Orientation reversal of manifolds

Dissertation

zur Erlangung des Doktorgrades (Dr. rer. nat.)
der Mathematisch-Naturwissenschaftlichen Fakultät
der Rheinischen Friedrich-Wilhelms-Universität Bonn

vorgelegt von
Daniel Müllner
aus München

Bonn, Oktober 2008

Angefertigt mit Genehmigung der Mathematisch-Naturwissenschaftlichen Fakultät der Rheinischen Friedrich-Wilhelms-Universität Bonn

1. Referent: Prof. Dr. Matthias Kreck (Bonn)
2. Referent: Prof. Dr. Carl-Friedrich Bödigheimer (Bonn)

Tag der Promotion: 5. 1. 2009

Diese Dissertation ist auf dem Hochschulschriftenserver der ULB Bonn `http://hss.ulb.uni-bonn.de/diss_online` elektronisch publiziert.

Erscheinungsjahr: 2009

Preface

There was a topologist, who
said, "I cannot reverse $\mathbb{C}P^2$!"
By a most intense stare
at the cup product square,
he found this is perfectly true.

Like many other topologists, I learned in my undergraduate studies about the complex projective spaces, and that $\mathbb{C}P^2$ is not oriented diffeomorphic to $-\mathbb{C}P^2$. My surprise over this fact abated over the time, but some of the initial air of mystery always stuck to the "chiral" manifolds. The idea for the topic of this thesis later came from the paper [Freedman et al.], where the authors consider formal linear combinations $\Sigma_i a_i M_i$ of (diffeomorphism classes of) manifolds M_i. Thinking about $\mathbb{C}P^2$, I wondered whether some manifolds appear twice in the index set and others only once. This led me to the question in which dimensions all manifolds have an orientation-reversing diffeomorphism.

In chemistry, a molecule is called chiral if it cannot be superimposed on its mirror image [Römpp]. Another definition which captures the properties of flexible and topologically complex molecules better is given by [Flapan]: A molecule "that can chemically change itself into its mirror image" is called achiral and chiral if it cannot. Chiral molecules have the same physical properties like melting and boiling points but they behave optically and chemically differently.

With the analogy to chiral molecules in mind, it seems a very natural question to ask whether an orientable manifold with its two orientations yields "the same" or "different" objects. Indeed, this analogy (not really a strong connection, though) to molecular chirality generated some of my motivation for this thesis.

Studying the chirality of manifolds, I was in the pleasant situation that my questions appealed to others as well, and referring to chiral molecules, I could also give non-mathematicians, in particular biologists, a taste of what I was doing. I hope that also the readers will discover their own liking for the topic in this thesis.

Acknowledgements

I would like to thank everybody who contributed to the completion of this thesis. First and foremost, I thank my advisor Prof. Matthias Kreck for his encouragement, his continuous support and the most helpful discussions with him. I also like to thank my fellow doctoral students and postdocs of the

workgroup in Heidelberg, Diarmuid Crowley, Thorsten Eppelmann, Martin Olbermann, Sadeeb Ottenburger, Jörg Sixt and Su Yang. They were both highly competent colleagues and friendly companions to me. I am particularly grateful to Diarmuid from whom I learned very much and who was always willing to help me and discuss any mathematical question. I thank Thorsten for the good teamwork in the joint lecture assistances.

After my move to Bonn, the members of the topology workgroup and the "Graduiertenkolleg" welcomed me warmly and provided an excellent environment for my ongoing work. I like to thank especially Prof. Carl-Friedrich Bödigheimer for his efforts to integrate me quickly in the Graduiertenkolleg and for acting as second referee of my thesis.

This thesis would be a much more meagre work without the discussions and help of the aforementioned people and others. I mention in particular Prof. Tom Goodwillie, Prof. Sigrid Böge and Martin Langer, whose hints and comments helped this thesis to progress at certain points. I am grateful to Sadeeb Ottenburger and Elisa Dultz for critical reading. Finally, I thank Elisa for her love and support in every way.

Contents

1

Introduction

In this work, the phenomenon of orientation reversal of manifolds is studied. We call an orientable manifold *amphicheiral* if it admits an orientation-reversing self-map and *chiral* if it does not. Below, this definition is extended by attributes, e. g. "topologically chiral" or "smoothly amphicheiral" that express various degrees of restriction on the orientation-reversing map. Many familiar manifolds like spheres or orientable surfaces are amphicheiral: in these cases mirror-symmetric embeddings into $\mathbb{R}^n$ exist, and reflection at the "equatorial" hyperplane reverses orientation. On the other hand, examples of chiral manifolds have been known for many decades, e. g. the complex projective spaces $\mathbb{C}P^{2k}$ or some lens spaces in dimensions congruent 3 mod 4. However, this phenomenon has not been studied systematically.

In the next chapter, we start with a survey of known results and examples of chiral manifolds. This cannot encompass every result which is related to chirality and amphicheirality of manifolds. Still we try to give a broad overview, state the most important results in this context and give reasons why the problems that are dealt with in the following chapters are relevant.

A fundamental question is in which dimensions there are chiral manifolds. The solution to this problem is the first main result of this work and the content of Chapter 3:

Theorem A

A single point, considered as an oriented 0-dimensional manifold, is chiral. In dimensions 1 and 2, every closed, orientable, smooth manifold admits an orientation-reversing diffeomorphism. In every dimension ≥ 3, there is a closed, connected, orientable, smooth manifold which does not admit a continuous map to itself with degree -1, i. e. it is chiral.

The construction of these chiral manifolds is divided into even and odd dimensions. First we construct odd-dimensional chiral manifolds in every dimension $n \geq 3$ as mapping tori of $(n-1)$-dimensional tori T^{n-1}. The fundamental group of the total space is a semidirect product of abelian groups. If we restrict the monodromy maps $H_1(T^{n-1}) \to H_1(T^{n-1})$ to certain maps, the effect of endomorphisms of the fundamental group on the orientation of the total space is easily controllable. This reduces the problem to an algebraic problem on the non-existence of solutions of certain matrix equations (Lemma 29) depending on the monodromy. A family of appropriate matrices for the monodromy is presented, and we prove the non-existence of solutions mostly by linear algebra and comparing eigenvalues but also by appealing in one step to the fundamental theorem of Galois theory.

Examples of chiral manifolds in even dimensions are then obtained by cartesian products of odd-dimensional chiral manifolds. Apart from the cup product structure in cohomology, we use a theorem by Hopf on the Umkehr homomorphism and Betti numbers to exclude degree −1 for all self-maps.

The odd-dimensional examples in Theorem A are Eilenberg-MacLane spaces, and the proof of chirality uses as a substantial ingredient that the effect of a self-map on homology is completely determined by the induced map on the fundamental group. Therefore, we next ask for obstructions other than the fundamental group und restrict the analysis to simply-connected manifolds.

In Chapter 4, it is shown that in dimensions 3, 5 and 6, a nontrivial fundamental group is a necessary characteristic of chiral manifolds. From dimension 7 on, we prove that there exist simply-connected chiral manifolds in every dimension:

Theorem B

In dimensions 3, 5 and 6, every simply-connected, closed smooth (or PL or topological) manifold is amphicheiral in the respective category. A closed, simply-connected, topological 4-manifold admits an orientation-reversing homeomorphism if its signature is zero. If the signature is nonzero, the manifold is chiral.

In every dimension ≥ 7 there is a closed, simply-connected, chiral smooth manifold.

The results in dimension 3 to 5 are obtained almost immediately from the powerful classification theorems for simply-connected manifolds in these dimension. The classification of simply-connected 6-manifolds needs more complicated invariants. We provide the necessary details on the invariants from the proof of the classification [Zhubr] and complete the argument by analysing the homology of the first Postnikov stage of the manifolds in question and the effect of automorphisms of the first Postnikov approximation.

For the evidence of simply-connected chiral manifolds in dimensions ≥ 7, examples in all dimensions except 9, 10, 13 and 17 can be constructed with methods used already in the previous chapters. Since simply-connected chiral manifolds in the remaining dimensions are more difficult to obtain, we split the proof of existence into two parts: (1) find a mechanism or an obstruction to orientation reversal in the partial homotopy type and (2) realise the obstruction by a simply-connected manifold.

The first step is done with the help of the Postnikov tower: In every instance, we construct an appropriate finite tower of principal $K(\pi, n)$-fibrations (or simply a single stage) and fix an element in the integral homology of one of the stages that is to be the image of the fundamental class of the manifold. Then we prove that (by the mechanism that lies in the particular construction) this homology class can never be mapped to its negative under any self-map of a single Postnikov stage or of the partial Postnikov tower.

In the second step, the obstruction is realised by proving that there is indeed a manifold with the correct partial homotopy type and the correct image of the fundamental class in the Postnikov approximation. This step involves bordism computations and surgery techniques.

For simply-connected chiral manifolds in dimensions 10 and 17, is is sufficient to construct a single Postnikov stage. The obstruction is manifest in the mod-3 Steenrod algebra in the cohomology of Eilenberg-MacLane spaces. The bordism computation in the second step is done in this and all further proofs with the help of the Atiyah-Hirzebruch spectral sequence. For the surgery step, we use the surgery theory of Kreck [Kreck99].

The examples in dimensions 9 and 13 require a more complicated setup of the Postnikov tower. Here, we construct a three-stage Postnikov tower by appropriate k-invariants. Together with the construction, we analyse the possible automorphisms of this Postnikov tower in each step. The analysis is made possible by rational homotopy theory. However, the information which is obtained from the rational homotopy type is not enough in our case, and we also include information about the automorphisms of the integral Postnikov tower.

Again, the Atiyah-Hirzebruch spectral sequence and Kreck's surgery theory are applied for the realisation part of the proof. Here, we extend a proposition in [Kreck99] in order to prove that surgery in rational homology in the middle dimension is possible in our setting.

Next, in order to further characterise the properties of manifolds which allow or prevent orientation reversal, we consider the question whether every manifold is bordant to a chiral one. This allows also an approximation to the (not mathematically precise) question "how many" manifolds are chiral or if "the majority" of manifolds is chiral or amphicheiral. The following statement is proved in Chapter 5.

Theorem C

In every dimension ≥ 3, every closed, smooth, oriented manifold is oriented bordant to a manifold of this type which is connected and chiral.

Summarising, we prove this by showing that the existing obstructions in our examples can be kept when we change the bordism class via connected sums of manifolds. A special case, for which an entirely new example is necessary, are nullbordant manifolds in dimension 4. We translate this problem into group homology and construct a series of finite groups G such that $H_4(G)$ contains an element of order > 2 which is invariant under all automorphisms of G. The proof is again completed by bordism and surgery arguments.

The majority of the theorems so far aimed at proving that certain manifolds or families of manifolds are chiral. The opposite problem, however, namely proving amphicheirality in nontrivial circumstances, is also an interesting question. In general, this is even more challenging since not only one obstruction to orientation reversal must be identified and realised but for the opposite direction every possible obstruction must vanish. Surgery theory is a framework for comparing diffeomorphism classes of manifolds, and smooth amphicheirality can be considered a showcase of surgery theory: Given the manifolds M and $-M$ it must be decided if M and $-M$ are oriented diffeomorphic. Surgery provides powerful theorems and some recipes for classification problems but not a generally applicable algorithm, so that a particular problem must still be solved individually. We carry out the surgery programme of [Kreck99] for some products of 3-dimensional lens spaces. We prove the following theorem.

Theorem D

Let r_1 and r_2 be coprime odd integers and let L_1 and L_2 be (any) 3-dimensional lens spaces with fundamental groups $\mathbb{Z}/r_1$ resp. $\mathbb{Z}/r_2$. Then the product $L_1 \times L_2$ admits an orientation-reversing self-diffeomorphism.

The question why these products constitute a relevant problem is discussed in the introduction of Chapter 6. The proof is facilitated by the fact that the products are known to be homotopically amphicheiral. This is not a necessary input to Kreck's surgery programme but we use it here since it simplifies the first part of the proof. We then carry out the bordism computation in the Atiyah-Hirzebruch spectral sequence. This uses the fact that we are dealing with a product manifold to a great extent, and we employ the module structure of the spectral sequence heavily. In the final surgery step, it is not necessary to analyse individual surgery obstructions, but we show that the obstruction group vanishes, using results by [Bak] and from the book [Oliver].

In Chapter 7, we add a new facet to the results of the previous chapters by showing that the order of an orientation-reversing map can be relevant. From the literature, we present examples of manifolds which admit an orientation-reversing diffeomorphism but none of finite order. We complement this with manifolds where the minimal order of an orientation-reversing map is finite:

Theorem E

For every positive integer k, there are infinitely many lens spaces which admit an orientation-reversing diffeomorphism of order 2^k but no orientation-reversing self-map of smaller order.

The nonexistence of orientation-reversing maps of smaller order is shown by a well-known formula for the degree of maps between lens spaces. Lens spaces with an orientation-reversing map of the desired order in infinitely many dimensions are given explicitly, and the map itself can be written by a simple formula in complex coordinates in the universal covering.

1.1 Conventions and notation

Throughout this work, all manifolds are compact and orientable. Also, boundaryless manifolds are understood without further notice (except for bounding manifolds in a bordism, but this will be clear from the context). With the exception of links (in the context of knot theory) and manifolds in the oriented bordism groups, every manifold is connected. Unless indicated otherwise, we consider smooth (i. e. differentiable of class C^∞) manifolds. Finally, all manifolds are required to be second-countable Hausdorff spaces.

The following list clarifies some notations, which otherwise follow common practice.

- If M is an oriented manifold, the same manifold with the opposite orientation is denoted $-M$ (as in bordism theory, not $\overline{M}$ as in algebraic geometry). Often, the initial orientation does not matter, and we speak of M and $-M$ for orientable manifolds, meaning that an arbitrary orientation for M is fixed.
- Care has been taken in using the equal sign. Often in mathematics, when this detail is not important, not only *equal* objects are related by = but also *isomorphic* objects. Since naturality is a crucial detail in some proofs, the equal sign is reserved in this text to correspondences which are canonical (i. e. do not depend on choices) or natural (i. e. functorial). Otherwise, isomorphisms are denoted as usual by $\cong$.
- An arrow with a tilde $\xrightarrow{\sim}$ also denotes an isomorphism. This is a little less standard notation than $\hookrightarrow$ for injective and $\twoheadrightarrow$ for surjective maps but there is no danger of confusion since weak equivalences in model categories, for which the symbol $\xrightarrow{\sim}$ is also used, do not occur in this text.
- Coefficients in homology and cohomology are always the integers $\mathbb{Z}$ if not stated otherwise. In order to distinguish relative (co-)homology from the notation with coefficient groups, the coefficient are separated by a semicolon (compare $H_*(A,B)$ and $H_*(A;\mathbb{Q})$).

  In the (co-)homology of Eilenberg-MacLane spaces, the "K" is often omitted. Thus, $H_*(\mathbb{Z}/3,3;\mathbb{Z}/3)$ denotes the integral homology of an Eilenberg-MacLane space $K(\mathbb{Z}/3,3)$ with $\mathbb{Z}/3$-coefficients.
- As a general rule, abelian groups are written additively and arbitrary groups multiplicatively. Thus, the trivial group is either denoted 0 or 1,

depending on the context. The neutral element is written 0, 1 or e.
Sometimes, it is preferable to write cyclic groups multiplicatively, hence there are two notations $(\mathbb{Z}/n, +)$ and $(C_n, \cdot)$.

- Even though the cartesian product of oriented manifolds is not commutative, we keep the notation $\prod_{i=1}^{n} M_i$ for the product $M_1 \times \ldots \times M_n$, taking care of the order of the factors.

References in the text often do not give credit to the original author of a proof but point to a source which is more accessible to non-experts or gives an overview over a certain topic.

1.2 Why the name "chiral manifold"?

There are two adjectives in the literature to describe manifolds whose orientation can be reversed: "symmetric" ([Rueff, p. 162], [Kirby, Problem 1.23]) and "amphicheiral" ([Siebenmann] and [Saveliev02] for 3-manifolds).

The term "symmetric manifold" does not seem to be a good choice to the author: It would easily be confused with the concept of a symmetric space, which by definition always is a Riemannian manifold. In a symmetric space, the symmetry maps reverse all geodesics through a given fixed point, so the symmetry maps do not reverse orientation if the symmetric space has even dimension. Besides, the converse "asymmetric manifold" is nowadays reserved for manifolds on which no finite group can act effectively [Puppe].

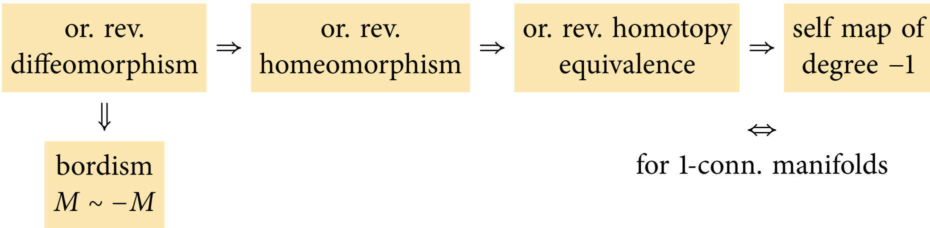
The attribute "amphicheiral", on the other hand, is perfectly legitimate for 3-manifolds: It is already used for knots and links, and the analogy to 3-manifold topology is even twofold because there are two different constructions that yield amphicheiral manifolds when applied to amphicheiral links. Firstly, when a 3-manifold admits a cyclic branched covering to an amphicheiral link in S^3, the manifold is amphicheiral. Secondly, when a 3-manifold is formed by surgery on a framed link, the manifold with the opposite orientation is obtained by surgery on the mirror image of this link, with the negative framing. Thus, surgery on an amphicheiral link with an appropriate framing yields an amphicheiral 3-manifold. Both relations between 3-manifolds (branched coverings and surgery) are discussed in more detail in Section 2.7.1.

Other logical descriptions would be "reversible" and "invertible". However, these terms have not been used in connection with orientation reversal of manifolds before. Furthermore, the parallels to knot theory would be rather misleading, as is also explained in Section 2.7.1.

For these reasons, the author chose to use the attribute "amphicheiral" for manifolds which admit an orientation-reversing self-map and the opposite "chiral" for those which do not.

1.3 Chirality in various categories

Requiring that a manifold admits a self-map of degree −1 is only a very weak form of amphicheirality. One can also ask for orientation-reversing self-homotopy equivalences or homeomorphisms. If smooth manifolds are dealt with, one can require an orientation-reversing diffeomorphism. The following figure lists all the types of maps that we consider, together with their interrelations.



The double arrows indicate that, e. g., an orientation-reversing homoeomorphism is automatically a homotopy equivalence, and likewise for the other types of maps. The bordism question is solved completely (the final step is due to Wall [Wall60]): A closed, smooth, oriented manifold is oriented bordant to its negative if and only if all its Pontrjagin numbers vanish.

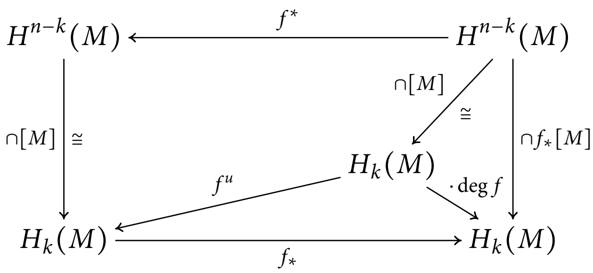
In the following, we indicate the category of maps which are considered by attributes like "topologically chiral", "homotopically amphicheiral" etc. The notions are self-explanatory. A "smoothly chiral manifold", e. g., is a differentiable manifold that does not admit an orientation-reversing diffeomorphism (but possibly an orientation-reversing homeomorphism).

The figure above indicated that there is no difference between orientation-reversing self-homotopy equivalences and maps of degree −1 for simply-connected manifolds. This is in fact the conclusion of the following lemma.

Lemma 1

A self-map $f : M \to M$ of a simply-connected, closed manifold M with degree ± 1 is a homotopy equivalence.

Proof. Let $f^u : H_*(M) \to H_*(M)$ be the Umkehr map, defined by the induced map $f^* : H^*(M) \to H^*(M)$ on cohomology and Poincaré duality on M. Let n be the dimension of M. The map $f_* \circ f^u : H_*(M) \to H_*(M)$ is multiplication by the degree of f, as can be seen from the following commutative diagram:



Since $\deg f = \pm 1$, we have $f^u \circ f_* = f_* \circ f^u = \pm \mathrm{id}_*$. Hence, the map f induces an isomorphism in homology. Since M is simply-connected and has the homotopy type of a CW-complex [Milnor59, Cor. 1], f is a homotopy equivalence by the Whitehead theorem for simply-connected CW-complexes [Bredon, Cor. 11.15]. □

A general goal of this work is to prove chirality and amphicheirality in the strongest possible sense. If chiral manifolds are produced, the aim is to exclude maps of degree −1 (or equivalently, orientation-reversing homotopy equivalences in the simply-connected case). In Chapter 6, when we prove that many products of 3-dimensional lens spaces are amphicheiral, we prove smooth amphicheirality. The various notions of chirality do not coincide, and in Chapter 3 we present (previously known) examples which are amphicheiral with respect to one type but chiral with respect to another category of maps.

2

Known examples and obstructions

In this chapter, we give an overview over the basic facts and most important results that exist about orientation reversal. Along with known examples of chiral manifolds, we collect the mechanisms which cause chirality. This yields a list of "obstructions" (in an informal sense) to orientation reversal, which will be extended by novel obstructions in subsequent chapters. We do not consider the items in this collection to be obstructions in a mathematically rigid sense, like certain cohomology classes in obstruction theory. The phenomenon of orientation reversal is too complex and heterogeneous for such an approach. Nevertheless, we find it useful to keep a list of topological concepts which are related to chirality. Moreover, not every hypothetical "obstruction" is admitted to the list but it must be proved to be "realised" by a chiral manifold.

We first review the straightforward results in dimension 0 to 2: A single point is chiral, and every 1- and 2-dimensional manifold is amphicheiral. The simplest chiral manifolds other than the point are detected by the intersection and linking forms. Their naturality properties can exclude orientation-reversing maps in dimensions congruent 0 and 3 mod 4, and we argue why the forms are not useful in dimensions congruent 1 and 2 mod 4.

Lens spaces play a prominent role in this work since we use them in many instances to prove and illustrate various results about chirality. Their oriented homotopy, homeomorphism and diffeomorphism classifications are reviewed. Subsequently, we deal with characteristic numbers since the Pontrjagin numbers are an obstruction to smooth amphicheirality and they give a complete answer to the question of chirality up to bordism. After this, we touch upon exotic spheres since they provide important examples for the distinction between the smooth and the topological category in Section 1.3.

A comparatively large part of this chapter is dedicated to 3-manifolds since the terms chiral and amphicheiral are adopted from this field. Since 3-manifolds are a subject of its own, we cannot provide a comprehensive summary of all tools and results. We mention briefly the Casson invariant and homology bordism but otherwise concentrate on the relations between amphicheiral links and amphicheiral 3-manifolds.

Another field of its own is the topology of 4-manifolds. Although there are results concerning smooth chirality and amphicheirality of 4-manifolds, especially the subject of smooth structures on 4-manifold is a highly specialised field of work, which goes beyond the scope of this thesis. Results on simply-connected 4-manifolds will be discussed later in Section 4.1.2.

2.1 Dimensions 0 to 2

A single point, considered as an orientable 0-dimensional manifold is chiral. This is an exceptional case because the tangent bundle is 0-dimensional and cannot be given an orientation. However, the following approach makes sense: Since $H_0(\mathrm{pt}) = H_0(\mathrm{pt}, \varnothing) \cong \mathbb{Z}$, a fundamental class can be assigned to the point. The group $H_0(\mathrm{pt})$ has a preferred generator which is represented by the map from the 0-simplex to the point. We call an oriented point the *positive* point if its fundamental class is this generator and the *negative* point if the fundamental class is the negative of the preferred generator. The orientation cannot be changed by a self-map $\mathrm{pt} \to \mathrm{pt}$ because the induced map on $H_0(\mathrm{pt})$ is always the identity.

Every closed, connected 1-dimensional manifold is homeomorphic to the circle S^1, which is clearly orientable and smoothly amphicheiral: simply consider the standard embedding as the unit circle in $\mathbb{R}^2$ and mirror the circle at any diameter. This obviously generalises to higher-dimensional spheres, where reflection at the equator reverses the orientation.

Recall that there is no difference between topological, piecewise linear (PL-) and smooth manifolds in dimensions up to three. More precisely, every topological manifold in these dimensions has a PL-structure, and every PL-manifold has a smooth structure. Furthermore, the refined structures are unique in the oriented sense: If two oriented PL-manifolds are oriented homeomorphic, they are even oriented PL-isomorphic, likewise for smooth manifolds and diffeomorphisms.

The diffeomorphism classes of closed, connected 2-manifolds are the connected sums of k tori ($k \geq 0$, the case $k = 0$ is the 2-sphere). All of these are smoothly amphicheiral because they can be embedded mirror-symmetrically into $\mathbb{R}^3$, as Figure 2.1 illustrates.

2.2 The cup product and the intersection form

The simplest examples of chiral manifolds in nonzero dimensions are given by the complex projective spaces $\mathbb{C}\mathrm{P}^{2n}$. Their cohomology ring is the truncated polynomial ring $\mathbb{Z}[t]/t^{2n+1}$ with one generator t in degree 2. Since the fundamental class $[\mathbb{C}\mathrm{P}^{2n}]$ is a generator of $H_{2n}(\mathbb{C}\mathrm{P}^{2n})$, we have $\langle t^{2n}, [\mathbb{C}\mathrm{P}^{2n}]\rangle = \pm 1$.

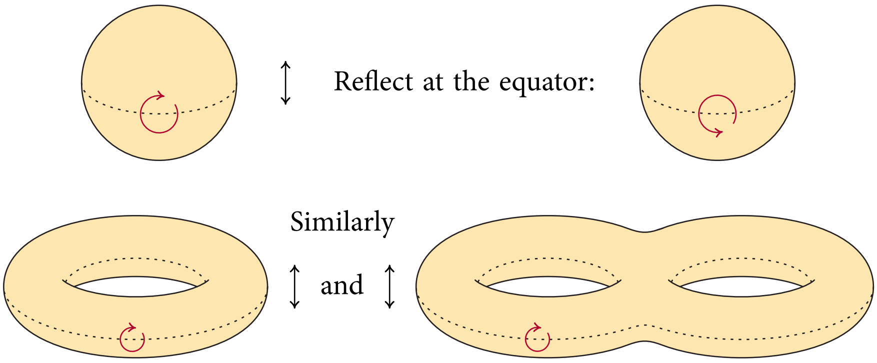


Figure 2.1: Mirror-symmetric embeddings of oriented surfaces

The sign depends on the orientation of $\mathbb{C}P^{2n}$ but not on the choice of t since t is raised to an even power. With the preferred orientation on $\mathbb{C}P^{2n}$, which is induced by the complex structure, the value is in fact +1 [MS, Thm. 14.1, Thm. 14.10, p. 170].

Suppose an orientation-preserving homotopy equivalence $f : \mathbb{C}P^{2n} \to -\mathbb{C}P^{2n}$ would exist. Since $f^*(t)$ is again a generator, we have $f^*(t) = \pm t$ and by the naturality of the Kronecker product

$$\langle t^{2n}, [\mathbb{C}P^{2n}]\rangle = \langle (f^*t)^{2n}, [\mathbb{C}P^{2n}]\rangle = \langle t^{2n}, [-\mathbb{C}P^{2n}]\rangle = -\langle t^{2n}, [\mathbb{C}P^{2n}]\rangle,$$

which is impossible. Thus, the complex projective spaces $\mathbb{C}P^{2n}$ in dimensions $4k$ are homotopically chiral. On the other hand, $\mathbb{C}P^{2n+1}$ is smoothly amphicheiral. The orientation is reversed by the map that conjugates the homogeneous coordinates.

Another point of view is the intersection form on middle cohomology. The element $t^n \in H^{2n}(\mathbb{C}P^{2n})$ intersects with itself with intersection number 1 since $\langle t^n \cup t^n, [\mathbb{C}P^{2n}]\rangle = 1$. In the following, we recall the definition and the main properties of the intersection form.

Denote the torsion subgroup of an abelian group A by $\operatorname{Tor} A := \operatorname{Tor}(A, \mathbb{Q}/\mathbb{Z})$. Moreover, let $A_{free} := A/\operatorname{Tor} A$. (We use this only for finitely generated abelian groups so that A_{free} is indeed a free abelian group, not only torsion-free.)

Theorem 2

*The **intersection form** on a closed, oriented 2k-dimensional manifold*

$$Q : H^k(M) \times H^k(M) \to \mathbb{Z}, \qquad Q(a,b) := \langle a \cup b, [M]\rangle$$

is a $(-1)^k$-symmetric bilinear form (i. e. it is symmetric if k is even and antisymmetric if k is odd). Since every homomorphism from a torsion group to $\mathbb{Z}$ is trivial, the intersection form is well-defined on the free quotient $H^k(M)_{free}$. This bilinear form is unimodular.

If $f : M \to N$ is a continuous map, we have $Q(f^*a, f^*b) = \deg(f) \cdot Q(a,b)$ for all $a, b \in H^k(N)$.

Proof. See e. g. [HatcherAT, Section 3.3, p. 249 ff.]. The naturality statement follows from the naturality properties of the cup product and the Kronecker pairing. □

Theorem 2 implies that a manifold is homotopically chiral if its intersection form Q is not isomorphic to its negative $-Q$. If k is odd, the intersection form on $A := H^k(M)_{free}$ is an antisymmetric bilinear form on a finitely generated free abelian group. By [MH, Cor. 3.5], A has even rank and there is a basis $(e_1, \ldots, e_{2m})$ of A such that Q has the form

$$\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix},$$

where I_m denotes the identity matrix of size m. The coordinate permutation $e_i \leftrightarrow e_{i+m}$ for all $i \le m$ changes this matrix to its negative. Thus, if the dimension of a manifold is congruent 2 mod 4, the intersection form is always isomorphic to its negative and cannot provide an obstruction to amphicheirality.

If k is even, the intersection form is symmetric. Unlike the antisymmetric case, symmetric unimodular bilinear forms over $\mathbb{Z}$ have not been classified. In particular, the number of positive (or equivalently, negative) definite forms grows rapidly with the rank [MH, p. 28]. Fortunately, this does not cause complications for the question of chirality because a definite form is never isomorphic to its negative. Indefinite symmetric unimodular form over $\mathbb{Z}$ are distinguished by their rank, signature and type [MH, Thm. 5.3]. Since the rank and the type of a form and its negative are the same, the signature is the only invariant which can distinguish a form Q from $-Q$. Summarising, we have

Proposition 3: [MH]

Let $Q : A \times A \to \mathbb{Z}$ be a symmetric unimodular bilinear form on a finitely generated free abelian group A. Then Q is isomorphic to $-Q$ if and only if its signature is zero.

The signature of an oriented manifold can be defined as the signature of its intersection form if the dimension is a multiple of 4. Otherwise, the signature is set to zero. Since a self-map of degree ± 1 induces an isomorphism in cohomology (see the proof of Lemma 1 and convert it from homology to cohomology), we have the following statement:

Corollary 4

A manifold with nonzero signature admits no self-map of degree -1.

Corollary 5

A $4k$-dimensional manifold with odd $(2k)$-th Betti number b_{2k} admits no self-map of degree -1.

Proof. Since the rank of the middle homology group is odd and the intersection form is symmetric in the present case, the signature must be nonzero. ☐

The observations in the two preceding lemmas and similar statements for the linking form were already made in 1938 by Rueff [Rueff].

We want to point out that the signature is the only algebraic obstruction to orientation reversal which can be obtained from the intersection form. This is not a mathematically rigid statement since the term “obstruction to amphicheirality” has not been given a mathematically well-defined meaning. Nevertheless, it should be clear what is meant by this statement: There might be chiral manifolds with signature 0 (in fact, there are), but there must be characteristics of these manifolds other than the intersection form that cause chirality.

In conclusion, we want to record the obstructions to orientation reversal from this section. Most generally, the *cup product structure* can be made responsible in the case of $\mathbb{C}P^{2n}$ since an even power of a cohomology element t that generates a cohomology group of rank 1 evaluates nontrivially on the fundamental class. More specifically, the *signature* of manifold is a homotopy-invariant obstruction. The point of view of the signature as a characteristic number will be taken up in Section 2.5.

2.3 The linking form

For odd-dimensional manifolds, the linking form is the analogue to the intersection form.

Theorem 6

Let M be a closed, oriented topological manifold of odd dimension $2k-1$. Then there is a nondegenerate, $(-1)^k$-symmetric bilinear form

$$L:\ \operatorname{Tor} H^k(X) \times \operatorname{Tor} H^k(X) \to \mathbb{Q}/\mathbb{Z},$$

which is called the ***linking form****. Furthermore, if $f: N \to M$ is a continuous map then $L(f^*a, f^*b) = \deg(f)\cdot L(a,b)$.*

Although this theorem is well-known, a proof of all properties in one piece is given in Appendix A.1. The cohomological version of the linking form is preferred because naturality can be handled more easily in this setting. A definition of the homological version can be found in [Ranicki, Ex. 12.44 (i)].

If the dimension of the manifold is congruent 1 mod 4, the linking form is antisymmetric. In analogy to the intersection form, an antisymmetric linking form is isomorphic to its negative. The proof is a little more complicated due to fact that there can be elements $x \in \operatorname{Tor} H^k(X)$ with $L(x,x) = \frac{1}{2}$.

Proposition 7

Let $L : G \times G \to \mathbb{Q}/\mathbb{Z}$ be a nondegenerate, antisymmetric bilinear form on a finite abelian group G. Then there is an isomorphism $f : G \to G$ such that $L(f(x), f(y)) = -L(x,y)$ for all $x, y \in G$.

Proof. According to [Wall62, Lemma 4(ii)], G is the direct sum of groups $G_i \cong \mathbb{Z}/\theta_i \oplus \mathbb{Z}/\theta_i$ with generators x_i, y_i and possibly a single direct summand $\mathbb{Z}/2$ with generator z. These summands are orthogonal with respect to L, i. e. we have $L(a,b) = 0$ for elements a and b of different summands[1)]. Furthermore, $L(z,z) = \frac{1}{2}$, and L has a matrix of the form

$$\begin{pmatrix} 0 & 1/\theta_i \\ -1/\theta_i & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1/\theta_i \\ -1/\theta_i & 1/2 \end{pmatrix}$$

on the summands G_i. In the first case, the form on G_i is reversed by interchanging x_i and y_i. In the second case, this effect is obtained by the base change $x_i' := x_i + 2y_i$, $y_i' := x_i + y_i$. □

Therefore, in analogy to the intersection form, we have the imprecise statement that the linking form cannot provide an algebraic obstruction to orientation reversal in dimensions congruent 1 mod 4. In dimensions congruent 3 mod 4, however, the linking form can be used to prove chirality, as in the following exemplary statement.

Lemma 8

Let M be a closed, oriented topological manifold of dimension $4k+3$. Suppose that $\operatorname{Tor} H^{2k+2}(M) \cong \mathbb{Z}/n$ and -1 is not a quadratic residue modulo n. Then M does not admit a self-map of degree -1.

Proof. Choose a generator $\alpha \in \operatorname{Tor} H^{2k+2}(M)$ and suppose $f : M \to M$ reverses orientation. Let $f^*\alpha = q\alpha$ with $q \in \mathbb{Z}/n$. Then

$$-L(\alpha,\alpha) = L(f^*\alpha, f^*\alpha) = L(q\alpha, q\alpha) = q^2 L(\alpha,\alpha) \in \mathbb{Q}/\mathbb{Z}.$$

Since the linking pairing is nondegenerate, $L(a,a)$ has order n in $\mathbb{Q}/\mathbb{Z}$. So $q^2 \equiv -1 \bmod n$, a contradiction. □

An application of this lemma is given by lens spaces: Every lens space in dimension $4k+3$ whose order of the (cyclic) fundamental group contains a factor 4 or a prime congruent 3 mod 4 is chiral in the strongest sense. We can thus add the *linking form* to our list of obstructions to orientation reversal.

---

1) Wall does not state that the $\mathbb{Z}/2$-summand is orthogonal to the others but this follows from the proof.

2.4 Lens spaces

Lens spaces form a very important class of manifolds for this work. They appear in many different situations, both in proofs and as illustrations of various aspects of chirality. Since the conventions about the parameters in lens spaces differ between sources, they are defined here.

Choose integers $n \geq 2$ and $t \geq 1$ and parameters $k_1, \ldots, k_n \in (\mathbb{Z}/t)^\times$. The lens space $L_t(k_1, \ldots, k_n)$ is defined as the quotient of the unit sphere $S^{2n-1} \subset \mathbb{C}^n$ under the free action of $\{\gamma \mid \gamma^t = 1\} \cong \mathbb{Z}/t$ by $\gamma(c_1, \ldots, c_n) = (\xi^{k_1} c_1, \ldots, \xi^{k_n} c_n)$. Here, ξ denotes the t-th root of unity $e^{2\pi i/t}$.

This lens space is a $(2n-1)$-dimensional closed, orientable, smooth, connected manifold with fundamental group $\mathbb{Z}/t$. It has a preferred orientation induced from the canonical orientation on $\mathbb{C}^n$ and the outer normal vector field of S^{2n-1}. Furthermore, its fundamental group has a preferred generator γ (if the fundamental group is nontrivial, i. e. if $t > 1$). The preferred generator of the fundamental group is given by the covering translation γ; alternatively, it can be described by any path from a basepoint x_0 to $\gamma(x_0)$ in S^{2n-1}. The choice of x_0 is irrelevant since the fundamental group is abelian.

The notation $L_t(k_1, \ldots, k_n)$ implies that the parameters k_i are relatively prime to t. This will be implicitly assumed in all statements in this work.

In some definitions, instead of the k_i their multiplicative inverses modulo t are used, e. g. [Milnor66, §12]. The classification theorems below are literally the same for both conventions, but the notation matters of course if individual lens spaces are identified.

The orientation of a lens space can be reversed by multiplying one of its parameters by -1; this corresponds to complex conjugation in the respective coordinate of $\mathbb{C}^n$ and preserves the preferred generator of the fundamental group. More precisely, write $L := L_t(k_1, \ldots, k_n)$ and $L' := L_t(l_1, \ldots, l_n)$ and let $l_i = -k_i$ for exactly one i, otherwise $l_i = k_i$. Then there is an orientation-reversing diffeomorphism $L \to L'$ which maps the preferred generator of $\pi_1(L)$ to the preferred generator of $\pi_1(L')$.

Lens spaces are classified (besides other concepts like simple homotopy type) up to oriented homotopy equivalence, homeomorphism and diffeomorphism.

Theorem 9: homotopy classification [Milnor66, 12.1], [Lück, Thm. 2.31]

The lens spaces $L_t(k_1, \ldots, k_n)$ and $L_{t'}(l_1, \ldots, l_n)$ are orientation-preserving homotopy equivalent if and only if $t = t'$ and there is $e \in (\mathbb{Z}/t)^\times$ such that $\prod_{i=1}^n k_i = e^n \cdot \prod_{i=1}^n l_i$ in $(\mathbb{Z}/t)^\times$. The same conditions apply for a map of degree 1 between the two lens spaces.

Corollary 10

Let L be a lens space of dimension $2n-1$ with fundamental group of order t. The following conditions are equivalent:

(a) L is homotopically amphicheiral.

(b) L admits a self-map of degree −1.

(c) −1 is an n-th power modulo t.

Note that the last condition is always fulfilled if n is odd. Hence, all lens spaces whose dimension is congruent 1 mod 4 are homotopically amphicheiral. In dimensions congruent 3 mod 4, if the order t of the fundamental group contains the factor 4 or a prime congruent 3 mod 4, -1 is not even a square mod t, so the lens space does not admit a self-map of degree -1.

The homeomorphism and diffeomorphism classifications of lens spaces agree, hence they also coincide with the PL classification.

Theorem 11: [Milnor66, Thm. 12.7]

Let $L := L_t(k_1, \ldots, k_n)$ and $L' := L_t(l_1, \ldots, l_n)$ be two $(2n-1)$-dimensional lens spaces with the same order of the fundamental group. The following statements are equivalent:

(a) L and L' are oriented homeomorphic.

(b) L and L' are oriented diffeomorphic.

(c) The sequences of parameters $(k_1, \ldots, k_n)$ and $(l_1, \ldots, l_n)$ can be converted into each other by the following operations:

(1) For a $k \in (\mathbb{Z}/t)^\times$, replace each k_i by kk_i.

(2) Permute the k_i.

(3) Replace an even number of the k_i by their negatives $-k_i$.

This theorem is proved with the *Reidemeister-Franz-torsion*. There exist several flavours of this torsion with values in different rings, see [Milnor66, §12], [Lück, Ch. 2.4] and [Ranicki97]. Milnor proves in fact only the classification up to oriented diffeomorphism. See [Lück, Thm. 2.1] and [Ranicki97] for comments on the homeomorphism invariance of the torsion, which was proved by Chapman [Chapman73], [Chapman74] after Milnor's paper. Alternatively, Lück's proof of the unoriented homeomorphism classification can be modified to yield the oriented statement. (The idea is to restrict the diffeomorphisms to those which preserve the preferred generator of the fundamental group; set $\alpha = \mathrm{id}$ in [Lück, Thm. 2.37] for this. Then combine this with the homotopy classification [Milnor66, 12.1], which also considers the generator of the fundamental group.)

Example 12

The lens space $L_5(1, 2)$ is smoothly amphicheiral. The lens space $L_5(1, 1)$ is smoothly chiral but homotopically amphicheiral.

Proof. The first lens space is smoothly amphicheiral since the parameters can be changed in the following way:

$$(1,2) \xmapsto[\text{mult. by } 2]{\text{operation (1)}} (2,4) \xmapsto[\text{mod } 5]{\text{congruence}} (2,-1) \xmapsto[\text{transposition}]{\text{operation (2)}} (-1,2)$$

The lens space $L_5(1,1)$ is homotopically amphicheiral by Theorem 9 since $2^2 \equiv -1 \bmod 5$. Since the two parameters are equal and all three operations (1), (2) and (3) produce again a pair of equal parameters, they cannot be converted to $(-1,1)$. □

The lens space $L_5(1,1)$ is an example where the categories in Section 1.3 differ because the orientation can be reversed by a homotopy equivalence but not by a homeomorphism.

At least in the types of torsion which Lück, Milnor and Ranicki describe, the Reidemeister-Franz-torsion is the same for a lens space and its negative. However, information about the preferred generator of the fundamental group can be recovered from the torsion. Together with the oriented homotopy classification [Milnor66, 12.1], this suffices to obtain the oriented homeomorphism classification. Therefore, we add to our informal list of obstructions to orientation reversal not the Reidemeister-Franz-torsion itself but suggest the item *"Reidemeister-Franz-torsion plus the oriented homotopy type"*. This is an obstruction to topological amphicheirality.

2.5 Characteristic numbers

In this section we review which characteristic numbers can distinguish between oppositely oriented manifolds. The overall reference for this section is [MS].

The Pontrjagin classes of a manifold $p_i(M) \in H^{4i}(M)$ are independent of the orientation because they are defined as Chern classes of the complexified tangent bundle. The complexified bundle has a canonical orientation which is determined by the complex structure and independent of the orientation of the underlying real bundle.

Therefore, the *Pontrjagin numbers* of a $4n$-dimensional manifold

$$p_I(M) := \langle p_{i_1} \cup \ldots \cup p_{i_r}, [M] \rangle \quad \text{for a partition } I = (i_1, \ldots, i_r) \text{ of } n$$

reverse their sign with the orientation of M. The Pontrjagin numbers are in general only oriented diffeomorphism invariants of the manifold. Hence, if a $4n$-dimensional manifold has a nonzero Pontrjagin number, it is smoothly chiral.

Pontrjagin numbers can also be considered with rational coefficients. An exceptional rational Pontrjagin number is the signature as a homotopy invariant (see Section 2.2), which can be expressed by Hirzebruch's signature theorem both as a Pontrjagin number and as the signature of the intersection form.

The Euler characteristic, the characteristic number corresponding to the Euler class, is independent of the orientation since it is the alternating sum of the Betti numbers. Stiefel-Whitney numbers (with values in $\mathbb{Z}/2$) are unoriented bordism invariants and thus independent of the orientation.

This is also the place for a comment on the difference between orientability and amphicheirality. A manifold is orientable if and only if its first Stiefel-Whitney class $w_1(M) \in H^1(M;\mathbb{Z}/2)$ is zero. This can be thought of as the answer to the "existence question for orientations". The "uniqueness question" has traditionally the answer that an orientable manifold has always two possible orientations. Another point of view is to ask not only about the orientation but about the orientable manifolds themselves. An amphicheiral orientable manifold defines a unique oriented manifold (unique up to orientation-preserving diffeomorphism, homeomorphism, ...), whereas there are two possibilities for a chiral manifold. While the existence question has a simple, definite answer, the uniqueness problem in this sense is apparently much more complicated, and the content of this thesis can be regarded as its fundamentals.

Two (oriented, closed, smooth) manifolds are oriented bordant if and only if they have the same Stiefel-Whitney and Pontrjagin numbers [Wall60]. A manifold with zero Pontrjagin numbers is nullbordant or an element of order 2 in the oriented bordism group. Thus, a manifold is oriented bordant to its negative if and only if all its Pontrjagin numbers vanish, as was remarked in Chapter 1. See also the stronger statement in Theorem 69.

An almost complex manifold has a canonical orientation given by the complex structure on its tangent bundle. Let τ denote the tangent bundle of an almost complex manifold M, equipped with a complex structure. If the dimension of M is $4n+2$, the conjugate bundle $\bar{\tau}$ has the same underlying real bundle with the opposite orientation, so that $\bar{\tau}$ defines an almost complex structure on the manifold $-M$. If the manifold has moreover a complex structure, the entire complex structure can be conjugated by conjugating each local chart in each local coordinate. This also reverses the orientation if the complex dimension is odd. By the identity for Chern classes (for any complex vector bundle ω)

$$c_k(\overline{\omega}) = (-1)^k c_k(\omega),$$

the Chern numbers c_I of M coincide with those of $-M$. Indeed, there are $2n+2$ minus signs in the formula

$$c_I(M) := \langle c_{i_1} \cup \ldots \cup c_{i_r}, [M^{4n+2}] \rangle \quad \text{for a partition } I = (i_1, \ldots, i_r) \text{ of } 2n+1,$$

namely $2n+1$ from the Chern classes and one from the fundamental class. Therefore, the Chern classes of an almost complex, $4n+2$-dimensional manifold cannot distinguish the orientations.

Given an almost complex manifold M of dimension $4n$, there is not necessarily an almost complex structure on $-M$. Thus, even the existence of a an almost complex structure is an obstruction to smooth amphicheirality. E. g., the manifold $-\mathbb{C}P^2$ does not have an almost complex structure. For the corresponding problem in the domain of complex manifolds, see [Beauville], [Kotschick92] and [Kotschick97]. We state one exemplary fact here.

Theorem 13: Part of [Kotschick97, Thm. 2]

Let X be a compact complex surface admitting a complex structure for $-X$. Then the signature of X vanishes.

The last theorem does not contribute anything new to our analysis of chirality since the signature is already an obstruction to self-maps of degree −1. However, Kotschick's work on complex structures on manifolds with opposite orientations provided a smoothly chiral, simply-connected 4-manifold with signature 0, see Theorem 44.

Summarising, we have seen that the *Pontrjagin numbers* are obstructions to smooth amphicheirality. The Euler characteristic, Stiefel-Whitney numbers and Chern numbers in dimensions congruent 2 mod 4 do not detect chiral manifolds. In dimensions congruent 0 mod 4, even the *existence of a complex or almost complex structure* can be a distinguishing feature for the orientation of smooth manifolds.

2.6 Exotic spheres

Exotic spheres are examples of manifolds whose differentiable structure forbids orientation reversal but which are topologically amphicheiral. By a homotopy n-sphere, we denote in the following always an oriented manifold. The starting point for this section is the following theorem by Kervaire and Milnor.

Theorem 14: [KM, Thm. 1.1]

The h-cobordism classes of homotopy n-spheres form an abelian group θ_n under the connected sum operation.

By the generalised Poincaré conjecture, all homotopy spheres are homeomorphic to the standard sphere (the work of Perelman, Freedman and Smale). Moreover, the h-cobordism theorem says that in dimension ≥ 5, h-cobordism classes of simply-connected manifolds coincide with oriented diffeomorphism classes. Thus, Theorem 14 can be formulated as follows.

Theorem 15

The oriented diffeomorphism classes of manifolds which are homeomorphic to the standard n-sphere ("exotic spheres") form an abelian group θ_n under the connected sum operation for $n \geq 5$.

The inverse of an element $\Sigma \in \theta_n$ is given by the manifold Σ with the opposite orientation [KM, Lemma 2.4]. Thus, the notation $-\Sigma$ for both the oppositely oriented manifold and the negative element in the group θ_n does not cause problems.

In dimensions ≤ 3 there are no exotic spheres, and the situation in dimension 4 is unknown. The groups θ_n in dimensions ≤ 17 are stated in Table 2.1.

Of course, all exotic spheres are topologically amphicheiral since they are homeomorphic to the standard sphere. However, Table 2.1 yields, e. g. that there are 13 different pairs of smoothly chiral 7-spheres $(\Sigma, -\Sigma)$. The standard sphere S^7 (the zero element in θ_7) is clearly smoothly amphicheiral, as well as the exotic sphere which has order 2 in $\theta_7 \cong \mathbb{Z}/28$. In some dimensions (8, 9, 14, 16, 17 in Table 2.1) there exist exotic spheres but all are amphicheiral. In dimensions $n \equiv 3 \bmod 4$, θ_n contains a large cyclic subgroup, yielding many chiral homotopy spheres. This is due to the following fact.

Theorem 16: [Levine, §3, especially Cor. 3.20]

For $n \geq 2$, the subgroup of all diffeomorphism classes of exotic $(4n-1)$-spheres which bound parallelisable $4n$-manifolds is a cyclic group of order

$$a_n \cdot 2^{2n-2}(2^{2n-1}-1) \cdot \left(\textit{numerator of } \tfrac{B_n}{4n}\right),$$

where $a_n = 1$ if n is even and $a_n = 2$ if n is odd. The symbol B_n denotes the n-th Bernoulli number ($B_2 = -\frac{1}{30}, B_3 = \frac{1}{42}, B_4 = -\frac{1}{30}, B_5 = \frac{5}{66}, \ldots$).

Besides lens spaces, exotic spheres are another instance where the categories of orientation-reversing maps in Section 1.3 differ. For the list of obstructions, we note therefore that the *smooth structure* can be an obstruction to smooth amphicheirality, even if the manifolds are topologically amphicheiral.

n	θ_n
≤ 6	0
7	$\mathbb{Z}/28$
8	$\mathbb{Z}/2$
9	$(\mathbb{Z}/2)^3$
10	$\mathbb{Z}/6$
11	$\mathbb{Z}/992$
12	0
13	$\mathbb{Z}/3$
14	$\mathbb{Z}/2$
15	$\mathbb{Z}/8128 \oplus \mathbb{Z}/2$
16	$\mathbb{Z}/2$
17	$(\mathbb{Z}/2)^4$

Table 2.1: The groups of homotopy spheres in dimensions ≤ 17 [KM], [Levine].

2.7 3-manifolds

Among the many specialised tools and results that exist for 3-manifold we present two which immediately detect chirality: the *Casson invariant* and *homology bordism*. We add them straight away to the list of obstructions.

The Casson invariant is a $\mathbb{Z}$-valued homeomorphism invariant for oriented integral homology 3-spheres [Saveliev99, esp. Ch. 12]. Its sign reverses with the orientation. Thus, if an integral homology sphere has Casson invariant $\neq 0$, it must be topologically chiral. The most prominent example for a manifold whose chirality is detected by the Casson invariant is the *Poincaré homology sphere*. There are many descriptions for this 3-manifold. It can be obtained, e. g., by identifying opposite faces of a solid dodecahedron in the appropriate way or by 1-surgery on the right-handed trefoil knot (see the section "Surgery" below). The Casson invariant is normed so that the value of the Poincaré homology sphere is -1 [Saveliev99, Ch. 17.5].

Another concept, which produces chiral 3-manifolds in abundance, is homology bordism. We quote from [Saveliev99, Ch. 11.4]: Two oriented integral homology 3-spheres are called *homology cobordant* "if there exists a smooth compact oriented 4-manifold W with boundary $\partial W = -\Sigma_0 \cup \Sigma_1$ such that the inclusion induced homomorphisms $H_*(\Sigma_i) \to H_*(W)$ are isomorphisms." The *homology cobordism group*, denoted $\Theta^3_{\mathbb{Z}}$, has the connected sum as group operation and S^3 as the neutral element. The Poincaré homology sphere has infinite order in $\Theta^3_{\mathbb{Z}}$. Even more, it is known that $\Theta^3_{\mathbb{Z}}$ contains a free abelian group of infinite rank. This provides us with a countable infinite number of homology 3-spheres, all of which are topologically chiral.

2.7.1 Knots and links

In this section, we review the connections between knot theory and the topology of 3-mainfolds. There are two different constructions to obtain 3-manifolds from a link in S^3: branched coverings and surgery. Both constructions justify the naming "amphicheiral" for manifolds with an orientation-reversing self-map. In the following, we deal only with tame knots, i. e. topological embeddings $S^1 \to S^3$ which are ambient isotopic[2)] to a simple closed polygonal curve in $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$.

The following definitions are standard in knot theory and can be found, e. g., in [BZ, Def. 2.1], [CF, Ch. I.4], [Conway, §5], and [HTW, p. 37]. A knot $K \subset S^3$ is called *amphicheiral* if there is an orientation-reversing homeomorphism of S^3 mapping the knot to itself. If this is not possible, the knot is called *chiral*.

[2)] Two embeddings $f_0, f_1 : X \to Y$ are called *ambient isotopic* if there is a homeomorphism $F : X \times I \to Y \times I$ such that $\mathrm{pr}_2 \circ F = \mathrm{id}_I$, $\mathrm{pr}_1 \circ F_{|t=0} = f_0$ and $\mathrm{pr}_1 \circ F_{|t=1} = f_1$. The analogous definition is valid in the PL-category.

The same concept applies to links. A knot K is called *reversible* or *invertible* if, having fixed an orientation of the knot, there is a homeomorphism $(S^3, K) \to (S^3, K)$ which preserves the orientation of S^3 but reverses the orientation of K. A knot without this property is called *irreversible*.

There are different concepts for equivalence of knots but they effect the same equivalence relation:

Theorem 17: [BZ, Thm. 1.10, Cor. 3.16]

Let k_0 and k_1 be PL-knots in S^3. The following assertions are equivalent:

(a) *There is an orientation-preserving homeomorphism $f : S^3 \to S^3$ such that $f(k_0) = k_1$.*

(b) *k_0 and k_1 are ambient isotopic.*

(c) *k_0 and k_1 are ambient PL-isotopic.*

The second relation matches the picture of pulling a string through 3-dimensional space but the first relation is used above for the definition of chirality and is more useful, e. g., in the proof of Proposition 19.

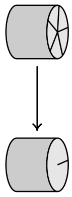
There are obvious refinements and combinations of the symmetry concepts like $(+)$-amphicheiral (if every homeomorphism of (S^3, K) preserves the orientation on K), $(-)$-amphicheiral (if the orientations of S^3 and K can be reversed together but not one at a time) and fully amphicheiral for amphicheiral, reversible knots. In the present work, however, it is sufficient to stay with the separate concepts of *chirality* and *reversibility* as defined above.

Branched coverings of the 3-sphere

Though "reversible" and "invertible" refer to orientation reversal of the knot itself, the orientation of the knot is irrelevant for surgery and branched coverings.

We use the definition of branched coverings in the PL-category. Let M, N be triangulated 3-manifolds and let L be a one-dimensional subcomplex in N. According to [PS, §22], a *branched covering* with covering manifold M, base N and branching set L is a continuous map $p : M \to N$ such that $K := p^{-1}(L)$ is a one-dimensional subcomplex in M and the restriction $p_{|M \setminus K} : M \setminus K \to N \setminus L$ is a covering map. Although the branching set of a branched covering can be an arbitrary 1-dimensional subcomplex, for the following definition of a cyclic branched covering it is required that L is a closed submanifold. A branched covering is k-fold *cyclic* if the restriction $p_{|K} : K \to L$ is a homeomorphism, and for every point $x \in K$, there are neighbourhoods $U \subset M$ of x and $V \subset N$ of $p(x)$ such that the projection is homeomorphic to an interval times the standard k-fold covering $z \mapsto z^k$ in the complex numbers. More precisely, let $B_r \subset \mathbb{C}$ be the open disk of radius r, then U and V are required to fit into the following diagram:

$$\begin{array}{ccccc} U & \xrightarrow{\text{homeomorph.}} & (-1,1)\times B_1 & & (x,z) \\ \downarrow{\scriptstyle p} & & \downarrow & & \downarrow \\ V & \xrightarrow{\text{homeomorph.}} & (-1,1)\times B_1 & & (x,z^k) \end{array} \tag{1}$$



The horizontal homeomorphisms are required to map $K \cap U$ and $L \cap U$ to $(-1,1) \times \{0\}$.

Lemma 18

Let $p : M \to N$ be a cyclic branched covering with branching set $L \subset N$. Every homeomorphism $f : (N, L) \to (N, L)$ is covered by a homeomorphism $\bar{f} : M \to M$.

Proof. Since $p : M \setminus K \to N \setminus L$ is a covering in the ordinary sense, the homeomorphism f on $N \setminus L$ lifts to a homeomorphism $\bar{f}_1$ of $M \setminus K$. Since $p_{|K} : K \to L$ is a homeomorphism, we can define $\bar{f}_2 : K \to K$ simply by $p^{-1} \circ f_{|L} \circ p$.

$$\begin{array}{ccc} M\setminus K & \xrightarrow{\bar{f}_1} & M\setminus K \\ {\scriptstyle \text{covering}}\downarrow{\scriptstyle \text{map}} & & \downarrow{\scriptstyle p} \\ N\setminus L & \xrightarrow{f_{|N\setminus L}} & N\setminus L \end{array} \qquad \begin{array}{ccc} K & \xrightarrow{\bar{f}_2} & K \\ {\scriptstyle \text{homeomorphism}}\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ L & \xrightarrow{f_{|L}} & L \end{array}$$

The maps $\bar{f}_1$ and $\bar{f}_2$ glue together to a continuous map $\bar{f} : M \to M$. Indeed, continuity has to be checked only at points $x \in K$. In a neighbourhood of every point in K, the topology is induced from the standard metric on $\mathbb{R} \times \mathbb{C}$ via the homeomorphisms in (1). By the pictures in (1) it is clear that the ε-δ-criterion for f "lifts" to $\bar{f}$ (in the appropriate sense), hence $\bar{f}$ is continuous everywhere. □

Proposition 19

Let the closed 3-manifold M have a map to S^3 which is a cyclic branched covering over an amphicheiral link. Then M is amphicheiral (by a diffeomorphism).

Proof. Let L denote the link and let $f : (S^3, L) \to (S^3, L)$ be an orientation-reversing homeomorphism. By Lemma 18, f is covered by a homeomorphism $\bar{f} : M \to M$ which clearly reverses the orientation outside the preimage of the branching set L. Since this submanifold has codimension 2, the orientation reversal on the open manifold $M \setminus p^{-1}(L)$ implies that the orientation is reversed on M. In dimensions ≤ 3, every homeomorphism can be smoothed (preserving the degree), so there is actually an orientation-reversing diffeomorphism. □

The condition "cyclic" in Proposition 19 is necessary: Hilden, Lozano and Montesinos showed that every closed, orientable 3-manifold can be obtained as a branched covering over S^3, branched over the figure-eight knot [HLM]. (The figure-eight knot is called "universal" because of this property.) Since chiral 3-manifolds exist and the figure-eight knot is clearly amphicheiral (Figure 2.2), Proposition 19 cannot hold for arbitrary branched coverings of S^3.

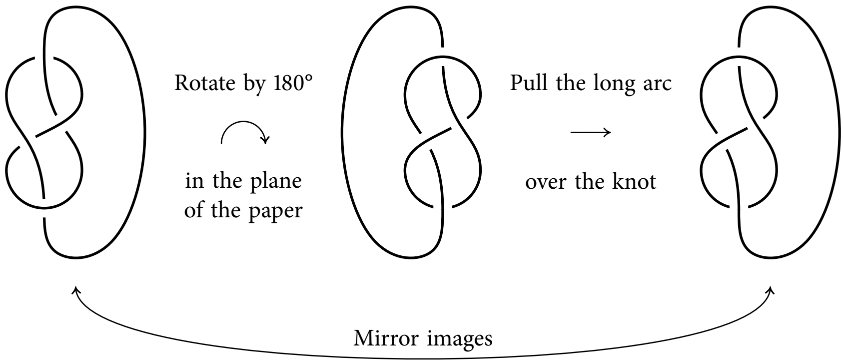


Figure 2.2: The figure-eight knot is amphicheiral [Flapan, p. 22].

Given a link L in S^3, a cyclic k-fold covering which is branched over L can be constructed as follows (for a more detailed description see [Flapan, p. 85 ff.]). Let V be a tubular neighbourhood of L, thus V is a union of solid tori with core L. Denote the complement $S^3 \setminus V$ by Z. Let S be a Seifert surface for L. Cut Z open along the surface $S \cap Z$ and glue k copies of the result together along the cut surface. This is possible since a Seifert surface is oriented, so there are two disjoint copies of the surface in the cut open manifold. The result X is a compact 3-manifold with a union of tori as boundary. We have a k-fold cyclic unbranched covering $X \to Z$, where the projection $\partial X \to \partial Z$ is the covering induced by meridional rotation about $2\pi/k$ on the tori in ∂X. On the other hand, we have the standard k-fold cyclic covering of the tubular neighbourhood V, whose branching set is the link L. This can be pictured again by the cylinders in diagram (1). (Close the cylinder to a solid torus, possibly with a Dehn twist. The spokes of the "wheel" in (1) correspond to a collar of the link in the Seifert surface.) The cyclic covering on the boundary corresponds to the covering $\partial X \to \partial Z$ on the boundary of X, so we can glue the solid tori back on X and obtain the desired closed 3-manifold.

We point out that this construction does not depend on an orientation of the link. The orientation of the constructed manifold is determined by an orientation on S^3.

In general, a branched cyclic covering is not uniquely determined by its base, the branching set and the branching index, but for a branched double covering of S^3, the result is unique up to fibre-preserving homeomorphism ([Flapan,

p. 78], see also [Rolfsen, 10.F.5–10.F.6]; for the way in which an orientation of the link L influences a branched cyclic covering, but not the orientation and not for double coverings, see [Rolfsen, 10.C.2]). We can therefore speak of the branched double covering of a link in S^3 without ambiguity.

Using the contrapositive of Proposition 19, one can show that some knots and links are chiral. The following exemplary facts are not at all new, but it is interesting to see that chirality of knots and links can be proved without any link invariants or link polynomials. These results of Rolfsen refer to an appropriate orientation on S^3, otherwise the negatively oriented manifolds are obtained. The links in question are displayed in Figure 2.3.

- The twofold covering of S^3 branched over the right-hand trefoil is the lens space $L_3(1,1)$ [Rolfsen, 10.D]. Since this lens space is chiral by Corollary 10, the trefoil knot must be, too.
- The twofold covering of S^3 branched over Whitehead's link is the lens space $L_8(1,5)$ [Rolfsen, 10.C.5]. Again, since this lens space is chiral, Whitehead's link must be, too.

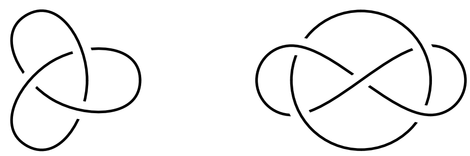


Figure 2.3: The right-hand trefoil and Whitehead's link

We have seen that branched cyclic coverings over an amphicheiral link on S^3 are always amphicheiral manifolds. In the opposite direction, however, chiral knots or links do not necessarily produce chiral manifolds.

Theorem 20: [Kanenobu]

There exist chiral (even prime) knots whose 2-fold branched covering spaces are diffeomorphically amphicheiral.

Surgery

The second construction of amphicheiral 3-manifolds from amphicheiral links is by surgery. A reference for the following explanations is [Saveliev99], in particular Ch. 2.2. For simplicity, the construction is described for a knot but can be extended to links in an obvious manner. Given a knot K in S^3, let $V \cong S^1 \times D^2$ be a tubular neighbourhood and $Z := S^3 \setminus V$. The result Q of the surgery along K is obtained by gluing a solid torus $S^1 \times D^2$ back to Z by a homeomorphism of tori $f : \partial(S^1 \times D^2) \to \partial Z$. The resulting manifold Q is determined up to homeomorphism by the isotopy class of the unoriented curve in ∂Z which is the image of a meridian $\{*\} \times \partial D^2$. This curve is determined by a pair of integers (p, q) as follows: A meridian m of V represents a generator of $H_1(Z) \cong \mathbb{Z}$, which is unique up to isotopy and is called the *canonical meridian*. This is complemented by a second curve l, unique up to isotopy, such that m and l generate $H_1(\partial V)$ and l is the zero element in $H_1(Z)$. The curve l is called the *canonical longitude*.

Furthermore, the curves m and l must be oriented, see [Saveliev99, Ch.2.2]. An orientation of Z is given by the standard orientation on $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$. If m and l intersect in a point $c \in \partial Z$, the orientations on m and l are chosen so that the triple (orientation of m, orientation of l, normal vector on c pointing inside Z) has the same orientation as K. The indeterminacy that (m, l) may be replaced by $(-m, -l)$ is irrelevant, see below.

The meridian $\{*\} \times \partial D^2$ is now isotopic to a curve that winds p times around m and q times around l for two relatively prime integers p, q. Since (p, q) and $(-p, -q)$ represent the same unoriented curve, the pair (p, q) can be represented by a reduced fraction $p/q \in \mathbb{Q} \cup \{\infty\}$. In the special case $p/q = 1/0$, the resulting manifold Q is the original sphere S^3. Quoting Saveliev, "surgeries of the type described are called *rational*. A surgery is called *integral* if $q = \pm 1$."

In the diagrams below, the numbers on each link component designate the winding number $p/1$ for integral surgery. Integral surgeries can also be described by framed links, see [Saveliev99, Ch. 3.1].

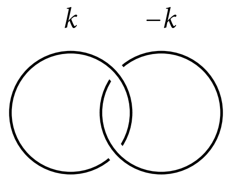
By [Saveliev99, Ch. 3.4], if a manifold Q is obtained by rational surgery on a link L, the manifold with the opposite orientation, $-Q$, is obtained from the mirror image of the link with the negative surgery coefficients. (Saveliev writes only about integral surgery since he deals with the linking form in the same chapter. His arguments are nevertheless valid for rational surgery, too.) The orientation on the link complement is reversed by a reflection, which turns the link into its mirror image. The opposite orientation of the link complement causes also the opposite orientation for the pairs (m, l) of meridian and longitude in the torus boundary components. Therefore, the surgery coefficients change their sign on each link component.

Proposition 21

Consider a link where every component has been assigned a rational number (the surgery coefficients). If the link is ambient isotopic to its mirror image with the negative of all surgery coefficients, the resulting manifold is amphicheiral. In particular, 0-surgery along an amphicheiral link always produces an amphicheiral manifold.

Example 22

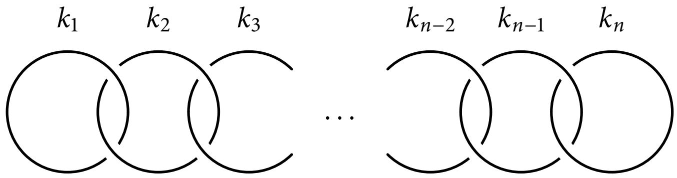
The following link is ambient isotopic to its mirror image with all framing coefficients reversed.



$(k \in \mathbb{Z})$

Therefore, surgery on this link must produce an amphicheiral manifold. In fact, we obtain the lens space $L_{k^2+1}(1, k)$*, which is amphicheiral because* $k \cdot (-k) \equiv 1 \bmod (k^2 + 1)$*. This is proved in* [Saveliev99, 2.3]*. In the special case* $k = 0$*, we have* $L_1(1, 0) = S^3$.

This example is in fact a special case of a chain of arbitrary length: Integral surgery on the link



$(k_i \in \mathbb{Z})$

gives the lens space $L_p(1, q)$, where $p := \varepsilon a_n / \operatorname{gcd}(a_n, b_n)$, $q := \varepsilon b_n / \operatorname{gcd}(a_n, b_n)$, the sign $\varepsilon = \pm 1$ is chosen to make $p \geq 0$ and a_n, b_n are recursively defined by

$$a_1 := k_n, \quad b_1 := -1, \qquad a_i := a_{i-1} k_{n-i+1} + b_{i-1}, \quad b_i := a_{i-1}.$$

If one sets $L_0(1, q) := S^2 \times S^1$, the formula above makes always sense, and the proof in [Saveliev99, 2.3] can be read to encompass the special cases $L_0(1, q)$ and $L_1(1, q)$, too. If $k_i = -k_{n-i+1}$ for all i, this construction yields more amphicheiral lens spaces since the link above is then ambient isotopic to its mirror image with all framing coefficients reversed.

3

Examples in every dimension ≥ 3

In this chapter, we prove our first main result: In every dimension greater than or equal to 3, there is a closed, orientable manifold that is chiral in the strongest sense, i. e. it does not admit a continuous map to itself with degree −1. Later, this theorem will be extended and is in fact contained in Theorem 70, but for a clearer line of thought, we prove the basic statement now.

The theorem is proved in two steps: First, a series of examples in every odd dimension is constructed (Section 3.1). Then we use cartesian products of chiral manifolds to produce even-dimensional chiral manifolds in all dimensions $n \equiv 2 \bmod 4$, $n \geq 6$ (Section 3.2). In dimensions congruent 0 modulo 4, many examples were already known, like the projective spaces $\mathbb{C}P^{2k}$ or any other manifold with nonzero signature.

Given these examples, the list of obstructions to orientation reversal can be extended by novel entries. Since we search for manifolds which are chiral in the strongest sense, the obstructions in this and in the subsequent chapters always forbid orientation-reversing homotopy equivalences or even self-maps. Thus, most of the obstructions will refer to properties which are already manifest in the homotopy type of the manifolds involved.

The new examples in odd dimensions are Eilenberg-MacLane spaces. Consider a self-map $f : X \to X$ of an Eilenberg-MacLane space. Since the homotopy class of f and the effect of f on the homology depends only on the endomorphism of the fundamental group $f_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$, we can add the *fundamental group and its endomorphisms* to the list of obstructions.

Another point of view is suggested by Lemma 28. The new examples are also mapping tori $F \to X \to S^1$ which are twisted in a way such that

- every self-map f of X is homotopic to a fibre-preserving map,
- the degree of f on X is given by the product of the degrees on the fibre F and the base S^1,
- the degrees of the maps on F and S^1 are coupled: if they are ±1 they are either both +1 or both −1.

Since the fibre and the base are both amphicheiral, the obstruction must lie in the *monodromy of a mapping torus*, or more generally spoken in the *twisting of a fibre bundle*.

For the chirality of the even-dimensional examples, we use the *cup product structure* in cohomology in a way which cannot be reduced to the intersection or linking form as in the simpler examples of Chapter 2.

3.1 Examples in every odd dimension ≥ 3

Theorem 23

In every odd dimension ≥ 3, there is a closed orientable manifold that does not admit a continuous map to itself with degree -1.

Examples of such manifolds will be provided by mapping tori of n-dimensional tori $T^n = S^1 \times \ldots \times S^1$. Although the base space S^1 and the fibre T^n are amphicheiral, the fibration is twisted in a way that makes orientation reversal impossible. We can exclude orientation-reversing maps by studying the automorphisms of the fundamental group. Therefore, we first work out some properties of the kind of groups which we will encounter as fundamental groups.

Lemma 24

Abelianisation is a right exact functor.

Although abelianisation as a functor is left adjoint to the inclusion of abelian groups into all groups, the simple category-theoretic argument "left adjoint functors are right exact" only applies to abelian categories. Therefore, the exactness property is checked *ad hoc*.

Proof. Clearly, abelianisation is functorial. Consider the following commutative diagram, where the middle row is assumed to be exact and the columns are exact by definition of the commutator subgroup and abelianisation. The surjectivity of the map $[B, B] \to [C, C]$ follows from the surjectivity of $B \to C$.

$$
\begin{array}{ccccccc}
 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \\
 & & [B,B] & \longrightarrow & [C,C] & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \\
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 1 \\
\downarrow & & \downarrow & \circledast & \downarrow & & \\
A^{ab} & \longrightarrow & B^{ab} & \longrightarrow & C^{ab} & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
1 & & 1 & & 1 & &
\end{array}
$$

The homomorphism $B^{ab} \to C^{ab}$ is surjective since the composition in the square ⊛ is surjective. The composition at B^{ab} is trivial since every element of A^{ab} lifts to A and the middle row is exact.

If an element $[b] \in B^{ab}$ maps to $e \in C^{ab}$, its representative $b \in B$ maps to a product of commutators $\prod_i [c_i, c_i'] \in C$. Let b_i resp. b_i' be preimages of c_i resp. c_i' in B. Then $b - \prod_i [b_i, b_i']$ has the same image $[b]$ in B^{ab} but maps to the neutral element in C. Hence it has a preimage $a \in A$. Its vertical image $[a] \in A^{ab}$ is a horizontal preimage for $[b] \in B^{ab}$. □

Lemma 25

Let the group G be a semidirect product of its subgroups N and H, i. e. there is a split extension

$$0 \to N \to G \overset{\curvearrowleft}{\to} H \to 0.$$

Since N is a normal subgroup of G, the commutators $[g, n]$ with $g \in G$, $n \in N$ lie in N. If these commutators generate N^{ab} then abelianisation induces an isomorphism $G^{ab} \to H^{ab}$.

Proof. Lemma 24 results in a diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & N^{ab} & \longrightarrow & G^{ab} & \longrightarrow & H^{ab} & \longrightarrow & 0. \end{array}$$

Each commutator $[g, n]$ is zero in G^{ab} but not necessarily in N^{ab}. Since N^{ab} is generated by these elements, the map $N^{ab} \to G^{ab}$ is the zero homomorphism. □

Corollary 26

Given a semidirect product $G \cong N \rtimes H$, if $[G, N] = N^{ab}$ and H is abelian, the normal subgroup N is the commutator subgroup $[G, G]$, and G is a semidirect product $G \cong [G, G] \rtimes G^{ab}$. The splitting map $G^{ab} \hookrightarrow G$ can be chosen as the old splitting map $H \hookrightarrow G$ precomposed with the isomorphism $G^{ab} \to H^{ab} = H$.

In our examples, H will be isomorphic to $\mathbb{Z}$. Every extension of a free group splits, so an exact sequence always yields a semidirect product. Define a homomorphism

$$\psi : H \to \mathrm{Out}(N), \quad h \mapsto [n \mapsto s(h) n s(h)^{-1}].$$

This definition might depend on the choice of a splitting $s : H \to G$. However, since two choices of $s(h)$ differ by an element of N, we obtain a well defined map ψ to the outer automorphism group.

If we restrict N to abelian groups, the distinction between $\mathrm{Aut}(N)$ and $\mathrm{Out}(N)$ becomes superfluous. Given a splitting map $s : H \to G$, we get a well-defined relation in G

$$s(h)n = \psi(h)(n)s(h) \quad \text{for all } h \in H,\ n \in N. \tag{1}$$

Now consider an endomorphism T of G. If $[G,N] = N^{ab}$, the normal subgroup N is preserved by T because it is the commutator subgroup $[G,G]$. Let p denote the given map $G \to H$. This defines another endomorphism $T_H := (p \circ T \circ s) : H \to H$. Note that T_H does not depend on the choice of s since T preserves $\ker p = [G,G]$.

Furthermore, T has to preserve the above relation (1):

$$\begin{aligned} & T(s(h)n) = T(\psi(h)(n)s(h)) \\ \Rightarrow \quad & T(s(h))T_{|N}(n)T(s(h))^{-1} = T_{|N}(\psi(h)(n)) \end{aligned}$$

From the relation $p \circ s = \mathrm{id}_H$ it follows that $T(s(h))$ and $s(T_H(h))$ differ by an element $x \in \ker p = N$, i.e. $T(s(h)) = s(T_H(h))x$. Since N is abelian, conjugation $xT_{|N}(n)x^{-1}$ by this element has no effect, and we get

$$\begin{aligned} & s(T_H(h))T_{|N}(n)s(T(h))^{-1} = T_{|N}(\psi(h)(n)) \\ \Rightarrow \quad & \psi(T_H(h))(T_{|N}(n)) = T_{|N}(\psi(h)(n)). \end{aligned}$$

The last line is independent of s. In summary, we have proved the following

Proposition 27

Let $0 \to N \to G \to H \to 0$ be an exact sequence of groups, with N being abelian and $H \cong \mathbb{Z}$. Define the homomorphism $\psi : H \to \mathrm{Aut}(N)$ as above by conjugation. Suppose also that $(\psi(h) - \mathrm{id}_N)$ is surjective for some $h \in H$[1].

A necessary condition for $T_{|N} : N \to N$ and $T_H : H \to H$ being induced from an endomorphism of G is

$$\psi(T_H(h))(T_{|N}(n)) = T_{|N}(\psi(h)(n)) \quad \text{for all } n \in N,\ h \in H.$$

Keeping this condition in mind for later, we now construct manifolds fitting into this algebraic setting. As mentioned above, we will consider mapping tori of n-dimensional tori $T^n := S^1 \times \ldots \times S^1$.

Let $f : T^n \to T^n$ be an orientation-preserving diffeomorphism and define the mapping torus M_f as the quotient space

$$M_f := T^n \times [0,1] \,\big/\, (x,0) \sim (f(x),1). \tag{2}$$

[1] Cf. Lemma 25, where a weaker condition was used.

M_f is a closed, connected, orientable manifold of dimension $n+1$. Note that M_f is a fibration, where the base space as well as the fibre are $K(\pi,1)$-manifolds. From the exact sequence of homotopy groups it follows that M_f itself is a $K(\pi,1)$-manifold, and its fundamental group fits into the exact sequence

$$\begin{array}{ccccccccc} 0 & \to & \pi_1(T^n) & \to & \pi_1(M_f,x_0) & \to & \pi_1(S^1) & \to & 0 \\ & & \wr\| & & & & \wr\| & & \\ & & \mathbb{Z}^n & & & & \mathbb{Z} & & \end{array}$$

The basepoint $x_0 \in M_f$ is given below, when we specify the CW-structure. For $\pi_1(T^n)$ and $\pi_1(S^1)$, the basepoint is not specified since these groups are abelian.

We orient the circle such that the positive generator $r \in H_1(S^1)$ follows the cycle $[0,1]/\{0,1\}$ in the positive direction. We now want to show that

$$\psi(r) = f_* : H_1(T^n) \to H_1(T^n).$$

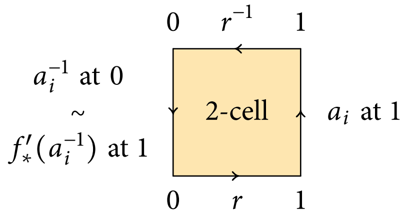
Note that we sometimes replace $\pi_1(S^1)$ and $\pi_1(T^n)$ by $H_1(S^1)$ and $H_1(T^n)$ resp. to highlight the fact that these are abelian groups.

Give S^1 its usual CW-structure with one 0-cell and one 1-cell, and T^n the corresponding product CW-structure. We replace f by a homotopic cellular map f' so that $M_{f'}$ is not necessarily a manifold but is homotopy equivalent to M_f. For convenience, let the basepoints of T^n, $M_{f'}$ and S^1 be their respective unique 0-cells. All fundamental groups in the following will refer to these basepoints.

The subcomplex $T^n \vee S^1 \subset M_{f'}$ has fundamental group $\pi_1(T^n) * \pi_1(S^1) \cong \mathbb{Z}^n * \mathbb{Z}$. Let a_i $(i = 1,\dots,n)$ be a system of generators for $\pi_1(T^n)$ and r the positive generator of $\pi_1(S^1)$. The 2-cells of the relative CW-complex $(M_{f'}, T^n \vee S^1)$ generate the relations

$$ra_i = f'_*(a_i)r,$$

as can be seen from the following scheme using the identification made in (2). Comparison with (1) shows that $\psi(r)$ is indeed equal to $f_* = f'_*$.



Now let $T : M_f \to M_f$ be a continuous map. The map T is homotopic to a basepoint-preserving map, so we assume this property w. l. o. g. Since M_f is a $K(\pi,1)$-manifold, the homotopy class of T and thus the effect on orientation is determined by the induced map on $\pi_1(M_f, x_0)$. We write T_* for the induced map on homology and homotopy in any degree.

Lemma 28

Let $f : T^n \to T^n$ be an orientation-preserving diffeomorphism such that $f_* - \mathrm{id}$ is surjective on $H_1(T^n)$. Let $T : M_f \to M_f$ be a continuous map. The induced map T_* on $H_{n+1}(M_f) \cong \mathbb{Z}$ is given by $\det((T_*)_{|N}) \cdot \det((T_*)_H)$, where in our case, $N = \pi_1(T^n) \cong \mathbb{Z}^n$ and $H = \pi_1(S^1) \cong \mathbb{Z}$.

Proof. The strategy is to show that T is homotopic to a fibre-preserving map and then to exploit naturality of the Serre spectral sequence.

Consider the diagram

$$\begin{array}{ccc} M_f & \xrightarrow{T} & M_f \\ \downarrow p & & \downarrow p \\ S^1 & \overset{t}{\dashrightarrow} & S^1 \end{array}$$

where p is the projection in our fibre bundle and t has to be defined.

Since there is a natural bijection $[M_f, S^1] \cong H^1(M_f)$, which is isomorphic to $\mathbb{Z}$, this diagram commutes up to homotopy, with the map t being any (basepoint-preserving) map with the correct degree.

By the homotopy lifting property, T is homotopic to a fibre-preserving map (and still preserving the basepoint), so we can replace T w. l. o. g. by this map.

Now we are in the situation of a commutative diagram

$$\begin{array}{ccccc} T^n & \hookrightarrow & M_f & \xrightarrow{p} & S^1 \\ \downarrow T_{|T^n} & & \downarrow T & & \downarrow t \\ T^n & \hookrightarrow & M_f & \xrightarrow{p} & S^1 \end{array}$$

so that we can apply the naturality of the Serre spectral sequence. To be precise, we consider the E^2 term of the homology spectral sequence for the fibration p. The only term with total degree at least $n+1$ is $E^2_{1,n} = H_1(S^1; H_n(T^n))$. *A priori,* the coefficients are local, but since we specified $f : T^n \to T^n$ as orientation-preserving, the coefficient group is in fact constant. Since there are no differentials from or to $E^2_{1,n}$, we have $E^2_{1,n} = E^\infty_{1,n}$. Since there are no other terms in degree $n+1$, we have a natural isomorphism $H_1(S^1; H_n(T^n)) \cong H_{n+1}(M_f) \cong \mathbb{Z}$. The word "natural" here refers to fibre-preserving maps of M_f, as always in the context of the Serre spectral sequence. Note that the map $t_* : H_1(S^1) \to H_1(S^1)$ coincides with $(T_*)_H$. Furthermore, $(T_{|T^n})_* = (T_*)_{|N}$ is given by the determinant of the map on $\pi_1(T^n) \cong \mathbb{Z}^n$ as is proved by the cohomology product structure of the n-torus.

The induced map on $H_1(S^1; H_n(T^n))$ is the tensor product of the two maps above, hence the lemma is proved. □

Having chosen a basis for $H_1(T^n) \cong \mathbb{Z}^n$, every invertible matrix $A \in SL(n, \mathbb{Z})$ can be realised as the induced map on $H_1(T^n)$ of an orientation-preserving diffeomorphism $f : T^n \to T^n$. Hence, we can construct a chiral $(n+1)$-manifold under the following circumstances:

Lemma 29

Suppose there is a matrix $A \in SL(n, \mathbb{Z})$ such that

(a) $\det(A - \mathrm{id}) = \pm 1$,

(b) the equation $AB = BA$ has no solution $B \in GL(n, \mathbb{Z})$, $\det B = -1$,

(c) the equation $A^{-1}B = BA$ has no solution $B \in SL(n, \mathbb{Z})$.

Then a mapping torus M_f with $f : T^n \to T^n$ realising A on $H_1(T^n) \cong \mathbb{Z}^n$ has no map onto itself with degree −1.

Proof. This is a consequence of Proposition 27 and Lemma 28. For the reader's convenience, we list the correspondence between the notations here and in Proposition 27:

$$A = f_* = \psi(1), \qquad A^{-1} = \psi(-1),$$
$$T_H(1) = \begin{cases} +1 \\ -1 \end{cases}, \qquad T_{|N} = B. \qquad \square$$

For odd n, this method fails because $B = -A$ is a solution for equation (b). This is the reason why this approach does not yield examples in even dimensions $n + 1$. For even $n \geq 2$, we construct an example in each dimension, thus proving Theorem 23.

It is shown that every matrix $A \in M(n \times n; \mathbb{Z})$, for n even, with characteristic polynomial

$$\chi_A(X) = X^n - X + 1$$

fulfills the lemma. Such a matrix is given, e. g., by the following scheme:

$$A := \left(\begin{array}{c|cc} 0 & \multicolumn{2}{c}{I_{n-1}} \\ \hline -1 & 1 & 0 \end{array}\right)$$

The value $\chi_A(0) = 1$ guarantees $A \in SL(n, \mathbb{Z})$, while $\chi_A(1) = 1$ ensures condition (a). Next we show that there is no solution to equation (b). The matrix A has no real eigenvalues. Indeed, $\chi_A(X)$ is always positive for real X, which can be shown easily. Since the coefficients are real, the zeros occur as pairwise conjugate complex numbers

$$\lambda_1, \bar{\lambda}_1, \ldots, \lambda_{n/2}, \bar{\lambda}_{n/2} \in \mathbb{C} \setminus \mathbb{R}.$$

We claim that the zeros are distinct, i. e. χ_A has no multiple zeros over $\mathbb{C}$. For this, we have to show that χ_A and χ'_A have no common zeros. We have

$$\chi'_A(X) = 0 \quad \Leftrightarrow \quad nX^{n-1} - 1 = 0 \quad \Leftrightarrow \quad X^{n-1} = \frac{1}{n}.$$

Insert this into $\chi_A(X) = 0$:

$$\frac{1}{n}X - X + 1 = 0 \quad \Leftrightarrow \quad X = \frac{n}{n-1}.$$

The latter is a contradiction to $X \notin \mathbb{R}$, hence all zeros of χ_A are distinct. Thus, A is diagonisable (over $\mathbb{C}$), and we have

$$\det A = \prod_{i=1}^{n/2} \lambda_i \bar{\lambda}_i = \prod_{i=1}^{n/2} |\lambda_i|^2.$$

Suppose now that $AB = BA$, $\det B = -1$. Expressing this algebraically, B lies in the centraliser of A, written $B \in Z(A)$. We use the following lemma, whose proof is postponed until we have finished Theorem 23, since the lemma is purely algebraic and valid in a general setting.

Lemma 30

Let $A \in M(n \times n; K)$ for any field K and suppose A has distinct eigenvalues in an algebraic closure $\overline{K}$. Then

$$B \in Z(A) \quad \Leftrightarrow \quad B \text{ is polynomial in } A \text{ (with coefficients in } K).$$

By this lemma, B is also diagonisable. Let $p \in \mathbb{Q}[X]$ be a polynomial such that $B = p(A)$. Then B has eigenvalues $p(\lambda_i), p(\bar{\lambda}_i)$ and determinant

$$\det B = \prod_{i=1}^{n/2} p(\lambda_i)\overline{p(\lambda_i)} = \prod_{i=1}^{n/2} |p(\lambda_i)|^2 \geq 0$$

This contradicts $\det B = -1$, hence equation (b) has no solution.

Now we show that there is no solution to equation (c). Suppose $BA = A^{-1}B$ for some $B \in SL(n, \mathbb{Z})$. Then $BAB^{-1} = A^{-1}$, i.e. A and A^{-1} are similar. Hence, they have the same eigenvalues. Since we have

$$\lambda \text{ is an eigenvalue of } A \quad \Leftrightarrow \quad \lambda^{-1} \text{ is an eigenvalue of } A^{-1},$$

we have

$$\chi_A(\lambda) = 0 \quad \Leftrightarrow \quad \chi_A(\lambda^{-1}) = 0.$$

Hence, $\chi_A(X) = X^n - X + 1$ and $X^n \chi_A(X^{-1}) = X^n - X^{n-1} + 1$ must have the same zeros. Note that we can neglect 0 as a possible eigenvalue/zero of the polynomials.

Since we have seen that all complex zeros of $\chi_A(X)$ are distinct, the second polynomial cannot have the same zeros for $n > 2$.

For $n = 2$, the two eigenvalues of A are in fact inverse to each other. One shows by hand that $BA = A^{-1}B$ implies that B is of the form

$$B = \begin{pmatrix} a & b \\ a+b & -a \end{pmatrix} \quad \text{with } a, b \in \mathbb{Z}.$$

Then $\det B = -(a^2 + ab + b^2) = -\frac{1}{2}(a^2 + (a+b)^2 + b^2)$, which is never positive. Thus, there is no solution to (c) in any case. With this argument, the proof of Theorem 23 is complete. □

We still have to prove Lemma 30. The implication "⇐" is obvious. For the opposite direction, we work over a splitting field $L \supseteq K$ of χ_A. Since A has distinct eigenvalues, all eigenspaces are one-dimensional. Furthermore, since B commutes with A, it respects these eigenspaces. Fix a basis of L^n for which A is diagonal. Then B is also diagonal with respect to this basis. Since A has distinct eigenvalues λ_i, the Vandermonde matrix $M = (m_{i,j})$ with $m_{i,j} := \lambda_i^j$ ($i = 1, \ldots, n$; $j = 0, \ldots, n-1$) is invertible. Thus, we can find coefficients $\mathbf{p} = (p_0, \ldots, p_{n-1}) \in L^n$ such that $M\mathbf{p} = \mathbf{b}$, where $\mathbf{b}$ is the vector consisting of the eigenvalues of B. This means exactly $B = p(A)$, where p is the polynomial with coefficient vector $\mathbf{p}$.

At this stage, B is a polynomial expression in A but maybe with coefficients in L, not just K. Since $L \supseteq K$ is a splitting field of a separable polynomial (i. e. a polynomial without multiple roots), it is a Galois extension [Lang, V.3 and V.4].

Let the Galois group $\mathrm{Aut}_K(L)$ act on matrices componentwise and on polynomials coefficientwise and let σ be an arbitrary element in the Galois group. Then

$$p(A) = B = B^\sigma = (p(A))^\sigma = p^\sigma(A^\sigma) = p^\sigma(A).$$

Since all eigenvalues of A are distinct, the characteristic polynomial χ_A of degree n is the minimal polynomial, and the powers $A^0, \ldots, A^{n-1}$ are linearly independent over L. Hence, the above equation implies that all coefficients of p are invariant under the action of σ. By the fundamental theorem of Galois theory, the coefficients lie in K. □

3.2 Products of chiral manifolds

The chiral manifolds that we have constructed in odd dimensions can be used to obtain examples in even dimensions. In dimensions which are divisible by four, the signature is a well-known obstruction to self-maps of degree −1, so the dimensions in which truly new information is obtained are those congruent 2 modulo 4 starting from 6.

We show that under certain conditions, products of chiral manifolds are again chiral. This method will be reused to prove the existence of simply-connected, chiral manifolds in all higher dimensions. In order to bundle similar ideas in one place, we do not stop when Theorem 23 is finally proved but continue with the construction of simply-connected chiral manifolds in all but a few dimensions. Chapter 4 is then dedicated to those more complicated cases.

Theorem 31

Let Σ be a rational homology sphere and M a closed, connected, orientable manifold of the same dimension which is not a rational homology sphere. If neither of these manifolds admits a map to itself of degree -1, then neither does the product $\Sigma \times M$.

The somewhat peculiar condition that M is *not* a rational homology sphere becomes clear in Lemma 33. This lemma forms the core of the argument and is itself an old and beautiful application of Poincaré duality.

Example 32

Let Σ be a lens space of dimension $n \equiv 3 \bmod 4$ with fundamental group of prime order $p \equiv 3 \bmod 4$, and let M be a chiral mapping torus of the same dimension, as constructed in the notes before. This yields examples of chiral manifolds in each dimension congruent 6 modulo 8.

Proof. We have $H_1(M) = \pi_1(M)^{ab} \cong \mathbb{Z}$, so M is not a rational homology sphere. □

Proof of Theorem 31. Let n be the dimension of Σ and M. By the Künneth theorem, we have

$$H^n(\Sigma \times M)/(\text{torsion}) \cong H^n(\Sigma) \oplus H^n(M) \cong \mathbb{Z}^2. \tag{3}$$

Consider the cohomology classes in $H^n(\Sigma)$ and $H^n(M)$ that are Kronecker dual to the fundamental classes $[\Sigma]$, $[M]$ and denote their images in the free quotient $H^n(\Sigma \times M)/(\text{torsion})$ by $[\Sigma]^*$ resp. $[M]^*$.

Let $T : \Sigma \times M \to \Sigma \times M$ be a continuous map. The effect on $H^n(\Sigma \times M)/(\text{torsion})$ is given (with respect to the basis $[\Sigma]^*$, $[M]^*$) by an integral matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since $[\Sigma]^* \cup [M]^*$ is a generator of $H^{2n}(\Sigma \times M) \cong \mathbb{Z}$, the mapping degree of T is given by $ad + (-1)^n bc$.

Now denote the usual inclusions and projections by

$$\begin{aligned} i_\Sigma &: \Sigma \to \Sigma \times M, & i_M &: M \to \Sigma \times M, \\ p_\Sigma &: \Sigma \times M \to \Sigma, & p_M &: \Sigma \times M \to M. \end{aligned}$$

Since the first isomorphism in (3) is induced by these inclusions and projections, a can be recovered, for example, as the degree of $p_\Sigma \circ T \circ i_\Sigma : \Sigma \to \Sigma$, and b is the degree of

$$f := (p_M \circ T \circ i_\Sigma) : \Sigma \to M.$$

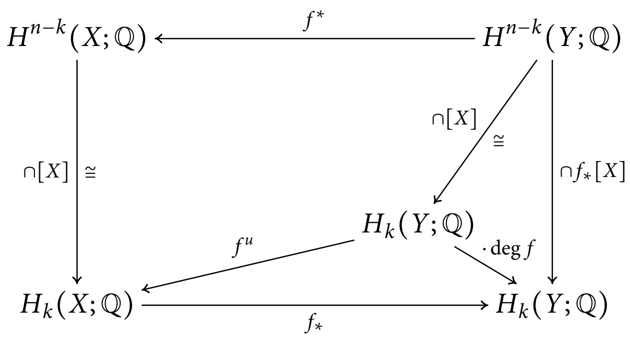
Now we use the following

Lemma 33: [Hopf, Satz IIIa]

Let $f : X \to Y$ be a map of n-dimensional closed, connected, orientable manifolds. If f has nonzero degree, then the Betti numbers of X are greater than or equal to the Betti numbers of Y.

This lemma goes back to Hopf's seminal paper on the Umkehr homomorphism. In modern mathematical language, a proof can be given in a few lines, see below. By the contrapositive of this lemma, $b = 0$, so the degree of T is equal to the product ad. Since neither of the factors can be -1 by assumption, T cannot reverse the orientation. □

Proof of Lemma 33. Let $f^u : H_*(Y;\mathbb{Q}) \to H_*(X;\mathbb{Q})$ be the Umkehr homomorphism, which is defined by the induced map $f^* : H^*(Y;\mathbb{Q}) \to H^*(X;\mathbb{Q})$ on cohomology and Poincaré duality on X and Y. Then $f_* \circ f^u : H_*(Y;\mathbb{Q}) \to H_*(Y;\mathbb{Q})$ is multiplication by the degree of f, as can be seen from the following commutative diagram. (A very similar diagram was shown in Section 1.3 to prove that a self-map of degree ± 1 of a simply-connected manifold is a homotopy equivalence.)



Hence, if $\deg f$ is nonzero, f_* must be surjective. □

If Σ and M have different dimensions, chirality of the product is even easier to prove.

Theorem 34

Let Σ be a rational homology sphere of dimension s and M a closed, connected, orientable manifold of different dimension $m \neq s$. Also require that $H^s(M;\mathbb{Q}) = 0$. If neither of these manifolds admits a map to itself of degree -1, then neither does the product $\Sigma \times M$.

Proof. By the rational Künneth theorem, we have

$$H^m(\Sigma \times M) \cong H^m(M) \oplus (H^s(\Sigma) \otimes M^{m-s}(M)),$$

all understood with rational coefficients. In degree s, we have $H^s(\Sigma \times M) \cong H^s(\Sigma)$ since $H^s(M) = 0$. Then, as before, we know that the degree of any map $T : \Sigma \times M \to \Sigma \times M$ is given by the product of the degrees of

$$p_\Sigma \circ T \circ i_\Sigma : \Sigma \to \Sigma \quad \text{and} \quad p_M \circ T \circ i_M : M \to M.$$

Since neither of those degrees can be -1, T cannot reverse orientation. □

Example 35

Let $d_1 > \ldots > d_k$ be positive integers and let L_j ($j = 1, \ldots, k$) be lens spaces of dimension $4d_j - 1$ which are homotopically chiral (e. g. when $|\pi_1(L_j)|$ is a prime congruent 3 mod 4). Then the product manifold $L_1 \times \ldots \times L_k$ admits no orientation-reversing self-map.

Proof. This follows by applying Theorem 34 several times. □

This last example finishes the construction of chiral manifolds in dimensions ≥ 3. (As was pointed out in Chapter 2, the point is a chiral 0-dimensional manifold, and all manifolds in dimensions 1 and 2 are amphicheiral.)

- Theorem 23 treats the odd dimensions ≥ 3.
- In dimensions which are divisible by 4, there are plenty of examples with nonzero signature.
- Dimension 6 is dealt with in Example 32.
- Finally, all dimensions which are congruent 2 mod 4 and at least 10 are handled by Example 35.

Example 35 allows a strong conclusion:

Theorem 36

Let $L = L_1 \times \ldots \times L_k$ be a product of lens spaces of pairwise different dimensions. Then L is homotopically chiral if and only if this holds for each single factor.

Proof. This follows immediately from Example 35 since in all other cases, L is clearly amphicheiral. □

Note that the condition of the theorem can be easily tested by Corollary 10.

We now want to apply this approach to simply-connected chiral manifolds and cover as many dimensions as possible. As "starting dimension" we cannot use 3 as in the examples with nontrivial fundamental group. Instead, we construct the "building blocks" from dimension 7 on.

Proposition 37

For every even integer $k \geq 4$, there is a $(k-2)$-connected $(2k-1)$-dimensional rational homology sphere that does not admit an orientation-reversing homotopy self-equivalence.

Corollary 38

This provides us with homotopically chiral, simply-connected manifolds in every dimension $n \equiv 3 \bmod 4$ starting from $n = 7$.

Proof. Let $n := 2k - 1$. We exhibit a closed, simply-connected n-dimensional manifold M with the following integral cohomology:

$$H^i(M) = \begin{cases} \mathbb{Z} & i = 0, n \\ \mathbb{Z}/6 & i = k \\ 0 & \text{else.} \end{cases}$$

Since -1 is not a quadratic residue modulo 6, the linking form immediately forbids a self-homotopy equivalence of this degree.

The modulus 6 was chosen only for simplicity; every multiple of 4 or $2p$, where p is a prime congruent 3 mod 4, would do. Note that our construction could provide an odd modulus only in dimensions with $k \in \{4, 8\}$, due to the Hopf invariant one problem (see [Kosinski, Appendix, Prop. 5.2])[2)]. We construct M as a linear S^{k-1}-bundle over S^k (that is, the sphere bundle of a Riemannian vector bundle of rank k). Let E be the total space of the pull-back of the tangent bundle TS^k under a smooth map $S^k \to S^k$ of degree 3. TS^k has Euler class $2[S^k]^*$, where $[S^k]^*$ denotes the Kronecker dual of the fundamental class. Because of naturality, E has Euler class $e = 6[S^k]^*$.

Let $M := SE$, the associated sphere bundle with respect to some Riemannian metric on E. The long exact Gysin sequence [HatcherAT, 4.D]

$$\ldots \to H^{i-k}(S^k) \xrightarrow{\cup e} H^i(S^k) \to H^i(M) \to H^{i-k+1}(S^k) \to \ldots$$

immediately gives the announced cohomology groups. M is simply-connected because the base space and the fibre are. □

Alternatively, M can be described as the boundary of a $2k$-dimensional handlebody with one handle of index k. Details of this construction can be found in [Kosinski, Ch. VI.12].

For completing the examples of simply-connected chiral manifolds in higher dimensions, we still need simply-connected homotopically chiral 7-manifolds that are *not* rational homotopy spheres. Explicitly, we define N_1 to be the connected sum of $S^3 \times S^4$ with a 7-manifold M^7 as constructed above. Similarly, $N_2 := (S^2 \times S^5)\#M^7$. These manifolds have the following homology groups:

[2)] Note that the arrow ϕ_* in the $\times$-shaped diagram on p. 231 must point in the opposite direction. Another reference is [HatcherVBKT, p. 93 in Section 3.2].

i	$H_i(N_1)$	$H_i(N_2)$
0	$\mathbb{Z}$	$\mathbb{Z}$
1	0	0
2	0	$\mathbb{Z}$
3	$\mathbb{Z} \oplus \mathbb{Z}/6$	$\mathbb{Z}/6$
4	$\mathbb{Z}$	0
5	0	$\mathbb{Z}$
6	0	0
7	$\mathbb{Z}$	$\mathbb{Z}$

They are chiral because of the linking form.

Corollary 39

In every dimension $n \equiv 2 \bmod 4$ starting from 14, there is a homotopically chiral simply-connected manifold.

Proof. In dimension 14, take the product of a simply-connected 7-dimensional rational homology sphere M^7 from Proposition 37 with either N_1 or N_2 and apply Theorem 31. In higher dimensions, use products of two rational homology spheres of different dimensions congruent 3 mod 4 and apply Theorem 34. □

Corollary 40

In every dimension $n \equiv 1 \bmod 4$ starting from 21, there is a homotopically chiral simply-connected manifold.

Proof. From dimension 25 on, we can take the product of the 14-dimensional manifold $M^7 \times N_2$ from the previous corollary with a rational homotopy sphere from Proposition 37. Note that, according to the rational Künneth theorem, $M^7 \times N_2$ has no rational homology in degree 11 (and of course not in higher degrees congruent 3 mod 4). Thus, Theorem 34 applies.

For dimension 21, consider $M^{21} := M^7 \times N_1 \times N_2$. Here, we can argue in a similar way as in the proof of Theorem 31, but we have to consider products of three manifolds instead of two. Lemma 33 shows that every map $M^7 \to N_1$ must have degree zero, likewise every map $M^7 \to N_2$ and $N_1 \to N_2$. One checks with the Künneth theorem that in our case, $H_7(M^{21};\mathbb{Q}) \cong \mathbb{Q}^3$, with basis $[M^7]^*$, $[N_1]^*$ and $[N_2]^*$. Every automorphism of $H_7(M^{21};\mathbb{Q})$ which is induced from a self-map of M^{21} is then given by a lower triangular (3×3)-matrix. Thus, the induced map on $[M^{21}]^* = [M^7]^* \cup [N_1]^* \cup [N_2]^*$ is given by the product of the three degrees, neither of which can be -1. □

4

Simply-connected chiral manifolds

In the previous chapter, it was shown that chiral manifolds exist in all dimensions greater than two. Apart from the well-known examples, where the intersection and linking forms posed obstructions to amphicheirality, the newly constructed chiral manifolds are all Eilenberg-MacLane spaces or contain them as factors. Since all the new examples depend so strongly on the fundamental group, it is natural to ask what other factors can influence chirality/amphicheirality. A fundamental question is in which dimensions there are *simply-connected* chiral manifolds.

This problem is all the more interesting as the answer differs in low dimensions from the case of arbitrary fundamental group in the last chapter.

Theorem 41

In dimensions 3, 5 and 6, every simply-connected, closed smooth (or PL or topological) manifold is amphicheiral in the respective category.

A closed, simply-connected, topological 4-manifold is topologically amphicheiral if its signature is zero. If the signature is nonzero, the orientation cannot be reversed, not even by a homotopy equivalence.

Classification results for simply-connected manifolds exist in dimensions up to 6. These far-reaching results are reviewed in Section 4.1, and apart from dimension 6 the corollaries about chirality and amphicheirality are obtained immediately. The classifying invariants in dimension 6 are considerably more complicated. In Section 4.1.4, the necessary arguments are provided to deduce that all simply-connected 6-manifolds are amphicheiral.

Together with Theorem 41, the main result of this chapter is the following theorem:

Theorem 42

In every dimension ≥ 7 there is a closed, simply-connected, smooth manifold which does not admit an orientation-reversing homotopy equivalence.

The aim was again to produce manifolds which are chiral in the strongest possible sense, so we present homotopically chiral, simply-connected manifolds.

In Section 3.2 (corollaries 38 to 40), such manifolds in all dimensions congruent 1, 2 and 3 mod 4, except 9, 10, 13 and 17, were already constructed. The complex projective spaces $\mathbb{C}P^{2k}$ clearly provide simply-connected examples in dimensions congruent 0 mod 4. Homotopically chiral, simply-connected manifolds in the remaining dimensions are proved to exist in Section 4.2 (dimensions 10 and 17) and Section 4.3 (dimensions 9 and 13).

Again, there are new obstructions to orientation reversal. For the 10- and 17-dimensional manifolds, the obstruction lies in the interplay between the cup product structure and *cohomology operations*. This may not come as a surprise since the linking form (see Appendix A.1) can also be regarded as a combination of the cup product and the Bockstein homomorphisms. We establish that other cohomology operations, in our case the mod-3 Steenrod operations, can be used to prove chirality in previously unknown cases.

In the case of the 9- and 13-dimensional examples, we construct a three-stage Postnikov approximation with appropriate k-invariants. In order to handle the automorphisms of a three-stage Postnikov tower, we use rational homotopy theory and the minimal model for the rational cohomology to make the problem accessible. The rational Postnikov tower, however, is not enough in our case, and we must also extract information from the integral Postnikov tower in order to restrict the possible automorphisms. For the list of obstruction to orientation reversal, we propose the item *"the structure of the rational minimal model plus information from the integral Postnikov tower"*.

We remark that the more general *"structure of the partial Postnikov tower"* is important both as an obstruction to orientation reversal and as a condition for amphicheirality. The Postnikov tower, from a certain stage on, is certainly an obstruction for the chiral manifolds in dimensions 9, 10, 13 and 17 which we construct. However, there is also a complementary proof in this chapter. In our review of simply-connected 6-manifolds, a crucial ingredient is the proof that there is *no* obstruction in the first stage of the Postnikov tower, a $K(G,2)$ (see Lemma 46).

4.1 Results in low dimensions

The results are deduced in the following subsections from the classification theorems in each dimension.

4.1.1 Dimension 3

By the Poincaré conjecture, which was proved by Perelman, every closed, smooth, simply-connected 3-manifold is diffeomorphic to S^3 (see [MT]). Hence, all closed, smooth, simply-connected 3-manifolds are smoothly amphicheiral. Since every topological 3-manifold admits a smooth structure, every closed, simply-connected topological (or PL) 3-manifold is topologically amphicheiral resp. PL-amphicheiral.

4.1.2 Dimension 4

Simply-connected topological 4-manifolds are classified up to homeomorphism by the intersection form. More precisely, we have the following statement

Theorem 43: Part (2) of [FQ, 10.1]

Let M, N be closed, simply-connected 4-manifolds and $h : H_2(M) \to H_2(N)$ be an isomorphism which preserves the intersection form. Moreover, suppose that the Kirby-Siebenmann invariants of M and N are equal. Then there is a homeomorphism $f : M \to N$, unique up to isotopy, such that $f_* = h$.

The Kirby-Siebenmann invariant is in general an element in $H^4(M, \partial M; \mathbb{Z}/2)$. It does not depend on the orientation of the manifold [Rudyak]. For a closed, connected 4-manifold, the Kirby-Siebenmann invariant is simply an element in $\mathbb{Z}/2$.

It was pointed out in Proposition 3 that a nondegenerate symmetric bilinear form Q over $\mathbb{Z}$ is isomorphic to its negative if and only if the signature is zero. By Poincaré duality and the universal coefficient theorems, $H_2(M)$ is a finitely generated free abelian group. If $H_2(M) \neq 0$, the orientation on M is determined by the intersection product. Indeed, any two elements $x, y \in H_2(M)$ with $Q(x, y) \neq 0$ determine the orientation because the intersection product reverses its sign with the orientation.

If $H_2(M) = 0$, this argument is not valid. However, M is then homeomorphic to the 4-sphere since the Kirby-Siebenmann invariant must vanish in this case (see [FQ, 10.2B]). (This is the 4-dimensional topological Poincaré conjecture [FQ, 7.1B].) Thus, M is topologically amphicheiral in this case, too.

With respect to smooth amphicheirality, the situation is more complicated since a 4-manifold can have many distinct differentiable structures. Of course, the signature is still an obstruction to amphicheirality. On the other hand, Kotschick found a closed, simply-connected, smooth 4-manifold with signature zero whose orientation cannot be reversed by a diffeomorphism.

Theorem 44: [Kotschick92, Thm. 3.7 and Rem. 3.9]

There exists a simply-connected, minimal compact complex surface of general type with signature zero which is not orientation-reversing diffeomorphic to another minimal, compact complex surface of general type.

4.1.3 Dimension 5

Barden classified all simply-connected, closed, oriented, smooth 5-manifolds [Barden, Thm. 2.3]. According to his results, a complete set of invariants is given by the isomorphism class of $H_2(M)$ and a nonnegative integer or infinity, $0 \le i(M) \le \infty$. There is an orientation-preserving diffeomorphism between two manifolds with the same invariants.

The invariant $i(M)$ is defined as follows: The second Stiefel-Whitney class of a simply-connected manifold can be regarded as a homomorphism

$$w : H_2(M) \to \mathbb{Z}/2.$$

If this map is nonzero, let $i(M)$ be the greatest integer i such that this map factors through the mod-2 reduction map $\mathbb{Z}/(2^i) \to \mathbb{Z}/2$. If the map can be lifted to $\mathbb{Z}$, let $i(M)$ be infinity. If w is the zero map, set $i(M)$ to zero.

Since neither the isomorphism class of $H_2(M)$ nor $i(M)$ depend on the orientation, every closed, simply-connected, smooth 5-manifold is (orientation-preserving) diffeomorphic to its negative. Since homology groups as well as the Stiefel-Whitney classes are invariant under homotopy equivalences of manifolds [MS, p. 131], Barden's invariants are invariants of the homotopy type. Now, let M be a simply-connected closed, topological 5-manifold. Since $H^4(M;\mathbb{Z}/2) \cong H_1(M;\mathbb{Z}/2) = 0$, the Kirby-Siebenmann invariant of M must vanish, so M has a PL-structure. Since PL/DIFF is 6-connected, every piecewise linear 5-manifold has a unique differentiable structure (see e. g. [FQ, 8.3]). All this implies that Barden's classification is the same for topological, PL and smooth manifolds, and it holds equally up to orientation-preserving homotopy equivalence, homeomorphism, combinatorial equivalence and diffeomorphism (see [Barden, Cor. 2.3.1], except for the topological case because Barden's paper was written prior to the work of Kirby and Siebenmann).

4.1.4 Dimension 6

Zhubr finished in [Zhubr] the classification of all simply-connected, closed, oriented 6-manifolds in the topological, PL and smooth category. He also achieved the classification up to homotopy type. It can be extracted from his results that every manifold of the above type is orientation-preserving homeomorphic (resp. combinatorially equivalent or diffeomorphic) to its negative. We describe the relevant parts of the classification and add the details for amphicheirality.

We adopt some of Zhubr's notations: Let $\widehat{\mathbb{N}}$ denote the set $\mathbb{N} \cup \{\infty\}$ (without zero). For any $m, n \in \widehat{\mathbb{N}}$, let $\rho_n : \mathbb{Z}/mn \to \mathbb{Z}/n$ be the reduction modulo n, with the special case $\mathbb{Z}/\infty = \mathbb{Z}$. Similarly, for $m \in \mathbb{N}$ and $n \in \widehat{\mathbb{N}}$, let $\iota_m : \mathbb{Z}/n \to \mathbb{Z}/mn$ be the multiplication by m.

Let M be a simply-connected, closed, oriented topological 6-manifold. Consider the following set of invariants:

- The third Betti number, divided by two, $r := \frac{1}{2} \operatorname{rk} H_3(M)$.
- The group $H_2(M)$, which is denoted shortly by G. Note that it is necessary to give an abstract group G for the question whether the invariants are realised by a manifold. A 6-manifold M and *some* isomorphism $H_2(M) \cong G$ is then obtained by the realisation part of the classification. However, we start from a given manifold M, and for our purpose it is appropriate to strictly identify $H_2(M)$ with G. This avoids complications where we otherwise would have to keep track of the choice of an isomorphism.
- The second Stiefel-Whitney class $w := w_2(M) \in H^2(M; \mathbb{Z}/2)$, which is regarded as a homomorphism $w : G \to \mathbb{Z}/2$. Define the "height" m as the maximum in $\widehat{\mathbb{N}}$ so that w can be extended to an $\omega \in \operatorname{Hom}(G, \mathbb{Z}/2^m)$ with $\rho_2 \omega = w$. The set of all such ω with maximal m is denoted by $U(w)$. (The height m was denoted by $i(M)$ in the previous section on Barden's work. If $w_2(M) = 0$, the definitions differ: $i(M) = 0$ but $m = \infty$, see [Zhubr, 1.13].)
- The homology class $\mu \in H_6(G, 2)$ that is the image of the fundamental class $[M]$ under the canonical homomorphism $H_6(M) \to H_6(G, 2)$. This homomorphism is induced from the identity $H_2(M) \to G$ under the canonical identifications $\operatorname{Hom}(H_2(M), G) = H^2(M; G) = [M, K(G, 2)]$.
- The Poincaré dual $p \in G$ of the first Pontrjagin class $p_1(M) \in H^4(M)$.
- The Poincaré dual $\Delta \in G/2G$ of the Kirby-Siebenmann triangulation class in $H^4(M; \mathbb{Z}/2)$.
- Two "exotic" invariants $\Gamma_\omega \in \mathbb{Z}/2^{m-1}$ and $\gamma_\omega \in G/2^{m-1}G$. Actually, these are functions $\Gamma : U(w) \to \mathbb{Z}/2^{m-1}$ and $\gamma : U(w) \to G/2^{m-1}G$ but their values at some arbitrary $\omega_0 \in U(w)$ determine the values at all other ω. For this reason, Zhubr does not write the invariants as functions but as single invariants Γ_ω, γ_ω which depend on the choice of $\omega \in U(w)$.

The classification theorem [Zhubr, Thm. 6.3] states that two manifolds M and M' with invariants $(b_3, G, w, \mu, p, \Delta, \Gamma_\omega, \gamma_\omega)$ and $(b_3', G', w', \mu', p', \Delta', \Gamma_\omega', \gamma_\omega')$ are oriented homeomorphic if and only if $r = r'$ and there exists an isomorphism $\varphi : G \to G'$ such that $w = \varphi^*(w')$, $\varphi_*(\mu) = \mu'$, $\varphi(p) = p'$, $\varphi(\Delta) = \Delta'$, $\Gamma_{\varphi^*\omega} = \Gamma_\omega'$ and $\varphi(\gamma_{\varphi^*\omega}) = \gamma_\omega'$.

Amphicheirality is proved by the following arguments:

1. The 6-manifold M with its orientation reversed has the invariants $(r, G, w, -\mu, -p, \Delta, -\Gamma_\omega, -\gamma_\omega)$. For the "standard" invariants $(r, G, w, \mu, p, \Delta)$ this is obvious from their well-known properties. E.g. the first Pontrjagin class is independent of the orientation but the Poincaré duality map

changes by a sign if the orientation is reversed. Thus, p reverses its sign if the orientation of M is reversed.

The minus sign for the invariants Γ_ω and γ_ω can be deduced from the following arguments: In [Zhubr], the symbols Γ_ω and γ_ω denote not only the invariants itself but also homomorphisms

$$\Gamma_\omega : t\Omega_6^{\mathrm{Spin}}(G,2,w) \to \mathbb{Z}/2^{m-1} \quad \text{and} \quad \gamma_\omega : t\Omega_6^{\mathrm{Spin}}(G,2,w) \to G/2^{m-1}G.$$

See [Zhubr, 1.8] for a definition of the bordism group $t\Omega_6^{\mathrm{Spin}}(G,2,w)$ and [Zhubr, 5.18 and 5.26][1)] for the definition of the homomorphisms. These homomorphisms do not depend on the manifold M but only on the data G, w and ω. The first two are independent of the orientation on M. Also, the freedom of choice for ω is not affected by the orientation, so we choose it to be the same element in $U(w)$ for both M and $-M$. The invariants $\Gamma_\omega(M)$ and $\gamma_\omega(M)$ are defined as the values of Γ_ω and γ_ω at M. The manifold M as an element of $t\Omega_6^{\mathrm{Spin}}(G,2,w)$ is the pair $(M, \mathrm{id}: H_2(M) \to G)$, see [Zhubr, 1.9]. The negative element in this bordism group is M with its orientation reversed, but the identification of $H_2(M)$ with G remains untouched. Altogether, the invariants $\Gamma_\omega(M)$ and $\gamma_\omega(M)$ change their signs when the orientation of M is reversed.

2. The homomorphism $\varphi := (-\mathrm{id}) : G \to G$ induces exactly the same transformation of invariants, i. e. $\varphi^*(w) = w$, $\varphi_*(\mu) = -\mu$, $\varphi(p) = -p$, $\varphi(\Delta) = \Delta$, $\Gamma_{\varphi^*\omega} = \Gamma_{-\omega} = -\Gamma_\omega$ and $\varphi(\gamma_{\varphi^*\omega}) = -\gamma_{-\omega} = -\gamma_\omega$. For w, p and Δ, this is again obvious. The relations $\Gamma_{-\omega} = -\Gamma_\omega$, $\gamma_{-\omega} = \gamma_\omega$ and $\varphi_*(\mu) = -\mu$ will be proved in the lemmas below.

Given the assertions above, this proves that every simply-connected, closed, oriented topological 6-manifold is amphicheiral. Moreover, Zhubr shows that the classification theorem for the smooth case is the same as above but with the invariant Δ always set to zero. Besides, the smooth and PL-classifications coincide since PL/DIFF is 6-connected. Thus, every simply-connected, closed, oriented, smooth (PL) 6-manifold is smoothly (resp. PL-)amphicheiral.

Lemma 45

For every $\omega \in U(w)$, we have $\gamma_{-\omega}(M) = \gamma_\omega(M)$ and $\Gamma_{-\omega}(M) = -\Gamma_\omega(M)$.

Proof. The theorem [Zhubr, 5.26] states that for any $x \in \mathrm{Hom}(G, \mathbb{Z}/2^{m-1})$, we have

$$\gamma_{\omega+\iota_2 x}(M) = \gamma_\omega(M) + \mu \cap (x^2 + \omega x).$$

For $x := -\rho_{2^{m-1}}(\omega)$, we have $\iota_2 x = -2\omega$ and thus

$$\gamma_{-\omega}(M) = \gamma_\omega(M) + \mu \cap \rho_{2^{m-1}}(\omega^2 - \omega^2) = \gamma_\omega(M).$$

1) Note that there are a few typographical errors in [Zhubr, 5.26]: The occurrences of $tO_6^{\mathrm{Spin}}(G,2,w)$ and $\Omega_6^{\mathrm{Spin}}(G,2,w)$ should be replaced by $t\Omega_6^{\mathrm{Spin}}(G,2,w)$. The domain of γ_ω is stated correctly e. g. at the end of Section 5.19.

The same theorem states also a formula for Γ_ω:

$$\langle x, \gamma_\omega(M)\rangle = \Gamma_{\omega+\iota_2 x}(M) - \Gamma_\omega(M) + R(\mu,\omega,x), \tag{1}$$

where $R(\mu,\omega,x)$ is defined in [Zhubr, 5.24] as

$$R(\mu,\omega,x) := \begin{cases} \langle\omega^2 x,\mu\rangle + \frac{3}{2}\langle\omega x^2,\mu\rangle + \langle x^3,\mu\rangle & \text{if } m=\infty,\\ \langle\omega^2 x,\mu\rangle + 3\iota_2^{-1}\langle\omega P(x),\mu\rangle + \langle x^3,\mu\rangle & \text{if } m\in\mathbb{N}.\end{cases}$$

The expression $P(x)$ denotes the Pontrjagin square, a nonstable cohomology operation $H^{2i}(X;\mathbb{Z}/2^{m-1}) \to H^{4i}(X;\mathbb{Z}/2^m)$. One of its properties [Zhubr, eq. (42)] is the relation $P(\rho_{2^{m-1}}y) = y^2$ for $y \in H^{2i}(X;\mathbb{Z}/2^m)$, so we have with $x := -\rho_{2^{m-1}}\omega$ as before

$$\begin{aligned} R(\mu,\omega,x) &= -\rho_{2^{m-1}}\langle\omega^3,\mu\rangle + 3\iota_2^{-1}\langle\omega^3,\mu\rangle - \rho_{2^{m-1}}\langle\omega^3,\mu\rangle \\ &= -2\rho_{2^{m-1}}\langle\omega^3,\mu\rangle + 3\iota_2^{-1}\langle\omega^3,\mu\rangle \end{aligned}$$

The last equation is valid for all $m \in \widehat{\mathbb{N}}$. Together with equation (1), this gives

$$-\langle\rho_{2^{m-1}}\omega, \gamma_\omega(M)\rangle = \Gamma_{-\omega}(M) - \Gamma_\omega(M) - 2\rho_{2^{m-1}}\langle\omega^3,\mu\rangle + 3\iota_2^{-1}\langle\omega^3,\mu\rangle.$$

According to [Zhubr, eq. (192)], the left hand side is equal to

$$-\left(2\Gamma_\omega(M) + \iota_2^{-1}\langle\omega^3,\mu\rangle\right),$$

so we have

$$\Gamma_{-\omega}(M) + \Gamma_\omega(M) = 2\rho_{2^{m-1}}\langle\omega^3,\mu\rangle - 4\iota_2^{-1}\langle\omega^3,\mu\rangle = 0. \qquad \square$$

Lemma 46

Let G be a finitely generated abelian group. The automorphism $-\mathrm{id}: G\to G$ induces

$$(-\mathrm{id})^k : H_{2k}(G,2) \to H_{2k}(G,2)$$

on the even homology groups in degrees 0 to 6.

Proof. 1. For $k=0$, this is clear since $K(G,2)$ is a connected space.

2. For $k=1$, the assertion is true since $H_2(G,2)$ is (though not canonically but naturally) isomorphic to G.

3. *The case $G\cong\mathbb{Z}$.* Since the homology of $K(\mathbb{Z},2)\simeq\mathbb{C}\mathrm{P}^\infty$ is free in every degree, it suffices to check the corresponding assertion in cohomology according to the universal coefficient theorem. In cohomology, the induced homomorphisms are the desired ones since $H^*(\mathbb{C}\mathrm{P}^\infty)$ is naturally isomorphic to the polynomial ring $\mathbb{Z}[t]$ with one generator in degree 2.

4. *If the assertion is true for abelian groups G_1 and G_2, it holds for $G_1 \oplus G_2$.* To prove this, recall that $H_0(G,2) \cong \mathbb{Z}$ and $H_1(G,2) = H_3(G,2) = 0$ for all abelian groups G, see [EMcL, Thm. 20.5]. Therefore, all Tor terms

$$\operatorname{Tor}(H_p(G_1,2), H_q(G_2,2))$$

with $p+q+1 \in \{2,4,6\}$ are equal to 0. This implies that the Künneth homomorphism, given by the cross product map

$$\bigoplus_{p+q=2k} H_p(G_1,2) \otimes H_q(G_2,2) \xrightarrow{\times} H_{2k}(G_1 \oplus G_2, 2)$$

is an isomorphism for $k \le 3$. Besides, only terms with even p and q contribute to the direct sum on the left hand side. Since the cross product is natural in both factors, the maps $(-\mathrm{id})^{p/2} \otimes (-\mathrm{id})^{q/2}$ on $H_p(G_1,2) \otimes H_q(G_2,2)$ amount to $(-\mathrm{id})^{(p+q)/2} = (-\mathrm{id})^k$ on $H_{2k}(G_1 \oplus G_2, 2)$.

5. The previous paragraphs have reduced the problem to the case of p-cyclic groups $G \cong \mathbb{Z}/p^r$ (p prime, $r \ge 1$) and $k \in \{2,3\}$. For p-cyclic groups, we compare the homology Serre spectral sequences of the path loop fibrations

$$\begin{array}{ccc} K(\mathbb{Z},1) \longrightarrow PK(\mathbb{Z},2) \simeq * \\ \downarrow \\ K(\mathbb{Z},2) \end{array} \quad \text{and} \quad \begin{array}{ccc} K(\mathbb{Z}/p^r,1) \longrightarrow PK(\mathbb{Z}/p^r,2) \simeq * \\ \downarrow \\ K(\mathbb{Z}/p^r,2). \end{array}$$

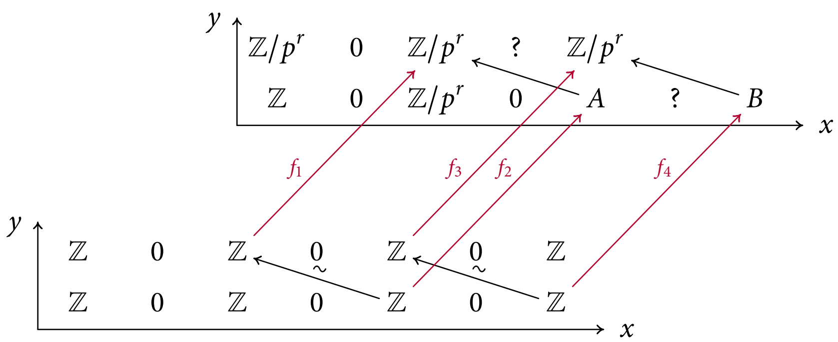
Reduction mod p^r induces a map $K(\mathbb{Z},2) \to K(\mathbb{Z}/p^r,2)$, which in turn induces by the path functor P a fibre-preserving map $PK(\mathbb{Z},2) \to PK(\mathbb{Z}/p^r,2)$. This is the starting point for a morphism of the E^2-stages of the Serre spectral sequence.

The diagram below shows the relevant part of the spectral sequences. In the lower half, we have

$$E^2_{x,y} \cong H_x(K(\mathbb{Z},2); H_y(K(\mathbb{Z},1))) = H_x(\mathbb{C}\mathrm{P}^\infty; H_y(S^1)),$$

and in the upper half

$$E^2_{x,y} \cong H_x(K(\mathbb{Z}/p^r,2); H_y(K(\mathbb{Z}/p^r,1))).$$



According to [EMcL, Thm. 21.1], we have

$$A \cong \begin{cases} \mathbb{Z}/p^r & \text{if } p \neq 2 \\ \mathbb{Z}/p^{r+1} & \text{if } p = 2 \end{cases} \quad \text{and} \quad B \cong \begin{cases} \mathbb{Z}/p^r & \text{if } p \neq 3 \\ \mathbb{Z}/p^{r+1} & \text{if } p = 3. \end{cases}$$

In any case, the $(4,1)$-entry in the upper diagram is $\mathbb{Z}/p^r$.

The arrows marked $f_1, \dots, f_4$ denote components of the spectral sequence morphism. We want to show that all of these are surjective. First, f_1 is surjective since it is the composition of the map $\mathbb{Z} \cong H_2(\mathbb{Z}, 2) \to H_2(\mathbb{Z}/p^r, 2) \cong \mathbb{Z}/p^r$ coming from the quotient $\mathbb{Z} \to \mathbb{Z}/p^r$ followed by a change of coefficients $\mathbb{Z} \to \mathbb{Z}/p^r$, which induces a surjection $H_2(\mathbb{Z}/p^r, 2) \to H_2(\mathbb{Z}/p^r, 2; \mathbb{Z}/p^r)$.

From the first "commuting parallelogram" we see that the composition

$$\mathbb{Z} \xrightarrow{f_2} A \cong \mathbb{Z}/p^s \to \mathbb{Z}/p^r$$

is surjective, where s is equal to r or $r+1$, depending on p. Since $1 \in \mathbb{Z}/p^r$ is not divisible by p, all of its preimages in $\mathbb{Z}/p^s$ are not, so each one is again a generator. This implies that a generator of $\mathbb{Z}/p^s$ lies in the image of f_2, so f_2 is surjective.

Exactly the same argument applies to f_3 and f_4 instead of f_1 and f_2. Since f_2 and f_4 are surjective, we can deduce the action $(-\mathrm{id})_*$ on $H_4(\mathbb{Z}/p^r, 2)$ and $H_6(\mathbb{Z}/p^r, 2)$ from that on $H_*(\mathbb{Z}, 2)$. □

4.2 Dimensions 10 and 17

Theorem 47

There exists a simply-connected (closed, smooth) 10-dimensional manifold that does not admit an orientation-reversing homotopy equivalence.

Proof. The strategy is the following: We prove existence of a 10-dimensional manifold M with $H^3(M; \mathbb{Z}/3) \cong \mathbb{Z}/3$ and $\langle i \cup P^1 i, \rho_3[M] \rangle \neq 0$, where i denotes a generator of the third cohomology group, $P^1 : H^3(M; \mathbb{Z}/3) \to H^7(M; \mathbb{Z}/3)$ is the first Steenrod power operation and ρ_3 is the reduction of integral coefficients modulo 3.

Then we can proceed by the familiar arguments which we already encountered when dealing with the linking form: If $T : M \to M$ is a homotopy equivalence, i is multiplied by some factor $k \in \mathbb{Z}/3$: $T^* i = k \cdot i$. Then we have

$$\begin{aligned} \deg T \cdot \langle i \cup P^1 i, \rho_3[M] \rangle &= \langle i \cup P^1 i, T_* \rho_3[M] \rangle = \langle (T^* i) \cup P^1 (T^* i), \rho_3[M] \rangle \\ &= k^2 \langle i \cup P^1 i, \rho_3[M] \rangle. \end{aligned}$$

k	$H^k(\mathbb{Z}/3,3;\mathbb{Z}/3)$	generators	$r(k) := \dim(H_k(\mathbb{Z}/3,3)\otimes\mathbb{Z}/3)$
0	$\mathbb{Z}/3$	1	1
1	0		0
2	0		0
3	$\mathbb{Z}/3$	ι	1
4	$\mathbb{Z}/3$	$\beta\iota$	0
5	0		0
6	0		0
7	$(\mathbb{Z}/3)^2$	$P^1\iota, \iota\cup\beta\iota$	2
8	$(\mathbb{Z}/3)^3$	$\beta P^1\iota, (\beta\iota)^2, P^1\beta\iota$	1
9	$\mathbb{Z}/3$	$\beta P^1\beta\iota$	0
10	$\mathbb{Z}/3$	$\iota\cup P^1\iota$	1

Table 4.1: The homology and cohomology of $K(\mathbb{Z}/3,3)$.

Thus, $\deg T \equiv k^2 \bmod 3$, and this is never congruent -1, so T cannot reverse orientation.

We will actually prove existence of a 2-connected manifold with integral homology $H_3(M)\cong\mathbb{Z}/3$. The obstruction we described must already be present in $K(\mathbb{Z}/3,3)$, the first stage of the Postnikov tower of M. That is, as a necessary prerequisite for our proof of Theorem 47, we have the following

Lemma 48

Let $\iota\in H^3(\mathbb{Z}/3,3;\mathbb{Z}/3)$ denote the canonical generator. There is a homology class $m\in H_{10}(\mathbb{Z}/3,3)$ such that $\langle\iota\cup P^1\iota,\rho_3 m\rangle\neq 0$ ($\in\mathbb{Z}/3$).

Later, m will be the image $f_*[M]$ of the fundamental class $[M]$ under a first Postnikov approximation $f: M\to K(\mathbb{Z},3)$, and i will be $f^*\iota$.

Proof of Lemma 48. The cohomology ring $H^*(\mathbb{Z}/3,3;\mathbb{Z}/3)$ is a module over the Steenrod algebra $\mathcal{A}_3$, generated by the canonical element $\iota\in H^3(\mathbb{Z}/3,3;\mathbb{Z}/3)$. As a free commutative algebra over $\mathbb{Z}/3$, it has generators $\Theta(\iota)$, where $\Theta\in\mathcal{A}_3$ runs through all "admissible monomials" of "excess" less than 3. (See [HatcherAT, Ch. 4.L] for reference and an explanation.) The first three columns in Table 4.1 list the cohomology groups together with their generators as a $(\mathbb{Z}/3)$ vector space up to degree 10. Given this information, we can already conclude the integral homology groups in degree ≤ 10. For the current lemma, the groups $H^*(\mathbb{Z}/3,3)\otimes\mathbb{Z}/3$ are sufficient. As means of computing them, we have the universal coefficient sequence

$$0\to H_k(X)\otimes\mathbb{Z}/3\to H_k(X;\mathbb{Z}/3)\to \mathrm{Tor}(H_{k-1}(X),\mathbb{Z}/3)\to 0, \qquad (2)$$

isomorphisms $H_k(\mathbb{Z}/3,3;\mathbb{Z}/3)\cong H^k(\mathbb{Z}/3,3;\mathbb{Z}/3)$ (since these groups are finitely generated) and the following lemma which we prove below.

Lemma 49

Let p be a prime number. $H_k(\mathbb{Z}/p, n)$ is a finite p-primary abelian group for all $n \geq 0$ and $k \geq 1$.

(Only the statement about finiteness is needed so far; the 3-primary part will be used later.)

Because of this lemma, $\operatorname{Tor}(H_{k-1}(\mathbb{Z}/3, 3), \mathbb{Z}/3)$ is isomorphic to the tensor product $H_{k-1}(\mathbb{Z}/3, 3) \otimes \mathbb{Z}/3$. In total, we have an exact sequence

$$0 \to H_k(\mathbb{Z}/3, 3) \otimes \mathbb{Z}/3 \to H^k(\mathbb{Z}/3, 3; \mathbb{Z}/3) \to H_{k-1}(\mathbb{Z}/3, 3) \otimes \mathbb{Z}/3 \to 0,$$

and working from $k = 0$ upwards, we can determine all the dimensions of the $\mathbb{Z}/3$ vector spaces. This information is collected in the last column of Table 4.1.

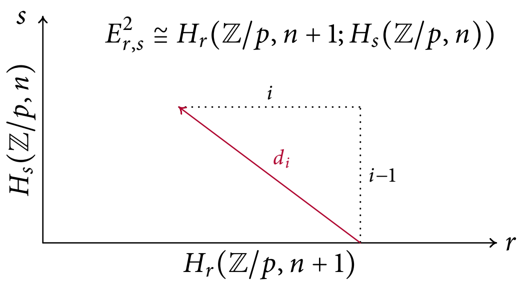
For $k = 10$, the left map in the universal coefficient sequence (2) is an isomorphism because the right term $\operatorname{Tor}(H_9(\mathbb{Z}/3, 3), \mathbb{Z}/3)$ vanishes. Since tensoring an abelian group with $\mathbb{Z}/3$ is always surjective, we have

$$\begin{aligned} H_{10}(\mathbb{Z}/3, 3) \twoheadrightarrow H_{10}(\mathbb{Z}/3, 3) \otimes \mathbb{Z}/3 \xrightarrow{\sim} H_{10}(\mathbb{Z}/3, 3; \mathbb{Z}/3) \\ \xrightarrow{\sim} \operatorname{Hom}(H^{10}(\mathbb{Z}/3, 3; \mathbb{Z}/3), \mathbb{Z}/3), \end{aligned}$$

and there is an element $m \in H_{10}(\mathbb{Z}/3, 3)$ such that $\langle \iota \cup P^1 \iota, \rho_3 m \rangle \neq 0$. □

Proof of Lemma 49. For $n = 0$ and $n = 1$, the statement is obviously true because a $K(\mathbb{Z}/p, 0)$ is given by a finite set and a $K(\mathbb{Z}/p, 1)$ is given by an infinite dimensional lens space, whose cohomology ring coincides with the ring $\mathbb{Z}[x]/px$ with $\deg x = 2$. For higher Eilenberg-MacLane spaces, we work inductively and consider the homology Serre spectral sequence for the path-loop fibration

$$K(\mathbb{Z}/p, n) \simeq \Omega K(\mathbb{Z}/p, n+1) \to PK(\mathbb{Z}/p, n+1) \to K(\mathbb{Z}/p, n+1).$$



Since the path space $PK(\mathbb{Z}/p, n+1)$ is contractible, its homology must vanish, so each $E^i_{r,s}$ (except $E^i_{0,0}$) must be zero for $i > r$. If an $E^2_{r,0} \cong H_r(\mathbb{Z}/p, n+1)$ was infinite, consider the least such r, $r > 0$. Then all differentials from $E^i_{r,0}$ ($i \geq 2$) go to finite groups, thus there is still an infinite kernel for each differential. Hence, $E^i_{r,0}$ is an infinite group for all i, contradicting the vanishing homology of the total space.

By the last conclusion and the inductive hypothesis, all terms $E^2_{r,s}$ for $s > 0$ are p-primary finite abelian groups. This class of groups is stable under subgroups and quotients, so each $E^i_{r,s}$ for $i \geq 2$ and $s > 0$ is p-primary finite abelian. If $E^2_{r,0} \cong H_r(\mathbb{Z}/p, n+1)$ contained, for some r, an element whose order is not divisible by p, all differentials had to be zero on this element. Hence, it would remain up to the E^∞-term, again contradicting the vanishing homology of the total space. □

For the further arguments, it is convenient to know the actual integral homology groups $H_k(\mathbb{Z}/3, 3)$. Direct inspection of Table 4.1 reveals that the Bockstein sequence

$$H_0(\mathbb{Z}/3,3;\mathbb{Z}/3) \xrightarrow{\beta} H_1(\mathbb{Z}/3,3;\mathbb{Z}/3) \xrightarrow{\beta} H_2(\mathbb{Z}/3,3;\mathbb{Z}/3) \xrightarrow{\beta} \dots$$
$$\dots \xrightarrow{\beta} H_{11}(\mathbb{Z}/3,3;\mathbb{Z}/3)$$

is exact, i. e. the 3-primary Bockstein cohomology of $K(\mathbb{Z}/3, 3)$ vanishes in degrees 1 to 10. (The last arrow is an injection because $\beta(\iota \cup P^1\iota) = \beta\iota \cup P^1\iota - \iota \cup \beta P^1\iota$ is nonzero.) According to [HatcherAT, Prop. 3E.3], this implies that there are no elements of order 9 in $H_k(\mathbb{Z}/3, 3)$ for $k = 1, \dots, 10$. Altogether, we have proved that the integral homology $H_k(\mathbb{Z}/3, 3)$ is isomorphic to $(\mathbb{Z}/3)^{r(k)}$ ($k = 1, \dots, 10$), where the multiplicity $r(k)$ is given by Table 4.1.

Now we continue with the proof of Theorem 47. We have shown that there is a homology class $m \in H_{10}(\mathbb{Z}/3, 3)$ with the desired properties, and we claim that there is a manifold M together with a map $f : M \to K(\mathbb{Z}/3, 3)$ such that $f_*[M] = m$. More restrictive, we are looking for a *framed* manifold M, although a spin manifold would be sufficient in the surgery step later. This task can be formulated as a bordism problem: Show that there is an element $(M, f) \in \Omega^{\mathrm{fr}}_{10}(K(\mathbb{Z}/3, 3))$ that maps to m under the Thom homomorphism

$$\Omega^{\mathrm{fr}}_{10}(K(\mathbb{Z}/3,3)) \to H_{10}(K(\mathbb{Z}/3,3))$$
$$(M, f) \mapsto f_*[M].$$

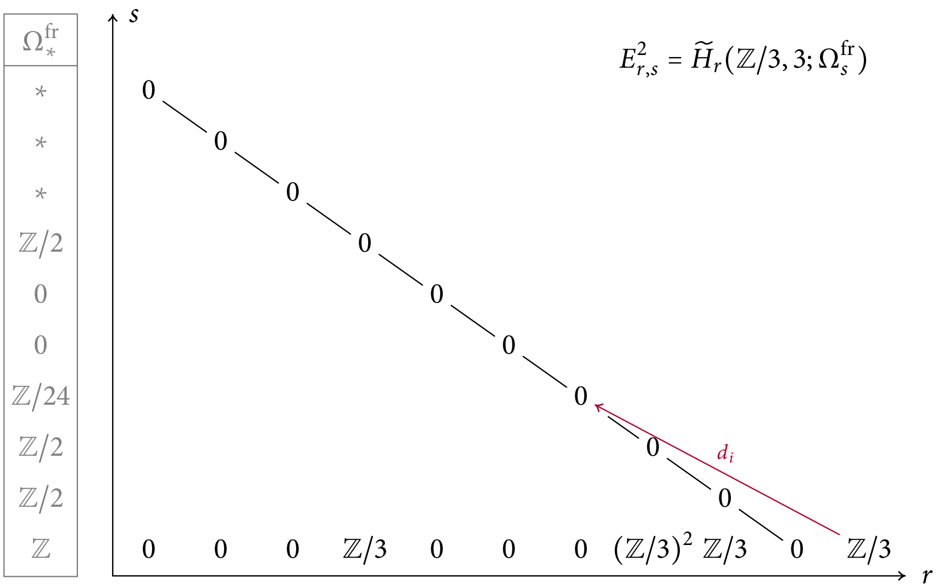
The Thom homomorphism factors through the edge homomorphism $E^\infty_{10,0} \hookrightarrow E^2_{10,0}$ in the Atiyah-Hirzebruch spectral sequence for the homology theory Ω^{fr}_*:

$$\Omega^{\mathrm{fr}}_{10}(K(\mathbb{Z}/3,3)) \twoheadrightarrow E^\infty_{10,0} \hookrightarrow E^2_{10,0} \cong H_{10}(\mathbb{Z}/3,3)$$

Thus, it is sufficient to prove surjectivity of the edge homomorphism.

Here and at several other places in this thesis, the Atiyah-Hirzebruch spectral sequence is used. The references which state and prove its properties in the way which is most useful for our purpose are [Conner, Ch. 1.7] and [Kochman, Ch. 4.2]. Conner only deals with the spectral sequence for oriented bordism but all statements and proofs carry over to framed bordism because both are generalised homology theories which satisfy the wedge axiom and have zero homology groups in negative degrees. Also what is said about the Ω_*-module structure remains valid since Ω_* and Ω_*^{fr} have analogous ring structures given by the cartesian product on the underlying manifolds.

Since there are no bordism groups of negative degree, the Atiyah-Hirzebruch spectral sequence is located in the first quadrant, and we have $E^\infty_{r,0} = E^{r+1}_{r,0}$. It is sufficient to show that all the intermediate inclusions $E^{i+1}_{10,0} = \ker d_i \subseteq E^i_{10,0}$ are in fact bijections, i. e. we want to show that all differentials starting from $E^2_{10,0}$ are zero. The diagram below shows the relevant part of the reduced Atiyah-Hirzebruch spectral sequence for $\Omega^{\mathrm{fr}}_{10}(K(\mathbb{Z}/3,3))$.



The diagram reveals that all terms $E^2_{r,9-r}$ on the 9-line are zero. Thus, the Thom map is surjective.

Now, we have a framed manifold M together with a map $f: M \to K(\mathbb{Z}/3,3)$ such that $f_*[M] = m$. We still need the correct third homology group. By [Kreck99, Prop. 4] (see below), (M, f) can be replaced by another manifold (M', f') (with the same image of the fundamental class) such that $f' : M' \to K(\mathbb{Z}/3,3)$ is a 5-equivalence. Hence, M' is 2-connected and $H_3(M')$ is isomorphic to $\mathbb{Z}/3$. □

Corollary 50

There exists a simply-connected (closed, smooth) 17-dimensional manifold that does not admit an orientation-reversing homotopy equivalence.

Proof. The 10-dimensional manifold whose existence was shown in Theorem 47 has nonzero Betti numbers only in degrees 0, 10 and possibly 5. By the argument which we have used several times before (Theorem 34), the product of this manifold with a 7-dimensional chiral rational homology sphere is chiral. □

In the following, we explain how [Kreck99, Prop. 4] can be applied in the proof of Theorem 47. If $\nu : M \to BO$ is the classifying map of the stable normal bundle of a manifold M, a framing of M is equivalent to the fibrewise homotopy class of a lift $\bar{\nu}$ to EO as in the following diagram:

$$\begin{array}{ccc} & & EO \\ & \overset{\bar{\nu}}{\nearrow} & \downarrow p \\ M & \xrightarrow{\ \nu\ } & BO \end{array}$$

The map p is the projection in the fibration $EO \to BO$ with contractible total space EO and fibre $O = \operatorname{colim}_n O(n)$. Let $B := K(\mathbb{Z}/3, 3) \times EO$ and consider the fibration over BO which is given by the projection to EO followed by p. The map $f \times \bar{\nu} : M \to B$ is a *normal B-structure* on M in the sense of [Kreck99, Section 2]. For the reader's convenience, we quote [Kreck99, Prop. 4] and detail how to get into the right context for applying it.

Proposition 51: [Kreck99, Prop. 4]

Let $\xi : B \to BO$ be a fibration and assume that B is connected and has finite $\left[\frac{m}{2}\right]$-skeleton. Let $\bar{\nu} : M \to B$ be a normal B-structure on an m-dimensional compact manifold M. Then, if $m \geq 4$, by a finite sequence of surgeries $(M, \bar{\nu})$ can be replaced by $(M', \bar{\nu}')$ so that $\bar{\nu}' : M' \to B$ is an $\left[\frac{m}{2}\right]$-equivalence.

Nearly all conditions of this proposition are fulfilled, only the requirements on B are too restrictive. However, the proof in [Kreck99] shows that it is sufficient if B is connected and has the homotopy type of a CW-complex with finite $\left[\frac{m}{2}\right]$-skeleton.

The space B has the homotopy type of a CW-complex by the following arguments:

- The space EO is homotopy equivalent to a CW-complex since the model used in our context is the union of Stiefel manifolds, which have a CW-structure (for the latter assertion see [Steenrod, Thm. IV.2.1]).
- Eilenberg-MacLane spaces can be constructed as CW-complexes.
- The cartesian product of two spaces with the homotopy type of a CW-complex has the homotopy type of a CW-complex by the following theorem.

Theorem 52: [FP, Thm. 5.4.2]

Let $p : Y \to X$ be a fibration with X path-connected and such that X and $F := p^{-1}(x)$ have the homotopy type of CW-complexes, for any $x \in X$. Then, Y has the type of a CW-complex.

For the finiteness of the CW-complex, there are easily controllable criteria if the group ring $\mathbb{Z}[\pi_1(B)]$ is Noetherian:

Theorem 53: [Wall65, Thm. A and additions on p. 61]

Let X be a space which is homotopy equivalent to a CW-complex. Denote its fundamental group by π. Consider the following sequence of conditions:

NF(1): The group π is finitely generated.

NF(2): The group π is finitely presented and $H_2(\widetilde{X})$ is finitely generated as a $\mathbb{Z}[\pi]$-module.

NF(n) ($n \geq 3$): NF($n-1$) holds and $H_n(\widetilde{X})$ is finitely generated over $\mathbb{Z}[\pi]$.

If $\mathbb{Z}[\pi]$ is Noetherian, X is homotopy equivalent to a complex with finite n-skeleton if and only if X satisfies NF(n).

As Wall points out further, $\mathbb{Z}[\pi]$ is Noetherian if π is a finite extensions of a polycyclic group. This includes finite groups and finitely generated abelian groups. Thus, the space $B = K(\mathbb{Z}/3, 3) \times EO$ on page 56 is homotopy equivalent to a CW-complex with finite 5-skeleton.

Also, all other spaces that will occur later in this work as total spaces B for Proposition 51 have the homotopy type of CW-complexes with finite k-skeleta, for the k which is required by the application. We list these spaces and the necessary arguments here to ensure that [Kreck99, Prop. 4] can later be applied without reservation.

- $P^4 \times EO$ on page 71. P^4 is part of a Postnikov system and homotopy equivalent to a CW-complex, see the remarks on page 66. Furthermore, P^4 is simply-connected and has finitely generated homology groups up to degree 4.
- $BSO \times K(\pi, 1)$ in Proposition 75. The classifying space BSO is a twofold covering of the CW-complex BO. It will be required that π is a finitely presented group. The groups π that actually appear in the application (Theorem 81) are polycyclic, so Theorem 53 applies.

  Alternatively, for a general finitely presented group, a standard CW-construction for the classifying space (1-cells correspond to generators, 2-cells to relations in π) yields a $K(\pi, 1)$ with finite 2-skeleton. Without the restrictions in Wall's theorem, one can then argue that BSO and $K(\pi, 1)$ are countable CW-complexes, and in this case the cartesian product (with the product topology) is again a CW-complex [HatcherAT, Thm. A.6].
- BSO in Lemma 72.
- $L \times EO$ in Theorem 82. L is a compact smooth manifold and thus a finite CW-complex.

4.3 Dimensions 9 and 13

In this section, examples of simply-connected, homotopically chiral manifolds are given in the last two missing dimensions. Before we start proving their existence, some preliminaries are necessary. Since repeated use of the universal coefficient theorems is made, it is convenient to enumerate them.

Proposition 54: [Munkres, Cor. 53.2, 55.3 and 56.4]

Let (X, A) be a topological pair and G an abelian group. For each $n \geq 0$, there are exact sequences

$$0 \leftarrow \mathrm{Hom}(H_n(X,A),G) \leftarrow H^n(X,A;G) \leftarrow \mathrm{Ext}(H_{n-1}(X,A),G) \leftarrow 0 \tag{UCT 1}$$

$$0 \leftarrow \mathrm{Hom}(H^n(X,A),G) \leftarrow H_n(X,A;G) \leftarrow \mathrm{Ext}(H^{n+1}(X,A),G) \leftarrow 0 \tag{UCT 2}$$

$$0 \rightarrow H_n(X,A) \otimes G \rightarrow H_n(X,A;G) \rightarrow \mathrm{Tor}(H_{n-1}(X,A),G) \rightarrow 0 \tag{UCT 3}$$

$$0 \rightarrow H^n(X,A) \otimes G \rightarrow H^n(X,A;G) \rightarrow \mathrm{Tor}(H^{n+1}(X,A),G) \rightarrow 0 \tag{UCT 4}$$

These sequences are natural with respect to homomorphisms which are induced by continuous maps. Moreover, the sequences split but not naturally. For the sequences (UCT 2) and (UCT 4) it is assumed that $H_i(X, A)$ is finitely generated for all $i \leq n + 1$[2].

Furthermore, some observations about principal $K(\pi, n)$-fibrations are needed. Such fibrations constitute the Postnikov tower of a simply-connected space. Let π be an abelian group. Quoting [GM, Ch. VI.B], a fibration with the fibre a $K(\pi, n)$ is called principal if the action of the fundamental group of the base on the fibre is trivial up to homotopy. The classifying space is a $K(\pi, n+1)$ [Baues, Cor. 5.2.3], and the universal fibration is (up to fibre homotopy equivalence) the path-loop fibration

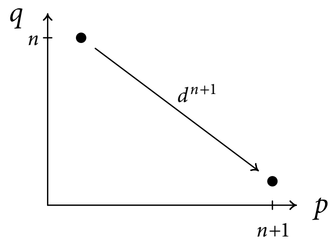
$$\begin{array}{ccc} K(\pi,n) \simeq \Omega K(\pi,n+1) & \longrightarrow & PK(\pi,n+1) \simeq * \\ & & \downarrow \\ & & K(\pi,n+1). \end{array}$$

Thus, up to fibre homotopy equivalence, principal $K(\pi, n)$-fibrations over a base space B are obtained as pull-backs of the path-loop fibration, and they are classified by the k-invariant $k^{n+1} \in [B, K(\pi, n+1)]$. Usually, the k-invariant is considered as a cohomology class $k^{n+1} \in H^{n+1}(B; \pi)$.

[2] In fact, Munkres requires that $H_i(X, A)$ is finitely generated for all i. With small adaptations of the proof it can be shown that the weaker condition suffices.

Given a principal $K(\pi, n)$-fibration $p\colon E \to B$ with k-invariant k^{n+1}, we would like to relate the k-invariant to the first possibly nonzero differential in the cohomology Serre spectral sequence

$$\begin{array}{ccc} d_{n+1} : E_{n+1}^{0,n} & \longrightarrow & E_{n+1}^{n+1,0} \\ \wr\Vert & & \wr\Vert \\ H^n(K(\pi,n)) & & H^{n+1}(B) \end{array}$$



Lemma 55

The k-invariant $k^{n+1} \in H^{n+1}(B; \pi)$ is the transgression of the canonical element $\Delta \in H^n(K(\pi, n); \pi)$.

Explanation and proof. The k-invariant is the principal obstruction to extending the trivial section over the basepoint

$$\begin{array}{c} \{*\} \subset E \\ \uparrow \\ \{*\} \subset B \end{array}$$

to all of B [GM, Lemma 6.2]. In terms of [Baues, 4.3.15], this obstruction is called the *characteristic cohomology class*

$$\bar{c}(p) \in H^{n+1}(B; \pi_n(F)),$$

where $F \simeq K(\pi, n)$ is the fibre. (Baues considers a more general case, where the fundamental group $\pi_1(B)$ may act nontrivially on $\pi_n(F)$ but in our context of principal fibrations, the action is trivial and all local coefficients are constant.) See also [Baues, 5.3.2] for a definition of the k-invariant.

By [Baues, Lemma 5.2.9], the characteristic class is the transgression of the fundamental class,

$$\bar{c}(p) = \tau(\Delta F).$$

By definition, the fundamental class $\Delta F \in H^n(F; \pi_n(F))$ is the canonical element, i. e. it maps to the identity under the isomorphisms

$$\begin{aligned} H^n(F; \pi_n(F)) &\cong \mathrm{Hom}(H_n(F), \pi_n(F)) && \text{(UCT 1)} \\ &\cong \mathrm{Hom}(\pi_n(F), \pi_n(F)) && \text{(Hurewicz isomorphism)} \end{aligned}$$ □

According to [McCleary, Thm. 6.8], the transgression τ coincides with the differential

$$d_{n+1} : E_{n+1}^{0,n} \to E_{n+1}^{n+1,0}$$

in the cohomology Serre spectral sequence. Note that we are working with coefficients in $\pi = \pi_n(F)$ here.

Since the fibre is $(n-1)$-connected, these E_{n+1}-terms are equal to the E_2-terms. Thus, for any coefficients G, the transgression is a homomorphism

$$\tau : H^n(F;G) \to H^{n+1}(B;G).$$

By now, the k-invariant was identified as the image of the canonical element under transgression with π-coefficients. We would like to relate this to the transgression homomorphism with *integer* coefficients $\tau : H^n(F) \to H^{n+1}(B)$. In order to distinguish the transgression homomorphism for the various coefficient groups, the latter is indicated by a subscript to τ in the following proposition.

Proposition 56

Suppose that π is a finitely generated free abelian group. Let $E \to B$ be a principal fibration with the fibre $F \simeq K(\pi, n)$. Assume that B is homotopy equivalent to a CW-complex and $H_i(B)$ is finitely generated for $i \leq n+2$. The map

$$\begin{array}{ccc} H^{n+1}(B;\pi) & \to & \operatorname{Hom}(H^n(F), H^{n+1}(B)) \\ k\text{-invariant} & \mapsto & \text{transgression in the spectral sequence} \\ k^{n+1} = \tau_\pi(\Delta) & \mapsto & (\tau_{\mathbb{Z}} = d_{n+1} : E_{n+1}^{0,n} \to E_{n+1}^{n+1,0}) \end{array}$$

coincides with the chain of natural isomorphisms

$$\begin{aligned} H^{n+1}(B;\pi) &\to H^{n+1}(B; H_n(F)) && \text{(Hurewicz)} \\ &\leftarrow H^{n+1}(B) \otimes H_n(F) && \text{(UCT 4)} \\ &\to H_n(F; H^{n+1}(B)) && \text{(UCT 3)} \\ &\to \operatorname{Hom}(H^n(F), H^{n+1}(B)). && \text{(UCT 2)} \end{aligned}$$

Note that all relevant Ext *and* Tor *groups in the universal coefficient theorems vanish because $H_{n-1}(F) = H^{n+1}(F) = 0$ and $H_n(F)$ is finitely generated free.*

In the following, $\pi_n(F) = \pi$ and $H_n(F)$ are always identified by the Hurewicz homomorphism, and we write only $H_n(F)$. The proof of Proposition 56 needs some preparation and is given on page 63 ff.

As Proposition 54 states, the universal coefficient maps used in Proposition 56 are natural. We also need the following facts (Lemmas 57 and 58) about the first map in the sequence (UCT 4).

Lemma 57

Let (X, A) be a topological pair and G an abelian group. The coefficient homomorphism $H^*(-) \otimes G \to H^*(-;G)$ commutes with the boundary homomorphisms in the long exact relative cohomology sequence, i. e. the *following square is commutative for all n:*

$$\begin{array}{ccc} H^n(A)\otimes G & \xrightarrow{\delta\otimes \mathrm{id}} & H^{n+1}(X,A)\otimes G \\ \downarrow & & \downarrow \\ H^n(A;G) & \xrightarrow{\delta} & H^{n+1}(X,A;G) \end{array}$$

Proof. Let $C_i(X)_{i\in\mathbb{Z}}$ be the singular chain complex of X or any complex of free abelian groups which is chain homotopy equivalent to it. Let

$$C^i(X;G) := \mathrm{Hom}(C_i(X),G)$$

denote the dual complex with coefficients in G. The coefficient homomorphism is defined on generators $c\otimes g$ with $c\in C^i(X)$, $g\in G$ by

$$\begin{aligned} C^i(X;\mathbb{Z})\otimes G &\to C^i(X;G) \\ c\otimes g &\mapsto c\cdot g = (x\mapsto c(x)\cdot g) \end{aligned}$$

Analogous notations and statements are valid for A and (X,A).

Consider the diagram chase for the boundary homomorphism δ. The following diagram makes it obvious that for computing $\delta(c\cdot g)$ instead of $\delta(c)$, every element can be tensored by g.

$$\begin{array}{ccccccccc} 0 & \longleftarrow & C^n(A;\mathbb{Z}) & \longleftarrow & C^n(X;\mathbb{Z}) & \longleftarrow & C^n(X,A;\mathbb{Z}) & \longleftarrow & 0 \\ & & \downarrow & c \leftarrow\!\shortmid\, c' & \downarrow & & \downarrow & & \\ & & & 0 \leftarrow\!\shortmid\, c'' \longleftarrow & & \shortmid\, \delta c & & & \\ 0 & \longleftarrow & C^{n+1}(A;\mathbb{Z}) & \longleftarrow & C^{n+1}(X;\mathbb{Z}) & \longleftarrow & C^{n+1}(X,A;\mathbb{Z}) & \longleftarrow & 0 \end{array}$$

leads to

$$\begin{array}{ccccccccc} 0 & \longleftarrow & C^n(A;G) & \longleftarrow & C^n(X;G) & \longleftarrow & C^n(X,A;G) & \longleftarrow & 0 \\ & & \downarrow & c\cdot g \leftarrow\!\shortmid\, c'\cdot g & \downarrow & & \downarrow & & \\ & & & 0 \leftarrow\!\shortmid\, c''\cdot g \longleftarrow & & \shortmid\, (\delta c)\cdot g & & & \\ 0 & \longleftarrow & C^{n+1}(A;G) & \longleftarrow & C^{n+1}(X;G) & \longleftarrow & C^{n+1}(X,A;G) & \longleftarrow & 0 \end{array}$$

(Note that the rows are exact since each C_i is free.)

Hence, we have $\delta(c\cdot g) = \delta(c)\cdot g$. □

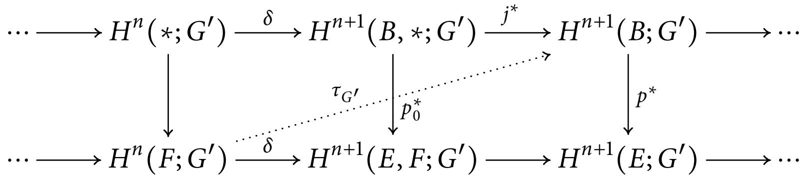
Lemma 58

Let $F\to E\xrightarrow{p} B$ be a fibration and G an abelian group. The coefficient homomorphism $H^*(-)\otimes G\to H^*(-;G)$ commutes with the transgression, i. e. the following square is commutative for all n:

$$\begin{array}{ccc} H^n(F)\otimes G & \xrightarrow{\tau_{\mathbb{Z}}\otimes \mathrm{id}} & H^{n+1}(B)\otimes G \\ \downarrow & & \downarrow \\ H^n(F;G) & \xrightarrow{\tau_G} & H^{n+1}(B;G) \end{array}$$

Proof. As in the previous proof, it is shown that in the diagram chase that defines the transgression, all elements can be tensored by $g \in G$ to obtain the transgression with G-coefficients.

Let p_0 denote the fibration $(E, F) \to (B, *)$. The map j denotes the inclusion $B \subset (B, *)$. The transgression (with any coefficients G') is then defined by the following scheme. The rows are the relative exact cohomology sequences, and the vertical maps are induced by the projections p and p_0.

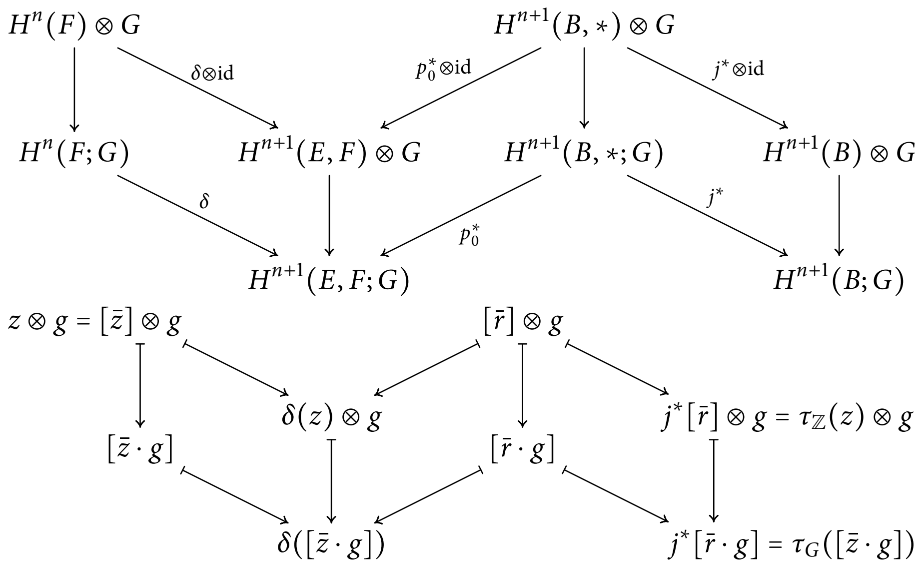


In a formula:

$$\begin{aligned} \tau : \delta^{-1}(\operatorname{im} p_0^*) &\to H^{n+1}(B;G')/j^*(\ker p_0^*) \\ z &\mapsto [j^*(r)], \end{aligned}$$

where $p_0^*(r) = \delta z$.

The transgression with coefficients in G on $z \cdot g \in H^n(F;G)$ can now be obtained by the same chain of elements as for z, except that everything is tensored with g. Indeed, let $\bar{z}$ and $\bar{r}$ be cocycles that represent the cohomology classes z and r resp. The following diagram displays the diagram chase for $\mathbb{Z}$- and G-coefficients side by side.



The bottom left quadrangle commutes because of Lemma 57. The other quadrangles commute because the coefficient map is natural. $\square$

Proof of Proposition 56. Consider the following commutative diagram, which relates the transgression to the various coefficient maps.

$$
\begin{array}{ccc}
\mathrm{Hom}(H^n(F), H^n(F)) & \xrightarrow{\mathrm{Hom}(\mathrm{id},\tau)} & \mathrm{Hom}(H^n(F), H^{n+1}(B)) \\
\text{(UCT 2)}\uparrow & & \uparrow \\
H_n(F; H^n(F)) & \xrightarrow[\text{coefficient change}]{\tau_*} & H_n(F; H^{n+1}(B)) \\
\text{(UCT 3)}\uparrow & & \uparrow \\
H^n(F) \otimes H_n(F) & \xrightarrow{\tau_{\mathbb{Z}} \otimes \mathrm{id}} & H^{n+1}(B) \otimes H_n(F) \\
\text{(UCT 4)}\downarrow & & \downarrow \\
H^n(F; H_n(F)) & \xrightarrow{\tau_{H_n(F)}} & H^{n+1}(B; H_n(F)) \\
\text{(UCT 1)}\downarrow & & \vdots \\
\mathrm{Hom}(H_n(F), H_n(F)) & \longrightarrow & \mathrm{Hom}(H_{n+1}(B), H_n(F))
\end{array}
$$

The vertical maps are coefficient maps, labelled accordingly. They are all isomorphisms except for the dotted arrow at the bottom right. The horizontal arrows marked $\tau_{\mathbb{Z}} \otimes \mathrm{id}$ and $\tau_{H_n(F)}$ are the transgressions with the respective coefficients. The square which connects them commutes by Lemma 58. Define all other horizontal maps as the maps which are induced by the coefficient isomorphisms, so commutativity in the other squares is a tautology.

Remembering the maps in the universal coefficient theorems, one easily sees that

- the homomorphism marked τ_* is induced by the coefficient change τ,
- the topmost homomorphism is given by $\mathrm{Hom}(\mathrm{id}, \tau)$.

Besides, the isomorphisms in the left column map $\mathrm{id} \in \mathrm{Hom}(H^n(F), H^n(F))$ to $\mathrm{id} \in \mathrm{Hom}(H_n(F), H_n(F))$.

As all relevant maps are identified now, we can, using Lemma 55, see that elements are mapped in the following way:

$$
\begin{array}{ccc}
\mathrm{id} & \mapsto & \tau \\
\updownarrow & & \updownarrow \\
\bullet & & \bullet \\
\updownarrow & & \updownarrow \\
\bullet & & \bullet \\
\updownarrow & & \updownarrow \\
\Delta & \mapsto & k^{n+1} \\
\updownarrow & & \\
\mathrm{id} & &
\end{array}
$$

Thus, the maps are exactly as stated in Proposition 56. □

4.3.1 The 9-dimensional example

Theorem 59

There exists a simply-connected, closed, smooth, 9-dimensional manifold which does not admit an orientation-reversing self-homotopy equivalence.

Summary of proof. We exhibit an obstruction to amphicheirality in the Postnikov tower. It is a combination of rational and integral information. First, we construct a candidate for the Postnikov approximation $P^4 \to P^3 \to P^2$ of the desired manifold M together with a candidate for the image of the fundamental class $m \in H_9(P^4)$. We show that there are very few automorphisms of $H_2(P^3)$ that can be induced from a self-homotopy equivalence $P^3 \to P^3$.

Let $P^4_{(0)} \to P^3_{(0)} \to P^2_{(0)}$ be the corresponding rational Postnikov tower and denote by $m_\mathbb{Q}$ the image of m in $H_9(P^4_{(0)})$. We show that $m_\mathbb{Q}$ cannot be reversed by a self-map of $P^4_{(0)}$ that induces one of the above automorphisms on H_2 (tensored with $\mathbb{Q}$).

A short bordism argument shows that there really is a 9-dimensional manifold M together with a map $g : M \to P^4$ inducing the correct image of the fundamental class, i. e. $g_*[M] = m$. By surgery, we alter M to M' so that $g' : M' \to P^3$ is a 4-equivalence and $g : M' \to P^4 \to P^4_{(0)}$ is rationally a 5-equivalence. Due to functoriality of the Postnikov approximations (see the remark below) P^3 and $P^4_{(0)}$, M' is homotopically chiral. □

Construction and automorphisms of P^3

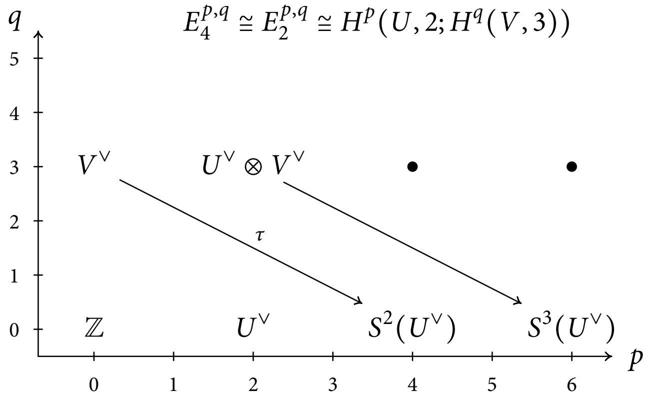
We start with a candidate for the Postnikov tower of fibrations $P^4 \to P^3 \to P^2$ of the desired manifold M. As the base, we choose $P^2 \cong K(U,2)$ with $U \cong \mathbb{Z}^3$. We fix a basis (a,b,c) of the dual group $U^\vee := \operatorname{Hom}(U,\mathbb{Z})$. Likewise, we let $V \cong \mathbb{Z}^3$ and fix a basis (A,B,C) of the dual group $V^\vee$. The space P^3 is defined as a principal fibration over P^2 with the fibre $K(V,3)$. By Proposition 56, there is a bijection between the possible k-invariants and the first differential in the Serre spectral sequence. This correspondence allows us to define the fibration by its transgression

$$\tau : V^\vee \to S^2(U^\vee)$$
$$A \mapsto bc, \quad B \mapsto 2ac, \quad C \mapsto 3ab.$$

Here, we have used that the base is homotopy equivalent to $(\mathbb{C}\mathrm{P}^\infty)^3$, whose cohomology algebra is the polynomial algebra $\mathbb{Z}[a,b,c] = S^*(U^\vee)$.

The cohomology of P^3 can be computed by the Serre spectral sequence. The fibre $K(V,3)$ has no other nontrivial cohomology groups in degree ≤ 5 apart from $H^0(\mathbb{Z}^3,3) \cong \mathbb{Z}$ and $H^3(\mathbb{Z}^3,3) \cong V^\vee \cong \mathbb{Z}^3$. This can either be proved by a very short argument with the spectral sequence for the path-loop fibration $K(V,2) \to * \to K(V,3)$ or by more general results about the homology of Eilenberg-MacLane spaces [EMcL, Section 23], the Künneth and universal coefficient theorems.

The following diagram shows the part of the spectral sequence that is necessary to compute $H^i(P^3)$ for $i \leq 5$. Zero entries in the E_2-page are left blank, while non-zero entries are either specified exactly or marked with a dot •.



A short computation immediately gives the following cohomology groups:

i	$H^i(P^3)$	generators
0	$\mathbb{Z}$	1
1	0	
2	$U^\vee$	a, b, c
3	0	
4	$\mathbb{Z}^3 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$	a^2, b^2, c^2, ac, ab
5	$\mathbb{Z}^2$	$2aA - bB, 3aA - cC$

Lemma 60

Let $T : P^3 \to P^3$ be a homotopy equivalence. Then the induced map on H^2 is necessarily of the form

$$\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix} \tag{3}$$

with respect to the basis (a, b, c).

Proof. Since P^2 is an Eilenberg-MacLane space $K(U, 2)$ and the projection $P^3 \to P^3$ induces an isomorphism on H^2 with any coefficients, the map T and the Postnikov fibrations can be complemented to a homotopy-commutative square

$$\begin{array}{ccc} P^3 & \xrightarrow{T} & P^3 \\ \downarrow & & \downarrow \\ P^2 & \longrightarrow & P^2 \end{array}$$

By the homotopy lifting property of a fibration, the map T is homotopic to a fibre-preserving map T'. This yields a restriction to the fibre, $T'_{|K(V,3)}$, in addition to the induced map on the base $K(U,2)$. For simplicity, we write the induced maps in cohomology simply as T^*. From the functoriality of the Serre spectral sequence, we get

$$T^*\tau(v) = \tau(T^*v) \tag{4}$$

for every $v \in V^\vee$.

Express the induced map on $H^2(P^3) = U^\vee$ by a matrix

$$M := \begin{pmatrix} g & h & i \\ k & l & m \\ p & q & r \end{pmatrix} \in M(3 \times 3; \mathbb{Z}).$$

By (4), we have

$$\tau(T^*C) = T^*(\tau(C)) = T^*(3ab) = 3(ga + kb + pc)(ha + lb + qc).$$

Since the right hand side is in the image of τ, the coefficients of a^2, b^2 and c^2 must be zero, i. e. $gh = kl = pq = 0$. Considering the images of A and B in the same manner, we obtain that in every row of M, the product of two arbitrary entries must vanish. Thus, in every row of M, there is at most one nonzero entry.

Since M is a unimodular matrix, it must be the product of a permutation matrix and a diagonal matrix with eigenvalues ±1. We want to show that the only possible permutation is the identity.

Suppose that the permutation is a transposition, e. g. $(a \leftrightarrow b)$. This would imply $\tau(T^*A) = T^*(\tau(A)) = T^*(bc) = \pm ac$ but only multiples of $2ac$ are in the image of τ. Likewise, the other transpositions $(b \leftrightarrow c)$ and $(a \leftrightarrow c)$ as well as the 3-cycles $(a \to b \to c)$ and $(c \to b \to a)$ are excluded. □

A note on functoriality

As was already indicated, the fibration $P^3 \to P^2$ shall eventually be the beginning of the Postnikov tower of a manifold M. Since automorphisms of M are considered it is crucial for the following arguments that the Postnikov approximations are functorial. In our context, the Postnikov approximations are always built as principal fibrations with the fibre an Eilenberg-MacLane space. Let $\mathcal{C}$ denote the category of spaces having the homotopy type of a simply-connected CW-complex with basepoint. By the following two technical prerequisites it can be shown inductively that every Postnikov stage can be constructed within $\mathcal{C}$:

- If $K(\pi, n)$ is chosen to be in $\mathcal{C}$, then so is its loop space by [Milnor59, Cor. 3].
- In a fibration with path-connected base space in $\mathcal{C}$ and the fibre in $\mathcal{C}$, also the total space lies in $\mathcal{C}$, see [FP, Thm. 5.4.2].

Thus, obstruction theory is available for the Postnikov spaces, and it follows almost immediately that given a map of spaces $X \to Y$, there is always an induced map on their Postnikov approximations $P^k_X \to P^k_Y$, unique up to homotopy, such that the diagram

$$\begin{array}{ccc} X & \longrightarrow & P^k_X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & P^k_Y \end{array}$$

commutes up to homotopy. The same argument holds for the rational Postnikov approximations.

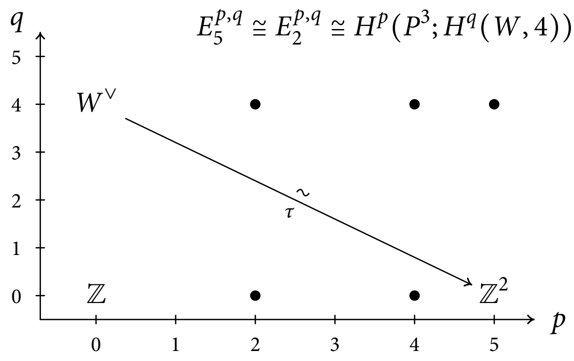
In the arguments given here, we do not relate the induced maps between successive stages in the tower of fibrations. (For $P^3 \to P^2$, it was done, though, explicitly and elementary in the proof of Lemma 60, and for $P^4_{(0)} \to P^3_{(0)} \to P^2_{(0)}$ we will refer to the properties of minimal models instead of spaces.) The full naturality statement involving all Postnikov stages at once would be [Kahn, Thm. 2.2].

Construction of P^4 and m

The next Postnikov stage, P^4, is again constructed as a pullback of the path-loop fibration. We choose the fibre as a $K(W,4)$ with $W \cong \mathbb{Z}^2$ and a basis α, β of the dual group $W^\vee$. The k-invariant is again determined by the transgression, which is chosen as the isomorphism

$$\tau \colon W^\vee \to H^5(P^3)$$
$$\alpha \mapsto 2aA - bB, \quad \beta \mapsto 3aA - cC.$$

The spectral sequence for this fibration immediately shows that $H^5(P^4) = 0$ and therefore $H_5(P^4;\mathbb{Q}) = 0$. (This result is needed later in Proposition 64.)



Lemma 61

There is a class $m \in H_9(P^4)$ such that

- *m is an element of infinite order,*

- *the image of m in $H_9(P^4_{(0)})$ is never mapped to its negative under any self-map of $P^4_{(0)}$ such that the induced map on $H^2(P^4_{(0)})$ is of the form* (3).

By $P^4_{(0)}$, we mean the rational localisation of P^4, as described in [GM, Ch. 7]. The above properties of m obviously remain if m is replaced by a nonzero multiple.

Proof. Consider the rational cohomology of $P^4_{(0)}$. The minimal model for it (uniquely determined up to isomorphism) is the free, graded-commutative, rational differential graded algebra

$$\mathfrak{M} := \mathbb{Q}[a', b', c', A', B', C', \alpha', \beta']$$

with degrees $|a'| = |b'| = |c'| = 2$, $|A'| = |B'| = |C'| = 3$ and $|\alpha'| = |\beta'| = 4$ and differentials

$$da' = db' = dc' = 0,$$
$$dA' = b'c', \quad dB' = 2a'c', \quad dC' = 3a'c',$$
$$d\alpha' = 2a'A' - b'B', \quad d\beta' = 3a'A' - c'C'.$$

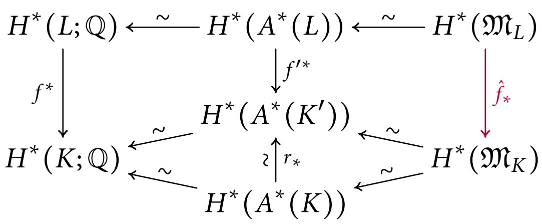
The generators are chosen so that $a' \in \mathfrak{M}^2$ maps to $a \in H^2(P^4_{(0)})$ under the isomorphisms

$$H^*(\mathfrak{M}) \cong H^*(P^4_{(0)}; \mathbb{Q}) \cong H^*(P^4; \mathbb{Q}), \tag{5}$$

and likewise for the other generators. This correspondence is natural. For the second isomorphism above, this follows immediately from the universal property of a localisation. However, since the minimal model is only determined up to some (noncanonical) isomorphism, the naturality of the first isomorphism must be stated carefully.

Lemma 62

Let K, L be simply-connected simplicial complexes. As in [GM, Ch. VIII.A], *denote by $A^*(K)$ the differential graded algebra of piecewise linear, polynomial differential forms on K. Choose minimal models $\mathfrak{M}_K \to A^*(K)$ and $\mathfrak{M}_L \to A^*(L)$. Let $f : K \to L$ be a map. Then there is an induced homomorphism $\hat{f} : \mathfrak{M}_L \to \mathfrak{M}_K$, unique up to homotopy, such that the following diagram commutes:*



Here, K' is a suitable rational subdivision of K, $r : A^*(K) \to A^*(K')$ the restriction of forms and $f' : K' \to L$ a simplicial map homotopic to f.

It follows from this lemma that the induced map $\hat{f}_*$ on cohomology exists and it is unique, so the isomorphisms in (5) are natural.

Proof. The horizontal maps in the left half are isomorphisms by the "piecewise linear deRham theorem" [GM, Ch. VIII.A]. The lower left triangle consists of isomorphisms and commutes because of [GM, Lemma 8.5] ("Naturality under subdivision"). In the right half, the map $H^*(\mathfrak{M}_K) \to H^*(A^*(K'))$ is simply defined to be the composition of the other two maps in the lower right triangle. Note that $\mathfrak{M}_K$ is then a minimal model for $A^*(K)$ as well as for $A^*(K')$ because $r: A^*(K) \to A^*(K')$ induces an isomorphism in cohomology. The horizontal maps in the right half are isomorphisms by the definition of a minimal model.

The upper left quadrangle commutes, since the horizontal maps are induced by the cochain map

$$\rho : A^*(K') \to C^*(K'; \mathbb{Q}),$$

defined by $\langle \rho(\omega), \Delta^n \rangle = \int_{\Delta^n} \omega$, and integration is natural:

$$\int_{\Delta^n} f'^* \omega = \int_{f'\Delta^n} \omega,$$

where $f'\Delta^n$ is the (oriented) simplex in L spanned by the images of the vertices in $\Delta^n \in K'$. (See also [GM, Ch. VIII.A].)

The existence of $\hat{f}$ and the commutativity of the upper right quadrangle is [GM, Cor. 10.11]. □

Consider the element $(d\alpha)\beta - ABC \in \mathfrak{M}^9$. It is easily verified that this element is closed, thus it represents a cohomology class $\overline{m}_{\mathbb{Q}} \in H^9(\mathfrak{M}) \cong H^9(P^4_{(0)})$. The cohomology class is nonzero since there is no expression in $\mathfrak{M}^8$ whose differential contains a summand ABC.

Let $m_{\mathbb{Q}} \in H_9(P^4_{(0)})$ be a homology class such that $\langle \overline{m}_{\mathbb{Q}}, m_{\mathbb{Q}} \rangle \in \mathbb{Q}$ is nonzero. The class $m_{\mathbb{Q}}$ itself might not be in the image of $H_9(P^4) \to H_9(P^4_{(0)})$ but a nonzero multiple of $m_{\mathbb{Q}}$ certainly is. We replace $m_{\mathbb{Q}}$ by this multiple and choose a preimage $m \in H_9(P^4)$.

Now consider an automorphism of $\mathfrak{M}$. Note that the differentials in every Hirsch extension which is used to build $\mathfrak{M}$ are injective, i. e. d is injective on the vector spaces $\mathbb{Q}\{A, B, C\}$ and $\mathbb{Q}\{\alpha, \beta\}$. For this reason, the automorphism of $\mathfrak{M}$ is completely determined by the restriction to the base degree $\mathfrak{M}^2 = \mathbb{Q}\{a, b, c\}$.

Let T_a be the automorphism of $\mathfrak{M}^2$ which is given by

$$a \mapsto -a, \quad b \mapsto b, \quad c \mapsto c.$$

The automorphism T_a extends uniquely to $\mathfrak{M}$ by

$$A \mapsto A, \quad B \mapsto -B, \quad C \mapsto -C,$$
$$\alpha \mapsto -\alpha, \quad \beta \mapsto -\beta.$$

It can be quickly checked that T_a fixes $\overline{m}_{\mathbb{Q}}$. Likewise, the automorphism T_b and T_c which reverse b resp. c fix $\overline{m}_{\mathbb{Q}}$. Hence, every automorphism T of P^4 that induces a diagonal matrix of the form (3) on $H^2(P^4_{(0)}) \cong H^2(\mathfrak{M}) \cong \mathfrak{M}^2$ fixes $\overline{m}_{\mathbb{Q}}$. Since the evaluation is natural, we have

$$\langle T_* m_{\mathbb{Q}}, \overline{m}_{\mathbb{Q}} \rangle = \langle m_{\mathbb{Q}}, T^* \overline{m}_{\mathbb{Q}} \rangle = \langle m_{\mathbb{Q}}, \overline{m}_{\mathbb{Q}} \rangle,$$

so $m_{\mathbb{Q}}$ cannot be reversed by T. The same clearly holds for m. □

Bordism argument

Proposition 63

There is a framed, 9-dimensional, closed, smooth manifold M together with a map $g : M \to P^4$ such that $g_*[M]$ is a nonzero multiple of $m \in H_9(P^4)$.

Proof. This proposition can be reformulated as follows: There is an element $[M, g] \in \Omega_9^{\mathrm{fr}}(P^4)$ that maps to a nonzero multiple of m under the Thom homomorphism

$$\Omega_9^{\mathrm{fr}}(P^4) \to H_9(P^4)$$
$$[M, g] \mapsto g_*[M].$$

The Thom homomorphism is in fact the edge homomorphism

$$\Omega_9^{\mathrm{fr}}(P^4) \twoheadrightarrow E^{\infty}_{9,0} \hookrightarrow E^2_{9,0} \cong H_9(P^4)$$

in the Atiyah-Hirzebruch spectral sequence for the homology theory Ω_*^{fr}.

The Atiyah-Hirzebruch spectral sequence for bordism homology theories lies in the first quadrant. Moreover, for framed bordism, all coefficient groups Ω_i^{fr} for $i > 0$ are finite abelian groups since $\Omega_i^{\mathrm{fr}} \cong \pi_i^s$ according to the Pontrjagin-Thom theorem. Thus, each of the (finitely many) differentials starting from $E^2_{9,0}$ has a nonzero multiple of m in the kernel, so a nonzero multiple of m survives to the E^{∞}-page. □

Surgery

So far, the manifold M can still be amphicheiral: even though its image of the fundamental class in P^4 is irreversible, the maps $M \to P^3$ and $M \to P^4_{(0)}$ are not necessarily functorial (not even up to homotopy). This crucial condition would hold if P^3 and $P^4_{(0)}$ were Postnikov approximations of M. Thus, we aim to replace M by surgery with a manifold M' such that the corresponding map $g' : M' \to P^4$ is a 4-equivalence and rationally a 5-equivalence. Note also that M' is then automatically simply-connected.

Since M is framed, its stable normal bundle $\nu : M \to BO$ is trivial. Thus, there is a lift of ν to the path space $EO \simeq PBO \simeq *$. Fix any such lift $\hat{\nu} : M \to EO$.

Together with the map g from the previous proposition, we have a fibration and a lift

$$\begin{array}{ccc} & & B \\ & \overset{\bar\nu}{\nearrow} & \downarrow \xi \\ M & \longrightarrow & BO \end{array} \quad := \quad \begin{array}{ccc} & & P^4 \times EO \\ & \overset{g\times\hat\nu}{\nearrow} & \downarrow \mathrm{ev}_1 \circ \mathrm{pr}_2 \\ M & \longrightarrow & BO \end{array}$$

The lift $\bar\nu$ is a *normal B-structure* on M in the language of [Kreck99, Section 2]. We refer to the discussion of [Kreck99, Prop. 4] on page 56 f. By this proposition, $[M, g]$ is bordant over P^4 to $[M', g']$ such that g' is a 4-equivalence. The proof of Theorem 59 is completed as soon as the following proposition is shown. (Denote the manifold to be dealt with again by M, which is M' from the current paragraph.)

Proposition 64

Let M be an m-dimensional, closed, smooth, simply-connected manifold with normal B-structure $\bar\nu : M \to B$ which is a $\left[\frac{m}{2}\right]$-equivalence. Assume that m is odd and at least 5. Also assume that $H_{[m/2]+1}(B;\mathbb{Q}) = 0$. Then, by a finite sequence of surgeries $(M, \bar\nu)$ can be replaced by $(M', \bar\nu')$ such that $\bar\nu' : M' \to B$ is again a 4-equivalence and additionally $\pi_{[m/2]+1}(B, M') \otimes \mathbb{Q} = 0$.

Let B' be the mapping cylinder of $\bar\nu$. Denote $[m/2]$ shortly by r. Compare the long exact sequences for relative homotopy and homology.

$$\begin{array}{ccccccccc} \pi_{r+1}(B)\otimes\mathbb{Q} & \longrightarrow & \pi_{r+1}(B', M)\otimes\mathbb{Q} & \longrightarrow & \pi_r(M)\otimes\mathbb{Q} & \xrightarrow{\bar\nu_*} & \pi_r(B)\otimes\mathbb{Q} & \longrightarrow & 0 \\ \downarrow & & \downarrow \wr & & \downarrow & & \downarrow & & \\ H_{r+1}(B;\mathbb{Q}) & \longrightarrow & H_{r+1}(B', M;\mathbb{Q}) & \longrightarrow & H_r(M;\mathbb{Q}) & \xrightarrow{\bar\nu_*} & H_r(B;\mathbb{Q}) & \longrightarrow & 0 \end{array}$$

The vertical maps are the respective Hurewicz maps. By the relative Hurewicz theorem, $H_r(B', M) = 0$ and $\pi_{r+1}(B', M) \to H_{r+1}(B', M)$ is an isomorphism. Since $H_{r+1}(B;\mathbb{Q}) = 0$ it suffices to make $\bar\nu_* : H_r(M;\mathbb{Q}) \to H_r(B;\mathbb{Q})$ injective. If this is the case, $H_{r+1}(B', M;\mathbb{Q}) \cong \pi_{r+1}(B', M) \otimes \mathbb{Q} = 0$. Since $H_r(M)$ is finitely generated, it suffices to decrease the rank of $\ker(H_r(M;\mathbb{Q}) \to H_r(B;\mathbb{Q}))$ by one in each surgery step. Instead of Proposition 64 we can thus prove the following statement.

Proposition 65

Let M be an m-dimensional, closed, smooth, simply-connected manifold with normal B-structure $\bar\nu : M \to B$ which is a $\left[\frac{m}{2}\right]$-equivalence. Assume that m is odd and ≥ 5. By attaching a $\left[\frac{m}{2}\right]$-handle to $M \times I$ which extends the B-structure, one can obtain that the rank of

$$\ker(\bar\nu'_* : H_{[m/2]}(M';\mathbb{Q}) \to H_{[m/2]}(B;\mathbb{Q}))$$

is one lower than for $\bar\nu_*$, while $\bar\nu'$ is still a $\left[\frac{m}{2}\right]$-equivalence.

As usual, the result of the surgery is denoted by M' and its normal B-structure by $\bar{\nu}' : M' \to B$.

Proof. Recall the abbreviation $r = \left[\frac{m}{2}\right]$. Let s be an element in the kernel of $\bar{\nu}_* : H_r(M) \to H_r(B)$ which is indivisible and of infinite order. By [Kreck99, Lemma 2 i)], s can be represented by an embedding $f : S^r \times D^{m-r} \hookrightarrow M$.

By [Kreck99, Lemma 2 ii)], the embedding can be chosen in a way such that the normal B-structure on M can be extended to the trace of the surgery by an r-handle attached along f. Denote the trace of the surgery by W. Since W is homotopy equivalent to both M with an $(r+1)$-cell attached and to M' with an $(m-r)$-cell attached (and in our case $m - r = r + 1$), the normal B-structure $\bar{\nu}'$ is still an r-equivalence.

It remains to compute the effect on the r-th homology groups. Consider the relative homology sequence of the pair (W, M):

$$H_{r+1}(W, M) \xrightarrow{\partial} H_r(M) \to H_r(W) \to H_r(W, M).$$

Since (W, M) is r-connected, $H_r(W, M)$ is zero. By [Kosinski, Lemma XI.10.1] $H_{r+1}(W, M) \cong \mathbb{Z}$ and the image of ∂ is generated by s.

Considering the analogous sequence for M', we have again a surjection $H_r(M') \twoheadrightarrow H_r(W)$ with the kernel generated by the image of the boundary map $\partial : H_{r+1}(W, M') \to H_r(M')$. Pictorially, the image of ∂ is generated by the homology class of the *meridian* $\{0\} \times S^{m-r-1} \hookrightarrow M$. In the following, we prove that the meridian is nullhomologous in $M \setminus S^r$ if and only if there is an element in $H_{m-r}(M)$ which intersects the embedded sphere S^r with intersection number one. This is the geometric meaning of Proposition 66 below, although its statement and proof are purely in terms of homology and we identify the image of the boundary map ∂ without appealing to its geometric meaning.

Denote the embedded image of $S^r \times D^{m-r}$ by $D\tau$ (like the unit disk bundle in the normal bundle alias a tubular neighbourhood τ of S^r). Its boundary in M is accordingly denoted by $S\tau$ and its interior by $\mathring{D}\tau$. By excision and homotopy invariance, the left vertical map in the commutative diagram below is an isomorphism.

$$\begin{array}{ccc} H_{r+1}(W, M') & \xrightarrow{\partial} & H_r(M') \\ \uparrow \wr & & \uparrow \\ H_r(M, M \setminus \mathring{D}\tau) & \xrightarrow{\partial} & H_r(M \setminus \mathring{D}\tau) \end{array}$$

Hence, if the lower boundary map is zero, the upper one is zero, too. The necessary information about the lower boundary map is given by the following proposition. It holds even for an embedding $S^r \hookrightarrow M$ with nontrivial normal bundle and arbitrary $r > 0$ but we apply it only in the setting with trivial normal bundle and $r = \left[\frac{m}{2}\right]$.

Proposition 66

Let M be a closed, oriented m-dimensional manifold and $S^r \hookrightarrow M$ an embedded r-sphere. The boundary map

$$\partial : H_{r+1}(M, M \setminus \mathring{D}\tau) \to H_r(M)$$

has image zero if and only if the homology class of S^r is indivisible and of infinite order.

This proposition finally proves Proposition 65 and thus Theorem 59: Since $s \in \ker(\bar{\nu}_*)$ is of infinite order and indivisible, we have a commutative diagram

$$\begin{array}{ccccc} H_r(M)/\mathbb{Z}s & \xrightarrow{\sim} & H_r(W) & \xleftarrow{\sim} & H_r(M') \\ & {\scriptstyle \bar{\nu}_*}\searrow & \downarrow & \swarrow{\scriptstyle \bar{\nu}'_*} & \\ & & H_r(B) & & \end{array}$$

□

Proof of Proposition 66. Let i be the inclusion $S^r \hookrightarrow M$. By the universal coefficient theorem (UCT 1), $i_*[S^r]$ is of infinite order and indivisible if and only if there is a cohomology class $\sigma \in H^r(M)$ such that $\langle \sigma, i_*[S^r] \rangle = 1$. This is in turn equivalent to $\langle i^*\sigma, [S^r] \rangle = 1$ in S^r. Since $H^r(S^r) \cong \mathbb{Z}$, this is finally equivalent to the statement

There exists $\sigma \in H^r(M)$ such that $i^*\sigma$ is a generator of $H^r(S^r) \cong \mathbb{Z}$.

Consider now the following commutative diagram with exact upper row:

$$\begin{array}{ccccc} H_{n-r}(M) & \longrightarrow & H_{n-r}(M, M \setminus \mathring{D}\tau) & \xrightarrow{\partial} & H_{n-r-1}(M \setminus \mathring{D}\tau) \\ & & \wr\uparrow \circledast & & \\ \wr\uparrow{\scriptstyle \cap[M]} & & H_{n-r}(D\tau, S\tau) & & \\ & & \wr\uparrow{\scriptstyle \cap[D\tau, S\tau]} & & \\ H^r(M) & \longrightarrow & H^r(D\tau) & & \\ & {\scriptstyle i^*}\searrow & \wr\downarrow & & \\ & & H^r(S^r) & & \end{array}$$

Regardless of which maps are described here, a simple diagram chase shows that the image of ∂ is zero if and only if i^* is surjective. The proof is completed by a description of the maps and the proof of commutativity.

The upper row is part of the relative homology sequence. The bottom triangle consists of the obvious restrictions induced by inclusions and therefore commutes. The map labelled $\circledast$ is the excision isomorphism. The remaining two vertical maps are Poincaré duality isomorphisms. As the last step, consider the upper left square.

$$\begin{array}{ccc}
H_{n-r}(M) & \longrightarrow & H_{n-r}(M, M \smallsetminus \mathring{D}\tau) \\
\uparrow {\scriptstyle \cap[M]} & \nearrow {\scriptstyle \cap[M, M\smallsetminus \mathring{D}\tau]} & \uparrow {\scriptstyle i^*} \\
 & & H_{n-r}(D\tau, S\tau) \\
 & & \uparrow {\scriptstyle \cap[D\tau, S\tau]} \\
H^r(M) & \xrightarrow{\ i^*\ } & H^r(D\tau)
\end{array}$$

The upper triangle commutes by the definition of the relative cap product. (See [Bredon, Ch. VI.5] for background information.) In the lower triangle, the map i denotes the inclusion of pairs $(D\tau, S\tau) \subset (M, M \smallsetminus \mathring{D}\tau)$ as well as the absolute inclusion $D\tau \subset M$. Note that i_* maps the fundamental class $[D\tau, S\tau]$ to the fundamental class $[M, M \smallsetminus \mathring{D}\tau]$. Thus we have for all $x \in H^r(M)$

$$i_*(i^* x \cap [D\tau, S\tau]) = x \cap i_*[D\tau, s\tau] = x \cap [M, M \smallsetminus \mathring{D}\tau] \qquad \square$$

4.3.2 Extension to dimension 13

Theorem 67

Let M be a manifold as in the previous section with all described properties. The product $N := M \times \mathbb{C}\mathrm{P}^2$ is a simply-connected closed, smooth, 13-dimensional manifold that does not admit an orientation-reversing self-homotopy equivalence.

Proof. Denote by P_N^k the k-th Postnikov stage of N and by $P_{N,(0)}^k$ its rational localisation. The homotopy sequence of the fibration $S^1 \to S^5 \to \mathbb{C}\mathrm{P}^2$ yields $\pi_3(\mathbb{C}\mathrm{P}^2) = 0$. Besides, $\mathbb{C}\mathrm{P}^2$ has the minimal algebra $\mathbb{Q}[x, X]$ with $|x| = 2$, $|X| = 5$ and $dX = x^3$. Thus, we are in a very similar situation as before:

- P_N^2 is homotopy equivalent to $(\mathbb{C}\mathrm{P}^\infty)^4$.
- P_N^3 is homotopy equivalent to $P^3 \times \mathbb{C}\mathrm{P}^\infty$.
- We have a basis (a, b, c, x) of $H^2(P_N^3)$.
- $H^4(P_N^3) \cong S^2(H^2(P_N^3)) = S^2(\mathbb{Z}\{a, b, c, x\})$
- Define a fibration $\widehat{P}_N^4 \to P_N^3$ with the fibre $K(V, 3)$ and k-invariant

$$k^4 : V^\vee \to S^2(\mathbb{Z}\{a, b, c, x\})$$
$$A \mapsto bc, \quad B \mapsto 2ac, \quad C \mapsto 3ab.$$

The space $\widehat{P}^4_N$ is not necessarily the fourth Postnikov stage of N (as P^4 was not necessarily the fourth Postnikov stage of M). However, its localisation $\widehat{P}^4_{N,(0)}$ is the correct rational Postnikov approximation, so we have $\widehat{P}^4_{N,(0)} = P^4_{N,(0)}$.

- The localisation $P^4_{N,(0)}$ has the minimal algebra $\mathfrak{M} \otimes \mathbb{Q}[x]$, where $\mathfrak{M}$ is the rational minimal algebra of $P^4_{(0)}$.
- The fundamental class of N is detected by

$$((d\alpha)\beta - ABC)x^2 \in H^{13}(P^4_{N,(0)}).$$

This shows that the proof is finished as soon as the following analogue to Lemma 60 is proved. □

Lemma 68

Let $T : P^3_N \to P^3_N$ be a homotopy equivalence. Then the induced map on H^2 is necessarily of the form

$$\begin{pmatrix} \pm 1 & 0 & 0 & * \\ 0 & \pm 1 & 0 & * \\ 0 & 0 & \pm 1 & * \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}$$

with respect to the basis (a, b, c, x).

Proof. Express the induced map on $H^2(P^3_N)$ by the matrix

$$\begin{pmatrix} g & h & i & j \\ k & l & m & n \\ p & q & r & s \\ t & u & v & w \end{pmatrix} \in M(4 \times 4; \mathbb{Z}).$$

Since $k^4(T^*C) = T^*(k^4(C))$ cannot have summands containing x, we have $th = tl = tq = tu = 0$. But since the second column must contain a nonzero entry, t must be zero. Likewise, we prove $u = 0$ and $v = 0$. From here on, the proof proceeds exactly as in Lemma 60. □

5

Bordism questions

So far, examples of chiral manifolds have been obtained in all dimensions where this is possible, both for manifolds with arbitrary fundamental group and for simply-connected manifolds. Since the dimension is a very coarse invariant for manifolds, it is an interesting question how dense (or sparse) chiral and amphicheiral manifolds are with respect to a finer differentiation. The most definite answer, of course, would be to regard chirality/amphicheirality itself as an invariant and to express it in terms of other invariants which are computable in some way. (The word "invariant" refers to homotopy/homeomorphism/diffeomorphism invariants of manifolds, depending on the category.)

This ultimate question, however, can presumably not be answered since there are too many possible obstructions to amphicheirality on all levels (homotopy, homeomorphism and diffeomorphism). This is illustrated in chapter 6 when we prove smooth amphicheirality for a certain class of manifolds in a nontrivial case. For these manifolds, all possible obstructions vanish and the "surgery programme" can be carried out, but the method shows that only slightly more general cases can be very difficult to handle. The author's feeling is that for many concrete families of manifolds, chirality is much easier to detect and to disprove than to classify the members in that family, but a general answer might be as impossible as a general classification of all manifolds.

Historically, the concept of bordism has proved very successful as a distinction of manifolds that is not too coarse and at the same time accessible to classification and computations. Therefore, it is an interesting question which oriented bordism classes contain chiral manifolds and which contain amphicheiral manifolds. Again, there are several flavours of chirality to consider. We can give definite answers (old and new) in one category in each direction (chiral vs. amphicheiral).

We start with the existing results: The question

When is a manifold bordant to a smoothly amphicheiral one?

has been solved previously. If a manifold has a nonzero Pontrjagin number, it cannot be smoothly amphicheiral. On the other hand, if all Pontrjagin numbers are zero, the following theorem states that the orientation of some manifold in the bordism class can be reversed even by an involution.

Theorem 69: [Kawakubo], [Rosenzweig, p. 5, ll. 13–20]

An element x of Ω_* has a representative which admits an orientation-reversing involution if and only if x is a torsion element of Ω_*.

The involutions considered in the two papers are constructed as diffeomorphisms. Moreover, the manifold representative in [Kawakubo] is connected.

In the opposite direction, recall that in low dimensions, the answer is immediate: a set of points is chiral if and only if its bordism class is nonzero; all one- and two-dimensional manifolds are smoothly amphicheiral. For all bordism classes in higher dimensions, the following result is proved in the next sections:

Theorem 70

In every dimension ≥ 3, every closed, smooth, oriented manifold is oriented bordant to a manifold of this type which is connected and homotopically chiral.

Summary of proof. The proof is split into three cases for different dimensions: Proposition 71 deals with all odd dimensions, and Proposition 74 handles the even dimensions greater than four. In dimension four, the bordism classes are detected by the signature [MS, Ch. 17,19], and manifold with nonzero signature is homotopically chiral.

Section 5.3 deals with the remaining single case: it is proved that there is a homotopically chiral 4-dimensional manifolds with signature zero (Theorem 81). □

Sections 5.1 and 5.2 do not contain new obstructions for our list. Instead, it is proved in these sections that the existing obstructions can be preserved when the bordism class of a manifold is changed. For the 4-dimensional manifolds with signature zero in Section 5.3, we exploit again the fact that the *fundamental group and its automorphisms* can pose obstructions to homotopical amphicheirality. The approach to the obstruction is purely algebraic using group homology. Then it is proved that the algebraic obstruction is realised geometrically by a manifold with the desired properties (dimension 4, simply-connected, signature 0).

5.1 Odd dimensions ≥ 3

Proposition 71

In every odd dimension ≥ 3 and every oriented bordism class, there is a connected manifold that admits no self-map of degree -1.

To show this, we first prove the following lemma.

Lemma 72

In every odd dimension ≥ 3 and every oriented bordism class, there is a simply-connected representative.

Proof of Lemma 72. Since $\Omega_3 = 0$ [Saveliev99, Cor. 2.5], the statement is trivial in dimension 3. Let M^n be a manifold of dimension $n \geq 4$. Since M is oriented, the stable normal bundle ν has a lift $\bar{\nu}$ to BSO. By [Kreck99, Prop. 4] M is bordant to a manifold M' such that the corresponding lift $\bar{\nu}'$ is at least a 2-equivalence. In particular, M' is simply-connected. □

Lemma 73

Let M^n be a closed, connected manifold of dimension $n \geq 3$ and let $f_1 : M \to K(\pi_1(M),1)$ induce an isomorphism on the fundamental group. Suppose that $f_{1*}[M] \in H_n(K(\pi_1(M),1))$, the image of the fundamental class, is never sent to its negative under any self-map of $K(\pi_1(M),1)$. Let N^n be a simply-connected manifold. Then $M\#N$ does not admit a self-map of degree -1.

Proof. We would like to have a map $M\#N \to M$ that has degree 1 and induces an isomorphism on π_1. Such a map can be defined by the scheme in Figure 5.1 on the following page. The fundamental group of M is not changed by this process because M is at least 3-dimensional.

Thus, the composition $f : M\#N \to M \xrightarrow{f_1} K(\pi_1(M),1)$ is the first Postnikov stage of $M\#N$. Since the Postnikov approximation is functorial up to homotopy and the image of the fundamental class $f_*[M\#N] = f_{1*}[M]$ cannot be mapped to its negative, $M\#N$ is chiral. □

Proof of Proposition 71. In Section 3.1, we constructed manifolds M that fulfill the conditions of Lemma 73 in every odd dimension $n \geq 3$. In fact, these manifolds are aspherical, so for the map f_1, we can use the identity on M. Proposition 71 now follows from the two previous lemmas. Note that no assumption about the bordism class of M is needed: Since N runs through all n-dimensional bordism classes, the connected sum $M\#N$ does as well. □

5.2 Even dimensions ≥ 6

Proposition 74

In every even dimension ≥ 6 and every oriented bordism class, there is a connected manifold that admits no self-map of degree -1.

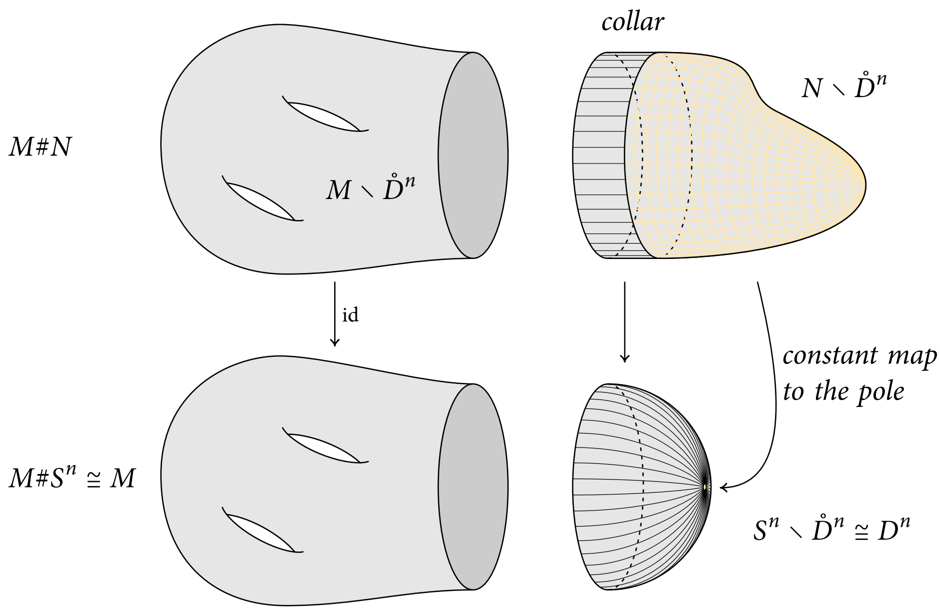


Figure 5.1: A map of degree 1 and an isomorphism on π_1.

Proof. The proposition follows again from Lemmas 72 and 73. Instead of an aspherical manifold M^n, however, we use a product $L^3 \times M^{n-3}$ of the following components:

- L^3 is a 3-dimensional lens space. Let r be the order of the fundamental group. Let $h : L^3 \to L^\infty$ be a map to an infinite-dimensional lens space which induces an isomorphism of the fundamental groups. We require that $h_*[L^3] \in H_3(L^\infty; \mathbb{Z}/r)$ is never mapped to its negative under any self-map of L^∞. This is the case, e. g., for every lens space L^3 such that $r = |\pi_1(L^3)|$ contains a prime factor congruent 3 modulo 4.
- M^{n-3} is an odd-dimensional aspherical chiral manifold, which was constructed in Section 3.1.

The map $f_1 := (\mathrm{id}_M \times h) : M \times L^3 \to M \times L^\infty$ induces an isomorphism on the fundamental groups, and the target is an Eilenberg-MacLane space. In the remaining proof it is shown that the image of the fundamental class

$$(\mathrm{id}_M \times h)_*[M \times L^3] \in H_n(M \times L^\infty; \mathbb{Z}/r)$$

is never mapped to its negative under any self-map of $M \times L^\infty$. This completes the requirements of Lemma 73.

We first determine all endomorphisms of $\pi_1(M \times L^\infty)$ and then study their effect on homology. Since all spaces are connected, we can fix any basepoint, and all self-maps of the respective spaces can be made basepoint-preserving by a homotopy. Since there is no danger of confusion, the basepoint is neglected in the notation of the fundamental group.

Recall the following facts from the construction of M: $G := \pi_1(M)$ is a semidirect product

$$0 \to \mathbb{Z}^k \to G \overset{\curvearrowleft}{\to} \mathbb{Z} \to 0$$

with $k = \dim(M) - 1$, and the right map is the abelianisation of G.

This exact sequence implies that all nontrivial elements of G have infinite order. Indeed, if an element maps to a nonzero number in $\mathbb{Z}$, it certainly has infinite order. On the other hand, if it maps to zero, then it is contained in the subgroup $\mathbb{Z}^k$.

Although there is no notion of the torsion subgroup in a general (non-abelian) group, we do have one in $\pi := \pi_1(M \times L^\infty) = G \times \mathbb{Z}/r$: The subset of all torsion elements in π constitutes the second factor $\mathbb{Z}/r$, hence it is a subgroup.

Denote by i_1, i_2, p_1 and p_2 the inclusions and projections for the two factors G and $\mathbb{Z}/r$ of π. Since the set of torsion elements is preserved under any endomorphism f of π, we can always complete the diagram below with a homomorphism f_T.

$$\begin{array}{ccc} \mathbb{Z}/r & \overset{f_T}{\dashrightarrow} & \mathbb{Z}/r \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle i_2} \\ \pi & \overset{f}{\longrightarrow} & \pi \end{array}$$

Denote the composition $p_1 \circ f \circ i_1$ by $f_G : G \to G$ and $p_2 \circ f \circ i_1$ by $f_S : G \to \mathbb{Z}/r$ ("S" like shear map). In the following, write the groups multiplicatively with neutral element e. We have for $g \in G$, $c \in C_r = \mathbb{Z}/r$

$$\begin{aligned} f(g,c) &= f(g,e) \cdot f(e,c) \\ &= (f_G(g), f_S(g)) \cdot (e, f_T(c)) \\ &= (f_G(g), f_S(g) \cdot f_T(c)). \end{aligned}$$

Therefore, any endomorphism f of π can be decomposed (in this order) into

$$\mathrm{id}_G \times f_T, \quad (\mathrm{id}_G, f_S \cdot \mathrm{id}_{C_r}) \quad \text{and} \quad f_G \times \mathrm{id}_{C_r}.$$

In the following, all homology and cohomology groups are with $\mathbb{Z}/r$-coefficients understood. Denote the image of the fundamental class $h_*[L^3] \in H_3(L^\infty)$ shortly by $[L]$. Since $(\mathrm{id}_M \times h)_*[M \times L^3]$ in $H_n(M \times L^\infty)$ is the homology cross product of the fundamental classes $[M]$ and $[L]$, it is clear that neither f_G nor f_T can reverse the product fundamental class. It is therefore sufficient to study the shear homomorphism $f_S : G \to \mathbb{Z}/r$ and its effect on homology.

For the rest of the proof, assume that $f = (\mathrm{id}_G, f_S \cdot \mathrm{id}_{C_r})$ for some homomorphism $f_S : G \to C_r$. Since we would like to exploit the cup product structure, we work in $\mathbb{Z}/r$-cohomology instead of homology.

Let $[L]^* \in H^3(L^\infty) \cong \mathbb{Z}/r$ be dual to $[L] = h_*[L^3] \in H_3(L^\infty)$. Likewise, let $[M]^* \in H^{n-3}(M)$ be dual to $[M]$. The cross product $[M]^* \times [L]^*$ evaluates on $(\mathrm{id}_M \times h)_*[M \times L^3]$ nontrivially. We identify $[M]^* \in H^{n-3}(M)$ with its image $p_1^*[M]^* \in H^{n-3}(M \times L^\infty)$ and likewise for $[L]^*$, so the cross product above can be written as a cup product $[M]^* \cup [L]^*$. The following two facts are shown:

- $f^*([M]^* \cup [L]^*)$ is a multiple $m \cdot ([M]^* \cup [L]^*)$, $m \in \mathbb{Z}/r$. (Here, the use of f^* for the induced map in cohomology is legitimate since $M \times L^\infty$ is an Eilenberg-MacLane space.)
- The coefficient m is never congruent -1 mod r.

By the particular form of f, we have the following commutative diagram

$$\begin{array}{ccc} G \times \mathbb{Z}/r & \xrightarrow{f} & G \times \mathbb{Z}/r \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_1} \\ G & \xrightarrow{\mathrm{id}_G} & G \end{array}$$

Thus, there is a corresponding diagram of Eilenberg-MacLane spaces which commutes up to homotopy. The induced diagram in cohomology is

$$\begin{array}{ccc} H^{n-3}(M \times L^\infty) & \xleftarrow{f^*} & H^{n-3}(M \times L^\infty) \\ \uparrow{\scriptstyle p_1^*} & & \uparrow{\scriptstyle p_1^*} \\ H^{n-3}(M) & \xleftarrow{\mathrm{id}} & H^{n-3}(M) \end{array}$$

Since $[M]^*$ is in the image of p_1^*, the above diagram then says that $f^*[M]^* = [M]^*$. The transformed fundamental class $[L]^*$, on the other hand, could *a priori* be any element of $H^3(M \times L^\infty) \cong \bigoplus_{i+j=3} H^i(M) \otimes H^j(L^\infty)$. However, only the part in $H^0(M) \otimes H^3(L^\infty)$ gives a nontrivial product with $[M]^*$.

Therefore, $f_*([M]^* \cup [L]^*) = [M]^* \cup f^*([L]^*)$ is a multiple of $[M]^* \cup [L]^*$. The factor is equal to the degree of the induced map on $H_3(L^\infty)$, given by the composition

$$L^\infty \xrightarrow{i_2} M \times L^\infty \xrightarrow{f} M \times L^\infty \xrightarrow{p_2} L^\infty.$$

Since it was assumed that $[L]$ is never mapped to $-[L]$, the proof is complete. □

5.3 Dimension 4 and signature 0

In this section, we prove that there are homotopically chiral 4-dimensional manifolds with signature zero. Since every simply-connected closed 4-manifold is amphicheiral, such a manifold must certainly have a nontrivial fundamental group. We pick up the idea that the obstruction to amphicheirality should already be manifest in the 1-type, as it obviously was in the aspherical odd-dimensional manifolds in Section 3.1.

Proposition 75

Let π be a finitely presented group with the following properties:

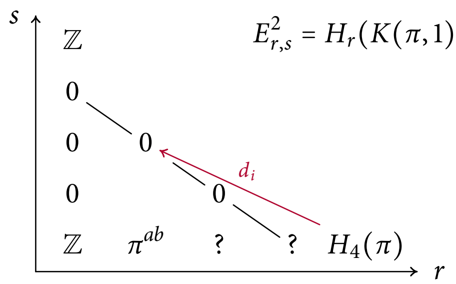
- *Every automorphism of π is an inner automorphism.*
- *There is an element $m \in H_4(\pi)$ of order greater than two.*

Then there is a (closed, connected, smooth, orientable) homotopically chiral 4-manifold with fundamental group π and signature zero.

Proof. Consider the oriented bordism group $\Omega_4(K(\pi,1))$. We show that there is an element $[M,f] \in \Omega_4(K(\pi,1))$ that maps to m under the Thom homomorphism

$$\begin{aligned}\Omega_4(K(\pi,1)) &\to H_4(K(\pi,1))\\ (M,f) &\mapsto f_*[M].\end{aligned}$$

For this, consider the Atiyah-Hirzebruch spectral sequence for $\Omega_*(K(\pi,1))$.



Since there is no differential from or to $E^2_{4,0}$, we have $E^\infty_{4,0} = E^2_{4,0}$, and the Thom homomorphism

$$\Omega_4(K(\pi,1)) \twoheadrightarrow E^\infty_{4,0} = E^2_{4,0} \cong H_4(\pi)$$

is surjective. Let (M',f') be a preimage of m.

By surgery below the middle dimension, (M',f') can be altered to (M,f) in the same bordism class such that $f : M \to K(\pi,1)$ is a 2-equivalence [Kreck99, Prop. 4][1)].

Now f is a first Postnikov approximation map for M, and as such it is functorial. Every homotopy equivalence of M induces an automorphism on $K(\pi,1)$. Since every automorphism of π is inner and inner automorphisms induce the identity on group homology [Brown, Prop. II.6.2], $m = f_*[M]$ is fixed under any automorphism of π. Thus, since $m \neq -m$, the fundamental

[1)] In [Kreck99, Prop. 4], take $B = BSO \times K(\pi,1)$ and $\xi : B \to BO$ the projection $BSO \to BO$ in the first factor and the constant map in the second factor.

class $[M]$ can never be sent to its negative under any homotopy equivalence of M.

The signature of M can be corrected to be zero by connected sum with several copies of $\mathbb{C}P^2$ or $-\mathbb{C}P^2$ (cf. Lemma 73 for a similar argument). □

In the rest of this section, we present an infinite set of finite groups that fulfill the requirements of Proposition 75. For this, a certain set of finite groups is studied, and it is shown that they all have only inner automorphisms (Proposition 77). For an infinite subset of these groups, we show that they also fulfill the condition on the fourth homology group (Proposition 80).

Lemma 76

Let $0 \to N \to G \overset{\curvearrowleft}{\to} Q \to 0$ be a split extension of groups, where Q is abelian. Denote by $\psi : Q \to \mathrm{Aut}(N)$ the action of Q on N by conjugation. Suppose that (1) ψ is injective and (2) the fixed point set N^Q consists only of the neutral element. Then G has trivial centre.

Proof. For $n \in N$, $q \in Q$, we have $[q, n] = \psi(q)(n)n^{-1}$. Thus, q is in the centre of G if and only if it is in the kernel of ψ. By condition (1), this only holds if $q = e$.

Write any other element $g \in G \smallsetminus Q$ as $g = nq$, $n \neq e$. Since Q is abelian, we have $[q', g] = [q', n]$ for any $q' \in Q$. By condition (2), this commutator does not vanish for some $q' \in Q$. □

Now the definitions and notations are introduced that lead to the desired fundamental groups. Let $p_1, \ldots, p_k$ be pairwise distinct odd primes. Let $N_i := C_{p_i}$, the cyclic group of order p_i (written multiplicatively). Let Q_i be the automorphism group of N_i. Since p_i is prime, it is a cyclic group of order $p_i - 1$.

Define G_i as the semidirect product

$$0 \to N_i \to G_i \overset{\curvearrowleft}{\to} Q_i \to 0,$$

where the operation $\psi_i : Q_i \to \mathrm{Aut}(N_i)$ is the identity.

Choose generators n_i of N_i and q_i of Q_i. Let $G(p_1, \ldots, p_k) := \times_{i=1}^k G_i$. Every element can be uniquely written in the form $n_1^{a_1} q_1^{b_1} \ldots n_k^{a_k} q_k^{b_k}$. Denote $G(p_1, \ldots, p_k)$ shortly by G, suppressing the parameters. There is no danger of confusion since one such group is fixed for the rest of this section.

Let $r_i \in \mathbb{Z}/p_i$ be the primitive root of p_i that corresponds to q_i. In other words, we have

$$q_i(n) = n^{r_i} \quad \text{for every } n \in N_i. \tag{1}$$

In particular, we have $r_i \notin \{[0], [1]\}$. Considered in G_i, equation (1) says

$$q_i n_i q_i^{-1} = n_i^{r_i}, \tag{2}$$

which is equivalent to

$$n_i q_i n_i^{-1} = n_i^{1-r_i} q_i. \tag{3}$$

Proposition 77

Every automorphism of G is inner.

This proposition extends [Huppert, Bsp. I.4.10], where the case of a single factor is proved, i. e. when $k = 1$ and $G = G_1$. For the sake of completeness, we state a small extension of the proposition.

Corollary 78

The group G is complete, i. e. it has trivial centre and every automorphism is inner.

Proof. Combine Lemma 76, the information $Z(G \times G') \cong Z(G) \times Z(G')$ for all groups G, G' and Proposition 77. □

Proof of Proposition 77. Let $\alpha : G \to G$ be an automorphism. First, it is shown that each cyclic subgroup N_i is invariant under α. Define exponents $a_{i,j} \in \mathbb{Z}/p_i$ and $b_{i,j} \in \mathbb{Z}/(p_i - 1)$ by

$$\alpha(n_i) =: \prod_{j=1}^{k} n_j^{a_{i,j}} q_j^{b_{i,j}}.$$

Let f_j be the composition $G \xrightarrow{\alpha} G \to G_j \to D_j$, where the second and third map are the projections. Applying f_j to equation (2) yields $q_j^{b_{i,j}} = q_j^{b_{i,j} r_i}$. Thus, $b_{i,j} \equiv b_{i,j} r_i \mod p_j - 1$, or equivalently,

$$p_j - 1 \mid b_{i,j}(r_i - 1) \quad \text{for all } i, j \in \{1, \dots, k\}. \tag{4}$$

Besides, we have $n_i^{p_i} = e$. Applying f_j to this equation yields $q_j^{b_{i,j} p_i} = e$, and thus

$$p_j - 1 \mid b_{i,j} p_i. \tag{5}$$

Since $r_i - 1$ and p_i are coprime, equations (4) and (5) imply $b_{i,j} \equiv 0 \mod p_j - 1$. Thus, the image of n_i in Q_j is trivial for all i, j.

This means that $\alpha(n_i) = \prod_j n_j^{a_{i,j}}$, i. e. α maps to the abelian subgroup $\bigoplus_j N_j$ of G. The order of $\alpha(n_i)$ is the product of all primes p_j such that $a_{i,j} \neq 0$. Hence, all $a_{i,j}$ with $i \neq j$ must be zero. This proves that $\alpha(n_i) \in N_i$.

In order to simplify the further arguments, define for each i an exponent $c_i \in \mathbb{Z}/p_i$ by $a_{i,i} =: r_i^{c_i}$. Let $\varepsilon_i : G \to G$ be the conjugation by $q_i^{-c_i}$. Then ε_i maps $n_i^{a_{i,i}}$ to n_i and leaves all other generators fixed.

Define an automorphism of G by $\varphi := \varepsilon_1 \circ \dots \varepsilon_k \circ \alpha$. The automorphism φ leaves all generators n_i fixed. In addition, it is sufficient to prove that φ is an inner automorphism in order to prove the proposition.

Now the images of the generators q_i are studied. Similar to before, write $\varphi(q_i)$ as $\prod_j n_j^{d_{i,j}} q_j^{e_{i,j}}$ with $d_{i,j} \in \mathbb{Z}/p_i$ and $e_{i,j} \in \mathbb{Z}/(p_i - 1)$. Applying φ to the relation (2) yields

$$\left(\prod_{j=1}^{k} n_j^{d_{i,j}} q_j^{e_{i,j}} \right) n_i \left(\prod_{j=1}^{k} q_j^{-e_{i,j}} n_j^{-d_{i,j}} \right) = n_i^{r_i}.$$

$$\begin{aligned}
\Leftrightarrow \quad & n_i^{d_{i,i}} q_i^{e_{i,i}} n_i q_i^{-e_{i,i}} n_i^{-d_{i,i}} = n_i^{r_i} \\
\Leftrightarrow \quad & n_i^{\left(r_i^{e_{i,i}}\right)} = n_i^{r_i} \\
\Leftrightarrow \quad & r_i^{e_{i,i}} \equiv r_i \bmod p_i \\
\Leftrightarrow \quad & e_{i,i} \equiv 1 \bmod p_i - 1
\end{aligned}$$

Similarly, applying φ to the relation $q_h n_i = n_i q_h$ for $h \neq i$ yields

$$\begin{aligned}
& \left(\prod_{j=1}^{k} n_j^{d_{h,j}} q_j^{e_{h,j}}\right) n_i = n_i \left(\prod_{j=1}^{k} n_j^{d_{h,j}} q_j^{e_{h,j}}\right) \\
\Leftrightarrow \quad & n_i^{d_{h,i}} q_i^{e_{h,i}} n_i q_i^{-e_{h,i}} n_i^{-d_{h,i}} = n_i \\
\Leftrightarrow \quad & n_i^{\left(r_i^{e_{h,i}}\right)} = n_i \\
\Leftrightarrow \quad & r_i^{e_{h,i}} \equiv 1 \bmod p_i \\
\Leftrightarrow \quad & e_{h,i} \equiv 0 \bmod p_i - 1 \quad \text{for all } h \neq i
\end{aligned}$$

This implies that $\varphi(q_i) = \left(\prod_j n_j^{d_{i,j}}\right) q_i$. Applying φ to the relation $q_h q_i = q_i q_h$ gives then

$$\left(\prod_{j=1}^{k} n_j^{d_{h,j}}\right) q_h \left(\prod_{j=1}^{k} n_j^{d_{i,j}}\right) q_i = \left(\prod_{j=1}^{k} n_j^{d_{i,j}}\right) q_i \left(\prod_{j=1}^{k} n_j^{d_{h,j}}\right) q_h.$$

Projection onto G_i yields for all $h \neq i$

$$\begin{aligned}
& n_i^{d_{h,i}} n_i^{d_{i,i}} q_i = n_i^{d_{i,i}} q_i n_i^{d_{h,i}} \\
\Leftrightarrow \quad & n_i^{d_{h,i}} = q_i n_i^{d_{h,i}} q_i^{-1} \\
\Leftrightarrow \quad & n_i^{d_{h,i}} = n_i^{r_i d_{h,i}} \\
\Leftrightarrow \quad & d_{h,i} \equiv r_i d_{h,i} \bmod p_i \\
\Leftrightarrow \quad & d_{h,i} \equiv 0 \bmod p_i
\end{aligned}$$

This holds for every $h \neq i$, so the image of q_i is finally restricted to the form $\varphi(q_i) = n_i^{d_{i,i}} q_i$.

Let $\gamma_i := d_i(1 - r_i)^{-1} \bmod p_i$. Conjugation by $n_i^{\gamma_i}$ maps q_i to

$$n_i^{\gamma_i} q_i n_i^{-\gamma_i} = n_i^{\gamma_i(1-r_i)} q_i = n_i^{d_{i,i}} q_i = \varphi(q_i)$$

(see equation (3)) and leaves all other generators fixed. Hence, the automorphism φ equals conjugation by $\prod_j n_j^{\gamma_j}$ for suitably defined $\gamma_1, \ldots, \gamma_k$. □

We now turn to the homology groups of G with constant integral coefficients. Wall computed in [Wall67] the integral homology of split extensions of finite cyclic groups by finite cyclic groups (in other words: of finite split metacyclic groups). We state his result in the special case of our groups G_i corresponding to the odd prime p_i [2)].

Proposition 79: special case of [Wall67, p. 253 ff.]

Let p_i be an odd prime. The finite split metacyclic group $G_i := C_{p_i} \rtimes C_{p_i-1}$ defined by an isomorphism $C_{p_i-1} \cong \operatorname{Aut}(C_{p_i})$ has the following homology groups with constant integral coefficients:

$$H_{2m-1}(G_i) \cong \begin{cases} \mathbb{Z}/(p_i-1) & \text{if } p_i - 1 \nmid m \\ \mathbb{Z}/p_i(p_i-1) & \text{if } p_i - 1 \mid m \end{cases} \qquad (m \geq 1)$$

$$H_{2m}(G_i) = 0 \qquad (m \geq 1)$$

Wall has to compute the differentials in a double complex explicitly in order to obtain his result. For the groups G_i (not for all groups which Wall considers), the following arguments give the homology quickly without computing any differential if one invests the Lyndon-Hochschild-Serre spectral sequence as a ready-made tool:

Alternative proof of Proposition 79. For every group extension

$$1 \to N \to G \to Q \to 1$$

and every left G-module M, there is the cohomological *Lyndon-Hochschild-Serre spectral sequence* with E_2-term

$$E_2^{r,s} \cong H^r(Q; H^s(N; M))$$

converging to $H^{r+s}(G; M)$ [Evens, Ch. 7.2]. This spectral sequence resides in the first quadrant. The N-module structure on M is simply defined by restriction from G to N. In our case, M is the trivial module $\mathbb{Z}$.

The coefficients $H^s(N; M)$ in the E_2-term are local coefficients, and the left action of Q on $H^s(N; M)$ is given by conjugation [Evens, Ch. 7.2], [Brown, Ch. II.6]: We have a map $Q \to \operatorname{Out}(N)$ given by conjugation with a preimage of $q \in Q$ in G (no matter which preimage). The induced map in cohomology is the action in the local coefficient system. Here again, it is used that inner automorphisms (in N) act trivially on the cohomology [Evens, Prop. 4.1.1].

By the definition of the group G_i, we know this action: We have $N = N_i = C_{p_i}$, $Q = Q_i \cong C_{p_i-1}$, and the chosen generator $q_i \in Q_i$ acts by the automorphism $n \mapsto n^{r_i}$. Fix a generator $x \in H_1(N) \cong \mathbb{Z}/p_i$. Denote by ρ_{p_i} the coefficient reduction modulo p_i, which in our case is an isomorphism $H_1(N) \xrightarrow{\sim} H_1(N; \mathbb{Z}/p_i)$.

Let $(\rho_{p_i} x)^*$ be the Kronecker dual in $H^1(N; \mathbb{Z}/p_i)$ and define

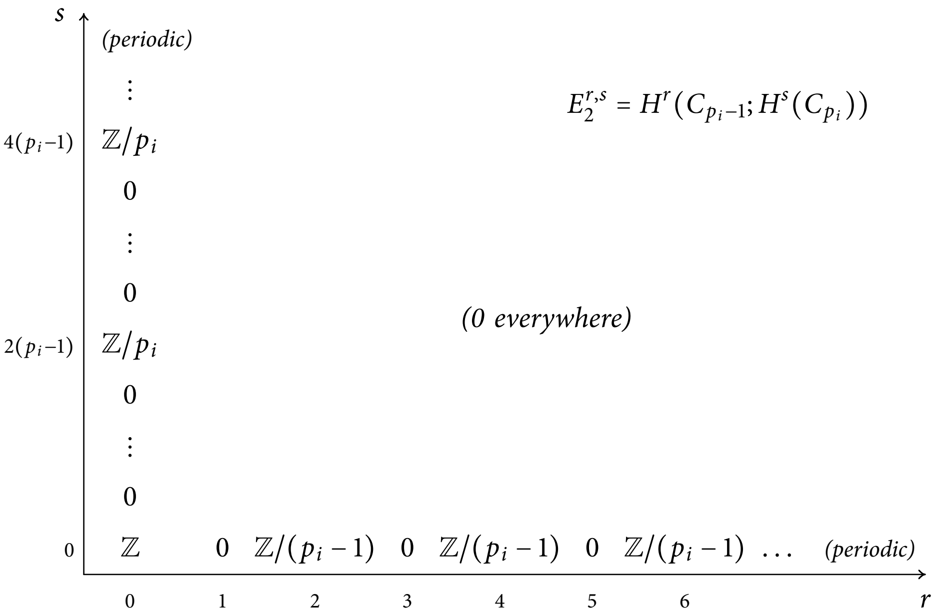
$$y := \beta(\rho_{p_i} x)^* \in H^2(N; \mathbb{Z}/p_i)$$

2) The correspondence between [Wall67, p. 253 ff.] and our symbols is the following: $x := n_i$, $y := q_i$, $r := p_i$, $s := p_i - 1$ and $t \equiv r_i^{-1} \bmod p_i$. Also note the typographical error on p. 254 (twice): The sum $\sum_{j=1}^{s-1} t^{jm}$ must start at $j = 0$.

as the image under the $\mathbb{Z}/p_i$ Bockstein homomorphism. In our case, the Bockstein homomorphism is again an isomorphism (which is a well-known fact from the cohomology of lens spaces), so y is in fact a generator for $H^2(N;\mathbb{Z}/p_i)$.

Since coefficient reduction, Kronecker duality and cohomology operations are natural (in the appropriate sense for duality), we conclude that q_i acts on $H^2(N;\mathbb{Z}/p_i) \cong H^2(N;\mathbb{Z})$ in the same way as on $H_1(N)$, which is by the map $y \mapsto r \cdot y$ (now written additively, as usual with abelian groups). By the product structure in the cohomology of finite cyclic groups, we conclude that h_i acts by multiplication with r^m on $H^{2m}(N)$. In particular, the action is trivial if $m \equiv 0 \mod p_i - 1$ and nontrivial otherwise.

We can now draw a diagram of the E_2-term of the Lyndon-Hochschild-Serre spectral sequence:



The base line is the cohomology of the quotient group $Q \cong C_{p_i-1}$. All entries $E_2^{r,s}$ with $r \geq 1$ and $s \geq 1$ are zero because the order p_i of every nontrivial element in N is invertible in the coefficient module $\mathbb{Z}/(p_i - 1)$ [Brown, Cor. III.10.2]. The first column is the zeroth cohomology $H^0(Q;H^s(N))$ with local coefficients. By [Brown, III.1.8], this is equal to the fixed points of $H^s(N)$ under the action of Q. If s is a multiple of $2(p_i - 1)$, the action is trivial, so $E_2^{0,s} \cong \mathbb{Z}/(p_i - 1)$. Otherwise, either the coefficient module is zero (if s is odd) or the action is free, so $E_2^{0,s} = 0$ in these cases.

There are no nontrivial homomorphisms $\mathbb{Z}/p_i \to \mathbb{Z}/(p_i - 1)$, so all higher differential vanish. By the universal coefficient theorem ((UCT 2) on page 58), the homology groups in Proposition 79 follow. □

Thus, we have $H_1(G_i) = \mathbb{Z}/2$ and $H_3(G_i) \cong \mathbb{Z}/6$ if $p_i = 3$ and $H_1(G_i) \cong H_3(G_i) \cong \mathbb{Z}/(p_i - 1)$ for all primes $p_i > 3$.

By the Künneth theorem, we get the following result:

Proposition 80

The group $G = G(p_1, \ldots, p_k)$ *has an element of order greater than 2 in* $H_4(G)$ *if and only if there are indices* $i, j \in \{1, \ldots, k\}$, $i \neq j$ *such that either* $p_i = 3$ *and* $p_j \equiv 1 \bmod 3$ *or* $\gcd(p_i - 1, p_j - 1) > 2$.

Proof. If the condition is fulfilled, $H_4(G)$ contains a summand

$$H_3(G_i) \otimes H_1(G_j) \cong \mathbb{Z}/6 \otimes \mathbb{Z}/(p_i - 1) \cong \mathbb{Z}/6$$

by the Künneth theorem in the first case (note that $p_j - 1$ is always even) and

$$H_3(G_i) \otimes H_1(G_j) \cong \mathbb{Z}/(p_i - 1) \otimes \mathbb{Z}/(p_j - 1) \cong \mathbb{Z}/\gcd(p_i - 1, p_j - 1)$$

in the second case. The necessity of the conditions (which is not needed for this work) can also be checked easily with the Künneth formula. □

The group $G(3,7) = G_3 \times G_7$ with 252 elements is the smallest group in this family. In fact, computer calculations with the computer algebra system GAP ([GAP], [HAP]) showed that $G(3,7)$ is the smallest finite group that has only inner automorphisms and an element of order greater than 2 in its 4th homology (with constant $\mathbb{Z}$-coefficients).

In summary, the following theorem was proved by Propositions 75, 77 and 80:

Theorem 81

There are infinitely many (with different fundamental groups) closed, connected, smooth, homotopically chiral 4-manifolds with signature zero.

6

Products of Lens spaces

In Theorem 36 it was shown that products of lens spaces of *different* dimensions are homotopically chiral if and only if each single factor is homotopically chiral. This leads to the question whether this is also true for lens spaces of the same dimension. Products of lens spaces are always amphicheiral if one of the factors is amphicheiral or if two factors are equal (i. e. homotopy equivalent, diffeomorphic, ...), since then two odd-dimensional factors can be interchanged and this reverses orientation.

For products of three-dimensional lens spaces, there are further results. These manifolds were classified up to unoriented homotopy equivalence by Huck and Metzler [HM], [Huck]. Their proofs are in fact sufficient to deduce the oriented statement. The results are stated and it is detailed how the proofs must be read in order to obtain the oriented classification in Appendix A.2. In Corollary 96, necessary and sufficient conditions are obtained when a product of three-dimensional lens spaces is homotopically amphicheiral. The conditions are numerical congruences which can be checked easily in each individual case.

Moreover, Metzler obtains diffeomorphisms of certain products of three-dimensional lens spaces in a constructive way [Metzler]. Again, he states only unoriented results but his proofs in fact produce orientation-preserving diffeomorphisms. Those results which are relevant for producing new amphicheiral products are stated in Appendix A.3, where it is also explained how to read the proofs so that they can be understood in the oriented sense. The conclusions about new, nontrivial orientation-reversing diffeomorphisms of products of lens spaces are summarised in the Propositions 101 and 103.

As mentioned above, Metzler really constructs diffeomorphisms between products of three-dimensional lens spaces. For this, he exploits the group structure on S^3 as the unit quaternions. One of the restrictions of his approach is that one of the lens space factors must always be a "standard" lens space $L_r(1,1)$.

In this chapter, we extend these results by showing that products of three-dimensional lens spaces can be smoothly amphicheiral in previously unknown cases. The result is the following theorem.

Theorem 82

Let r_1 and r_2 be coprime odd integers and let L_1 and L_2 be (any) 3-di-

mensional lens spaces with fundamental groups $\mathbb{Z}/r_1$ resp. $\mathbb{Z}/r_2$. Then the product $L_1 \times L_2$ is smoothly amphicheiral.

This result intersects with Metzler's but neither is a subset of the other. In fact, the approach here is completely different. We use the surgery theory of [Kreck99] to establish orientation-preserving diffeomorphisms between $L_1 \times L_2$ and $-L_1 \times L_2$. The argumentation is facilitated by the fact that it is known from Corollary 96 that $L_1 \times L_2$ is homotopically amphicheiral. This is however not a crucial ingredient, and the exact technical condition ($L_1 \times L_2$ and its negative are bordant over their normal 3-type) can be proved independently. Since the result is available, though, we adopt it gratefully and cut the first step in Kreck's surgery programme short.

Proof of Theorem 82. **Preliminaries**
The surgery technique which is used in this chapter in presented in [Kreck99], and we first explain the necessary preliminaries to get into the correct context. In [Kreck99], all spaces are equipped with basepoints, and all maps preserve basepoints. We will stick to this convention. A very important detail is the way in which a manifold is assigned a classifying map for its stable normal bundle. Given a manifold M^n, it can always be embedded in an $\mathbb{R}^{r+n}$ by the Whitney embedding theorem if r is large enough. The *normal Gauss map* for this embedding is a map from M to $G_{r,n}$, the Grassmannian of r-planes in $\mathbb{R}^{r+n}$. It is defined by assigning a point $p \in M$ the r-plane that is orthogonal to the tangent plane of M at p in $\mathbb{R}^{r+n}$.

In the following, we work with a very specific model for BO, the classifying space for stable real vector bundles. Let BO be the colimit over the Grassmannian manifolds $G_{k,l}$ for $k, l \geq 0$, where

- $G_{k,l}$ is embedded in $G_{k,l+1}$ by the map induced from
$$\mathbb{R}^{k+l} \to \mathbb{R}^{k+l+1}, \quad (v_1, \ldots, v_{r+n}) \mapsto (0, v_1, \ldots, v_{r+n}),$$
- $G_{k,l}$ is embedded in $G_{k+1,l}$ by mapping a plane $V \subset \mathbb{R}^{k+l}$ to the plane $V \oplus \langle e_{r+n+1} \rangle \subset \mathbb{R}^{k+l+1}$.

(It is easy to verify that both stabilisation steps commute. For more details, see [Switzer, 11.36–11.55]). It will become clear below why it is necessary to choose this model for BO and not any space that is homotopy equivalent to it.

In Kreck's surgery theory, one works with the *stable normal Gauss map*, which is the composition
$$\nu : M \to G_{r,n} \to BO$$
corresponding to an embedding $M^n \hookrightarrow \mathbb{R}^{r+n}$. Choosing another embedding yields a map that is homotopic to ν.

Given a fibration $B \to BO$, a B-structure on M is defined as a lift $\bar{\nu} : M \to B$ of the stable normal Gauss map ν. At first sight, this notion depends on the classifying map ν, i. e. on the embedding $M^n \hookrightarrow \mathbb{R}^{r+n}$: A priori, there is no correspondence between B-structures for different maps $M \to BO$, even if they are homotopic. Another choice of ν, say ν', which is homotopic to ν, gives rise to another lift $\bar{\nu}'$ by the homotopy lifting property of a fibration. All possibilities for $\bar{\nu}'$ are of course homotopic but the *fibrewise* homotopy class depends on the choice of the homotopy $\nu \sim \nu'$.

This problem is resolved by the arguments in [Stong, Ch. II]: Stong argues that if the codimension r is large enough, two different embeddings $M^n \to \mathbb{R}^{n+r}$ are regularly homotopic and "any two such regular homotopies are homotopic through regular homotopies leaving endpoints fixed" [Stong, p. 15][1)]. By the lifting property for fibrations, this provides a *fibre-preserving* homotopy over ν' between two lifts $\bar{\nu}_1'$ and $\bar{\nu}_2'$ that were obtained from $\bar{\nu}$ along two different homotopies $\nu \sim \nu'$. Thus, there is a canonical correspondence between fibrewise homotopy classes of lifts $M \to B$ for any two maps $M \to BO$ that are the stable Gauss maps of actual embeddings $M \to \mathbb{R}^{n+r}$.

As a consequence of this, though it is not necessary to fix a map $M \to BO$, only those classifying maps $M \to BO$ shall be allowed that come from an embedding. If the last statement is to make sense, the specific model for BO as the colimit of Grassmannians should be chosen.

Consequences of these considerations become apparent when products of manifolds are studied: Given a fibration $B \to BO$ and two manifolds with B-structures

$$\begin{array}{ccc} & & B \\ & {\scriptstyle \bar{\nu}_1}\nearrow & \downarrow \\ M_1 & \xrightarrow{\nu_1} & BO, \end{array} \qquad\qquad \begin{array}{ccc} & & B \\ & {\scriptstyle \bar{\nu}_2}\nearrow & \downarrow \\ M_2 & \xrightarrow{\nu_2} & BO, \end{array}$$

we would like to define a "product B-structure" on $M_1 \times M_2$. To justify the reference to a more complicated approach below, we explain why a simpler idea does not give the desired result easily. In his explanations to this topic, Stong suggests to appeal to the H-space structure of BO corresponding to the Whitney sum of vector bundles [Stong, p. 24 f.]. After an H-space map has been fixed in the homotopy class, one could try to lift this map to B and "define" a B-structure on $M_1 \times M_2$ by

$$\begin{array}{ccccc} & & B \times B & \overset{\text{lift}}{\dashrightarrow} & B \\ & {\scriptstyle \bar{\nu}_1 \times \bar{\nu}_2}\nearrow & \downarrow & & \downarrow \\ M_1 \times M_2 & \xrightarrow{\nu_1 \times \nu_2} & BO \times BO & \xrightarrow[\text{structure}]{\text{H-space}} & BO \end{array}$$

1) At the cited place, Stong works in fact with so-called (B_r, f_r)-structures over BO_r and stabilises at a later stage to BO. This difference, however, does not compromise the arguments that are given in this text.

Since it is not clear that the composite in the bottom row of this diagram is induced by an embedding of $M_1 \times M_2$, the composite $M_1 \times M_2 \to B$ does not necessarily induce a sensible B-structure on the product. On the other hand, if the product of two embeddings $M_1 \times M_2 \hookrightarrow \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \cong \mathbb{R}^{N_1+N_2}$ is chosen for the classifying map to BO, it is not clear what the product of B-structures should be. The author of this thesis did not succeed in finding an H-space structure on the Grassmannian model for BO that always takes the stable normal Gauss maps of two embeddings to the stable normal Gauss map of the embedding of the product.

Product structures in bordism theory, however, can be defined in a similar but different way. Kochman explains in his book [Kochman, Ch. 1] very carefully how the classifying normal map for products of manifolds can be constructed. He also details how to define the product of B-structures based on this, if the spaces that form B in the colimit admit "multiplication maps" with certain properties. Kochman gives several examples, including the case of framings; which is needed here ($B = EO$). The downside of this approach is that an embedding of the manifold into a finite-dimensional real vector space is part of the structure. More precisely, it is an equivalence class of embeddings but only stabilisation by extra coordinates is allowed; very simple operations like permutation of the coordinates, rotation or deformation of the embedding lead out of the equivalence class. (There are bordism relations in these cases but no equivalences of embeddings.)

In the course of this proof, we will refer to products of framed manifolds, the ring structure on the framed bordism groups Ω_*^{fr} and the Ω_*^{fr}-module structure on the homology theory $\Omega_*^{\mathrm{fr}}(-)$ without mentioning the technical details but always implicitly refer to [Kochman] for a careful setup of these structures.

As a last item of fine print, we mention the way that orientations are fixed in [Kreck99]. Usually, an orientation on the stable normal bundle (and thus, after agreeing on conventions, on the manifold itself) can be given by choosing one of two possible lifts

$$\begin{array}{ccc} & & BSO \\ & \nearrow & \downarrow \\ M & \xrightarrow{\nu} & BO. \end{array}$$

The orientation corresponding to a specific lift can be seen directly if BSO is defined as the double covering of BO built as the colimit of oriented Grassmannians. In [Kreck99] however, already the r-plane which is the fibre of the universal r-plane bundle $\gamma_r \to BO_r$ over the basepoint is oriented. For unorientable manifolds, this gives a local orientation at the basepoint. For orientable manifolds, with which we are dealing here exclusively, the only difference is that an orientation is not given when a lift of the stable normal map ν is chosen but is already predefined by the embedding of the manifold into an $\mathbb{R}^N$. Correspondingly, the stable normal maps for M and $-M$ must be different (but one can easily be obtained from the other, if desired: e. g. if the vector $u \in \mathbb{R}^N$

($u \neq 0$) is tangent to $M \subset \mathbb{R}^N$ at the basepoint $0 \in \mathbb{R}^N$, then mirroring $\mathbb{R}^N$ at the plane orthogonal to u gives an embedding of M which corresponds to the negative orientation.)

Bordism computation

Given a manifold M with a lift of the stable normal Gauss map $\bar{\nu}: M \to EO$ and a map to another space $f: M \to X$, write $(M, \bar{\nu}, f)$ for the element in the bordism group $\Omega_*^{\mathrm{fr}}(X)$. In Kreck's notation, this bordism group would be written as $\Omega_*(X \times EO)$ with the fibration $X \times EO \to BO$ (constant in the first factor) understood. Given a map $g: X \to Y$, the induced map on the bordism groups is given by $g_*(M, \bar{\nu}, f) = (M, \bar{\nu}, g \circ f)$, i. e. only the last data is affected. Elements in $\Omega_*^{\mathrm{fr}} = \Omega_*^{\mathrm{fr}}(\mathrm{pt})$ are written $(M, \bar{\nu})$, omitting the constant map to the point.

Now we return to the specific setting of Theorem 82. Let L_1 and L_2 be lens spaces with fundamental groups $\mathbb{Z}/r_1$ resp. $\mathbb{Z}/r_2$. Let L be the product $L_1 \times L_2$. Since r_1 and r_2 are required to be coprime, we know by Corollary 96 that there is an orientation-preserving homotopy equivalence $T: -L \to L$.

Choose embeddings of L_1, L_2 and $-L_1$ and denote the classifying maps of the stable normal bundles by ν_1, ν_2 and ν_3 resp. Since every closed oriented 3-manifold is parallelisable [Stiefel, Satz 21], there are lifts $\bar{\nu}_1$, $\bar{\nu}_2$ and $\bar{\nu}_3$ in the fibration $EO \to BO$.

We want to prove the following

Proposition 83

There are framings $\bar{\nu}_w: L_1 \to EO$, $\bar{\nu}_x: -L_1 \to EO$ and $\bar{\nu}_y, \bar{\nu}_z: L_2 \to EO$ such that $(L_1 \times L_2, \bar{\nu}_w \times \bar{\nu}_y, \mathrm{id})$ and $(-L_1 \times L_2, \bar{\nu}_x \times \bar{\nu}_z, T)$ coincide in the framed bordism homology group $\Omega_6^{\mathrm{fr}}(L)$.

Proof. For the beginning, we choose framings $\bar{\nu}_1$, $\bar{\nu}_2$ and $\bar{\nu}_3$ on L_1, L_2 and $-L_1$ that will later be adapted.

There is a standard procedure to translate a tangential framing of a manifold into a framing of the stable normal bundle [Stong, p. 23 f.][2)]. A generator of $\Omega_3^{\mathrm{fr}} \cong \mathbb{Z}/24$ is given by the Lie group $SU(2) \cong S^3$ with tangential framing any left invariant vector field [Gershenson, p. 128f.], [BS, §6]. For the correct identification of the generator, also the correct orientation on S^3 must be chosen. This detail can be neglected here since only a generator of Ω_6^{fr} is needed, no matter which one. We denote it by (S^3, α).

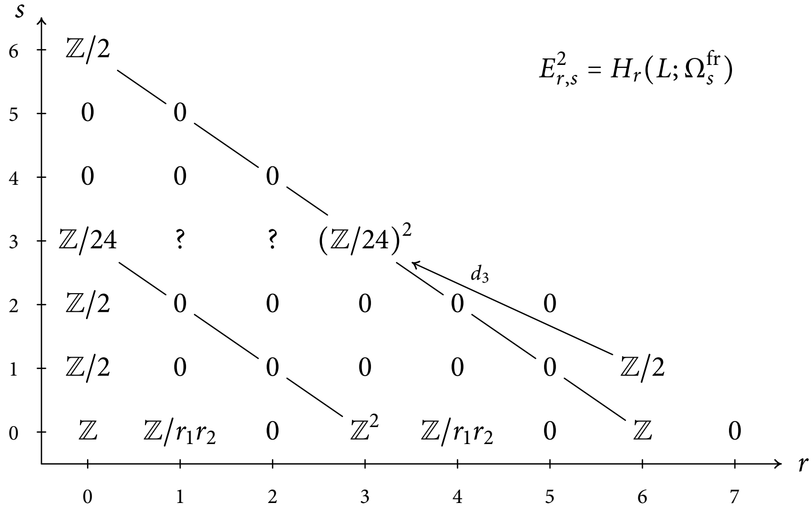
[2)] Stong defines a map $BO \to BO$ by giving a map on each Grassmannian: $I_{n,N}: G_{n,N} \to G_{N,n}$ maps each n-plane in $\mathbb{R}^{n+N}$ to its orthogonal N-plane. With the conventions that are used here on page 92, the maps $I_{n,N}$ do not glue together under the two stabilisations that are used to build BO from the Grassmannians. Instead, each $I_{n,N}$ has to be composed with the map induced from $\mathbb{R}^{n+N} \to \mathbb{R}^{n+N}$, $(x_1, \dots, x_{n+N}) \mapsto (x_{n+N}, x_{n+N-1}, \dots, x_1)$, i. e. the order of the coordinates must be reversed. Stong is not explicit about the convention which he uses for the stabilisation $BO = \mathrm{colim}_{n,N}\, G_{n,N}$. However, a similar correction map must always be applied since the general idea is that the two stabilisation directions “grow” each $\mathbb{R}^{n+N}$ at different coordinates, and this must be compensated when a plane is mapped to its orthogonal complement.

Thus, by (multiple) connected sum with (S^3, α), any framed 3-manifold can be changed to any element in Ω_3^{fr} by changing only the framing and preserving the underlying manifold. Therefore, the framings on L_1, L_2 and $-L_1$ can be chosen such that these lens spaces represent the zero element in Ω_3^{fr}.

In order to prove Proposition 83, we determine the bordism group $\Omega_6^{\mathrm{fr}}(L)$ with the help of the Atiyah-Hirzebruch spectral sequence as far as necessary. Recall that there is a splitting for every space X

$$\Omega_k^{\mathrm{fr}}(X) \cong \widetilde{\Omega}_k^{\mathrm{fr}}(X) \oplus \Omega_k^{\mathrm{fr}},$$

where the inclusion of Ω_k^{fr} and the projection to it are induced by the inclusion $\mathrm{pt} \to X$ and the constant map $X \to \mathrm{pt}$ respectively. The reduced bordism group $\widetilde{\Omega}_k^{\mathrm{fr}}(X)$ is defined as the kernel of the second map. This implies that the zero-column of the Atiyah-Hirzebruch spectral sequence always splits off as a direct summand of $\Omega_*^{\mathrm{fr}}(X)$ in the E^∞-page and that there are no differentials from or to the zero-column on any page. By the Pontrjagin-Thom theorem (see e. g. [Kochman, Cor. 1.5.11]), the framed bordism groups $\Omega_k^{\mathrm{fr}}(\mathrm{pt}) \cong E_{0,k}^\infty$ are isomorphic to the stable homotopy groups of spheres. The following figure shows the relevant part of the Atiyah-Hirzebruch spectral sequence.



From this diagram, we conclude an exact sequence

$$\mathbb{Z}/2 \xrightarrow{d_3} (\mathbb{Z}/24)^2 \to \widetilde{\Omega}_6^{\mathrm{fr}}(L) \xrightarrow{\text{Thom}} \mathbb{Z} \to 0,$$

and $\Omega_6^{\mathrm{fr}}(L) \cong \widetilde{\Omega}_6^{\mathrm{fr}}(L) \oplus \mathbb{Z}/2$ because the leftmost column splits off.

The map d_3 denotes here and in the following paragraphs always the differential from $E^2_{6,1}$ to $E^2_{3,3}$. The right map in the sequence above is the Thom homomorphism

$$\begin{aligned} \widetilde{\Omega}_6^{\mathrm{fr}}(L) \subset \Omega_6^{\mathrm{fr}}(L) &\to H_6(L) \\ (M, f) &\longmapsto f_*[M]. \end{aligned}$$

The orientation-reversing homotopy equivalence $T\colon -L \to L$ was chosen so that the images of $(L, \bar{\nu}_1 \times \bar{\nu}_2, \mathrm{id})$ and $(-L, \bar{\nu}_3 \times \bar{\nu}_2, T)$ under the Thom map coincide. Moreover, these elements are in the reduced bordism group $\widetilde{\Omega}_6^{\mathrm{fr}}(L)$ due to their initial framings.

The E^∞-terms in the Atiyah-Hirzebruch spectral sequence for a CW-complex X are the quotients $E^\infty_{r,s} = J_{r,s}/J_{r-1,s+1}$ in a filtration

$$0 \subset J_{0,n} \subset \ldots \subset J_{r-1,s+1} \subset J_{r,s} \subset \ldots \subset J_{r+s,0} = \Omega^{\mathrm{fr}}_{r+s}(X).$$

The subgroup $J_{r,s}$ is the image of the map $\Omega^{\mathrm{fr}}_{r+s}(X^r) \to \Omega^{\mathrm{fr}}_{r+s}(X)$, which is induced by the inclusion of the r-skeleton. It is actually an Ω^{fr}_*-submodule because the module structure preserves the filtration (see [Conner, Ch. 1.7] for the analogous case of oriented bordism).

Since the images of $(L, \bar{\nu}_1 \times \bar{\nu}_2, \mathrm{id})$ and $(-L, \bar{\nu}_3 \times \bar{\nu}_2, T)$ under the Thom homomorphism coincide, the difference between these elements is in the subgroup $J_{3,3}$. We want to show that this difference is zero for suitable framings on L and $-L$.

Lemma 84

The submodule $J_{3,3} \subset \Omega^{\mathrm{fr}}_6(L)$ is equal to the image of the module map

$$\Omega^{\mathrm{fr}}_3(L) \otimes \Omega^{\mathrm{fr}}_3 \to \Omega^{\mathrm{fr}}_6(L).$$

Proof. The module structure yields a commutative diagram

$$\begin{array}{ccc} \Omega^{\mathrm{fr}}_3(L) \otimes \Omega^{\mathrm{fr}}_3 & = & J_{3,0} \otimes \Omega^{\mathrm{fr}}_3 \\ \downarrow & & \downarrow \\ \Omega^{\mathrm{fr}}_6(L) & \hookleftarrow & J_{3,3} \end{array}$$

Thus, the lemma is equivalent to surjectivity of the right vertical arrow. For this, consider the following morphism of exact sequences, again given by the module structure.

$$\begin{array}{ccccccccc} & & J_{2,1} \otimes \Omega^{\mathrm{fr}}_3 & \longrightarrow & J_{3,0} \otimes \Omega^{\mathrm{fr}}_3 & \longrightarrow & E^\infty_{3,0} \otimes \Omega^{\mathrm{fr}}_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & J_{2,4} & \longrightarrow & J_{3,3} & \longrightarrow & E^\infty_{3,3} & \longrightarrow & 0 \end{array}$$

By the "four-lemma" (a weak version of the five-lemma aiming at surjectivity of the middle vertical arrow) it is sufficient to prove surjectivity of the outer vertical arrows.

The modules $J_{2,1}$ and $J_{2,4}$ on the left hand side are equal to Ω^{fr}_3 and Ω^{fr}_6 resp. since the E^∞-entries at positions $(2,1)$, $(1,2)$, $(2,4)$ and $(1,5)$ are all zero. Thus, the left vertical map is the multiplication $\Omega^{\mathrm{fr}}_3 \otimes \Omega^{\mathrm{fr}}_3 \to \Omega^{\mathrm{fr}}_6$. This is a surjection since the Pontrjagin-Thom map is a ring isomorphism [Kochman, Thm. 1.5.4], and in the stable homotopy groups of spheres, we have that $\nu^2 \in \pi_6^s = \mathbb{Z}/2$ is the nonzero element with $\nu \in \pi_3^s$ the generator that is represented by the quaternionic Hopf map $S^7 \to S^4$.

The right vertical map fits into a commutative square

$$\begin{array}{ccc} E^\infty_{3,0} \otimes \Omega_3^{\mathrm{fr}} & \longrightarrow & E^2_{3,0} \otimes \Omega_3^{\mathrm{fr}} \\ \downarrow & & \downarrow \\ E^\infty_{3,3} & \leftarrow - - - - & E^2_{3,3} \end{array} \tag{1}$$

Since there are no differentials leaving the $E^*_{3,0}$-entries, the upper horizontal map is surjective. The right vertical map is the module map on the E^2-term, which is given by

$$\begin{array}{ccccc} H_3(L;\Omega_0^{\mathrm{fr}}) \otimes \Omega_3^{\mathrm{fr}} & \to & H_3(L;\Omega_0^{\mathrm{fr}} \otimes \Omega_3^{\mathrm{fr}}) & \to & H_3(L;\Omega_3^{\mathrm{fr}}) \\ \wr\| & & \wr\| & & \wr\| \\ \mathbb{Z}^2 \otimes \mathbb{Z}/24 & \twoheadrightarrow & (\mathbb{Z}/24)^2 & \xrightarrow{\sim} & (\mathbb{Z}/24)^2 \end{array}$$

see [Conner, Lemma 7.1]. The left arrow in the upper row is the coefficient change map in homology and a surjection in the present case. The right arrow is the module map on Ω_*^{fr}, which is an isomorphism since $\Omega_0^{\mathrm{fr}} \cong \mathbb{Z}$.

The lower horizontal map in the diagram (1) exists since the only possibly nonzero differential involving $E^k_{3,3}$ $(k \geq 2)$ is the differential marked d_3 in the spectral sequence. This gives a surjection $E^2_{3,3} \twoheadrightarrow E^\infty_{3,3}$. Since the subsequent identifications $E^{k+1}_{*,*} \cong H(E^k_{*,*}, d_k)$ are module homomorphisms, the diagram (1) commutes.

Since all three arrows of the composition from top to bottom in diagram (1) are surjective, the left vertical map is surjective, too. This proves Lemma 84. □

By the Atiyah-Hirzebruch spectral sequence, the third bordism group $\Omega_3^{\mathrm{fr}}(L)$ fits into an exact sequence

$$\begin{array}{ccccccccc} 0 & \to & \Omega_3^{\mathrm{fr}} & \to & \Omega_3^{\mathrm{fr}}(L) & \to & H_3(L) & \to & 0. \\ & & \wr\| & & & & \wr\| & & \\ & & \mathbb{Z}/24 & & & & \mathbb{Z}^2 & & \end{array}$$

Let $\bar{\nu}_a$ and $\bar{\nu}_b$ be any framings of L_1 resp. L_2 and denote the inclusions of L_1 and L_2 into L by i_1 resp. i_2. A set of generators for $\Omega_3^{\mathrm{fr}}(L)$ is then given by $(L_1, \bar{\nu}_a, i_1)$, $(L_2, \bar{\nu}_b, i_2)$ and (S^3, α).

In the following, we admit only framings $\bar{\nu}_a$ and $\bar{\nu}_b$ such that $(L_1, \bar{\nu}_a)$ is an even element in $\Omega_3^{\mathrm{fr}} \cong \mathbb{Z}/24$, likewise for L_2. We call framings having this property "even". Note that a framing $\bar{\nu}$ on a 3-manifold M is even if and only if $(M, \bar{\nu}) \times (S^3 \alpha) = 0$ in Ω_6^{fr}. The restriction to even framings has the effect that the cross products $g_1 := (L_1, \bar{\nu}_a, i_1) \times (S^3, \alpha)$ and $g_2 := -(L_2, \bar{\nu}_b, i_2) \times (S^3, \alpha)$ lie in the reduced bordism group $\widetilde{\Omega}_6^{\mathrm{fr}}(L)$. The minus sign in the definition of g_2 is deliberate since we will work in the following with the flipped product $(S^3, \alpha) \times (L_2, \bar{\nu}_b, i_2)$, which is also a representative of g_2. Note that g_1 and g_2 do not depend on the choice of the framings $\bar{\nu}_a$ and $\bar{\nu}_b$, as long as these framings are even.

By Lemma 84, the subgroup $J_{3,3} \subset \Omega_6^{\mathrm{fr}}(L)$ is generated by g_1, g_2 and $g_3 := (S^3, \alpha) \times (S^3, \alpha)$. The last element g_3 also generates Ω_6^{fr}.

Since the difference $(L, \bar{\nu}_1 \times \bar{\nu}_2, \mathrm{id}) - (-L, \bar{\nu}_3 \times \bar{\nu}_2, T)$ lies in $J_{3,3} \cap \widetilde{\Omega}_6^{\mathrm{fr}}(L)$, there are $p, q \in \mathbb{Z}/24$ such that

$$T_*(-L, \bar{\nu}_3 \times \bar{\nu}_2, \mathrm{id}) = (L, \bar{\nu}_1 \times \bar{\nu}_2, \mathrm{id}) + p \cdot g_1 + q \cdot g_2. \tag{2}$$

The coefficients p and q may not be unique, depending on the differential d_3, but at least there are values which fulfill equation (2). There are different strategies for the last step of the proof, depending on the parity of p and q. The three cases below should be understood non-exclusively. If more than one case applies, each procedure works.

Case 1: p is even. Choose $\bar{\nu}_a = \bar{\nu}_1$ and write equation (2) as

$$\begin{aligned}(-L, \bar{\nu}_3 \times \bar{\nu}_2, T) &= (L_1, \bar{\nu}_1, i_1) \times (L_2, \bar{\nu}_2, i_2) + p \cdot (L_1, \bar{\nu}_1, i_1) \times (S^3, \alpha) + q \cdot g_2 \\ &= (L_1, \bar{\nu}_1, i_1) \times \big((L_2, \bar{\nu}_2, i_2) + p \cdot (S^3, \alpha)\big) + q \cdot g_2\end{aligned}$$

The sum $(L_2, \bar{\nu}_2, i_2) + p \cdot (S^3, \alpha)$ is bordant to $(L_2, \bar{\nu}_y, i_2)$ with a new framing $\bar{\nu}_y$. Since $\bar{\nu}_2$ is an even framing and p an even number, the new framing $\bar{\nu}_y$ is even. Choose $\bar{\nu}_b = \bar{\nu}_y$ and write

$$\begin{aligned}(-L, \bar{\nu}_3 \times \bar{\nu}_2, T) &= (L_1, \bar{\nu}_1, i_1) \times (L_2, \bar{\nu}_y, i_2) + q \cdot (S^3, \alpha) \times (L_2, \bar{\nu}_y, i_2) \\ &= \big((L_1, \bar{\nu}_1, i_1) + q \cdot (S^3, \alpha)\big) \times (L_2, \bar{\nu}_y, i_2)\end{aligned}$$

This step changes the framing of L_1 to a new framing $\bar{\nu}_x$, so finally we have

$$(-L, \bar{\nu}_3 \times \bar{\nu}_2, T) = (L_1, \bar{\nu}_x, i_1) \times (L_2, \bar{\nu}_y, i_2) = (L, \bar{\nu}_x \times \bar{\nu}_y, \mathrm{id}),$$

as desired.

Case 2: q is even. This is analogous to the previous case with the roles of p and q switched. Choose first $\bar{\nu}_b = \bar{\nu}_2$ and change the framing of L_1 to another even framing $\bar{\nu}_x$. Then choose $\bar{\nu}_a = \bar{\nu}_x$ and change the framing of L_2 in order to dispose of the generator g_1.

Case 3: p and q are odd. Consider the subgroup $M' := J_{3,3} \cap \widetilde{\Omega}_6^{\mathrm{fr}}(L)$ in $\Omega_6^{\mathrm{fr}}(L)$. This subgroup is invariant under automorphisms of L because both constituents $J_{3,3}$ and $\widetilde{\Omega}_6^{\mathrm{fr}}(L)$ are. Another set of generators for M', which is more suitable for working on the left hand side of the equation, is

$$h_1 := (-L_1, \bar{\nu}_3, i_1) \times (S^3, \alpha) \quad \text{and} \quad h_2 := (S^3, \alpha) \times (L_2, \bar{\nu}_2, i_2).$$

There is still the small issue that coefficients for this set of generators are not unique modulo 24 if the differential $d_3 : E^3_{6,1} \to E^3_{3,3} \cong (\mathbb{Z}/24)^2$ is nonzero. Since $E^3_{6,1} = \mathbb{Z}/2$, this is overcome by reducing everything modulo 12. Accordingly, we consider $M := M'/12M'$, which is isomorphic to $(\mathbb{Z}/12)^2$. Two bases (g_1, g_2) and (h_1, h_2) have been identified for M as a $\mathbb{Z}/12$-module.

Let $\tau \in M(2 \times 2; \mathbb{Z}/12)$ be the matrix of the homomorphism T_* with basis (h_1, h_2) in the domain and (g_1, g_2) in the target understood. Since T is a homotopy equivalence, τ is an invertible matrix. Thus, not all entries of τ are even. Let the i-th column have at least one odd entry. Then we can write

$$T_* h_i = p' \cdot g_1 + q' \cdot g_2, \tag{3}$$

again in M', with $p', q' \in \mathbb{Z}/24$. The coefficients p', q' may not be uniquely determined but at least one of them is odd.

Adding equation (3) to equation (1) changes the framing on the left hand side (it is still in the reduced bordism group) and changes the coefficients on the right hand side such that at least one of them is odd. With this change, the problem is reduced to case 1 or 2.

This finishes the proof of Proposition 83. □

Surgery step

In the previous section it was shown that L and $-L$ are framed bordant over L itself. This is the starting point of the actual surgery step, in which the connectedness of the inclusions of L and $-L$ into the bounding manifold is increased stepwise to make it finally an s-cobordism. The proof follows the technique of [Kreck99], and the part of its main theorem which is needed here, is excerpted below as Theorem 85.

Let $w_k \in H^k(BO; \mathbb{Z}/2)$ be the k-th Stiefel-Whitney class of the universal bundle. Let $w_k(B)$ be the class which is pulled back by the projection $B \to BO$. In our case, $w_k(B) = 0$ for all $k > 0$ because the projection factors through EO, which has vanishing cohomology.

The following theorem is [Kreck99, Thm. 4] restricted to dimension 6 ($q = 3$), manifolds M_0, M_1 without boundary and $w_{q+1}(B) = 0$. It is stated with the simplifications that follow immediately from the restrictions, and only with the part of the conclusion that is needed here.

Theorem 85: Special case of [Kreck99, Thm. 4]

Let M_0 and M_1 be closed, connected, smooth 6-dimensional manifolds with the same Euler characteristic. Suppose there are normal 3-smoothings in a fibration $B \to BO$. Let W together with a normal structure $\bar{\nu}$ be a B-bordism between M_0 and M_1. Then M_0 and M_1 are oriented diffeomorphic if and only if $\theta(W, \bar{\nu}) \in L_7(\pi_1(B), w_1(B))$ vanishes.

Here, we set $M_0 = L$, $M_1 = -L$. A k-smoothing is a lift of the stable normal bundle $\bar{\nu}: M_i \to B$ over BO which is a $(k+1)$-equivalence. We have not only 3-smoothings to B but even full homotopy equivalences

$$(\mathrm{id}, \bar{\nu}_w \times \bar{\nu}_y): L \to (L \times EO) = B \qquad \text{and} \qquad (T, \bar{\nu}_x \times \bar{\nu}_z): -L \to (L \times EO).$$

In the previous section of the proof, the existence of a B-bordism was shown. The symbol $L_{2q+1}(\pi, w)$ denotes a certain abelian group. It depends on the parity of q, a group π and a homomorphism $w: \pi \to \mathbb{Z}/2$. The orientation character $w_1(B)$ can equivalently be regarded as an element in $H^1(B, \mathbb{Z}/2)$ or as a homomorphism $\pi_1(B) \to \mathbb{Z}/2$, and is zero in the present case. In the group $L_7(\pi_1(B), w_1(B))$, an obstruction $\theta(W, \bar{\nu})$ is defined. We do not analyse the obstruction $\theta(W, \bar{\nu})$ and the way it is obtained from W but show in the following that its containing group $L_7(\pi_1(B), w_1(B))$ is trivial. This step concludes the proof of Theorem 82. □

Proposition 86

Let π be a finite cyclic group of odd order and $w: \pi \to \mathbb{Z}/2$ the trivial map. Then the group $L_{2q+1}(\pi, w)$ vanishes for all $q \in \mathbb{N}_0$.

The groups $L_{2q+1}(\pi, w)$ were defined in [Kreck99, p. 733]. The definition is reproduced along with the proof below.

Proof. As above, let π be a group and $w: \pi \to \mathbb{Z}/2$ be a homomorphism. Denote the integral group ring of π by $\Lambda = \mathbb{Z}[\pi]$. There is an anti-involution $\bar{\ }: \Lambda \to \Lambda$ defined by $\bar{g} = w(g)g^{-1}$ for $g \in \pi$.

Set ε to $(-1)^q$. An ε-quadratic form over Λ is given by a left Λ-module together with an ε-hermitian sesquilinear form $\lambda: V \times V \to \Lambda$ and a so-called quadratic refinement, which is a map $\mu: V \to \Lambda/\langle x - \varepsilon\bar{x}\rangle$. (Here, $\langle x - \varepsilon\bar{x}\rangle$ means the additive subgroup of Λ consisting of all elements of the specified form.) For a better understanding of the concept, we quote all the required properties from [Kreck99, p. 725], even though they are not required to be remembered in the further proof.

i) For fixed $v \in V$, the map $V \to \Lambda$, $w \mapsto \lambda(w, v)$ is a Λ-homomorphism.
ii) $\lambda(v, w) = \varepsilon\overline{\lambda(w, v)}$.
iii) $\lambda(v, v) = \mu(v) + \varepsilon\overline{\mu(v)}$ (which is actually a well-defined element in Λ).
iv) $\mu(v + w) = \mu(v) + \mu(w) + \lambda(v, w) \in \Lambda/\langle x - \varepsilon\bar{x}\rangle$.
v) $\mu(x \cdot v) = x \cdot \mu(v) \cdot \bar{x}$ for $x \in \Lambda$, $v \in V$.

An important case is the ε-hyperbolic form H_ε^r, which is the r-fold orthogonal sum of H_ε, and H_ε is the form on $\Lambda \oplus \Lambda$ with standard basis e and f and $\lambda(e, f) = 1$, $\lambda(f, e) = \varepsilon$, $\lambda(e, e) = \lambda(f, f) = 0$ and $\mu(e) = \mu(f) = 0$.

A finitely generated, free Λ-module is called *based* if it is equipped with an equivalence class of bases, where two bases are equivalent if the matrix of base changes vanishes in the Whitehead group $\mathrm{Wh}(\pi)$. An isomorphism between based Λ-modules is called a *simple isomorphism* if the matrix of the isomorphism with respect to the given bases vanishes in $\mathrm{Wh}(\pi)$.

An element in $L_{2q+1}(\pi, w)$ is now represented by a hyperbolic form H_ε^r on Λ^{2r} together with a free, based direct summand $V \subset \Lambda^{2r}$ of rank r. Furthermore, it is required that λ and μ vanish on V. Two equivalence relations among those pairs (H_ε^r, V) are introduced to define $L_{2q+1}(\pi, w)$:

- Stabilisation by orthogonal sum with $(H_\varepsilon, \Lambda \times \{0\})$.
- The action of a certain group $RU^\varepsilon(\Lambda)$ on stable equivalence classes of objects.

  The group $RU^\varepsilon(\Lambda)$ is the colimit of groups $RU^\varepsilon(\Lambda^r) \subset \mathrm{Aut}(\Lambda^{2r})$, which are generated by the map $e_i \mapsto \varepsilon f_i$, $f_i \mapsto e_i$ and those simple isometries of H_ε^r preserving $\Lambda^r \times \{0\}$ and inducing a simple isomorphism on $\Lambda^r \times \{0\}$. An element $A \in RU^\varepsilon(\Lambda^r)$ acts by mapping (H_ε^r, V) to $(H_\varepsilon^r, A(V))$.

  The exact definition of $RU^\varepsilon(\Lambda)$ is not important in the following; it should be remembered that all representatives of elements are simple isometries of H_ε^r.

The resulting set $L_{2q+1}(\pi, w)$ is a monoid under orthogonal sum, and it can be shown that it is actually an abelian group with zero element the class of $0 \sim (H_\varepsilon^r, (\Lambda \times \{0\})^r)$.

According to [Kreck99, p. 733], there is an exact sequence

$$0 \to L_{2q+1}^s(\pi, w) \to L_{2q+1}(\pi, w) \to \mathrm{Wh}(\pi), \tag{4}$$

where the left group is one of Wall's surgery obstruction groups. If π is a finite group of odd order and w is trivial, $L_{2q+1}^s(\pi, w)$ is zero by [Bak], so we are only concerned with the image of L_{2q+1} in the Whitehead group.

The map to the Whitehead group is given by the following: Let $(\alpha_1, \dots, \alpha_r)$ be the chosen basis of V (up to simple isomorphism) and denote the dual elements in the free module H_ε^r by $(\beta_1, \dots, \beta_r)$ (so we have $\lambda(\alpha_i, \beta_j) = \delta_{ij}$). The image in the Whitehead group is the Whitehead torsion of the base change between the standard basis of H_ε^r and the basis $(\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_r)$. This is the definition given in [Kreck99] but since a permutation of coordinates has trivial Whitehead torsion, the standard basis can as well be compared with the basis $(\alpha_1, \beta_1, \alpha_2, \beta_2, \dots, \alpha_r, \beta_r)$. Denote the matrix of such a base change by A. Instead of a base change, consider A now as an automorphism of the hyperbolic form $H_\varepsilon^r = (\Lambda^{2r}, \lambda)$ with respect to the standard basis. The advantage of the modified definition for the map to the Whitehead group is that the representative A is the matrix of an isometry of the hyperbolic form.

Let $L \in M(2r \times 2r; \Lambda)$ be the matrix of λ with respect to the standard basis. This means that $\lambda(v, w)$ is given by

$$\mathbf{v}^t L \overline{\mathbf{w}} \in \Lambda$$

if $\mathbf{v}, \mathbf{w}$ denote the coordinate (row) vectors of v resp. w with respect to the standard basis. Since A is an isometry, we have $A^t L \overline{A} = L$, and after transposition

$$A^* L^t A = L^t.$$

Here, A^* denotes as usual the conjugate transpose of the matrix A. Conjugation refers to the involution on Λ, which was denoted by an overline before. Considering these matrices in the Whitehead group, we have $\tau(A^*) + \tau(L^t) + \tau(A) = \tau(L^t)$, hence $\tau(A^*) + \tau(A) = 0$. This result is also contained in [CS, Lemma 6.2].

The definition of the involution on $\Lambda = \mathbb{Z}[\pi]$ in the surgery context involved the orientation character w. In the present case, however, w is trivial, so the involution is simply induced by the map $g \mapsto g^{-1}$ on π. This involution (conjugation) on π induces an involution on the matrix rings over $\mathbb{Z}[\pi]$ given by the conjugate transpose, as above. The involution on $GL_n(\mathbb{Z}[\pi])$ continues through all the steps in the definition of the Whitehead group, so an involution of the Whitehead group is obtained, which is also referred to as the *standard involution* [Oliver, Ch. 5c]. If this involution on $\mathrm{Wh}(\pi)$ is also denoted by a star, we can thus write $\tau(A)^* + \tau(A) = 0$. If f denotes the map from $L_{2q+1}(\pi, w)$ to $\mathrm{Wh}(\pi)$ in (4), the following result can be stated:

$$f(x)^* + f(x) = 0 \text{ for all } x \in L_{2q+1}(\pi, w). \tag{5}$$

If π is a finite group, then $\mathrm{Wh}(\pi)$ is a finitely generated abelian group ([Oliver, Thm. 2.5 (iii)] with $A = \mathbb{Q}[\pi]$ and $I = \mathfrak{A} = \mathbb{Z}[\pi]$). The subgroup $SK_1(\mathbb{Z}[\pi])$ of $\mathrm{Wh}(\pi)$ is a finite group [Oliver, Thm. 2.5 (ii)], and the quotient $\mathrm{Wh}(\pi)/SK_1(\mathbb{Z}[\pi])$ is torsion free [Oliver, Thm. 7.4]. Thus, $SK_1(\mathbb{Z}[\pi])$ is the torsion subgroup of $\mathrm{Wh}(\pi)$. If π is finite cyclic, $SK_1(\mathbb{Z}[\pi])$ is zero [Oliver, Thm. 5.6].

For the present proof, the following proposition is most helpful:

Proposition 87: [Oliver, Cor. 7.5]

Let π be a finite group such that $SK_1(\pi) = 0$. Then conjugation acts trivially on $\mathrm{Wh}(\pi)$.

Together with (5), this implies $2f(x) = 0$ and hence $f(x) = 0$. □

7

Orientation-reversing diffeomorphisms of minimal order

Another question in connection with orientation reversal is the following:

If a manifold is amphicheiral, what is the minimal order of an orientation-reversing diffeomorphism?

Siebenmann presented a 3-manifold that admits an orientation-reversing diffeomorphism but none of finite order [Siebenmann, p. 176]. Another example is obtained by combining two theorems: Kreck proved in [Kreck07] that there are infinitely many closed simply-connected smooth 6-manifolds on which no finite group can act effectively. But it was shown in Chapter 4 that every such manifold is smoothly amphicheiral. We can conclude

Proposition 88

There are infinitely many closed simply-connected smooth 6-manifolds which admit an orientation-reversing diffeomorphism but none of finite order.

In view of diffeomorphisms of finite order, let $f : M \to M$ be an orientation-reversing diffeomorphism of order $2^k \cdot l$ with l odd. Then f^l is an orientation-reversing diffeomorphism of order 2^k. Thus, only powers of two are important for the minimal order of an orientation-reversing diffeomorphism, and we ask the following question:

Given $k \geq 1$, is there a manifold which admits an orientation-reversing diffeomorphism of order 2^k but none of order 2^{k-1}?

This question can be answered in the affirmative. The key to one direction of the solution is the following result about lens spaces.

Proposition 89: see e. g. [Lück, Thm. 2.31]

Let L be a lens space of dimension $2n-1$ with fundamental group $\mathbb{Z}/r$. There is a self-map of L with degree $d \in \mathbb{Z}$ and automorphism $x \mapsto e \cdot x$ of the fundamental group if and only if $e^n \equiv d \bmod r$.

Corollary 90

A lens space of dimension $2n-1$ with fundamental group of order $r>2$ does not admit an orientation-reversing self-map whose order is a divisor of n.

As immediate examples, we can conclude that no 3-dimensional lens space admits an orientation-reversing involution and no 7-dimensional lens space admits an orientation-reversing diffeomorphism of order less than 8.

To complement Corollary 90, we construct lens spaces with orientation-reversing diffeomorphisms of minimal order. Let L be a lens space of dimension $p-2$ with prime fundamental group $\mathbb{Z}/p$, $p \geq 5$. Let $p-1 = 2^k \cdot l$ be the factorisation into even and odd parts. By Corollary 90, L has no orientation-reversing diffeomorphism of order 2^{k-1}.

Since p is prime, the group of multiplicative units in $\mathbb{Z}/p$ is a cyclic group of order $p-1$. Let $c \in (\mathbb{Z}/p)^\times$ be a primitive root mod p, i. e. a generator of this group. In the following, abbreviate $(p-1)/2$ by n. Consider the lens space $L := L_p(c, c^2, \ldots, c^n)$. As usual, L is formed as the quotient of the unit sphere $S^{p-2} \subset \mathbb{C}^n$ under the $\mathbb{Z}/p$-action

$$(z_1, \ldots, z_n) \mapsto \left(\exp\left(\tfrac{2\pi i c}{p}\right) \cdot z_1, \ldots, \exp\left(\tfrac{2\pi i c^n}{p}\right) \cdot z_n\right).$$

The diffeomorphism

$$\widetilde{f} : S^{p-2} \to S^{p-2}$$
$$(z_1, z_2, \ldots, z_n) \mapsto (z_2, \ldots, z_n, \overline{z_1})$$

preserves the $\mathbb{Z}/p$-orbits. (Here it is needed that $c^n \equiv -1 \bmod p$ because both sides of the equation are the unique element of order two in the cyclic group of units.) Moreover, $\widetilde{f}$ reverses the orientation, so it induces an orientation-reversing diffeomorphism f on the lens space L. From the definition follows $\widetilde{f}^{2n} = \widetilde{f}^{p-1} = \mathrm{id}$. This implies that f^l is an orientation-reversing diffeomorphism of order 2^k. (Indeed, the order cannot be smaller, as was shown above.)

By Dirichlet's theorem, the arithmetic progression

$$2^k+1, \quad 3 \cdot 2^k + 1, \quad 5 \cdot 2^k + 1, \quad \ldots$$

contains infinitely many primes. Thus, for every positive integer k, there are suitable primes p, and we get the following

Theorem 91

For every positive integer k, there are infinitely many lens spaces which admit an orientation-reversing diffeomorphism of order 2^k but no orientation-reversing self-map of smaller order.

A

Appendix

A.1 The linking form

In this section, the linking form is defined and its properties, as stated in Theorem 6, are proved. The statement was the following:

Theorem 6

Let M be a closed, oriented topological manifold of odd dimension $2k-1$. Then there is a nondegenerate, $(-1)^k$-symmetric bilinear form

$$L\colon \operatorname{Tor} H^k(X) \times \operatorname{Tor} H^k(X) \to \mathbb{Q}/\mathbb{Z},$$

which is called the ***linking form****. Furthermore, if $f : N \to M$ is a continuous map then $L(f^*a, f^*b) = \deg(f) \cdot L(a,b)$.*

Recall that a bilinear form is called nondegenerate if the two insertion homomorphisms

$$\operatorname{Tor} H^k(X) \to \operatorname{Hom}(\operatorname{Tor} H^k(X), \mathbb{Q}/\mathbb{Z}) \tag{1}$$

given by $a \mapsto L(a, \cdot)$ and $a \mapsto L(\cdot, a)$ are isomorphisms. Also recall the notation $A_{free} := A/\operatorname{Tor} A$ for any finitely generated abelian group A. Before the lemma is proved, we need the following lemma.

Lemma 92

Let A be a finitely generated abelian group. Since every map f from a torsion group to $\mathbb{Z}$ is zero, every map $A \to \mathbb{Z}$ descends to a map $\bar{f}$ on the quotient A_{free}. Then the map

$$\begin{aligned} i\colon \operatorname{Hom}(A, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} &\to \operatorname{Hom}(A_{free}, \mathbb{Q}/\mathbb{Z}) \\ f \otimes q &\mapsto q \cdot \bar{f} \end{aligned}$$

is an isomorphism.

Proof. For the proof of injectivity, let $\sum_j f_j \otimes q_j \neq 0$. Then there exists an $a \in A$ such that $\sum f_j(a) \otimes q_j \not\equiv 0$ in $\mathbb{Q}/\mathbb{Z}$. But this implies $\sum q_j \cdot f_j(a) \notin \mathbb{Z}$, so $\sum q_j \bar{f_j}$ is nonzero in $\mathrm{Hom}(A_{free}, \mathbb{Q}/\mathbb{Z})$.

In order to prove surjectivity, we need that A is finitely generated. Let g be a map in $\mathrm{Hom}(A_{free}, \mathbb{Q}/\mathbb{Z})$ and choose generators $a_1, \ldots, a_m$ of A_{free}. Choose $r_j, s_j \in \mathbb{Z}$ such that $g([a_j]) \equiv r_j/s_j \bmod \mathbb{Z}$. Let $S := \prod s_j$ and define $f : A \to \mathbb{Z}$ by $f := S \cdot g$. Then $f \otimes S^{-1}$ is a preimage for g. □

Proof of Theorem 6. Consider the following diagram, which will be explained below.

$$
\begin{array}{ccccc}
H^{k-1}(M) \otimes \mathbb{Q}/\mathbb{Z} & \hookrightarrow & H^{k-1}(M;\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\ \beta\ } \!\!\!\!\to & \mathrm{Tor}\, H^k(M) \\
\downarrow{\scriptstyle g\otimes 1} \!\!\!\!\! \downarrow & & & & \\
\mathrm{Hom}(H_{k-1}(M), \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} & & {\scriptstyle h}\downarrow{\scriptstyle \cong} & & {\scriptstyle \Phi}\,\vdots \\
{\scriptstyle i}\downarrow{\scriptstyle \cong} & & & & \\
\mathrm{Hom}(H_{k-1}(M)_{free}, \mathbb{Q}/\mathbb{Z}) & \hookrightarrow & \mathrm{Hom}(H_{k-1}(M), \mathbb{Q}/\mathbb{Z}) & \twoheadrightarrow & \mathrm{Hom}(\mathrm{Tor}\, H_{k-1}(M), \mathbb{Q}/\mathbb{Z})
\end{array}
$$

The first row of this diagram is one of the universal coefficient theorems ((UCT 4) on page 58) and therefore a natural exact sequence. Following the construction of this sequence, one can identify the right map as the Bockstein homomorphism β for the coefficient sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$.

For the second row, consider the exact sequence

$$0 \to \mathrm{Tor}\, H_{k-1}(M) \to H_{k-1}(M) \to H_{k-1}(M)_{free} \to 0 \tag{2}$$

and apply the contravariant functor $\mathrm{Hom}(\,\cdot\,, \mathbb{Q}/\mathbb{Z})$. We obtain the long exact Ext-sequence [Munkres, Ex. 52.4, Ex. 41.4]

$$
\begin{aligned}
0 \to \mathrm{Hom}(H_{k-1}(M)_{free}, \mathbb{Q}/\mathbb{Z}) &\to \mathrm{Hom}(H_{k-1}(M), \mathbb{Q}/\mathbb{Z}) \to \\
&\to \mathrm{Hom}(\mathrm{Tor}\, H_{k-1}(M), \mathbb{Q}/\mathbb{Z}) \to \mathrm{Ext}(H_{k-1}(M)_{free}, \mathbb{Q}/\mathbb{Z}) \to \ldots
\end{aligned}
$$

Since $\mathbb{Q}/\mathbb{Z}$ is divisible, the Ext-term is zero, and we have the desired short exact sequence. Since the sequence (2) is natural in M, also the second row in the diagram is natural.

The downward homomorphism g is the Kronecker map. It is surjective with kernel $\mathrm{Ext}(H_{k-2}(M), \mathbb{Z})$ [Munkres, Lemma 45.7, Cor. 53.2]. Tensoring with $\mathbb{Q}/\mathbb{Z}$ preserves surjectivity. The map h is also the Kronecker map. Here, the coefficient group is divisible, so the Ext-term vanishes, and we have an isomorphism. The Kronecker map is natural in a mixed co-/contravariant sense but this will become clear when we deal with it later.

The map i is the isomorphism from Lemma 92. It relies on the fact that M is compact and therefore $H_{k-1}(M)$ is finitely generated.

Let $C^k = C^k(M)$ $(k \in \mathbb{N}_0)$ denote as usual the singular cochain complex of M. The left square in the diagram commutes since both ways an element

$[\alpha] \otimes [q] \in H^{k-1}(M) \otimes \mathbb{Q}/\mathbb{Z}$ with representatives $\alpha \in C^k$, $q \in \mathbb{Q}$ is sent to the homomorphism

$$\begin{aligned} H_{k-1}(M) &\to \mathbb{Q}/\mathbb{Z} \\ [b] &\mapsto q \cdot \alpha(b) \pmod{\mathbb{Z}}. \end{aligned}$$

Having checked the commutativity, there is, by an easy diagram chase, a unique isomorphism

$$\begin{aligned} \Phi\colon \operatorname{Tor} H^k(M) &\to \operatorname{Hom}(\operatorname{Tor} H_{k-1}(M), \mathbb{Q}/\mathbb{Z}) \\ \varepsilon &\mapsto h(\eta) \quad \text{if } \beta\eta = \varepsilon \end{aligned}$$

commuting with the existing maps in the diagram.

Now consider the following homomorphism of exact sequences (cf. [Ranicki, Ex. 12.44 (i)])

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Tor} H^k(M) & \longrightarrow & H^k(M) & \longrightarrow & H^k(M) \otimes \mathbb{Q} \\ & & \cap[M] \big\downarrow \Delta & & \cap[M] \big\downarrow \cong & & \cap[M] \big\downarrow \cong \\ 0 & \longrightarrow & \operatorname{Tor} H_{k-1}(M) & \longrightarrow & H_{k-1}(M) & \longrightarrow & H_{k-1}(M) \otimes \mathbb{Q} \end{array}$$

The middle and right vertical maps are isomorphisms, according to Poincaré duality. Since the torsion subgroup of any abelian group is preserved by all homomorphisms, the left vertical map is well-defined. Denote this map by Δ. By the five-lemma, it is an isomorphism.

The linking isomorphism is now given by the isomorphism Φ, composed with

$$\operatorname{Hom}(\Delta, \mathrm{id})\colon \operatorname{Hom}(\operatorname{Tor} H_{k-1}(M), \mathbb{Q}/\mathbb{Z}) \to \operatorname{Hom}(\operatorname{Tor} H^k(M), \mathbb{Q}/\mathbb{Z}).$$

This finishes the construction of the linking form. It is expressed by the first insertion homomorphism in (1), so this is an isomorphism by construction. We still have to check that it is $(-1)^k$-symmetric and natural. By the symmetry, also the second insertion map in (1) is then an isomorphism, so the linking form is nondegenerate.

For the symmetry, the map $\operatorname{Tor} H^k(X) \to \operatorname{Hom}(\operatorname{Tor} H^k(X), \mathbb{Q}/\mathbb{Z})$ must be studied on cochain level. Consider $[\varepsilon] \in H^{k-1}(X; \mathbb{Q}/\mathbb{Z})$ with representative $\varepsilon \in C^{k-1}(M, \mathbb{Q}/\mathbb{Z})$. Let $\bar{\varepsilon} \in C^{k-1}(M, \mathbb{Q})$ be a lift to $\mathbb{Q}$ (which exists since $C^{k-1}(M, \mathbb{Q}/\mathbb{Z}) = \operatorname{Hom}(C_{k-1}, \mathbb{Q}/\mathbb{Z})$ and $C_{k-1}(M)$ is free). The Bockstein image of $[\varepsilon]$ is defined as $\beta[\varepsilon] = [\delta\bar{\varepsilon}]$, where $\delta\bar{\varepsilon}$ actually lies in the subgroup $C^k(M, \mathbb{Z}) \subset C^k(M, \mathbb{Q})$.

Now let $[\eta]$ be an element in $\operatorname{Tor} H^k(M)$, which is represented by $\eta \in C^k$, and choose ε such that $\beta[\varepsilon] = [\eta]$. Likewise, let $[\kappa] \in \operatorname{Tor} H^k(M)$ and choose $\gamma \in C^{k-1}(M, \mathbb{Q}/\mathbb{Z})$ such that $\beta[\gamma] = [\kappa]$. The cup product $\bar{\varepsilon} \cup \kappa$ makes sense if both cochains are regarded as rational cochains and the cup product refers to the ring structure of $\mathbb{Q}$.

By the first part of the proof, the linking form $L([\eta],[\kappa])$ is given by the rational number $\langle\bar{\varepsilon}\cup\kappa,[M]\rangle$[1], and all choices are irrelevant if it is considered modulo $\mathbb{Z}$.

By the coboundary formula for the cup product, we have

$$\begin{aligned} L([\eta],[\kappa]) &= \langle\bar{\varepsilon}\cup\kappa,[M]\rangle \qquad (\mathrm{mod}\ \mathbb{Z}) \\ &= \langle\bar{\varepsilon}\cup(\delta\bar{\gamma}),[M]\rangle \\ &= (-1)^k\langle(\delta\bar{\varepsilon})\cup\bar{\gamma}-\delta(\bar{\varepsilon}\cup\bar{\gamma}),[M]\rangle \\ &= (-1)^k\langle(\delta\bar{\varepsilon})\cup\bar{\gamma},[M]\rangle-(-1)^k\langle\bar{\varepsilon}\cup\bar{\gamma},\partial[M]\rangle \\ &= (-1)^k\cdot(-1)^{k(k-1)}\langle\bar{\gamma}\cup(\delta\bar{\varepsilon}),[M]\rangle \\ &= (-1)^k L([\kappa],[\eta]). \end{aligned}$$

Naturality is also readily checked. In fact, given a map $f: N\to M$ and $[\eta],[\kappa]\in \operatorname{Tor} H^k(M)$, choose $\bar{\varepsilon}\in C^{k-1}(M,\mathbb{Q})$ as before. Then we have $\beta[f^*\varepsilon]=f^*[\eta]$, and thus

$$\begin{aligned} L(f^*[\eta],f^*[\kappa]) &= \langle f^*\bar{\varepsilon}\cup f^*\kappa,[N]\rangle \\ &= \langle\bar{\varepsilon}\cup\kappa,f_*[M]\rangle \\ &= \deg f\cdot\langle\bar{\varepsilon}\cup\kappa,[M]\rangle \\ &= \deg f\cdot L([\eta],[\kappa]). \end{aligned}$$ □

When the linking form is antisymmetric, this poses restrictions on the homology of manifolds, very similar to restrictions obtained from the intersection form, e. g.

Corollary 93

There is no closed, oriented topological manifold M of dimension $4k+1$ which has $\operatorname{Tor} H^{2k+1}(M)\cong\mathbb{Z}/n$ and $n>2$.

Proof. Let α be a generator of $\operatorname{Tor} H^{2k+1}(M)$. We have $L(\alpha,\alpha)=-L(\alpha,\alpha)$ because the linking form is antisymmetric. Since the pairing is nondegenerate, $L(\alpha,\alpha)$ has order $n>2$, a contradiction. □

Also, there are immediate consequences concerning the chirality of manifolds, see e. g. Lemma 8

[1] The square bracket notation is ambiguous here (though standard): $[\eta]$ means the cohomology class of the cochain η but $[M]$ is the fundamental class of the manifold M.

A.2 Homotopy equivalences between products of lens spaces

Note on the literature. There are at least three articles containing the full classification of products of 3-dimensional lens spaces up to unoriented homotopy equivalence. Unfortunately, all three articles claim conflicting results.

The first work was written by Sieradski [Sieradski] and was corrected by Huck in [Huck]. In [HM], Huck and Metzler give necessary conditions that products of lens spaces of arbitrary but equal dimension are homotopy equivalent. They also announce that these conditions are sufficient in the case of 3-dimensional lens spaces. In a subsequent paper [Huck], Huck constructs the claimed homotopy equivalences. However, this last paper wrongly states an additional condition in the main theorem[2)]. This condition only appears in the statement of the theorem and is not backed up in the proof, so it can safely be removed from the statement without compromising the otherwise valid conclusions of [Huck] and [HM].

The correct theorem appears in the second-newest paper [HM], and we refer to this classification in the following.

Careful inspection of the proofs in [HM] and [Huck] reveals that very few modifications are necessary in order to improve the statements to the oriented versions. In the following, we state the adapted theorems and give the necessary information to complement the proofs.

Theorem 94: oriented version of [HM, p. 13]

If there is an oriented homotopy equivalence between two products of lens spaces of dimension $2n-1$

$$\prod_{i=1}^{s} L^{2n-1}_{m_i}(r_1(i),\dots,r_n(i)) \quad \textit{and} \quad \prod_{i=1}^{s} L^{2n-1}_{m'_i}(r'_1(i),\dots,r'_n(i)),$$

the following two conditions hold:

1. *the fundamental groups are isomorphic:*

$$\bigoplus_{i=1}^{s} \mathbb{Z}/m_i \cong \bigoplus_{i=1}^{s} \mathbb{Z}/m'_i$$

(Note that this implies $\gcd(m_1,\dots,m_s) = \gcd(m'_1,\dots,m'_s)$*. Abbreviate this by* $\gcd$*.)*

[2)] To correct the statement, remove the phrase “If −1 is a square mod p_j or” from the theorem, likewise in the German statement on p. 68.

2. *(a) if n is odd, the congruence relation*

$$\prod_{i=1}^{s} r_1(i)\cdot\ldots\cdot r_n(i) \equiv k^n \cdot \prod_{i=1}^{s} r_1'(i)\cdot\ldots\cdot r_n'(i) \mod \gcd$$

for some $k \in \mathbb{Z}$;
(b) if n is even, for each maximal prime power divisor $p_j^{x_j}$ *of* gcd *the congruence relation*

$$\prod_{i=1}^{s} r_1(i)\cdot\ldots\cdot r_n(i) \equiv \varepsilon_j k^n \cdot \prod_{i=1}^{s} r_1'(i)\cdot\ldots\cdot r_n'(i) \mod p_j^{x_j},$$

where the signs $\varepsilon_j \in \{+1,-1\}$ *for the different prime divisors are determined as follows:*

The sequences $(m_1,\ldots,m_2)$ *and* $(m_1',\ldots,m_2')$ *determine sequences of corresponding maximal prime power divisors*

$$p_j^{x_{ij}} \mid m_i \quad \textit{resp.} \quad p_j^{x_{ij}'} \mid m_i'.$$

By condition 1, $(x_{1j}',\ldots,x_{sj}')$ *is a permutation of* $(x_{1j},\ldots,x_{sj})$*, and the permutation is uniquely determined if these prime power exponents are pairwise distinct. We get the following conditions on the signs* ε_j *in the system of congruence equations above: If* $x_{ij} = x_{lj}$ *for some* $i \neq l$*, then the sign* ε_j *is arbitrary; otherwise, the sign has to be equal to the sign of the corresponding permutation of prime power exponents.*

Proof. The proof in [HM] can be carried over almost unchanged. One only has to restrict the signs at the right places from ±1 to +1, so we state the necessary changes to this paper.

Every map f between products of lens spaces induces a map $\widetilde{f}$ of their universal coverings (which are products of spheres S^{2n-1}). The key information is that the degree of $\widetilde{f}$ is given by the determinant $|(d_{ij})|$ in equation (2) (p. 16). Accordingly, if a homotopy equivalence preserves the orientation, the determinant is +1, not ±1 (p. 17, l. 6). This implies that the ± signs in the following two displayed equations (lines 11 and 13) can be removed.

Further examination of the proof shows that this was the only place where the sign of the degree entered the formulas. Therefore, some ± signs can be removed from the statement of the theorem, too: there is no sign in condition 2. (a) (p. 13), and $\varepsilon = \pm 1$ is replaced by $\varepsilon = 1$ (p. 13, l. 14).

At last, note two small typographical errors: In equation (1) on p. 14, replace "mod n" by "mod m'" (see e. g. [Olum, Thm. V] for the correct statement), and on p. 19, l. 12, the condition $(p-1) \mid (n-1)$ should be $(p-1) \nmid (n-1)$. □

Theorem 95: oriented version of [Huck, main theorem]

In the case $n = 2$ (i. e. 3-dimensional lens spaces), the conditions in Theorem 94 are sufficient.

Proof. Here, the proof in [Huck] can be carried over without any change. The statement of the theorem has to be adapted, of course, so that the conditions equal 1 and 2 (b) of Theorem 94 above. Also, the homotopy equivalences of a single factor, $L_{e_1}(r') \simeq L_{e_1}(\pm k^2 r')$ (p. 72, l. −11), must be restricted to the orientation-preserving ones, i. e. $L_{e_1}(r') \simeq L_{e_1}(k^2 r')$. (Huck uses a slightly different notation: $L_{e_1}(r')$ in [Huck] is written $L_{e_1}(1, r')$ in this text.) All other homotopy equivalences in the proof preserve the orientation, although Huck does not emphasise this.

Huck composes every homotopy equivalence of maps which he calls "elementare Homogenitätstransformationen". Their degree is given by the determinant of a matrix

$$\begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix}$$

(top of p. 78). The maps in a certain subset of these homotopy equivalences ("spezielle Homogenitätstransformationen", bottom of p. 78) have degree 1 since their matrices of partial degrees are shear matrices $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$.

With the more complicated transformations, Huck explicitly constructs maps of degree +1. The relevant information is on p. 81 (last displayed equation with right hand side 1), p. 86, l. −10 and −7 and p. 89, l. −11. Note also a typographical error which is relevant in this context: the correction term on p. 87, l. −6 is $-s(\mathrm{kgV})^2$, not $-s'(\mathrm{kgV})^2$.

Every occurrence of a homotopy equivalence can thus be understood in the orientation-preserving sense. □

From Theorem 94 and Theorem 95, we can deduce, when products of 3-dimensional lens spaces are amphicheiral.

Corollary 96

Consider a product of 3-dimensional lens spaces

$$L := L_{m_1}(1, r_1) \times \ldots \times L_{m_s}(1, r_s).$$

The sequence $(m_1, \ldots, m_s)$ determines, for each prime p_j, a sequence of corresponding maximal prime power divisors $p_j^{x_{ij}}$ of m_i. The product L is homotopically amphicheiral if and only if for each maximal prime power divisor $p_j^{y_j}$ of $\gcd := \gcd(m_1, \ldots, m_s)$, we have $x_{ij} = x_{lj}$ for some $i \neq l$, or -1 is a square $\mathrm{mod}\ p_j^{y_j}$.

A.3 Diffeomorphisms between products of lens spaces

Metzler constructs in his dissertation [Metzler] various diffeomorphisms between products of 3-dimensional lens spaces. He uses the Lie group structure on S^3 as the unit quaternions to construct sophisticated diffeomorphisms between products of S^3 which respect the equivalence relations given by the respective covering transformations. A common feature of his results is that one of the lens space factors must always be $L_m(1,1)$ or $L_m(1,-1)$, otherwise he has no constructions.

As with the results of Huck, Metzler and Sieradski in the previous section, Metzler states and proves his results only with regard to unoriented diffeomorphism. His methods, however—at least in the theorems which are cited here and whose proofs were carefully checked—produce orientation-preserving diffeomorphisms. Evidence for the fact that Metzler really means the unoriented statements is given by the proof of his Theorem 5 (quoted as Theorem 99 below): Here, the orientation changes during the proof (without any hint, and rightly so, since Metzler does not need it for his conclusions). However, the orientation changes twice in total, thus the theorem finally yields an orientation-preserving diffeomorphism.

We state Metzler's results that are of interest for our work and indicate why they are true even with respect to oriented diffeomorphisms. We also draw the conclusions about amphicheirality of products of lens spaces that can be obtained from Metzler's results.

Theorem 97: [Metzler, Satz 1]

$L_m(1,1) \times L_n(1,r)$ is diffeomorphic to $S^3 \times L_{mn}(1,1)$ for $(m,n) = 1$.

(As usual, (m,n) denotes the greatest common divisor.) Since the second product is clearly smoothly amphicheiral, the statement above holds with regard to both orientation-preserving and orientation-reversing diffeomorphisms. We can conclude

Corollary 98

The product of 3-dimensional lens spaces $L_m(1,1) \times L_n(1,r)$ is smoothly amphicheiral if $(m,n) = 1$.

Obviously, the products $L_m(1,-1) \times L_n(1,r)$ and $L_m(1,s) \times L_n(1,\pm1)$ are then amphicheiral, too. In the following, we do not mention the extensions that result from simply reversing the orientation or interchanging the factors. Also the isomorphisms that only change the preferred generator of the fundamental group ($L_n(r,s) \cong L_n(kr,ks)$ for all $k \in (\mathbb{Z}/n)^\times$) are not taken into account.

Theorem 99: [Metzler, Satz 5]

$L_m(1,r) \times L_n(1,1)$ is diffeomorphic to $L_m(1,1) \times L_n(1,r)$ for $(r, mn) = 1$.

Here, it is important to know if the diffeomorphism can be chosen to be orientation-preserving. Metzler does not consider this question in his dissertation but by tracing his construction it can be seen that he indeed gives an orientation-preserving diffeomorphism. In fact, the diffeomorphism in Theorem 99 is given on the universal covering as a composition of five diffeomorphisms $f_1, \dots, f_5 : S^3 \times S^3 \to S^3 \times S^3$.

The diffeomorphisms f_1 to f_3 are all given by the scheme

$$f_i : S^3 \to S^3, \quad (z_1, z_2) \mapsto (g(z_1,z_2) \cdot z_1 \cdot h(z_1,z_2), z_2) \quad \text{or} \quad (z_1, g(z_1,z_2) \cdot z_2 \cdot h(z_1,z_2)) \tag{3}$$

for $z_1, z_2 \in S^3 \subset \mathbb{H}$, certain maps $g, h : S^3 \times S^3 \to S^3$ and multiplication understood in the quaternions $\mathbb{H}$. Hence, the induced map on $H^3(S^3 \times S^3)$ is given (with the obvious basis) by a triangular matrix $\left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$, so f_1 to f_3 preserve the orientation.

The map f_4 is the product of the identity in the first factor and a map of degree -1 in the second factor (compare [Metzler, eq. (37)]), hence it reverses the orientation. This is compensated by f_5, which simply interchanges the two spheres. Altogether, Metzler constructs an orientation-preserving diffeomorphism.

Since the parameter r in Theorem 99 can be changed freely modulo m in one product and modulo n in the other, we can vary the parameter modulo (m, n):

Lemma 100

The products $L_m(1,1) \times L_n(1,r_1)$ and $L_m(1,1) \times L_n(1,r_2)$ are oriented diffeomorphic if $r_1 - r_2$ is a multiple of (m, n).

Proof. Implicitly, we have $(r_1, n) = 1$. Let $g := (m, n)$. Choose $x, y \in \mathbb{Z}$ such that $xn + y\frac{m}{g} = 1 - r_1$. Then we have

$$(r_1 + xn, \tfrac{m}{g}) = 1$$
$$\text{and} \quad (r_1 + xn, n) = (r_1, n) = 1 \quad \Rightarrow \quad (r_1 + xn, g) = 1.$$

Both lines together imply $(r_1 + xn, m) = 1$. Hence, we can replace r_1 by $s_1 := r_1 + xn$ and have thus $L_n(1, r_1) = L_n(1, s_1)$, $(s_1, n) = 1$ and additionally $(s_1, m) = 1$. Likewise, we replace r_2 by s_2 such that $(s_2, m) = (s_2, n) = 1$.

Let a, b be integers such that $s_2 - s_1 = an + bm$. We have the following chain of orientation-preserving diffeomorphisms

$$L_m(1,1) \times L_n(1,s_1) = L_m(1,1) \times L_n(1, s_1 + an) \to L_m(1, s_1 + an) \times L_n(1,1)$$
$$= L_m(1, s_1 + an + bm) \times L_n(1,1) = L_m(1, s_2) \times L_n(1,1) \to L_m(1,1) \times L_n(1, s_2)$$

The two arrows in this formula are given by Theorem 99. For the first arrow, we have to check that $(s_1 + an, mn) = 1$ in order to apply Theorem 99. Since $(s_1, n) = 1$ and $(s_2, m) = 1$, we have

$$(s_1 + an, n) = 1$$
$$\text{and} \quad (s_1 + an, m) = (s_2 - bm, m) = (s_2, m) = 1$$

Both lines together imply $(r + bn, mn) = 1$, so Theorem 99 applies. For the second arrow, we already ensured $(s_2, mn) = 1$. □

This enables us to extend Corollary 98 to the case $(m, n) = 2$.

Proposition 101

The product of 3-dimensional lens spaces $L_m(1,1) \times L_n(1,r)$ is smoothly amphicheiral if $(m,n) \leq 2$.

Proof. By Lemma 100, the product $L_m(1,1) \times L_n(1,r)$ is clearly oriented diffeomorphic to $L_m(1,1) \times L_n(1,-r)$. □

Theorem 102: [Metzler, Satz 7], restricted to two factors

The product $L_{m_1 n}(1, r_1) \times L_{m_2}(1,1)$ is diffeomorphic to $L_{m_1}(1,1) \times L_{m_2 n}(1, r_2)$ if $(m_1, n) = (m_2, n) = 1$ and

$$r_1 \equiv \begin{cases} 1 & \text{mod } m_1 \\ -1 & \text{mod } n, \end{cases} \qquad r_2 \equiv \begin{cases} 1 & \text{mod } m_2 \\ -1 & \text{mod } n. \end{cases}$$

Again, it is important to know whether the constructed diffeomorphism preserves the orientation. Metzler proves the theorem with a composite of two maps on the universal covering $f_2 \circ f_1 : S^3 \times S^3 \to S^3 \times S^3$. Both maps were designed according to the scheme (3), so they preserve the orientation. In the first half of the proof, he also states the covering transformations on $S^3 \times S^3$ (on both ends of the map) carefully, thereby identifying the fundamental groups $\mathbb{Z}/(m_1 n) \oplus \mathbb{Z}/m_2$ and $\mathbb{Z}/m_1 \oplus \mathbb{Z}/(m_2 n)$. By checking these actions one sees that Metzler really generates the space $L_{m_1}(1,1) \times L_{m_2 n}(1, r_2)$ with its canonical orientation on the right hand side and not, e. g. the unoriented diffeomorphic space $L_{m_1}(1,-1) \times L_{m_2 n}(1, r_2)$.

This shows that Theorem 102 can in fact be read with an orientation-preserving diffeomorphism understood, which gives us some amphicheiral products of lens spaces that were not yet covered by Theorems 97 and 99. The other theorems in [Metzler] do not add to the present results.

Proposition 103

Let a, b, c be pairwise coprime integers and let $r \equiv 1 \bmod a$. Then the product $L := L_{ab}(1, r) \times L_{ac}(1,1)$ is smoothly amphicheiral.

Proof. By Lemma 100, L is oriented diffeomorphic to $L_2 := L_{ab}(1, r_2) \times L_{ac}(1,1)$ with

$$r_2 \equiv \begin{cases} 1 & \text{mod } a \\ -1 & \text{mod } b. \end{cases}$$

Thus, we can apply Theorem 102 with $m_1 = a$, $n = b$ and $m_2 = ac$. This gives a diffeomorphism to $L_a(1,1) \times L_{abc}(1, r_3)$ with

$$r_3 \equiv \begin{cases} 1 & \text{mod } a \\ -1 & \text{mod } b \\ 1 & \text{mod } c. \end{cases}$$

Since a, b and c are pairwise coprime, we can apply Lemma 100 again and obtain a diffeomorphism to $L_a(1,1) \times L_{abc}(1, r_4)$ with

$$r_4 \equiv \begin{cases} 1 & \text{mod } a \\ 1 & \text{mod } b \\ -1 & \text{mod } c. \end{cases}$$

Theorem 102, applied this time with $m_1 = a$, $n = c$ and $m_2 = ab$, gives $L_5 := L_{ac}(1, r_5) \times L_{ab}(1,1)$ with

$$r_5 \equiv \begin{cases} 1 & \text{mod } a \\ -1 & \text{mod } b. \end{cases}$$

By Lemma 100, L_2 is oriented diffeomorphic to $L_{ab}(1,1) \times L_{ac}(1,1)$, and L_5 is oriented diffeomorphic to $L_{ac}(1,1) \times L_{ab}(1,1)$. Since these are products of odd-dimensional manifolds with their factors interchanged, this proves that L is amphicheiral. □

References

[Bak] Anthony Bak, *Odd dimension surgery groups of odd torsion groups vanish*, Topology **14** (1975), no. 4, 367–374.

[Barden] Dennis Barden, *Simply connected five-manifolds*, Ann. of Math. (2) **82** (1965), 365–385.

[Baues] Hans-Joachim Baues, *Obstruction theory on homotopy classification of maps*, Lecture Notes in Mathematics, vol. 628, Springer-Verlag, Berlin, 1977.

[Beauville] Arnaud Beauville, *Surfaces complexes et orientation*, Astérisque **126** (1985), 41–43.

[Bredon] Glen E. Bredon, *Topology and geometry*, Graduate Texts in Mathematics, vol. 139, Springer-Verlag, New York, 1993.

[Brown] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1982.

[BS] James C. Becker and Reinhard E. Schultz, *Fixed-point indices and left invariant framings*, Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977). Part I, Lecture Notes in Math., vol. 657, Springer, Berlin, 1978, pp. 1–31.

[BZ] Gerhard Burde and Heiner Zieschang, *Knots*, de Gruyter Studies in Mathematics, vol. 5, Walter de Gruyter & Co., Berlin, 1985.

[CF] Richard H. Crowell and Ralph H. Fox, *Introduction to knot theory*, Graduate Texts in Mathematics, vol. 57, Springer-Verlag, New York, 1977. Reprint of the 1963 original.

[Chapman73] Thomas A. Chapman, *Compact Hilbert cube manifolds and the invariance of Whitehead torsion*, Bull. Amer. Math. Soc. **79** (1973), 52–56.

[Chapman74] ______, *Topological invariance of Whitehead torsion*, Amer. J. Math. **96** (1974), 488–497.

[Conner] Pierre E. Conner, *Differentiable periodic maps*, 2nd ed., Lecture Notes in Mathematics, vol. 738, Springer, Berlin, 1979.

[Conway] John H. Conway, *An enumeration of knots and links, and some of their algebraic properties*, Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967), Pergamon, Oxford, 1970, pp. 329–358.

[CS] Diarmuid Crowley and Jörg Sixt, *Stably diffeomorphic manifolds and* $l_{2q+1}(\mathbb{Z}[\pi])$, Preprint, 2008.

[EMcL] Samuel Eilenberg and Saunders Mac Lane, *On the groups $H(\Pi, n)$. II. Methods of computation*, Ann. of Math. (2) **60** (1954), 49–139.

[Evens] Leonard Evens, *The cohomology of groups*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1991.

[Flapan] Erica Flapan, *When topology meets chemistry: A topological look at molecular chirality*, Outlooks, Cambridge University Press, Cambridge, 2000.

[FP] Rudolf Fritsch and Renzo A. Piccinini, *Cellular structures in topology*, Cambridge Studies in Advanced Mathematics, vol. 19, Cambridge University Press, Cambridge, 1990.

[FQ] Michael H. Freedman and Frank Quinn, *Topology of 4-manifolds*, Princeton Mathematical Series, vol. 39, Princeton University Press, Princeton, NJ, 1990.

[Freedman et al.] Michael H. Freedman, Alexei Kitaev, Chetan Nayak, Johannes K. Slingerland, Kevin Walker, and Zhenghan Wang, *Universal manifold pairings and positivity*, Geom. Topol. **9** (2005), 2303–2317 (electronic).

[GAP] The GAP Group, *GAP—Groups, Algorithms, and Programming*, Version 4.4.10, 2007, `http://www.gap-system.org`.

[Gershenson] Hillel H. Gershenson, *A problem in compact Lie groups and framed cobordism*, Pacific J. Math. **51** (1974), 121–129.

[GM] Phillip A. Griffiths and John W. Morgan, *Rational homotopy theory and differential forms*, Progress in Mathematics, vol. 16, Birkhäuser Boston, Mass., 1981.

[HAP] Graham Ellis, *HAP—Homological Algebra Programming*, Version 1.8.6, `http://hamilton.nuigalway.ie/Hap/www`.

[HatcherAT] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.
`http://www.math.cornell.edu/~hatcher/AT/ATpage.html`.

[HatcherVBKT] ______, *Vector bundles and K-Theory*, Version 2.0, January 2003.
`http://www.math.cornell.edu/~hatcher/VBKT/VBpage.html`.

[HLM] Hugh M. Hilden, María Teresa Lozano, and José María Montesinos, *On knots that are universal*, Topology **24** (1985), no. 4, 499–504.

[HM] Günther Huck and Wolfgang Metzler, *Über den Homotopietyp von Linsenraumprodukten*, Fund. Math. **125** (1985), no. 1, 11–22.

[Hopf] Heinz Hopf, *Zur Algebra der Abbildungen von Mannigfaltigkeiten*, J. Reine Angew. Math. **163** (1930), 71–88.

[HTW] Jim Hoste, Morwen Thistlethwaite, and Jeff Weeks, *The first 1,701,936 knots*, Math. Intelligencer **20** (1998), no. 4, 33–48.

[Huck] Günther Huck, *Homotopieäquivalenzen zwischen Produkten aus dreidimensionalen Linsenräumen*, Fund. Math. **128** (1987), no. 2, 67–90.

[Huppert] Bertram Huppert, *Endliche Gruppen I*, Die Grundlehren der Mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin, 1967.

[Kahn] Donald W. Kahn, *Induced maps for Postnikov systems*, Trans. Amer. Math. Soc. **107** (1963), 432–450.

[Kanenobu] Taizo Kanenobu, *A note on 2-fold branched covering spaces of* S^3, Math. Ann. **256** (1981), no. 4, 449–452.

[Kawakubo] Katsuo Kawakubo, *Orientation reversing involution*, J. Math. Kyoto Univ. **16** (1976), no. 1, 113–115.

[Kirby] Rob Kirby, *Problems in low dimensional manifold theory*, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976). Part 2, Proc. Sympos. Pure Math., vol. 32, Amer. Math. Soc., Providence, R.I., 1978, pp. 273–312.

[KM] Michel A. Kervaire and John W. Milnor, *Groups of homotopy spheres. I*, Ann. of Math. (2) 77 (1963), 504–537.

[Kochman] Stanley O. Kochman, *Bordism, stable homotopy and Adams spectral sequences*, Fields Institute Monographs, vol. 7, American Mathematical Society, Providence, RI, 1996.

[Kosinski] Antoni A. Kosinski, *Differential manifolds*, Pure and Applied Mathematics, vol. 138, Academic Press Inc., Boston, MA, 1993.

[Kotschick92] Dieter Kotschick, *Orientation-reversing homeomorphisms in surface geography*, Math. Ann. **292** (1992), no. 2, 375–381.

[Kotschick97] ———, *Orientations and geometrisations of compact complex surfaces*, Bull. London Math. Soc. **29** (1997), no. 2, 145–149.

[Kreck99] Matthias Kreck, *Surgery and duality*, Ann. of Math. (2) **149** (1999), no. 3, 707–754.

[Kreck07] ———, *Simply connected asymmetric manifolds*, Preprint, 2007.

[Lang] Serge Lang, *Algebra*, 3rd ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002.

[Levine] Jerome P. Levine, *Lectures on groups of homotopy spheres*, Algebraic and geometric topology (New Brunswick, N.J., 1983), Lecture Notes in Math., vol. 1126, Springer, Berlin, 1985, pp. 62–95.

[Lück] Wolfgang Lück, *A basic introduction to surgery theory*, Topology of high-dimensional manifolds, No. 1, 2 (Trieste, 2001), ICTP Lect. Notes, vol. 9, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002, pp. 1–224.

[McCleary] John McCleary, *A user's guide to spectral sequences*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 58, Cambridge University Press, Cambridge, 2001.

[Metzler] Wolfgang Metzler, *Diffeomorphismen zwischen Produkten mit dreidimensionalen Linsenräumen als Faktoren*, Dissertationes Math. Rozprawy Mat. **65** (1969), 60.

[MH] John W. Milnor and Dale Husemoller, *Symmetric bilinear forms*, Springer-Verlag, New York, 1973. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73.

[Milnor59] John W. Milnor, *On spaces having the homotopy type of a CW-complex*, Trans. Amer. Math. Soc. **90** (1959), 272–280.

[Milnor66] ________, *Whitehead torsion*, Bull. Amer. Math. Soc. **72** (1966), 358–426.

[MS] John W. Milnor and James D. Stasheff, *Characteristic classes*, Annals of Mathematics Studies, vol. 76, Princeton University Press, Princeton, N. J., 1974.

[MT] John Morgan and Gang Tian, *Ricci flow and the Poincaré conjecture*, Clay Mathematics Monographs, vol. 3, American Mathematical Society, Providence, RI, 2007.

[Munkres] James R. Munkres, *Elements of algebraic topology*, Addison-Wesley Publishing Company, Menlo Park, CA, 1984.

[Oliver] Robert Oliver, *Whitehead groups of finite groups*, London Mathematical Society Lecture Note Series, vol. 132, Cambridge University Press, Cambridge, 1988.

[Olum] Paul Olum, *Mappings of manifolds and the notion of degree*, Ann. of Math. (2) **58** (1953), 458–480.

[PS] Viktor V. Prasolov and Alexei B. Sossinsky, *Knots, links, braids and 3-manifolds*, Translations of Mathematical Monographs, vol. 154, American Mathematical Society, Providence, RI, 1997.

[Puppe] Volker Puppe, *Do manifolds have little symmetry?*, J. Fixed Point Theory Appl. **2** (2007), no. 1, 85–96.

[Ranicki] Andrew Ranicki, *Algebraic and geometric surgery*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2002. Electronic version of September 2008.
`http://www.maths.ed.ac.uk/~aar/surgery.pdf`.

[Ranicki97] ________, *Notes on Reidemeister torsion* (1997). Unpublished notes.
`http://www.maths.ed.ac.uk/~aar/papers/torsion.pdf`.

[Rolfsen] Dale Rolfsen, *Knots and links*, AMS Chelsea Publishing, Providence, RI, 2003. Corrected reprint of the 1976 original.

[Römpp] Jürgen Falbe and Manfred Regitz, *Römpp Lexikon Chemie*, 10. völlig überarbeitete Auflage, Bd. 1, Georg Thieme Verlag, Stuttgart, 1996.

[Rosenzweig] Harry L. Rosenzweig, *Bordism of involutions on manifolds*, Illinois J. Math. **16** (1972), 1–10.

[Rudyak] Yuli B. Rudyak, *Piecewise linear structures on topological manifolds* (2001). `http://arxiv.org/abs/math/0105047v1`.

[Rueff] Marcel Rueff, *Beiträge zur Untersuchung der Abbildungen von Mannigfaltigkeiten*, Compositio Math. **6** (1939), 161–202.

[Saveliev99] Nikolai Saveliev, *Lectures on the topology of* 3*-manifolds*: *An introduction to the Casson invariant*, de Gruyter Textbook, Walter de Gruyter & Co., Berlin, 1999.

[Saveliev02] ______, *Invariants for homology* 3*-spheres*, Encyclopaedia of Mathematical Sciences, vol. 140, Springer-Verlag, Berlin, 2002.

[Siebenmann] Laurent Siebenmann, *On vanishing of the Rohlin invariant and nonfinitely amphicheiral homology* 3*-spheres*, Topology Symposium, Siegen 1979 (Proc. Sympos., Univ. Siegen, Siegen, 1979), Lecture Notes in Math., vol. 788, Springer, Berlin, 1980, pp. 172–222.

[Sieradski] Allan J. Sieradski, *Non-cancellation and a related phenomenon for the lens spaces*, Topology **17** (1978), no. 1, 85–93.

[Steenrod] Norman E. Steenrod, *Cohomology operations*, Annals of Mathematics Studies, vol. 50, Princeton University Press, Princeton, N.J., 1962.

[Stiefel] Eduard Stiefel, *Richtungsfelder und Fernparallelismus in* n*-dimensionalen Mannigfaltigkeiten*, Comment. Math. Helv. **8** (1935), no. 1, 305–353.

[Stong] Robert E. Stong, *Notes on cobordism theory*, Mathematical notes, Princeton University Press, Princeton, N.J., 1968.

[Switzer] Robert M. Switzer, *Algebraic topology—homotopy and homology*, Classics in Mathematics, Springer-Verlag, Berlin, 2002. Reprint of the 1975 original.

[Wall60] C. T. C. Wall, *Determination of the cobordism ring*, Ann. of Math. (2) **72** (1960), 292–311.

[Wall62] ______, *Killing the middle homotopy groups of odd dimensional manifolds*, Trans. Amer. Math. Soc. **103** (1962), 421–433.

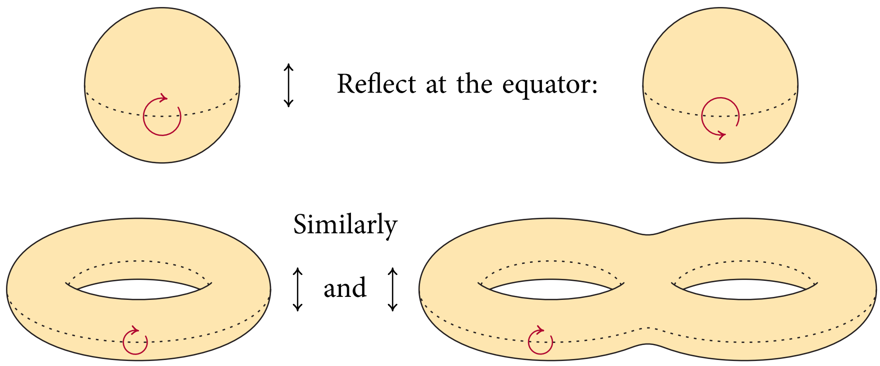
[Wall65] ______, *Finiteness conditions for CW-complexes*, Ann. of Math. (2) **81** (1965), 56–69.

[Wall67] ______, *Resolutions for extensions of groups*, Proc. Cambridge Philos. Soc. **57** (1961), 251–255.

[Zhubr] Alexey V. Zhubr, *Closed simply connected six-dimensional manifolds: proofs of classification theorems*, St. Petersburg Math. J. **12** (2001), no. 4, 605–680.

Summary

We study the phenomenon of orientation reversal of manifolds. An orientable manifold is called *amphicheiral* if it admits an orientation-reversing self-map and *chiral* if it does not. Many familiar manifolds like spheres or orientable surfaces are amphicheiral: they can be embedded mirror-symmetrically into $\mathbb{R}^n$, as the following figure illustrates.



On the other hand, examples of chiral manifolds have been known for many decades, e. g. the complex projective spaces $\mathbb{C}P^{2k}$ or some lens spaces in dimensions congruent 3 mod 4. However, this phenomenon has not been analysed systematically.

Chiral manifolds can be studied in various categories by restricting the orientation-reversing map to homotopy equivalences, homeomorphisms or diffeomorphisms. The various notions of chirality do not coincide, and we extend the definition of chiral and amphicheiral manifolds by attributes, e. g. "topologically chiral" or "smoothly amphicheiral" that express the various restrictions on the orientation-reversing map.

We start with a survey of known results and examples of chiral manifolds, observing the basic facts that the point in dimension 0 is chiral and every closed, orientable 1- and 2-dimensional manifold is amphicheiral. A fundamental question is whether there are chiral manifolds in every dimension ≥ 3, and we prove this as the first main result. Our general aim is to produce manifolds which are chiral in the strongest sense, so we construct manifolds in every dimension ≥ 3 which do not admit a self-map of degree -1.

The obstruction to orientation reversal in the constructed manifolds lies in the fundamental group since, e. g., the odd-dimensional examples are Eilenberg-MacLane spaces, and the proof of chirality uses as a substantial ingredient that the effect of a self-map on homology is completely determined by the induced map on the fundamental group. Therefore, we next ask for obstructions other than the fundamental group und restrict the analysis to simply-connected manifolds.

In dimensions 3, 5 and 6, every simply-connected (closed, orientable, smooth) manifold is amphicheiral by a diffeomorphism, and a topological 4-manifolds is amphicheiral if and only if its signature is zero. In all dimensions ≥ 7, we prove the existence of a simply-connected manifold which does not allow a self-map of degree -1.

Next, in order to further characterise the properties of manifolds which allow or prevent orientation reversal, we consider the question whether every manifold is bordant to a chiral one. This allows also an approximation to the (not mathematically precise) question "how many" manifolds are chiral or if "the majority" of manifolds is chiral or amphicheiral. We prove that in every dimension ≥ 3, every closed, smooth, oriented manifold is oriented bordant to a manifold of this type which is connected and chiral.

The majority of the theorems so far aimed at proving that certain manifolds or families of manifolds are chiral. The opposite problem, however, namely proving amphicheirality in nontrivial circumstances, is also an interesting question. In general, this is even more challenging since not only one obstruction to orientation reversal must be identified and realised but for the opposite direction every possible obstruction must vanish. By using surgery theory, we prove the following theorem: Every product of 3-dimensional lens spaces whose orders of the fundamental groups are odd and coprime admits an orientation-reversing self-diffeomorphism.

In the last chapter, we add a new facet to the results by showing that the order of an orientation-reversing map can be relevant: For every positive integer k, there are infinitely many lens spaces which admit an orientation-reversing diffeomorphism of order 2^k but no orientation-reversing self-map of smaller order.

Conclusion

With analogies to chiral molecules in chemistry and chiral knots in knot theory in mind, it seems a very natural question whether an orientable manifold with its two orientations yields "the same" or "different" objects. Although the manifolds one usually imagines (spheres and 2-dimensional surfaces) are amphicheiral, we show that chiral manifolds exist in every dimension greater than two. Furthermore, the results give a little insight into the variety of mechanisms that can obstruct orientation reversal, particularly in the homotopy type. Aiming in the opposite direction, we prove that products of lens spaces can have orientation-reversing diffeomorphisms in nontrivial circumstances.