Optimal transport and geometric analysis in Heisenberg groups

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Introduction

This thesis is located at the interface between analysis, differential geometry and probability theory just after recent developments including optimal transport as a significant tool in the study of metric measure spaces. We will especially examine the subRiemannian Heisenberg group $\mathbb{H}_n$ for n a positive integer and report new results in this area.

The generalization of geometric analysis theorems – such as functional inequalities – from the Euclidean or Riemannian setting to metric spaces is an ambitious program. A lot of people with different mathematical background working on this generalization. Consequently, the type of metric spaces they consider differ a lot because there are very few things that can be done without special assumptions. In metric geometry the considered spaces are often geodesic spaces because they partially recover the structure of the Riemannian manifolds. Geometers consider specific versions of them as CAT spaces, Alexandrov spaces or δ-hyperbolic spaces (see [21, 51]). Another example of metric spaces are the ones satisfying a weak $(1, 1)$-local Poincaré inequality for a measure that is assumed to be doubling. These spaces are not necessarily geodesic but they contain "a lot of curves". They provide a minimal setting for quasiconformal geometry [55, 57], present a first-order calculus [102, 57] as well as Sobolev spaces and even have their own differential structures [24]. A last example are the countably rectifiable metric spaces, initially defined by Federer in [39] and better understood since [67] and [5]. Although progresses on the theory of abstract metric spaces are interesting on their own, it is an essential problem to recognize examples of metric spaces satisfying these theories. It is one of the goals of this thesis to classify the position of the Heisenberg groups $\mathbb{H}_n$ with respect to the recent theories of Lott, Sturm and Villani. These authors used optimal transport in order to define a second-order calculus on metric spaces. They established a definition and a theory of metric measure spaces with a "lower Ricci curvature bound".

The wide class of metric spaces that is targeted in this thesis are the subRiemannian manifolds (see [84]) and particularly the Carnot groups (stratified nilpotent Lie group) with the Carnot-Carathéodory distance (see [52, 56]). The last ones present a rich structure with dilations and invariance under translation. However, in this thesis we will restrict the study to the Heisenberg group (and some related spaces) that is in some sense the easiest Carnot group. The study of its particular geometry gives an insight of possible behaviors for the other Carnot groups. But it is not certain that our results hold in greater generality. Indeed, our approach relies too much on the knowledge of the geodesics of $\mathbb{H}_n$ while currently the geodesics in subRiemannian geometry (also of Carnot group) are really problematic and bad-known. In a famous paper of 1995, Montgomery [82] (see also [84, 83]) proved that the geodesics (meaning the curves of shortest length) of a subRiemannian manifold can be *abnormal* geodesics: not every geodesic of a subRiemannian manifold is a *normal* Pontryagin extremal as it was thought (and even stated) before.

We would like to review some of the numerous perspectives on the Carnot and Heisenberg groups. The Carnot groups have first been studied with respect to their subelliptic operator. The Lie algebra of Carnot groups have a graded basis and the Lie algebra is Lie bracket-generated by vectors fields on the first grade. This condition on manifolds with vector fields is known as Hörmander condition after his famous paper [58] of 1967. Hörmander proved that under this condition the subelliptic operator $\Delta_G = \sum_{i=1}^{k} X_i^2$ is hypoelliptic if the vector fields $(X_1, \ldots, X_k)$ and finitely many bracket-generated vector fields span the whole tangent space in each point. Hypoellipticity means for Δ_G that if g is a smooth function, f solving $\Delta_G f = g$ is smooth as well. This result aroused of great interest among the mathematical community. For instance, new proofs of this theorem created new perspectives: Kohn [68] used pseudodifferential operators and Malliavin [79] inaugurated what is now called Malliavin calculus (see also [87]). Starting from the Hörmander theorem, the subelliptic operator Δ_G (called Kohn operator for the Heisenberg group) has been considered in term of evolution equations and harmonic analysis as the natural replacement of the Laplace operator for Carnot groups. The book of Folland and Stein [45] proposed to study Hardy spaces in this setting recovering the classical theorems on Hardy spaces. The program was continued by Rothschild and Stein [95]. Jerison [60] proved the local Poincaré inequality for $\mathbb{R}^n$ with bracket generating vector fields (see also [106]). In his paper as in [95], the Carnot groups play the role of local approximating models of the general spaces. The last chapters of Stein's book [103] give a nice overview on these developments.

After Malliavin [79] worked on the general case of vector fields with the Hörmander condition, Gaveau [49] studied the subelliptic diffusion in the Heisenberg group with stochastic methods. He obtained an explicit expression for the density of the fundamental solutions and for the solutions of the equation using a computation of the Lévy area (see [112]). Furthermore, he developed estimates for these functions. Other estimates appear later as in [13]. A stochastic treatment of the subelliptic diffusion can also be found in [31] where a central limit theorem for Carnot groups is proved (see also [53] where the theorem is proved for dynamic random walks).

Another trend on Carnot groups is represented by the seminal paper of Gromov "Carnot-Carathatéodory spaces seen from within" [52] where the author presents geometric ideas in the intrinsic point of view. In this approach one does not deduce results from the definition with vector fields of the natural distance but from the distance itself via its own metric properties. One can suppose that it was the kind of philosophy adopted previously by Pansu in another very important paper [93]. In this paper it is proved that every quasi-isometry of quaternionic or the Cayley hyperbolic spaces has bounded distance from an isometry. One of the tools developed in this paper is a (Pansu-)Rademacher theorem proving that Lipschitz maps between Carnot groups are almost everywhere Pansu-differentiable. Here, the definition of Pansu-differentiability is inspired by the intrinsic geometry of Carnot groups and their dilations. As noticed by Semmes [101], this Pansu-Rademacher theorem applied to maps between Euclidean spaces and the Heisenberg group has a terrific consequence: the range of a Lipschitz map defined on $\mathbb{R}^d$ for $d \geq 2$ to $\mathbb{H}_n$ has d-dimensional Hausdorff measure 0. Actually this result has a counterpart for the discrete Heisenberg group and his Cayley graph in computer science since the paper by Cheeger and Kleiner [28]. These authors proved a conjecture in relation to problems of biLipschitz embeddings of graphs in Banach spaces. The remark of Semmes implies that $\mathbb{H}_n$ is not rectifiable in the sense of Federer in [39] that has been later studied by Ambrosio and Kirchheim [67, 5]. More generally, Carnot groups seem to require a specific rectifiability theory and a geometric measure theory. The first significant step has been done by Franchi, Serapioni and Serra Cassano for the Heisenberg group [46]. These authors extends this setting to the classical De Giorgi's rectifiability divergence theorems [33]: the sets of finite perimeter have a countably rectifiable border in a sense specific to $\mathbb{H}_n$. It opened the door to a theory of rectifiability of codimension 1 that has been continued in [47] and [6] for Carnot groups of step 2 and general Carnot groups. As far as we know, no special definition or work has been found for rectifiability of other dimensions (or codimensions) except for dimension 1. Indeed, Ferrari, Franchi and Pajot have generalized a Theorem of Peter Jones [62] about the so-called geometric traveling salesman problem to $\mathbb{H}_1$. This theory has relations to the analysis of singular integrals defined on 1-dimensional sets [61].

Before we present our main results, we would like to introduce the use of optimal transport in metric geometry. Optimal transport is well-adapted to the poor structure of a general Polish metric space (X, d) because the formulation of this theory is essentially metric (or even more general) and the weak topology of the measures on these spaces does not require a rich structure on X. This space of measures is called the Wasserstein space $\mathcal{P}_2(X)$ in the modern terminology and optimal transport permits to give a distance –the Wasserstein distance– to this topology. Most of the time in the recent development in geometry, X is geodesic which implies that $\mathcal{P}_2(X)$ is geodesic as well. Moreover, if X has a special differential structure $\mathcal{P}_2(X)$ might also have a nice tangent structure. The breakthrough on this topic are the papers of Otto ([63, 92] the first one with Jordan and Kinderlehrer) where the Wasserstein space $\mathcal{P}_2(\mathbb{R}^n)$ is considered for the first time formally as an infinite dimensional Riemannian manifold. Otto realized that the solutions of the heat equation are densities of measures describing a special curve on $\mathcal{P}_2(\mathbb{R}^n)$. The relative entropy $\int \rho \ln \rho$ can be regarded as a function on this formal manifold and the diffusion curve moves with a speed and direction determined by the gradient of this function (the vector field $\frac{-\nabla \rho}{\rho}$). This discovery initiated the study of the gradient flow of different functionals in the Wasserstein spaces starting with the Rényi entropy recovering the porous medium equation [92]. People continued this approach in various spaces X, following various definitions of the gradient flow on $\mathcal{P}_2(X)$, sometimes with numerical aspirations. Nowadays, the most documented book on this subject is probably the book by Ambrosio, Gigli and Savaré [4] where this theory is developed in fine analysis for Hilbert spaces.

As $\mathcal{P}_2(\mathbb{R}^n)$ is a kind of Riemannian manifold, one should by regard the geodesics of this manifold and consider the behavior of functionals along the geodesics of the Wasserstein space. It turns out that on Riemannian manifolds the concavity of certain functionals, namely the entropies of Rényi and Bolzmann, are in some sense equivalent to the fact that the Ricci curvature of these manifolds has a lower bound. Cordero-Erausquin, McCann and Schmuckenschläger proved in [29] that the entropy is (roughly speaking) convex for Riemannian manifolds with a bound. Sturm and von Renesse [111] proved the converse implication. The previous concordance between these properties led to a very exciting treatment of Ricci curvature of metric measure spaces. Lott and Villani [77, 78] and Sturm [104, 105] independently proposed very similar definitions of a metric measure space with curvature bounded below by K. One of the essential points of this theory is the stability of these bounds with respect to the measured Gromov-Hausdorff topology (see [50]). This result resonates with the sequence of papers by Cheeger and Colding [25, 26, 27]. These authors show that a limit of Riemannian manifolds with a uniform bound on the Ricci curvature provides similar results as the Riemannian manifolds with the same bound. The limit metric space shall now be understood not only as a limit but as a space with an intrinsic synthetic curvature bounded below. A second important part of this theory is the coherence with the Bakry-Émery theory [11] (see also [10]). Indeed, a Riemannian manifold with an elliptic operator satisfies the Bakry-Émery condition $CD(K, N)$ if and only if this manifold with the invariant measure satisfies $CD(K, N)$ in the sense of optimal transport (this is the reason why the name Curvature-Dimension is used in this theory). Moreover one can recover log-Sobolev inequalities (which is one of the initial aims of the Bakry-Émery theory) using the new synthetic Ricci curvature bounds. Although the Bakry-Émery theory provides a calculus that makes sense in many settings, it must be formulated correctly for each example. This provokes that a comparison with the Ricci bounds obtained by optimal transport can not systematically be done. However, it makes sense to consider Bakry-Émery calculus in the subRiemannian setting.

The Curvature-Dimension condition is also coherent with the theorems on the growth of balls such as the Bishop-Gromov theorem or the Bonnet-Myers theorem. The growth satisfies the same estimates as manifolds with the same bounds. Actually the conclusion of an angular variant of the Bishop-Gromov theorem can be turned into an alternative definition for a metric measure space with a lower bound on the Ricci curvature. This has been done by Ohta [89] and Sturm [105] where it is called "Measure Contraction Property (MCP)". These papers were the first systematic studies of this property in the general setting. However, $MCP(K, N)$ has already been considered in the special setting of Alexandrov spaces (as in [71]) and also briefly proposed by Gromov [50] and by Cheeger and Colding [25]. Unfortunately, $MCP(K, N)$ is not really significant if the dimension parameter N is different from the topological dimension of the considered space ($MCP(K, \infty)$ does not even exist). In particular, it seems that one can not recover functional inequalities such as log-Sobolev inequalities from MCP.

At the beginning of this thesis, there were so far we know essentially two works at the intersection between optimal transport and subRiemannian geometry, namely the one by Ambrosio and Rigot [7] and the extension by Rigot [94]. Ambrosio and Rigot proved the existence and uniqueness of the solutions to the Monge problem in the Heisenberg groups for the Carnot-Carathéodory distance and the Korányi distance. The paper is an extension of this work to the H-type groups. These results are nice and satisfactory because they are intrinsic (using the Pansu-differentiablity) and correspond faithfully to the theorems of Brenier [19] and McCann [80] obtained for $\mathbb{R}^n$ and compact manifolds. During the thesis, Agrachev and Lee [2] and Figalli and Rifford [43] obtained important generalizations to several classes of subRiemannian mannifolds where the abnormal geodesics do play a very significant role. Maybe unfortunately, their proofs rely on an extrinsic point of view. Except for the Monge problem, we recently learnt from a paper by Khesin and Lee [66] about the possibility to represent the subelliptic diffusion on compact manifolds with bracket generating vector fields by a Wasserstein gradient flow. The quite algebraic proof is carried out in a "smooth" Wasserstein space. In [42] Figalli and the author answered an open problem about the absolute continuity of the measure interpolated by optimal transport in the Heisenberg groups (a second proof by Figalli and Rifford appeared later in [43]). It will be presented in this thesis. We will also report on the results of [64] where we deal with the synthetic Ricci curvature bounds MCP and CD.

Let us now review the main author's results of this thesis:

- Theorem 2.3.6 established in a joint work with Figalli [42], positively answers an open question [7, Section 7 (c)] by Ambrosio and Rigot. Actually, if $(\mu_s)_{s\in[0,1]}$ is a geodesic segment of $\mathcal{P}_2(\mathbb{H}_n)$ and μ_1 is absolutely continuous, then the intermediate measures μ_s $(s \in]0,1[)$ are absolutely continuous as well. Theorem 2.3.6 also provides an above estimate on the density of these measures. The specificity of this proof is based on the fact that it is different from the classical proof on manifolds that can not be adapted. The two main ingredients for this new proof are a contraction estimate (essentially equivalent to $MCP(0, 2n+3)$) and the uniqueness of the geodesics proved by Ambrosio and Rigot.

- Theorem 3.4.5 and Theorem 3.5.12 specify for which parameters (K,N) the Curvature-Dimension condition CD and the Measure Contraction Property MCP are satisfied. It appears that CD does not hold for any pair (K,N) while $MCP(K,N)$ holds only for $K \leq 0$ and $N \in [1,+\infty[$ greater than the critical value $2n+3$. This dimension $2n+3$ is quite unexpected because it is neither the topological dimension of $\mathbb{H}_n$ nor its Hausdorff dimension, that are $2n+1$ and $2n+2$. It is also surprising that no condition CD holds while $MCP(0,2n+3)$ is satisfied. Actually, the mismatch between the topological and the "contraction" dimensions permit us to prove that the geodesic Brunn-Minkowski inequality BM is false in $\mathbb{H}_n$ which is enough for the proof because BM is an intermediate property between CD and MCP.

- In Section 1.8, Theorem 4.5.1 presents the solutions of the subelliptic equation $\Delta_{\mathbb{H}}\rho_s = \partial_s\rho_s$ as a Wasserstein gradient flow of the relative entropy Ent_∞. Conversely, Theorem 4.5.2 shows that some gradient flows satisfying a particular condition are solutions of the subelliptic equation. The nice aspect of these results is that the classical proof making use of the convexity of the entropy functional along the geodesics can not hold here because $CD(K,\infty)$ does not hold in $\mathbb{H}_n$. The proof is based on the information about the gradient flow of Ent_∞ on the manifolds approximating $\mathbb{H}_n$.

- Section 1.8 provides an example of a compact subset Ω of $\mathbb{H}_1$ that does not satisfy the geometric traveling salesman problem criterion by Ferrari, Franchi and Pajot. This condition on compact sets $E \subset \mathbb{H}_1$ is known to be sufficient for covering E by a rectifiable curve. But Ω is precisely defined as the support of a rectifiable curve ω. This implies that the criterion by Ferrari, Franchi and Pajot is not a necessary condition for a compact set to be covered by a rectifiable curve.

In this thesis, we also make some remarks extending the main results to other metric spaces such as the Grušin plane (Theorem 3.5.13), Alexandrov spaces (Theorem 3.2.9) or the Albanese torus (Theorem 4.5.5). An extension of the method permits us to deny a multiplicative Brunn-Minkowski inequality in $\mathbb{H}_n$ for the exponents $N > 2n + 1$ (extensions of Theorem 3.5.12). However, all the possible easy extensions have not been considered. Beside these main results and related results, there are some examples, remarks or calculations in this thesis that are either new or have not been written to our knowledge.

We are now in a position to comment on the plane of this report. In Chapter 1 we define $\mathbb{H}_n$ and some related spaces and specify their basic geometric features and estimates (especially of $\mathbb{H}_1$). We determine the geodesics of these spaces which permits us to prepare the MCP results of Chapter 3 by computing contraction estimates for $\mathbb{H}_n$ and the Grušin plane. In the last section of this chapter, section 1.8, we present the set $\Omega = \omega([0, 1])$ related to the geometric traveling salesman problem in $\mathbb{H}_1$. In Chapter 2 we present the theory of optimal transport for general metric spaces, for $\mathbb{R}^n$ and finally for $\mathbb{H}_n$. We give some exotic examples of transport plans and answer the open question by Ambrosio and Rigot by using the estimates of Chapter 1. Chapter 3 is devoted to different definitions of curvature lower bounds for metric measure spaces, including Alexandrov spaces, the Bakry-Émery criterion, the Measure Contraction Property and the Curvature-Dimension condition by Lott-Villani and Sturm. It turns out that MCP is the only one of these properties that holds for the Heisenberg group. The proof that CD is not satisfied is based on the contradiction of the generalized "geodesic" Brunn-Minkowski inequality. Almost at the end of Chapter 3 we state the critical dimension for the "multiplicative" Brunn-Minkowski inequality to hold in $\mathbb{H}_n$. Finally, in Chapter 4 we prove the equivalence (under certain conditions) of subelliptic diffusions in $\mathbb{H}_1$ and Wasserstein gradient flows of the entropy in $\mathcal{P}_2(\mathbb{H}_1)$.

Chapter 1

The Heisenberg group and other related metric spaces

In this chapter we first introduce the Heisenberg group and some related spaces (Section 1.1 and Section 1.2). Then try to get an intuition on the horizontal curves and the dimension of the subspaces of $\mathbb{H}_1$ (Section 1.3 and Section 1.4). Then we compute the geodesics and state estimates on the contraction of sets along the geodesics (Section 1.5, 1.6 and 1.7). In particular we state Theorem 1.7.7 that is a key estimate for the main results of Chapter 2 and Chapter 3. Section 1.8 is devoted to the geometric traveling salesman problem in $\mathbb{H}_1$. We prove one of our main theorems, namely Theorem 1.8.4 about the couterexample curve ω.

In this chapter we often consider $\mathbb{H}_n$ with $n = 1$. Nevertheless we also state the corresponding results for $n > 1$ that we need in the next chapters.

1.1 The Heisenberg group $\mathbb{H}_n$

Let n be a non-negative integer. As a set $\mathbb{H}_n$ can be written in the form $\mathbb{R}^{2n+1} = \mathbb{C}^n \times \mathbb{R}$ and an element of $\mathbb{H}_n$ can also be written as $(z;t) = (z_1, \cdots, z_n; t)$ where $z_k := x_k + \mathbf{i}y_k \in \mathbb{C}^n$ for $1 \le k \le n$ and $t \in \mathbb{R}$. The group structure of $\mathbb{H}_n$ is given by

$$(z_1, \cdots, z_n; t) \cdot (z_1', \cdots, z_n'; t') = \left(z_1 + z_1', \cdots, z_n + z_n'; t + t' - \frac{1}{2} \sum_{k=1}^{n} \Im(z_k \overline{z_k'}) \right)$$

where $\Im$ denotes the imaginary part of a complex number. The Heisenberg group $\mathbb{H}_n$ is then a Lie group with neutral element $0_{\mathbb{H}} := (0;0)$. The inverse element of $(z;t)$ is $(-z;-t)$. Throughout this report, $\mathrm{tran}_p : \mathbb{H}_n \to \mathbb{H}_n$ will be the left translation

$$\mathrm{tran}_p(q) = p \cdot q.$$

This map is affine. Indeed

$$(x_1, y_1, \cdots, x_n, y_n, t) \cdot (x_1', y_1', \cdots, x_n', y_n', t') =$$

$$\begin{pmatrix} x_1 \\ y_1 \\ \vdots \\ x_n \\ y_n \\ t \end{pmatrix} + \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 1 & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 1 & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \\ -\frac{1}{2}y_1 & \frac{1}{2}x_1 & \cdots & -\frac{1}{2}y_n & \frac{1}{2}x_n & 1 \end{pmatrix} \begin{pmatrix} x_1' \\ y_1' \\ \vdots \\ x_n' \\ y_n' \\ t' \end{pmatrix} \tag{1.1}$$

and the determinant of the linear part is 1. 1. It follows that the Haar measure of $\mathbb{H}_n$ is the Lebesgue measure $\mathcal{L}^{2n+1}$ of $\mathbb{R}^{2n+1}$ which is left (and actually also right) invariant. For $\lambda > 0$, we denote by dil_λ the dilation

$$\mathrm{dil}_\lambda(z; t) = (\lambda z; \lambda^2 t)$$

where $\lambda \geq 0$. The measure behaves also well under dilations:

$$\mathcal{L}^{2n+1}(\mathrm{dil}_\lambda(E)) = \lambda^{2n+2}\mathcal{L}^{2n+1}(E) \tag{1.2}$$

if $\lambda \geq 0$ and $E \subset \mathbb{R}^{2n+1}$ is a measurable set.

In order to define the Carnot-Carathéodory metric (or Carnot-Carathéodory distance, see (1.8)), we consider the Lie algebra associated to $\mathbb{H}_n$. This is the vector space of left-invariant vector fields. A basis for this vector space is given by $(\mathbf{X}_1, \cdots, \mathbf{X}_n, \mathbf{Y}_1, \cdots, \mathbf{Y}_n, \mathbf{T})$ where

$$\begin{aligned} \mathbf{X}_k &= \partial_{x_k} - \frac{1}{2} y_k \partial_t \\ \mathbf{Y}_k &= \partial_{y_k} + \frac{1}{2} x_k \partial_t \\ \mathbf{T} &= \partial_t. \end{aligned}$$

For $n = 1$ we will write $\mathbf{X}$ and $\mathbf{Y}$ instead of $\mathbf{X}_1$ and $\mathbf{Y}_1$. Roughly speaking, the Carnot-Carathéodory distance between two points p and q is the infimum of the lengths of the horizontal curves connecting p and q. By a horizontal curve we mean an absolutely continuous curve γ from an interval $I \subset \mathbb{R}$ to $\mathbb{R}^{2n+1} \simeq \mathbb{H}_n$ whose derivative $\gamma'(s)$ is spanned by

$$\{\mathbf{X}_1(\gamma(s)), \cdots, \mathbf{X}_n(\gamma(s)), \mathbf{Y}_1(\gamma(s)), \cdots, \mathbf{Y}_n(\gamma(s))\}$$

in almost every point $s \in I$. The length of this curve is then

$$\mathrm{length}_c(\gamma) = \int_0^r \|\gamma'(s)\|_{\mathbb{H}}\, ds \tag{1.3}$$

where $\|\sum_{k=1}^n (a_k \mathbf{X}_k + b_k \mathbf{Y}_k)\|_{\mathbb{H}}^2 = \sum_{k=1}^n (a_k^2 + b_k^2)$. By convention the length of a non-horizontal curve is $+\infty$.

Example 1.1.1 (An horizontal curve). We exhibit an horizontal curve $\gamma_{x,t}$ of finite length between $0_{\mathbb{H}}$ and $(x, 0, t) \in \mathbb{H}_1$. It is made of five line segments of $\mathbb{R}^3$. We will not specify the parametrization (take any absolutely continuous one). For this example we will note $\sqrt{t} = \pm\sqrt{|t|}$ the real number of square $|t|$ that have the same sign as t. The first of the five segments goes from $0_{\mathbb{H}}$ to $(x, 0, 0)$ and is tangent to $\mathbf{X}$. On this segment the vector field $\mathbf{X}$ is actually constant and equal to $\frac{\partial}{\partial x} - \frac{0}{2}\frac{\partial}{\partial t}$. The second segment goes from $(x, 0, 0)$ to $(x, -\sqrt{t}, -\sqrt{t}\frac{x}{2})$. This segment is tangent to $\mathbf{Y}$ and this vector field is constant along the line and equals $\frac{\partial}{\partial y} + \frac{x}{2}\frac{\partial}{\partial t}$. The next three segments connect the points $(x + \sqrt{t}, -\sqrt{t}, \sqrt{t}\frac{\sqrt{t}-x}{2})$, $(x + \sqrt{t}, 0, t)$ and eventually $(x, 0, t)$. They are respectively tangent to $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{X}$ and these vector fields are constant along the three segments. The trajectory of the $z = x + \mathbf{i}y$ coordinate in $\mathbb{C}$ is in fact quite easy: it goes along a line segment from $0_{\mathbb{C}}$ to $(x, 0)$, and from there draw a square of side $\sqrt{|t|}$.

A computation yields $\mathrm{length}_c(\gamma) = |x| + 4\sqrt{|t|}$. It is exactly the length of the projected curve in $\mathbb{C}$. We will explain his phenomenon in 1.3.

1.1.1 Some geometric transformations

We will become more familiar with the Heisenberg group by considering its symmetries. In this subsection, we see that some transformations preserve the length of the horizontal curves and that some other scale it.

For simplicity we will sometime (as in this subsection) only consider $\mathbb{H}_n$ in the case $n = 1$. However, all the main results of this report are true (with the correct adaptation) in higher dimensions (see for instance Remark 4.5.3). It should also be noticed that some of the result of this thesis are only stated in the special case of $\mathbb{H}_1$ (for instance in Section 1.8 because the reference paper [40] is written for $\mathbb{H}_1$, or in Chapter 4 for simplicity and because initially in [75] the estimates of the fundamental solution $\mathfrak{h}$ in are considered in $\mathbb{H}_1$ (but see Remark 4.5.3)).

The Carnot-Carathéodory metric (as defined in Subsection 1.1.2) and also the Lebesgue have a good behavior under the action of translations tran_p and dilations dil_λ. It is due to the symmetries of the horizontal distribution. From the fact that $\mathbf{X}$ and $\mathbf{Y}$ are left-invariant, we get that

$$\mathrm{length}_c(\mathrm{tran}_p(\gamma)) = \mathrm{length}_c(\gamma).$$

From the identities

$$D\,\mathrm{dil}_\lambda(p).\mathbf{X} = \lambda\mathbf{X}(\mathrm{dil}_\lambda(p)) \quad \text{and} \quad D\,\mathrm{dil}_\lambda(p).\mathbf{Y} = \lambda\mathbf{Y}(\mathrm{dil}_\lambda(p)), \tag{1.4}$$

where D is the operator giving the total derivative of a map, we get

$$\mathrm{length}_c(\mathrm{dil}_\lambda(\gamma)) = \lambda\,\mathrm{length}_c(\gamma).$$

Define now sym by

$$\mathrm{sym}(x, y, t) = (x, -y, -t).$$

Then

$$D\,\mathrm{sym}(p).\mathbf{X} = \mathbf{X}(\mathrm{sym}(p)) \quad \text{and} \quad D\,\mathrm{sym}(p).\mathbf{Y} = -\mathbf{Y}(\mathrm{sym}(p)). \tag{1.5}$$

Therefore for any horizontal curve

$$\mathrm{length}_c(\mathrm{sym}(\gamma)) = \mathrm{length}_c(\gamma).$$

We finally introduce the rotations

$$\mathrm{rot}_\theta(z;t) = (e^{\mathbf{i}\theta} z; t) \tag{1.6}$$

for any $\theta \in \mathbb{R}$. We have still

$$\mathrm{length}_c(\mathrm{rot}_\theta(\gamma)) = \mathrm{length}_c(\gamma)$$

since

$$\begin{aligned} D\,\mathrm{rot}_\theta(p).\mathbf{X} &= \cos(\theta)\mathbf{X}(\mathrm{rot}_\theta(p)) + \sin(\theta)\mathbf{Y}(\mathrm{rot}_\theta(p)) \\ D\,\mathrm{rot}_\theta(p).\mathbf{Y} &= \cos(\theta)\mathbf{Y}(\mathrm{rot}_\theta(p)) - \sin(\theta)\mathbf{X}(\mathrm{rot}_\theta(p)). \end{aligned}$$

Hence we have

$$\|D\,\mathrm{rot}_\theta(p).\mathbf{X}\|_{\mathbb{H}} = \|\mathbf{X}(p)\|_{\mathbb{H}} = 1$$

and the corresponding equation for $\mathbf{Y}$.

However, the horizontal vector fields $\mathbf{X}$ and $\mathbf{Y}$ are not invariant under rotations. So for $|z| > 0$, we introduce

$$\begin{aligned} \mathbf{R}(z;t) &= \cos(\theta)\mathbf{X} + \sin(\theta)\mathbf{Y} = \frac{x}{|z|}\frac{\partial}{\partial x} + \frac{y}{|z|}\frac{\partial}{\partial y} \\ \mathbf{\Theta}(z;t) &= \sin(-\theta)\mathbf{X} + \cos(\theta)\mathbf{Y} = \frac{x}{|z|}\frac{\partial}{\partial y} - \frac{y}{|z|}\frac{\partial}{\partial x} + \frac{|z|}{2}\mathbf{T} \end{aligned}$$

where $z = |z|e^{\mathbf{i}\theta}$. As one can easily check $\|a\mathbf{R} + b\mathbf{\Theta}\|_{\mathbb{H}} = \sqrt{a^2+b^2}$ and we have the nice relations

$$D\,\mathrm{rot}_\theta(\mathbf{R}) = \mathbf{R} \quad \text{and} \quad D\,\mathrm{rot}_\theta(\mathbf{\Theta}) = \mathbf{\Theta}. \tag{1.7}$$

Example 1.1.2 (Connectivity of $\mathbb{H}_1$). We show that there is an horizontal curve of finite length between any two points p and q of the Heisenberg group. Assume first that $p = 0_{\mathbb{H}}$ and consider the horizontal curve $\gamma_{|z|,t}$ of Example 1.1.1 where $q = (z;t)$. If $\Im(z) = 0$ we are done. Otherwise we consider $\mathrm{rot}_\theta(\gamma_{|z|,t})$ where $z = |z|e^{i\theta}$. If $p \neq 0_{\mathbb{H}}$, there is a horizontal curve between $0_{\mathbb{H}}$ and $p^{-1}\cdot q$. Just translate it with tran_p. Because rot_θ and tran_p preserve the length, the curves we have built have length $|Z| + 4\sqrt{T}$ where $(Z;T) = p^{-1}\cdot q$.

Remark 1.1.3. For $n > 1$ the geometric transformations tran_p and dil_λ have the same properties. The rotation rot_θ must be defined for $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ by $\mathrm{rot}_\theta(z_1,\ldots,z_n;t) = (e^{\mathbf{i}\theta_1}z_1, \ldots, e^{\mathbf{i}\theta_n}z_n;t)$ and the length is still invariant under this transformation. The same remark holds for $\mathrm{sym}_k(z_1,\ldots,z_k,\ldots,z,n;t) = (z_1,\ldots,\overline{z_k},\ldots,z_n;-t)$.

Examples 1.1.1 and 1.1.2 can be adapted to $\mathbb{H}_n$. If p and q are in $\mathbb{H}_n$ and $p^{-1}\cdot q = (Z;T) \in \mathbb{H}_n$, there is an horizontal curve between p and q of length $|Z| + 4\sqrt{T}$.

1.1.2 Carnot-Carathéodory distance

The Carnot-Carathéodory distance between p and q of $\mathbb{H}_n$ is

$$d_c(p,q) := \inf \int \|\gamma'(s)\|_{\mathbb{H}}\, ds = \inf \mathrm{length}_c(\gamma) \tag{1.8}$$

where the infimum is taken over all horizontal curve (on some real interval) connecting p and q. As we have seen in Example 1.1.2 and Remark 1.1.3, there is at least one horizontal curve of finite length between any two points such that d_c is finite. The axioms of a distance are not difficult to prove. From relations (1.5), (1.7) and (1.4) we get

Proposition 1.1.4. *For $p \in \mathbb{H}_1$, $\lambda > 0$ and $\theta \in \mathbb{R}$, the transformations* sym, tran_p *and* rot_θ *are isometries of* $(\mathbb{H}_1, d_c)$*. The dilation* dil_λ *multiplies the distance by λ.*

Remark 1.1.5. Proposition 1.1.4 also holds in dimension $n > 1$ for the transformation of Remark 1.1.3.

We compare now the distance to the Euclidean one.

Proposition 1.1.6. *For any set $\Omega \subset \mathbb{R}^3$ bounded with respect to the Euclidean norm, there exists two positive constants $c < C$ (depending only on Ω) such that if $(p, q) \in \Omega^2$*

$$c|p-q| < d_c(p,q) < C|p-q|^{1/2}.$$

Proof. We first suppose that for any $(z;t) \in \Omega$, $\max(|z|,|t|) < 1$. Then for $p = (z;t)$ and $q = (z';t')$ in Ω thanks to Example 1.1.2 we know that:

$$\begin{aligned} d_c((z;t),(z';t')) &\le d_c(0_{\mathbb{H}}, (z'-z; t'-t+\frac{1}{2}\Im(z\overline{z'})) \\ &\le |z-z'| + 4\sqrt{|t-t'| + |\frac{1}{2}\Im(z\overline{z'})|}. \end{aligned}$$

In Ω, $|z'-z| \le \sqrt{2}|z-z'|^{1/2}$ and $|\Im(z\overline{z'})| = |\Im(z(\overline{z'-z}))| \le |z-z'|$. We have eventually on Ω

$$d_c(p,q) \le 5\sqrt{2}|p-q|^{1/2}.$$

The proof of the other estimate is a little more tricky. Let $\gamma(s) = (x,y,t)(s)$ be a horizontal curve from $p \in \Omega$ to $q \in \Omega$. Because $|p-q| < 2\sqrt{2}$, we know from the first part of this proof that $d_c(p,q) < 10 \cdot 2^{1/4}$. Thus we can assume $\mathrm{length}_c(\gamma) < 12$. We will now estimate the Euclidean length of γ (as a curve of $\mathbb{R}^3$) that we denote by $\mathrm{length}_{\mathbb{R}^3}$. For almost every time s

$$\dot{\gamma} = a(s)\mathbf{X} + b(s)\mathbf{Y} = a(s)\frac{\partial}{\partial x} + b(s)\frac{\partial}{\partial y} - \frac{1}{2}(a(s)y(s) - b(s)x(s))\frac{\partial}{\partial t}.$$

Then

$$\begin{aligned} \mathrm{length}_{\mathbb{R}^3}(\gamma) &= \int \sqrt{a^2+b^2+\frac{(ay-bx)^2}{4}} && (1.9) \\ &\le \int \sqrt{(a^2+b^2)(1+\max_s(|z(s)|^2))} && (1.10) \\ &\le \sqrt{1+\max_s(|z(s)|^2)}\,\mathrm{length}_c(\gamma). && (1.11) \end{aligned}$$

But we can estimate $|z(s)|$ because

$$|z(s)| \le 1 + \int_{s_0}^{s} \sqrt{\dot{\gamma_x}^2 + \dot{\gamma_y}^2} \le 1 + \mathrm{length}_c(\gamma) < 13.$$

It follows that

$$|p-q| \leq \sqrt{1+13^2}\,\mathrm{length}_c\,\gamma.$$

For $c_1 = 1/\sqrt{170}$, $C_1 = 5\sqrt{2}$ and for any curve short enough we can write

$$c_1|p-q| < d_c(p,q) < C_1|p-q|^{1/2}.$$

If Ω is now bounded with $\max_{\{(z,t)\in\Omega\}}(|z|,|t|^{1/2}) \leq M$ where $M \geq 1$, we use the dilation $\mathrm{dil}_{1/M}$ and prove that $c(\Omega) = c_1/M$ and $C(\Omega) = \sqrt{M}C_1$ satisfy the desired conclusion. $\square$

From these estimates, we get more information about the topology of $\mathbb{H}_1$. Because Proposition 1.1.6 also holds with a similar proof for $\mathbb{H}_n$, we state the next corollary for a general n.

Corollary 1.1.7. *The Heisenberg group with the Carnot-Carathéodory distance* $(\mathbb{H}_n, d_c)$ *has the same topology as* $(\mathbb{R}^{2n+1}, |.-.|)$. *In particular it is locally compact. Moreover,* $\mathbb{H}_n$ *is a Polish space (that is complete and separable).*

1.1.3 Equivalent distances and estimates of d_c

The geometry provided by the Carnot-Carathéodory distance has a rich structure but it is unfortunately possible to compute $d_c(p,q)$ only for special pairs (p,q). However, there are some more easy-to-work equivalent metric that are carrying the same ideas: left-translation invariance and good dilation behavior. The standard way to make these metric is to define their from a *homogeneous norm*. It is function $\|.\|$ that satisfies the following

- $\|.\|$ is a continuous function (of $\mathbb{R}^{2n+1}$) vanishing only in $0_{\mathbb{H}}$.
- $\|\,\mathrm{dil}_\lambda\, p\| = \lambda\|p\|$.
- $\|p^{-1}\| = \|p\|$.

Then the metric can be refund as $d(p,q) := \|p^{-1}\cdot q\|$. It is left invariant, vanish uniquely when $p = q$ verify $d(\mathrm{dil}_\lambda\, p, \mathrm{dil}\,\lambda q) = d(p,q)$ but does not generally satisfy the triangle inequality. A weak-triangle inequality does occur.

Proposition 1.1.8. *For a map d constructed from an homogeneous norm, we can find a constant $C > 0$ such that*

$$d(p,r) \leq C(d(p,q) + d(q,r)). \tag{1.12}$$

Such a function d is called a *quasi-metric*.

Proof. We have to show that the map $(a,b) \to \frac{\|a^{-1}b\|}{\|a\|+\|b\|}$ has a maximum on $(\mathbb{H}_1)^2\backslash\{0_{\mathbb{H}}\}$. We can use dilations to reduce this set to the compact set $\{\|a\| + \|b\| = 1\}$. On this set there is a maximum because the map is continuous. $\square$

Proposition 1.1.9. *All quasi-metric constructed from homogeneous norm as above are equivalent. The Carnot-Carathéodory metric is a representantative of this equivalence class.*

Proof. The proof is very similar to the equivalence of the norms of the real vector space $\mathbb{R}^{2n+1}$. We consider two homogeneous norm $\|.\|$ and $\|.\|'$ and the sphere $\mathcal{S}^{\mathbb{H}} = \{p \in \mathbb{H}_n, \|p\| = 1\}$. On this compact set the continuous map $\|.\|'$ has a minimum m and a maximum M. With obvious notation we have

$$md(p,q) \leq d'(p,q) \leq Md(p,q)$$

which achieve the proof of the equivalence.

We now want to prove that $\|.\|_c = d_c(0_{\mathbb{H}}, .)$ is an homogeneous norm. The dilation property is certainly true (Proposition 1.1.4 and Remark 1.1.3). Using the isometries rot_θ and sym_1 and choosing θ such that

$$\mathrm{rot}_\theta \circ \mathrm{sym}_1 \circ \mathrm{rot}_\theta^{-1}(z;t) = (-z;-t)$$

we obtain that $\|p\| = \|p^{-1}\|$. The norm $\|.\|_c$ is continuous because of the continuity of d_c in the $\mathbb{R}^{2n+1}$ topology (Corollary 1.1.7). $\square$

Example 1.1.10. The function $e(z;t) = |z| + 4|t|^{1/2}$ is an homogeneous norm. It is the estimate of the Carnot-Carathéodory distance provided in Example 1.1.2.

Example 1.1.11. The homogeneous norm $\|(z;t)\|_\infty := \max\left(|z|, |t|^{1/2}\right)$ provide a true distance d_∞, that is a quasi-metric with $C = 1$ in (1.12).

Proof. For this we have to prove $\|(z;t)\cdot(z';t')\|_\infty \leq \|(z;t)\|_\infty \|(z';t')\|_\infty$. We call $m := \|(z;t)\|_\infty$ and $m' = \|(z';t')\|_\infty$. Thus $|z + z'| \leq m + m'$ is obvious. The second estimate is

$$|t + t' - \frac{1}{2}\sum_{k=1}^{n} \Im(z_k \overline{z'_k})|^{1/2} \leq |m^2 + m'^2 + 2mm'|^{1/2} \leq m + m'.$$

$\square$

Example 1.1.12. The Korányi-Reimann distance constructed from $\|(z;t)\|_{KR} = \left(|z|^4 + 16t^2\right)^{1/4}$ is a true distance (a quasi-metric with constant $C = 1$ in (1.12)). The proof of this fact is quite tricky. We repeat the proof that we found in [70].

Proof. Here, $|.|$ is the complex norm, so $\|(a,b)\|_{KR}^2 = ||a|^2 + 4\mathbf{i}b|$.

$$\begin{aligned}
\|(z;t)\cdot(z';t')\|_{KR}^2 &= \|(z+z';t+t' - \frac{1}{2}\sum_{k=1}^{n}\Im(z_k\overline{z'_k})\|_{KR}^2 \\
&= \left| |z+z'|^2 + \mathbf{i}(4t + 4t' - 2\sum_{k=1}^{n}\Im(z_k\overline{z'_k})) \right| \\
&\leq \left||z|^2 + 4\mathbf{i}t\right| + \left||z'|^2 + 4\mathbf{i}t'\right| + 2\sum_{k=1}^{n}\left|\Re(z_k\overline{z'_k}) - \mathbf{i}\Im(z_k\overline{z'_k})\right| \\
&\leq \|(z;t)\|_{KR}^2 + \|(z';t')\|_{KR}^2 + 2\|(z;t)\|_{KR}\|(z;t)\|_{KR} \\
&\leq \left(\|(z;t)\|_{KR} + \|(z';t')\|_{KR}\right)^2.
\end{aligned}$$

In fact

$$2\sum_{k=1}^{n}\left|\Re(z_k\overline{z'_k}) - \mathbf{i}\Im(z_k\overline{z'_k})\right| = 2\sum_{k=1}^{n}|\Re(\bar{z}_k z'_k) + \mathbf{i}\Im(\bar{z}_k z'_k)| = 2\sum_{k=1}^{n}|\bar{z}_k z'_k| \leq 2|z||z'|$$

as a consequence of the Cauchy-Schwarz inequality. $\square$

1.2 Subgroups and quotients of $\mathbb{H}_1$

We give here the definition of some spaces related to $\mathbb{H}_1$ that we will be used in the sequel.

1.2.1 Linear subgroups

The multiplicative law of $\mathbb{H}_1$ is not so far from being the classical addition of $\mathbb{R}^3$. In fact $(z;t)\cdot(z';t') = (z+z';t+t')$ if and only if $\Im(z\overline{z'}) = 0$, which happens exactly when z and z' are real collinear. It is also the only situation where $(z;t)$ and $(z';t')$ commutate.

The only linear 2-planes that are also subgroups are the ones that contain $\mathbf{T}$. We call them *vertical planes*. The restriction of d_c on these planes is equivalent to the restriction of d_∞ (Example 1.1.11), so it is simply equivalent to $\max(|z-z'|, |t-t'|^{1/2})$.

The linear lines of $\mathbb{R}^3$ are all contained in some linear vertical plane. Then the restriction of the product on them is just $+$ of $\mathbb{R}^3$. If $(z;t)$ is a non-zero vector, the distance between $(\lambda z;\lambda t)$ and $(\mu z;\mu t)$ depends only on $|\lambda-\mu|$ and is equal to $d_c(0, |\lambda-\mu|(z;t))$.

1.2.2 The Euclidean plane

The center of $\mathbb{H}_n$ is

$$L = \{(z;t) \in \mathbb{H}_n \mid z = 0\}.$$

It will play an important role in the cut locus problem for instance in Section 1.5. It is obviously a normal group and the quotient $\mathbb{H}_1/L$ is simply $\mathbb{R}^2$. The map $Z : (z;t) \in \mathbb{H}_1 \to z \in \mathbb{R}^2$ gives a way to represent this quotient. We will get much information on the metric of $\mathbb{H}_1$ (for example in Section 1.3) just by this projection.

1.2.3 The discrete Heisenberg group $\mathbb{H}_1^{\mathbb{Z}}$

Another subgroup is the discrete Heisenberg group

$$\mathbb{H}_1^{\mathbb{Z}} = \operatorname{span}\{(1,0,0),(0,1,0)\}.$$

We adopt the same multiplicative notation as for $\mathbb{H}_1$ such that for $k \in \mathbb{Z}$ and an element $p \in \mathbb{H}_1^{\mathbb{Z}}$, the element p^k is $\underbrace{p\cdot\ldots\cdot p}_{k}$ if k is positive. Otherwise it is the inverse element of p^{-k}.

Lemma 1.2.1. *The discrete Heisenberg group $\mathbb{H}_1^{\mathbb{Z}}$ is the subset of the points (x,y,t) such that x, y and $t+\frac{xy}{2}$ are integers.*

Proof. First of all $(0,0,1)$ is in $\mathbb{H}_1^{\mathbb{Z}}$ because it is the commutator of $(1,0,0)$ and $(0,1,0)$. Because $(0,0,1)$ is in the center L of $\mathbb{H}_1$, any element of $\mathbb{H}_1^{\mathbb{Z}}$ can be written $\{(1,0,0)^x\cdot(0,1,0)^y\cdot(0,0,1)^t\} = (x,y,t+\frac{xy}{2})$ where x, y and t are in $\mathbb{Z}$. Then x, y and $\left(t+\frac{xy}{2}\right)+\frac{xy}{2} = t+xy$ are in $\mathbb{Z}$ as we wish.

Conversely consider (x,y,t) such that x, y and $t+\frac{xy}{2}$ are integers. Then $t-\frac{xy}{2} = t+\frac{xy}{2}-xy \in \mathbb{Z}$ and the element is spanned by $(1,0,0)$ and $(0,1,0)$. Namely $(x,y,t) = (1,0,0)^x\cdot(0,1,0)^y\cdot(0,0,1)^{t-\frac{xy}{2}}$ $\square$

On $\mathbb{H}_1^{\mathbb{Z}}$, we will consider the graph length d_{HZ} on the Cayley graph. Two points p and q have distance one if and only if

$$p^{-1} \cdot q \in \{(1,0,0),(0,1,0),(-1,0,0),(0,-1,0)\}.$$

They are said to be neighbour. The distance $d_{HZ}(p,q)$ is recursively defined to be 1 plus the distance between p and the closest neighbour of q.

1.2.4 The Albanese torus $\mathbb{T}$

The Albanese torus $\mathbb{T}$ is obtained as the space $\mathbb{H}_1/\mathbb{H}_1^{\mathbb{Z}}$ of left cosets $\dot{p} = p \cdot \mathbb{H}_1^{\mathbb{Z}}$. A fundamental domain for the action is $[0,1[^3$. Because the discrete Heisenberg group is not normal (for example $(\frac{1}{3},0,0) \cdot \mathbb{H}_1^{\mathbb{Z}} \neq \mathbb{H}_1^{\mathbb{Z}} \cdot (\frac{1}{3},0,0)$), this torus will not inherit a group structure. However, the structure of vector space induced by $\mathbf{X}, \mathbf{Y}$ and $\mathbf{T}$ is preserved because these fields are invariant under left-translations. As a consequence, the distance induced by the quotient, that is

$$d_{\mathbb{T}}(\tilde{p},\tilde{q}) = \min_{p' \in p \cdot \mathbb{H}_1^{\mathbb{Z}}} d_c(p',q) \tag{1.13}$$

can also be seen as the sub-Riemannian distance induced by the quotient distribution $(\mathbf{X}_{\mathbb{T}}, \mathbf{Y}_{\mathbb{T}})$

$$d_{\mathbb{T}}(\tilde{p},\tilde{q}) = \inf_{\gamma' = a(s)\mathbf{X}_{\mathbb{T}} + b(s)\mathbf{Y}_{\mathbb{T}}} \int \sqrt{a^2+b^2}(s)ds.$$

Because the Lebesgue measure is invariant under left translation of the Heisenberg group, its quotient $\mathcal{L}_{\mathbb{T}}$ is the natural measure on $\mathbb{T}$. It is a probability measure and up to a constant it is the Hausdorff measure of $(\mathbb{T}, d_{\mathbb{T}})$. For convenience, we may avoid to write the indices $_{\mathbb{T}}$. Note that $\mathbb{T}$ is compact which is an advantage over $\mathbb{H}_1$.

1.2.5 The Grušin plane G

We consider the action of $\mathcal{S}^1$ on $\mathbb{H}_1$ by the family of isometries $\{\mathrm{rot}_\theta\}_{\theta\in[-\pi,\pi[}$. The topological quotient is the half-plane $G^+ = \mathbb{R}^+ \times \mathbb{R}$. The cylindrical projection $\Upsilon : (z;t) \in \mathbb{H}_1 \to (|z|;t)$ allows us to investigate this projection class in an easy way. The distance between two elements of G^+ is the minimum of the distances between two representatatives of these classes. Note that these classes are Euclidean circles of $\mathbb{R}^3$ centered in L and orthogonal to this line. There is a way to see this distance as the distance induced continuously from a Riemannian metric on $G^{+*} =]0,+\infty[\times\mathbb{R}$. For that we will use the $(\mathbf{R},\mathbf{\Theta})$ frame of subsection 1.1.1. Remind that it is only defined out of L (for points $(z;t)$ with $|z|>0$).

Because $D\Upsilon\mathbf{R}(z;t) = \frac{\partial}{\partial r}$ and $D\Upsilon\mathbf{\Theta}(z;t) = \frac{|z|}{2}\frac{\partial}{\partial t}$, the length of a curve γ staying in $\mathbb{H}_1 \setminus L$ that goes from one circle to another is equal to the length of its projection $\Upsilon(\gamma)$ in G^+ computed in the orthonormal frame

$$(\frac{\partial}{\partial r}, \frac{r}{2}\frac{\partial}{\partial t}).$$

It will be obvious after Section 1.5.2, that the geodesics of $\mathbb{H}_1$ can be approach by other curves that don't cross L. That is why the distance on $G^{+*} =]0,+\infty[\times\mathbb{R}$

induced by the cylindrical projection Υ corresponds to the Riemannian metric with orthonormal basis

$$(\frac{\partial}{\partial r}, \frac{r}{2}\frac{\partial}{\partial t}).$$

The Grušin plane is the metric space that we obtain by gluing two copies of G^+ along $r = 0$. It is then $\mathbb{R}^2$ equipped with the subRiemannian metric computed in the frame $(\mathbf{R}_G, \mathbf{T}_G)$ where

$$(\mathbf{R}_G, \mathbf{T}_G)(r,t) = (\frac{\partial}{\partial r}, r\frac{\partial}{\partial t})$$

on the whole $\mathbb{R}^2$. We did not choose $\frac{r}{2}\frac{\partial}{\partial t}$ (both choices are isometric. To see this consider, the isometry $I_G : (r,t) \to (r, 2t)$) as suggested above because the equations and the parametrization of the geodesics are less convenient in this way (especially in Subsection 1.6.3).

1.2.6 Approximating manifolds

It is possible to define Riemannian manifolds that approximate in a reasonable sense the Heisenberg group. Hence we will denote $\mathbb{H}_1^\varepsilon$ the space $\mathbb{R}^3$ with the orthonormal frame $(\mathbf{X}, \mathbf{Y}, \varepsilon\mathbf{T})$. The scalar product is therefore defined by

$$\langle a\mathbf{X}(p) + b\mathbf{Y}(p) + c\mathbf{T}(p), a'\mathbf{X}(p) + b'(p)\mathbf{Y} + c'\mathbf{T}(p)\rangle_\varepsilon = aa' + bb' + \frac{1}{\varepsilon^2}cc'.$$

In this expression, we can see that the part in ε degenerates when $\varepsilon \to 0$. In fact

$$\|a\mathbf{X}(p)+b\mathbf{Y}(p)+c\mathbf{T}(p)\|_{\mathbb{H}}^2 = \lim_{\varepsilon\to 0} \|a\mathbf{X}(p)+b\mathbf{Y}(p)+c\mathbf{T}(p)\|_\varepsilon^2 = \lim_{\varepsilon\to 0}(a^2+b^2+c^2/\varepsilon^2).$$

The Laplace-Beltrami operator (we will say Laplace operator) of $\mathbb{H}_1^\varepsilon$ is $\Delta_\varepsilon = \mathbf{X}^2 + \mathbf{Y}^2 + (\varepsilon\mathbf{T})^2$ while the standard subelliptic operator associated to $\mathbb{H}_1$ is $\Delta_{\mathbb{H}} = \mathbf{X}^2 + \mathbf{Y}^2$. We denote the gradients of a function with the same index convention by $\nabla_\varepsilon f = \mathbf{X}f\mathbf{X} + \mathbf{Y}f\mathbf{Y} + (\varepsilon\mathbf{T})f(\varepsilon\mathbf{T})$ and $\nabla_{\mathbb{H}} f = \mathbf{X}f\mathbf{X} + \mathbf{Y}f\mathbf{Y}$. Similarly the divergence operator is $\mathrm{div}_\varepsilon(a\mathbf{X} + b\mathbf{Y} + c\varepsilon\mathbf{T}) = \mathbf{X}a + \mathbf{Y}b + \varepsilon\mathbf{T}c$ while $\mathrm{div}_{\mathbb{H}}(a\mathbf{X} + b\mathbf{Y}) = \mathbf{X}a + \mathbf{Y}b$. Note that $\mathrm{div}_{\mathbb{H}}$ only acts on the so-called horizontal vector fields. For these fields it equals div_ε independently of $\varepsilon > 0$. The Riemannian volume vol_ε is left-invariant with respect to translation. So up to a constant it is the Lebesgue measure of $\mathbb{R}^3$.

The same definitions make also sense for $\mathbb{T}^\varepsilon$ approximating the Albanese torus and for the approximating manifolds $\mathbb{H}_n^\varepsilon$ of $\mathbb{H}_n$. In Chapter 3 we will see that basically when $\varepsilon > 0$ tends to 0, $(\mathbb{H}_n^\varepsilon, d_\varepsilon)$ tends to $(\mathbb{H}_n, d_c)$ in a special topology of metric spaces, namely the Gromov-Hausdorff topology.

1.3 A naïve understanding of $\mathbb{H}_1$

In this section, we insist on the link between $\mathbb{H}_1$ and $\mathbb{R}^2$ and the role played by the complex projection Z. (The similar link exists between $\mathbb{H}_n$ and $\mathbb{R}^{2n}$).

1.3.1 Horizontal curves, lengths and distances

We defined in Section 1.1 an horizontal curve of $\mathbb{H}_1$ as an absolutely continuous curve γ of $\mathbb{R}^3$ whose derivative $\gamma'(s)$ can be written for almost every s as

$$\dot{\gamma}(s) = a(s)\mathbf{X}(\gamma(s)) + b(s)\mathbf{Y}(\gamma(s)).$$

In the Euclidean basis this becomes

$$\dot{\gamma}(s) = a(s)\frac{\partial}{\partial x} + b(s)\frac{\partial}{\partial y} + \frac{1}{2}\left(x(s)b(s) - y(s)a(s)\right)$$

where $\gamma(s) = (x(s), y(s), t(s))$. Then for almost every s

$$\begin{cases} a(s) = \dot{x} \\ b(s) = \dot{y} \\ \quad \dot{t} = \dfrac{x\dot{y} - y\dot{x}}{2} \end{cases}$$

and for an horizontal curve and for $s < s'$

$$t(s') = t(s) + \int_s^{s'} d\mathcal{A}(\gamma) \tag{1.14}$$

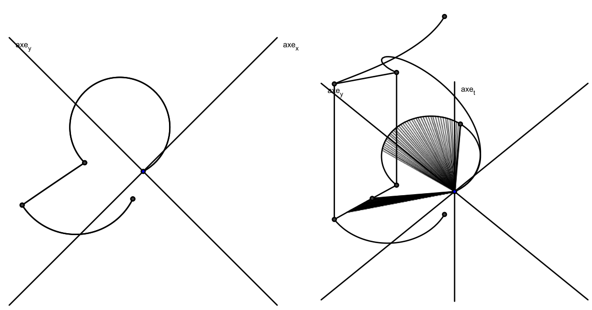

Figure 1.1: Horizontal lift of a planar curve

where $d\mathcal{A} = \frac{xdy - ydx}{2}$ is the algebraic area differential form. Alternatively we could define an horizontal curve of $\mathbb{H}_1$ as an absolutely continuous curve verifying (1.14). Now, we observe that it is enough to know $\gamma(s_0)$ at some time s_0 and the projected curve $\gamma^{\mathbb{C}} = Z(\gamma)$ to characterize an horizontal curve. We use for that (1.14) where γ is replaced by $\gamma^{\mathbb{C}}$. If α is an absolutely continuous curve of $\mathbb{R}^2$ (we will say *planar curve*), we will denote by $\mathrm{Lift}_p(\alpha)$ the horizontal curve with projection α such that $\mathrm{Lift}_p(\alpha)(s_0) = p$ for some initial time s_0 and p satisfying $\alpha(s_0) = Z(p)$. The map Lift will be called *horizontal lift* or $\mathbb{H}$-lift

Lemma 1.3.1. *Let γ be an horizontal curve. Then*

$$\mathrm{length}_c(\gamma) = \mathrm{length}^{\mathbb{C}}(Z(\gamma))$$

where $\text{length}^{\mathbb{C}}$ *is the usual Euclidean length of* $\mathbb{R}^2$*.*

Similarly for a planar curve α*,*

$$\text{length}^{\mathbb{C}}(\alpha) = \text{length}_c(\text{Lift}(\alpha)).$$

Proof. If $\dot{\gamma}(s) = a(s)\mathbf{X} + b(s)\mathbf{Y}$ then $Z(\dot{\gamma})(s) = a(s)\frac{\partial}{\partial x} + b(s)\frac{\partial}{\partial y}$. The length of both is $\int \sqrt{a^2 + b^2}$. $\square$

1.3.2 Commutation relations

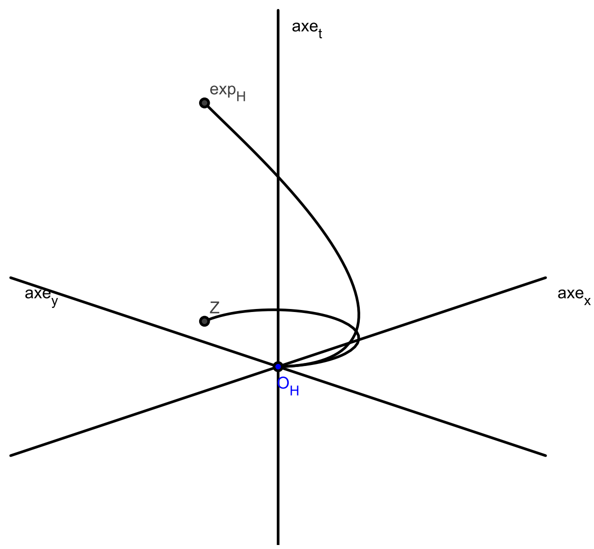

Figure 1.2: Lift of an arc of circle

The complex projection Z almost commutates with dil_λ, tran_p, rot_θ and sym. In fact we have the following rules:

$$Z(\text{dil}_\lambda(z;t)) = \text{dil}^{\mathbb{C}}_\lambda(z) \quad \text{and} \quad Z(\text{tran}_p(z;t)) = \text{tran}^{\mathbb{C}}_{Z(p)}(z)$$
$$Z(\text{rot}_\theta(z;t)) = \text{rot}^{\mathbb{C}}_\theta(z) \quad \text{and} \quad Z(\text{sym}(z;t)) = \text{sym}^{\mathbb{C}}(z)$$

where

$$\text{dil}^{\mathbb{C}}_\lambda(z) = \lambda z$$
$$\text{tran}^{\mathbb{C}}_{a+\mathbf{i}b}(z) = a + \mathbf{i}b + z$$
$$\text{rot}^{\mathbb{C}}_\theta(z) = e^{\mathbf{i}\theta} z$$

and $\text{sym}^{\mathbb{C}}$ is the complex conjugation $z \to \bar{z}$. As a consequence we have similar relations for Lift_p (defined just above):

$$\text{dil}_\lambda(\text{Lift}_p(\alpha)) = \text{Lift}_{\text{dil}_\lambda(p)}(\text{dil}^{\mathbb{C}}_\lambda(\alpha))$$
$$\text{tran}_q(\text{Lift}_p(\alpha)) = \text{Lift}_{q\cdot p}(\text{tran}^{\mathbb{C}}_{Z(p)}(\alpha))$$
$$\text{rot}_\theta(\text{Lift}_p(\alpha)) = \text{Lift}_{\text{rot}_\theta}(p)(\text{rot}^{\mathbb{C}}_\theta(\alpha))$$

1.3.3 Parallelogram rule

Now, we give a naïve interpretation of the product of $\mathbb{H}_1$ written on the form

$$(x,y,t)\cdot(x',y',t') = (z+z';t+t'+\frac{xy'-yx'}{2}).$$

In this subsection we try to see it as a the parallelogram rule. For that, we first consider the set of planar curves starting from $0_{\mathbb{H}}$ and defined on segments $[0,\tau]$ for some $\tau \geq 0$. We denote this set by $\mathcal{PC}$ and consider it with the catenation of curves $*$. The catenated curve $\alpha_1 * \alpha_2$ is obtained as the catenation of α_1 with the translated curve $\alpha_1(\tau_1)+\alpha_2$. Then if α_1 and α_2 are defined on $[0,\tau_1]$ and $[0,\tau_2]$ respectively, $\alpha_1 * \alpha_2$ is defined on $[0,\tau_1+\tau_2]$. Observe that the curve $s \in [0,0] \to 0$ is the unique neutral element. But $(\mathcal{PC},*)$ is not a group yet. We obtain a group when we identify the curves with the same two ends. The quotient is commutative and just isomorphic to $(\mathbb{R}^2,+)$. Another equivalence will bring something more interesting : the relation $\alpha_1 \sim \alpha_2$ will be

$$\begin{cases} \alpha_1(\tau_1) = \alpha_2(\tau_2) \\ \displaystyle\int_0^{\tau_1}(x_1\dot{y_1} - y_1\dot{x_1})ds = \int_0^{\tau_2}(x_2\dot{y_2} - y_2\dot{x_2})ds \end{cases} \tag{1.15}$$

where $\alpha_i = (x_i,y_i)$ for $i \in \{1,2\}$. Then $F : \alpha \in \mathcal{PC} \to (\alpha(\tau), \frac{1}{2}\int_0^\tau (x\dot{y} - y\dot{x}))$ is onto and two curves have the same image if and only if they are equivalent. The point $F(\alpha)$ is actually $\mathrm{Lift}_{0_{\mathbb{H}}}(\alpha)(\tau)$, that is the end point of the horizontal lift starting from $0_{\mathbb{H}}$ and F induces a bijection between the equivalence classes and $\mathbb{H}_1$.

Proposition 1.3.2. *The equivalence relation $\sim$ is compatible with the catenation $*$ and $(\mathcal{PC},*)/\sim$ that we denote by $(\widetilde{\mathcal{PC}},\tilde{*})$ is isomorphic to $(\mathbb{H}_1,\cdot)$.*

Proof. We compute now the equivalence class of $\alpha_1 * \alpha_2$ for α_1 and α_2 two elements of $\mathcal{PC}$. The third coordinate of $F(\alpha_1 * \alpha_2)$ is the half of

$$\int_0^{\tau_1+\tau_2}(\alpha_1 * \alpha_2)_x(\alpha_1 \dot{*} \alpha_2)_y - (\alpha_1 * \alpha_2)_y(\alpha_1 \dot{*} \alpha_2)_x =$$
$$\int_0^{\tau_1}(x_1\dot{y_1} - y_1\dot{x_1})ds + \int_0^{\tau_2}\left[(x_1(\tau_1)+x_2(s))\dot{y_2}(s) - (y_1(\tau_1)+y_2(s))\dot{x_2}(s)\right]ds =$$
$$\int_0^{\tau_1}(x_1\dot{y_1} - y_1\dot{x_1})ds + \int_0^{\tau_2}(x_2\dot{y_2} - y_2\dot{x_2})ds + [x_1(\tau_1)y_2(\tau_2) - x_2(\tau_2)y_1(\tau_1)]$$

Then for $F(\alpha_1) = (X_1,Y_1,T_1)$ and $F(\alpha_2) = (X_2,Y_2,T_2)$, we have proved that

$$F(\alpha_1 * \alpha_2) = (X_1+X_2, Y_1+Y_2, T_1+T_2+\frac{1}{2}(X_1Y_2 - X_2Y_1)).$$

This expression only depends on $F(\alpha_1)$ and $F(\alpha_2)$ which means only on the classes of α_1 and α_2. Then the equivalence relation is compatible with $*$ and the quotient multiplicative structure is isomorphic to $(\mathbb{H}_1,\cdot)$. ☐

On the figure 1.3 we see that the algebraic area swept by $\alpha_1 * \alpha_2$ is the one swept by each curve plus the algebraic area of the triangle $0\alpha_1(\tau_1)(\alpha(\tau_1)+\alpha_2(\tau_2))$ that is $\frac{(x_1(\tau_1)y_2(\tau_2)-x_2(\tau_2)y_1(\tau_1))}{2}$.

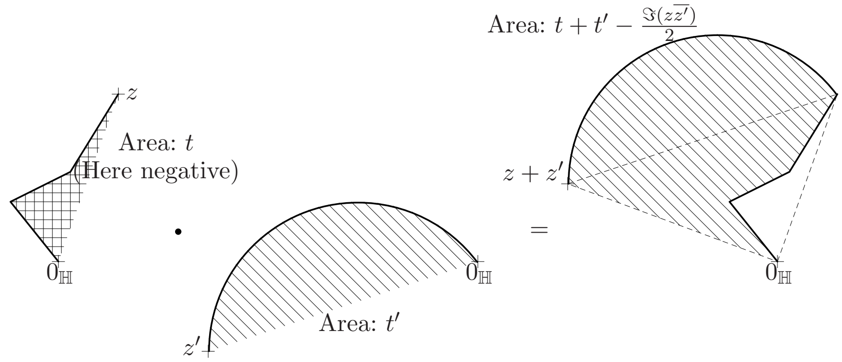

Figure 1.3: The area swept by the catenation of two curves

We continue now our naïve interpretation of $\mathbb{H}_1$ and are now interested in the metric aspect. As we explained $\mathbb{R}^2$ can be seen as a quotient of $\mathcal{PC}$. Then we recover the Euclidean norm by taking the minimum of the length of the curves in an equivalence class : the length of the straight line is the norm of the class. For the Heisenberg group, it is exactly the same : the homogeneous norm (Subsection 1.1.3) $\|.\|_c = d_c(0_{\mathbb{H}}, \cdot)$ of $F(\alpha) = \mathrm{Lift}_{0_{\mathbb{H}}}(\alpha)(\tau)$ is the shortest length in $\mathbb{C}$ for an equivalent curve β in the class of α. Indeed any horizontal curve γ starting in $0_{\mathbb{H}}$ goes to $F(\alpha)$ if and only if its complex projection $\beta = Z(\gamma)$ is in the class of α and the planar length of β is $\mathrm{length}_c(\gamma)$. In Subsection 1.5.2, we will see that the β of minimizing length is an arc of circle.

More generally the Carnot-Carathéodory distance between the classes of α_1 and α_2 is the minimum length for an horizontal curve γ from $F(\alpha_1)$ to $F(\alpha_2)$. If we denote the planar projection by $\beta = Z(\gamma)$, the horizontal lift $\mathrm{Lift}_0(\alpha_1 * \beta)$ starts from $0_{\mathbb{H}}$ goes by $F(\alpha_1)$ at time τ_1 and finishes in $F(\alpha_2)$. Actually we want $F(\alpha_1 * \beta) = F(\alpha_2)$ which is also

$$F(\beta) = F(\alpha)^{-1} \cdot F(\alpha_2) = F(\bar{\alpha_1} * \alpha_2)$$

where $\bar{\alpha} : s \in [0, \tau] \to \alpha(\tau - s) - \alpha(\tau)$. The distance between $F(\alpha_1)$ and $F(\alpha_2)$ is then the minimum length of a curve β in the class of $\bar{\alpha_1} * \alpha_2$.

Remark 1.3.3. The action on $\mathcal{PC}$ of the planar transformations of Subsection 1.3.2 has the expected interpretation on $(\widetilde{\mathcal{PC}}, \tilde{*}) = (\mathbb{H}_1, \cdot)$. For example if you dilate a planar curve α with $\lambda = 1/2$, the curve you obtain will sweep an algebraic area four time smaller than the first one. Then this transformation leaves the equivalence $\sim$ invariant and the quotient map is given by $\mathrm{dil}_{1/2}$. Generally for $\alpha \in \mathcal{PC}$, $\lambda > 0$ and $\theta \in \mathbb{R}$:

$$\begin{aligned} F(\mathrm{dil}^{\mathbb{C}}_{\lambda}(\alpha)) &= \mathrm{dil}_{\lambda}(F(\alpha)) \\ F(\mathrm{rot}^{\mathbb{C}}_{\theta}(\alpha)) &= \mathrm{rot}_{\theta}(F(\alpha)) \\ F(\mathrm{sym}^{\mathbb{C}}(\alpha)) &= \mathrm{sym}(F(\alpha)). \end{aligned}$$

1.4 Hausdorff dimensions of some subsets of $\mathbb{H}_1$.

In order to give a better idea of the strange geometry of the Heisenberg group, we will compute the Hausdorff dimension of the affine subspaces of $\mathbb{H}_1$ and of some other sets.

The Hausdorff measures $(\mathcal{H}_d^m)_{m\in[0,\infty[}$ of a metric space (X,d) are a family of outer measure what are defined by

$$\mathcal{H}_d^m(E) = \sup_{\varepsilon>0} \inf \left\{ \sum_{i\in\mathbb{N}} \operatorname{diam}(E_i)^m : E \subset \bigcup_{i\in\mathbb{N}} E_i \text{ and } \operatorname{diam}(B_i) \leq \varepsilon \right\}$$

where $\operatorname{diam}(B_i)$ is the diameter of B_i. The function $m \to \mathcal{H}_d^m(E)$ is decreasing and take the values $+\infty$ and 0 except maybe at the critical point m_0 where $\mathcal{H}_d^{m_0}(E)$ can be $+\infty$, 0 or a finite value. This critical value m_0 is the Hausdorff dimension of E. It is invariant in a equivalence class of distances. For the Heisenberg group, a good distance is d_∞ of Example 1.1.11 because it makes the computations of the dimension easier.

One can compute the Hausdorff by using measures (actually outer measures that are nonnegative countably subadditive set function defined on all subsets of a metric space). The next lemma sometime called Moran lemma will involve the so-called local Ahlfors n-regularity of a metric space. It can be defined as follow : there is a constant $C \geq 1$ and a constant $T > 0$ such that for every ball $B(p,R)$ whose radius R satisfies $0 < R < T$ we have

$$C^{-1}R^n \leq \mu(B(p,R)) \leq CR^n.$$

We also explain what is Borel regularity : the open set are measurable and every set is contained in a Borel set with the same measure. Let us now state the lemma.

Lemma 1.4.1 (Moran)**.** *If μ is a Borel regular measure on a metric space X satisfying the local Ahlfors n-regular property then the Hausdorff dimension of X is n.*

A proof of this lemma can be found in [57]. It essentially require some covering theorems. Let us now look at some computations of the dimension.

Example 1.4.2. The Hausdorff dimension of $\mathbb{H}_1$ is 4.

Proof. We use for this the Lebesgue outer measure. It is Borel regular thanks to corollary 1.1.7. We have already observed that the translations do not change the Lebesgue measure and that a dilation $\operatorname{dil}_\lambda$ multiplies it by λ^4. Then

$$\mathcal{L}^3(B(p,R)) = \mathcal{L}^3(\operatorname{tran}_p \circ \operatorname{dil}_R(B(0,1))) = R^4\mathcal{L}^3(B(0,1)).$$

But $\mathcal{L}^3(B(0,1))$ is finite and non-zero (d_c is equivalent to d_∞ whose balls are cylinders). Then the Hausdorff dimension of $\mathbb{H}_1$ is 4. $\square$

For the same reason $\mathbb{H}_n$ has dimension $2n+2$.

Before we begin the computation of the Hausdorff of the not trivial linear subspaces, we will reduce the computation to some representative cases. We already stressed that the translations of $\mathbb{H}_1$ are affine maps with maximal rank.

Then $\mathbb{H}_1$ acts by left translations on the affine subspaces of $\mathbb{R}^3$ of given rank (1 or 2). Each translation is also an isometry such that in an orbit the Hausdorff dimension does not change. The rotations rot_θ are other maps that are both linear diffeomorphism and isometries of $\mathbb{H}_1$.

Let H_1 (respectively H_2) the set of the lines and H_2 (respectively the planes) of $\mathbb{R}^3$. We denote the orbit of a set E (line or plane) under the action of translations by $\mathrm{Orb}_{\mathrm{tran}}(E)$ and under translations and rotations by $\mathrm{Orb}_{\mathrm{tran}}^{\mathrm{rot}}(E)$.

Lemma 1.4.3. *Let $l, l' \in H_1$, $(p,p') \in l \times l'$ and $a\mathbf{X}(p)+b\mathbf{Y}(p)+c\mathbf{T}(p)$ directing the line l in p (respectively $a'\mathbf{X}(p')+b'\mathbf{Y}(p')+c'\mathbf{T}(p')$ directing l' in p'). then $l' \in \mathrm{Orb}_{\mathrm{tran}}(l)$ if and only if (a,b,c) and (a',b',c') are collinear. Moreover, in this case $\mathrm{tran}_{p\cdot p'^{-1}}(l') = l$.*

Proof. Consider $l_0 = \mathrm{tran}_{p^{-1}}(l)$ and $l'_0 = \mathrm{tran}_{p'^{-1}}(l')$. Both lines are going through $0_{\mathbb{H}}$. They are directed in this point by $a\mathbf{X}(0)+b\mathbf{Y}(0)+c\mathbf{T}(0)$ and $a'\mathbf{X}(0)+b'\mathbf{Y}(0)+c'\mathbf{T}(0)$ because the vector fields $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{T}$ are left-invariant. Then if (a,b,c) and (a',b',c') are collinear, the lines are on the same orbit. Assume by contradiction that l_0 and l'_0 are not collinear and that nevertheless there is a $q \in \mathbb{H}_1$ such that $\mathrm{tran}_q(l) = l'$. Then for $q_0 = p'_0 \cdot q \cdot p_0^{-1}$, $\mathrm{tran}_{q_0}(l_0) = l'_0$. Since $0_{\mathbb{H}}$ is in l'_0, we know that q_0^{-1} is in l_0. Hence $q_0 \in l_0$ because l_0 is a subgroup of $\mathbb{H}_1$. Finally $l'_0 = \mathrm{tran}_{q_0}(l_0) = l_0$ which is a contradiction of our assumption that l_0 and l'_0 are not collinear. $\square$

Remark 1.4.4. Apply this lemma to $l = l'$ and $p \neq p'$: you see that on a line the coordinates of a tangent vector in the frame $(\mathbf{X}, \mathbf{Y}, \mathbf{T})$ are all collinear. In particular a line whose direction has a $\mathbf{T}$ coordinate equal to 0 is horizontal. We call it a $\mathbb{H}$-*line*. The $\mathbb{H}$-line are horizontal lifts of the lines of $\mathbb{R}^2$ because they are horizontal and their Z-projections are planar lines. Conversely one can easily check that there is a $\mathbb{H}$-line going through any point p in any direction $a\mathbf{X}(p)+b\mathbf{Y}(p)$. Hence by using the uniqueness, the horizontal lift of a planar line is a $\mathbb{H}$-line.

Therefore it is possible to represent the orbits of H_1 under the action of $\mathbb{H}_1$ by the lines going through $0_{\mathbb{H}}$. If one add now the action of the rotations, a set of representantatives are the lines going through $0_{\mathbb{H}}$ that are spanned by $a\mathbf{X}+b\mathbf{T}$ with $a \geq 0$ and $a^2+b^2=1$ (and $b=1$ if $a=0$).

We recall that a vertical plane is a plane that contains a line directed by $\mathbf{T}$.

Lemma 1.4.5. *A plane P is in $\mathrm{Orb}_{\mathrm{tran}}(\mathbb{C}\times\{0\})$ or it is vertical. Two vertical planes are in the same orbit if and only if they are parallel in $\mathbb{R}^3$ but all vertical planes are in $\mathrm{Orb}_{\mathrm{tran}}^{\mathrm{rot}}(\mathbb{R}\times\{0\}\times\mathbb{R})$.*

Proof. Let P be a non vertical plane and $p \in P$. Then there are a and b such that $\mathbf{X}(p)+a\mathbf{T}(p)$ and $\mathbf{Y}(p)+b\mathbf{T}(p)$ span P in p. In $q = (-2b, 2a, 0)$, the plane $\mathbb{C}\times\{0\}$ is tangent to $(\frac{\partial}{\partial x} - a\frac{\partial}{\partial t}) + a\mathbf{T} = \mathbf{X}(q)+a\mathbf{T}(q)$ and to $\frac{\partial}{\partial y} - b\frac{\partial}{\partial t} + b\mathbf{T} = \mathbf{Y}(q)+b\mathbf{T}(q)$. Then $\mathrm{tran}_{q\cdot p^{-1}}(P) = \mathbb{C}\times\{0\}$.

A translation of $\mathbb{H}_1$ is a translation of $\mathbb{C}$ for the two first coordinates and something more intricate for the t-coordinate. Two parallel vertical plane are then obviously in the same orbit.

We can translate any two vertical planes to two other containing the center L. A rotation of one of them on the second finish the proof. $\square$

Proposition 1.4.6. *Every plane $P \in H_2$ has Hausdorff dimension* 3. *The $\mathbb{H}$-lines have dimension* 1 *and the other lines have dimension* 2.

Proof. Because of Lemma 1.4.5, for the planes it is enough to compute the dimensions of $\mathbb{C} \times \{0\}$ and $\mathbb{R} \times \{0\} \times \mathbb{R}$. For the lines Lemma 1.4.3 it is enough to make it for the one going through $0_{\mathbb{H}}$ and directed by $a\mathbf{X} + b\mathbf{T}$. For $a = 0$, we are considering L. For $b = 0$, it is a $\mathbb{H}$-line. In what follows, we will often use the Hausdorff dimension of a set with respect to the distance d_∞ defined in example 1.1.11. It does not change anything because this distance is equivalent to d_c.

$\mathbb{R} \times \{0\} \times \mathbb{R}$ As explained in Subsection 1.2.1, this set is isomorphic to $\mathbb{R}^2$ and for points $(x;t)$ and $(x';t')$ of this vertical plane,

$$d_\infty((x;t),(x';t')) = \|(x - x', t - t')\|_\infty = \max(|x - x'|, |t - t|^{1/2}).$$

Any ball of radius R is then a rectangle of area $2R \cdot 2R^2 = 4R^3$

$\mathbb{C} \times \{0\}$ We prove that the dimension is 3 because for all $0 < r < R$, the usual Lebesgue measure on $\mathbb{C} \times \{0\}$ is local Ahlfors 3-regular on the annulus $\{(z;0) \in \mathbb{H}_1,\, r \geq |z| \geq R\}$. We use one more time the metric d_∞. What is the ball with center $(z;0)$ and radius R?

$$d_\infty((z;0),(z';0)) = \|(z - z'; \frac{1}{2}\Im(z\overline{z'})\|_\infty \leq R \Longleftrightarrow \begin{cases} |z - z'| \leq R \\ \frac{1}{2}\Im(z\overline{z'}) \leq R^2 \end{cases}.$$

Then this ball is the intersection of the Euclidean circle of radius R and a band (intersection of two half-plane) with center $(z;0)$ and of width $\frac{R^2}{|z|}$. With the Moran lemma we conclude that the dimension of $\mathbb{C} \times \{0\}$ is 3.

Lines As explained in Subsection 1.2.1 the distance d_∞ between $\lambda(a,0,b)$ and $\mu(a,0,b)$ is $\|(\lambda - \mu)(a,0,b)\|_\infty$. But

$$d_\infty(\nu(a,0,b)) = \max(|\nu a|, |\nu b|^{1/2}) = \begin{cases} |\nu a| & \text{if } |\nu| \geq |b/a^2| \\ |\nu b|^{1/2} & \text{if } |\nu| \leq |b/a^2| \end{cases}$$

Thus if $b \neq 0$ the balls of radius $R \leq \frac{|b|}{|a|}$ have a one dimensional Lebesgue measure $\mathcal{L}^1$ equal to $2\sqrt{a^2 + b^2}\frac{R^2}{|b|}$. With the Moran lemma, we conclude that the dimension is 2.

In the case of $\mathbb{H}$-lines ($b = 0$), the restriction of d_∞ is isometric to the distance on $\mathbb{R}$ and the dimension is 1.

$\square$

Remark 1.4.7. As a consequence of Proposition 1.4.6, the dimension of L, that is the center of $\mathbb{H}_1$ is 2. A more direct proof is to consider that d_∞ restricted to L is of the form $d^{1/2}$ where (L, d) is isomorphic to $\mathbb{R}$ with the classical distance. It follows from the general theory that the Hausdorff dimension of (L, d_c) is $2 = 1/(1/2)$, that is the dimension of $\mathbb{R}$ through the exponent $1/2$.

Remark 1.4.8. Any line $\{(\mu z;\mu t)\in\mathbb{H}_n \mid \mu\in\mathbb{R}\}$ of the n-th Heisenberg group is also a subgroup of $\mathbb{H}_n$. For $t=0$ it is isometric to $\mathbb{R}$ because $d_c((\mu z;0),(\lambda z;0)) = |t-t'|\cdot|z|$. Else

$$\begin{aligned} d_c((\mu z;\mu t),(\lambda z;\lambda t)) =& d_c(0_{\mathbb{H}},((\mu-\lambda)z;(\mu-\lambda)t)) \\ \sim& 2\sqrt{\pi|\mu-\lambda|\cdot|t|} \end{aligned} \tag{1.16}$$

when $|\mu-\lambda|$ tends to 0. Actually we will see in Section 1.5 that $d_c(0_{\mathbb{H}},(0;t)) = 2\sqrt{\pi|t|}$ and $d_c((0;t),(z;t)) = |z|$ what proves (1.16).

Let us compute the dimension for a last surface that is different from the planes.

Example 1.4.9. We consider $\{(z;t)\in\mathbb{H}_1,\, |z|=1\}$. We take the d_∞ metric so that the distance between $(e^{i\theta},t)$ and $(e^{i\theta'},t')$ is

$$\max\left(|e^{i\theta'}-e^{i\theta}|, |t'-t+\frac{1}{2}\sin(\theta-\theta')|^{1/2}\right).$$

Now, we will prove that the Hausdorff dimension of this set is 3 using on the cylinder the surface volume μ defined for any $\mathcal{C}^2$-submanifold of $\mathbb{R}^3$. That measure is Borel regular. Let us consider now the ball B_R with center $(e^{i\theta},t)$ and radius $R<1$. Thus

$$\begin{aligned} B_R =& \left\{(e^{i\theta'},t'),\, |e^{i\theta}-e^{i\theta'}|\le R\right\}\bigcap \\ & \left\{(e^{i\theta'},t'),\, t-R^2-\frac{1}{2}\sin(\theta-\theta')\ \le\ t'\le t+R^2-\frac{1}{2}\sin(\theta-\theta')\right\} \end{aligned}$$

By puzzling, we find that B_R have the same area as

$$\left\{(e^{i\theta'},t'),\, |e^{i\theta}-e^{i\theta'}|\le R\right\}\bigcap\left\{(e^{i\theta'},t'),\, t-R^2\le t'\le t+R^2\right\}$$

so that $\mu_{cy}(B_R) = 8R^2\cdot\arcsin(R/2)$. Finally, we have for $R<1$,

$$4R^3\le\mu(B_R)\le 2\pi R^3.$$

There is a result of Gromov in [52] saying that every set in $\mathbb{H}_1$ with topological dimension 2 has Hausdorff dimension greater or equal to 3. In fact every embedded smooth surface of $\mathbb{R}^3$ has exactly dimension 3 as suggests the following coarea formula that can be found in [56]

Proposition 1.4.10. *Let f be a smooth function and u a nonnegative measurable function f of $\mathbb{H}_n$. Then*

$$\int_{\mathbb{H}_n} u(p)\|\nabla_{\mathbb{H}}f(p)\|_{\mathbb{H}}d\mathcal{H}^{2n+2} = \int_0^{+\infty}\int_{\{f=t\}} u(q)d\mathcal{H}^{2n+1}(q)dt.$$

In a recent paper [12], Balogh, Tyson and Warhurst solve the problem to know what are the possible pairs (α,β) where α is the Euclidean and β the subRiemannian Hausdorff dimension of a subset of $\mathbb{H}_n$. They solved actually more generally the problem in the setting of Carnot groups. But the original open problem of Gromov to describe the pairs (α,β) for smooth submanifolds of a Carnot group is still open.

1.5 Geodesics

We begin with some definitions. A geodesic in a metric space (X, d) is a curve γ defined on an interval $I \subset \mathbb{R}$ such that for any four points s, t, s', t' of I,

$$|t' - s'|d(\gamma(s), \gamma(t)) = |t - s|d(\gamma(s'), \gamma(t')).$$

If I is $[0, 1]$, this definition is equivalent to the following : for any s and t,

$$d(\gamma(s), \gamma(t)) = |t - s|d(\gamma(0), \gamma(1)).$$

A metric space is called geodesic if there is a geodesic γ connecting each pair of points (p, q). In a geodesic metric space a s-intermediate point between p and q is any point $\gamma(s)$ such that γ is a geodesic with $\gamma(0) = p$ and $\gamma(1) = q$.

A curve such that in the neighbourhood of every time s the restriction of the curve is a geodesic is called a local geodesic. It is sometime just called geodesic.

Continuous non-decreasing reparametrizations of such curves as before are often called geodesics too. In particular in a geodesic space, if $\tilde{\gamma}$ is a reparametrization, then the distance $d(\tilde{\gamma}(a), \tilde{\gamma}(b))$ is not only smaller than but it is equal to the metric length of $\tilde{\gamma}$. This metric length is defined by

$$\lim_{\varepsilon \to 0} \inf_{\sigma} \sum d(\tilde{\gamma}(\sigma_i), \tilde{\gamma}(\sigma_{i+1})) \tag{1.17}$$

where $(\sigma_i)_{0 \leq i \leq n}$ is a partition starting in $\sigma_0 = a$ and ending in $\sigma_n = b$ such that $|\sigma_{i+1} - \sigma_i| \leq \varepsilon$ for every $i < n$. As is proved by Korányi in [69], in the Heisenberg group the length of an absolutely continuous curve of $\mathbb{R}^3$ is the same if you compute it with the metric formula (1.17) or with the subRiemannian one (1.3). In particular, if the curve is not horizontal, the length is infinite. Actually it is also the length of a curve computed with the Koranyi-Reimann distance d_{KR} of Subsection 1.1.3.

In this section, we will prove that the metric space $(\mathbb{H}_n, d_c)$ is geodesic. Metric geodesics of $\mathbb{H}_n$ (as defined in the begining of this section) are certainly absolutely continuous because of metric estimates such as Proposition 1.1.6. Because of the above definitions and results, the length of these geodesics is $d_c(p, q)$. Therefore the infimum in the definition of the Carnot-Carthéodory distance (1.8) is in fact a minimum.

1.5.1 Dido's problem

In this subsection, we suppose that we know the planar isoperimetric problem and that its solutions are circles. We consider now a very old variant of this problem called Dido's problem [107]. It is related to the foundation of Carthage in Tunisia. It is written that Queen Dido and her followers arrived on a coast by the sea and that the local inhabitant allow her to stay in as much land as can be encompassed in an oxhide. Then Dido made a rope by cutting the oxhide into fine strips and encircle a wide domain of land. Finding the way to limit this piece of land is a variant of the isoperimetric problem and the optimal way is to make an arc of circle. However, the full circle is not optimal because it does not take advantage of the fact that the coast is a natural border. This classical problem of calculus of variation can be reformulated in the following way: consider the curves $\alpha : [0, 1] \to \mathbb{R}^2$ of given length l such that $\alpha(0) = 0_{\mathbb{C}}$ and $\Im(\alpha(1)) = 0$.

Then the problem is to maximize the algebraic area $\frac{1}{2}\int_0^1 \dot{\alpha} \times \alpha$. Actually in our problem of geodesics in the Heisenberg group we will be interested in the dual problem : find the shortest curve enclosing a given area. We can formulate the dual problem in this way: a curve α starts in $0 \in \mathbb{C}$ is defined on $[0,1]$, ends on the real axis ($\Im(\alpha(1)) = 0$) and has the algebraic area $\frac{1}{2}\int_0^1 \dot{\alpha} \times \alpha$. Under these constraints we want to minimize $\int_0^1 |\alpha'|$, that is the length of the curve.

We present here the solutions of Dido's problem and we will see one variant in the next paragraph. The key idea is to close the curve α by connecting it with its symmetric curve with respect to the real line. We obtain a closed curve whose swept area is two times the initial one.

$$\frac{1}{2}\int_0^1 \dot{\alpha} \times \alpha + \frac{1}{2}\int_1^0 \dot{\bar{\alpha}} \times \bar{\alpha} = 2 \cdot \frac{1}{2}\int_0^1 \dot{\alpha} \times \alpha$$

(Here, $\bar{\alpha}$ is the complex conjugated curve. It is not a curve with inverse parametrization as in Subsection 1.3.3.) The length of this curve is also twice the initial one. If the new curve is a circle, its length is the minimum among all the curve enclosing the same area. This fact is in particular true among the curves symmetric with respect to $y = 0$. It follows that the solution of the authentic Dido's problem is an half of circle. If we now consider the sign of the algebraic area, there are for a given starting point and a given area (positive or negative) exactly two solutions to the problem. These solutions are symmetric with respect to the starting point $0_{\mathbb{C}}$.

In the second version, we fix the two ends of the curve. Let us assume for example $\alpha(0) = 0_{\mathbb{C}}$ and $\alpha(1) = x$ for a given $x \in \mathbb{R}^*$. There is an unique arc of circle from the first to the second point that encloses the given algebraic area: for a positive area, the area between the line and the arc of circle is a strictly increasing and continuous function of the radius, for a negative area, it is strictly decreasing. We will prove now that this unique arc of circle is the shortest possible curve. Compare our candidate with another curve and connect both of them with the rest of the circle. Hence we have two closed curves enclosing the same area and one of them is a circle. The length of the circle is smaller. The arc of circle is then also shorter that the curve. We proved that the arc of circle of given area is the shortest curve in this restrictive version of Dido's problem. In the critical case $x = 0$, the problem is the classical isoperimetric problem. An infinity of circles are solution.

1.5.2 Geodesics of $\mathbb{H}_1$

The problem of the geodesics in $\mathbb{H}_1$ is very similar to Dido's problem. Let us first explicit what is the relation between geodesics and the minimizing curves in (1.8). After we will see the link with Dido's problem.

We have already explain that geodesics minimize the length in (1.8). Take it now in the other sense and reparametrize with constant speed on $[0,1]$ a curve γ that shall minimize the length. This new curve $\tilde{\gamma}$ has the same length and is minimizing too. It is even a geodesic. Actually any restriction of $\tilde{\gamma}$ to $[a,b] \subset [0,1]$ minimizes the length between its ends. If it does not, neither does the initial curve! With the constant speed parametrization, the distance $d_c(\tilde{\gamma}(a), \tilde{\gamma}(b))$ is also the time difference $|b-a|$ multiplied with the speed $d_c(\tilde{\gamma}(0), \tilde{\gamma}(0))$ which is the definition of a geodesic. Then the curves minimizing the length in the definition of the Carnot-Carathéodory distance are the absolutely continuous reparametrizations of geodesics between the same end points.

Finally we want to find a minimum in (1.8). In order to exhibit the relation with Dido's problem, we will use the complex projection Z, the horizontal lift Lift and the general philosophy of Section 1.3. The horizontal curves from $p = (z; t)$ to $q = (z'; t')$ are exactly the $\mathbb{H}$-lifts starting in p of those of the absolutely continuous planar curves connecting $z = Z(p)$ to $z' = Z(q)$ that enclose an algebraic area $t' - t$. Minimizing the length of these curves is the same as minimizing the length in this family of planar curves. This variational problem is strongly related to Dido's problem. In fact if α is a planar curve defined on $[a, b]$ from z to z', the area constraint is

$$t' - t = \frac{1}{2}\int_a^b \dot{\alpha} \times \alpha = \frac{1}{2}\int_a^b (\alpha - \dot{\alpha(a)}) \times (\alpha - \alpha(a)) + \frac{1}{2}(\alpha(b) - \alpha(a)) \times \alpha(a)$$

which is equivalent to

$$\frac{1}{2}\int_a^b (\alpha - \dot{\alpha(a)}) \times (\alpha - \alpha(a)) = t - t' - (\frac{1}{2}\alpha(b) \times \alpha(a)).$$

We made this change of origin by translation in order to see that just as in Dido's problem, the area between the curve α and the segment $[\alpha(a), \alpha(b)]$ represented by the left-hand side is a given area just depending on the ends p and q of the lifted curve $\mathrm{Lift}_p(\alpha)$. This area is exactly $t' - t - \frac{1}{2}\Im(z\overline{z'})$, that is the third coordinate of $p^{-1} \cdot q$ because up to a translation, connecting p to q is the same as connecting $0_{\mathbb{H}}$ to $p^{-1} \cdot q$. We can then claim after Subsection 1.5.1:

Proposition 1.5.1. *The geodesics of $\mathbb{H}_1$ are the horizontal lifts of the arc of circles parametrized with constant speed. These are just local geodesics if and only if the arc makes more than a full circle. The $\mathbb{H}$-lines are also geodesics and correspond to the degenerated case of the horizontal lift of a line.*

Let us say more about this proposition : the arc of circle we have considered in Dido's problem are part of a circle. Observe that if you turn two time on a circle of radius R, you have an area equal to $2 \cdot \pi R^2$ and the length squared is $(2\pi R)^2$. The quotient is $1/2\pi$. A circle of radius $\sqrt{2}R$ has the same area but its optimal isoperimetric quotient is $1/\pi$. A similar phenomenon occurs each time you consider an arc of circle making more than a full circle.

In Dido's problem, the case of an area equal to zero is solved by a segment. The horizontal lift of these solutions is a $\mathbb{H}$-line as explained in Remark 1.4.4.

Now, we will give the equations of these geodesics. Because translations are isometries, it is enough to make it for the geodesics starting from $0_{\mathbb{H}}$. If $v \in \mathbb{C}$ and $\varphi \in \mathbb{R}$,

$$\alpha_{v,\varphi}(s) = \begin{cases} v\frac{e^{i\varphi s}-1}{i\varphi} & \text{if } \varphi \neq 0 \\ sv & \text{else} \end{cases}$$

is the only constant speed parametrization of an arc of circle with tangent vector v in 0 and that draw an angle equal to φ on the time interval $[0, 1]$. It is not difficult to see that the algebraic area swept on $[0, s]$ is

$$\frac{1}{2}\int_0^s \alpha_{v,\varphi} \times \dot{\alpha_{v,\varphi}} = |v|^2 \left(\frac{\varphi s - \sin(\varphi s)}{2\varphi^2}\right)$$

or 0 if $\varphi = 0$ because it is the area of an angular sector plus the area of a triangle. Then we can parametrize the geodesics (local or global) starting in 0 with the two parameters $v \in \mathbb{C}$ and $\varphi \in \mathbb{R}$ (see Figure 1.4).

$$\gamma_{v,\varphi}(s) = \begin{cases} (v\frac{e^{i\varphi s}-1}{i\varphi}, |v|^2\left(\frac{\varphi s - \sin(\varphi s)}{2\varphi^2}\right)) & \text{if } \varphi \neq 0 \\ (sv, 0) & \text{otherwise.} \end{cases} \tag{1.18}$$

A geodesic $\gamma_{v,\varphi}$ is global on a segment $[a,b]$ if and only if $|b-a| \leq 2\pi/|\varphi|$ because on these intervals the projected curve $Z(\gamma_{v,\varphi})$ makes less than a circle. In particular $\mathbb{H}$-lines are global because they are done with $\varphi = 0$.

We define now the $\mathbb{H}$-exponential $\exp^{\mathbb{H}}$ map thanks to the point attained at time 1 by the geodesic $\gamma_{v,\varphi}$.

$$\exp^{\mathbb{H}}(v,\varphi) = \begin{cases} (v\frac{e^{i\varphi}-1}{i\varphi}, |v|^2\left(\frac{\varphi - \sin(\varphi)}{2\varphi^2}\right)) & \text{if } \varphi \neq 0 \\ (v, 0) & \text{otherwise.} \end{cases} \tag{1.19}$$

The notation exp is inspired from the Riemannian geometry where $\exp_p(\overrightarrow{v})$ is the end point of the unique constant-speed geodesic, parametrized on $[0,1]$, starting in p with velocity vector $\overrightarrow{v}$. In the case of the Heisenberg group, for any $p \in \mathbb{H}_1$ and any $v \in \mathbb{C}$, the curve $p \cdot \gamma_{v,\varphi}$ is geodesic tangent to $\Re(v)\mathbf{X}(p) + \Im(v)\mathbf{Y}(p)$ at time 0 and its end-point is $p \cdot \exp^{\mathbb{H}}(v,\varphi)$. However, there is not an unique geodesic tangent to $\Re(v)\mathbf{X}(p) + \Im(v)\mathbf{Y}(p)$ in 0 such that one have to parametrize these geodesics with φ. We write $\exp^{\mathbb{H}}$ and not $\exp_{\mathbb{H}}$ as in [7] where it appears for the first time, in relation to Theorem 2.2.4 because our convention about the definition of $\mathbb{H}_1$ is somewhat different. The same remark holds for $\exp^{\mathbb{H}}$ on $\mathbb{H}_n$ that will be defined in the next subsection.

1.5.3 Geodesics of $\mathbb{H}_n$

We prove now that $\mathbb{H}_n$ have also geodesics. For that we will not try to minimize the length but the energy of the curve :

$$E(\gamma) = \int_0^1 \|\dot{\gamma}\|_c^2.$$

Because of the Cauchy-Schwarz inequality we have

$$E(\gamma) \cdot \int_0^1 1^2 \geq \text{length}_c^2(\gamma)$$

with equality if 1 and $\|\dot{\gamma}\|$ are collinear what happens exactly when γ has a constant speed. Then a curve minimizing the energy for two fixed ends also minimizes the length and minimizing curves for the length can minimize the energy if you reparametrize them with constant speed. With our terminology curves minimizing the energy are exactly geodesics because they have constant speed. The energy is then the square of the length.

The projected curve on the n first coordinates is now a curve in $\mathbb{C}^n$ between some points of $\mathbb{C}^n$ that for simplicity we assume equal to 0 and some other $(z_1, \cdots, z_n) \in \mathbb{C}^n$. This projected curve $\alpha = (\alpha_1, \cdots, \alpha_n)$ allows us to know the original γ by using the horizontal lift.

$$\gamma(s) = \text{Lift}_{0_{\mathbb{H}}}(\alpha)(s) = \left(\alpha_1(s), \cdots, \alpha_n(s), \sum_{i=1}^n \frac{1}{2}\int_0^s \alpha_i \times \dot{\alpha_i}\right).$$

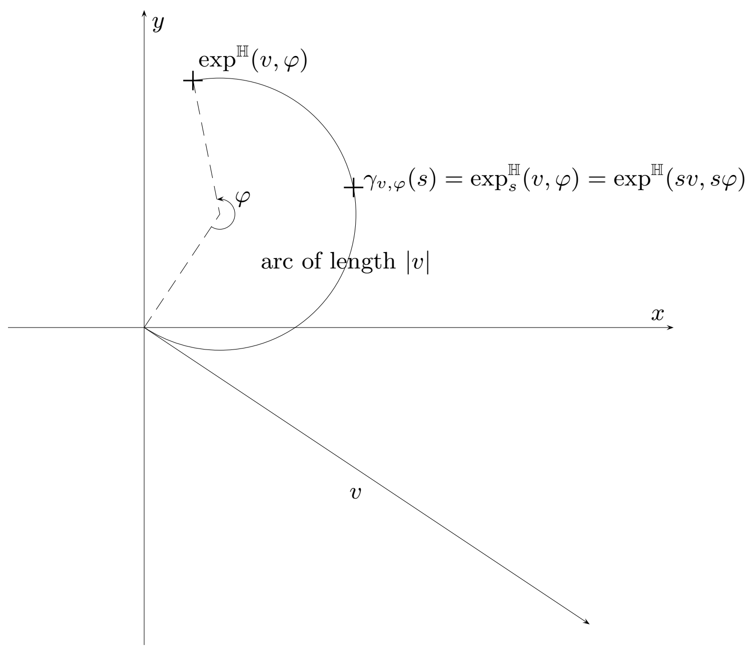

Figure 1.4: The exponential map $\exp^{\mathbb{H}}$

Moreover, $\|\dot\gamma\|^2_{\mathbb{H}} = |\dot\alpha|^2_{\mathbb{C}^n} = \sum_{i=1}^n |\dot\alpha_i|^2$ so that the length and the energy of a horizontal curve γ are simply the ones of $\alpha = Z(\gamma)$. Thus our new problem is for fixed ends $(0, z_i)_{0\le i\le n}$ and a given area

$$t = \sum_{i=1}^n \mathcal{A}(\alpha_i)$$

to minimize the energy of γ which is

$$E(\gamma) = \sum_{i=1}^n E(\alpha_i).$$

For a given n-tuple of areas $(\mathcal{A}_1, \cdots, \mathcal{A}_n)$ whose $\mathcal{A}$ is equal to t, we know from the subsection 1.5.2 that there is an optimal curve. It is the curve α whose coordinates α_i are arc of circles with the correct end and algebraic area $\mathcal{A}_i$, parametrized with constant speed on $[0,1]$. These coordinates are α_{v_i,φ_i} for some $(v_i,\varphi_i) \in \mathbb{C}\times\mathbb{R}$ and each z_i is then $Z\left(\exp^{\mathbb{H}}(v_i,\alpha_i)\right)$. This curve α has the minimum energy for a given n-tuple $(\mathcal{A}_1, \cdots, \mathcal{A}_n)$. This energy is

$$\sum_{i=1}^n |v_i|^2$$

because $|v_i|$ is the length of α_i (v_i is the initial speed and the curve is defined on $[0,1]$). But $z_i = v_i \frac{e^{i\varphi_i}-1}{i\varphi_i}$, so

$$|v_i|^2 = |z_i|^2 \frac{\varphi_i^2}{2(1-\cos(\varphi))}.$$

Now, we write also $\mathcal{A}$ as a function of $|z_i|$ and φ_i.

$$\mathcal{A} = \sum_{i=1}^{n} \mathcal{A}_i = \sum_{i=1}^{n} |v_i|^2 \frac{\varphi_i - \sin(\varphi_i)}{2\varphi_i^2} = \sum_{i=1}^{n} |z_i|^2 \frac{\varphi - \sin(\varphi)}{4(1-\cos(\varphi))}.$$

Hence we have a global energy E depending only on $(\varphi_1, \cdots, \varphi_n)$ and we want to minimize it under the constraint $\mathcal{A}(\varphi_1, \cdots, \varphi_n) = t$. Actually for the indices i such that $z_i = 0$, α_i is constant equal to 0. Without loss of generality, we can assume that $z_i \neq 0$ for any $i \in \{1, \ldots, n\}$ and $t > 0$. Then it is enough to study the variation of E on $]0, +\infty[^n$ because the $\mathcal{A}_i(\varphi_i)$ are even and non negative on $]0, \infty[$. Hence there are two Lagrange multipliers λ and μ with $(\lambda, \nu) \neq (0,0)$ such that for every φ_i

$$\lambda \frac{\partial E}{\partial \varphi_i} = \mu \frac{\partial \mathcal{A}}{\partial \varphi_i}.$$

Thus

$$\varphi_i \left(\frac{\lambda}{2}\right) \frac{2(1-\cos\varphi_i) - \varphi_i \sin\varphi_i}{(1-\cos\varphi_i)^2} = \left(\frac{\mu}{4}\right) \frac{2(1-\cos\varphi_i) - \varphi_i \sin\varphi_i}{(1-\cos\varphi_i)^2}.$$

We obtain then $\varphi_1 = \cdots = \varphi_n$. Eventually we get geodesics of the form

$$\gamma_{v,\varphi}(s) = \left(\frac{e^{i\varphi s}-1}{i\varphi} v, |v|^2 \left(\frac{\varphi s - \sin(\varphi s)}{2\varphi^2}\right)\right) \in \mathbb{C}^n \times \mathbb{R} \tag{1.20}$$

and $(sv, 0)$ if $\varphi = 0$ where $v = (v_1, \cdots, v_n) \in \mathbb{C}^n$ and a $\mathbb{H}$-exponential map

$$\exp^{\mathbb{H}}(v, \varphi) = \begin{cases} (\frac{e^{i\varphi}-1}{i\varphi} v, |v|^2 \left(\frac{\varphi - \sin(\varphi)}{2\varphi^2}\right)) & \text{if } \varphi \neq 0 \\ (v, 0) & \text{otherwise.} \end{cases} \tag{1.21}$$

We define $\exp_s^{\mathbb{H}}(v, \varphi)$ as $\exp^{\mathbb{H}}(sv, s\varphi) = \gamma_{v,\varphi}(s)$.

As in the case $n = 1$ illustrated by Figure 1.4, the curve $\gamma_{v,\varphi}$ is in $0_{\mathbb{H}}$ at time $s = 0$ and it is tangent to $v \in \mathbb{C}^n$. The angle φ indicates the circular angle that each complex coordinate z_i draw in $\mathbb{C}$ on the time interval $[0,1]$.

We set $D_1 := (\mathbb{C}^n \backslash \{0\}) \times] - 2\pi, 2\pi[$ and similarly $D_s := (\mathbb{C}^n \backslash \{0\}) \times] - 2s\pi, 2s\pi[$. From the results of this section, we know that $\exp^{\mathbb{H}}$ is one-to-one on D_1 and $\exp^{\mathbb{H}}(D_1) = \mathbb{H}_n \backslash L$. Moreover, for any $s \in [-1, 0[\cup]0, 1]$, the map $\exp_s^{\mathbb{H}}$ is one-to-one from D_1 to $\exp^{\mathbb{H}}(D_{|s|})$. Note also that $\exp^{\mathbb{H}}$ is an analytic map on D_1. We will see in Section 1.7 that $\exp^{\mathbb{H}}$ is actually a diffeomorphism by computing the Jacobian determinant of $\exp^{\mathbb{H}}$.

We state now Proposition 1.5.1 in a more formal and general way.

Proposition 1.5.2. *The geodesics of $\mathbb{H}_n$ are the curves $p \cdot \gamma_{v,\varphi}$. On $[a, b]$ with $b > a$, the geodesic is global and unique if $(b-a)|\varphi| < 2\pi$. It is global but not unique if $(b-a)|\varphi| = 2\pi$ (change v in any v' with $|v_i| = |v'_i|$ for each i) and it is just locally geodesic if $(b-a)|\varphi| > 2\pi$.*

1.6 Geodesics in other spaces

We report the reader to Subsection 1.2 for the definitions of the spaces considered in this section.

1.6.1 Geodesics in the discrete Heisenberg group

We consider here the discrete Heisenberg group with the distance d_{HZ}. There are projections and horizontal lift relations between $\mathbb{H}_1^{\mathbb{Z}}$ and $\mathbb{Z}^2$ exactly as between $\mathbb{H}_1$ and $\mathbb{R}^2$ (Section 1.3). The geodesics in the discrete case are sequences of points, but one can recover the exact definition of geodesic by adding vertices (of length 1) between the neighbours (the points at distance 1).

We will consider the geodesics between $0_{\mathbb{H}}$ and a point $(m, n, t) \in \mathbb{H}_1^{\mathbb{Z}}$ for m, n and t non-negative. The other cases follow immediately from this one. Because the length in the graph $\mathbb{Z}^2$ between $(0, 0)$ and (m, n) is $m + n$ and because the projection $Z : (m, n, t) \to (m, n)$ is 1-Lipschitz from $\mathbb{H}_1^{\mathbb{Z}}$ to $\mathbb{Z}^2$, any path of length $m + n$ from $0_{\mathbb{H}}$ to some point (m, n, t) is a geodesic. In fact if $t \leq \frac{mn}{2}$ there is such a path and the critical case $t = \frac{mn}{2}$ is obtained as the horizontal lift of the path that goes m time on the right and then n time up.

If $t > \frac{mn}{2}$, we can identify some other geodesics, namely pieces of square. These curves correspond to the solution of the discrete Dido's problem just as arcs of circle are solutions of the usual Dido's problem. In the discrete isoperimetric problem, one try to minimize A/l^2 where A is the algebraic area of closed curve and l its length. The solutions are squares with isoperimetric constant $1/16$. Suppose that $m \geq n$ and that $t = c^2 - \frac{mn}{2}$ for some $c \geq m$. Then the geodesic is unique from $0_{\mathbb{H}}$ to (x, y, t) and it is a piece of square of side length c. The displacement in $\mathbb{H}_1^{\mathbb{Z}}$ is the horizontal lift of the following sequence in $\mathbb{Z}^2$. We have to go $c - n$ down then c right, then c up and finally $c - m$ left.

If t is not of the type discussed above, the geodesics from $0_{\mathbb{H}}$ to (m, n, t) are not unique but it is possible to know the distance to zero. For instance for $t = c(c + 1) - \frac{mn}{2}$ where $c \geq m \geq n$, pieces of rectangles whose side lengths are c and $c + 1$ are geodesics. Some geodesics are little variation close to such rectangle paths.

It turns out that the distance of (m, n, t) to $0_{\mathbb{H}}$ is equal to

$$f\left(\max(|m|, |n|), \min(|m|, |n|), |t|\right)$$

where

$$f(m,n,t) = \begin{cases} m+n & \text{if } t \leq \frac{mn}{2} \\ 2(m+c)-(m+n) & \text{if } \frac{mn}{2} \leq \frac{mc}{2} < t + \frac{mn}{2} \leq \frac{m(c+1)}{2} \leq \frac{m^2}{2} \\ 4c+2-(m+n) & \text{if } \frac{m^2}{2} \leq \frac{c^2}{2} < t + \frac{mn}{2} \leq \frac{c(c+1)}{2} \\ 4(c+1)-(m+n) & \text{if } \frac{m^2}{2} \leq \frac{c(c+1)}{2} < t + \frac{mn}{2} \leq \frac{(c+1)^2}{2} \end{cases}$$

1.6.2 Geodesics on the Albanese torus

We give here two estimates on the length of the geodesics of $\mathbb{T}$. From equation (1.13), and the observation that the distance between two points of $\mathbb{H}_1^{\mathbb{Z}}$ is greater than 1, we see that if $d_c(p, q) < \frac{1}{2}$, the distance between the cosets $\tilde{p}$ and $\tilde{q}$ of p and q is exactly $d_c(p, q)$. Then the projections of the geodesics of length $|v|$ starting from p on $\mathbb{T}$ are minimizing geodesics at least if $|v| < 1/2$.

It is possible from any point p of $\mathbb{H}_1$ to roughly reach some point of $\mathbb{H}_\mathbb{Z}$ with a curve of length smaller than $\sqrt{2}/2 + \sqrt{2\pi} < 3$. For that we can reach a point of $\mathbb{Z}^2 \times \mathbb{R}$ and then lift a circle of $\mathbb{C}$ with area smaller than $1/2$. Then the projection on $\mathbb{T}$ of any geodesic of length greater than 3 is no longer a minimizing geodesic of $\mathbb{T}$. Indeed we just have proved that if p and q are the end points of a geodesic of $\mathbb{H}_1$ of length greater that 3, there is an element $p' \in p \cdot \mathbb{H}_1^\mathbb{Z}$ such that $d_c(p', q) < 3$. Then $d_c(\tilde{p}, \tilde{q}) = d_c(\tilde{p'}, \tilde{q}) < 3$ and the projection in $\mathbb{T}$ of the geodesic is not globally minimal.

1.6.3 Geodesics in the Grušin plane

Recall Subsection 1.2.5 for the notations. The motivation for this subsection arises partially from the paper of Agrachev, Boscain and Sigalotti [1] where the authors give for the Grušin plane the equations of the geodesics and determine the cut locus. In this paper the computations are left to the reader and the authors investigate more general cases. Nevertheless the graphics in [1] are very useful for the understanding of G and its geodesics.

We will then compute the geodesics of the Grušin plane using a very powerful tool of optimal control: the Pontryagin Maximum Principle (PMP). It is a very much evolved theorem of calculus of variations. In particular it is a way to obtain symplectic differential equations in the cotangent bundle for those curves whose energy variation is equal to zero. The theorem works actually for a wide class of problems and its specification to differential geometry allows to study distances defined from vector fields, such as the Grušin plane and the Heisenberg group.

The Pontryagin Maximum Principle (PMP) is quite difficult to enunciate. We will not do it, see for example [15]. Sometime like in the Heisenberg group, it is possible to find the geodesics with some special arguments of geometry and without too much computations. For example we obtained the geodesics of $\mathbb{H}_1$ through the projection Z and thanks to the isoperimetric geometry of $\mathbb{R}^2$. Here, we could do the same and find the geodesics of G^+ and G through the cylindrical projection Υ and the fact that we know the geodesics of $\mathbb{H}_1$. Instead of that we will just guess them with basic geometry and check after our guess with the PMP.

Let us start with G^+. We know that this space is geodesic because of the following facts. The distance between two points p and q is the one between the two circle classes $\Upsilon^{-1}(p)$ and $\Upsilon^{-1}(q)$ of $\mathbb{H}_1$. These circles are compact and d_c is continuous in the topology of $\mathbb{R}^3$ so that there are two points whose distance is the distances between the circles. We consider the geodesic γ between them, we project it and we obtain a geodesic of G^+, that is $\Upsilon(\gamma)$. So the geodesics of G^+ are projections of geodesics of $\mathbb{H}_1$ but one should notice that all projections of geodesics of $\mathbb{H}_1$ are not necessarily geodesics in G^+.

Some projections of curves by Υ are certainly geodesics, namely the one of the geodesics of $\mathbb{H}_1$ that go through L. All geodesics γ of $\mathbb{H}_1$ from $(0;t) \in L$ to a point of the circle class $\Upsilon^{-1}(p)$ have the same length since the circle $\mathcal{S}^1$ acts by rotation under these geodesics. As the cylindrical projection $\Upsilon(\gamma)$ does not depend on the geodesic γ we choose, and as $\{(0;t)\}$ is a circle class, this curve, that is $\Upsilon(\gamma)$ is a geodesic of G^+.

We will now explain that each geodesic of G^+ is in fact part of a curve $\Upsilon(\gamma)$ where γ is a geodesic of $\mathbb{H}_1$ going through L. We consider two circle classes $\Upsilon^{-1}(p)$ and $\Upsilon^{-1}(q)$ on the one hand and on the other hand a geodesic going through L. The planar projection of them on $\mathbb{R}^2$ are three circles. Two have center $0_{\mathbb{C}}$ and the other goes through $0_{\mathbb{C}}$. If we now change the radius of the third circle (still going through 0), we also change the algebraic area that it sweeps between its intersections with the first and the second circle. This area can be chosen as the difference between the third coordinates of $\Upsilon^{-1}(p)$ and $\Upsilon^{-1}(q)$. Hence the construction we have suggested just proves that it is possible to connect any p and q of G^+ with a geodesic of the type we have described, that is the projection of a geodesic going through L.

Let us now give an idea of what are the geodesics in G between two points that lie in different copies of G^+. Suppose for instance that we start on $G^- =]-\infty, 0] \times \mathbb{R}$ and consider a curve connecting $p \in G^+$. Once we have attained a point $(0, t)$ of the midline $\{r = 0\}$, by using sym^G define just below, we can symmetrize any part of the curve in G^- to a part in G^+ without changing the length. Hence the minimal geodesics of G between points in different half-plane will be the catenation of two geodesics each one being (up to symmetry) of the type described for G^+.

It is quite difficult to make this argument rigorous. That is why we will rather find the geodesics of G with our analytic tool, that is the PMP.

We will compute the local geodesics starting from $(0, 0)$ or from $(-1, 0)$. It is enough to compute the geodesics from these points because usuals transformations send $(0, 0)$ on the points of $\{r = 0\}$ (translation) and $(-1, 0)$ on the other points (symmetry, translation, dilation). These transformations are the isometries

$$
\begin{aligned}
\mathrm{sym}^G &: (r, t) \mapsto (-r, t) \\
\mathrm{sym}_2^G &: (r, t) \mapsto (r, -t) \\
\mathrm{trans}_\tau^G &: (r, t) \mapsto (r, t + \tau)
\end{aligned}
$$

and the dilation

$$
\mathrm{dil}_\lambda^G : (r, t) \mapsto (\lambda r, \lambda^2 t)
$$

that is a isometry between G and $(G, \lambda^{-1} d_G)$.

In our case the Pontryagin theorem just states that the local geodesics are the solutions of the usual Hamiltonian gradient, extending the case of the Riemannian manifolds. Actually G is locally Riemannian in almost every point and the singular set is quite small because it is just $\{r = 0\}$. In this sense the PMP is almost a too powerful tool. The Hamiltonian is defined on the cotangent bundle by

$$
\begin{aligned}
H(\zeta, \lambda) &= \frac{1}{2}[(\lambda(\zeta)(\mathbf{R}_G)^2 + (\lambda(\zeta)(\mathbf{T}_G))^2)] \\
&= \frac{1}{2}(\lambda_r^2 + (r\lambda_t)^2)
\end{aligned}
$$

where $\lambda = (\lambda_r(\zeta), \lambda_t(\zeta))$ is a linear form defined on $T^*\mathbb{R}^2_\zeta$ and $\zeta = (r, t) \in \mathbb{R}^2$. From the book [84] by Montgomory or from the introduction by Boscain and Piccoli [15] on optimal control we know that the symplectic equations are satisfied by the locally minimal geodesics. This result is classic for manifolds. There are other locally minimal geodesics (called abnormal geodesics, see [82]) that appear in subRiemannian geometry. It is not the case for the Grušin plane or the Heisenberg group where all locally minimal geodesics (with constant speed) satisfies the symplectic equation. The system of equations is given in the chart $T^*\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2$ by

$$\begin{cases} \dot{r} = \frac{\partial H}{\partial \lambda_r} \\ \dot{t} = \frac{\partial H}{\partial \lambda_t} \\ \dot{\lambda_r} = -\frac{\partial H}{\partial r} \\ \dot{\lambda_t} = -\frac{\partial H}{\partial t} \end{cases}$$

In our case, this yields

$$\begin{cases} \dot{r} = \lambda_r \\ \dot{t} = r^2 \lambda_t \\ \dot{\lambda_r} = -r\lambda_t^2 \\ \dot{\lambda_t} = 0 \end{cases} \tag{1.22}$$

Geodesics from a non-singular point

Here, we will compute the equations of the local minimal geodesics starting from $(-1, 0)$. The Hamiltonian H is constant along the geodesics. For the choice $H = \frac{1}{2}$, we obtain the geodesics with arc-length parametrization. For this choice $-r\lambda_t$ and λ_r are the sine and the cosine of some angle because the sum of their square is 1. We denote this angle at the initial time $s = 0$ by φ.

$$\begin{cases} -(-1)\lambda_t(0) = \sin(\varphi) \\ \qquad\quad \lambda_r(0) = \cos(\varphi). \end{cases}$$

Because of (1.22), we have also $\lambda_t(s) = \sin(\varphi)$ for every time s. The differential system (1.22) is trivial in the cases $\varphi = 0, \pi$. The solution is given by

$$r = -1 \pm s; \qquad t = 0 \tag{1.23}$$

Let us then consider the general case $\varphi \in]-\pi, 0[\cup]0, \pi[$. From the first and the third equation of the symplectic system (1.22) we get the harmonic differential equation

$$\ddot{r} = -\lambda_t^2 r$$

that we solve by $r(s) = A\sin(\lambda_t s - \vartheta)$ where the integration constants satisfies the initial conditions

$$\begin{cases} r(0) = -1 \\ \dot{r}(0) = \lambda_r(0) = \cos(\varphi). \end{cases}$$

We then have $A = \frac{1}{\sin(\vartheta)}$ and $\text{cotan}(\vartheta) = \text{cotan}(\varphi)$. It follows

$$r(s) = \frac{\sin(\lambda_t s - \vartheta)}{\sin(\vartheta)}$$

and $\vartheta = \varphi + k\pi$ for some integer k. Because the expression of $r(t, \vartheta)$ does not depend on k, we can assume $\vartheta = \varphi$. We get

$$r(s) = \frac{\sin(\lambda_t s - \varphi)}{\sin(\varphi)}$$
$$\lambda_t = \sin(\varphi)$$
$$\dot{t} = \lambda_t r^2.$$

After linearization, integration and taking account the fact that $y(0) = 0$, we have

$$t(s) = \frac{s}{2\lambda_t} - \frac{\sin(2\lambda_t s - 2\varphi) + \sin(2\varphi)}{4\lambda_t^2}.$$

The locally minimal geodesics starting from $(-1, 0)$ with arc-length parametrization are then given by

$$\begin{cases} r(s) = \frac{\sin(\alpha s - \varphi)}{\alpha} \\ t(s) = \frac{s}{2\alpha} - \frac{\sin(2\alpha s - 2\varphi) + \sin(2\varphi)}{4\alpha^2}. \end{cases} \tag{1.24}$$

where $\alpha = \sin(\varphi)$ and $\alpha \neq 0$ (α is just a new notation for λ_t). The parameter φ very nicely corresponds to the angle that makes the tangent vector $\dot{\zeta}(0)$ with the x-axis in $\mathbb{R}^2$. On the critical set $\varphi \in \{k\pi \mid k \in \mathbb{Z}\}$, the equations (1.23) continue differentially in φ and s the equations of (1.24). More synthetically, (1.23) and (1.24) provide a map $E^{G,1}$ from $(\varphi, s) \in \mathbb{R} \times [0, +\infty[$ in the Grušin plane, $\mathcal{C}^\infty$ on $\mathbb{R} \times]0, +\infty[$ such that for every $\varphi \in \mathbb{R}$, the curve $E^{G,1}_\varphi = E^{G,1}(\varphi, \cdot)$ is the only arc-length locally minimal geodesic starting from $(-1, 0)$ and making an angle φ with $\{t = 0\}$ at time $s = 0$.

Geodesics from a singular point

With a similar calculus we compute the locally minimal geodesics starting from $(0, 0)$. Their equations are

$$\begin{cases} r(s) = \pm\frac{\sin(\beta s)}{\beta} \\ t(s) = \frac{s}{2\beta} - \frac{\sin(2\beta s)}{4\beta^2}. \end{cases} \tag{1.25}$$

where β is a parameter. For $\beta = 0$ we interpret the system as

$$r(s) = \pm s \quad \text{and} \quad t = 0.$$

Similarly to $E^{G,1}$, we define $E^{G,2,+}(\beta, s)$ and $E^{G,2,-}(\beta, s)$ as the solutions of (1.25) for $(\beta, s) \in \mathbb{R} \times]0, +\infty[$. Note that if $v \in \mathbb{C}$ has modulus 1, the curve $E^{G,2,+}_\beta : s \in [0, \pi/|\beta|] \to G$ is exactly the cylindrical projection of the geodesic $\gamma_{v,2\beta}$ of $\mathbb{H}_1$ defined in (1.18). More precisely with I_G defined in Subsection 1.2.5, on this interval $E^{G,2,+}_\beta(s) = I_G \circ \Upsilon(\gamma_{v,2\beta})(s)$. We will see in the next paragraph that as $\gamma_{v,2\beta}$, the Grušin curve is no longer globally geodesic for $s \geq \pi/|\beta|$.

For a calculation of the geodesics starting from $(0, 0)$ without using optimal control and more about the link with the Heisenberg group, see [37].

Cut locus

We study now how long the geodesics are minimal. For that we start with an exception: from the equations of the locally minimal geodesics we could compute that when $\alpha \neq 0$ (or $\beta \neq 0$) the t-coordinate is strictly monotone. It follows that the only way to link two points with the same t-coordinate is to use a geodesic with equation $r(s) = r(0) \pm s; \quad t(s) = t(0)$. The locally minimal geodesics with $\alpha = 0$ or $\beta = 0$ are globally minimal. Nevertheless we define the cut locus of a given point $(r, t) \in G$ as the set of the points (r', t') for which there exists a local geodesic starting from (r,t) that stops to be the unique minimal geodesic when it goes through (r',t').

Let us consider now the geodesic starting from $(0,0)$. For a given $\beta \neq 0$ (for simplicity we suppose $\beta > 0$). The curve reaches $(0, \frac{\pi}{2\beta^2})$ at time $\frac{\pi}{\beta}$. We can quickly find all the other geodesics that reaches this point. These are the geodesics with parameter $\sqrt{k}\beta$ for $k \in \mathbb{N}\backslash\{0\}$. They arrive in $(0, \frac{\pi}{2\beta^2})$ at time $\sqrt{k}\frac{\pi}{\beta}$. It proves that the geodesic with parameter β is globally minimal at least until $s = \sqrt{k}\beta$. We prove now that it is no longer true for greater times s. The important remark is that in (1.25), both signs $+$ and $-$ arrive on $(0, \frac{\pi}{2\beta^2})$ at the time $\frac{\pi}{\beta}$. Then a curve from $[0,s]$ in $\mathbb{R}^2$ obtained with the parameter β by changing the sign on the time $\frac{\pi}{\beta}$ has the same length like the ones with parameter β keeping the same sign. But the first curve is not even locally minimal. It is then also not a global geodesic and all curves with the same length between the same ends are also not global geodesics. We have proved that the cut locus of a curve starting from $(0,0)$ with parameter $\beta \neq 0$ is the point $(0, \frac{\pi}{2\beta^2})$.

The discussion for the geodesics starting from $(-1,0)$ is the same but the computations are a little more intricate. We want to show that for $\sin(\varphi) = \alpha \neq 0$ the locally minimal geodesic parametrized by φ is globally minimal until $s = \frac{\pi}{|\alpha|}$ but not for greater times. For simplicity and because of the symmetries, we can assume $0 < \varphi < \pi$ and then $\alpha > 0$. At time $\frac{\pi}{\alpha}$, the geodesic arrives in $(1, \frac{\pi}{2\alpha^2})$. The other geodesics with parameter φ' cross the line $r = 1$ each time when $\sin(\alpha' s - \varphi') = \sin(\varphi')$ which happens exactly if $\alpha' s = (2k+1)\pi$ or $\alpha' s - \varphi' = \varphi' + 2k\pi$ for some integer k. We will consider the intersection case $\varphi' = \frac{\pi}{2}$ as a third case. We begin with the first case. The equality on the t-coordinate yields

$$\frac{\pi}{2\alpha^2} = \frac{s}{2\alpha'} - \frac{\sin(2\alpha' s - 2\varphi') + \sin(2\varphi')}{4\alpha'^2}$$

for a time s verifying $2\alpha' s = (4k+2)\pi$. It implies $\frac{\pi}{2\alpha^2} = \frac{(2k+1)\pi}{2\alpha'^2}$ and further $\sin(\varphi') = \sqrt{2k+1}\sin(\varphi)$ (like for φ, we also suppose that φ' have a non-negative sine). Then with these curves we reach the point of coordinates $(1, \frac{\pi}{2\alpha^2})$ at time $s = \sqrt{2k+1}\frac{\pi}{\alpha}$. The shortest locally minimal geodesic is given for $k = 0$. This parameter k is the one of the curve of parameter φ and also that of the curve of parameter $\pi - \varphi$. Note that these locally minimal geodesic are different except in the special (third) case $\varphi = \frac{\pi}{2}$. In the first case, combining the two geodesics of same length, we know that the geodesic in no longer minimal for $s > \frac{\pi}{\alpha}$.

For the second case ($\alpha' s = 2\varphi' + 2k\pi$), assuming also $0 < \varphi' < \pi$, when the curve crosses $r = 1$ the t-coordinate is

$$\begin{aligned}&\frac{s}{2\alpha'} - \frac{\sin(2\alpha' s - 2\varphi') + \sin(2\varphi')}{4\alpha'^2}\\ =&\frac{\varphi' + k\pi}{\alpha'^2} - \frac{2\sin(2\varphi')}{4\alpha'^2} \leq \frac{2\varphi' + 2k\pi}{\alpha'^2}.\end{aligned}$$

If the t-coordinate is $\frac{\pi}{2\alpha^2}$, then $\alpha' \leq \alpha\sqrt{\frac{2\varphi' + 2k\pi}{\pi/2}}$. It follows

$$s \geq \sqrt{\pi/2}\frac{\sqrt{2\varphi' + 2k\pi}}{\alpha} = \frac{\pi}{\alpha}\sqrt{\frac{\varphi'}{\pi} + k}. \tag{1.26}$$

The time s is just possibly shorter than $\frac{\pi}{\alpha}$ if $k = 0$. But, as we will see, it is still longer.

The t-coordinate equality provides

$$\frac{\pi}{2\alpha^2} = \frac{2\varphi' - \sin(2\varphi')}{2\alpha'^2}.$$

For $\varphi' < \pi/2$, a precise study of the right-hand term shows that it is non-decreasing and then smaller than $\pi/2$ (this proves also that $\exp^{\mathbb{H}}(v, \cdot)$ has a increasing t-coordinate on $[0, \pi]$, see (1.19)). It follows that $\pi/2 \leq \varphi' < \pi$. But for $\varphi' > \pi/2$, the curve we are considering is $E^{G,1}_{\varphi'}(\sigma)$ for $\sigma \in [0, s]$. It is not globally geodesic because $s > \frac{2\varphi'}{\alpha'} > \frac{\pi}{\alpha'}$ and we prove in the first case that the cut locus is attained for $\sigma \leq \frac{\pi}{\alpha'}$.

Then we have proved that for $0 < \alpha < 1$, both $E^{\alpha,+}$ and $E^{\alpha,-}$ are global geodesics on $[0, \frac{\pi}{\alpha}]$ and that $\frac{\pi}{2\alpha^2} = E^{G,1}_{\varphi}(\frac{\pi}{\alpha}) = E^{G,1}_{\pi-\varphi}(\frac{\pi}{\alpha})$ is in the cut locus of $(-1, 0)$

In the third case $\varphi = \pi/2$ there is a unique global geodesic to $(1, \pi/2)$. However, it is not globally geodesic on $[0, s]$ for any fixed $s > \pi/2$. The reason is simply that using dil^G_λ and tran^G_τ, one can observe that $E^{G,1}_{\pi/2}(s)$ is in the cut locus of $E^{G,1}_{\pi/2}(s - \pi/2)$. The segment $[s - \pi/2, s]$ is then maximal for the geodesic minimality. Hence the curve is not a global geodesic on $[0, s]$ for any fixed $s > \pi/2$, but it is a global geodesic on $[0, \pi/2]$.

1.6.4 Geodesics on the approximating manifolds.

Here, as for the geodesics in the Heisenberg group $\mathbb{H}_n$, a geodesic minimizes the energy of the curves with fixed ends. Let us take a curve γ with

$$\dot{\gamma} = a(s)\mathbf{X}(\gamma(s)) + b(s)\mathbf{Y}(\gamma(s)) + c(s)\mathbf{T}(\gamma(s))$$

in almost every $s \in [0, 1]$, $\gamma(0) = 0_{\mathbb{H}}$ and $\gamma(1) = (z; t)$. In $\mathbb{H}^\varepsilon_1$ it has energy

$$\int_0^1 a(s)^2 + b(s)^2 + \frac{1}{\varepsilon^2} c(s)^2 ds.$$

Because $\mathbf{X}$ and $\mathbf{Y}$ are invariant under the third direction, the curves with

$$\dot{\gamma} = a(s)\mathbf{X}(\gamma(s)) + b(s)\mathbf{Y}(\gamma(s)) + \left(\int_0^1 c\right) \mathbf{T}(\gamma(s))$$

have a smaller energy and the same ends as the former one. The next step in minimizing the energy is to chose $a(s)$ and $b(s)$ in such a way that the horizontal curve with these controls is a geodesic from $0_{\mathbb{H}}$ to $(x, y, t - \varepsilon \int_0^1 c)$. Then the energy of this last curve is

$$d_c(0_{\mathbb{H}}, (z; t'))^2 + \frac{|t - t'|^2}{\varepsilon^2}$$

where $t' = t - \int_0^1 c(s) ds$. It is then a one parameter problem to minimize the energy. We will solve it for $z = 0$ and without loss of generality we can suppose $t \geq 0$. Then the function to minimize is

$$4\pi|t'| + \frac{(t - t')^2}{\varepsilon^2}$$

The minimizing t' is certainly positive, it is $t' = 0$ if $t \leq 2\pi\varepsilon^2$ and $t - 2\pi\varepsilon^2$ in the other case. The distance to $(0, 0, t)$, the square root of the energy is

$$d_\varepsilon(0_{\mathbb{H}}, (0; t)) = \begin{cases} \frac{|t|}{\varepsilon} & \text{if } |-t| \leq 2\pi\varepsilon^2 \\ 2\sqrt{\pi(|t| - \pi\varepsilon^2)} & \text{if } |-t| \geq 2\pi\varepsilon^2 \end{cases}$$

and the geodesics starting from $0_{\mathbb{H}}$ are defined as

$$s \to \exp_\varepsilon(s(a\mathbf{X} + b\mathbf{Y} + c\mathbf{T}))$$

where $\exp_\varepsilon$ is the usual Riemannian exponential map. Here, it is precisely

$$\exp_\varepsilon(a\mathbf{X} + b\mathbf{Y} + c\mathbf{T}) = \exp^{\mathbb{H}}(a\mathbf{X} + b\mathbf{Y}, \frac{c}{\varepsilon^2}) \cdot (0; c).$$

As we saw before, the geodesics of the manifold have two components. The first one is a geodesic of $\mathbb{H}_1$ and the second is a constant growth on the third coordinate. One can see that the cut locus of the manifold is $\{0\} \times (]-\infty, -2\pi\varepsilon^2] \cup [2\pi\varepsilon^2 \infty[)$ and that it is attained for $c = \pm 2\pi\varepsilon^2$.

1.7 Contraction along geodesics

In this section we will consider the contractions of the Heisenberg group and the Grušin plane. Contractions are maps that, for a given fixed point c, called the center of contraction, and a given ratio s between 0 and 1, map any point p connected to c by a unique geodesic, to a point p_s on this geodesic. The ratio s sets that the distance between p_s and c is s time the distance from p_s to c, which determine uniquely p_s. For instance a ratio equal to $1/2$ means that we take the midpoint of p and c. In $\mathbb{R}^n$, the contractions are simply dilations but the dilations dil_λ of $\mathbb{H}_n$ are not contractions.

1.7.1 Contraction in $\mathbb{H}_n$

We introduce two helpful maps for this thesis: the intermediate-points map $\mathcal{M}$ and the geodesic-inversion map $\mathcal{I}$. We know from Subsection 1.5.2 (and Proposition 1.5.2 for $\mathbb{H}_n$) that there is a unique geodesic from p to q if and only if $Z(p) \neq Z(q)$ or $p = q$ where $Z(z; t) = z$ as before. We will denote the open set $\{(p, q) \in (\mathbb{H}_n)^2 \mid Z(p) \neq Z(q)\} = \{(p, q) \in (\mathbb{H}_n)^2 \mid p^{-1} \cdot q \notin L\}$ by U. On this set we define our first map.

Definition 1.7.1. We define the *intermediate-points map* $\mathcal{M}$ from the set $U \times [0, 1]$ to $\mathbb{H}_n$ by

$$\mathcal{M}(p, q, s) = \mathrm{tran}_p \circ \exp_s^{\mathbb{H}} \circ (\exp^{\mathbb{H}})^{-1} \circ \mathrm{tran}_{p^{-1}}(q).$$

We will use now the notations on the geodesics, maps and domains of $\mathbb{H}_n$ that were introduced just before Proposition 1.5.2. The point $\mathcal{M}(p, q, s)$ is actually the unique s-intermediate point between p and q. It is a s-intermediate point when $p = 0_{\mathbb{H}}$ because $\exp_s^{\mathbb{H}} \circ (\exp^{\mathbb{H}})^{-1}(\gamma_{v,\varphi}(1))$ is $\gamma_{v,\varphi}(s)$ for $(v, \varphi) \in D_1$. The general case follows from the left-invariance of the Carnot-Carathéodory metric. Moreover, $\mathcal{M}(p, q, s)$ is the unique s-intermediate point between p and q because there is a unique geodesic from p to q (the pair (p, q) is in U) and because the s-intermediate points in a geodesic space lie on the geodesics connecting two points.

Thanks to the regularity of $\exp^{\mathbb{H}}$ and recalling that tran_p is affine, we have the following regularity lemma.

Lemma 1.7.2. *The map $\mathcal{M}$ is measurable. It is C^∞ on $U \times]0,1[$. The curve $s \in [0,1] \to \mathcal{M}(p,q,s)$ is the unique geodesic from p to q.*

Let us now introduce the geodesic-inversion map $\mathcal{I}$.

Definition 1.7.3. We define the *geodesic-inversion map* $\mathcal{I}$ on $\mathbb{H}_n \backslash L$ by $\mathcal{I}(p) = \exp^{\mathbb{H}}_{-1} \circ (\exp^{\mathbb{H}})^{-1}(p)$.

The name comes from the fact that using the fact that $\exp^{\mathbb{H}}$ is one-to-one on D_1, for $(v, \varphi) \in D_1$ and $s \in [-1, 1]$:

$$\begin{aligned}
\mathcal{I}(\gamma_{v,\varphi}(s)) &= \mathcal{I}(\exp^{\mathbb{H}}_s(v, \varphi)) \\
&= \exp^{\mathbb{H}}_{-1} \circ (\exp^{\mathbb{H}})^{-1}(\exp^{\mathbb{H}}(sv, s\varphi)) \\
&= \exp^{\mathbb{H}}_{-1}(sv, s\varphi) \\
&= \gamma_{v,\varphi}(-s)
\end{aligned}$$

It follows that $\mathcal{I} \circ \mathcal{I}$ is the identity on $\mathbb{H}_n \backslash L$. Note that $\gamma_{v,\varphi}$ is a local geodesic on $[-s, s]$ with $\gamma_{v,\varphi}(0) = 0_{\mathbb{H}}$. That is why for any $p \in \mathbb{H}_n$ we will call $(p, \mathcal{I}(p))$ a pair of $\mathcal{I}$-conjugate points. We now establish the connection between $\mathcal{M}$ and $\mathcal{I}$.

Lemma 1.7.4. *Let p be in $\mathbb{H}_n \backslash L$. Then $\mathcal{M}(\mathcal{I}(p), p, 1/2)$ is well defined and is the point $0_{\mathbb{H}}$ if and only if the φ-coordinate of $(\exp^{\mathbb{H}})^{-1}(p)$ verifies $|\varphi| < \pi$, i.e when $p \in \exp^{\mathbb{H}}(D_{1/2})$.*

Proof. Therefore we have to see when $\mathcal{M}(\mathcal{I}(p), p, 1/2)$ exists and is the point $0_{\mathbb{H}}$. The point p is $\exp^{\mathbb{H}}(v, \varphi)$ for some $|\varphi| < 2\pi$. Moreover, the definition of $\mathcal{I}$ implies that $\mathcal{I}(p) = \exp^{\mathbb{H}}_{-1}(v, \varphi)$. Therefore we have to say when $\mathcal{M}(\gamma_{v,\varphi}(-1), \gamma_{v,\varphi}(1), 1/2)$ exists and if it is $0_{\mathbb{H}}$.

It follows from equation (1.20) that the z-coordinates of $\gamma_{v,\varphi}(-1)$ and $\gamma_{v,\varphi}(1)$ are equal if and only if $|\varphi| = \pi$. Therefore $(\gamma_{v,\varphi}(-1), \gamma_{v,\varphi}(1)) \in U$ if and only if $|\varphi| \neq \pi$. In this case there is a unique geodesic δ defined on $[-1, 1]$ between the two points and we can define the midpoint

$$\delta(0) = \mathcal{M}(\delta(-1), \delta(1), 1/2) = \mathcal{M}(\gamma_{v,\varphi}(-1), \gamma_{v,\varphi}(1), 1/2).$$

We only know that on this interval $\gamma_{v,\varphi}$ is a local geodesic.

If $|\varphi| < \pi$ then $2|\varphi| < 2\pi$. In this case the curve δ is the restriction of $\gamma_{v,\varphi}$ to $[-1, 1]$ because by Proposition 1.5.2 both maps are the unique geodesic defined on $[-1, 1]$ that goes from $\mathcal{I}(p)$ to p. The midpoint is then $\delta(0) = \gamma_{v,\varphi}(0) = 0_{\mathbb{H}}$.

If $\pi < |\varphi| < 2\pi$ we make a proof by contradiction. Assume that $\delta(0) = 0_{\mathbb{H}}$. Then by Proposition 1.5.2, the curve $\delta \mid_{[0,1]}$ is the unique geodesic from $0_{\mathbb{H}}$ to $p = \gamma_{v,\varphi}(1)$ and $s \in [0, 1] \to \delta(-s)$ is the unique geodesic between $0_{\mathbb{H}}$ and $\mathcal{I}(p) = \gamma_{v,\varphi}(-1)$ (both have a Z projection making an absolute angle $|\varphi|$ smaller than 2π). It follows that δ is $\gamma_{v,\varphi}$ on $[0, 1]$ and $[-1, 0]$. This contradicts the fact that $|\varphi| > \pi$: for $2|\varphi| > 2\pi$, the restriction to $[-1, 1]$ of $\gamma_{v,\varphi}$ is not a geodesic because its complex projection makes more than one full circle and consequently it can not be δ. Hence $\mathcal{M}(p, \mathcal{I}(p), 1/2)$ is not $0_{\mathbb{H}}$. $\square$

As mentioned at the end of Section 1.5, we present the computation of the Jacobian determinant. To prove that $\exp^{\mathbb{H}}$ is a diffeomorphism from D_1 to $\mathbb{H}_n \setminus L$, we only need to prove that the Jacobian of $\exp^{\mathbb{H}}$ does not vanish. This fact is mentioned in [7] where the authors state that $\exp^{\mathbb{H}}$ is a diffeomorphism and the result of the calculation is given for $\mathbb{H}^1$ in the paper of Monti (see [85]). We now give all the details of this computation for every $n \in \mathbb{N}\setminus\{0\}$ because for the next chapters we do not only need the fact that the Jacobian determinant does not vanish, but also its exact value.

Proposition 1.7.5. *The Jacobian determinant of* $\exp^{\mathbb{H}}$ *is given by*

$$\mathrm{Jac}(\exp^{\mathbb{H}})(v,\varphi) = \begin{cases} 2^{2n}|v|^2\left(\frac{\sin(\varphi/2)}{\varphi}\right)^{2n-1}\frac{\sin(\varphi/2)-(\varphi/2)\cos(\varphi/2)}{\varphi^3} & \text{for } \varphi \neq 0, \\ |v|^2/12 & \text{otherwise.} \end{cases}$$

It does not vanish on D_1.

Proof. We recall the expression of $\exp^{\mathbb{H}}$:

$$\exp^{\mathbb{H}}(v,\varphi) = \begin{cases} (\frac{e^{i\varphi}-1}{i\varphi}v; |v|^2\left(\frac{\varphi-\sin(\varphi)}{2\varphi^2}\right)) & \text{if } \varphi \neq 0 \\ (v;0) & \text{else.} \end{cases}$$

where $|v|^2 = |v_1|^2+\cdots+|v_n|^2$. We start by calculating $\mathrm{Jac}(\exp^{\mathbb{H}}) = \det(D\exp^{\mathbb{H}})$ for $\varphi \neq 0$. The case $\varphi = 0$ is obtained as a limit.

We first have to compute the real derivative of $\exp^{\mathbb{H}}$, i.e. the derivative of $\exp^{\mathbb{H}}$ as a map from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{2n+1}$. We write $D\exp^{\mathbb{H}}$ as a matrix $\left(\begin{smallmatrix} P & C \\ R & q \end{smallmatrix}\right)$ where the block P is made of the $2n$ first rows and columns. If we identify complex numbers with 2×2 matrices ($a+ib$ is $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$), we can write P as an $n\times n$ complex matrix $\frac{e^{i\varphi}-1}{i\varphi}I_n$ where I_n is the identity matrix of $M_n(\mathbb{C})$. The column C is $(\frac{e^{i\varphi}}{\varphi}+i\frac{e^{i\varphi}-1}{\varphi^2})v$ seen as a $\mathbb{R}^{2n}$ vector, the row R is $(x_1\frac{\varphi-\sin(\varphi)}{\varphi^2}, y_1\frac{\varphi-\sin(\varphi)}{\varphi^2}, \cdots, x_n\frac{\varphi-\sin(\varphi)}{\varphi^2}, y_n\frac{\varphi-\sin(\varphi)}{\varphi^2})$, and the real number q is $|v|^2\left(\frac{\sin(\varphi)}{\varphi^3}-\frac{1+\cos(\varphi)}{2\varphi^2}\right)$.

It is difficult to compute directly the determinant of $\left(\begin{smallmatrix} P & C \\ R & q \end{smallmatrix}\right)$ in any point. Because of this we now prove that if $|v| = |v'|$, the determinants $\mathrm{Jac}(\exp^{\mathbb{H}})(v,\varphi)$ and $\mathrm{Jac}(\exp^{\mathbb{H}})(v',\varphi)$ are also the same. Let T be a unitary $\mathbb{C}$-linear map so that $T(v) = v'$. Consider now T' defined by $T'(v,\varphi) = (T(v),\varphi)$. Then it is not difficult to see that $\exp^{\mathbb{H}}\circ T' = T'\circ\exp^{\mathbb{H}}$. It follows that $(\mathrm{Jac}(\exp^{\mathbb{H}})\circ T')\cdot\det_{\mathbb{R}}(T') = \det_{\mathbb{R}}(T')\cdot\mathrm{Jac}(\exp^{\mathbb{H}})$ and hence we have $\mathrm{Jac}(\exp^{\mathbb{H}})(v,\varphi) = \mathrm{Jac}(\exp^{\mathbb{H}})(v',\varphi)$. We use this relation to simplify the computation by choosing $v' = (0,\cdots,0,|v|)$. With this new vector v', most of the entries of C and R are equal to zero, so we can calculate the determinant of $D\exp^{\mathbb{H}} = \left(\begin{smallmatrix} P & C \\ R & q \end{smallmatrix}\right)$ by blocks. We get that $\mathrm{Jac}(\exp^{\mathbb{H}})(v,\varphi)$ is the product of

$$\begin{vmatrix} \sin(\varphi)/\varphi & (\cos(\varphi)-1)/\varphi \\ (1-\cos(\varphi))/\varphi & \sin(\varphi)/\varphi \end{vmatrix}^{n-1}$$

with

$$\begin{vmatrix} \sin(\varphi)/\varphi & (\cos(\varphi)-1)/\varphi & |v|(\frac{\cos(\varphi)}{\varphi}-\frac{\sin(\varphi)}{\varphi^2}) \\ (1-\cos(\varphi))/\varphi & \sin(\varphi)/\varphi & |v|(\frac{\sin(\varphi)}{\varphi}+\frac{\cos(\varphi)-1}{\varphi^2}) \\ |v|\frac{\varphi-\sin(\varphi)}{\varphi^2} & 0 & |v|^2\left(\frac{\sin(\varphi)}{\varphi^3}-\frac{1+\cos(\varphi)}{2\varphi^2}\right) \end{vmatrix}.$$

This is just

$$2^{n-1}\left(\frac{2\sin^2(\varphi/2)}{\varphi^2}\right)^{n-1}|v|^2\begin{vmatrix}\sin(\varphi)/\varphi & (\cos(\varphi)-1)/\varphi & \frac{\cos(\varphi)}{\varphi}\\ (1-\cos(\varphi))/\varphi & \sin(\varphi)/\varphi & \frac{\sin(\varphi)}{\varphi}\\ \frac{\varphi-\sin(\varphi)}{\varphi^2} & 0 & \frac{1-\cos(\varphi)}{2\varphi^2}\end{vmatrix}$$

which is

$$2^{2n}|v|^2\left(\frac{\sin(\varphi/2)}{\varphi}\right)^{2n-1}\frac{\sin(\varphi/2)-(\varphi/2)\cos(\varphi/2)}{\varphi^3}.$$

The continuous limit at $\varphi=0$ is $|v|^2/12$.

It remains to show that $\mathrm{Jac}(\exp^{\mathbb{H}})$ does not vanish on D_1. This is clear for $\varphi=0$. Otherwise we have to prove that the odd function $f(u):=\sin(u)-u\cos(u)$ does not vanish for $u\in]0,\pi[$. $f(0)=0$. The first derivative of f is the map $f'(u)=u\sin(u)$ which is positive on $]0,\pi[$. On this interval f is non-decreasing and does not vanish. □

We recall that for $0<|s|\leq 1$ we have $\exp_s^{\mathbb{H}}(v,\varphi)=\exp^{\mathbb{H}}(sv,s\varphi)$, so we get the following corollary.

Corollary 1.7.6. *Let $0<|s|\leq 1$. The Jacobian determinant of $\exp_s^{\mathbb{H}}$ on D_1 is*

$$\mathrm{Jac}(\exp_s^{\mathbb{H}})(v,\varphi)=\begin{cases}2^{2n}s|v|^2\left(\frac{\sin\frac{s\varphi}{2}}{\varphi}\right)^{2n-1}\frac{\sin\frac{s\varphi}{2}-\frac{s\varphi}{2}\cos\frac{s\varphi}{2}}{\varphi^3} & \text{for } \varphi\neq 0,\\ s^{2n+3}|v|^2/12 & \text{otherwise.}\end{cases}$$

We now state a key estimate for this thesis. It first appeared in [64] and we will use for proving the main results of Chapter 2 and Chapter 3. Indeed, in Theorem 2.3.6 it will replace the Monge-Mather shortening principle and this estimate is essentially equivalent to $MCP(0,2n+3)$ that we prove in Theorem 3.4.5. The critical exponent is also one of the main ingredient in the proof of Theorem 3.5.12.

Theorem 1.7.7. *Let $p\in\mathbb{H}_n$ and E a measurable set. Then $\mathcal{M}_s(p,E\setminus p\cdot L)$ is measurable and for any $s\in[0,1]$,*

$$\mathcal{L}^{2n+1}\big(\mathcal{M}_s(p,(E\setminus p\cdot L))\big)\geq s^{2n+3}\mathcal{L}^{2n+1}(E).$$

Moreover, the exponent $2n+3$ in the right-hand side term is optimal, in the sense that it can not be replaced by a smaller exponent N.

Proof. Let E be a measurable set with non-zero measure and $s\in]0,1[$. We set $N=2n+3$. It should be noticed that $\mathcal{M}_s(p,q)$ is not defined for $q\in p\cdot L$. That is not a problem because the set that we want to contract is $E\setminus p\cdot L$. Because of the left-invariance of d_c and $\mathcal{L}$ we only need to prove the estimate for $p=0_{\mathbb{H}}$. The map $\mathcal{M}^s_{0_{\mathbb{H}}}:=\mathcal{M}(0_{\mathbb{H}},\cdot,s)$ is one-to-one on $\mathbb{H}_n\backslash L$ and it equals $\exp_s^{\mathbb{H}}\circ(\exp^{\mathbb{H}})^{-1}$. If we denote $F:=\mathcal{M}^s_{0_{\mathbb{H}}}(E\setminus L)$, then we have:

$$\mathcal{L}^{2n+1}(F)=\int_{E\setminus L}\mathrm{Jac}(\mathcal{M}_{0_{\mathbb{H}},s})(q)d\mathcal{L}^{2n+1}(q). \tag{1.27}$$

From the expression of $\mathcal{M}^s_{0_\mathbb{H}}$ on $\mathbb{H}_n \backslash L$ we get that $\mathrm{Jac}(\mathcal{M}^s_{0_\mathbb{H}}) = \frac{\mathrm{Jac}(\exp^\mathbb{H}_s)}{\mathrm{Jac}(\exp^\mathbb{H})} \circ (\exp^\mathbb{H})^{-1}$. But we know the expression of these Jacobian determinants by Proposition 1.7.5 and Corollary 1.7.6. Hence it is enough to prove that

$$\frac{\mathrm{Jac}(\exp^\mathbb{H}_s)}{\mathrm{Jac}(\exp^\mathbb{H})}(v,\varphi) = s\left(\frac{\sin(s\varphi/2)}{\sin\varphi/2}\right)^{2n-1}\left(\frac{\sin(s\varphi/2)-(s\varphi/2)\cos(s\varphi/2)}{\sin(\varphi/2)-(\varphi/2)\cos(\varphi/2)}\right) \geq s^N \tag{1.28}$$

when $(v,\varphi) \in D_1$. However, for $\varphi = 0$ this relation must be replaced to

$$\frac{\mathrm{Jac}(\exp^\mathbb{H}_s)}{\mathrm{Jac}(\exp^\mathbb{H})}(v,0) = s^{2n+3} \geq s^N \tag{1.29}$$

which is obviously true. Both sides of (1.28) are 0 at 0 and 1 at 1. It is the same if we raise these expressions to the power of $1/N$. Hence, we want to prove that $s \to \left(\frac{\mathrm{Jac}(\exp^\mathbb{H}_s)}{\mathrm{Jac}(\exp^\mathbb{H})}\right)^{1/N}(v,\varphi)$ lies above the diagonal between $(0,0)$ and $(1,1)$. That is in particular true if this function is concave in s for each $(v,\varphi) \in D_1$. This last assertion is equivalent to the $1/N$-concavity ($1/N$-concavity means positivity and concavity when raised to the power of $1/N$) on $]0,\pi[$ of the function g_{2n-1} defined for $k \in \mathbb{N}$ by

$$g_k(u) = u\sin^k(u)(\sin(u) - u\cos(u)).$$

In the next lemma, we will prove a stronger statement: g_k is $1/(k+4)$-concave. It follows that g_{2n-1} is $1/N$-concave because $N = 2n+3$.

Lemma 1.7.8. *For all $k \in \mathbb{N}$ the function g_k is $(k+4)^{-1}$-concave on $]0,\pi[$.*

Proof. We will prove this lemma by induction. We begin by proving that g_0 is $1/4$-concave. For simplicity we will denote $g = g_0$. This function is positive because it is the product of $\mathrm{Id} : u \to u$ with the function f that we met in the proof of Proposition 1.7.5. Its first derivative is $g'(u) = (1+u^2)\sin(u) - u\cos(u)$ and its second derivative is $g''(u) = 3u\sin(u) + u^2\cos(u)$. After differentiating one more time it follows that g is concave on $[\alpha,\pi]$ where α can be calculated to be smaller than 2.46. It is true that $1/4$-concavity is a weaker statement than concavity but we want it on all $[0,\pi]$. It is equivalent to the negativity of $(g''g - g'^2) + \frac{1}{4}g'^2$. A first step is to prove the weaker relation $g''g - g'^2 \leq 0$ which is the differential version of log-concavity (g positive and $\log(g)$ concave). Both factors of g are log-concave : Id is concave and

$$f''f - f'^2 = (\sin u + u\cos u)(\sin u - u\cos u) - (u\sin u)^2 = \sin^2 u - u^2 \leq 0.$$

It follows that g is log-concave. Alternatively we can write

$$g''g - g'^2 = (\mathrm{Id})^2(f''f - f'^2) + (\mathrm{Id}''\,\mathrm{Id} - \mathrm{Id}'^2)f^2$$

where both terms of the sum are negative on $]0,\pi[$. For $1/4$-concavity, we have to prove the negativity of $(g''g - g'^2) + \frac{1}{4}g'^2$, which is

$$\begin{aligned} &u^2\left[\sin^2(u) - u^2\right] + [0-1](\sin(u) - u\cos(u))^2 \\ +&\frac{1}{4}\left[(1+u^2)\sin(u) - u\cos(u)\right]^2 \end{aligned} \tag{1.30}$$

for $u \in [0, \pi]$. It is quite difficult to prove that this expression is negative. We replace the previous expression by a pointwise greater polynomial. To do this, we replace cos and sin in each term by the begining of their Taylor series. We start with $\frac{1}{4}g'^2(u)$. It is constructed from g' which is positive for $u \in [0, \pi]$. On this interval, we have:

$$0 \leq (1+u^2)\sin(u) - u\cos(u) \leq (1+u^2)(u - u^3/6 + u^5/120) - u(1 - u^2/2).$$

For $0 \leq u \leq 2\sqrt{2}$, we have

$$\sin(u) - u\cos(u) \geq (u - u^3/6) - u(1 - u^2/2 + u^4/24) = u^3/3 - u^5/24 \geq 0$$

and finally, for $u \in [0, \pi]$ we have

$$0 \leq \sin(u) \leq u - u^3/6 + u^5/120.$$

We can then estimate (1.30) for $u \leq 2\sqrt{2}$:

$$\begin{aligned}
&u^2\left[\sin^2(u) - u^2\right] - (\sin(u) - u\cos(u))^2 \\
&+ \frac{1}{4}\left[(1+u^2)\sin(u) - u\cos(u)\right]^2 \\
=&u^2\left[(u - u^3/6 + u^5/120)^2 - u^2\right] - (u^3/3 - u^5/24)^2 \\
&+ \frac{1}{4}((1+u^2)(u - u^3/6 + u^5/120) - u(1-u^2/2))^2 \\
=&-\frac{1}{30}u^8 + \frac{421}{57600}u^{10} - \frac{17}{28800}u^{12} + \frac{1}{57600}u^{14} \\
\leq&u^8\left(\left(\frac{8}{57600} - \frac{17}{28800}\right)(u^2)^2 + \frac{421}{57600}u^2 - \frac{1}{30}\right) \leq 0.
\end{aligned}$$

So we have 1/4-concavity of g on $[0, 2\sqrt{2}]$. But we already proved that g is concave on $[2.46, \pi]$. Thus g is 1/4-concave on $[0, \pi]$ which is the reunion of the two intervals.

Let us now prove by induction that g_{k+1} is $1/(k+5)$-concave. For this let us assume that g_k is $1/(k+4)$-concave for some integer k. Then $g_{k+1} = g_k \cdot \sin$. We have now to prove the negativity of

$$\begin{aligned}
&\left((g_k\sin)''(g_k\sin) - (g_k\sin)'^2\right) + \frac{1}{k+5}(g_k\sin)'^2 \\
=&(g_k''g_k - g_k'^2)\sin^2 + (-\sin\sin - \cos^2)g_k^2 + \frac{1}{k+5}(g_k\sin)'^2 \\
=&(g_k''g_k - g_k'^2)\sin^2 - g_k^2 + \frac{g_k'^2\sin^2 + 2g_kg_k'\sin\cos + g_k^2\cos^2}{k+5} \\
=&(g_k''g_k - g_k'^2 + \frac{g_k'^2}{k+4})\sin^2 - \frac{g_k'^2\sin^2}{k+4} - g_k^2 + \frac{g_k'^2\sin^2 + 2g_kg_k'\sin\cos + g_k^2\cos^2}{k+5} \\
=&(g_k''g_k - g_k'^2 + \frac{g_k'^2}{k+4})\sin^2 + \frac{-g_k'^2\sin^2}{(k+4)(k+5)} + g_k^2\left(\frac{\cos^2}{k+5} - 1\right) + \frac{2g_kg_k'\sin\cos}{k+5}.
\end{aligned}$$

The first term T_1 in the previous sum is negative because of the $1/(k+4)$-concavity of g_k. The second term T_2 is clearly negative. The third term T_3 is also negative. It remains to prove that $|T_4| \le |T_2| + |T_3|$ where T_4 is the last term. We compare $|T_4|^2$ and $(2\sqrt{|T_2||T_3|})^2 \le (|T_2| + |T_3|)^2$:

$$\begin{aligned}
&4|T_2||T_3| - T_4^2 \\
=&4\left[\frac{g_k'^2 \sin^2}{(k+4)(k+5)}\right]\left[g_k^2\left(1 - \frac{\cos^2}{k+5}\right)\right] - \left[\frac{2g_k g_k' \sin\cos}{k+5}\right]^2 \\
=&4g_k^2 g_k'^2 \left[\frac{\frac{k+5-\cos^2}{k+4} - \cos^2}{(k+5)^2}\right]\sin^2 \ge 0.
\end{aligned}$$

□

For the optimality of $2n+3$, we set now $N < 2n+3$. Let p be the point $(1, 0, \cdots, 0; 0) = \exp^{\mathbb{H}}((1, 0, \cdots, 0), 0)$ and E_r the (Euclidian) ball $\mathcal{B}(p, r)$ with center p and radius $r < 1$. For a fixed s in $]0, 1[$, we define the set F_r by $\mathcal{M}^s_{0_{\mathbb{H}}}(E_r)$. As $E_r \cap L = \emptyset$ we have still the change of variable (1.27). But $\mathrm{Jac}(\mathcal{M}_{0_{\mathbb{H}},s})(p) < s^N$ and by continuity, we can find a radius $r > 0$ small enough such that $\mathrm{Jac}(\mathcal{M}_{0_{\mathbb{H}},s})(q) < s^N$ holds for every $q \in E_r$. For this choice of r we get that $s^N \mathcal{L}^{2n+1}(E_r) > \mathcal{L}^{2n+1}(F_r)$ which contradicts the estimate.

□

Remark 1.7.9. The exponent $2n+3$ in Theorem 1.7.7 can appear surprising because we should have expected the topological dimension $(2n+1)$ or the Hausdorff dimension $(2n+2)$ instead of $2n+3$. We now illustrate how this exponent arises for the unit ball $\mathcal{B}^{\mathbb{H}}_1$, of $\mathbb{H}_1$. For $0 < s < 1$, the contraction $\mathcal{M}_{0_{\mathbb{H}},s}(\mathcal{B}^{\mathbb{H}}_1)$ is certainly contained in the Heisenberg ball $\mathcal{B}^{\mathbb{H}}_s$ with center $0_{\mathbb{H}}$ and radius s. This ball is the dilatation $\delta_s(\mathcal{B}^{\mathbb{H}}_1)$ of the unit ball and its volume is $s^4\mathcal{L}(\mathcal{B}^{\mathbb{H}}_1)$. Nevertheless, the best relation in $\mathbb{H}_1$ says that $\mathcal{L}(\mathcal{M}_{0_{\mathbb{H}},s}(\mathcal{B}^{\mathbb{H}}_1)) \ge s^5\mathcal{L}(\mathcal{B}^{\mathbb{H}}_1)$. Rescaling, we get $\mathcal{L}(\delta_{1/s}(\mathcal{M}_{0_{\mathbb{H}},s}(\mathcal{B}^{\mathbb{H}}_1))) \ge s\mathcal{L}(\mathcal{B}^{\mathbb{H}}_1)$ where $\delta_{1/s}(\mathcal{M}_{0_{\mathbb{H}},s}(\mathcal{B}^{\mathbb{H}}_1))$ is a subset of $\mathcal{B}^{\mathbb{H}}_1$. It is possible to interpret the factor s appearing in this expression by writing down an explicit expression for this subset. It is actually the subset of points whose angle φ in the (v, φ)-coordinate is between $-s2\pi$ and $s2\pi$. Indeed φ is linearly increasing on geodesic paths starting from $0_{\mathbb{H}}$. Moreover, the dilation $\delta_{1/s}$ does not change the value of φ. It is possible to calculate that the Lebesgue measure of $\mathcal{L}(\delta_{1/s}(\mathcal{M}_{0_{\mathbb{H}},s}(\mathcal{B}^{\mathbb{H}}_1))$ is equivalent to $s\frac{\pi^2}{12}$ for s close to 0, which justifies the factor s. See the figure 1.5 which shows the set $\{y = 0\}$. The sets $\mathcal{B}^{\mathbb{H}}_1$ and $\delta_{1/s}(\mathcal{M}_{0_{\mathbb{H}},s}(\mathcal{B}^{\mathbb{H}}_1))$ are then obtained by rotating this figure around the axe $L = \{(0, 0)\} \times \mathbb{R}$.

1.7.2 Contraction in the Grušin plane

Before we estimate the rate of contraction from center $(-1, 0)$, we need to compute $\mathrm{Jac}(E^{G,1})$. See the system (1.24) for the expression of $E^{G,1}$.

Proposition 1.7.10. *The value of the Jacobian determinant is*

$$\mathrm{Jac}(E^{G,1})(\varphi, s) = \frac{\sin(\alpha s) - \cos(\varphi)\cos(\alpha s - \varphi)\alpha s}{\sin^3(\varphi)}$$

where $\alpha = \sin(\varphi)$ as before.

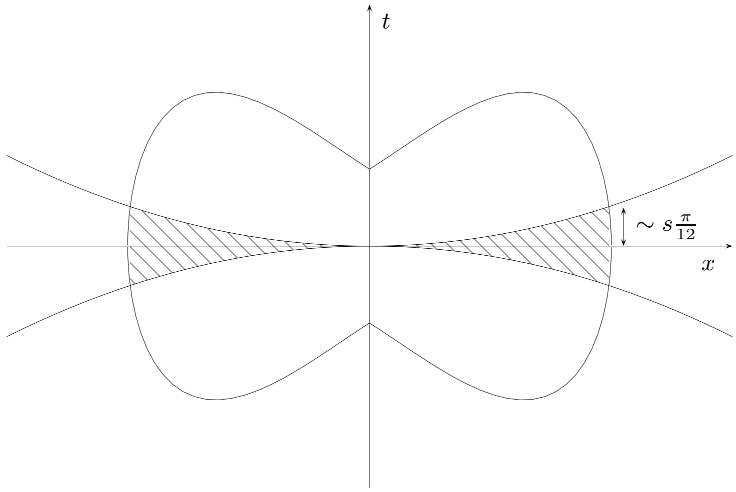

Figure 1.5: The sets $\mathcal{B}_1^{\mathbb{H}}$ and $\delta_{1/s}(\mathcal{M}_{0_{\mathbb{H}},s}(\mathcal{B}_1^{\mathbb{H}}))$.

Proof. First of all, we write $E^{G,1}(\varphi,s)$ as the composition of $e : (\varphi,\tau) \mapsto \left(\frac{\sin(\tau)}{\sin(\varphi)}, \frac{(2\tau-\sin(2\tau))+(2\varphi-\sin(2\varphi))}{4\sin^2(\varphi)}\right)$ and $R(\varphi,s) = (\varphi, \sin(\varphi)s-\varphi)$. The determinant of $E^{G,1}$ is then $\sin(\varphi)$ times the value of the Jacobian determinant of e at the point $R(\varphi,s)$. We will compute it and start writing $De(\varphi,\tau)$:

$$\begin{pmatrix} \frac{-\cos(\varphi)\sin(\tau)}{\sin^2(\varphi)} & \frac{\cos(\tau)}{\sin(\varphi)} \\ 1-\frac{\cos(\varphi)[(2\tau-\sin(2\tau))+(2\varphi-\sin(2\varphi))]}{2\sin^3(\varphi)} & \frac{\sin^2(\tau)}{\sin^2(\varphi)} \end{pmatrix}$$

The useful fact in this computation is the fact that the partial derivative of $2h-\sin(2h)$ under h is $4\sin^2(h)$. Then the determinant of the last matrix is

$$\frac{\cos(\tau)\cos(\varphi)[(2\tau-\sin(2\tau))+(2\varphi-\sin(2\varphi))]}{2\sin^4(\varphi)} - \frac{\cos(\tau)\sin^3\varphi+\cos(\varphi)\sin^3(\tau)}{\sin^4(\varphi)}.$$

But in the second term

$$\begin{aligned} &\cos(\tau)\sin^3\varphi+\cos(\varphi)\sin^3(\tau) \\ =&\sin(\tau+\varphi)-[\cos(\tau)\sin(\varphi)\cos^2(\varphi)+\cos(\varphi)\sin(\tau)\cos^2(\tau)] \\ =&\sin(\tau+\varphi)-\cos(\tau)\cos(\varphi)\left[\frac{\sin(2\varphi)}{2}+\frac{\sin(2\tau)}{2}\right]. \end{aligned}$$

We achieve the calculation of $\mathrm{Jac}(e)$. It is

$$\mathrm{Jac}(e) = \frac{\cos(\varphi)\cos(\tau)[\tau+\varphi]-\sin(\tau+\varphi)}{\sin^4(\varphi)}.$$

It simply follows that

$$\mathrm{Jac}(E^{G,1})(\varphi,s) = \frac{\cos(\varphi)\cos(\alpha s-\varphi)[\alpha s]-\sin(\alpha s)}{\sin^3(\varphi)}.$$

$\square$

The question is now to determine when $\mathrm{Jac}(E^{G,1})$ is 0 and what is its sign in function of φ and s. The sign of the denominator is the one of $\alpha = \sin(\varphi)$. The numerator can be written $u\cos(\varphi)\cos(u-\varphi) - \sin(u)$ with $u = \alpha s$. But

$$\begin{aligned}\cos(\varphi)\cos(u-\varphi) &= \frac{\cos(u)+\cos(2\varphi-u)}{2} \\ &= \frac{(1+\cos(u))-(1-\cos(2\varphi-u))}{2} \\ &= \cos^2(u/2)-\sin^2(\varphi-u/2).\end{aligned}$$

Then

$$\mathrm{Jac}(E^{G,1})(\varphi,s) = \frac{2\cos(u/2)[(u/2)\cos(u/2)-\sin(u/2)] - u\sin^2(u/2-\varphi)}{\alpha^3}$$

where $\alpha = \sin(\varphi)$ and $u = \sin(\varphi)s = \alpha s$. We observe that this function is even in φ because α and u are odd. It follows that we can restrict our study to $\alpha = \sin(\varphi) > 0$.

For a fixed φ the smaller $s > 0$ such that $\mathrm{Jac}(E^{G,1})(\varphi,\cdot)$ vanish is the time for the so-called *first conjugate point*.

Proposition 1.7.11. *The Jacobian determinant* $\mathrm{Jac}(E^{G,1})(\varphi,\cdot)$ *is non-positive for* $s < \frac{\pi}{|\alpha|}$*. The first conjugate point corresponds to a time* $s \in [\frac{\pi}{|\alpha|}, \frac{2\pi}{|\alpha|}]$*. For* $|\alpha| = 1$*, the first conjugate point is* $(1, \frac{\pi}{2\alpha}) = E^{G,1}(\arcsin(\alpha), \frac{\pi}{|\alpha|})$*.*

Remark 1.7.12. As in Riemannian geometry, in the Grušin plane the first conjugate point happens after the geodesic met the cut locus at time $s = \frac{\pi}{|\alpha|}$. For $|\alpha| = 1$, both locus collapse, i.e the time for the first conjugate point is exactly $\frac{\pi}{|\alpha|}$.

Proof. Assume $\alpha > 0$. For $0 < u < \pi$, $\cos(u/2)$ is non-negative and the function $(u/2)\cos(u/2) - \sin(u/2)$ is non-positive. Then $\mathrm{Jac}(E^{G,1})(\varphi,\cdot)$ is non-positive for $0 < s < \frac{\pi}{\alpha}$.

For $s = \frac{\pi}{\alpha}$ it is the same except for a special case : if $\varphi = \pi/2 + k\pi$, we have $\sin^2(u/2-\varphi) = 0$ because $u/2 - \varphi = -k\pi$ and the Jacobian determinant vanishes.

Take now the value of $\mathrm{Jac}(E^{G,1})$ for $u = \alpha s = 2\pi$. It is $\frac{2\pi\cos(\varphi)^3}{\alpha^3}$. It is non-negative except if $\alpha = \sin(\varphi) = 1$. In this case the value of $\mathrm{Jac}(E^{G,1})$ is non-negative for $s \in]\frac{\pi}{\alpha}, \frac{2\pi}{\alpha}[$ because it is equal to $\frac{\sin(\alpha s)}{\alpha^3}$. We conclude for $\alpha > 0$ with the intermediate value theorem.

We can easily deduce the corresponding results for $\alpha < 0$ because of the parity of the Jacobian determinant. $\square$

We want to consider now the contraction maps F_q. It is a family of maps with one quotient parameter $q \in [0,1]$ so that F_0 is identically equal to $(0,1)$. More precisely F_q is defined by

$$F_q = E^{G,1} \circ \delta_q \circ (E^{G,1})^{-1}$$

where $\delta_q(s,\varphi) = (qs,\varphi)$ and the map $E^{G,1}$ is restricted to the domain where it parametrizes G (for s smaller as the cut locus time). The map F_q is just taking a point on the geodesic between $(-1,0)$ and the point we map. This point is obtained after walking on this geodesic with a time quotient q. Hence a point $(r,t) \in G$ the curve $q \to F_q(r,t)$ is a constant-speed geodesic parametrized on $[0,1]$. Such a map is defined for any (r,t) except on the cut locus $\{1\} \times ([\frac{\pi}{2}, +\infty[\cup [-\frac{\pi}{2}, -\infty[)$ of $(-1,0)$. For a given point we want to estimate the map $q \mapsto \mathrm{Jac}(F_q)$ from below. In comparison to $\mathbb{R}^n$, we would like this function to be estimated by $q \mapsto q^n$. This comparison of the volume has recently inspired the definition of the Measure Contraction Property that we will introduce in Chapter 3 in relation with Ricci curvature. As we will see, it is possible to find a dimension parameter N such that every point (r,t) not being in the cut locus verifies for every $q \in [0,1]$

$$\mathrm{Jac}(F_q)(r,t) \geq q^N.$$

It what follows, the smaller admissible dimension will be determined as the maximum of a certain function of two variables.

The Jacobian determinant of F_q in $E^{G,1}(\varphi,s)$ is actually simply

$$\frac{q\,\mathrm{Jac}(E^{G,1})(\varphi,qs)}{\mathrm{Jac}(E^{G,1})(\varphi,s)}.$$

Now we observe for $N \geq 1$.

$$\begin{aligned} &\frac{q\,\mathrm{Jac}(E^{G,1})(\varphi,qs)}{\mathrm{Jac}(E^{G,1})(\varphi,s)} \geq q^N \\ \Leftrightarrow &\frac{\mathrm{Jac}(E^{G,1})(\varphi,qs)}{(qs)^{N-1}} \leq \frac{\mathrm{Jac}(E^{G,1})(\varphi,s)}{s^{N-1}} \end{aligned}$$

Our goal is then to find the smaller N such that $\frac{\mathrm{Jac}(E^{G,1})(\varphi,s)}{s^{N-1}}$ increases on $[0, \frac{\pi}{\alpha}[$. By taking the logarithm of the opposite of this function (that should decrease) and by derivating, we obtain that all N that bounds from above the function

$$1 + \frac{s\frac{\partial\,\mathrm{Jac}(E^{G,1})(\varphi,s)}{\partial s}}{\mathrm{Jac}(E^{G,1})(\varphi,s)}$$

are admissible. Then the supremum (we will prove it exists) of this function is the optimal exponent we are looking for. With the notation $u = \alpha s$ (used in order to have a rectangular domain $(\varphi,u) \in]0,\pi[\times]0,\pi[$) the function is

$$h(\varphi,u) = 1 + \frac{u\cos(u) - u\cos(\varphi)[\cos(u-\varphi) - u\sin(u-\varphi)]}{\sin(u) - u\cos(\varphi)\cos(u-\varphi)}$$

that can rewrite as

$$\begin{aligned} h(\varphi,u) &= 2 + \frac{u\cos(u) - \sin(u) + u^2\sin(u-\varphi)\cos(\varphi)}{\sin(u) - u\cos(\varphi)\cos(u-\varphi)} \\ &= 2 + \frac{u\cos(u) - \sin(u) + u^2\frac{\sin(u)+\sin(u-2\varphi)}{2}}{2\cos(u/2)[\sin(u/2) - (u/2)\cos(u/2)] + u\sin^2(u/2-\varphi)} \end{aligned}$$

Lemma 1.7.13. *The function h is bounded from above.*

Proof. The quotient $h-2$ is made of two functions that are continuous on $]0,\pi[\times]0,\pi[$, and the denominator is strictly positive on this domain (see the proof of Proposition 1.7.11). However, the problem is that the denominator of h vanishes on the border for $u=0$ and for $(\varphi,u)=(\pi/2,\pi)$. This point does not matter because the numerator is non-positive, such that around $(\pi/2,\pi)$ the function h is locally bounded from above.

We split the locus $u=0$ in $(\varphi,u)\in]0,\pi]\times\{0\}$ and $(\varphi,u)=(0,0)$. Close to each point $(0,\varphi)$, with $\varphi\neq 0$ the numerator is equivalent to $-u^2\sin(2\varphi)/2$ and the denominator decreases like u. We can then extend the function in this point with the value $2+0$. The real problem is around $(0,0)$. In fact h has no limit in this point. We will show that it has a limit superior and it will be enough for the proof. When (u,φ) tends to $(0,0)$, $u\cos(u)-\sin(u)+u^2\frac{\sin(u)}{2}$ is equivalent to $u^3/6$ and $2\cos(u/2)[\sin(u/2)-(u/2)\cos(u/2)]$ to $u^3/12$. Moreover, $u^2\frac{\sin(u-2\varphi)}{2}$ in the numerator of $h-2$ is equivalent to $u^2(u/2-\varphi)$ and $u\sin^2(u/2-\varphi)$ to $u(u/2-\varphi)^2$. Generally it is not allowed to sum equivalence relations, However, if we assume $u\geq 2\varphi$, all terms are positive and we can make this addition. If contrarily $u<2\varphi$, we observe that around $(0,0)$, $h(u,u-\varphi)$ is smaller than $h(u,\varphi)$. Hence it is enough to estimate the function on the cone $0\leq 2\varphi\leq u$. Under this constraint when (φ,u) tends to $(0,0)$, the function $h-2$ is equivalent to

$$\frac{2+12(1-\frac{2\varphi}{u})}{1+6(1-\frac{2\varphi}{u})^2}.$$

This is a function of $\frac{2\varphi}{u})^2\in[0,1]$. The maximum is achieved for $u=3\varphi$ and it is 3. Then the limit superior of h around $(0,0)$ is 5 and the function is bounded above. $\square$

Remark 1.7.14. It seems, looking at the graph of h that the limit superior 5 in $(0,0)$ is also the sharpest bound for h. We make the conjecture that the Grušin plane equipped with the Lebesgue measure satisfies $MCP(0,5)$ and does not satisfy $MCP(0,N)$ for $N<5$ (for the definition of MCP see Chapter 3). Actually it would be interesting to find (if it exists) a geometric metric space with a lot of isometries and invariant measure (for example the Hausdorff measure) such that the sharpest contraction exponent is not an integer.

Using the transformations of the Grušin plan that we introduced in Subsection 1.6.3, we can define contractions from any point. Actually if f is a composition of maps dil^G, tran^G, sym_1^G and sym_2^G, then the map $F_q^{f(-1,0)}=f\circ F_q\circ f^{-1}$ is the contraction map of center $f(q)$: it is not defined on $f\left(\{1\}\times\left(]-\infty,-\frac{\pi}{2}]\cup[\frac{\pi}{2},+\infty[\right)\right)$ because on there is more than one minimal geodesic between $f(q)$ and any point of this set. However, out of this set $F_q^{f(-1,0)}$ maps a point (r,t) to the intermediate point on the geodesic between $f(q)$ and (r,t), respecting the distance ratio q. Then the Jacobian determinant of $F_q^{f(-1,0)}$ in (r,t) is $\mathrm{Jac}(F_q)(f^{-1}(r,t))$. It is uniformly greater than q^N for the same N.

Remark 1.7.15. We will not estimate the Jacobian determinant of $F_q^{(0,0)}=E^{G,2}\circ\delta_q\circ(E^{G,2})^{-1}$, the contraction of center $(0,0)$. However, we directly know from the relation between $\mathbb{H}_1$ and G and Theorem 1.7.7 that

$$|r_q|\,\mathrm{Jac}(F_q^{(0,0)})(r,t)\geq q^5|r| \tag{1.31}$$

where r_q is the r-coordinate of $F_q^{(0,0)}(r,t)$. Actually $\mathcal{L}_r = |r|\mathcal{L}$ is up to a constant the push-forward measure of the Lebesgue measure $\mathcal{L}^3$ under the cylindrical projection Υ. For sets E and $F := F_q^{(0,0)}(E)$, inequality (1.31) writes simply

$$\mathcal{L}_r(F) \geq q^5 \mathcal{L}_r(E).$$

1.7.3 Local Poincaré inequality

We report here a proof of the Poincaré inequality on some manifolds with a bounded Ricci curvature that we adapt from [96, 5.6.3] to the Heisenberg group $\mathbb{H}_n$. The proof make use of the contraction estimate of this section.

Proposition 1.7.16. *In the Heisenberg group $\mathbb{H}_n$, the following Poincaré inequality holds:*

$$\int_{\mathcal{B}(p,r)} |f(q) - f_{\mathcal{B}}| \, d\mathcal{L}(q) \leq \frac{2^{2n+3}}{n} r \int_{\mathcal{B}(p,2r)} |\nabla_{\mathbb{H}} f(q)|_{\mathbb{H}} \, d\mathcal{L}(q)$$

where $\mathcal{B}(p,r)$ is a Carnot-Carathéodory ball of center p and radius r and $f_{\mathcal{B}} = \frac{1}{\mathcal{L}(\mathcal{B})} \int_{\mathcal{B}} f(q) \, d\mathcal{L}(q)$.

Proof. We will need here the intermediate-points map $\mathcal{M}(p,q,s)$. It is well defined for any $(p,q,s) \in U \times [0,1]$, i.e. if $q \notin p \cdot L$. Note that $\mathcal{L}((\mathbb{H}_n)^2 \backslash U) = 0$.

For $(p,q) \in U$, the curve $s \in [0,1] \mapsto \mathcal{M}(p,q,s)$ is the unique geodesic between p and q defined on $[0,1]$. An easy consequence is that

$$\mathcal{M}(p,q,s) = \mathcal{M}(q,p,1-s). \tag{1.32}$$

We now fix a ball $\mathcal{B}$ of radius r and f a smooth function. We start by estimating:

$$\begin{aligned}
\int_{\mathcal{B}} |f(p) - f_{\mathcal{B}}| \, d\mathcal{L}(p) &\leq \frac{1}{\mathcal{L}(\mathcal{B})} \int_{\mathcal{B}} \int_{\mathcal{B} \backslash p \cdot L} |f(p) - f(q)| \, d\mathcal{L}(p,q) \\
&\leq \frac{1}{\mathcal{L}(\mathcal{B})} \int_{\mathcal{B}} \int_{\mathcal{B} \backslash p \cdot L} \int_0^1 d_c(p,q) |\nabla_{\mathbb{H}} f(\mathcal{M}(p,q,s))| \, ds \, d\mathcal{L}(p,q) \\
&\leq \frac{2}{\mathcal{L}(\mathcal{B})} \int_{\mathcal{B}} \int_{\mathcal{B} \backslash p \cdot L} \int_{1/2}^1 d_c(p,q) |\nabla_{\mathbb{H}} f(\mathcal{M}(p,q,s))| \, ds \, d\mathcal{L}(p,q)
\end{aligned}$$

To obtain the previous inequality we break the set $U \times [0,1]$ into two pieces $U \times [0,1/2[$ and $U \times [1/2,1]$. The integrals on these pieces are the same. For this we just have to use the change of variable $(p,q,s) \to (q,p,1-s)$ and the relation (1.32). Saloff-Coste write in [96] that this trick is taken from [55].

Applying the contraction estimate $\mathrm{Jac}(\mathcal{M}_{x,s}(y)) \geq s^{2n+3}$, we can write:

$$\begin{aligned}
&\int_{\mathcal{B}}\int_{\mathcal{B}\setminus p\cdot L}\int_{1/2}^{1} d_c(p,q)|\nabla_{\mathbb{H}} f(\mathcal{M}(p,q,s))|\,ds\,d\mathcal{L}(p,q)\\
\leq &\int_{1/2}^{1}\frac{1}{s^{2n+3}}\int_{\mathcal{B}}\int_{\mathcal{B}\setminus p\cdot L} d_c(p,q)|\nabla_{\mathbb{H}} f(\mathcal{M}_{p,s}(q))|\,\mathrm{Jac}(\mathcal{M}_{p,s}(q))d\mathcal{L}(p,q)\,ds\\
\leq &\int_{1/2}^{1}\frac{1}{s^{2n+3}}\int_{\mathcal{B}}\int_{\mathcal{M}_{p,s}(\mathcal{B}\setminus p\cdot L)} 2r|\nabla_{\mathbb{H}} f(m)|d\mathcal{L}(m)\,d\mathcal{L}(p)\,ds\\
\leq &\int_{1/2}^{1}\frac{1}{s^{2n+3}}\int_{\mathcal{B}}\int_{\mathcal{B}_{2r}} 2r|\nabla_{\mathbb{H}} f(m)|d\mathcal{L}(m)\,d\mathcal{L}(p)\,ds\\
\leq &\frac{2^{2n+2}-1}{2n+2}2r\mathcal{L}(\mathcal{B})\int_{\mathcal{B}_{2r}}|\nabla_{\mathbb{H}} f(m)|d\mathcal{L}(m)\\
\leq &\frac{2^{2n+2}}{n}r\mathcal{L}(\mathcal{B})\int_{\mathcal{B}_{2r}}|\nabla_{\mathbb{H}} f(m)|d\mathcal{L}(m).
\end{aligned}$$

In this calculation, we use the fact that $m = \mathcal{M}_{p,s}(q)$ is included in the ball $\mathcal{B}_{2r}$ with the same center as $\mathcal{B}$ but with radius $2r$ instead of r. It is impossible that $m \notin \mathcal{B}_{2r}$ because we would have $d(p,q) = d(p,m)+d(m,q) > (2r-r)+(2r-r) = \mathrm{diam}(\mathcal{B})$.

If we now come back to the beginning of the proof, we have

$$\int_{\mathcal{B}}|f(p)-f_{\mathcal{B}}|\,d\mathcal{L}(p) \leq \frac{2^{2n+3}r}{n}\int_{\mathcal{B}_{2r}}|\nabla_{\mathbb{H}} f(m)|d\mathcal{L}(m)$$

which is the proposition we want.

Remark 1.7.17. Thanks to the contraction estimate of Subsection 1.7.2, a similar proof also works for the Grušin plane with the Lebesgue measure $\mathcal{L}^2$, so a local Poincaré inequality holds in this space too.

□

We reproduct now a second proof taken from [55, 11.3] where the authors write (for Carnot groups) a ameliorated version of the first proof of a Poincaré inequality in the Heisenberg group that was initially found (also for Carnot groups) by Varopoulos (see [106]).

Proof. As above f is a smooth function and $\mathcal{B}$ a ball with radius r. We still denote by $\mathcal{B}_{2r}$ the ball with same center and radius $2r$. For every $q \notin L$ we denote by γ_q the geodesic between $0_{\mathbb{H}}$ and q. For $q \in L$, γ_q will also be defined as a fixed geodesic between $0_{\mathbb{H}}$ and q, but the choice will not be unique (such a simplification could also have been done for the first proof). As the Carnot-Carthéodory metric is left-invariant, $p\cdot\gamma_q$ is a geodesic between p and $p\cdot q$. It follows that

$$|f(p)-f(p\cdot q)| \leq |q|_c\int_0^1|\nabla_{\mathbb{H}} f(p\cdot\gamma_q(s))|\,ds$$

where we recall that $|q|_c = d_c(0_{\mathbb{H}},q)$. Then by the left invariance of the Lebesgue measure and denoting $\chi_{\mathcal{B}}$ the characteristic function of $\mathcal{B}$

$$\begin{aligned}\int_{\mathcal{B}} |f(p) - f_{\mathcal{B}}| \, d\mathcal{L}(p) &\leq \frac{1}{\mathcal{L}(\mathcal{B})} \int_{\mathcal{B}} \int_{\mathcal{B}} |f(p) - f(w)| \, d\mathcal{L}(p,w) \\ &\leq \int_{\mathbb{H}} \int_{\mathbb{H}} \chi_{\mathcal{B}}(p) \chi_{\mathcal{B}}(p \cdot q) |f(p) - f(p \cdot q)| \, d\mathcal{L}(p,q) \\ &\leq \int_{\mathbb{H}} \int_{\mathbb{H}} \chi_{\mathcal{B}}(p) \chi_{\mathcal{B}}(p \cdot q) |q|_c \int_0^1 |\nabla_{\mathbb{H}} f(p \cdot \gamma_q(s))| \, ds \, d\mathcal{L}(p,q).\end{aligned}$$

Invoking the right invariance of the Lebesgue measure and denoting by $\mathcal{B}h$ the right translation of $\mathcal{B}$ by h (it is not a ball) we obtain

$$\begin{aligned}&\int_{\mathbb{H}} \chi_{\mathcal{B}}(p) \chi_{\mathcal{B}}(p \cdot q) |\nabla_{\mathbb{H}} f(p \cdot \gamma_z(s))| \, d\mathcal{L}(p) \\ =& \int_{\mathbb{H}} \chi_{\mathcal{B}\gamma_q(s)}(p \cdot \gamma_q(s)) \chi_{\mathcal{B}q^{-1} \cdot \gamma_q(s)}(p \cdot q \cdot q^{-1} \cdot \gamma_q(s)) |\nabla_{\mathbb{H}} f(p \cdot \gamma_q(s))| \, d\mathcal{L}(p) \\ =& \int_{\mathbb{H}} \chi_{\mathcal{B}\gamma_q(s)}(m) \chi_{\mathcal{B}q^{-1} \cdot \gamma_q(s)}(\zeta) |\nabla_{\mathbb{H}} f(m)| \, d\mathcal{L}(m) \\ \leq& \chi_{\mathcal{B}_{2r}}(q) \int_{\mathcal{B}_{2r}} |\nabla_{\mathbb{H}} f(m)| \, d\mathcal{L}(m)\end{aligned}$$

because $\mathcal{B}\gamma_q(s) \subset \mathcal{B}_{2r}$ and if $|q|_c > 2r$ then $\mathcal{B}\gamma_q(s) \cap \mathcal{B}q^{-1} \cdot \gamma_q(s) = \emptyset$.

$$\begin{aligned}\int_{\mathcal{B}} |f(p) - f_{\mathcal{B}}| \, d\mathcal{L}(p) &\leq \frac{1}{\mathcal{L}(\mathcal{B})} \int_{\mathbb{H}} |q|_c \int_0^1 \left(\chi_{\mathcal{B}_{2r}}(q) \int_{\mathcal{B}_{2r}} |\nabla_{\mathbb{H}} f(m)| \, dm \right) ds \, d\mathcal{L}(q) \\ &= \frac{1}{\mathcal{L}(\mathcal{B})} \int_{\mathcal{B}_{2r}} \int_{\mathcal{B}_{2r}} |q|_c \cdot |\nabla_{\mathbb{H}} f(m)| \, d\mathcal{L}(m) \, d\mathcal{L}(q) \\ &= Cr \int_{\mathcal{B}_{2r}} |\nabla_{\mathbb{H}} f(m)| \, d\mathcal{L}(m).\end{aligned}$$

The proof is complete. □

Remark 1.7.18. Here, we discuss about the value of $C = \frac{1}{r\mathcal{L}(\mathcal{B})} \int_{\mathcal{B}_{2r}} |q| \, d\mathcal{L}(q)$ appearing in the previous proof with $|q|_c = d_c(0,q)$. This constant is not computed in [55]. Before this, we insist on the fact that neither this constant nor the constant $\frac{2^{2n+3}}{n}$ of the first proof is optimal. The dilation dil_r acts on the measure and the distance in such a way that C is independent of r and equals $\frac{2^{2n+3}}{\mathcal{L}(\mathcal{B}_1)} \int_{\mathcal{B}_1} |q|_c \, d\mathcal{L}(q)$ (the Jacobian determinant of dil_r is r^{2n+2} in each point and it multiplies the distances by r). We remind the coarea formula of the Heisenberg group (Proposition 1.4.10)

$$\int_{\mathbb{H}} u(q) |\nabla_{\mathbb{H}} F(q)| \, d\mathcal{H}^{2n+2}(q) = \int_0^{+\infty} \int_{\{F=t\}} u(q) \, d\mathcal{H}^{2n+1}(q) \, dt \tag{1.33}$$

where $\mathcal{H}^k$ is the k-dimensional Hausdorff measure associated to d_c, F is smooth and u non-negative and measurable. Here, we first choose $F(q) = d(0,q)$ (that is not smooth in $0_{\mathbb{H}}$ but verify (1.33) by approximation) and $u = \chi_{\mathcal{B}_1}$. As it is proved in [85], we have $|\nabla_{\mathbb{H}} F(q)| = 1$ for every $q \neq 0_{\mathbb{H}}$. We obtain

$$\int_{\mathcal{B}_1} 1 \, d\mathcal{H}^{2n+2}(q) = \int_0^1 \mathcal{H}^{2n+1}(S_{\mathbb{H}}(t)) \, dt$$

where $\mathcal{S}^{\mathbb{H}}(t)$ is the Carnot-Carathéodory sphere of radius t. Its $2n+1$-dimensional measure is t^{2n+1} times the measure of $S^{\mathbb{H}}(1)$. It follows that

$$\mathcal{H}^{2n+2}(\mathcal{B}_1) = \frac{\mathcal{H}^{2n+1}(\mathcal{S}^{\mathbb{H}}(1))}{2n+2}.$$

We now apply a second time the coarea formula to $G(q) = F^2(q)/2$ whose gradient's norm is $F(q)$. Then

$$\begin{aligned}\int_{\mathcal{B}_1} |q|_c \, d\mathcal{H}^{2n+2}(q) &= \int_0^{\frac{1}{2}} \mathcal{H}^{2n+1}(\mathcal{S}^{\mathbb{H}}(\sqrt{2t}))\, dt \\ &= \frac{\mathcal{H}^{2n+1}(\mathcal{S}^{\mathbb{H}}(1))}{2n+3}.\end{aligned}$$

As $\mathcal{H}^{2n+2}$ and $\mathcal{L}$ are equal up to a real factor, we obtain the constant

$$\begin{aligned}C &= \frac{2^{2n+3}}{\mathcal{L}(\mathcal{B}_1)} \int_{\mathcal{B}_1} |q|_c \, d\mathcal{L}(q) \\ &= \frac{2^{2n+3}}{\mathcal{H}^{2n+2}(\mathcal{B}_1)} \int_{\mathcal{B}_1} |q|_c \, d\mathcal{H}^{2n+2}(q) \\ &= \frac{(2n+2)2^{2n+3}}{2n+3} = \frac{(n+1)2^{2n+4}}{2n+3}.\end{aligned}$$

We just made this computation for sake of completeness. Again, these constants are not optimal. Nevertheless when n grows, the constant given using the contraction map is better than the classical one when n goes to infinity. The first one is $\frac{2^{2n+3}}{n}$ while the second is $\frac{(n+1)2^{2n+4}}{2n+3}$.

1.8 The geometric traveling salesman problem in the Heisenberg group

In this section, we will define a counterexample to the converse statement of the main result in [40]. It will be a curve $\omega[0,1]$ whose Z projection looks like a fractal limit of the doted curves in Figure 1.6.

For a given metric space (X, d), the geometric traveling salesman problem is the attempt to characterize compact subsets $E \subset X$ that are contained in a rectifiable curve of X, i.e. a curve of finite length as in (1.17). The characterization arises as the finiteness of a double summation over balls with different centers and radius (see below). This theory has been introduced in $\mathbb{R}^2$ by Peter Jones [62] and it has been completed by Okikiolu [91] who gave the reverse implication for the Euclidean spaces of greater dimension. In order to give the characterization of Jones, we must first define what is a dyadic net of a compact subset E in a metric space (X, d). It is an increasing sequence $(\Delta_k)_{k\in\mathbb{Z}}$ of subsets of E such that for any $k \in \mathbb{Z}$,

- for all $x_1, x_2 \in \Delta_k$, the points are the same or $d(x_1, x_2) > 2^{-k}$,
- for any $x \in E$ there exists $y \in \Delta_k$ such that $d(x, y) \leq 2^{-k}$.

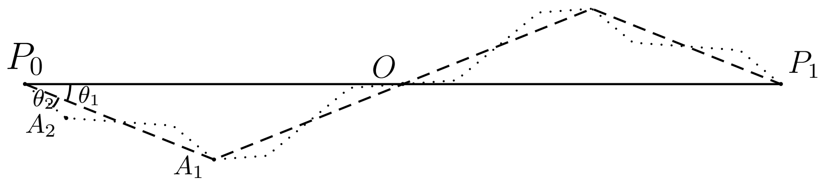

Figure 1.6: The counter-example curve.

Actually for any compact set E, there exists such a dyadic net $(\Delta_k)_{k\in\mathbb{Z}}$. In this section it will not matter what is the choice for the dyadic net. We define

$$B_X^{\Delta}(E) = \sum_{k\in\mathbb{Z}} 2^{-k} \sum_{x\in\Delta_k} \beta_X^2(x, A\cdot 2^{-k})(E) \tag{1.34}$$

where $A > 1$ is a constant to be specified (it will be 5) and $\beta_X(x,r)(E)$ depends on the ambient space. For $\mathbb{R}^n$, it is

$$\min_{l \text{ is a line}} \frac{\max_{y\in E\cap\mathcal{B}(x,r)} d(y,l)}{r}.$$

Here, we consider in fact the maximum distance to Euclidean lines of the points of E that are included in $\mathcal{B}(x,r)$. The minimum of this quantity over l is $\beta_{\mathbb{R}^n}(x,r)(E)$. A set that is "flat" around x at scale r will have a small β number. We give a version of Peter Jones' theorem formulated in the survey [98]. The original theorem is given for dyadic squares instead of a dyadic net.

Theorem 1.8.1. *There exists a constant $C > 0$ (independent of the dyadic net Δ) such that for any compact subset $E \subset \mathbb{R}^n$ with $B_{\mathbb{R}^n}^{\Delta}(E) < +\infty$, there is a Lipschitz curve $\Gamma = \gamma([0,1]) \supset E$ satisfying the following inequality*

$$\mathcal{H}^1(\Gamma) \leq C\left(\operatorname{diam}(E) + B_{\mathbb{R}^n}^{\Delta}(E)\right)$$

and whatever Γ is,

$$B_{\mathbb{R}^n}^{\Delta}(E) \leq C\mathcal{H}^1(\Gamma).$$

In [99] Schul proved that the constant C in the previous result is independent of the dimension n while in the original proof of Theorem 1.8.1 C depends exponentially on the dimension. It permitted him to prove a similar theorem for separable Hilbert spaces. From there it is natural to try to prove the same type of result in other metric spaces. In general metric spaces (X,d) their is an article by Haolama [54] where the author uses the Menger curvature in the definition of the β_X numbers. There is namely no definitely good definition of lines in (X,d) for the geometric traveling salesman problem. In the case of the first Heisenberg group $\mathbb{H}_1$, Ferrari, Franchi and Pajot [40] obtain the exact counterpart of the beginning of Theorem 1.8.1 by using $\mathbb{H}$-lines (see Remark 1.4.4) in the definition of $\beta_{\mathbb{H}}(x,r)$. Precisely

$$\beta_{\mathbb{H}}(x,r)(E) = \min_{\mathbb{H}\text{-line}} \frac{\max_{y\in E\cap \mathcal{B}^{\mathbb{H}}(x,r)} d_c(y,l)}{r}$$

where the balls $\mathcal{B}^{\mathbb{H}}(x,r)$ are the balls of $\mathbb{H}_1$. It is observed in [40] that the $\mathbb{H}$-lines are the left-translations $\mathrm{tran}_p(l_0)$ of the lines l_0 going through $0_{\mathbb{H}}$ in the plane $\mathbb{C}\times\{0_{\mathbb{R}}\}$, that is the $\mathbb{H}$-line going through $0_{\mathbb{H}}$.

The authors show that if the quantity $B^{\Delta}_{\mathbb{H}}(E)$ of (1.34) is finite, there is a rectifiable curve δ covering E. Equivalently there is Lipschitz curve δ_2 that reparametrizes δ and satisfies $\delta_2([0,1])\subset E$. We give here a discrete version of this theorem – In the original theorem $B_{\mathbb{H}}$ is defined by integrating the $\beta^2_{\mathbb{H}}$ on $\mathbb{H}_1\times\mathbb{R}^+$.

Theorem 1.8.2 ([40])**.** *Let E be a compact subset of $\mathbb{H}_1$ and Δ a dyadic net. Then if $B^{\Delta}_{\mathbb{H}}(K) < +\infty$ there is a Lipschitz curve $\Gamma=\gamma([0,1])$ such that $E\subset\Gamma$. Moreover, Γ can satisfy*

$$\mathcal{H}^1(\Gamma)\le C\left(\mathrm{diam}(E)+B^{\Delta}_{\mathbb{H}}(E)\right)$$

where the constant C is independent of E and of its the dyadic net.

They also prove that for regular enough curves of finite length, $B^{\Delta}_{\mathbb{H}}$ is finite.

Proposition 1.8.3 ([40])**.** *Let $\delta:[0,1]\to\mathbb{H}_1$ be $\mathcal{C}^{1,\alpha}$-curve, i.e. δ is an horizontal curve and $Z(\delta)$ is a $\mathcal{C}^{1,\alpha}$ planar curve of $\mathbb{C}$. Then*

$$B^{\Delta}_{\mathbb{H}}(\delta([0,1])) < +\infty.$$

The previous theorem suggests that it should be possible to characterize any compact set K included in a rectifiable curve with $B_{\mathbb{H}}(K)<+\infty$. This would in particular happen for for rectifiable curves themselves. Our curve $\omega([0,1])$ is a counter-example to this statement.

Theorem 1.8.4. *There is a Lispchitz curve $\omega:[0,1]\to\mathbb{H}_1$ such that for any dyadic net Δ of $\Omega=\omega([0,1])$,*

$$B^{\Delta}_{\mathbb{H}}(\Omega)=+\infty.$$

In the first part of this section, we complete our point of view on curves of $\mathbb{H}_1$ that we explained in 1.3 and we state two useful lemmas. The second part is the construction of the curve and in the third one we use the lemmas for proving Theorem 1.8.4: the curve is really a counterexample to the Jones' result adapted to the Heisenberg group.

1.8.1 Closed horizontal curves

If α and β are two curves such that the end point of α is the starting point of β, we defined in Subsection 1.3 $\alpha*\beta$ as the catenation of the two curves. The curve $\alpha*\beta$ is defined on $[0,b+b']$ if b and b' are the end times of α and β respectively. In what follows we will be more permissive: the intervals will possibly be $[a,b]$ with $a\neq 0$ and we will write $\alpha\beta$ instead of $\alpha*\beta$. For α defined on $[a,b]$ let moreover $\bar{\alpha}$ be defined on $[-b,-a]$ by $\bar{\alpha}(s)=\alpha(-s)$.

Lemma 1.8.5. *Let $z \in \mathbb{C}$, $z' \in \mathbb{C}$ and (α_1, α_2) two planar curves going from z to z', defined respectively on $[a_1, b_1]$ and $[a_2, b_2]$. Then the algebraic area swept by the catenation $\overline{\alpha_2}\alpha_1$ is equal to the third coordinate of*

$$[\mathrm{Lift}(\alpha_1)(b_1) - \mathrm{Lift}(\alpha_2)(b_2)] - [\mathrm{Lift}(\alpha_1)(a_1) - \mathrm{Lift}(\alpha_2)(a_2)]$$

for any $\mathbb{H}$-lift $\mathrm{Lift}(\alpha_1)$ and $\mathrm{Lift}(\alpha_2)$ of α_1 and α_2 respectively.

Proof. We first assume that both lifts $\mathrm{Lift}(\alpha_1)$ and $\mathrm{Lift}(\alpha_2)$ start in a same point p with $Z(p) = z$. Then $\overline{\mathrm{Lift}(\alpha_2)}\,\mathrm{Lift}(\alpha_1)$ is a lift of $\overline{\alpha_2}\alpha_1$ and it follows that it encloses an algebraic area equal to the third coordinate of

$$\begin{aligned}[\mathrm{Lift}(\alpha_1)(b_1) - \mathrm{Lift}(\alpha_2)(b_2)] - [0] =& [\mathrm{Lift}(\alpha_1)(b_1) - \mathrm{Lift}(\alpha_2)(b_2)] \\ & - [\mathrm{Lift}(\alpha_1)(a_1) - \mathrm{Lift}(\alpha_2)(a_2)].\end{aligned}$$

The third coordinate difference between two $\mathbb{H}$-lifts of a same planar curve is a constant because of equation (1.14). The conclusion follows by making a vertical translation of $\mathrm{Lift}(\alpha_1)$ or $\mathrm{Lift}(\alpha_2)$. ☐

1.8.2 Geometric Lemmas

In this subsection we will often use the exponent $^{\mathbb{C}}$ for $Z(\cdot)$. For example, we will write $l^{\mathbb{C}}$ and $q^{\mathbb{C}}$ for the complex projections of l and q respectively.

The orthogonal projection on a line of $\mathbb{C}$ has no obvious horizontally lifted counterpart in $\mathbb{H}_1$ as we will see now.

Definition 1.8.6. Let $p \in \mathbb{H}_1$ and l be a $\mathbb{H}$-line. The $\mathbb{C}$-*projection* of p on l is the only point $p^l \in l$ such that $p^{l,\mathbb{C}} := (p^l)^{\mathbb{C}}$ is the orthogonal projection of $p^{\mathbb{C}}$ on $l^{\mathbb{C}}$.

Now, let ζ be a planar line. The *lifted-$\mathbb{C}$-projection* of p on ζ is the only point $p^{\zeta} \in \mathbb{H}_1$ such that

- $p^{\zeta,\mathbb{C}} := (p^{\zeta})^{\mathbb{C}}$ is the orthogonal projection of $p^{\mathbb{C}}$ on the line ζ
- p and p^{ζ} are on a $\mathbb{H}$-line

We give an example. The line of equation

$$x = 2 \quad \text{and} \quad t = 3 + y$$

is a $\mathbb{H}$-line. Its complex projection is $x = 2$. The $\mathbb{C}$-projection of the origin $0_{\mathbb{H}} = (0, 0, 0)$ on this line is $(2, 0, 3)$. The lifted-$\mathbb{C}$-projection on $x = 2$ is $(2, 0, 0)$ because $y = t = 0$ is a $\mathbb{H}$-line and its complex projection is orthogonal to $x = 2$.

Notice that like in the previous example, for a given $\mathbb{H}$-line l and a point $p \in \mathbb{H}_1$, the point $p^{l^{\mathbb{C}}}$ is a well-defined point of $\mathbb{H}_1$ and that it is not always on l. If it is then $p^{l^{\mathbb{C}}} = p^l$ and this point also realizes the distance of p to l. In the next lemma, we give pieces of information about the metric projection of a point to a $\mathbb{H}$-line in the general case.

Lemma 1.8.7. *Let p be a point of $\mathbb{H}_1$ an l a $\mathbb{H}$-line. There is a point q on l that minimizes the distance to p. In $q^{\mathbb{C}}$ the Z-projection of the unique geodesic between p and q make a right angle with $l^{\mathbb{C}}$.*

Proof. It is easier to understand this proof with a look at Figure 1.7. It represents the situation seen from above, which is equivalent to the planar figure obtained by Z-projection. Nevertheless the names of the points and curves are the names of the figure in $\mathbb{H}_1$. There are many analytical or geometric ways to convince that the distance of p to a point of the $\mathbb{H}$-line tends to ∞ at the ends of this line. With a standard compactness argument, we find a point q on l that minimizes the distance to p and let γ be the geodesic from p to q. We will apply now Lemma 1.8.5. For the first curve α_1, we connect $\alpha := \gamma^{\mathbb{C}}$ with a part of $l^{\mathbb{C}}$ going from $q^{\mathbb{C}}$ to $p^{l,\mathbb{C}} = p^{l^{\mathbb{C}},\mathbb{C}} \in l^{\mathbb{C}}$, the orthogonal projection of $p^{\mathbb{C}}$ on $l^{\mathbb{C}}$. The second curve (α_2 in Lemma 1.8.5) is the segment line from $p^{\mathbb{C}}$ to $p^{l,\mathbb{C}}$. The lemma brings us the following information: our closed curve $\overline{\alpha_2}\alpha_1$ encloses an algebraic area whose value $\mathcal{T}$ is the difference between the third coordinates of p^l and $p^{l^{\mathbb{C}}}$. The Euclidean transposition to our minimizing problem is then equivalent to finding the shortest curve from $p^{\mathbb{C}}$ to $l^{\mathbb{C}}$ such that the algebraic area covered by a moving radius centered in $p^{l,\mathbb{C}}$ is exactly the given quantity $\mathcal{T}$.

The following symmetrization argument using the symmetry with respect to the line $l^{\mathbb{C}}$ and Dido's problem conclude the proof: the shortest symmetric curve from $p^{\mathbb{C}}$ to its symmetric point with respect to $l^{\mathbb{C}}$ that covers the area $2\mathcal{T}$ is an arc of circle. The solution is unique if $p^{\mathbb{C}} \notin l^{\mathbb{C}}$ and the curve makes a right angle with $l^{\mathbb{C}}$. $\square$

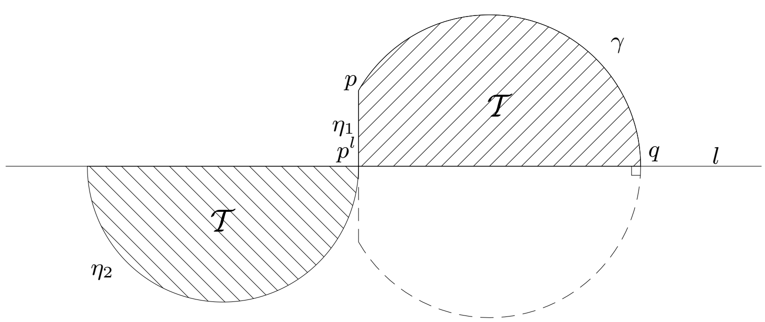

Figure 1.7: Projection lemmas

Remark 1.8.8. Another proof could use the Heisenberg gradient of the distance [7, 85].

We estimate now the distance of a point to a $\mathbb{H}$-line.

Lemma 1.8.9. *Let p be a point of $\mathbb{H}_1$ and l a $\mathbb{H}$-line. Then the distance of p to the line l is comparable to the Euclidean distance between the projections $p^{\mathbb{C}}$*

and $l^{\mathbb{C}}$ plus the distance of the point $p^{l^{\mathbb{C}}}$ obtained by lifted-$\mathbb{C}$-projected to l. In fact

$$\max\left(d_c(p^{\mathbb{C}}, l^{\mathbb{C}}), \frac{d_c(p^{l^{\mathbb{C}}}, l)}{\sqrt{2}}\right) \leq d_c(p, l) \leq d_c(p^{\mathbb{C}}, l^{\mathbb{C}}) + d_c(p^{l^{\mathbb{C}}}, l).$$

Proof. We use the same notations as in Lemma 1.8.7. We have in fact to compare the length of γ to the sum of the lengths of two curves: η_1, the $\mathbb{H}$-line segment from p to $p^{l^{\mathbb{C}}}$ and η_2 one of the two possible shortest curves from $p^{l^{\mathbb{C}}}$ to l. The connexion η of the two previous curves goes from p to l. It follows that the length of η is greater than the one of γ. For the other estimate, we just need to remark than each of the η_i is up to a constant smaller than γ. It is obvious for η_1 with constant 1. For η_2 we require one more time Lemma 1.8.5 and the Dido's problem with a symmetrization in a similar way as in Lemma 1.8.7. We observe that $\eta_2^{\mathbb{C}}$ describes an half circle and enclose an algebraic area $\mathcal{T}$ as it is represented on Figure 1.7. We obtain that η_2 has a length smaller than $\sqrt{2}$ the one of α: when we symmetrize $\eta_2^{\mathbb{C}}$ we obtain a circle of area $2\mathcal{T}$. The curve $\alpha^{\mathbb{C}}$ connected with its symmetrization enclose the same area. It minimizes the length if it is an half of circle. The quotient of the lengths of a circle and an half circle with the same area is $\sqrt{2}$. □

We estimate the distance of two points to a $\mathbb{H}$-line.

Lemma 1.8.10. *Let p_1 and p_2 be two points being on a same $\mathbb{H}$-line and denote another $\mathbb{H}$-line by l. Then*

$$d(p_1, l) + d(p_2, l) \geq \frac{d(p_1^{\mathbb{C}}, l^{\mathbb{C}}) + d(p_2^{\mathbb{C}}, l^{\mathbb{C}}) + \sqrt{|\mathcal{U}(p_1^{\mathbb{C}} p_1^{l,\mathbb{C}} p_2^{l,\mathbb{C}} p_2^{\mathbb{C}})|}}{2}$$

where $\mathcal{U}(p_1^{\mathbb{C}} p_1^{l,\mathbb{C}} p_2^{l,\mathbb{C}} p_2^{\mathbb{C}})$ is the algebraic area of the trapezoid $p_1^{\mathbb{C}} p_1^{l,\mathbb{C}} p_2^{l,\mathbb{C}} p_2^{\mathbb{C}}$.

Proof. First of all $d(p_i^{\mathbb{C}}, l^{\mathbb{C}}) \leq d(p_i, l)$ for $i \in \{1, 2\}$ and we can sum these two relations. It is then enough to prove $d_c(p_1, l) + d_c(p_2, l) \geq \sqrt{|\mathcal{U}(p_1^{\mathbb{C}} p_1^{l,\mathbb{C}} p_2^{l,\mathbb{C}} p_2^{\mathbb{C}})|}$. For that we use Lemma 1.8.5 where we consider the two following curves (in fact their complex projections): On the one hand the $\mathbb{H}$-line segment of l from p_1^l to p_2^l and on the other hand the $\mathbb{H}$-polygonal line from $p_1^{l^{\mathbb{C}}}$ to $p_2^{l^{\mathbb{C}}}$ going through p_1 and p_2. Then the algebraic area of the trapezoid is the third coordinate of

$$[p_1^{l^{\mathbb{C}}} - p_1^l] - [p_2^{l^{\mathbb{C}}} - p_2^l]$$

where the sign minus is the difference between two vectors of $\mathbb{R}^3$. Let $\mathcal{T}_i$ be the third coordinate of $[p_i^{l^{\mathbb{C}}} - p_i^l]$ for $i \in \{1, 2\}$ and write simply $\mathcal{U}$ instead of $\mathcal{U}(p_1^{\mathbb{C}} p_1^{l,\mathbb{C}} p_2^{l,\mathbb{C}} p_2^{\mathbb{C}})$. Then there is a i such that $|\mathcal{T}_i| \geq \frac{|\mathcal{U}|}{2}$. For this i we know exactly that the distance of $p_i^{l^{\mathbb{C}}}$ to l is $\sqrt{2\pi|\mathcal{T}_i|}$ (Dido's problem or see the end of Lemma 1.8.9). Therefore and because of Lemma 1.8.9, we have $d_c(p_i, l) \geq \frac{d_c(p_i^{l^{\mathbb{C}}}, l)}{\sqrt{2}}$ and finally

$$d_c(p_1, l) + d_c(p_2, l) \geq \frac{1}{\sqrt{2}} \sqrt{2\pi \frac{|\mathcal{U}|}{2}} \geq \sqrt{|\mathcal{U}|}.$$

□

1.8.3 Construction of $\omega([0,1])$

As we saw in section 1.3, the absolutely continuous curves of $\mathbb{H}_1$ are exactly the horizontal lifts of the absolutely continuous curves of $\mathbb{C}$. We will describe our curve ω as the $\mathbb{H}$-lift starting in $\omega(0) = (-1, 0, 0)$ of a planar curve $\omega^{\mathbb{C}}$. This curve is a Von-Koch-like fractal with finite length that we obtain as a limit of certain polygonal lines $(\omega_n^{\mathbb{C}})_{n\in\mathbb{N}}$ (see Figure 1.6 for a representation of $\omega_0^{\mathbb{C}}$, $\omega_1^{\mathbb{C}}$ and $\omega_2^{\mathbb{C}}$). Before we explain the recursive way to build the curves, we precise that ω and the ω_n will go from $(-1,0,0)$ to $(1,0,0)$. The direct consequence is that $\omega^{\mathbb{C}}$ and the $\omega_n^{\mathbb{C}}$ go from -1 to 1 in $\mathbb{C}$.

For the construction of $(\omega_n^{\mathbb{C}})_{n\in\mathbb{N}}$, we require a sequence $(\theta_n)_{n\geq 1}$ of non-negative angles that tends to 0. We start from the simple line segment $\omega_0^{\mathbb{C}}$: $s \in [0,1] \mapsto (-1+2s, 0, 0)$ and we obtain $\omega_{n+1}^{\mathbb{C}}$ from $\omega_n^{\mathbb{C}}$ in the way we describe below. The curve $\omega_n^{\mathbb{C}}$ is made of 4^n segments having the same length. Let us denote this length by l_n and the total length by $L_n = 4^n \cdot l_n$. On the $n+1$ step we change every segment line by a polygonal line made of 4 segments, having the same beginning and the same end. These four segments have length $\frac{l_n}{4\cos\theta_{n+1}}$ and all make with the former line segment an angle θ_{n+1} (see Figure 1.6). There are two ways to respect these conditions. However, the construction is unique if we precise the orientation: when the time grows the first of the 4 small segments make a negative angle with respect to the segment of length l_n.

The important remark is that replacing the segment by the polygonal line of 4 segments, we do not change the swept algebraic area.

Let us define the value of the angles θ_n. In all this construction, it will be $\theta_n = \frac{C}{n}$ where $C = 0.2$. We prove now that $\omega^{\mathbb{C}}$ is well-defined as the limit of $\left(\omega_n^{\mathbb{C}}\right)_{n\in\mathbb{N}}$ where each $\omega_n^{\mathbb{C}}$ is parametrized with constant speed on $[0,1]$.

Proposition 1.8.11. *The sequence of curves $(\omega_n^{\mathbb{C}})_{n\in\mathbb{N}}$ tends to a rectifiable curve $\omega^{\mathbb{C}} : [0,1] \to \mathbb{H}_1$ parametrized with constant speed.*

Proof. The speed of the curves $\omega_n^{\mathbb{C}}$ is exactly the length L_n and this quantity is also the best Lipschitz constant of $\omega^{\mathbb{C}}$. Let us prove the uniform convergence. The curves $\omega_n^{\mathbb{C}}$ and $\omega_{n+1}^{\mathbb{C}}$ meet at every time $\frac{\sigma}{4^n} \in [0,1]$ where $\sigma = 0, \cdots, 4^n$. Between two subsequent meetings the curve ω_{n+1} always repeats the same motion pattern while ω_n is a segment. On $[\frac{\sigma}{4^n}, \frac{\sigma+1}{4^n}]$ the curves are the more distant when the first of the four segments is done, exactly at time $\frac{\sigma}{4^n} + \frac{1}{4^{n+1}}$. The maximum distance is also attained at time $\frac{\sigma}{4^n} + \frac{3}{4^{n+1}}$. From this observation we deduce

$$\|\omega_n^{\mathbb{C}} - \omega_{n+1}^{\mathbb{C}}\| = (\sin\theta_n) l_{n+1}.$$

The quotient between l_n and l_{n+1} is $\frac{1}{4\cos(\theta_{n+1})}$. Because all θ_n have a cosine greater than 0.5, this quotient is smaller than 1/2. We conclude that the series

$$\sum_{n=0}^{+\infty} \|\omega_{n+1}^{\mathbb{C}} - \omega_n^{\mathbb{C}}\| \leq \sum_{n=0}^{+\infty} (\sin\theta_n) l_0 \cdot 2^{-n}$$

converge.

In the next lemma we prove that $L := \limsup_{n\to+\infty} L_n < +\infty$. As a direct consequence $\omega^{\mathbb{C}}$ will be L-Lipschitz. We recall that $\theta_n = \frac{C}{n}$ where $C = 0.2$ and with a few trigonometry we see that $L_n = \frac{2}{\prod_{m=1}^{n} \cos\theta_m}$.

Lemma 1.8.12. *We have $L \leq 2.4 = 1.2 \cdot L_0$. Moreover, L is the optimal Lipschitz constant and the length of $\omega^{\mathbb{C}}$.*

Proof. Because of the convexity of log, if $(1-x) \in [e^{-1}, 1]$, then

$$\log(1-x) \geq \frac{-x}{1-e^{-1}} \geq -2x.$$

It is possible to apply it to $x = \theta^2/2$ because $\theta \leq C \leq \sqrt{2-2e^{-1}}$. Then we have

$$\begin{aligned}
\log\left(\frac{1}{\prod_{n=1}^{N} \cos\theta_n}\right) &= -\sum_{n=1}^{N} \log(\cos\theta_n) \\
&\leq -\sum_{n=1}^{N} \ln(1-\frac{\theta_n^2}{2}) \\
&\leq \sum_{n=1}^{N} \theta_n^2 \\
&\leq C^2 \frac{\pi^2}{6} \leq 0.08.
\end{aligned}$$

Then we have $L \leq L_0 \exp(0.08) \leq 1.2 \cdot L_0$.

Thus L is the optimal Lipschitz constant for $\omega^{\mathbb{C}}$. Indeed for $m \geq n$ the distance between $\omega^{\mathbb{C}}(\frac{\sigma}{4^n})$ and $\omega^{\mathbb{C}}(\frac{\sigma+1}{4^n})$ is $L_n/4^n$ because

$$\omega^{\mathbb{C}}(\frac{\sigma}{4^n}) = \omega_m^{\mathbb{C}}(\frac{\sigma}{4^n}) = \omega_n^{\mathbb{C}}(\frac{\sigma}{4^n}).$$

It follows also from the same observation that L is the length of $\omega^{\mathbb{C}}$. □

□

We defined ω as the lift of $\omega^{\mathbb{C}}$ starting from $(-1,0,0)$ and ω_n the one of $\omega_n^{\mathbb{C}}$ starting from $(-1,0,0)$. All these curves are parametrized with constant speed on $[0,1]$.

Lemma 1.8.13. *The curves ω_n and ω_{n+1} exactly meet on the points $\frac{\sigma}{4^n}$ for $\sigma = 0, \cdots, 4^n$.*

Proof. The property is surely true for $\sigma = 0$ because $\omega_{n+1}(0) = \omega_n(0) = (-1,0,0)$. Let σ be an integer smaller than $4^n - 1$. We assume that on $[0, \frac{\sigma}{4^n}]$ the curves ω_n and ω_{n+1} only meet at the times $\frac{\sigma'}{4^n}$ for $\sigma' = 0, \cdots, \sigma$. Let us now exam what happen on $[\frac{\sigma}{4^n}, \frac{\sigma+1}{4^n}]$. The curves are both starting from $\omega_n(\frac{\sigma}{4^n}) = \omega_{n+1}(\frac{\sigma}{4^n})$ and respectively lift $\omega_n^{\mathbb{C}}$ and $\omega_{n+1}^{\mathbb{C}}$. The previous planar curves meet at $\frac{\sigma}{4^n}$, at $\frac{\sigma+1}{4^n}$ and at the mid point $\frac{\sigma}{4^n} + \frac{1}{2\cdot 4^n}$. Then these are the only possible meeting points for ω_n and ω_{n+1} on $[\frac{\sigma}{4^n}, \frac{\sigma+1}{4^n}]$. Now, We consider two $\mathbb{H}$-lift, starting from $\omega_{n+1}(\frac{\sigma}{4^n})$ and we will use Lemma 1.8.5 for them. On the one hand we lift horizontally $\omega_{n+1}^{\mathbb{C}}$ on $\left[\frac{\sigma}{4^n}, \frac{\sigma}{4^n} + \frac{1}{2\cdot 4^n}\right]$ and on the other hand we lift $\omega_n^{\mathbb{C}}$ on the same interval. Both planar curves arrive in the same point and the associated closed planar curve sweeps the positive area $(\frac{l_n^2 \cdot \tan(\theta_{n+1})}{4})$ of a triangle. This quantity is the difference for the third coordinate of the end points of the lifts. We have

$$\omega_{n+1}(\frac{\sigma}{4^n}+\frac{1}{2\cdot 4^n}) \neq \omega_n(\frac{\sigma}{4^n}+\frac{1}{2\cdot 4^n}).$$

If we make the similar operation lifting $\omega^{\mathbb{C}}_{n+1}$ and $\omega^{\mathbb{C}}_n$ on $\left[\frac{\sigma}{4^n}, \frac{\sigma+1}{4^n}\right]$, we contrarily obtain an algebraic area equal to zero and can conclude that

$$\omega_{n+1}(\frac{\sigma+1}{4^n}) = \omega_n(\frac{\sigma+1}{4^n}).$$

□

A corollary of this lemma is that for any integer $m \geq n$, $\omega(\frac{\sigma}{4^n}) = \omega_m(\frac{\sigma}{4^n})$.

Remark 1.8.14. In the previous lemma, we remarked that $\omega_{n+1}(\frac{\sigma}{4^n}+\frac{1}{2\cdot 4^n})$ has the same first coordinates as $\omega_n(\frac{\sigma}{4^n}+\frac{1}{2\cdot 4^n})$ but the t-coordinate difference is $\frac{l_n^2\cdot\tan(\theta_{n+1})}{4^2}$. Then the Carnot-Caratheodory distance between them is greater than $\frac{K}{4^n\cdot\sqrt{n}}$ for some constant K. It is an indication that the linear segments of ω_n are not such a good approximation of ω where a good approximation would have been to be smaller than $\frac{K'}{4^n\cdot n}$ for some constant K'. This is a decisive observation and a good reason for believing in Theorem 1.8.4.

Remark 1.8.15. An amazing observation is that $\omega^{\mathbb{C}}$ is not derivable in any point $\frac{\sigma}{4^n}$ for any n and $\sigma \leq 4^n$. Around these points, the curve is making a spiral because $\sum_{m=n}^{+\infty}\theta_m = +\infty$. However, $\omega^{\mathbb{C}}$ is a Lipschitz curve and is then almost everywhere derivable. In fact it seems that for a time $s \in [0,1]$, written $\overline{0, a_1a_2\cdots}^4$ in basis 4, the curve $\omega^{\mathbb{C}}$ is derivable in s if and only if the series $\sum_{m=1}^{+\infty}\frac{\varepsilon(\overline{a_m}^4)}{m}$ converge. Here, ε is defined by

$$\varepsilon(0) = \varepsilon(3) = 1 \quad \text{and} \quad \varepsilon(1) = \varepsilon(2) = -1.$$

1.8.4 Counterexample for the inverse implication in [40]

We prove in this subsection that $B^{\Delta}_{\mathbb{H}}(\omega([0,1]))$ is infinite. With the notations of the beginning of this section, the first step will consist in estimating the cardinal of Δ_k. In the second step, we will estimate from below the value of $\beta_{\mathbb{H}}(x, A\cdot 2^{-k})$ for a $x \in \Delta_k$. For this we will require the geometric lemmas of Section 1.8.2.

Because of the second property of the net, $\omega \subset \bigcup_{x\in\Delta_k}\mathcal{B}^{\mathbb{H}}(x, 2^{-k})$. The projection of a ball for the Heisenberg metric on the complex plane is a ball of $\mathbb{R}^2$ with the same radius. That is why

$$\omega^{\mathbb{C}} \subset \bigcup_{x\in\Delta_k}\mathcal{B}^{\mathbb{C}}(x^{\mathbb{C}}, 2^{-k}).$$

If we perform a second projection on the real axis, we obtain that the segment $[-1,1]$ is covered by a family of segments of length 2^{-k+1} which is indexed by Δ_k. We conclude that the cardinal of Δ_k is greater than 2^k.

In this paragraph, we examine what is the right fractal scale of the portion of $\omega([0,1])$ intercepted a ball $\mathcal{B}^{\mathbb{H}}(x, A\cdot 2^{-k})$ with center in Δ_k. Let us compare $A\cdot 2^{-k}$ to $\frac{L_\infty}{4^n} \leq \frac{2.4}{4^n}$ and assume $A = 5$ for the rest of this proof. We observe that for every $k > 0$ and $n = \lceil k/2 \rceil$, $\frac{2.4}{4^n}$ is smaller than $A \cdot 2^{-k}$. It follows that there is a $\sigma \in \{0, 1, \cdots, 4^n - 1\}$ such that $\omega\left([\frac{\sigma}{4^n}, \frac{\sigma+1}{4^n}]\right) \subset \mathcal{B}(x, A \cdot 2^{-k})$.

If we rescale correctly the last portion of curve using the similitudes of the Heisenberg group (Subsection 1.1.1), we obtain a curve that could have been ω if we had chosen the sequence of angle $(\theta_{n+m})_{m=1}^{+\infty}$. In particular this curve includes the set Λ_θ made of the five points

$$\left\{(-1; 0), (-\frac{1 + i\tan(\theta)}{2}; \frac{\tan(\theta)}{2}), (0; \frac{\tan(\theta)}{2}), (\frac{1 + i\tan(\theta)}{2}; \frac{\tan(\theta)}{2}), (1; 0)\right\}$$

for $\theta = \theta_{n+1}$. We are interested in the maximal distance of one point of Λ_θ to a given $\mathbb{H}$-line l. We denote this distance by $d_\theta(l)$ and D_θ is the minimum of $d_\theta(l)$ over all the $\mathbb{H}$-lines l. We noticed that there is a similitude mapping Λ_θ on a part of $\omega \cap \mathcal{B}(x, A \cdot 2^{-k})$. This map multiplies the distances by $\frac{l_n}{2}$ where we recall that l_n is the length of the 4^n segments composing ω_n. Then the distance of $\omega \cap \mathcal{B}(x, A \cdot 2^{-k})$ to the closest $\mathbb{H}$-line is greater than $\frac{l_n}{2} D_\theta$ and

$$\begin{aligned}\beta_{\mathbb{H}}(x, A \cdot 2^{-k}) &\geq \frac{l_n}{2} \cdot \frac{D_\theta}{A \cdot 2^{-k}} \\ &\geq \frac{2.4 \cdot D_\theta}{4^n \cdot A \cdot 2^{-k}} \\ &\geq \frac{D_\theta}{A}. \end{aligned} \tag{1.35}$$

Proposition 1.8.16. *Let $\theta < 0.2$ be a positive angle and l a $\mathbb{H}$-line. Then the maximum distance of one of the five points of Λ_θ to l is greater than $K \cdot \sqrt{\theta}$ for some constant K independent of l and θ. In other words*

$$D_\theta \geq K\sqrt{\theta}.$$

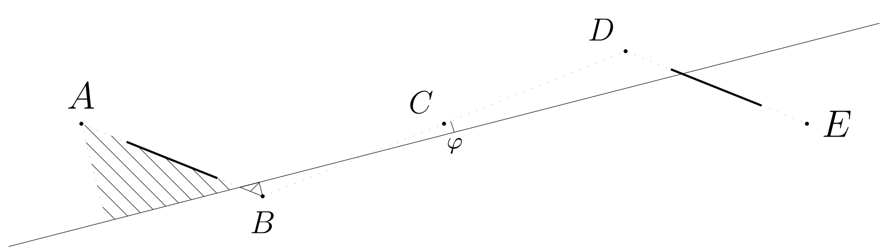

Figure 1.8: The five points are far from a $\mathbb{H}$-line.

Proof. In this proof the points of $\mathbb{H}_1$ will be denoted with capital letters. We will write A, B, C, D, E where we would have wrote a, b, c, d, e before (and A is different from the real constant $A = 5$ introduced before). Let us first denote the five points by A, B, C, D, E where $A = (-1, 0, 0)$ and $E = (1, 0, 0)$ like on Figure 1.8. Thanks to the two geometric lemmata, Lemma 1.8.9 and Lemma 1.8.10, we will just have to consider the projections

$$\begin{aligned}
A^{\mathbb{C}} &= -1 \\
B^{\mathbb{C}} &= -\frac{1}{2} - i\frac{\tan(\theta)}{2} \\
C^{\mathbb{C}} &= 0 \\
D^{\mathbb{C}} &= \frac{1}{2} + i\frac{\tan(\theta)}{2} \\
E^{\mathbb{C}} &= 1
\end{aligned}$$

and a planar line $l^{\mathbb{C}}$ together with the fact that some points are on a same $\mathbb{H}$-line. It is the case of the couples (A, B), (D, E) and (A, E). The three points B, C and D are also on a same $\mathbb{H}$-line.

In this proof, we will sort the possible planar lines $l^{\mathbb{C}}$ by the geometric angle $\varphi \in [0, \frac{\pi}{2}]$ they make with the line $(B^{\mathbb{C}}D^{\mathbb{C}})$. If $\varphi \geq \sqrt{\theta}$, then one of the point $B^{\mathbb{C}}$ or $D^{\mathbb{C}}$ is more distant than $l_\theta \sin\sqrt{\theta}$ to the line $l^{\mathbb{C}}$ where l_θ is the distance between $B^{\mathbb{C}}$ and $C^{\mathbb{C}}$ (it is also the distance between B and C in $\mathbb{H}_1$ or between $A^{\mathbb{C}}$ and $B^{\mathbb{C}}$ for example in $\mathbb{C}$). Then because of Lemma 1.8.9, the distance of the line l to the farest point is greater than $\frac{1}{2} \cdot (\sqrt{\theta}\frac{2}{\pi})$.

If $\varphi \in [\frac{\theta}{4}, \sqrt{\theta}]$, we consider one of the segment $[B^{\mathbb{C}}C^{\mathbb{C}}]$ or $[C^{\mathbb{C}}D^{\mathbb{C}}]$ that the line $l^{\mathbb{C}}$ does not intersect. Let assume for example, $l^{\mathbb{C}}$ does not intersect $[B^{\mathbb{C}}C^{\mathbb{C}}]$. Then the area of the trapezoid obtained when we project $B^{\mathbb{C}}$ and $C^{\mathbb{C}}$ on $l^{\mathbb{C}}$ is greater that $\frac{l_\theta^2 \sin(\varphi)\cdot\cos(\varphi)}{2} \geq \frac{\sin(2\varphi)}{16}$. But $2\varphi \leq 2\sqrt{0.2} \leq \frac{\pi}{2}$. It follows that $\sin(2\varphi) \geq \frac{2\cdot 2\varphi}{\pi}$ and

$$\sqrt{|\mathcal{U}(B^{\mathbb{C}}, B^{\mathbb{C},l}, C^{\mathbb{C},l}, C^{\mathbb{C}})|} \geq \sqrt{\frac{\varphi}{4\pi}} \geq \sqrt{\frac{\theta}{16\pi}},$$

which thanks to Lemma 1.8.10 provides a lower bound for the distance to l with the right exponent of θ.

The last case, $\varphi \in [0, \frac{\theta}{4}]$ is the more intricate. Here, the line $l^{\mathbb{C}}$ can be very close to $(B^{\mathbb{C}}D^{\mathbb{C}})$. We will prove that it composes a great enough area when projecting orthogonally one of the segments $[A^{\mathbb{C}}B^{\mathbb{C}}]$ or $[D^{\mathbb{C}}E^{\mathbb{C}}]$ on $l^{\mathbb{C}}$. Unlike in the previous case, $l^{\mathbb{C}}$ can intersect both $[A^{\mathbb{C}}B^{\mathbb{C}}]$ and $[C^{\mathbb{C}}D^{\mathbb{C}}]$. Let assume for a while that $l^{\mathbb{C}}$ can not intersect the central segment of $[A^{\mathbb{C}}B^{\mathbb{C}}]$ and the central segment of $[D^{\mathbb{C}}E^{\mathbb{C}}]$ where we mean by central segment the points on the segment obtained as barycenter of the ends with coefficients between $\frac{1}{4}$ and $\frac{3}{4}$. This assumption is true and we postpone it to Lemma 1.8.17. Assume for example that $l^{\mathbb{C}}$ does not intercept the central segment of $[A^{\mathbb{C}}B^{\mathbb{C}}]$. Then projecting $A^{\mathbb{C}}$ and $B^{\mathbb{C}}$ on $l^{\mathbb{C}}$, we compose a trapezoid (self-intersecting in the more difficult case as on Figure 1.8). The angle ψ between $l^{\mathbb{C}}$ and $(A^{\mathbb{C}}B^{\mathbb{C}})$ is included in $[2\theta - \varphi, 2\theta + \varphi]$. This angle ψ is then greater than $\frac{7\theta}{4}$ and smaller than $\frac{\pi}{4}$. Hence we can estimate the algebraic area of the trapezoid in a similar way as in the previous case.

$$
\begin{aligned}
|\mathcal{U}(A^{\mathbb{C}}B^{\mathbb{C}}B^{\mathbb{C},l}A^{\mathbb{C},l})| &\geq \left(\frac{3\cdot l_\theta}{4}\right)^2 \frac{\sin(2\psi)}{4} - \left(\frac{l_\theta}{4}\right)^2 \frac{\sin(2\psi)}{4} \\
&\geq \frac{\sin(2\psi)}{32} \\
&\geq \frac{2\cdot(2\psi)}{\pi\cdot 32} \geq \frac{7\theta}{32\pi}.
\end{aligned}
$$

Then we have $\sqrt{|\mathcal{U}(A^{\mathbb{C}}B^{\mathbb{C}}B^{\mathbb{C},l}A^{\mathbb{C},l})|} \geq \sqrt{\theta\frac{7}{32\pi}}$ and Lemma 1.8.10 concludes the proof.

Lemma 1.8.17. *A planar line $l^{\mathbb{C}}$ that makes an angle $\varphi < \frac{\theta}{4}$ with $(B^{\mathbb{C}}D^{\mathbb{C}})$ can not intercept both the central segments of $[A^{\mathbb{C}}B^{\mathbb{C}}]$ and the one of $[D^{\mathbb{C}}E^{\mathbb{C}}]$.*

Proof. We argue by contradiction and assume that $l^{\mathbb{C}}$ intercepts both the central segment of $[A^{\mathbb{C}}B^{\mathbb{C}}]$ and the central segment of $[D^{\mathbb{C}}E^{\mathbb{C}}]$. We can suppose that $l^{\mathbb{C}}$ goes through $C^{\mathbb{C}}$. Actually as $[A^{\mathbb{C}}B^{\mathbb{C}}]$ is the image of $[D^{\mathbb{C}}E^{\mathbb{C}}]$ by central symmetry, the image $l'^{\mathbb{C}}$ of $l^{\mathbb{C}}$ by the same symmetry has the same property as $l^{\mathbb{C}}$. Namely it goes through the central segments. Moreover, because both central segments of $[A^{\mathbb{C}}B^{\mathbb{C}}]$ and $[D^{\mathbb{C}}E^{\mathbb{C}}]$ are convex, the parallel lines between $l^{\mathbb{C}}$ and $l'^{\mathbb{C}}$ also intercept these two sets. That is why we can assume that $l^{\mathbb{C}}$ is one of the two lines making an angle φ with $(B^{\mathbb{C}}D^{\mathbb{C}})$ and going through $C^{\mathbb{C}}$. It's not difficult to convince oneself that $l^{\mathbb{C}}$ can not cross the central segment of $[A^{\mathbb{C}}B^{\mathbb{C}}]$. Indeed, assume that we divide uniformly $[A^{\mathbb{C}}B^{\mathbb{C}}]$ in four equal parts and join the five points with $C^{\mathbb{C}}$, the greatest of the four angles is the one involving the line $(B^{\mathbb{C}}C^{\mathbb{C}})$. Then it is greater than $\theta/4$ which is the angle average and it is also greater than φ. This implies a contradiction. □

□

By (1.35) and Proposition 1.8.16, we finally get

$$
\begin{aligned}
B^{\Delta}_{\mathbb{H}}(\omega([0,1])) \geq &\sum_{k\in\mathbb{N}} 2^{-k} \sum_{x\in\Delta_k} \beta^2_{\mathbb{H}}(x, A\cdot 2^{-k})(\omega([0,1])) \\
\geq &\sum_{k\in\mathbb{N}} 2^{-k}2^k \left(\frac{D_{\theta_{\lceil k/2\rceil+1}}}{A}\right)^2 \\
\geq &C\sum_{k\in\mathbb{N}} \frac{1}{\lceil k/2\rceil+1} \geq +\infty.
\end{aligned}
$$

Chapter 2

Optimal transport

Optimal transport is appeared on the XVIII century with the very applied problem of Monge for master builder: Find the best way to minimize the mean distance resulting from the displacement of an amount of soil (déblais) to a given construction (remblais). Nowadays the developments starting from this question have found a very wide field of application e.g. in economy, statistics, analysis and geometry. The "déblais" and "remblais" have become probability measure in special metric spaces and a particular interest occurs on the probability measure interpolated by optimal transport. The question stated by Ambrosio and Rigot in [7, Section 7 (c)] attests of this interest: are the measures interpolated between an absolutely continuous probability measure of $\mathbb{H}_n$ and another probability measure, also absolutely continuous measures? This chapter is an extensive version of [42] where Figalli and the author solved this problem, answering with yes. With respect to [42], there will be more introducing definitions and examples.

2.1 Monge and Kantorovich problems

In this section we present the two problems at the origin of mass transport in the context of metric spaces (X, d). We give some examples when X is the Euclidean space and state the Brenier-McCann theorem about existence and uniqueness of an optimal transport map in $\mathbb{R}^n$.

2.1.1 Statement of the problems

Let (X, d) be a Polish metric space. The L^2 Monge optimal transport problem is to find for two given probability measures μ_0 and μ_1 a map T satisfying $T_\#\mu_0 = \mu_1$ and that minimizes

$$C_2^M(T) = \frac{1}{2}\int_X d^2(p, T(p))d\mu_0(p) \tag{2.1}$$

where M stands for "Monge". We will denote by $C_2^M(\mu_0, \mu_1) = \inf_T C_2^M(T)$ the infimum of this problem. If T satisfies the push-forward condition $T_\#\mu_0 = \mu_1$ it is called a *transport map*. If it minimizes the cost C_2^M, it is an *optimal transport map*. The existence of an optimal map strongly relies on the geometry of X

and on the probability measures μ_0 and μ_1. Note that if μ_0 has an atomic part and μ_1 does not give any mass to the points of X, it does not even exist any transport map between the two measures.

Remark 2.1.1. The original problem of Monge [81] was actually a L^1 problem (in (2.1) replace the distance squared by the distance) but the L^2 is more adapted to the the modern developments begining with Brenier [18, 19] where d is a geodesic distance of a special space. It is also more suitable for dual formulations. In the rest of this thesis, we will most of time simply write "the Monge problem" instead of "L^2 Monge's optimal transport problem".

The usual approach for solving the Monge problem has been to consider the relaxed version due to Kantorovich [65] (more than 150 years after Monge). As we will see the Kantorovich's optimal transport problem is more linear. The space of candidates to be a minimizer is never empty and it is convex. Moreover, the new problem has a more symmetric formulation. The starting observation is that (2.1) can be rewritten as

$$\frac{1}{2}\int_{X\times X} d^2(p,q))d\pi_T(p,q) \tag{2.2}$$

where $\pi_T = (\mathrm{Id} \otimes T)_{\#}\mu_0$ is the *plan* associated to the map T. Here, and generally in optimal transport, a plan is a probability measure on the product space X^2. Roughly speaking , in the Kantorovich's problem, plans replace the maps of Monge's problem. Therefore it consists in finding an *optimal transport plan* which is a plan that would realize the minimum of

$$C_2(\pi) = \frac{1}{2}\int_{X\times X} d^2(p,q))d\pi(p,q) \tag{2.3}$$

under the transport constraints $p_{\#}\pi = \mu_0$ and $q_{\#}\pi = \mu_1$ where p and q are taken for the first and second coordinates map of $X \times X$. We call $C_2(\mu_0, \mu_1)$ the infimum of (2.3) under this constraint. For two measurable sets A and B, $\pi(A \times B)$ has to be interpreted as the mass that leaves A and arrives on B. Then the first marginal equality $p_{\#}\pi = \mu_0$ means that the mass that leaves X and goes somewhere is distributed following the law of μ_0. The second marginal equality means that the mass arriving from somewhere to X is distributed under the law of ν.

As one can easily check π_T is a transport plan if and only if T is a transport map. It is then a plan such that the mass starting from p does not "split" and μ_0-almost certainly go to $T(p)$. The tensorial product $\mu_0 \otimes \mu_1$ is always a transport map. For this transport plan, the mass in p split and is mapped in X following the distribution μ_1 independently of p. Moreover, any convex combination of two transport maps is a transport map too.

It can happen that any transport map π between two probability measures μ_0 and μ_1 provides an infinite cost $C_2(\pi)$. Then every transport plan is optimal and $C_2(\mu_0, \mu_1) = +\infty$. In order to avoid this degenerate situation, one can assume that μ_0 and μ_1 are in the space

$$\mathcal{P}_2(X) = \left\{ \mu \in \mathcal{P}(X) \mid \int d^2(p,p_0)d\mu(p) < +\infty \right\}$$

for some $p_0 \in X$ where $\mathcal{P}(X)$ is the space of the probability measures of X. In fact as μ is a probability measure, if the integral in the definition is finite for some p_0 it is finite for any $p_0' \in X$ such that the definition of $\mathcal{P}_2(X)$ does not depend on p_0 . The space $\mathcal{P}_2(X)$ is the so-called "Wasserstein space" or "L^2-Wasserstein space" relatively to the exponent in the definition. We will use this wide-accepted appellation although there is discussion about the name "Wasserstein". See e.g. the bibliographical notes in Chapter 6 of [109]. Let us now show why $C_2(\mu_0, \mu_1) < +\infty$ when the measures are in $\mathcal{P}_2(X)$. In fact

$$\begin{aligned}C_2(\mu_0 \otimes \mu_1) &= \int \frac{d(p,q)^2}{2} d\mu_0(p) d\mu_1(q) \\ &\leq \int \left(d(p,p_0)^2 + d(p_0,q)^2\right) d\mu_0(p) d\mu_1(q) < +\infty.\end{aligned}$$

Usual strategy for the existence and uniqueness of solutions

The problem of existence and uniqueness of solutions of the Monge problem has been considered in the setting of Euclidean spaces (Brenier, [18]), compact Riemannian manifolds (McCann , [80]), the Wiener space (Feyel and Üstünel, [41]), loop groups (Fang and Shao, [38]), Alexandrov spaces (Bertrand, [14]) and Finsler manifolds (Ohta, [90]). The common strategy of proof may be divided into three steps.

- There is an optimal transport plan. It is most of the time a consequence of the topology of the probability space $\mathcal{P}(X)$. Particularly, in the case of Polish spaces, Prokhorov's theorem, a theorem of weak compactness permits to prove

  Proposition 2.1.2. *Let (X, d) be a Polish metric space and μ_0 and μ_1 two probability measures. Then there is a transport plan π that minimizes the cost C_2 of the Kantorovich problem.*

  This proposition appears for instance in [108, Theorem 4.1].

- Prove that π is concentrated on the graph of a measurable map T. Because of the transport constraint, there is a unique possible way to concentrate a probability measure such as π on $\text{graph}(T) = \{(p, T(p)) \in X^2\}$. This unique plan is π_T defined above. This step relies on the following theorem (see [109, Theorem 5.9(ii)]):

  Theorem 2.1.3. *With the notations of Proposition 2.1.2, if $C_2(\mu_0, \mu_1)$ is finite, there exists a c-convex function $\phi : X \to \mathbb{R}$ such π-almost certainly, the inequality*

  $$\phi(p) + \phi^c(q) + \frac{d^2(p,q)}{2} \geq 0 \qquad (2.4)$$

  is an equality.

  For a definition of c-convex function see the end of this subsection. Theorem 2.1.3 tells that π is concentrated on the c-subdifferential of ϕ, namely on the set $\partial\phi = \{(p,q) \in X^2 \mid (2.4) \text{ is an equality}\}$. If (p, q) is in the c-subdifferential of ϕ, the function $f_q = \phi(\cdot) + \frac{d^2(\cdot,q)}{2}$ has a minimum in p. The derivate of f_q in p, if it exists must be 0. The general hope is that it is possible from this information to determinate q as a function of p. More precisely it is enough to prove that there is a map T such that μ_0-almost surely the intersection $(\{p\} \times X) \cap \partial\phi$ is the single set $\{(p, T(p))\}$.

The technical difficulty of this central step depends on the possibility of a metric differentiable structure on X and also on the differentiability of f_q that can result of the differentiabily properties of the distance squared d^2 and of the c-convex function ϕ.

- We can now prove the uniqueness. Let us take two optimal transport maps from μ_0 and μ_1. Thanks to the linearity of C_2, the cost associated to the plan $\pi = \frac{\pi_T + \pi_{T'}}{2}$ equals $C_2(\pi_T)$ and $C_2(\pi_{T'})$. Because of the last item, it must be concentrated in the graph of a map T_1. But a decomposition of the measure π with respect to μ_0 is

$$\pi = \int_X \frac{\delta_{T(p)} + \delta_{T'(p)}}{2} d\mu_0(p),$$

so μ_0-almost surely $\frac{\delta_{T(p)} + \delta_{T'(p)}}{2}$ is a Dirac mass of X. It follows that μ_0-almost surely $T_1(p) = T(p) = T(p')$ and the optimal plan $\pi = \pi_T = \pi'_T$ is unique.

The paper of Ambrosio and Rigot [7] started the study of existence and uniqueness of solutions to the Monge problem in subRiemannian geometry with the Heisenberg group and its Carnot-Carathéodory distance using the Pansu differentiability [93] of Lipschitz maps defined on $\mathbb{H}_n$. More recently Agrachev and Lee [2] and Figalli and Rifford [43] succeeded in extending the theorem of existence and uniqueness of solutions to more general subRiemannian manifolds. There approach use the differentiability of the subRiemannian distance for the extrinsic Riemannian geometry (see Subsection 2.3.5).

c-convex functions

Before we define the c-convex functions, we should explain that in the appellations c-convex, c-transform and c-subdifferential, c stand for the cost function of the problem, indeed

$$c(p, q) = \frac{d(p, q)^2}{2}$$

so that $C_2(\pi) = \int_{X \times X} c(p, q) d\pi(p, q)$. Theorem 2.1.3 where appears a c-convex function ϕ is in fact part of the more general duality theory of Kantorovich where the cost functions c can be functions on product measure spaces $(X, \mu_0) \times (Y, \mu_1)$ (see [108, Chapter 1]).

Let $\mathcal{F}_X$ be the set of the functions from X to $\mathbb{R} \cup \{+\infty\}$ that are not identically infinite. We set $\mathrm{Dom}(\phi) = \{p \in X \mid \phi(p) < +\infty\}$ and call it the domain of ϕ. Thus the domain of a function of $\mathcal{F}_X$ is not empty.

The c-transform of a function $\psi \in \mathcal{F}_X$ is

$$\psi^c(p) = \sup_{q \in X} (-c(p, q) - \psi(q)) = -\inf_{q \in X} \left(\frac{d(p, q)^2}{2} + \psi(q) \right).$$

A function $\phi \in \mathcal{F}_X$ is said to be c-convex if it is the c-transform ψ^c of a function of $\psi \in \mathcal{F}_X$. One can prove that if ϕ is c-convex, then $\phi^{cc} = \phi$. There is between a c-convex function ϕ and its c-transform ϕ^c a special relation

$$\phi(p) + \phi^c(q) + c(p,q) \geq 0.$$

The set of pairs (p,q) such that $\phi(p) + \phi^c(q) + c(p,q) = 0$ is called the c-subdifferential $\partial^c\phi$ of ϕ. It is a subset of $X \times X$ but we can see it as a multivalued map and we introduce consequently the following notation

$$\partial^c\phi(p) = \left\{(p,q) \in X^2 \mid \phi(p) + \phi^c(q) + c(p,q) = 0\right\} = \partial^c\phi \cap (\{p\} \times X).$$

2.1.2 Optimal transport in $\mathbb{R}^n$

In $\mathbb{R}^n$, the Brenier-McCann theorem (Theorem 2.1.10) states that the Monge-Kantorovich problem has a unique solution if we suppose μ_0 absolutely continuous (and in the easiest version μ_0 and μ_1 concentrated on a compact set).

Some exotic examples of optimal transport in $\mathbb{R}^n$

Here we give some simple examples of pairs (μ_0, μ_1) for which an optimal transport plan is known but such that the Brenier-McCann hypothesis are not satisfied.

Example 2.1.4 (Contraction on a point). Suppose that μ_0 is a Dirac mass δ_p and μ_1 is any measure. The transport plan $\delta_p \otimes \mu_1$ is optimal because it the unique possible one. It is not inherited from a map T transporting μ_0 on μ_1 (except if μ_1 is a Dirac mass too). Then the Monge problem has not a solution in this case.

Example 2.1.5 (Orthogonal spaces). Suppose that μ_0 is concentrated on $\mathbb{R}^m \times \{0_{\mathbb{R}^{n-m}}\}$ and μ_1 is concentrated on $\{0_{\mathbb{R}^m}\} \times \mathbb{R}^{n-m}$. Let π be a transport plan between μ_0 and μ_1. Then because of the Pythagorean theorem, $\pi(p,q)$-almost surely $|p-q|^2 = |p|^2 + |q|^2$ such that

$$\begin{aligned} C_2(\pi) &= \frac{1}{2}\int_{X\times X} |p|^2 d\pi(p,q) + \frac{1}{2}\int_{X\times X} |q|^2 d\pi(p,q) \\ &\frac{1}{2}\int_X |p|^2 d\mu_0(p) + \frac{1}{2}\int_X |q|^2 d\mu_1(q). \end{aligned}$$

The cost does not depend on the coupling π. Every transport plan is optimal

Example 2.1.6 (Translation). Consider μ_0 a measure on $\mathbb{R}^n$ (non necessarily absolutely continuous) and $v \in \mathbb{R}^n$. We will prove that $\tau_v(p) = p + v$ is the unique optimal map between μ_0 and $\mu_1 = \tau_\#\mu_0$ (and π_{τ_v} is the unique optimal plan). The cost associated to τ_v is $C_2^M(\tau_v) = \frac{1}{2}\int |v|^2 d\mu_0 = |v|^2/2$. Let now π be a transport plan between μ_0 and μ_1. Then $\int(q-p)d\pi(p,q)$, that is the mean deplacement vector is v. Indeed

$$\int (q-p)d\pi(p,q) = \int qd(\tau_{v\#}\mu_0)(q) - \int pd\mu_0(p) = \int vd\mu_0 = v.$$

Moreover, $c_0(v) = \frac{|v|^2}{2}$ is a convex function of $\mathbb{R}^n$. Hence because of the Jensen theorem

$$C_2(\pi) = \int c_0(q-p)d\pi(p,q) \geq c_0(\int (q-p)d\pi(p,q)) = c_0(v) = C_2^M(\tau_v).$$

It follows that τ_v and π_{τ_v} are optimal. For the uniqueness of these optimal map and plan, we consider the equality case in the Jensen inequality : $q-p$ has to be constant $\pi(p,q)$-almost surely because c_0 is strictly convex. Then this constant vector must be v.

c-convex functions of $\mathbb{R}^n$

It's generally not possible to determinate if a function is c-convex. In the special setting of $c(p,q) = |p-q|^2/2$ in $\mathbb{R}^n$, there an easiest statement equivalent to the definition of c-convex functions.

Lemma 2.1.7. *The c-convex functions of $\mathbb{R}^n$ are the functions $\phi \in \mathcal{F}_{\mathbb{R}^n}$ such that $\frac{|p|^2}{2} + \phi(p)$ is l.s.c. (lower semi-continuous) and convex on $\mathbb{R}^n$.*

Proof. We consider the basic bijection τ on the space of functions $\mathcal{F}_{\mathbb{R}^n}$ defined by
$$F(\phi)(p) = \frac{|p|^2}{2} + \phi(p).$$
Then from the definition of a c-transform, we observe that

$$\begin{aligned}
F(\psi^c)(p) &= \frac{|p|^2}{2} - \inf(\frac{|p-q|^2}{2} + \psi(q)) \\
&= -\inf_{q\in X}(\frac{|p-q|^2}{2} + \psi(q) - \frac{|p|^2}{2}) \\
&= -\inf_{q\in X}(-\langle p \mid q\rangle + F(\psi)(q)) \\
&= \sup_{q\in X}(\langle p \mid q\rangle - F(\psi)(q)).
\end{aligned}$$

Thus we recognize that in $\mathcal{F}_{\mathbb{R}^n}$ seen as the image set under the transformation F, the c-transform becomes the Legendre transformation. A function ϕ is then exactly c-convex, if $F(\phi)$ is the Legendre transformation of some function of $\mathcal{F}_{\mathbb{R}^n}$. It is well-known that the set of Legendre transformated functions is in fact the set of l.s.c. (i.e. the preimage of any $]x,+\infty]$ is open) convex functions of $\mathcal{F}_{\mathbb{R}^n}$ (see [20] or [108, Proposition 2.5]). Hence $\phi\mathcal{F}_{\mathbb{R}^n}$ is c-convex if and only if $\phi(p) + \frac{|p|^2}{2}$ is l.s.c. and convex. □

The functions of $\mathcal{F}_{\mathbb{R}^n}$ such that $\phi(p)+\lambda\frac{|p|^2}{2}$ is convex for some real λ are called semiconvex functions with constant λ. Of course convex function are semiconvex with any constant $\lambda \geq 0$. The linear definition of semiconvex functions of constant λ is then

$$\phi(sp + (1-s)q) \leq s\phi(p) + (1-s)\phi(q) + \frac{\lambda|p-q|^2}{2}s(1-s)$$

for all $(p,q,s) \in \mathbb{R}^n \times \mathbb{R}^n \times [0,1]$. A differential definition for enough differentiable functions is
$$D^2\phi \geq -\lambda \operatorname{Id}_n .$$
As a consequence of Lemma 2.1.7 above, a c-convex function of $\mathbb{R}^n$ is a l.s.c. semiconvex function with constant 1 that is not identically $+\infty$.

The λ-convex functions of $\mathbb{R}^n$ are locally Lipschitz and the Rademacher Theorem apply

Theorem 2.1.8 (Rademacher)**.** *A locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ is Lebesgue-almost everywhere differentiable.*

The λ-convex functions have even good differentiability properties at order 2 as the Alexandrov theorem states:

Theorem 2.1.9 (Alexandrov)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally λ-convex function. Then at almost every point p, the function f is differentiable and there exists a symmetric linear map $A_p : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$f(p+v) = f(p) + \langle \nabla f(p) \mid v \rangle + \frac{\langle A_p v \mid v \rangle}{2} + o(|v|^2)$$

as $v \to 0$.

The detailed proofs of both previous theorems can be found in [36].

Brenier-McCann Theorem

We are now able to state and prove the theorem.

Theorem 2.1.10 (Brenier-McCann)**.** *Let μ_0 and μ_1 be two probability measures of $\mathbb{R}^n$. We suppose that μ_0 is absolutely continuous and that*

$$C_2(\mu_0, \mu_1) < +\infty.$$

Then there is a c-convex function $\phi \in \mathcal{F}_{\mathbb{R}^n}$ such that

$$T(p) = p + \nabla\phi(p)$$

is an optimal transport map from μ_0 to μ_1. Moreover, π_T is the unique optimal transport plan.

Conversely assume that $\phi \in \mathcal{F}_{\mathbb{R}^n}$ is a c-convex function and $\mu_0 \in \mathcal{P}_2(\mathbb{R}^n)$ is an absolutely continuous probability measure with $\mu_0(\operatorname{Dom}\phi) = 1$. Assume furthermore

$$\int \frac{|\nabla(\phi)|^2}{2} d\mu_0 < +\infty.$$

Then $T : p \to p + \nabla\phi(p)$ is the optimal transport map from μ_0 to $T_{\#}\mu_0$.

Remark 2.1.11. We give for this theorem the names of Brenier and McCann because Brenier [18] proved the existence of a "monotone" transport map (in Theorem 2.1.10 T is the monotone map) and McCann stated the theorem in context of optimal transport and gave it a more geometrical aspect. In particular he stated the theorem on compact Riemannian manifolds [80]. For a more complete statement of the theorem see [108].

Proof. Let π be an optimal plan between μ_0 and μ_1 and $\phi \in \mathcal{F}_{\mathbb{R}^n}$ a c-convex function such that π is concentrated on $\partial^c\phi$. Thanks to Theorem 2.1.8 as a semiconvex function with constant 1, ϕ is almost everywhere differentiable on $\overset{\circ}{\operatorname{Dom}}(\phi)$. The cost $c_q = \frac{|\cdot - q|^2}{2}$ is smooth and can be differentiate in every p. Let A be the set of differentiation of ϕ. Therefore on A we can differentiate every

$f_q = \phi(\cdot) + \frac{|\cdot - q|^2}{2}$. If $p \in A$ and $q \in \partial^c\phi(p)$ the derivative of f_q in p has to be 0. Then for $q \in \partial^c\phi(p)$, we obtain

$$\nabla\phi(p) + p - q = 0 \tag{2.5}$$

or more simply $q = p + \nabla\phi(p)$. Note that

$$\mu_0(A) = \mu_0(\overset{\circ}{\mathrm{Dom}}(\phi)) = \mu_0(\mathrm{Dom}(\phi)) - \mu_0(\partial(\mathrm{Dom}(\phi))) = 1 - 0.$$

We have on the one hand $\mu_0(\mathrm{Dom}(\phi)) = 1$ because $\partial^c\phi \subset \mathrm{Dom}\,\phi \times \mathbb{R}^n$, thus $\mu_0 = p_\# \pi$ is concentrated on $\mathrm{Dom}\,\phi$. On the other hand $\mathrm{Dom}(\phi)$ is a convex set as the domain of a semiconvex function, so the measure of the absolutely continuous mass μ_0 of the border $\partial(\mathrm{Dom}(\phi))$ is 0. Then π is concentrated on $(A \times \mathbb{R}^n) \cap \partial^c\phi$ which is as we proved above exactly the graph of the function $T(p) = p + \nabla\phi(p)$. Then the scheme proposed in Subsection 2.1.1 about the existence and uniqueness of a solution to the Monge problem applies to the plan π. It equals π_T and T is the unique solution of the Monge problem.

For the converse part, let μ_1 be the push-forward measure $T_\#\mu_0$ where $T(p) = p + \nabla\phi(p)$. Then the cost $C_2(\mu_0, \mu_1)$ is finite because

$$C_2^M(T) = \frac{1}{2}\int |\nabla\phi(p)|^2 d\mu_0(p) < +\infty.$$

We have also $\mu_1 \in \mathcal{P}_2(\mathbb{R}^n)$ because

$$\int \frac{|q - p_0|^2}{2} d\mu_1(q) = \int \frac{|p + \nabla\phi(p) - p_0|^2}{2} d\mu_0(p) < +\infty.$$

Suppose that π is a transport plan from μ_0 to μ_1. We can assume that $C_2(\pi) < +\infty$. In a first time we also assume that ϕ and ϕ^c are integrable with respect to μ_0 and μ_1 respectively and that the integrals are not $-\infty$, which we will prove after. Thus from

$$\phi(p) + \phi^c(q) + \frac{|p - q|^2}{2} \geq 0$$

we obtain

$$C_2(\pi) \geq -\int \phi(p) d\mu_0(p) - \int \phi^c(q) d(\mu_1)(q).$$

Our goal is to prove that the lower bound on the right-hand left is finite and is $C_2(\pi_T)$. It is in fact enough to prove that $\mu_0(p)$-almost surely

$$\phi(p) + \phi^c(T(p)) + \frac{|p - T(p)|^2}{2} = 0 \tag{2.6}$$

and integrate the relation with respect to μ_0.

We know from the definition of the c-transform that

$$\phi^{cc}(p) = \phi(p) = \sup_q \left(-\frac{|p - q|^2}{2} - \phi^c(q) \right).$$

In fact if $p \in \overset{\circ}{\mathrm{Dom}}(\phi)$ this sup is attained by some point q. It is a consequence of the bijection shown in Lemma 2.1.7 and the fact that this property holds for the

Legendre transformation (see [108, 2.1.3.]). Moreover, because $\mu_0(\overset{\circ}{\mathrm{Dom}}\,\phi) = 0$, the Rademacher Theorem implies that ϕ is derivable μ_0-almost surely. Then $\mu_0(p)$-almost surely $\nabla\phi(p)$ exists and there is a q that maximizes $-\frac{|p-q|^2}{2} - \phi^c(q)$. For this q the derivative of $p \to \phi(p) + \phi^c(q) + \frac{|p-q|^2}{2}$ vanishes such that $q = T(p)$. Then relation (2.6) is satisfied as we wanted and after integrating it with respect to μ_0, the result follows.

It remains to prove that ϕ and ϕ^c are integrable. Let (p_0, q_0) be in $\mathrm{Dom}(\phi) \times \mathrm{Dom}(\phi^c)$. Then

$$\frac{|p-q|^2}{2} \leq |p|^2 + |q|^2.$$

It follows

$$\begin{aligned}
\phi(p) &= \sup_q \left(-\frac{|p-q|^2}{2} - \phi^c(q) \right) \\
&= \sup_q \big(-(|p|^2 + |q|^2) - \phi^c(q) \big) \\
&= \big(-|q_0|^2 - \phi^c(q_0) \big) - |p|^2
\end{aligned}$$

and $\phi(p)$ is μ_0-integrable with $\int \phi(p) d\mu_0(p) \in \mathbb{R} \cup +\infty$. By interchanging the roles of ϕ and ϕ^c, we have also $\int \phi^c(q) d\mu_1(q) \in \mathbb{R} \cup +\infty$. □

Let us illustrate the second part of Theorem 2.1.10 with some examples. Before that, we make two remarks

Remark 2.1.12. Note that $T(p) = p + \nabla\phi(p)$ in Theorem 2.1.10 can be seen as the gradient of $\phi(p) + \frac{|p|^2}{2}$, indeed a proper l.s.c. convex function as proved in Lemma 2.1.7. For example the converse implication in Theorem 2.1.10 is roughly speaking, that the gradient of any convex function pushes forward optimally any absolutely continuous measure. This presentation may look like easier. Nevertheless we chosen to present the result in this way because it is more geometric and closer to Theorem 2.2.4, the corresponding theorem in the Heisenberg group. In this theorem appears the exponential map $\exp^{\mathbb{H}}$ that actually also appear in the Brenier-McCann theorem because T can be written $T(p) = \exp_p(\nabla\phi(p))$ where $\exp_p(v)$ is the exponential map of Riemannian geometry (in $\mathbb{R}^n$ simply $p + v$). This expression $T(p) = \exp_p(\nabla\phi(p))$ is also what appear in [80] for compact Riemannian manifolds.

Remark 2.1.13. In the converse part of Theorem 2.1.10, the condition $\mu_0 \in \mathcal{P}_2(\mathbb{R}^n)$ is not necessary and can be replace by "$\mu_0 \in \mathcal{P}(\mathbb{R}^n)$ is absolutely continuous". In fact for any $k \in \mathbb{N}$, the theorem applies to μ_0^k defined by

$$\mu_0^k(A) = \mu_0(A \cap [-k,k]^n)/\mu_0([-k,k]^n),$$

so T is optimal between μ_0^k and $\mu_1^k = T_\#\mu_0^k$. But $(\mu_i^k)_{k\in\mathbb{N}}$ weakly converges to μ_i for every $i \in \{0,1\}$. Therefore because of the stability theorem of optimal plan (see [109, Theorem 5.19]), the optimal plans $((\mathrm{Id} \otimes T)_\#\mu_0^k)_{k\in\mathbb{N}}$ up to a subsequence, weakly converge to an optimal plan between μ_0 and μ_1. But this limit is π_T. This proves that π_T is an optimal plan.

Example 2.1.14. In $\mathbb{R}$, the derivative of a convex function is non-decreasing and conversely, any locally integrable non-decreasing function has a convex integral. Then the optimal transport from μ_0 absolutely continuous to μ_1 is a non-decreasing map T. Conversely if μ_0 is absolutely continuous and T is non-decreasing T is an optimal map from μ_0 to μ_1. Actually as one can see for example in [108], this monotone map is still optimal if $\mu_0(\{p\}) = 0$ for any $p \in \mathbb{R}$ even if it is not absolutely continuous .

One can also prove that for a non-decreasing map T, the optimal plan π_T is also optimal for the original Monge problem in $\mathbb{R}$ (see Remark 2.1.1) where the cost to minimize is

$$C_1^M(T) = \int |T(p) - p| d\mu_0(p).$$

We will actually meet the original Monge problem in the next section and we will need the following observation: if for all $(p, q) \in \operatorname{supp} \mu_0 \otimes \mu_1$ we have $q \geq p$ (that is $\operatorname{supp}(\mu_0)$ is totally "on the left" of $\operatorname{supp}(\mu_1)$) then every plan is optimal. Let π be a coupling of μ_0 and μ_1. Then $C_1(\pi) = \int |q - p| d\pi(p, q) = \int (q-p) d\pi(p, q) = \int q d\mu_1(q) - \int p d\mu_0(p)$ independently of π. Every π is optimal as we said.

Example 2.1.15 (Translations on $\mathbb{R}^n$). We have already proved the optimality of the translation maps $T(p) = p + v$ in Example 2.1.6 but we can recover it from Theorem 2.1.10 using the linear (and then c-convex) function $\phi(p) = \langle v \mid p \rangle$ whose derivative is the constant map $p \to v$. The geometric argument of Example 2.1.6 is in fact better in this case because it also apply to non absolutely-continuous measures μ_0. Nevertheless it is also possible to recover the optimality for non-absolutely continuous measures directly from Proposition 2.1.3 and the elements of the proof of Theorem 2.1.10 because $\phi = \langle v \mid \cdot \rangle$ is smooth.

Example 2.1.16 (Dilations). Consider $\phi(p) = s\frac{|p|^2}{2} - \frac{|p|^2}{2}$. This function is semi-convex with constant 1 if and only if $s \geq 0$. Then $\nabla \phi(p) = p(s-1)$ what means that T is the dilation $T(p) = sp$. As a consequence of the Brenier-McCann theorem T is optimal. Particularly if $s = 0$, we recover the obvious fact that constant maps to a point are optimal (Example 2.1.4). We will see in Section 2.3, that the optimality of the dilation for $s \in [0, 1]$ can be recovered as a geometric consequence of the optimality of the contraction to 0. Let us insist on the fact that T is not optimal for $s = -1$. For example if μ_0 is the uniform measure on a ball $\mathcal{B}(c, r)$ of $\mathbb{R}^n$, $T_\# \mu_0$ is the uniform measure on the ball $\mathcal{B}(-c, r)$ and the optimal transport is not T but simply the translation $p \to p - 2c$. With a similar argument, we see that the optimal transport maps between a measure and another measure obtained by a rotation is a priori not this rotation.

The dilation with center m and quotient s is also optimal. It is associated to the c-convex function $s\frac{|p-m|^2}{2} - \frac{|p-m|^2}{2}$.

2.2 Optimal transport in the Heisenberg group

2.2.1 Examples of optimal transport in $\mathbb{H}_n$

As we did in subsection 2.1.2 for $\mathbb{R}^n$, we give some examples of optimal transport maps or plan in $\mathbb{H}_n$ that does not require theory but just little geometric arguments.

Example 2.2.1 (Contraction). If μ_1 is a Dirac mass, there is a unique transport plan and it is optimal.

Example 2.2.2 (One dimensional transport on L). As we already mentioned in Chapter 1, the center $L = \{(z;t) \in \mathbb{H}_n \mid z = 0\}$ with the restriction of d_c is isometric to $(\mathbb{R}, \sqrt{d_{Euc}})$. Thus the L^2 Monge problem for measures concentrated on L (or on some $(z;t) \cdot L$) isometrically consists on minimizing $\int_{\mathbb{R}\times\mathbb{R}} \sqrt{|t-t'|}^2 d\pi(t,t')$ for π a transport plan between two given probability measures of $\mathbb{R}$. This optimization problem is exactly the original L^1 Monge problem on $\mathbb{R}$. As a consequence of what we said in Example 2.1.14, if there is a $z \in \mathbb{C}^n$ such that $\{z\} = Z(\operatorname{supp}\mu_0) = Z(\operatorname{supp}\mu_1)$, that is μ_0 and μ_1 are concentrated on $(z;0) \cdot L$ and if there is a transport plan π such that $\pi((z;t),(z;t'))$-almost surely $t' \geq t$, π is optimal. In particular if for all $((z;t),(z;t')) \in (\operatorname{supp}\mu_0) \times (\operatorname{supp}\mu_1)$ we have $t' \geq t$, any transport plan from μ_0 to μ_1 is optimal.

Example 2.2.3 (Lifts of optimal transports on $\mathbb{R}^{2n}$). Let μ_0 a probability measure of $\mathbb{H}_n$ and let m_0 be the projection $Z_\#\mu_0$ on $\mathbb{C}^n = \mathbb{R}^{2n}$. Consider now a map $T_\mathbb{C} : \mathbb{C}^n \to \mathbb{C}^n$ that is an optimal transport map between m_0 and $m_1 = (T_\mathbb{C})_\# m_0$. Then we can lift the optimal transport. More precisely there is a map T of $\mathbb{H}_n$ which is optimal between μ_0 and $T_\#\mu_0$ such that $Z(T) = T_\mathbb{C}(Z)$. In other words we have the commutation relation $T_\mathbb{C}(z) = Z(T(z;t))$. We now define T and will after check the assumption.

$$T(z;t) = (z;t).\exp^{\mathbb{H}}(T_\mathbb{C}(z) - z, 0).$$

Because $\exp^{\mathbb{H}}(z,0) = (z;0)$ it is also $T(z;t) = (z;t).(T_\mathbb{C}(z) - z;0)$ and if $T_\mathbb{C}(z) = z + \nabla\phi(z)$ for some c-convex function ϕ, we can write $T(z;t) = (z;t)\cdot(\nabla\phi(z);0)$. Actually $d_c((z;t),T(z;t)) = |z - T_\mathbb{C}(z)|$. It follows that

$$C_2^M(T) = C_2^M(T_\mathbb{C})$$

Suppose that π is a transport plan between μ_0 and $T_\#\mu_0$. Then $(Z\otimes Z)_\#\pi$ is a coupling between $m_0 = Z_\#\mu_0$ and $Z_\#(T_\#\mu_0)$. This second measure is simply $(Z\circ T)_\#\mu_0 = (T_\mathbb{C}\circ Z)_\#\mu_0 = m_1$. The cost associated to the coupling π is greater than the one of $(Z\otimes Z)_\#\pi$ because Z is 1-Lipschitz (Lemma 1.3.1). Hence it is also greater than $C_2^M(T_\mathbb{C}) = C_2^M(T)$.

A similar optimal transport plan of $\mathbb{H}_n$ can be built from an optimal plan π of $\mathbb{C}^n$ too (not just for optimal plans of the form π_T).

2.2.2 The theorem of Ambrosio and Rigot

Before we state the theorem of Ambrosio and Rigot, we define the approximate differentiability of a function. A function $f : \mathbb{R}^{2n+1} \to \mathbb{R}$ has an approximate differential at $p \in \mathbb{R}^{2n+1}$ if there exists a function $h : \mathbb{R}^{2n+1} \to \mathbb{R}$ differentiable at p such that the set $\{f = h\}$ has density 1 at p with respect to the Lebesgue measure. In this case the approximate derivatives of f at p are defined as

$$\begin{aligned}(\tilde{\mathbf{X}}f(p) + \mathbf{i}\tilde{\mathbf{Y}}f(p), \tilde{\mathbf{T}}f(p)) :=& (\mathbf{X}h(p) + \mathbf{i}\mathbf{Y}h(p), \mathbf{T}h(p)) \\ =& (\mathbf{X}_1h(p) + \mathbf{i}\mathbf{Y}_1h(p), \dots, \mathbf{X}_nh(p) + \mathbf{i}\mathbf{Y}_nh(p), \mathbf{T}h(p)).\end{aligned}$$

It is not difficult to show that this definition makes sense. Note that $\mathbf{X}$ an $\mathbf{Y}$. stand for the n-vector $(\mathbf{X}_1, \dots, \mathbf{X}_n)$ and $(\mathbf{Y}_1, \dots, \mathbf{Y}_n)$ (and note as $\mathbf{X}_1$ and $\mathbf{Y}_1$ of $\mathbb{H}_1$ as sometime in this thesis).

Theorem 2.2.4. *[7, Theorem 5.1 and Remark 5.9] Let μ_0 and μ_1 be two Borel probability measures on $\mathbb{H}_n$. Assume that μ is absolutely continuous with respect to $\mathcal{L}^{2n+1}$ and that*

$$\int d_c(p, 0_{\mathbb{H}})^2 d\mu_0(p) + \int d_c(0_{\mathbb{H}}, q)^2 d\mu_1(q) < +\infty.$$

Then there exists a unique optimal transport plan from μ_0 to μ_1, and this plan is induced by a map T. If $\mathrm{supp}(\mu_1)$ is compact, T is given by

$$T(p) := p \cdot \exp^{\mathbb{H}}(\mathbf{X}\phi(p) + \mathbf{i}\mathbf{Y}\phi(p), \mathbf{T}\phi(p)) \quad \text{for } \mu_0\text{-a.e. } x \in \mathbb{H}_n \tag{2.7}$$

for some $(d_c^2/2)$-convex and locally Lipschitz map ϕ. Whatever without any assumption on $\mathrm{supp}(\mu_1)$ there exists a function c-convex ϕ which is approximately differentiable μ_0-a.e. such that the optimal transport plan is concentrated on the graph of

$$T(p) := p \cdot \exp^{\mathbb{H}}(\tilde{\mathbf{X}}\phi(p) + \mathbf{i}\tilde{\mathbf{Y}}\phi(p), \tilde{\mathbf{T}}\phi(p)).$$

Conversely, if T is representable as in (2.7) for some map ϕ such that

$$\begin{cases} \mathbf{X}\phi(p), \mathbf{Y}\phi(p), \mathbf{T}\phi(p) \ \text{exist} \\ \phi(p) = \max\limits_{q \in \mathbb{H}_n} -\dfrac{d_c^2(p,q)}{2} - \phi^c(q) \quad \text{for } \mu_0\text{-a.e } p \in \mathbb{H}_n \end{cases} \tag{2.8}$$

and if

$$\int_{\mathbb{H}_n} d_c(p, 0_{\mathbb{H}})^2 + d_c(0_{\mathbb{H}}, T(p))^2 d\mu_0(p) < +\infty$$

then T is the optimal transport map between μ_0 and $\mu_1 = T_{\#}\mu_0$.

Remark 2.2.5. This formulation of the theorem is slightly different from the original statement by Ambrosio and Rigot because we are using different notations. In particular the angles in the map $\exp_{\mathbb{H}}$ of [7] are parametrized between $-\pi/2$ and $\pi/2$ while the map $\exp^{\mathbb{H}}$ of this report is defined on $[-2\pi, 2\pi]$. Moreover, in their convention the basis of the Lie Algebra appears another way such that

$$[\mathbf{X}_k, \mathbf{Y}_k] = -4\mathbf{T}$$

for any $k \in \{1, \ldots, n\}$. Another difference is that in this paper the important functions are the opposite of the c-convex functions, namely the c-concave maps.

Notice that the curve $s \in [0,1] \to p \cdot \exp_s^{\mathbb{H}}(\mathbf{X}\phi(p) + \mathbf{i}\mathbf{Y}\phi(p), \mathbf{T}\phi(p))$ is a curve starting in p and tangent to the horizontal vector $\nabla_{\mathbb{H}}\phi(p)$ in this point. Actually on the one side $\exp_s^{\mathbb{H}}(\mathbf{X}\phi(p)+\mathbf{i}\mathbf{Y}\phi(p), \mathbf{T}\phi(p))$ starts in $0_{\mathbb{H}}$ and is tangent to $\sum_{i=1}^n \mathbf{X}_i\phi(p)\mathbf{X}_i(0_{\mathbb{H}}) + \mathbf{Y}_i\phi(p)\mathbf{Y}_i(0_{\mathbb{H}})$. On the other side the vector fields $\mathbf{X}_i$ and $\mathbf{Y}_i$ are left invariant under tran_p. See also Remark 2.1.12

The strategy of the proof is the same as the one exposed in Subsection 2.1.1. It relies on the Pansu-Rademacher theorem on Pansu-differentiability of Lipschitz functions [93]. A phenomenon occurs that make the proof more intricate than in $\mathbb{R}^n$: the distance squared to $0_{\mathbb{H}}$ in not differentiable on L (but see Lemma 2.3.4). Unlike $\mathbb{R}^n$ there is no result like Lemma 2.1.7 the permit to identify easily c-convex functions. Using Theorem 2.2.4, we give now an example of optimal transport map that is different from the ones we presented before.

Example 2.2.6 (Non-intuitive optimal transport map). For simplicity we take $n = 1$ and consider the map

$$T(z;t) = (0, t + \frac{|z|^2}{2\pi}) = (z;t) \cdot \exp^{\mathbb{H}}(\mathbf{i}\frac{\pi}{2}z, \pi).$$

The range of this map is L but p and $T(p) \in L$ are not on a $\mathbb{H}$-line as in Example 2.2.3. In fact $T(z;t) = (z;t) \cdot \exp^{\mathbb{H}}(\mathbf{i}\frac{\pi}{2}z, \pi)$ means that the geodesic between p and $T(p)$ is the horizontal lift of an half circle spanned between $Z(p)$ and $0_{\mathbb{H}}$. As one can verify $(\mathbf{i}\frac{\pi}{2}z, \pi) = (\mathbf{X}\phi + \mathbf{i}\mathbf{Y}\phi, \mathbf{T}\phi)$ for $\phi(z;t) = \pi t$. We want to prove that T and ϕ satisfy the converse part in Theorem 2.2.4. The function πt is differentiable. Let us now prove that it is c-convex. For that we will first compute ϕ^c and then check that $\phi^{cc} = \phi$ with the supremum in the definition of ϕ^{cc} achieved as in equation (2.8).

$$-\phi^c(z';t') = \inf_{(z;t)} \left(\frac{d_c^2((z;t),(z';t'))}{2} + \pi t \right) \tag{2.9}$$

If $z' \neq 0$, we take $(z;t) = (z';t') \cdot \exp(Cz', -\pi)$ (we go down in the third coordinate thanks to an half circle). Then $d((z;t),(z';t')) = C|z'|$ and $t = t' - \frac{C^2|z'|^2}{2} - \frac{C|z'|^2}{\pi}$ which correspond to t' minus the area of the half circle minus the area of a triangle. Then letting $C \to +\infty$ we see that $\phi^c(z';t') = +\infty$.

If $z' = 0$, we try first to minimize $(\frac{d_c^2((z;t),(z';t'))}{2} + \pi t)$ for a fixed distance $d = d_c((z;t),(z';0))$. Then we want to minimize t starting from $(z';t')$. In planar formulation, we search a curve starting in $0_{\mathbb{C}}$ with length d that maximizes the algebraic area (with a minus coefficient). The solution is given by the Dido problem and is a half circle. Then the area is $-d^2/(2\pi)$ and $t = t' - d^2/(2\pi)$. The infimum in (2.9) is then the infimum under $d = d_c((z;t),(z';0))$ of

$$\pi \left(t' - d^2((z;t),(z';t'))/(2\pi)\right) + \frac{d^2((z;t),(z';t'))}{2}$$

which is simply $\pi t'$ independently of d. It follows that

$$\phi^c(z;t) = \begin{cases} -\pi t' & \text{if } z' = 0 \\ +\infty & \text{if } z' \neq 0 \end{cases}$$

Hence we can now try to compute

$$\phi^{cc}(z;t) = -\inf_{t' \in \mathbb{R}} \left(\frac{d_c^2((z;t),(0;t'))}{2} - \pi t' \right).$$

For $z \neq 0$, the associated planar question is : what is the best way to reach $0_{\mathbb{C}}$ starting from z when one want to minimize $\frac{d_c^2}{2} - \pi(t'-t)$ where d_c is the length of the curve and $t'-t$ the algebraic area. In fact the best way is to draw a half circle and we will obtain 0. For other curves the quotient between d^2 and the area is greater than the Dido isoperimetric constant 2π. Then $\phi^{cc}(z;t) = \phi(z,t) = \pi t$ for $z \neq 0$. If $z = 0$, $t = t'$ is the minimum so $\phi^{cc} = \phi$ and ϕ satisfies the conditions of the Theorem 2.2.4.

2.2.3 Some examples of c-convex functions of $\mathbb{H}_n$

In this subsection we describe two special types of c-convex functions of $\mathbb{H}_n$: on the one hand some c-convex functions in relation with the c-convex functions of $\mathbb{C}^n$, on the other hand some smooth c-convex functions.

About c-convex functions obtained from c-convex functions of $\mathbb{C}^n$

We use the letter $\tilde{c}$ for $\tilde{c}(z,z') = |z-z'|^2/2$ if the points are in $\mathbb{C}^n$ and $c(p,q) = d_c(p,q)^2/2$ for points of $\mathbb{H}_n$. Let $\psi_{\mathbb{C}}$ be a function of $\mathbb{C}^n$ and ψ the function defined on $\mathbb{H}_n$ by $\psi(z;t) = \psi_{\mathbb{C}}(z)$. We would like to compute the c-transform of ψ. We can estimate it as follows

$$\begin{aligned}
\psi^c(q) &= \sup_{p\in\mathbb{H}_n}(-\psi(p) - \frac{d_c(p,q)^2}{2})\\
&\leq \sup_{q\in\mathbb{H}_n}(-\psi_{\mathbb{C}}(Z(p)) - \frac{|Z(p)-Z(q)|^2}{2})\\
&\leq \sup_{z\in\mathbb{C}^n}(-\psi_{\mathbb{C}}(Z(p)) - \frac{|Z(p)-z|^2}{2}) = \psi_{\mathbb{C}}^{\tilde{c}}(Z(q))
\end{aligned}$$

because of the definition of ψ and because Z is 1-Lipschitz (Lemma 1.3.1). Therefore $\psi^c(q)$ is smaller than $\psi_{\mathbb{C}}^c(Z(q))$. Actually both functions are the same because for a fixed $q \in \mathbb{H}$ and any $z \in \mathbb{C}^n$, there is a (unique) $p = (z;t)$ such that $d_c(q,p) = |Z(q) - z|$. In fact this point is the one that is obtained when one lifts horizontally the segment $[Z(q), z]$ with by a $\mathbb{H}$-line starting from q. We have $p = q \cdot \exp^{\mathbb{H}}(z - Z(q), 0)$.

So it is easy to obtain conjugated functions ϕ and ϕ^c of $\mathbb{H}_n$ from c-convex functions of $\mathbb{C}$ and two points (p,q) are in the c-subdifferential $\partial^c\phi$ if and only if $(Z(p), Z(q)) \in \partial^c\phi_{\mathbb{C}}$ and $q = p\cdot\exp^{\mathbb{H}}(Z(q) - Z(p), 0)$ (p and q are on a same $\mathbb{H}$-line). The optimal transport that are concentrated on such subdifferentials are in fact the one we described in Example 2.2.3.

Let us just illustrate the situation for the easy example of μ_0 a probability measure of $\mathbb{H}_1$ and T the 1/2-dilation of $\mathbb{C}$. As we already saw in Example 2.1.16 T is associated to the c-convex function $\phi_{\mathbb{C}}(z) = -|z|^2/4$, so $T(z) = z - z/2 = z/2$. Then the function $\phi(z;t) = -|z|^2/4$ is a c-concave function of $\mathbb{H}_1$ and

$$T(p) = p\cdot\exp^{\mathbb{H}}(\mathbf{X}\phi(p) + \mathbf{iY}\phi(p), \mathbf{T}\phi(p)) = p\cdot\exp^{\mathbb{H}}(\frac{\partial}{\partial z}\phi_{\mathbb{C}}(Z(p)), 0)$$

is the optimal transport between μ_0 and $T_\#\mu_0$. But the $\mathbb{H}$-line starting from $(z;t)$ in direction $-z/2$ is $s \in \mathbb{R} \to (sz;t)$ (the projection on $\mathbb{C}$ sweeps a 0 algebraic area). Then $T(z;t) = (z/2;t)$ is the optimal transport map between μ_0 and $T_\#\mu_0$.

Some smooth c-convex functions

It is not known so much about the regularity of the c-convex functions. However, it is possible to prove the following

Proposition 2.2.7. *If ψ is smooth function of $\mathbb{H}_n$ with a compact support, then for $s > 0$ small enough, the function $s\psi$ is c-convex.*

Before we prove it we will need a lemma

Lemma 2.2.8. *Let ζ be a function from $\mathbb{H}_n$ to $\mathbb{C}^n$ and θ a real function. We assume that both are smooth with a compact support. Then for $s > 0$ small enough the function $F_s : p \to p\cdot\exp_s^{\mathbb{H}}(\zeta,\theta)$ is a diffeomorphism on its range.*

Proof. For s small, the map F_s is not far from identity. With relation (1.1) one can check easily that it is also the case for the differential DF_s. The determinant of this map is then close to 1. For s small enough it does not vanish.

We will explain a little longer why F_s is one-to-one for s small enough. In this proof it will be useful to consider $\mathbb{H}_n$ with the distance d_ε of the approximating manifold $\mathbb{H}_n^\varepsilon$. Here $\varepsilon > 0$ is fixed (not necessarily small). Notice that the differentiable structure of $\mathbb{H}_n^\varepsilon$ is the one of $\mathbb{R}^{2n+1}$ such that ζ and θ are smooth and supported on a compact set of $\mathbb{H}_n^\varepsilon$. We get on the one hand

$$|\zeta(p) - \zeta(p')| + |\theta(p) - \theta(p')| \leq C_1 d_\varepsilon(p, p')$$

where C_1 is a constant, p and p' are any points of $\mathbb{H}_n$. On the other hand the exponential map $\exp^{\mathbb{H}}$ is smooth from $\mathbb{R}^{2n+1}$ to $\mathbb{H}_n^\varepsilon$. Then for vectors $v, v' \in \mathbb{C}^n$ and numbers $\varphi, \varphi' \in \mathbb{R}$ included in a bounded set we have

$$d_\varepsilon(\exp^{\mathbb{H}}(v, \varphi), \exp^{\mathbb{H}}(v', \varphi')) \leq C_2 \left(|v - v'| + |\varphi - \varphi'|\right)$$

for some constant C_2. Suppose now that $F_s(p) = F_s(p')$. Then from the definition $p'^{-1} \cdot p = \exp_s^{\mathbb{H}}(\zeta', \theta') \cdot \left(\exp_s^{\mathbb{H}}(\zeta, \theta)\right)^{-1}$ such that

$$\begin{aligned} d_\varepsilon(p, p') &= d_\varepsilon(\exp^{\mathbb{H}}(s\zeta, s\theta), \exp^{\mathbb{H}}(s\zeta', s\theta')) \\ &\leq sC_2 \left(|\zeta - \zeta'| + |\theta - \theta'|\right) \\ &\leq sC_2 C_1 d_\varepsilon(p, p'). \end{aligned}$$

It follows that for $s < (C_1 C_2)^{-1}$ the map F_s is one-to-one. $\square$

We make now the proof of Proposition 2.2.7.

Proof. Let s small enough in the sense of Lemma 2.2.8 where $\zeta = \mathbf{X}\psi + \mathbf{i}\mathbf{Y}\psi := (\mathbf{X}_1\psi + \mathbf{i}\mathbf{Y}_1\psi \cdots \mathbf{X}_n\psi + \mathbf{i}\mathbf{Y}_n\psi)$ and $\theta = \mathbf{T}\psi$. The function $s\psi$ will be c-convex if

$$s\psi(p) = \sup_{q \in \mathbb{H}_n} \left(-c(p, q) - (s\psi)^c(q)\right)$$

where

$$(s\psi)^c(q) = \sup_{p \in \mathbb{H}_n} \left(-c(p, q) - s\psi(p)\right)$$

The previous supremum is achieved because ψ is smooth with a compact support. If p maximizes the quantity on the right-hand side then the same analysis as the one of Ambrosio and Rigot in [7] provides $q = p \cdot \exp^{\mathbb{H}}(s\mathbf{X}\psi(p) + \mathbf{i}s\mathbf{Y}\psi(p), s\mathbf{T}\psi(p))$. Hence from Lemma 2.2.8, there a unique maximizer p. Therefore the map F_s of Lemma 2.2.8 is a bijection of $\mathbb{H}_n$ and the inequality

$$(s\psi)^c(q) + s\psi(p) + c(p, q) \geq 0$$

is an equality if and only if $q = F_s(p)$. For $p \in \mathbb{H}_n$ fixed we compute now

$$(s\psi)^{cc}(p) = \sup_{q \in \mathbb{H}_n} \left(-c(p, q) - (s\psi)^c(q)\right).$$

For any $q \in \mathbb{H}_n$ we have the above estimate

$$-c(p, q) - (s\psi)^c(q) \leq s\psi(p)$$

with equality if $q = F_s(p)$. It follows $(s\psi)^{cc} = s\psi$ and $s\psi$ is c-convex. $\square$

Remark 2.2.9. For s small enough, the kind of c-convex functions $s\psi$ as in Proposition 2.2.7 satisfies condition (2.8) in Theorem 2.2.4. Indeed, we saw in the proof that the max is achieved. It follows that $s\psi$ can be used in order to build an optimal transport map starting from any given absolutely continuous measure of $\mathcal{P}_2(\mathbb{H}_n)$. This map will be F_s of the proof defined by $p \to p \cdot \exp^{\mathbb{H}}(s\nabla_{\mathbb{H}}\psi(p), s\mathbb{T}\psi(p))$.

2.3 A problem by Ambrosio and Rigot

For a Polish metric space (X, d), the so-called Wasserstein distance W (or L^2-Wasserstein distance) is a distance on $\mathcal{P}_2(X)$ that is directly related to optimal transport. We will now say more about the geometry it provides. The Wasserstein distance is defined by $W(\mu_0, \mu_1) = \sqrt{2C_2(\mu_0, \mu_1)}$, so

$$W(\mu_0, \mu_1)^2 = \inf_\pi \int d^2(p, q)d\pi(p, q)$$

where π is a optimal transport. The function $W : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{+\infty\}$ is symmetric, $W(\mu_0, \mu_1) \geq 0$ with equality if and only if $\mu_0 = \mu_1$ and the triangle inequality holds. The previous property is a consequence of a coupling technic called gluing lemma (see [108, Chapter 7]). Moreover, W is finite on $\mathcal{P}_2(X)$ because this set was exactly defined as the set of probability measures μ such that $W(\mu, \delta_{p_0})$, that is the distance to a Dirac measure δ_{p_0} is finite. With the triangle inequality, the distance between two measures of $\mathcal{P}_2(X)$ is finite.

If the Polish space (X, d) is a geodesic space (remain Section 1.5), there is a nice way for building geodesics in X using the following lemma. From this it follows that $\mathcal{P}_2(X)$ is geodesic if X itself is geodesic.

Lemma 2.3.1. *[109, Proposition 7.16] Let (X, d) be a Polish geodesic space and π a Borel probability measure on $X \times X$. Then there is a Borel probability measure Π on the set of geodesics of X (with the uniform distance on $\mathcal{C}([0, 1], X)$) such that*

$$(e_{0,1})_\# \Pi = \pi$$

where $e_{0,1}(\gamma) = (\gamma(0), \gamma(1))$.

The letter e in $e_{0,1}$ is the first letter of "evaluation map". For $s, t \in [0, 1]$ we will also use e_s and $e_{s,t}$ defined by $e_{s,t}(\gamma) = (e_s(\gamma), e_t(\gamma)) = (\gamma(s), \gamma(t))$. Lemma 2.3.1 is useful for geodesics when π is an optimal transport map between μ_0 and μ_1. Then the curve $(\mu_s)_{s\in[0,1]}$ defined as $\mu_s = (e_s)_\#\Pi$ is a geodesic of $\mathcal{P}_2(X)$ and the transport map $\pi_{s,t} := (e_{s,t})_\#\Pi$ are optimal. We have for $0 \leq s \leq t \leq 1$

$$\begin{aligned} W(\mu_s, \mu_t)^2 &\leq \int d^2(p, q)d\pi_{s,t} \\ &\leq \int d^2(p, q)d(e_{s,t})_\#\Pi \\ &\leq \int d^2(\gamma(s), \gamma(t))d\Pi(\gamma) \\ &\leq \int (t - s)^2 d^2(\gamma(0), \gamma(1))d\Pi(\gamma) \\ &\leq (t - s)^2 W^2(\mu_0, \mu_1). \end{aligned}$$

Then $W(\mu_s,\mu_t) \leq |t-s|W(\mu_0,\mu_1)$ and similarly $W(\mu_0,\mu,s) \leq |s|W(\mu_0,\mu_1)$ and $W(\mu_t,\mu_1) \leq |s|W(\mu_t,\mu_1)$. It follows from this and the triangle inequality

$$\begin{aligned}
W(\mu_0,\mu_1) &\leq W(\mu_0,\mu_s) + W(\mu_s,\mu_t) + W(\mu_t,\mu_1) \\
&\leq sW(\mu_0,\mu_1) + (t-s)W(\mu_0,\mu_1) + (1-t)W(\mu_0,\mu_1) \\
&\leq W(\mu_0,\mu_1).
\end{aligned}$$

The previous inequality is in fact an equality so that

$$W(\mu_s,\mu,t) = |t-s|W(\mu_0,\mu_1)$$

for every $s,t \in [0,1]$ which is the definition of a geodesic. All geodesics of $\mathcal{P}_2(X)$ have in fact the previous form as is proved in [109, Chapter 7] in a greater generality.

Proposition 2.3.2. *Let $(\mu_s)_{s\in[0,1]}$ be a geodesic of a Polish space X. Then there is a measure Π on the Polish space $\mathcal{C}([0,1],X)$ such that $\mu_s = (e_s)_\#\Pi$ and $(e_{s,t})_\#\Pi$ is optimal between μ_s and μ_t. Moreover, Π is concentrated on the set of geodesics.*

An easy way to remember this proposition is " a geodesic of measures is a measure on the geodesics ".

2.3.1 Examples of geodesics of $\mathcal{P}_2(\mathbb{H}_n)$

In this subsection, we present examples of geodesics of $\mathcal{P}_2(\mathbb{H}_n)$ thanks to some optimal plans of the Heisenberg group presented in Section 2.2. We insist on the fact that there are possibly several geodesics between two measures and explain that it is not the case under the assumption of absolutely continuity

Example 2.3.3 (A first example). The measures Π with $(e_{0,1})_\#\pi$ in Lemma 2.3.1 are often unique. It happens when $\pi(p,q)$-almost surely there is a unique geodesic between p and q. Let us give a very simple example. If μ_0 is a measure of $\mathcal{P}_2(\mathbb{H}_n)$ and $v \in \mathbb{C}^n$ then $T : p \to p \cdot \exp^{\mathbb{H}}(v,0)$ is an optimal transport map as we have seen in Example 2.2.3. But $s \in [0,1] \to p \cdot \exp^{\mathbb{H}}(sv,0)$ is the unique geodesic between p and $T(p)$ so that $((T_s)_\#\mu_0)_{s\in[0,1]}$ for $T_s(p) = p \cdot \exp^{\mathbb{H}}(sv,0)$ is the unique possible geodesic corresponding to the optimal plan π_T. The measure on geodesics Π is therefore $(\gamma_v)_\#\mu_0$ where $\gamma_v(p)$ is the curve $s \in [0,1] \to p \cdot \exp^{\mathbb{H}}(sv,0)$. Moreover, we exhibited the unique geodesic between the two ends because $(\mathrm{Id} \otimes T)_\#\mu_0$ is the unique optimal plan between them.

Example of two measures connected by infinitely many geodesics and other remarks

The second example is related to the transport between measures concentrated on L as presented in Example 2.2.2. What are the geodesics and intermediate measures in this case? Let as before $\mu_0 \in \mathcal{P}_2(\mathbb{H}_1)$ be concentrated on L and T a measurable map such that $T(t) \geq t$ (we identify L with $\mathbb{R}$ as before). Then T is an optimal transport plan between μ_0 and $\mu_1 = T_\#\mu_0$. The geodesics between p and $T(p)$ are horizontal lifts of circles of radius $\sqrt{\frac{T(t)-t}{\pi}}$ (and perimeter $2\sqrt{(T(t)-t)\pi}$) and the mass transported from p to $T(p)$ will travel along these geodesics.

A possible choice for a geodesic between μ_0 and μ_1 is to take a unique geodesic for each starting point $(0;t)$. For example

$$T_s((0;t)) = (0;t)\cdot \exp^{\mathbb{H}}(2\sqrt{(T(t)-t)\pi}e^{i\theta(t)}, 2\pi)$$

where θ is a measurable function from $\mathbb{R}$ to $\mathbb{R}$ that indicates the direction of the geodesic starting from $(0;t)$. Thus $((T_s)_\#\mu_0)_{s\in[0,1]})$ is a geodesic. Another possibility is to transport the mass starting from $(0;t)$ according to an angular distribution independent of t. For this role take $u \in \mathcal{P}(S^1)$, a probability measure on the set of complex numbers of modulus 1 and define Π as $f_\#(\mu_0 \otimes u)$ with $f(t, e^{i\theta})$ the geodesic starting from $(0;t)$ and being tangent to $2\sqrt{(T(t)-t)\pi}e^{i\theta}$. Finally any probability measure on $\mathbb{R}\times S^1$ with first marginal μ_0 provides actually a geodesic between μ_0 and μ_1. Note that for any optimal transport plan different from $\pi_T = (\mathrm{Id} \otimes T)$ there will be other geodesics.

Let us continue with this example and the optimal transport provided by T. Take a convenient Π, possibly one those presented before. Thus the transport plan $\pi_{1/2,1} = (e_{1/2,1})_\#\Pi$ is optimal between $\mu_{1/2}$ and μ_1. The restriction to $[1/2,1]$ of $(e_s)_\#\Pi$ is a geodesic between these two measures but it is possible to make it more precise and recover what is Π. Actually $\mu_{1/2}(p)$-almost surely, p is the midpoint of a geodesic between $(0;t)$ and $(0;T(t))$. It means that $z = Z(p)$ is the midpoint of a full circle beginning in $0_\mathbb{C}$ of area $T(t)-t$. From this information, we know that $T(t)-t = \frac{\pi}{4}|z|^2$ and we can localize t, $T(t)$ and the geodesic going through p. Let call it $\gamma(p)$. It follows that $\Pi = \gamma_\#\mu_{1/2}$. Furthermore we have $\pi_{1/2,1} = (e_{1/2,1})_\#\gamma_\#\mu_{1/2}$. But $e_{1/2}\circ\gamma(p) = p$ and writing $U(p) = e_1\circ\gamma(p)$ it follows that U is an optimal transport map from $\mu_{1/2}$ to μ_1. In fact we can write the expression of U. It is $U(z;t) = (0; t + \frac{|z|^2}{2\pi})$ and we met it in Example 2.2.6.

This is an illustration of the fact that it is possible to find some non-intuitive optimal transport plans as interpolated transport plans between two intermediate measures. With the same transport as before we learn for example that the map $U_1 : (z;t) \to (\mathbf{i}z; t + |z|^2(\frac{2+\pi}{4}))$ is an optimal transport map between $\mu_{1/4}$ and $\mu_{3/4}$. An additional remark about this fact is that contrarily to $\mathbb{R}^n$ the optimal transport maps of $\mathbb{H}_1$ have not all the differential with positive eigenvalues (These are $\mathbf{i}$, $-\mathbf{i}$ and 1 for U_1) that makes it impossible to apply in $\mathbb{H}_n$ the arithmetico-geometric inequality for matrices as is done in [30] for example. This type of transport maps (differential with positive eigenvalues), called monotone happens in $\mathbb{R}^n$ (the gradient of a convex function is monotone, see also Remark 2.1.11) and permits to state nicely functional inequalities.

Uniqueness of the geodesics starting from an absolutely continuous measure

In the proof of [7], Ambrosio and Rigot have to compute the differential of d_c. Unfortunately d_c is not smooth, possibly not in (p,q) where $p \neq q$. Actually $d_c(0_\mathbb{H}, \cdot)$ is not differentiable in any point of L. However, there are left and right derivatives along vectors $\mathbf{X}_j$. The left and right derivatives $\mathbf{X}_j^-\phi$ and $\mathbf{X}_j^+\phi$ are defined to be the left and right derivatives of $s \to \phi(p\cdot(0,\cdots,0,s+\mathbf{i}0,\cdots,0;0))$ in 0. In [7] Ambrosio and Rigot prove

Lemma 2.3.4. *For any $q \in L^*$ and $j \in \{1,\cdots,n\}$ we have*

$$\mathbf{X}_j^+(q) = -1 \quad \textit{and} \quad \mathbf{X}_j^-(q) = 1.$$

It applies to the following lemma that is a simple variant of [7, Lemma 4.7].

Lemma 2.3.5. *Let $\phi \in \mathcal{F}_{\mathbb{H}_n}$ be approximately differentiable in $p \in \mathrm{Dom}(\phi)$. Then $\partial^c\phi(p) \cap (p \cdot L^*) = \emptyset$.*

In this lemma L^* denotes $\{(0;t) \mid t \neq 0\}$. Approximate differentiability is defined just before Theorem 2.2.4.

Proof. Let ϕ and p be as in the statement and ϕ_2 differentiable in p be such that $A = \{\phi = \phi_2\}$ has density 1 in p. We assume by contradiction that one can find $q \in \partial^c\phi(p) \cap p \cdot L^*$. By definition of the c-subdifferential we have

$$\phi(p \cdot h) - \phi(p) \geq \frac{d_c(p,q)^2 - d_c(p \cdot h, q)^2}{2}$$

for all $h \in \mathbb{H}_n$. We suppose now that $p \cdot h \in A \setminus \{p\}$ and we let it go (in fact a sequence $(h_j)_{j\in\mathbb{N}}$) to 0 such that $h = \|h\|(1,0,\cdots,0;0) + o(\|h\|)$ where $\|h\|$ is the Euclidean norm of $\mathbb{R}^{2n+1}$. This phenomenon may occur because A has density 1 in p. As a consequence there is a sequence of points tending to p in any half-cone centred in p. We get

$$\frac{\phi(p \cdot h) - \phi(p)}{\|h\|} \geq \frac{d_c(q^{-1} \cdot p, 0_{\mathbb{H}})^2 - d_c(q^{-1} \cdot p \cdot h, 0_{\mathbb{H}})^2}{2\|h\|}. \tag{2.10}$$

The left hand side goes to $\widetilde{\mathbf{X}}_1\phi(p) = \mathbf{X}_1\phi_2(p)$ when $\|h\|$ goes to 0. We have $q^{-1} \cdot p \in L^*$ by assumption and it follows from [7, Lemma 3.16] that the right hand side goes to $d_c(q^{-1} \cdot p)X_1^+ d_c(q^{-1} \cdot p) = -d_c(q^{-1} \cdot p)$ where d_c is here used for $d_c(0_{\mathbb{H}}, \cdot)$. We now assume that $h = \|h\|(-1,0,\cdots,0;0) + o(\|h\|)$ when $h \in A \setminus \{0_{\mathbb{H}}\}$ tend to $0_{\mathbb{H}}$. We have again inequality (2.10) but this time h is moving in the direction of $-\mathbf{X}_1$ so the right hand side goes to $d_c(q^{-1} \cdot p)(-X_1^-)d_c(y^{-1} \cdot x) = -d_c(q^{-1} \cdot p)$ and the other side to $-\widetilde{\mathbf{X}_1}\phi(p) = -\mathbf{X}_1\phi_2(p)$. Hence both $\mathbf{X}_1\phi_2(p)$ and $-\mathbf{X}_1\phi_2(p)$ are smaller than the non-positive $-d_c(q^{-1} \cdot p)$ which implies a contradiction to the fact that $q \in \partial^c(c) \cap p \cdot L^*$. $\square$

Under the hypotheses of Theorem 2.2.4, in $\mu_0(p)$-almost every p the function ϕ is approximately differentiable and $(p, T(p)) \in \partial^c(\phi)$ where T is the optimal transport map. Then the previous lemma shows that μ_0-almost surely $T(p) \notin p \cdot L^*$ which brings that there is almost surely an unique geodesic between p and $T(p)$ (Section 1.5). Because of the uniqueness of the transport plan under the hypothesis of Theorem 2.2.4, it follows that $(\mu_s)_{s\in[0,1]}$ with $\mu_s = (T_s)_{\#}\mu_0$ and

$$T_s(p) = p \cdot \exp^{\mathbb{H}}(s\widetilde{\mathbf{X}}_1\phi(p) + \mathbf{i}s\widetilde{\mathbf{Y}}_1\phi(p), \cdots, s\widetilde{\mathbf{X}}_n\phi(p) + \mathbf{i}s\widetilde{\mathbf{Y}}_n\phi(p); s\widetilde{\mathbf{T}}\phi(p))$$

is the unique geodesic in $\mathcal{P}_2(\mathbb{H}_n)$ between μ_0 and μ_1.

2.3.2 Statement of the problem

Take $\mu_0 \in \mathcal{P}_2(\mathbb{H}_n)$ an absolutely continuous measure and $\mu_1 \in \mathcal{P}_2(\mathbb{H}_n)$. Let $(\mu_s)_{s\in[0,1]}$ be a geodesic. Then with the notations of Theorem 2.2.4 we have

$$\mu_s := T_{s\#}\mu \quad \text{with} \quad T_s(x) := x \cdot \exp^{\mathbb{H}}(s\tilde{\mathbf{X}}\varphi(x) + \mathbf{i}s\tilde{\mathbf{Y}}\varphi(x), s\tilde{\mathbf{T}}\varphi(x)).$$

In [7, Section 7 (c)] the following open problem is raised: are all measures μ_s absolutely continuous for $s \in [0, 1[$?

This question is motivated by the fact that the above property holds in the Euclidean and the Riemannian setting (see [109, Chapter 8]). A positive answer to the above question is given in [42]. Subsequently, except some variations, we reproduct in this chapter the content of this paper.

By [109, Theorem 7.29] we know that for any time $s \in [0, 1)$ the map T_s is μ_0-essentially injective (i.e. its restriction to a set with full μ_0-measure is injective), and there exists an inverse transport map S_s uniquely defined up to μ_s-negligible sets such that $S_s \circ T_s = \mathrm{Id}$ μ_0-a.e. (and so $S_{s\#}\mu_s = \mu_0$). Actually Theorem 7.29 in [109] holds because $\mathbb{H}_n$ is non-branching which means that there is a unique way to continue pieces of geodesics to longer geodesic (see Section 1.5).

Our main result is the following:

Theorem 2.3.6. *Let $(\mu_s)_{s\in[0,1]}$ be a geodesic of the Wasserstein space $\mathcal{P}_2(\mathbb{H}_n)$ and assume that μ_0 has density ρ with respect to $\mathcal{L}^{2n+1}$. Then for any $s \in [0, 1)$ the measure μ_s is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^{2n+1}$, and its density is bounded by*

$$\frac{1}{(1-s)^{2n+3}} \rho \circ T_s^{-1}|_{T_s(A)}, \tag{2.11}$$

where T_s is the (μ_0-almost uniquely defined) optimal transport map from μ_0 to μ_s, and A is any set of full μ_0-measure on which T_s is injective.

We remark that the usual way to prove the absolute continuity of the intermediate measures is to use the Monge-Mather shortening principle (see [109, Chapter 8]). In Subsection 2.3.3 we will see that this approach cannot work for the Heisenberg group. We will also give an example of an optimal transport $(\mu_t)_{t\in[0,1]}$ such that the measure at time 1/2 is concentrated on a set of Hausdorff dimension 1, while the sets of dimension 1 are negligible for μ_0 and μ_1. These "bad" results show that strange phenomena can occur in the Heisenberg case, and this made less clear the answer to the absolute continuity question.

However, in Subsection 2.3.4 we will see that the absolutely continuity is a consequence of the following two properties: the contraction estimate (Theorem 1.7.7), and the fact that the optimal transport map exists and the Wasserstein geodesic is unique (paragraph after Lemma 2.3.5).

2.3.3 Failure of the Monge-Mather shortening principle

A good presentation of the Monge-Mather shortening principle can be found in [109, Chapter 8]. For what follows we just need to consider is in the case of geodesic spaces.

Let (X, d) be a geodesic space, and denote by $\mathcal{H}_d$ the Hausdorff measure for d. The idea of the shortening lemma is the following: fix a Borel set K, and take 4 points $a, b, p, q \in K$. Suppose that we want to transport a and b on p and q (this is an informal way to say that we want to transport the measure $\frac{1}{2}(\delta_a + \delta_b)$ onto $\frac{1}{2}(\delta_p + \delta_q)$), and assume that for the quadratic cost it is optimal to send a on p and b on q, that is

$$d^2(a, p) + d^2(b, q) \leq d^2(a, q) + d^2(b, p). \tag{2.12}$$

Consider now two constant-speed geodesics $\alpha, \beta : [0,1] \to X$ from a to p and from b to q respectively, and suppose that we can prove the following estimate: there is a constant $C(K,s)$ (depending only on K and on the time $s \in [0,1]$) such that

$$C(K,s)d(\alpha(s),\beta(s)) \geq d(a,b). \tag{2.13}$$

Then, given any Wasserstein geodesic $(\mu_s)_{s\in[0,1]}$ such that $\mu_0(K) = \mu_1(K) = 1$, if μ_0 is absolutely continuous with respect to $\mathcal{H}_d$ one can easily prove that also μ_s is absolutely continuous with respect to $\mathcal{H}_d$. Actually if T is optimal, for $\mu_0 \otimes \mu_0$-almost every (a,b), the inequality (2.12) holds for $p = T(a)$ and $q = T(b)$ ([109, Theorem 5.9 (ii)]), then almost surely $C(K,s)d(T_s(a), T_s(b)) \geq d(a,b)$ and there is a set $A \subset K$ with $\mu_0(A) = 1$ such that T_s is injective on A and T_s^{-1} is $C(K,s)^{-1}$-Lipschitz from $T_s(A)$ to A. Then if a Borel set B has Hausdorff measure 0, the set $S_s(B) = T_s^{-1}(B)$ has also Hausdorff measure 0. But μ_0 is absolutely continuous with respect to the Hausdorff measure. It follows that $\mu_s(B) = \mu_0(T_s^{-1}(B)) = 0$ and μ_s is absolutely continuous.

We will now just prove the estimate (2.13) for optimal transport in $\mathbb{R}^n$. Here $\alpha(s) = sp + (1-s)a$ and $\beta(s) = sq + (1-s)b$. It follows that

$$\begin{aligned}
|\alpha(s) - \beta(s)|^2 &\leq |s(p-q) + (1-s)(a-b)|^2 \\
&\geq s^2|p-q|^2 + (1-s)^2|a-b|^2 + 2s(1-s)\langle p-q \mid a-b\rangle \\
&\geq (1-s)^2|a-b|^2
\end{aligned}$$

The previous inequality follows from condition (2.12). Actually

$$\begin{aligned}
2\langle p-q \mid a-b\rangle &= 2\langle (p-a) + (a-b) + (b-q) \mid a-b\rangle \\
&= (|a-b|^2 + 2\langle p-a \mid a-b\rangle) + (|a-b|^2 + 2\langle b-q \mid a-b\rangle) \\
&= (|p-b|^2 - |p-a|^2) + (|a-q|^2 - |b-q|^2) \geq 0.
\end{aligned}$$

The Heisenberg group $(\mathbb{H}_n, d_c)$ with the Lebesgue measure satisfy the above framework because as we mentioned in Chapter 1 the $2n+2$-dimensional Hausdorff measure $\mathcal{H}^{2n+2}_{d_c}$ and the Lebesgue measure $\mathcal{L}^{2n+1}$ are the same up to a constant (both are the Haar measure of the group). In particular absolute continuity with respect to $\mathcal{L}^{2n+1}$ or with respect to $\mathcal{H}^{2n+2}_{d_c}$ are the same.

Horizontal right translation

We saw in Example 2.1.6 that right translations by an horizontal vector provide an optimal transport in the Heisenberg group.

Let μ_0 be the restriction of $\mathcal{L}^{2n+1}$ to $]0,1[^{2n+1}$, and consider the horizontal vector $u = (1,0,\ldots,0;0)$. The intermediate map T_s is given for any $s \in [0,1]$ by the map $a \mapsto a\cdot(s,0,\ldots,0;0)$. More precisely, writing a as $(x+\mathbf{i}y, z_2, \ldots, z_n; t)$, we have

$$T_s(a) = ((x+s) + \mathbf{i}y, z_2, \ldots, z_n; t - \frac{sy}{2}). \tag{2.14}$$

We observe that T_s is affine on $\mathbb{R}^{2n+1}$ with Jacobian determinant 1, so the measure $\mu_s = T_{s\#}\mu_0$ is absolutely continuous. However, as we will show, the shortening principle does not hold.

Fix $a \in]0,1[^{2n+1}$, and let

$$a_\varepsilon := a + \varepsilon(\mathbf{i}, 0, \ldots, 0; \frac{x}{2} + s) = (x + \mathbf{i}(y+\varepsilon), z_2, \ldots, z_n; t + \frac{\varepsilon x}{2} + \varepsilon s)$$

with ε small enough so that $a_\varepsilon \in]0,1[^{2n+1}$. Then, using (2.14) twice,

$$\begin{aligned} T_s(a_\varepsilon) &= a_\varepsilon \cdot (s, \ldots, 0; 0) \\ &= ((x+s) + \mathbf{i}(y+\varepsilon), z_2, \ldots, z_n; (t + \frac{\varepsilon x}{2} + \varepsilon s) - \frac{s(y+\varepsilon)}{2}) \\ &= ((x+s) + \mathbf{i}(y+\varepsilon), z_2, \ldots, z_n; (t - \frac{sy}{2}) + \frac{\varepsilon(x+s)}{2}) \\ &= T_s(a) \cdot v_\varepsilon \end{aligned}$$

where v_ε is the horizontal vector $(\mathbf{i}\varepsilon, 0, \ldots, 0; 0)$. Therefore as in Remark 1.4.8

$$d_c(a, a_\varepsilon) = d_c(0_{\mathbb{H}}, a^{-1} \cdot a_\varepsilon) = d_c\left(0_{\mathbb{H}}, (\mathbf{i}\varepsilon, 0, \ldots, 0; \varepsilon s)\right) \sim 2\sqrt{\pi|\varepsilon| s}$$

as $\varepsilon \to 0$, while

$$d_c(T_s(a), T_s(a_\varepsilon)) = d_c(0, v_\varepsilon) = |\varepsilon|.$$

Thus we see that the shortening principle cannot hold. Moreover, from this example one can also see that there is no hope to find a decomposition of $]0,1[^{2n+1}$ into a family of countable Borel sets such that on each set the shortening principle holds, possibly with a different constant (if such a weaker condition holds, one can still prove quite easily the absolute continuity of the interpolation).

Dimension of the support of a special optimal transport

We consider the following transportation problem: the two measures μ_0 and μ_1 are concentrated on the vertical line

$$L := \{(z;t) \in \mathbb{H}_n \mid z = 0_{\mathbb{C}^n})\},$$

with μ_0 concentrated on the negative part $L^- = L \cap \{t \leq 0\}$ and μ_1 on the positive one $L^+ = L \cap \{t \geq 0\}$. Recall from Example 2.2.2 that in this situation all transport plans are optimal.

Let us investigate a concrete example: identifying $L = \{0_{\mathbb{C}^n}\} \times \mathbb{R}$ with $\mathbb{R}$, let μ_0 and μ_1 be $\mathcal{L}^1\lfloor_{[-1,0]}$ and $\mathcal{L}^1\lfloor_{[0,1]}$ respectively. An optimal transport plan is given by $\pi_T = (\mathrm{Id}, T)_\# \mu_0$, where the transport map is $T : (0_{\mathbb{C}^n}; t) \mapsto (0_{\mathbb{C}^n}; -t)$.

There is a multiple choice of geodesics between $(0_{\mathbb{C}^n}; t)$ and $T(0_{\mathbb{C}^n}; t)$ as we saw in Section 2.3.1. To construct a Wasserstein geodesic, we select the unique geodesic between $(0_{\mathbb{C}^n}; t)$ and $(0_{\mathbb{C}^n}; -t)$ whose midpoint is on the horizontal half-line $\{(r, 0, \ldots, 0; 0) \mid r \in [0, +\infty)\}$. This midpoint is exactly $(2\sqrt{\frac{2|t|}{\pi}}, 0, \ldots, 0; 0)$ because it is obtain after lifting an half-circle of radius $\sqrt{\frac{2|t|}{\pi}}$.

Using these geodesics of $\mathbb{H}_n$, we have actually defined a Wasserstein geodesic $(\mu_s)_{s\in[0,1]}$ between μ_0 and μ_1 which satisfies the following property: although μ_0 and μ_1 are absolutely continuous with respect to the 2-dimensional Hausdorff measure (induced by the distance d_c), the intermediate measure $\mu_{1/2}$ is concentrated on the horizontal line $\{(r, 0, \ldots, 0; 0) \mid r \in \mathbb{R}\}$ whose dimension is 1 (Proposition 1.4.6). This observation could suggest that one can find a measure μ_0 absolutely continuous with respect to the Lebesgue measure such that $\mu_{1/2}$ is not absolutely continuous because concentrated on a set of lower dimension. As announced before, we will prove in Section 2.3.4 that this cannot happen.

2.3.4 Proof of Theorem 2.3.6

The starting point for the proof of the theorem is Theorem 1.7.7. Given $p, q \in \mathbb{H}_n$ and $s \in]0,1[$, recall that $\mathcal{M}^s(x,y)$ is the set of points m such that

$$d_c(p,m) = s d_c(p,q), \qquad d_c(m,q) = (1-s) d_c(p,q).$$

For $E \subset \mathbb{H}_n$, we denote by $\mathcal{M}_s(E,q)$ the set

$$\mathcal{M}^s(E,q) := \bigcup_{p \in E} \mathcal{M}^s(p,q).$$

We remark that, for fixed q, for $\mathcal{L}^{2n+1}$-a.e. p is not in $q \cdot L^*$ such that the set $\mathcal{M}^s(p,q)$ is a single point and the curve $s \mapsto \mathcal{M}^s(p,q)$ is the unique geodesic between p and q defined on $[0,1]$.

Proposition 2.3.7. *[64, Section 2] Let $q \in \mathbb{H}_n$ and E a measurable set. Then $\mathcal{M}^s(E,q)$ is measurable and for any $s \in [0,1]$,*

$$\mathcal{L}^{2n+1}\big(\mathcal{M}^s(E,q)\big) \geq (1-s)^{2n+3} \mathcal{L}^{2n+1}(E).$$

Remark 2.3.8. The very little difference between the previous proposition and Theorem 1.7.7 is the fact that we consider $\mathcal{M}^s(E,y)$ and not $\mathcal{M}^s(E \setminus (q \cdot L^*), q)$. In fact $\mathcal{M}^s(q \cdot L^*, q)$ has measure 0. If we suppose up to a translation $q = 0_{\mathbb{H}}$, it is the set $\{(z;t) \in \mathbb{H}_n^* \mid |z_1|^2 = \cdots = |z_n|^2 = C(n,s) \cdot t\}$ where $C(n,s)$ only depends on n and s. This set has Hausdorff dimension $2n+1$ and not $2n+2$ as explained in Section 1.4.

The idea of the proof is now the following: first we approximate the target measure μ_1 by a sequence of discrete measures, and using Proposition 2.3.7 we prove the absolute continuity of the interpolation in the case of a discrete target measure. Then we pass to the limit, and we finally get the upper bound on the density of the interpolation.

Let $\mu_1^k = \frac{1}{k}\sum_{i=1}^k \delta_{q_i}$ be a sequence weakly converging to μ_1, and denote by T^k the optimal transport map between $\mu_0 = \rho\mathcal{L}^{2n+1}$ and μ_1^k. As in the begining of Subsection 2.3.2 for $(\mu_s)_{s\in[0,1]}$ and T_s, the curve $(\mu_s^k)_{s\in[0,1]}$ denotes the unique Wasserstein geodesic between μ_0 and μ_1^k, and T_s^k is the transport map from μ_0 to μ_s^k.

We remark that, if we prove the estimate in (2.11) with a certain set A of full μ_0-measure, then the bound will obviously be true also for any set containing A. Thus, up to a replacement of A with $A \cap \{\rho > 0\}$, we can assume that $A \subset \{\rho > 0\}$, so that μ_0 and $\mathcal{L}^{2n+1}$ are equivalent on A.

For each $i = 1, \ldots, k$, let $A_i \subset A$ be the set of points $x \in A$ such that $T^k(p) = q_i$. The sets A_i are mutually disjoint and $\mu_0\big(\mathbb{H}_n \setminus \cup_{i=1}^k A_i\big) = 0$.

Let us fix i. Since $T^k(A_i) = q_i$, the curve $s \mapsto T_s^k(p)$ is the unique geodesic from p to q_i for $\mathcal{L}^{2n+1}$-a.e. $p \in A_i$. Therefore there exists $B_i \subset A_i$ such that $\mathcal{L}^{2n+1}(A_i \setminus B_i) = 0$ and $s \mapsto T_s^k(p)$ is the unique geodesic from p to q_i for all $p \in B_i$. Consider now $E \subset B_i$. By the uniqueness of the geodesics from E to q_i we have

$$\mathcal{M}^s(E, q_i) = T_s^k(E).$$

We can therefore apply Proposition 2.3.7 to obtain that, for any $E \subset B_i$

$$\mathcal{L}^{2n+1}\big(T_s^k(E)\big) \geq (1-s)^{2n+3} \mathcal{L}^{2n+1}(E).$$

Since $\mathcal{L}^{2n+1}(A_i \setminus B_i) = 0$, the above estimate is still true if $E \subset A_i$. Recalling now that the sets A_i are disjoint and T_s^k is essentially injective, we easily obtain

$$\forall E \subset A, \qquad \mathcal{L}^{2n+1}\big(T_s^k(E)\big) \geq (1-s)^{2n+3}\mathcal{L}^{2n+1}(E).$$

Indeed it suffices to take $E \subset A$, split it as $E_i = E \cap A_i$, write the estimate for E_i and add all the estimates for $i = 1, \dots, k$. The above property can also be stated by saying that, for any $F \subset T_s^k(A)$,

$$\mathcal{L}^{2n+1}\big(F\big) \geq (1-s)^{2n+3}\mathcal{L}^{2n+1}((T_s^k)^{-1}(F) \cap A),$$

or equivalently

$$\int_A g(T_s^k(p))\, d\mathcal{L}^{2n+1}(p) \leq \frac{1}{(1-s)^{2n+3}} \int_{\mathbb{H}_n} g(q)\, d\mathcal{L}^{2n+1}(q) \tag{2.15}$$

for all $g \in C_c(\mathbb{H}_n)$, with $g \geq 0$. Since the Wasserstein geodesic between μ_0 and μ_1 is unique, by the stability of the optimal transport we have that, for any fixed s, the sequence μ_s^k weakly converges to μ_s, and the optimal transport maps T_s^k from μ_0 to μ_s^k converge in μ_0-measure to T_s from μ_0 to μ_s (see [109, Chapter 7 and Corollary 5.21]).

Thus, up to a subsequence, we can assume that $T_s^k \to T_s$ μ_0-a.e., which in particular implies that $T_s^k \to T_s$ for $\mathcal{L}^{2n+1}$-a.e. $p \in A$. We can therefore pass to the limit in (2.15), obtaining

$$\int_A g(T_s(x))\, d\mathcal{L}^{2n+1}(p) \leq \frac{1}{(1-s)^{2n+3}} \int_{\mathbb{H}_n} g(y)\, d\mathcal{L}^{2n+1}(q) \tag{2.16}$$

for all $g \in C_c(\mathbb{H}_n)$, $g \geq 0$. Moreover, arguing by approximation and using the monotone convergence theorem, we obtain that (2.16) holds for any measurable function $g \geq 0$ (in this case, both sides of the equation can be infinite).

From this fact we can directly conclude that T_s sends a set with positive Lebesgue measure into a set with positive Lebesgue measure, which implies that μ_s is absolutely continuous.

In order to prove the bound on the density of μ_s, we consider in (2.16)

$$g(q) := \chi_{T_s(A)}(y) h(y) \rho \circ T_s^{-1}(y),$$

with $h \geq 0$. In this way we get

$$\begin{aligned}
\int_{T_s(A)} h(q)\, d\mu_s(q) &= \int_A h(T_s(p))\, d\mu_0(p) \\
&= \int_A h(T_s(p))\rho(p)\, d\mathcal{L}^{2n+1}(p) \\
&\leq \frac{1}{(1-s)^{2n+3}} \int_{\mathbb{H}_n} h(q)\rho \circ T_s^{-1}(q))\, d\mathcal{L}^{2n+1}(q).
\end{aligned}$$

From the arbitrariness of h and the fact that μ_s is concentrated on $T_s(A)$ the bound follows.

2.3.5 A possible rehabilitation of the Monge-Mather principle

There is a variant of the Monge-Mather shortening principle that could be useful to prove the absolute continuity of the intermediate measures. A proof, by Figalli and Rifford [43] written after [42] treats of more general spaces than $\mathbb{H}_n$ and uses arguments that are close to this possible variant. We describe it now: the distance d_c play actually two roles in the Monge-Mather shortening principle. On the one hand it is involved in the cost comparison (2.12) and the geodesics, on the other hand and it appears in the Lipschitz estimate (2.13). It is actually possible to replace d_c by another distance d' for the second role and the new estimate would imply the absolute continuity of μ_s with respect to $\mathcal{H}_{d'}$, provided μ_0 is $\mathcal{H}_{d'}$-absolutely continuous. But in our case the $(2n+1)$-dimensional Hausdorff distance obtained from the Riemannian approximating distance d_ε for some $\varepsilon > 0$ (see Subsection 1.2.6) is exactly the Lebesgue measure of $\mathbb{R}^{2n+1}$. So it would be enough to prove

$$C(K,s)d_\varepsilon(\alpha(s),\beta(s)) \geq d_\varepsilon(a,b) \tag{2.17}$$

for four end-points a, b, p, q in K where (2.12) holds for d_c. The problem of different dimensions for the $\mathbb{H}$-lines and other lines that we have met before for the horizontal translation, will then no longer exist: the restriction of d_ε to any line of $\mathbb{R}^{2n+1}$ is locally equivalent to the Euclidean distance. However, we begin to give an example showing that it does not work so easily and that this variant is false if nothing is changed. We will give afterward an insight in the proof of Figalli and Rifford and propose, as a conjecture a second variant of the Monge-Mather shortening principle that could prove the absolute continuity of the intermediate measures.

So we show that the first variant of the Monge-Mather shortening principle does not immediately work, as long as the geodesics have different lengths. Again, it consists in proving (2.13) for two geodesics of $\mathbb{H}_n$ with the four ends in K such that the cost condition (2.12) is satisfied. Up to geometric transformations, what follows is a counterexample for any open set K because it take place in a ball ($\mathcal{B}^{\mathbb{H}}(0_{\mathbb{H}}, 4)$ for example)

Consider the one-parameter family of quadruple $(a_\lambda, b_\lambda, p_\lambda, q_\lambda)_{\lambda\in(0,1]}$, with

$$\begin{aligned}
a_\lambda &= (-\lambda, 0, \cdots, 0; 0) = \mathrm{dil}_\lambda(a_1)\\
b_\lambda &= (-\mathbf{i}\lambda, 0, \cdots, 0; 2\lambda^2) = \mathrm{dil}_\lambda(b_1)\\
p_\lambda &= (\lambda, 0, \cdots, 0; 0) = \mathrm{dil}_\lambda(x_1)\\
q_\lambda &= (\mathbf{i}\lambda, 0, \cdots, 0; 2\lambda^2) = \mathrm{dil}_\lambda(y_1)
\end{aligned}$$

Under the dilation dil_λ, the distance between points is just multiplied by λ:

$$d_c(\mathrm{dil}_\lambda(m), \mathrm{dil}_\lambda(n)) = \lambda d_c(m,n).$$

Since we can verify $d_c^2(a_1,p_1) + d_c^2(b_1,q_1) \leq d_c^2(a_1,q_1) + d_c^2(b_1,p_1)$, the similar relation holds for any $\lambda > 0$ and the optimal transport send a_λ on p_λ and b_λ on q_λ. The corresponding geodesics are

$$\begin{aligned}
\alpha(s) &= (-\lambda + 2s\lambda, 0\cdots, 0; 0),\\
\beta(s) &= (\mathbf{i}(-\lambda + 2s\lambda), 0\cdots, 0; 2\lambda^2),
\end{aligned}$$

with midpoints $\alpha(1/2) = (0, \cdots, 0; 0)$ and $\beta(1/2) = (0, \cdots, 0; 2\lambda^2)$. Thus, as $\lambda \to 0$, the Riemannian distance d_ε between the midpoints is equivalent to $C\lambda^2$ for some constant C, while $d_\varepsilon(a_\lambda, b_\lambda)$ is equivalent to $C'\lambda$ because it is also the distance of $a_\lambda^{-1} \cdot b_\lambda = (\lambda + \mathbf{i}\lambda, 0, \cdots, 0; \frac{5}{2}\lambda^2)$ to $0_{\mathbb{H}}$.

Although this fact shows that we cannot hope to prove a shortening principle with d_ε and geodesics on an open set, the following statement could however be true: fix $0 < m < M$, and let α and β be two geodesics with length between m and M such that

$$d_c^2(\alpha(0), \alpha(1)) + d_c^2(\beta(0), \beta(1)) \leq d_c^2(\alpha(0), \beta(1)) + d_c^2(\beta(0), \alpha(1)).$$

Then the estimate (2.13) holds for d_ε and a constant $C(K, s, m, M)$.

In [43] the authors prove a statement close to the previous conjecture. For an optimal transport under the hypothesis of Theorem 2.2.4, a measure Π as in Lemma 2.3.1 is concentrated on a set S of geodesics such that for every $(\alpha, \beta) \in S^2$ the cost estimate (2.12). Then they obtain that on $S_k = S \cap \{\mathrm{length}_c\, \alpha > 1/k\}$ the Lipschitz estimate (2.13) is locally satisfied for d_ε. It is not clear if the previous variant of the shortening lemma really holds or if the result of Figalli and Rifford only works because α and β are part of an "optimal" bunch of geodesics. It is not sure that any two geodesics of length greater than $1/k$ will satisfy a Lipschitz estimate (2.13) for d_ε.

Very briefly, the proof of Figalli and Rifford relies on the semiconcavity of the distance squared outside of the diagonal set $\{(p, q) \in \mathbb{H}_n \times \mathbb{H}_n \mid p = q\}$ and on differentiability properties of the semiconcave functions. They are able to recognize the optimal transport from μ_s to μ_0 as a map $F(d\phi_{k,s}(x))$ where $\phi_{k,s}$ is semiconcave and the maps $d\phi_{k,s}$ and F are locally Lipschitz on $e_s(S_k)$ and $T^*(\mathbb{R}^{n+1})$ respectively. In the last sentence e_s is the evaluation map as before and the distance on $\mathbb{H}_n$ is d_ε.

Chapter 3

Curvature bounds for the Heisenberg group

In this chapter we treat of different notions of synthetic curvature bounds in metric spaces and confront them with the Heisenberg group. Basically a synthetic curvature bound for a metric space is a property that is equivalent in the Riemannian case to having a lower or an upper bound on one of the curvature tensors. While the sectional curvature or the Ricci curvature can not be computed in non-smooth settings, this property must make sense in the metric setting too. One also expect a synthetic curvature bound to provide theorems that are similar to the classical theorems of Riemannian geometry. We will consider the Alexandrov spaces (the generic name for two classes of metric spaces with a synthetic sectional curvature bounded from below or from above), the criterion of Bakry-Émery, the Measure Contraction Property (MCP) and the Curvature-Dimension CD (three synthetic Ricci curvatures bounded from below). A part of the results satisfied by a Riemannian manifold as the local Poincaré inequality or growth estimates on the balls also hold for the Heisenberg group. However, it turns out that the Heisenberg group only satisfy a Measure Contraction Property (Theorem 3.4.5) whose definition relies on the contraction maps along geodesic (see Section 1.7). The main result of this chapter is the fact that the curvature-dimension condition CD does not hold in the Heisenberg group (Theorem 3.5.12), which with Theorem 3.4.5 has been proved in [64]. In this chapter we will also detailed the known facts that $\mathbb{H}_n$ is not an Alexandrov space and does not satisfy the Bakry-Émery criterion.

3.1 Ricci curvature of manifolds

We make here a short reminder about the definitions of sectional and Ricci curvature. Although they are defined in each textbook in differential geometry ([48, 34]) it is not really easy to have a precise intuition of what it is (especially the Ricci curvature). That is why we begin with a rough presentation and will give precise formulas afterward. The sectional curvature $\mathrm{Sec}_p(\sigma)$ has been introduced by Riemann as the Gauss curvature of the submanifolds of dimension 2 that are obtained when one consider in p the geodesics that are tangent to a given subspace $\sigma \subset T_pM$. It only depends on p and σ and $\mathrm{Sec}_p(v,w)$ is defined as $\mathrm{Sec}_p(\mathrm{span}(v,w))$. The Ricci curvature $\mathrm{Ric}_p(v)$ is a quadratic tensor that associate a tangential direction v with roughly speaking the mean of the scalar curvatures $\mathrm{Sec}_p(\sigma)$ for $v \in \sigma \subset T_pM$. More precisely if $(\frac{v}{\|v\|_g}, e_2, \ldots, e_n)$ is an orthonormal basis of (TM_p, g),

$$\mathrm{Ric}_p(v) = \|v\|_g^2 \sum_{j=2}^{n} \mathrm{Sec}_p(\frac{v}{\|v\|_g}, e_j).$$

Notice that $\mathrm{Ric}_p(v)$ can be positive even if $\mathrm{Sec}_p(\frac{v}{\|v\|_g}, e_j) \leq 0$ for some j. We precise that $\mathrm{Ric}_p(v)$ does not depend on the choice of the orthonormal base $(e_j)_{j=1}^n$.

A lot of results in Riemannian geometry and geometric analysis have been obtained under the assumption that Sec_p is negative or Ric_p is positive. More generally for any real number κ it can be supposed that $\mathrm{Sec}_p - \kappa$ (resp. $\mathrm{Ric}_p(\frac{v}{\|v\|_g}) - (n-1)\kappa\,\mathrm{Id}$) is negative (resp. is a positive quadratic form). Because of its definition the assumptions concerning Ric are always weaker than some assumptions on Sec. For example if the sectional curvature is uniformly greater that κ, the Ricci curvature $\mathrm{Ric}_p(v) = \|v\|_g^2 \,\mathrm{Ric}(\frac{v}{\|v\|_g})$ is greater than $\sum_{j=2}^{n} \kappa = (n-1)\kappa$.

The more usual way to compute explicitly the scalar and Ricci curvatures is to consider the Riemann tensor

$$\mathrm{Riem}_p(u, v, w, z) = g_p(\nabla_v \nabla_u w - \nabla_u \nabla_v w + \nabla_{[u,v]} w, z) \tag{3.1}$$

where ∇ is the Levi-Civita connection. Then the definition of the scalar curvature of the plane σ is

$$\mathrm{Sec}_p(e_1, e_2) = \mathrm{Riem}_p(e_1, e_2, e_1, e_2) \tag{3.2}$$

where $e_1(p), e_2(p)$ are any two spanning vectors $(\mathrm{span}(e_1(p), e_2(p)) = \sigma \subset TM_p)$ such that $g_p(e_i, e_j) = \delta_{i,j}$ for any $i, j \in \{1, 2\}$. Note that the definitions of Riem_p uses vector fields u, v, w and z but Sec_p is a function on the 2-planes included in TM_p for a fixed p. Actually the differential form Riem_p is a tensor, which means that it only depends on the vectors $u(p)$, $v(p)$, $w(p)$ and $z(p)$ of TM_p. However, for four vectors $u, v, w, z \in TM_p$, if one want to compute $\mathrm{Riem}_p(u, v, w, z)$ in the formula (3.1), one first have to extend the vectors to vector fields of TM. The value of $\mathrm{Riem}_p(u, v, w, z)$ will not depend on the way it is done.

If $(\frac{v}{\|v\|_g}, e_2, \ldots, e_n)$ is an orthonormal basis of (TM_p, g_p), Ric_p is defined as

$$\mathrm{Ric}_p(v) = \sum_{j=2}^{n} \|v\|_g^2 \,\mathrm{Riem}(\frac{v}{\|v\|_g}, e_j, \frac{v}{\|v\|_g}, e_j).$$

3.1.1 Ricci curvature of the approximating manifolds

We will now apply these definitions to the approximating manifolds $\mathbb{H}_1^\varepsilon$ of Section 1.2.6. Basically the idea is that if these manifolds have a lower bound for the Ricci curvature it shall be also true for $\mathbb{H}_1$. We will see that such a lower bound does not exist uniformly in ε. As in a large part of this thesis $\mathbf{X}$ (resp. $\mathbf{Y}$) will stand for $\mathbf{X}_1$ (resp. $\mathbf{Y}_1$). We first fix $\varepsilon > 0$. Because of the left-invariance of the metric $\mathrm{Ric}_p(\mathbf{X})$, $\mathrm{Ric}_p(\mathbf{Y})$ and $\mathrm{Ric}_p(\varepsilon\mathbf{T})$ do not depend on p and it suffices to determine what is Ric in $0_{\mathbb{H}}$.

We already know the Lie brackets between the vectors. Hence the first step is to compute the Levi-Civita connection between them, namely $\nabla_{\mathbf{X}}\mathbf{Y}$, $\nabla_{\mathbf{X}}\varepsilon\mathbf{T}$ and $\nabla_{\mathbf{Y}}\varepsilon\mathbf{T}$. For that we recall the relation defining the Levi-Civita connection (called Kozul identity in [23] where the computation is also made).

$$\begin{aligned}\langle \nabla_V U, W\rangle = \frac{1}{2}\,(&U\langle V, W\rangle + V\langle W, U\rangle - W\langle U, V\rangle \\ &-\langle [U,W],V\rangle - \langle [V,W],U\rangle - \langle [U,V],W\rangle)\,.\end{aligned}$$

For three vectors of $(\mathbf{X},\mathbf{Y},\varepsilon\mathbf{T})$, the first line in the previous identity vanish because the scalar products are constant. Let us index the frame $(\mathbf{X},\mathbf{Y},\varepsilon\mathbf{T})$ as (U_1,U_2,U_3) and denote $\langle [U_i,U_j],U_k\rangle$ by α_{ijk}. Then $\langle \nabla_{U_i}U_j,U_k\rangle = -\frac{1}{2}(\alpha_{jki} + \alpha_{ikj} + \alpha_{jik})$. The only non zero brackets are $[\mathbf{X},\mathbf{Y}]$ and $[\mathbf{Y},\mathbf{X}]$ so that all α's are 0 except $\alpha_{123} = \frac{1}{\varepsilon}$ and $\alpha_{213} = -\frac{1}{\varepsilon}$. Then the non-zero connections are

$$\nabla_{\mathbf{X}}\mathbf{Y} = -\nabla_{\mathbf{Y}}\mathbf{X} = \frac{1}{2\varepsilon}\varepsilon\mathbf{T}$$

and

$$\nabla_{\mathbf{X}}(\varepsilon\mathbf{T}) = \nabla_{(\varepsilon\mathbf{T})}\mathbf{X} = -\frac{1}{2\varepsilon}\mathbf{Y} \quad \text{and} \quad \nabla_{\mathbf{Y}}(\varepsilon\mathbf{T}) = \nabla_{(\varepsilon\mathbf{T})}\mathbf{Y} = \frac{1}{2\varepsilon}\mathbf{X}.$$

Then

$$\begin{aligned}\mathrm{Sec}(\mathbf{X},\mathbf{Y}) &= \langle \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{X} - \nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{X} + \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{X} \mid \mathbf{Y}\rangle \\ &= \langle 0 + \nabla_{\mathbf{X}}(\frac{1}{2\varepsilon}\varepsilon\mathbf{T}) + \frac{1}{\varepsilon}\nabla_{(\varepsilon\mathbf{T})}\mathbf{X} \mid \mathbf{Y}\rangle \\ &= -\frac{1}{4\varepsilon^2} - \frac{1}{4\varepsilon^2} = -\frac{3}{4\varepsilon^2}.\end{aligned}$$

Similar computations shows that

$$\mathrm{Sec}(\mathbf{X},\varepsilon\mathbf{T}) = \frac{1}{4\varepsilon^2} \quad \text{and} \quad \mathrm{Sec}(\mathbf{Y},\varepsilon\mathbf{T}) = \frac{1}{4\varepsilon^2}.$$

Then

$$\mathrm{Ric}(\mathbf{X}) = \mathrm{Ric}(\mathbf{Y}) = -\frac{1}{2\varepsilon^2} \quad \text{and} \quad \mathrm{Ric}(\varepsilon\mathbf{T}) = \frac{1}{2\varepsilon^2}.$$

Moreover $\mathbf{X}$, $\mathbf{Y}$ and $\varepsilon\mathbf{T}$ are eigenvectors of Ric seen as symmetric operator on $(TM_{0_{\mathbb{H}}}, g_{0_{\mathbb{H}}})$ because for all θ, $D(\mathrm{rot}_\theta)_{0_{\mathbb{H}}}$ is an isometry such that the only possible eigenspaces are $\mathbb{R}^3$, $\{t=0\}$ and $\{x=y=0\}$. It follows that the Ricci tensor has matrix

$$\frac{1}{2\varepsilon^2}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the $(\mathbf{X},\mathbf{Y},\varepsilon\mathbf{T})$ frame.

Observe that when ε goes to 0, the Ricci curvature lower bound $-\frac{1}{2\varepsilon^2}$ tends to $-\infty$. It is not a proof that $\mathbb{H}_1$ has not a synthetic Ricci curvature lower bound but this is quite coherent with this evidence. In Subsection 3.5.2 we will make this observation again in the more theoretical background of the Curvature-Dimension CD.

Remark 3.1.1. for $\mathbb{H}_n$ the frame $(\mathbf{X}_1,\mathbf{Y}_1,\ldots,\mathbf{X}_n,\mathbf{Y}_n,\mathbf{T})$ is also made of eigenvectors of Ric. The associated eigenvalues are, independently of n, $\frac{1}{2\varepsilon^2}$ for $\mathbf{T}$ and $-\frac{1}{2\varepsilon^2}$ for the other vectors.

3.2 Alexandrov spaces

In [3] Alexandrov (or Aleksandrov) found a way to deal with some metric spaces as if they were manifolds with a lower (resp. an upper) bound. These spaces were called after the name of this mathematician, *Alexandrov space of curvature* $\geq \kappa$ (resp. *Alexandrov space of curvature* $\leq \kappa$). Note that some variants of the second ones are also called $CAT(\kappa)$ spaces (where A stand for Alexandrov). However, in this thesis, we are essentially interested in the spaces with lower bounds for some curvature, so under them the Alexandrov space of curvature $\geq \kappa$. A milestone on this subject is the long and intricate article [22] by Burago, Gromov and Perelman. The book on metric geometry [21] is an easier introduction. In the first part of this section we will prove in a very basic way that $\mathbb{H}_1$ is not an Alexandrov space. This result is certainly well-know but to our knowledge there is no reference for it in the mathematical literature. Moreover it make it clear that the only case to really check is $\kappa = 0$, which will also be the case for CD in Section 3.5.

In Chapter 2 we already mentioned the theorem of Bertrand [14] about optimal transportation on Alexandrov spaces that we will state in this section. In Subsection 3.2.2, we will show that it is possible to state the absolute continuity of the intermediate measures as we have done in Theorem 2.3.6 for $\mathbb{H}_n$. These results rely on lower curvature bounds because it makes use of contraction estimates (similar to the ones of Section 1.7), which will be interpreted as positive curvature with MCP in Section 3.4. In the special case of positive synthetic sectional curvature (Alexandrov space with curvature ≥ 0) a easier proof by Figalli only relies on the definition of these spaces. Therefore it seems that it is an essential assumption to have a curvature bounded from below for proving the absolute continuity of measures interpolated by optimal transport.

3.2.1 Definition

The definition of Alexandrov spaces is based on the model spaces of Riemannian geometry. The functions σ and τ associated to the contraction maps of the model space are interesting for this whole chapter because they also appear in the definitions of MCP and CD (see Section 3.4).

Model spaces

The model spaces are manifolds with constant sectional curvature. They are described in any textbook in Riemannian geometry (e.g. [48, 34]). The model space of curvature 0 and dimension n is the Euclidean space $\mathbb{R}^n$. We introduce for the model spaces contractions along geodesics as we did in Section 1.7 for the Heisenberg group and the Grušin plane with $\mathcal{M}_p^s$ and the two maps E^G. In $\mathbb{R}^n$ the contraction of center $0_{\mathbb{R}^n}$ and ratio s are simply the dilations $x \to sx$. These are diffeormophisms with constant Jacobian determinant equal to s^n. For the other model space, the Jacobian determinant of the contraction maps are no longer constant but depends on the distance to the contraction center.

The other model spaces are the scaled hyperbolic plane $(\mathbb{H}^n, (-\kappa)^{-1/2} d_H)$ ($\mathbb{H}^n$ is not the Heisenberg group $\mathbb{H}_n$) with constant sectional curvature κ (for some $\kappa < 0$) and for $\kappa > 0$ the scaled sphere $(\mathbb{S}^n, \kappa^{-1/2} d_S)$. We denote the model spaces of dimension n and sectional curvature $\kappa \in \mathbb{R}$ by S^n_κ and simply by S_κ if $n=2$. Let moreover d_κ be the distance of S^n_κ and $E^{\kappa,n}_{p,s}$ the contraction of center $p \in S^n_\kappa$ and ratio s. The behaviour of these contractions is well-known. If $p \in S_\kappa$ is a given fixed point and $\exp_p$ the Riemannian exponential in p then the contraction map $E^{\kappa,n}_{p,s}(q) = \exp_p \circ (s \exp_p^{-1})(q)$ is defined everywhere except on the cut-locus of p. This cut locus is just a point for the sphere and the empty set for the other model spaces. If $q = \exp_p(v)$ and $w \in (TS^n_\kappa)_q$ is orthogonal to $D\exp_p(v).v$, then

$$\|DE^{\kappa,n}_{p,s}(q).w\|_\kappa = \frac{\sigma_\kappa(s d_\kappa(p,q))}{\sigma_\kappa(d_\kappa(p,q))}\|w\|_\kappa$$

with

$$\sigma_\kappa(d) = \begin{cases} (1/\sqrt{\kappa})\sin(\sqrt{\kappa}d) & \text{if } \kappa > 0 \\ d & \text{if } \kappa = 0 \\ (1/\sqrt{-\kappa})\sinh(\sqrt{-\kappa}d) & \text{if } \kappa < 0 \end{cases}. \tag{3.3}$$

With the same notations if now w is $D\exp_p(v).v$, then $\|DE^{\kappa,n}_{p,s}(q).w\|_\kappa$ is simply $s\|w\|$. Let us now establish the Jacobian determinant of the contraction maps of S^n_κ. It does not depend on the center of contraction p but only on the distance $d_\kappa(p,q)$ to the center of contraction p. For the Euclidean case ($\kappa = 0$), as said before, it is $s^n = s\left(\frac{\sigma_\kappa(sd_\kappa(p,q))}{\sigma_\kappa(d_\kappa(p,q))}\right)^{n-1}$. More generally this formula holds so that the contraction Jacobian is $\frac{\tau_{\kappa,n}(sd_\kappa(p,q))}{\tau_{\kappa,n}(d_\kappa(p,q))}$ for

$$\tau_{\kappa,n}(d) = \begin{cases} d\left((1/\sqrt{\kappa})\sin(\sqrt{\kappa}d)\right)^{n-1} & \text{if } \kappa > 0 \\ d^n & \text{if } \kappa = 0 \\ d\left((1/\sqrt{-\kappa})\sinh(\sqrt{-\kappa}d)\right)^{n-1} & \text{if } \kappa < 0 \end{cases}. \tag{3.4}$$

The definition of Alexandrov spaces uses the notion of *comparison triangle* in the model spaces S_κ. A comparison triangle of a triangle $\{a,b,c\} \subset X$ in $\tilde{X}$ is a triangle $\tilde{a}\tilde{b}\tilde{c}$ with the same sidelengths as abc. If $\tilde{X}$ is a model space S_κ, every metric embedding is the same up to global isometries of S_κ. Moreover for $\kappa > 0$ only small enough triangle have a comparison triangle in the sphere S_κ.

A geodesic metric space (X,d) is an Alexandrov space of curvature $\geq \kappa$ if for every point p there is a neighborhood U_p with the following properties

1. Every triangle included in (U_p,d) has a comparison triangle $\tilde{a}\tilde{b}\tilde{c}$ in S_κ.

2. For every triangle abc of (U_p,d) and a comparison triangle $\tilde{a}\tilde{b}\tilde{c}$, if α is a geodesic from b to c and $\tilde{\alpha}$ is a geodesic from $\tilde{b}$ to $\tilde{c}$, both parametrized with constant speed, for every $s \in [0,1]$,

$$d(a,\alpha(s)) \geq d_\kappa(\tilde{a},\tilde{\alpha}(s)).$$

A first fact is that any model space S^n_κ is an Alexandrov space of curvature $\geq \kappa'$ for any $\kappa' \leq \kappa$. In fact any triangle of S^n_κ is included in a geodesically embedded copy of S_κ in S^n_κ and it can be easily proved thanks to the cosine formula that this space is an Alexandrov space of curvature $\geq \kappa'$. Actually a Riemannian manifold M is an Alexandrov space of curvature $\geq \kappa$ if and only if for any $p \in M$ and any plane $\sigma \subset TM_p$, we have $\mathrm{Sec}_p(\sigma) \geq \kappa$.

About the Heisenberg group

The Heisenberg group $\mathbb{H}_1$ is not an Alexandrov space with curvature ≥ 0 as prove the following triangle $a = (0;0)$, $b = (0;\mathcal{A})$ and $c = (0;-\mathcal{A})$. The comparison triangle of abc in $\mathbb{R}^2$ is a isosceles right triangle because the geodesic between the points have squared length equal (up to the constant 4π) to the caught areas which are $\mathcal{A}$, $\mathcal{A}$ and $2\mathcal{A}$. The point $m = (\sqrt{\frac{8\mathcal{A}}{\pi}};0)$ correspond to $\alpha(1/2)$ for one of the geodesic between b and c. Then $d(a,m) = \sqrt{\frac{8\mathcal{A}}{\pi}}$ while $d_0(\tilde{a},\tilde{m}) = d_0(\tilde{b},\tilde{m}) = d_0(\tilde{c},\tilde{m}) = \frac{d(b,c)}{2} = \sqrt{2\pi\mathcal{A}}$ is greater.

For any $\kappa \in \mathbb{R}$, the Heisenberg group is also not an Alexandrov spaces with curvature $\geq \kappa$. In the case $\kappa \geq 0$, it is a direct consequence of the previous paragraph. If $\kappa < 0$ (let say $\kappa = -1$) we have to observe that small triangles of the Heisenberg group are compared to small triangles of the hyperbolic space and that these triangles have almost Euclidean ratios. When the parameter $\mathcal{A}$ goes to 0 in the last example and with the same notations, $d_\kappa(\tilde{a},\tilde{m})$ is equivalent to $d(b,c)/2 = \sqrt{2\pi\mathcal{A}}$ while $d(a,m)$ is still $\sqrt{\frac{8\mathcal{A}}{\pi}}$.

The definition of Alexandrov spaces of curvature bounded from above (or CAT spaces) is similar to the other one. Here in point 2. of the definition, distances to the opposite side have to be smaller than the corresponding ones in the comparison triangle. The Heisenberg space is also not an Alexandrov space of curvature bounded above because of the triangle $a = (0;0)$, $b = (2,\pi/2)$ and $c = (-2,\pi/2)$. The points b and c are reached from a thanks to the $\mathbb{H}$-lifts of two half circles of length π. The midpoint of the side $[bc]$ is $m = (0;\pi/2)$. It has distance $\sqrt{2}\pi$ to a. It is easy to check that it is greater that the distance $d_0(\tilde{a},\tilde{m})$ in the Euclidean comparison triangle. Scaling the triangle abc with the dilations dil_λ for small λ's, we obtain counterexamples for the other model spaces.

Remark 3.2.1. Argument using scaling with the dilations dil_λ will also be used in the extensions of Theorem 3.5.12 where we will prove that $CD(K,N)$ is not satisfied for $K \neq 0$.

Remark 3.2.2. A more theoretical way to prove that the Heisenberg group is not an Alexandrov space is to mention the well-known fact (see [21]) that for Alexandrov spaces the Hausdorff dimension equals the topological dimension. The topological dimension of $\mathbb{H}_n$ is $2n+1$ and the Haudorff dimension is $2n+2$. Then the Heisenberg group is not an Alexandrov space.

Globalization theorem

A theorem due to Toponogov states that for any Alexandrov space (X,d) with curvature $\geq \kappa$, the whole space realizes the conditions 1. and 2. of the definition. Precisely every triangle abc of X has a comparison triangle in S^κ and the comparison inequality of the distances 2. holds for every $s \in [0,1]$. It is particularly true when the model space is a sphere $(\mathbb{S},\kappa^{-1}d_S)$ such that the Alexandrov spaces with strictly lower bound κ are bounded (if it were not the case, some triangles would not have a comparison triangle in $(\mathbb{S},\kappa^{-1}d_S)$).

3.2.2 Transport in Alexandrov spaces and problem of absolute continuity

In [14], Bertrand proves for Alexandrov spaces the existence and uniqueness of solutions to the Monge problem. Bertrand follows the scheme explained in Subsection 2.1.1 and his proof relies on the fact that the structure of the Alexandrov spaces is close to the one of the Riemannian manifolds in the sense of the paper [22] by Burago, Gromov and Perelman. In his paper Bertrand does not address the problem of absolute continuity for the intermediate measures. Nevertheless in the book [109, Open problem 8.21], Villani asks whether it is possible to follow the Monge-Mather shortening principle and find a Lipschitz estimate as (2.13) for the Alexandrov spaces. As shown in Subsection 2.3.3, the absolute continuity would follow. In the bibliographic notes of [109, Chapter 8] Villani mentions a direct method by Figalli showing that the answer is yes when the lower bound κ is positive. We will explain in detail this computation in Lemma 3.2.5. In the case of a non-positive κ the problem is still open. However, in Remark 3.2.6 we propose a sufficient geometric inequality that would imply a positive answer. Whatever κ is, as explained by Figalli and the author in [42, Theorem 1.3] it is possible to exactly follow the proof of Theorem 2.3.6 (Theorem 1.2 in [42]) that was stated for $\mathbb{H}_n$ in the same paper and obtain the absolute continuity of the intermediate measures. Indeed, as $\mathbb{H}_n$ the Alexandrov spaces satisfy the two following properties: on the one hand estimates on the contraction maps and on the other hand if T is the optimal transport map between an absolutely continuous measure μ_0 and another measure μ_1 the geodesic between p and $T(p)$ is $\mu_0(p)$-almost surely unique. The proof of [42, Theorem 1.3] has been written by Schulte in his master thesis.

Now reproduct now the theorem of Bertrand.

Theorem 3.2.3. *Let (X, d) be a finite dimensional Alexandrov space of dimension n and $\mathcal{H}^n$ be the corresponding Hausdorff measure. Let μ_0, μ_1 be probability measures on X with compact supports such that μ_0 is absolutely continuous with respect to $\mathcal{H}^n$.*

Under these assumptions, Kantorovitch problem admits a solution, and any optimal plan is supported in the graph of a Borel function T. This map T is also a minimizer of Monge's problem and satisfies for μ-almost every $p \in X$,

$$T(x) = \exp(\nabla\phi(p)),$$

where ϕ is a d^2-convex function.

Moreover, up to modifications on negligible sets, the map $\nabla\phi$ is unique, and hence so is the optimal map T.

Remark 3.2.4. In this theorem appear functions like ∇ and exp. They are allowed by the fact that an Alexandrov space with lower curvature is "almost-everywhere a Riemannian manifold". We refer the reader to [14] for more details.

The Monge-Mahter shortening principle

We state some easy estimates of the Alexandrov spaces. The next lemma is attributed to Figalli in the notes of Chapter 8 of [109].

Lemma 3.2.5. *Let (X, d) be an Alexandrov space with curvature ≥ 0 and a, b, p, q four points. Let α (resp. β) be a geodesic from a to p (resp. from b to q). Then*

$$d(\alpha(s), \beta(s))^2 \geq (1-s)^2 d(a,b)^2 + s^2 d(p,q)^2 + s(1-s)\{a : b : p : q\}$$

where $\{a : b : p : q\} := [d^2(a,q) + d^2(b,p)] - [d^2(a,p) + d^2(b,q)]$. It follows for $\{a : b : p : q\}$ positive that $d(\alpha(s), \beta(s)) \geq (1-s)d(a,b)$ and $d(\alpha(s), \beta(s)) \geq sd(p,q)$.

Proof. On the triangles of $\mathbb{R}^2$, the median equality is an equality between the square length of the median and the square length of the sides. It can be generalized for a segment between a vertices and any point of the opposite side and become an inequality on Alexandrov spaces with positive curvature. We consider the median inequality for the points $\alpha_s = \alpha(s)$ and $\beta_s = \beta(s)$ on the triangles $ap\beta_s$ with $\alpha_s \in [ap]$ and on the triangles bqa and bqp with $\beta_s \in [bq]$. One have then

$$d(\alpha_s, \beta_s)^2 \geq (1-s)d(a, \beta_s)^2 + sd(p, \beta_s)^2 - s(1-s)d(a,p)^2 \tag{3.5}$$

$$d(\beta_s, a)^2 \geq (1-s)d(b,a)^2 + sd(q,a)^2 - s(1-s)d(b,q)^2 \tag{3.6}$$

$$d(\beta_s, p)^2 \geq (1-s)d(b,p)^2 + sd(q,p)^2 - s(1-s)d(b,q)^2 \tag{3.7}$$

The result follows from (3.5)+$(1-s)\times$(3.6)+$s\times$(3.7). ☐

This estimate and the Monge-Mather shortening principle presented in Subsection 2.3.3 provide that in an Alexandrov space with positive curvature the transport interpolated measures μ_s between two measures μ_0 and μ_1, one of them being absolutely continuous is also absolutely continuous for $s \in]0,1[$. This result applies in particular to the Alexandrov spaces of Lemma 3.2.5.

Remark 3.2.6. For non-positive κ, it may be difficult to do a similar computation as in Lemma 3.2.5 and it is quite clear that inequality (2.13) can not hold on the whole hyperbolic space. Open problem 8.21 of [109] asks if it is possible to get it on bounded parts of Alexandrov spaces of curvature $\geq \kappa$ (especially if $\kappa < 0$). In particular we would like to stress that the answer is yes if for each bounded domain of the hyperbolic plane S^{-1}, there exists positive functions f and g defined on $[0,1]$ such that the system of equations

$$\begin{cases} d(a,m)^2 \geq f(1-s)d(a,b)^2 + f(s)d(a,c)^2 - g(s)d(b,c)^2 \\ g(s) \leq f(s)f(1-s) \\ f(s) + f(1-s) \leq 1 \end{cases}$$

is satisfied for any triangle abc of the domain. Indeed, with the notation of Lemma 3.2.5 we obtain in this case

$$\begin{aligned} d(\alpha(s), \beta(s))^2 \geq & f(1-s)^2 d(a,b)^2 + f(s)^2 d(p,q)^2 \\ & + f(s)f(1-s)[d^2(a,q) + d^2(b,p)] \\ & - g(s)d^2(a,p) - g(s)(f(s) + f(1-s))d^2(b,q) \end{aligned}$$

which is enough for a Lipschitz estimate because the sum of the two last lines is positive. A possibility may be to choose $f(s) = s/C$ and $g(s) = s(1-s)/C^2$.

Proof using an estimate on the contraction map

Even for $\kappa < 0$ it is possible to get an interesting estimate when $a = b$ in the shortening principle. We have then two points p and q related to a common point $a = b$.

Lemma 3.2.7. *In an Alexandrov space (X, d) with curvature $\geq \kappa$, let a, p and q be three points and α, β two geodesics from a to p and q respectively. Then $d(\alpha(s), \beta(s))$ is greater than $d(\tilde{\alpha}(s), \tilde{\beta}(s))$ where $\tilde{a}\tilde{p}\tilde{q}$ is a comparison triangle of apq and $\tilde{\alpha}$ and $\tilde{\beta}$ are the curves parameterizing the sides.*

Proof. In the triangle apq we have $d(\alpha(s), q) \geq d_\kappa(\tilde{\alpha}(s), \tilde{q})$. We take now a comparison triangle of $a\alpha(s)q$ and observe that $d(\alpha(s), \beta(s))$ is greater than the comparative distance. This distance is taken in the comparative triangle of $a\alpha(s)q$ whose side lengths are $d(a, \alpha(s))$, $d(a, q)$ and $d(\alpha(s), q)$. It is then greater than $d_\kappa(\tilde{\alpha}(s), \tilde{\beta}(s))$ because this distance is taken with a comparison triangle with lengths $d(a, \alpha(s))$, $d(a, q)$ and $d_\kappa(\tilde{\alpha}(s), \tilde{q})$. Actually we use two times the fact that for a triangle of a model space with two fixed sides, the length of the third side is a monotone function of the angle which is a consequence of the sine theorem on the model spaces. □

From there Kuwae and Shioya [71] get an interesting estimate on the volume of contracted sets

Proposition 3.2.8. *Let A be a measurable set and p a point of an Alexandrov space (X, d) of curvature $\geq \kappa$ and Hausdorff dimension n. Then*

$$\mathcal{H}_d^n(E_{\kappa,n}^{s,p}(A)) = \int_A \frac{\tau_{\kappa,n}(sd(p,q))}{\tau_{\kappa,n}(d(p,q))} d\mathcal{H}_d^n(q)$$

The proof relies on the fact that $E_{\kappa,n}^{s,p}$ is injective almost everywhere on X and that at almost every a the space X is locally close to be isometric to $\mathbb{R}^n$.

Proposition 3.2.8 is one of the two elements for the next theorem, the counterpart of Theorem 2.3.6 for Alexandrov spaces. The second element is that under the hypothesis of Theorem 3.2.3, there is μ_0-almost surely a unique geodesic between x and $T(x)$ which together with Proposition 2.3.2 prove the assumption. For more details on this uniqueness see the paper of Bertrand [14] or [100, Corollary 3.1.9].

Theorem 3.2.9 (Theorem 1.3 in [42])**.** *Let (X, d) be an n-dimensional, complete Alexandrov space with curvature $\geq \kappa$. Let μ_0 and μ_1 be two compactly supported probability measures, with μ_0 absolutely continuous with respect to the n-dimensional Hausdorff measure $\mathcal{H}_d^n$. Denote by μ_s the unique Wasserstein geodesic between μ_0 and μ_1. Then, for any $s \in [0, 1[$, the measure μ_s is absolutely continuous with respect to $\mathcal{H}_d^n$, and its density is bounded by*

$$\frac{\tau_{\kappa,n}\left(\frac{d(x,T_s^{-1}(x))}{s}\right)}{\tau_{\kappa,n}\left((1-s)\frac{d(x,T_s^{-1}(x))}{s}\right)} \rho \circ T_s^{-1}(x)|_{T_s(B)}.$$

Here T_s is the (μ_0-almost uniquely defined) optimal transport map from μ_0 to μ_s, B is any set of full μ_0-measure on which T_s is injective and ρ is the density of μ_0.

Remark 3.2.10. In [100] is explained how to relax the assumption on the support of μ_0 and μ_1.

3.3 The Bakry-Émery criterion

On Riemannian manifolds (M, g) of dimension n it is usual to consider certain elliptic operators $L = \Delta - \nabla V \cdot \nabla$ where Δ is the Laplace-Beltrami operator and V is a regular function on M called potential. Then $e^{-V(p)} d\,\mathrm{vol}(p)$ is an invariant measure of the L-diffusion. While some properties of the heat semi-group depend on the Ricci tensor lower bounds (for example for a compact manifold), the similar properties remains true if we consider a modified Ricci tensor adapted to this diffusion. This tensor implicates a dimension factor $N \geq n$ and it is defined as

$$\mathrm{Ric}_{N,V} = \mathrm{Ric} + \nabla^2 V - \frac{\nabla V \otimes \nabla V}{N - n}.$$

The lower bound condition $\mathrm{Ric}_{N,V} \geq K$ writes then

$$\mathrm{Ric}(v) + (\nabla^2 V)(v, v) - \frac{\langle \nabla V \mid v \rangle^2}{N - n} \geq K g_p(v, v)$$

for any point $p \in M$ and any vector $v \in TM_p$. This assumption is equivalent to the Bakry-Émery criterion

$$\Gamma_2(f, f) \geq \frac{(Lf)^2}{N} + K |\nabla f|^2 \tag{3.8}$$

exposed (the first time) in [11] and explained in detail in [10] (see also [109]). In this criterion, Γ_2 is indirectly defined from the operator L through

$$\Gamma_1(f, g) = \frac{1}{2} \left(L(fg) - f L(g) - L(f) g \right).$$

In fact Γ_2 is obtained by replacing in the formulas of Γ_1 the products of type $f L(g)$ by $\Gamma_1(f, L(g))$. Thus

$$\Gamma_2(f, g) = \frac{1}{2} \left(L(\Gamma_1(f, g)) - \Gamma_1(f, L(g)) - \Gamma_1(L(f), g) \right).$$

Note that for the elliptic operator we consider ($L = \Delta - \nabla V \cdot \nabla$), the so-called "carré du champ" operator $\Gamma_1(f, g)(p)$ is simply $\langle \nabla f \mid \nabla g \rangle_p$. Actually the more general Bakry-Émery criterion for other elliptic operators L uses $\Gamma_1(f, f)$ in (3.8) at the place of $|\nabla f|^2$. Under our hypothesis on the form of L, the "carré du champs itéré" is

$$\Gamma_2(f, f)(p) = L \frac{|\nabla f|^2}{2} - \langle \nabla f \mid \nabla (Lf) \rangle_p.$$

In the case of $\mathbb{H}_n$, it is possible to consider the subelliptic (and hypoelliptic) operator $L_{\mathbb{H}} = \Delta_{\mathbb{H}} - \langle \nabla V \mid \nabla_{\mathbb{H}} \cdot \rangle_{\mathbb{H}}$ with a smooth potential V. In this sections we will first consider $V \equiv 0$ and then $V = V_{\mathbb{H}}$ related to the subelliptic diffusion at time 1 by $\mathfrak{h}_1 = \mathfrak{h}(1, \cdot) = e^{-V_{\mathbb{H}}}$ where $\mathfrak{h}$ will be presented in Subsection 3.3.2. This second choice is quite natural because it generalizes the classical feature of the Ornstein-Uhlenbeck operator on $\mathbb{R}^n$ whose invariant distribution is a Gaussian mass.

3.3.1 Computation of Γ_2

We can then compute Γ_1 and Γ_2 of $\Delta_{\mathbb{H}}$. We get $\Gamma_1 = \langle \nabla_{\mathbb{H}} f \mid \nabla_{\mathbb{H}} g\rangle$. Now

$$\begin{aligned}
2\Gamma_2(f,f) &= \Delta_{\mathbb{H}}(\langle \nabla_{\mathbb{H}} f \mid \nabla_{\mathbb{H}} f\rangle) - 2\langle \nabla_{\mathbb{H}} f \mid \nabla_{\mathbb{H}}\Delta_{\mathbb{H}} f\rangle\\
&= \Delta_{\mathbb{H}}(\mathbf{X}f)^2 - 2(\mathbf{X}f\mathbf{X}\Delta f) + \Delta_{\mathbb{H}}(\mathbf{Y}f)^2 - 2(\mathbf{Y}f\mathbf{Y}\Delta_{\mathbb{H}} f)\\
&= 2(\Delta_{\mathbb{H}}\mathbf{X}f)(\mathbf{X}f) + 2(\mathbf{X}^2 f)^2 + 2(\mathbf{YX}f)^2 - 2[\mathbf{X}^3 f\mathbf{X}f + (\mathbf{XY}^2 f\mathbf{X}f)]\\
&\quad + 2(\Delta_{\mathbb{H}}\mathbf{Y}f)(\mathbf{Y}f) + 2(\mathbf{Y}^2 f)^2 + 2(\mathbf{XY}f)^2 - 2[\mathbf{Y}^3 f\mathbf{Y}f + (\mathbf{YX}^2 f\mathbf{Y}f)]\\
&= 2(\mathbf{X}^2 f)^2 + 2(\mathbf{Y}^2 f)^2 + 2(\mathbf{XY}f)^2 + 2(\mathbf{YX}f)^2\\
&\quad + 2(\mathbf{X}^2\mathbf{Y}f\mathbf{Y}f - (\mathbf{YX}^2 f\mathbf{Y}f) + \mathbf{Y}^2\mathbf{X}f\mathbf{X}f - (\mathbf{XY}^2 f\mathbf{X}f))\\
&= 2(\mathbf{X}^2 f)^2 + 2(\mathbf{Y}^2 f)^2 + (\mathbf{XY}+\mathbf{YX}f)^2 + (\mathbf{XY}-\mathbf{YX}f)^2\\
&\quad + 2(\mathbf{X}^2\mathbf{Y}f\mathbf{Y}f - (\mathbf{YX}^2 f\mathbf{Y}f) + \mathbf{Y}^2\mathbf{X}f\mathbf{X}f - (\mathbf{XY}^2 f\mathbf{X}f))\\
&= 2(\mathbf{X}^2 f)^2 + 2(\mathbf{Y}^2 f)^2 + (\mathbf{XY}+\mathbf{YX})^2(f) + (\mathbf{T}f)^2 + 4\mathbf{XT}f\mathbf{Y}f - 4\mathbf{YT}f\mathbf{X}f
\end{aligned}$$

We examine the Bakry-Émery criterion. As explained in [32], there is no $K \in \mathbb{R}$ such that $\Gamma_2(f,f) \geq K\Gamma_1(f,f)$. We give for that the counterexample of a function f that is C^∞, with compact support with $f(z;t) = t^2$ locally around $0_{\mathbb{H}}$. Then in the neighborhood of $0_{\mathbb{H}}$, we have $\mathbf{X}(f) = -yt$ and

$$\mathbf{TX}f = -y \quad\text{and}\quad \mathbf{YX}f = -t - \frac{xy}{2} \quad\text{and}\quad \mathbf{X}^2 f = \frac{y^2}{2}.$$

from $\mathbf{Y}f = xt$ we also have

$$\mathbf{TY} = x \quad\text{and}\quad \mathbf{XY} = t - \frac{xy}{2} \quad\text{and}\quad \mathbf{Y}^2 f = \frac{x^2}{2}.$$

Then the criterion $\Gamma_2(f,f) \geq K\Gamma_1(f,f)$ around $0_{\mathbb{H}}$ becomes

$$(\frac{x^4}{4} + \frac{y^4}{4}) + \frac{1}{2}x^2y^2 + \frac{1}{2}4t^2 - 4xyt \geq K(x^2+y^2)t^2. \tag{3.9}$$

Take now $x = y$ and $t = xy$ such that $2t^2 - 4xyt$ is $-2x^4$. It follows that that the left-hand side is $-x^4$ while the right-hand side is $2Kx^6$. Thus the condition is not satisfied around $0_{\mathbb{H}}$ for any $K \in \mathbb{R}$.

3.3.2 The "carré du champs itéré" Γ_2 for another operator

Consider now $L_= \Delta_{\mathbb{H}} - \langle \nabla V \mid \nabla_{\mathbb{H}}\cdot\rangle_{\mathbb{H}}$ with a smooth potential V. We will first prove that the Bakry-Émery criterion is not satisfied for this operator. We will introduce a special potential $V = V_{\mathbb{H}}$ related to the subelliptic heat equation on $\mathbb{H}_n$. This potential will permit us to equip $\mathbb{H}_n$ with a canonical probability measure $e^{-V_{\mathbb{H}}}\mathcal{L}$. We will meet this metric measure space and the subelliptic diffusion later in Section 3.5 and Chapter 4.

For the operator $L_= \Delta_{\mathbb{H}} - \langle \nabla V \mid \nabla_{\mathbb{H}}\rangle_{\mathbb{H}}$ the "carré du champs" operator Γ_1 is the same as the one of $\Delta_{\mathbb{H}}$. But we have

$$\begin{aligned}
2\Gamma_2(f,f) &= L(\langle \nabla_{\mathbb{H}} f \mid \nabla_{\mathbb{H}} f\rangle) - 2\langle \nabla_{\mathbb{H}} f \mid \nabla_{\mathbb{H}} Lf\rangle\\
&= \Delta_{\mathbb{H}}(\langle \nabla_{\mathbb{H}} f \mid \nabla_{\mathbb{H}} f\rangle) - 2\langle \nabla_{\mathbb{H}} f \mid \nabla_{\mathbb{H}}\Delta_{\mathbb{H}} f\rangle + 2A(f)
\end{aligned}$$

where

$$A(f) = \langle \nabla_{\mathbb{H}} f \mid \nabla_{\mathbb{H}} \langle \nabla_{\mathbb{H}} f \mid \nabla_{\mathbb{H}} V \rangle\rangle - \langle \nabla_{\mathbb{H}} V \mid \nabla_{\mathbb{H}} \langle \nabla_{\mathbb{H}} f \mid \nabla_{\mathbb{H}} f \rangle\rangle$$

is the difference between the function $\Gamma_2(L)$ and $\Gamma_2(\Delta_{\mathbb{H}})$. Let us justify that the same counterexample $f(z;t) = t^2$ as in Subsection 3.3.1 also works when one adds $A(f)$ on the left-hand side of 3.9. We have

$$\begin{aligned}
A(f) =& \mathbf{X}f\mathbf{X}\left(\mathbf{X}f\mathbf{X}V + \mathbf{Y}f + \mathbf{Y}V\right) + \mathbf{Y}f\mathbf{Y}\left(\mathbf{X}f\mathbf{X}V + \mathbf{Y}f + \mathbf{Y}V\right)\\
&+ \mathbf{X}V\mathbf{X}\left((\mathbf{X}f)^2 + (\mathbf{Y}f)^2\right) + \mathbf{Y}V\mathbf{X}\left((\mathbf{X}f)^2 + (\mathbf{Y}f)^2\right)\\
=&(-yt)\mathbf{X}\left(-yt\mathbf{X}V + xt\mathbf{Y}V\right) + (xt)\mathbf{Y}\left(-yt\mathbf{X}V + xt\mathbf{Y}V\right)\\
&+ \mathbf{X}V\mathbf{X}(t^2(x^2+y^2)) + \mathbf{Y}V\mathbf{Y}(t^2(x^2+y^2))\\
=&(-yt)\left(-yt\mathbf{X}^2V + xt\mathbf{X}\mathbf{Y}V\right) + (xt)\left(-yt\mathbf{Y}\mathbf{X}V + xt\mathbf{Y}^2V\right)\\
&+ (-yt)\left(\frac{y^2}{2}\mathbf{X}V + (t - \frac{xy}{2})\mathbf{Y}V\right) + (xt)\left(-\frac{xy}{2} - t\mathbf{X}V + \frac{x^2}{2}\mathbf{Y}V\right)\\
&+ \mathbf{X}V(-yt(x^2+y^2) + 2t^2x) + \mathbf{Y}V(xt(x^2+y^2) + 2t^2y).
\end{aligned}$$

Hence for $x = y$ and $t = xy$ as in Subsection 3.3.1, $A(f) = O(|x|^5)$ when (x, y, t) tends to 0. Then if we add $A(f)$ in the left-hand side of (3.9) it is still equivalent to $-x^4$ and the Bakry-Émery criterion is not satisfied by L.

We want now to introduce a special potential $V_{\mathbb{H}}$ obtained from the subelliptic diffusion of the operator $\Delta_{\mathbb{H}}$. It is defined by $e^{-V_{\mathbb{H}}} = \mathfrak{h}(1, \cdot)$ where $\mathfrak{h}_s = \mathfrak{h}(s, \cdot)$ is the solution of the subelliptic heat equation

$$\frac{\partial}{\partial s} f_s = \Delta_{\mathbb{H}} f_s \tag{3.10}$$

starting from a Dirac measure in $0_{\mathbb{H}}$ at time 0. This equation is one of the more basic examples of the Hörmander [58] theory operator built as the sum of squared vector field. Then $\mathfrak{h}_1$ is smooth strictly positive and $V_{\mathbb{H}}$ is smooth too. The potential $V_{\mathbb{H}}$ is then of the type considered before and $L_{\mathbb{H}} = \Delta_{\mathbb{H}} - \langle \nabla V \mid \nabla_{\mathbb{H}} \cdot \rangle_{\mathbb{H}}$ does not satisfy the Bakry-Émery criterion. However, Hong-Quan Li proved that a log-Sobolev inequality holds in $(\mathbb{H}_1, e^{-V_{\mathbb{H}}}\mathcal{L})$ as we will see in Section 3.5. We will now say more about equation 3.10 and the associated stochastic equation. We will also consider the corresponding equations for the approximating manifolds.

The usual stochastic process associated to the Heisenberg group has been studied for more than fifty years beginning with Paul Lévy and his Lévy area. However, the relation with the diffusion on the Heisenberg group has been noticed only later. A founding article on this subject is the paper by Gaveau [49] in 1977. The stochastic equation

$$dX_s = \sum_{i=1}^{n} \left(\mathbf{X}(X_s)dB_{1,i} + \mathbf{Y}(X_s)dB_{2,i}\right).$$

corresponds to the subelliptic heat equation (3.10). The stochastic process $(X_s)_{s\geq 0}$ can be described without special knowledges on stochastic differential equation by considering the Lévy area of a Brownian motion and the solution of (3.10) are explicitly given by intricate formulas.

We first describe what is $(X_s)_{s\geq 0}$ for initial value $X_0 = 0_{\mathbb{H}}$ and begin with $n = 1$. The projection $Z(X_s)$ is a Wiener process (up to a time scaling) on $\mathbb{C}$ and the t-coordinate $t(X_s)$ is the algebraic algebra swept by this Brownian motion, $Z(X_s)$ at time $s > 0$. Actually because almost surely a Brownian path is not absolutely continuous, it seems (and it is) not possible to apply formula (1.14) for the algebraic area. Lévy approach to this problem has been to consider a stochastic integral extending the definition of the algebraic area. It is the so-called Lévy area. Therefore it make sense to say that X_s is the coupling of a Brownian motion and of its Lévy area. For $n > 1$, the Brownian motion $(X^1, \cdots, X^n)$ takes place in $\mathbb{C}^n$ and the Lévy area is simply the sum of the Lévy area of each X^i.

The law of X_s with initial value $X_0 = 0_{\mathbb{H}}$ is absolutely continuous. The formula for its density $\mathfrak{h}(s, \cdot)$ is states in [49], using the Lévy formula (see [112]):

$$\mathfrak{h}(s,(z;t)) = \frac{1}{8\pi^2 s^2}\int_{\mathbb{R}} \exp\left(\frac{\lambda}{s}(it - \frac{|z|^2}{4}\coth\lambda)\right)\frac{\lambda}{\sinh\lambda}d\lambda.$$

One can check that $\mathfrak{h}$ is a C^∞ function on $]0, +\infty[\times(\mathbb{C}^n \times \mathbb{R})$ which is not surprising because $\mathfrak{h}(s,(z;t))$ is the solution to the associated subelliptic partial differential equation

$$\frac{\partial}{\partial s} f_s = \Delta_{\mathbb{H}} f_s$$

and the theorem of Hörmander [58] explains that the solution are smooth for non-negative times. Here the function $\mathfrak{h}$ satisfies moreover some symmetry properties

$$\begin{cases} \mathfrak{h}(s,(e^{i\theta_1}z_1,\ldots,e^{i\theta_n}z_n;t)) = \mathfrak{h}(s,(z;t)) \\ \mathfrak{h}(\lambda s,(\sqrt{\lambda}z,\lambda t)) = \lambda^{-2}\mathfrak{h}(s,(z;t)) \end{cases} \tag{3.11}$$

due to the invariant actions of the rotations and dilations, and

$$\mathfrak{h}(s+s',q) = \int \mathfrak{h}(s,p)\times\mathfrak{h}(s',p^{-1}\cdot q)d\mathcal{L}(p).$$

Because of the group structure the solutions for other initial distributions are obtain thanks to the Heisenberg convolution defined by

$$f *_{\mathbb{H}} g(q) = \int f(p^{-1}\cdot q)g(p)d\mathcal{L}(p).$$

In fact $\mathfrak{h}$ is a solution of (3.10) exactly like the Gaussian functions are in $\mathbb{R}^n$ solution of the heat equation.

The diffusion on the approximating manifolds $\mathbb{H}_n^\varepsilon$ is paradoxically more difficult to describe. The stochastic equation is

$$dX_s = \varepsilon\mathbf{T}(X_s)dB_3 + \sum_{i=1}^n\left(\mathbf{X}(X_s)dB_{1,i} + \mathbf{Y}(X_s)dB_{2,i}\right),$$

and it is associated to the heat equation

$$\frac{\partial}{\partial s} f = \Delta_\varepsilon f := \Delta_{\mathbb{H}} f + (\varepsilon\mathbf{T})^2 f \tag{3.12}$$

where Δ_ε is the Laplace-Beltrami operator of $\mathbb{H}_\varepsilon$. It is direct to check that $\mathfrak{h}^\varepsilon = u^\varepsilon *_{\mathbb{H}} \mathfrak{h}$ is the solution of this equation, where $*_{\mathbb{H}}$ is the Heisenberg convolution as before and $u^\varepsilon(s,\cdot)$ a degenerated Gaussian measure concentrated on $L = \{0_{\mathbb{H}}\} \times \mathbb{R}$. An expression for the density of u^ε at time s is

$$u^\varepsilon(s) = \frac{1}{\sqrt{4\pi s\varepsilon^2}} \exp(\frac{-t^2}{4s\varepsilon^2}) \tag{3.13}$$

Note that if f and g are regular enough we have $\mathbf{X}(f *_{\mathbb{H}} g)(q) = (\mathbf{X}f) *_{\mathbb{H}} g(q) = (f *_{\mathbb{H}} (\mathbf{X}g))(q)$ and the same rule with $\mathbf{Y}$ and $\mathbf{T}$ because of the left-invariance of these vector fields with respect to the product. The convolution does not implicate the time parameter s, so

$$\frac{\partial}{\partial s}(f *_{\mathbb{H}} g) = (\frac{\partial}{\partial s} f *_{\mathbb{H}} g) + (f *_{\mathbb{H}} \frac{\partial}{\partial s} g).$$

Then at least formally equation 3.12 holds for $\mathfrak{h}^\varepsilon$ and even if u^ε is not smooth, it can be made true. A process which is solution of the stochastic version is actually the Heisenberg product (in this case just a Euclidean vector sum on the t-coordinate) of X_s with a Brownian motion U_s^ε. This last one is normalized such that $\mathrm{Var}(U_s^\varepsilon) = 2s\varepsilon^2$ where Var is the usual variance of $\mathbb{R}$ that we identify with L.

Remark 3.3.1. Exactly as it is possible to approximate the Gaussian distribution on $\mathbb{R}^n$ by random walks on $\mathbb{Z}^n$, it is possible to approximate $\mathfrak{h}(1,\cdot)$ by random walks on the discrete Heisenberg group $\mathbb{H}_1^{\mathbb{Z}}$. The scaling is made by using the dilations dil_λ of Subsection 1.1.1. See [31, 53] and the references therein.

3.4 The Measure contraction property MCP

We have seen that Alexandrov spaces are a nice generalization of Riemannian manifolds with a lower bound on the sectional curvature. In general metric measure spaces, there are two conditions which can be thought of as replacements for the Ricci curvature bounds of differential geometry: the geometric curvature-dimension $CD(K,N)$ and the measure contraction property $MCP(K,N)$. In our case where the geodesic between two points is almost surely unique, curvature-dimension $CD(K,N)$ is more restrictive than the measure contraction property $MCP(K,N)$, although it was not clear for a long time whether the two properties are equivalent. Moreover, in this situation (when there is almost surely a unique normal geodesic between two points), the measure contraction property implies a Poincaré inequality and the doubling property for metric measure spaces. This is shown in [110] and [78]. Metric measure spaces verifying a weak Poincaré inequality and the doubling property have proved to be a perfect setting for analysis with minimal hypotheses. A good reference on this new theory is the book by Heinonen (see [57]). It is possible to define a differentiable structure on such space, as proved in the Cheeger's paper [24] or to define Sobolev spaces with interesting properties (see [24],[55] and [102]). Another area of application of the Poincaré inequality is conformal geometry where it enables to analyze the quasi-conformal maps between metric spaces (see the survey article [17]). Some of the more famous examples of doubling metric measure spaces with a Poincaré inequality are Euclidean spaces and more generally complete manifolds with non-negative Ricci curvature, Carnot groups including $\mathbb{H}_n$ (see [106]), the boundary of hyperbolic buildings (see [16]), some Cantor-like sets with worm-holes (see [72] and the erratum [73]).

We now give the definition of the curvature-dimension $CD(K,N)$ and of the measure contraction property $MCP(K,N)$. In Sections 3.4 and 3.5 we will prove that $\mathbb{H}_n$ does not satisfy $CD(K,N)$ (Theorem 3.5.12) for any K,N but satisfies $MCP(0,2n+3)$ where the bound $2n+3$ is sharp (Theorem 3.4.5). The case where $K \neq 0$ in not really interesting in the Heisenberg group. We will see why and which properties hold after the proof of Theorem 3.5.12. Let (X,d,μ) be a metric measure space. The curvature-dimension condition $CD(K,N)$ is a geometric condition on the optimal transportation of mass between any pair of absolutely continuous probability measures on (X,d,μ).

The definition of $CD(K,N)$ in [105] uses special functions of the geometry of the model space S_κ^N where $(N-1)\kappa = K$. These functions $\tau_{\kappa,n}$ have been defined in (3.4).

Before we define the curvature-dimension condition, we also need to explain what is the relative Rényi entropy functional Ent_N. For a measure μ with density ρ with respect to ν, it is:

$$\mathrm{Ent}_N(\mu \mid \nu) = -\int_X \rho^{1-1/N} d\nu.$$

The functional Ent_N is a relative entropy because it is defined with respect to ν. When it is clear what the reference measure is, we will possibly write $\mathrm{Ent}_N(\mu)$ instead of $\mathrm{Ent}_N(\mu \mid \nu)$. For $N = +\infty$, we denote the relative Bolzmann entropy by Ent_∞. It is defined as

$$\mathrm{Ent}_\infty(\mu \mid \nu) = \int_X \rho \ln(\rho) \nu. \tag{3.14}$$

Standard Hypothesis 3.4.1. In the next two sections about the Measure contraction property MCP and the Curvature-dimension condition CD, all the metric measure spaces (X,d,ν) will be Polish, locally compact, geodesic with $(\nu \otimes \nu)(p,q)$-almost surely unique between p and q. We suppose also that the space is non-branching which means that two geodesics with a common part are both included in a same local geodesic. Moreover ν is not identically 0, it is finite on the balls and is defined on the Borel σ-algebra. Although the original definitions in [89, 104, 105, 77, 78] are given for more general hypothesis, we will in this report give equivalent definitions for metric measure spaces as explained here. First the spaces we are considering in this report are all of this form, second these definitions will be easier to understand.

Definition 3.4.2. Let $K \in \mathbb{R}$ and $N \in [1, +\infty[$ and set $\kappa = K/(N-1)$. We say that a metric measure space (X,d,ν) as in Standard Hypothesis 3.4.1 satisfies the curvature-dimension condition $CD(K,N)$ if and only if for each pair $\mu_0, \mu_1 \in \mathcal{P}_2(X)$ of absolutely continuous measures with respect to ν, there exists an optimal transport plan π and a geodesic $(\mu_s)_{s\in[0,1]}$ of absolutely continuous measures of $\mathcal{P}_2(X)$ such that

$$\begin{aligned}\mathrm{Ent}_N(\mu_s) \leq &\int \left(\frac{\tau_{\kappa,N}((1-s)d(p,q))}{\tau_{\kappa,N}(d(p,q))}\right)^{1/N} (-\rho_0^{-1/N}(p))d\pi(p,q)\\ &+ \int \left(\frac{\tau_{\kappa,N}(sd(p,q))}{\tau_{\kappa,N}(d(p,q))}\right)^{1/N} (-\rho_1^{-1/N}(q))d\pi(p,q)\end{aligned}$$

for all $s \in [0,1]$. Here we denoted the density of μ_s with respect to ν by ρ_s.

For $N = +\infty$ the relation has to be changed in

$$\mathrm{Ent}_\infty(\mu_s) \leq (1-s)\,\mathrm{Ent}_\infty(\mu_0) + s\,\mathrm{Ent}_\infty(\mu_1) - K\frac{s(1-s)}{2}W^2(\mu_0,\mu_1)$$

where $(\mu_s)_{s\in[0,1]}$ is a geodesic of $\mathcal{P}_2(X)$.

The property $CD(0,N)$ is easier to understand than the general case because for $\kappa = 0$ the coefficient $\left(\frac{\tau_{\kappa,N}(sd(p,q))}{\tau_{\kappa,N}(d(p,q))}\right)^{1/N}$ is simply s independently of $d(p,q)$. Thus in the particular case $K = 0$ the definition becomes :

The curvature-dimension condition $CD(0,N)$ holds in (X,d,μ) (as in Standard Hypothesis 3.4.1) if for every pair (μ_0,μ_1) of absolutely continuous measure of $\mathcal{P}_2(X)$, there is a geodesic $(\mu_s)_{s\in[0,1]}$ connecting μ_0 and μ_1 such that $\mathrm{Ent}_N(\mu_s)$ is a convex function of $[0,1] \to \mathbb{R}$.

Remark 3.4.3. A faithful transcription of Definition 3.4.2 would be "for any $s \in [0,1]$, there is a geodesic" and not "There is a geodesic such that any $s \in [0,1]$". In fact Figalli and Villani [44] proved that under Standard Hypothesis 3.4.1, the definitions are the same.

We will see in Theorem 3.5.12 that this property does not hold in the Heisenberg group and in Theorem 3.5.13 that it does not hold for the Grušin plane.

The measure contraction property $MCP(K,N)$ (see [105], [78], [89]) is a condition on metric measure spaces (X,μ,d). Its formulation is much simpler if there exists a measurable map

$$\mathcal{N} : (p,q,s) \in X \times X \times [0,1] \to X$$

such that for every $p \in X$ and μ-a.e $q \in X$, the curve $s \in [0,1] \to \mathcal{N}(p,q,s)$ is the unique normal geodesic from p to q. Then the space (X,d,μ) satisfies $MCP(K,N)$ if and only if for almost every $p \in X$, every $s \in [0,1]$ and every μ-measurable set E

$$\int \frac{\tau_{\kappa,N}(sd(p,q))}{\tau_{\kappa,N}(d(p,q))}\mathcal{N}_{p,s}^{-1}(q)d\mu(q) \leq \mu(E) \tag{3.15}$$

where $\mathcal{N}_{p,s}(q) := \mathcal{N}(p,q,s)$ and $\kappa = K/(N-1)$ as before.

In the special case $K = 0$, the coefficient $\frac{\tau_{\kappa,N}(sd(p,q))}{\tau_{\kappa,N}(d(p,q))}$ is simply s^N and the estimate becomes

$$s^N\mu(\mathcal{N}_{p,s}^{-1}(E)) \leq \mu(E).$$

The following proposition proved in [89] and [105] is the main property that is expected for a synthetic Ricci curvature bound for metric spaces: the coherence with the Riemannian case.

Proposition 3.4.4. *Let M be a Riemannian manifold of dimension n. Then $MCP(K, n)$ holds for this manifold with its Riemannian volume if and only if the Ricci curvature of M is uniformly greater than K on M.*

Let us enunciate some spaces with a MCP. The next theorems are simply the consequence of contraction estimates that we have stated before in this report.

Theorem 3.4.5. *The measure contraction property $MCP(K, N)$ holds in $\mathbb{H}_n$ if and only if $N \geq 2n + 3$ and $K \leq 0$.*

Proof. It is proved in [89, 105] that $MCP(0, N)$ implies $MCP(K, N)$ for any negative K and this result can also be proved directly from the definition. Furthermore in these papers is proved that spaces satisfying $MCP(K, N)$ for a non-negative K are bounded. It is not the case of the Heisenberg group. From there the theorem is a direct consequence of Theorem 1.7.7. □

As we explained in the introduction of this section, the measure contraction property implicates a weak local Poincaré inequality as in [57]. For a proof see [78, 110]. These proofs essentially use the same approach as in Subsection 1.7.3 where we proved the Poincaré inequality for the Heisenberg group.

Theorem 3.4.6. *The measure contraction property $MCP(0, N)$ holds in G for some $N \geq 1$.*

Proof. This a direct consequence of the definition of MCP and of the results of 1.7.2. □

Theorem 3.4.7. *The measure contraction property $MCP((N-1)\kappa, N)$ holds for any Alexandrov space of curvature $\geq \kappa$.*

Proof. This a direct consequence of the estimate of Kuwae and Shioya in Proposition 3.2.8. □

Theorem 3.4.7 is very comforting because it exactly correspond to the Riemannian relations between the different curvatures. As explained in Section 3.1 any Riemannian manifold of dimension n with sectional curvature greater than κ has Ricci curvature greater than $(n-1)\kappa$. The previous theorem is the right counterpart of it for metric geometry because Alexandrov spaces are considered as spaces with a lower bound on the sectional curvature. This result strengthens the interpretation of MCP as a synthetic Ricci curvature bound.

3.5 The Curvature-Dimension $CD(K, N)$

The definition of this condition has been given in Section 3.4. Here we will examine it for the Heisenberg group and the Grušin plane. They are some reason to think that this condition could hold in these spaces and some other to think that it does not for any K and N. Before proving in Theorem 3.5.12 and Theorem 3.5.13 that the second alternative is true, we expose these arguments.

3.5.1 Arguments for $CD(K,N)$

The first argument is that CD and MCP are the same type of properties based on measures or sets displaced along geodesic. If MCP is true, one can reasonably suppose the CD holds too as it is the case for Riemannian manifolds.

Proposition 3.5.1. *Let M be a Riemannian manifold of dimension n with its Riemannian volume. Then the following statements are equivalent*

(i) The Ricci curvature is uniformly bounded below by K,

(ii) the Measure contraction property $MCP(K,n)$ holds,

(iii) the Curvature-Dimension condition $CD(K,n)$ holds.

Moreover for $N \geq 1$, the two further statements are equivalent

(i') The Ricci curvature is uniformly bounded below by K and $N \geq n$,

(ii') the Curvature-Dimension condition $CD(K,N)$ holds.

The second argument is a sort of continuation of the first one. We restrict for a while our question to $CD(0,N)$ on $\mathbb{H}_n$ and will consider the geodesics of Examples 2.2.1 and 2.2.3. Remind that in Example 2.2.1 μ_1 is a Dirac measure. Without loss of generality we assume that $\mu_1 = \delta_{0_\mathbb{H}}$. We may suppose moreover that μ_0 is an absolutely continuous measure of the Wasserstein space $\mathcal{P}_2(\mathbb{H}_n)$ and denote by $(\mu_s)_{s\in[0,1]}$ the unique geodesic between the two measures. Although μ_1 is not absolutely continuous, in a space with $CD(K,N)$ the functional Ent_N is supposed to be convex on each $[0,s_1]$ for $s_1 < 1$ because on such segments, the extremities are absolutely continuous and the geodesic is unique. Let ρ_s be the density of μ_s and T_s the optimal transport map between μ_0 and μ_s. Then

$$\begin{aligned}
\mathrm{Ent}_{2n+3}(\rho_s \mid \mathcal{L}) &= -\int_{\mathcal{M}^s_{0_\mathbb{H}}(\mathbb{H}_1)} \rho_s^{1-1/(2n+3)}(y)\,dq \\
&= -\int_{\mathbb{H}_1} (\rho_s \circ T_s)^{1-1/5}(p)\,\mathrm{Jac}(T_s)(x)\,dp \\
&= -\int \left(\rho_s \circ T_s\,\mathrm{Jac}(T_s)\right)^{1-1/(2n+3)}\mathrm{Jac}(T_s)^{1/(2n+3)} \\
&= -\int \rho_0^{1-1/(2n+3)}(\mathrm{Jac}(T_s))^{1/(2n+3)}.
\end{aligned}$$

But T_s is μ_0 almost everywhere $\mathcal{M}^s_{0_\mathbb{H}}$ and we already computed the Jacobian determinant of this map in the proof of Theorem 1.7.7. We proved that it is concave in Lemma 1.7.8. It follows that Ent_{2n+3} is convex along $(\mu_s)_{s\in[0,1[}$. The concavity of the contraction map is actually a stronger property than MCP. Notice that the previous computation can be made for any geodesic $(\mu_s)_{s\in[0,1]}$ with optimal transport maps T_s. It shows that $CD(0,N)$ is related to the $1/N$-concavity of $s \to \mathrm{Jac}(T_s)$ for the geodesics.

The relative entropy Ent_{2n+3} is also convex along the geodesics of the type presented in Example 2.2.3. Surprisingly the property even holds for Ent_{2n}. Here $\mu_0^\mathbb{C}$ and $\mu_1^\mathbb{C}$ are two absolutely continuous measures of $\mathcal{P}_2(\mathbb{C}^n)$. Let $T^\mathbb{C}$ be the optimal transport map between $\mu_0^\mathbb{C}$ and $\mu_1^\mathbb{C}$ and $(\mu_s^\mathbb{C})$ the unique geodesic between them with optimal transport map $T_s^{\mathbb{C}}$. Then for an absolutely continuous measure $\mu_0 \in \mathcal{P}_2(\mathbb{H}_n)$ such that $Z_{\#}\mu_0 = \mu_0^{\mathbb{C}}$, we have seen that $T_s(p) = p \cdot \exp^{\mathbb{H}}(T_s^{\mathbb{C}}(p) - p, 0) =$ is an optimal transport map between μ_0 and $\mu_1 = T_{\#}\mu_0$. But in $\mathbb{H}_n$ seen as $\mathbb{C}^n \times \mathbb{R}$,

$$T_s(z;t) = (T_s^{\mathbb{C}}(z), t - \frac{1}{2}\sum_{k=1}^{n} z_k \overline{(T_s^{\mathbb{C}}(z))_k - z_k}).$$

In the Euclidean case the optimal transport map is $\mu_0^{\mathbb{C}}$ almost everywhere differentiable (see [108]). Because of the previous relation it also holds for T_s and μ_0-almost surely, $\mathrm{Jac}(T_s)(z;t) = \mathrm{Jac}(T_s^{\mathbb{C}})(z)$. The Euclidean space $\mathbb{C}^n$ satisfies $CD(0, 2n)$ because it is a manifold with curvature 0 so that $s \to \mathrm{Jac}(T_s^{\mathbb{C}})(z)$ is $1/(2n)$-convex (see last paragraph). So $s \to \mathrm{Jac}(T_s(z)$ is also $1/(2n)$-convex and we conclude that Ent_{2n} is convex along all the lifted geodesics.

We now mention that some spaces satisfying the $CD(K, N)$ condition satisfy a logarithmic Sobolev inequality (log-Sobolev inequality) and that it is also the case of $\mathbb{H}_1$ and G. A definition for log-Sobolev inequalities in metric spaces can be found in [109, Chapter 30]. For the Heisenberg group $\mathbb{H}_1$ with a measure reference ν the log-Sobolev inequality is satisfied if for any smooth non-negative function f with $\int f^2 d\nu = 1$, the inequality

$$\int f^2 \ln(f^2) d\nu \leq C \int \|\nabla_{\mathbb{H}} f\|_{\mathbb{H}}^2 d\nu$$

is satisfied for a fixed constant C.

Proposition 3.5.2. *Let (X, d, ν) be a space as in Standard Hypothesis 3.4.1 such that $CD(K, N)$ is satisfied for a non-negative K, then a log-Sobolev inequality holds in X.*

Li recently proved in [75] (see also [32])

Theorem 3.5.3. *Let f a smooth function of $\mathbb{H}_1$ with compact support. Then there is constant C such that*

$$P_s \|\nabla_{\mathbb{H}} f\|_{\mathbb{H}} \leq C \|\nabla_{\mathbb{H}}(P_s f)\|_{\mathbb{H}}$$

for any s and any point of $\mathbb{H}_1$. Here P_s is the subelliptic heat semigroup obtained thanks to the convolution $*_{\mathbb{H}}\mathfrak{h}_s$ as in Subsection 3.3.2 and with $\mathfrak{h}_s = \mathfrak{h}(s, \cdot)$.

As a consequence, a log-Sobolev inequality holds in $\mathbb{H}_n$ with the measure $\mathfrak{h}_1 = e^{-V_{\mathbb{H}}} d\mathcal{L}$ defined in Subsection 3.3.2.

From Theorem 3.5.3 follows a corollary on the Grušin plane G. See Section 1.2 for the notations.

Corollary 3.5.4. *The following log-Sobolev inequality holds for G*

$$\int f^2 \ln(f^2) d\nu_G \leq C \int \|\nabla_G f\|_G^2 d\mathcal{L}_r$$

where f is any smooth non negative function with $\int f^2 d\nu_G = 1$, the measure ν_G is specified in Remark 3.5.5 and C is independent of the function f.

Remind from Section 1.2 that G is the plane $\mathbb{R}^2$ with coordinates (r,t) equipped with the subRiemannian frame $(\partial_r, \frac{r}{2}\partial_t)$. The subRiemannian gradient of $\mathbb{H}_1$ can be decomposed on the horizontal polar frame $(\mathbf{\Theta}, \mathbf{R})$ defined in Subsection 1.1.1.

$$\nabla_{\mathbb{H}} f = (\mathbf{X}f)\mathbf{X} + (\mathbf{Y}f)\mathbf{Y} = (\mathbf{R}f)\mathbf{R} + (\mathbf{\Theta}f)\mathbf{\Theta}$$

Its squared norm is

$$\|\nabla_{\mathbb{H}} f\|_{\mathbb{H}}^2 = (\mathbf{X}f)^2 + (\mathbf{Y}f)^2 = (\mathbf{R}f)^2 + (\mathbf{\Theta}f)^2.$$

In the Grušin plane with a given measure ν_G, let f be a function as in the statement of the corollary. The log-Sobolev inequality is

$$\int f^2 \ln(f^2) \leq C \int \|\nabla_G f\|_G^2$$

where

$$\nabla_G f = (\partial_r f)\partial_r + (\frac{r}{2}\partial_t f)\frac{r}{2}\partial_t$$

and

$$\|\nabla_G f\|^2 = (\partial_r f)^2 + (\frac{r}{2}\partial_t f)^2.$$

We remind that Υ is the map $(x,y,t) \mapsto (\sqrt{x^2+y^2}, t)$ from $\mathbb{H}_1 \backslash L$ onto the half Grušin plan G^{+*}. Then

$$D\Upsilon(x,y,t).(\mathbf{R}) = \partial_r(r,t)$$
$$D\Upsilon(x,y,t).(\mathbf{\Theta}) = \frac{r}{2}\partial_t(r,t)$$

where $r = \sqrt{x^2+y^2}$. As we noticed in Subsection 1.6.3, Υ preserves the length of the horizontal curves in $\mathbb{H}_1$ because the lengths can be calculated thanks to the "orthonormal" frames $(\mathbf{R}, \mathbf{\Theta})$ in the Heisenberg group and equivalently with $(\partial_r, \frac{r}{2}\partial_t)$ of the Grušin plane for the projected curve. Another consequence is that for a given curve γ of the half Grušin plane G^{+*}, there is a unique horizontal lift of this curve in the Heisenberg group.

Let now f_G be a function on G. We assume it is smooth non negative and

$$\int_{G^{+*}} f_G^2(x) d\Upsilon_{\#}(e^{-V_{\mathbb{H}}}\mathcal{L})(x) + \int_{G^{-*}} f_G^2(x) d\Upsilon'_{\#}(e^{-V_{\mathbb{H}}}\mathcal{L})(x) = 2$$

where $\Upsilon'(x,y,t) := (-r,t)$ maps on the left half Grušin plane G^{-*}. Let ν_G be the measure $\frac{\Upsilon_{\#}(e^{-V_{\mathbb{H}}}\mathcal{L}) + \Upsilon'_{\#}(e^{-V_{\mathbb{H}}}\mathcal{L})}{2}$ on G. We prove now the log-Sobolev inequality with this measure.

Let f_+ and f_- be two functions of $\mathbb{H}_1$ defined with a cylindrical symmetry by $f_1(x,y,t) = f_G(\sqrt{x^2+y^2}, t)$ and $f_2(x,y,t) = f_G(-\sqrt{x^2+y^2}, t)$. Although these functions are not smooth on L, the log-Sobolev inequality can be applied for their normalized form $f_+/\sqrt{\int_{G^{+*}} f_+^2}$ and $f_-/\sqrt{\int_{G^{-*}} f_-^2}$ by using approximation arguments. Thus

$$\begin{aligned}\int_{G^{+*}} f_G^2 \ln(f_G^2) d\Upsilon_\#(e^{-V_{\mathbb{H}}}\mathcal{L}) &= \int f_+^2 \ln(f_+^2) d(e^{-V_{\mathbb{H}}}\mathcal{L}) \\ &\le C \int \|\nabla_{\mathbb{H}_1} f_+\|_{\mathbb{H}_1}^2 d(e^{-V_{\mathbb{H}}}\mathcal{L}) + \int_{\mathbb{H}_1} f_+^2 \ln\left(\int_{\mathbb{H}_1} f_+^2\right) \\ &= C \int \|\nabla_G f_G\|_G^2 d\Upsilon_\#(e^{-V_{\mathbb{H}}}\mathcal{L}) + \int_{\mathbb{H}_1} f_+^2 \ln\left(\int_{\mathbb{H}_1} f_+^2\right).\end{aligned}$$

It is true because $\mathbf{R}f_+(x,y,t) = \partial_r f_G(r,t)$ and $\mathbf{\Theta} f_+ = 0 + \frac{r}{2}\partial_t f_+ = \frac{r}{2}\partial_t f_G$. In the same way we obtain

$$\int_{G^{-*}} f_G^2 \ln(f_G^2) d\Upsilon'_\#(e^{-V_{\mathbb{H}}}\mathcal{L}) \le C \int \|\nabla_G f_G\|_G^2 d\Upsilon'_\#(e^{-V_{\mathbb{H}}}\mathcal{L}) + \int_{\mathbb{H}_1} f_-^2 \ln\left(\int_{\mathbb{H}_1} f_-^2\right)$$

and finally

$$\int f_G^2 \ln(f_G^2) d\nu_G \le C \int \|\nabla_G f_G\|_G^2 d\nu_G.$$

It means that we have for G and ν_G a log-Sobolev inequality with the same constant as for $\mathbb{H}_1$ and $e^{-V_{\mathbb{H}}}\mathcal{L}$.

Remark 3.5.5. The measure ν_G of Corollary 3.5.4 is symmetric with respect to $\{r = 0\}$. The measure on the right side is the law at time 1 of the coupled process (R, L) where R and L are respectively the Lévy area and the Bessel process associated to a 2-dimensional Brownian motion (up to a time scaling constant). Actually we have seen in Subsection 3.3.2 that $e^{-V_{\mathbb{H}}} d\mathcal{L}$ is the law of the coupling of a Brownian motion and its Lévy area at time 1. The 2-dimensional Bessel process is the norm of a two dimensional Brownian motion.

Hence one can consider the log-Sobolev inequalities on $\mathbb{H}_1$ (see [32] for the greater dimensions) and G as a positive evidence for the synthetic Ricci curvature $CD(K,N)$ in these spaces (with the modified reference measures $e^{-V_{\mathbb{H}}}$ or ν_G).

3.5.2 Arguments against $CD(K,N)$

Contrarily to MCP, the synthetic Ricci curvature CD is well-adapted to the theory of Bakry-Émery as shows the following proposition. See Section 3.3 for the notations.

Proposition 3.5.6. *Let (M,g) be a Riemannian manifold of dimension n with the measure $e^{-V}\operatorname{vol}_g$ where V is smooth . Let $L = \Delta - \langle \nabla_{\mathbb{H}} V \mid \cdot\rangle$ and consider the operators Γ_1 and Γ_2 Then for $N \ge 1$ and $K \in \mathbb{R}$ the two statements are equivalent*

- *(i) For any smooth function with compact support*

$$\Gamma_2(f) \ge K\|\nabla_g f\|^2 - \frac{1}{N}(Lf)^2.$$

- *(ii) the metric measure space $(M, d_g, \operatorname{vol}_g)$ satisfies $CD(K,N)$.*

The Heisenberg group is certainly not a Riemannian manifold but the formalism of Bakry-émery make sense on it. We have proved in Section 3.3 that the criterion of Bakry-Émery ((i) in the proposition) is not satisfied. Although the relation between the Bakry-Émery and the synthetic Ricci curvature theories is not established for general metric measure spaces, this remark tend to prove that no CD condition holds in $\mathbb{H}_n$.

Approximation of $\mathbb{H}_n$ by $\mathbb{H}_n^\varepsilon$.

A great advantage of CD is the stability under convergence. Basically for a certain topology on metric measure spaces introduced by Gromov (see [50]), the limit of a sequence of metric measure spaces with a synthetic curvature bound CD satisfies CD too. This gives a precise sense to the computation we have made in Subsection 3.1.1 where we observed that approximating manifolds $\mathbb{H}_n^\varepsilon$ have a Ricci curvature lower bounds tending to $-\infty$ when ε goes to 0. For this argument again CD in the Heisenberg groups we will present the distance $\mathbb{D}$ introduced by Sturm in [104]. We will then show that the approximating manifolds converge to $\mathbb{H}_n$ in this sense. But before that we state the convergence theorem.

They are different versions of this result depending on the authors (Lott and Villani or Sturm) and on the exact definition of CD. It is not really a problem because we will not need to apply any of these theorems. We reproduct here Theorem 3.1 of Sturm in [105]. For other related results, see [109, Chapter 29]

Theorem 3.5.7. *Let $((M_i, d_i, \nu_i))_{i\in\mathbb{N}}$ be a sequence of normalized metric measure spaces, where for each $i \in \mathbb{N}$ the space (M_i, d_i, ν_i) satisfies the curvature-dimension condition $CD(K_i, N_i)$ and has diameter $\leq L_i$. Assume that, as $i \to +\infty$,*

$$(M_i, d_i, \nu_i) \to (M, d, \nu)$$

for the $\mathbb{D}$ distance and $(K_i, N_i, L_i) \to (K, N, L)$ for some $(K, N, L) \in \mathbb{R}^3$ satisfying $KL^2 < (N-1)\pi^2$. Then the space (M, d, ν) satisfies the curvature-dimension condition $CD(K, N)$ and has diameter $\leq L$.

As the previous theorem happens for bounded spaces we will prove in Proposition 3.5.8 the convergence of the approximating Albanese torus $\mathbb{T}^\varepsilon$ to $\mathbb{T}$. These torus $\mathbb{T}^\varepsilon$ satisfy $CD(-\frac{1}{2\varepsilon^2}, 3)$ which would imply that $\mathbb{T}$ satisfy $CD(-\infty, 3)$. But this property does not exist (alternatively is satisfied by any space). For sake of completeness and although Theorem 3.5.7 does not apply to unbounded spaces, we will also prove the convergence for the Heisenberg group (see Proposition 3.5.9).

We now define now the distance $\mathbb{D}$. Let (X, d, ν) and (X', d', ν') be two metric measure spaces. We assume that they are bounded. Then the distance $\mathbb{D}$ between them is defined by

$$\mathbb{D}\left((X, d, \nu), (X', d', \nu')\right) = \inf_{(Z, d_Z)} W(\nu^Z, \nu'^Z)$$

where (X, d) and (X', d') are isometrically embedded in (Z, d_Z) and $W(\nu^Z, \nu'^Z)$ stands for the Wassertein distance of $\mathcal{P}_2(Z)$ between the embedded measures obtained as push-forward of ν and ν'.

Inspired by the ideas of Gromov exposed in his book [50], in [104, Theorem 3.16], Sturm proved that the set of compact metric probability measure spaces with an uniform bound on the diameter and a common doubling constant is a compact set for $\mathbb{D}$. In particular it is complete: every Cauchy sequence of these metric spaces has a limit.

Recall that we defined the Albanese torus $(\mathbb{T}, d_{\mathbb{T}})$ and the approximating manifolds $(\mathbb{T}^\varepsilon, d_{\mathbb{T}^\varepsilon})$ in Section 1.2. Both are compact. A fundamental domain for these space is $[0,1[^3$. As reference measure, we take $\mathcal{L}^3$ on $[0,1[^3$ because the Lebesgue measure is the Haar measure of $\mathbb{H}_1$ and $\mathcal{L}^3([0,1[^3) = 1$.

Proposition 3.5.8. *For any $\varepsilon > 0$ we have*

$$\mathbb{D}(\mathbb{T}^\varepsilon, \mathbb{T}) \leq \pi\varepsilon$$

where the spaces are taken with their usual distances and the Lebesgue measure.

Before the proof we make a comparison of the distances of $\mathbb{H}_1$ and $\mathbb{H}_1^\varepsilon$. From the definition in Subsection 1.2.6 we get $d_\varepsilon \leq d_c$. The second estimate use the fact established in Section 1.6.4 that for $(v, \varphi) \in \mathbb{C} \times [-2\pi, 2\pi]$ the following "Pythagorean" equality holds:

$$d_\varepsilon(0_{\mathbb{H}}, \exp^{\mathbb{H}}(v,\varphi) \cdot (0_{\mathbb{H}}; \varphi\varepsilon^2))^2 = d_c(0_{\mathbb{H}}, \exp^{\mathbb{H}}(v,\varphi))^2 + d_\varepsilon(0_{\mathbb{H}}, (0; \varphi\varepsilon^2))^2.$$

But

$$d_\varepsilon(0_{\mathbb{H}}, \exp^{\mathbb{H}}(v,\varphi) \cdot (0; \varphi\varepsilon^2)) \leq d_\varepsilon(0_{\mathbb{H}}, \exp^{\mathbb{H}}(v,\varphi)) + d_\varepsilon(0_{\mathbb{H}}, (0; \varphi\varepsilon^2))$$

so

$$d_c(0_{\mathbb{H}}, \exp^{\mathbb{H}}(v,\varphi))^2 \leq d_\varepsilon(0_{\mathbb{H}}, \exp^{\mathbb{H}}(v,\varphi)^2 + 2 d_\varepsilon(0_{\mathbb{H}}, \exp^{\mathbb{H}}(v,\varphi)) d_\varepsilon(0_{\mathbb{H}}, (0; \varphi\varepsilon^2)$$

But $d_\varepsilon(0_{\mathbb{H}}, (0; \varphi\varepsilon^2)$ is $\varepsilon|\varphi|$ and one get the following estimate that is independent from the coordinates

$$d_c \leq \sqrt{d_\varepsilon(d_\varepsilon + 2\pi\varepsilon)}.$$

The two previous estimates are also available for $\mathbb{H}_n$ and $\mathbb{H}_n^\varepsilon$. It is also the case for the distances $d_{\mathbb{T}}$ and $d_{\mathbb{T}^\varepsilon}$ on the Albanese torus $\mathbb{T}$ and its approximating manifold $\mathbb{T}^\varepsilon$. We now give the proof of Proposition 3.5.8

Proof. We have $d_{\mathbb{T}^\varepsilon} \leq d_{\mathbb{T}} \leq d_{\mathbb{T}^\varepsilon} \sqrt{1 + \frac{2\pi\varepsilon}{d_{\mathbb{T}^\varepsilon}}} \leq d_{\mathbb{T}^\varepsilon} + \pi\varepsilon$. Then we define a distance d^Z on $Z = \mathbb{T} \sqcup \mathbb{T}^\varepsilon$ as follow

$$d^Z(p,q) = \begin{cases} d_{\mathbb{T}}(p,q) & \text{if } (p,q) \in \mathbb{T} \times \mathbb{T} \\ d_{\mathbb{T}^\varepsilon}(p,q) & \text{if } (p,q) \in \mathbb{T}^\varepsilon \times \mathbb{T}^\varepsilon \\ d_{\mathbb{T}^\varepsilon}(p',q) + \pi\varepsilon & \text{if } (p,q) \in \mathbb{T} \times \mathbb{T}^\varepsilon \\ d_{\mathbb{T}^\varepsilon}(p,q') + \pi\varepsilon & \text{if } (p,q) \in \mathbb{T}^\varepsilon \times \mathbb{T} \end{cases}$$

where p' and q' are the copies in $\mathbb{T}^\varepsilon$ of $p \in \mathbb{T}$ and $q \in \mathbb{T}$ respectively. We check that the function d^Z is a distance of Z. The triangle inequality $d^Z(p,q) \leq d^Z(p,m) + d^Z(m,q)$ is the only difficult point. It holds for points p, m and q all in $\mathbb{T}$ or all in $\mathbb{T}^\varepsilon$. We have to see that it holds if m or q are in another part of Z than the two other points. If $p, q \in \mathbb{T}$ and $m \in \mathbb{T}^\varepsilon$ and m'' is the point corresponding to m in $\mathbb{T}$,

$$\begin{aligned} d^Z(p,q) = d_{\mathbb{T}}(p,q) &\leq d_{\mathbb{T}}(p,m'') + d_{\mathbb{T}}(m'',q) \\ &\leq (d_{\mathbb{T}^\varepsilon}(p',m) + \pi\varepsilon) + (d_{\mathbb{T}^\varepsilon}(m,q') + \pi\varepsilon) \\ &\leq d^Z(p,m) + d^Z(m,q). \end{aligned}$$

he proof is easier if $p, q \in \mathbb{T}^\varepsilon$ and $m \in \mathbb{T}$. Let then m' the point corresponding to m in $\mathbb{T}^\varepsilon$. Then

$$d^Z(p,q) = d_{\mathbb{T}^\varepsilon}(p,q) \leq d_{\mathbb{T}^\varepsilon}(p,m') + d_{\mathbb{T}^\varepsilon}(m',q) \leq d^Z(p,m) + d^Z(m,q).$$

We suppose now $p, m \in \mathbb{T}$ and $q \in \mathbb{T}^\varepsilon$. Then

$$\begin{aligned} d^Z(p,q) = d_{\mathbb{T}^\varepsilon}(p',q) + \pi\varepsilon &\leq d_{\mathbb{T}^\varepsilon}(p',m') + d_{\mathbb{T}^\varepsilon}(m',q) + \pi\varepsilon \\ &\leq d_{\mathbb{T}}(p,m) + d_{\mathbb{T}^\varepsilon}(m',q) + \pi\varepsilon \\ &\leq d^Z(p,m) + d^Z(m,q). \end{aligned}$$

If $p, m \in \mathbb{T}^\varepsilon$ and $q \in \mathbb{T}$, the proof is easier. Let q' be the point corresponding to q in $\mathbb{T}^\varepsilon$. Then

$$\begin{aligned} d^Z(p,q) \leq d_{\mathbb{T}^\varepsilon}(p,q') + \pi\varepsilon &\leq d_{\mathbb{T}^\varepsilon}(p,m) + d_{\mathbb{T}^\varepsilon}(m,q') + \pi\varepsilon \\ &\leq d^Z(p,m) + d^Z(m,q) \end{aligned}$$

We take now the trivial deterministic transport plan $p \to p'$ between $\mathbb{T}$ and $\mathbb{T}^\varepsilon$. This is the best coupling for this (Z, d^Z) because $\pi\varepsilon = d^Z(p,p')$ is the shortest distance between two points of $\mathbb{T}$ and $\mathbb{T}^\varepsilon$. Then

$$\mathbb{D}(\mathbb{T}, \mathbb{T}^\varepsilon) \leq \sqrt{\int (\pi\varepsilon)^2} = \pi\varepsilon.$$

□

In this context of a distance between metric spaces with probability measure, it can be useful to change the usual Haar measure of the group $\mathbb{H}_n$ for the diffusion probability measures $\mathfrak{h}_1 = e^{-V_{\mathbb{H}}}$ defined in Subsection 3.3.2. Indeed $\mathbb{D}$ is only defined between spaces with a probability measures. This change of measure enables to compare $\mathbb{H}_n$ and the $\mathbb{H}_n^\varepsilon$ directly without considering the quotient torus $\mathbb{T}$ and $\mathbb{T}^\varepsilon$. The distance $\mathbb{D}$ makes still sense for non-compact metric spaces but it is a distance that takes infinite values.

Proposition 3.5.9. *For any $\varepsilon > 0$, we have*

$$\mathbb{D}((\mathbb{H}_n^\varepsilon, \mathfrak{h}_1^\varepsilon \mathcal{L}), (\mathbb{H}_n, \mathfrak{h}_1 \mathcal{L})) \leq \sqrt{8\varepsilon\sqrt{\pi}} + \pi\varepsilon$$

where $\mathbb{H}_n$ and $\mathbb{H}_n^\varepsilon$ have the Carnot-Carathéodory distance d_c and the approximating distance d_ε.

Proof. We introduce an intermediate space $(\mathbb{H}_n, \mathfrak{h}_1^\varepsilon)$ made of the Heisenberg group with d_c and $\mathfrak{h}_1^\varepsilon$ the diffusion distribution of $\mathbb{H}^\varepsilon$ at time 1. First this space is close to $(\mathbb{H}_n, \mathfrak{h}_1)$. Indeed, e embed the two Heisenberg groups into themselves with the identity map and we have to estimate a classical Wasserstein distance of two measures $\mathfrak{h}_1$ and $\mathfrak{h}_1^\varepsilon$. We noticed that $\mathfrak{h}_1^\varepsilon$ is the law of $X \cdot U^\varepsilon$ where X and U^ε are two independent random variables, the law of X_1 is $\mathfrak{h}_1$ and the one of U^ε is u^ε described in Subsection 3.13. Then the law of $(X, X \cdot U^\varepsilon)$ is a coupling of $\mathfrak{h}_1$ and $\mathfrak{h}_1^\varepsilon$. The cost related to this coupling is

$$\sqrt{\mathbb{E}\left[d_c(0, U^\varepsilon)^2\right]} = \sqrt{8\varepsilon\sqrt{\pi}}.$$

We now estimate the second part. Exactly as in Proposition 3.5.8

$$\mathbb{D}((\mathbb{H}_n^\varepsilon, \mathfrak{h}_1^\varepsilon), (\mathbb{H}_n, \mathfrak{h}_1^\varepsilon)) \leq \pi\varepsilon$$

and we have proved the proposition. □

3.5.3 The generalized Brunn-Minkowski inequalities, Failure of CD in $\mathbb{H}_n$

The classical Brunn-Minkowski inequality in $\mathbb{R}^n$ (see [39, 3.2.41] for instance) is a very useful geometric lower bound on the measure of the Minkowski sum (i.e the usual sum of two sets in $\mathbb{R}^n$) of two compact sets in $\mathbb{R}^n$. This inequality is equivalent to the following statement: given two compact sets K_0 and K_1, in $\mathbb{R}^n$ and $s \in [0,1]$ then

$$(\mathcal{L}^n)^{1/n}(sK_1 + (1-s)K_0) \geq s(\mathcal{L}^n)^{1/n}(K_1) + (1-s)(\mathcal{L}^n)^{1/n}(K_0) \qquad (3.16)$$

with $sK_1 + (1-s)K_0 = \{sk_1 + (1-s)k_0 \in \mathbb{R}^n \mid k_1 \in K_1 \quad k_0 \in K_0\}$. The generalization of $sK_1 + (1-s)K_0$ to geodesic metric space use the geodesics from K_0 to K_1. We consider the set of the s-intermediate points from a point k_0 in K_0 to a point k_1 in K_1. We call this set the s-intermediate set and denote it by "$sK_1 + (1-s)K_0$". The s-intermediate points were defined in the begining of Section 1.5.

Let (X, d, μ) be a geodesic metric measure space and N be greater than 1. We say that the generalized Brunn-Minkowski inequality $BM(0,N)$ holds in (X, d, μ) if the inequality

$$\mu^{1/N}(\text{“}sK_1 + (1-s)K_0\text{”}) \geq s\mu^{1/N}(K_1) + (1-s)\mu^{1/N}(K_0) \qquad (3.17)$$

is true for every pair (K_0, K_1) of compact sets of non-zero measure (where $\mu(\text{“}sK_1 + (1-s)K_0\text{”})$ will denote the outer measure of "$sK_1 + (1-s)K_0$" if the latter is not measurable).

The following statement is a consequence of [105, Proposition 2.1, Theorem 5.4].

Proposition 3.5.10. *In a metric measure space (X, d, ν) satisfying Standard Hypothesis 3.4.1, the two following implications hold*

$$CD(0,N) \Rightarrow BM(0,N) \Rightarrow MCP(0,N).$$

Therefore in order to prove that $CD(0,N)$ does not hold in $\mathbb{H}_n$, it is enough to prove that no Brunn-Minkowski inequality holds in this space. That is what we will do

In $\mathbb{H}_n$ it is useful to interpret the s-intermediate set using the intermediate-points map $\mathcal{M}$. Suppose that K_0 and K_1 are two compact sets such that $K \times K_1 \subset U$. We recall that $U = \{(p,q) \in \mathbb{H}_n \mid p^{-1} \cdot q \notin L\}$. Then "$sK_1 + (1-s)K_0$" is simply

$$\mathcal{M}^s(K_0, K_1) = \{\mathcal{M}^s(p,q) \in \mathbb{H}_n \mid (p,q) \in K_0 \times K_1\}.$$

Lemma 3.5.11. *There are two compact sets K and K' of $\mathbb{H}_n$ such that*

$$\mathcal{L}^{2n+1}(K) = \mathcal{L}^{2n+1}(K') > \mathcal{L}^{2n+1}(\mathcal{M}^{1/2}(K, K')).$$

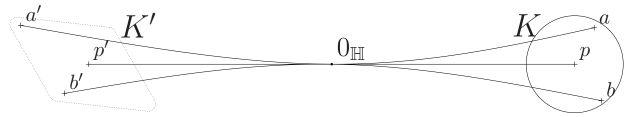

Figure 3.1: Steps of an heuristic proof–1

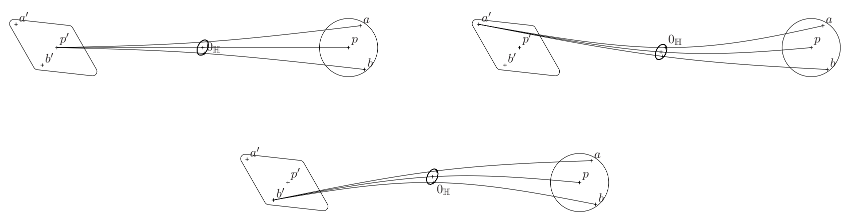

Figure 3.2: Steps of an heuristic proof–2

Let N be a dimension greater than 1. We can raise the inequality in Lemma 3.5.11 to the power $1/N$ and using (3.17) we obtain as a corollary the following theorem.

Theorem 3.5.12. *The generalized Brunn-Minkowski inequality $BM(0, N)$ and the curvature-dimension $CD(0, N)$ do not hold for any N.*

We now give a proof of Lemma 3.5.11.

Proof. On Figures 3.1 3.2 and 3.3 are schemed the different steps of the proof. First on Figure 3.1 one can see the construction of the sets: K is a small ball and its geodesic inverse K' has the same size. Then on Figure 3.2 appears the fact that the contracted sets with different contraction centers in K' look like an ellipsoid containing $0_{\mathbb{H}}$. Finally on Figure 3.3 is represented the midset $\mathcal{M}^{1/2}(K', K)$ as the reunion of these contracted sets. It looks like one of these ellipsoid but with double size.

Let us start the rigorous proof and firstly consider a part of $\mathbb{H}$-line of $\mathbb{H}_n$: the curve of parameter $((x, \cdots, 0), 0)$ on the interval $x \in [-1, 1]$. On the $\mathbb{H}$-lines the φ parameter is 0 such that $p' = (-1, 0, \cdots, 0)$ and $p = (1, 0, \cdots, 0)$ is a pair of $\mathcal{I}$-conjugate point (the geodesic-inversion $\mathcal{I}$ is defined in Section 1.7.1). On U the midpoint map $\mathcal{M}^{1/2}$ is single and smooth as explained in Lemma 1.7.2. We recall that Lemma 1.7.4 exactly tells us when the midpoint of two $\mathcal{I}$-conjugate points is $0_{\mathbb{H}}$: it is the case for p and p' and for any pair of $\mathcal{I}$-conjugate points with one element in $\exp^{\mathbb{H}}(D_{1/2})$. Our counterexample consists on the one hand of a small compact ball $K_r := \mathcal{B}(p, r)$ with center p and (Euclidian) radius r and on the other hand of

$$K'_r = \mathcal{I}(K_r) = \{\mathcal{I}(a) \in \mathbb{H}_n \mid a \in K_r\}.$$

Figure 3.3: Steps of an heuristic proof–3

We then consider the set $\mathcal{M}^{1/2}(K'_r, K_r)$ of midpoints between K_r and K'_r. By continuity we can choose r small enough such that $K_r \subset \exp^{\mathbb{H}}(D_{1/2})$ and $K_r \times K'_r \subset U$. Hence because of Lemma 1.7.4, for any $a \in K_r$, the midpoint $\mathcal{M}^{1/2}(\mathcal{I}(a), a)$ is $0_{\mathbb{H}}$.

We have to show that K'_r has the same measure as K_r and this measure is greater than the measure of $\mathcal{M}^{1/2}(K'_r, K_r)$. The first claim is actually straightforward: $\exp^{\mathbb{H}}$ and $\exp^{\mathbb{H}}_{-1}$ are diffeomorphisms and have the same Jacobian determinant up to sign (Corollary 1.7.6) on $(\exp^{\mathbb{H}})^{-1}(K_r)$. Hence

$$\begin{aligned}\mathcal{L}^{2n+1}(K'_r) &= \mathcal{L}^{2n+1}(\exp^{\mathbb{H}}_{-1}((\exp^{\mathbb{H}})^{-1}(K_r))) \\ &= \mathcal{L}^{2n+1}(\exp^{\mathbb{H}}((\exp^{\mathbb{H}})^{-1}(K_r))) = \mathcal{L}^{2n+1}(K_r).\end{aligned}$$

The key to the second claim is the fact that

$$\mathcal{M}^{1/2}(K'_r, K_r) = \bigcup_{a,b\in K_r} \mathcal{M}^{1/2}(\mathcal{I}(a), b) = \bigcup_{a,b\in K_r} \mathcal{M}^{1/2}(\mathcal{I}(a), a + (b-a)). \tag{3.18}$$

The mid-set $\mathcal{M}^{1/2}(K'_r, K_r)$ shall have a small measure because each mid-point $\mathcal{M}^{1/2}(\mathcal{I}(a), a+(b-a))$ is close to $0_{\mathbb{H}} = \mathcal{M}^{1/2}(\mathcal{I}(a), a)$. We will use differentiation tools to quantify this idea. By Lemma 1.7.2, $\mathcal{M}^{1/2}$ is C^∞-differentiable on U. For any $q \in \mathbb{H}_n \backslash L$ let $\mathcal{M}^{1/2}_q$ be the map $\mathcal{M}(q, \cdot, 1/2)$. We now write

$$\begin{aligned}&\mathcal{M}^{1/2}(\mathcal{I}(a), a + (b-a)) \qquad\qquad (3.19)\\ &\quad = 0 + D\mathcal{M}^{1/2}_{\mathcal{I}(a)}(a).(b-a) \\ &\qquad + \left[\mathcal{M}^{1/2}\left(\mathcal{I}(a), a + (b-a)\right) - D\mathcal{M}^{1/2}_{\mathcal{I}(a)}(a).(b-a)\right] \\ &\quad = D\mathcal{M}^{1/2}_{p'}(p).(b-a) + \left[\left(D\mathcal{M}^{1/2}_{\mathcal{I}(a)}(a) - D\mathcal{M}^{1/2}_{p'}(p)\right).(b-a)\right] \\ &\qquad + \left[\mathcal{M}^{1/2}\left(\mathcal{I}(a), a + (b-a)\right) - D\mathcal{M}^{1/2}_{\mathcal{I}(a)}(a).(b-a)\right].\end{aligned}$$

For a and b close to p, the two last terms of the previous sum are small and can be bounded using the continuity of $D\mathcal{M}^{1/2}_{\mathcal{I}(a)}(a)$ and the Taylor development of order two of $\mathcal{M}^{1/2}(\cdot,\cdot)$ on $K'_r \times K_r$. When r tends to zero,

$$\begin{aligned}&\sup_{a,b\in K_r} \Big| \left(D\mathcal{M}^{1/2}_{\mathcal{I}(a)}(a) - D\mathcal{M}^{1/2}_{p'}(p)\right).(b-a) \\ &\quad + \mathcal{M}^{1/2}(\mathcal{I}(a), a + (b-a)) - D\mathcal{M}^{1/2}_{\mathcal{I}(a)}(a).(b-a)\Big| = o(r).\end{aligned}$$

Therefore, as $K_r - K_r = \{q \in \mathbb{R}^{2n+1} \mid q = a - b \quad a, b \in \mathcal{B}(p,r)\} = \mathcal{B}(0,2r)$, the relations (3.18) and (3.19) give the following set inclusion

$$\mathcal{M}^{1/2}(K'_r, K_r) \subset D\mathcal{M}^{1/2}_{p'}(p).(\mathcal{B}(0,2r)) + \mathcal{B}(0, \varepsilon(r) r) \tag{3.20}$$

where $\varepsilon(r)$ is a non-negative function which tends to zero when r tends to zero. We observe now that the measure of the right-hand set is equivalent to the measure of $D\mathcal{M}_{p'}^{1/2}(p).(\mathcal{B}(0,2r))$. Because of the left-invariance of the whole setting of the Heisenberg group, the contraction along a $\mathbb{H}$-line does not depend on the contraction center (Here p'). Here $\mathrm{Jac}(\mathcal{M}_{p'}^{1/2})(p)$ has the same value as $\mathrm{Jac}(\mathcal{M}_{0_{\mathbb{H}}}^{1/2}) = \mathrm{Jac}(\exp_{1/2}^{\mathbb{H}} \circ (\exp^{\mathbb{H}})^{-1})$ taken at the point $p'^{-1} \cdot p$ which is $((2,0,\cdots,0),0) = \exp^{\mathbb{H}}((2,0,\cdots,0),0)$. This Jacobian determinant was calculated on equation (1.29). On the $\mathbb{H}$-line the φ-coordinate of $\exp_{-1}^{\mathbb{H}}(p'^{-1} \cdot p)$ is 0 such that the Jacobian determinant is $s^{2n+3} = \frac{1}{2^{2n+3}}$ (the worth concentration in Theorem 1.7.7). It follows that

$$\mathcal{L}^{2n+1}(D\mathcal{M}_{p'}^{1/2}(p).(\mathcal{B}(0,2r))) = \frac{2^{2n+1}}{2^{2n+3}}\mathcal{L}^{2n+1}(\mathcal{B}(p,r)) = \frac{1}{4}\mathcal{L}^{2n+1}(K_r).$$

Hence by (3.20) and the remark that follows it, we get that

$$\mathcal{L}^{2n+1}(\mathcal{M}^{1/2}(K'_r,K_r)) \leq \frac{1}{4}\mathcal{L}^{2n+1}(K_r)(1+o(r))$$

when r tends to zero. Choosing now a small enough r, the lemma is proved. □

Extensions of Theorem 3.5.12

The same argument also prove that $CD(0,+\infty)$ is not satisfied by the Heisenberg group because this condition provides a special infinite dimensional Brunn-Minkowski inequality

$$\nu(\mathcal{M}^s(A,B)) \geq \nu(A)^{1-s} \times \nu(B)^s$$

which is false for the same sets K and K' as in Lemma 3.5.11.

For a fixed N, Lemma 3.5.11 does not only yield that $CD(0,N)$ does not hold. This also implies that $CD(K,N)$ does not hold for any $K>0$ because this condition is less demanding than $CD(0,N)$. Alternatively, spaces verifying $CD(K,N)$ with $K>0$ are bounded.

Also for any $K<0$, the curvature-dimension bound $CD(K,N)$ does not hold. We argue by contradiction. Assume that $CD(K,N)$ holds in the space $(\mathbb{H}_n, d_c, \mathcal{L}^{2n+1})$ for some $K<0$. Then the "scaled space" property from [105] tells us that $(\mathbb{H}_n, \lambda^{-1}d_c, \lambda^{-(2n+2)}\mathcal{L}^{2n+1})$ verifies $CD(\lambda^2 K, N)$ for all $\lambda>0$. But this space is exactly isomorphic to our metric measure space via the dilation dil_λ. Hence $CD(K,N)$ would hold in $(\mathbb{H}_n, d_c, \mathcal{L}^{2n+1})$ for every non-positive K. In Theorem 3.5.12 we proved that the Heisenberg group does not satisfy $CD(0,N)$. Therefore there are μ_0 and μ_1 two absolutely continuous measures of $\mathcal{P}_2(\mathbb{H}_n)$ and $(\mu_s)_{s\in[0,1]}$ the geodesic between them such that $\mathrm{Ent}_N(\mu_s) > (1-s')\,\mathrm{Ent}_N(\mu_0) + s'\,\mathrm{Ent}_N(\mu_1)$ for a fixed $s' \in]0,1[$. Because of Theorem 2.2.4 and Subsection 2.3.1 the optimal transportation map π between μ_0 and μ_1 is unique and the geodesic $(\mu_s)_{s\in[0,1]}$ is unique too. In Definition 3.4.2, the coefficient $\left(\frac{\tau_{K,N}(s'd)}{\tau_{K,N}(d)}\right)^{1/N}$ converges to s' when K goes to 0. Thus Lebesgue's theorem provides that

$$\int \left(\frac{\tau_{K,N}((1-s')d(p,q))}{\tau_{K,N}(d(p,q))}\right)^{1/N} (-\rho_0^{-1/N}(p))d\pi(p,q)$$

tends to $(1-s')\int(-\rho_0^{-1/N}(p))d\pi(p,q) = (1-s')\operatorname{Ent}_N(\mu_0)$ while

$$\int \left(\frac{\tau_{K,N}(s'd(p,q))}{\tau_{K,N}(d(p,q))}\right)^{1/N}(-\rho_1^{-1/N}(q))d\pi(p,q)$$

tends to $s'\operatorname{Ent}_N(\mu_1)$. Then from our assumption that $CD(K,N)$ holds for any $K<0$, letting K tend to 0 we get $\operatorname{Ent}_N(\mu_{s'}) \le (1-s')\operatorname{Ent}_N(\mu_0) + s'\operatorname{Ent}_N(\mu_1)$ which is a contradiction. It follows that $CD(K,N)$ does not hold in $\mathbb{H}_n$ for any K. Moreover the same argument of scaling works for $N=+\infty$ as well.

We now consider the Heisenberg group with another reference measure ν. We assume for instance that ν is a probability measure, absolutely continuous with respect to $\mathcal{L}^{2n+1}$ and with a continuous density η. Note that the diffusion measure $e^{-V_{\mathbb{H}}}\mathcal{L}^{2n+1}$ of Subsection 3.3.2 is of this type. We can still prove that $CD(0,N)$ is false. We can assume without loss of generality (up to translate) that $\eta(0_{\mathbb{H}})>0$ and we take the sets K_r and K'_r that we dilate thanks to $\operatorname{dil}_\lambda$. As η is continuous we have

$$\nu(\mathcal{M}^{1/2}(\operatorname{dil}_\lambda(K_r),\operatorname{dil}_\lambda(K'_r)) = \nu(\operatorname{dil}_\lambda(\mathcal{M}^{1/2}(K_r,K'_r)) < \nu(\operatorname{dil}_\lambda(K_r))$$

for λ small enough such that $BM(0,N)$ and $CD(0,N)$ don't hold. Moreover, $CD(K,N)$ is also not satisfied for $K\neq 0$. A way to prove it is to consider the restricted probability measures ν_A defined by $\nu_A(B)=\nu(A\cap B)/\nu(A)$ for $A=\operatorname{dil}_\lambda(K_r)$ and $A=\operatorname{dil}_\lambda(K'_r)$ and introduce the entropy of the midpoint between them. For λ small enough the inequality in Definition 3.4.2 will not hold because η is continuous and non-zero in $0_{\mathbb{H}}$ such that the metric measure space $(\mathbb{H}_n,d_c,\nu)$ is locally close to $(\mathbb{H}_n,d_c,\eta(0_{\mathbb{H}})\mathcal{L}^{2n+1})$.

The generalized Brunn-Minkowski inequality is a *geodesic* generalization (we interpreted "$(1-s)K_0+sK_1$" as the set of the s-intermediate points). An other version of the Brunn-Minkowski inequality in $\mathbb{R}^n$, equivalent to (3.16) provides a *multiplicative* Brunn-Minkowski inequality

$$\mathcal{L}^n(F+F')^{1/n} \ge \mathcal{L}^n(F)^{1/n} + \mathcal{L}^n(F')^{1/n}. \tag{3.21}$$

The method in Lemma 3.5.11 also apply to the multiplicative Brunn-Minkowski inequality of $\mathbb{H}_n$ defined replacing $F+F'$ in (3.21) by $F\cdot F' = \{a\cdot b\in\mathbb{H}^n \mid a\in F \quad b\in F'\}$. About this inequality [86], Monti proves that

$$\mathcal{L}^3(F\cdot F')^{1/4} \ge \mathcal{L}^3(F)^{1/4} + \mathcal{L}^3(F')^{1/4}$$

does not hold in $\mathbb{H}_1$ (4 is the Hausdorff dimension of $\mathbb{H}_1$) using an argument based on the non-optimality of the unit ball in the isoperimetric inequality for $\mathbb{H}_1$ (on this subject, see the book [23]). Another proof for $\mathbb{H}_n$ of Hausdorff dimension $2n+2$ is the following: Take F to be the set K_r defined above and denote by F' the set $\{b\in\mathbb{H}^n \mid \exists c\in F,\, c\cdot b = 0_{\mathbb{H}}\}$ of inverse elements (it is simply $-F$ because $(z,t)^{-1}=(-z,-t)$). Using the methods of this section we get that $F\cdot F'$ is very close to $D\operatorname{tran}_{p'}(p).(\mathcal{B}(0,2r))$. The measure of the previous set is $2^{2n+1}\mathcal{L}^{2n+1}(F)$ because, as we said just in the beginning of the thesis, $\operatorname{Jac}(\operatorname{tran}_{p'})=1$ in every point. As $\mathcal{L}^{2n+1}(F)=\mathcal{L}^{2n+1}(F')$ it follows that for r small enough

$$\mathcal{L}^{2n+1}(F\cdot F')^{\frac{1}{2n+2}} < \mathcal{L}^{2n+1}(F)^{\frac{1}{2n+2}} + \mathcal{L}^{2n+1}(F')^{\frac{1}{2n+2}} \tag{3.22}$$

and the multiplicative Brunn-Minkowski inequality is false for Hausdorff dimension (i.e. $2n+2$). In the paper by Leonardi and Masnou (see [74]), the authors show that the multiplicative Brunn-Minkowski inequality is true for the topological dimension (i.e. $2n+1$). They explain that there could be in principle a $N \in]2n+1, 2n+2[$ such that the multiplicative Brunn-Minkowski inequality holds in $\mathbb{H}^n$: in fact if this equality holds for N, then it holds for $N' < N$. We proved in (3.22) that the sets F and F' defined here are a counterexample to the multiplicative Brunn-Minkowski inequality with dimension $N = 2n+2$. They are actually also counterexamples for any $N > 2n+1$. It follows that $2n+1$ is the largest dimension for which the multiplicative Brunn-Minkowski inequality is true.

Failure of CD in the Grušin plane

It is possible to make a similar argument as in Lemma 3.5.11 for the Grušin plane G with the same consequence on the curvature-dimension condition.

Theorem 3.5.13. *The curvature-dimension $CD(K,N)$ does not hold for any $N \geq 1$ and $K \in \mathbb{R}$ in the Grušin plane.*

Proof. Just as before for the Heisenberg group, it is enough to prove that $CD(0,N)$ is not satisfied. Indeed, the Grušin plane has dilations dil^G_λ playing the same role as dil_λ for $\mathbb{H}_n$.

We want to estimate the Lebesgue measure of "$sA + (1-s)B$" for two sets A and B with a known Lebesgue measure and prove that this midset is small. We will prove that the weakest Brunn-Minkowski inequality $BM(0,+\infty)$ – corresponding to $CD(0,+\infty)$– does not hold. More precisely, we will find two sets A and B such that

$$\sqrt{\mathcal{L}^2(A)\mathcal{L}^2(B)} > \mathcal{L}^2\left(\text{“}\frac{A+B}{2}\text{”}\right)$$

where "$\frac{A+B}{2}$" is the set of the points in the middle of a geodesic from a point of A to of point of B. Here we suppose A and B compact in order to avoid measurability problems for "$\frac{A+B}{2}$". We define the map F_{-1} exactly as we did in Subsection 1.7.2 for F_q with $q \in [0,1]$. This map plays the role played by $\mathcal{I}$ in $\mathbb{H}_n$. It takes a point in the (r,t)-coordinates and maps it to the other end of a local geodesic with midpoint $(-1,0)$. More precisely

$$F_{-1}(E^{G,1}(\varphi,s)) = E^{G,1}(\varphi,-s) = E^{G,1}(\pi+\varphi,s).$$

About the minimality of the geodesics, using the geometric transformations of G and Subsection 1.7.2, it is not difficult to prove the next lemma.

Lemma 3.5.14. *Let s_0 ans s_1 be two real numbers with $s_1 - s_0 \geq 0$. Then the map $s \in [s_0, s_1] \mapsto E^{G,1}(\varphi,s)$ is a globally minimal geodesic if and only if $s_1 - s_0 \leq \frac{\pi}{|\alpha|}$ where $\alpha = \sin(\varphi)$. For $\alpha = 0$, the geodesic is always globally minimal*

Therefore for $p = E^{G,1}(\varphi,s)$ close to $(0,0)$, i.e. for (φ,s) close to $(0,1)$, there is a unique geodesic between p and $F_{-1}(p) = (-2,0)$ and the midpoint is $(-1,0)$. Let $\varepsilon > 0$ be a small parameter. We denote now the Euclidean ball with center $(-2,0)$ and radius $\varepsilon > 0$ by A and let B be $F_{-1}(A)$. Using the

Jacobian determinant of $E^{G,1}$ in Proposition 1.7.10 we find that the Jacobian determinant of F_{-1} has a norm equivalent to

$$\frac{s\varphi\frac{\varphi^2+(1-s)^2\varphi^2}{2}-\frac{(s\varphi)^3}{6}}{s\varphi\frac{\varphi^2+(1+s)^2\varphi^2}{2}-\frac{(s\varphi)^3}{6}}$$

in $E^{G,1}(\pi+\varphi,1)$ when (φ,s) tends to $(0,1)$. Therefore the measure of B is equivalent to $\frac{\mathcal{L}^2(A)}{7}$ when ε tends to zero. In a similar way like in the Heisenberg group, we obtain that "$\frac{A+B}{2}$" is included in a set whose measure is equivalent (when ε tends to 0) to the one of the following set : the set you obtain when you contract with quotient $1/2$ and center $(0,0)=F_{-1}(-2,0)$ the Euclidian ball with center $(-2,0)$ and radius 2ε. The measure of this set is equivalent to a product : $4\mathcal{L}^2(A)$ (the volume of $\mathcal{B}((-2,0),2\varepsilon)$) times $1/2^4$ (the Jacobian determinant of the contraction because (1.31) is an equality in the contraction direction we consider).

Then $\sqrt{\mathcal{L}^2(A)\mathcal{L}^2(B)}\sim\frac{\mathcal{L}^2(A)}{\sqrt{7}}$ and $\mathcal{L}^2\left(\text{“}\frac{A+B}{2}\text{”}\right)$ is smaller to a function equivalent to $2^{-2}\mathcal{L}^2(A)$ when ε goes to 0, the radius of A tends to zero. It follows that the infinite dimensional geodesic Brunn-Minkowski inequality does not hold when we consider the Grušin plane with the Lebesgue measure. $\square$

Chapter 4

Gradient flow in the Heisenberg group

This chapter is devoted to a new approach of the subelliptic heat diffusion in the Heisenberg group. It is now well-known that the heat diffusion on manifolds M with a lower bound on the Ricci curvature can be represented by a special curve in the Wasserstein space $\mathcal{P}_2(M)$. Roughly speaking this curve tends to move in directions minimizing the entropy functional Ent_∞ defined in 3.14. Conversely the so-called gradient flows of the entropy are one-parameter families of measures whose density evolves in a way solving the heat equation. In this chapter we prove that in the case of the Heisenberg group with the Carnot-Carathéodory distance this concordance still remain whereas we replace the heat equation by the subelliptic "heat equation" $\Delta_{\mathbb{H}}\rho_s = \partial_s\rho_s$ (Theorem 4.5.1 and Theorem 4.5.2). However, in Theorem 4.5.2, we were not able to get rid of a strange assumption on the weak differentiability with respect to the vector field $\mathbf{T}$.

The interesting point in these results is the fact that contrarily to the case of manifolds with a lower Ricci bound, in the Heisenberg group Ent_∞ has no convexity properties along the optimal transport. Indeed, we proved it in Chapter 3 when we considered $CD(K, +\infty)$.

4.1 Definitions

4.1.1 Absolutely continuous curves

A curve $(\gamma_s)_{s\in I}$ in a metric space (X, d) is said to be absolutely continuous on I if there exists a $m \in L^1(\mathbb{R})$ such that for any $a < b$ in I,

$$d(\gamma(a), \gamma(b)) \leq \int_a^b m(s)ds.$$

It is proved in [9] that if γ is absolutely continuous, for almost every $s \in I$ the metric derivative

$$|\dot{\gamma_s}| := \lim_{|h|\to 0} \frac{d(\gamma_{s+h}, \gamma_s)}{|h|}$$

exists and

$$l(\gamma) = \int_a^b |\dot{\gamma_s}| ds \tag{4.1}$$

equals the metric length of the curve γ between a and b defined at the beginning of Section 1.5. In a geodesic space $l(\gamma) \geq d(a,b)$. If $l(\gamma) = d(a,b)$, some reparametrizations of γ are (constant-speed) geodesics. We denote the space of absolutely continuous curv by $AC(X)$. Let $AC_2(X) \subset AC(X)$ be the subspace of absolutely continuous curves such that $|\dot{\gamma_s}|^2$ is locally integrable.

4.1.2 Gradient flow

In this chapter, curves will move in Wassertein spaces $(\mathcal{P}_2, W)$ and we will study the slope of the functional entropy Ent_∞ introduced in (3.14). It is defined as

$$\mathrm{Slope}(\mathrm{Ent}_\infty)(\mu) = \max\left(0, \limsup_{\nu\to\mu} \frac{\mathrm{Ent}_\infty(\mu) - \mathrm{Ent}_\infty(\nu)}{W(\mu,\nu)}\right).$$

This quantity is positive and quantify how much the entropy can locally decrease. For the slope in the Wasserstein space $\mathcal{P}_2(\mathbb{H}_n)$ we will write Slope and for the Wasserstein spaces $\mathcal{P}_2(\mathbb{H}_n^\varepsilon)$ it will be $\mathrm{Slope}^\varepsilon$. As $d_c \geq d_\varepsilon$ we have for the Wasserstein spaces $W \geq W^\varepsilon$ where W^ε is the distance of $\mathcal{P}_2(\mathbb{H}_n^\varepsilon)$. So $\mathrm{Slope}^\varepsilon(\mathrm{Ent}_\infty)(\mu) \geq \mathrm{Slope}(\mathrm{Ent}_\infty)(\mu)$ in every μ of finite entropy.

We will use in this chapter a very metric definition for the gradient flow of the entropy. It refers to v) in Theorem 5.3 of [8] or to ii) of Proposition 23.2 of [109]. A curve of the Wasserstein space $(\mu_s)_{s\in I}$ is said to be a gradient flow of the entropy if

- it is an absolutely continuous curve of $\mathcal{P}_2(X)$ and for almost every $s \in I$,

$$|\dot{\mu_s}| = \mathrm{Slope}(\mathrm{Ent}_\infty)(\mu_s),$$

- the function $E(s) = \mathrm{Ent}_\infty(\mu_s)$ is absolutely continuous and for almost every $s \in I$,

$$\dot{E}(s) = -\,\mathrm{Slope}(\mathrm{Ent}_\infty)(\mu_s) \cdot |\dot{\mu_s}|.$$

Remark 4.1.1. Of course then $\dot{E}(s) = -|\dot{\mu_s}|^2$ for almost every s, but in the first formulation it is easier to recognize a chain rule derivation where it appears that the curve falls off in the direction of the slope. Nevertheless it follows from this remark that $(\mu_s)_{s\in I}$ is in $AC^2(X)$.

4.1.3 Functional spaces, Tangent spaces

Our approach of the gradient flow is based on the approximation of our space by the Wasserstein spaces, $\mathcal{P}_2(\mathbb{H}_n^\varepsilon)$, of the approximating manifolds defined in Subsection 1.2.6. There are known results on the gradient flow in this case because each manifold $\mathbb{H}_n^\varepsilon$ has a lower bound for the Ricci curvature, indeed $(\frac{-1}{2\varepsilon^2})$ (see Subsection 3.1.1) and the case of manifolds with a lower bound on the Ricci curvature has been studied by Erbar [35], Savaré [97] and Villani [109]. Roughly speaking, they showed under some conditions that the density ρ_s of the measures evolving as a gradient flow of Ent_∞ is solution of the heat equation

$\Delta_\varepsilon \rho_s = \frac{\partial}{\partial s}\rho_s$ or the conversely that solutions of the heat equation are gradient flows.

We will show in Section 4.5 that under certain assumptions on the regularity of μ_s, the gradient flows of Ent_∞ in $\mathcal{P}_2(\mathbb{H}_n)$ are solution of the subelliptic heat equation and that the converse statement is also true. The essential fact in this result is that $\mathbb{H}_n$ has no lower curvature bound whereas this condition is generally useful in the case of Riemannian manifolds. In Section 4.2, we will state some known results for manifold with an uniform lower bound for the Ricci curvature applied to the approximating manifolds $\mathbb{H}_1^\varepsilon$.

The vector spaces we will introduce now are defined using weak formulations that require test functions. In the sequel the space of test functions will be $\mathcal{C}_c^{+\infty}(\mathbb{R}^{2n+1})$, the space of smooth functions of $\mathbb{R}^{2n+1}$ with a compact support. The next definitions are done for functions of $\mathbb{H}_1 = \mathbb{R}^3$ and $\mathbf{X}$ (resp. $\mathbf{Y}$) is written instead of $\mathbf{X}_1$ (resp. $\mathbf{Y}_1$). Nevertheless if one understand $\mathbf{X}$ as the sum over k of the vector fields $\mathbf{X}_k$ and the same for $\mathbf{Y}$, it is easy to deduce the corresponding definitions for $\mathbb{H}_n$ which also make sense. Similarly $\mathcal{L}$ is the Lebesgue measure $\mathcal{L}^3$ but it can be understood as $\mathcal{L}^{2n+1}$ as well.

Let f be a function of $L^1_{\text{loc}}(\mathbb{H}_1)$ (a $\mathcal{L}$-locally integrable function on $\mathbb{H}_1$). If a function u of L^1_{loc} satisfies

$$\int u\psi d\mathcal{L} + \int f\mathbf{X}\psi d\mathcal{L} = 0$$

for any $\psi \in \mathcal{C}_c^{+\infty}(\mathbb{R}^3)$, it will be the weak $\mathbf{X}$-derivative of f and we will write it $\mathbf{X}f$. Note that these notations are coherent with the the usual ones when f is smooth. Actually in this case, $f\psi$ is a test function (smooth with compact support). Then

$$\begin{aligned}\int \mathbf{X}(f\psi)d\mathcal{L} &= \int (\partial_x - \frac{1}{2}y\partial_t)(f\psi)d\mathcal{L} \\ &= \int (\partial_x(f\psi)d\mathcal{L} - \frac{1}{2}\int\left(\int\int (y\partial_t)(f\psi)dtdx\right)dy = 0\end{aligned}$$

But the first integral is also $\int(\mathbf{X}f)\psi d\mathcal{L} + \int f(\mathbf{X}\psi)d\mathcal{L}$ as we want. We define in the same way the weak derivatives $\mathbf{Y}f$ and $\mathbf{T}f$.

A test horizontal vector field will be a field $\xi_\mathbb{H} = \psi_\mathbf{X}\mathbf{X} + \psi_\mathbf{Y}\mathbf{Y}$ where both coordinates are test functions. Let $\mathcal{C}_c^{+\infty}(T\mathbb{H}_1)$ be the space of these vector fields. For any measurable horizontal vector fields we use the norms

$$\|u\|_{L^2_\mathbb{H}} = \sqrt{\int \|u\|^2_\mathbb{H} d\mathcal{L}} \quad \text{and} \quad \|u\|_{L^1_\mathbb{H}} = \int \|u\|_\mathbb{H} d\mathcal{L}$$

of the Lebesgue spaces $L^2_\mathbb{H} = L^2(T\mathbb{H}_1)$ and $L^1_\mathbb{H} = L^1(T\mathbb{H}_1)$ and similarly, further Lebesgue spaces, $L^2_\mathbb{H}(\mu)$ and $L^1_\mathbb{H}(\mu)$ are defined with respect to the measure μ. Then if for a given function $f \in L^1_{\text{loc}}$ there exists an horizontal vector field $u \in L^1_{\mathbb{H},\text{loc}}$ such that for any horizontal test vector field $\xi_\mathbb{H}$,

$$\int f \operatorname{div}_\mathbb{H} \xi_\mathbb{H} := \int f(\mathbf{X}\psi_\mathbf{X} + \mathbf{Y}\psi_\mathbf{Y}) = -\int \langle u \mid \xi_\mathbb{H}\rangle_\mathbb{H}, \tag{4.2}$$

we will call it a local integrable weak horizontal gradient of f and denote u by $\nabla_\mathbb{H} f$. Then the weak derivatives $\mathbf{X}f$ and $\mathbf{Y}f$ exist and $\nabla_\mathbb{H} f = \mathbf{X}f\mathbf{X} + \mathbf{Y}f\mathbf{Y}$. We note furthermore $W^{1,1}_{\rm loc}(\mathbb{H}_1)$ the space of locally integrable functions functions f with a locally integrable weak horizontal gradient $\nabla_{\mathbb{H}} f$. Note that in (4.2), the operator div can be chosen as $\operatorname{div}_\varepsilon$ instead of $\operatorname{div}_{\mathbb{H}}$ because for $\xi_{\mathbb{H}} \in \mathcal{C}^{+\infty}_c(T\mathbb{H}_1)$

$$\operatorname{div}_{\mathbb{H}}(\xi_{\mathbb{H}}) = \operatorname{div}_\varepsilon(\xi_{\mathbb{H}}).$$

Fix now some $\varepsilon > 0$. A test vector field is a field $\xi = \xi_{\mathbb{H}} + \psi_\varepsilon(\varepsilon \mathbf{T})$ where $\xi_{\mathbb{H}} \in \mathcal{C}^{+\infty}_c(T\mathbb{H}_1)$ and $\psi_\varepsilon \in \mathcal{C}^{+\infty}_c(\mathbb{R}^3)$. Test vector fields are in fact simply smooth vector fields with compact support in $\mathbb{H}_1 = \mathbb{R}^3$ but we write them in the basis $(\mathbf{X}, \mathbf{Y}, \varepsilon\mathbf{T})$ because the interesting spaces are $L^2_\varepsilon = L^2(\mathbb{H}^\varepsilon_1)$ and $L^1_\varepsilon = L^1(\mathbb{H}^\varepsilon_1)$ with norms

$$\|u\|_{L^2_\varepsilon} = \sqrt{\int (\psi^2_{\mathbf{X}} + \psi^2_{\mathbf{Y}} + \psi^2_\varepsilon) d\mathcal{L}} \quad \text{and} \quad \|u\|_{L^1_\varepsilon} = \int \|u\|_\varepsilon d\mathcal{L}$$

and the corresponding Lebesgue spaces $L^2_\varepsilon(\mu)$ and $L^1_\varepsilon(\mu)$ where the measure has changed. Then if for a given function $f \in L^1_{\rm loc}$ there exists a vector field $u \in L^1_{\varepsilon,\rm loc}$ such that for any test vector field ξ

$$\int f \operatorname{div}_\varepsilon \xi := \int f(\mathbf{X}\psi_{\mathbf{X}} + \mathbf{Y}\psi_{\mathbf{Y}} + \varepsilon\mathbf{T}\psi_\varepsilon) = -\int \langle u \mid \psi\rangle_\varepsilon,$$

u is called a locally integrable weak gradient of f and we denote it by $\nabla_\varepsilon f$.

Let $W^{1,1}_{\rm loc}(\mathbb{H}^\varepsilon_1)$ be the space of the functions f with a locally integrable weak gradient $\nabla_\varepsilon f$. If $f \in W^{1,1}_{\rm loc}(\mathbb{H}^\varepsilon_1)$, the weak derivatives $\mathbf{X}f$ and $\mathbf{Y}f$ and $\mathbf{T}f$ exist and $\nabla_\varepsilon f = \nabla_{\mathbb{H}} f + \varepsilon\mathbf{T}f\varepsilon\mathbf{T}$. Conversely if $\mathbf{X}f$, $\mathbf{Y}f$ and $\mathbf{T}f$ exist and are locally integrable, f is in $W^{1,1}_{\rm loc}(\mathbb{H}^\varepsilon_1)$. Observe that if f has a weak gradient $\nabla_\varepsilon f$ for some ε, $\mathbf{T}f$ is well-defined and integrable, so there is also a weak gradient $\nabla_{\varepsilon'} f$ for $\varepsilon' \neq \varepsilon$. It is simply $\nabla_{\mathbb{H}} f + (\varepsilon'\mathbf{T})f(\varepsilon'\mathbf{T})$. Then $W^{1,1}_{\rm loc}(\mathbb{H}^\varepsilon_1) = W^{1,1}_{\rm loc}(\mathbb{H}^{\varepsilon'}_1)$ and we will simply denote this space by $W^{1,1}_{\rm loc}$. Moreover, note that

$$W^{1,1}_{\rm loc} = W^{1,1}_{\rm loc}(\mathbb{H}_1) \cap \{f \mid \mathbf{T}f \text{ exists and is locally integrable.}\}$$

This remark is at the origin of the assumption we make in Theorem 4.5.2 about the weak $\mathbf{T}$-derivability of the measures. Indeed, for the proof we need the measures to be in $W^{1,1}_{\rm loc}$.

For a given absolutely continuous measure μ, we define now the tangent space of $\mathcal{P}_2(\mathbb{H}^\varepsilon_1)$ at μ, the Hilbert space $\operatorname{Tan}_\varepsilon(\mu)$ as the space of the vector fields $\nabla_\varepsilon\psi$ where ψ is a test function, completed in the Hilbert space $L^2_\varepsilon(\mu)$. More formally

$$\operatorname{Tan}_\varepsilon(\mu) = \overline{\{\nabla_\varepsilon\psi \mid \psi \in \mathcal{C}^\infty_c\}}^{L^2_\varepsilon(\mu)}.$$

Similarly the tangent space $\operatorname{Tan}_{\mathbb{H}}(\mu)$ of $\mathcal{P}_2(\mathbb{H}_1)$ at μ is defined by

$$\operatorname{Tan}_{\mathbb{H}}(\mu) = \overline{\{\nabla_{\mathbb{H}}\psi \mid \psi \in \mathcal{C}^\infty_c\}}^{L^2_{\mathbb{H}}(\mu)}.$$

4.2 Some results concerning the approximating manifolds and their Wasserstein spaces

We state here, for the approximating manifolds $\mathbb{H}^\varepsilon_1$ a proposition that Erbar [35, Proposition 3.2.2] proved for Riemannian manifolds. It is the translation of the same statement for Euclidean spaces by Ambrosio and Savaré [8, Theorem 4.16]. The proof of Erbar also strongly relies on Theorem 23.13 of [109] whose proof is long and difficult.

Proposition 4.2.1. *Let $\varepsilon > 0$ and $d\mu = \rho d\mathcal{L}$ a probability measure. Then the following statement are equivalent:*

(i) $\mathrm{Slope}^\varepsilon(\mathrm{Ent}_\infty)(\mu) < +\infty$

(ii) $\rho \in W^{1,1}_{loc}(\mathbb{H}_1^\varepsilon)$, $\nabla_\varepsilon \rho = \rho w^\varepsilon$ *$\mathcal{L}$-almost surely for some* $w^\varepsilon \in L^2_\varepsilon(\mu)$.

In this case $w^\varepsilon \in \mathrm{Tan}_\varepsilon(\mu)$ *and* $\mathrm{Slope}(\mathrm{Ent}_\infty)(\mu) = \|w^\varepsilon\|_{L^2_\varepsilon(\mu)}$.

Remark 4.2.2. The vector field w^ε is sometime simply written "$\nabla_\varepsilon \rho/\rho$". Actually as $\mu = \rho\mathcal{L}$, the function ρ is μ-almost surely non-zero such that $\nabla_\varepsilon\rho/\rho$ make sense in $L^2_\varepsilon(\mu)$. It shows that there is a unique possible $w^\varepsilon \in L^2_\varepsilon(\mu)$. However, in order to avoid bad interpretations of this quotient we will rather use w^ε.

Remark 4.2.3. If statement *(ii)* is true for some ε, it also holds for other $\varepsilon' > 0$. It follows that if the slope is finite in $\mathcal{P}_2(\mathbb{H}_1^\varepsilon)$, it is also finite in the other Wasserstein spaces $\mathcal{P}_2(\mathbb{H}_1^{\varepsilon'})$ even for $\varepsilon' > \varepsilon$.

We state now a mixing of a propositions by Villani [109, Theorem 23.13] and by Erbar [35]. In fact the second point (4.4) is not in [109, Theorem 23.13]. It is obtained from it, approximating $v^\varepsilon_{s'}$ by the gradient of c-convex functions and using [35, Lemma 2.1.3]. The first point is a mixing of Proposition 2.5 of [35] and [109, Theorem 13.8]. It is proved exactly in the same way as we will prove Proposition 4.3.1.

Proposition 4.2.4. *Let $\varepsilon > 0$ and $(\mu_s)_{s\in I}$ an AC^2-curve of $\mathcal{P}_2(\mathbb{H}_1^\varepsilon)$. Assume that for almost every $s \in I$, the slope* $\mathrm{Slope}^\varepsilon(\mathrm{Ent}_\infty)(\mu_s)$ *is finite. Let $w^\varepsilon_s \in L^2_\varepsilon(\mu_s)$ be the corresponding vector with respect to Proposition 4.2.1. Then there is a subset $I'_\varepsilon \subset I$ of full-measure such that for any $s' \in I'_\varepsilon$ there is a vector field $v^\varepsilon_{s'} \in \mathrm{Tan}_\varepsilon(\mu_{s'})$ satisfying the two following statements*

- *For every test function $\psi \in \mathcal{C}^\infty_c$,*

$$\frac{\partial}{\partial s}\mu_s(\psi)(s') = \int \langle v^\varepsilon_{s'} \mid \nabla_\varepsilon\psi\rangle_\varepsilon d\mu_{s'}, \tag{4.3}$$

- *the entropy evolves in a way such that*

$$\mathrm{Ent}_\infty(\mu_s) \geq \mathrm{Ent}_\infty(\mu_{s'}) + \int \langle (s-s')v^\varepsilon_{s'} \mid w^\varepsilon_{s'}\rangle_\varepsilon d\mu_{s'} + o(|s-s'|)s \tag{4.4}$$

  when s goes to s'.

Moreover for any μ and ν in $\mathcal{P}_2(\mathbb{H}_1^\varepsilon)$ such that $\mathrm{Slope}^\varepsilon(\mathrm{Ent}_\infty)(\mu) < +\infty$,

$$\mathrm{Ent}_\infty(\nu) \geq \mathrm{Ent}_\infty(\mu) - W^\varepsilon(\mu,\nu)\sqrt{\int \|w^\varepsilon\|^2_\varepsilon d\mu} - \frac{1}{2\cdot 2\varepsilon^2}W^\varepsilon(\mu,\nu)^2. \tag{4.5}$$

Remark 4.2.5. Here $d\mu_s = \rho_s d\mathcal{L}$. In the proposition it is possible to change $\mathcal{L}$ in vol_ε, the Riemannian volume of $\mathbb{H}_1^\varepsilon$ and ρ_s in ρ^ε_s, the density with respect to this volume. Then $v^\varepsilon_{s'}$ must be multiplied by the same constant. This form is closer to the standard statement for the Riemannian manifolds.

Remark 4.2.6. It was the idea of Jordan, Kinderlehrer and Otto [63] to consider the Wasserstein space as if it were a Riemannian manifold with infinite dimension. The vector field w_s^ε has to be understood as the gradient of Ent_∞ (a function on this manifold) and v_s^ε as the velocity vector of the curve $(\mu_s)_{s\in I}$. Then $\int \langle v_{s'}^\varepsilon \mid w_{s'}^\varepsilon\rangle_\varepsilon d\mu_{s'}$ is the scalar product in the tangent space of $\mathcal{P}_2$ at μ. It should be the derivative of Ent_∞ in direction v_s^ε. But in $\mathcal{P}_2$, singular measures with infinite entropy are dense. It is a basic reason why we have in (4.4) a subgradient inequality instead of a gradient equality.

Remark 4.2.7. For Riemannian manifolds M for which K is a lower bound of $\mathrm{Ric}(p)$ for any $p \in M$, the bound $\frac{-1}{2\cdot 2\varepsilon^2}$ in (4.5) should be replaced by $\frac{1}{2}K$.

In Proposition 4.3.1 we will prove a similar statement to (4.3) for the "true" Heisenberg group $\mathbb{H}_1$. In Section 4.4 we will let ε go to zero in (4.5) and get a result on the slope of the entropy in $\mathcal{P}_2(\mathbb{H}_1)$, the Wasserstein space of the "true" Heisenberg group. Inequality (4.4) will also be interpreted in the context of $\mathcal{P}_2(\mathbb{H}_1)$.

4.3 Speed and velocity

Equality (4.3) in Proposition 4.2.4 shows that for the AC^2-curves $(\mu_s)_{s\in I}$ of the Wasserstein spaces $\mathcal{P}_2(\mathbb{H}_1^\varepsilon)$ (where it is known that there is a metric speed $|\dot\mu_s|$) it is possible to define a velocity (speed and direction) thanks to a vector field v_s^ε. We show that there is a similar velocity for $\mathcal{P}_2(\mathbb{H}_1)$, the Wasserstein space of the Heisenberg group.

Proposition 4.3.1. *Let $(\mu_s)_{s\in I}$ be an AC^2-curve of $(\mathcal{P}_2(\mathbb{H}_1), W)$. Then there is a subset $I' \subset I$ of full measure such that in any $s' \in I'$ there is a vector field $v_{s'} \in \mathrm{Tan}(\mu_{s'})$ so that*

$$\partial_{s'}\mu_{s'} + \mathrm{div}_{\mathbb{H}}(v_{s'}\mu_{s'}) = 0 \tag{4.6}$$

in a the weak sense. It means that for every function $\psi \in \mathcal{C}_c^{+\infty}$,

$$\left(\frac{\partial}{\partial s}\int \psi d\mu_s\right)|_{s'} = \int \langle \nabla_{\mathbb{H}}\psi | v_{s'}\rangle_{\mathbb{H}} d\mu_{s'}.$$

Moreover, $\|v_{s'}\|_{L^2(\mu_{s'})} \leq |\dot\mu_{s'}|$ for any $s' \in I'$.

Proof. Let $\psi \in \mathcal{C}_c^{+\infty}$. We assume also that ψ it is a 1-Lipschitz function of $(\mathbb{H}_1, d_c)$. Then for $s < t$ in I, by using an optimal transport plan with respect to the 1-Wasserstein cost, we get

$$\left|\int_{\mathbb{H}} \psi d\mu_s - \int_{\mathbb{H}} \psi d\mu_t\right| \leq W_1(\mu_s, \mu_t) \leq W(\mu_s, \mu_t).$$

Then $\zeta^\psi(s) := \int_{\mathbb{H}} \psi d\mu_s$ is an absolutely continuous function. It is derivable for almost every $s \in I$. Let $\pi_{s,t}$ be an optimal transport plan between μ_s and μ_t. We define now Ψ by

$$\Psi(x,y) = \begin{cases} \frac{|\psi(p)-\psi(q)|}{d_c(p,q)} & \text{if } p \neq q \\ \|\nabla_{\mathbb{H}}\psi(p)\|_{\mathbb{H}} & \text{else.} \end{cases}$$

Then Ψ is bounded above by 1 and it is upper semi-continuous. Let $s' \in I$ such that ζ^ψ is differentiable and the metric derivative $|\dot{\mu}_{s'}|$ exists. We get

$$\begin{aligned}
\left|\frac{d}{ds}\int \psi d\mu_s\right|(s') &\leq \liminf_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\left|\int \psi d\mu_s' - \int \psi d\mu_{s'+\varepsilon}\right| \\
&\leq \liminf_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\int |\psi(q)-\psi(p)| d\pi_{s',s'+\varepsilon}(p,q) \\
&\leq \left(\liminf_{\varepsilon\downarrow 0}\sqrt{\int \Psi(p,q)^2 d\pi_{s',s'+\varepsilon}(p,q)}\frac{\sqrt{\int d(p,q)^2 d\pi_{s',s'+\varepsilon}}}{\varepsilon}\right) \\
&\leq \left(\liminf_{\varepsilon\downarrow 0}\sqrt{\int \Psi(p,q)^2 d\pi_{s',s'+\varepsilon}(x,y)}\frac{W(\mu_{s'},\mu_{s'+\varepsilon})}{\varepsilon}\right) \\
&\leq \left(|\dot{\mu}_{s'}|\liminf_{\varepsilon\downarrow 0}\sqrt{\int \Psi(p,q)^2 d\pi_{s',s'+\varepsilon}(p,q)}\right).
\end{aligned}$$

Since Ψ is upper semi-continuous and $\pi_{s',s'+\varepsilon}$ weakly converges to $(\mathrm{Id}\otimes\mathrm{Id})_{\#}\mu_{s'}$ (see [109, Theorem 5.19]) when $\varepsilon \downarrow 0$, we get

$$\begin{aligned}
\left|\frac{d}{ds}\int \psi d\mu_s\right| &\leq |\dot{\mu}_s|\sqrt{\int |\Psi(x,x)|^2 d\mu_s(x)} \\
&= |\dot{\mu}_s|\sqrt{\int \|\nabla_{\mathbb{H}}\psi(x)\|^2_{\mathbb{H}} d\mu_s(x)}. \qquad (4.7)
\end{aligned}$$

This is the key estimate of the proof. We have assumed that ψ is 1-Lipschitz. The estimate also hold without this assumption.

We already know that for any $\psi \in \mathcal{C}_c^{+\infty}$, the function ζ^ψ is differentiable at almost every $s \in I$. We will now prove that for almost every $s \in I$, every function ζ^ψ is differentiable. We use the fact that there is sequence $(\psi_k)_{k\in\mathbb{N}}$ of test functions such that $\nabla_{\mathbb{H}}\psi_k$ is dense in $\mathrm{Tan}(\mu_s)$. Moreover, one assume that the sequence $(\nabla_{\mathbb{H}}\psi_k)_{k\in\mathbb{N}}$ is dense in $\{\nabla_{\mathbb{H}}\psi \mid \psi \in \mathcal{C}_c^{+\infty}\}$ for the norm of $L^\infty(T\mathbb{H}_1)$. The functions $\zeta^k = \int \psi_k d\mu_s$ are countable and derivable in almost every $s \in I$. Thus there is a set $I_1' \subset I$ with full measure in I such that in each $s' \in I'$, the metric derivative $|\dot{\mu}_{s'}|$ exists and the ζ^k are differentiable.

For $s' \in I_1'$ we define the bounded operator $T_{s'}$ on $\{\nabla_{\mathbb{H}}\psi_k \mid k \in \mathbb{N}\}$ by $T_{s'}(\nabla_{\mathbb{H}}\psi_k) = \frac{d}{ds}\zeta^k(s')$. This set is dense in $\mathrm{Tan}(\mu_{s'})$ so that we can extend $T_{s'}$ on $\mathrm{Tan}(\mu_{s'})$ and represent it by a vector field $v_{s'} \in \mathrm{Tan}(\mu_{s'})$:

$$T_{s'}(w) = \int \langle w|v_{s'}\rangle_{\mathbb{H}} d\mu_{s'}.$$

We will show that for any $\psi \in \mathcal{C}_c^{+\infty}$

$$T_{s'}(\nabla_{\mathbb{H}}\psi) = \lim_{\varepsilon\to 0}\frac{\zeta^\psi(s'+\varepsilon)-\zeta^\psi(s')}{|\varepsilon|}.$$

For every $k \in \mathbb{N}$

$$\limsup_{\varepsilon\to 0}\left\|T_{s'}(\nabla_{\mathbb{H}}\psi) - \frac{\zeta_{s'+\varepsilon}-\zeta_{s'}}{\varepsilon}\right\| \leq \|T_{s'}(\nabla_{\mathbb{H}}\psi) - T_{s'}(\nabla_{\mathbb{H}}\psi_k)\|$$
$$+\left\|T_{s'}(\nabla_{\mathbb{H}}\psi_k) - \frac{\zeta^k_{s'+\varepsilon}-\zeta^k_{s'}}{\varepsilon}\right\| + \left\|\frac{\zeta^{(\psi_k-\psi)}(s'+\varepsilon) - \zeta^{(\psi_k-\psi)}(s')}{\varepsilon}\right\|.$$

As the curves $\zeta^{(\psi_k-\psi)}(s')$ are absolutely continuous, by using the estimate (4.7) its differentiability set we can estimate the previous expression by

$$\|T_{s'}(\nabla_{\mathbb{H}}\psi) - T_{s'}(\nabla_{\mathbb{H}}\psi_k)\| + \left\|T_{s'}(\nabla_{\mathbb{H}}\psi_k) - \frac{\zeta^k_{s'+\varepsilon}-\zeta^k_{s'}}{\varepsilon}\right\|$$
$$+\frac{1}{\varepsilon}\sqrt{\int_{s'}^{s'+\varepsilon}|\dot{\mu}_s|^2 ds}\sqrt{\int_{s'}^{s'+\varepsilon}\|\nabla_{\mathbb{H}}(\psi_k-\psi)\|^2_{L^2(\mu_u)}du}.$$

For a given k, we let ε go to 0 and we obtain

$$\limsup_{\varepsilon\to 0}\left\|T_{s'}(\nabla_{\mathbb{H}}\psi) - \frac{\zeta_{s'+\varepsilon}-\zeta_{s'}}{\varepsilon}\right\| \leq 2|\dot{\mu}_{s'}|\cdot\|\nabla_{\mathbb{H}}(\psi_k-\psi)\|_\infty$$

on the $s' \in I'_1$ that are also Lebesgue points of $\tau \to |\dot{\mu_\tau}|^2$. Using the density of the $\nabla_{\mathbb{H}}\psi_k$ in $L^\infty(T\mathbb{H}_1)$, we have the first part of the proposition.

We will now prove $\|v_{s'}\|_{L^2(\mu_{s'})} \leq |\dot{\mu}_{s'}|$. For $s' \in I'$, we consider $T_{s'}(\nabla'_{\mathbb{H}}\psi_k)$ where $(\nabla_{\mathbb{H}}\psi'_k)_{k\in\mathbb{N}}$ is a sequence tending to $v_{s'}$ in $\mathrm{Tan}(\mu_{s'})$. On the one hand this sequence tends to

$$\lim_{k\to\infty}\int\langle v_{s'} \mid \nabla_{\mathbb{H}}\psi'_k\rangle_{\mathbb{H}} d\mu_{s'} = \int\|v_{s'}\|^2_{\mathbb{H}} d\mu_{s'}.$$

On the other hand, from estimate (4.7) it is smaller than

$$|\dot{\mu}_{s'}|\lim_{k\to\infty}\sqrt{\int\|\nabla_{\mathbb{H}}\psi'_k\|^2_{\mathbb{H}}d\mu_{s'}} = |\dot{\mu}_{s'}|\sqrt{\int\|v_{s'}\|^2_{\mathbb{H}}d\mu_{s'}}.$$

Then $\|v_{s'}\|_{L^2(\mu_{s'})}$ is smaller than $|\dot{\mu}_{s'}|$.

□

Remark 4.3.2. If one carefully read the proof, the set I' in Proposition 4.3.1 can be chosen as the intersection of the following sets: the set where $|\dot{\mu}_s|$ exists, the differentiation set of the functions ζ^k and the Lebesgue points of $\tau \to |\dot{\mu}_s|^2$.

4.4 Slope

After the previous Section we can represent the velocity of $(\mu_s)_{s\in I}$ by a vector field of $\mathrm{Tan}(\mu_{s'})$. Proposition 4.4.1 makes the picture more precise and permits to identify "the gradient of the entropy" as a vector field. For this proof we will not only assume that the slope of the entropy in $\mathcal{P}_2(\mathbb{H}_1)$ is finite but also that the slope of Ent_∞ is finite in $\mathcal{P}_2(\mathbb{H}_1^\varepsilon)$ for some ε. Proposition 4.4.1 has to be read in relation with Proposition 4.2.1.

Proposition 4.4.1. *Let μ be an absolutely continuous probability measure of density ρ and $\varepsilon > 0$. Assume that* $\mathrm{Slope}(\mathrm{Ent}_\infty)(\mu) < +\infty$. *Then there is an horizontal vector field $w^{\mathbb{H}} \in L^2_{\mathbb{H}}(\mu)$ such that $\nabla_{\mathbb{H}}\rho = \rho w^{\mathbb{H}}$ $\mathcal{L}$-almost surely. Moreover, the two followings statement are equivalent*

(i) $\mathrm{Slope}^\varepsilon(\mathrm{Ent}_\infty)(\mu) < +\infty$,

(ii) The weak gradient $\mathbf{T}\rho$ exists and there is $w^{\mathbf{T}} \in L^2(\mu)$ such that $\mathbf{T}\rho = \rho w^{\mathbf{T}}$ ($\mathcal{L}$-almost everywhere).

If these assumptions hold, there is actually as in Proposition 4.2.1 a vector field $w^\varepsilon \in \mathrm{Tan}_\varepsilon(\mu)$ such that $\nabla_\varepsilon\rho = \rho w^\varepsilon$ and $\mathrm{Slope}^\varepsilon(\mathrm{Ent}_\infty)(\mu) = \|w^\varepsilon\|_{L^2_\varepsilon(\mu)}$. *Moreover, $w^{\mathbb{H}} \in \mathrm{Tan}_{\mathbb{H}}(\mu)$, $w^{\mathbb{H}} + \varepsilon w^{\mathbf{T}}(\varepsilon\mathbf{T}) = w^\varepsilon$ and*

$$\mathrm{Slope}(\mathrm{Ent}_\infty)(\mu) = \|w^{\mathbb{H}}\|_{L^2_{\mathbb{H}}(\mu)}.$$

Remark 4.4.2. As in Remark 4.2.2, $w^{\mathbb{H}}$ could be written "$\nabla_{\mathbb{H}}\rho/\rho$" and $w^{\mathbf{T}}$ should be understood as "$\mathbf{T}\rho/\rho$".

Proof. We will first compute the slope of the entropy in some smooth directions. Let $\psi \in \mathcal{C}_c^{+\infty}$ and μ be an absolutely continuous probability measure of density ρ. Let $F_s(p) = p \cdot \exp^{\mathbb{H}}(s\mathbf{X}\psi(p) + s\mathbf{i}\mathbf{Y}\psi(p), s\mathbf{T}\psi(p))$ as in Proposition 2.2.7 and Lemma 2.2.8. For s small enough we proved in this lemma that F_s is smooth, one-to-one and that $\mathcal{J}_s = \mathrm{Jac}(F_s)$ does not vanish, In Remark 2.2.9 we noticed that if $\mu \in \mathcal{P}_2(\mathbb{H}_1)$ is absolutely continuous, F_s is the optimal transport map between μ and $\mu_s = (F_s)_\#\mu$. Furthermore $(\mu_s)_{s\in]0,s'[}$ is a geodesic in $\mathcal{P}_2(\mathbb{H}_1)$ for s' small enough and the speed of this curve is $\|\nabla_{\mathbb{H}}\psi\|_{L^2_{\mathbb{H}}(\mu)}$. The density of $\mu_s = (F_s)_\#\mu$ in $F_s(p)$ is $\rho(p)\mathcal{J}_s(p)^{-1}$ and the entropy of this measure is then

$$\begin{aligned}
\mathrm{Ent}_\infty((F_s)_\#\mu) &= \int_{\mathbb{H}} \rho(p)\mathcal{J}_s(p)^{-1}\ln(\rho\mathcal{J}_s(p)^{-1})\mathcal{J}_s(p)d\mathcal{L}(p)\\
&= \int_{\mathbb{H}} \rho(p)\ln(\rho\mathcal{J}_s(p)^{-1})d\mathcal{L}(p)\\
&= \mathrm{Ent}_\infty(\mu) - \int_{\mathbb{H}} \rho(p)\ln(\mathcal{J}_s(p))d\mathcal{L}(p).
\end{aligned}$$

One can differentiate under the integral sign because $\frac{\dot{\mathcal{J}}_s}{\mathcal{J}_s}$ is a smooth function on a compact set, so it is bounded. Therefore $\frac{d}{ds}\mathrm{Ent}_\infty((F_s)_\#\mu) = \int -\rho\frac{\dot{\mathcal{J}}_s}{\mathcal{J}_s}d\mathcal{L}$. In $s = 0$ we have $\mathcal{J} = 1$ and for $\varepsilon > 0$ we have $\dot{\mathcal{J}} = \mathrm{div}_\varepsilon(\nabla_{\mathbb{H}}\psi)$. This is also $\mathrm{div}_{\mathbb{H}}(\nabla_{\mathbb{H}}\psi)$ as we noticed Subsection 4.1.3. Then we get

$$\frac{d}{ds}\mathrm{Ent}_\infty(\mu_s)\mid_{s=0} = -\int \rho\,\mathrm{div}_{\mathbb{H}}\nabla_{\mathbb{H}}\psi d\mathcal{L}.$$

Thus we know exactly the speed of $(\mu_s)_{s>0}$ and how the entropy decreases with respect to time. It follows that we know the slope of the entropy along this curve:

$$\begin{aligned}
\lim_{s\to 0}\frac{|\mathrm{Ent}_\infty(F_s)_\#\mu - \mathrm{Ent}_\infty(\mu)|}{d(\mu,(F_s)_\#\mu)} &= \frac{\frac{d}{ds}\mathrm{Ent}_\infty((F_s)_\#\mu)}{|\dot{(F_s)}_\#\mu|}\\
&= \frac{|\int\rho\,\mathrm{div}_{\mathbb{H}}\nabla_{\mathbb{H}}\psi|}{\|\nabla_{\mathbb{H}}\psi\|_{L^2_{\mathbb{H}}(\mu)}}.
\end{aligned}$$

It is possible to make a similar computation for $U_s(p) = p\cdot\exp^{\mathbb{H}}(s\zeta(p), s\theta(p))$ where as in Lemma 2.2.8, ζ is a smooth $\mathbb{C}$-valued function with compact support and $\theta \in \mathcal{C}_c^{+\infty}$. Indeed, because of this lemma the map U_s is also smooth, one-to-one and $\mathrm{Jac}(U_s)$ does not vanish. Let $\xi_{\mathbb{H}}(p)$ be $\Re(\zeta)\mathbf{X} + \mathbf{i}\Im(\zeta)\mathbf{Y} \in \mathcal{C}_c^{+\infty}(T\mathbb{H}_1)$. In each p, $\xi_{\mathbb{H}}(p)$ is the speed vector of the curve $U_s(p)$ of $\mathbb{H}_1$. Then the metric speed of $((U_s)_{\#}\mu)_{s>0}$ is smaller than $\|\xi_{\mathbb{H}}\|_{L^2_{\mathbb{H}}(\mu)}$ (contrarily to the previous case, here we have only an inequality). As before the derivate in time of $\mathrm{Ent}_\infty(\mu_s)$ is $-\int \rho \operatorname{div}_{\mathbb{H}}(\xi_{\mathbb{H}})d\mathcal{L}$. Note that if the entropy grows, which we want to avoid, we can replace ζ by $-\zeta$ and $\xi_{\mathbb{H}}$ by $-\xi_{\mathbb{H}}$. Therefore the following inequality with slope and speed on the right hand-side holds:

$$\left|\int \rho \operatorname{div}_{\mathbb{H}}(\xi_{\mathbb{H}})d\mathcal{L}\right| \leq \mathrm{Slope}(\mathrm{Ent}_\infty)(\mu)\cdot\|\xi_{\mathbb{H}}\|_{L^2_{\mathbb{H}}(\mu)}.$$

Because the slope $\mathrm{Slope}(\mathrm{Ent}_\infty)(\mu)$ is finite, the Riesz representation theorem provides an horizontal vector field $w^{\mathbb{H}} \in L^2_{\mathbb{H}}(\mu)$ with $\|w^{\mathbb{H}}\|_{L^2_{\mathbb{H}}(\mu)} \leq \mathrm{Slope}(\mathrm{Ent}_\infty)(\mu)$ such that for any $\xi_{\mathbb{H}} \in \mathcal{C}_c^{+\infty}(T\mathbb{H}_1)$,

$$-\int \rho \operatorname{div}_{\mathbb{H}}(\xi_{\mathbb{H}})d\mathcal{L} = \int \langle w^{\mathbb{H}} \mid \xi_{\mathbb{H}}\rangle_{\mathbb{H}} d\mu.$$

From there, one observes that $\rho \in W^{1,1}_{\mathrm{loc}}(\mathbb{H}_1)$ with $\nabla_{\mathbb{H}}\rho = \rho w^{\mathbb{H}}$.

For the second part of the proposition it is clear from Proposition 4.2.1 that *(ii)* is a consequence of *(i)*. Then we assume that the weak gradient $\mathbf{T}\rho$ exists and that there is $w^{\mathbf{T}} \in L^2(\mu)$ such that $\mathbf{T}\rho = \rho w^{\mathbf{T}}$. With the first part of the proposition, $\rho \in W^{1,1}_{\mathrm{loc}}(\mathbb{H}^\varepsilon)$ and $w^\varepsilon := w^{\mathbb{H}} + \varepsilon w^{\mathbf{T}}(\varepsilon\mathbf{T}) \in L^2_\varepsilon(\mu)$ satisfy

$$\rho w^\varepsilon = \rho w^{\mathbb{H}} + \rho\varepsilon w^{\mathbf{T}}(\varepsilon\mathbf{T}) = \nabla_{\mathbb{H}}\rho + \varepsilon\mathbf{T}\rho(\varepsilon\mathbf{T}) = \nabla_\varepsilon\rho.$$

Then Proposition 4.2.1 states that $\mathrm{Slope}^\varepsilon(\mathrm{Ent}_\infty)(\mu) < +\infty$ and $w^\varepsilon \in \mathrm{Tan}_\varepsilon(\mu)$. Therefore it is possible to approach this vector field in $L^2(\mathbb{H}_1^\varepsilon)$ by a sequence $(\nabla_\varepsilon\psi_k)_{k\in\mathbb{N}}$ where every $\psi_k \in \mathcal{C}_c^{+\infty}$. It follows that $(\nabla_{\mathbb{H}}\psi_k)_{k\in\mathbb{N}}$ tends to the horizontal part of w^ε. Hence $w^{\mathbb{H}}$ is in the tangent space $\mathrm{Tan}_{\mathbb{H}}(\mu)$.

We already know the inequality $\|w^{\mathbb{H}}\|_{L^2_{\mathbb{H}}(\mu)} \leq \mathrm{Slope}(\mathrm{Ent}_\infty)(\mu)$, we will prove the opposite inequality thanks to inequality (4.5) in Proposition 4.2.4. In this inequality, we first replace every $W^\varepsilon(\mu,\nu)$ by $W(\mu,\nu)$. It is allowed because the second is greater. Then we write ε as a function of $W^\varepsilon(\mu,\nu)$, actually we state $\varepsilon = W(\mu,\nu)^{1/3}$. But

$$\|w^{\mathbb{H}}\|_{L^2_{\mathbb{H}}} \leq \|w^\varepsilon\|_{L^2_\varepsilon} \leq \sqrt{\|w^\varepsilon\|^2_{L^2_\varepsilon} + (\varepsilon\|w^{\mathbf{T}}\|_{L^2(\mu)})^2} \leq \|w^{\mathbb{H}}\|_{L^2_{\mathbb{H}}} + \frac{1}{2}\frac{(\varepsilon\|w^{\mathbf{T}}\|_{L^2(\mu)})^2}{\|w^{\mathbb{H}}\|_{L^2_{\mathbb{H}}}}$$

because the graph of $\sqrt{\ }$ is under the tangent line in $\|w^{\mathbb{H}}\|^2_{L^2_{\mathbb{H}}}$. It follows

$$\begin{aligned}\mathrm{Ent}_\infty(\nu) \geq{}& \mathrm{Ent}_\infty(\mu) - W(\mu,\nu)\left[\|w^{\mathbb{H}}\|_{L^2_{\mathbb{H}}} + \frac{W(\mu,\nu)^{2/3}}{2\|w\|_{L^2_{\mathbb{H}}}}\|w^{\mathbf{T}}\|^2_{L^2(\mu)}\right]\\ &- W(\mu,\nu)^{4/3}\\ \geq{}& \mathrm{Ent}_\infty(\mu) - W(\mu,\nu)\|w^{\mathbb{H}}\|_{L^2_{\mathbb{H}}} - O(W(\mu,\nu)^{4/3}).\end{aligned}$$

Thus $\mathrm{Slope}(\mathrm{Ent}_\infty)(\mu) \leq \|w^{\mathbb{H}}\|_{L^2_{\mathbb{H}}}$ and the equality follows. □

4.5 Heat equations on the Heisenberg group

In this section we prove two theorems that justify the belief that the gradient flows of the entropy in the Heisenberg group are exactly the solutions of the subelliptic diffusion

$$\Delta_{\mathbb{H}}\rho_s = \partial_s \rho_s.$$

Actually Gaveau [49] proved that there is a fundamental solution $\mathfrak{h}_s$ such that the solutions of the equation on $]0,+\infty[$ are given by a convolution

$$\rho_s(p) = (\rho_0 *_{\mathbb{H}} \mathfrak{h}_s)(p) = \int_{\mathbb{H}} \mathfrak{h}_s(q^{-1}\cdot p)\rho_0(q)d\mathcal{L}(q). \tag{4.8}$$

We already introduced $\mathfrak{h}_s$ in Chapter 2 with a little different notation $\mathfrak{h}(s,\cdot)$. Note that $\mathfrak{h}$ depends on the dimension of $\mathbb{H}_n$, which does not appear in this notation (we should write $\mathfrak{h}^n$).

The density of probability of $\mathfrak{h}_s$ is non negative, smooth in time s and space p. In $\mathbb{H}_1$ it is simply the law of (B_s, L_s) where $(B_s)_{s\geq 0}$ is a scaled Brownian motion starting in $0_{\mathbb{C}}$ and $L_s = \frac{1}{2}\int B_s \times dB_s$ is the Lévy area "the algebraic area swept by the Brownian motion" (there is a stochastic meaning although the algebraic area only exists for smooth enough paths). An expression for $\mathfrak{h}_s$ is:

$$\mathfrak{h}_s(z,t) = \frac{1}{8\pi^2 s^2}\int_{\mathbb{R}} \exp\left(\frac{\lambda}{s}(it - \frac{|z|^2}{4}\coth\lambda)\right)\frac{\lambda}{\sinh\lambda}d\lambda. \tag{4.9}$$

As we already mentioned, $\mathfrak{h}_s$ is real and non-negative. Because of the rapid decay we can justify the differentiation under the integral sign and obtain

$$\begin{aligned}
\mathbf{X}\mathfrak{h}_s &= \frac{1}{16\pi^2 s^3}\int_{\mathbb{R}}(\lambda(-iy - x\coth\lambda))\exp\left(\frac{\lambda}{s}(it - \frac{|z|^2}{4}\coth\lambda)\right)\frac{\lambda}{\sinh\lambda}d\lambda\\
\mathbf{Y}\mathfrak{h}_s &= \frac{1}{16\pi^2 s^3}\int_{\mathbb{R}}(\lambda(ix - y\coth\lambda))\exp\left(\frac{\lambda}{s}(it - \frac{|z|^2}{4}\coth\lambda)\right)\frac{\lambda}{\sinh\lambda}d\lambda\\
\mathbf{T}\mathfrak{h}_s &= \frac{1}{8\pi^2 s^3}\int_{\mathbb{R}} i\lambda\exp\left(\frac{\lambda}{s}(it - \frac{|z|^2}{4}\coth\lambda)\right)\frac{\lambda}{\sinh\lambda}d\lambda\\
\Delta_{\mathbb{H}}\mathfrak{h}_s &= \frac{1}{32\pi^2 s^4}\int_{\mathbb{R}}(|z|^2\lambda^2(\coth^2(\lambda)-1))\exp\left(\frac{\lambda}{s}(it - \frac{|z|^2}{4}\coth\lambda)\right)\frac{\lambda}{\sinh\lambda}d\lambda\\
&\quad - \frac{1}{16\pi^2 s^3}\int_{\mathbb{R}}\lambda\coth(\lambda)\exp\left(\frac{\lambda}{s}(it - \frac{|z|^2}{4}\coth\lambda)\right)\frac{\lambda}{\sinh\lambda}d\lambda = \partial_s\mathfrak{h}_s.
\end{aligned}$$

In [13], Beals, Greiner and Gaveau obtain a fine estimate of the decay of $\mathfrak{h}_s$ (also in $\mathbb{H}_n$):

$$\mathfrak{h}_s(z,t) \leq C\frac{\exp\left(-\frac{d_c^2(z,t)}{4s}\right)}{s^{3/2}\sqrt{s + |z|d_c(z,t)}}. \tag{4.10}$$

Actually a similar estimate has already been obtained for $\mathbb{H}_1$ by Hueber and Müller in [59] and they proved that the reverse inequality also holds with another constant C. For estimate (4.10) these authors use a contour in $\mathbb{C}$ obtaining that is possible to replace in (4.9) the integration over λ by the integration over $\lambda + \mathbf{i}\tau$

where $|\tau| < \pi$ is fixed and λ goes on $\mathbb{R}$. From there, they examine the minimum of $\Re(\exp(f_{z,t}))$ for τ fixed and $\lambda \in \mathbb{R}$ where $f_{z,t}(\lambda+\mathbf{i}\tau) = \lambda(it - \frac{|z|^2}{4}\coth(\lambda+\mathbf{i}\tau))$ is the phase appearing in $\mathfrak{h}_1$ (and the derivatives of $\mathfrak{h}_1$) and $|\tau| < \pi$ a well-chosen parameter chosen in function of the ratio $t/|z|^2$. They deduce then the estimate on $\mathfrak{h}_s$ from the special relations (3.11). With the same method they proved the estimate

$$\begin{aligned}\left|\frac{1}{s^{n+1}}\int_{\mathbb{R}}\exp\left(\frac{\lambda}{s}(it - \frac{|z|^2}{4}\coth\lambda)\right)\left(\frac{\lambda}{\sinh\lambda}\right)^n d\lambda\right| \leq \\ C\min\left(\frac{d_c(z,t)}{|z|}, 1+\frac{d_c(z,t)^2}{s}\right)^{n-1}\frac{\exp(-\frac{d_c^2(z,t)}{4s})}{s^{n+1/2}\sqrt{s+|z|d_c(z,t)}}.\end{aligned} \tag{4.11}$$

As Hong-Quan Li noticed in [75, Lemma 3.2], it is possible to use the same technic for estimating the integrals

$$\int_{\mathbb{R}}\lambda\exp\left(\frac{\lambda}{s}(it - \frac{|z|^2}{4}\coth\lambda)\right)\frac{\lambda}{\sinh\lambda}d\lambda$$

and

$$\int_{\mathbb{R}}(\cosh\lambda)\exp\left(\frac{\lambda}{s}(it - \frac{|z|^2}{4}\coth\lambda)\right)\left(\frac{\lambda}{\sinh\lambda}\right)^2 d\lambda$$

where the holomorphic factors λ and $\cosh(\lambda)$ appear in addition to the factors in (4.9) or (4.11). These factors don't change the analysis of Beals, Gaveau and Greiner that essentially relies on the phase $f_{z,t}$. Therefore the upper bound is the same up to the constant C and from the expression of $\mathbf{X}\mathfrak{h}_s$ and $\mathbf{Y}\mathfrak{h}_s$, Hong-Quan Li obtains then

$$|\nabla_{\mathbb{H}}\mathfrak{h}_s| \leq C d_c(z,t)\frac{\exp(-\frac{d_c^2(z,t)}{4s})}{s^{5/2}\sqrt{s+|z|d_c(z,t)}}.$$

Actually this remark extends to $\Delta_{\mathbb{H}}\mathfrak{h}_s$ and we have

$$\begin{aligned}|\Delta_{\mathbb{H}}\mathfrak{h}_s| \leq & C\left(\frac{|z|^2}{s^4}\left|\int_{\mathbb{R}}\lambda^2\exp(\frac{\lambda}{s}(it - \frac{|z|^2}{4}\coth\lambda))\frac{\lambda}{\sinh\lambda}d\lambda\right|\right. \\ & + \frac{|z|^2}{s^4}\left|\int_{\mathbb{R}}\cosh^2(\lambda)\exp(\frac{\lambda}{s}(it - \frac{|z|^2}{4}\coth\lambda))\left(\frac{\lambda}{\sinh\lambda}\right)^3 d\lambda\right| \\ & \left. + \frac{1}{s^3}\left|\int_{\mathbb{R}}\cosh(\lambda)\exp(\frac{\lambda}{s}(it - \frac{|z|^2}{4}\coth\lambda))\left(\frac{\lambda}{\sinh\lambda}\right)^2 d\lambda\right|\right) \\ \leq & C\left(|z^2|\frac{d_c(z,t)^2}{|z|^2}\frac{\exp(-\frac{d_c^2(z,t)}{4s})}{s^{7/2}\sqrt{s+|z|d_c(z,t)}}\right. \\ & + \frac{|z|^2}{s}\frac{\exp(-\frac{d_c^2(z,t)}{4s})}{s^{5/2}\sqrt{s+|z|d_c(z,t)}} \\ & \left. + \min\left(1+\frac{d_c(z,t)^2}{s}, \frac{d_c(z,t)}{|z|}\right)\frac{\exp(-\frac{d_c^2(z,t)}{4s})}{s^{5/2}\sqrt{s+|z|d_c(z,t)}}\right) \\ \leq & C\left(d_c(z,t)^2 + s\right)\frac{\exp(-\frac{d_c^2(z,t)}{4s})}{s^{7/2}\sqrt{s+|z|d_c(z,t)}}\end{aligned}$$

where d_c is the d_c distance to $0_{\mathbb{H}}$.

If we now assume that the support of ρ_0 is compact and included in a ball of center $0_{\mathbb{H}}$ and radius d_0 we get new (rough) estimates for the decays of ρ_s, $|\nabla_{\mathbb{H}}\rho_s|$ and $|\Delta_{\mathbb{H}}\rho_s|$ simply considering the definition of the convolution (4.8), the following decay of $\mathfrak{h}_s$ and its derivatives. For s greater that some $s_0 > 0$ and $\eta > 0$ there is a constant C_η depending on η and s_0 and a constant C depending on s_0 such that

$$\begin{aligned} C_\eta^{-1} \exp\left(-\frac{(d_c(z,t)+d_0)^2}{4s}(1+\eta)\right) s^{-2} &\leq |\rho_s| \\ &\leq C\exp\left(-\frac{(d_c(z,t)-d_0)^2}{4s}\right) s^{-2}, \end{aligned} \tag{4.12}$$

$$|\mathbf{T}\rho_s| = |\rho_0 *_{\mathbb{H}} (\mathbf{T}\mathfrak{h}_s)| \leq \rho_0 *_{\mathbb{H}} |\mathbf{T}\mathfrak{h}_s| \leq C\exp\left(-\frac{(d_c(z,t)-d_0)^2}{4s}\right) s^{-3}, \tag{4.13}$$

$$\begin{aligned} |\nabla_{\mathbb{H}}\rho_s| = |\rho_0 *_{\mathbb{H}} (\nabla_{\mathbb{H}}\mathfrak{h}_s)| &\leq \rho_0 *_{\mathbb{H}} |\nabla_{\mathbb{H}}\mathfrak{h}_s| \\ &\leq C_\eta \exp\left(-\frac{(d_c(z,t)-d_0)^2}{4s}(1-\eta)\right) s^{-5/2}, \end{aligned} \tag{4.14}$$

$$\begin{aligned} |\Delta_{\mathbb{H}}\rho_s| = |\rho_0 *_{\mathbb{H}} (\Delta_{\mathbb{H}}\mathfrak{h}_s)| &\leq \rho_0 *_{\mathbb{H}} |\Delta_{\mathbb{H}}\mathfrak{h}_s| \\ &\leq C_\eta \exp\left(-\frac{(d_c(z,t)-d_0)^2}{4s}(1-\eta)\right) s^{-3}. \end{aligned} \tag{4.15}$$

Theorem 4.5.1. *Let $(\rho_s)_{s\in]0,+\infty[}$ be a solution of the subelliptic heat equation*

$$\begin{cases} \Delta_{\mathbb{H}}\rho_s = \partial_s\rho_s \\ \rho_0 d\mathcal{L} = \mu_0 \end{cases}$$

in $\mathbb{H}_1$ where μ_0 has a compact support. The curve $(\mu_s)_{s\geq 0}$ of measures $\rho_s d\mathcal{L} = \mu_s$ is a gradient flow of the entropy Ent_∞.

Proof. Let $\psi \in \mathcal{C}_c^{+\infty}$. We recall that ρ_s is smooth in space and time for $s > 0$. Moreover, $\psi\rho_s$ is in $\mathcal{C}_c^{+\infty}$ and its support is in the support of ψ. Then

$$\begin{aligned} \frac{d}{ds}\int_{\mathbb{H}} \psi\rho_s d\mathcal{L} &= \int_{\mathbb{H}} \psi\partial_s\rho_s d\mathcal{L} = \int_{\mathbb{H}} \psi\Delta_{\mathbb{H}}\rho_s d\mathcal{L} \\ &= \int_{\mathbb{H}} \langle\nabla_{\mathbb{H}}\psi \mid -\nabla_{\mathbb{H}}\rho_s\rangle_{\mathbb{H}} d\mathcal{L} = \int_{\mathbb{H}} \langle\nabla_{\mathbb{H}}\psi \mid -\frac{\nabla_{\mathbb{H}}\rho_s}{\rho_s}\rangle_{\mathbb{H}} d\mu_s. \end{aligned} \tag{4.16}$$

Hence

$$\partial_s(\mu_s) + \mathrm{div}_{\mathbb{H}}\left(-\frac{\nabla_{\mathbb{H}}\rho_s}{\rho_s}\mu_s\right) = 0 \tag{4.17}$$

holds for every s. We prove now that $-\frac{\nabla_{\mathbb{H}}\rho_s}{\rho_s}$ is in $L^2_{\mathbb{H}}(\mu_s)$. Indeed the integral $\int \frac{\|\nabla_{\mathbb{H}}\rho_s\|^2_{\mathbb{H}}}{\rho_s} d\mu_s$ is finite because we can estimate the numerator from above with (4.14) and the denominator from below with (4.12). Moreover, the domination is such that $\|\frac{\nabla_{\mathbb{H}}\rho_s}{\rho_s}\|_{L^2_{\mathbb{H}}(\mu_s)}$ is continuous on I.

Starting from (4.17) because of the Mass Conservation Formula [109, Chapter 1] there is a probability measure Π on $\mathcal{C}(I, \mathbb{H}_1)$, the space of curves over $\mathbb{H}_1$ with the Borel sigma-field, satisfying two conditions:

- the curve γ is $\Pi(\gamma)$-almost certainly is an integral curve of the smooth vector field $-\nabla_{\mathbb{H}}\rho_s/\rho_s$,
$$\dot{\gamma}_s = -\frac{\nabla_{\mathbb{H}}\rho_s}{\rho_s}$$
- the law of the point $\gamma(s)$ with respect to Π is μ_s.

Then

$$\begin{aligned}
W^2(\mu_s, \mu_t) &\leq \int_{\mathcal{C}(I,\mathbb{H}_1)} d_c^2(\gamma_s, \gamma_t) d\Pi(\gamma) \\
&\leq \int_{\mathcal{C}(I,\mathbb{H}_1)} \left((t-s) \int_s^t \left\| \frac{\nabla_{\mathbb{H}}\rho_\tau}{\rho_\tau}(\gamma_\tau) \right\|_{\mathbb{H}}^2 d\tau \right) d\Pi(\gamma) \\
&\leq (t-s) \int_s^t \left(\int_{\mathbb{H}_1} \left\| \frac{\nabla_{\mathbb{H}}\rho_\tau}{\rho_\tau} \right\|_{\mathbb{H}}^2 d\mu_\tau \right) d\tau \\
&\leq (t-s)^2 \sup_{\{\tau \in [s,t]\}} \|\nabla_{\mathbb{H}}\rho_\tau/\rho_\tau\|^2_{L^2_{\mathbb{H}}(\mu_\tau)}.
\end{aligned}$$

Hence $(\mu_s)_{s\in I}$ is locally Lipschitz and therefore this curve is absolutely continuous. The norm $\|\nabla_{\mathbb{H}}\rho_s/\rho_s\|_{L^2_{\mathbb{H}}(\mu_\tau)}$ is continuous such that letting t go to s we get

$$|\dot{\mu}_s| \leq \|\nabla_{\mathbb{H}}\rho_s/\rho_s\|_{L^2_{\mathbb{H}}(\mu_s)}$$

at every time s where $|\dot{\mu}_s|$ exists.

We can identify the Slope of Ent_∞. Actually by using (4.12) and (4.13) one get that

$$\|\nabla_\varepsilon\rho_s/\rho_s\|_{L^2_\varepsilon(\mu_s)} = \sqrt{\|\nabla_{\mathbb{H}}\rho_s/\rho_s\|^2_{L^2_{\mathbb{H}}(\mu_s)} + \varepsilon^2 \int \left(\frac{\mathbf{T}\rho_s}{\rho_s}\right)^2 \rho_s d\mathcal{L}}$$

is finite. But this quantity is $\mathrm{Slope}^\varepsilon(\mathrm{Ent}_\infty)(\mu_s)$ such that condition *(ii)* of Proposition 4.4.1 is satisfied and as ρ_s does not vanish the vector field $w_s^{\mathbb{H}}$ of Proposition 4.4.1 is $\nabla_{\mathbb{H}}\rho_s/\rho_s$. From there in the Wasserstein space of the "true" Heisenberg group we have $\mathrm{Slope}(\mathrm{Ent}_\infty)(\mu_s) = \|\nabla_{\mathbb{H}}\rho_s/\rho_s\|_{L^2_{\mathbb{H}}(\mu_s)}$.

We compute now $\dot{E}(s)$ where $E(s) = \mathrm{Ent}_\infty(\mu_s)$ as in Section 4.1. Firstly

$$\partial_s(\rho_s \ln(\rho_s)) = (1 + \ln(\rho_s))\partial_s\rho_s = (1 + \ln(\rho_s))\Delta_{\mathbb{H}}\rho_s. \tag{4.18}$$

Using (4.12) and (4.15) we see that about every time $s_0 > 0$, (4.18) is dominated independently of s by a function of $L^1(\mathbb{H}_1)$ (use for instance $X \to |1 + \ln(X)| < X + X^{-1/2}$). Therefore it is allowed to derivate under the integral sign and we obtain

$$\dot{E}(s) = \int (1 + \ln(\rho_s))\Delta_{\mathbb{H}}\rho_s.$$

We want now to justify the partial integration

$$\dot{E}(s) = -\int \langle \nabla_{\mathbb{H}}(1 + \ln(\rho_s)) \mid \nabla_{\mathbb{H}}\rho_s \rangle \tag{4.19}$$

which is less certain as the one in (4.16) because the supports of $1+\ln(\rho_s)$, $\mathbf{X}\rho_s$ and $\mathbf{Y}\rho_s$ are infinite. However, if for a smooth function f we integrate $\mathbf{X}f = \partial_x f - \frac{y}{2}\partial_t f$ on $[-R,R]^3$, we get

$$\begin{aligned}\left|\int_{[-R,R]^3} \mathbf{X} f d\mathcal{L}\right| &= \left|\int_{\{x=R\}\times[-R,R]^2} f - \int_{\{x=R\}\times[-R,R]^2} f\right. \\ &\quad \left. -\frac{1}{2}\int_{[-R,R]^2\times\{t=R\}} yf + \frac{1}{2}\int_{[-R,R]^2\times\{t=-R\}} yf\right| \\ &\leq \int_{\partial[-R,R]^3} |f|\end{aligned}$$

where $\partial[-R,R]^3$ is the border of $[-R,R]^3$. We are interested in $f = (1+\ln(\rho_s))\mathbf{X}\rho_s$. As we have (4.14) and because d_c is Lipschitz-equivalent to d_{KR} (see Subsection 1.1.3), the previous integral tends to zero as R goes to infinity. A similar computation holds for $\left|\int_{[-R,R]^3} \mathbf{Y}((1+\ln(\rho_s))\mathbf{Y}\rho_s)\right|$. It also tends to zero when R goes to infinity. From there the partial integration in (4.19) is justified. It follows

$$\dot{E}_s = \int \left\langle \frac{\nabla_{\mathbb{H}}\rho_s}{\rho_s} \mid \frac{\nabla_{\mathbb{H}}\rho_s}{\rho_s} \right\rangle_{\mathbb{H}} \rho_s d\mathcal{L} = \int \left\|\frac{\nabla_{\mathbb{H}}\rho_s}{\rho_s}\right\|^2 d\mu_s = \left\|\frac{\nabla_{\mathbb{H}}\rho_s}{\rho_s}\right\|^2_{L^2_{\mathbb{H}}(\mu_s)}.$$

Because of the differentiability properties of the gradient flow, we have

$$\dot{E}_s \leq \mathrm{Slope}(\mathrm{Ent}_\infty)(\mu_s)\cdot|\dot{\mu}_s| \tag{4.20}$$

in almost every $s\in I$. However, in this proof we have shown that is almost every $s\in I$

$$\begin{cases} |\dot{\mu}_s| \leq \left\|\frac{\nabla_{\mathbb{H}}\rho_s}{\rho_s}\right\|_{L^2_{\mathbb{H}}(\mu_s)} \\ \mathrm{Slope}(\mu_s) = \left\|\frac{\nabla_{\mathbb{H}}\rho_s}{\rho_s}\right\|_{L^2_{\mathbb{H}}(\mu_s)} \\ \dot{E}(s) = \left\|\frac{\nabla_{\mathbb{H}}\rho_s}{\rho_s}\right\|^2_{L^2_{\mathbb{H}}(\mu_s)} \end{cases}$$

such that inequality (4.20) is an equality for almost every $s\in I$. Thus $(\mu_s)_{s\in I}$ is a gradient flow of Ent_∞. ☐

Theorem 4.5.2. *Let $(\mu_s)_{s\in I}$ be a gradient flow of Ent_∞ in $\mathcal{P}_2(\mathbb{H}_1)$ as defined in Section 4.1. Assume that for almost every $s\in I$, there exists a weak derivative $\mathbf{T}\rho_s$ and a function $w_s^{\mathbf{T}}\in L^2(\mu_s)$ such that $\mathbf{T}\rho_s = w_s^{\mathbf{T}}\rho_s$. Then the density $(\rho_s)_{s\in I}$ satisfies the "subelliptic heat equation"*

$$\partial_s\rho_s = \Delta_{\mathbb{H}}\rho_s.$$

Proof. Let $(\mu_s)_{s\in I}$ be a gradient flow of the entropy in $\mathcal{P}_2(\mathbb{H}_1)$. This curve is absolutely continuous (even in AC^2). Hence $\mathrm{Slope}(\mathrm{Ent}_\infty)(\mu_s) = |\dot{\mu}_s|$ is finite in almost every $s\in I$. With the assumption about $w^{\mathbf{T}}$ we get that Proposition

4.4.1 applies, in particular for any $\varepsilon > 0$ we have $\rho_s \in W^{1,1}_{\mathrm{loc}}(\mathbb{H}_1^\varepsilon)$ in almost every s. Therefore Proposition 4.2.1 and Proposition 4.2.4 apply too. Actually $(\mu_s)_{s\in I}$ is a gradient flow of Ent_∞ such that with Remark 4.1.1 it is a AC^2-curve of $\mathcal{P}_2(\mathbb{H}_1)$. Consequently it is a AC^2-curve of $\mathcal{P}_2(\mathbb{H}_1^\varepsilon)$ too.

From Proposition 4.4.1, we know that $\mathrm{Slope}(\mathrm{Ent}_\infty)(\mu_s) = \|w_s^{\mathbb{H}}\|_{L^2_{\mathbb{H}}}$ in almost every s where $w_s^{\mathbb{H}} \in \mathrm{Tan}_{\mathbb{H}}(\mu_s)$ is the horizontal part of $w_s^\varepsilon \in \mathrm{Tan}_\varepsilon(\mu_s)$ with $\mathrm{Slope}^\varepsilon(\mathrm{Ent}_\infty)(\mu_s) = \|w_s^\varepsilon\|_{L^2_\varepsilon}$. We fix now some $s' \in I'_\varepsilon \cap I'$ with the notation of Proposition 4.2.4 and Proposition 4.3.1. Then we would like to interpret the scalar product $\int \langle v_{s'}^\varepsilon \mid w_{s'}^\varepsilon\rangle_\varepsilon d\mu_{s'}$ appearing in (4.4) of Proposition 4.2.4 in terms of the Wasserstein space $\mathcal{P}_2(\mathbb{H}_1)$. First of all in the tangent space $\mathrm{Tan}_\varepsilon(\mu_s)$ we approach the vector field $w_{s'}^\varepsilon$ by a sequence $(\nabla_\varepsilon\psi_k)_{k\in\mathbb{N}}$. Then because of in Proposition 4.2.4, we see that $\int\langle w_{s'}^\varepsilon \mid v_{s'}^\varepsilon\rangle_\varepsilon d\mu_{s'}$ is the limit of $\frac{\partial}{\partial s}\mu_s(\psi_k)(s')$. But we know from Proposition 4.3.1 that the previous derivative is also $\int\langle\nabla_{\mathbb{H}}\psi_k|v_{s'}\rangle_{\mathbb{H}} d\mu_{s'}$ such that the limit is simply $\int\langle w_{s'}^{\mathbb{H}} \mid v_{s'}\rangle_{\mathbb{H}} d\mu_{s'}$. Indeed $(\nabla_{\mathbb{H}}\psi_k)_{k\in\mathbb{N}}$ tends to to the horizontal part w_s^ε.

Then in almost every $s' \in I$, (4.4) writes

$$\mathrm{Ent}_\infty(\mu_s) - \mathrm{Ent}_\infty(\mu_{s'}) \geq (s - s')\int\langle w_{s'}^{\mathbb{H}}|v_{s'}\rangle_{\mathbb{H}} d\mu_{s'} + o(|s-s'|).$$

But $(\mu_s)_{s\in I}$ is a gradient flow. It satisfies $\dot{E}_{s'} = -\left(\mathrm{Slope}(\mathrm{Ent}_\infty)(\mu_{s'})\right)\cdot|\dot{\mu}_{s'}|$ where $|\dot{\mu}_{s'}| \geq \|v_{s'}\|_{L^2_{\mathbb{H}}(\mu_{s'})}$ and $\mathrm{Slope}(\mathrm{Ent}_\infty)(\mu_{s'}) = \|w_{s'}^{\mathbb{H}}\|_{L^2_{\mathbb{H}}(\mu_s)}$. Then because of the Cauchy-Schwarz inequality, the only possibility is that $v_{s'}$ and $w_{s'}^{\mathbb{H}}$ are negatively collinear in $\mathrm{Tan}_{\mathbb{H}}(\mu_{s'})$. Moreover, as the gradient flows satisfy $\mathrm{Slope}(\mathrm{Ent}_\infty)(\mu_{s'}) = |\dot{\mu}_{s'}|$, we have simply $v_{s'} = -w_{s'}^{\mathbb{H}}$ in almost every $s' \in I$. Replacing $v_{s'}$, one can therefore rewrite the continuity equation of Proposition 4.3.1:

$$\partial_{s'}\mu_{s'} + \mathrm{div}_{\mathbb{H}}(-w_{s'}^{\mathbb{H}}\mu_{s'}).$$

Because of relation $\nabla_{\mathbb{H}}\rho_{s'} = \rho_{s'}w_{s'}^{\mathbb{H}}$ it is also

$$\partial_s\mu_s + \mathrm{div}_{\mathbb{H}}(-\nabla_{\mathbb{H}}\rho_{s'}d\mathcal{L}) \tag{4.21}$$

where $\nabla_{\mathbb{H}}\rho_{s'}$ is the weak gradient of $\rho_{s'}$. Remind that (4.21) means that for any $\psi \in \mathcal{C}_c^{+\infty}$,

$$\left(\frac{\partial}{\partial s}\int\psi d\mu_s\right)|_{s'} = \int\langle\nabla_{\mathbb{H}}\psi| - \nabla_{\mathbb{H}}\rho_{s'}\rangle_{\mathbb{H}} d\mathcal{L}. \tag{4.22}$$

We know from the proof of Proposition 4.3.1 that the integral on the left-hand side, namely ζ^ψ is absolutely continuous in s. It follows that we can integrate (4.22) on an intervall $[\sigma,\tau]$ and obtain

$$\int\psi\rho_\tau d\mathcal{L} - \int\psi\rho_\sigma d\mathcal{L} = -\int_\sigma^\tau\int\langle\nabla_{\mathbb{H}}\psi|\nabla_{\mathbb{H}}\rho_{s'}\rangle_{\mathbb{H}} d\mathcal{L}ds'$$

for any $\psi \in \mathcal{C}_c^{+\infty}$. We recognize a weak formulation of the "subelliptic heat equation". By using classical references about hypoelliptic operator as [95] or [103] and the references therein, this concludes the proof. ☐

Remark 4.5.3. As we already mentioned, Theorem 4.5.1 and Theorem 4.5.2 are also true in $\mathbb{H}_n$. In order to make the proofs clearer, we made the proof for

$\mathbb{H}_1$ writing $\mathbf{X}$ and $\mathbf{Y}$ for the horizontal vectors. However, these vector can be changed by a sum of $\mathbf{X}_k$ or of $\mathbf{Y}_k$ each time one need it. The only possible problem concern the decay of $\mathfrak{h}_s$ and of its derivative for dimension n. Also in this case, there is an expression of $\mathfrak{h}_s$ these satisfies analogous estimates (see [76]). These estimates are sufficient for the proof.

Remark 4.5.4. It is also possible to make the proofs in the Albanese torus $\mathbb{T}$ approximating it by $\mathbb{T}^\varepsilon$ (see Section 1.2). The fact that $\mathbb{T}$ is compact change small elements in the proofs. The partial integration are still right in this case and one does not require rapid decay. Furthermore the assumption on the support of μ_0 in Theorem 4.5.1 is unnecessary. It follows that one can mix Theorem 4.5.1 and Theorem 4.5.2 in the following way

Theorem 4.5.5. *Let $(\mu_s)_{s\in I}$ be a curve of $\mathcal{P}_2(\mathbb{T})$ and ρ_s the density curve of μ_s with respect to $\mathcal{L}_\mathbb{T}$. Both statements are equivalent*

- *The density curve $(\rho_s)_{s\in]0,+\infty[}$ is a solution of the subelliptic heat equation of $\mathbb{T}$,*
$$\Delta_\mathbb{T}\rho_s = \partial_s\rho_s,$$
where $\Delta_\mathbb{T} = \mathbf{X}_\mathbb{T}^2 + \mathbf{Y}_\mathbb{T}^2$.

- *The curve $(\mu_s)_{s\in I}$ of measures with density $\rho_s = d\mu_s/d\mathcal{L}$ is a gradient flow of the entropy* Ent_∞ *and the weak derivative $\mathbf{T}\rho_s$ exists with $w^\mathbf{T}$ defined by $w_s^\mathbf{T}\rho_s = \mathbf{T}\rho_s$ in $L^2_\mathbb{T}$.*

Remark 4.5.6. In [66, Section 6], Khesin and Lee prove a similar result to Theorem 4.5.1. There paper takes place in the wide class of bracket-generating distribution τ on a connected and compact manifold M (it includes the Albanese torus). They also approximate their metric space by Riemannian manifolds completing the horizontal tangent space by the other directions. However, they does not prove Theorem 4.5.1 for the Heisenberg group that is not compact. Their proof is more algebraic than the proofs of this section and the Wasserstein space they are considering is a "smooth" Wassersein space. It is restricted to smooth measure and the distance is possibly different because the authors begin to give a tangential structure to $\mathcal{P}_2(M_\tau)$ defining the length of curves on the Wasserstein space and then defining the distance. From there the definition of the gradient flow is different: the solution of the subelliptic equation goes in the "smooth" direction with the greatest slope but a rough $\mathrm{Slope}(\mathrm{Ent}_\infty)$ is not defined. In their paper, there is no analogous result to Theorem 4.5.2.

Appendix A

Résumé en français

Introduction

Le groupe de Heisenberg $\mathbb{H}_n$ apparaît dans de divers domaines mathématiques ou plus généralement scientifiques et techniques. Il s'agit en effet d'un espace de référence en théorie du contrôle et en géométrie sous-riemannienne tout comme $\mathbb{R}^n$ est l'espace de référence de la géométrie riemannienne. Ces espaces ont entre autres choses, en commun le fait de vérifier une inégalité de Poincaré locale avec une mesure canonique doublante. Ceci constitue un cadre très apprécié pour l'analyse dans les espaces métriques mesurés (voir [57]). Plus que cela, le groupe de Heisenberg permet de mesurer le degré de généralité des théories sur les espaces métriques car il se prête assez bien aux calculs. Dans cette veine Ambrosio et Rigot [7] ont étendu à $\mathbb{H}_n$, une grande part des résultats connus dans le cas riemannien au sujet du transport de mesure. En particulier il existe un unique transport optimal d'une mesure absolument continue sur une deuxième mesure et ce plan de transport est donné par une application. Dans ce résumé de thèse, nous allons donner des résultats complémentaires correspondants aux (quatre) résultats principaux de la thèse ou à des versions simplifiées de ceux-ci.

Nous ferons tout d'abord l'analyse de certaines courbes du groupe de Heisenberg. D'une part on verra ce que sont les géodésiques de cet espace (en particulier les $\mathbb{H}$-droites) et d'autre part on présentera une courbe horizontale curieuse, ω qui constitue un contre-exemple au sujet du problème du voyageur de commerce géométrique dans le groupe de Heisenberg. En effet la généralisation initiée par Ferrari, Franchi et Pajot [40] d'un théorème euclidien de Jones [62] n'est valable que dans le sens direct. La réciproque est fausse car ω est de longueur finie alors que l'intégrale des nombres $\beta_{\mathbb{H}}(x, r)$ diverge. Ces nombres $\beta_{\mathbb{H}}(x, r)$ mesurent l'éloignement de ω dans la boule de centre x et rayon r par rapport à la $\mathbb{H}$-droite la plus proche.

La propriété MCP (Measure Contraction Property) démontrée à partir de l'analyse des géodésiques sera de première importance pour deux des trois résultats relatifs au transport de mesure dans le groupe de Heisenberg. Figalli et l'auteur ont résolu dans [42] une question posée par Ambrosio et Rigot à la fin de leur article [7, partie 7] : tout comme sur les variétés riemanniennes, les mesures qu'interpolent les transports optimaux partant d'une mesure absolument continue, sont aussi absolument continus. L'estimée des contractions joue un rôle central dans cette démonstration.

Par ailleurs il sera question de l'application récente qui a été faite du transport de mesure pour définir ce qu'est un espace métrique mesuré dont la courbure est minorée. On doit ce développement passionnant à Lott et Villani [77, 78] ainsi qu'à Sturm [104, 105]. Ces auteurs ont tiré parti de l'équivalence qui existe pour les variétés riemanniennes de dimension inférieure à N entre, avoir une courbure de Ricci uniformément supérieure à K, et avoir une fonctionnelle entropie convexe dans un certain sens le long du transport de mesure. Cette seconde propriété porte le nom de courbure-dimension $CD(K, N)$ (nom emprunté à Bakry et Émery [11] désignant une propriété apparentée mais différente) et a un sens dans les espaces métriques mesurés. Rien n'indique de façon directe que $\mathbb{H}_n$ doive ou ne doive pas vérifier la condition de courbure-dimension. On a montré dans [64] que cette propriété est fausse pour $\mathbb{H}_n$ quels que soient les paramètres K et N. La propriété $MCP(K, N)$ déjà mentionnée auparavant est tout comme $CD(K, N)$, une inégalité géométrique qu'un espace métrique mesuré peut vérifier ou non et qu'on interprète comme une courbure de Ricci minorée par K. Dans le cas de $\mathbb{H}_1$, la propriété sera vraie si et seulement si $K \leq 0$ et $N \geq 5$. De façon surprenante, alors que la définition de MCP engage des idées proche du transport de mesure, cette propriété peut être vraie, tandis que CD ne l'est pas (MCP est généralement plus faible). Par ailleurs la dimension 5, optimale, est assez inattendue : ce n'est ni la dimension topologique (qui est 3) ni la dimension de Hausdorff (qui est 4) du groupe de Heisenberg.

Le dernier résultat de ce résumé concerne la diffusion sous-elliptique dans $\mathbb{H}_1$ et sa présentation comme flot du gradient dans l'espace de Wasserstein $\mathcal{P}_2(\mathbb{H}_1)$. Celui-ci est l'espace des mesures de $\mathbb{H}_1$ considéré avec la distance du transport de mesure, dite distance de Wasserstein. Il apparaît qu'en se déplaçant continument dans cet espace de manière à abaisser aux mieux l'entropie de Bolzmann Ent_∞ des mesures considérées, on trouve une courbe de mesures dont la densité est solution de l'équation de la chaleur sous-elliptique.

Les résultats mentionnés se trouvent aux emplacements suivants dans la thèse. La courbe ω et sa qualité de contre-exemple sont présentés en la sous-section 1.8. La résolution de la question d'Ambrosio et Rigot dans $\mathbb{H}_n$ apparaît au Théorème 2.3.6 où on utilise une inégalité essentiellement équivalente à $MCP(0, 2n+3)$. Le traitement des courbures de Ricci $CD(K, N)$ et $MCP(K, N)$ dans $\mathbb{H}_n$ est faite au chapitre 3 (voir Theorem 3.4.5 et Theorem 3.5.12 avec les extensions). On y examine tout les $K \in \mathbb{R}$ et $N \in [0, +\infty]$ au contraire de ce résumé où on s'est restreint à $K = 0$ pour $\mathbb{H}_1$. Le dernier résultat principal au sujet du flot de gradient de l'entropie se trouve divisé en deux théorèmes (Theorem 3.5.12 et Theorem 3.5.13) au chapitre 4.

Ce résumé comporte quatre parties chacune d'entre elles avec un des résultats principaux. Dans la première partie après avoir défini $\mathbb{H}_1$, on verra quelles sont ses géodésiques et on présentera la courbe ω relative au problème du voyageur de commerce géométrique. On donnera aussi des indications sur la preuve de $MCP(0, 5)$. Dans la deuxième partie il sera question des définitions du transport de mesure et de CD. On citera le théorème de Ambrosio et de Rigot sur le transport de mesure dans le groupe de Heisenberg, puis on expliquera comment montrer l'absolue continuité des mesures interpolées lors de ce transport. La troisième partie continuera avec une comparaison des courbures de Ricci synthétiques MCP et CD dans le groupe de Heisenberg et une démonstration de ce que $CD(0,N)$ n'est pas vraie dans $\mathbb{H}_1$. On finira avec la correspondance entre flot de gradient de l'entropie et la diffusion sous-elliptique dans la dernière partie.

A.1 Le groupe de Heisenberg, courbes et géodésiques

A.1.1 Premiers contacts avec $\mathbb{H}_1$

À $\mathbb{H}_1$ sont associés la distance de Carnot-Carathéodory d_c ainsi que $\mathcal{L}$, la mesure de Lebesgue de $\mathbb{R}^3$. On peut en effet présenter $\mathbb{H}_1$ comme $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ si bien qu'un élément courant sera noté $(z;t)$ avec $z = x + \mathbf{i}y$. Le produit du groupe est le suivant:

$$(z;t) \cdot (z';t') = (z + z'; t + t' - \frac{1}{2}\Im(z\overline{z'}))$$

où $\Im$ est la fonction partie imaginaire. Avec ce produit $\mathbb{H}_1$ est un groupe de Lie d'élément neutre $(0_{\mathbb{C}}; 0_{\mathbb{R}})$ et l'inverse de $(z;t)$ est $(-z;-t)$. Dans l'algèbre de Lie des vecteurs invariants par translation à gauche on utilisera la base

$$\mathbf{X} = \partial_x - \frac{1}{2}y\partial_t \quad , \quad \mathbf{Y} = \partial_y + \frac{1}{2}x\partial_t \quad , \quad \mathbf{T} = \partial_t.$$

L'ensemble $L = \{(z;t) \in \mathbb{H}_1 \mid z = 0\}$ est à la fois sous-groupe dérivé et centre du groupe. Comme nous le verrons, L joue aussi un rôle important pour la géométrie de $(\mathbb{H}_1, d_c)$.

Soit maintenant $\tau_p(q) = p \cdot q$ la translation à gauche par p. On remarque qu'il s'agit d'une transformation affine qui préserve le volume de $\mathbb{R}^3$. Son déterminant vaut en effet 1. Cela fait de $\mathcal{L}$ la mesure de Haar du groupe. La distance d_c que nous allons définir maintenant est elle aussi invariante par translation à gauche.

Entre deux points, d_c est définie comme l'infimum des longueurs des courbes reliant ces deux points. La fonctionnelle longueur dont il est question a une définition spécifique que nous allons donner par la suite et qui s'exprime pour les courbes absolument continues de $\mathbb{R}^3$ qui vérifient la condition d'horizontalité

$$\gamma_t' = \frac{1}{2}(\gamma_x \gamma_y' - \gamma_y \gamma_y') \qquad \text{pour presque tout } s \in [s_0, s_1] \tag{A.1}$$

où $(\gamma_x, \gamma_y, \gamma_t)$ sont les coordonnées de γ. Cette condition exprime le fait que la troisième coordonnée doit croître proportionnellement à l'aire balayée par le vecteur $\overrightarrow{0_{\mathbb{C}}g}$ où $g = (\gamma_x, \gamma_y)$ désigne la projection de la courbe γ sur $\mathbb{C}$. On a $g = Z(\gamma)$ où $Z : (z;t) \to z$ est la projection complexe. Plus classiquement, on présente habituellement de façon équivalente les courbes horizontales comme celles qui sont tangentes en presque tout temps s au sous-espace engendré par $\mathbf{X}(\gamma(s))$ et$\mathbf{Y}(\gamma(s))$.

La longueur d'une courbe horizontale est alors précisemment $\int_{s_0}^{s_1} \sqrt{\gamma_x'^2 + \gamma_y'^2}$, à savoir la longueur euclidienne de la courbe projetée $g(s)$. Une courbe non-horizontale sera considérée de longueur infinie. Récapitulons:

$$d_c(p,q) = \inf_{(\gamma(s_0),\gamma(s_1))=(p,q)} \begin{cases} \int_{s_0}^{s_1} \sqrt{\gamma_x'^2 + \gamma_y'^2} & \text{si (A.1)} \\ +\infty & \text{sinon.} \end{cases} \tag{A.2}$$

L'infimum dans la formule (A.2) est en fait atteint par au moins une courbe. Une telle courbe sera appelée géodésique dans ce qui suivra et le nom ne sera pas donné abusivement puisque quitte à la reparamétrer, la courbe considérée sera un plongement isométrique d'un segment de $\mathbb{R}$ dans $(\mathbb{H}_1, d_c)$. Le fait qu'il existe des courbes qui minimisent la longueur peut se déduire de la connaissance des solutions au problème de Didon, une variante du problème isopérimétrique plan. Il s'agit de comparer pour une courbe $g(s) \in \mathbb{C}$ non fermée, la longueur de la courbe à l'aire algébrique qu'elle entoure : les solutions sont uniques et sont des arcs de cercles (voir figure A.1). Dans le problème d'infimum (A.2), on cherche à minimiser la longueur des courbes allant de $p = (z_p; t_p)$ à $q = (z_q; t_q)$ et satisfaisant la condition d'horizontalité (A.1). Alors on peut représenter bijectivement les courbes γ par leurs projections complexes g à partir du moment où :

- $\gamma(0) = p$
- La courbe projetée g joint z_p à z_q.
- Cette courbe balaie une aire algébrique de valeur $t_q - t_p$.

Minimiser la longueur de γ ou de façon équivalente celle de g revient donc exactement à résoudre le problème de Didon : la courbe g de plus petite longueur est un arc de cercle. En menant de p le relevé horizontal de cette courbe, on arrive en q (du fait de la condition sur l'aire algébrique) et la courbe relevée γ est une géodésique de $\mathbb{H}_1$.

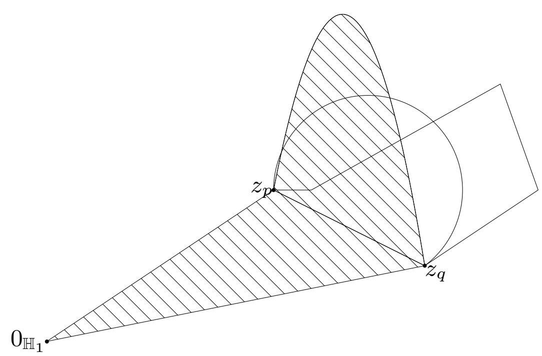

Figure A.1: Parmi les courbes balayant une aire donnée, l'arc de cercle est la plus courte.

Suite à ces explications rapides, nous pouvons donner les équations explicites des géodésiques partant de l'origine $0_{\mathbb{H}} = (0; 0)$. Nous paramétrons ces courbes par longueur d'arc sur le segment $[0, 1]$. Suivant le principe indiqué à la figure A.2, on obtient tous les arcs de cercle en spécifiant leur vecteur tangent à l'origine (c'est le vecteur $v \in \mathbb{C}$) ainsi que leur angle d'ouverture $\varphi \in [-\pi, \pi]$. Il y a une certaine diversité dans les arcs de cercle : parmi les cas particuliers

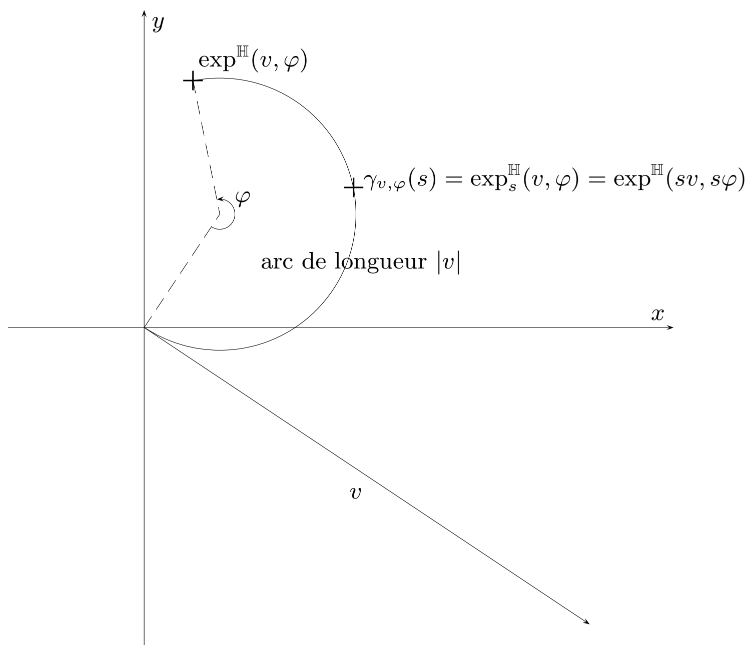

Figure A.2: Projection de $\gamma_{v,\varphi}$ sur $\mathbb{C} \times \{0\}$ dans $\mathbb{H}_1$.

on trouve les cercles entiers dont l'ouverture d'angle est de 2π ou -2π (selon le sens de rotation), les demi-cercles qui sont tels que $|\varphi| = \pi$ ou encore les segments de droite dont l'ouverture d'angle est 0. On appèle ces derniers segments géodésiques et leur prolongations géodésiques sont appelées $\mathbb{H}$-droites ou droites horizontales. Ces dernières ont la particularité d'être globalement minimales. Ce sont par ailleurs de véritables droites de $\mathbb{R}^3$. Au contraire des $\mathbb{H}$-droites les autres géodésiques sont seulement localement minimales. Pour $|\varphi| > 2\pi$, les arcs de cercle considŕés font plus d'un tour et, parmi les courbes balayant la même aire algébrique ils ne sont pas de longueur minimale : L'arc géodésique minimal est alors obtenu à comme le relevé d'un arc de cercle de rayon plus grand. On a finalement les équations des géodésiques en fonction des paramètres v et φ.

$$\gamma_{v,\varphi}(s) = \begin{cases} \left(v\frac{e^{i\varphi s}-1}{i\varphi}, |v|^2 \left(\frac{\varphi s - \sin(\varphi s)}{2\varphi^2} \right) \right) & \text{si } \varphi \neq 0 \\ (sv, 0) & \text{si } \varphi = 0. \end{cases}$$

La coordonnée complexe suit une paramétrisation d'arc de cercle tandis que la troisième coordonnée se déduit du calcul de l'aire algébrique balayée par cet arc en fonction de s.

Les géodésiques partant du point p sont simplement les courbes $\gamma_{v,\varphi}^p := p \cdot \gamma_{v,\varphi}$. La raison en est que la translation à gauche τ_p conserve l'horizontalité des courbes (comme on peut le vérifier) et ne fait que translater dans $\mathbb{C}$ la projection complexe de ces-dernières. Il s'en suit que τ_p préserve les longueurs et la distance de $\mathbb{H}_1$.

Cette paramétrisation des géodésiques partant de l'origine est similaire à la démarche qu'on a lorsqu'on définit les coordonées sphériques d'un point. En s'en inspirant on peut donc considérer (v, φ) comme des coordonnées sphériques de $\mathbb{H}_1$ et l'application

$$\exp^{\mathbb{H}}(v, \varphi) := \gamma_{v,\varphi}(1) = \begin{cases} \left(v\frac{e^{i\varphi}-1}{i\varphi}, |v|^2\frac{\varphi-\sin(\varphi)}{2\varphi^2}\right) & \text{si } \varphi \neq 0 \\ (sv, 0) & \text{si } \varphi = 0. \end{cases}$$

comme une application de changement de coordonnées sphérique-carthésien. On a alors la proposition:

Proposition A.1.1. *L'application* $\exp^{\mathbb{H}}$ *est un difféomorphisme de classe* $\mathcal{C}^\infty$ *de* $\mathbb{C}^* \times]-2\pi, 2\pi[=: \mathcal{D}$ *sur* $\mathbb{H}_1 \setminus L$.

Dans cette proposition on a exclu les relevés horizontaux des cercles complets ($\varphi = \pm 2\pi$) car pour une longueur $|v|$ donnée, ils atteignent tous le même point de L quel que soit l'argument du complexe v. On rappelle que $L = \{(z; t) \in \mathbb{H}_1 \mid z = 0\}$ et que $\mathbb{C}^*$ est une notation pour $\mathbb{C} \setminus \{0_{\mathbb{C}}\}$. La lettre $\mathcal{D}$ est mise pour domaine des coordonnées sphériques.

De façon analogue, pour tout $s \in]0, 1]$, l'application $\exp_s^{\mathbb{H}}$ définie par

$$\exp_s^{\mathbb{H}}(v, \varphi, \cdot) = \gamma_{v,\varphi}(s) = \exp^{\mathbb{H}}(sv, s\varphi)$$

est un difféomorphisme de $\mathcal{D}$ sur son image.

A.1.2 Problème du voyageur de commerce géométrique

Dans cette sous-section nous allons brièvement présenter $\omega([0, 1])$. C'est une courbe horizontale de $\mathbb{H}_1$ de longueur finie mais qui se laisse difficilement approcher par les $\mathbb{H}$-droites dans un sens que nous allons préciser, celui du voyageur de commerce géométrique [40].

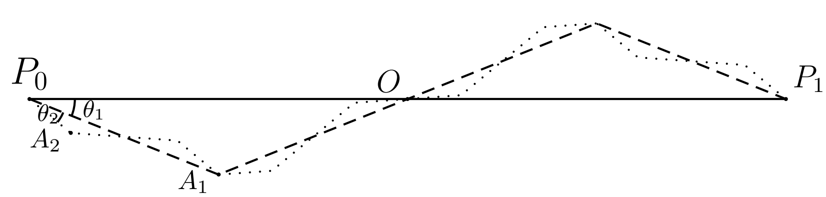

Figure A.3: La courbe $\omega([0, 1])$

La courbe est en fait une courbe "fractale" obtenue à chaque étape par remplacements successifs des segments horizontaux par une ligne brisée faite de quatre segments horizontaux. Sur la Figure A.3 sont représentées les projections sur $\mathbb{C}$ des trois premières courbes (ω_0, ω_1 et ω_2) de la suite $(\omega_n)_{n\in\mathbb{N}}$ qui converge vers ω. Rappelons que Z est la projection complexe $(z;t) \to z$. A partir de ω_0, le simple segment horizontal de P_0 à P_1, on construit ω_1 une ligne brisée pamétrée à vitesse contante dont la projection sur $\mathbb{C}$ fait constamment avec celle de ω_0 un angle $\theta_1 = 0.2$. Il faut observer que $Z(\omega_1)$ délimite avec le segment $Z(\omega_0)$ deux triangles isocèles, chacun d'un côté du segment et dont les aires sont égales. Cela se traduit dans $\mathbb{H}_1$ par le fait que pour $\omega_0(0) = \omega_1(0)$ on a aussi $\omega_0(1) = \omega_1(1)$. De la même façon $Z(\omega_{n+1})$ se construit à partir de $Z(\omega_n)$ en formant des triangles d'aires égales à gauche et à droite de chacun des segments de $Z(\omega_n)$ et dont l'angle isocèle vaut $\theta_n = \frac{0.2}{n}$. Il s'en suit qu'en adoptant une paramétrisation à vitesse constante sur $[0,1]$ on a pour tout $\sigma \in \{0,1,\ldots,4^n\}$

$$\omega_n(\frac{\sigma}{4^n}) = \omega_{n+1}(\frac{\sigma}{4^n}).$$

Il s'avère que cette construction converge et que la courbe limite est de longueur finie. En effet de l'étape n à $n+1$ on multiplie la longueur par $\cos(\theta_n)^{-1}$. Puisque $\cos(\theta) = 1 - \frac{\theta^2}{2} + o(\theta^2)$ au voisinage de 0, la convergence de la suite des longueurs résulte de celle de la série $\sum \frac{1}{n^2}$.

Voici quelques définitions avant d'énoncer le résultat principal de [40]. Les nombres $\beta_{\mathbb{H}}(p,r)(E)$ sont définis par

$$\min_{l\ \mathbb{H}\text{-droite}} \frac{\max_{q\in E\cap\mathcal{B}(p,r)} d_c(q,l)}{r}.$$

On considère donc la distance maximale entre une droite horizontale et les points q de E contenus dans $\mathcal{B}(p,r)$. Le minimum de cette quantité (après normalisation) prise sur toutes les droites horizontales est $\beta_{\mathbb{H}}(p,r)(E)$. On définit alors

$$\mathbf{B}_{\mathbb{H}}(E) = \int_{p\in\mathbb{H}_1}\int_{r>0} \frac{\beta^2_{\mathbb{H}}(p,r)(E)}{r^4} dr d\mathcal{L}(p).$$

On peut donc maintenant formuler le théorème de Ferrari, Franchi et Pajot.

Théorème A.1.2. *(i) Soit $E \subset \mathbb{H}_1$ compact. Alors E est contenu dans une courbe Γ de longueur finie $l(\Gamma)$ si*

$$\mathbf{B}_{\mathbb{H}}(E) < +\infty.$$

De plus, $\inf_{\Gamma\supset E} l(\Gamma) < C(\operatorname{diam}(E) + \beta_{\mathbb{H}}(E))$ *(où l est la longueur et C est une constante absolue).*

(ii) Si Γ est une géodésique de $\mathbb{H}_1$, alors

$$\beta_{\mathbb{H}}(\Gamma) < l(\Gamma)$$

où C est une constante absolue.

Ce théorème est la réplique dans le groupe de Heisenberg d'un théorème euclidien du à Peter Jones [62] (voir aussi [91, 99]). Cependant dans ce théorème la partie réciproque *(ii)* est valable pour toute les courbes rectifiables Γ. Notre courbe $\Omega = \omega([0,1])$ démontre que cette réciproque n'est pas vraie dans la même généralité pour $\mathbb{H}_1$. Cela résulte d'une minoration minutieuse des nombres $\beta(p,r)(\Omega)$. Si cette analyse est fastidueuse, il est assez facile d'estimer grossièrement la grandeur de ces nombres.

Pour des boules de rayon fixé r, les premières courbes ω_n semble être des droites et c'est pour n de l'ordre de $-\ln_4(r)$ que ω_n approche raisonablement les virages effectués par ω. On imagine alors que la boule intercepte précisément un des segments de ω_{n-1}. Ce segment horizontal de taille r semble alors assez raisonnable comme $\mathbb{H}$-droite approchant ω_n dans $\mathcal{B}(p,r)$. On prend alors comme distance caractéristique celle prise entre les points à mis-parcours sur chacune des courbes : le milieu du segment pour ω_{n-1} et le points obtenu après le premier triangle isocèle pour ω_n. Ces points on la même cordonnée complexe. On les note $(z;t)$ et $(z;t')$ et on remarque que $|t-t'|$ est l'aire des triangles isocèles, de l'ordre de $r^2\theta_n$. Ainsi la distance entre les points est de l'ordre de $r\sqrt{\theta_n}$ et $\beta^2_{\mathbb{H}}(p,r)$ de celui de θ_n. En estimant que les p concerné par cette estimation balaient une volume d'ordre r^3, on arrive a l'intégrale

$$\int_{1>r>0} \frac{1}{r\ln_4(1/r)}dr$$

qui diverge.

A.1.3 Deux applications auxiliaires

Nous définissons ici deux applications qui nous seront utiles dans les prochaines parties. Il s'agit de l'application point-intermédiaire $\mathcal{M}$ et de l'inverse géodésique $\mathcal{I}$.

L'application $\mathcal{M}$ a pour argument (p,q,s) un élément de $\mathbb{H}_1\times\mathbb{H}_1\times[0,1]$ mais on utilisera aussi pour $\mathcal{M}(p,q,s)$ l'écriture $\mathcal{M}^s(p,q)$ ou encore $\mathcal{M}^s_p(q)$. Lorsque z_p et z_q sont distincts, $\mathcal{M}^s(p,q)$ sera défini de façon univoque comme le point m pris sur la géodésique de p à q en respectant les proportions $d_c(p,m)=sd_c(p,q)$ et $d_c(m,q)=(1-s)d_c(p,q)$. Il s'en suit que

$$\mathcal{M}(p,q,s)=\tau_p\circ\exp^{\mathbb{H}}_s\circ\left(\exp^{\mathbb{H}}\right)^{-1}\circ\tau_p^{-1}(q)$$

ou bien encore

$$\mathcal{M}(p,q,s)=\gamma^p_{(\exp^{\mathbb{H}})^{-1}(p^{-1}\cdot q)}(s)$$

où on reconnaîtra dans $\gamma^p_{(\exp^{\mathbb{H}})^{-1}(p^{-1}\cdot q)}$ la géodésique normalisée allant de p à q.

Remarque A.1.3. L'application $\mathcal{M}$ n'est pas définie quand $z_p=z_q$ car dans ce cas, il y a une infinité de géodésiques entre p et q (autant que de cercles d'aire t_q-t_p passant par z_p) d'où une indétermination.

L'inverse géodésique $\mathcal{I}$ est pour ainsi dire l'application qui à un point p associe $\mathcal{I}(p)$ de façon à ce que la géodésique de p à $\mathcal{I}(p)$ ait pour milieu le point origine $0_{\mathbb{H}}$. Ainsi lorsque $\mathcal{I}$ est bien définie, on a l'identité

$$\mathcal{I}(p)=\exp^{\mathbb{H}}(-\left(\exp^{\mathbb{H}}\right)^{-1}(v,\varphi)).$$

C'est à dire que pour $p=\exp^{\mathbb{H}}(v,\varphi)$ on aura $\mathcal{I}(p)=\exp^{\mathbb{H}}(-v,-\varphi)$.

Remarque A.1.4. Concrètement cette application n'est bien définie que sur $\exp^{\mathbb{H}}_{1/2}(\mathbb{C}\times[-2\pi,2\pi])=\exp^{\mathbb{H}}(\mathbb{C}\times[-\pi,\pi])$. On peut en fait voir que cet ensemble est le fermé constitué des point compris entre les deux paraboloïdes d'équations $|z|^2 = \pm 2|t|/\pi$: en dehors, il n'y a pas de courbe d'extrémité p et de milieu $0_{\mathbb{H}}$ qui soit globalement géodésique.

Le dernier point que nous voudrions évoquer dans cette partie est le calcul du déterminant jacobien de $\exp^{\mathbb{H}}$. C'est en fait un élément de la démonstration de la proposition A.1.1 car on y voit que $\mathrm{Jac}(\exp^{\mathbb{H}})$ ne s'annule pas. Mais la valeur exacte de ce déterminant nous importe aussi beaucoup pour la suite.

Proposition A.1.5. *Le jacobien de* $\exp^{\mathbb{H}}$ *vaut*

$$\mathrm{Jac}(\exp^{\mathbb{H}})(v,\varphi) = \begin{cases} 4|v|^2 \left(\frac{\sin(\varphi/2)}{\varphi}\right) \frac{\sin(\varphi/2)-(\varphi/2)\cos(\varphi/2)}{\varphi^3} & \textit{pour } \varphi \neq 0, \\ |v|^2/12 & \textit{sinon.} \end{cases}$$

Pour $0 < s < 1$, *celui de* $\exp^{\mathbb{H}}_s$ *est*

$$\mathrm{Jac}(\exp^{\mathbb{H}}_s)(v,\varphi) = \begin{cases} 4s|v|^2 \left(\frac{\sin\frac{s\varphi}{2}}{\varphi}\right) \frac{\sin\frac{s\varphi}{2}-\frac{s\varphi}{2}\cos\frac{s\varphi}{2}}{\varphi^3} & \textit{pour } \varphi \neq 0, \\ s^5|v|^2/12 & \textit{sinon.} \end{cases}$$

A.1.4 Propriété de Contraction de Mesure MCP

Nous allons maintenant définir la Propriété de Contraction de Mesure pour certains espaces métriques dont fait partie le groupe de Heisenberg. Par la suite nous donnerons les étapes du calcul qui permet d'établir $MCP(0,5)$ pour $\mathbb{H}_1$. La propriété MCP n'a été effectivement considérée comme prolongement de la courbure de Ricci dans les espaces métriques mesurés qu'à partir des articles de Sturm [105] et de Ohta [89]. Leurs définitions sont presque identiques mais différentes et ont l'avantage d'inclure a priori des espaces pour lesquels le nombre de géodésiques entre deux points est illimité. Cependant les espaces connus vérifiants une $MCP(K,N)$ sont moins sophistiqués. La définition exacte étant difficile à s'approprier, nous donnons ici une formulation plus simple dans le cas où on peut associer à (X,d,ν) une application mesurable

$$\mathcal{N} : X \times X \times [0,1] \to X$$

telle que pour $\nu \otimes \nu$-presque tout couple de points (p,q), la géodésique de p à q est unique et correspond à $(\mathcal{N}(p,q,s))_{s\in[0,1]}$ (l'application $\mathcal{M}$ est bien sûr une telle application dans le cas de $\mathbb{H}_1$). L'espace métrique mesuré vérifie alors $MCP(0,N)$ si et seulement si pour presque tout point p, on a pour tout ensemble ν-mesurable E et pour s parcourant $[0,1]$:

$$s^N \nu(\mathcal{N}_{p,s}^{-1}(E)) \leq \nu(E)$$

où $\mathcal{N}_{p,s} = \mathcal{N}(p,q,s)$. Lorsque l'inverse de $\mathcal{N}_{p,s}$ est mesurable, on peut opter pour une formulation plus directe du type

$$\nu(\mathcal{N}_{p,s}(F)) \geq s^N \nu(F)$$

qui met très clairement en évidence que $\mathbb{R}^N$ vérifie $MCP(0,N)$ (dans ce cas on a égalité). Cette formulation est en particulier possible sur $\mathbb{H}_1$ car $\mathcal{M}_p^s$ est un homéomorphisme de $\{(z_q;t_q) \in \mathbb{H}_1 \mid z_q \neq z_p\}$ sur son image et car $p \cdot L$, l'ensemble complémentaire est de mesure nulle. Dans le groupe de Heisenberg, on peut également tirer parti de la dérivabilité de la contraction et du bon accord de la structure différentielle avec la mesure de référence de l'espace métrique mesuré. Il est en effet suffisant de vérifier une minoration du Jacobien de $\mathcal{M}_p^s$: pour presque tout couple de points (p,q) et pour tout $s \in [0,1]$ on souhaite avoir

$$\mathrm{Jac}(\mathcal{M}_p^s)(q) \geq s^N.$$

Enfin l'invariance de la mesure et des distances par translation à gauche permet une dernière simplification de l'énoncé. Pour montrer $MCP(0,5)$ dans $\mathbb{H}_1$, il suffit de prouver la dernière inégalité pour $p = 0_{\mathbb{H}_1}$ et $N = 5$ ce que nous allons faire maintenant. On connaît sur $\mathbb{H}_1 \setminus L$ une expression interessante de $\mathcal{M}_s^{0_{\mathbb{H}}}$ à savoir $\exp_s^{\mathbb{H}} \circ \left(\exp^{\mathbb{H}}\right)^{-1}$. Le déterminant jacobien de la contraction de rapport s au point $\exp^{\mathbb{H}}(v,\varphi)$ est ainsi

$$\left(\frac{\mathrm{Jac}(\exp_s^{\mathbb{H}})}{\mathrm{Jac}(\exp^{\mathbb{H}})}\right)(v,\varphi)$$

et c'est cette quantité que l'on voudrait être supérieure à s^5 pour tout s et tout point de coordonnée sphérique $(v,\varphi) \in \mathcal{D}$. De façon équivalente on voudrait montrer que $\mathrm{Jac}(\exp_s^{\mathbb{H}})^{1/5}$ est supérieure à la fonction affine $s\,\mathrm{Jac}(\exp^{\mathbb{H}})^{1/5}$. Puisque les deux fonctions sont égales en 0 et en 1 il est suffisant de prouver que

$$\mathrm{Jac}(\exp_s^{\mathbb{H}})^{1/5}(v,\varphi) = \left(4s|v|^2\left(\frac{\sin\frac{s\varphi}{2}}{\varphi}\right)\frac{\sin\frac{s\varphi}{2} - \frac{s\varphi}{2}\cos\frac{s\varphi}{2}}{\varphi^3}\right)^{1/5}$$

est concave en s et ce bien sûr pour tout $(v,\varphi) \in \mathcal{D}$. Du fait que φ décrit $]-2\pi, 2\pi[$ et de par la symétrie de la fonction en φ, la démonstration se réduit à montrer que

$$F(x) = x\sin(x)(\sin(x) - x\cos(x))$$

est 1/5-concave sur $[0,\pi]$. On sait que pour des fonctions suffisamment dérivables, être 1/5-concave équivaut à ce que $F''F - F'^2 + \frac{F'^2}{5} \leq 0$ tandis que la log-concavité (concavité du logarithme de la fonction) équivaut seulement à $F''F - F'^2 \leq 0$. Or cette log-concavité est aussi une simple conséquence des log-concavités de $a(x) = x$, de $b = \sin$ et de $c(x) = \sin(x) - x\cos(x)$ sur $[0,\pi]$ car $\ln(abc) = \ln(a) + \ln(b) + \ln(c)$. Cela nous incite à écrire $F''F - F'^2 + \frac{F'^2}{5}$ sous la forme :

$$\left[(a''a - a'^2)b^2c^2 + a^2(b''b - b'^2)c^2 + a^2b^2(c''c - c^2)\right] + \frac{F'^2}{5}.$$

où on a remplacé

$$F''F - F'^2$$

par l'expression entre crochets (ce qui permet une deuxième fois de déduire la log-concavité de F à partir de celle de ces facteurs). On est donc réduit à montrer que dans l'expression

$$\frac{F'^2}{5} - \left[(bc)^2 + (ac)^2 + (x^2 - \sin^2(x))(ab)^2\right],$$

le terme positif $\frac{F'^2}{5}$ ne parvient pas à compenser le terme relatif à la log-concavité de F. On obtient après une étude approfondie des deux termes de signes opposés la négativité recherchée (voir [64]). Plus aisément, on peut vérifier par un développement limité en $s = 0$ que l'exposant 5 est la plus petite puissance pour laquelle on peut obtenir la négativité de l'expression.

A.2 Transport optimal de mesure dans $\mathbb{H}_1$

A.2.1 Généralités et définitions

Le transport optimal de masse connaît un regain d'intérêt depuis une vingtaine d'année car son utilisation a débouché sur de nouvelles applications dans divers domaines. Dans cet exposé nous parlerons de l'emploi qui en est fait en géométrie depuis les travaux de Lott et Villani [77, 78] ainsi que de Sturm [104, 105]. Ces auteurs ont réussi à définir pour les espaces métriques mesurés une propriété qui prolonge de façon convaincante celle d'avoir une courbure de Ricci uniformément bornée inférieurement, propriété qui elle n'a de sens que pour les variétés riemanniennes. Il s'agit de la courbure-dimension que l'on note $CD(K, N)$ où $K \in \mathbb{R}$ est la courbure et $N \geq 1$ est un paramètre-dimension. Nous allons répéter les arguments de [64] où il est montré que le triplet $(\mathbb{H}_1, d_c, \mathcal{L})$ ne vérifie aucun $CD(K, N)$ quels que soit les paramètres K et N. La partie principale du travail qui consiste à montrer $CD(0, N)$ n'est vraie pour aucun N est obtenue par la négation d'une inégalité de Brunn-Minkowski généralisée (voir partie A.3 et [64]). Cependant nous allons tout d'abord définir correctement le transport de masse et évoquer sa réalisation dans le groupe de Heisenberg. Les résultats connus à ce sujet sont dus à Ambrosio et Rigot [7] et ont été récemment complétés par Figalli et l'auteur de ce résumé dans [42].

Le point de départ habituel pour expliquer le transport optimal de masse est le problème de Monge-Kantorovich. Il s'agit, s'étant donné un espace métrique (X, d) et deux mesures de probabilité boréliennes μ_0 et μ_1 sur X, de considérer le problème d'optimisation:

$$\inf_{\pi} \int_{X \times X} d^2(p, q) d\pi(p, q). \tag{A.3}$$

L'infimum est pris sur les mesures π de $X \times X$ qui sont des couplages de μ_0 et μ_1, c'est à dire dont les marginales (les projections sur X) sont μ_0 et μ_1. La fonction coût $d^2(p, q)$ est celle qui apparaît le plus souvent dans l'intégrale lorsque il s'agit de géométrie ; elle peut prendre d'autres valeurs $c(p, q)$ dans le cas de la théorie générale. La racine carré de (A.3) est appelée distance de Wasserstein ; on la notera $W(\mu_0, \mu_1)$. Le nom de distance est justifié lorsqu'on se restreint à l'espace $\mathcal{P}_2(X)$ dit de Wasserstein, constitué des mesures de probabilité dont le second moment est fini ($\int_X d^2(o, p) d\mu(p) < +\infty$ pour un $o \in X$ ou de façon équivalente pour tout o): en effet si les deux mesures sont dans cet espace, $W(\mu_0, \mu_1)$ sera nécessairement finie. Il s'avère par ailleurs que lorsque l'espace (X, d) est géodésique, il en va de même de $(\mathcal{P}_2(X), W)$. Nous allons décrire maintenant plus en détail comment cela se réalise dans $\mathbb{H}_1$ en commençant par un théorème d'Ambrosio et Rigot (voir [7] et aussi [42]). Ce résultat fait intervenir la différentiabilité approximative dont on peut trouver une description dans [4]. Il n'est cependant pas nécessaire pour la suite de comprendre cette notion en détail. On pourra se contenter de savoir qu'il s'agit d'une extension de la différentiabilité ordinaire.

Proposition A.2.1. *Soit $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{H}_1)$. On suppose par ailleurs que ces deux mesures sont absolument continues par rapport à $\mathcal{L}$. Alors il existe un unique couplage optimal π entre μ_0 et μ_1 et ce couplage est induit par une application de transport T, i.e. $\pi = (\mathrm{Id} \otimes T)_{\#}\mu_0$. Si μ_0 est à support compact,* *on sait de plus qu'il existe une fonction ψ différentiable en μ_0-presque tout point p telle que*

$$T(p) := p \cdot \exp^{\mathbb{H}}(\mathbf{X}\psi(p) + \mathbf{i}\mathbf{Y}\psi(p), \mathbf{T}\psi(p)).$$

Même si μ_0 n'est pas à support compact, il existe une fonction ψ approximativement différentiable en μ_0-presque tout point telle que

$$T(p) := p \cdot \exp^{\mathbb{H}}(\tilde{\mathbf{X}}\psi(p) + \mathbf{i}\tilde{\mathbf{Y}}\psi(p), \tilde{\mathbf{T}}\psi(p)).$$

À partir de là on peut construire une courbe de mesures qui est géodésique dans $\mathcal{P}_2(\mathbb{H}_1)$. Illustrons cela par une image : on peut comparer les mesures μ_0 et μ_1 à deux nids de fourmis (vivant dans le groupe de Heisenberg!) pour lesquels la densité en fourmi représenterait la densité des mesures et où les fourmis sont indistinctes. Le plan de transport π correspond à la façon optimale pour les fourmis de passer d'une configuration à une autre : il minimise la somme des carrés des distances parcourues par les insectes. Comme on l'a dit dans la proposition précédente, ce plan est unique ce qui signifie que le nid n'a qu'un seul choix de déplacement global ; chaque fourmi de μ_0 sait exactement où elle doit se rendre au temps 1. Entre les temps 0 et 1 chaque insecte se rend à vitesse constante de son point de départ à son point d'arrivée. Si on arrête le mouvement à l'instant $s \in [0,1]$, on peut observer une nouvelle configuration μ_s. En terme mathématique nous sommes en train de parler de $T_{s\#}\mu_0$ où

$$T_s(p) := x \cdot \exp_s^{\mathbb{H}}(\tilde{\mathbf{X}}\psi(p) + \mathbf{i}\tilde{\mathbf{Y}}\psi(p), \tilde{\mathbf{T}}\psi(p)).$$

On vient de mettre en évidence des plans de transport entre μ_0 et μ_s d'une part, entre μ_s et μ_1 d'autre part : on prend ceux induits par la trajectoire des fourmis. Le premier engendre un coût de $s^2W^2(\mu_0,\mu_1)$ et le second de $(1-s)^2W^2(\mu_0,\mu_1)$. Du fait de l'inégalité triangulaire $W(\mu_0,\mu_1) \le W(\mu_0,\mu_s) + W(\mu_s,\mu_1)$, on en déduit que les deux transports signalés sont optimaux et que la courbe $(\mu_s)_{s\in[0,1]}$ est géodésique dans l'espace de Wasserstein $\mathcal{P}_2(\mathbb{H}_1)$.

En adaptant [7, Lemme 4.7] on peut voir que pour μ_0-presque tout point la courbe $\gamma^p_{\tilde{\mathbf{X}}\varphi(p)+\mathbf{i}\tilde{\mathbf{Y}}\varphi(p),\tilde{\mathbf{T}}\varphi(p)}$ qui relie p à $T(p)$ est l'unique géodésique possible entre ces deux points. Ainsi, les fourmis qui comme on l'a dit ont un but défini de façon unique, ne peuvent-elles emprunter qu'un seul chemin chacune. À l'échelle globale, cela signifie que entre les deux mesures la géodésique est unique dans l'espace de Wasserstein. Cette unicité est a priori une propriété fausse si on part d'une mesure μ_0 qui n'est pas absolument continue. On peut s'en convaincre en considérant le transport entre deux mesures concentrées sur L. Entre deux points distincts de L, il y a en effet une infinité de géodésiques et cela se répercute à l'échelle des mesures.

Remarque A.2.2. On sait depuis [42] que les mesures μ_s pour $s<1$ sont elles-même absolument continue ce qui fait du sous-espace $\mathcal{P}_2^{ac}(\mathbb{H}_1) \subset \mathcal{P}_2(\mathbb{H}_1)$ des mesures absolument continues un espace géodésiquement complet. Cette question concernant l'absolue continuité avait été posée dans [7]. Comme on verra dans la sous-partie A.2.2, on ne pouvait pas y répondre en utilisant la technique utilisée sur les variété riemanniennes où là aussi les mesures interpolées par le transport sont absolument continues.

Expliquons maintenant ce qu'on entend par la propriété $CD(0,N)$. Pour une variété riemannienne de dimension N équipée de son volume riemannien, cette propriété est équivalente au fait que la courbure de Ricci est positive en tout point. Pour la définition on prend un espace métrique mesuré (X, d, ν), on considère de nouveau le transport entre mesures absolument continues et on analyse la façon dont l'entropie de Rényi de ces mesures évolue au cours du temps. Cette entropie est une fonctionnelle définie par

$$\mathrm{Ent}_N(\mu \mid \nu) = \begin{cases} -\int_X \rho^{1-1/N} d\nu & \text{si } d\mu = \rho d\nu \\ +\infty & \text{si } \mu \text{ n'est pas absolument continue} \end{cases}.$$

Pour $N = +\infty$ on considère l'entropie de Bolzmann:

$$\mathrm{Ent}_\infty(\mu \mid \nu) = \begin{cases} \int_X \rho \ln(\rho) d\nu & \text{si } d\mu = \rho d\nu \\ +\infty & \text{si } \mu \text{ n'est pas absolument continue} \end{cases}.$$

L'entropie est une façon de mesurer la répartition de la mesure : une mesure μ_s qui est répartie de façon plutôt uniforme par rapport à ν à une entropie petite tandis que une mesure très concentrée autour de certain points a une grande entropie. Un calcul simple illustre et quantifie cela : pour un ensemble de ν-mesure V, la mesure uniformément répartie sur cet ensemble a une entropie qui vaut $-V^{1/N}$. La définition de la courbure dimension s'exprime alors ainsi :

Définition A.2.3. Soit $N \in [1, +\infty]$. On dit que l'espace métrique (X, d, ν) vérifie $CD(0, N)$ si pour tout couple (μ_0, μ_1) de mesures absolument continues, il existe une géodésique $(\mu_s)_{s\in[0,1]}$ de $\mathcal{P}_2^{ac}$, paramétrée à vitesse constante, telle que pour tout s,

$$\mathrm{Ent}_N(\mu_s \mid \nu) \leq (1 - s) \mathrm{Ent}_N(\mu_0 \mid \nu) + s \mathrm{Ent}_N(\mu_1 \mid \nu).$$

En terme de fourmis se déplaçant dans un espace avec courbure-dimension $CD(0, N)$, on s'attend à ce qu'au cours du trajet la fourmilière se répartisse de façon plus uniforme et plus large qu'elle ne l'est dans ses positions initiales et finales. Les fourmis s'éloignent les unes des autres pour que relativement à cet éloignement certaines se regroupent de nouveau à la fin, peut-être à différents endroits.

Comme nous l'annonçons depuis tout à l'heure, ce comportement n'est pas celui qui a cours dans le groupe de Heisenberg. Nous en ferons la preuve dans la partie A.3.

Remarque A.2.4. Dans le cas de $\mathbb{H}_1$ où il y a unicité de la géodésique dans $\mathcal{P}_2^{ac}(\mathbb{H}_1)$, on voit assez vite que $CD(0, N)$ est équivalente à la convexité de la fonctionnelle entropie le long des géodésiques. Contrairement à ce qui semble, la condition $CD(0, N)$ est plus faible dans le cas général. Prenons l'exemple des mesures interpolées au temps $1/2$ par $\mu_{1/4}$ et $\mu_{3/4}$: pour le transport apparent, il s'agit de $\mu_{1/2}$ mais ce qu'impose $CD(0, N)$ est que *il existe* une géodésique (non nécessairement celle qu'on connaît) avec les bonnes interpolations sur l'entropie.

A.2.2 Absolue continuité au cours du transport

Le principe de raccourcissement de Monge-Mather est décrit de façon détaillée dans le livre de Cédric Villani [109, Chapitre 8]. On utilise ce principe pour montrer une inégalité sur un transport de mesure $(T_{s\#}\mu_0)_{s\in[0,1]}$. De cette inégalité de racourcissement et sous l'hypothèse que μ_0 est absolument continue par rapport à la mesure de Hausdorff de l'espace, on peut conclure à l'absolue continuité des mesures intermédiaires. On part du constat suivant: pour $\mu_0 \otimes \mu_0$-presque tout couple de point (a, b), les courbes $T_s(a)$ et $T_s(b)$ ne peuvent pas se rencontrer à un temps $s < 1$ fixé. Si cela arrivait on pourrait racourcir le transport en " mélangeant" les courbes : la fin de chacune des courbes (pour les temps supérieurs à s) serait remplacée par celle de l'autre courbe. Une version quantitative de l'injectivité qu'on évoque, du type

$$d(T_s(a), T_s(b)) \geq C d(a, b) \tag{A.4}$$

permettrait de déduire

$$\mathcal{H}^n_d(T_s(E)) \geq C^n \mathcal{H}^n_d(E) \tag{A.5}$$

où $\mathcal{H}^n_d$ désigne la mesure de Hausdorff n-dimensionelle pour la distance d.

À partir de là, si on suppose qu'un ensemble F a une mesure de Hausdorff nulle, celle de $T_s^{-1}(F)$ est tout autant nulle car $F = T_s(T_s^{-1}(F))$. Par absolue continuité de μ_0 par rapport à $\mathcal{H}^n_d$, on déduit $\mu_0(T_s^{-1}(F)) = 0$. L'application T_s étant une application de transport de μ_0 sur μ_s, il s'en suit finalement $\mu_s(F) = 0$. On considère désormais les bouts de cette chaîne logique on reconnaît que μ_s est absolument continue par rapport à $\mathcal{H}^n_d$.

Dans le groupe de Heisenberg cette argumentation basée sur l'inégalité lipschitzienne (A.4) répondrait positivement à la question d'Ambrosio et Rigot : on pourrait déduire que μ_s est absolument continue par rapport à la mesure de Lebesgue car celle-ci est à une contante près égale à la mesure de Hausdorff 4-dimensionnelle de $(\mathbb{H}_1, d_c)$. En fait contrairement à ce qui se passe dans le cas riemannien, il est montré dans [42] que l'inégalité (A.4) est tout à fait fausse pour le groupe de Heisenberg : un transport optimal aussi simple que la multiplication à droite par le vecteur $(1, 0, 0)$ suffit à nier la majoration (A.4).

Il faut donc trouver autre chose pour $\mathbb{H}_1$. La démonstration de [42] reprend le schéma précédent au niveau de l'inégalité (A.5) qu'elle démontre à cela près que C est remplacée par la constante $(1 - s)^5$. On obtient cette majoration grâce à la souplesse du transport de mesure qui permet un passage à la limite opportun. On envisage en fait la mesure μ_1 comme la limite faible d'une suite de mesures discrètes $\mu_1^k = \frac{1}{k} \sum_{i=1}^k \delta_{y_i}$. Pour chacun des transports optimaux $(T^k_{s\,\#}\mu_0)_{s \in [0,1]}$ de μ_0 à μ_1^k on peut montrer comme nous allons le voir l'inégalité

$$\mathcal{L}(T_s^k(E)) \geq (1 - s)^5 \mathcal{L}(E), \tag{A.6}$$

comme une conséquence de $MCP(0, 5)$ et de l'injectivité du transport T_s^k. En invoquant la compacité des plans de transport et l'unicité du transport de μ_0 à μ_1, la minoration de $\mathcal{L}(T_s^k(E))$ dans (A.6) passe à la limite et on peut ainsi remplacer T_s^k par T_s et avoir (A.5). Comme on l'a expliqué auparavant, cela suffit pour montrer l'absolue continuité des mesures pour $s < 1$. Nous rapportons le lecteur à la lecture des théorèmes de ([109, Chapitre 7 et Corollaire 5.21]) pour la justification de ce passage à la limite.

Revenons cependant à l'inégalité (A.6) et voyons son rapport avec MCP. C'est pour $k = 1$ que le lien est le plus apparent : on considère le transport de μ_0 sur la mesure $\mu_1^1 = \delta_{y_1}$. Il n'y a qu'un seul plan de transport possible et celui-ci est nécessairement optimal ; on transporte chaque élement de volume sur y_1 et cela se fait μ_0-presque sûrement le long de la μ_0-presque sûrement unique géodésique qui mène à y_1. De fait on a donc presque sûrement

$$T_s^1(q) = \mathcal{M}_{y_1}^{1-s}(q)$$

et l'estimation sur les volumes résulte de $MCP(0,5)$.

Pour $k > 1$, on peut démontrer (A.6) en se rappelant l'injectivité du transport. On commence par écrire E comme la réunion disjointes d'ensembles $(E_j)_{j=1}^k$ sur lesquels l'application T_s^k est la contraction $\mathcal{M}_{y_j}^s$. Alors on doit comparer la mesure de la réunion des $\mathcal{M}_{y_j}^s(E_j)$ à $\mathcal{L}(E) = \sum \mathcal{L}(E_j)$. Puisque MCP fournit pour chaque E_j une inégalité du type (A.6), il suffit de souligner que l'on peut sommer ces inégalités. Le transport étant injectif, les $T_k^s(E_j) = \mathcal{M}_{y_j}^s(E_j)$ sont en effet disjoints et on a la majoration désirée.

A.3 Courbure-dimension dans $\mathbb{H}_1$: espoirs et déception

Comme nous l'avons expliqué dans l'introduction, les propriétés CD et MCP prolongent les bornes sur la courbure de Ricci en ce sens qu'elles sont équivalentes avec les propriétés de courbure de Ricci uniformément minorée dans le cas où l'espace considéré est une variété riemannienne de dimension N, équipée du volume riemannien. Cela ne serait pas suffisant si c'était la seule chose : pourquoi sinon, ne pas proclamer qu'un espace métrique qui n'est pas une variété ne vérifie pas de borne de Ricci synthétique? Lott, Sturm et Villani ont démontré bien plus que l'équivalence, à commencer par des théorèmes qui sont usuellement obtenus pour les mêmes hypothèses de courbure sur les variétés riemannienne : théorème de Bishop-Gromov, théorème de Bonnet-Myers mais aussi inégalité de Poincaré locale (sous l'hypothèse d'unicité presque certaine des géodésiques entre deux points) en particulier. Un autre point fort de la théorie est la compatibilité avec les distances entre espaces métriques mesurés. En particulier pour la distance décrite par Sturm [104], une distance mélangeant les idées de la distance de Gromov-Hausdorff et celles du transport de masse, une suite convergente de variétés riemanniennes vérifiant uniformément $CD(K,N)$ et dont le diamètre est majoré, aura pour limite un triplet (X, d, ν) qui satisfaira le même $CD(K,N)$ et la même borne sur le diamètre. Pour plus de renseignement sur les propriétés de CD et MCP, on pourra se reporter aux articles fondateurs de la théorie [77, 78, 104, 105] ou au livre formidablement détaillé de Villani [109].

Nous allons maintenant comparer les propriétés $MCP(0,5)$ et $CD(0,5)$. Tout d'abord, si l'on suppose la presque sure existence et unicité d'une géodésique entre deux points, $MCP(K,N)$ est toujours une conséquence de $CD(K,N)$. Nous allons réciproquement voir pourquoi il est raisonnable de penser qu'un espace vérifiant $MCP(0,5)$ puisse aussi satisfaire $CD(0,5)$. Si on reprend la définition A.2.3, on voit qu'avec CD, il s'agit de la convexité de l'entropie Ent_5 le long des géodésiques de $\mathcal{P}_2(\mathbb{H}_1)$. Le fait est que dans $\mathbb{H}_1$, des familles importantes de transport de mesure s'effectue avec la convexitité de cette entropie : la première est celle des transports par contraction. Pour la convexité de l'entropie, on exploite une propriété de contraction de mesure renforcée. La seconde famille de transport est celle des transports de mesure qui sont les relevés horizontaux du transport optimaux du plan $\mathbb{R}^2$.

Nous sommes tout d'abord à même de prouver la convexité de l'entropie, pour ces géodésiques particulières qui se terminent par une mesure de Dirac δ_y et que nous avons déjà rencontrées dans la partie A.2.2. L'argument décisif ne sera pas à proprement MCP mais la $1/5$ concavité du jacobien de la contraction sur y, une propriété plus forte qui nous a permis de démontrer MCP. Reprenons donc les notations précédentes : $(\mu_s)_{s\in[0,1[}$ est une géodésique d'extrémité $\mu_1^1 = \delta_{y_1}$. Toutes les mesures sont absolument continues sauf δ_{y_1} dont l'entropie est infinie. On calcule donc l'entropie pour $s < 1$

$$\begin{aligned}
\mathrm{Ent}_5(\rho_s \mid \mathcal{L}) &= -\int_{\mathcal{M}_s^y(\mathbb{H}_1)} \rho_s^{1-1/5}(y)\,dy \\
&= -\int_{\mathbb{H}_1} (\rho_s \circ T_s)^{1-1/5}(x)\,\mathrm{Jac}(T_s)(x)\,dx \\
&= -\int \left(\rho_s \circ T_s\,\mathrm{Jac}(T_s)\right)^{1-1/5}\mathrm{Jac}(T_s)^{1/5} \\
&= -\int \rho_0^{1-1/5}(\mathrm{Jac}(T_s))^{1/5}.
\end{aligned}$$

Avant de conclure, justifions ces égalités. On obtient la deuxième ligne de la première par un changement de variable possible du fait que T_s est $\mathcal{C}^\infty$ sur $\mathbb{H}_1 \setminus qL$. On passe ensuite à la troisième ligne par une manipulation algébrique et on conclut grâce à l'identité $\rho_0 = \rho_s(T_s(x))\,\mathrm{Jac}(T_s)$ qui dérive de la relation de mesure image $\mu_s = T_{s\#}\mu_0$. À partir de cette expression de ρ_s et de la concavité de $\mathrm{Jac}(T_s)^{1/5}$ que l'on connaît depuis la partie A.1.4, on voit donc que l'entropie de Rényi de dimension 5 est convexe le long des géodésiques "de contraction".

Rien ne s'oppose à faire la même démonstration pour les mesures discrètes μ_1^k de tout à l'heure ce qui laisse envisager qu'on puisse de nouveau passer l'inégalité à la limite et conclure à la propriété $CD(0,5)$. Mais cette fois-là la démonstration ne peut pas se faire faute de passage à la limite valide. Nous allons d'ailleurs montrer au théorème A.3.1 qu'aucune relation $CD(0,N)$ ne se vérifie dans $\mathbb{H}_1$.

La deuxième classe de transport optimal d'entropie convexe ne repose pas sur les propriétés du type MCP mais sur l'essence particulière de $\mathbb{H}_1$. Le théorème de Ambrosio et Rigot contient en fait une seconde partie que nous n'avons pas cité dans la proposition A.2.1. On y apprend quelles sont les fonctions ψ qui donnent lieu à un transport optimal. On peut en particulier montrer que les fonctions $\psi(z;t) = \theta(z)$ pour lesquelles $\theta(z) + \frac{|z|^2}{2}$ est convexe sur $\mathbb{C}$ sont de celles-là. Or dans le transport de mesure sur les espaces euclidiens, le transport de Brenier [18], ce sont précisément de telles fonctions θ qui indiquent les transports optimaux : les applications de transport sont alors en effet de la forme $T(x) = x + \nabla\theta$. Par ailleurs la courbure de Ricci de $\mathbb{R}^d$ étant 0, l'espace euclidien vérifie $CD(0,N)$ pour $N \geq d$ et donc l'entropie est convexe le long du transport optimal.

Ainsi donc en notant comme précédemment $Z : (z;t) \to z$ la projection sur $\mathbb{C}$, les transports optimaux $(\mu_s)_{s\in[0,1]}$ de $\mathbb{H}_1$ hérités des fonctions $\psi(z,t)$ dont on vient de donner la forme, se projettent en des transports optimaux $(Z_\sharp\mu_s)_{s\in[0,1]}$ de $\mathbb{R}^2$ dont l'application de transport est donnée par θ. On a même plus car la convexité de l'entropie Ent_N quand $N \geq 2$ pour ces transports de $\mathbb{H}_1$ est héritée de celle qu'on sait vérifiée pour des transport de mesure donnés par l'application $T(x) = x + \nabla\theta$. Ainsi pour la classe de transport optimaux présenté ici, on a la convexité de l'entropie 2-dimensionnelle ce qui est beaucoup mieux que ce que nous laisse espérer $MCP(0,5)$.

Cependant bien que large, les deux familles de transports présentée ici sont loins de constituer l'ensemble des transports optimaux de $\mathbb{H}_1$. Dans le théorème suivant, nous présentons un transport particulier pour lequel l'entropie n'est pas convexe quelle que soit sa dimension N.

Théorème A.3.1. *Quelle que soit la dimension $N \in [1, +\infty]$, la propriété $CD(0, N)$ est fausse dans le groupe de Heisenberg.*

Fixons $N \geq 1$. Nous allons donc considérer un transport de mesure pour lequel l'entropie Ent_N n'est pas convexe. Cet exemple ne dépendra en fait même pas de la dimension N. Soit $r > 0$ un paramètre réel qui a vocation à être petit. on considère B_r la boule euclidienne (de $\mathbb{R}^3$) centrée en $(1,0,0)$ et $I_r := \mathcal{I}(B_r)$ son conjugué géodésique. Les deux ensembles ont un même volume $V_r = \frac{4}{3}\pi r^3$ car les paramètres sphériques (v, φ) de $\mathcal{D}$ qui décrivent ces ensembles sont opposés et car $\mathrm{Jac}(\exp^{\mathbb{H}})(-v, -\varphi) = \mathrm{Jac}(\exp^{\mathbb{H}})(v, \varphi)$. Le mesures que nous nous proposons de transporter l'une sur l'autre sont simplement les mesures uniformément distribuées sur ces ensembles:

$$\mu_0 = \mathbf{1}_{B_r}/V_r \quad , \quad \mu_1 = \mathbf{1}_{I_r}/V_r.$$

Les entropies de μ_0 et μ_1 sont égales et valent $-(V_r)^{1/N}$ ($-\ln(V_r)$ pour $N = +\infty$). Nous allons montrer que $\mathrm{Ent}_N(\mu_{1/2}) > -(V_r)^{1/n}$ pour r suffisamment petit, ce qui suffira à nier la convexité de la fonctionnelle Ent_N. De part la structure du transport de mesure, $\mu_{1/2}$ se concentre sur $M_r = \mathcal{M}^{1/2}(B_r, I_r)$, l'ensemble des milieux des géodésiques reliant les points de B_r à ceux de son conjugué géodésique :

$$M_r = \{\mathcal{M}(p, \mathcal{I}(q)) \mid (p,q) \in (B_r)^2\}.$$

On a donc

$$\begin{aligned}
\mathrm{Ent}_N(\mu_{1/2}) &= -\int_{M_r} \rho_{1/2}^{1-1/N} \\
&= \mathcal{L}(M_r) \int_{M_r} -\rho_{1/2}^{1-1/N}(x) \frac{dx}{\mathcal{L}(M_r)} \\
&\geq \mathcal{L}(M_r) \left(- \left(\int_{M_r} \rho_{1/2}(x) \frac{dx}{\mathcal{L}(M_r)} \right)^{1-1/N} \right) \\
&\geq -(\mathcal{L}(M_r))^{1/N}
\end{aligned}$$

Pour $N = +\infty$ on trouve de même $\mathrm{Ent}_N(\mu_{1/2}) \geq -\ln(\mathcal{L}(M_r))$. Ce calcul basé sur l'inégalité de Jensen nous apprend qu'entre les différentes mesures de probabilité concentrées sur un ensemble, celle dont l'entropie est la plus faible est la mesure uniforme. Cela est en accord avec notre présentation utilisant les fourmis: plus le nid est bien réparti et plus les fourmis prennent de la place, plus l'entropie est basse.

Il suffit donc pour terminer la preuve de montrer que $\mathcal{L}(M_r) < V_r$.

Essayons de comprendre ce qu'est l'ensemble M_r. C'est la superposition (mathématiquement parlant la réunion) des ensembles $\mathcal{M}_{1/2}(\mathcal{I}(p), B_r)$ lorsque p décrit B_r. Cette réunion est loin d'être disjointe et c'est aussi pourquoi l'ensemble M_r est relativement petit. En fait $0_{\mathbb{H}}$ est contenu dans chacun des $\mathcal{M}^{1/2}(\mathcal{I}(p), B_r)$ en temps que milieu de p et $\mathcal{I}(p)$. Avec un peu de calcul différentiel (voir [64]), on a plus précisément

$$\mathcal{M}^{1/2}(\mathcal{I}(p), B_r) \subset D\mathcal{M}^{1/2}_{\mathcal{I}(p)}(p).(B_r - p) + B(0_{\mathbb{H}}, o(r)). \tag{A.7}$$

Ici les opérations + et − sont prises au sens de $\mathbb{R}^3$ et $B(0, o(r))$ est une boule euclidienne centrée en $0_{\mathbb{H}}$ dont le rayon est négligeable devant r. Cette inclusion est en fait également vraie lorsqu'on remplace uniformément $D\mathcal{M}^{1/2}_{\mathcal{I}(p)}(p)$ par $D\mathcal{M}^{1/2}_{(-1,0,0)}(1, 0, 0)$ qui lui est proche et correspond à $p = (1, 0, 0)$, le centre de la boule B_r. Le $o(r)$ dans (A.7) est certes remplacé par une fonction plus grande, mais cette grandeur est encore négligeable par rapport à r, uniformément en p. On peut désormais faire la réunion de des relations ensemblistes et obtenir

$$M_r \subset D\mathcal{M}^{1/2}_{(-1,0,0)}(1, 0, 0).(B_r - B_r) + B(0, o(r)).$$

L'ensemble $B_r - B_r$ n'est rien d'autre que la boule euclidienne de rayon $2r$ et son volume est tout simplement $8V_r$. Si on prend l'image de cet ensemble par une application affine de déterminant $1/2^5$ (voir la proposition A.1.5) on obtient un ellipsoïde de volume $V_r/4$. L'ensemble qui nous intéresse et qui contient M_r est le $o(r)$-voisinage tubulaire d'un ellipsoïde de cette sorte. Son volume équivaut ainsi à $V_r/4$. On conclut alors au fait que $\mathcal{L}(M_r) < V_r$ pour r suffisamment petit ce qui, comme on l'a déjà souligné, suffit à la démonstration.

A.4 Flot de gradient dans le groupe de Heisenberg

Les géodésiques ne sont pas les seules courbes intéressantes de l'espace de Wasserstein. Jordan, Kinderlehrer et Otto [63] ont eu les premiers l'intuition de considérer le flot de la chaleur de $\mathbb{R}^n$ comme une courbe l'espace de Wasserstein euclidien, $\mathcal{P}_2(\mathbb{R}^n)$. Ils ont constaté que de façon formelle, $\mathcal{P}_2(\mathbb{R}^n)$ était une variété de dimension infinie et que la trajectoire de la chaleur dans l'espace des mesures était une courbe intégrale du champs de gradient (formel) de $-\operatorname{Ent}_\infty$ (malgré le signe, on parle du flot de gradient de l'entropie ou flot de gradient de $\operatorname{Ent}_\infty$). En effet la fonctionnelle entropie est une fonction réelle de $\mathcal{P}_2(\mathbb{R}^n)$ et il apparaît que la chaleur diffuse de façon à minimiser au mieux l'entropie à tout instant.

Dès lors on a cherché à justifier rigoureusement cette approche et à étendre cette observation à d'autres fonctionnelles ou classes de fonctionnelles. Ce faisant il est apparu que l'on pouvait traiter plus facilement les fonctionnelles présentant des propriétés de convexités le long des géodésique de le l'espace de Wasserstein (voir par exemple [8]). Comme nous le savons pour les entropies et dans le cas des variétés riemanniennes, cette convexité traduit une courbure de Ricci minorée. Alors qu'il est par exemple possible de définir le flot de gradient de $\operatorname{Ent}_\infty$ sur les variétés à courbure de Ricci minorée [35], les espaces d'Alexandrov [88] et les espace de Hilbert [4], il semblait délicat d'envisager un même travail pour le groupe de Heisenberg qui ne vérifie pas CD. Cependant en approchant $\mathbb{H}_1$ par des variétés riemanniennes $\mathbb{H}_1^\varepsilon$ (celle de la sous-section 1.2.6) pour $\varepsilon > 0$ tendant vers 0 on obtient tout de même des résultats intéressants. Il est à noter que la courbure de Ricci de $\mathbb{H}_1^\varepsilon$ peut être au mieux minorée par $-\frac{1}{2\varepsilon^2}$, une quantité qui tend vers $-\infty$.

L'opérateur de diffusion naturellement associé à la géométrie de $\mathbb{H}_1$ est

$$\Delta_{\mathbb{H}} = \mathbf{X}^2 + \mathbf{Y}^2.$$

C'est un opérateur hypoelliptique car $\mathbf{X}$ et $\mathbf{Y}$ vérifient la condition de Hörmander [58]: $\mathbf{X}$, $\mathbf{Y}$ et leur crochet de Lie $\mathbf{T}$ engendre l'algèbre de Lie entière. Ainsi la diffusion sous-elliptique de $\Delta_{\mathbb{H}}$ est bien maîtrisée. Le semi-groupe de la chaleur s'obtient par la convolution avec une gaussienne généralisée dont on connaît une expression et des estimées depuis l'article de Gaveau [49].

La définition de flot de gradient que nous avons adoptée est particulièrement générale. Elle stipule qu'un flot de gradient $(\mu_s)_{s\in I}$ doit avoir une vitesse métrique $|\dot{\mu_s}|$ (un réel positif) au sens de [4] en tout temps et que celle-ci doit être égale à la pente de l'entropie $\mathrm{Slope}(\mathrm{Ent}_\infty)$, définie par

$$\mathrm{Slope}(\mathrm{Ent}_\infty)(\mu) = \max\left(0, \limsup_{\nu\to\mu} \frac{\mathrm{Ent}_\infty(\mu) - \mathrm{Ent}_\infty(\nu)}{W(\mu,\nu)}\right).$$

De plus en presque tout temps s la dérivé de $\mathrm{Ent}_\infty(\mu_s)$ (qu'on suppose absolument continue) doit valoir $-\mathrm{Slope}(\mathrm{Ent}_\infty)(\mu_s)\cdot|\dot{\mu_s}|$.

Avec cette définition et en approchant $\mathbb{H}_1$ par les variétés $\mathbb{H}_1^\varepsilon$ on obtient la correspondance entre flot de gradient et diffusion sous-elliptique dans les deux sens. Cependant la démonstration du théorème nécessite des hypothèses supplémentaires: compacité du support à un instant initial et existence d'une dérivée faible pour ρ dans la direction $\mathbf{T}$.

Théorème A.4.1. *Soit $(\rho_s)_{s\in]0,+\infty[}$ la solution de l'équation de la chaleur sous-elliptique*

$$\begin{cases} \Delta_{\mathbb{H}}\rho_s = \partial_s\rho_s \\ \rho_0 d\mathcal{L} = \mu_0 \end{cases}$$

dans $\mathbb{H}_1$ pour μ_0 à support compact. Alors la courbe $(\mu_s)_{s\geq 0}$ des mesures $\rho_s d\mathcal{L} = \mu_s$ est un flot de gradient de l'entropie Ent_∞ *dans $\mathcal{P}_2(\mathbb{H}_1)$.*

Réciproquement soit $(\mu_s)_{s\in I}$ un flot de gradient de Ent_∞ *dans $\mathcal{P}_2(\mathbb{H}_1)$. On suppose que pour $s \in I$, il existe une dérivée faible $\mathbf{T}\rho_s$ telle que*

$$\int \frac{(\mathbf{T}\rho_s)^2}{\rho_s} < +\infty.$$

Alors la fonction $(\rho_s)_{s\in I}$ est solution de l'équation de la chaleur sous-elliptique.

Appendix B

Zusammenfassung auf Deutsch

Einleitung

Die Heisenberg-Gruppe $\mathbb{H}^n$ taucht in mehreren mathematischen oder allgemein wissenschaftlichen und technischen Gebieten auf. Es handelt sich hierbei nämlich um einen Referenzraum der Kontrolltheorie und der Sub-Riemmanschen Geometrie, so wie auch $\mathbb{R}^N$ ein Referenzraum der Riemannschen Geometrie ist. Den genannten Räumen ist unter anderem gemein, dass sie eine lokale Poincaré Ungleichung mit einer kanonischen Doubling-Eigenschaft erfüllen. Dies erlaubt uns, Analysis mit möglichst wenig Struktur zu betreiben(siehe [57]). Darüber hinaus ermöglicht uns die Heisenberg-Gruppe, die Allgemeingültigkeit der Theorien über die metrischen Maßräume einzuschätzen, da sie sich relativ gut für explizite Rechnungen eignet. Auf diese Weise konnten Ambrosio und Rigot einen großen Teil der Resultate zum Massentransport auf Riemmannschen Mannigfaltigkeiten auf $\mathbb{H}_n$ erweitern. Speziell gibt es einen eindeutigen optimalen Transport von einem absolutstetigem Maß zu einem zweiten Maß; die Abbildung dieses Massentransports ist durch eine Abbildung gegeben. In dieser Zusammenfassung geben wir ergänzende Ergebnisse zum Massentransport in der Heisenberg-Gruppe, welche den vier Hauptresultate der Doktorarbeit, bzw. vereinfachten Versionen der selbigen entsprechen.

Zuerst werden wir einige Kurven der Heisenberg-Gruppe analysieren. Zum einen werden wir sehen, was die Geodäten dieses Raumes (besonders die $\mathbb{H}$-Geraden) sind, und zum anderen untersuchen wir eine eigenartige horizontale Kurve, ω, welche in der Heisenberg-Gruppe ein Gegenbeispiel zum geometrischen Problem des Handlungsreisenden darstellt. Tatsächlich gilt die von Ferrari, Franchi und Pajot eingeführte Verallgemeinerung [40] eines euklidischen Theorems von Jones nur im direkten Sinne. Die Reziproke ist falsch, da ω endlicher Länge ist, während das Integral der Zahlen $\beta_{\mathbb{H}}(p,r)$ divergiert. Die Zahlen $\beta_{\mathbb{H}}(p,r)$ messen in der Kugel mit Zentrum p und Radius r die Entfernung von ω zur nächsten $\mathbb{H}$-Geraden.

Die MCP-Eigenschaft (Measure Contraction Property), welche ausgehend von der Analyse der Geodäten bewiesen wird, wird von großer Bedeutung für zwei der drei Ergebnisse zum Massentransport in der Heisenberg-Gruppe sein.

Figalli und der Autor haben in [42] eine von Ambrosio und Rigot am Ende ihres Artikels [7, partie 7] gestellte Frage gelöst: Wie auch auf den Riemannschen Mannigfaltigkeiten sind die Maße, welche die von einem absolutstetigen Maß ausgehenden optimalen Transporte interpolieren, ebenfalls absolutstetig. Die Abschätzung der Verzerrungen spielt eine zentrale Rolle in diesem Beweis.

Außerdem wird es um die seit kurzem bekannte Anwendung des Massentransports gehen, die ermöglicht, einen metrischen Maßraum mit von unten beschränkter Krümmung zu definieren. Diese überaus interessante Ausführung verdanken wir Lott und Villani [77, 78], sowie Sturm [104, 105]. Im Riemannschen Fall ist es bekannt das N-Dimensionale Mannigfaltigkeiten genau dann eine untere Ricci Schranke aufweisen, wenn ein Entropiefunktional während des Massentransports konvex wird. Diese zweite Eigenschaft trägt die Bezeichnung Krümmungs-Dimension $CD(K, N)$ (ursprünglich genutzt von Bakry und Émery [11] um eine ähnliche aber andere Eigenschaft zu benennen) und erhält ihren Sinn in den metrischen Maßräumen. Nichts weist direkt darauf hin, dass $\mathbb{H}_n$ die Krümmungs-Dimension erfüllen muss oder nicht erfüllen kann. Wir haben in [64] gezeigt, dass diese Eigenschaft für $\mathbb{H}_n$ unabhängig von den Parametern K und N falsch ist.

Die schon zuvor genannte Eigenschaft $MCP(K, N)$ ist, so wie auch die Bedingung $CD(K, N)$, eine geometrische Ungleichung, die ein metrischer Maßraum erfüllt oder nicht und die man wie eine nach unten durch K beschränkte Ricci-Krümmung interpretieren kann. Im Fall der $\mathbb{H}_1$ ist die Eigenschaft genau dann erfüllt wenn $K \leq 0$ und $N \geq 5$. Obwohl die Definition von MCP der des Massentransports sehr ähnlich ist, kann diese Eigenschaft erstaunlicherweise erfüllt sein, während sie bei CD nicht gilt (MCP ist generell schwächer). Außerdem ist die optimale Dimension 5 ziemlich unerwartet, denn es ist weder die topologische Dimension 3, noch die Hausdorff Dimension 4 der Heisenberg-Gruppe.

Das letzte Ergebnis dieser Zusammenfassung betrifft die subelliptische Diffusion in $\mathbb{H}_1$ und ihre Abbildung als Gradientenfluss im Wasserstein Raum $\mathcal{P}_2(\mathbb{H}_1)$. Dies ist der Raum der Wahrscheinlichkeitsmaße mit dem Abstand des Massentransports, dem sogenannten Wasserstein Abstand. Es scheint, dass man bei stetiger Bewegung in diesem Raum, bei welcher man die Bolzmann-Entropie Ent_∞ der betrachteten Maße so weit wie möglich senkt, eine Maß-Kurve findet, deren Dichte die Lösung der subelliptischen Wärmeleitungsgleichung ist.

Die genannten Resultate situieren sich wie folgt in der Doktorarbeit. Die Kurve ω und ihre Eigenschaft als Gegenbeispiel werden in dem Unterpunkt 1.8 dargestellt. Die Lösung der Frage von Ambrosio und Rigot in $\mathbb{H}_n$ erscheint im Theorem 2.3.6, wo wir eine im Wesentlichen zu $MCP(0, 2n+3)$ äquivalente Ungleichung benutzen. Die Behandlung der Ricci-Krümmungen $CD(K, N)$ und $MCP(K, N)$ in $\mathbb{H}_n$ erfolgt im Kapitel 3 (siehe Theorem 3.4.5 und Theorem 3.5.12 mit den Erweiterungen). Wir untersuchen dort, im Gegensatz zu der Zusammenfassung, wo wir uns auf $K = 0$ für $\mathbb{H}_1$ beschränkt haben, alle $K \in \mathbb{R}$ und $N \in [0, +\infty]$. Das letzte wesentliche Ergebnis zum Gradientenfluss der Entropie ist in zwei Sätze geteilt: (Theorem 3.5.12 und Theorem 3.5.13), welche beide im Kapitel 4 aufgeführt sind.

Diese Zusammenfassung besteht aus vier Teilen, von denen jeder eines der zentralen Ergebnisse beinhaltet. Im ersten Teil definieren wir zunächst $\mathbb{H}_1$. Anschließend betrachten wir ihre Geodäten und stellen die Kurve ω im Hinblick auf des geometrische Problem des Handlungsreisenden vor. Zudem geben wir

Hinweise auf den $MCP(0,5)$-Beweis. Im zweiten Teil behandelt die Definitionen des Massentransports und von CD. Wir führen das Theorem von Ambrosio und Rigot zum Massentransport in der Heisenberg-Gruppe an und erklären im Folgenden, wie man die Absolutstetigkeit der im Laufe dieses Transportes interpolierten Maße zeigen kann. Der dritte Teil fährt mit einem Vergleich der synthetischen Ricci-Krümmungen MCP und CD in der Heisenberg-Gruppe und einem Beweis, dass $CD(0,N)$ in $\mathbb{H}_1$ nicht gilt, fort. Wir schließen im letzten Teil mit der Übereinstimmung zwischen dem Gradientenfluss der Entropie und der subellipischen Diffusion ab.

B.1 Die Heisenberg-Gruppe, Kurven und Geodäten

B.1.1 Erste Eigenschaften von $\mathbb{H}_1$

Sei $\mathbb{H}_1$ die Heisenberg-Gruppe versehen mit dem Carnot-Carathéodory Abstand d_c, sowie $\mathcal{L}$ das $\mathbb{R}^3$ Lebesgue Maß. Betrachte $\mathbb{H}_1$ als $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$. Dann wird ein Element $(z;t) \in \mathbb{H}_1$ durch $z = x + \mathbf{i}y$ beschrieben. Das Gruppenprodukt wird durch

$$(z;t) \cdot (z';t') = (z + z'; t + t' - \frac{1}{2}\Im(z\overline{z'}))$$

gegeben, wobei $\Im$ dem Imaginärteil entspricht. Hiermit ist $\mathbb{H}_1$ eine Lie Gruppe mit neutralem Element $(0_{\mathbb{C}}, 0_{\mathbb{R}})$ und inversem Element $(-z;-t)$. Für die Lie Algebra der linksinvarianten Vektorfelder benutzt man

$$\mathbf{X} = \partial_x - \frac{1}{2}y\partial_t \quad , \quad \mathbf{Y} = \partial_y + \frac{1}{2}x\partial_t \quad , \quad \mathbf{T} = \partial_t$$

als Basis. Die Menge $L = \{(z;t) \in \mathbb{H}_1 \mid z = 0\}$ ist gleichzeitig die Kommutatorgruppe und das Zentrum der Gruppe. Im Folgenden wird L auch in der Geometrie von $(\mathbb{H}_1, d_c)$ eine wichtige Rolle spielen.

Sei nun $\tau_p(q) = p \cdot q$ die Linkstranslation um p. Man bemerkt, dass es sich um eine affine Transformation mit Determinante 1 handelt, die folglich das Volumen von $\mathbb{R}^3$ erhält. Dann ist $\mathcal{L}$ das Haar Maßder Gruppe. Auch der im folgenden definierte Abstand d_c ist linksinvariant.

Der Abstand d_c zwischen zwei Punkte wird als das Infimum der Länge der diese Punkte verbindenden Kurven definiert. Diese Länge wird nur für in $\mathbb{R}^3$ absolutstetigen Kurven definiert, die folgende Bedingung erfüllen:

$$\gamma_t' = 2(\gamma_y\gamma_x' - \gamma_x\gamma_y') \qquad \text{für fast alle } s \in [s_0, s_1], \tag{B.1}$$

wobei $(\gamma_x, \gamma_y, \gamma_t)$ die Koordinaten von γ sind. Diese Bedingung bedeutet, daßdie dritte Koordinate proportial zu der durch $\overrightarrow{0_{\mathbb{C}}g}$ überstrichene algebraische Fläche ist, wobei $g = (\gamma_x, \gamma_y)$ die Projektion von γ auf $\mathbb{C}$ ist. Für $Z : (z;t) \to z$ gilt weiterhin $g = Z(\gamma)$. Kurven mit der Eigenschaft (B.1) werden horizontale Kurven genannt. Gewöhnlicherweise sind die horizontalen Kurven diejenigen, die in fast jeder Zeit s zu dem durch $\mathbf{X}(\gamma(s))$ und $\mathbf{Y}(\gamma(s))$ erzeugten Untervektorraum tangential sind. Die Länge einer horizontalen Kurve ist dann genau $\int_{s_0}^{s_1} \sqrt{\gamma_x'^2 + \gamma_y'^2}$, und folglich die Länge der in $\mathbb{C}$ projizierten Kurve g. Die Länge einer nicht horizontalen Kurve kann man als unendlich definieren. Zusammengefasst

$$d_c(p,q) = \inf_{(\gamma(s_0),\gamma(s_1))=(p,q)} \begin{cases} \int_{s_0}^{s_1} \sqrt{\gamma_x'^2 + \gamma_y'^2} & \text{wenn (B.1)} \\ +\infty & \text{sonst.} \end{cases} \tag{B.2}$$

Das Infimum wird mindestens von eine Kurve angenommen. Eine solche Kurve wird im folgenden Geodäte genannt. Es wird dadurch gerechtfertigt, dass es sich um eine isometrische Einbettung von $\mathbb{R}$ in $(\mathbb{H}_1, d_c)$ handelt. Die Existenz von Geodäten in $\mathbb{H}_1$ kann man aus dem Dido Problem folgern, das eine Erweiterung des isoperimetischen Problems ist. Es geht darum für eine nicht abgeschlossene Komplexe Kurve g, die Länge und die übergtrichene algebraische Fläche zu vergleichen. Die eindeutigen Lösungen sind Kreisbögen (siehe Figur B.1). In dem Variationsproblem (B.2) versucht man, die Länge von Kurven zwischen $p = (z_p; t_p)$ und $q = (z_q; t_q)$ zu minimieren, wobei die Bedingung (B.1) erfüllt sein muss. Notwendige und hinreichende Bedingungen, um solche Kurven γ durch ihre komplexen Projektionen g darzustellen sind

- $\gamma(0) = p$
- Die projizierte Kurve g verbindet z_p mit z_q.
- Die überstrichene algebraische Fläche dieser Kurve ist $t_q - t_p$.

Die Länge von γ oder äquivalenterweise die von g zu minimieren führt genau darauf zurück, das Dido Problem zu lösen. Die Kurve g mit der kleinsten Länge ist ein Kreisbogen. Wenn man aus p die horizontale Aufhebung dieser Kurve nimmt, kommt man in q an (wegen der Flächengleichung) und die aufgehobene Kurve γ ist eine Geodäte von $\mathbb{H}_1$.

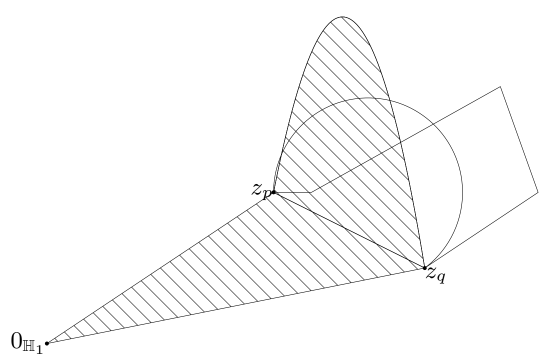

Figure B.1: Unter den Kurven, die eine gegebene Fläche überstreichen ist der Kreisbogen die Kürzeste.

Nach diesen kurzen Erklärungen, können wir die expliziten Gleichungen der Geodäten angeben, die von $0_{\mathbb{H}} = (0,0)$ ausgehen. Wir parametrisieren diese

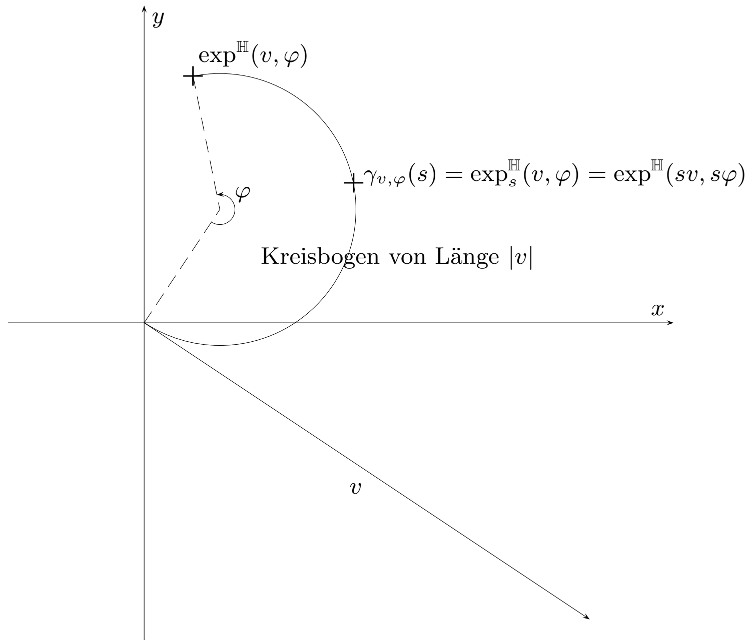

Figure B.2: Projektion von $\gamma_{v,\varphi}$ auf $\mathbb{C} \times \{0\}$ in $\mathbb{H}_1$.

Kurven mit konstanter Geschwindigkeit auf $[0, 1]$. Nach dem Prinzip von Figur B.2, werden alle Kreisbögen beschrieben, indem man den Tangentialvektor in 0 ($v \in \mathbb{C}$) und die Winkelöffnung $\varphi \in [-\pi, \pi]$ angibt. Unter allen verschiedenen Kreisbögen gibt es als Spezialfall die Kreislinien mit Winkelöffnung -2π oder 2π (je nach Rotationrichtung), die Halbkreislinien für die $|\varphi| = \pi$ gilt und die Segmente mit Winkelöffnung 0. Letztere nennt man horizontale Segmente und ihre Verlängerungen als Geodäten horizontale Geraden oder $\mathbb{H}$-Geraden.

Wenn $|\varphi| > 2\pi$, machen die Kreissegmente mehr als eine Runde und sind nicht die kürzesten Kreissegmente unter denen, die dieselbe algebraische Fläche umfangen. Letztlich hat man die Geodätengleichungen als Funktion von v und φ.

$$\gamma_{v,\varphi}(s) = \begin{cases} \left(v\frac{e^{i\varphi s}-1}{i\varphi}, |v|^2 \frac{\varphi s - \sin(\varphi s)}{2\varphi^2} \right) & \text{wenn } \varphi \neq 0 \\ (sv, 0) & \text{wenn } \varphi = 0. \end{cases}$$

Die komplexe Koordinate ist die Kurve einer Kreislinie, während die dritte Koordinate lässt sich als umfangene algebreaische Fläche als eine Funktion von s berechnen.

Die Geodäten, die aus p gehen sind einfach die Kurven ($\gamma_{v,\varphi}^p = p \cdot \gamma_{v,\varphi}$). Der Grund darum ist, dass die Linkstranslation τ_p die horizontaligkeit der Kurven (wie man es überprüfen kann) behält und, dass es für die projektierte Kurve in $\mathbb{C}$ nur eine Translation ist. Es folgt daraus, dass τ_p die Längen behält und, dass eine Isometrie von $\mathbb{H}_1$ ist.

Diese Parametrisierung der Geodäten, die aus $0_{\mathbb{H}}$ gehen ähnelt sich an der Weise, wie man die sphärischen Koordinaten eines Punktes findet. Aus diesem Vergleich kann man (v, φ) als die sphärischen Koordinaten eines Punktes von $\mathbb{H}_1$ betrachten und die Abbildung

$$\exp^{\mathbb{H}}(v, \varphi) := \gamma_{v,\varphi}(1) = \begin{cases} \left(i\frac{e^{-i\varphi}-1}{\varphi}v, 2\frac{\varphi-\sin(\varphi)}{\varphi^2}|v|^2\right) & \text{wenn } \varphi \neq 0 \\ (sv, 0) & \text{wenn } \varphi = 0. \end{cases}$$

wäre eine Koordinatenwechsel Abbildung von den sphrärischen zu den kartesianischen Koordinaten. Es gilt dann die Proposition

Proposition B.1.1. *Die Abbildung* $\exp^{\mathbb{H}}$ *ist ein* $\mathcal{C}^\infty$*-Diffeomorphismus von* $\mathcal{D} := \mathbb{C}^* \times]-2\pi, 2\pi[$ *zu* $\mathbb{H}_1 \setminus L$.

Entsprechend, die für jede $s \in]0, 1]$ durch

$$\exp_s^{\mathbb{H}}(v, \varphi, \cdot) = \gamma_{v,\varphi}(s) = \exp^{\mathbb{H}}(sv, s\varphi)$$

definierte Abbildung $\exp_s^{\mathbb{H}}$ ist ein Diffeomorphismus von $\mathcal{D}$ auf seinem Abbild.

B.1.2 Das geometrische Problem des Handlungsreisenden

In diesem Absatz präsentieren wir kurz die Kurve $\omega([0, 1])$. Das ist eine horizontale Kurve in $\mathbb{H}_1$ mit endlicher Länge, die sich im Sinne des geometrischen Handlungsreisenden Problems nur schwer von $\mathbb{H}_1$-Geraden approximieren lässt [40].

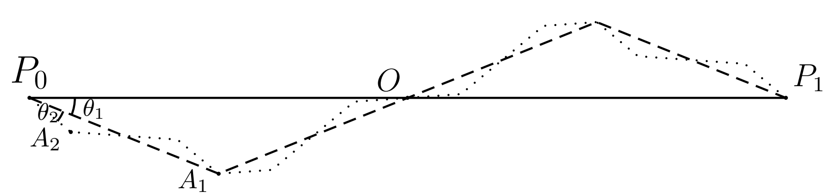

Figure B.3: Die Kurve $\omega([0, 1])$

Es handelt sich um eine fraktale Kurve, die man iterativ konstruiert. In jedem Schritt wird ein horizontales Segment durch eine Linie ersetzt, die aus vier horizontalen Segmenten besteht. In Figur B.3 sind die Projektionen in die komplexe Ebene der ersten drei approximierenden Kurven $(\omega_1, \omega_2, \omega_3)$ der Folge $(\omega_n)_{n\in\mathbb{N}}$, die gegen ω konvergiert, dargestellt. Sei von nun an Z die Projektion auf die Komplexe Ebene $(z, t) \mapsto t$. Ausgehend von ω_0, der Strecke von P_0 nach P_1 konstruiert man ω_1 als stückweise lineare Kurve mit denselben Endpunkten und konstanter Geschwindigkeit auf $[0,1]$, so dass die komplexen Projektionen von ω_0 und ω_1 in jedem Punkt einen Winkel $\theta_1 = 0,2$ einschließen. Die Kurven $Z(\omega_1)$ und $Z(\omega_2)$ schließen zwei gleichschenklige Dreiecke ein, deren Flächen übereinstimmen. In die Geometrie von $\mathbb{H}_1$ übersetzt bedeuted das, dass sobald $\omega_0(0) = \omega_1(0)$ gilt, auch $\omega_0(1) = \omega_1(1)$ gilt. Iterativ konstruiert man dann ω_{n+1} aus ω_n, indem man für jedes Segment der $Z(\omega_n)$ zwei gleichschenklige Dreiecke gleicher Fläche bildet, die einen Winkel $\theta_n = \frac{0,2}{n}$ einschließen. Daraus folgt, dass sobald man konstante Geschwindigkeit auf $[0,1]$ voraussetzt, dass für alle $\sigma \in \{0,1,\ldots,4^n\}$ gilt

$$\omega_n(\frac{\sigma}{4^n}) = \omega_{n+1}(\frac{\sigma}{4^n}).$$

Man zeigt dann, dass die so konstruierte Folge konvergiert, und dass die Grenzkurve ω von endlicher Länge ist. Bei jedem Schritt verändert sich nämlich die Länge der Kurve um einen Faktor $\cos(\theta_n)^{-1}$. Weil $\cos(\theta) = 1 - \frac{\theta^2}{2} + o(\theta^2)$, folgt die Konvergenz der Längen aus der Konvergenz der Reihe $\sum \frac{1}{n^2}$.

Es folgen einige Definitionen, bevor wir zum Hauptresultat aus [40] kommen. Die Zahlen $\beta_{\mathbb{H}}(p,r)(E)$ sind definiert als

$$\beta_{\mathbb{H}}(p,r)(E) = \inf_{l\ \mathbb{H}\text{-gerade}} \left(\frac{\max_{p \in \mathcal{B}(p,r) \cap E} d_c(p,l)}{r} \right).$$

Man betrachtet also den maximalen Abstand zwischen einer horizontalen Gerade und den Punkten aus E, die in $\mathcal{B}(p,r)$ enthalten sind. Das Minimum dieser Grösse (nach einer Normalisierung) über alle horizontalen Geraden ist $\beta_{\mathbb{H}}(p,r)(E)$. Man definiert also

$$\mathbf{B}(p,r)(E) = \int_{p \in \mathbb{H}_1} \int_{r>0} \frac{\beta^2_{\mathbb{H}}(p,r)}{r^4} dr d\mathcal{L}(p).$$

Jetzt können wir den Satz von Ferrari, Franchi und Pajot formulieren:

Satz B.1.2. *(i) Sei $E \subset \mathbb{H}_1$ kompakt. Dann ist E enthalten in einer Kurve Γ endlicher Länge $l(\Gamma)$, wenn*

$$\mathbf{B}(E) \leq +\infty.$$

Außerdem $\inf_{\Gamma \supset E} l(\Gamma) \leq C(\mathrm{diam}(E) + \mathbf{B}(E))$ *(wobei C eine feste Konstante ist).*

(ii) Falls γ eine Geodäte in $\mathbb{H}_1$ ist, dann $\mathbf{B}(E) < Cl(\Gamma)$, wobei C wieder eine feste Konstante ist.

Dieser Satz ist das Analogon eines euklidischen Satzes von Peter Jones [62] (siehe auch [91, 99]) in der Heisenberg-Gruppe. In dem Eulidischen Rahmen gilt die zweite Aussage *(ii)* für alle rektifieziertbaren Kurven Γ. Unsere Kurve $\Omega = \omega([0,1])$ zeigt dass diese Aussage in dieser Allgemeinheit in der Heisenberg-Gruppe nicht gilt. Das folgt aus eine genauen Analyse unterer Schranken der Zahlen $\beta_{\mathbb{H}}(p,r)$. Wenn auch die genaue Analyse etwas schwerfällig ist, so sind grobere Abschätzungen recht einfach zu verstehen. In Kugeln der Grössenordnung r sind die ersten ω_n recht grobe Approximationen von ω. Eine vernünftige Annäherung erfolgt erst for n von der Grössenordnung $-\log_4(r)$. Nehmen wir also an, dass die Kugel genau eins der Segmente von ω_{n-1} unterteilt. Das horizontale Segment der Länge r ist dann eine recht gute Aproximation von ω_n durch eine $\mathbb{H}_1$-Gerade. Man nimmt also als charakteristischen

Abstand zwischen ω_n und der Approximation den Abstand der Mittelpunkte der Kurven. Diese haben dieselben komplexen Projektionen, liegen aber auf verschiedenen Höhen, weil ω_n eine zusätzliche Fläche umschließt. Wenn wir die unterschiedlichen Höhen mit t und t' bezeichnen, ist der Abstand $|t-t'|$ durch die Fläche eines der gleichschenkligen Dreiecke gegeben. Diese ist von der Grössenordnung $r^2\theta_n$. Also ist der Abstand in $\mathbb{H}_1$ zwischen diesen Punkten von der Grössenordnung $r\sqrt{\theta_n}$ und $\beta_{\mathbb{H}}(p,r)$ ist folglich von der Größenordunung θ_n. Wenn man nun beachtet, dass die Punkte p, für die diese Abschätzung gilt, eine Fläche der Grössenordnung r^3 überstreichen, bekommt man das folgende divergente Integral:

$$\int_{1>r>0} \frac{1}{r \ln_4(1/r)}.$$

B.1.3 Zwei nützliche Abbildungen

Hier definieren wir zwei Abbildungen, die nützlich in den nächsten Teilen sein werden. Es handelt sich um die Zwischenpunktabbildung $\mathcal{M}$ und das geodätische Inverse $\mathcal{I}$.

Die Abbildung $\mathcal{M}$ ist für $(p,q,s) = \mathbb{H}_1 \times \mathbb{H}_1 \times [0,1]$ definiert. Wir verwenden aber auch die Bezeichnungen $\mathcal{M}^s(p,q)$ sowie $\mathcal{M}^s_p(q)$ für $\mathcal{M}(p,q,s)$. Wenn z_p und z_q verschieden sind, ist $\mathcal{M}(p,q,s)$ eindeutig als ein Punkt m auf der Geodäten von p bis q bestimmt, wobei die Abstände die Relationen $d_c(p,m) = sd_c(p,q)$ und $d_c(m,q) = (1-s)d_c(p,q)$ erfüllen müssen. Es folgt dann, dass

$$\mathcal{M}(p,q,s) = \tau_p \circ \exp^{\mathbb{H}}_s \circ \left(\exp^{\mathbb{H}}\right)^{-1} \circ \tau_p^{-1}(q)$$

und

$$\mathcal{M}(p,q,s) = \gamma^p_{(\exp^{\mathbb{H}})^{-1}(p^{-1}\cdot q)}(s).$$

Hier erkennt man in $\gamma^p_{(\exp^{\mathbb{H}})^{-1}(p^{-1}\cdot q)}$ die normalisierte Geodäte von p bis q.

Bemerkung B.1.3. Die Abbildung $\mathcal{M}$ in nicht wohldefiniert, wenn $z_p = z_q$, denn es gibt in diesem Fall unendlich viele Geodäten zwischen p und q (eine für jede Kreislinie mit Fläche $t_q - t_p$ durch z_p).

Das geodätische Inverse $\mathcal{I}$ ist die Abbildung, die einen Punkt p auf $\mathcal{I}(p)$ abbildet, so dass $0_{\mathbb{H}}$ der Mittelpunkt der Geodäten zwischen p und $\mathcal{I}(p)$ ist. Die Abbildung $\mathcal{I}$ ist wohldefiniert und es gilt

$$\mathcal{I}(p) = \exp^{\mathbb{H}}(-\left(\exp^{\mathbb{H}}\right)^{-1}(p)).$$

Dann wenn $p = \exp^{\mathbb{H}}(v,\varphi)$, hat man $\mathcal{I}(p) = \exp^{\mathbb{H}}(-v,-\varphi)$.

Bemerkung B.1.4. Genauer gesagt ist diese Abbildung nur auf $\exp^{\mathbb{H}}_{1/2}(\mathbb{C} \times [-2\pi, 2\pi]) = \exp^{\mathbb{H}}(\mathbb{C} \times [-\pi,\pi])$ wohl definiert. Man kann sehen, dass dies die abgeschlossene Menge zwischen den zwei Paraboloïden, die durch die Gleichung $|z|^2 = \pm 2|t|/\pi$ gegeben sind, ist. Außerhalb dieser Menge gibt es keine Kurven mit Ende p und Mittelpunkt $0_{\mathbb{H}}$, die globale Geodäten sind.

Zuletzt geben wir noch die explizite Form der Jacobi Determinante von $\exp^{\mathbb{H}}$. Es ist ein Teil des Beweises von Proposition B.1.1. Dort wird gezeigt, dass $\mathrm{Jac}(\exp^{\mathbb{H}})$ nirgendwo verschwindet.

Proposition B.1.5. *Die Jacobi-Determinante von* $\exp^{\mathbb{H}}$ *ist*

$$\mathrm{Jac}(\exp_1^{\mathbb{H}})(v,\varphi) = \begin{cases} 4|v|^2 \left(\frac{\sin(\varphi/2)}{\varphi}\right) \frac{\sin(\varphi/2)-(\varphi/2)\cos(\varphi/2)}{\varphi^3} & \text{für } \varphi \neq 0, \\ |v|^2/12 & \text{sonst.} \end{cases}$$

Für $0 < s < 1$ *ist die Jacobi-Determinante von* $\exp_s^{\mathbb{H}}$ *gegeben durch*

$$\mathrm{Jac}(\exp_s^{\mathbb{H}})(v,\varphi) = \begin{cases} 4s|v|^2 \left(\frac{\sin\frac{s\varphi}{2}}{\varphi}\right) \frac{\sin\frac{s\varphi}{2} - \frac{s\varphi}{2}\cos\frac{s\varphi}{2}}{\varphi^3} & \text{für } \varphi \neq 0, \\ s^5|v|^2/12 & \text{sonst.} \end{cases}$$

B.1.4 Massenkontraktionseigenschaft MCP

Wir definieren die Maßkontraktionseigenschaft für einige metrische Maßräume, wie unter anderem die Heisenberg-Gruppe. Im Folgenden werden wir die Rechnungsschritte angeben, die ermöglichen $MCP(0,5)$ in $\mathbb{H}_1$ zu beweisen.

Die Maßkontraktionseigenschaft MCP wird erst seit dem Erscheinen der Artikel von Sturm [105] und Ohta [89] als Erweiterung der Ricci Krümmung in metrischen Maßräumen betrachtet. Sie führen beide auf eine sehr ähnliche Art eine solche Massenkontraktionseigenschaft ein, die den Vorteil hat für eine sehr allgemeine Klasse von Räumen wohldefiniert zu sein. Insbesondere macht die Eigenschaft auch dann Sinn, wenn zwischen zwei Punkten immer unendlich viele Geodäten existieren. Die Räume, von denen man weiß, dass sie diese Eigenschaft erfüllen sind aber meistens weit weniger kompliziert.

Da die allgemeine Definition schwerer zu formulieren ist, geben wir hier eine einfachere Formulierung für den Fall, in dem eine messbare Abbildung

$$\mathcal{N} : X \times X \times [0,1] \to X$$

existiert, so dass es für $\nu\otimes\nu$-fast jedes Punktepaar (p,q) eine eindeutige Geodäte von p bis q gibt, die durch $(\mathcal{N}(p,q,s))_{s\in[0,1]}$ gegeben ist (Für die Heisenberg-Gruppe ist natürlich $\mathcal{M}$ eine passende Abbildung). Dann erfüllt der metrische Maßraum (X,d,ν) die Eigenschaft $MCP(0,N)$ genau dann, wenn für fast jeden Punkt p, für jede ν-messbare Menge E und für alle $s \in [0,1]$ gilt:

$$s^N \nu(\mathcal{N}_{p,s}^{-1}(E)) \leq \nu(E),$$

wobei $\mathcal{N}_{p,s} = \mathcal{N}(p,q,s)$. Wenn die Inverse Abbildung $\mathcal{N}_{p,s}$ messbar ist, kann man auch die direktere Formulierung

$$\nu(\mathcal{N}_{p,s}(F)) \geq s^N \nu(F)$$

benutzen, die anschaulischer zeigt, dass R^N die Eigenschaft $MCP(0,N)$ erfüllt (man hat dann eine Gleichheit). Diese Formulierung ist insbesondere im Fall von $\mathbb{H}_1$ möglich, weil $\mathcal{M}_p^s$ ein Homeomorphismus von $\{(z_q;t_q) \in \mathbb{H}_1 \mid z_q \neq z_p\}$ auf sein Bild ist und, weil die komplementäre Menge Maß null hat.

In der Heisenberg-Gruppe, kann man auch die Differenzierbarkeit der Kontraktionabbildung benutzen. Es ist nämlich hinreichend, eine unterere Schranke für die Jacobi-Determinante von $\mathcal{M}_p^s$ zu haben. Für fast jedes Paar (p,q) und für jede Zeit $s \in [0,1]$ will man

$$\mathrm{Jac}(\mathcal{M}_p^s)(q) \geq s^N$$

haben. Die Invarianz unter Linkstranslationen macht hier einige Rechnungen einfacher. Um $MCP(0,5)$ in $\mathbb{H}_1$ zu beweisen, genügt es, die letzte Ungleichung für $p = 0_{\mathbb{H}_1}$ und $N = 5$ zu beweisen. Das ist genau der Ansatz, den wir verfolgen werden. Man kennt für $\mathcal{M}_s^{0_{\mathbb{H}}}$ auf $\mathbb{H}_1 \setminus L$ den expliziten Ausdruck $\exp_s^{\mathbb{H}} \circ \left(\exp^{\mathbb{H}}\right)^{-1}$. Die Jacobi-Determinante der s-Mittelpunkt Abbildung ist dann für $s \in (0,1)$ im Punkt $\exp^{\mathbb{H}}(v,\varphi)$ gegeben durch:

$$\left(\frac{\operatorname{Jac}(\exp_s^{\mathbb{H}})}{\operatorname{Jac}(\exp^{\mathbb{H}})}\right)(v,\varphi).$$

Wir wollen zeigen, dass diese Funktion für alle s und jeden Punkt mit sphärischen Koordinaten $(v,\varphi) \in \mathcal{D}$ grösser als s^5 ist. Es ist äquivalent zu zeigen, dass $\operatorname{Jac}(\exp_s^{\mathbb{H}})^{1/5}$ grösser als die affine Funktion $s \operatorname{Jac}(\exp^{\mathbb{H}})^{1/5}$ ist. Da beide gleich in 0 und 1 sind, reicht es hierfür zu zeigen, dass

$$\operatorname{Jac}(\exp_s^{\mathbb{H}})^{1/5}(v,\varphi) = 4s|v|^2 \left(\frac{\sin\frac{s\varphi}{2}}{\varphi}\right) \frac{\sin\frac{s\varphi}{2} - \frac{s\varphi}{2}\cos\frac{s\varphi}{2}}{\varphi^3}$$

konkav in s ist und, dass es für jede $(v,\varphi) \in \mathcal{D}$ gilt. Da φ das Interval $]-2\pi, 2\pi[$ durchläuft, und die Funktion in φ symmetrisch ist, besteht der Beweis darin zu zeigen, dass

$$F(x) = x\sin(x)(\sin(x) - x\cos(x))$$

$1/5$-konkav auf $[0,\pi]$ ist. Bei hinreichend oft differenzierbaren Funktionen weiß man, dass eine Funktion F genau dann $1/5$-konkav ist, wenn $F''F - F'^2 + \frac{F'^2}{5} \leq 0$ gilt. Bei log-konkaven Funktionen ist die Bedingung $F''F - F'^2 \leq 0$. Diese Log-Konkavität ist aber hier eine einfache Folge der Log-Konkavität der Faktoren $a(x) = x$, $b = \sin$ und $c(x) = \sin(x) - x\cos(x)$ auf $[0,\pi]$. Die Log-Konkavität eines Produkts, folgt leicht aus der Log-Konkavität der Faktoren, denn $\ln(abc) = \ln(a) + \ln(b) + \ln(c)$. Wenn man also dies in den Ausdruck $F''F - F'^2 + \frac{F'^2}{5}$ einsetzt, erhält man:

$$\left[(a''a - a'^2)b^2c^2 + a^2(b''b - b'^2)c^2 + a^2b^2(c''c - c^2)\right] + \frac{F'^2}{5}.$$

Um die $1/5$-Konkavität zu zeigen, genügt es also zu beweisen, dass in dem Ausdruck

$$\frac{F'^2}{5} - \left[(bc)^2 + (ac)^2 + (x^2 - \sin^2(x))(ab)^2\right],$$

der positive Term $\frac{F'^2}{5}$ nicht groß genug ist, um die Summe positiv zu machen. Dieses Resultat erhält man durch eine detaillierte Untersuchung der beiden Terme (siehe [64]). Umgekehrt erhält man durch eine Reihenentwicklung um den Punkt $s = 0$, dass 5 tatsächlich der optimale Exponent ist.

B.2 Optimaler Massentransport in $\mathbb{H}_1$

B.2.1 Definitionen

Die Theorie des optimalen Massentransport hat seit etwa zwanzig Jahren eine gesteigerte Aufmerksamkeit genossen, seit neue Anwendungen in diversen Feldern der Mathematik entdeckt wurden. In dieser Zusammenfassung beschreiben wir die Anwendung in der Geometrie, wie sie in den Arbeiten von Lott und Villani [77, 78] sowie Sturm [104, 105] beschrieben werden. Diesen Autoren ist es gelungen auf eine überzeugende Art die Eigenschaft einer unteren Ricci-Schranke zu haben, für metrische Maßraüme zu definieren - eine Eigenschaft, die bis dahin nur für Riemannsche Mannigfaltigkeiten definiert war. Es handelt sich hierbei um die Krümmungs-Dimensions-Bedingung $CD(K, N)$, wobei $K \in \mathbb{R}$ für die Krümmung und $N \in [1, +\infty]$ für die Dimension steht. Wir geben außerdem die Argumente aus [64], die zeigen, dass für das Tripel $(\mathbb{H}_1, d_c, \mathcal{L})$ für kein Paar von Parametern K und N eine solche Bedingung erfüllt ist. Der wichtigste Schritt, der darin besteht $CD(0, N)$ zu wiederlegen, ist vollzogen indem man zeigt, dass eine generalisierte Brunn-Minkowski Ungleichung nicht gilt (siehe Abschnitt B.3 sowie [64]). Dennoch wollen wir zunächst den Massentransport allgemein definieren und seine Realisierung in der Heisenberg-Gruppe besprechen. Die bekannten Resultate stammen von Ambrosio und Rigot [7] und wurden in [42] von Figalli und dem Autor vervollständigt.

Als Startpunkt wählen wir das optimale Transportproblem von Monge-Kantorovich. Gegeben einen metrischen Raum (X, d) sowie zwei borelsche Wahrscheinlichkeitsmaße μ_0 und μ_1 auf X betrachtet man das folgende Variationsproblem:

$$\inf_{\pi} \int_{X \times X} d^2(p, q) d\pi(p, q). \tag{B.3}$$

Hier nimmt man das Infimum über alle Wahrscheinlichkeitsmaße π auf $X \times X$, deren Marginale μ_0 und μ_1 sind. Solche Maße nennt man auch Kopplungen. In geometrischen Anwendungen ist die Kostenfunktion $d^2(p, q)$ die am häufigsten betrachtete. Andere Kostenfunktionen $c(p, q)$ spielen aber durchaus auch eine Rolle in der allgemeinen Theorie. Die Wurzel von (B.3) nennt man Wasserstein-Abstand und bezeichnet sie mit $W(\mu_0, \mu_1)$. Der Wasserstein Abstand ist tatsächlich eine Metrik, wenn man ihn auf den Wasserstein-Raum $\mathcal{P}_2(X)$, der aus den Wahrscheinlichkeitsmaßen mit endlichem zweiten Moment besteht, beschränkt (ein Maß hat endliches zweites Moment, wenn das Intergral $\int_X d^2(o, p) d\mu(p) < +\infty$ für ein $o \in X$ oder äquivalent für alle o). Für solche Maße ist der Wasserstein-Abstand automatisch endlich. Es zeigt sich, dass dieser Wasserstein Raum mit der Wasserstein-Metrik ein geodätischer Raum ist, wenn das für den Grundraum X gilt.

Wir stellen nun die Theorie im Spezialfall von $X = \mathbb{H}_1$ vor, wie in der Arbeit von Ambrosio und Rigot (siehe [7] und [42]). Diese Resultate benötigen den Begriff der approximativen Differenzierbarkeit, wie er etwa in [4] beschrieben wird. Für das Folgende ist es aber nicht unbedingt notwendig diesen Begriff genau zu verstehen – man sollte nur beachten, dass es sich um eine Erweiterung des gewöhnlichen Differenzierbarkeitsbegriffs handelt.

Proposition B.2.1. *Sei $\mu_0, \mu_1 \in \mathcal{P}_2(\mathbb{H}_1)$. Wir nehmen außerdem an, dass dieses Maß absolutstetig bezüglich des Lebesgue-Maßes sind. Dann existiert eine eindeutige optimale Kopplung π von μ_0 und μ_1, und diese Kopplung ist durch eine Abbildung T gegeben, d.h. $\pi = (\mathrm{Id} \otimes T)_{\#}\mu_0$. Falls μ kompakten Träger hat, dann existiert außerdem eine in μ_0 fast jedem Punkt differenzierbare Funktion ψ, so dass*

$$T(p) := p \cdot \exp^{\mathbb{H}}(\mathbf{X}\psi(p) + \mathbf{i}\mathbf{Y}\psi(p), \mathbf{T}\psi(p)).$$

Auch in dem Fall, wo μ_0 keinen kompakten Träger hat, existiert ein Abbildung ψ, die allerdings nur approximativ differenzierbar in μ_0 fast jedem Punkt sein *muss, so dass*

$$T(p) := p \cdot \exp^{\mathbb{H}}(\tilde{\mathbf{X}}\psi(p) + \mathbf{i}\tilde{\mathbf{Y}}\psi(p), \tilde{\mathbf{T}}\psi(p)).$$

Ausgehend von diesem Resultat, kann man eine geodätische Kurve von Maßen in $\mathcal{P}_2(\mathbb{H}_1)$ konstruieren. Illustrieren wir das an folgendem Bild: Vergleichen wir die beiden Maße μ_0 und μ_1 mit zwei Ameisenhaufen (die in der Heisenberg-Gruppe leben!). Die Dichte der Maße entspricht hierbei der Dichte an Ameisen (die wir als ununterscheidbar annehmen). Der optimale Transportplan π entspricht nun der optimalen Wanderung der Ameisen von der einen Konfiguration in die andere: Er minimiert die Summe der quadratischen Abstände, die die Insekten zurücklegen müssen. Die Proposition besagt also, dass ein solcher minimierender Plan der globalen Bewegung existiert und eindeutig ist, jede einzelne Ameise weiß schon zum Zeitpunkt 0 genau, wo sie sich zum Zeitpunkg 1 befinden wird. Zwischen den Zeitpunkten 0 und 1 bewegt sich jedes Insekt mit gleichförmiger Geschwindigkeit entlang einer Geodäten von seinem Start- zu seinem Zielpunkt. Wenn man diese Bewegung zu einem Zwischenzeitpunkt $s \in [0,1]$ anhält, ergibt sich eine weitere Konfiguration μ_s. Mathematisch ausgedrückt handelt es sich um $T_{s\#}\mu_0$, wobei

$$T_s(p) := x \cdot \exp_s^{\mathbb{H}}(\tilde{\mathbf{X}}\psi(p) + \mathbf{i}\tilde{\mathbf{Y}}\psi(p), \tilde{\mathbf{T}}\psi(p)).$$

Auf diese Weise sind zwei weitere Transportpläne zwischen μ_0 und μ_s bzw. zwischen μ_s und μ_1 beschrieben: Man folgt einfach den Pfaden der Ameisen. Der erste Transport hat Transportkosten $s^2W^2(\mu_0, \mu_1)$ und der zweite Transportkosten $(1-s)^2W^2(\mu_0, \mu_1)$. Aus der Tatsache, dass wegen der Dreiecksungleichung gilt $W(\mu_0, \mu_1) \leq W(\mu_0, \mu_s) + W(\mu_s, \mu_1)$ folgt, dass diese beiden Transportpläne wieder optimal sind und dass die Kurve $(\mu_s)_{s\in[0,1]}$ eine Geodäte des Wassersteinraums $\mathcal{P}_2(\mathbb{H}_1)$ darstellt.

Mit einem ähnlichen Argument wie [7, Lemme 4.7] kann man sehen, dass für μ fast jeden Punkt p die Kurve $\gamma^p_{\tilde{\mathbf{X}}\varphi(p)+\mathbf{i}\tilde{\mathbf{Y}}\varphi(p),\tilde{\mathbf{T}}\varphi(p)}$, die p mit $T(p)$ verbindet, die einzige Geodäte zwischen diesen beiden Punkten ist. Das heißt für die Ameisen, die ja ein eindeutiges Ziel vor Augen haben, dass sie nur einen einzigen Weg wählen können. Auf dem Niveau der Maße bedeuted diese Beobachtung, dass es nur diese Geodäte zwischen μ_0 und μ_1 gibt. Diese Aussage ist im Allgemeinen nicht richtig, wenn das Startmaß μ_0 nicht absolutstetig ist. Als Gegenbeispiel kann man etwa den Transport zwischen zwei auf L konzentrierten Maßen betrachten. Zwischen zwei verschiedenen Punkten auf L gibt es immer unendlich viele Geodäten und diese Eigenschaft überträgt sich auf das Niveau der Maße.

Bemerkung B.2.2. In dem Artikel [42] wird gezeigt, dass die Zwischenmaße μ_s für $0 < s < 1$ auch absolutstetig sind. Diese Beobachtung zeigt, dass der Unterraum $\mathcal{P}_2^{ac}(\mathbb{H}_1) \subset \mathcal{P}_2(\mathbb{H}_1)$ der absolutstetigen Maße in $\mathcal{P}_2(\mathbb{H}_1)$ ein vollständiger geodätischer Raum ist. Die Frage bezüglich der Absolutstetigkeit war in [7] gestellt worden. Im Absatz B.2.2 werden wir sehen, dass die Techniken, die man verwendet um diese Aussage im Riemannschen Fall zu zeigen, sich nicht auf die Heisenberg-Gruppe anwenden lassen.

Wir erklären nun, was man unter der Bedingung $CD(0,N)$ versteht. Für eine N-dimensionale Riemannsche Mannigfaltigkeit mit dem Volumenmaß gilt diese Eigenschaft genau dann, wenn die Ricci-Krümmung in jedem Punkt nichtnegativ ist. Für die allgemeine Definition betrachte einen metrischen Maßraum

(X, d, ν). Wir betrachten wieder den optimalen Transport zwischen absolutstetigen Maßen und man analysiert wie die Rényi Entropie dieser Maße sich im Laufe der Zeit verändern. Dieses Entropiefunktional ist definiert durch

$$\mathrm{Ent}_N(\mu \mid \nu) = \begin{cases} -\int_X \rho^{1-1/N} d\nu & \text{wenn } d\mu = \rho d\nu \\ +\infty & \text{wenn } \mu \text{ nicht absolutstetig ist} \end{cases}.$$

Für $N = +\infty$ nimmt man die Boltzmann-Entropie:

$$\mathrm{Ent}_\infty(\mu \mid \nu) = \begin{cases} \int_X \rho \ln(\rho) d\nu & \text{wenn } d\mu = \rho d\nu \\ +\infty & \text{wenn } \mu \text{ nicht absolutstetig ist} \end{cases}.$$

Grob gesprochen misst die Entropie, wie sehr die Masse eines Maßes auf den Raum verteilt ist: Ein Maß, dass die Masse recht gleichmäßig bezüglich ν verteilt, hat eine eher kleine Entropie, während ein Maß, dass die Masse um wenige Punkte konzentriert, eine große Entropie hat. Folgende einfache Rechnung macht diese Aussage quantitativer: Für eine Menge vom ν-Maß V ergibt sich für die Entropie $-V^{1/N}$. Die Definition der Krümmungs-Dimension Bedingung lautet dann wie folgt:

Definition B.2.3. Sei $N \in [1, +\infty]$. Der metrische Maßraum (X, d, ν) erfüllt $CD(0, N)$, wenn für jedes Paar (μ_0, μ_1) absolutstetiger Maße eine Geodäte $(\mu_s)_{s \in [0,1]}$ in $\mathcal{P}_2^{ac}$ mit konstanter Geschwindigkeit existiert, so dass für jedes s gilt

$$\mathrm{Ent}_N(\mu_s \mid \nu) \leq (1-s)\,\mathrm{Ent}_N(\mu_0 \mid \nu) + s\,\mathrm{Ent}_N(\mu_1 \mid \nu).$$

Für die Ameisen, die sich auf dem Raum bewegen heißt diese Bedingung, dass entlang ihres Weges die Ameisen gleichmässiger über den Raum verteilt sind als zu Beginn und Ende ihrer Reise. Entlang des Weges entfernen sie sich voneinander und kommen erst gegen Ende des Weges an anderen Orten wieder Näher zueinander.

Unser Resultat über die Heisenberg-Gruppe sagt nun, dass dieses Verhalten in der Heisenberg-Gruppe nicht auftritt. Der Beweis findet sich im Abschnitt B.3.

Bemerkung B.2.4. Da in der Heisenberg-Gruppe die Eindeutigkeit der Geodäten in $\mathcal{P}_2^{ac}(\mathbb{H}_1)$ gilt, kann man sich schnell davon überzeugen, dass die Bedingung $CD(0, N)$ äquivalent zur Konvexität der Entropie entlang dieser Geodäten ist. Das bedeutet, dass die Bedingung $CD(0, N)$ hier etwas schwächer ist als im allgemeinen Fall. Wenn man zum Beispiel den Mittelpunkt der Maße $\mu_{1/4}$ und $\mu_{3/4}$: für den betrachteten Transport handelt es sich hierbei offenbar um $\mu_{1/2}$. Die allgemeine Bedingung $CD(0, N)$ fordert aber nur, dass eine Geodäte *existiert*, die die richtigen Interpolationseigenschaften hat - nicht unbedingt aber, dass es sich dabei um die Geodäte handelt, die man kennt.

B.2.2 Absolutstetigkeit entlang des Transports

Das Verkürzungsprinzip von Monge-Mather ist detailliert im Buch von Cédric Villani [109, Chapitre 8] beschrieben. Wir benutzen dieses Prinzip um eine Ungleichung über den Massentransport in $(T_{s\#}\mu_0)_{s \in [0,1]}$ zu zeigen. Aus dieser Ungleichung kann man unter der Hypothese, dass das Startmaß absolutstetig bezüglich des Hausdorff-Maßes des Raums ist die Absolutstetigkeit der Zwischenmaße folgern. Wir beginnen mit folgender Beobachtung: Für $\mu_0 \otimes \mu_0$ fast alle Punkt-Paare (a, b) treffen sich die Kurven $T_s(a)$ und $T_s(b)$ fast sicher nicht zu einem fixierten Zeitpunkt $s < 1$. Würde dies nämlich passieren, so könnte man den Transport verkürzen indem man die Kurven mischt: Das Ende jeder Kurve könnte durch das Ende der anderen ersetzt werden. Auf diese Weise entstünde ein besserer Transportplan. Nehmen wir für den Moment an, es gilt eine Quantitative Version dieser Beobachtung vom Typ

$$d(T_s(a), T_s(b)) \geq Cd(a, b). \tag{B.4}$$

Daraus können wir schlussfolgern, dass

$$d\mathcal{H}_d^n(T_s(E)) \geq C^n \mathcal{H}_d^n(E). \tag{B.5}$$

Hierbei bezeichnet $\mathcal{H}_d^n$ das n-dimensionale Hausdorff-Maß für den Abstand d.

Davon ausgehend gilt für jede Menge F vom Hausdorff-Massß 0, dass automatisch auch $T_s^{-1}(F)$ Hausdorff-Massß 0 hat, denn $F = T_s(T_s^{-1}(F))$. Aus der Absolutstetigkeit von μ bezüglich $\mathcal{H}_d^n$ folgt dann $\mu_0(T_s^{-1}(F)) = 0$. Da die Abbildung T_s einen optimalen Transportplan zwischen μ_0 und μ_s erzeugt folgt also $\mu_s(F) = 0$. Insgesamt können wir also festhalten, dass μ_s absolutstetig bezüglich $\mathcal{H}_d^n$ ist. Insbesondere würde eine Ungleichung vom Typ (B.4) die Frage von Ambrosio und Rigot positiv beantworten, da für die Heisenberg-Gruppe das Lebesgue-Maß mit dem vierdimensionalen Hausdorff-Maß übereinstimmt. In [42] wird aber gezeigt, das die Ungleichung (B.4), im Gegensatz zum klassischen Fall einer Riemannschen Mannigfaltigkeit, in der Heisenberg-Gruppe nicht gilt: Die Rechtsmultiplikation mit dem Vektor $(1, 0, 0)$ genügt als einfaches Gegenbeispiel.

Für $\mathbb{H}_1$ muss man also einen anderen Ansatz finden. Die Beweisidee in [42] greift die vorherigie Strategie auf dem Niveau der Ungleichung (B.5) auf. Eine ähnliche Version, in der nur die Konstante C durch $(1 - s)^5$ erstzt ist, wird gezeigt. Das ist wegen der Stabilität der Massentransport Techniken möglich, die es erlauben den notwendigen Grenzübergang zu rechtfertigen. Das Maß μ_1 wird als schwacher Limes einer Folge diskreter Maße $\mu_1^k = \frac{1}{k}\sum_{i=1}^{k} \delta_{y_i}$ betrachtet. Für jeden einzelnen dieser optimalen Transportpläne $(T_s^k{}_{\#}\mu_0)_{s\in[0,1]}$ von μ_0 nach μ_1^k kann man, wie unten erklärt, eine Ungleichung

$$\mathcal{L}(T_s^k(E)) \geq (1 - s)^5 \mathcal{L}(E), \tag{B.6}$$

als Konsequenz von $MCP(0, 5)$ und der Injektivität der Transportungleichung folgern.

Es wird dann mit Hilfe einiger Standart-Resultate über Massentransport gezeigt, dass diese untere Abschätzung für $\mathcal{L}(T_s^k(E))$ auch im Limes gilt. Genau wie vorher zeigt das dann die Absolutstetigkeit der Maße für $s < 1$. Wir verweisen den interessierten Leser für die detaillierte Rechtfertigung dies Grenzübergangs zu ([109, Chapitre 7 et Corollaire 5.21]).

Kommen wir dennoch noch einmal zu der Ungleichung (B.6) zurück, um den Zusammenhang zu der Bedingung MCP darzustellen. Am einfachsten geht das im Fall $k = 1$: Betrachten wir dafür den Transport von μ_0 zu $\mu_1^1 = \delta_{y_1}$. Offenbar gibt es nur einen möglichen Transportplan und dieser ist notwendigerweise optimal: Jedes Volumenelement wird zu y_1 transportiert und dieser Transport verläuft entlang von μ_0-fast sicher eindeutigen Geodäten. Man erhält also μ_1 fast sicher

$$T_s^1(q) = \mathcal{M}_{y_1}^{1-s}(q)$$

und die Volumenabschätzung folgt somit aus $MCP(0,5)$.

Im allgemeinen Fall $k > 1$ kann man die Ungleichung (B.6) zeigen, indem man sich an die Injektivität des Transports erinnert. Schreibe dafür zunächst E als disjunkte Vereinigung von Mengen $(E_j)_{j=1}^k$ auf denen die Abbildung T_s^k mit der Mittelpunktabbildung $\mathcal{M}_{y_j}^s$ übereinstimmt. Da MCP für jede der Mengen E_j eine Ungleichung vom Typ (B.6) liefert, genügt es, sich davon zu überzeugen, dass man diese Ungleichungen summieren kann. Da aber der Transport injektiv ist, sind die Mittelmengen $T_k^s(E_j) = \mathcal{M}_{y_j}^s(E_j)$ disjunkt und man erhält die gewünschte Abschätzung.

B.3 Krümmungsdimension in $\mathbb{H}_1$: Hoffnungen und Enttäuschungen

Wie wir bereits oben erklärt haben, generalisieren die Krümmungs-Bedingungen CD und MCP Krümmungsschranken im Sinne, dass die definierenden Kriterien im Fall Riemannscher Mannigfaltigkeiten äquivalent zu solchen Schranken sind. Allerdings ist dies nicht die einzige Rechtfertigung für die Aussage, dass dieses Konzept eine sinnvolle Verallgemeinerung von Krümmungsschranken für metrischen Maßräume ist. Lott, Sturm und Villani haben gezeigt, dass metrische Maßräume, die diese Bedingungen erfüllen, auch automatisch weitere Eigenschaften besitzen, die im Riemannschen Rahmen von Krümmungsschranken impliziert werden. Unter diesen Eigenschaften sind der Satz von Bishop-Gromov, der Satz von Bonnet-Myers aber auch insbesondere eine lokale Poincaré Ungleichung (zumindest, wenn die Zusatzforderung nach fast sicherer Eindeutigkeit der Geodäten erfüllt ist).

Ein weiterer wesentlicher Punkt ist das Stabilitätsverhalten dieser Eigenschaften unter der Konvergenz von metrischen Maßräume. Insbesondere gilt für die Abstandsfunktion, die von Sturm in [104] beschrieben wird –ein Abstandsbegriff, der die Idee des Gromov-Hausdorff Abstands mit der des Massentransports verbindet– dass eine konvergente Folge Riemannscher Mannigfaltigkeiten, die gleichmäßig $CD(K,N)$ erfüllt und außerdem beschränkte Durchmesser aufweist, gegen einen metrischen Maßraum (X,d,ν) konvergiert, ebenfalls $CD(K,N)$ erfüllt und derselben Schranke für den Durchmesser genügt. Für mehr Details zu dieser Theorie verweisen wir auf die ursprünglichen Arbeiten [77, 78, 104, 105] oder das sehr detaillierte Buch von Villani [109]. Wir werden nun die beiden Eigenschaften $MCP(0,5)$ und $CD(05)$ vergleichen. Zunächst ist hierbei festzuhalten, dass falls fast sicher eindeutige Geodäten zwischen zwei Punkten existieren, $CD(K,N)$ immer $MCP(K,N)$ impliziert. Zunächst scheint es vernünftig anzunehmen, dass auch die Umkehrung gültig ist. Die Definition B.2.3 von CD fordert die Konvexität der Entropie Ent_5 entlang der Geodäten von $\mathcal{P}_2(\mathbb{H}_1)$. Tatsächlich ist es so, das in $\mathbb{H}_1$ wichtige Beispiele für Transportpläne diese Eigenschaft erfüllen: Das erste Beispiel hierfür ist Transport durch Kontraktion. Um hier die Konvexität der Entropie zu sehen, verwendet man eine Verstärkte Maß-Kontraktionseigenschaft. Eine zweite Klasse von

Beispielen umfasst alle Transportpläne die Hochhebungen von optimalen Transportplänen in der Ebene $\mathbb{R}^2$ sind.

Wir sind in der Lage die Konvexität der Entropie für diejenigen Transportpläne zu zeigen, die in einem Dirac-Maß δ_y enden, die wir ja schon im Absatz B.2.2 kennengelernt haben. Das Argument ist nicht genau MCP sondern die $1/5$ Konkavität der Jacobi-Determinante der Kontraktion zu y, eine etwas stärke Eigenschaft, die uns erlaubt hat MCP zu beweisen. Widerholen wir die Notation von oben: $(\mu_s)_{s\in[0,1[}$ ist eine Geodäte mit einem Ende $\mu_1^1 = \delta_{y_1}$. Alle Maße sind absolutstetig mit Ausnahme von μ_1^1, was eine unendliche Entropie aufweist. Berechnen wir also die Entropie für $s < 1$:

$$
\begin{aligned}
\mathrm{Ent}_5(\rho_s \mid \mathcal{L}) &= -\int_{\mathcal{M}_s^y(\mathbb{H}_1)} \rho_s^{1-1/5}(y)\,dy \\
&= -\int_{\mathbb{H}_1} (\rho_s \circ T_s)^{1-1/5}(x)\,\mathrm{Jac}(T_s)(x)\,dx \\
&= -\int (\rho_s \circ T_s\,\mathrm{Jac}(T_s))^{1-1/5}\,\mathrm{Jac}(T_s)^{1/5} \\
&= -\int \rho_0^{1-1/5}(\mathrm{Jac}(T_s))^{1/5}.
\end{aligned}
$$

Bevor wir zum Resultat kommen, diskutieren wir kurz die Rechtfertigung dieser Rechnung. Die zweite Zeile folgt aus der der ersten durch eine Variablensubstitution, die gilt, weil T_s auf $\mathbb{H}_1 \setminus qL$ $\mathcal{C}^\infty$ ist. Die dritte Zeile folgt durch einige algebraische Umformungen aus der zweiten und die Schlussfolgerung folgt aus der Gleichung $\rho_0 = \rho_s(T_s(x))\,\mathrm{Jac}(T_s)$, die aus der Defintion des Bildmaßes folgt $\mu_s = T_{s\#}\mu_0$. Ausgehend von dieser Beziehung und von der Konkavität von $\mathrm{Jac}(T_s)^{1/5}$, die wir ja schon im Absatz B.1.4 gezeigt hatten, sieht man dann, dass die 5-dimensionale Renyi Entropie entlang dieser Kontraktions-Geodäten konvex ist.

Dieselbe Beweisführung bleibt auch im Falle diskreter Maße μ_1^k gültig und man könnte annehmen, dass sich die allgemeine Aussage wie eben durch Approximation mit solchen diskreten Maßen gewinnen ließe. Allerdings scheitert dieser Beweisversuch an dem Fehlen eines gültigen Arguments für den Grenzübergang. Wir werden insbesonder im Satz B.3.1 zeigen, dass in $\mathbb{H}_1$ keine Eigenschaft $CD(0, N)$ gilt.

Die zweite Klasse von optimalen Transportplänen, entlang derer die Entropie konvex ist, hängt nicht direkt mit MCP zusammen sondern mit den besonderen Eigenschaften von $\mathbb{H}_1$. Der Satz von Ambrosio und Rigot enthält noch eine weitere Aussage, die wir oben in B.2.1 noch nicht erwähnt hatten. Es handelt sich hierbei um eine Klassifikation derjenigen Abbildungen, die als optimale Transport Abbildungen in Frage kommen. Insbesonder kann man zeigen, dass die Funktionen von der Form $\psi(z;t) = \theta(z)$ für die $\theta(z) + \frac{|z|^2}{2}$ konvex auf $\mathbb{C}$ ist, solche Abbildungen sind. Im euklidischen Fall ist bekannt, dass diese Abbildungen θ schon alle optimalen Transportpläne umfassen [18]: Alle optimalen Transportabbildung aus einem absolutstetigen Maß sind von der Form $T(x) = x + \nabla\theta$. Da nun die Ricci Krümmung der $\mathbb{R}^d$ überall 0 ist, erfüllt der euklidische Raum auch $CD(0, N)$ für $N \geq d$ und somit ist die Entropie entlang eines optimalen Transports konvex.

Daher werden die optimalen Transportpläne $(\mu_s)_{s\in[0,1]}$ dieser Form in $\mathbb{H}_1$ von der Projektionsabbildung $Z : (z;t) \to z$ auf optimale Transportpläne

$(Z_\sharp \mu_s)_{s\in[0,1]}$ von $\mathbb{R}^2$ in $\mathbb{C}$ abgebildet, die gegeben sind durch die Abbildung θ. Man weiß sogar mehr, weil die Konvexität der Entropie Ent_N für $N \geq 2$ für Transportpläne der Form $T(x) = x + \nabla\theta$ zeigen lässt. Daher hat man für die optimalen Transportpläne fr $\psi(z;t) = \theta(z)$ sogar die Konvexität der zweidimensionalen Entropie – ein viel stärkeres Resultat als $MCP(0,5)$ vermuten lassen würde.

Trotzdem beinhalten diese beiden großen Klassen von Transportplänen bei Weitem nicht alle optimalen Transportpläne in $\mathbb{H}_1$. Im folgenden Satz stellen wir einen bestimmten Massentransport vor, für den die Entropie für keine Dimension N konvex ist.

Satz B.3.1. *Die Eigenschaft $CD(0,N)$ gilt in der Heisenberg-Gruppe für kein $N \in [1,+\infty]$.*

Halten wir hierfür ein $N \geq 1$ fest. Wir konstruieren nun einen Transport für den die Entropie Ent_N nicht konvex ist. Dieses Beispiel hängt nicht einmal von der Dimension ab.

Sei $r > 0$ hierfür ein (kleiner) Parameter. Betrachte B_r, den euklidischen Ball vom Radius (in $\mathbb{R}^3$) mit Mittelpunkt $(1,0,0)$ und $I_r := \mathcal{I}(B_r)$ seine geodätische Konjugierte. Diese beiden Mengen haben dasselbe Volumen $V_r = \frac{4}{3}\pi r^3$, denn die sphärischen Koordinaten, die diese beiden Mengen beschreiben stimmen bis auf Vorzeichen überein und es gilt die Gleichheit der Jacobi Determinanten $\mathrm{Jac}(\exp^{\mathbb{H}})(-v,-\varphi) = \mathrm{Jac}(\exp^{\mathbb{H}})(v,\varphi)$. Die Maße die wir nun betrachten, sind einfach die uniformen Verteilungen auf diesen beiden Mengen.

$$\mu_0 = \mathbf{1}_{B_r}/V_r \quad , \quad \mu_1 = \mathbf{1}_{I_r}/V_r.$$

Für alle N stimmt die Entropie Ent_N von μ_0 und μ_1 überein und nimmt den Wert $-(V_r)^{1/N}$ beziehungsweise $-\ln(V_r)$ für $N = +\infty$ an. Wir zeigen nun, dass $\mathrm{Ent}_N(\mu_{1/2}) > -(V_r)^{1/n}$ für hinreichend kleine r, was genügt um die Konvexität von Ent_N zu widerlegen. Die Konstruktion der Maße μ_0 und μ_1 ergibt sofort, dass $\mu_{1/2}$ in $M_r = \mathcal{M}^{1/2}(B_r, I_r)$, der Menge der Mittelpunkte zwischen den Punkten in B_r und den geodätisch konjugierten konzentriert ist.

$$M_r = \{\mathcal{M}(p, \mathcal{I}(q)) \mid (p,q) \in (B_r)^2\}.$$

Daher gilt

$$\begin{aligned}
\mathrm{Ent}_N(\mu_{1/2}) &= -\int_{M_r} \rho_{1/2}^{1-1/N} \\
&= \mathcal{L}(M_r) \int_{M_r} -\rho_{1/2}^{1-1/N}(x) \frac{dx}{\mathcal{L}(M_r)} \\
&\geq \mathcal{L}(M_r) \left(-\left(\int_{M_r} \rho_{1/2}(x) \frac{dx}{\mathcal{L}(M_r)} \right)^{1-1/N} \right) \\
&\geq -(\mathcal{L}(M_r))^{1/N}
\end{aligned}$$

Für $N = +\infty$ findet man außerdem $\mathrm{Ent}_N(\mu_{1/2}) \geq -\ln(\mathcal{L}(M_r))$. Diese Rechnung auf Grundlage der Jensen'schen Ungleichung zeigt, dass unter den Wahrscheinlichkeitsmaßen, die auf einer gegebenen Menge konzentriert sind, die uniformen Verteilungen diejenigen mit der schwächsten Entropie sind. Das stimmt auch mit unserem Bild der Ameisen überein: Je gleichmäßiger sich die Ameisen verteilen und je weiter sie sich ausbreiten, umso kleiner wird die Entropie. Um den Beweis zu beenden genügt es nun zu zeigen, dass $\mathcal{L}(M_r) < V_r$.

Analysieren wir hierfür noch einmal genauer die Menge M_r. Das ist die Überlagerung (mathematisch gesprochen die Vereinigung) von Mittelmengen $\mathcal{M}_{1/2}(\mathcal{I}(p), B_r)$, wenn p den Ball B_r durchläuft Diese Vereinigung ist bei weitem nicht disjunkt und daher ist die Menge M_r auch relativ klein.

Genauer gesagt ist $0_{\mathbb{H}}$ in jeder Menge $\mathcal{M}^{1/2}(\mathcal{I}(p), B_r)$ enthalten, wenn der Zeitpunkt in der Mitte von p und $\mathcal{I}(p)$ liegt. Mit ein wenig Analysis (siehe [64]) erhält man genauer gesagt:

$$\mathcal{M}^{1/2}(\mathcal{I}(p), B_r) \subset D\mathcal{M}^{1/2}_{\mathcal{I}(p)}(p).(B_r - p) + B(0_{\mathbb{H}}, o(r)). \tag{B.7}$$

Hier sind die Operationen $+$ und $-$ im Sinne von $\mathbb{R}^3$ zu verstehen und $B(0, o(r))$ ist eine euklidische Kugel mit Mittelpunkt $0_{\mathbb{H}}$ und einem Radius, der klein gegen r ist. Diese Inklusion bleibt wahr, wenn man gleichmäßig $D\mathcal{M}^{1/2}_{\mathcal{I}(p)}(p)$ durch $D\mathcal{M}^{1/2}_{(-1,0,0)}(1, 0, 0)$ ersetzt, was wiederum nahe bei $p = (1, 0, 0)$ liegt, dem Mittelpunkt der Kugel B_r. In diesem Fall ist bleibt das $o(r)$ in (B.7) wahr, sobald man den Rest durch eine größere Funktion ersetzt, die trotzdem uniform in p klein gegen den Radius bleibt. Jedoch kann man die Inklusionen vereinigen und bekommt

$$M_r \subset D\mathcal{M}^{1/2}_{(-1,0,0)}(1, 0, 0).(B_r - B_r) + B(0, o(r)).$$

Die Menge $B_r - B_r$ ist einfach die euklidische Kugel mit Radius $2r$. Folglich ihr sein Volumen $8V_r$. Man erhält ein Ellipsoid mit Volumen $V_r/4$ indem man das Bild von eine affine Abbildung mit Determinante $1/2^5$ nimmt. Die M_r enthaltende Menge, die wir betrachten ist der $o(r)$-Schlauch um so ein Ellipsoid mit Volumen äquivalent zu $V_r/4$. Für r klein genug folgt daraus $\mathcal{L}(M_r) < V_r$, was für den Beweis genügt, wie schon aufgewiesen.

B.4 Gradientenfluss in der Heisenberg-Gruppe

Die Geodäten sind nicht die einzigen interessanten Kurven des Wasserstein-Raumes. Als erster haben Jordan, Kinderlehrer und Otto [63], die Idee gehabt, den Wärmeleitungsfluss als eine Kurve des euklidischen Wasserstein-Raumes, $\mathcal{P}_2(\mathbb{R}^n)$ zu betrachten. Sie haben festgestellt, dass es sich formal bei $\mathcal{P}_2(\mathbb{R}^n)$ um eine unendlichdimensionale Mannigfaltigkeit handelt und, dass die Evolution der Wärme in diesem Maßraum eine Integralkurve des formalen Gradienten von $-\operatorname{Ent}_\infty$ entspricht. Trotz des Vorzeichnens bezeichnet man diese Entwicklung als Gradientenfluss der Entropie. Das Entropiefunktionnal ist nämlich eine reale Funktion vo $\mathcal{P}_2(\mathbb{R}^n)$ und es scheint, dass die Wärme auf die Weise verteilt, dass sie zu jedem Zeitpunkt die Entropie am besten minimiert.

Seitdem wurden Versuch unternommen, diese Beobachtung rigoros zu machen, und sie auf weitere Funktionale bzw. weitere Klassen von Funktionalen zu erweitern. Hierbei hat es sich als einfacher erwiesen, Funktionale zu betrachten, die entlang der Massentransporten Konvexitätseigenschaften aufweisen (siehe etwa [4]). Wie schon erwähnt, ist diese Konvexität für die Entropien auf eine nach unten beschränkte Ricci Krümmung zurückzuführen. Während es möglich ist, den Gradientenfluss von Ent_∞ zum Beispiel für Mannigfaltigkeiten mit einer unteren Schranke [35, 96], Alexandrov Räume [88] und Hilber Räume [4] zu definieren, scheint es schwieriger zu sein, dieselbe Arbeit für die Heisenberg-Gruppe zu beabsichtigen. Trotzdem kann man interessante Resultate finden indem die $\mathbb{H}_1$ durch Manigfaltigkeiten $\mathbb{H}_1^\varepsilon$ approximiert wird, wobei $\varepsilon > 0$ gegen Null geht. Diese Manigfaltigkeiten haben konstante unteren Ricci Krümmung Schranke $\frac{-1}{2\varepsilon^2}$, die gegen $-\infty$ divergiert.

Der natürliche Operator zu $\mathbb{H}_1$ wird durch

$$\Delta_{\mathbb{H}} = \mathbf{X}^2 + \mathbf{Y}^2$$

definiert. Es handelt sich um einem hypoelliptischen Operator, weil $\mathbf{X}$ und $\mathbf{Y}$ die sogenannte Hörmander-Bedingung erfüllen : $\mathbf{X}$, $\mathbf{Y}$und ihrer Lie Klammer $\mathbf{T}$ erzeugen die ganze Lie Algebra. Diese subelliptische Diffusion ist gut verstanden. Die Wärmeleitungshalbgruppe erhält man durch die Faltung mit einer generalisierten Gaußschen Verteilung, deren explizite Form und einige damit verbundene Abschätzungen seit dem Artikel von Gaveau [49] bekannt sind.

Die Definition, die wir für den Gradientenfluss angenommen haben ist besonders allgemein. Diese Definition fordert, dass der Fluss $(\mu_s)_{s\in I}$ in fast jeden Zeitpunkt eine metrische Geschwindigkeit $|\dot{\mu}_s|$ (im Sinne von [9]) haben muss und, dass diese Geschwindigkeit mit der Steigung der Entropie übereinstimmen muss, welche wie folgt definiert ist:

$$\mathrm{Slope}(\mathrm{Ent}_\infty)(\mu) = \max\left(0, \limsup_{\nu\to\mu} \frac{\mathrm{Ent}_\infty(\mu) - \mathrm{Ent}_\infty(\nu)}{W(\mu,\nu)}\right).$$

Außerdem muss die Ableitung von $\mathrm{Ent}_\infty(\mu_s)$ –die absolutstetig sein muss– gleich $-\mathrm{Slope}(\mathrm{Ent}_\infty)(\mu_s)\cdot|\dot{\mu}_s|$ sein.

Mit dieser Definition und durch die Approximation von $\mathbb{H}_1$ durch $\mathbb{H}_1^\varepsilon$ erhält man den Zusammmenhang zwischen Gradientenfluss und subelliptische Diffusion in beide Richtungen. Dennoch benötigt der Beweis dieses Satzes zwei zusätzlichen Voraussetzungen: einerseits die Kompaktheit des Trägers zum Zeitpunkt 0, anderseits die Existenz eine Ableitung in der dritten Koordinate.

Satz B.4.1. *Sei $(\rho_s)_{s\in]0,+\infty[}$ die Lösung der subelliptischen Wärmeleitungsgleichung:*

$$\begin{cases} \Delta_{\mathbb{H}}\rho_s = \partial_s\rho_s \\ \rho_0 d\mathcal{L} = \mu_0 \end{cases} \tag{B.8}$$

in $\mathbb{H}_1$, wobei μ_0 einen kompakten Träger habe. Dann ist die Kurve $(\mu_s)_{s\geq 0}$ der Maße mit Dichte $(\rho_s)_{s\geq 0}$ ein Gradientenfluss der Entropie Ent_∞.

Umgekehrt sei $(\mu_s)_{s\in I}$ ein Gradientenfluss der Entropie in $\mathcal{P}_2(\mathbb{H}_1)$. Angenommen es gibt für jede $s\in I$ eine schwache Ableitung $\mathbf{T}\rho_s$ mit

$$\int \frac{(\mathbf{T}\rho_s)^2}{\rho_s} < +\infty,$$

dann ist ρ_s Lösung der elliptischen Wärmeleitungsgleichung.

Bibliography

[1] A. Agrachev, U. Boscain, and M. Sigalotti. A Gauss-Bonnet-like formula on two-dimensional almost-Riemannian manifolds. *Discrete Contin. Dyn. Syst.*, 20(4):801–822, 2008.

[2] A. Agrachev and P. Lee. Optimal transportation under non-holomic constraints. *preprint*, 2007.

[3] A. D. Aleksandrov. *Vnutrennyaya Geometriya Vypuklyh Poverhnosteĭ.* OGIZ, Moscow-Leningrad,], 1948.

[4] L. Ambrosio, N. Gigli, and G. Savaré. *Gradient flows in metric spaces and in the space of probability measures.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, second edition, 2008.

[5] L. Ambrosio and B. Kirchheim. Rectifiable sets in metric and Banach spaces. *Math. Ann.*, 318(3):527–555, 2000.

[6] L. Ambrosio, B. Kleiner, and E. Le Donne. Rectifiability of sets of finite perimeter in Carnot groups: existence of a tangent hyperplane. *preprint*, 2008.

[7] L. Ambrosio and S. Rigot. Optimal mass transportation in the Heisenberg group. *J. Funct. Anal.*, 208(2):261–301, 2004.

[8] L. Ambrosio and G. Savaré. Gradient Flows of Probability Measures. In *Handbook of differential equations, evolutionary equations*, volume 3, pages 1–136. 2006.

[9] L. Ambrosio and P. Tilli. *Topics on analysis in metric spaces*, volume 25 of *Oxford Lecture Series in Mathematics and its Applications.* Oxford University Press, Oxford, 2004.

[10] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, and G. Scheffer. *Sur les inégalités de Sobolev logarithmiques*, volume 10 of *Panoramas et Synthèses [Panoramas and Syntheses].* Société Mathématique de France, Paris, 2000. With a preface by Dominique Bakry and Michel Ledoux.

[11] D. Bakry and M. Émery. Diffusions hypercontractives. In *Séminaire de probabilités, XIX, 1983/84*, volume 1123 of *Lecture Notes in Math.*, pages 177–206. Springer, Berlin, 1985.

[12] Z. Balogh, J. Tyson, and B. Warhurst. Sub-Riemannian vs. euclidean dimension comparison and fractal geometry on carnot groups. *preprint*, 2007.

[13] R. Beals, B. Gaveau, and P.C. Greiner. Hamilton-Jacobi theory and the heat kernel on Heisenberg groups. *J. Math. Pures Appl. (9)*, 79(7):633–689, 2000.

[14] J. Bertrand. Existence and uniqueness of optimal maps on Alexandrov spaces. *preprint*, 2007.

[15] U. Boscain and B. Piccoli. A short Introduction to Optimal Control. In *Controle non lineaire et applications*, Travaux en cours, pages 19–66. Hermann, 2005.

[16] M. Bourdon and H. Pajot. Poincaré inequalities and quasiconformal structure on the boundary of some hyperbolic buildings. *Proc. Amer. Math. Soc.*, 127(8):2315–2324, 1999.

[17] M. Bourdon and H. Pajot. Quasi-conformal geometry and hyperbolic geometry. In *Rigidity in dynamics and geometry (Cambridge, 2000)*, pages 1–17. Springer, Berlin, 2002.

[18] Y. Brenier. Décomposition polaire et réarrangement monotone des champs de vecteurs. *C. R. Acad. Sci. Paris Sér. I Math.*, 305(19):805–808, 1987.

[19] Y. Brenier. Polar factorization and monotone rearrangement of vector-valued functions. *Comm. Pure Appl. Math.*, 44(4):375–417, 1991.

[20] H. Brezis. *Analyse fonctionnelle*. Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

[21] D. Burago, Y. Burago, and S. Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[22] Yu. Burago, M. Gromov, and G. Perel′man. A. D. Aleksandrov spaces with curvatures bounded below. *Uspekhi Mat. Nauk*, 47(2(284)):3–51, 222, 1992.

[23] L. Capogna, D. Danielli, S. D. Pauls, and J. T. Tyson. *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*, volume 259 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2007.

[24] J. Cheeger. Differentiability of Lipschitz functions on metric measure spaces. *Geom. Funct. Anal.*, 9(3):428–517, 1999.

[25] J. Cheeger and T. H. Colding. On the structure of spaces with Ricci curvature bounded below. I. *J. Differential Geom.*, 46(3):406–480, 1997.

[26] J. Cheeger and T. H. Colding. On the structure of spaces with Ricci curvature bounded below. II. *J. Differential Geom.*, 54(1):13–35, 2000.

[27] J. Cheeger and T. H. Colding. On the structure of spaces with Ricci curvature bounded below. III. *J. Differential Geom.*, 54(1):37–74, 2000.

[28] J. Cheeger and B. Kleiner. Generalized differential and bi-Lipschitz nonembedding in L^1. *C. R. Math. Acad. Sci. Paris*, 343(5):297–301, 2006.

[29] D. Cordero-Erausquin, R. J. McCann, and M. Schmuckenschläger. A Riemannian interpolation inequality à la Borell, Brascamp and Lieb. *Invent. Math.*, 146(2):219–257, 2001.

[30] D. Cordero-Erausquin, B. Nazaret, and C. Villani. A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities. *Adv. Math.*, 182(2):307–332, 2004.

[31] P. Crépel and A. Raugi. Théorème central limite sur les groupes nilpotents. *Ann. Inst. H. Poincaré Sect. B (N.S.)*, 14(2):145–164, 1978.

[32] Bakry D., Baudoin F., Bonnefont M., and Chafai D. On gradient bounds for the heat kernel on the Heisenberg group. *preprint*, 2008.

[33] E. De Giorgi. Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio ad r dimensioni. *Ann. Mat. Pura Appl. (4)*, 36:191–213, 1954.

[34] M. P. do Carmo. *Riemannian geometry*. Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.

[35] M. Erbar. Die Wärmeleitungsgleichung auf Mannigfaltigkeiten als Gradientenfluss im Wassersteinraum. Master's thesis, Bonn Universität, 2008.

[36] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[37] R. R. Faĭzullin. On a connection between the nonholonomic metric on the Heisenberg group and the Grushin metric. *Sibirsk. Mat. Zh.*, 44(6):1377–1384, 2003.

[38] S. Fang and J. Shao. Optimal transport maps for Monge-Kantorovich problem on loop groups. *J. Funct. Anal.*, 248(1):225–257, 2007.

[39] H. Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[40] F. Ferrari, B. Franchi, and H. Pajot. The geometric traveling salesman problem in the Heisenberg group. *Rev. Mat. Iberoam.*, 23(2):437–480, 2007.

[41] D. Feyel and A. S. Üstünel. Solution of the Monge-Ampère equation on Wiener space for general log-concave measures. *J. Funct. Anal.*, 232(1):29–55, 2006.

[42] A. Figalli and N. Juillet. Absolute continuity of Wasserstein geodesics in the Heisenberg group. *J. Funct. Anal.*, 255(1):133–141, 2008.

[43] A. Figalli and L. Rifford. Mass Transportation on sub-Riemannian Manifolds. *preprint*, 2008.

[44] A. Figalli and C. Villani. Strong displacement convexity on Riemannian manifolds. *Math. Z.*, 257(2):251–259, 2007.

[45] G. B. Folland and E. M. Stein. *Hardy spaces on homogeneous groups*, volume 28 of *Mathematical Notes*. Princeton University Press, Princeton, N.J., 1982.

[46] B. Franchi, R. Serapioni, and F. Serra Cassano. Rectifiability and perimeter in the Heisenberg group. *Math. Ann.*, 321(3):479–531, 2001.

[47] B. Franchi, R. Serapioni, and F. Serra Cassano. On the structure of finite perimeter sets in step 2 Carnot groups. *J. Geom. Anal.*, 13(3):421–466, 2003.

[48] S. Gallot, D. Hulin, and J. Lafontaine. *Riemannian geometry*. Universitext. Springer-Verlag, Berlin, third edition, 2004.

[49] B. Gaveau. Principe de moindre action, propagation de la chaleur et estimées sous elliptiques sur certains groupes nilpotents. *Acta Math.*, 139(1-2):95–153, 1977.

[50] M. Gromov. *Structures métriques pour les variétés riemanniennes*, volume 1 of *Textes Mathématiques [Mathematical Texts]*. CEDIC, Paris, 1981. Edited by J. Lafontaine and P. Pansu.

[51] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

[52] M. Gromov. Carnot-Carathéodory spaces seen from within. In *Sub-Riemannian geometry*, volume 144 of *Progr. Math.*, pages 79–323. Birkhäuser, Basel, 1996.

[53] N. Guillotin-Plantard and R. Schott. Dynamic random walks on Heisenberg groups. *J. Theoret. Probab.*, 19(2):377–395, 2006.

[54] I. Hahlomaa. Menger curvature and Lipschitz parametrizations in metric spaces. *Fund. Math.*, 185(2):143–169, 2005.

[55] P. Hajłasz and P. Koskela. Sobolev met Poincaré. *Mem. Amer. Math. Soc.*, 145(688):x+101, 2000.

[56] J. Heinonen. Calculus on Carnot groups. In *Fall School in Analysis (Jyväskylä, 1994)*, volume 68 of *Report*, pages 1–31. Univ. Jyväskylä, Jyväskylä, 1995.

[57] J. Heinonen. *Lectures on analysis on metric spaces*. Universitext. Springer-Verlag, New York, 2001.

[58] L. Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119:147–171, 1967.

[59] H. Hueber and D. Müller. Asymptotics for some Green kernels on the Heisenberg group and the Martin boundary. *Math. Ann.*, 283(1):97–119, 1989.

[60] D. Jerison. The Poincaré inequality for vector fields satisfying Hörmander's condition. *Duke Math. J.*, 53(2):503–523, 1986.

[61] P. W. Jones. Square functions, Cauchy integrals, analytic capacity, and harmonic measure. In *Harmonic analysis and partial differential equations (El Escorial, 1987)*, volume 1384 of *Lecture Notes in Math.*, pages 24–68. Springer, Berlin, 1989.

[62] P. W. Jones. Rectifiable sets and the traveling salesman problem. *Invent. Math.*, 102(1):1–15, 1990.

[63] R. Jordan, D. Kinderlehrer, and F. Otto. The variational formulation of the Fokker-Planck equation. *SIAM J. Math. Anal.*, 29(1):1–17 (electronic), 1998.

[64] N. Juillet. Ricci curvature bounds and geometric inequalities in the Heisenberg group. *preprint SFB611 Bonn*, 2006.

[65] L. V. Kantorovich. On mass transportation. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 312(Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 11):11–14, 2004.

[66] B. Khesin and Lee P. A nonholonomic moser theorem and otimal mass transport. *preprint*, 2008.

[67] B. Kirchheim. Rectifiable metric spaces: local structure and regularity of the Hausdorff measure. *Proc. Amer. Math. Soc.*, 121(1):113–123, 1994.

[68] J. J. Kohn. Pseudo-differential operators and hypoellipticity. In *Partial differential equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971)*, pages 61–69. Amer. Math. Soc., Providence, R.I., 1973.

[69] A. Korányi. Geometric properties of Heisenberg-type groups. *Adv. in Math.*, 56(1):28–38, 1985.

[70] A. Korányi and H. M. Reimann. Quasiconformal mappings on the Heisenberg group. *Invent. Math.*, 80(2):309–338, 1985.

[71] K. Kuwae and T. Shioya. On generalized measure contraction property and energy functionals over Lipschitz maps. *Potential Anal.*, 15(1-2):105–121, 2001. International Conference on Potential Analysis 1998 (Hammamet).

[72] T. J. Laakso. Ahlfors Q-regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality. *Geom. Funct. Anal.*, 10(1):111–123, 2000.

[73] T. J. Laakso. Erratum to: "Ahlfors Q-regular spaces with arbitrary $Q > 1$ admitting weak Poincaré inequality" [Geom. Funct. Anal. **10** (2000), no. 1, 111–123; MR1748917 (2001m:30027)]. *Geom. Funct. Anal.*, 12(3):650, 2002.

[74] G. P. Leonardi and S. Masnou. On the isoperimetric problem in the Heisenberg group $\mathbb{H}^n$. *Ann. Mat. Pura Appl. (4)*, 184(4):533–553, 2005.

[75] H.-Q. Li. Estimation optimale du gradient du semi-groupe de la chaleur sur le groupe de Heisenberg. *J. Funct. Anal.*, 236(2):369–394, 2006.

[76] H.-Q. Li. Estimations asymptotiques du noyau de la chaleur sur les groupes de Heisenberg. *C. R. Math. Acad. Sci. Paris*, 344(8):497–502, 2007.

[77] J. Lott and C. Villani. Ricci curvature for metric-measure spaces via optimal transport. *to appear in Ann of Math.*, 2006.

[78] J. Lott and C. Villani. Weak curvature conditions and functional inequalities. *J. Funct. Anal.*, 245(1):311–333, 2007.

[79] P. Malliavin. Stochastic calculus of variation and hypoelliptic operators. In *Proceedings of the International Symposium on Stochastic Differential Equations (Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1976)*, pages 195–263, New York, 1978. Wiley.

[80] R. J. McCann. Polar factorization of maps on Riemannian manifolds. *Geom. Funct. Anal.*, 11(3):589–608, 2001.

[81] G. Monge. Mémoire sur la théorie des déblais et des remblais. *Histoire de l'académie Royale des Sciences de Paris*, 1781.

[82] R. Montgomery. Abnormal minimizers. *SIAM J. Control Optim.*, 32(6):1605–1620, 1994.

[83] R. Montgomery. A survey of singular curves in sub-Riemannian geometry. *J. Dynam. Control Systems*, 1(1):49–90, 1995.

[84] R. Montgomery. *A Tour of Subriemannian Geometries, Their Geodesics and Applications*, volume 91 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[85] R. Monti. Some properties of Carnot-Carathéodory balls in the Heisenberg group. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 11(3):155–167 (2001), 2000.

[86] R. Monti. Brunn-Minkowski and isoperimetric inequality in the Heisenberg group. *Ann. Acad. Sci. Fenn. Math.*, 28(1):99–109, 2003.

[87] D. Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006.

[88] S.-I. Ohta. Gradient flows on Wasserstein spaces over compact Alexandrov spaces. *to appear in Am. J. Math.*, 2006.

[89] S.-I. Ohta. On the measure contraction property of metric measure spaces. *Comment. Math. Helv.*, 82(4):805–828, 2007.

[90] S.-I. Ohta. Finsler interpolation inequalities. *preprint*, 2008.

[91] K. Okikiolu. Characterization of subsets of rectifiable curves in $\mathbf{R}^n$. *J. London Math. Soc. (2)*, 46(2):336–348, 1992.

[92] F. Otto. The geometry of dissipative evolution equations: the porous medium equation. *Comm. Partial Differential Equations*, 26(1-2):101–174, 2001.

[93] P. Pansu. Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un. *Ann. of Math. (2)*, 129(1):1–60, 1989.

[94] S. Rigot. Mass transportation in groups of type H. *Commun. Contemp. Math.*, 7(4):509–537, 2005.

[95] L. P. Rothschild and E. M. Stein. Hypoelliptic differential operators and nilpotent groups. *Acta Math.*, 137(3-4):247–320, 1976.

[96] L. Saloff-Coste. *Aspects of Sobolev-type inequalities*, volume 289 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2002.

[97] G. Savaré. Gradient flows and diffusion semigroups in metric spaces under lower curvature bounds. *C. R. Math. Acad. Sci. Paris*, 345(3):151–154, 2007.

[98] R. Schul. Analyst's traveling salesman theorems. A survey. In *In the tradition of Ahlfors-Bers. IV*, volume 432 of *Contemp. Math.*, pages 209–220. Amer. Math. Soc., Providence, RI, 2007.

[99] R. Schul. Subsets of rectifiable curves in Hilbert space—the analyst's TSP. *J. Anal. Math.*, 103:331–375, 2007.

[100] B. Schulte. Optimaler Massentransport auf Alexandrov-Räumen. Master's thesis, Bonn Universität, 2008.

[101] S. Semmes. On the nonexistence of bi-Lipschitz parameterizations and geometric problems about A_∞-weights. *Rev. Mat. Iberoamericana*, 12(2):337–410, 1996.

[102] N. Shanmugalingam. Newtonian spaces: an extension of Sobolev spaces to metric measure spaces. *Rev. Mat. Iberoamericana*, 16(2):243–279, 2000.

[103] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[104] K.-T. Sturm. On the geometry of metric measure spaces. I. *Acta Math.*, 196(1):65–131, 2006.

[105] K.-T. Sturm. On the geometry of metric measure spaces. II. *Acta Math.*, 196(1):133–177, 2006.

[106] N. T. Varopoulos. Fonctions harmoniques sur les groupes de Lie. *C. R. Acad. Sci. Paris Sér. I Math.*, 304(17):519–521, 1987.

[107] Vergil. *Aeneis*. Reclam, 2003.

[108] C. Villani. *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2003.

[109] C. Villani. *Optimal transport, old and new*. Springer (to appear in Grundlehren der mathematischen Wissenschaften), 2008.

[110] M.-K. von Renesse. On Local Poincaré via Transportation. *to appear in Math. Z.*, 2006.

[111] Max-K. von Renesse and Karl-Theodor Sturm. Transport inequalities, gradient estimates, entropy, and Ricci curvature. *Comm. Pure Appl. Math.*, 58(7):923–940, 2005.

[112] M. Yor. Remarques sur une formule de Paul Lévy. In *Seminar on Probability, XIV (Paris, 1978/1979) (French)*, volume 784 of *Lecture Notes in Math.*, pages 343–346. Springer, Berlin, 1980.

Curiculum Vitae

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Abstracts

In dieser Doktorarbeit betrachten wir die Heisenberg-Gruppe $\mathbb{H}_n = \mathbb{R}^{2n+1}$ mit dem Carnot-Caratéodory Abstand d_c und dem Lebesgue Maß $\mathcal{L}^{2n+1}$. In Kapitel 1, im Zusammenhang mit dem geometrischen Problem des Handlungsreisenden, konstruieren wir eine Kurve ω endlicher Länge, die jedoch nicht das Kriterium von Ferrari, Franchi und Pajot erfüllt, das sicher stellt, dass eine Menge im Bild einer rektifizierbaren Kurve enthalten ist. Wir erhalten des Weiteren eine feine Abbschätzung der Jacobi-Determinante für diejenigen Abbildungen, die eine Menge entlang von Geodäten zu einem Punkt kontrahieren. Dies ist im Wesentlichen äquivalent zu der Massenkontraktionseigenschaft $MCP(0, 2n+3)$. Mit Hilfe dieser Abschätzung beantworten wir eine Frage von Ambrosio und Rigot zum Optimaltransport in $\mathbb{H}_n$ positiv. Wir beweisen nämlich im Kapitel 2, dass die Maße auf einer Geodäte des Wasserstein-Raumes absolutstetig ist, sobald ein Ende dieser Kurve absolutstetig ist. In Kapitel 3 zeigen wir, dass die Krümmungs-Dimension-Bedingung $CD(K, N)$ durch kein $K \in \mathbb{R}$ und kein $N \in [1, +\infty]$ erfüllt wird. Wir betrachten außerdem für $\mathbb{H}_n$ andere metrische Definitionen der Krümmung. Abschließend weisen wir in Kapitel 4 auf den Zusammenhang zwischen dem Wasserstein-Gradientenfluss der Bolzmann-Entropie und subelliptischer Diffusion hin.

Schlüsselwörter: Heisenberg-Gruppe, Optimaltransport, Ricci Krümmung, Problem des Handlungdreisenden, Gradientenfluss
Klassifikation: 28A33, 28A75, 28A80, 32Q10, 53C17, 60J60

On considère le groupe de Heisenberg $\mathbb{H}_n = \mathbb{R}^{2n+1}$ avec la distance de Carnot-Carathéodory d_c et la mesure de Lebegue $\mathcal{L}^{2n+1}$. Dans le premier chapitre, dans le cadre du problème du voyageur de commerce géométrique de $\mathbb{H}_1$, on construit une courbe de longueur finie qui pourtant ne vérifie pas le critère de Ferrari, Franchi et Pajot établissant qu'un ensemble est contenu dans une courbe rectifiable. On montre aussi une inégalité sur le déterminant jacobien des applications de contraction sur un point qui suivent les géodésiques. Cette inégalité est essentiellement équivalente à la Propriété de Contraction de Mesure $MCP(0, 2n + 3)$. Grâce à cette propriété on répond positivement au Chapitre 2 à une question d'Ambrosio et Rigot à propos du transport de mesure dans $\mathbb{H}_n$ (travail en commun avec Figalli). Il s'avère en effet que les mesures traversées par une géodésique de l'espace de Wasserstein sont absolument continues dès qu'une extrémité de la géodésique l'est. Au Chapitre 3 on démontre que la Courbure-Dimension $CD(K, N)$ définie par transport de mesure n'est pas vérifiée pour $\mathbb{H}_n$ et que cela vaut quels que soient les paramètres $K \in \mathbb{R}$ et $N \in [1, +\infty]$. On discute aussi d'autres propriétés de courbures dans le cas du groupe de Heisenberg. Le Chapitre 4 est dédié à la correspondance entre l'équation de la chaleur sous-elliptique et le flot de gradient de l'entropie de Bolzmann dans l'espace de Wasserstein.

Mot-Clefs: Groupe de Heisenberg, transport optimal, courbure de Ricci, problème du voyageur de commerce, flot de gradient
Classification: 28A33, 28A75, 28A80, 32Q10, 53C17, 60J60

Abstract

In this thesis we consider the Heisenberg group $\mathbb{H}_n = \mathbb{R}^{2n+1}$ with its Carnot-Carathéodory distance d_c and the Lebesgue measure $\mathcal{L}^{2n+1}$. In Chapter 1, in relation with the geometric traveling salesman problem in $\mathbb{H}_1$, we construct a curve of finite length that does not satisfy the criterion of Ferrari, Franchi and Pajot about sets contained in the range of a rectifiable curve. We also prove a sharp Jacobian estimate of that maps that contract sets to a point going along geodesics. This is essentially equivalent to the Measure Contraction Property $MCP(0, 2n+3)$. With this estimate we answer positively a question by Ambrosio and Rigot about optimal transport in $\mathbb{H}_n$ (common work with Figalli). Indeed, in Chapter 2 we prove the absolute continuity of the measure of $\mathbb{H}_n$ on a Wasserstein geodesic starting from an absolutely continuous measure. In Chapter 3, we prove that the Curvature-Dimension $CD(K, N)$ condition defined by optimal transport does not hold for any $K \in \mathbb{R}$ and $N \in [1, +\infty]$. We also discuss other metric curvature properties in the case of $\mathbb{H}_n$. Finally Chapter 4 is devoted to the concordance of the subelliptic "heat" equation and the Wasserstein gradient flow of the Bolzmann entropy.

Keywords

Heisenberg group, optimal transport, Ricci curvature, traveling salesman problem, gradient flow

Classification

28A33, 28A75, 28A80, 32Q10, 53C17, 60J60