.8 Regularized determinants on compact manifolds

On a n-dimensional orientable connected closed Riemannian manifold, the determinant of the Laplace operator is defined by a zeta regularization method. We know that the spectrum of the Laplacian Δ consists of a discrete set $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$ of eigenvalues and that the corresponding eigenfunctions $\{\phi_i\}_{i\in\mathbb{N}}$ form an orthonormal basis of $L^2(M)$. Let $N(\lambda) = \sum_{\lambda_j \leq \lambda} 1$, it is well known that it satisfies Weyl's asymptotic formula $N(\lambda) \sim \frac{\omega_n \operatorname{vol}(M)}{(2\pi)^n} \lambda^{n/2}$ as $\lambda \to \infty$, where $\omega_n = \frac{(2\pi)^{n/2}}{\Gamma(n/2)}$. This implies in particular that $\lambda_j^{n/2} \sim \frac{(2\pi)^n j}{\omega_n \operatorname{vol}(M)}$, for $j \gg 1$.

The spectral zeta function associated to Δ is defined by the series:

$$\zeta_\Delta(s) = \sum_{\lambda_j > 0} \lambda_j^{-s}.$$

Since for $j \gg 1$, $\lambda_j^{-s} \sim (C(n)j)^{-2s/n}$ with $C(n) = 2\pi/w_n \operatorname{vol}(M)$, it is clear that the series converges absolutely on $\operatorname{Re}(s) > n/2$ and that the convergence is uniform on compact subsets of the same half plane. Using Mellin transform we obtain

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \sum_{j=1}^\infty e^{-\lambda_j t} t^{s-1} dt = \frac{1}{\Gamma(s)} \int_0^\infty (\operatorname{Tr}(e^{-t\Delta}) - 1) t^{s-1} dt.$$

Let $F(s) = \frac{1}{\Gamma(s)} \int_1^\infty \sum_{j=1}^\infty e^{-\lambda_j t} t^{s-1} dt$. It converges absolutely and uniformly on compact subsets of $\mathbb{C}$. On the other hand the heat kernel has an asymptotic expansion at the diagonal as $t \to 0$:

$$\sum_{j=0}^\infty e^{-t\lambda_j} \phi_j(x)^2 = (4\pi t)^{-n/2} \left(\sum_{j=0}^k t^j u_j(x,x) \right) + O(t^{k-\frac{n}{2}+1}).$$

The coefficients are universal polynomials in the curvature and its derivatives; see [16]. The expansion of the heat kernel induces an asymptotic expansion on the trace of the heat operator as $t \to 0^+$:

$$\operatorname{Tr}(e^{-t\Delta}) \sim t^{-n/2} \sum_{j=0}^\infty a_j t^j.$$

The numbers a_j are called the heat invariants. We use this asymptotic expansion to obtain:

$$\int_0^1 (\operatorname{Tr}(e^{-t\Delta}) - 1) t^{s-1} dt = -\frac{1}{s} + \sum_{k=0}^N \frac{a_k}{k + s - \frac{n}{2}} + O(1),$$

as $t \to 0$. Thus, we can continue $\zeta_\Delta(s)$ to a meromorphic function on $\mathbb{C}$ by using

$$\zeta_\Delta(s) = \frac{1}{\Gamma(s)} \left\{ -\frac{1}{s} + \sum_{k=0}^N \frac{a_k}{k + s - \frac{n}{2}} + \text{ analytic in s} \right\}.$$

The pole at $s = 0$ in the brackets cancels with the zero of $\frac{1}{\Gamma(s)} = s e^{\gamma s} \prod_{n=1}^\infty (1 + \frac{s}{n}) e^{-\frac{s}{n}}$. In this way we obtain an extension of $\zeta_\Delta(s)$ that is analytic at $s = 0$. The regularized determinant of Δ is then defined by:

$$\det \Delta = \exp\left(-\frac{d}{ds} \zeta_\Delta(s)\big|_{s=0} \right). \tag{1.17}$$

In the same way, one can define a regularized determinant for every self-adjoint non-negative elliptic operator P on a closed manifold.

1.9 Relative determinants

The notion of relative determinant was introduced by W. Müller in [30]. The relative determinant is defined for two self-adjoint, nonnegative linear operators, H_1 and H_0, in a separable Hilbert space $\mathcal{H}$ satisfying the following assumptions:

1. For each $t > 0$, $e^{-tH_1} - e^{-tH_0}$ is a trace class operator.

2. As $t \to 0$, there is an asymptotic expansion of the relative trace of the form:

$$\operatorname{Tr}(e^{-tH_1} - e^{-tH_0}) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{k(j)} a_{jk} t^{\alpha_j} \log^k t,$$

   where $-\infty < \alpha_0 < \alpha_1 < \cdots$ and $\alpha_k \to \infty$. Moreover, if $\alpha_j = 0$ we assume that $a_{jk} = 0$ for $k > 0$.

3. $\operatorname{Tr}(e^{-tH_1} - e^{-tH_0}) = h + O(e^{-ct})$, as $t \to \infty$, where $h = \dim \operatorname{Ker} H_1 - \dim \operatorname{Ker} H_0$.

These properties allow us to define the relative zeta function as:

$$\zeta(s; H_1, H_0) = \frac{1}{\Gamma(s)} \int_0^{\infty} (\operatorname{Tr}(e^{-tH_1} - e^{-tH_0}) - h) t^{s-1} dt.$$

The relative determinant is then defined as:

$$\det(H_1, H_0) := e^{-\zeta'(0; H_1, H_0)}.$$

In a more general setting, condition 3) is replaced by an asymptotic expansion as $t \to \infty$. Then, in order to define the relative zeta function, the integral has to be split in two parts, see [30].

Chapter 2

Trace class property of relative heat operators

In this chapter we give a proof of Theorem 2.3, which says that the difference of the heat operators for g and h is trace class. As we know, none of the heat operators $e^{-t\Delta_h}$ nor $e^{-t\Delta_g}$ is trace class; which is the reason why we consider their difference. The trace class property is very important because it allows us to study the scattering theory of a pair of operators. In addition, it is the first step to define the relative determinant of the pair.

Let (M, g), M_0, Z as well as Δ_g, $\Delta_{Z,D}$, and Δ_1 be as in Chapter 1. Here again we consider the surface (M, g) only with one cusp so that it can be decomposed as $M = M_0 \cup Z$ with M_0 compact and $Z = [1, \infty) \times S^1$.

Recall that the product of a trace class operator with a bounded operator is trace class, the product of two Hilbert-Schmidt operators is a trace class operator and the product of a Hilbert-Schmidt operator with a bounded operator is Hilbert-Schmidt.

Let T be a trace class operator and $T = RS$ with R, S Hilbert-Schmidt. Then

$$\|T\|_1 \leq \|R\|_2 \|S\|_2,$$

where $\|\cdot\|_1$ denotes the trace norm and $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm, see [21] for the corresponding definitions. For an integral operator R with integral kernel $r(z, z')$ the Hilbert-Schmidt norm is giving by:

$$\|R\|_2^2 = \int_M \int_M |r(z, z')|^2 dA(z) dA(z').$$

In this chapter we need to make many estimates for which we repeatedly use the following lemmas:

Lemma 2.1. *For any $a > 0$, and $b, n, m \in \mathbb{R}$, we have that:*

$$\int_n^m e^{-ax^2 - bx} dx = \frac{e^{b^2/4a}}{\sqrt{a}} \int_{\sqrt{a}(n+\frac{b}{2a})}^{\sqrt{a}(m+\frac{b}{2a})} e^{-v^2} dv \leq \frac{\sqrt{\pi} e^{b^2/4a}}{\sqrt{a}}.$$

For any $c > 0$, $0 < t \leq T$, $k, \ell \geq 0$ with $k + \ell > 2$ we have:

$$\int_1^\infty \int_1^\infty y^{-k} y'^{-\ell} e^{-\frac{c}{t} \log(y/y')^2} dy dy' \leq \sqrt{t} e^{(1-k)^2 t/c}. \tag{2.1}$$

Proof. Both results follow just from a change of variables. For the second part, for y' fixed, put $v = \log(y/y')$, so $y'e^v = y$, and since $1 \le y' < \infty$, then $-\infty < -\log(y') \le 0$, thus

$$\int_1^\infty \int_1^\infty y^{-k} y'^{-\ell} e^{-\frac{c}{t}\log(y/y')^2} dy dy' = \int_1^\infty \int_{-\log(y')}^\infty y'^{-\ell-k+1} e^{(1-k)v} e^{-\frac{c}{t}v^2} dv dy'$$
$$\le \int_1^\infty y'^{-\ell-k+1} \int_{-\infty}^\infty e^{(1-k)v} e^{-\frac{c}{t}v^2} dv dy' \ll \sqrt{t} e^{(1-k)^2 t/c}.$$

□

Note that for $a, b, c > 0$ the function $f(t) = t^{-a} e^{-ct^{-b}}$ is bounded in $(0, \infty)$ and $\lim_{t\to 0} f(t) = 0$.

Lemma 2.2. *Let $\varphi \in C^\infty(M)$, $\psi = e^{-2\varphi} - 1$ and $\widetilde{\psi} = e^{2\varphi} - 1$. If $\varphi|_Z(y,x)$, $\Delta_g \varphi|_Z(y,x)$ and $|\nabla_g \varphi|_g|_Z(y,x)$ are $O(y^{-k})$ as $y \to \infty$, then so are $\psi|_Z(y,x)$, $\Delta_g \psi|_Z(y,x)$, $|\nabla_g \psi|_g|_Z(y,x)$ and the analogues functions corresponding to $\widetilde{\psi}$.*

Proof. Let $z = (y,x) \in Z$. Since $\varphi = O(y^{-k})$, as $y \to \infty$, there exist constants $c > 0$ and $N > 1$ such that $|\varphi(y,x)| \le c y^{-k}$, for $y > N$. Then there exists a constant $c_1 > 0$ such that for $y \ge N$:

$$|\widetilde{\psi}(x,y)| = |e^{2\varphi} - 1| \le \sum_{\ell=1}^\infty \frac{(2|\varphi|)^\ell}{\ell!} \le \sum_{\ell=1}^\infty \frac{(2c)^\ell}{y^{k\ell}\ell!} \le \frac{1}{y^k} \sum_{\ell=1}^\infty \frac{(2c)^\ell}{y^{k\ell-k}\ell!}$$
$$\le \frac{1}{y^k} \sum_{\ell=1}^\infty \frac{(2c)^\ell}{N^{k\ell-k}\ell!} = \frac{N^k}{y^k} \sum_{\ell=1}^\infty \frac{(2c)^\ell}{N^{k\ell}\ell!} \le y^{-k}(N^k e^{2c/N^k}).$$

For the other statements, notice that

$$\nabla_g \psi = \nabla_g e^{-2\varphi} = -2e^{-2\varphi} \nabla_g \varphi$$
$$\Delta_g \psi = \Delta_g e^{-2\varphi} = -2e^{-2\varphi} \Delta_g \varphi - 4e^{-2\varphi} |\nabla_g \varphi|^2.$$

□

Later we will also use the fact that the Riemannian distance in the cusp Z satisfies $d_g(z,z') \ge |\log(y/y')|$, for $z = (y,x)$, $z' = (y',x')$, see for example [28].

2.1 Trace class property for relative heat operators of conformal transformations

Take g as background metric on M. Let h be a conformal transformation of the metric g by a conformal factor $e^{2\varphi}$. We assume that on the cusp Z the functions $\varphi(y,x)$, $|\nabla_g \varphi(y,x)|$ and $\Delta_g \varphi(y,x)$ are $O(1/y)$ as y goes to infinity. Recall that this implies that g and h are quasi-isometric and h has bounded curvature.

Let $e^{-t\Delta_h}$ and $e^{-t\Delta_g}$ denote the heat operators corresponding to the Laplacians Δ_h and Δ_g, respectively. Let $K_h(z,z',t)$ and $K_g(z,z',t)$ denote their kernels. As we explained in Chapter 1 we need to consider the unitary map $T : L^2(M, dA_g) \to L^2(M, dA_h)$, $T(f) = e^{-\varphi} f$. Recall the expression for $T^{-1}\Delta_h T$ given by equation (1.4) which implies:

$$\Delta_g - T^{-1}\Delta_h T = (1 - e^{-2\varphi})\Delta_g + e^{-2\varphi}(-2\langle \nabla_g \varphi, \nabla_g \cdot \rangle_g + \Delta_g \varphi + |\nabla_g \varphi|_g^2), \tag{2.2}$$
$$T\Delta_g T^{-1} - \Delta_h = (e^{2\varphi} - 1)\Delta_h - 2e^{2\varphi}\langle \nabla_h \varphi, \nabla_h \cdot \rangle_h + (\Delta_g \varphi - |\nabla_g \varphi|_g^2). \tag{2.3}$$

Note that the functions $e^{-2\varphi}-1$ and $1-e^{2\varphi}$ are $O(y^{-1})$ as $y\to\infty$ and that the functions $e^{2\varphi}$ and $e^{-2\varphi}$ are bounded on M. In this section we use these facts, the decay at infinity of the function φ and its derivatives and the estimates for heat kernels and their derivatives given in Chapter 1.

Theorem 2.3. *Let $h=e^{2\varphi}g$, and assume that on the cusp Z the functions $\varphi(y,x)$, $|\nabla_g\varphi(y,x)|$ and $\Delta_g\varphi(y,x)$ are $O(1/y)$ as $y\to\infty$. Then for any $t>0$ the operator*

$$T^{-1}e^{-t\Delta_h}T-e^{-t\Delta_g}$$

is trace class.

To prove this statement we follow a procedure similar to that used by Müller and Salomonsen in [32]. We use Duhamel's principle which was stated in Section 1.6. Recall how it is given in terms of the operators:

$$T^{-1}e^{-t\Delta_h}T-e^{-t\Delta_g}=\int_0^t T^{-1}e^{-s\Delta_h}T(\Delta_g-T^{-1}\Delta_hT)e^{-(t-s)\Delta_g}\,ds. \tag{2.4}$$

Let $\|\cdot\|$ denote the operator norm and $\|\cdot\|_{1,g}$, ($\|\cdot\|_{1,h}$, resp.), denote the trace norm in $L^2(M,dA_g)$, (in $L^2(M,dA_h)$, resp.), then:

$$\begin{aligned}\|T^{-1}e^{-t\Delta_h}T-e^{-t\Delta_g}\|_{1,g}\le&\int_0^{t/2}\|e^{-s\Delta_h}\|\ \|(\Delta_g-T^{-1}\Delta_hT)e^{-(t-s)\Delta_g}\|_{1,g}\,ds\\&+\int_{t/2}^t\|e^{-s\Delta_h}(T\Delta_gT^{-1}-\Delta_h)\|_{1,h}\ \|e^{-(t-s)\Delta_g}\|\,ds\\ \le\int_0^{t/2}\|(\Delta_g-T^{-1}\Delta_hT)e^{-(t-s)\Delta_g}\|_{1,g}\,ds&+\int_{t/2}^t\|e^{-s\Delta_h}(T\Delta_gT^{-1}-\Delta_h)\|_{1,h}\,ds\end{aligned} \tag{2.5}$$

When considering the trace of the operator in the right hand side of equation (2.4) as an integral using heat kernels and their estimates one has to take two aspects into account. One is related with the time singularity at $t=0$ and the other one is related with the convergence of the space integral. The idea of breaking up the integral in equation (2.5) comes from the need to avoid the time singularities coming from the heat kernel $K_h(z,z',s)$ ($K_g(z,z',t-s)$) close to $s=0$ ($t-s=t$) that do not integrate to something finite in a neighborhood of 0 (of t).

Equation (2.5) reduces the proof of the theorem to the following Lemma:

Lemma 2.4. *Let $0<a<b<\infty$, under the same conditions of theorem 2.3 we have that for each $t\in[a,b]$, the operators*

$$(\Delta_g-T^{-1}\Delta_hT)e^{-t\Delta_g}\quad \textit{and}\quad e^{-t\Delta_h}(T\Delta_gT^{-1}-\Delta_h)$$

are trace class and each trace norm is uniformly bounded on $t\in[a,b]$.

In the proof we will use a choice of auxiliary function ϕ repeatedly, where $\phi\in C^\infty(M)$, satisfies $\phi>0$ and

$$\phi(y,x)=y^{-1/2},\quad (y,x)\in Z. \tag{2.6}$$

Let M_ϕ and M_ϕ^{-1} denote the operators multiplication by ϕ and ϕ^{-1}, respectively.

The proof of Lemma 2.4 is presented in the following two subsections:

2.1.1 Trace class property of $(\Delta_g - T^{-1}\Delta_h T)e^{-t\Delta_g}$

Let us write

$$(\Delta_g - T^{-1}\Delta_h T)e^{-t\Delta_g} = ((\Delta_g - T^{-1}\Delta_h T)e^{-(t/2)\Delta_g} M_\phi^{-1}) \circ (M_\phi e^{-(t/2)\Delta_g}).$$

Let us prove that for every $t > 0$, $(\Delta_g - T^{-1}\Delta_h T)e^{-t\Delta_g} M_\phi^{-1}$ and $M_\phi e^{-t\Delta_g}$ are Hilbert-Schmidt operators. The reason to include the auxiliary function ϕ is that the heat operator $e^{-t\Delta_g}$ itself is not Hilbert-Schmidt but when composed with M_ϕ it is.

The operator $(\Delta_g - T^{-1}\Delta_h T)e^{-t\Delta_g} M_\phi^{-1}$ is Hilbert-Schmidt.

We use equation (2.2) to write the operator as:

$$\begin{aligned}(\Delta_g - T^{-1}\Delta_h T)e^{-t\Delta_g} M_\phi^{-1} = ((1 - e^{-2\varphi(z)})\Delta_g)e^{-t\Delta_g} M_\phi^{-1} \\ + e^{-2\varphi}(-2\langle \nabla_g \varphi, \nabla_g \cdot \rangle_g + (\Delta_g \varphi + |\nabla_g \varphi|_g^2))e^{-t\Delta_g} M_\phi^{-1}.\end{aligned}$$

We start by proving the Hilbert-Schmidt property for $((1 - e^{-2\varphi(z)})\Delta_g)e^{-t\Delta_g} M_\phi^{-1}$. In order to do that we just need to prove that the following integral is finite:

$$\int_M \int_M |(1 - e^{-2\varphi(z)})\Delta_{g,z} K_g(z, z', t)\phi(z')^{-1}|^2 dA_g(z) dA_g(z').$$

Let us use the decomposition of M as $M = M_0 \cup Z$ to split the integral as:

$$\begin{aligned}\int_M \int_M \cdots \ dA_g(z)dA_g(z') = \int_{M_0} \int_{M_0} \cdots \ dA_g(z)dA_g(z') + \int_{M_0} \int_{Z} \cdots \ dA_g(z)dA_g(z') \\ + \int_{Z} \int_{M_0} \cdots \ dA_g(z)dA_g(z') + \int_{Z} \int_{Z} \cdots \ dA_g(z)dA_g(z') \quad (2.7)\end{aligned}$$

and let us prove that each integral is finite. We use the estimates for the derivatives of heat kernel $K_g(z, z', t)$ given in (1.14), the decay of the function $1 - e^{-2\varphi(z)}$ at infinity, which by Lemma 2.2 is the same decay as the one of the function φ, and the definition of the function $i(z)$ given in (1.6). To estimate the resulting integrals we use the lemmas given at the beginning of this chapter. For the sake of simplicity let us just write c instead of $2c$ for the constant in the exponential factor of the estimates of the heat kernels.

For the first term in the sum in equation (2.7) which involves $z \in M_0$ and $z' \in M_0$ we have:

$$\begin{aligned}\int_{M_0} \int_{M_0} |(1 - e^{-2\varphi(z)})\Delta_{g,z} K_g(z, z', t)\phi(z')^{-1}|^2 dA_g(z) dA_g(z') \\ \ll \int_{M_0} \int_{M_0} t^{-4} e^{-\frac{c}{t} d_g^2(z,z')} \ dA_g(z) \ dA_g(z') \ll t^{-4}.\end{aligned}$$

For the second term in the sum in equation (2.7) which involves $z' \in M_0$ and $z \in Z$ we have:

$$\begin{aligned}&\int_{M_0} \int_{Z} |(1 - e^{-2\varphi(z)})\Delta_{g,z} K_g(z, z', t)\phi(z')^{-1}|^2 dA_g(z) dA_g(z') \\ &\ll t^{-4} \int_{M_0} \int_{S^1} \int_1^\infty \frac{1}{y^3} \ e^{-\frac{c}{t} d_g^2((y,x),z')} \ dy \ dx \ dA_g(z') \ll t^{-4}.\end{aligned}$$

The third term in the sum in equation (2.7) involves variables $z \in M_0$ and $z' \in Z$. In this case we use that the Riemannian distance satisfies $d_g(z,z') \geq d_g(\partial Z, z') \geq |\log(y')|$ from which we infer:

$$\begin{aligned}
&\int_Z \int_{M_0} |(1-e^{-2\varphi(z)})\Delta_{g,z} K_g(z,z',t)\phi(z')^{-1}|^2 dA_g(z) dA_g(z') \\
&\quad \ll t^{-4} \int_{S^1} \int_1^\infty \int_{M_0} e^{-\frac{c}{t} d_g^2(z,(y',x'))}\, dA_g(z)\, dx'\, dy' \ll t^{-4} \int_1^\infty e^{-\frac{c}{t}(\log(y'))^2}\, dy' \\
&\quad = t^{-4} \int_0^\infty e^{-\frac{c}{t}u^2} e^u\, du \ll t^{-7/2} e^{t/c'}.
\end{aligned}$$

Finally, the last term in the sum in equation (2.7) in which the variables z, z' lie in Z we have:

$$\begin{aligned}
&\int_Z \int_Z |(1-e^{-2\varphi(z)})\Delta_{g,z} K_g(z,z',t)\phi(z')^{-1}|^2 dA_g(z) dA_g(z') \\
&\qquad \ll t^{-4} \int_1^\infty \int_1^\infty y^{-3} e^{-\frac{c}{t}(\log(y/y'))^2}\, dy\, dy' \ll t^{-7/2} e^{\frac{t}{c}}.
\end{aligned}$$

Thus we obtain:

$$\|(1-e^{-2\varphi})\Delta_g e^{-t\Delta_g} M_\phi^{-1}\|_2^2 \ll t^{-4}\left(1 + t^{1/2} e^{t/c}\right).$$

Now we prove that the operators $e^{-2\varphi}\langle \nabla_g \varphi, \nabla_g \cdot \rangle_g e^{-t\Delta_g} M_\phi^{-1}$ and $e^{-2\varphi}(\Delta_g \varphi + |\nabla_g \varphi|_g^2)) e^{-t\Delta_g} M_\phi^{-1}$ are Hilbert-Schmidt. Their integral kernels are given by

$$e^{-2\varphi(z)}\langle \nabla_{g,z}\varphi(z), \nabla_{g,z} K_g(z,z',t)\rangle_g \phi^{-1}(z') \text{ and } e^{-2\varphi(z)}(\Delta_g \varphi(z) + |\nabla_{g,z}\varphi(z)|_g^2) K_g(z,z',t)\phi^{-1}(z'),$$

respectively. For which we have respectively the following estimates:

$$|e^{-2\varphi(z)}\langle \nabla_{g,z}\varphi(z), \nabla_{g,z} K_g(z,z',t)\rangle_g \phi^{-1}(z')|^2 \ll t^{-3} i(z) i(z') |\nabla_g \varphi(z)|^2 e^{-\frac{c}{t} d_g^2(z,z')} \phi^{-1}(z')^2$$

$$\begin{aligned}
&|e^{-2\varphi(z)}(\Delta_g \varphi(z) + |\nabla_{g,z}\varphi(z)|_g^2 K_g(z,z',t)\phi^{-1}(z')|^2 \\
&\qquad \ll t^{-2}(|\Delta_g \varphi(z)| + |\nabla_g \varphi(z)|_g^2)^2 i(z) i(z') e^{-\frac{c}{t} d_g^2(z,z')} \phi^{-1}(z')^2.
\end{aligned}$$

We split the integrals on $M \times M$ in the same way as in equation (2.7), and the integrals obtained are very similar to those carried out in the previous part for the operator $(1-e^{-2\varphi})\Delta_g e^{-t\Delta_g}$. The main difference occurs in the power of t. For the operator $e^{-2\varphi}\langle \nabla_g \varphi, \nabla_g \cdot \rangle_g e^{-t\Delta_g} M_\phi^{-1}$ we use the estimates in equation (1.13) and the decay of the function $|\varphi|$ at infinity. Let us check the following two cases:

When $z \in M_0$, $z' \in Z$:

$$\begin{aligned}
&\int_Z \int_{M_0} |e^{-2\varphi(z)}\langle \nabla_{g,z}\varphi(z), \nabla_{g,z} K_g(z,z',t)\rangle_g \phi^{-1}(z')|^2 dA_g(z) dA_g(z') \\
&\qquad \ll t^{-3} \int_1^\infty y'^2 e^{-\frac{c}{t}(\log(y'))^2} \frac{dy'}{y'^2} \ll t^{-5/2} e^{t/c'}.
\end{aligned}$$

When $z \in Z$, $z' \in Z$:

$$\begin{aligned}
&\int_Z \int_Z |e^{-2\varphi(z)}\langle \nabla_{g,z}\varphi(z), \nabla_{g,z} K_g(z,z',t)\rangle_g \phi^{-1}(z')|^2 dA_g(z) dA_g(z') \\
&\quad \ll t^{-3} \int_1^\infty \int_1^\infty y y' \frac{1}{y^2} y' e^{-\frac{c}{t}(\log(y/y'))^2} \frac{dy}{y^2} \frac{dy'}{y'^2} = t^{-3} \int_1^\infty \int_1^\infty y^{-3} e^{-\frac{c}{t}(\log(y/y'))^2} dy dy' \ll t^{-5/2} e^{\frac{t}{c}}.
\end{aligned}$$

Now for the operator $e^{-2\varphi}(\Delta_g\varphi + |\nabla_g\varphi|_g^2))e^{-t\Delta_g}M_\phi^{-1}$ we use the estimate of the heat kernel given in equation (1.8) and the decay of the functions involving φ. Let us check only the integral on $Z\times Z$. For $z\in Z$ we have $(\Delta_g\varphi(z) + |\nabla_{g,z}\varphi(z)|_g^2)^2 \ll (y^{-1}+y^{-2})^2 \ll y^{-2}$. Then

$$\int_Z\int_Z |e^{-2\varphi(z)}(\Delta_g\varphi(z) + |\nabla_{g,z}\varphi(z)|_g^2)K_g(z,z',t)\phi^{-1}(z')|^2 dA_g(z)dA_g(z')$$
$$\ll t^{-2}\int_1^\infty\int_1^\infty y^{-3}e^{-\frac{c}{t}(\log(y/y'))^2}dydy' \ll t^{-3/2}e^{\frac{t}{c}}.$$

Thus in the same way as above we obtain:

$$\|e^{-2\varphi}\langle\nabla_g\varphi, \nabla_g\,\cdot\,\rangle_g e^{-t\Delta_g}M_\phi^{-1}\|_2^2 \ll t^{-3}\left(1+t^{1/2}e^{t/c}\right)$$
$$\|e^{-2\varphi}(\Delta_g\varphi + |\nabla_g\varphi|_g^2)e^{-t\Delta_g}M_\phi^{-1}\|_2^2 \ll t^{-2}\left(1+t^{1/2}e^{t/c}\right)$$

The operator $M_\phi e^{-t\Delta_g}$ is Hilbert-Schmidt

In order to prove this, we have to prove that the following integral is finite:

$$\int_M\int_M |\phi(z)K_g(z,z',t)|^2 dA_g(z)dA_g(z').$$

We decompose the integral as in equation (2.7), and prove that each integral is finite. We use here the estimates for $K_g(z,z',t)$ given in (1.8) and the definition of the functions ϕ and $i(z)$. Again, for the sake of simplicity we just write c instead of $2c$ in the exponential factor of the heat estimates. The computations are very similar to those in the previous case.

For the first term in which $z,z'\in M_0$ we have:

$$\int_{M_0}\int_{M_0} |\phi(z)K_g(z,z',t)|^2 dA_g(z)dA_g(z') \ll \int_{M_0}\int_{M_0} t^{-2}e^{-\frac{c}{t}d_g^2(z,z')}\ dA_g(z)\ dA_g(z') \ll t^{-2}.$$

For the second term which involves $z'\in M_0$ and $z\in Z$ we have:

$$\int_{M_0}\int_Z |\phi(z)K_g(z,z',t)|^2 dA_g(z)dA_g(z')$$
$$\ll t^{-2}\int_{M_0}\int_{S^1}\int_1^\infty \frac{1}{y^2}\ e^{-\frac{c}{t}d_g^2((y,x),z')}\ dy\ dx\ dA_g(z') \ll t^{-2}.$$

For the third term in the sum in which $z\in M_0$ and $z'\in Z$ we have:

$$\int_Z\int_{M_0} |\phi(z)K_g(z,z',t)|^2 dA_g(z)dA_g(z') \ll t^{-2}\int_1^\infty \frac{1}{y'}e^{-\frac{c}{t}(\log(y'))^2}\ dy'$$
$$= t^{-2}\int_0^\infty e^{-\frac{c}{t}u^2}\ du \ll t^{-3/2}.$$

Finally, the last term in which the variables z,z' lie in Z satisfies:

$$\int_Z\int_Z |\phi(z)K_g(z,z',t)|^2 dA_g(z')dA_g(z) \ll \int_1^\infty\int_1^\infty \frac{1}{y}\ y\ y' t^{-2}e^{\frac{-c}{t}(\log(y/y'))^2}\frac{dy'}{y'^2}\frac{dy}{y^2} \tag{2.8}$$
$$= t^{-2}\int_1^\infty\int_1^\infty \frac{1}{y^2}\frac{1}{y'}e^{\frac{-c}{t}(\log(y/y'))^2}\ dy\ dy' \le t^{-3/2}e^{c't}. \tag{2.9}$$

Therefore

$$\|M_\phi e^{-t\Delta_g}\|_2^2 \ll t^{-2} + t^{-3/2}e^{t/4c}.$$

In this way we have that $(\Delta_g - T^{-1}\Delta_h T)e^{-t\Delta_g}$ is a trace class operator and

$$\begin{aligned}\|(\Delta_g - T^{-1}\Delta_h T)e^{-t\Delta_g}\|_{1,g} &\leq \|(\Delta_g - T^{-1}\Delta_h T)e^{-(t/2)\Delta_g}M_\phi^{-1}\|_2 \cdot \|M_\phi e^{-(t/2)\Delta_g}\|_2 \\ &\ll (t^{-2}+t^{-3}+t^{-4})^{1/2}\left(1+t^{1/2}e^{t/c}\right)^{1/2}\left(t^{-2}+t^{-3/2}e^{t/c'}\right)^{1/2}\end{aligned}$$

the last expression is integrable for t in compact subsets of $(0,\infty)$.

2.1.2 Trace class property of $e^{-t\Delta_h}(T\Delta_g T^{-1} - \Delta_h)$

The proof is very similar to the proof for $(\Delta_g - T^{-1}\Delta_h T)e^{-t\Delta_g}$ since the heat kernels satisfy the same estimates, and the metrics are quasi-isometric. Let us write:

$$e^{-t\Delta_h}(T\Delta_g T^{-1} - \Delta_h) = (e^{-(t/2)\Delta_h}M_\phi)\circ(M_\phi^{-1}e^{-(t/2)\Delta_h}(T\Delta_g T^{-1} - \Delta_h)),$$

where $\phi \in C^\infty(M)$ is as above. Then we have to prove that for every $t>0$, the operators $e^{-t\Delta_h}M_\phi$ and $M_\phi^{-1}e^{-t\Delta_h}(T\Delta_g T^{-1} - \Delta_h)$ are Hilbert-Schmidt, i.e. that their kernels are square integrable.

The operator $M_\phi^{-1}e^{-t\Delta_h}(T\Delta_g T^{-1} - \Delta_h)$ is Hilbert-Schmidt

First of all let us consider the kernel of the operator $e^{-t\Delta_h}(T\Delta_g T^{-1} - \Delta_h)$. Let $f \in C_c^\infty(M)$. Then we have:

$$\begin{aligned}(e^{-t\Delta_h}(T\Delta_g T^{-1} - \Delta_h)f)(z) &= \int_M K_h(z,z',t)\cdot(T\Delta_{g,z'}T^{-1} - \Delta_{h,z'})f(z')dA_h(z') \\ &= \int_M ((T\Delta_{g,z'}T^{-1} - \Delta_{h,z'})K_h(z,z',t))\cdot f(z')dA_h(z')\end{aligned}$$

since the operators $T\Delta_{g,z'}T^{-1}$ and Δ_h are symmetric on $L^2(M, dA_h)$. Now, use equation (2.3):

$$M_\phi^{-1}(T\Delta_g T^{-1} - \Delta_h)e^{-t\Delta_h} = M_\phi^{-1}e^{-t\Delta_h}\{(e^{2\varphi}-1)\Delta_h - 2e^{2\varphi}\langle\nabla_h\varphi, \nabla_h\,\cdot\,\rangle_h + (\Delta_g\varphi - |\nabla_g\varphi|_g^2)\}.$$

It follows that $M_\phi^{-1}e^{-t\Delta_h}(T\Delta_g T^{-1} - \Delta_h)$ is Hilbert-Schmidt if the following functions

1. $\phi(z)^{-1}(e^{2\varphi}(z')-1)\Delta_{h,z'}K_h(z,z',t)$,
2. $\phi(z)^{-1}e^{2\varphi(z')}\langle\nabla_{h,z'}\varphi, \nabla_{h,z'}K_h\rangle_h$ and
3. $\phi(z)^{-1}(\Delta_g\varphi(z') - |\nabla_{g,z'}\varphi|_g^2)K_h(z,z',t)$

are in $L^2(M\times M, dA_h dA_h)$.

We split the integral in the same way as in equation (2.7) and use the estimates on the heat kernel $K_h(z,z',t)$ and its derivatives given in equations (1.12), (1.13) and (1.14). We also use that since φ is bounded on M and since $dA_h = e^{2\varphi}dA_g$, then for any function $f \in L^1(M, dA_h)$ we have:

$$\int_M |f|dA_h \ll \int_M |f|dA_g.$$

For the first function listed above, the integrals are almost the same as the ones corresponding to the operator $(1-e^{-2\varphi})\Delta_g e^{-t\Delta_g} M_\phi^{-1}$. We obtain:

$$\int_M \int_M |\phi(z)^{-1}(e^{2\varphi(z')}-1)\Delta_{h,z'} K_h(z,z',t)|^2 dA_h(z) dA_h(z') \ll t^{-4} + t^{-7/2} e^{t/c}$$

for some constant $c > 0$. Similarly for the other two functions we get bounds by $t^{-3}(1+t^{1/2}e^{t/c})$ and $t^{-2}(1+t^{1/2}e^{t/c})$, respectively. Combining these estimates we obtain:

$$\|M_\phi^{-1} e^{-t\Delta_h}(T\Delta_g T^{-1} - \Delta_h)\|_2^2 \ll (t^{-4}+t^{-3}+t^{-2})(1+t^{1/2}e^{t/c}).$$

$e^{-t\Delta_h} M_\phi$ is Hilbert-Schmidt

The kernel of $e^{-t\Delta_h} M_\phi$ is $K_h(z,z',t)\phi(z')$. Since the estimates for the kernel $K_h(z,z',t)$ are, up to some constants, the same as the estimates for $K_g(z,z',t)$, we can use exactly the same proof as for the operator $M_\phi e^{-t\Delta_g}$ to show that:

$$\|e^{-t\Delta_h} M_\phi\|_2^2 \ll t^{-2}(1+\sqrt{t}e^{\frac{t}{c}}).$$

Finally, for the operator $e^{-t\Delta_h}(T\Delta_g T^{-1} - \Delta_h)$ we obtain:

$$\begin{aligned}\|e^{-t\Delta_h}(T\Delta_g T^{-1} - \Delta_h)\|_{1,h} &\le \|e^{-(t/2)\Delta_h} M_\phi\|_2 \cdot \|M_\phi^{-1} e^{-(t/2)\Delta_h}(T\Delta_g T^{-1} - \Delta_h)\|_2 \\ &\ll t^{-1}(t^{-4}+t^{-3}+t^{-2})^{1/2}\left(1+\sqrt{t}e^{t/c}\right)\end{aligned}$$

This expression is clearly integrable for t on compact subsets of $(0,\infty)$.

This finishes the proofs of Lemma 2.4 and Theorem 2.3.

2.2 Operators on the cusp

Proposition 2.5. *The operator $e^{-t\Delta_g} - e^{-t\Delta_{Z,D}}$ is trace class for all $t > 0$, where $e^{-t\Delta_{Z,D}}$ is considered as acting on $L^2(M, dA_g)$.*

This is a corollary of Proposition 6.4 in [28]. The statement of that proposition can be rewritten in our notation as follows:

Assume that M can be decomposed as $M = M_0 \cup Z$ with $Z = [1,\infty) \times S^1$. Let P_0 be the orthogonal projection of $L^2(M, dA_g)$ onto $L^2([1,\infty), y^{-2}dy)$. Then for every $t > 0$, $e^{-t\Delta_g} - e^{-t\Delta_{1,0}} P_0$ is a trace class operator.

To see that Proposition 2.5 follows from this statement, recall what we explained in section 1.1: the operator $\Delta_{Z,D}$ can be decomposed as $\Delta_{Z,D} = \Delta_{1,0} \oplus \Delta_{Z,1}$, where the heat operator $e^{-t\Delta_{Z,1}}$ is trace class. So we have:

$$\|e^{-t\Delta_g} - e^{-t\Delta_{Z,D}}\|_1 = \|e^{-t\Delta_g} - e^{-t\Delta_{1,0}}\|_1 + \|e^{-t\Delta_{Z,1}}\|_1$$

Now, let us consider the operator $\Delta_{a,0}$ for $a > 1$. To see that $e^{-t\Delta_g} - e^{-t\Delta_{a,0}}$ is trace class, we will proceed by writing the difference as

$$e^{-t\Delta_g} - e^{-t\Delta_{a,0}} = e^{-t\Delta_g} - e^{-t\Delta_{1,0}} + e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}.$$

By the proposition above, the first difference is trace class, so it suffices to show that $e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}$ is trace class.

Proposition 2.6. *For any $a > 1$ and $t > 0$ the operator $e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}$ acting on $L^2([1,\infty), y^{-2}dy)$ is trace class and the trace is given by:*

$$\mathrm{Tr}(e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}) = -\frac{1}{\sqrt{4\pi t}}\ e^{-t/4}\log(a).$$

As an operator on $L^2([a,\infty), y^{-2}dy)$ the trace is given by:

$$\mathrm{Tr}(e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}) = -\frac{e^{-t/4}}{\sqrt{4\pi}} Erf(\log(a)/\sqrt{t}),$$

where $Erf(s) = \int_0^s e^{-v^2}dv$.

Proof. This proof uses a classical method and the explicit expression for each heat kernel, see Section 1.5.2. Recall equation (1.9):

$$p_a(y,y',t) = \frac{e^{-t/4}}{\sqrt{4\pi t}}\ (yy')^{1/2}\left\{e^{-(\log(y/y'))^2/4t} - e^{-(\log(yy')-\log(a^2))^2/4t}\right\},$$

for $y, y' > a$; for $1 \le y \le a$, $p_a(y,y',t) = 0$. First note that $e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}$ is a Hilbert Schmidt operator:

$$\begin{aligned}\int_1^\infty\int_1^\infty |p_a(y,y',t) - p_1(y,y',t)|^2\frac{dy'}{y'^2}\frac{dy}{y^2} &= \int_a^\infty\int_a^\infty |p_a(y,y',t) - p_1(y,y',t)|^2\frac{dy'}{y'^2}\frac{dy}{y^2}\\ &+ \int_1^a\int_1^\infty |p_1(y,y',t)|^2\frac{dy'}{y'^2}\frac{dy}{y^2} + \int_a^\infty\int_1^a |p_1(y,y',t)|^2\frac{dy'}{y'^2}\frac{dy}{y^2}\end{aligned} \tag{2.10}$$

For the first integral in the right hand side of the previous equation we use that $(\log(y/a) + \log(y'/a))^2 \ge \log(y/a)^2$ and a similar expression for y' to obtain:

$$\begin{aligned}&\int_a^\infty\int_a^\infty |p_a(y,y',t) - p_1(y,y',t)|^2\frac{dy'}{y'^2}\frac{dy}{y^2}\\ &= \frac{e^{-t/2}}{4\pi t}\int_a^\infty\int_a^\infty (yy')^{-1}(-e^{-\frac{\log(yy'/a^2)^2}{4t}} + e^{-\frac{\log(yy')^2}{4t}})^2 dy'dy\\ &\le \frac{e^{-t/2}}{4\pi t}\int_a^\infty\int_a^\infty (yy')^{-1}(e^{-\frac{(\log(y/a)+\log(y'/a))^2}{2t}} + e^{-\frac{\log(yy')^2}{2t}})dy'dy\\ &\le \frac{e^{-t/2}}{4\pi t}\int_a^\infty\int_a^\infty (y^{-1}e^{-\frac{\log(y/a)^2}{4t}}y'^{-1}e^{-\frac{\log(y'/a)^2}{4t}} + y^{-1}e^{-\frac{\log(y)^2}{4t}}y'^{-1}e^{-\frac{\log(y')^2}{4t}})dy'dy.\end{aligned}$$

All the integrals involved are clearly finite. For the other terms in the right hand side of equation (2.10), using the symmetry of the kernel $p_1(y,y',t) = p_1(y',y,t)$ we have that

$$\begin{aligned}&\int_1^a\int_1^\infty |p_1(y,y',t)|^2\frac{dy'}{y'^2}\frac{dy}{y^2} + \int_a^\infty\int_1^a |p_1(y,y',t)|^2\frac{dy'}{y'^2}\frac{dy}{y^2} \le 2\int_1^a\int_1^\infty |p_1(y,y',t)|^2\frac{dy'}{y'^2}\frac{dy}{y^2}\\ &\ll \frac{e^{-t/2}}{4\pi t}\int_1^a\int_1^\infty (yy')^{-1}e^{-(\log(y/y'))^2/2t}dy'dy + \frac{e^{-t/2}}{4\pi t}\int_1^a\int_1^\infty (yy')^{-1}e^{-(\log(yy'))^2/2t}dy'dy\end{aligned}$$

The second integral in the right hand side of last equation is computed in the same way as above. For the first integral in the right hand side, we make the usual change of variables: $u = \log(y/y')$ to obtain:

$$\int_1^a \int_1^\infty (yy')^{-1} e^{-(\log(y/y'))^2/2t} dy' dy = \int_1^a y^{-1} dy \int_{-\infty}^\infty e^{-u^2/2t} du \ll \sqrt{t}.$$

Therefore the operator $e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}$ is Hilbert-Schmidt, for every $t > 0$.

The second step is to decompose the difference as the following sum:

$$e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}} = e^{-t/2\Delta_{a,0}} M_\phi \cdot M_\phi^{-1}(e^{-t/2\Delta_{a,0}} - e^{-t/2\Delta_{1,0}}) + (e^{-t/2\Delta_{a,0}} - e^{-t/2\Delta_{1,0}}) M_\phi^{-1} \cdot M_\phi e^{-t/2\Delta_{1,0}},$$

where M_ϕ is multiplication by the function ϕ defined in equation (2.6). We shall prove that each term is Hilbert Schmidt. Let us start with $e^{-t/2\Delta_{a,0}} M_\phi$ and let us set $s = t/2$:

$$\|e^{-s\Delta_{a,0}} M_\phi\|_2^2 \le \frac{e^{-s/2}}{4\pi s} \int_a^\infty \int_a^\infty y'^{-2} y^{-1} \Big(e^{-\frac{\log(y/y')^2}{2s}} + e^{-\frac{\log(yy'/a^2)^2}{2s}} \\ + 2e^{-\frac{\log(y/y')^2}{4s}} e^{-\frac{\log(yy'/a^2)^2}{4s}}\Big)\, dy' dy$$

The first integral in the right hand side is estimated by $C\sqrt{t}$, as in equation (2.1). For the other terms we have:

$$\int_a^\infty \int_a^\infty y'^{-2} y^{-1} \Big(e^{-\frac{\log(yy'/a^2)^2}{2s}} + 2e^{-\frac{\log(y/y')^2}{4s}} e^{-\frac{\log(yy'/a^2)^2}{4s}}\Big)\, dy' dy \\ \ll \int_a^\infty \int_a^\infty y'^{-2} y^{-1} e^{-\frac{\log(yy'/a^2)^2}{4s}}\, dy' dy \ll \int_a^\infty y^{-1} e^{-\frac{\log(y/a)^2}{4s}}\, dy < \infty.$$

For $M_\phi^{-1}(e^{-t/2\Delta_{a,0}} - e^{-t/2\Delta_{1,0}})$, let $s = t/2$ then in the same way as above we obtain

$$\begin{aligned} \|M_\phi^{-1}(e^{-t/2\Delta_{a,0}} - e^{-t/2\Delta_{1,0}})\|_2^2 &\le \frac{e^{-s/2}}{4\pi s} \int_a^\infty \int_a^\infty y'^{-1} \Big(e^{-\frac{\log(yy'/a^2)^2}{2s}} + e^{-\frac{\log(yy')^2}{2s}}\Big) dy' dy \\ &\quad + \int_1^a \int_1^\infty |p_1(y,y',s)|^2 \frac{dy'}{y'^2} \frac{dy}{y} + \int_1^\infty \int_1^a |p_1(y,y',s)|^2 \frac{dy'}{y'^2} \frac{dy}{y} \\ &= \frac{e^{-s/2}}{4\pi s} \Big(\int_a^\infty \int_a^\infty y'^{-1} e^{-\frac{\log(yy'/a^2)^2}{2s}} dy' dy + \int_1^\infty \int_1^\infty y'^{-1} e^{-\frac{\log(yy')^2}{2s}} dy' dy \\ &\quad + \int_1^a \int_1^\infty y'^{-1} e^{-\frac{\log(y/y')^2}{2s}} dy' dy + \int_1^a \int_1^\infty y'^{-1} e^{-\frac{\log(y/y')^2}{2s}} dy dy' \Big). \end{aligned} \tag{2.11}$$

The fact that both integrals in the third line in equation (2.11) are finite follows from:

$$\begin{aligned} \int_a^\infty \int_a^\infty y'^{-1} e^{-\frac{\log(yy'/a^2)^2}{2s}} dy' dy &\le \int_a^\infty e^{-\frac{\log(y/a)^2}{4s}} \int_a^\infty y'^{-1} e^{-\frac{\log(y'/a)^2}{4s}} dy' dy \\ &= \left(\int_{\log(a)}^\infty e^{-\frac{u^2}{4s}} e^u du \right) \left(\int_{\log(a)}^\infty e^{-\frac{u'^2}{4s}} du' \right) < \infty. \end{aligned} \tag{2.12}$$

The finiteness of the fourth line of equation (2.11) follows from a similar argument as in equation (2.12).

The estimate of the Hilbert Schmidt norm of the other operators follows in the same way. This finishes the proof of the trace class property. Now, let us compute the trace:

$$
\begin{aligned}
&\mathrm{Tr}(e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}) = \int_1^\infty (p_a(y,y,t) - p_1(y,y,t))\frac{dy}{y^2} \\
&= -\int_1^a p_1(y,y,t)\frac{dy}{y^2} + \int_a^\infty (p_a(y,y,t) - p_1(y,y,t))\frac{dy}{y^2} \\
&= -\frac{e^{-t/4}}{\sqrt{4\pi t}}\int_1^a (1 - e^{-(\log(y^2))^2/4t})\frac{dy}{y} + \frac{e^{-t/4}}{\sqrt{4\pi t}}\int_a^\infty (e^{-(\log(y^2))^2/4t} - e^{-(\log(y^2)-\log(a^2))^2/4t})\frac{dy}{y} \\
&= \frac{e^{-t/4}}{\sqrt{4\pi t}}\left\{-\log(a) + \int_1^\infty e^{-(\log(y))^2/t}\frac{dy}{y} - \int_a^\infty e^{-(\log(y/a))^2/t}\frac{dy}{y}\right\} = -\frac{e^{-t/4}}{\sqrt{4\pi t}}\log(a),
\end{aligned}
$$

where the two integrals in the last line cancel after the change of variables $v = y/a$ in the latest one and noticing that dy/y is invariant under such change. If we consider $e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}$ as an operator acting on $L^2([a,\infty), y^{-2}dy)$ we have that

$$
\mathrm{Tr}(e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}) = \int_a^\infty (p_a(y,y,t) - p_1(y,y,t))\frac{dy}{y^2} = -\frac{e^{-t/4}}{\sqrt{4\pi t}}\int_1^a e^{-(\log(y))^2/t}\frac{dy}{y}.
$$

This finishes the proof of the Proposition. $\square$

Remark 2.7. *The trace of $e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}}$ as an operator on $L^2([a,\infty), y^{-2}dy)$ has an asymptotic expansion for small values of t. This follows from Proposition 2.6 and the fact that $Erf(x)$ has an expansion for $x \gg 1$. Taking into account only the first term we have that $Erf(x) = \frac{\sqrt{\pi}}{2} + O(x^{-1})$, as $x \to \infty$ from which we infer that:*

$$
\mathrm{Tr}(e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}})_{L^2([a,\infty),y^{-2}dy)} = -\frac{1}{4} + O(\sqrt{t}) \qquad \text{as } t \to 0.
$$

Remark 2.8. *Let us study now the case when the manifold M can be decomposed as $M = M_0 \cup Z_a$ with $a \geq 1$ and we want to compare the operators $e^{-t\Delta_g}$ and $e^{-t\Delta_{1,0}}$. In this case we could consider the operator $e^{-t\Delta_{1,0}}$ acting on $L^2(M, dA_g)$ in the way explained in subsection 1.5.2. However it is more convenient and accurate to consider the extended space:*

$$
\begin{aligned}
L^2(M, dA_g) \oplus L^2([1,a], y^{-2}dy) &= L^2(M_0, dA_g) \oplus L^2_0(Z_a) \oplus L^2([a,\infty), y^{-2}dy) \oplus L^2([1,a], y^{-2}dy) \\
&= L^2(M_0, dA_g) \oplus L^2_0(Z_a) \oplus L^2([1,\infty), y^{-2}dy) \qquad (2.13)
\end{aligned}
$$

where $L^2_0(Z_a)$ is the space defined in equation (1.2). Then the operators $e^{-t\Delta_g}$ and $e^{-t\Delta_{1,0}}$ act on the extended space by being null where they are not defined. In this way we have that

$$
\mathrm{Tr}(e^{-t\Delta_g} - e^{-t\Delta_{1,0}})_{L^2(M)\oplus L^2([1,a])} = \mathrm{Tr}(e^{-t\Delta_g} - e^{-t\Delta_{a,0}})_{L^2(M)} + \mathrm{Tr}(e^{-t\Delta_{a,0}} - e^{-t\Delta_{1,0}})_{L^2([1,\infty))} \quad (2.14)
$$

where for the sake of simplicity we dropped the densities in the notation of the L^2 spaces.

Chapter 3

Expansion of relative heat traces for small time

In Chapter 2 we proved that under suitable conditions $T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_g}$ is a trace class operator. In this chapter we prove the existence of an expansion in t up to order two of the relative heat trace $\mathrm{Tr}(T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_g})$ for small time. Our goal was to prove existence of a complete asymptotic expansion of the relative heat trace as $t \to 0$ under the same conditions of Theorem 2.3; namely that φ and its derivatives up to order two decay as y^{-1} at infinity. We were not able to completely realize this, but we did identify a condition for the existence of the expansion up to order two which is enough to define the relative determinant of the pair (Δ_h, Δ_g). We give an explicit sufficient rate of decay, but we expect this could be improved.

3.1 Expansion of the trace of $T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_g}$

Let (M, g) be as in Chapter 1. For the sake of simplicity we assume that (M, g) has only one cusp $Z \cong [1, \infty)$ with the hyperbolic metric on it. We take g as the background metric on M. Let h be a conformal transformation of the metric g by a conformal factor $e^{2\varphi}$. To start with, let us assume that for $(y, x) \in Z$, the functions $\varphi(y, x)$ and $\Delta_g\varphi(y, x)$ are $O(y^{-1})$ as $y \to \infty$.

Let $e^{-t\Delta_h}$ and $e^{-t\Delta_g}$ denote the heat operators corresponding to the Laplacians Δ_h and Δ_g, respectively. Let $K_h(z, z', t)$ and $K_g(z, z', t)$ denote their kernels.

As before, for $\alpha > 1$ let

$$M_\alpha := M_0 \cup ([1, \alpha] \times S^1), \quad Z'_\alpha = [1, \alpha] \times S^1, \quad Z_\alpha = [\alpha, \infty) \times S^1.$$

We start by constructing the kernel of a parametrix $Q_h(z, w, t)$ for the heat operator associated to Δ_h by patching together suitable heat kernels over $Z'_3 = M_3 \cap Z = [1, 3] \times S^1$. The construction is similar to the one in Section 1.5.1. Let us consider the following kernels:

- $K_{1,h}(z, w, t)$: the heat kernel for $\Delta_{1,h}$ on the complete cusp $\widetilde{Z} = \mathbb{R}^+ \times S^1$, as was defined at the end of Section 1.5.1.

- $K_{Z,h}(z, w, t)$: the heat kernel for $\Delta_{Z,h}$, the self-adjoint extension of the Laplacian on (Z, h) obtained after imposing Dirichlet boundary conditions at $\{1\} \times S^1$. By equation (1.15) the kernel $K_{Z,h}$ is given by $K_{1,h} + p_{h,D}$, where the term $p_{h,D}$ comes from the boundary condition.

- For the compact part we consider a closed manifold W containing M_2 isometrically. Let $\Delta_{W,h}$ be the Laplacian on W and $K_{W,h}(z,w,t)$ be the kernel of the corresponding heat operator $e^{-t\Delta_W}$.

For any two constants $1 < b < c$, let $\phi_{(b,c)}$ be a smooth function on $[1,\infty)\times S^1$ that is constant in the second variable, is non-decreasing in the first variable, and satisfies $\phi_{(b,c)}(y,x) = 0$ for $y \le b$, and $\phi_{(b,c)}(y,x) = 1$ for $y \ge c$. Let $\psi_2 = \phi_{(\frac{5}{4},2)}$ and $\psi_1 = 1-\psi_2$; then $\{\psi_1,\psi_2\}$ is a partition of unity on $[1,2]\times S^1$. Let $\varphi_2 = \phi_{(1,\frac{9}{8})}$ and $\varphi_1 = 1-\phi_{(\frac{5}{2},3)}$, so that $\varphi_i = 1$ on the support of ψ_i, $i = 1,2$. Extend these functions to M in the obvious way. Note that $|\nabla_h\varphi_i(z)| \ll 1$ and $|\Delta_h\varphi_i(z)| \ll 1$, for $i = 1,2$. For this choice of functions we have that:

- $\operatorname{supp}\nabla_h\varphi_1 \subseteq [\frac{5}{2},3]\times S^1$, and, $\operatorname{supp}\psi_1 \subseteq M_2$.
- $\operatorname{supp}\nabla_h\varphi_2 \subseteq [1,\frac{9}{8}]\times S^1$, and, $\operatorname{supp}\psi_2 \subseteq [\frac{5}{4},\infty)\times S^1$.

Now, we put:

$$Q_h(z,w,t) = \varphi_1(z)K_{W,h}(z,w,t)\psi_1(w) + \varphi_2(z)K_{1,h}(z,w,t)\psi_2(w). \tag{3.1}$$

From the properties of the heat kernels, $K_{W,h}$ and $K_{1,h}$, and the construction of the gluing functions it is easy to see that $Q_h(z,w,t) \to \delta_{w-z}$, as $t \to 0$.

Lemma 3.1. *There exist constants $C \ge 0$ and $c > 0$ such that*

$$\left|\left(\frac{\partial}{\partial t} + \Delta_{h,z}\right) Q_h(z,w,t)\right| \le Ce^{-c/t}, \qquad \textit{for} \qquad 0 < t \le 1.$$

Proof. Here we use the estimates for the heat kernels given in Chapter 1 (see (1.12), (1.13) and (1.14) as well as Theorem 2.3) and the equivalence of the geodesic distances d_g and d_h.

$$\begin{aligned}(\frac{\partial}{\partial t}+\Delta_{h,z})Q_h(z,w,t) &= \varphi_1(z)((\frac{\partial}{\partial t}+\Delta_{h,z})K_{W,h})\psi_1(w) + (2\langle\nabla\varphi_1,\nabla_z K_{W,h}\rangle + (\Delta_h\varphi_1)K_{W,h})\psi_1(w)\\ &+ \varphi_2(z)((\frac{\partial}{\partial t}+\Delta_{h,z})K_{1,h})\psi_2(w) + (2\langle\nabla\varphi_2,\nabla_z K_{1,h}\rangle + (\Delta_h\varphi_2)K_{1,h})\psi_2(w).\end{aligned}$$

Because the kernels satisfy the heat equation it follows that:

$$\begin{aligned}\left|\left(\frac{\partial}{\partial t} + \Delta_{h,z}\right) Q_h(z,w,t)\right| &\ll |((\langle\nabla\varphi_1,\nabla_z K_{W,h}\rangle + (\Delta_h\varphi_1)K_{W,h})\psi_1(w)|\\ &+ |((\langle\nabla\varphi_2,\nabla_z K_{1,h}\rangle + (\Delta_h\varphi_2)K_{1,h})\psi_2(w)|.\end{aligned}$$

Note that $\left|\left(\frac{\partial}{\partial t} + \Delta_{h,z}\right) Q_h(z,w,t)\right|$ has compact support in z. Let us consider the following terms separately:

$$\begin{aligned}S_1 &:= |((\langle\nabla\varphi_1,\nabla_z K_{W,h}\rangle + (\Delta_h\varphi_1)K_{W,h})\psi_1(w)|,\\ S_2 &:= |((\langle\nabla\varphi_2,\nabla_z K_{1,h}\rangle + (\Delta_h\varphi_2)K_{1,h})\psi_2(w)|.\end{aligned}$$

Notice that $S_1 \ne 0$ if $z \in \operatorname{supp}\nabla\varphi_1$ and $w \in \operatorname{supp}\psi_1$. In this case $d_g(z,w) \ge \log((5/2)/2)) = \log(5/4) > 0$, so that taking $c_1' = c\log(5/4)$ we obtain:

$$\begin{aligned}S_1 &\le (|\nabla\varphi_1(z)|\ |\nabla_z K_{W,h}(z,w,t)| + |\Delta_h\varphi_1(z)|\ |K_{W,h}(z,w,t)|)\chi_{\operatorname{supp}\psi_1}(w)\\ &\ll t^{-3/2}e^{-cd_g^2(z,w)/t} + t^{-1}e^{-cd_g^2(z,w)/t}\\ &\le (t^{-3/2}+t^{-1})e^{-c_1'/t} \ll e^{-c_1'/2t},\end{aligned}$$

since $(t^{-3/2}+t^{-1})e^{-c_1'/2t}$ is bounded for $0 < t \leq 1$. In the same way as above, note that $S_2 \neq 0$ if $z \in \operatorname{supp} \nabla\varphi_2$ and $w = (v,u) \in \operatorname{supp}\psi_2 = [\frac{5}{4}, \infty) \times S^1$. In this case $d_g(z,w) \geq \log(v/(9/8)) \geq \log(10/9) > 0$. Therefore:

$$\begin{aligned} S_2 &\leq (|\nabla\varphi_2(z)|\; |\nabla_z K_{1,h}(z,w,t)| + |\Delta_h\varphi_2(z))|\; |K_{1,h}(z,w,t)|)\chi_{\operatorname{supp}\psi_2}(w) \\ &\ll (i(z)i(w))^{1/2}t^{-3/2}e^{-cd_g^2(z,w)/t} + (i(z)i(w))^{1/2}t^{-1}e^{-cd_g^2(z,w)/t} \\ &\ll v^{1/2}e^{-c(\log(8v/9))^2/2t}(t^{-3/2}+t^{-1})e^{-c_2'/2t} \ll \sqrt{v}e^{-c(\log(8v/9))^2/2}e^{-c_2'/4t} \ll e^{-c_2'/4t}, \end{aligned}$$

where $c_2' = c\log(10/9)$. This finishes the proof of the lemma. $\square$

Remark 3.2. *Note that*

$$\left(\frac{\partial}{\partial t} + \Delta_{h,z}\right) Q_h(z,w,t)\bigg|_{w=z} = 0.$$

In order that the expression above does not vanish we need that

$$d_g(z,w) \geq \min\{\log(5/4), \log(10/9)\} > 0.$$

To see this, consider the following:

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta_{h,z}\right) Q_h(z,w,t) &= (2\langle\nabla\varphi_1(z), \nabla_z K_{W,h}(z,w,t)\rangle + (\Delta_h\varphi_1(z))K_{W,h}(z,w,t))\psi_1(w) \\ &\quad + (2\langle\nabla\varphi_2(z), \nabla_z K_{1,h}(z,w,t)\rangle + (\Delta_h\varphi_2(z))K_{1,h}(z,w,t))\psi_2(w) = 0 \end{aligned}$$

unless the following conditions are satisfied:

- $z \in \operatorname{supp}\nabla\varphi_1 \subseteq [\frac{5}{2}, 3] \times S^1$ *and* $w \in \operatorname{supp}\psi_1 \subseteq M_2$. *This implies that* $d_g(z,w) \geq \log(5/4)$.
- $z \in \operatorname{supp}\nabla\varphi_2 \subseteq [1, \frac{9}{8}] \times S^1$ *and* $w \in \operatorname{supp}\psi_2 \subseteq Z_{\frac{5}{4}}$. *This implies that* $d_g(z,w) \geq \log(10/9)$.

We now prove that in the expression for the trace we can exchange the heat kernel K_h for the parametrix Q_h built above.

Lemma 3.3. *There exist constants $C \geq 0$ and $c_3 > 0$ such that, for any $0 < t \leq 1$:*

$$\int_M |Q_h(z,z,t) - K_h(z,z,t)| dA_h(z) \leq Ce^{-\frac{c_3}{t}}.$$

Proof. Applying Duhamel's principle (see equation (1.16)) to the heat kernel K_h and the parametrix Q_h we infer:

$$Q_h(z,z',t) - K_h(z,z',t) = \int_0^t \int_M K_h(z,w,s)\left(\frac{\partial}{\partial t} + \Delta_{h,w}\right) Q_h(w,z',t-s)\; dA_h(w)\; ds.$$

Using Remark 3.2 we have that:

$$\begin{aligned} &\int_M |Q_h(z,z,t) - K_h(z,z,t)| dA_h(z) \\ &\leq \int_0^t \int_M \int_M \left|K_h(z,w,s)\left(\frac{\partial}{\partial t} + \Delta_{h,w}\right) Q_h(w,z,t-s)\right|\; dA_h(w)\; dA_h(z)\; ds \\ &= \int_0^t \left(\int_{M_2}\int_{[\frac{5}{2},3]\times S^1} \cdot\; dA_h(w)\; dA_h(z) + \int_{Z_{\frac{5}{4}}}\int_{[1,\frac{9}{8}]\times S^1} \cdot\; dA_h(w)\; dA_h(z)\right)\; ds. \end{aligned}$$

For the first integral in the right hand side we have:

$$\int_0^t \int_{M_2} \int_{[\frac{5}{2},3]\times S^1} \left|K_h(z,w,s)\left(\frac{\partial}{\partial t}+\Delta_{h,w}\right)Q_h(w,z,t-s)\right| \, dA_h(w)\, dA_h(z)\, ds$$
$$\ll \int_0^t \int_{M_2} \int_{[\frac{5}{2},3]\times S^1} i(z)^{1/2} s^{-1} e^{-\frac{c_2}{s}} e^{-\frac{c'}{t-s}} \, dA_h(w)\, dA_h(z)\, ds$$
$$\ll \left(\int_0^t e^{-\frac{c_2}{2s}} e^{-\frac{c'}{t-s}} \, ds\right)\left(\int_{\frac{5}{2}}^3 \frac{dv}{v^2}\right) \ll te^{-\frac{c_3}{t}} \ll e^{-\frac{c_3}{t}}$$

since $0 < t \leq 1$.

For the second integral, recall that $\operatorname{supp}\psi_2 \subset [5/4,\infty)\times S^1$. Thus:

$$\int_0^t \int_{Z_{\frac{5}{4}}} \int_{[1,\frac{9}{8}]\times S^1} \left|K_h(z,w,s)\left(\frac{\partial}{\partial t}+\Delta_{h,w}\right)Q_h(w,z,t-s)\right| \, dA_h(w)\, dA_h(z)\, ds$$
$$\ll \int_0^t \int_{Z_{\frac{5}{4}}} \int_{[1,\frac{9}{8}]\times S^1} i(z)^{1/2} s^{-1} e^{-\frac{c}{s}d_g^2(z,w)} e^{-\frac{c_1}{t-s}} \, dA_h(w)\, dA_h(z)\, ds$$
$$\ll \int_0^t \int_{\frac{5}{4}}^{\infty} \int_1^{\frac{9}{8}} y^{1/2} e^{-\frac{c_2}{2s}} e^{-\frac{c_1}{t-s}} \frac{dv}{v^2}\frac{dy}{y^2}\, ds$$
$$\leq \left(\int_0^t e^{-\frac{c_2}{2s}} e^{-\frac{c_1}{t-s}} \, ds\right)\left(\int_{\frac{5}{4}}^{\infty} y^{-3/2}\, dy\right)\left(\int_1^{\infty}\frac{dv}{v^2}\right) \leq te^{-\frac{c_3}{t}} \leq e^{-\frac{c_3}{t}}.$$

Therefore we obtain that:

$$\int_M |Q_h(z,z,t)-K_h(z,z,t)| dA_h(z) \ll e^{-\frac{c_3}{t}}.$$

□

Thus in the heat trace we can replace $K_h(z,w,t)$ by $Q_h(z,w,t)$. Since the function $e^{-2\varphi}$ is bounded, the derivatives of the gluing functions φ_1 and φ_2 with respect to the metric g satisfy the same bounds as the derivatives with respect to the metric h. Then we can perform the same construction for the kernel $K_g(z,w,t)$ to replace it by $Q_g(z,w,t)$. Let us state the main result of this chapter:

Theorem 3.4. *Let $\varphi|_Z(z)$, $\Delta_g\varphi|_Z(z)$, and, $|\nabla_g\varphi|_g|_Z(z)$ with $z=(y,x)$, be $O(y^{-32})$ as $y\to\infty$. Then there is an expansion of the relative heat trace:*

$$\operatorname{Tr}(T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_g}) = a_0 t^{-1} + a_1 + a_2 t + O(t^2), \text{ as } t\to 0. \tag{3.2}$$

Proof. First of all recall that the kernel of $T^{-1}e^{-t\Delta_h}T$ is given by $e^{\varphi(z)}K_h(z,z',t)e^{\varphi(z')}$. Then the relative heat trace is given by:

$$\operatorname{Tr}(T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_g}) = \int_M (K_h(z,z,t)e^{2\varphi(z)} - K_g(z,z,t))\, dA_g(z).$$

Let us start by using Lemma 3.3:

$$
\begin{aligned}
&\left|\int_M (K_h(z,z,t)e^{2\varphi(z)} - K_g(z,z,t))dA_g(z) - \int_M (Q_h(z,z,t)e^{2\varphi(z)} - Q_g(z,z,t))dA_g(z)\right| \\
&\qquad = \left|\int_M (K_h(z,z,t) - Q_h(z,z,t))e^{2\varphi(z)} - K_g(z,z,t) + Q_g(z,z,t)dA_g(z)\right| \\
&\qquad \leq \int_M |Q_h(z,z,t) - K_h(z,z,t)|dA_h(z) + \int_M |Q_g(z,z,t) - K_g(z,z,t)|dA_g(z) \ll e^{-c_3/t}.
\end{aligned}
$$

Therefore in order to prove Theorem 3.4 we can replace the heat kernels by the corresponding parametrices. Let $a > 1$ and let us decompose the integral as a sum:

$$
\int_M Q_h(z,z,t)e^{2\varphi(z)} - Q_g(z,z,t)dA_g(z) = I_0(t) + I_1(t) + I_2(t),
$$

where

$$
I_0(t) = \int_M \psi_1(z)(K_{W,h}(z,z,t)e^{2\varphi(z)} - K_{W,g}(z,z,t))\ dA_g(z), \tag{3.3}
$$

$$
I_1(t) = \int_{[1,a]\times S^1} \psi_2(z)(K_{1,h}(z,z,t)e^{2\varphi(z)} - K_{1,g}(z,z,t))\ dA_g(z) \text{ and,} \tag{3.4}
$$

$$
I_2(t) = \int_{Z_a} \psi_2(z)(K_{1,h}(z,z,t)e^{2\varphi(z)} - K_{1,g}(z,z,t))\ dA_g(z). \tag{3.5}
$$

For $I_0(t)$ we use the asymptotic expansion of the kernels $K_{W,h}$ and $K_{W,g}$:

$$
K_{W,h}(z,z,t) = t^{-1}\sum_{k=0}^{N} a_k(h,z)t^k + R_N(h,z,t) \quad \text{and} \quad K_{W,g}(z,z,t) = t^{-1}\sum_{k=0}^{N} a_k(g,z)t^k + R_N(g,z,t).
$$

For any N the remainder terms $R_N(h,z,t)$ and $R_N(g,z,t)$ are uniformly bounded in a compact set therefore they can be integrated. In this way the integral I_0 has a complete asymptotic expansion in t:

$$
\begin{aligned}
I_0(t) = \int_{M_2} &\psi_1(z)(K_{W,h}(z,z,t)e^{2\varphi(z)} - K_{W,g}(z,z,t))\ dA_g(z) \\
&= t^{-1}\sum_{k=0}^{N}\int_{M_2} \psi_1(z)(a_k(h,z)e^{2\varphi(z)} - a_k(g,z))t^k\ dA_g(z) \\
&\qquad + \int_{M_2} \psi_1(z)(R_N(h,z,t)e^{2\varphi(z)} - R_N(g,z,t))\ dA_g(z).
\end{aligned}
$$

The other two integrals can be rewritten as traces of the operators:

$$
A(t) = M_{\chi_{Z'_a}} M_{\psi_2}(T^{-1}e^{-t\Delta_{1,h}}T - e^{-t\Delta_{1,g}}) \ \text{ and } \ B(t) = M_{\chi_{Z_a}} M_{\psi_2}(T^{-1}e^{-t\Delta_{1,h}}T - e^{-t\Delta_{1,g}}),
$$

respectively. We will prove in Proposition 3.5 that for $0 < t \leq 1$, taking $a = t^{-1/9}$ and assuming that $\varphi((y,x))$ and $\Delta_g\varphi((y,x))$ are $O(y^{-1})$ as $y \to \infty$, $\mathrm{Tr}(A(t)) = I_1(t)$ has a complete asymptotic expansion of the form:

$$
I_1(t) \sim t^{-1}\sum_{j=0}^{\infty} b_j t^j.
$$

For $B(t)$, also taking $a = t^{-1/9}$ and under the conditions that $\varphi((y,x))$, $|\nabla_g \varphi((y,x))|$ and $\Delta_g \varphi((y,x))$ are $O(y^{-32})$ as $y \to \infty$, we will prove in Proposition 3.7 that $|\operatorname{Tr}(B(t))| \ll t^2$. The idea in this part is to assume that $\varphi((y,x))$ decays at infinity as y^{-k} and to use Duhamel's principle in a similar way as in the proof of Theorem 2.3 to estimate the trace norm of $B(t)$. The condition $k = 32$ comes from requiring that $I_2(t) = \operatorname{Tr}(B(t)) = O(t^2)$.

Proving these two facts will complete the proof of the theorem. $\square$

Proposition 3.5. *Under the conditions of Theorem 2.3 we have that there is a complete asymptotic expansion as $t \to 0$ of the integral $I_1(t)$ in equation (3.4), with $a = t^{-1/9}$. The asymptotic expansion has the following form:*

$$\int_{[1,a]\times S^1} \psi_2(z)(K_{1,h}(z,z,t)e^{2\varphi(z)} - K_{1,g}(z,z,t))\, dA_g(z) \sim t^{-1}\sum_{j=0}^{\infty} b_j t^j.$$

Proof. In order to deal with the integral

$$\int_1^a \int_{S^1} \psi_2(z)(K_{1,h}(z,z,t)e^{2\varphi(z)} - K_{1,g}(z,z,t))\, dA_g(z),$$

we first recall what $K_{1,h}$ and $K_{1,g}$ are. Recall that h was extended to the complete cusp $\widetilde{Z}$ and that $K_{1,h}(z,w,t)$ denotes the heat kernel for Δ_h on $\widetilde{Z}$. The idea of the proof of Proposition 3.5 is to use the local asymptotic expansion of the corresponding heat kernels and find a uniform bound on the remainder term.

Let us consider the universal covering of $\widetilde{Z}$: Let $\hat{Z} = \mathbb{R}^+ \times \mathbb{R}$, $\pi : \hat{Z} \to \widetilde{Z}$ be the quotient function, and, $\Gamma = \mathbb{Z}$ be the group of deck transformations. The metric h on $\widetilde{Z}$ induces a metric $\hat{h}$ on $\hat{Z}$, that has the same curvature properties as h. In addition, $\hat{h} = e^{2\hat{\varphi}}\hat{g}_0$, where $\hat{g}_0$ is the lift of g_0 to $\hat{Z}$ and is precisely the hyperbolic metric on $\mathbb{H}$, and the function $\hat{\varphi}$ is a lift of $\widetilde{\varphi}$ ($\widetilde{\varphi}$ the extension of φ to $\widetilde{Z}$), $\hat{\varphi} = \widetilde{\varphi} \circ \pi$. It follows that $\hat{h}$ and $\hat{g}_0$ are quasi-isometric. Therefore by Proposition 2.1 in [32] it follows that the injectivity radius of $\hat{h}$ is bounded from below by a positive constant independent of the point. In this way $(\hat{Z}, \hat{h})$ has bounded geometry. Let k_h denote the heat kernel for $\Delta_{\hat{h}}$ in $\hat{Z}$. Then from the results in [11] we have the following estimate:

$$k_h(\widetilde{z}, \widetilde{w}, t) \le C t^{-1} e^{-\frac{c d^2(\widetilde{z},\widetilde{w})}{t}},$$

where $\widetilde{z}, \widetilde{w} \in \hat{Z}$ and $0 < t \le 1$. Before we continue we need the following lemma:

Lemma 3.6. *$K_{1,h}(z,w,t) = \sum_{m\in\mathbb{Z}} k_h(\widetilde{z}, \widetilde{w}+m, t)$, where $\pi(\widetilde{z}) = z$, $\pi(\widetilde{w}) = w$.*

Proof. Let $H(z,w,t) := \sum_{m\in\mathbb{Z}} k_h(\widetilde{z}, \widetilde{w}+m, t)$. In order to prove this lemma, it suffices to prove that H satisfies the following defining properties of the heat kernel:

1. $\left(\frac{\partial}{\partial t} + \Delta_h\right) H(z,w,t) = 0$
2. $H(z,w,t) \to \delta_{w-z}$, as $t \to 0$.
3. $H(z,w,t) = H(w,z,t)$.

First equation:

$$\left(\frac{\partial}{\partial t}+\Delta_h\right)H(z,w,t)=\left(\frac{\partial}{\partial t}+\Delta_{\hat h}\right)\sum_{m\in\mathbb{Z}}k_h(z,w+m,t)=\sum_{m\in\mathbb{Z}}\left(\frac{\partial}{\partial t}+\Delta_{\hat h}\right)k_h(z,w+m,t)=0$$

where we can exchange the series and the derivatives because of the uniform convergence of the series in C^2. Since $(\hat Z,h)$ is complete and with bounded curvature, the heat kernel k_h and its derivatives satisfy the estimates given in [11]. The uniform convergence in C^2 follows from these estimates.

Second equation: Let $F\subset\hat Z$ be a fundamental domain for $\Gamma=\mathbb{Z}$, let $f\in C_c^\infty$, and let $\widetilde f$ be a lift of f, so that $f\circ\pi=\widetilde f$, then

$$\begin{aligned}\int_Z H(z,w,t)f(w)dA_h(w)&=\int_F\sum_{m\in\mathbb{Z}}k_h(\widetilde z,\widetilde w+m,t)\widetilde f(\widetilde w)dA_{\hat h}(\widetilde w)=\sum_{m\in Z}\int_{F+m}k_h(\widetilde z,\widetilde w,t)\widetilde f(\widetilde w)dA_{\hat h}(\widetilde w)\\&=\int_{\hat Z}k_h(\widetilde z,\widetilde w,t)\widetilde f(\widetilde w)dA_{\hat h}(\widetilde w)\to\widetilde f(\widetilde z)=f(z),\quad\text{as } t\to 0.\end{aligned}$$

Third equation: Let $\widetilde z,\widetilde w\in\hat Z$ be fixed. Then,

$$H(z,w,t)=\sum_{m\in\mathbb{Z}}k_h(\widetilde z,\widetilde w+m,t)=\sum_{m\in\mathbb{Z}}k_h(\widetilde w+m,\widetilde z,t)=\sum_{m\in\mathbb{Z}}k_h(\widetilde w,\widetilde z+m,t)=H(w,z,t).$$

$\square$

Continuing with the proof of Proposition 3.5, notice that we can perform the above construction for the kernel $K_{1,g}$. Then the integral $I_1(t)$ becomes:

$$\begin{aligned}I_1(t)&=\int_1^a\int_{S^1}\psi_2(z)(K_{1,h}(z,z,t)e^{2\varphi(z)}-K_{1,g}(z,z,t))\,dA_g(z)\\&=\int_1^a\int_0^{2\pi}\widetilde\psi_2(\widetilde z)\left(\sum_{m\in\mathbb{Z}}k_h(\widetilde z,\widetilde z+m,t)e^{2\hat\varphi(\widetilde z+m)}-\sum_{l\in\mathbb{Z}}k_g(\widetilde z,\widetilde z+l,t)\right)\,dA_{\hat g}(\widetilde z),\end{aligned}$$

because $F=\mathbb{R}^+\times[0,2\pi]$ is a fundamental domain for Γ and the domain corresponding to Z'_a in F is $[1,a]\times[0,2\pi]$. Thus

$$\begin{aligned}I_1(t)&=\int_1^a\int_0^{2\pi}\sum_{m\in\mathbb{Z}}\widetilde\psi_2(\widetilde z)(k_h(\widetilde z,\widetilde z+m,t)e^{2\hat\varphi(\widetilde z+m)}-k_g(\widetilde z,\widetilde z+m,t))\,dA_{\hat g}(\widetilde z)\\&=\int_1^a\int_0^{2\pi}\widetilde\psi_2(\widetilde z)(k_h(\widetilde z,\widetilde z,t)e^{2\hat\varphi(\widetilde z)}-k_g(\widetilde z,\widetilde z,t))\,dA_{\hat g}(\widetilde z)\\&+\int_1^a\int_0^{2\pi}\widetilde\psi_2(\widetilde z)\sum_{m\neq 0}(k_h(\widetilde z,\widetilde z+m,t)e^{2\hat\varphi(\widetilde z+m)}-k_g(\widetilde z,\widetilde z+m,t))\,dA_{\hat g}(\widetilde z).\end{aligned}\tag{3.6}$$

We will start by estimating the second term on the right hand side of (3.6). Note that $\hat\varphi=\widetilde\varphi\circ\pi$ implies that the function $e^{2\hat\varphi}$ is bounded. This, the fact that the metrics $\hat h$ and $\hat g$ are quasi-isometric and the estimate on the heat kernel k_h imply that:

$$\sum_{m\neq 0}k_h(\widetilde z,\widetilde z+m,t)e^{2\hat\varphi(\widetilde z+m)}\ll t^{-1}\sum_{m\neq 0}\exp\left(-\frac{c_1d_{\hat g}^2(\widetilde z,\widetilde z+m)}{t}\right).\tag{3.7}$$

From the explicit expression of the hyperbolic distance in the upper half plane, we have that

$$d_{\hat{g}}((\widetilde{x},\widetilde{y}),(\widetilde{x}+m,\widetilde{y}))=\cosh^{-1}\left(1+\frac{m^2}{2\widetilde{y}^2}\right).$$

Since for $s\geq 1$ we have that $\cosh^{-1}(s)=\log(s+\sqrt{s^2-1})$, we obtain

$$d_{\hat{g}}((\widetilde{x},\widetilde{y}),(\widetilde{x}+m,\widetilde{y}))=\log\left(1+\frac{m^2}{2\widetilde{y}^2}+\frac{|m|}{\widetilde{y}}\sqrt{\frac{m^2}{4\widetilde{y}^2}+1}\right)\geq\log\left(1+\frac{m^2}{2\widetilde{y}^2}\right).$$

For $\widetilde{y}=y\in[1,a]$, we have $\frac{1}{2a^2}\leq\frac{1}{2y^2}\leq 1$ and $\log(1+\frac{m^2}{2\widetilde{y}^2})\geq\log(1+\frac{1}{2a^2})$. Thus

$$e^{-\frac{c_1 d_{\hat{g}}^2(\widetilde{z},\widetilde{z}+m)}{t}}\leq e^{-\frac{c_1\log(1+1/2a^2)^2}{2t}}e^{-\frac{c_1\log(1+m^2/2\widetilde{y}^2)^2}{2t}}.$$

For $0\leq s\leq 1$, we have that $\log(1+s)\geq s^2/2$. Applying this to $s=(2a^2)^{-1}$, we obtain

$$\sum_{m\neq 0}e^{-\frac{c_1 d_{\hat{g}}^2(\widetilde{z},\widetilde{z}+m)}{t}}\leq e^{-\frac{c_1}{2^6a^8t}}\sum_{m\neq 0}e^{-\frac{c_1\log(1+\frac{m^2}{2\widetilde{y}^2})^2}{2t}}\leq e^{-\frac{c_2}{a^8t}}\sum_{m\neq 0}e^{-\frac{c_1\log(1+\frac{m^2}{2a^2})^2}{2t}},\tag{3.8}$$

with c_2 a positive constant. In order to estimate the series, we compare it with an integral using the fact that $e^{-\frac{c_1\log(1+\frac{m^2}{2a^2})^2}{2t}}$ is a decreasing function of m. We proceed in the following way:

$$\begin{aligned}\sum_{m\neq 0}e^{-\frac{c_1\log(1+\frac{m^2}{2a^2})^2}{2t}}&\ll\int_1^\infty e^{-\frac{c_1\log(1+\frac{u^2}{2a^2})^2}{2t}}du\leq\int_1^{\sqrt{2}a}e^{-\frac{c_1\log(1+\frac{u^2}{2a^2})^2}{2t}}du+\int_{\sqrt{2}a}^\infty e^{-\frac{2c_1\log(\frac{u}{\sqrt{2}a})^2}{t}}du\\&\ll(\sqrt{2}a-1)+a\int_0^\infty e^{-\frac{2c_1v^2}{t}}e^v dv\ll a(1+\sqrt{t}e^{ct})\ll a,\end{aligned}\tag{3.9}$$

where for one integral we used that $e^{-x}\leq 1$, for all $x\geq 0$, and for the other integral we used the change of variables $v=log(\frac{u}{\sqrt{2}a})$; in the middle step we used that for $x\geq 1$, $\log(x+1)^2\geq\log(x)^2$. Now we can use (3.7) and the bounds above to estimate the second term in the right hand side of equation (3.6):

$$\begin{aligned}&\int_1^a\int_0^{2\pi}|\widetilde{\psi}_2(\widetilde{z})\sum_{m\neq 0}(k_h(\widetilde{z},\widetilde{z}+m,t)e^{2\hat{\varphi}(\widetilde{z}+m)}-k_g(\widetilde{z},\widetilde{z}+m,t))|\,dA_{\hat{g}}(\widetilde{z})\\&\qquad\ll t^{-1}\int_1^a\int_0^{2\pi}|\widetilde{\psi}_2(\widetilde{z})\sum_{m\neq 0}e^{-\frac{c_1 d_{\hat{g}}^2(\widetilde{z},\widetilde{z}+m)}{t}}|dA_{\hat{g}}(\widetilde{z})\\&\qquad\ll t^{-1}e^{-\frac{c_2}{a^8t}}\int_1^a\sum_{m\neq 0}e^{-\frac{c_1\log(1+\frac{m^2}{2a^2})^2}{2t}}\frac{dy}{y^2}\ll t^{-1}ae^{-\frac{c_2}{a^8t}}.\end{aligned}$$

Now taking $a=t^{-1/9}$ we get $a^8t=t^{1/9}$. Therefore $e^{-\frac{c_2}{a^8t}}=e^{-\frac{c_2}{t^{1/9}}}$ and we obtain:

$$\begin{aligned}&\int_1^a\int_0^{2\pi}|\widetilde{\psi}_2(\widetilde{z})\sum_{m\neq 0}(k_h(\widetilde{z},\widetilde{z}+m,t)e^{2\hat{\varphi}(\widetilde{z}+m)}-k_g(\widetilde{z},\widetilde{z}+m,t))|\,dA_{\hat{g}}(\widetilde{z})\\&\qquad\ll t^{-1}t^{-1/9}e^{-\frac{c_2}{t^{1/9}}}\ll t^{-10/9}e^{-\frac{c_2}{t^{1/9}}}\ll e^{-\frac{c_2}{2t^{1/9}}}.\end{aligned}$$

Let us see now what happens with the first term in the right hand side of equation (3.6):

$$\int_1^a \int_0^{2\pi} \widetilde{\psi}_2(\widetilde{z})(k_h(\widetilde{z},\widetilde{z},t)e^{2\hat{\varphi}(\widetilde{z})} - k_g(\widetilde{z},\widetilde{z},t))dA_{\hat{g}}(\widetilde{z})$$
$$= \int_1^{t^{-1/9}} \int_0^{2\pi} \widetilde{\psi}_2(\widetilde{z})(k_h(\widetilde{z},\widetilde{z},t)e^{2\hat{\varphi}(\widetilde{z})} - k_g(\widetilde{z},\widetilde{z},t))dA_{\hat{g}}(\widetilde{z}).$$

The kernel $k_h(\widetilde{z},\widetilde{z},t)$, as well as $k_g(\widetilde{z},\widetilde{z},t)$, has a uniform local asymptotic expansion as $t \to 0$ of the usual form:

$$k_h(\widetilde{z},\widetilde{z},t) = t^{-1}\sum_{k=0}^{N} a_k(\hat{h},\widetilde{z})t^k + R_N(\hat{h},\widetilde{z},t) \quad \text{and} \quad k_g(\widetilde{z},\widetilde{z},t) = t^{-1}\sum_{k=0}^{N} a_k(\hat{g},\widetilde{z})t^k + R_N(\hat{g},\widetilde{z},t)$$

for any $N \geq 0$. For the remainder terms there is a constant $C > 0$ such that

$$|R_N(\hat{h},\widetilde{z},t)| \leq Ct^N \quad \text{and} \quad |R_N(\hat{g},\widetilde{z},t)| \leq Ct^N \tag{3.10}$$

independent of $\widetilde{z}$. Replacing the corresponding expansion in the previous integral we obtain:

$$\int_1^{t^{-1/9}} \int_0^{2\pi} \widetilde{\psi}_2(\widetilde{z})t^{-1}\left(\sum_{k=0}^{N} a_k(\hat{h},\widetilde{z})e^{2\hat{\varphi}(\widetilde{z})} - a_k(\hat{g},\widetilde{z})\right)t^k dA_{\hat{g}}(\widetilde{z})$$
$$+ \int_1^{t^{-1/9}} \int_0^{2\pi} \widetilde{\psi}_{t^{-1/9},1}(\widetilde{z})(R_N(\hat{h},\widetilde{z},t)e^{2\hat{\varphi}(\widetilde{z})} - R_N(\hat{g},\widetilde{z},t))dA_{\hat{g}}(\widetilde{z}).$$

The first term can be integrated without any problem to obtain:

$$\int_1^{t^{-1/9}} \int_0^{2\pi} \widetilde{\psi}_2(\widetilde{z})t^{-1}\left(\sum_{k=0}^{N} a_k(\hat{h},\widetilde{z})e^{2\hat{\varphi}(\widetilde{z})} - a_k(\hat{g},\widetilde{z})\right)t^k dA_{\hat{g}}(\widetilde{z}) = t^{-1}\sum_{k=0}^{N} b_k t^k.$$

As for the remainder terms, using equation (3.10) we have:

$$\left|\int_1^{t^{-1/9}} \int_0^{2\pi} \widetilde{\psi}_2(\widetilde{z})(R_N(\hat{h},\widetilde{z},t)e^{2\hat{\varphi}(\widetilde{z})} - R_N(\hat{g},\widetilde{z},t))dA_{\hat{g}}(\widetilde{z})\right|$$
$$\leq \int_1^{t^{-1/9}} \int_0^{2\pi} (|R_N(\hat{h},\widetilde{z},t)e^{2\hat{\varphi}(\widetilde{z})}| + |R_N(\hat{g},\widetilde{z},t)|)dA_{\hat{g}}(\widetilde{z}) \ll t^N \int_1^{t^{-1/9}} \frac{dy}{y^2} \ll t^N,$$

since $0 < t \leq 1$. This finishes the proof of Proposition 3.5. □

Proposition 3.7. *Under the conditions of Theorem 3.4, for $0 < t \leq 1$, and for $a = t^{-1/9}$ we have that:*

$$|\operatorname{Tr}(M_{\chi_{Z_a}} M_{\psi_2}(T^{-1}e^{-t\Delta_{1,h}}T - e^{-t\Delta_{1,g}}))| \ll t^2.$$

Proof. To prove Proposition 3.7 we want to apply Duhamel's principle on the cusp Z. However the heat operators involved in the trace correspond to Laplacians in the complete cusp $\widetilde{Z}$. Therefore in order to make the computations easier we first replace them by the heat operators $e^{-t\Delta_{Z,h}}$ and

$e^{-t\Delta_{Z,g}}$ described at the beginning of this section and apply Duhamel's principle to $e^{-t\Delta_{Z,h}}$ and $e^{-t\Delta_{Z,g}}$. We have to take into account more terms, but we avoid the problem of the singularity at $y = 0$. Using equations (1.10) and (1.15) to replace the respective kernels we obtain:

$$\begin{aligned}\mathrm{Tr}(M_{\chi_{Z_a}} M_{\psi_2}(T^{-1}e^{-t\Delta_{1,h}}T - e^{-t\Delta_{1,g}})) &= \mathrm{Tr}(M_{\chi_{Z_a}} M_{\psi_2}(T^{-1}e^{-t\Delta_{Z,h}}T - e^{-t\Delta_{Z,g}})) \\ &- \int_M \chi_{Z_a}(z)\psi_2(z)(p_{h,D}(z,z,t)e^{2\varphi(z)} - p_{1,D}(z,z,t))dA_g(z).\end{aligned}$$

From equation (1.11) and $\mathrm{supp}(\psi_2) = Z_{5/4}$ it follows that:

$$\begin{aligned}&\left|\int_M \psi_2(z)(p_{h,D}(z,z,t)e^{2\varphi(z)} - p_{1,D}(z,z,t))dA_g(z)\right| \ll \int_{Z_{\frac{5}{4}}} t^{-1}y(e^{-\frac{cd_h(z,\partial Z)}{t}} + e^{-\frac{c'd_g(z,\partial Z)}{t}})dA_g(z) \\ &\ll \int_{\frac{5}{4}}^{\infty} t^{-1}ye^{-\frac{c_1\log(y)^2}{t}}\frac{dy}{y^2} \leq t^{-1}e^{-\frac{c_1\log(5/4)^2}{2t}}\int_{\frac{5}{4}}^{\infty} y^{-1}e^{-\frac{c_1\log(y)^2}{2t}}dy \ll e^{-\frac{c_1\log(5/4)^2}{4t}}.\end{aligned}$$

We now continue with the estimation of the trace of the operator $M_{\chi_{Z_a}} M_{\psi_2}(T^{-1}e^{-t\Delta_{Z,h}}T - e^{-t\Delta_{Z,g}})$. The kernel of $T^{-1}e^{-t\Delta_{Z,h}}T - e^{-t\Delta_{Z,g}}$ as operator on $L^2(M, dA_g)$ is given by

$$e^{\varphi(z)}K_{Z,h}(z,w,t)e^{\varphi(w)} - K_{Z,g}(z,w,t),$$

that for $z = w$ takes the form $K_{Z,h}(z,z,t)e^{2\varphi(z)} - K_{Z,g}(z,z,t)$. From the usual form of Duhamel's principle in equation (1.16) we infer:

$$\begin{aligned}&K_{Z,h}(z,w,t)e^{2\varphi(w)} - K_{Z,g}(z,w,t) = \\ &\qquad\int_0^t\int_M K_{Z,h}(z,z',s)e^{2\varphi(z')}(\Delta_{Z,g} - \Delta_{Z,h})K_{Z,g}(z',w,t-s)dA_g(z')\ ds.\end{aligned}$$

Then taking $z = w$ in the equation above and using the conformal transformation of Laplacians we obtain:

$$\begin{aligned}\mathrm{Tr}(M_{\chi_{Z_a}} M_{\psi_2}(T^{-1}e^{-t\Delta_{Z,h}}T - e^{-t\Delta_{Z,g}})) &= \int_{Z_a}\psi_2(z)(K_{Z,h}(z,z,t)e^{2\varphi(z)} - K_{Z,g}(z,z,t))dA_g(z) \\ = \int_{Z_a}\psi_2(z)\int_0^t\int_Z K_{Z,h}(z,z',s)&e^{2\varphi(z')}(1 - e^{-2\varphi(z')})\Delta_{Z,g}K_{Z,g}(z',z,t-s)dA_g(z')\ ds\ dA_g(z).\end{aligned}$$

Remember that $\mathrm{supp}(\psi_2) = Z_{5/4}$ and let us first assume that $a > 5/4$, so $4a/5 > 1$. Split the integral as the sum of the following terms:

1. $J_1 = \int_0^t\int_{Z_a}\int_{[1,\frac{4a}{5}]\times S^1} \cdot\, dA_g(z')dA_g(z)ds.$

2. $J_2 = \int_0^{t/2}\int_{Z_a}\int_{Z_{\frac{4a}{5}}} \cdot\, dA_g(z')dA_g(z)ds.$

3. $J_3 = \int_{t/2}^{t}\int_{Z_a}\int_{Z_{\frac{4a}{5}}} \cdot\, dA_g(z')dA_g(z)ds.$

Let $k \geq 1$ and suppose that $\varphi(y,x) = O(y^{-k})$ as $y \to \infty$. Note that according to Lemma (2.2), $\psi = 1 - e^{-2\varphi}$ and $\widetilde{\psi} = e^{2\varphi} - 1$ have the same order as φ. Then for J_1 we have:

$$J_1 = \int_0^t \int_{Z_a} \int_{[1,\frac{4a}{5}]\times S^1} \psi_2(z)(K_{1,h}(z,z',s) + p_{h,D}(z,z',s))e^{2\varphi(z')}$$

$$\psi(z')\Delta_{Z,g}(K_{1,g}(z',z,t-s) + p_{1,D}(z',z,t-s))\ dA_g(z')\ dA_g(z)\ ds.$$

Note that on this region $a \leq y < \infty$ and $1 \leq y' \leq \frac{4a}{5}$. Thus $1 < \frac{5}{4} \leq \frac{y}{y'}$, so $\log(y/y')$ is bounded away from 0. Using the

$$|J_1| \ll \int_0^t \int_a^\infty \int_1^{\frac{4a}{5}} s^{-1}(t-s)^{-2} y(e^{-\frac{c\log(y/y')^2}{s}} + e^{-\frac{c\log(y)^2}{s}} e^{-\frac{c\log(y')^2}{s}})$$

$$y'^{-k+1}(e^{-\frac{c\log(y/y')^2}{t-s}} + e^{-\frac{c\log(y)^2}{t-s}} e^{-\frac{c\log(y')^2}{t-s}})\frac{dy'}{y'^2}\frac{dy}{y^2}ds$$

$$\ll at^{-2}\int_0^{t/2}\int_a^\infty s^{-1}y^{-1}(e^{-\frac{c\log(5y/4a)^2}{s}} + e^{-\frac{c\log(y)^2}{s}})dyds$$

$$+ at^{-1}\int_{t/2}^t\int_a^\infty (t-s)^{-2}y^{-1}(e^{-\frac{c\log(5y/4a)^2}{t-s}} + e^{-\frac{c\log(y)^2}{t-s}})dyds.$$

Since $y \geq a > \frac{5}{4}$ we have an estimate in s:

$$e^{-\frac{c\log(5y/4a)^2}{s}} + e^{-\frac{c\log(y)^2}{s}} \leq e^{-\frac{c\log(5/4)^2}{2s}}(e^{-\frac{c\log(5y/4a)^2}{2s}} + e^{-\frac{c\log(y)^2}{2s}})$$

and $\int_a^\infty y^{-1}e^{-\frac{c\log(5y/4a)^2}{2s}}dy = \int_{\frac{5}{4}}^\infty v^{-1}e^{-\frac{c\log(v)^2}{2s}}dv \ll \sqrt{s}$. We get a similar estimate for $t-s$, and together these give:

$$|J_1| \ll at^{-2}\int_0^{t/2} s^{-1}e^{-\frac{c\log(5/4)^2}{2s}}\int_{\frac{5}{4}}^\infty y^{-1}e^{-\frac{c\log(y)^2}{2s}}dyds$$

$$+ at^{-1}\int_{t/2}^t (t-s)^{-2}e^{-\frac{c\log(5/4)^2}{2(t-s)}}\int_{\frac{5}{4}}^\infty y^{-1}e^{-\frac{c\log(y)^2}{2(t-s)}}dyds$$

$$\ll at^{-2}\int_0^{t/2} s^{-1/2}e^{-\frac{c\log(5/4)^2}{2s}}ds + at^{-1}\int_{t/2}^t (t-s)^{-3/2}e^{-\frac{c\log(5/4)^2}{2(t-s)}}ds$$

$$\ll at^{-2}e^{-\frac{c\log(5/4)^2}{4t}}\int_0^{t/2}ds + at^{-1}e^{-\frac{c\log(5/4)^2}{2t}}\int_{t/2}^t ds \ll a(t^{-1}+1)e^{c_1/t} \ll ae^{-\frac{c'}{t}},$$

for some constants $c_1, c' > 0$, where we also used that for any $b > 0$ the function $f(s) = s^{-1}e^{-\frac{b}{s}} \ll 1$ on $\mathbb{R}^+$.

For J_2, let us use that the variable $z' \in Z_{\frac{4a}{5}}$ to multiply the inside the integral by the characteristic function $\chi_{Z_{\frac{4a}{5}}}(z')$. Then, denoting again $1 - e^{2\varphi}$ by ψ we have:

$$J_2 = \int_0^{t/2}\int_{Z_a}\int_{Z_{\frac{4a}{5}}} \psi_2(z)K_{Z,h}(z,z',s)e^{2\varphi(z')}$$

$$\chi_{Z_{\frac{4a}{5}}}(z')\psi(z')\Delta_{Z,g}K_{Z,g}(z',z,t-s)dA_g(z')dA_g(z)ds.$$

Writing this integral in terms of traces of the corresponding operators we infer:

$$|J_2| = \left| \int_0^{t/2} \mathrm{Tr}(M_{\psi_2} e^{-s\Delta_{Z,h}} M_{e^{2\varphi}} M_{\chi_{Z_{\frac{4a}{5}}}} M_\psi \Delta_{Z,g} e^{-(t-s)\Delta_{Z,g}}) ds \right|$$

$$\ll \int_0^{t/2} \|M_{\chi_{Z_{\frac{4a}{5}}}} M_\psi \Delta_{Z,g} e^{-(t-s)\Delta_{Z,g}}\|_1 ds = \int_{t/2}^{t} \|M_{\chi_{Z_{\frac{4a}{5}}}} M_\psi \Delta_{Z,g} e^{-s\Delta_{Z,g}}\|_1 ds.$$

To obtain a bound, we use a similar method to the one used in Chapter 2 to prove the trace class property. Let us use the auxiliary function ϕ defined by equation (2.6). Then for the trace norm of the operator $M_{\chi_{Z_{\frac{4a}{5}}}} M_\psi \Delta_{Z,g} e^{-s\Delta_{Z,g}}$ we have that:

$$\|M_{\chi_{Z_{\frac{4a}{5}}}} M_\psi \Delta_{Z,g} e^{-s\Delta_{Z,g}}\|_1 \leq \|M_{\chi_{Z_{\frac{4a}{5}}}} M_\psi \Delta_{Z,g} e^{-s/2\Delta_{Z,g}} M_\phi^{-1}\|_2 \|M_\phi e^{-s/2\Delta_{Z,g}}\|_2.$$

The terms in the right hand side can be estimated as follows:

$$\|M_{\chi_{Z_{\frac{4a}{5}}}} M_\psi \Delta_{Z,g} e^{-s/2\Delta_{Z,g}} M_\phi^{-1}\|_2^2 = \int_Z \int_Z |\chi_{Z_{\frac{4a}{5}}}(z)\psi(z)\Delta_{Z,g} K_{Z,g}(z,z',s/2)\phi(z')^{-1}|^2 dA_g(z') dA_g(z)$$
$$= \int_{Z_{\frac{4a}{5}}} \int_Z |\psi(z)\Delta_{Z,g} K_{Z,g}(z,z',s/2)\phi(z')^{-1}|^2 dA_g(z') dA_g(z)$$
$$\ll \int_{\frac{4a}{5}}^\infty \int_1^\infty y^{-2k} y y' s^{-4} (e^{-\frac{4c}{s}(\log(y/y'))^2} + e^{-\frac{4c}{s}(\log(yy'))^2}) y' \frac{dy'}{y'^2} \frac{dy}{y^2}$$
$$= s^{-4} \int_{\frac{4a}{5}}^\infty \int_1^\infty y^{-2k-1} e^{-\frac{4c}{s}(\log(y'/y))^2}\ dy'\ dy + s^{-4} \int_{\frac{4a}{5}}^\infty \int_1^\infty y^{-2k-1} e^{-\frac{4c}{s}(\log(y'))^2}\ dy'\ dy.$$

For the first integral in the right hand side, let us fix y and let us make the change of variables $v = \log(y'/y)$, $y' = ye^v$, $dy' = ye^v dv$. Then we obtain:

$$s^{-4} \int_{\frac{4a}{5}}^\infty \int_{-\log(y)}^\infty y^{-2k} e^v e^{\frac{-4c}{s}v^2}\ dv\ dy \ll s^{-4} e^{\frac{s}{4c}} \sqrt{s} \int_{\frac{4a}{5}}^\infty y^{-2k} \int_{-\infty}^\infty e^{-v^2}\ dv\ dy \ll s^{-7/2} a^{-2k+1} e^{\frac{s}{4c}}.$$

For the second integral, we obtain in a similar way:

$$s^{-4} \int_{\frac{4a}{5}}^\infty \int_1^\infty y^{-2k-1} e^{-\frac{4c}{s}(\log(y'))^2}\ dy'\ dy \ll s^{-7/2} e^{\frac{s}{4c}} a^{-2k}.$$

Thus,

$$\|M_{\chi_{Z_{\frac{4a}{5}}}} M_\psi \Delta_{Z,g} e^{-s/2\Delta_{Z,g}} M_\phi^{-1}\|_2 \ll s^{-7/4}(a^{-k} + a^{-k+1/2}).$$

For the operator $M_\phi e^{-s/2\Delta_{Z,g}}$, using equation (2.9) we have:

$$\|M_\phi e^{-s/2\Delta_{Z,g}}\|_2^2 \ll \int_1^\infty \int_1^\infty s^{-2} y^{-1} y y' (e^{-\frac{2c}{s}(\log(y/y'))^2} + e^{-\frac{2c}{s}(\log(yy'))^2})^2 \frac{dy'}{y'^2} \frac{dy}{y^2}$$
$$\ll \int_1^\infty \int_1^\infty s^{-2} y'^{-1} y^{-2} (e^{-\frac{4c}{s}(\log(y/y'))^2} + e^{-\frac{4c}{s}(\log(yy'))^2}) dy' dy$$
$$\ll s^{-2}\sqrt{s} e^{s/4c} + s^{-2} \int_1^\infty y'^{-1} e^{-\frac{4c}{s}(\log(y'))^2} dy' \ll s^{-3/2}(1 + e^{s/4c}).$$

Since $s \leq t \leq 1$ we have that $\|M_\phi e^{-s/2\Delta_{Z,g}}\|_2 \ll s^{-3/4}$. It follows that:

$$|J_2| \ll \int_{t/2}^t s^{-7/4}(a^{-k} + a^{-k+1/2}) \cdot s^{-3/4} ds \ll a^{-k+1/2} t^{-3/2}.$$

Now, for J_3 we have:

$$J_3 = \int_{t/2}^t \int_{Z_a} \int_{Z_{\frac{4a}{5}}} \psi_2(z) K_{Z,h}(z,z',s) e^{2\varphi(z')} \chi_{Z_{\frac{4a}{5}}}(z') \\ (\Delta_{Z,g} - \Delta_{Z,h})_{z'} K_{Z,g}(z',z,t-s) dA_g(z') dA_g(z) ds.$$

Remember that $\Delta_{Z,g} - \Delta_{Z,h} = (e^{2\varphi(z')} - 1)\Delta_{Z,h} = \widetilde{\psi}(z')\Delta_{Z,h}$, so the previous equation becomes:

$$J_3 = \int_{t/2}^t \int_{Z_a} \int_{Z_{\frac{4a}{5}}} \psi_2(z) K_{Z,h}(z,z',s) \chi_{Z_{\frac{4a}{5}}}(z') \widetilde{\psi}(z') (\Delta_{Z,h} K_{Z,g}(z',z,t-s)) e^{-2\varphi(z)} dA_h(z') dA_h(z) ds$$

$$= \int_{t/2}^t \int_{Z_a} \int_{Z_{\frac{4a}{5}}} \psi_2(z) (\Delta_{Z,h} K_{Z,h}(z,z',s) \widetilde{\psi}(z')) \chi_{Z_{\frac{4a}{5}}}(z') K_{Z,g}(z',z,t-s) e^{-2\varphi(z)} dA_h(z') dA_h(z) ds$$

$$= \int_{t/2}^t \int_{Z_a} \int_{Z_{\frac{4a}{5}}} \psi_2(z) e^{-2\varphi(z)} K_{Z,g}(z,z',t-s) \chi_{Z_{\frac{4a}{5}}}(z') (\Delta_{Z,h} \widetilde{\psi}(z') K_{Z,h}(z',z,s)) dA_h(z') dA_h(z) ds.$$

Writing this in terms of the corresponding operators we obtain:

$$J_3 = \int_{t/2}^t \mathrm{Tr}(M_{\psi_2} M_{e^{-2\varphi}} e^{-(t-s)\Delta_{Z,g}} M_{\chi_{Z_{\frac{4a}{5}}}} \Delta_{Z,h} M_{\widetilde{\psi}} e^{-s\Delta_{Z,h}}) ds,$$

$$|J_3| \leq \int_{t/2}^t \|M_{\chi_{Z_{\frac{4a}{5}}}} \Delta_{Z,h} M_{\widetilde{\psi}} e^{-s\Delta_{Z,h}}\|_1 \; ds.$$

We are now working in $L^2(M, dA_h)$ therefore to simplify notation we do not write the subindex h in the trace and the Hilbert-Schmidt norms.

$$\|M_{\chi_{Z_{\frac{4a}{5}}}} \Delta_{Z,h} M_{\widetilde{\psi}} e^{-s\Delta_{Z,h}}\|_1 \leq \|M_{\chi_{Z_{\frac{4a}{5}}}} \Delta_{Z,h} M_{\widetilde{\psi}} e^{-s\Delta_{Z,h}/2} M_{\phi^{-1}}\|_2 \|M_\phi e^{-s\Delta_{Z,h}/2}\|_2$$

The kernel of the operator $M_{\chi_{Z_{\frac{4a}{5}}}} \Delta_{Z,h} M_{\widetilde{\psi}} e^{-s\Delta_{Z,h}/2} M_{\phi^{-1}}$ is $\chi_{Z_{\frac{4a}{5}}}(z')(\Delta_{Z,h}(\widetilde{\psi}(z') K_{Z,h}(z',z,s))\phi(z)^{-1}$. Using the decay assumptions on φ and its derivatives, we have that:

$$|\Delta_{Z,h}(\widetilde{\psi} K_{Z,h})|^2 \ll |\widetilde{\psi}\Delta_{Z,h} K_{Z,h}|^2 + |K_{Z,h}\Delta_{Z,h}\widetilde{\psi}|^2 + 2|\langle \nabla\widetilde{\psi}, \nabla K_{Z,h}\rangle|^2 \\ \ll y'^{-2k+1} y (s^{-4} + s^{-2} + s^{-3})(e^{-\frac{c}{s}(\log(y/y'))^2} + e^{-\frac{c}{s}(\log(yy'))^2})^2.$$

Since for $0 < s < 1$ we have that $s^{-4} + s^{-2} + s^{-3} \ll s^{-4}$, we can estimate the Hilbert-Schmidt norm by:

$$\|M_{\chi_{Z_{\frac{4a}{5}}}} \Delta_{Z,h} M_{\widetilde{\psi}} e^{-s\Delta_{Z,h}/2} M_{\phi^{-1}}\|_2^2 = \int_Z \int_Z |\chi_{Z_{\frac{4a}{5}}}(z')\widetilde{\psi}(z')\Delta_{h,z'} K_h(z',z,s/2)\phi(z)^{-1}|^2 dA_h(z') dA_h(z)$$

$$\ll s^{-4} \int_1^\infty \int_{\frac{4a}{5}}^\infty y^2 \; y'^{-2k+1} (e^{-\frac{2c}{s}(\log(y/y'))^2} + e^{-\frac{2c}{s}(\log(yy'))^2})^2 \frac{dy'}{y'^2} \frac{dy}{y^2}$$

$$\ll s^{-4} \int_{\frac{4a}{5}}^\infty \int_1^\infty (y'^{-2k-1} e^{-\frac{4c}{s}(\log(y/y'))^2} + y'^{-2k-1} e^{-\frac{4c}{s}(\log(y))^2}) \; dy \; dy'$$

$$\ll (a^{-2k+1} + a^{-2k}) s^{-7/2} e^{s/4c} \ll a^{-2k+1} s^{-7/2}.$$

We finally obtain:
$$\|M_\phi^{-1} e^{-s/2\Delta_{Z,h}} \widetilde{\psi}\Delta_h\|_2 \le a^{-k+1/2} s^{-7/4}.$$

For the operator $e^{-s/2\Delta_{Z,h}} M_\phi$, the proof goes in the same way as for the operator $M_\phi e^{-s/2\Delta_{Z,g}}$. At the end we obtain:

$$\|e^{-s\Delta_{Z,h}} M_\phi\|_2 = \left(\int_Z \int_Z |K_{Z,h}(z,z',s/2)\phi(z')|^2 dA_h(z') dA_h(z)\right)^{1/2} \ll s^{-3/4}.$$

In this way:
$$|J_3| \ll \int_{t/2}^{t} a^{-k+1/2} s^{-7/4} s^{-3/4} ds \ll a^{-k+1/2} t^{-3/2}.$$

Therefore for $0 < t < 1$, we obtain:

$$|\operatorname{Tr}(M_{\psi_2}(T^{-1} e^{-t\Delta_{Z,h}} T - e^{-t\Delta_{Z,g}}))| \ll a^{-k+1/2} t^{-3/2} + a e^{-c'/t} \ll a^{-k+1/2} t^{-3/2}.$$

We know that $t^{-3/2} \ge 1$, thus for $a = t^{-1/9}$, the condition $a^{-k+1/2} t^{-3/2} \le t^\alpha$, for $\alpha \ge 1$, becomes: $\frac{k}{9} - \frac{1}{18} - \frac{3}{2} \ge \alpha$, thus, $k \ge 9\alpha + 14$. Then, for $\alpha = 2$, we need $k \ge 32$; here is where 32 comes in. If $1 \le a \le \frac{5}{4}$, then $1 \le t^{-1/9} \le \frac{5}{4}$ and $\left(\frac{4}{5}\right)^9 \le t \le 1$. Since $\operatorname{Tr}(M_{\chi_{Z_a}} M_{\psi_2}(T^{-1} e^{-t\Delta_{1,h}} T - e^{-t\Delta_{1,g}}))$ is continuous on $[\left(\frac{4}{5}\right)^9, 1]$ the statement of the Proposition also holds when $1 \le a \le \frac{5}{4}$. This finishes the proof of the Proposition. □

Remark 3.8. *In the proof of Proposition 3.5 we took $a = t^{-1/9}$, but what we need is that $a^8 t = t^\kappa$ for some $\kappa > 0$. So we could take $a = t^{-\beta}$ with $0 < \beta < 1/8$. This will allow us to weaken the decay of the function φ at infinity in Proposition 3.7. However, the decay still must be greater than 29 for this to work. Also note that we can obtain a higher order expansion if we require higher decay at infinity of the functions φ and $\Delta_g \varphi$.*

To compute the coefficients in the expansion (3.2) remember that the coefficients in the local expansion of the heat kernels are given by universal functions. Thus we have that:

$$\begin{aligned}
&\operatorname{Tr}(T^{-1} e^{-t\Delta_h} T - e^{-t\Delta_g}) = \int_M K_h(z,z,t) e^{2\varphi(z)} - K_g(z,z,t)\ dA_g(z)\\
&= \int_M \psi_{t^{-1/9},1}(z) t^{-1} \sum_{\ell=0}^{2} (a_\ell(h,z) e^{2\varphi(z)} - a_\ell(g,z)) t^\ell\ dA_g(z) + O(t^2)\\
&= \int_{M_{2t^{-1/9}}} \psi_{t^{-1/9},1}(z) \left(\frac{t^{-1}}{4\pi}(e^{2\varphi(z)} - 1) + \frac{1}{12\pi}(K(h,z) e^{2\varphi(z)} - K(g,z))\right.\\
&\qquad \left. + t(a_2(h,z) e^{2\varphi(z)} - a_2(g,z))\right)\ dA_g(z) + O(t^2),
\end{aligned}$$

where $K(g,z)$ and $K(h,z)$ denote the Gaussian curvatures corresponding to each metric. So, as $t \to 0$ we have:

$$
\begin{aligned}
&\mathrm{Tr}(T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_g}) \\
&= \int_M \left\{ \frac{t^{-1}}{4\pi}(e^{2\varphi(z)} - 1) + \frac{1}{12\pi}(K(h,z)e^{2\varphi(z)} - K(g,z)) + t(a_2(h,z)e^{2\varphi(z)} - a_2(g,z)) \right\} \, dA_g(z) \\
&\quad + O(t^2 + e^{-c/t^{1/9}} + e^{-c'/t}) \\
&= \frac{t^{-1}}{4\pi}\int_M e^{2\varphi(z)} - 1 \; dA_g(z) + \frac{1}{12\pi}\int_M K(h,z)e^{2\varphi(z)} - K(g,z) \; dA_g(z) \\
&\quad + \; t \int_M a_2(h,z)e^{2\varphi(z)} - a_2(g,z) \; dA_g(z) + O(t^2).
\end{aligned}
$$

From Gauss-Bonnet's theorem follows that the constant term in the expansion vanishes. Therefore we finally obtain:

$$
\begin{aligned}
\mathrm{Tr}(T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_g}) = \frac{t^{-1}}{4\pi}(A_h - A_g) + \; t \; \left(\int_M a_2(h,z)dA_h(z) - \int_M a_2(g,z)dA_g(z) \right) \\
+ O(t^2), \text{ as } t \to 0, \quad (3.11)
\end{aligned}
$$

where A_h and A_g denote the area of M with respect to the metrics h and g, respectively.

3.2 Expansion for other relative heat traces.

In this section we consider surfaces with several cusps. Let (M,g) be a Riemannian surface of genus p with m cusps. So, (M,g) admits a decomposition of the form $M = M_0 \cup Z_{a_1} \cup \cdots Z_{a_m}$, where $a_i \geq 1$ for $1 \leq i \leq m$, M_0 is a compact surface with boundary and the metric in each cusp Z_{a_i} is hyperbolic. We assume that $a_i \geq 1$ but this is not really necessary, it can be $a_i > 0$. Let $\bar{\Delta}_{a,0}$ be the direct sum $\oplus_{j=1}^m \Delta_{a_j,0}$ of the Dirichlet Laplacians $\Delta_{a_j,0}$ defined in Chapter 1. Proposition 6.4 in [28] establishes that the operator $e^{-t\Delta_g} - e^{-t\bar{\Delta}_{a,0}}$ is trace class. For its trace there is the following asymptotic expansion as $t \to 0$:

$$
\mathrm{Tr}(e^{-t\Delta_g} - e^{-t\bar{\Delta}_{a,0}}) = \frac{A_g}{4\pi}t^{-1} + (\frac{\gamma m}{2} + \sum_{j=1}^m \log(a_j))\frac{1}{\sqrt{4\pi t}} + \frac{m\log(t)}{2\sqrt{4\pi t}} + \frac{\chi(M)}{6} + \frac{m}{4} + O(\sqrt{t}) \quad (3.12)
$$

A close examination of the proof of equation (3.12) in [28] shows that the term $\sum_{j=1}^m \frac{\log(a_j)}{\sqrt{4\pi t}}$ can be replaced by $e^{-t/4}\sum_{j=1}^m \frac{\log(a_j)}{\sqrt{4\pi t}}$.

In Chapter 5 it will be convenient to consider the relative determinant of the pair $(\Delta_g, \bar{\Delta}_{1,0})$. To that purpose we consider the trace $\mathrm{Tr}(e^{-t\Delta_g} - e^{-t\bar{\Delta}_{1,0}})$. Recall that in Remark 2.8 we explained that the trace is taken in an extended L^2 space. From equation (2.13) follows that in this case the extended space is given by

$$
L^2(M, dA_g) \oplus \oplus_{j=1}^m L^2([1,a_j], y^{-2}dy) = L^2(M_0, dA_g) \oplus \oplus_{j=1}^m (L_0^2(Z_{a_j}) \oplus L^2([1,\infty), y^{-2}dy)). \quad (3.13)
$$

Thus, using Proposition 2.6 and equations (2.14) and (3.12) we obtain the following asymptotic expansion as $t \to 0$:

$$
\mathrm{Tr}(e^{-t\Delta_g} - e^{-t\bar{\Delta}_{1,0}}) = \frac{A_g}{4\pi}t^{-1} + \frac{\gamma m}{2\sqrt{4\pi t}} + \frac{m\log(t)}{2\sqrt{4\pi t}} + \frac{\chi(M)}{6} + \frac{m}{4} + O(\sqrt{t}). \quad (3.14)
$$

This together with equation (3.11) gives:

$$\operatorname{Tr}(T^{-1}e^{-t\Delta_h}T - e^{-t\bar{\Delta}_{1,0}}) = \frac{A_h}{4\pi}t^{-1} + \frac{\gamma m}{2}\frac{1}{\sqrt{4\pi t}} + \frac{m\log(t)}{2\sqrt{4\pi t}} + \frac{\chi(M)}{6} + \frac{m}{4} + O(\sqrt{t}). \tag{3.15}$$

where the transformation T is the identity in the space $\oplus_{j=1}^{m} L^2([1, a_j], y^{-2}dy)$.

Chapter 4

Polyakov's formula for the relative determinant, extremals

In [33] the authors proved that on compact surfaces, with and without boundary and under suitable restrictions, the regularized determinant of the Laplace operator has an extremum. In this chapter we discuss the generalization of the extremal property of determinants given by OPS to certain cases of surfaces with asymptotically cusp ends using the relative determinant introduced by W. Müller in [30]. We study the relative determinant as a function of the metric inside a conformal class, considering non-compact deformations with good decay and proof Polyakov's formula for the relative determinant. Relative determinants on surfaces with cusps were studied by W. Müller in [30], and by R. Lundelius in [26] in terms of "heights".

4.1 Definition of the relative determinant

For the definition of the relative regularized determinant we borrow the definition from [30], as it was recalled in Section 1.9. Let (M, g), $h := e^{2\varphi}g$ where φ and its derivatives up to order two have a suitable decay at infinity. We use the notation introduced in Chapter 1. As before we consider positive Laplacians. Let us recall how the area element, the Laplace operator and the Gaussian curvature change under conformal transformations: $dA_h = e^{2\varphi}dA_g$, $\Delta_h = e^{-2\varphi}\Delta_g$, and $K_h = e^{-2\varphi}(\Delta_g\varphi + K_g)$. Since relative determinants on surfaces with cusps were already studied by W. Müller in [30], here we restrict our attention to the definition and the properties of the following relative determinants:

- $\det(\Delta_h, \Delta_g)$, and
- $\det(\Delta_h, \Delta_{1,0})$.

In Chapter 2 we proved that the operators, $T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_g}$, $e^{-t\Delta_g} - e^{-t\Delta_{1,0}}$ and $T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_{1,0}}$ are trace class, where $T : L^2(M, dA_g) \to L^2(M, dA_h)$, $f \mapsto e^{-\varphi}f$. In Chapter 3 we proved the corresponding asymptotic expansions of their traces as $t \to 0$. See equations (3.12) to (3.15).

Remember the condition for the existence of the expansion of the trace of $T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_g}$ for small values of t: the conformal factor $\varphi|_Z(y, x)$, as well as its derivatives up to second order, should decay as y^{-32}, as $y \to \infty$. The expansion for this case is:

$$\mathrm{Tr}(T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_{1,0}}) = \frac{A_h}{4\pi}t^{-1} + \frac{\gamma m}{2}\frac{1}{\sqrt{4\pi t}} + \frac{m\log(t)}{2\sqrt{4\pi t}} + \frac{1}{12\pi}\int_M K_h\, dA_h + \frac{m}{4} + O(\sqrt{t}) \quad \text{as } t \to 0.$$

Let us take $m = 1$ and let us fix the notation:

$$a_0 = \frac{A_h}{4\pi} \quad a_{10} = \frac{\gamma}{4\sqrt{\pi}}, \quad a_{11} = \frac{1}{4\sqrt{\pi}}, \quad a_2 = \frac{\chi(M)}{6} + \frac{1}{4}.$$

For the asymptotic expansion of the relative heat trace for big t, the trace class property together with the fact that $\sigma_{ac}(\Delta_{1,0}) = [1/4, \infty)$ and Lemma 2.22 in [30] give the existence of a constant $C_1 > 0$ such that

$$\mathrm{Tr}(T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_{1,0}}) = 1 + O(e^{-C_1 t}), \quad \text{as } t \to \infty, \tag{4.1}$$

where the value 1 in the right hand side comes from $\dim \mathrm{Ker}\, \Delta_h - \dim \mathrm{Ker}\, \Delta_{1,0} = 1 - 0$ and the trace is taken in $L^2(M, dA_g)$.

Following [30], we see that the conditions of Theorem 3.4 suffice to define the relative determinant of $(\Delta_h, \Delta_{1,0})$. We start by defining the relative zeta function as:

$$\zeta(s; \Delta_h, \Delta_{1,0}) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}(\mathrm{Tr}(T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_{1,0}}) - 1)dt, \tag{4.2}$$

for $\mathrm{Re}(s) > 1$. It follows from the asymptotic expansions (3.15) and (4.1) that the function $\zeta(s; \Delta_h, \Delta_{1,0})$ has a meromorphic continuation to the complex plane, which we denote again by ζ. To see that there is a meromorhic extension and that it is regular at $s = 0$, consider

$$\begin{aligned}
&\zeta(s; \Delta_h, \Delta_{1,0}) = \zeta_1(s) + \zeta_2(s) \\
&= \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}(\mathrm{Tr}(T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_{1,0}}) - 1)dt + \frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}(\mathrm{Tr}(T^{-1}e^{-t\Delta_h}T - e^{-t\Delta_{1,0}}) - 1)dt \\
&= \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}(a_0 t^{-1} + (a_{10} + a_{11}\log t)t^{-1/2} + a_2 - 1 + O(\sqrt{t}))dt + \frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}f(t)dt,
\end{aligned}$$

where $f(t) = O(e^{-c_1 t})$, as $t \to \infty$, thus the term $\frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}f(t)dt$ is analytic at $s = 0$. For $\mathrm{Re}(s) > 1$, we have that:

$$\begin{aligned}
\zeta_1(s) &= \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}(a_0 t^{-1} + (a_{10} + a_{11}\log t)t^{-1/2} + a_2 - 1 + \vartheta(t))dt \\
&= \frac{1}{\Gamma(s)}\left(\frac{a_0}{s-1} + \frac{a_{10}}{s-1/2} - \frac{a_{11}}{(s-1/2)^2} + \frac{a_2 - 1}{s} + \vartheta_1(s)\right),
\end{aligned}$$

where $\vartheta(t) = O(\sqrt{t})$) and $\vartheta_1(s)$ is a function that is analytic at $s = 0$.

Therefore, we can define the regularized relative determinant of (Δ_h, Δ_0) as:

$$\det(\Delta_h, \Delta_0) = \exp\left(-\frac{d}{ds}\zeta(s; \Delta_h, \Delta_0)\Big|_{s=0}\right). \tag{4.3}$$

4.2 Polyakov's formula

The main tool to study extremal properties of determinants is Polyakov's formula. This formula relates the determinant of a given metric to the determinant of a conformal perturbation of it. In this section we establish a variational formula for $\zeta(s; \Delta_h, \Delta_{1,0})$ that implies Polyakov's formula for relative determinants. The formula obtained is the same as the one for regularized determinants on compact surfaces given in [33]. The proof of the variational formula and Polyakov's formula follows the main lines of the corresponding proof in [33] but we focus in the technical details that allow us to perform each step in the main proof.

In order to study the variation of the relative regularized determinant and the variation of the respective relative zeta function at the metric h we need to consider the following set of functions:

$$\mathcal{F}_{32} := \{\psi \in C^\infty(M) |\ \psi(z) \text{ and } \Delta_g \psi(z) \text{ are } O(i(z)^{-32}) \text{ as } y = i(z) \to \infty\}.$$

Remember that if $\psi(z) = O(i(z)^{-32})$ as $y = i(z) \to \infty$, so are $1 - e^{2\psi(z)}$ and $1 - e^{-2\psi(z)}$ and their derivatives up to second order. Now, for $\psi \in \mathcal{F}_{32}$ and $u \in \mathbb{R}$, let us consider:

$$h_u := e^{2(\varphi + u\psi)} g = e^{2u\psi} h$$
$$\Delta_u := \Delta_{h_u} = e^{-2u\psi} \Delta_h, \quad dA_u := dA_{h_u} = e^{2u\psi} dA_h,$$
$$T_u : L^2(M, dA_u) \to L^2(M, dA_h), f \mapsto f e^{u\psi}.$$

T_u is an unitary map, since for $f \in L^2(M, dA_u)$, $\int_M |T_u f|^2 dA_h = \int_M |f|^2 dA_u$. Let us consider the following functional:

$$\begin{aligned} F : \mathcal{F}_{32} \to \mathbb{C}, F_s(\varphi + u\psi) &:= \zeta(s; \Delta_u, \Delta_{1,0}) \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} (\mathrm{Tr}(T_u e^{-t\Delta_u} T_u^{-1} - T e^{-t\Delta_{1,0}} T^{-1}) - 1) dt, \end{aligned}$$

where the trace is taken in $L^2(M, dA_h)$. The variation of ζ at φ in the direction of ψ is defined as:

$$\frac{\delta\zeta}{\delta\psi}(s; \Delta_h, \Delta_{1,0}) := \left.\frac{\partial}{\partial u} F_s(\varphi + u\psi)\right|_{u=0}$$

Before we proceed with the computation of the derivative in the equation above, we need the following lemmas:

Lemma 4.1.
$$\left.\frac{d}{du} \mathrm{Tr}(T_u e^{-t\Delta_u} T_u^{-1} - T e^{-t\Delta_{1,0}} T^{-1})\right|_{u=0} = -t\, \mathrm{Tr}(\dot{\Delta}_h e^{-t\Delta_h}),$$

where $\dot{\Delta}_h \equiv \frac{\partial}{\partial u} \left.\Delta_u\right|_{u=0} = -2\psi\Delta_h$.

Proof. Let $H_u = T_u \Delta_u T_u^{-1}$. Then H_u is a family of self-adjoint operators acting on $L^2(M, dA_h)$. Note that $e^{-tH_u} = T_u e^{-t\Delta_u} T_u^{-1}$. It is also clear that:

$$\frac{d}{du} \mathrm{Tr}(T_u e^{-t\Delta_u} T_u^{-1} - T e^{-t\Delta_{1,0}} T^{-1}) = \mathrm{Tr}\left(\frac{d}{du} e^{-tH_u}\right).$$

Let $u_1, u_2 > 0$, with $u_1 > u_2$. Let us apply Duhamel's principle in terms of the operators:

$$
\begin{aligned}
e^{-tH_{u_1}} - e^{-tH_{u_2}} &= \int_0^t \frac{\partial}{\partial s}\left(e^{-sH_{u_1}} e^{-(t-s)H_{u_2}}\right)\, ds \\
&= \int_0^t -e^{-sH_{u_1}} H_{u_1} e^{-(t-s)H_{u_2}} + e^{-sH_{u_1}} H_{u_2} e^{-(t-s)H_{u_2}}\, ds.
\end{aligned}
$$

Dividing by $u_1 - u_2$ the previous equation becomes:

$$
\frac{e^{-tH_{u_1}} - e^{-tH_{u_2}}}{u_1 - u_2} = -\int_0^t e^{-sH_{u_1}} \left(\frac{H_{u_1} - H_{u_2}}{u_1 - u_2}\right) e^{-(t-s)H_{u_2}}\, ds,
$$

and letting $u_2 \to u_1$, we obtain:

$$
\frac{d}{du}\, e^{-tH_u}\Big|_{u=u_1} = -\int_0^t e^{-sH_{u_1}} \left(\frac{d}{du} H_u\Big|_{u=u_1}\right) e^{-(t-s)H_{u_1}}\, ds.
$$

Therefore we get:

$$
\begin{aligned}
\frac{d}{du}\operatorname{Tr}(T_u e^{-t\Delta_u} T_u^{-1} - T e^{-t\Delta_{1,0}} T^{-1}) &= -\int_0^t \operatorname{Tr}(e^{-sH_u}\left(\frac{d}{du} H_u\right) e^{-(t-s)H_u})\, ds \\
&= -\int_0^t \operatorname{Tr}(\left(\frac{d}{du} H_u\right) e^{-tH_u})\, ds = -t \operatorname{Tr}\left(\dot{H}_u e^{-tH_u}\right). \qquad (4.4)
\end{aligned}
$$

Let us compute the derivative $\dot{H}_u$:

$$
\begin{aligned}
\frac{d}{du} H_u &= \left(\frac{d}{du} T_u\right) \Delta_u T_u^{-1} + T_u \left(\frac{d}{du}\Delta_u\right) T_u^{-1} + T_u \Delta_u \left(\frac{d}{du} T_u^{-1}\right) \\
&= \psi T_u \Delta_u T_u^{-1} + T_u \left(\frac{d}{du}\Delta_u\right) T_u^{-1} - T_u \Delta_u \psi T_u^{-1}.
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
\operatorname{Tr}\left(\dot{H}_u e^{-tH_u}\right) &= \operatorname{Tr}\left(\psi T_u \Delta_u e^{-t\Delta_u} T_u^{-1}\right) + \operatorname{Tr}\left(T_u \dot{\Delta}_u e^{-t\Delta_u} T_u^{-1}\right) - \operatorname{Tr}\left(T_u \Delta_u \psi e^{-t\Delta_u} T_u^{-1}\right) \\
&= \operatorname{Tr}\left(\psi \Delta_u e^{-t\Delta_u}\right) + \operatorname{Tr}\left(\dot{\Delta}_u e^{-t\Delta_u}\right) - \operatorname{Tr}\left(\Delta_u \psi e^{-t\Delta_u}\right).
\end{aligned}
$$

From the rate of decay assumed for ψ and $\Delta_g \psi$ we have that the operators $\psi e^{-t\Delta_u}$ and $\Delta_u \psi e^{-t\Delta_u}$ are trace class; the proof follows in the same way as the proofs in Chapter 2. Now we use the fact that for a bounded operator A and a trace class operator B we have $\operatorname{Tr}(AB) = \operatorname{Tr}(BA)$. Using that $e^{-t\Delta_u}\Delta_u$ is bounded for all $t > 0$ we obtain:

$$
\operatorname{Tr}\left(\Delta_u \psi e^{-t\Delta_u}\right) = \operatorname{Tr}\left(e^{-\frac{t}{2}\Delta_u} \Delta_u \psi e^{-\frac{t}{2}\Delta_u}\right) = \operatorname{Tr}\left(\psi e^{-t\Delta_u} \Delta_u\right) = \operatorname{Tr}\left(\psi \Delta_u e^{-t\Delta_u}\right).
$$

In this way we get:

$$
\operatorname{Tr}\left(\dot{H}_u e^{-tH_u}\right) = \operatorname{Tr}\left(\dot{\Delta}_u e^{-t\Delta_u}\right) = -2 \operatorname{Tr}\left(\psi \Delta_u e^{-t\Delta_u}\right).
$$

Taking $u = 0$ in the previous equation together with equation (4.4) implies the statement of the lemma. □

Lemma 4.2. *For any $t > 0$, the operator $\psi e^{-t\Delta_h}$ is trace class.*

Proof. Let us use the semigroup property to decompose the operator $\psi e^{-t\Delta_h}$ as

$$\psi e^{-t\Delta_h} = \psi e^{-(t/2)\Delta_h} M_{\phi^{-1}} M_\phi e^{-(t/2)\Delta_h},$$

where ϕ is a smooth function on M such that $\phi(y, x) = y^{-1/2}$, for $(y, x) \in Z$ and where M_ϕ denotes the multiplication operator by ϕ. Each of the operators $\psi e^{-t/2\Delta_h} M_{\phi^{-1}}$ and $M_\phi e^{-t/2\Delta_h}$ is Hilbert-Schmidt. The proof of the Hilbert-Schmidt property for $M_\phi e^{-t/2\Delta_h}$ is the same as the proof for $M_\phi e^{-t/2\Delta_g}$ in Chapter 2. For the operator $\psi e^{-t/2\Delta_h} M_{\phi^{-1}}$ the proof is similar. We just need to verify that

$$\int_M \int_M |\psi(z) K_h(z, z', t/2) \phi(z')^{-1}|^2 dA_h(z) dA_h(z') < \infty.$$

The integrals obtained after estimating the heat kernel are of the same kind as those obtained in Chapter 2. $\square$

In this way we have for the variation of the relative zeta function that:

$$\begin{aligned}\frac{\delta\zeta}{\delta\psi}(s; \Delta_h, \Delta_{1,0}) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left.\frac{d}{du}(\operatorname{Tr}(T_u e^{-t\Delta_u} T_u^{-1} - T e^{-t\Delta_{1,0}} T^{-1}) - 1)\right|_{u=0} dt \\ &= \frac{-1}{\Gamma(s)} \int_0^\infty t^s \operatorname{Tr}((-2\psi \Delta_h e^{-t\Delta_h}) dt = \frac{-2}{\Gamma(s)} \int_0^\infty t^s \frac{\partial}{\partial t} \operatorname{Tr}(\psi e^{-t\Delta_h}) dt,\end{aligned}$$

where the last equality follows from:

$$\begin{aligned}\frac{\partial}{\partial t}(\operatorname{Tr} \psi e^{-t\Delta_h}) = \int_M \psi(z) \frac{\partial}{\partial t} K_h(z, z, t) dA_h(z) \\ = - \int_M \psi(z) \Delta_h K_h(z, z, t) dA_h(z) = - \operatorname{Tr}(\psi \Delta_h e^{-t\Delta_h}).\end{aligned}$$

Remember that in equation (4.2), the constant 1 stands for $\dim \operatorname{Ker} \Delta_h - \dim \operatorname{Ker} \Delta_{1,0}$. In this case we have $\dim \operatorname{Ker} \Delta_u - \dim \operatorname{Ker} \Delta_{1,0} = 1$, i.e. it will be independent of u. Now,

$$\frac{\partial}{\partial t} \psi e^{-t\Delta_h} = \frac{\partial}{\partial t}\ \psi(e^{-t\Delta_h} - P_{\operatorname{Ker}(\Delta_h)}).$$

Therefore

$$\frac{\delta\zeta}{\delta\psi}(s; \Delta_h, \Delta_{1,0}) = \frac{-2}{\Gamma(s)} \int_0^\infty t^s \frac{\partial}{\partial t} \operatorname{Tr}(\psi(e^{-t\Delta_h} - P_{\operatorname{Ker}(\Delta_h)})) dt. \tag{4.5}$$

Before we proceed with the computation of the above integral, we need the following lemmas:

Lemma 4.3. *There exists a constant $c > 0$ such that:*

$$\operatorname{Tr}(\psi(e^{-t\Delta_h} - P_{\operatorname{Ker}(\Delta_h)})) = O(e^{-ct}), \text{ as } t \to \infty.$$

Proof. Let $t > 1$ and let us write:

$$\psi(e^{-t\Delta_h} - P_{\mathrm{Ker}(\Delta_h)}) = \psi e^{-\frac{1}{2}\Delta_h}(e^{-(t-\frac{1}{2})\Delta_h} - P_{\mathrm{Ker}(\Delta_h)}),$$

where we used that $e^{-\frac{1}{2}\Delta_h} P_{\mathrm{Ker}(\Delta_h)} = P_{\mathrm{Ker}(\Delta_h)}$. Now, by Lemma 4.2 we have that $\psi e^{-\frac{1}{2}\Delta_h}$ is trace class. On the other hand, using the spectral theorem we have for $f \in L^2(M, dA_h)$:

$$e^{-t\Delta_h} f - P_{\mathrm{Ker}(\Delta_h)} f = e^{-t(\Delta_h - P_{\mathrm{Ker}(\Delta_h)})} f.$$

Note that $\sigma_{\mathrm{ess}}(\Delta_h) = [1/4, \infty)$ implies that 0 is an isolated eigenvalue of Δ_h and $\sigma(\Delta_h - P_{\mathrm{Ker}(\Delta_h)}) \subseteq [c_1, \infty)$ for some $c_1 \in (0, 1/4]$. Thus $\|e^{-t(\Delta_h - P_{\mathrm{Ker}(\Delta_h)})}\|_{L^2(M,h)} \leq e^{-c_1 t}$ for any $t > 0$. If $t > 1$, $t - \frac{1}{2} > 0$ and for the trace we obtain:

$$\begin{aligned} |\operatorname{Tr}(\psi(e^{-t\Delta_h} - P_{\mathrm{Ker}(\Delta_h)}))| &\leq \|\psi e^{-\frac{1}{2}\Delta_h}(e^{-(t-\frac{1}{2})\Delta_h} - P_{\mathrm{Ker}(\Delta_h)})\|_1 \\ &\leq \|\psi e^{-\frac{1}{2}\Delta_h}\|_1 \|e^{-(t-\frac{1}{2})(\Delta_h - P_{\mathrm{Ker}(\Delta_h)})}\|_{L^2(M,h)} \ll e^{-c_1 t}. \end{aligned}$$

This proves Lemma 4.3. $\square$

Lemma 4.4. *For $0 < t \leq 1$ the trace of the operator $\psi(e^{-t\Delta_h} - P_{\mathrm{Ker}(\Delta_h)})$ has the following expansion:*

$$\operatorname{Tr}(\psi(e^{-t\Delta_h} - P_{\mathrm{Ker}(\Delta_h)})) = \int_M \psi(z) \left(\frac{1}{4\pi t} + \frac{K_h(z)}{12\pi} - \frac{1}{A_h} \right) dA_h + O(t)$$

as $t \to 0$.

Proof. In order to prove Lemma 4.4 we use a method similar to the one used in Section 3.1 to prove the existence of the expansion of the relative heat trace $\operatorname{Tr}(e^{-t\Delta_h} - e^{-t\Delta_g})$ for small t. We start by considering the parametrix kernel $Q_h(z, z', t)$ defined by equation (3.1):

$$Q_h(z, w, t) = \varphi_1(z) K_{W,h}(z, w, t) \psi_1(w) + \varphi_2(z) K_{1,h}(z, w, t) \psi_2(w),$$

where the functions φ_i and ψ_i, $i = 1, 2$, are defined in Section 3.1. Lemma 3.3 gives a constant $c_3 > 0$ that allows us to replace the heat kernel $K_h(z, z', t)$ by $Q_h(z, z', t)$:

$$\operatorname{Tr}(\psi(e^{-t\Delta_h} - P_{\mathrm{Ker}(\Delta_h)})) = \int_M \psi(z)(K_h(z, z, t) - \frac{1}{A_h}) dA_h(z)$$

$$\begin{aligned} &\left| \int_M \psi(z)(K_h(z, z, t) - \frac{1}{A_h}) dA_h(z) - \int_M \psi(z)(Q_h(z, z, t) - \frac{1}{A_h}) dA_h(z) \right| \\ &\qquad\qquad \ll \int_M |K_h(z, z, t) - Q_h(z, z, t)| dA_h(z) = O(e^{-c_3/t}). \end{aligned}$$

With this we can restrict our attention to $\int_M \psi(z)(Q_h(z, z, t) - \frac{1}{A_h}) dA_h(z)$ and split the integral as the sum of the following two terms:

$$\begin{aligned} L_1(t) &= \int_{M_2} \psi(z) \psi_1(z)(K_{W,h}(z, z, t) - \frac{1}{A_h}) dA_h(z) \\ L_2(t) &= \int_{Z_{\frac{5}{4}}} \psi(z) \psi_2(z)(K_{1,h}(z, z, t) - \frac{1}{A_h}) dA_h(z). \end{aligned}$$

For $L_1(t)$ we use the asymptotic expansion of the kernel $K_{W,h}(z,z,t)$ on the compact manifold W to obtain:

$$L_1(t) = \int_{M_2} \psi(z)\psi_1(z)(K_{W,h}(z,z,t) - \frac{1}{A_h})dA_h(z)$$
$$= \int_{M_2} \psi(z)\psi_1(z)\left(\frac{1}{4\pi t} + \frac{K_h(z)}{12\pi} - \frac{1}{A_h} + R_1(z,t)\right)dA_h(z). \quad (4.6)$$

For $L_2(t)$, we consider the same construction as in the proof of Proposition 3.5. Let us summarize the idea. For the details see the proof of Proposition 3.5. We first extend the metric h to a metric $\widetilde{h}$ on the complete cusp $\widetilde{Z} = (0,\infty) \times S^1$ and then we lift $\widetilde{h}$ to a metric $\hat{h}$ on the universal cover $\hat{Z} = \mathbb{R}^+ \times \mathbb{R}$ of the complete cusp $\widetilde{Z}$. We also lift the functions ψ and ψ_2 to functions on $\hat{Z}$ satisfying $\hat{\psi} = \widetilde{\psi} \circ \pi$ and $\hat{\psi}_2 = \widetilde{\psi}_2 \circ \pi$. By Lemma 3.6 we have that

$$K_{1,h}(z,w,t) = \sum_{m\in\mathbb{Z}} k_h(\widetilde{z}, \widetilde{w} + m, t),$$

where $\pi(\widetilde{z}) = z$ and $\pi(\widetilde{w}) = w$. We also have that $F = \mathbb{R}+ \times [0,2\pi]$ is a fundamental domain for the the group of deck transformations $\Gamma = \mathbb{Z}$. Now, let $a > 5/4$ and let us split the integral $L_2(t)$ as the sum $L_2 = J_1(t) + J_2(t) + J_3(t)$, where the J_i, $i = 1,2,3$, are given by:

$$J_1(t) = \int_{\frac{5}{4}}^{\infty} \int_0^{2\pi} \hat{\psi}(\widetilde{z})\hat{\psi}_2(\widetilde{z})(k_h(\widetilde{z}, \widetilde{z}, t) - \frac{1}{A_h})dA_{\hat{h}}(\widetilde{z}),$$
$$J_2(t) = \int_{\frac{5}{4}}^{a} \int_0^{2\pi} \hat{\psi}(\widetilde{z})\hat{\psi}_2(\widetilde{z}) \sum_{m\neq 0} k_h(\widetilde{z}, \widetilde{z} + m, t)dA_{\hat{h}}(\widetilde{z}),$$
$$J_3(t) = \int_{a}^{\infty} \int_0^{2\pi} \hat{\psi}(\widetilde{z})\hat{\psi}_2(\widetilde{z}) \sum_{m\neq 0} k_h(\widetilde{z}, \widetilde{z} + m, t)dA_{\hat{h}}(\widetilde{z}).$$

For J_1 we use the local asymptotic expansion of the heat kernel $k_h(\widetilde{z}, \widetilde{z}, t)$:

$$J_1(t) = \int_{\frac{5}{4}}^{\infty} \int_0^{2\pi} \hat{\psi}(\widetilde{z})\hat{\psi}_2(\widetilde{z}) \left(\frac{1}{4\pi t} + \frac{K_{\hat{h}}(\widetilde{z})}{12\pi} - \frac{1}{A_h} + R_{1,1}(\widetilde{z}, t)\right) dA_{\hat{h}}(\widetilde{z}) \quad (4.7)$$

We know that $|R_{1,1}(\widetilde{z}, t)| = O(t)$, uniformly in $\widetilde{z}$, see [10].

For $J_2(t)$ we use the same kind of estimates as in the proof of Proposition 3.5. We use namely that the metric $\hat{h}$ is quasi-isometric to the hyperbolic metric in the upper half plane. Therefore the heat kernel k_h satisfies the estimate:

$$k_h(\widetilde{z}, \widetilde{w}, t) \ll t^{-1} e^{-\frac{c_1 d_{\hat{g}}(\widetilde{z}, \widetilde{w})}{t}},$$

where $d_{\hat{g}}$ is the hyperbolic distance. Thus the series can be estimated in the same way as in equation (3.8). Then using equation (3.9) we obtain:

$$J_2(t) \ll \int_{\frac{5}{4}}^{a} y^{-32} e^{-\frac{c_2}{a^8 t}} \sum_{m\neq 0} e^{-\frac{c_1 \log(1+\frac{m^2}{2a^2})^2}{2t}} \frac{dy}{y^2}$$
$$\ll e^{-\frac{c_2}{a^8 t}} \int_{\frac{5}{4}}^{a} y^{-32} \int_1^{\infty} e^{-\frac{c_1 \log(1+\frac{u^2}{2a^2})^2}{2t}} du \, \frac{dy}{y^2} \ll a e^{-\frac{c_2}{a^8 t}}. \quad (4.8)$$

Now, for J_3 we have:

$$J_3(t) = \int_a^\infty \int_0^{2\pi} \hat{\psi}(\widetilde{z})\hat{\psi}_2(\widetilde{z}) \sum_{m\neq 0} k_h(\widetilde{z}, \widetilde{z}+m, t) dA_{\hat{h}}(\widetilde{z})$$
$$\leq \int_{Z_a} \psi(z)\psi_2(z) K_{1,h}(z,z,t) dA_h(z) \ll \int_a^\infty t^{-1} y^{-32} y \frac{dy}{y^2} = t^{-1} \int_a^\infty y^{-33} dy \ll t^{-1} a^{-32}. \quad (4.9)$$

As in Chapter 3, let us take $a = t^{-1/9}$. Then putting equations (4.6) (4.7) (4.8) and (4.9) together we obtain:

$$\begin{aligned}\operatorname{Tr}(\psi(e^{-t\Delta_h} - P_{\operatorname{Ker}(\Delta_h)})) &= \int_M \psi(z)(K_h(z,z,t) - \frac{1}{A_h}) dA_h(z) \\ &= \int_{M_2} \psi(z)\psi_1(z) \left(\frac{1}{4\pi t} + \frac{K_h(z)}{12\pi} - \frac{1}{A_h} + R_1(z,t) \right) dA_h(z) \\ &\quad + \int_{\frac{5}{4}}^\infty \int_0^{2\pi} \hat{\psi}(\widetilde{z})\hat{\psi}_2(\widetilde{z}) \left(\frac{1}{4\pi t} + \frac{K_{\hat{h}}(\widetilde{z})}{12\pi} - \frac{1}{A_h} + R_{1,1}(\widetilde{z},t) \right) dA_{\hat{h}}(\widetilde{z}) + O(t^2),\end{aligned}$$

where $O(t^2)$ is clearly independent of z. Now we know that $|R_1(z,t)| \ll t$ and $|R_{1,1}(\widetilde{z},t)| \ll t$ uniformly in z. Therefore we can make the following estimate:

$$\int_{M_2} \psi(z)\psi_1(z) R_1(z,t) dA_h(z) + \int_{\frac{5}{4}}^\infty \int_0^{2\pi} \hat{\psi}(\widetilde{z})\hat{\psi}_2(\widetilde{z}) R_{1,1}(\widetilde{z},t) dA_{\hat{h}}(\widetilde{z}) \ll t.$$

In this way we conclude that:

$$\operatorname{Tr}(\psi(e^{-t\Delta_h} - P_{\operatorname{Ker}(\Delta_h)})) = \int_M \psi(z) \left(\frac{1}{4\pi t} + \frac{K_h(z)}{12\pi} - \frac{1}{A_h} \right) dA_h(z) + O(t).$$

This finishes the proof of Lemma 4.4. □

Going back to the variation of the relative zeta function, let us apply integration by parts in equation (4.5) to obtain for $\operatorname{Re}(s) > 0$:

$$\frac{\delta\zeta}{\delta\psi}(s; \Delta_h, \Delta_{1,0}) = \frac{2s}{\Gamma(s)} \int_0^\infty t^{s-1} \operatorname{Tr}(\psi(e^{-t\Delta_h} - P_{\operatorname{Ker}(\Delta_h)})) dt.$$

Let us now split this integral as:

$$\frac{\delta\zeta}{\delta\psi}(s; \Delta_h, \Delta_{1,0}) = \frac{2s}{\Gamma(s)} \left(\int_0^1 t^{s-1} \operatorname{Tr}(\psi(e^{-t\Delta_h} - P_{\operatorname{Ker}(\Delta_h)})) dt + \int_1^\infty t^{s-1} \operatorname{Tr}(\psi(e^{-t\Delta_h} - P_{\operatorname{Ker}(\Delta_h)})) dt \right) \quad (4.10)$$

and let us study each term separately.

For the second term on the right hand side in equation (4.10) it follows from Lemma 4.3 that $\int_1^\infty t^{s-1} \operatorname{Tr}(\psi(e^{-t\Delta_h} - P_{\operatorname{Ker}(\Delta_h)})) dt$ is an entire function of s. Since $\Gamma(s)^{-1} \sim s$, we infer

$$\frac{d}{ds} \frac{2s}{\Gamma(s)} \int_1^\infty t^{s-1} \operatorname{Tr}(\psi(e^{-t\Delta_h} - P_{\operatorname{Ker}(\Delta_h)})) dt \,\Big|_{s=0} = 0$$

For the first term on the right hand side of (4.10), we use Lemma 4.4. Thus, for $Re(s) > 1$ and taking the corresponding extensions, we have:

$$\begin{aligned}
\frac{\delta\zeta_1}{\delta\psi}(s;\Delta_h,\Delta_{1,0}) &= \frac{2s}{\Gamma(s)}\int_0^1 t^{s-1}\left\{\int_M \psi(z)\left(\frac{1}{4\pi t}+\frac{K_h(z)}{12\pi}-\frac{1}{A_h}\right)dA_h + O(t)\right\}\, dt \\
&= \frac{2s}{\Gamma(s)}\left\{\frac{1}{4\pi}\frac{1}{s-1}\int_M \psi(z)dA_h + \frac{1}{s}\int_M \psi(z)(\frac{K_h(z)}{12\pi}-\frac{1}{A_h})dA_h + \text{ analytic in } s\right\} \\
&= \frac{2s}{\Gamma(s)}\left\{\frac{1}{s}\int_M \psi(z)(\frac{K_h(z)}{12\pi}-\frac{1}{A_h})dA_h + \text{ analytic in } s \text{ near } 0\right\}.
\end{aligned}$$

We consider now the derivative with respect to s at $s = 0$. We first take into account that $\frac{1}{\Gamma(s)} = s + O(s^2)$. Then

$$\begin{aligned}
\frac{d}{ds}\frac{\delta\zeta_1}{\delta\psi}(s;\Delta_h,\Delta_{1,0})\Big|_{s=0} &= \frac{d}{ds}\,\frac{s}{\Gamma(s)}\left(\frac{1}{s}\int_M 2\psi(z)(\frac{K_h(z)}{12\pi}-\frac{1}{A_h})dA_h + \text{ analytic in } s\right)\Big|_{s=0} \\
&= \frac{d}{ds}\,\frac{1}{\Gamma(s)}\left(\int_M 2\psi(z)(\frac{K_\varphi(z)}{12\pi}-\frac{1}{A_h})dA_h \ + (\widetilde{a}_1 s+\dots)\right)\Big|_{s=0} \\
&= \int_M 2\psi(z)\left(\frac{K_h(z)}{12\pi}-\frac{1}{A_h}\right)dA_h.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\delta}{\delta\psi}\log\det(\Delta_h,\Delta_{1,0}) &= -\frac{\delta}{\delta\psi}\frac{d}{ds}\zeta(s;\Delta_h,\Delta_0)\big|_{s=0} = -\int_M 2\psi(z)\left(\frac{K_h(z)}{12\pi}-\frac{1}{A_h}\right)dA_h \\
&= -\int_M 2\psi(z)\left(\frac{1}{12\pi}e^{-2\varphi}(\Delta_g\varphi+K_g)-\frac{1}{A_h}\right)e^{2\varphi}dA_g \\
&= -\frac{1}{6\pi}\int_M \psi(z)(\Delta_g\varphi+K_g)dA_g + \frac{1}{A_h}\int_M 2\psi e^{2\varphi}dA_g \\
&= -\frac{1}{6\pi}\int_M \psi(\Delta_g\varphi+K_g)\; dA_g + \frac{\delta}{\delta\psi}\log A_h.
\end{aligned}$$

Using that

$$\begin{aligned}
\frac{1}{2}\,\frac{\partial}{\partial u}\int_M |\nabla_g(\varphi+u\psi)|^2\; dA_g\Big|_{u=0} &= \frac{1}{2}\,\frac{\partial}{\partial u}\langle d_g(\varphi+u\psi), d_g(\varphi+u\psi)\rangle\Big|_{u=0} \\
&= \frac{1}{2}\left(\langle d_g^* d_g\varphi,\psi\rangle + \langle\psi, d_g^* d_g\varphi\rangle\right) = \langle\psi,\Delta_g\varphi\rangle, \\
\frac{\partial}{\partial u}\int_M K_g\;(\varphi+u\psi)\; dA_g\Big|_{u=0} &= \int_M K_g\;\psi\; dA_g,
\end{aligned}$$

for all ψ in the domain of F we obtain:

$$\log\det(\Delta_h,\Delta_{1,0}) = -\frac{1}{12\pi}\int_M |\nabla_g\varphi|^2\; dA_g - \frac{1}{6\pi}\int_M K_g\;\varphi\; dA_g + \log A_h + C.$$

Notice that if $\varphi = 0$ we have $\Delta_h = \Delta_g$ and from the previous equation we obtain $C = \log\det(\Delta_g,\Delta_{1,0})$. In this way, we have proved Polyakov's formula:

Theorem 4.5. *Let (M,g) be a surface with cusps and let $h = e^{2\varphi}g$ be a conformal transformation of g with $\varphi \in \mathcal{F}_{32}$. For the corresponding relative determinants we have the following formula:*

$$\log\det(\Delta_h,\Delta_{1,0}) = -\frac{1}{12\pi}\int_M |\nabla_g\varphi|^2\ dA_g - \frac{1}{6\pi}\int_M K_g\ \varphi\ dA_g + \log A_h + \log\det(\Delta_g,\Delta_{1,0}). \quad (4.11)$$

4.3 Extremal properties of the relative determinant

Following [33] we choose a functional Φ related to $\det(\Delta_h,\Delta_{1,0})$ that is translation invariant and such that maximizing $\log\det(\Delta_h,\Delta_{1,0})$ is the same as minimizing Φ for metrics of constant area. It is convenient to choose the same functional as in [33]:

$$\Phi(\varphi) = \frac{1}{2}\int_M |\nabla_g\varphi|^2\ dA_g + \int_M K_g\ \varphi\ dA_g - \pi\chi(M)\log\left(\int_M e^{2\varphi}dA_g\right). \quad (4.12)$$

It is translation invariant since for any constant a we have:

$$\begin{aligned}\Phi(\varphi+a) &= \frac{1}{2}\int_M |\nabla_g(\varphi+a)|^2\ dA_g + \int_M K_g\ (\varphi+a)\ dA_g - \pi\chi(M)\log\left(e^{2a}\int_M e^{2\varphi}dA_g\right)\\ &= \Phi(\varphi) + a\left(\int_M K_g\ dA_g - 2\pi\chi(M)\right) = \Phi(\varphi).\end{aligned}$$

From

$$\begin{aligned}-6\pi\log\det(\Delta_h,\Delta_{1,0}) &= \frac{1}{2}\int_M |\nabla_g\varphi|^2\ dA_g + \int_M K_g\ \varphi\ dA_g - 6\pi\log(A_h)\\ &= \Phi(\varphi) + \pi\chi(M)\log\left(\int_M e^{2\varphi}dA_g\right) - 6\pi\log(A_h),\end{aligned}$$

it follows that

$$\Phi(\varphi) = -6\pi\log\det(\Delta_h,\Delta_{1,0}) + \pi(6-\chi(M))\log(A_h).$$

Then, under the constraint $A_h = 1$ we have

$$\Phi(\varphi) = -6\pi\log\det(\Delta_h,\Delta_{1,0}).$$

Notice that Φ is convex, if $\chi(M) \leq 0$. Therefore, Φ may attain a minimum.

Let us analyze the functional without requiring the constraint $A_h = 1$. Assume that $\chi(M) \leq 0$ and that φ minimizes Φ. If this happens we have that $\frac{\delta\Phi}{\delta\psi}(\varphi) = 0$, for all $\psi \in \mathcal{F}_{32}$.

$$\begin{aligned}\frac{\delta\Phi}{\delta\psi}(\varphi) &= -6\pi\frac{\delta}{\delta\psi}\log\det(\Delta_h,\Delta_{1,0}) + \pi(6-\chi(M))\frac{\delta\log(A_h)}{\delta\psi}\\ &= \int_M \psi(\Delta_g\varphi + K_g)\ dA_g + -6\pi\frac{\delta\log(A_h)}{\delta\psi} + \pi(6-\chi(M))\frac{\delta\log(A_h)}{\delta\psi}\\ &= \int_M \psi(\Delta_g\varphi + K_g)\ dA_g - \pi\chi(M)\frac{1}{A_h}\int_M 2\psi e^{2\varphi}dA_g\\ &= \int_M \psi[\Delta_g\varphi + K_g - \pi\chi(M)\frac{2}{A_h}\ e^{2\varphi}]dA_g = 0.\end{aligned}$$

By elliptic regularity we have:

$$\Delta_g \varphi + K_g - \frac{2\pi\chi(M)}{A_h}\ e^{2\varphi} = 0,$$

thus

$$K_h = e^{-2\varphi}(\Delta_g \varphi + K_g) = \frac{2\pi\chi(M)}{\int_M e^{2\varphi} dA_g}.$$

The left hand side in the last equation is independent of $x \in M$. Therefore if φ minimizes Φ it follows that K_h should be constant. If $A_h = 2\pi(2p + m - 2)$, it follows that $K_h \equiv -1$, where p is the genus of M and m is the number of cusps.

On the other hand if $K_h \equiv$ constant we have that:

$$\begin{aligned} \frac{\delta\Phi}{\delta\psi}(\varphi) &= \int_M e^{2\varphi}\psi K_h dA_g - \frac{\pi\chi(M)}{A_h}\int_M 2\psi e^{2\varphi} dA_g \\ &= \int_M \frac{e^{2\varphi}\psi}{A_h}(K_h A_h - 2\pi\chi(M)) dA_g = 0, \end{aligned}$$

because of Gauss-Bonnet theorem and the constant value of the Gaussian curvature ($K_h A_h = \int_M K_h\ dA_h$).

Remark 4.6. *About the existence of a maximizer of the relative determinant, consider starting with a metric τ on M of negative constant curvature $K_\tau = -1$, and taking the conformal class*

$$\mathrm{Conf}_{1,32}(\tau) = \{h | h = e^{2\psi}\tau,\ \textit{with } \psi \in \mathcal{F}_{32} \textit{ and } A_h = 2\pi(2p + m - 2)\}.$$

Since τ itself is the maximizer and $\tau \in \mathrm{Conf}_{1,32}(\tau)$, the maximizer trivially exists inside the conformal class. However, if we start with a general metric g on M that is hyperbolic only in the cusp Z, the differential equation for the curvature on the cusp will be:

$$-e^{2\varphi} = \Delta_g \varphi - 1.$$

This implies that in the cusp the function φ should decay as y^{-1}, being in this case the function φ outside the conformal class under consideration. Therefore in order to have a maximizer of the relative determinant inside the conformal class we need to be able to define the determinant for Laplacians whose metrics have conformal factors $e^{2\varphi}$ with φ having a decay of y^{-1} at infinity.

Chapter 5

Boundedness and comparison

In this chapter we specialize to hyperbolic surfaces. We use Selberg's trace formula and use the work of L. Bers in [1] and of J. Jorgenson and R. Lundelius in [19] to prove that the relative determinant tends to zero when one approaches the boundary of the the moduli space of hyperbolic surfaces of fixed genus q with m cusps, $\mathcal{M}_{q,m}$. This fact implies that the relative determinant is bounded as a function on the moduli space.

Let (M,τ) be a Riemann surface of genus q with m cusps, where τ is a hyperbolic metric of constant negative unitary curvature. To each element $[\tau] \in \mathcal{M}_{q,m}$ we associate the relative determinant $\det(\Delta_\tau, \bar{\Delta}_{1,0})$. Let us recall the operator $\bar{\Delta}_{1,0}$ defined in Section 1.1. Let us denote by $\Delta_{1,0}$ the self-adjoint extension of the operator

$$-y^2\frac{\partial^2}{\partial y^2} : C_c^\infty((1,\infty)) \to L^2([1,\infty), y^{-2}dy)$$

obtained after imposing Dirichlet boundary conditions at $y=1$. Let $\bar{\Delta}_{1,0}$ be the operator defined as the direct sum $\oplus_{j=1}^m \Delta_{1,0}$. The operator $\bar{\Delta}_{1,0}$ acts on a subspace of $\oplus_{j=1}^m L^2([1,\infty), y_j^{-2}dy_j)$. If (M,τ) can be decomposed as $M = M_0 \cup Z_{a_1} \cup \cdots Z_{a_m}$, with $a_j \geq 1$; then the difference $e^{-t\Delta_\tau} - e^{-\bar{\Delta}_{1,0}}$ is taken in the extended L^2 space given by equation (3.13):

$$L^2(M, dA_\tau) \oplus \oplus_{j=1}^m L^2([1,a_j], y^{-2}dy) = L^2(M_0, dA_\tau) \oplus \oplus_{j=1}^m (L_0^2(Z_{a_j}) \oplus L^2([1,\infty), y^{-2}dy)).$$

We want to see how $\det(\Delta_\tau, \bar{\Delta}_{1,0})$ behaves as we let the class $[\tau]$ approach the 'boundary' of the moduli space, where by 'boundary' we mean the set $\overline{\mathcal{M}_{q,m}} \setminus \mathcal{M}_{q,m}$. We assume surfaces connected, although the limit M may not be connected. In order to prove our statements we use some of the results of Jorgenson and Lundelius in [19]. Therein they define a determinant for Laplacians on hyperbolic Riemann surfaces of finite volume, non-connected in general. We compare both determinants. Part of the problem is to understand the kind of degenerations under consideration.

Let us start by recalling Selberg's trace formula [39] as it is presented in [17], applied to the function $h(r) = e^{-t(\frac{1}{4}+r^2)}$ and its Fourier transform $g(u) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{t}{4}} e^{-\frac{u^2}{4t}}$.

Let Γ be a Fuchsian group of the first kind. Let $\Gamma \setminus \mathbb{H} = M$ be the associated surface, let Δ be the Laplacian on M and let $\lambda_j^2 = \frac{1}{4} - r_j^2$ be the sequence of eigenvalues of Δ. We do not include the contribution of the elliptic elements, because we consider groups without elliptic elements. In this case Selberg's trace formula applied to the heat operator takes the form:

$$\sum_j e^{-t(\frac{1}{4}+r_j^2)} - \frac{1}{4\pi}\int_{\mathbb{R}} e^{-t(\frac{1}{4}+\lambda^2)}\frac{\phi'}{\phi}(\frac{1}{2}+i\lambda)d\lambda + \frac{e^{-\frac{t}{4}}}{4}\operatorname{Tr}(\Phi(\frac{1}{2}))$$
$$= \frac{\operatorname{Area}(M)}{4\pi}\int_{\mathbb{R}} e^{-t(\frac{1}{4}+\lambda^2)}\lambda\tanh(\pi\lambda)d\lambda + \frac{e^{-\frac{t}{4}}}{\sqrt{4\pi t}}\sum_{k=1}^{\infty}\sum_{\{\gamma\}_\Gamma}\frac{\ell(\gamma)}{2\sinh(\frac{k\ell(\gamma)}{2})}e^{-\frac{(k\ell(\gamma))^2}{4t}}$$
$$-\frac{m}{\pi}\int_{\mathbb{R}} e^{-t(\frac{1}{4}+\lambda^2)}\frac{\Gamma'}{\Gamma}(1+i\lambda)d\lambda + \frac{m}{4}e^{-\frac{t}{4}} - m\log(2)\frac{e^{-\frac{t}{4}}}{\sqrt{4\pi t}}, \tag{5.1}$$

where the sum runs over the primitive hyperbolic conjugacy classes γ with length $\ell(\gamma)$, m is the number of inequivalent cusps, $\Phi(s)$ is the scattering matrix and $\phi(s) = \det\Phi(s)$.

In the notation of [19] the hyperbolic heat trace $\mathrm{HTr}K_M(t)$ and the regularized trace $\mathrm{STr}K_M(t)$ are given by

$$\mathrm{HTr}\ \mathrm{K}_M(t) = \frac{e^{-\frac{t}{4}}}{\sqrt{16\pi t}}\sum_{k=1}^{\infty}\sum_{\{\gamma\}\Gamma}\frac{\ell(\gamma)}{\sinh(\frac{k\ell(\gamma)}{2})}e^{-\frac{(k\ell(\gamma))^2}{4t}}$$
$$\mathrm{STrK}_M(t) = \mathrm{HTr}\ \mathrm{K}_M(t) + \operatorname{Area}(M)K_{\mathbb{H}}(t,0),$$

where

$$K_{\mathbb{H}}(t,0) = \frac{1}{4\pi}\int_{\mathbb{R}} e^{-t(\frac{1}{4}+\lambda^2)}\lambda\tanh(\pi\lambda)d\lambda.$$

With the help of these expressions, the authors in [19] define a hyperbolic zeta function and a hyperbolic determinant:

$$\zeta_{M,\mathrm{hyp}}(s) = \frac{1}{\Gamma(s)}\int_0^{\infty}(\mathrm{STrK}_M(t) - d)t^{s-1}dt \quad \text{and} \quad \det{}_{\mathrm{hyp}}\Delta_\tau := \exp(-\zeta'_{\mathrm{hyp}}(0)), \tag{5.2}$$

where d is the number of connected components of M as well as the dimension of $\operatorname{Ker}(\Delta_\tau)$.

We want to see now how the hyperbolic determinant $\det_{\mathrm{hyp}}\Delta_\tau$ relates to the relative determinant $(\Delta_\tau, \bar{\Delta}_{1,0})$. In order to do that let us first consider $P(t)$ be the contribution of the parabolic elements to the trace formula. We know that $P(t)$ is given by

$$P(t) = \int_{\mathbb{R}} e^{-t(\frac{1}{4}+r^2)}\frac{\Gamma'}{\Gamma}(1+ir)\ dr,$$

and for which we have the following lemma:

Lemma 5.1. *$P(t)$ has the following asymptotic expansions:*

$$P(t) \sim -\frac{\pi}{2}\frac{\log(t)}{t} + \frac{\sqrt{\pi}}{2\sqrt{t}}(-B_1 + \gamma - \log(4) + \pi) + t^{-1/2}\sum_{j=1}^{\infty} b_j t^{j/2}, \qquad \text{as } t \to 0,$$

where B_1 is the first Bernoulli number and γ in this case denoted the Euler constant. As $t \to \infty$, we have that $P(t) = O(e^{-\frac{t}{4}})$.

Proof. The proof of Lemma 5.1 easily follows from the formula

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)} = \frac{1}{2z} + \log(z) - \int_0^\infty \left(\frac{1}{2} - \frac{1}{u} + \frac{1}{e^u - 1}\right) du,$$

for $\mathrm{Re}(z) > 0$, and from Stirling's formula:

$$\log(\Gamma(z)) = (z - \frac{1}{2})\log(z) - z + \frac{1}{2}\log(2\pi) + \sum_{r=1}^{\infty} \frac{(-1)^{r-1}B_r}{2r(2r-1)z^{2r-1}},$$

for $|\arg(z)| \leq \frac{\pi}{2} - \theta$, where B_r is the r-th Bernoulli number. □

The relation between the two determinants is given by the following proposition:

Proposition 5.2. *For the relative determinant and the hyperbolic determinant we have the following relation:*

$$\det(\Delta_\tau, \bar{\Delta}_{1,0}) = C \det_{hyp}(\Delta_\tau),$$

where C is a constant that depends only on the number of cusps of M.

Proof. Equation (2.2) in [29] implies the following formula:

$$\mathrm{Tr}(e^{-t\Delta_\tau} - e^{-t\bar{\Delta}_{a,0}}) = \sum_k e^{-t\lambda_k} - \frac{1}{4\pi}\int_{\mathbb{R}} e^{-t(\frac{1}{4}+r^2)} \frac{\phi'}{\phi}(\frac{1}{2} + ir)\, dr \\ + \frac{e^{-\frac{t}{4}}}{4}(\mathrm{Tr}(\Phi(\frac{1}{2})) + m) + \frac{e^{-\frac{t}{4}}}{\sqrt{4\pi t}} \sum_{j=1}^{m} \log(a_j), \quad (5.3)$$

where the operator $\bar{\Delta}_{a,0}$ was given in Definition 1.1. The equation above differs from equation (2.2) in [29] by the term $e^{-\frac{t}{4}}m/4$ that comes from the boundary condition of the operator $\bar{\Delta}_{a,0}$. Equation (5.3) and Proposition 2.6 imply that

$$\mathrm{Tr}(e^{-t\Delta_\tau} - e^{-t\bar{\Delta}_{1,0}}) = \sum_j e^{-t\lambda_j} - \frac{1}{4\pi}\int_{\mathbb{R}} e^{-t(\frac{1}{4}+r^2)} \frac{\phi'}{\phi}(\frac{1}{2} + ir)\, dr + \frac{e^{-\frac{t}{4}}}{4}(\mathrm{Tr}(\Phi(\frac{1}{2})) + m).$$

Combining this equation with Selberg's trace formula in equation (5.1) gives

$$\mathrm{Tr}(e^{-t\Delta_\tau} - e^{-t\bar{\Delta}_{1,0}}) - \mathrm{STrK}_M(t) = -\frac{m}{\pi}P(t) - \frac{m\log(2)}{\sqrt{4\pi t}} e^{-\frac{t}{4}} + \frac{m}{2} e^{-\frac{t}{4}}, \quad (5.4)$$

Let consider the following auxiliary function:

$$\xi(s) = \frac{m}{\Gamma(s)} \int_0^\infty \left\{ -\frac{1}{\pi}P(t) + e^{-\frac{t}{4}} \left(\frac{1}{2} - \frac{\log(2)}{\sqrt{4\pi t}}\right)\right\} t^{s-1} dt. \quad (5.5)$$

From Lemma 5.1 it follows that the function $\xi(s)$ has a meromorphic continuation to $\mathbb{C}$ that is analytic at $s = 0$. In this way we have that $\zeta(s; \Delta_\tau, \bar{\Delta}_{1,0}) = \zeta_{M,\mathrm{hyp}}(s) + \xi(s)$, thus,

$$\det(\Delta_\tau, \bar{\Delta}_{1,0}) = e^{-\xi'(0)} \det_{\mathrm{hyp}}(\Delta_\tau).$$

The constant $C = e^{-\xi'(0)}$ depends only on the number of cusps of M. □

Now let us consider how to approach the "boundary" of the moduli space. For this we refer to [1]. Let us recall the notation and the main theorem in this paper. Let $G = \mathrm{SL}(2,\mathbb{R})/\{\pm I\}$. Every Fuchsian group Γ satisfying the condition $\mathrm{mes}(G/\Gamma) \leq \mu$ has a signature $\sigma = (p, n; \nu_1, \cdots, \nu_n)$, where p and n are integers, the ν_j are integers or the symbol ∞, and $p \geq 0$, $n \geq 0$, $2 \leq \nu_1 \leq \cdots \leq \nu_n \leq \infty$. Let

$$X(\sigma) = \{[\Gamma] : [\Gamma] \text{ is a conjugacy class of Fuchsian groups } \Gamma \text{ with signature } \sigma\}$$

The spaces $X(\sigma)$, with their natural topologies, are metrizable. The topology of $X(\sigma)$ can be derived from the Teichmüller topology, see [1] for the details. The theorem that is of our interest is the following:

Theorem 5.3. *(L. Bers) The subset of $X(\sigma)$ corresponding to groups Γ such that $\ell(\gamma) \geq 2 + \epsilon > 2$ for all hyperbolic $\gamma \in \Gamma$ is compact.*

This implies that the only possible deformations are obtained is by pinching smallest geodesics, i.e., we can approach the boundary of the moduli space by deforming hyperbolic elements in the group.

For example if we consider hyperbolic elements of the form $\begin{pmatrix} 1+\epsilon & b \\ 0 & \frac{1}{1+\epsilon} \end{pmatrix}$ they degenerate to $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ that is parabolic.

Recall now that the goal of this chapter is to prove the following theorem:

Theorem 5.4. $\det(\Delta_\tau, \bar{\Delta}_{1,0})$ *tends to zero as $[\tau]$ approaches the 'boundary' of the moduli space.*

As we said before, we use the results in [19]. Thus let us describe their notation and explain their results: Let $\{M_l\}_{l \in I \subset \mathbb{R}^p_+}$ be a degenerating family of hyperbolic Riemann surfaces of finite volume (each surface M_l is assumed to have m cusps) with p pinching geodesics. This means that for each $l = (l_1, \cdots, l_p) \in I$ the cutoff cylinders $C_{l_k,\epsilon}$ are embedded in M_l for every $0 < \epsilon < 1/2$. A fundamental domain for the cutoff cylinder $C_{l_k,\epsilon}$ inside the fundamental domain for the complete cylinder C_{l_k} in $\mathbb{H}$ would be

$$\{\rho \exp(i\alpha) : 1 \leq \rho < \exp(l_k),\ \cot^{-1}(\epsilon/(2l_k)) < \alpha < \pi - \cot^{-1}(\epsilon/(2l_k))\}.$$

From Gauss-Bonnet follows that the area of the surfaces is kept invariant during the deformation i.e., $A_{l_j} = A_{l_k} = c$, c is a constant. Let $\mathrm{DH}(\Gamma_l) \subset \mathrm{H}(\Gamma_l)$ be a set of representatives of primitive non-conjugated hyperbolic classes corresponding to the geodesics that we are pinching. Proposition 2.1 in [19] yields that the degenerating heat trace for $t > 0$ equals:

$$\mathrm{DTr}K_{M_l}(t) = \frac{e^{-t/4}}{\sqrt{16\pi t}} \sum_{\mathrm{DH}(\Gamma_l)} \sum_{n=1}^{\infty} \frac{\ell(\gamma)}{\sinh(n\ell(\gamma)/2)} e^{-(n\ell(\gamma))^2/4t}.$$

Let M be the Riemann surface that is the limit of the degenerating family $\{M_l\}$ then M is not necessarily connected and the number of cusps of M is $m + 2p$. Theorem 2.2 in [19] states that:

$$\lim_{l \to 0} (\mathrm{HTr}K_{M_l}(t) - \mathrm{DTr}K_{M_l}(t)) = \mathrm{HTr}K_M(t).$$

Their next step is to separate (in the trace) the small eigenvalues of the Laplacian on M_l because some of them may degenerate to 0 (since the limit surface M may not be connected, the eigenvalue

0 of the Laplacian on M has multiplicity equal to the number of connected components of M). Let $0 < \alpha < 1/4$ be such that α is not an eigenvalue of M and consider:

$$\mathrm{HTr}K^{\alpha}_{M_l}(t) := \mathrm{HTr}K_{M_l}(t) - \sum_{\lambda_{n,l}\le\alpha} e^{-\lambda_{n,l}t}$$

From this definition we have that:

$$\mathrm{STr}K^{\alpha}_{M_l}(t) = \mathrm{HTr}K^{\alpha}_{M_l}(t) + A_l K_{\mathbb{H}}(t,0) = STrK_{M_l}(t) - \sum_{\lambda_{j,l}\le\alpha} e^{-t\lambda_{j,l}}, \qquad \text{and}$$

$$\mathrm{STr}K^{\alpha}_{M}(t) = \mathrm{HTr}K_{M}(t) - \sum_{\lambda_j(M)\le\alpha} e^{-t\lambda_j(M)} + AK_{\mathbb{H}}(t,0),$$

where A denotes the area of the limit surface M that satisfies $A = A_l$, for any $l \in I$.

Now for the given manifold M_*, Jorgenson and Lundelius consider the truncated hyperbolic zeta function:

$$\zeta^{\alpha}_{\mathrm{hyp}\ M_*}(s) = \frac{1}{\Gamma(s)}\int_0^{\infty} \mathrm{STr}K^{\alpha}_{M_*}(t)t^{s-1}\ dt = \frac{1}{\Gamma(s)}\int_0^{\infty}(\mathrm{STr}K_{M_*}(t) - d_*)t^{s-1}\ dt - \sum_{0<\lambda_{j,*}\le\alpha}\lambda_{j,*}^{-s},$$

where d_* is the dimension of the kernel of Δ_*, and the corresponding determinant is:

$$\log\det{}^{\alpha}_{\mathrm{hyp}}\Delta_{M_*} = -\left.\frac{\partial}{\partial s}\zeta^{\alpha}_{\mathrm{hyp}\ M_*}\right|_{s=0}.$$

Let us see now how $\det^{\alpha}_{\mathrm{hyp}}\Delta_{M_l}$ relates to $\det(\Delta_{M_l}, \bar{\Delta}_{1,0})$. Notice that the operator $\bar{\Delta}_{1,0}$ remains constant through the degeneration and recall that it only has continuous spectrum equal to $[\frac{1}{4}, \infty)$ with multiplicity m. At the moment we are not yet concerned with the relative determinant $\det(\Delta_M, \bar{\Delta}_{1,0})$ but rather with the behavior of the relative determinant of the degenerating surfaces. Equation (5.4) applied to M_l can be rewritten as:

$$\mathrm{Tr}(e^{-t\Delta_{M_l}} - e^{-t\bar{\Delta}_{1,0}}) - \mathrm{STr}K^{\alpha}_{M_l}(t) = m\left(-\frac{1}{\pi}P(t) + (\frac{1}{2} - \frac{\log(2)}{\sqrt{4\pi t}})e^{-\frac{t}{4}}\right) + \sum_{\lambda_{j,l}\le\alpha} e^{-t\lambda_{j,l}}$$

Writing this in terms of zeta functions we obtain:

$$\zeta(s, \Delta_{M_l}, \bar{\Delta}_{1,0}) - \zeta^{\alpha}_{\mathrm{hyp}\ M_l}(s) = \xi(s) + \sum_{\lambda_{j,l}\le\alpha}\lambda_{j,l}^{-s},$$

where $\xi(s)$ is as in equation (5.5). Taking the meromorphic continuations and differentiating we obtain that:

$$\log\det{}^{\alpha}_{\mathrm{hyp}}\Delta_{M_l} = \log\det(\Delta_{M_l}, \bar{\Delta}_{1,0}) + mc - \sum_{\lambda_{j,l}\le\alpha}\log(\lambda_{j,l}), \tag{5.6}$$

where for $\xi(s)$ we used again Lemma 5.1 and the fact that from (5.5) is clear that $\xi'(0) = c\ m$, where c is a constant independent of l. We wanted to use now Corollary 4.3 in [19]. However, there is a misprint in a sign in their result. For this reason we decided to refer to Theorem 4.1 in [19] and keep track of the signs. Theorem 4.1 in [19] establishes:

$$\lim_{l\to 0}\left(\zeta^{\alpha}_{\mathrm{hyp}\ M_l}(s) - \frac{1}{\Gamma(s)}\int_0^{\infty}\mathrm{DTr}K_{M_l}(t)t^{s-1}dt - \zeta^{\alpha}_{\mathrm{hyp}\ M}(s)\right) = 0 \tag{5.7}$$

In order to deal with the second term in the left-hand side of equation (5.7) we follow Remark 4.2 in [19]:

$$\int_0^\infty \mathrm{DTr} K_{M_l}(t) t^{s-1} dt = \sum_{\mathrm{DH}(\Gamma_l)} \sum_{n=1}^\infty \frac{1}{\sqrt{16\pi}} \frac{\ell(\gamma)}{\sinh(n\ell(\gamma)/2)} \int_0^\infty e^{-t/4} e^{-(n\ell(\gamma))^2/4t} t^{s-3/2} dt$$
$$= \sum_{\mathrm{DH}(\Gamma_l)} \sum_{n=1}^\infty \frac{1}{\sqrt{16\pi}} \frac{\ell(\gamma)}{\sinh(n\ell(\gamma)/2)} K_{s-1/2}(1/2, n\ell(\gamma)/2),$$

where $K_s(a,b) := \int_0^\infty e^{-a^2t - b^2/t} t^{s-1} dt$ is the K-Bessel function. Now, taking into account $1/\Gamma(s)$ and differentiating we obtain:

$$\frac{d}{ds} \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{DTr} K_{M_l}(t) t^{s-1} dt \bigg|_{s=0} = \sum_{\mathrm{DH}(\Gamma_l)} \sum_{n=1}^\infty \frac{1}{\sqrt{16\pi}} \frac{\ell(\gamma)}{\sinh(n\ell(\gamma)/2)} K_{-1/2}(1/2, n\ell(\gamma)/2)$$
$$= \sum_{\mathrm{DH}(\Gamma_l)} \sum_{n=1}^\infty \frac{1}{\sqrt{16\pi}} \frac{\ell(\gamma)}{\sinh(n\ell(\gamma)/2)} K_{1/2}(n\ell(\gamma)/2, 1/2) = \sum_{\mathrm{DH}(\Gamma_l)} \sum_{n=1}^\infty \frac{e^{-n\ell(\gamma)}}{n(1-e^{-n\ell(\gamma)})},$$

where we used that $K_{1/2}(a,b) = \sqrt{\pi} e^{-2ab}/b$. This together with equation (5.7) gives:

$$\lim_{l\to 0} \big(\log \det{}_{\mathrm{hyp}} \Delta_{M_l} + \sum_{\gamma \in \mathrm{DH}(\Gamma_l)} \sum_{n=1}^\infty \frac{e^{-n\ell(\gamma)}}{n(1-e^{-n\ell(\gamma)})} \big) = \log \det{}_{\mathrm{hyp}}^\alpha \Delta_M.$$

Let us replace $\log \det_{\mathrm{hyp}} \Delta_{M_l}$ in the expression above using equation (5.6):

$$\lim_{l\to 0} \big(\log \det(\Delta_{M_l}, \bar{\Delta}_{1,0}) + mc + \sum_{\gamma \in \mathrm{DH}(\Gamma_l)} \sum_{n=1}^\infty \frac{e^{-n\ell(\gamma)}}{n(1-e^{-n\ell(\gamma)})} - \sum_{0<\lambda_{j,l}\leq \alpha} \log(\lambda_{j,l}) \big)$$
$$= \log \det{}_{\mathrm{hyp}}^\alpha \Delta_M. \quad (5.8)$$

In order to study the behavior of $\log \det(\Delta_{M_l}, \bar{\Delta}_{1,0})$ we need to know the behavior of the series in the left-hand side of equation (5.8) as $l \to 0$. Recall that $\ell(\gamma) \to 0$ as $l \to 0$. Let us remark here that the asymptotic expansion for the series given in [19] is wrong. For the correct expression we refer to [45, page 308]: if $\mathrm{Re}(s) > 0$ we have:

$$\sum_{n=1}^\infty \frac{e^{-ns\ell(\gamma)}}{n(1-e^{-n\ell(\gamma)})} = \Big(\frac{\pi^2}{6\ell(\gamma)} + (s - \frac{1}{2}) \log(1 - e^{-s\ell(\gamma)}) \Big) + O(1),$$

as $\ell(\gamma) \to 0^+$. Taking $s = 1$ we see that

$$\lim_{l\to 0} \sum_{\gamma \in DH(\Gamma_l)} \sum_{n=1}^\infty \frac{e^{-n\ell(\gamma)}}{n(1-e^{-n\ell(\gamma)})} = \infty$$

For the sum of the logarithm of the small eigenvalues we have the following: some of the small eigenvalues of the family $\{M_l\}$ may degenerate. For the eigenvalues of M, $0 = \lambda_j(M)$, that come from degeneration we know that for any $0 < \alpha < \frac{1}{4}$, α not an eigenvalue of M, there is a l_0 such that for all $0 < l \leq l_0$, $\lambda_{l,j} \leq \alpha$. This is due to the convergence of any finite number of eigenvalues. Thus $\lim_{l\to 0} \sum_{0<\lambda_{j,l}\leq\alpha} \log(\lambda_{j,l}) = -\infty$. In this way we have:

$$\lim_{l\to 0} \sum_{\gamma\in \mathrm{DH}(\Gamma_l)} \sum_{n=1}^{\infty} \frac{-e^{-n\ell(\gamma)}}{n(1-e^{-n\ell(\gamma)})} - \sum_{0<\lambda_{j,l}\leq\alpha} \log(\lambda_{j,l}) = \infty,$$

since the term cm and the hyperbolic α-regularized determinant of the limit surface are both finite, it follows that

$$\lim_{l\to 0} \log(\det(\Delta_{M_l}, \bar{\Delta}_{1,0})) = -\infty.$$

This finishes the proof of Theorem 5.4.

Chapter 6

Splitting formula for the relative determinant

Splitting formulas for determinants have been widely studied. They have been proved in the setting of compact manifold by Burghelea, Friedlander and Kappeler in [5], and in other settings by many other authors. For example, in the setting of manifolds with cylindrical ends they were studied by Müller and Müller in [31] and Loya and Park in [25]. The purpose of this chapter is to use the Dirichlet-to-Neumann operator for the Laplacian on a manifold with cusps to obtain a splitting formula for the relative determinant $\det(\Delta_g, \Delta_{\beta,0})$.

6.1 Dirichlet-to-Neumann operator for Δ_g

In this section we give the definition of the Dirichlet-to-Neumann operator $\mathcal{N}(z)$ and its main properties. We also study the limit operator as the parameter z goes to zero.

6.1.1 Definition and properties

In this part we decompose the manifold (M, g) as $M = M_\alpha \cup Z_\alpha$ where $\alpha \geq 1$ and Z_α is isometric to $[\alpha, \infty) \times S^1$ with the hyperbolic metric.

As before, let Δ_g be the self adjoint Laplacian on M, let $\beta \geq \alpha$ and $\Delta_{Z_\beta,D}$ be the self adjoint Dirichlet Laplacian on Z_β obtained by imposing Dirichlet boundary conditions at $\{\beta\} \times S^1$. Similarly, let $\Delta_{M_\beta,D}$ be the self adjoint extension of the Laplacian on M_β obtained after imposing Dirichlet boundary conditions at $\partial M_\beta = \{\beta\} \times S^1$. We will explicitly compute a part of the Dirichlet-to-Neumann operator $\mathcal{N}(z)$ on $\Sigma_\beta \simeq \{\beta\} \times S^1$, for any value of β. The metric on Σ_β is given by $g_{\Sigma_\beta} = \beta^{-2} dx \oplus dx$, the eigenvalues for the Laplacian Δ_{Σ_β} are $\{4\pi^2 n^2 \beta^2\}_{n \in \mathbb{Z}}$ and the corresponding eigenfunctions are $\{\beta \exp(2\pi i n x)\}_{n \in \mathbb{Z}}$.

Let z be in the resolvent set of Δ_g, $\rho(\Delta_g)$. Then the Dirichlet-to-Neumann operator,

$$\mathcal{N}(z) : C^\infty(\Sigma_\beta) \to C^\infty(\Sigma_\beta),$$

is defined as follows. Let $f \in C^\infty(\Sigma_\beta)$ and let $\tilde{f}$ be the unique square integrable solution to the problem

$$\begin{cases} (\Delta_g - z)\tilde{f} = 0 & \text{in } M \setminus \Sigma_\beta \\ \tilde{f} = f & \text{on } \Sigma_\beta. \end{cases}$$

Let n^+ denote the inwards unit normal vector field at Σ_β on M_α and n^- the one on $\mathbb{Z}_\beta$. Then $\mathcal{N}(z)f$ is defined by the following equation

$$\mathcal{N}(z)f := -\left(\frac{\partial}{\partial n^+}\left(\tilde{f}\,|_{M_\beta}\right) + \frac{\partial}{\partial n^-}\left(\tilde{f}\,|_{Z_\beta}\right)\right)$$

Theorem 2.1 in [7] establishes that for $z \in \mathbb{C}\setminus[0,\infty)$, the Dirichlet-to-Neumann operator is a 1st-order elliptic, invertible, pseudodifferential operator, whose principal symbol is a scalar, $\mathrm{sym}_p(\mathcal{N}(z))(x,\eta) = 2\sqrt{g_x(\eta,\eta)}$, $(x,\eta) \in T^*M$. In addition, the function $z \mapsto \mathcal{N}(z)$ is holomorphic as function of z. In particular, $\mathcal{N}(z)$ has continuous extensions $H^1(\Sigma_\beta) \to L^2(\Sigma_\beta) \to H^{-1}(\Sigma_\beta)$. Then we can think of $\mathcal{N}(z)$ as an operator $\mathcal{N}(z) : H^1(\Sigma_\beta) \subset L^2(\Sigma_\beta) \to L^2(\Sigma_\beta)$ on $L^2(\Sigma_\beta)$. Furthermore, for $f \in C^\infty(\Sigma_\beta)$ we have that:

$$\mathcal{N}(z)^{-1}f(x) = \int_{\Sigma_\beta} G(x,y,z)f(y)d\mu(y), \tag{6.1}$$

where $G(x,y,z)$ is the Schwartz kernel of $(\Delta_g - z)^{-1}$ on M, see Theorem 2.1 in [7]. This expression is equivalent to:

$$\mathcal{N}(z)^{-1}f = \rho_{\Sigma_\beta} \circ (\Delta_g - z)^{-1} \circ i_{\Sigma_\beta}(f),$$

where $i_{\Sigma_\beta}(f) = f \otimes \delta_{\Sigma_\beta}$, in the distributional sense, this means $f \otimes \delta_{\Sigma_\beta}(\varphi) = \int_{\Sigma_\beta} \varphi \cdot f$ for any $\varphi \in C^\infty(M)$. For convenience of the reader we reproduce the proof of equation (6.1) as it is given in [7], but for functions and using our notation. Let $f \in C^\infty(\Sigma_\beta)$ and $\varphi \in C_0^\infty(M)$, then

$$(\Delta_g - z)^{-1}(\delta_{\Sigma_\beta} \otimes f)(\varphi) = \int_{\Sigma_\beta}\int_M \varphi(w)G(z,w,v)f(v)dA_g(w)dA_{\Sigma_\beta}(v) = \langle \varphi, u\rangle,$$

where $u \in L^2(M)$ is defined by $u(w) = \int_{\Sigma_\beta} G(z,w,v)f(v)dA_{\Sigma_\beta}(v)$. Then from the previous equation it is clear that in the distributional sense $\langle u, \varphi\rangle = \langle (\Delta_g - z)^{-1}(\delta_{\Sigma_\beta} \otimes f), \varphi\rangle$, therefore $u = (\Delta_g - z)^{-1}(\delta_{\Sigma_\beta} \otimes f)$, and, $(\Delta_g - z)u = \delta_{\Sigma_\beta} \otimes f$. In particular $(\Delta_g - z)u = 0$, on $M \setminus \Sigma_\beta$. Now use Green's formulas and smoothness of φ to obtain:

$$\begin{aligned}
&\langle (\Delta_g - \bar{z})\varphi, u\rangle_{L^2(M)} = \langle \varphi, (\Delta_g - z)u\rangle_{L^2(M)} = ((\Delta_g - z)u)(\varphi) = \int_M ((\Delta_g u)\varphi - (zu)\varphi)dA_g \\
&= \int_{M_\beta} (u(\Delta_g\varphi) - (\Delta_g u)\varphi)dA_g + \int_{Z_\beta} (u(\Delta_g\varphi) - (\Delta_g u)\varphi)dA_g \\
&= \int_{\Sigma_\beta} (u\frac{\partial\varphi}{\partial n^+} - \frac{\partial u}{\partial n^+}\varphi + u\frac{\partial \varphi}{\partial n^-} - \frac{\partial u}{\partial n^-}\varphi)dA_{\Sigma_\beta} = \int_{\Sigma_\beta} -(\frac{\partial u}{\partial n^+} + \frac{\partial u}{\partial n^-})\varphi dA_{\Sigma_\beta} \\
&= \int_{\Sigma_\beta} (\mathcal{N}(u|_{\Sigma_\beta}))\varphi\, dA_{\Sigma_\beta}.
\end{aligned}$$

Therefore $(\Delta_g - z)u = \delta_{\Sigma_\beta} \otimes (\mathcal{N}(u|_{\Sigma_\beta}))$. Then it follows that $\mathcal{N}(u|_{\Sigma_\beta}) = f$, thus $\mathcal{N}^{-1}f = u|_{\Sigma_\beta}$.

Now, remember that $0 \in \sigma(\Delta_g)$ is an isolated eigenvalue. Thus the Dirichlet-to-Neumann operator $\mathcal{N}(z)$ is actually defined for z in a neighborhood of zero and it makes sense to consider its limit as z approaches zero. Indeed, we show that it exists for $z = 0$ and that the dependence on z is continuous. In order to do this, let us split the problem in the classical way: let $\mathcal{N}(z) = \mathcal{N}_1(z) + \mathcal{N}_2(z)$, where for $i = 1,2$ $\mathcal{N}_i(z)$ is defined as follows.

Let $f \in C^\infty(\Sigma_\beta)$, then let $\varphi_1 \in C^\infty(M_\beta \setminus \Sigma_\beta) \cap C^0(M_\beta)$ be the unique solution to the problem

$$\begin{cases} (\Delta - z)\varphi_1 = 0 & \text{in } M_\beta \setminus \Sigma_\beta \\ \varphi_1 = f & \text{on } \Sigma_\beta. \end{cases}$$

Put $\mathcal{N}_1(z)f = -\frac{\partial \varphi_1}{\partial n^+}$. Similarly, let $\varphi_2 \in C^\infty(Z_\beta) \cap L^2(Z_\beta)$ be the unique square integrable solution to the problem:

$$\begin{cases} (\Delta - z)\varphi_2 = 0 & \text{in } Z_\beta \\ \varphi_2 = f & \text{on } \Sigma_\beta \end{cases}$$

Put $\mathcal{N}_2(z)f = -\frac{\partial \varphi_2}{\partial n^-}$.

Using the usual method of separation of variables in the cusp we can compute the operator $\mathcal{N}_2(z)$ explicitly. This explicit expression of $\mathcal{N}_2(z)$ is useful to compute the limit of the operator as $z \to 0$.

Proposition 6.1. *Let $f \in C^\infty(\Sigma_\beta)$. Write $z = s(1-s)$. Then for $\mathcal{N}_2(z)f$ we have*

$$\text{If} \quad \mathrm{Re}(s) > \frac{1}{2}, \quad \mathcal{N}_2(s(1-s))f = -(1-2s)c_0(f)\beta - sf + \beta\sqrt{\Delta_{\Sigma_\beta}}\, \frac{K_{s+\frac{1}{2}}(\beta\sqrt{\Delta}_{\Sigma_\beta})}{K_{s-\frac{1}{2}}(\beta\sqrt{\Delta}_{\Sigma_\beta})} f. \tag{6.2}$$

$$\text{If} \quad \mathrm{Re}(s) < \frac{1}{2}, \quad \mathcal{N}_2(s)f(x) = -sf(x) + \beta\sqrt{\Delta_{\Sigma_\beta}}\, \frac{K_{s+\frac{1}{2}}(\beta\sqrt{\Delta_{\Sigma_\beta}})}{K_{s-\frac{1}{2}}(\beta\sqrt{\Delta_{\Sigma_\beta}})} f(x), \tag{6.3}$$

where $c_0(f)$ is the projection of f on the kernel of Δ_{Σ_β}.

If $\mathrm{Re}(s) = \frac{1}{2}$, $f \in \mathrm{Dom}(\mathcal{N}_2(z))$ only if it satisfies $\int_{\Sigma_\beta} f dA_{\Sigma_\beta} = 0$. In this case we have:

$$\mathcal{N}_2(s)f = -sf + \beta\sqrt{\Delta_{\Sigma_\beta}}\, \frac{K_{s+\frac{1}{2}}(\beta\sqrt{\Delta_{\Sigma_\beta}})}{K_{s-\frac{1}{2}}(\beta\sqrt{\Delta_{\Sigma_\beta}})} f, \tag{6.4}$$

where K_ν is the modified Bessel function of order ν.

Proof. On Z_β the Laplacian is given by $\Delta_g = -y^2\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}\right)$. Therefore the Fourier expansions of $\varphi_2(y,x)$ and $f(x)$ have the following forms:

$$\varphi_2(y,x) = \sum_{n\in\mathbb{Z}} a_n(y)\beta e^{2\pi i n x} \qquad \text{where} \quad a_n(y) = \int_0^1 \varphi_2(y,x)\beta e^{-2\pi i n x}\frac{dx}{\beta^2}$$

$$f(x) = \sum_{n\in\mathbb{Z}} c_n\beta e^{2\pi i n x} \qquad \text{where} \quad c_n = \int_0^1 f(x)\beta e^{-2\pi i n x}\frac{dx}{\beta^2}.$$

Using separation of variables, the problem becomes:

$$\begin{cases} (-y^2\frac{d^2}{dy^2} + y^2 4\pi^2 n^2\beta^2 - z)a_n(y) = 0 & \\ a_n(\beta) = c_n, & \text{for } n \in \mathbb{Z}. \end{cases}$$

Set $z = s(1-s)$ with $s \in \mathbb{C}$. Then for $n \neq 0$, two linear independent solutions of the equation

$$\left(-y^2\frac{d^2}{dy^2} + 4\pi^2 n^2\beta^2 y^2 - s(1-s)\right)a_n(y) = 0 \tag{6.5}$$

are $y^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|\beta y)$ and $y^{\frac{1}{2}}I_{s-\frac{1}{2}}(2\pi|n|\beta y)$, where $K_{s-\frac{1}{2}}$ and $I_{s-\frac{1}{2}}$ are the modified Bessel functions. We discard $I_{s-\frac{1}{2}}$ because it is not square integrable on $[1,\infty)$ for any value of s. Thus,

$$\varphi_2(y,x) = \sum_{n\in\mathbb{Z}} a_n(y)e^{2\pi inx} = b_{0,1}y^s\beta + b_{0,2}y^{1-s}\beta + \sum_{n\neq 0} b_n y^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|\beta y)\beta e^{2\pi inx}.$$

I.e. for $n \neq 0$, $a_n(y) = b_n y^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|\beta y)$, where b_n and $b_{0,1}, b_{0,2}$ are constants determined by the boundary and the square integrable conditions.

Case $\mathrm{Re}(s) > \frac{1}{2}$**.** In this case $b_{0,1} = 0$ and $y^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|\beta y)$ is square integrable on $[1,\infty[$. Then we have:

$$\varphi_2(y,x) = b_{0,2}y^{1-s}\beta + \sum_{n\neq 0} b_n y^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|\beta y)\beta e^{2\pi inx},$$

where $a_0(y) = b_{0,2}y^{1-s}$ and $a_n(y) = b_n y^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|\beta y)$. The boundary condition $\varphi_2(\beta,x) = f(x)$ is equivalent to $a_n(\beta) = c_n$. Thus $b_{0,2} = c_0\beta^{s-1}$ and

$$b_n = \frac{c_n}{\beta^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|\beta^2)}.$$

In this way we obtain:

$$\varphi_2(y,x) = c_0\beta^{s-1}y^{1-s}\beta + \sum_{n\neq 0} \frac{c_n}{\beta^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|\beta^2)}\; y^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|\beta y)\beta e^{2\pi inx},$$

$$\begin{aligned} y\frac{\partial}{\partial y}\;\varphi_2(y,x) &= (1-s)yc_0\beta^{s-1}y^{-s}\beta \\ &\quad + y\sum_{n\neq 0}\frac{c_n\beta e^{2\pi inx}}{\beta^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|\beta^2)}\left(\frac{y^{-\frac{1}{2}}}{2}K_{s-\frac{1}{2}}(2\pi|n|\beta y) + y^{\frac{1}{2}}\frac{d}{dy}K_{s-\frac{1}{2}}(2\pi|n|\beta y)\right). \end{aligned}$$

Let us use here the following equation:

$$\frac{d}{dy}K_{s-\frac{1}{2}}(2\pi|n|\beta y) = (s-\frac{1}{2})y^{-1}K_{s-\frac{1}{2}}(2\pi|n|\beta y) - 2\pi|n|\beta K_{s+\frac{1}{2}}(2\pi|n|\beta y).$$

Then we have:

$$\begin{aligned} y\frac{\partial}{\partial y}\;\varphi_2(y,x) &= (1-s)yc_0\beta^{s-1}y^{-s}\beta \\ &\quad + y\sum_{n\neq 0}\frac{c_n\beta e^{2\pi inx}}{\beta^{\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|\beta^2)}\left(sy^{-\frac{1}{2}}K_{s-\frac{1}{2}}(2\pi|n|\beta y) - 2\pi|n|\beta y^{\frac{1}{2}}K_{s+\frac{1}{2}}(2\pi|n|\beta y)\right) \end{aligned}$$

$$\begin{aligned} y\;\frac{\partial}{\partial y}\;\varphi_2(y,x)\Big|_{y=\beta} &= (1-s)c_0\beta + \beta\sum_{n\neq 0} c_n\left(s\beta^{-1} - 2\pi|n|\beta\frac{K_{s+\frac{1}{2}}(2\pi|n|\beta^2)}{K_{s-\frac{1}{2}}(2\pi|n|\beta^2)}\right)\beta e^{2\pi inx} \\ &= (1-2s)c_0\beta + sf(x) - \beta\sqrt{\Delta_{\Sigma_\beta}}\;\frac{K_{s+\frac{1}{2}}}{K_{s-\frac{1}{2}}}(\beta\sqrt{\Delta_{\Sigma_\beta}})f(x), \end{aligned}$$

where we have chosen the positive square root of the eigenvalues to define $\sqrt{\Delta_{\Sigma_\beta}}$. Recall that the term c_0 is the projection of f on the kernel of Δ_{Σ_β}. Then we obtain equation (6.2).

Case $\mathrm{Re}(s) = \frac{1}{2}$**.** This is an interesting case. The computations are the same as in the previous case but the square integrability condition implies that the zero term in the Fourier expansion of the solution φ_2 should be null, thus

$$\varphi_2(y,x) = \sum_{n\neq 0} b_n y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|\beta y)\beta e^{2\pi inx}.$$

and the condition $a_0 = c_0$ gives $c_0 = 0$. This means that only in the case when $c_0 = 0$ will there exist a solution to the problem. Hence for f to be in the domain of $\mathcal{N}_2(s(1-1))$, f should satisfy $c_0(f) = \int_{\Sigma_\beta} f dA_{\Sigma_\beta} = 0$. For such functions f equation (6.4) holds.

Case $\mathrm{Re}(s) < \frac{1}{2}$**.** In this case $b_{0,2} = 0$ and $b_{0,1} = c_0\beta^{-s}$. Then:

$$\varphi_2(y,x) = c_0\beta^{-s}y^s\beta + \sum_{n\neq 0} \frac{c_n\beta e^{2\pi inx}}{\beta^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|\beta^2)}\ y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi|n|\beta y),$$

$$y\ \frac{\partial}{\partial y}\ \varphi_2(y,x)\bigg|_{y=\beta} = sf - \beta\sum_{n\neq 0} 2\pi|n|\beta \frac{K_{s+\frac{1}{2}}(2\pi|n|\beta^2)}{K_{s-\frac{1}{2}}(2\pi|n|\beta^2)}\ c_n\beta e^{2\pi inx}.$$

Thus we obtain equation (6.3). □

Remark 6.2. *For $z < 0$ the operator $\mathcal{N}(z)$ is positive. This follows from the non-negativity of the Laplacian Δ_g and from the construction of $\mathcal{N}(z)$. Recall that the Schwartz kernel of $\mathcal{N}(z)^{-1}$ is the same as the Schwartz kernel of $(\Delta_g - z)^{-1}$. We have $\Delta_g \geq 0$. If $z < 0$, then $(\Delta_g - z) > 0$, and $(\Delta_g - z)^{-1} > 0$. Therefore $\mathcal{N}(z)^{-1} > 0$.*

6.1.2 Existence and properties for $\mathcal{N}$

Lemma 6.3. *For every $f \in C^\infty(\Sigma_\beta)$ there exists a unique solution $\tilde{f} \in C^\infty(M \setminus \Sigma_\beta) \cap L^2(Z_\beta) \cap C^0(M)$ to the problem:*

$$\begin{cases} \Delta_g \tilde{f} = 0 & \text{in } M \setminus \Sigma_\beta \\ \tilde{f} = f & \text{on } \Sigma_\beta. \end{cases}$$

In addition, using the notation introduced above we have that:

$$\mathcal{N}_2 f := -\ y\frac{\partial}{\partial y}\varphi_2(y,x)\bigg|_{y=\beta} = \beta\sqrt{\Delta_{\Sigma_\beta}} f.$$

Proof. As in the proof of Lemma 3.1 in [31], the uniqueness of the solution $\varphi_1 \in C^\infty(M_\beta \setminus \Sigma_\beta) \cap C^0(M_\beta)$ of the Dirichlet problem on M_β follows from the invertibility of $\Delta_{M_\beta,D}$. The uniqueness of the solution on Z_β also follows from the invertibility of $\Delta_{Z_\beta,D}$. To see the existence and uniqueness on Z_β more explicitly let us follow the same procedure as above taking $z = 0$. One way to obtain $z = 0$ is to take $s = 1$ in equation (6.5). In this case the square integrable condition gives

$$\varphi_2(y,x) = \sum_{n\in\mathbb{Z}} a_n(y)e^{2\pi inx} = b_{0,2}\beta + \sum_{n\neq 0} b_n y^{\frac{1}{2}} K_{\frac{1}{2}}(2\pi|n|\beta y)\beta e^{2\pi inx}.$$

We know that $K_{\frac{1}{2}}(r) = \sqrt{\frac{\pi}{2}} r^{-\frac{1}{2}} e^{-r}$. Then for $n \neq 0$ we have $a_n(y) = \frac{b_n}{2\sqrt{|n|\beta}} e^{-2\pi|n|\beta y}$. The boundary condition $\varphi_2(\beta, x) = f(x)$, which is equivalent to $a_n(\beta) = c_n$, gives $b_0 = c_0$ and $b_n = c_n 2\sqrt{|n|\beta} e^{2\pi|n|\beta^2}$. Then

$$\varphi_2(y,x) = c_0\beta + \sum_{n\neq 0} c_n e^{2\pi|n|\beta^2} e^{-2\pi|n|\beta y} \beta e^{2\pi i n x}.$$

Taking the inward derivative we have:

$$y\frac{\partial}{\partial y}\varphi_2(y,x)\Big|_{y=\beta} = \beta\sum_{n\neq 0} -2\pi|n|\beta \cdot c_n\ \beta e^{2\pi i n x} = -\beta\sqrt{\Delta_{\Sigma_\beta}} f.$$

The other way to obtain $z = 0$ is taking $s = 0$ in equation (6.5). In this case we have:

$$\varphi_2(y,x) = \sum_{n\in\mathbb{Z}} a_n(y) e^{2\pi i n x} = b_{0,1}\beta + \sum_{n\neq 0} b_n y^{\frac{1}{2}} K_{-\frac{1}{2}}(2\pi|n|\beta y)\beta e^{2\pi i n x}.$$

As above, and using that $K_{-\frac{1}{2}} = K_{\frac{1}{2}}$, we obtain:

$$\varphi_2(y,x) = c_0\beta + \sum_{n\neq 0} c_n e^{2\pi|n|\beta^2} e^{-2\pi|n|\beta y} \beta e^{2\pi i n x}.$$

Thus for $s = 0$ and for $s = 1$, we obtain the same solution of the Dirichlet problem on Z_β. Since $\varphi_1|_{\Sigma_\beta} = \varphi_2|_{\Sigma_\beta}$, we have that the solution $\tilde{f}$ is continuous on M. Taking the inward derivative we have

$$y\frac{\partial}{\partial y}\varphi(y,x)\Big|_{y=\beta} = \beta\sum_{n\neq 0} -2\pi|n|\beta \cdot c_n\ \beta e^{2\pi i n x} = -\beta\sqrt{\Delta_{\Sigma_\beta}} f.$$

In this way we obtain:

$$\mathcal{N}_2 f = \beta\sqrt{\Delta_{\Sigma_\beta}} f. \tag{6.6}$$

$\square$

Remark 6.4. *For $z \in \rho(\Delta_g)$, the resolvent set of Δ_g, it is well known that $\mathcal{N}_1(z)$ is a 1st order invertible elliptic pseudodifferential operator. The limit, $\mathcal{N}_1$, as $z \to 0$, it is well known to be a 1st order elliptic pseudodifferential operator, but it is non-invertible, see for example [5] and [44, Section 7.11]. Therefore the operator $\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2$ is non-invertible. However it is non-negative and* $\dim(\mathrm{Ker}(\mathcal{N})) = 1$.

Lemma 6.5. *Let $f \in C^\infty(\Sigma_\beta)$. Then $\mathcal{N}(z)f$ depends continuously of z in a small enough neighborhood of $z = 0$, and*

$$\lim_{z\to 0}\mathcal{N}(z)f = \mathcal{N}f$$

Proof. The proof that $\lim_{z\to 0}\mathcal{N}_1(z)f = \mathcal{N}_1 f$ is the same as for Lemma 3.3 in [31]. For the convenience of the reader we repeat here the argument with our notation. For $f \in C^\infty(\Sigma_\beta)$, let $\varphi_1(z)$ be the unique function in $C^\infty(M_\beta \setminus \Sigma_\beta)$ satisfying $(\Delta_g - z)\varphi_1(z) = 0$, $\varphi_1(z)|_{\Sigma_\beta} = f$ and

$$\varphi_1(z) = \tilde{f} - (\Delta_{M_\beta, D} - z)^{-1}((\Delta_{M_\beta} - z)(\tilde{f})),$$

where $\tilde{f} \in C^\infty(M_\beta)$ is any extension of f. Since $\Delta_{M_\beta,D}$ is invertible, the formula also holds for $z = 0$. From this representation of $\varphi_1(z)$, it follows immediately that $\mathcal{N}_1(z)f$ converges to $\mathcal{N}_1 f$ as $z \to 0$.

Now let us take the limit of $\mathcal{N}_2(z)$ as $s \to 1$. To do that we use equation (6.2) to obtain:

$$\lim_{s\to 1} \mathcal{N}_2(s(1-s))f = c_0\beta - f + \beta\sqrt{\Delta_{\Sigma_\beta}}\,\frac{K_{\frac{3}{2}}(\sqrt{\Delta_{\Sigma_\beta}})}{K_{\frac{1}{2}}(\sqrt{\Delta_{\Sigma_\beta}})}f.$$

Using the expression $K_{\frac{3}{2}}(u) = \sqrt{\frac{\pi}{2}}u^{-3/2}e^{-u}(u+1)$, we have that $\frac{K_{\frac{3}{2}}(2\pi|n|\beta^2)}{K_{\frac{1}{2}}(2\pi|n|\beta^2)} = \frac{2\pi|n|\beta^2+1}{2\pi|n|\beta^2}$. Thus,

$$\begin{aligned}\lim_{s\to 1} \mathcal{N}_2(s(1-s))f &= c_0\beta - f + \sum_{n\neq 0}(2\pi|n|\beta^2+1)c_n\beta e^{2\pi inx} \\ &= \beta\sum_{n\neq 0} 2\pi|n|\beta c_n\beta e^{2\pi inx} = \beta\sqrt{\Delta_{\Sigma_\beta}}f = \mathcal{N}_2(0)f.\end{aligned}$$

For the limit when $s \to 0$ we have:

$$\begin{aligned}\lim_{s\to 0} \mathcal{N}_2(s(1-s))f &= \lim_{s\to 0} -sf(x) + \beta\sum_{n\neq 0} 2\pi|n|\beta\frac{K_{s+\frac{1}{2}}(2\pi|n|\beta^2)}{K_{s-\frac{1}{2}}(2\pi|n|\beta^2)}\ c_n\beta e^{2\pi inx} \\ &= \beta\sum_{n\neq 0} 2\pi|n|\beta\ c_n\beta e^{2\pi inx} = \beta\sqrt{\Delta_{\Sigma_\beta}}f.\end{aligned}$$

Thus it follows that

$$\lim_{s\to 1} \mathcal{N}_2(s(1-s))f = \lim_{s\to 0} \mathcal{N}_2(s(1-s))f = \mathcal{N}_2(0)f = \beta\sqrt{\Delta_{\Sigma_\beta}}f.$$

$\square$

6.2 Splitting formula for the relative determinant

We want to have a splitting formula for the relative determinant that relates $\det(\Delta_g, \Delta_{\beta,0})$ to the regularized determinant of the Dirichlet-to-Neumann operator $\mathcal{N}(0)$. We will use this formula in Chapter 7 to prove compactness of isospectral sets of metrics inside a conformal class with compact support on surfaces with cusps. For $z \in \rho(\Delta_g)$ Corollary 4.6 in [7] establishes the following splitting formula for complete surfaces, which we rewrite using his notation:

$$\det(\mathcal{L} - z, \mathcal{L}_{0,D} - z) = \det \mathcal{N}(z), \tag{6.7}$$

where $\mathcal{L}$ is the self-adjoint extension of the Laplacian on M and $\mathcal{L}_{0,D}$ is the self-adjoint extension of the Laplacian on $M \setminus \Sigma$ with Dirichlet boundary conditions on Σ. Let $\lambda > 0$, put $z = -\lambda$ and let us denote $\mathcal{N}(-\lambda)$ by $R(\lambda)$. Then $R(\lambda) > 0$ and it has the same properties as $\mathcal{N}(-\lambda)$. Let us take $\Sigma = \Sigma_\beta$. In our case equation (6.7) has the form:

$$\det(\Delta_g + \lambda, \Delta_{Z_{\beta,D}} + \lambda)(\det(\Delta_{M_{\beta,D}} + \lambda))^{-1} = \det R(\lambda) = \det \mathcal{N}(-\lambda). \tag{6.8}$$

We now want to take the limit on both sides of equation (6.8) as $\lambda \to 0^+$.

Lemma 6.6. *As $\lambda \to 0^+$ we have the following decomposition:*

$$\log\det(\Delta_g+\lambda, \Delta_{Z_\beta,D}+\lambda) - \log\det(\Delta_{M_\beta,D}+\lambda) = \log\lambda + \log\det(\Delta_g, \Delta_{Z_\beta,D}) - \log\det\Delta_{M_\beta,D} + o(1).$$

Proof. Let us go back to the definition of $\log\det(\Delta_g+\lambda, \Delta_{Z_\beta,D}+\lambda)$:

$$\begin{aligned}
\log\det(\Delta_g, \Delta_{Z_\beta,D}) &= -\frac{d}{ds}\zeta(s;\Delta_g,\Delta_{Z_\beta,D})\Big|_{s=0},\\
\log\det(\Delta_g+\lambda, \Delta_{Z_\beta,D}+\lambda) &= -\frac{d}{ds}\zeta(s;\Delta_g+\lambda,\Delta_{Z_\beta,D}+\lambda)\Big|_{s=0},
\end{aligned}$$

where $\zeta(s;\Delta_g,\Delta_{Z_\beta,D})$ and $\zeta(s;\Delta_g+\lambda,\Delta_{Z_\beta,D}+\lambda)$ are respectively the meromorphic continuations of the functions

$$\begin{aligned}
&\frac{1}{\Gamma(s)}\int_0^\infty (\mathrm{Tr}(e^{-t\Delta_g} - e^{-t\Delta_{Z_\beta,D}}) - 1)t^{s-1}dt \quad \text{and}\\
&\frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}(e^{-t\Delta_g} - e^{-t\Delta_{Z_\beta,D}})e^{-t\lambda}t^{s-1}dt.
\end{aligned}$$

We use the same notation for the function and its analytic continuation. Note that the second integral above converges because of the asymptotic expansions of the relative heat traces for small and large t. Further,

$$\begin{aligned}
&\zeta(s;\Delta_g+\lambda,\Delta_{Z_\beta,D}+\lambda) = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}(e^{-t\Delta_g} - e^{-t\Delta_{Z_\beta,D}})e^{-t\lambda}t^{s-1}dt\\
&= \frac{1}{\Gamma(s)}\int_0^\infty \left\{(\mathrm{Tr}(e^{-t\Delta_g} - e^{-t\Delta_{Z_\beta,D}}) - 1)e^{-t\lambda} + e^{-t\lambda}\right\} t^{s-1}dt\\
&= \frac{1}{\Gamma(s)}\int_0^\infty (\mathrm{Tr}(e^{-t\Delta_g} - e^{-t\Delta_{Z_\beta,D}}) - 1)e^{-t\lambda}t^{s-1}dt + \frac{1}{\Gamma(s)}\Gamma(s)\lambda^{-s}\\
&= \lambda^{-s} + \frac{1}{\Gamma(s)}\int_0^\infty (\mathrm{Tr}(e^{-t\Delta_g} - e^{-t\Delta_{Z_\beta,D}}) - 1)\left\{1 - t\lambda + \frac{t^2\lambda^2}{2!} - \dots\right\} t^{s-1}dt\\
&= \lambda^{-s} + \zeta(s,\Delta_g,\Delta_{Z_\beta,D}) + \frac{\lambda}{\Gamma(s)}\int_0^\infty (\mathrm{Tr}(e^{-t\Delta_g} - e^{-t\Delta_{Z_\beta,D}}) - 1)\frac{e^{-t\lambda}-1}{\lambda}t^{s-1}dt.
\end{aligned}$$

The last integral converges in a half plane. Therefore due to the asymptotic expansions, it has an analytic continuation that is holomorphic at $s=0$. So,

$$\begin{aligned}
\frac{d}{ds}\zeta(s;\Delta_g+\lambda,\Delta_{Z_\beta,D}+\lambda)\Big|_{s=0} &= -\log\lambda + \frac{d}{ds}\zeta(s;\Delta_g,\Delta_{Z_\beta,D})\Big|_{s=0} - o(1), \text{ as } \lambda\to 0^+,\\
-\log\det(\Delta_g+\lambda,\Delta_{Z_\beta,D}+\lambda) &= -\log\lambda - \log\det(\Delta_g,\Delta_{Z_\beta,D}) - o(1), \text{ as } \lambda\to 0^+,
\end{aligned}$$

as desired. Similarly, $\log\det(\Delta_{M_\beta,D}+\lambda) = \log\det(\Delta_{M_\beta,D}) + o(1)$ as $\lambda\to 0^+$, follows in the same way as above from:

$$\zeta_{\Delta_{M_\beta,D}+\lambda}(s) = \frac{1}{\Gamma(s)}\int_0^\infty \mathrm{Tr}(e^{-t\Delta_{M_\beta,D}})e^{-t\lambda}t^{s-1}dt.$$

This finishes the proof of the lemma. $\square$

Now let us take care of the middle side of equation (6.8). First of all recall that for $\lambda > 0$, $R(\lambda)$ is a 1st order elliptic, invertible, self-adjoint pseudodifferential operator, therefore its zeta determinant is well defined. Set $R = R(0)$. We saw that $R \geq 0$, $\operatorname{Ker} R = \mathbb{C}$ and $\lim_{\lambda \to 0} R(\lambda) = R$ in the strong sense. Let $0 < \mu_1(\lambda) \leq \mu_2(\lambda) \leq \mu_3(\lambda) \leq \dots$ be the eigenvalues of $R(\lambda)$. Then

$$\begin{aligned} \mu_1(\lambda) &\to 0, && \text{as } \lambda \to 0, \\ \mu_i(\lambda) &\geq c > 0, && \text{for } i \geq 2,\ \lambda \geq 0. \end{aligned}$$

The regularized determinant of R, $\det^* R$, is defined as usual by the meromorphic continuation of

$$\zeta_R^*(s) = \sum_{\mu_i > 0} \mu_i(0)^{-s}.$$

Lemma 6.7. *As $\lambda \to 0^+$ there is the following asymptotic expansion:*

$$\log \det R(\lambda) = \log \mu_1(\lambda) + \log \det{}^* R + o(1). \tag{6.9}$$

Proof. Let $\operatorname{Ker}(R)$ be the kernel of R, $\mathcal{H} = (\operatorname{Ker}(R))^{\perp}$ be its orthogonal complement and $P : L^2(\Sigma_\beta) \to \operatorname{Ker}(R)$ and $P^{\perp} : L^2(\Sigma_\beta) \to \mathcal{H}$ be the corresponding orthogonal projections. By definition:

$$\log \det R(\lambda) := - \left. \frac{d}{ds} \right|_{s=0} \zeta_{R(\lambda)}(s) = - \left. \frac{d}{ds} \right|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty \operatorname{Tr}(e^{-tR(\lambda)}) t^{s-1} dt.$$

The first thing to do is to separate the first eigenvalue. For that, let γ be a contour in $\mathbb{C}$ contained in $\rho(R(\lambda))$ and surrounding the spectrum of $R(\lambda)$, for all $\lambda \geq 0$ small enough. Then:

$$\begin{aligned} e^{-tR(\lambda)} &= \frac{1}{2i\pi} \int_\gamma e^{-t\xi} (R(\lambda) - \xi)^{-1} d\xi \\ &= \frac{1}{2i\pi} \int_{\gamma_1} e^{-t\xi} (R(\lambda) - \xi)^{-1} d\xi + \frac{1}{2i\pi} \int_{\gamma_2} e^{-t\xi} (R(\lambda) - \xi)^{-1} d\xi, \end{aligned}$$

where γ_1 is a contour surrounding $\{\mu_1(\lambda), 0\}$ and γ_2 surrounds the half line $[c, \infty)$, where $\mu_2(\lambda) \geq c$ for all $\lambda > 0$. From the assumptions is clear that γ_1 and γ_2 can be chosen without overlapping and independently of λ. It is also clear that:

$$\frac{1}{2i\pi} \int_{\gamma_1} e^{-t\xi} (R(\lambda) - \xi)^{-1} d\xi = e^{-t\mu_1(\lambda)} P(\lambda).$$

Therefore

$$\zeta_{R(\lambda)}(s) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\mu_1(\lambda)} t^{s-1} dt + \frac{1}{\Gamma(s)} \int_0^\infty \operatorname{Tr} \left(\int_{\gamma_2} e^{-t\xi} (R(\lambda) - \xi)^{-1} d\xi \right) t^{s-1} dt.$$

There are two ways to approach the proof. One is using the convergence of the resolvent and it was suggested by R. Mazzeo; the other one is following the argument in [25].

As it was explained above, we know that $R(\lambda)$ is a family of 1st order pseudodifferential operators. Hence they have bounded extensions to operators from $H^1(\Sigma_\beta)$ to $L^2(\Sigma_\beta)$. Since $H^1(\Sigma_\beta) \subset L^2(\Sigma_\beta)$, $R(\lambda)$ is a bounded operator acting on a subspace of $L^2(\Sigma_\beta)$ into $L^2(\Sigma_\beta)$ that depends continuously on λ. The resolvent of $R(\lambda)$ also depends continuously of λ. Since R has 0 as eigenvalue, the resolvent $(R-\xi)^{-1}$ has a pole at $\xi = 0$ and can be written as:

$$(R-\xi)^{-1} = -\xi^{-1}P + A(\xi),$$

with $A(\xi)$ a holomorphic operator in ξ. On the other hand we have $\mu_1(\lambda) > 0$ for $\lambda > 0$. Therefore we have that $(R(\lambda)-\xi)^{-1}$ is continuous in λ close to 0 and holomorphic in ξ far from $\sigma(R(\lambda))$. When integrating over γ_2 we are actually dealing with the operators $P(\lambda)^{\perp}R(\lambda)$ or $P^{\perp}R(\lambda)$. From general results about resolvents we have that $(P(\lambda)^{\perp}R(\lambda)-\xi)^{-1}$ converges continuously to $(P^{\perp}R-\xi)^{-1}$ as $\lambda \to 0^+$ for $\xi \in \rho(R(\lambda))$. Now:

$$\begin{aligned} e^{-tP(\lambda)^{\perp}R(\lambda)} &= \frac{1}{2i\pi}\int_{\gamma_2} e^{-t\xi}(P(\lambda)^{\perp}R(\lambda)-\xi)^{-1}d\xi, \\ e^{-tP^{\perp}R} &= \frac{1}{2i\pi}\int_{\gamma_2} e^{-t\xi}(P^{\perp}R-\xi)^{-1}d\xi. \end{aligned}$$

From the preceding expressions it is clear that $e^{-tP(\lambda)^{\perp}R(\lambda)}$ converges to $e^{-tP^{\perp}R}$. Therefore $\mathrm{Tr}(e^{-tP(\lambda)^{\perp}R(\lambda)})$ depends continuously on λ and so does the zeta function. In this way we obtain:

$$\log\det P(\lambda)^{\perp}R(\lambda) = \log\det P^{\perp}R + o(1), \text{ as } \lambda \to 0^+.$$

This finishes the proof of equation (6.9).

The other method we have of proving equation (6.9) is using the approach of [25]. We use that $R(\lambda)^{-1} = \rho_{\Sigma_\beta}\circ(\Delta_g+\lambda)^{-1}\circ i_{\Sigma_\beta}$. Let $Q(\lambda) := (\Delta_g+\lambda)^{-1} - \frac{1}{\lambda}P$ and $T(\lambda) := \rho_{\Sigma_\beta}\circ Q(\lambda)\circ i_{\Sigma_\beta}$, so that $R(\lambda)^{-1} = \frac{1}{\lambda}P + T(\lambda)$. The main point is that:

$$R = \begin{cases} 0, & \text{on } \mathrm{Ker}(R) \\ T^{-1} & \text{on } \mathcal{H} = (\mathrm{Ker}(R))^{\perp} \end{cases}$$

with $T = P^{\perp}T(0)P^{\perp}$. This implies that:

$$-\frac{d}{ds}\bigg|_{s=0}\zeta_{P^{\perp}R(\lambda)}(s) = \log\det{}^*R + o(1), \text{ as } \lambda \to 0^+.$$

This also finishes the proof of the Lemma. $\square$

Let us now introduce some notation. Remember that $R(\lambda) = \mathcal{N}(z)$, with $\lambda = -z > 0$, and

$$R(\lambda)^{-1}f = \rho_{\Sigma_\beta}\circ(\Delta_g+\lambda)^{-1}\circ i_{\Sigma_\beta}(f).$$

Let $\mu > 0$ and let P_μ be the spectral projection on $[0,\mu]$. Then $R(\lambda)^{-1}$ can be decomposed as

$$R(\lambda)^{-1} = \rho_{\Sigma_\beta}\circ P_\mu(\Delta_g+\lambda)^{-1}\circ i_{\Sigma_\beta} + \rho_{\Sigma_\beta}\circ(I-P_\mu)(\Delta_g+\lambda)^{-1}\circ i_{\Sigma_\beta}$$

Let $Q_\mu(\lambda) := \rho_{\Sigma_\beta}\circ P_\mu(\Delta_g+\lambda)^{-1}\circ i_{\Sigma_\beta}$. Then the kernel of $Q_\mu(\lambda)$ in terms of the spectral decomposition of Δ_g on M is given by:

$$K_{Q_\mu(\lambda)}(x,y,\lambda) = \sum_{\lambda_j\le\mu}\frac{1}{\lambda_j+\lambda}\varphi_j(x)\overline{\varphi_j(y)} + \frac{1}{2\pi}\int_0^{\mu}\frac{1}{\lambda+1/4+r^2}E(x,\frac{1}{2}+ir)E(y,\frac{1}{2}-ir)dr,$$

for $x, y \in \Sigma_\beta$. We can write $R(\lambda)^{-1} = Q_\mu(\lambda) + \tilde{Q}_\mu(\lambda)$ with

$$\tilde{Q}_\mu(\lambda) = \rho_{\Sigma_\beta} \circ (I - P_\mu)(\Delta_g + \lambda)^{-1} \circ i_{\Sigma_\beta}.$$

Now further decompose $Q_\mu(\lambda)$ as $Q_{\mu,1}(\lambda) + Q_{\mu,2}(\lambda)$, where $Q_{\mu,1}(\lambda)$ is given by:

$$Q_{\mu,1}(\lambda)f = \frac{1}{\lambda}\frac{1}{A_g}\int_{\Sigma_\beta} f(y)d\mu(y), \quad \text{with} \quad K_{Q_{\mu,1}(\lambda)}(x,y,\lambda) = \frac{1}{\lambda}\frac{1}{A_g},$$

and $Q_{\mu,2}(\lambda)$ is the operator whose kernel is:

$$K_{Q_{\mu,2}(\lambda)}(x,y,\lambda) = \sum_{0<\lambda_j\leq\mu} \frac{1}{\lambda+\lambda_j}\varphi_j(x)\overline{\varphi_j(y)} + \frac{1}{2\pi}\int_0^\mu \frac{1}{\lambda+1/4+r^2}E(x,\frac{1}{2}+ir)E(y,\frac{1}{2}-ir)dr. \tag{6.10}$$

We have the following Lemma:

Lemma 6.8. *There is a constant $C > 0$ such that*

$$\|\rho_{\Sigma_\beta} \ \circ \ (I - P_\mu)(\Delta_g + \lambda)^{-1} \circ i_{\Sigma_\beta}\|_{L^2(\Sigma_\beta)} \leq C,$$

for all $\lambda > 0$.

Proof. The proof goes as in Lemma 3.5 in [31].

We know that $i_{\Sigma_\beta} : L^2(\Sigma_\beta) \to H^{-1}(M)$ and $\rho_{\Sigma_\beta} : H^1(M) \to L^2(\Sigma_\beta)$ are continuous and dual to each other. We want to see that $(I - P_\mu)(\Delta_g + \lambda)^{-1} : H^{-1}(M) \to H^1(M)$ is bounded by a constant independent of λ.

Let us see that $(\Delta_g + \lambda)^{-1} : H^{-1}(M) \to H^1(M)$ is bounded: Let $\lambda > 0$ such that $\lambda \in \rho(\Delta_g)$ we have that $(\Delta_g + \lambda)^{-1} : \mathrm{dom}(\Delta_g + \lambda)^{-1} \subset L^2(M) \to H^2(M)$ is bounded. For $f \in H^{-1}(M)$ we have

$$\begin{aligned}
\|(I - P_\mu)(\Delta_g + \lambda)^{-1}f\|_{H^1} &= \|(\Delta_g + I)^{1/2}(I - P_\mu)(\Delta_g + \lambda)^{-1}f\|_{L^2} \\
&= \|(\Delta_g + I)(I - P_\mu)(\Delta_g + \lambda)^{-1}(\Delta_g + I)^{-1/2}f\|_{L^2} \\
&\leq \|(\Delta_g + I)(I - P_\mu)(\Delta_g + \lambda)^{-1}\|_{L^2}\|(\Delta_g + I)^{-1/2}f\|_{L^2} \ll \|f\|_{H^{-1}}
\end{aligned}$$

The fact that $(\Delta_g + I)(I - P_\mu)(\Delta_g + \lambda)^{-1}$ is bounded by a constant independent of $\lambda > 0$ follows from the spectral theorem. For $\phi \in L^2(M)$ it is ease to see that:

$$(\Delta_g + I)(I - P_\mu)(\Delta_g + \lambda)^{-1}\phi \leq \left(1 + \frac{1}{\mu}\right)\phi,$$

for any $\lambda > 0$. □

Now we study the behavior of $\log \mu_1(\lambda)$ as $\lambda \to 0^+$. For that we have the following Proposition:

Lemma 6.9.

$$\log \mu_1(\lambda) = \log \lambda + \log\left(\frac{A_g}{\ell_\beta}\right) + o(1), \tag{6.11}$$

as $\lambda \to 0^+$, where $A_g = \mathit{area}(M)$, and $\ell_\beta = \mathit{length}(\Sigma_\beta)$.

Proof. First observe that:

$$\frac{1}{\mu_1(\lambda)} = \|R(\lambda)^{-1}\|.$$

Now, let us study $\|R(\lambda)^{-1}\|$ as $\lambda \to 0^+$, where the norm is the operator norm in $L^2(\Sigma_\beta)$. From the expression for $K_{Q_{\mu,2}(\lambda)}$ in equation (6.10) we have:

$$\lim_{\lambda\to 0} K_{Q_{\mu,2}(\lambda)}(x,x',\lambda) = \sum_{0<\lambda_j\leq\mu} \frac{1}{\lambda_j}\varphi_j(x)\overline{\varphi_j(x')} + \frac{1}{2\pi}\int_0^\mu \frac{1}{1/4+r^2}E(x,\frac{1}{2}+ir)E(x',\frac{1}{2}-ir)dr.$$

Thus $\|Q_{\mu,2}(\lambda)\|$ remains bounded as $\lambda \to 0^+$. We also have that:

$$\|Q_{\mu,1}(\lambda)\| = \frac{1}{\lambda}\frac{\ell_\beta}{A_g}.$$

Since $R(\lambda)^{-1} = Q_{\mu,1}(\lambda) + Q_{\mu,2}(\lambda) + \tilde{Q}_\mu(\lambda)$, using Lemma 6.8 it follows that:

$$\|R(\lambda)^{-1}\| = \frac{\ell_\beta}{\lambda A_g} + O(1), \text{ as } \lambda \to 0^+.$$

This equation together with $\frac{1}{\mu_1(\lambda)} = \|R(\lambda)^{-1}\|$ gives:

$$\frac{1}{\mu_1(\lambda)} = \frac{\ell_\beta}{\lambda A_g} + O(1), \text{ as } \lambda \to 0^+.$$

Now remember the expansion for the logarithm:

$$\log(a+x) = \log(a) + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{na^n}x^n = \log(a) + \frac{1}{a}x - \frac{1}{2a^2}x^2 + \dots.$$

Then writing $\frac{1}{\mu_1(\lambda)} = \frac{\ell_\beta}{\lambda A_g} + u(\lambda)$, with $u(\lambda) = O(1)$ as $\lambda \to 0$, we obtain:

$$\begin{aligned}\log\left(\frac{\ell_\beta}{\lambda A_g} + u(\lambda)\right) &= \log\left(\frac{\ell_\beta}{\lambda A_g}\right) + \frac{\lambda A_g}{\ell_\beta}u - \frac{1}{2}\left(\frac{\lambda A_g}{\ell_\beta}\right)^2 u^2 + \dots \\ &= \log\left(\frac{\ell_\beta}{\lambda A_g}\right) + O(\lambda) \text{ as } \lambda \to 0^+.\end{aligned}$$

Then equation (6.11) follows straight from:

$$\log\left(\frac{1}{\mu_1(\lambda)}\right) = -\log(\mu_1(\lambda)) = \log\left(\frac{\ell_\beta}{A_g}\right) - \log\lambda + O(\lambda) \text{ as } \lambda \to 0^+.$$

□

Putting everything together we obtain the splitting formula, that is the main result of this chapter:

Theorem 6.10. *For the relative determinant of the Laplace operator on a surface with cusps (M, g), and the regularized determinant of the Dirichlet-to-Neumann operator on $\Sigma_\beta = \{\beta\} \times S^1 \subset M$, we have the following splitting formula:*

$$\frac{\det(\Delta_g, \Delta_{Z_\beta,D})}{\det(\Delta_{M_{\beta,D}})} = \frac{A_g}{\ell_\beta}\det{}^* R,$$

where A_g denotes the area of M and ℓ_β denoted the length of Σ_β.

Proof. We start with the splitting formula for $\lambda > 0$, and $\lambda \in \rho(\Delta_g)$:

$$\log\det(\Delta_g + \lambda, \Delta_{Z_\beta,D} + \lambda) - \log\det(\Delta_{M_{\beta,D}} + \lambda) = \log\det R(\lambda)$$

From the previous lemmas we have that:

$$\begin{aligned}\log\det(\Delta_g, \Delta_{Z_\beta,D}) &+ \log\lambda - \log\det(\Delta_{M_\beta,D} + \lambda) + o(1)\\ &= \log\mu_1(\lambda) + \log\det{}^* R + o(1) = \log\lambda + \log\left(\frac{A_g}{\ell_\beta}\right) - O(\lambda) + \log\det{}^* R + o(1)\end{aligned}$$

Letting $\lambda \to 0$, we finally obtain:

$$\log\det(\Delta_g, \Delta_{Z_\beta,D}) - \log\det(\Delta_{M_{\beta,D}}) = \log\left(\frac{A_g}{\ell_\beta}\right) + \log\det{}^* R.$$

That is the same as the equation in the statement of the theorem. □

Remark 6.11. *If we further decompose the operator $\Delta_{Z_\beta,D}$ as $\Delta_{\beta,0} \oplus \Delta_{Z_\beta,1}$ we obtain:*

$$\log\det(\Delta_g, \Delta_{\beta,0}) - \log\det(\Delta_{Z_\beta,1}) - \log\det(\Delta_{M_{\beta,D}}) = \log\left(\frac{A_g}{\ell_\beta}\right) + \log\det{}^* R. \tag{6.12}$$

Chapter 7

Compactness of isospectral sets of conformal metrics

In this chapter we consider the isospectral problem for a surface with cusps restricting our attention to a conformal class of metrics. Moreover we assume that the conformal factors have support in a fixed compact set. We partially generalize the result of B. Osgood, R. Phillips, and P. Sarnak in [34] that states that on a closed surface every set of isometry classes of isospectral metrics is sequentially compact in the C^∞-topology. The generalization is partial in the sense that we consider only a fixed conformal class of metrics. Concerning the variation in the moduli space of surfaces of constant negative curvature W. Müller proved that the resonance set $\sigma(\Gamma)$ of a hyperbolic surface of finite area $\Gamma \setminus \mathbb{H}$ determines the surface in the moduli space up to finitely many possibilities [29, Thm. 8.10]. In particular the resonance set determines the topological type (p, m), (p is the genus and m is the number of cusps), and the length spectrum of $\Gamma \setminus \mathbb{H}$. Here we prove that, given a fixed compact set $K \subset M$, inside a "K-compactly supported" conformal class, sets of isospectral metrics are compact in the C^∞-topology.

Let us start by review OPS's proof of compactness of isospectral sets of metrics on closed surfaces in the C^∞-topology. In this setting, two metrics g_1 and g_2 are called isospectral if the spectra of the Laplacians Δ_{g_1} and Δ_{g_2} are the same including multiplicities. In particular, the regularized determinant $\det \Delta$ and the heat invariants a_j for $j \geq 0$ have the same values at g_1 and g_2. Recall that the heat invariants are the coefficients of the asymptotic expansion of the heat trace for small t.

To define the notion of convergence they fix a background metric g_0. Associated to g_0, there is the Levi-Civita connection and the covariant derivative that allow us to differentiate in the whole tensor algebra. A sequence of metrics $\{g_n\}_{n\in\mathbb{N}}$ converges to a metric g in C^k if $\|g_n - g\|_{C^k} \to 0$, as $n \to \infty$. A sequence of isometry classes of metrics $\hat{g}_n$ converges to an isometry class $\hat{g}$ if there are representatives $h_n \in \hat{g}_n$, $h \in \hat{g}$, such that $h_n \to h$, as $n \to \infty$. Now, let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence of functions in $C^k(M)$ and let σ be a fixed metric on M. Then $\rho_n\sigma \to \rho\sigma$ in C^k as metrics if and only if $\rho_n \to \rho$ in C^k as functions. Moreover, if the metrics $\sigma_n \to \sigma$ in C^∞, and the function $\rho_n \to \rho$ in C^k, then the metrics $\rho_n\sigma_n \to \rho\sigma$ in C^k.

After defining convergence and isospectrality, OPS consider a sequence of isospectral isometry classes of metrics $\{\hat{g}_n\}_{n\in\mathbb{N}}$ and pick representatives g_n. For each g_n there is a metric of constant curvature τ_n such that $g_n = e^{2\varphi_n}\tau_n$. In this way, they associate to each $\hat{g}_n$ a hyperbolic isometry class $\hat{\tau}_n$. They use that for each n, $\det \Delta_{\hat{\tau}_n} \geq \det \Delta_{\hat{g}_n} = \text{constant} > 0$ and Mumford's compactness theorem to prove that there exists a subsequence of $\{\hat{\tau}_n\}_{n\in\mathbb{N}}$ that converges to an element $\hat{\tau}$ in the moduli space. To have compactness of the conformal factors $\{\varphi_n\}_{n\in\mathbb{N}}$, they prove that for each $k\in\mathbb{N}$ the k-th Sobolev norms $\|\varphi_n\|_k$ are uniformly bounded. Compactness in the C^∞-topology follows then from Rellich's Lemma and the Sobolev embedding theorems on M. The constant value of the determinant is used to prove uniform boundedness of the first Sobolev norm. For the higher Sobolev norms, they use the constant values of the heat invariants.

Now let (M,g) be a surface of fixed genus p and a fixed number of cusps m. We usually take $m=1$ to make the proofs simpler but the statements hold for general m. We take g as the background Riemannian metric. Let us decompose M as $M=M_0\cup_{\Sigma_\alpha}Z_\alpha$ where M_0 is compact with boundary Σ_α and the metric on $Z_\alpha=[\alpha,\infty)\times S^1$ is the usual hyperbolic metric.

For $s>0$ and $f\in H^s(M,g)$ recall the definition of the Sobolev norms:

$$\|f\|_{H^s}:=\|(\Delta+I)^{s/2}f\|_{L^2}$$

Now, let K be a compact subset of M. Then there is a $\beta\geq\alpha$ such that $K\subset M_\beta$ and such that $K\cap\Sigma_\beta=\emptyset$. Fix that β and let us define the "K-compactly supported" conformal class of g as the set

$$[g]_K=\{e^{2\varphi}g\mid\varphi\in C_c^\infty(M),\ \operatorname{supp}\varphi\subset K\}. \tag{7.1}$$

Then for every metric in $h\in[g]_K$, (M,h) is a surface with cusps and the cusp (if $m=1$) is contained in $M\setminus M_\beta$.

Since we restricted to a conformal class, the notion of convergence of metrics reduces to the convergence of the conformal factors:

Definition 7.1. *A sequence of metrics $\{g_n\}_{n\in\mathbb{N}}$, with $g_n=e^{2\varphi_n}g$ converges to a metric h in C^k if and only if the sequence of function $\{\varphi_n\}_{n\in\mathbb{N}}$ converges to a function φ in C^k.*

Now we explain what we mean by isospectrality of surfaces with cusps (M,g). Let us start by defining what are resonances: Let $R(s)=(\Delta_g-s(1-s))^{-1}$ be the resolvent of the Laplacian Δ_g for $\mathrm{Re}(s)>1/2$ and $s\neq\bar{s}$. The resolvent $R(s)=(\Delta_g-s(1-s))^{-1}$ regarded as operator from $C_c^\infty(M)$ to $L^2_{loc}(M)$ admits a meromorphic extension to $\mathbb{C}$. The poles of the meromorphic continuation are called resonances. For each pole ρ one can define its multiplicity $n(\rho)$. If $\lambda_j=s_j(1-s_j)$ is an eigenvalue of Δ_g, then s_j is a resonance. The complement of the set of poles that correspond to eigenvalues are poles of the scattering matrix, see [29].

Definition 7.2. *The resonance set of Δ_g is the union of the poles of the scattering matrix and of the set $\{s_j|s_j(1-s_j)$ is an eigenvalue$\}$.*

One reason to consider resonances is that the following trace formula holds ([29, (2.2)]):

$$\begin{aligned}\mathrm{Tr}(e^{-t\Delta_g}-e^{-t\bar{\Delta}_{\alpha,0}})&=\int_M(K_g(z,z,t)-\sum_{j=1}^m p_{\alpha_j}(z,z,t))dA_g(z)\\&=\sum_k e^{-\lambda_k t}-\frac{1}{4}\int_{-\infty}^{\infty}e^{-(1/4+\lambda^2)t}\frac{\phi'}{\phi}(1/2+i\lambda)d\lambda\\&\qquad+\frac{1}{4}e^{-t/4}(\mathrm{Tr}(C(1/2))+m)+\frac{e^{-t/4}}{\sqrt{4\pi t}}\sum_{j=1}^m\log(\alpha_j),\end{aligned}\tag{7.2}$$

where the term $\frac{m}{4}e^{-t/4}$ comes from the extra term of $p_{\alpha_j}(z,z,t))$ determined by the Dirichlet extension, i.e. the term $\frac{e^{-t/4}}{\sqrt{4\pi t}}(yy')^{1/2}e^{-(\log(yy')-\log(\alpha^2))^2/4t}$ in equation (1.9). Now by Theorem 5.11 in [29] the integral that involves the logarithmic derivative of the scattering matrix can be rewritten as follows:

$$
\begin{aligned}
&-\frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-(1/4+\lambda^2)t}\frac{\phi'}{\phi}(1/2+i\lambda)d\lambda \\
&\quad = \frac{\log(q)}{(4\pi)^{3/2}}\frac{e^{-t/4}}{\sqrt{t}} + \frac{1}{4}\sum_{\rho} n(\rho)\{e^{-t\rho(1-\rho)}\mathrm{Erfc}(\sqrt{t}(\sqrt{t}(1/2-\rho)) + e^{-t\bar{\rho}(1-\bar{\rho})}\mathrm{Erfc}(1-\bar{\rho}))\}, \quad (7.3)
\end{aligned}
$$

where ρ runs over all zeros and poles of $\phi(s)$ in $\mathrm{Re}(s), 1/2$, $n(\rho)$ denotes either the order of the pole ρ or the negative of the order of the zero ρ, q is a well-determined constant and Erfc is the complementary error function, see [29, (5.13)].

We are ready now to define isospectral metrics:

Definition 7.3. *Two cusp metrics g_1 and g_2 on M are isospectral if their resonance sets including multiplicities are the same.*

For the definition of isospectrality, the continuous spectra are irrelevant since for two surfaces with cusps (M,g_1) and (M,g_2), $\sigma_c(\Delta_{g_1}) = \sigma_c(\Delta_{g_2}) = [1/4,\infty)$ with multiplicity the number of cusps.

Remark 7.4. *Let (M,g_1) and (M,g_2) be two surfaces with cusps that are isospectral. Then, equations (7.2) and (7.3) imply that the corresponding traces of the heat operators coincide. Under the same hypothesis, the fact that the determinants of the scattering matrices are the same follows from Theorem 3.31 in [29], which expresses the determinant of the scattering matrix as the following Weierstrass product:*

$$\phi(s) = \phi(1/2)q^{s-1/2}\prod_{\rho}\frac{s-1+\bar{\rho}}{s-\rho},$$

where ρ runs over all poles of $\phi(s)$, counted with the order and q is the same constant of equation (7.3). It is also clear that the eigenvalues of the Laplacians coincide. In this way, we also have that the corresponding relative determinants satisfy

$$\det(\Delta_{g_1},\bar{\Delta}_{a,0}) = \det(\Delta_{g_2},\bar{\Delta}_{a,0}),$$

for any $a=(a_1,\dots,a_m)$ with $\min\{a_j, 1\le j\le m\}$ big enough.

The main theorem in this chapter is:

Theorem 7.5. *Let (M,g) be a surface with cusps, let $K\subset M$ be a fixed compact subset of M and let $[g]_K = \{e^{2\varphi}g \mid \varphi\in C_c^\infty(M),\ \operatorname{supp}\varphi\subset K\}$ be the K-compactly supported conformal class of g. Then isospectral sets in $[g]_K$ are compact in the C^∞-topology.*

Remark 7.6. *As in the compact case, the proof of Theorem 7.5 uses Sobolev embedding theorems and Rellich's Lemma. We refer the reader to Appendix A and the references therein for the statements of these theorems. It is well known that due to the nullity of the injectivity radius that neither Sobolev embedding theorems nor Rellich's lemma hold on surfaces with cusps in their standard form. This is one of the reasons why we restrict our attention to transformations of the metric that take place inside a fixed compact set and use the theorems for compact manifolds.*

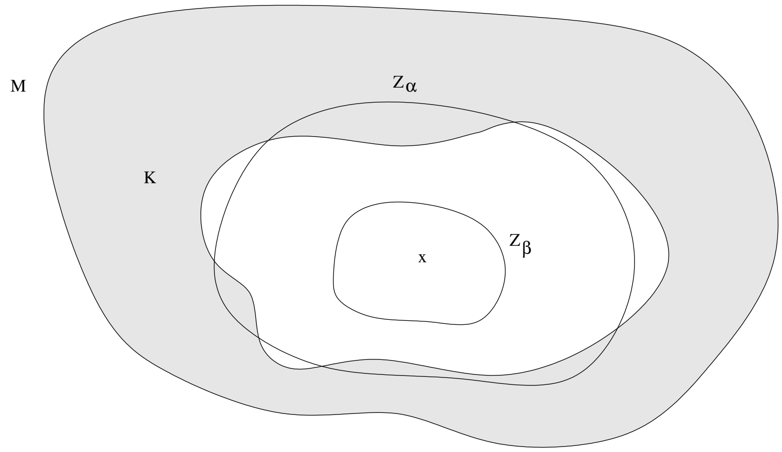

Figure 7.1: Cusps and the compact set K. K is shown in gray.

Proof. First of all we need to compactify M to a Riemannian manifold that contains K isometrically. It is convenient at this point to change the coordinates in the cusp by the transformation $z \to w = e^{iz}$. Then Z_α becomes $\{w \in \mathbb{C} : 0 < |w| \leq e^{-\alpha}\} =: D^*_{e^{-\alpha}}$ and the metric on it becomes

$$g|_{D^*_{e^{-\alpha}}} = \log(|w|^{-1})^{-2}|w|^{-2}|dw|^2.$$

Let us keep the old notation in these new coordinates. Then for $b \geq \alpha$, $M_b = M_0 \cup (D^*_{e^{-\alpha}} \setminus D^*_{e^{-b}}) \cup \Sigma_b$ and we could also denote $D^*_{e^{-b}}$ by Z_b. Let $f \in C^\infty(M)$ satisfy

$$f(w) := \begin{cases} |\log(|w|)||w| & \text{if } w \in D^*_{e^{-\beta-2}} (\cong Z_{\beta+2}) \\ 1 & \text{if } w \in M_{\beta+1}, \end{cases}$$

and put

$$\sigma = f(z)^2 \cdot g.$$

Then take $\widetilde{M} = M \cup \{0\}$ the one-point compactification of M (m-point compactification if M has m cusps). The metric σ on M extends to a smooth metric on $\widetilde{M}$ which we denote again by σ. Thus $(\widetilde{M}, \sigma)$ is a closed manifold that contains M_β isometrically and that has the same genus as M.

Now let $\{g_n\}_{n\in\mathbb{N}} \subset [g]_K$ be a sequence of isospectral of metrics on M conformal to g. Notice that since the metrics in the sequence are isospectral their areas A_{g_n} are the same and by the Gauss-Bonnet theorem we have that $A_{g_n} = 2\pi(2p+m-2)$. Since $g_n \in [g]_K$, there exists a function $\varphi_n \in C^\infty_c(M)$ such that $g_n = e^{2\varphi_n} g$ and $\operatorname{supp} \varphi_n \subset K$, for each $n \in \mathbb{N}$. Now put

$$\widetilde{g}_n := e^{2\varphi_n}\sigma.$$

Then the metrics $\widetilde{g}_n$ are conformal to σ on $\widetilde{M}$. The fact that $K \subsetneq M_{\beta+1} = M \setminus D^*_{e^{-\beta-1}}$ and $\sigma|_{M_{\beta+1}} = g|_{M_{\beta+1}}$ imply that the values $A_{g_n} - A_g(D^*_{e^{\beta+1}})$ are constant. Then the areas $A_{\widetilde{g}_n}$ of $(\widetilde{M}, \widetilde{g}_n)$ have all the same value; this follows from:

$$A_{\widetilde{g}_n} = A_{g_n} - A_g(D^*_{e^{\beta+1}}) + A_\sigma(\widetilde{M} \setminus M_{\beta+1}).$$

Therefore we can renormalize the metrics $\widetilde{g}_n$ such that $A_{\widetilde{g}_n} = 1$.

As in the previous paragraph, the definitions of K and σ, the condition $\operatorname{supp}\varphi_n \subset K$ for all $n \in \mathbb{N}$ and the locality of the Laplacians Δ_g and Δ_σ imply that

$$\|\varphi_n\|^2_{H^k(\widetilde{M},\sigma)} = \|\varphi_n\|^2_{H^k(M,g)}. \tag{7.4}$$

Notice here that compactness of $\{\varphi_n\}_{n\in\mathbb{N}}$ in $C^\infty(\widetilde{M},\sigma)$, together with $\operatorname{supp}\varphi_n \subset K \Subset M$, for all $n \in \mathbb{N}$ implies compactness of $\{\varphi_n\}_{n\in\mathbb{N}}$ in $C^\infty(M,g)$. In order to prove compactness in $C^\infty(M,g)$ we therefore need to prove that for each $k \geq 1$:

$$\|\varphi_n\|_{H^k(\widetilde{M},\sigma)} \leq C(k) \qquad \text{for all } n \in \mathbb{N},$$

where $C(k)$ is a constant that may depend on k. The kth Sobolev norm can also be written as

$$\|\varphi_n\|^2_{H^k(\widetilde{M},\sigma)} = \sum_{l=0}^{k} \int_{\widetilde{M}} |\nabla^l_\sigma \varphi_n(x)|^2 dA_\sigma,$$

where for the sake of simplicity ∇^l_σ is denoted just by ∇^l.

In Lemmas 7.7 and 7.8 we prove that if $\{g_n\}_{n\in\mathbb{N}}$ is isospectral then $\det\Delta_{\widetilde{g}_n}$ is constant and the heat invariants of the metrics $\widetilde{g}_n$ are the same for all n. The theorem will then follow from the results of [34]. In Lemma 7.9 we prove the uniform bound in the first Sobolev norm. The proof follows the same lines as in [34] but we repeat it here for convenience and completeness. □

Lemma 7.7. *Given a sequence $\{g_n\}_{n\in\mathbb{N}}$ of isospectral metrics in a conformal class $[g]_K$, let $\{\widetilde{g}_n\}_{n\in\mathbb{N}}$ be the associated sequence of metrics on $\widetilde{M}$ defined above. Then the regularized determinants $\det\Delta_{\widetilde{g}_n}$ are constant, i.e. their value is independent of n.*

Proof. Let h be any metric in $[g]_K$ and let $\widetilde{h} = f^2 \cdot h$. Recall that $\widetilde{M} = M \cup \{0\}$ was defined above as the one-point compactification of M. Then for the relative determinant of $(\Delta_h, \Delta_{\beta,0})$ and the determinant of $\Delta_{\widetilde{h}}$ we have the following splitting formulas:

$$\log\det(\Delta_h, \Delta_{\beta,0}) - \log\det\Delta_{Z_\beta,1} - \log\det\Delta_{(M_\beta,h),D} = \log\left(\frac{A_h(M)}{\ell(\Sigma_\beta,h)}\right) + \log\det{}^* R_h$$

and

$$\log\det\Delta_{(\widetilde{M},\widetilde{h})} - \log\det\Delta_{(M_\beta,\widetilde{h}),D} - \log\det\Delta_{(\widetilde{M}\setminus M_\beta,\widetilde{h}),D} = \log\left(\frac{A_{\widetilde{h}}(\widetilde{M})}{\ell(\Sigma_\beta,\widetilde{h})}\right) + \log\det{}^* R_{\widetilde{h}},$$

where the first formula is the one given by equation (6.12) and follows straight forward from Theorem 6.10, and the second formula is the well known splitting formula for a closed surface, as in [5]. Subtracting the equations we obtain:

$$\begin{aligned}\log\det\Delta_{(\widetilde{M},\widetilde{h})} &- \log\det(\Delta_h,\Delta_{\beta,0}) + \log\det\Delta_{Z_\beta,1} - \log\det\Delta_{(\widetilde{M}\setminus M_\beta,\widetilde{h}),D} \\ &= \log\left(\frac{A_{\widetilde{h}}(\widetilde{M})}{\ell(\Sigma_\beta,\widetilde{h})}\right) - \log\left(\frac{A_h(M)}{\ell(\Sigma_\beta,h)}\right) + \log\det{}^* R_{\widetilde{h}} - \log\det{}^* R_h.\end{aligned}$$

From the definition of f we have that $\widetilde{h} = h$ on $M_{\beta+1}$, and $f \equiv 1$ in a neighborhood of Σ_β. So we have that $\ell(\Sigma_\beta, h) = \ell(\Sigma_\beta, \widetilde{h})$. On the other hand, the Dirichlet-to-Neumann operators are the same for both metrics. To see this, notice that given a function $u \in C^\infty(\Sigma_\beta)$, the unique solution to the problem $\Delta_g \widetilde{u} = 0$ on $M \setminus \Sigma_\beta$ with $\widetilde{u}|_{\Sigma_\beta} = u$ will also be a solution of $\Delta_h \widetilde{u} = e^{-2\varphi}\Delta_g \widetilde{u} = 0$ on $M \setminus \Sigma_\beta$ satisfying the same boundary condition. Then we have:

$$\log\det\Delta_{(\widetilde{M},\widetilde{h})} - \log\det(\Delta_h, \Delta_{\beta,0}) - \log\det\Delta_{(\widetilde{M}\setminus M_\beta,\widetilde{h}),D} = \log(A_{\widetilde{h}}(\widetilde{M})) - \log(A_h(M)) + c \quad (7.5)$$

where c is a constant that does not depend on h. Now, let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of isospectral metrics in $[g]_K$ and let $\{\widetilde{g}_n\}_{n\in\mathbb{N}}$ be as in the proof of Theorem 7.5. Recall that $A_{g_n}(M)$, $A_{\widetilde{g}_n}(\widetilde{M})$, $\det(\Delta_{g_n}, \Delta_{\beta,0})$ are constants independent of n. Moreover, $\widetilde{g}_n|_{\widetilde{M}\setminus M_\beta} = \sigma|_{\widetilde{M}\setminus M_\beta}$. Therefore $\det\Delta_{(\widetilde{M}\setminus M_\beta,\widetilde{g}_n),D}$ is also constant. Then replacing h and $\widetilde{h}$ by g_n and $\widetilde{g}_n$ respectively, in equation (7.5) we obtain:

$$\log\det(\Delta_{\widetilde{g}_n}) = \text{constant}.$$

This finishes the proof of the lemma. $\square$

Lemma 7.8. *The heat invariants corresponding to the metrics of the sequence $\{\widetilde{g}_n\}_{n\in\mathbb{N}}$ are the same for any $n \in \mathbb{N}$ if we start with an isospectral sequence $\{g_n\}_{n\in\mathbb{N}}$.*

Proof. Let h be any of the metrics g_n we are considering. Let us start by constructing the kernel of a parametrix H_h for the heat operator $e^{-t\Delta_h}$ on the surface with cusps (M, h), as we explained in chapter 1 and in a similar way as in chapter 3. Namely we use the standard method of gluing the heat kernel on the complete hyperbolic cusp $(0,\infty)\times S^1$, denoted by K_1 and independent of h, with the heat kernel on $(\widetilde{M}, \widetilde{h})$, denoted by $K_{2,\widetilde{h}}$, restricted to $M_{\beta+2}$.

Let us recall how we defined the gluing functions: For any two constants $1 < b < c$, let $\phi_{(b,c)}$ be as defined in Chapter 1, so that $\phi_{(b,c)}(y,x) = 0$ for $y \leq b$, and $\phi_{(b,c)}(y,x) = 1$ for $y \geq c$. Let $\psi_1 = \phi_{(\beta+\frac{5}{4},\beta+2)}$, and $\psi_2 = 1 - \psi_1$; then $\{\psi_1, \psi_2\}$ is a partition of unity on $[\beta+1, \beta+2]\times S^1$. Let $\phi_1 = \phi_{(\beta,\beta+1)}$ and $\phi_2 = 1 - \phi_{(\beta+\frac{5}{2},\beta+3)}$, so that $\phi_i = 1$ on the support of ψ_i, $i = 1, 2$. The parametrix we are considering is:

$$H_h(z,z',t) = \phi_1(z)K_1(z,z',t)\psi_1(z') + \phi_2(z)K_{2,\widetilde{h}}(z,z',t)\psi_2(z').$$

As in Lemma 3.3, we can prove that there exist constants $C, c > 0$ such that:

$$\int_M |K_h(z,z,t) - H_h(z,z,t)|dA_h(z) \leq Ce^{-\frac{c}{t}}$$

for $0 < t \leq 1$. Then for small t we can replace the heat kernel K_h for the parametrix H_h. Let

$$p_\beta(z,z',t) := \frac{(yy')^{1/2}}{\sqrt{4\pi t}}e^{-\frac{t}{4}}(e^{-\frac{(\log(y/y'))^2}{4t}} - e^{-\frac{(\log(yy'/\beta^2))^2}{4t}})$$

and $p_\beta(z,z',t) := 0$ elsewhere. We thereby derive an analog to equation (8.14) in [28, page 283], exactly as it is done there:

$$\int_M (K_h(z,z,t) - p_\beta(z,z,t))\, dA_h(z) = \int_{Z_{\beta+1}} (K_1(z,z,t) - p_\beta(z,z,t))\, dA_h(z) + \int_{M_{\beta+1}} K_{2,\widetilde{h}}(z,z,t)\, dA_h(z) + O(e^{-\frac{c}{t}}), \text{ as } t \to 0.$$

For a metric $\widetilde{g}_n$ on $\widetilde{M}$ the heat invariants are, by definition, the coefficients in the asymptotic expansion of the trace of the heat kernel as $t \to 0$:

$$\int_{\widetilde{M}} K_{2,\widetilde{g}_n}(z,z,t)\; dA_{\widetilde{g}_n} \sim \frac{1}{t}\sum_{j=0}^{\infty} a_j(\widetilde{g}_n)t^j, \qquad \text{as } t \to 0.$$

The goal of this lemma is to prove that $a_j(\widetilde{g}_n) = a_j(\widetilde{g}_m)$ for any $n, m \in \mathbb{N}$, and for all $j \geq 0$. This will follow from the equality of the asymptotic expansions for small values of t of the integrals

$$\int_{\widetilde{M}} K_{2,\widetilde{g}_n}(z,z,t)\; dA_{\widetilde{g}_n} \quad \text{and} \quad \int_{\widetilde{M}} K_{2,\widetilde{g}_m}(z,z,t)\; dA_{\widetilde{g}_m} \tag{7.6}$$

for any $n, m \in \mathbb{N}$. We can split the integral over $\widetilde{M}$ as an integral over $M_{\beta+1}$ and one over $\widetilde{M}\setminus M_{\beta+1}$. Given two metrics g_n and g_m as in the statement of the lemma, we have that on $\widetilde{M}\setminus M_{\beta+1}$, $\widetilde{g}_n = \widetilde{g}_m$. Since relative to any coordinate system, the coefficients of the asymptotic expansion of the heat kernel are given by universal polynomials in terms of the metric tensor and its covariant derivatives, we have that $a_j(z, \widetilde{g}_n) = a_j(z, \widetilde{g}_m)$, for $z \in \widetilde{M} \setminus M_{\beta+1}$. On $\widetilde{M} \setminus M_{\beta+1}$ we have that $dA_{\widetilde{g}_n} = dA_{\widetilde{g}_m}$. Therefore:

$$\int_{\widetilde{M}\setminus M_0} K_{2,\widetilde{g}_n}(z,z,t)\; dA_{\widetilde{g}_n}(z) = \int_{\widetilde{M}\setminus M_0} K_{2,\widetilde{g}_m}(z,z,t)\; dA_{\widetilde{g}_n}(z).$$

By assumption, K_1 and $p_\beta(z,z,t)$ are independent of g_n and g_m. Therefore:

$$\begin{aligned}
&\int_{M_{\beta+1}} K_{2,\widetilde{g}_n}(z,z,t)\; dA_{\widetilde{g}_n}(z) - \int_{M_{\beta+1}} K_{2,\widetilde{g}_m}(z,z,t)\; dA_{\widetilde{g}_m}(z)\\
&\quad \sim_{t\to 0} \int_M (K_{g_n}(z,z,t) - p_\beta(z,z,t))\; dA_{g_n}(z) - \int_{Z_{\beta+1}} (K_1(z,z,t) - p_\beta(z,z,t))\; dA_{g_n}(z)\\
&\quad\quad - \int_M (K_{g_m}(z,z,t) - p_\beta(z,z,t))\; dA_{g_m}(z) + \int_{Z_{\beta+1}} (K_1(z,z,t) - p_\beta(z,z,t))\; dA_{g_m}(z)\\
&\quad\quad = \int_M (K_{g_n}(z,z,t) - p_\beta(z,z,t))\; dA_{g_n}(z) - \int_M (K_{g_m}(z,z,t) - p_\beta(z,z,t))\; dA_{g_m}(z) = 0,
\end{aligned}$$

where the last equality follows from the fact that the metrics are isospectral and from equations (7.2) and (7.3). So, we have proved that the asymptotic expansions as $t \to 0$ for the integrals in (7.6) are the same. From the definition of the heat invariants it follows that:

$$a_j(\widetilde{g}_n) = a_j(\widetilde{g}_m), \qquad \text{for all } j \geq 0, \;\; \text{and } n, m \in \mathbb{N}\,.$$

□

Let us prove now the uniform bound on the first Sobolev norms of the metrics $\{\varphi_n\}_{n\in\mathbb{N}}$ in $(\widetilde{M}, \sigma)$. The proof only requires the constant value of the determinants $\det \Delta_{\widetilde{g}_n}$ and the constant value of the areas $A_{\widetilde{g}_n}$. In fact, the proof is exactly the same as in [34], but restricted to a conformal class. For convenience and completeness we repeat the proof here adapted to the restriction to a conformal class.

Lemma 7.9. *([34]) For all $n \in \mathbb{N}$ we have that $\|\varphi_n\|_{H^1(\widetilde{M},\sigma)} \ll 1$.*

Proof. We take $(\widetilde{M}, \sigma)$ as the background Riemannian manifold. We assume that $\widetilde{M}$ admits a metric of negative curvature, so that it is Euler characteristic is negative. We know that in the conformal class of the metric σ, $[\sigma] = \{e^{2\varphi}\sigma \mid \varphi \in C^\infty(\widetilde{M})\}$, there is a unique hyperbolic metric, τ of unit area, $\tau = e^{2\psi}\sigma$. Thus, the metrics $\widetilde{g}_n$ are conformal to the uniform metric τ:

$$\widetilde{g}_n = e^{2(\varphi_n - \psi)}\tau.$$

Polyakov's formula for regularized determinants on closed surfaces gives ([34, (1.13)]):

$$\log\det(\Delta_{\widetilde{g}_n}) = -\frac{1}{6\pi}\left\{\frac{1}{2}\int_{\widetilde{M}} |\nabla_\tau(\varphi_n - \psi)|^2 dA_\tau + \int_{\widetilde{M}} K_\tau\ (\varphi_n - \psi) dA_\tau\right\} + \log(A_{\widetilde{g}_n}) + \log\det(\Delta_\tau).$$

This is equivalent to

$$-6\pi\log\det(\Delta_{\widetilde{g}_n}) = \frac{1}{2}\int_{\widetilde{M}} |\nabla_\tau(\varphi_n - \psi)|^2 dA_\tau + 2\pi(2-2p)\int_{\widetilde{M}} (\varphi_n - \psi) dA_\tau - 6\pi\log\det(\Delta_\tau).$$

Let $\psi_n := \varphi_n - \psi$. Since $A_\tau = \int_{\widetilde{M}} dA_\tau = 1$ and

$$A_{\widetilde{g}_n} = \int_{\widetilde{M}} dA_{\widetilde{g}_n} = \int_{\widetilde{M}} e^{2\varphi_n} dA_\sigma = \int_{\widetilde{M}} e^{2(\varphi_n - \psi)} dA_\tau = 1,$$

we can apply Jensen's inequality to obtain:

$$\exp\left(\int_{\widetilde{M}} 2\psi_n\ dA_\tau\right) \le \int_{\widetilde{M}} \exp(2\psi_n)\ dA_\tau, \quad \text{thus} \quad \frac{1}{2}\int_{\widetilde{M}} \psi_n\ dA_\tau \le \log\left(\int_{\widetilde{M}} e^{2\psi_n}\ dA_\tau\right) = 0. \tag{7.7}$$

In [33], the authors proved that inside a conformal class $[\sigma]$, among all metrics of unit area, the functional logarithm of the determinant attains its maximum at the metric of constant curvature. Thus:

$$\log\det(\Delta_\tau) \ge \log\det(\Delta_{\widetilde{g}_n}).$$

From Lemma 7.7 we have that $\log\det(\Delta_{\widetilde{g}_n})$ is constant. Therefore there exists a constant $C > 0$ such that $C \ge \log\det(\Delta_\tau) - \log\det(\Delta_{\widetilde{g}_n}) \ge 0$. Then,

$$C \ge 6\pi(\log\det(\Delta_\tau) - \log\det(\Delta_{\widetilde{g}_n})) = \frac{1}{2}\int_{\widetilde{M}} |\nabla_\tau\psi_n|^2 dA_\tau + 2\pi(2-2p)\int_{\widetilde{M}} \psi_n\ dA_\tau \ge 0. \tag{7.8}$$

We restrict now to surfaces for which $p \ge 1$. Then, from equation (7.7) it follows that the term $2\pi(2-2p)\int_{\widetilde{M}} \psi_n\ dA_\tau$ is positive. Therefore

$$\frac{1}{2}\int_{\widetilde{M}} |\nabla_\tau\psi_n|^2 dA_\tau \le C - 2\pi(2-2p)\int_{\widetilde{M}} \psi_n\ dA_\tau \le C. \tag{7.9}$$

Thus,

$$\int_{\widetilde{M}} |\nabla_\tau\psi_n|^2\ dA_\tau \ll 1. \tag{7.10}$$

In order to see that the previous equation implies $\int_{\widetilde{M}} |\nabla_\sigma\varphi_n|^2\ dA_\sigma \ll 1$, notice that there is a constant $C_{\sigma,\tau} \ge 0$ such that

$$\int_{\widetilde{M}} |\nabla_\sigma\psi_n|^2\ dA_\sigma \le C_{\sigma,\tau}\int_{\widetilde{M}} |\nabla_\tau\psi_n|^2\ dA_\tau \ll 1,$$

and

$$\|\nabla_\sigma \varphi_n\|_{L^2(\widetilde{M},\sigma)} \leq \|\nabla_\sigma(\varphi_n - \psi)\|_{L^2(\widetilde{M},\sigma)} + \|\nabla_\sigma \psi\|_{L^2(\widetilde{M},\sigma)} \ll 1.$$

The next step is to prove that

$$\int_{\widetilde{M}} |\varphi_n|^2 dA_\sigma \ll 1.$$

In order to do this, use Trudinger's inequality and the fact that τ is a metric of unit area, thus

$$1 = \int_{\widetilde{M}} e^{2\psi_n}\ dA_\tau \leq C \exp\left(c_1 \int_{\widetilde{M}} |\nabla_\tau \psi_n|^2\ dA_\tau + c_2 \int_{\widetilde{M}} \psi_n\ dA_\tau \right).$$

Then we have $\frac{-1}{c_2}\log(C' e^{c_1'}) \leq \int_{\widetilde{M}} \psi_n\ dA_\tau \leq 0$. Therefore

$$\left| \int_{\widetilde{M}} \psi_n\ dA_\tau \right| \ll 1. \tag{7.11}$$

To show that the $L^2(\widetilde{M},\sigma)$-norms of the functions ψ_n are uniformly bounded, use equations (7.10), (7.11), and the min-max principle in the following way. If $\psi_n \perp 1$, i.e. $\int_{\widetilde{M}} \psi_n\ dA_\tau = 0$, one has

$$\frac{\int_{\widetilde{M}} |\nabla_\tau \psi_n|^2\ dA_\tau}{\int_{\widetilde{M}} |\psi_n|^2\ dA_\tau} \geq \lambda_1(\Delta_\tau),$$

so

$$\int_{\widetilde{M}} |\psi_n|^2\ dA_\tau \leq \frac{1}{\lambda_1(\Delta_\tau)} \int_{\widetilde{M}} |\nabla_\tau \psi_n|^2\ dA_\tau \ll 1.$$

If ψ_n is not orthogonal to the constant functions, then decompose it as $\psi_n = \hat{\psi}_n + c(\psi_n)$ with $c(\psi_n)$ the projection of ψ_n on the kernel of Δ_τ, and $\hat{\psi}_n \perp 1$. Then $\nabla_\tau \psi_n = \nabla_\tau \hat{\psi}_n$, and $\|\psi_n\|_2^2 = \|\hat{\psi}_n\|_2^2 + c(\psi_n)^2 A_\tau^2 \ll 1$. Thus, we have

$$\int_{\widetilde{M}} |\psi_n|^2\ dA_\tau \ll 1.$$

Now use that $dA_\sigma = \rho\ dA_\tau$, for a positive function ρ, to obtain

$$\int_{\widetilde{M}} |\psi_n|^2\ dA_\sigma \leq \frac{1}{\min \rho} \int_{\widetilde{M}} |\psi_n|^2\ dA_\tau \ll 1.$$

Finally, the triangle inequality for the metric σ gives

$$\|\varphi_n\|_2 \leq \|\varphi_n - \psi\|_2 + \|\psi\|_2 \ll 1.$$

Putting everything together we obtain:

$$\|\varphi_n\|_{H^1(\widetilde{M},\sigma)} \ll 1, \quad \text{for all } n \in \mathbb{N}.$$

$\square$

The uniform estimates for the higher Sobolev norms follow in the same way as in [34]. The idea of the proof is the following. The constant value of all the heat invariants, $a_j(\widetilde{g}_n) =$ constant for all $n \in \mathbb{N}$, implies uniform bounds for the corresponding curvatures and all their derivatives (the proof of this implication is tiresome and full of technicalities). Then, using the equation for the conformal change of the curvature, which in this case is $-e^{2\psi_n} = \Delta_\tau \psi_n + K_{\psi_n}$, one obtains uniform estimates for all the Sobolev norms of the conformal factors ψ_n, therefore for all the Sobolev norms of the functions φ_n, $n \in \mathbb{N}$. This finishes the proof of compactness of isospectral sets of metrics in this case.

Remark 7.10. *To extend Theorem 7.5 to include non compactly supported deformations we need to solve several problems. The first problem is to find a suitable weighted Sobolev space where the Sobolev embeddings and the Rellich's Lemma hold. On the other hand, from equation (7.9) is clear that we need the existence of a maximizer of* $\det(\Delta_., \Delta_{1,0})$ *inside the conformal class. As we noticed in Chapter 4, Remark 4.6 this is only possible if the function* φ *in the conformal factor decays in the cusps as* y^{-1}*, as* $y \to \infty$*. Therefore we should be able to define the relative determinant for this wider class of metrics. We also need a complete asymptotic expansion of the relative heat trace for small* t *since this asymptotic expansion is where the constancy of the heat invariants and their relation with the higher derivatives of the function* φ *comes from. Those are the necessary ingredients to obtain uniform boundedness of the Sobolev norms of the functions* φ_n*. Once we have all that, we need to improve the bounds such that the weighted Sobolev norms are uniformly bounded.*

Appendix A

Sobolev spaces

Sobolev embedding theorems and Rellich's Lemma on closed manifolds are a key tool in the proof of compactness (up to diffeomorphism) of isospectral sets of metrics on closed surfaces, [34]. In the first part of this appendix we give the statement of these theorems. It is well known that for Sobolev spaces defined on surfaces with cusps these theorems do not hold any more. In the second part we give a brief description of the Sobolev spaces defined on surfaces with cusps.

A.1 Closed manifolds

In this part we give a brief description of Sobolev spaces for closed manifolds and state their main properties that are used in this thesis. We refer the reader to [24] and [43]. In [24] the authors define Sobolev spaces in the setting of Hermitian vector bundles with connection on a Riemannian manifold. The following definitions and results are presented as they are stated in [43, Chapter 4].

We assume that the theory of Sobolev spaces in $\mathbb{R}^n$ is well known to the reader. Let us recall the definition of the Sobolev space in $\mathbb{R}^n$:

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) | (1+|\xi|^2)^{s/2}\hat{u}(\xi) \in L^2(\mathbb{R}^n)\}$$

where $\hat{u}$ denotes the Fourier transform of u. The sth-Sobolev norm of u is given by

$$\|u\|_s^2 = \int_{\mathbb{R}} (1+|\xi|)^{2s} |\hat{u}(\xi)|^2 d\xi.$$

Let M be a compact manifold, and let $u \in \mathcal{D}'(M)$. We say that $u \in H^s(M)$ provided that on any coordinate patch $U \subset M$, any $\psi \in C_0^\infty(U)$, the element $\psi u \in \mathcal{E}'(U)$ belongs to $H^s(U)$, if U is identified with its image in R^n. By the invariance under coordinate changes, it suffices to work with any single coordinate cover of M.

The Sobolev embedding theorems have several parts. Let us state only the part that is of interest for us:

Theorem *Let M be a smooth compact manifold of dimension n, then*

1. **Sobolev embedding.** *If $u \in H^s(M)$, then $u \in C^k(M)$, provided $s > \frac{n}{2} + k$. The inclusion $H^s(M) \subset C^k(M)$ is continuous.*

2. **Rellich's Lemma** *Given $s \in \mathbb{R}$, the inclusion $H^{s+\sigma}(M) \to H^s(M)$ is compact for all $\sigma > 0$.*

The theorems above are also valid on complete open manifolds with bounded geometry. However it is well known that they are not valid in general.

A.2 Surfaces with cusps

In this part we refer to [36] and [32]. There, the authors define and study Sobolev spaces for Riemannian manifolds with bounded curvature. See these references for the details.

Let M be a Riemannian manifold with bounded curvature. We also assume that M is complete. Let $k \in \mathbb{N}$, then the Sobolev space $H^{2k}(M)$ is defined as:

$$H^{2k}(M) := \{f \in L^2(M) | \Delta_g^l f \in L^2(M), \text{ for all } l = 1, \dots, k\},$$

with the norm:

$$\|f\|_{H^{2k}} := \|(\Delta_g + I)^k f\|_{L^2}. \tag{A.1}$$

The closure of $C_0^\infty(M)$ in $H^{2k}(M)$ is denoted by $H_0^{2k}(M)$ and $C^\infty(M) \cap H^{2k}(M)$ is dense in $H^{2k}(M)$. Let

$$\tilde{C}_k^\infty(M) = \{f \in C^\infty(M) | (\Delta_g + I)^k f \in L^2(M)\},$$

then $H^{2k}(M)$ is the completion of $\tilde{C}_k^\infty(M)$ with respect to the norm defined in equation (A.1). Now for $s \in \mathbb{R}$, let $(\Delta_g + I)^{s/2}$ be defined by the spectral theorem. Let

$$\tilde{C}_s^\infty(M) = \{f \in C^\infty(M) | (\Delta_g + I)^{s/2} f \in L^2(M)\},$$

and define $H^s(M)$ as the completion of $\tilde{C}_s^\infty(M)$ respect to the norm:

$$\|f\|_{H^s} := \|(\Delta_g + I)^{s/2} f\|_{L^2}. \tag{A.2}$$

Lemma 3.2 in [32] establishes that if M is complete, then $H^{2k}(M) = H_0^{2k}(M)$, for any $k \in \mathbb{N}$.

Now let (M, g) be a surface with cusps, as it was described in Chapter 1. Let $0 \leq s' \leq s$, for $\lambda \geq 0$ we have that $(1+\lambda)^{s'} \leq (1+\lambda)^s$. Therefore $\|f\|_{H^{s'}} \leq \|f\|_{H^s}$. In this way we have that $H^s(M) \subset H^{s'}(M)$.

For $s > 0$, we claim that $H^{-s}(M) \cong (H_0^s)^*(M)$. First notice that $H^s(M) \subset H_0^{-s}(M)$. To see that, let $f, \varphi \in H^s(M)$ and let ϕ_f be the corresponding element in $(H^s(M))'$. Then for $\varphi \in C_0^\infty(M)$ we have that $\phi_f(\varphi) = \langle f, \varphi \rangle_{H^s(M)}$, and $|\phi_f(\varphi)| \leq \|f\|_{H^s} \|\varphi\|_{H^s}$. Now,

$$\begin{aligned} \langle \phi_f, \varphi \rangle_{H^{-s}} &= \langle (\Delta_g + I)^{-s/2} \phi_f, (\Delta_g + I)^{-s/2} \varphi \rangle_{L^2} = \langle \phi_f, (\Delta_g + I)^{-s} \varphi \rangle_{L^2} = \langle f, (\Delta_g + I)^{-s} \varphi \rangle_{H^s} \\ |\langle \phi_f, \varphi \rangle_{H^{-s}}| &\leq \|f\|_{H^s} \|(\Delta_g + I)^{-s} \varphi\|_{H^s} = \|f\|_{H^s} \|\varphi\|_{H^{-s}}. \end{aligned}$$

To see the isomorphism, consider the pairing $C_0^\infty(M) \times C_0^\infty(M) \to \mathbb{C}$ given by:

$$(f, \varphi) := \langle f, \varphi \rangle_{L^2} = \int_M f\varphi \, dA_g.$$

This pairing has a continuous extension to $H_0^{-s} \times H_0^s \to \mathbb{C}$, since for $f, \varphi \in C_0^\infty$ we have:

$$(f, \varphi) = \langle f, \varphi \rangle_{L^2} = \langle (\Delta_g + I)^{-s/2} f, (\Delta_g + I)^{s/2} \varphi \rangle_{L^2}.$$

Let $f \in H_0^{-s}$ and let $\varphi \in C_0^\infty(M)$. Then:

$$\begin{aligned} |\langle f, \varphi \rangle_{L^2}| &= |\langle (\Delta_g + I)^{-s/2} f, (\Delta_g + I)^{s/2} \varphi \rangle_{L^2}| \leq \|(\Delta_g + I)^{-s/2} f\|_{L^2} \|(\Delta_g + I)^{s/2} \varphi\|_{L^2} \\ &= \|f\|_{H^{-s}} \|\varphi\|_{H^s}. \end{aligned}$$

If $s > 0$ we have that $H^s \subset L^2 \subset H^{-s}$. The operator $\Delta_g : H^2(M) \to L^2(M)$ is naturally continuous.

Lemma A.1. *The operators $\Delta_g + I : H_0^1(M) \to H_0^{-1}(M)$ and $(\Delta_g + I)^{-1} : H_0^{-1}(M) \to H_0^1(M)$ are continuous.*

Proof. Let us start with $\Delta_g + I$. Let $f \in C_0^\infty(M)$. Then

$$\|(\Delta_g + I)f\|_{H^{-1}} = \|(\Delta_g + I)^{-1/2}(\Delta_g + I)f\|_{L^2} = \|(\Delta_g + I)^{1/2} f\|_{L^2} = \|f\|_{H^1}$$

Now let $f \in H_0^1(M)$. Then there exists $\{f_k\} \subset C_0^\infty(M)$ so that $f_k \longrightarrow f$ in $H^1(M)$, that is,

$$\|f_k - f\|_{H^1(M)} = \|(\Delta_g + I)^{1/2}(f_k - f)\|_{L^2} \to 0, \text{ as } k \to \infty,$$

in particular $f_k \to f$ in L^2.

On the other hand, $\|(\Delta_g+I)(f_k - f_j)\|_{H^{-1}} = \|f_k - f_j\|_{H^1} \to 0$ as $k, j \to \infty$ therefore $\{(\Delta_g+I)f_k\}$ it is a Cauchy sequence in $H^{-1}(M)$ and there exists $\psi \in H^{-1}(M)$ so that $(\Delta_g + I)f_k \to \psi$ in $H^{-1}(M)$, thus $\|(\Delta_g + I)f_k - \psi\|_{H^{-1}} \to 0$. We just have to prove that $(\Delta_g + I)f = \psi$ in $H^{-1}(M)$, i.e. in the distributional sense. Let $\varphi \in C_0^\infty(M)$:

$$\langle(\Delta_g + I)f, \varphi\rangle = \langle f, (\Delta_g + I)\varphi\rangle = \lim_{k\to\infty}\langle f_k, (\Delta_g + I)\varphi\rangle = \lim_{k\to\infty}\langle(\Delta_g + I)f_k, \varphi\rangle = \langle\psi, \varphi\rangle.$$

Thus, $\Delta_g + I : H_0^1(M) \to H_0^{-1}(M)$ is continuous.

To prove continuity of $(\Delta_g + I)^{-1} : H_0^{-1}(M) \to H_0^1(M)$ we proceed almost in the same way as in the previous case. Let $f \in C_0^\infty(M)$. Then

$$\|(\Delta_g + I)^{-1}f\|_{H^1} = \|(\Delta_g + I)^{1/2}(\Delta_g + I)^{-1}f\|_{L^2} = \|(\Delta_g + I)^{-1/2}f\|_{L^2} = \|f\|_{H^{-1}}.$$

Now let $f \in H_0^{-1}(M)$. Then there exists $\{f_k\} \subset C_0^\infty(M)$ so that $f_k \to f$ in $H^{-1}(M)$. That is,

$$\|f_k - f\|_{H^{-1}} = \|(\Delta_g + I)^{-1/2}(f_k - f)\|_{L^2} \to 0, \text{ as } k \to \infty.$$

In the same way as above, we have that $\{(\Delta_g + I)^{-1}f_k\}$ is a Cauchy sequence in $H^1(M)$ and there exists $\psi \in H^1(M)$ so that $(\Delta_g + I)f_k \to \psi$ in $H^1(M)$, thus

$$\|(\Delta_g + I)^{-1}f_k - \psi\|_{H^1} = \|(\Delta_g + I)^{-1/2}f_k - (\Delta_g + I)^{1/2}\psi\|_0 \to 0.$$

As before, we have to prove that $(\Delta_g + I)^{-1}f = g$ in $H^1(M)$. To do this, we proceed a little bit differently from the previous case. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ so that for all $k > N$

$$\begin{aligned}
\|(\Delta_g + I)^{-1}f - \psi\|_{H^1} &= \|(\Delta_g + I)^{-1/2}f - (\Delta_g + I)^{1/2}\psi\|_{L^2} \\
&\le \|(\Delta_g + I)^{-1/2}(f - f_k)\|_{L^2} + \|(\Delta_g + I)^{-1/2}f_k - (\Delta_g + I)^{1/2}\psi\|_{L^2} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus $(\Delta_g + I)^{-1}f = \psi$ and $(\Delta_g + I)^{-1} : H_0^{-1}(M) \to H_0^1(M)$ is continuous. $\square$

Appendix B

Spectral shift functions

In this part, we compute the spectral shift function associated to the Laplacian on the surface with cusps and the Dirichlet Laplacian on the cusps.

The general theory of spectral shift functions associates a function of a real variable to a pair of operators satisfying certain conditions. Let us mention two important aspects of spectral shift functions. The first is a trace formula. An important part of the theory is devoted to understanding what conditions on a pair of operators (A, B) and a function φ make the formula

$$\operatorname{Tr}(\varphi(A) - \varphi(B)) = \int_{\mathbb{R}} \varphi'(\lambda)\xi(\lambda)d\lambda$$

hold. The second aspect of the theory that we want to mention here is the relationship of spectral shift functions to the scattering matrix $S(\lambda)$ that is given by the equation:

$$\det S(\lambda) = e^{-2\pi i \xi(\lambda)}$$

for almost every $\lambda \in \mathbb{R}$.

For the details, we refer to [3] [4] and [30]. Let H and H_0 be two self-adjoint operators acting on a separable Hilbert space $\mathcal{H}$ such that their difference $H - H_0$ is trace class. For $z \in \rho(H_0)$, the perturbation determinant is defined as

$$\Delta_{H/H_0}(z) = \det(I + (H - H_0)R_z(H(0))) = \det((H - z)(H_0 - z)^{-1}).$$

It is analytic in both half planes $\operatorname{Im}(z) > 0$ and $\operatorname{Im}(z) < 0$. The trace class property of $(H - H_0)R_z(H(0))$ implies that $\Delta_{H/H_0}(z) \to 1$ as $|\operatorname{Im}(z)| \to \infty$. Let $\varsigma(z) := \log \Delta_{H/H_0}(z) = \int_{\mathbb{R}} \frac{\xi(t)}{t-z} dt$, for $\operatorname{Im}(z) \neq 0$. The branch of the logarithm is fixed by the condition $\varsigma(z) \to 0$ as $\operatorname{Im}(z) \to \infty$. After analyzing the function $\varsigma(z)$ close to the real line one obtains that:

$$\xi(\lambda) = \xi(\lambda; H, H_0) = \pi^{-1} \lim_{\varepsilon \to 0^+} \arg \Delta_{H/H_0}(\lambda + i\varepsilon)$$

for almost every $\lambda \in \mathbb{R}$. The spectral shift function $\xi(\lambda)$ is real valued, belongs to $L^1(\mathbb{R})$ and satisfies

$$\operatorname{Tr}(H - H_0) = \int_{\mathbb{R}} \xi(\lambda)d\lambda, \qquad \|\xi\|_1 \leq \|H - H_0\|_1.$$

There is an invariance principle for spectral shift functions (named after the invariance principle in Scattering theory). This principle relates the spectral shift function associated to a pair $(F(H), F(H_0))$, for a suitable function F, with the one associated to a the pair (H, H_0), see [4]:

$$\xi(\lambda; H, H_0) = \epsilon\, \xi(F(\lambda); F(H), F(H_0)), \qquad \epsilon = \mathrm{sgn} F',$$

where the spectral shift function on the left-hand side of the previous equation is obviously integrable only with a suitable weight. To finish this mini introduction let us include Proposition 2.1 in [30]:

Proposition B.1. *([30]) Let H, H_0 be two nonnegative self-adjoint operators in $\mathcal{H}$ and assume that $e^{-tH} - e^{-tH_0}$ is a trace class operator for $t > 0$. Then there exists a unique real valued locally integrable function $\xi(\lambda) = \xi(\lambda; H, H_0)$ on $\mathbb{R}$ such that for each $t > 0$, $e^{-t\lambda}\xi(\lambda) \in L^1(\mathbb{R})$ and the following conditions hold:*

1. $\mathrm{Tr}(e^{-tH} - e^{-tH_0}) = -t \int_0^\infty e^{-t\lambda}\xi(\lambda) d\lambda$.
2. *For every $\varphi \in \{f : \mathbb{R} \to \mathbb{R} \,|\, f \in L^1$ and $\int_{\mathbb{R}} |\hat{f}(p)|(1 + |p|)dp < \infty\}$, $\varphi(H) - \varphi(H_0)$ is a trace class operator and*
$$\mathrm{Tr}(\varphi(H) - \varphi(H_0)) = \int_{\mathbb{R}} \varphi'(\lambda)\xi(\lambda)d\lambda.$$
3. $\xi(\lambda) = 0$ *for* $\lambda < 0$.

B.1 Spectral shift function for a surface with cusps

Now let (M, g) be a surface with cusps as described in Chapter 1, where $Z_{a_j} \simeq [a_j, \infty) \times S^1$ with $a_j > 0$, $j = 1, \dots, m$. Let us consider the pair of self-adjoint operators $(\Delta_g, \Delta_{0,D})$, where Δ_g is the Laplacian on the surface with cusps and $\Delta_{0,D}$ denotes the operator $\oplus_{j=1}^m \Delta_{Z_j,D}$, where $\Delta_{Z_j,D}$ is the self-adjoint extension of $-y_j^2 \left(\frac{\partial^2}{\partial y_j^2} + \frac{\partial^2}{\partial x^2}\right) : C_c^\infty(Z_{a_j}) \to L^2(Z_{a_j}, y_j^{-2} dy\, dx)$, with respect to Dirichlet boundary conditions at $\{a_j\} \times S^1$. In what follows we compute $\xi(\lambda; \Delta_g, \Delta_{0,D})$.

Let us start decomposing the spectral shift function into its discrete and continuous parts:

$$\xi(\lambda, \Delta_g, \Delta_{0,D}) = \xi_d(\lambda) + \xi_c(\lambda).$$

For the discrete part, remember that we can decompose the operator $\Delta_{0,D}$ as $\Delta_{0,D} = \oplus_{j=1}^m (\Delta_{a_j,0} \oplus \Delta_{Z_j,1})$ where the operators $\Delta_{Z_j,1}$ have only point spectrum. Therefore we have that

$$\xi_d(\lambda) = -N_g(\lambda) + \sum_{j=1}^m N_j(\lambda),$$

where $N_g(\lambda) = \sum_{\lambda_i \le \lambda} 1$ and $\lambda_0 = 0 < \lambda_1 \le \lambda_2 \le \dots$ are the eigenvalues of Δ_g and $N_j(\lambda)$ is the counting function corresponding to $\Delta_{Z_j,1}$.

Now let us proceed with the continuous part. Let Δ_{ac} denote the absolute continuous part of Δ_g and let $\bar{\Delta}_{a,0}$ denote $\oplus_{j=1}^m \Delta_{a_j,0}$. We know that for a suitable class of functions; we have that

$$\mathrm{Tr}(\varphi(\Delta_{ac}) - \varphi(\bar{\Delta}_{a,0})) = \int_{\mathbb{R}} \varphi'(\lambda)\xi_c(\lambda)d\lambda. \tag{B.1}$$

Let $Y \gg \max\{a_j\}$, let $Z_{j,Y} = [Y, \infty)$ and let $M_Y := M \setminus \bigcup_{j=1}^m Z_{j,Y}$. Then we have

$$\mathrm{Tr}(\varphi(\Delta_{ac}) - \varphi(\bar{\Delta}_{a,0})) = \lim_{Y\to\infty} \int_{M_Y} K_{c,\varphi}(z,z)\ dA_g(z) - \int_{M_Y} K_{0,\varphi}(z,z)\ dA_g(z), \tag{B.2}$$

where $K_{c,\varphi}(z,z')$ and $K_{0,\varphi}(z,z')$ are the kernels of the operators $\varphi(\Delta_{ac})$ and $\varphi(\bar{\Delta}_{a,0})$, respectively. We use equations (B.1) and (B.2) to compute the spectral shift function explicitly. Using the spectral decomposition of the Laplacian we get:

$$K_{c,\varphi}(z,z') = \frac{1}{4\pi}\int_{-\infty}^{\infty} \varphi(\frac{1}{4}+\lambda^2)\ \sum_{j=1}^m E^j(z,\frac{1}{2}+i\lambda)E^j(z',\frac{1}{2}-i\lambda)d\lambda,$$
$$K_{0,\varphi}(z,z') = \frac{1}{4\pi}\int_{-\infty}^{\infty} \varphi(\frac{1}{4}+\lambda^2)\sum_{j=1}^m e^j(y,\frac{1}{2}+i\lambda)e^j(y',\frac{1}{2}-i\lambda)d\lambda,$$

where E^j is the Eisenstein series associated with cusp Z_{a_j} and e^j is a function that satisfies the following equations in the cusp Z_{a_j}:

$$\begin{aligned} -y^2\frac{d^2}{dy^2}e^j(y,s) &= s(1-s)e^j(y,s),\\ e^j(a_j,s) &= 0,\\ e^j(y,s) &= y^s + \psi(y,s), \end{aligned}$$

with $\psi(y,s) \in L^2(Z_{a_j})$, for $\mathrm{Re}(s) > \frac{1}{2}$. The solution to this problem is:

$$e^j(y,s) = y^s - a_j^{2s-1}y^{1-s}.$$

Integrating this function over the corresponding truncated cusp we obtain:

$$\int_{a_j}^Y |e^j(y,\frac{1}{2}+i\lambda)|^2 dA_g(y) = \int_{a_j}^Y (2y - a_j^{2i\lambda}y^{1-2i\lambda} - a_j^{-2i\lambda}y^{1+2i\lambda})\frac{dy}{y^2}$$
$$= \left\{2\log(y) + \frac{a_j^{2i\lambda}}{2i\lambda}y^{-2i\lambda} - \frac{a_j^{-2i\lambda}}{2i\lambda}y^{2i\lambda}\right\}\Bigg|_{a_j}^Y = 2\log(Y) + \frac{a_j^{2i\lambda}}{2i\lambda}Y^{-2i\lambda} - \frac{a_j^{-2i\lambda}}{2i\lambda}Y^{2i\lambda} - 2\log(a_j).$$

Now we want to compute the corresponding integral for the Eisenstein series associated to the cusp Z_{a_j}. In order to do this, define $\widetilde{E}^j_Y$ by:

$$\widetilde{E}^j_Y(z,s) = \begin{cases} E^j(z,s) & \text{if } z \in M \setminus \bigcup_{j=1}^m Z_{j,Y},\\ E^j(z,s) - \delta_{ij}y_i^s - C_{ij}(s)y_i^{1-s} & \text{if } z \in Z_{i,Y}. \end{cases}$$

From Proposition 7.13 in [28], it follows that for $0 < \mathrm{Re}(s) < 1$,

$$\int_{M_Y} E^i(z,s)E^j(z,s')dA_g(z) = \langle \widetilde{E}^i(z,s), \widetilde{E}^j(z,s')\rangle + O(e^{-cY})$$

as $Y \to \infty$. Now let us use Lemma 7.23 in [28]:

$$\langle \widetilde{E}^i(z, \frac{1}{2} + i\lambda), \widetilde{E}^j(z, \frac{1}{2} + i\lambda') \rangle = \delta_{ij} \frac{Y^{i(\lambda-\lambda')} - Y^{-i(\lambda-\lambda')}}{i(\lambda - \lambda')}$$
$$+ \; Y^{-i(\lambda-\lambda')} \sum_{k=1}^{m} C_{ik}(\frac{1}{2} + i\lambda) \frac{C_{kj}(\frac{1}{2} - i\lambda) - C_{kj}(\frac{1}{2} - i\lambda')}{i(\lambda - \lambda')}$$
$$+ \; \frac{1}{i(\lambda + \lambda')} (Y^{i(\lambda+\lambda')} C_{ji}(\frac{1}{2} - i\lambda') - Y^{-2i(\lambda-\lambda')} C_{ij}(\frac{1}{2} + i\lambda))$$

and let $\lambda' \to \lambda$ to obtain

$$\langle \widetilde{E}^i(z, \frac{1}{2} + i\lambda), \widetilde{E}^j(z, \frac{1}{2} + i\lambda) \rangle = \delta_{ij} 2\log(Y) + \; \left(C(\frac{1}{2} + i\lambda) \frac{d}{ds} C(\frac{1}{2} - i\lambda) \right)_{i,j}$$
$$+ \; \frac{1}{2i\lambda} (Y^{2i\lambda} C_{ji}(\frac{1}{2} - i\lambda) - Y^{-2i\lambda} C_{ij}(\frac{1}{2} + i\lambda)).$$

In this way we obtain that:

$$\begin{aligned}\int_{M_Y} |E(z,s)|^2 dA_g(z) &= \sum_{j=1}^{m} \int_{M_Y} |E^j(z,s)|^2 dA_g(z) \\ &= 2m \log(Y) + \operatorname{Tr}\left(C(\frac{1}{2} + i\lambda) \frac{d}{ds} C(\frac{1}{2} - i\lambda) \right) \\ &\quad + \; \frac{1}{2i\lambda} \left(Y^{2i\lambda} \operatorname{Tr}(C(\frac{1}{2} - i\lambda)) - Y^{-2i\lambda} \operatorname{Tr}(C(\frac{1}{2} + i\lambda)) \right) + O(e^{-cY}).\end{aligned}$$

Put everything together for equation (B.2); we have

$$\begin{aligned}&\int_{M_Y} K_{c,\varphi}(z,z) dA_g(z) - \int_{M_Y} K_{0,\varphi}(z,z) dA_g(z) \\ &\quad = \int_{M_Y} \frac{1}{4\pi} \int_{\mathbb{R}} \varphi\left(\frac{1}{4} + \lambda^2\right) \left(\sum_{j=1}^{m} |E^j(z, \frac{1}{2} + i\lambda)|^2 - \sum_{j=1}^{m} |e^j(z, \frac{1}{2} + i\lambda)|^2 \right) dA_g(z) \\ &\quad = \frac{1}{4\pi} \int_{\mathbb{R}} \varphi\left(\frac{1}{4} + \lambda^2\right) \{2m\log(Y) + \operatorname{Tr}\left(C(\frac{1}{2} + i\lambda) \frac{d}{ds} C(\frac{1}{2} - i\lambda) \right) \\ &\qquad + \frac{1}{2i\lambda} \left(Y^{2i\lambda} \operatorname{Tr}(C(\frac{1}{2} - i\lambda)) - Y^{-2i\lambda} \operatorname{Tr}(C(\frac{1}{2} + i\lambda)) \right) - 2m\log(Y) \\ &\qquad - \sum_{j=1}^{m} \left\{ \frac{1}{2i\lambda} (a_j^{2i\lambda} Y^{-2i\lambda} - a_j^{-2i\lambda} Y^{2i\lambda}) - 2\log(a_j) \right\} + O(e^{-cY}) \} d\lambda \\ &= \frac{1}{4\pi} \int_{\mathbb{R}} \varphi\left(\frac{1}{4} + \lambda^2\right) \left\{ \operatorname{Tr}\left(C(\frac{1}{2} + i\lambda) \frac{d}{ds} C(\frac{1}{2} - i\lambda) \right) + \frac{1}{2i\lambda} Y^{2i\lambda} (\operatorname{Tr}(C(\frac{1}{2} - i\lambda)) + \sum_{j=1}^{m} a_j^{-2i\lambda}) \right. \\ &\qquad \left. - \frac{1}{2i\lambda} Y^{-2i\lambda} (\operatorname{Tr}(C(\frac{1}{2} + i\lambda)) + \sum_{j=1}^{m} a_j^{2i\lambda}) + \sum_{j=1}^{m} 2\log(a_j) + O(e^{-cY}) \right\} d\lambda.\end{aligned}$$

Hence we obtain:

$$\begin{aligned}
&\mathrm{Tr}(\varphi(\Delta_{ac})-\varphi(\bar{\Delta}_{a,0}))\\
&\quad=\frac{1}{4\pi}\int_{\mathbb{R}}\varphi\left(\frac{1}{4}+\lambda^2\right)\lim_{Y\to\infty}\left\{\mathrm{Tr}\left(C(\frac{1}{2}+i\lambda)\frac{d}{ds}C(\frac{1}{2}-i\lambda)\right)+\sum_{j=1}^{m}2\log(a_j)\right.\\
&\qquad\left.+\frac{1}{2i\lambda}Y^{2i\lambda}(\mathrm{Tr}(C(\frac{1}{2}-i\lambda))+\sum_{j=1}^{m}a_j^{-2i\lambda})\quad-\frac{1}{2i\lambda}Y^{-2i\lambda}(\mathrm{Tr}(C(\frac{1}{2}+i\lambda))+\sum_{j=1}^{m}a_j^{2i\lambda})+O(e^{-cY})\right\}d\lambda\\
&\quad=\frac{1}{4\pi}\int_{\mathbb{R}}\varphi\left(\frac{1}{4}+\lambda^2\right)\left\{\mathrm{Tr}\left(C(\frac{1}{2}+i\lambda)\frac{d}{ds}C(\frac{1}{2}-i\lambda)\right)+\sum_{j=1}^{m}2\log(a_j)\right\}d\lambda\\
&\qquad+\lim_{Y\to\infty}\frac{1}{4\pi}\int_{\mathbb{R}}\varphi\left(\frac{1}{4}+\lambda^2\right)\left\{\frac{1}{2i\lambda}Y^{2i\lambda}(\mathrm{Tr}(C(\frac{1}{2}-i\lambda))+\sum_{j=1}^{m}a_j^{-2i\lambda})\right.\\
&\qquad\qquad\left.-\frac{1}{2i\lambda}Y^{-2i\lambda}(\mathrm{Tr}(C(\frac{1}{2}+i\lambda))+\sum_{j=1}^{m}a_j^{2i\lambda})\right\}d\lambda.
\end{aligned}$$

Write $Y^{2i\lambda}=\cos(2\lambda\log Y)+i\sin(2\lambda\log Y)$. Then the last integral is the difference of the two following terms

$$\frac{1}{4\pi}\int_{\mathbb{R}}\varphi\left(\frac{1}{4}+\lambda^2\right)(\mathrm{Tr}(C(\frac{1}{2}-i\lambda))+\mathrm{Tr}(C(\frac{1}{2}+i\lambda))+\sum_{j=1}^{m}a_j^{2i\lambda}+a_j^{-2i\lambda})\frac{\sin(2\lambda\log Y)}{2\lambda}d\lambda,\quad\text{and}$$

$$\frac{1}{4i\pi}\int_{\mathbb{R}}\varphi\left(\frac{1}{4}+\lambda^2\right)\frac{\mathrm{Tr}(C(\frac{1}{2}+i\lambda))-\mathrm{Tr}(C(\frac{1}{2}-i\lambda))+\sum_{j=1}^{m}a_j^{2i\lambda}-a_j^{-2i\lambda}}{2\lambda}\cos(2\lambda\log Y)\ d\lambda.$$

Taking the limit as $Y\to\infty$ we obtain:

$$\begin{aligned}
\mathrm{Tr}(\varphi(\Delta_{ac})-\varphi(\bar{\Delta}_{a,0}))=\frac{1}{4\pi}\int_{\mathbb{R}}\varphi\left(\frac{1}{4}+\lambda^2\right)\left\{\mathrm{Tr}\left(C(\frac{1}{2}+i\lambda)\frac{d}{ds}C(\frac{1}{2})-i\lambda)\right)+\sum_{j=1}^{m}2\log(a_j)\right\}d\lambda\\
+\varphi\left(\frac{1}{4}\right)\frac{\mathrm{Tr}(C(\frac{1}{2}))+m}{4}.
\end{aligned}$$

Now we are ready to compute the spectral shift function. We first consider $\varphi\in C_c^\infty(\mathbb{R})$ such that $\varphi(1/4)=0$. Then the previous equation becomes:

$$\mathrm{Tr}(\varphi(\Delta_{ac})-\varphi(\bar{\Delta}_{a,0}))=\frac{1}{4\pi}\int_{\mathbb{R}}\varphi\left(\frac{1}{4}+\lambda^2\right)\left\{\mathrm{Tr}\left(C(\frac{1}{2}+i\lambda)\frac{d}{ds}C(\frac{1}{2})-i\lambda)\right)+\sum_{j=1}^{m}2\log(a_j)\right\}d\lambda.$$

Now let us use the fact that the functions $\varphi\left(\frac{1}{4}+\lambda^2\right)$, $\mathrm{Tr}\left(C(\frac{1}{2}+i\lambda)\frac{d}{ds}C(\frac{1}{2}-i\lambda)\right)$ and the constants are even, thus so it is the integrand. Therefore we obtain:

$$\mathrm{Tr}(\varphi(\Delta_{ac})-\varphi(\bar{\Delta}_{a,0}))=\frac{1}{2\pi}\int_0^\infty\varphi\left(\frac{1}{4}+\lambda^2\right)\left\{\mathrm{Tr}\left(C(\frac{1}{2}+i\lambda)\frac{d}{ds}C(\frac{1}{2})-i\lambda)\right)+\sum_{j=1}^{m}2\log(a_j)\right\}d\lambda.$$

Let us also call $\rho(\lambda) = \mathrm{Tr}\left(C(\frac{1}{2}+i\lambda)\frac{d}{ds}C(\frac{1}{2}-i\lambda)\right) + \sum_{j=1}^{m} 2\log(a_j)$. Then the change of variables $\lambda' = 1/4 + \lambda^2$ and integration by parts give:

$$\mathrm{Tr}(\varphi(\Delta_{ac}) - \varphi(\bar{\Delta}_{a,0})) = \frac{1}{4\pi}\int_{1/4}^{\infty} \varphi(\lambda')\frac{\rho(\sqrt{\lambda'-1/4})}{\sqrt{\lambda'-1/4}}d\lambda' = -\frac{1}{4\pi}\int_{1/4}^{\infty}\varphi'(\lambda')\int_{1/4}^{\lambda'}\frac{g(\sqrt{\widetilde{\lambda}-1/4})}{\sqrt{\widetilde{\lambda}-1/4}}\,d\widetilde{\lambda}\,d\lambda',$$

where we used $\frac{d}{d\lambda'}\int_{1/4}^{\lambda'}\frac{\rho(\sqrt{\widetilde{\lambda}-1/4})}{\sqrt{\widetilde{\lambda}-1/4}}d\widetilde{\lambda} = \frac{\rho(\sqrt{\lambda'-1/4})}{\sqrt{\lambda'-1/4}}$.

In this way, we have for the spectral shift function that:

$$\widetilde{\xi_c}(\lambda') := \begin{cases} -\frac{1}{4\pi}\int_{1/4}^{\lambda'}\frac{\rho(\sqrt{\widetilde{\lambda}-1/4})}{\sqrt{\widetilde{\lambda}-1/4}}\,d\widetilde{\lambda} - \frac{(\mathrm{Tr}(C(\frac{1}{2}))+m)}{4} & \text{if } \lambda' \geq \frac{1}{4} \\ 0 & \text{if } \lambda' < \frac{1}{4}. \end{cases}$$

Now let $\varphi \in C_c^\infty(\mathbb{R})$. Then

$$\begin{aligned} \int_{1/4}^{\infty}\varphi'(\lambda')\widetilde{\xi_c}(\lambda')d\lambda' &= \varphi(\lambda')\widetilde{\xi_c}(\lambda')\Big|_{\lambda'=1/4}^{\infty} + \frac{1}{4\pi}\int_{1/4}^{\infty}\varphi(\lambda')\frac{g(\sqrt{\lambda'-1/4})}{\sqrt{\lambda'-1/4}}d\lambda' \\ &= \varphi(1/4)\frac{(\mathrm{Tr}(C(\frac{1}{2}))+m)}{4} + \frac{1}{4\pi}\int_{1/4}^{\infty}\varphi(\lambda')\frac{g(\sqrt{\lambda'-1/4})}{\sqrt{\lambda'-1/4}}d\lambda'. \end{aligned}$$

In the last integral, make the inverse substitution $\lambda = \sqrt{\lambda'-1/4}$ to get:

$$\begin{aligned} \int_{1/4}^{\infty}\varphi'(\lambda')\widetilde{\xi_c}(\lambda')d\lambda' &= \varphi(1/4)\frac{(\mathrm{Tr}(C(\frac{1}{2}))+m)}{4} + \frac{1}{2\pi}\int_0^\infty \varphi(1/4+\lambda^2)\rho(\lambda)\,d\lambda \\ &= \mathrm{Tr}(\varphi(\Delta_{ac}) - \varphi(\bar{\Delta}_{a,0})) = \int_{\mathbb{R}}\varphi'(\lambda')\xi_c(\lambda')\,d\lambda'. \end{aligned}$$

Since $\widetilde{\xi_c}(\lambda') = 0$ for $\lambda' < 1/4$, for any $\varphi \in C_c^\infty(\mathbb{R})$ we have that

$$\int_{\mathbb{R}}\frac{d\varphi}{d\lambda'}(\lambda')(\widetilde{\xi_c}(\lambda') - \xi_c(\lambda'))d\lambda' = 0.$$

By ellipticity of $\frac{d}{d\lambda'}$, it follows that $\widetilde{\xi_c} - \xi_c \in C^\infty(\mathbb{R})$ and

$$\frac{d}{d\lambda'}(\widetilde{\xi_c}(\lambda') - \xi_c(\lambda')) = 0,$$

thus $\widetilde{\xi_c}(\lambda') - \xi_c(\lambda') = c$, a constant function. Recall that $\xi_c(\lambda')$ is the absolutely continuous part of the spectral shift function $\xi(\lambda;\Delta,\Delta_0)$ and $\sigma_{ac}(\Delta) = \sigma_{ac}(\bar{\Delta}_{a,0}) = [1/4,\infty)$. It follows from the properties of the spectral shift function that $\xi_c(\lambda') = 0$ for $\lambda' < 1/4$. Since $\widetilde{\xi_c}(\lambda') = 0$, for $\lambda' < 1/4$ we have that $c = 0$, therefore $\xi_c(\lambda') = \widetilde{\xi_c}(\lambda')$. Finally we obtain:

$$\xi_c(\lambda) = -\frac{1}{2\pi}\int_0^{\sqrt{\lambda-1/4}}\mathrm{Tr}\left(C(\frac{1}{2}+i\widetilde{\lambda})\frac{d}{ds}C(\frac{1}{2}-i\widetilde{\lambda})\right)\,d\widetilde{\lambda} - \frac{(\mathrm{Tr}(C(\frac{1}{2}))+m)}{4} - \frac{\sqrt{\lambda-1/4}}{2\pi}\sum_{j=1}^{m}\log a_j,$$

if $\lambda \geq \frac{1}{4}$, and otherwise $\xi_c(\lambda) = 0$.

B.2 Spectral shift function for $(\Delta, \Delta_{1,0})$

Now consider the Laplacian $\Delta_{1,0}$ that is the self adjoint extension of the direct sum of

$$-y_j^2 \frac{\partial^2}{\partial y_j^2} : C_0^\infty([1,\infty)) \to L^2([1,\infty), \frac{dy_j}{y_j^2})$$

obtained by imposing Dirichlet boundary conditions at $y_j = 1$, for all $1 \leq j \leq m$. In this case the kernel of $\varphi(\Delta_{1,0})$ is

$$\hat{K}_{1,0,\varphi}(z,z') = \frac{1}{4\pi} \int_{-\infty}^{\infty} \varphi(\frac{1}{4} + \lambda^2) \sum_{j=1}^{m} \hat{e}^j(y, \frac{1}{2} + i\lambda) \hat{e}^j(y', \frac{1}{2} - i\lambda) d\lambda,$$

where $\hat{e}^j$ satisfy the following equations on $[1,\infty)$:

$$\begin{aligned} -y^2 \frac{d^2}{dy^2} \hat{e}^j(y,s) &= s(1-s)\hat{e}^j(y,s), \\ \hat{e}^j(1,s) &= 0, \\ e^j(y,s) &= y^s + \varphi(y,s). \end{aligned}$$

Then in the previous proof the only changes are that $a_j = 1$, for $1 \leq j \leq m$, and that the term involving $\log(a_j)$ becomes null. Therefore we obtain:

$$\xi_c(\lambda; \Delta, \Delta_{1,0}) := \begin{cases} -\frac{1}{2\pi} \int_0^{\sqrt{\lambda - 1/4}} \mathrm{Tr}\left(C(\frac{1}{2} + i\tilde{\lambda}) \frac{d}{ds} C(\frac{1}{2} - i\tilde{\lambda}) \right) d\tilde{\lambda} - \frac{(\mathrm{Tr}(C(\frac{1}{2})) + m)}{4} & \text{if } \lambda \geq \frac{1}{4} \\ 0 & \text{if } \lambda < \frac{1}{4}. \end{cases}$$

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