A diffeomorphism classification of 5- and 7-dimensional non-simply-connected homogeneous spaces

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Abstract

In this thesis we classify non-simply-connected smooth closed 5- and 7-dimensional orientable manifolds with finite cyclic fundamental group up to diffeomorphism.
The manifolds which we consider admit transitive actions of Lie groups and are diffeomorphic to the total space of certain principal $U(1)$-fibre bundles over a product of complex projective spaces.

Contents

1 Introduction

The classification of smooth manifolds up to diffeomorphism is a central problem in mathematics. For manifolds of dimension greater than or equal to four it's in general not possible to give a complete answer since one fails to classify their fundamental groups ([Nov]).
If the interest lies in smooth manifolds with fixed dimension and fundamental group which maybe fulfill additional homotopical or (differential-) topological properties one could try to solve the classification problem abstractly under these restricted conditions. We state some classical and pioneering examples which go into this direction:

In 1962 Stephen Smale succeeded in classifying closed smooth simply-connected 5-manifolds with vanishing second Stiefel-Whitney class [S] and three years later Denis Barden completed the classification of smooth closed simply-connected 5-manifolds [Bar].
In 1963 Michel A. Kervaire and John W. Milnor reduced the classification of smooth closed simply-connected oriented n-manifolds which have the same homology as the n-sphere, where n is greater than four, to the study of the stable homotopy groups of spheres [K-M].

In this thesis we classify two classes of closed, smooth, orientable, non-simply-connected 5-and 7-dimensional homogeneous manifolds with finite cyclic fundamental group.

This work is split into three parts and in the following we give a summary of each one which includes the statements of its main results.

Chapter 2

This chapter deals with differential topological properties of 7-manifolds which in 1981 E. Witten has introduced as homogeneous spaces [W]. These manifolds are constructed in the following way:

Let

$$\begin{aligned} G &:= SU(3) \times SU(2) \times U(1), \\ H &:= SU(2) \times U(1) \times U(1) \end{aligned}$$

and $i : SU(2) \to G$ be the homomorphism which sends $A \in SU(2)$ to

$$\left(\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right).$$

Let Φ be a Lie group homomorphism from H to G with finite kernel and the properties:

a) $\Phi(A,1,1) = i(A)$,

b) the image of the restriction of Φ to

$$\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\} \times U(1) \times U(1)$$

lies in

$$\left\{ \left(\begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x^{-2} \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, z \right) | x, y, z \in U(1) \right\}.$$

We denote the set of such Lie group homomorphisms by $\mathcal{F}$. If $\Psi \in \mathcal{F}$, then $\frac{G}{\Psi(H)}$ is the left coset space $\{[g\Psi(H)]|g \in G\}$ considered as a smooth manifold and we call it a *Witten space*. We denote the set $\{\frac{G}{\Phi(H)}|\Phi \in \mathcal{F}\}$ by $\mathcal{W}$.

In the first half of the second chapter we compute basic (differential-) topological invariants of these manifolds which will be used in this and the last part of this thesis. Furthermore we parametrise $\mathcal{W}$ by the set of coprime triples, i.e. an element in $\mathcal{W}$ is denoted by M^{pqr}, for some coprime $p, q, r \in \mathbb{Z}$. It turns out that the fundamental group of the Witten space M^{pqr} is isomorphic to $\mathbb{Z}/\gcd(p,q)$ (where we define $\gcd(0,0)$ to be 0) and that M^{pqr} and M^{pq1} are diffeomorphic. We introduce the following notation:

$$M^{pq} := M^{pq1}.$$

By construction M^{pq} has a transitive G-action, thus it's clear what we mean if we speak about Riemannian metrics on M^{pq} which are homogeneous w.r.t. G (which exist since the isotropy group is compact). The main result of the second chapter is

Theorem 1.0.1. *(A classification of the non-simply-connected Witten spaces)*
Let s be a natural number greater than 1 and M^{pq}, $M^{p'q'}$ be two Witten manifolds with $\pi_1(M^{pq}) \cong \pi_1(M^{p'q'}) \cong \mathbb{Z}/s$. Then the following statements are equivalent:

1) There exist homogeneous metrics m_1 and m_2 on M^{pq} and $M^{p'q'}$ w.r.t. $SU(3) \times SU(2) \times U(1)$ s.t. (M^{pq}, m_1) and (M^{pq}, m_2) are isometric.

2) M^{pq} and $M^{p'q'}$ are diffeomorphic.

3) $|p| = |p'|$ and $|q| = |q'|$.

The proof applies an equivariant diffeomorphism classification result and makes use of the so called σ-invariant which Michael F. Atiyah and Isadore M. Singer have defined for smooth cyclic group actions on compact manifolds ([A-S]).

In contrast to the non-simply-connected Witten spaces the classification of the simply-connected ones by Matthias Kreck and Stephan Stolz [Kr-St] is more complicated. In particular they led to the first examples of homeomorphic but non-diffeomorphic homogeneous spaces.

Chapter 3

The main part of this chapter is dedicated to the study of 5-dimensional manifolds which are defined in a similar way as the Witten spaces:

Let

$$\begin{aligned} A &:= SU(2) \times SU(2) \times U(1) \text{ and} \\ T^2 &:= U(1) \times U(1). \end{aligned}$$

Let Ψ be a Lie group homomorphism from T^2 to A with finite kernel and the property that the image of Ψ lies in

$$\left\{ \left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, z \right) | x, y, z \in U(1) \right\}.$$

By $\mathcal{L}$ we denote the set of smooth manifolds which we obtain as left coset spaces of the form $\frac{A}{\Psi(T^2)}$, where Ψ is a homomorphism as described above.

We compute invariants of these manifolds which are analogues to those of the Witten spaces. As for the Witten spaces there is a parametrisation of the elements in $\mathcal{L}$ by three coprime integers, i.e. an element in $\mathcal{L}$ is denoted by N^{pqr}, for some coprime $p, q, r \in \mathbb{Z}$ and as in the case of the Witten spaces it turns out that $\pi_1(N^{pqr}) \cong \mathbb{Z}/\gcd(p,q)$ and that N^{pqr} is diffeomorphic to N^{pq1}. We denote N^{pq1} by N^{pq} and orient N^{pq} as explained on p. 65.

Theorem 1.0.2. *(A diffeomorphism classification of certain 5-manifolds)*
Let r be a natural number greater than 1 s.t. $gcd(r,6) = 1$ and N^{pq}, $N^{p'q'} \in \mathcal{L}$ being oriented with $\pi_1(N^{pq}) \cong \pi_1(N^{p'q'}) \cong \mathbb{Z}/r$. Let further (m,n), (m',n') be pairs of integral numbers s.t. $m\frac{q}{r} + n\frac{p}{r} = 1 = m'\frac{q'}{r} + n'\frac{p'}{r}$. Then there exists an orientation preserving diffeomorphism between N^{pq} and $N^{p'q'}$ if and only if there exist $\epsilon, \epsilon', \delta \in \{\pm 1\}$ and $k, k' \in \mathbb{Z}/r$ s.t.

$$\begin{aligned} pq &= \delta p'q' \\ (\epsilon m + k\frac{p}{r})(\epsilon n - k\frac{q}{r}) &\equiv \delta(\epsilon' m' + k'\frac{p'}{r})(\epsilon' n' - k'\frac{q'}{r}) mod\ r, \\ \frac{q}{r}(\epsilon m + k\frac{p}{r}) - \frac{p}{r}(\epsilon n - k\frac{q}{r}) &\equiv \frac{q'}{r}(\epsilon' m' + k'\frac{p'}{r}) - \frac{p'}{r}(\epsilon n' - k'\frac{q'}{r}) mod\ r. \end{aligned}$$

The previous theorem is an application of the following

Theorem 1.0.3. *(A diffeomorphism classification of 5-manifolds)*
Let $r \in \mathbb{N}$ greater than 1 s.t. $gcd(r,6) = 1$ and N, N' be oriented smooth closed 5-dimensional spin manifolds with $\pi_1(N) \cong \pi_1(N') \cong \mathbb{Z}/r$ and $\pi_2(N) \cong \pi_2(N') \cong \mathbb{Z}$. Then N and N' are diffeomorphic if and only if the R-torsions of N and N' are equivalent (see [M, p. 405]) and if there is a generator u of $H^1(N;\mathbb{Z}/r)$, an isomorphism

$$\alpha : H^1(N;\mathbb{Z}/r) \xrightarrow{\sim} H^1(N';\mathbb{Z}/r)$$

and $z \in H^2(N;\mathbb{Z})$ and $z' \in H^2(N';\mathbb{Z})$ which project to generators of $\frac{H^2(N;\mathbb{Z})}{torsion}$ and $\frac{H^2(N';\mathbb{Z})}{torsion}$ resp. s.t.

$$\begin{aligned}
\langle u(\beta_r(u))^2, [N]_{\mathbb{Z}/r}\rangle &\equiv \langle \alpha(u)(\beta_r(\alpha(u)))^2, [N']_{\mathbb{Z}/r}\rangle \textit{ mod } r \\
\langle u\beta_r(u)z, [N]_{\mathbb{Z}/r}\rangle &\equiv \langle \alpha(u)\beta_r(\alpha(u))z', [N']_{\mathbb{Z}/r}\rangle \textit{ mod } r \\
\langle uz^2, [N]_{\mathbb{Z}/r}\rangle &\equiv \langle \alpha(u)z'^2, [N']_{\mathbb{Z}/r}\rangle \textit{ mod } r, \\
\langle \rho_r(p_1(N))u, [N]_{\mathbb{Z}/r}\rangle &\equiv \langle \rho_r(p_1(N'))\alpha(u), [N']_{\mathbb{Z}/r}\rangle \textit{ mod } r, \\
\sigma(g \in \pi_1(N'), \tilde{N}') &= \sigma(\tilde{\alpha}(g), \tilde{N}) \textit{ for all } g \in \pi_1(N') \setminus \{0\},
\end{aligned}$$

where $\tilde{\alpha} : \pi_1(N') \xrightarrow{\sim} \pi_1(N)$ is the isomorphism that corresponds to α under the Kronecker isomorphism, β_r is the mod-r-Bockstein homomorphism and ρ_r the mod-r-reduction in cohomology.

The σ-invariants are the same σ-invariants which we have applied in the proof of Theorem 1.0.1.

The so called *modified surgery theory* which Matthias Kreck has developed in [Kr.1] implies a strategy for proving this theorem. We will explain it in the course of this chapter and prove Theorem 1.0.3.

Furthermore we show that the non-simply-connected manifolds in $\mathcal{L}$ which fulfill the homotopical assumptions of the previous theorem don't represent all diffeomorphism classes of such manifolds.

Chapter 4

The non-simply-connected Witten spaces are interesting examples of 7-manifolds with cyclic fundamental group and second homotopy group isomorphic to $\mathbb{Z}$. If one is interested in classifying such manifolds, then one could try to apply the modified surgery. The first step is the determination of the so called *normal 2-type* which in the *spin^c* case is

$$(K(\mathbb{Z}/r, 1) \times K(\mathbb{Z}, 2) \times BSpin, \xi),$$

where $\xi : K(\mathbb{Z}/r, 1) \times K(\mathbb{Z}, 2) \times BSpin \to BO$ is a fibration depending on the second Stiefel-Whitney class of the tangent bundle of the manifold. The next step is to decide whether two normal 2-smoothings of manifolds under consideration are normally bordant. This of course requires a study of the corresponding bordism group. We will carry out the last step for manifolds of the prescribed type where the order of the fundamental group is coprime to 6.

Theorem 1.0.4. *(A bordism classification of certain 7-manifolds)*
Let $r \in \mathbb{N}$ greater than 1 s.t. $gcd(r,6) = 1$ and M, M' be closed smooth oriented spin 7-manifolds with normal 2-type equal to

$$(L_r^\infty \times \mathbb{C}P^\infty \times BSpin, \xi),$$

where $\xi : L_r^\infty \times \mathbb{C}P^\infty \times BSpin \to BO$ is a certain fibration (see above). Furthermore let $g := f \times \nu_{sp} : M \to L_r^\infty \times \mathbb{C}P^\infty \times BSpin$ and $g' := f' \times \nu'_{sp} : M' \to L_r^\infty \times \mathbb{C}P^\infty \times BSpin$ be normal 2-smoothings, where ν_{sp} resp. ν'_{sp} is the classifying map of the unique spin bundle over M resp. M'.
Then (M, g) and (M', g') represent the same element in $\Omega_7(L_r^\infty \times \mathbb{C}P^\infty \times BSpin, \xi)$ if and only if

$$\begin{aligned} \left\langle \rho_r(p_1(M)) f^*(v_1 z_r), [M]_{\mathbb{Z}/r} \right\rangle &\equiv \left\langle \rho_r(p_1(M')) f'^*(v_1 z_r), [M']_{\mathbb{Z}/r} \right\rangle (r), \\ \left\langle \rho_r(p_1(M)) \beta_r f^*(v_1) f^*(z_r), [M]_{\mathbb{Z}/r} \right\rangle &\equiv \left\langle \rho_r(p_1(M')) \beta_r f'^*(v_1) f'^*(z_r), [M']_{\mathbb{Z}/r} \right\rangle (r), \\ f^*([M]) &= f'^*([M']), \end{aligned}$$

where ρ_r is the mod-r-reduction in cohomology, v_1 is a generator of $H^1(L_r^\infty; \mathbb{Z}/r) \subset H^2(L_r^\infty \times \mathbb{C}P^\infty \times BSpin; \mathbb{Z})$ and z_r is the mod-r-reduction of the standard generator of $H^2(\mathbb{C}P^\infty; \mathbb{Z}) \subset H^2(L_r^\infty \times \mathbb{C}P^\infty \times BSpin; \mathbb{Z})$.

As an application of the previous theorem we obtain

Theorem 1.0.5. *(A bordism classification of Witten spaces)*
Let r be a natural number with $gcd(r,6) = 1$ and M^{pq}, $M^{p'q'}$ be oriented spin Witten manifolds with $\pi_1(M^{pq}) \cong \pi_1(M^{p'q'}) \cong \mathbb{Z}/r$ and (m,n), $(m',n') \in \mathbb{Z}^2$ s.t. $m\frac{q}{r} + n\frac{p}{r} = 1 = m'\frac{q'}{r} + n'\frac{p'}{r}$. There are normal 2-smoothings $f \times \nu_{sp} : M^{pq} \to L_r^\infty \times \mathbb{C}P^\infty \times BSpin$ and $f' \times \nu'_{sp} : M^{p'q'} \to L_r^\infty \times \mathbb{C}P^\infty \times BSpin$ s.t. $[(M^{pq}, f \times \nu_{sp})] = [(M^{p'q'}, f' \times \nu'_{sp})] \in \Omega_7(L_r^\infty \times \mathbb{C}P^\infty \times BSpin, \xi)$ if and only if there exist triples (s, ϵ, k) and (s', ϵ', k') in

$$(\mathbb{Z}/r)^* \times \{\pm 1\} \times \mathbb{Z}/r \ \ s.t$$

$$(1) \qquad s^2\frac{q}{r} \equiv s'^2\frac{q'}{r} \ mod\ r,$$

$$(2) \qquad s(k\frac{q}{r} - \epsilon n) \equiv s'(k'\frac{q'}{r} - \epsilon' n') \ mod\ r,$$

$$(3) \qquad s(\epsilon m + k\frac{p}{r})^2(k\frac{q}{r} - \epsilon n) \equiv s'(\epsilon' m' + k'\frac{p'}{r})^2(k'\frac{q'}{r} - \epsilon' n') \ mod\ r,$$

$$(4) \qquad s(\epsilon m + k\frac{p}{r})((\epsilon m + k\frac{p}{r})\frac{q}{r} - 2(\epsilon n - k\frac{q}{r})\frac{p}{r}) \equiv s' \cdot$$
$$\cdot(\epsilon' m' + k'\frac{p'}{r})((\epsilon' m' + k'\frac{p'}{r})\frac{q'}{r} - 2(\epsilon' n' - k'\frac{q'}{r})\frac{p'}{r}) \ mod\ r,$$

$$(5) \qquad \frac{s^3}{r^2}(2pq(\epsilon m + k\frac{p}{r}) - p^2(\epsilon n - k\frac{q}{r})) \equiv \frac{s'^3}{r^2}(2p'q'(\epsilon' m' + k'\frac{p'}{r})$$
$$-p'^2(\epsilon' n' - k'\frac{q'}{r})) \ mod\ r,$$

$$(6) \qquad s^4\frac{p^2q}{r^3} \equiv s'^4\frac{p'^2q'}{r^3} \ mod\ r.$$

Although we do not apply Theorem 1.0.4. in the context of a classification program it might be useful for later work.
Furthermore it's interesting to compare the previous result with the diffeomorphism classification of the Witten spaces, since it reveals non-trivial surgery obstructions which lie in a monoid called $l_8(\mathbb{Z}[\mathbb{Z}/r])$ (see [Kr, p. 725]).

We would like to stress that all the proofs which appear in this thesis were done by the author and not copied from somewhere else.

Notations and Conventions

- In this thesis the natural numbers start with 1, i.e. $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0$ denotes the natural numbers with 0.
- The positive real numbers are denoted by $\mathbb{R}_{>0}$.
- In this work it will always be clear from the context in which category we are working, e.g. in the category of groups, smooth (oriented) manifolds or (real or complex) vector bundles. Depending on the category the symbol $\cong$ relates objects which are isomorphic.

- Let N be a connected smooth manifold. If we have chosen a specific cell decomposition for N, then we write

$$N = e_0 \cup \cdots .$$

For example $S^2 = e_0 \cup e_2$.

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2 Witten spaces

In 1981 E. Witten introduced a certain class of 7-dimensional smooth manifolds in the framework of a physical theory [W]. In this chapter we construct these so called *Witten spaces*, give two topological characterisations and shed some light on some of their geometrical properties. Furthermore we compute some of their basic invariants which will play a role in the next and the last chapter of this work.

2.1 A definition and a parametrisation of the Witten spaces

We denote the 12-dimensional Lie group

$$SU(3) \times SU(2) \times U(1)$$

by G and the 5-dimensional Lie group

$$SU(2) \times U(1) \times U(1)$$

by H.

Let $i : SU(2) \to G$ be the homomorphism which sends $A \in SU(2)$ to

$$\left(\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, 1 \right).$$

In the following we describe the maps from H to G which Witten has considered in order to construct the manifolds to play a role in [W]:

Let Φ be a Lie group homomorphism from H to G with finite kernel and the properties:

- $\Phi(A, 1, 1) = i(A)$,
- the image of the restriction of Φ to

  $$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \times U(1) \times U(1)$$

  lies in

  $$S := \left\{ \left(\begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x^{-2} \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, z \right) | x, y, z \in U(1) \right\}.$$

The set consisting of such homomorphisms which we denote by $\mathcal{F}$ is not empty:
The map

$$\begin{array}{rcl}\Psi : H & \to & G, \\ (A, z_1, z_2) & \mapsto & \left(\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} z_1 & 0 \\ 0 & z_1^{-1} \end{pmatrix}, z_2 \right)\end{array}$$

clearly lies in $\mathcal{F}$.

Definition 2.1.1. *Let $\Phi \in \mathcal{F}$. Then the homogeneous space*

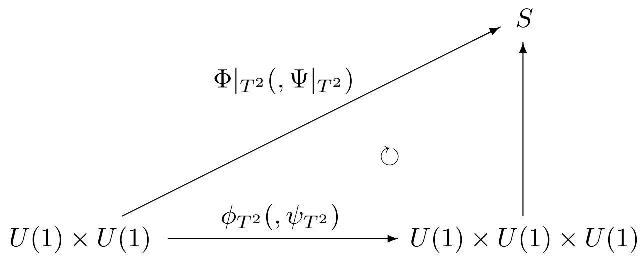
$$\frac{G}{\Phi(H)} := \{[g\Phi(H)] | g \in G\}$$

*considered as a smooth manifold is called a **Witten space**. We denote the set of all Witten spaces by $\mathcal{W}$.*

Remark 2.1.2. *Let $l \in G$ then $\dfrac{G}{l\Phi(H)l^{-1}}$ is diffeomorphic to a Witten space.*

Classification of images

Let Φ, $\Psi \in \mathcal{F}$. We identify $\{1\} \times U(1) \times U(1)$ with $U(1) \times U(1)$ and denote the restriction of Φ (resp. Ψ) to $U(1) \times U(1)$ by $\Phi|_{T^2}$ (resp. $\Psi|_{T^2}$):



The right vertical map sends (x, y, z) to

$$\left(\begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x^{-2} \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}, z \right)$$

and is clearly an isomorphism. The homomorphism ϕ_{T^2} (resp. ψ_{T^2}) is the unique one which makes the above diagram commutative.
Since

$$\Phi\left(SU(2) \times \{1\} \times \{1\}\right) = \Psi(SU(2) \times \{1\} \times \{1\})$$

it follows that

$$\operatorname{im}(\Phi) = \operatorname{im}(\Psi) \quad \Leftrightarrow \quad \operatorname{im}(\phi_{T^2}) = \operatorname{im}(\psi_{T^2}). \tag{2.1}$$

We denote the set of Lie group homomorphisms from $U(1)\times U(1)$ to $U(1)\times U(1)\times U(1)$ with finite kernel by $\bar{\mathcal{F}}$. Of course the above triangle gives rise to a 1-1 correspondence between $\mathcal{F}$ and $\bar{\mathcal{F}}$.

Let ϕ, $\psi \in \bar{\mathcal{F}}$. We further denote the differential of ϕ (resp. ψ) at the neutral element e of $U(1)\times U(1)$ by $(d\phi)_e$ (resp. $(d\psi)_e$) and by $\exp_{U(1)\times U(1)}$ (resp. $\exp_{U(1)\times U(1)\times U(1)}$) we mean the Lie exponential map of $U(1)\times U(1)$ (resp. $U(1)\times U(1)\times U(1)$). If we speak of the Lie algebra of $\operatorname{im}(\psi)$, then we mean the Lie subalgebra in the Lie algebra of $U(1)\times U(1)\times U(1)$ which is associated to the Lie subgroup $\operatorname{im}(\psi)$.

Lemma 2.1.3. *im(ϕ)=im(ψ) if and only if im$(d\phi)_e$=im$(d\psi)_e$.*

Proof. The image of ϕ (resp. ψ) is a compact and connected Lie subgroup of $U(1)\times U(1)\times U(1)$. In this case the Lie exponential map $\exp_{U(1)\times U(1)\times U(1)}$ maps the Lie algebra of $\operatorname{im}(\phi)$ (resp. $\operatorname{im}(\psi)$) surjectively onto $\operatorname{im}(\phi)$ (resp. $\operatorname{im}(\psi)$). If the Lie algebras of $\operatorname{im}(\phi)$ and $\operatorname{im}(\psi)$ are not the same, then the fact that $\exp_{U(1)\times U(1)\times U(1)}$ is a local diffeomorphism around 0 implies that the images of the corresponding Lie algebras are not the same. Summarizing these considerations yield:

$$\operatorname{im}(\phi) = \operatorname{im}(\psi) \Leftrightarrow \text{ The Lie algebras of } \operatorname{im}(\phi) \text{ and } \operatorname{im}(\psi) \text{ are the same.}$$

Let γ be an arbitrary Lie group homomorphism from $U(1)\times U(1)$ to $U(1)\times U(1)\times U(1)$ and $(d\gamma)_e$ its differential at the neutral element. Then we have the following fact

$$\gamma \circ \exp_{U(1)\times U(1)} = \exp_{U(1)\times U(1)\times U(1)} \circ (d\gamma)_e \tag{2.2}$$

([He, Lemma 1.12., p. 110]). This equation allows us to identify the Lie algebra of $\operatorname{im}(\phi)$ (resp. $\operatorname{im}(\psi)$) with the image of $(d\phi)_e$ (resp. $(d\psi)_e$). ■

Now we equip the Lie algebras of $U(1)\times U(1)$ and $U(1)\times U(1)\times U(1)$ with the standard bases and identify them with $\mathbb{R}^2$ resp. $\mathbb{R}^3$ with their standard bases.

The equation (2.2) which we have used in the proof of Lemma 2.1.3. implies that $\phi(z_1, z_2)$ is of the form $(z_1^{a_1} z_2^{b_1}, z_1^{a_2} z_2^{b_2}, z_1^{a_3} z_2^{b_3})$, where

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{Z}^3.$$

Thus the representation matrix of $(d\phi)_e$ w.r.t. the chosen basis is

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}$$

and since kerϕ is finite it has rank two.

Some notations and a convention

- Let's denote by $\mathrm{Gr}_2^{\mathbb{Z}}(\mathbb{R}^3)$ the set of 2-dimensional real subvectorspaces of $\mathbb{R}^3$ with the property that they possess a basis with integer coordinates.
- We denote by $\mathrm{Gr}_2(\mathbb{Z}^3)$ the set of all submodules L of $\mathbb{Z}^3$ with the property that $\mathbb{Z}^3/L \cong \mathbb{Z}$.
- The set of epimorphisms from $\mathbb{Z}^3$ to $\mathbb{Z}$ is denoted by $\mathcal{E}$. We say that two elements of $\mathcal{E}$ are equivalent if they differ by a sign. The resulting quotient set is denoted by $\mathcal{E}_\pm$.
- Let $n \in \mathbb{N}$ and $s_1, \ldots, s_n \in \mathbb{Z}$. The integers $s_1, \ldots, s_n$ are called **coprime** if their greatest common divisor is 1, i.e. $\gcd(s_1, \ldots, s_n) = 1$. By a generalisation of *Bézout's identity* ([J-J]) $s_1, \ldots, s_n$ are coprime if and only if there exist $t_1, \ldots, t_n \in \mathbb{Z}$ s.t. $\sum_{i=1}^n t_i s_i = 1$.
- The set $\{(p, q, r) \in \mathbb{Z}^3 | p, q, r \text{ are coprime}\}$ is denoted by I. We say that two elements of I are equivalent if they differ by a sign. The resulting quotient set is denoted by $I_\pm$.
- Let $\mathbb{Z}$ and $\mathbb{Z}^3$ be equipped with the standard bases then we denote the kernel of the map from $\mathbb{Z}^3$ to $\mathbb{Z}$ which sends $\begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}$ to $as_1 + bs_2 + cs_3$ by $\ker(a, b, c)$.

Let $L \in \mathrm{Gr}_2(\mathbb{Z}^3)$, v_L be one of the two possible isomorphisms between $\mathbb{Z}^3/L$ and $\mathbb{Z}$ and $\mathrm{pr}_L : \mathbb{Z}^3 \to \mathbb{Z}^3/L$ be the linear quotient map. Then it's clear that

$$\alpha_2 := v_L \circ \mathrm{pr}_L : \mathbb{Z}^3 \to \mathbb{Z}$$

defines an element in $\mathcal{E}_\pm$ which doesn't depend on the choice of the isomorphism v_L.

Lemma 2.1.4. *The following chain of maps is a chain of bijections:*

$$\{im(d\phi)_e | \phi \in \bar{\mathcal{F}}\} = Gr_2^{\mathbb{Z}}(\mathbb{R}^3) \xrightarrow{\alpha_1} Gr_2(\mathbb{Z}^3) \xrightarrow{\alpha_2} \mathcal{E}_\pm \xrightarrow{\alpha_3} I_\pm,$$

where

$$\begin{aligned} \alpha_1(V) &= V \cap \mathbb{Z}^3, \\ \alpha_3(s) &= \textit{the representation matrix of } s \textit{ w.r.t. the standard basis of } \mathbb{Z}^3 \textit{ and } \mathbb{Z}. \end{aligned}$$

Proof. First of all one observes that all the maps are well defined.
The map α_3 is clearly a bijection. The map from $\mathcal{E}_\pm$ to $\mathrm{Gr}_2(\mathbb{Z}^3)$ which sends an epimorphism to its kernel is the inverse of α_2.
Let $L \in \mathrm{Gr}_2(\mathbb{Z}^3)$. There is a unique 2-dimensional (real) sub-vectorspace $L_\mathbb{R}$ of $\mathbb{R}^3$ which contains L. The map from $\mathrm{Gr}_2(\mathbb{Z}^3)$ to $\mathrm{Gr}_2^\mathbb{Z}(\mathbb{R}^3)$ which sends L to $L_\mathbb{R}$ is the inverse of α_1. ∎

Definition 2.1.5. *Let* $\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} =: \boldsymbol{e}, \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} =: \boldsymbol{f} \in \mathbb{Z}^3$.

- *We call the map from $U(1)$ to $S \subset G$ which maps*
  z_1 *to* $\left(\begin{pmatrix} z_1^{e_1} & 0 & 0 \\ 0 & z_1^{e_1} & 0 \\ 0 & 0 & z_1^{-2e_1} \end{pmatrix}, \begin{pmatrix} z_1^{e_2} & 0 \\ 0 & z_1^{-e_2} \end{pmatrix}, z_1^{e_3} \right)$
  the homomorphism which is ***induced*** *by* $\boldsymbol{e}$.
- *We call the map from $U(1) \times U(1)$ to $S \subset G$ which maps*
  (z_1, z_2) *to* $\left(\begin{pmatrix} z_1^{e_1} z_2^{f_1} & 0 & 0 \\ 0 & z_1^{e_1} z_2^{f_1} & 0 \\ 0 & 0 & z_1^{-2e_1} z_2^{-2f_1} \end{pmatrix}, \begin{pmatrix} z_1^{e_2} z_2^{f_2} & 0 \\ 0 & z_1^{-e_2} z_2^{-f_2} \end{pmatrix}, z_1^{e_3} z_2^{f_3} \right)$
  the homomorphism which is ***induced*** *by* $(\boldsymbol{e}, \boldsymbol{f})$.

Corollary 2.1.6. *Let i be the embedding of $SU(2)$ into G which we have defined on* *p. 9.* *The map*

$$\begin{aligned} \mathcal{P} : I_\pm &\rightarrow \{im(\Phi) | \Phi \in \mathcal{F}\}, \\ (p, q, r) &\mapsto im(\mu_\psi(i \times \psi)), \end{aligned}$$

is well defined and a bijection, where ψ is the homomorphism which is induced by two linearly independent elements of $ker(p, q, r)$ and

$$\begin{aligned} \mu_\psi : im(i) \times im(\psi) &\rightarrow G, \\ (n, u) &\mapsto nu \end{aligned}$$

is the multiplication map which is a homomorphism since $im(i)$ and S commute.

Proof. This follows from the fact (2.1) on p. 11, Lemma 2.1.3. and Lemma 2.1.4. ∎

Let $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ be the basepoints of the spheres S^5 resp. S^3 written in complex coordinates. Furthermore we denote the homogeneous left coset space

$$\frac{SU(3)}{i(SU(2))}$$

simply by
$$\frac{SU(3)}{SU(2)}.$$
Let
$$\boldsymbol{a} := \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \boldsymbol{b} := \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{Z}^3$$
be a basis of $\ker(p,q,r)$ and ψ the homomorphism from $U(1)\times U(1)$ to S which is induced by $(\boldsymbol{a},\boldsymbol{b})$, i.e.
$$\psi(z_1,z_2) = \left(\begin{pmatrix} z_1^{a_1} z_2^{b_1} & 0 & 0 \\ 0 & z_1^{a_1} z_2^{b_1} & 0 \\ 0 & 0 & z_1^{-2a_1} z_2^{-2b_1} \end{pmatrix}, \begin{pmatrix} z_1^{a_2} z_2^{b_2} & 0 \\ 0 & z_1^{-a_2} z_2^{-b_2} \end{pmatrix}, z_1^{a_3} z_2^{b_3} \right).$$
This homomorphism induces a $U(1)\times U(1)$-action on $\frac{SU(3)}{SU(2)} \times SU(2) \times U(1)$, just given by right multiplication. The maps
$$\begin{aligned} \Delta_1 : \frac{SU(3)}{SU(2)} &\to S^5, \\ [L] &\mapsto L \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and} \\ \Delta_2 : SU(2) &\to S^3, \\ A &\mapsto A \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$
are the standard diffeomorphisms.
The pushforward of the previously described action via the diffeomorphism $\Delta_1 \times \Delta_2 \times id_{U(1)}$ is the following action on $S^5 \times S^3 \times U(1)$:
$$\begin{aligned} F_{\boldsymbol{a},\boldsymbol{b}} : U(1)\times U(1)\times (S^5\times S^3\times U(1)) &\to S^5\times S^3\times U(1) \\ (z_1,z_2,((x_1,x_2,x_3),(x_4,x_5),x_6)) &\mapsto (z_1^{-2a_1} z_2^{-2b_1}(x_1,x_2,x_3), z_1^{a_2} z_2^{b_2}(x_4,x_5), \\ &\quad z_1^{a_3} z_2^{b_3} x_6). \end{aligned}$$
On the other hand the $U(1)\times U(1)$-action which is given by
$$\begin{aligned} G_{\boldsymbol{a},\boldsymbol{b}} : U(1)\times U(1)\times (S^5\times S^3\times U(1)) &\to S^5\times S^3\times U(1) \\ (z_1,z_2,((x_1,x_2,x_3),(x_4,x_5),x_6)) &\mapsto (z_1^{2a_1} z_2^{2b_1}(x_1,x_2,x_3), z_1^{2a_2} z_2^{2b_2}(x_4,x_5), \\ &\quad z_1^{2a_3} z_2^{2b_3} x_6) \end{aligned}$$
is the pushforward of the $U(1)\times U(1)$-action on $\frac{SU(3)}{SU(2)} \times SU(2) \times U(1)$ that comes from the homomorphism ψ_D which is induced by
$$\left(\begin{pmatrix} -a_1 \\ 2a_2 \\ 2a_3 \end{pmatrix}, \begin{pmatrix} -b_1 \\ 2b_2 \\ 2b_3 \end{pmatrix} \right) = (\boldsymbol{Da}, \boldsymbol{Db}),$$

where $\boldsymbol{D}$ is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \in \mathrm{Gl}(3, \mathbb{Q}).$$

For a technical reason which we will give in section 2.4. we define M^{pqr} to be

$$\frac{G}{\mathrm{im}(\mu_{\psi_{\boldsymbol{D}}}(i \times \psi_{\boldsymbol{D}}))}.$$

Definition/Lemma 2.1.7. *Let $(a, b, c) \in I_{\pm}$. We define M^{abc} to be*

$$\frac{G}{im(\mu_\gamma(i \times \gamma))},$$

where γ is the homomorphism from $U(1) \times U(1)$ to G which is induced by two linearly independent elements of $\boldsymbol{D}\cdot ker(a, b, c)$. There is the following parametrisation of $\mathcal{W}$:

$$\kappa : I_{\pm} \quad \to \quad \mathcal{W}, \quad (a, b, c) \mapsto M^{abc},$$

i.e. κ is a bijection.

Proof. This follows from Corollary 2.1.6. and the fact that $\boldsymbol{D} \in \mathrm{Gl}(3, \mathbb{Q})$ induces an automorphism of $\mathrm{Gr}_2^{\mathbb{Z}}(\mathbb{R}^3)$. ■

2.2 The universal covering spaces

From now on we always assume that $p, q, r \in \mathbb{Z}$ are coprime.

Let $M^{pqr} \in \mathcal{W}$. From Corollary 2.1.6. and Definition 2.1.7. we know that it is sufficient to find two linearly independent elements of $\boldsymbol{D}\ker(p, q, r)$ in order to determine the unique subgroup H_{pqr} of G s.t.

$$\frac{G}{H_{pqr}} = M^{pqr}.$$

If $p = q = 0$, then $r = 1$. Two linearly independent elements of $\boldsymbol{D}\ker(0, 0, 1)$ are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$. By the construction of the Witten spaces it follows that

$$\begin{aligned} M^{001} &= \frac{\frac{SU(3)}{SU(2)}}{U(1)} \times \frac{SU(2)}{U(1)} \times U(1) \\ &\cong \mathbb{C}P^2 \times \mathbb{C}P^1 \times U(1). \end{aligned}$$

Now we assume that $q \neq 0$.

There are the following linearly independent elements of $\ker(p,q,r)$:

$$\boldsymbol{u} := \begin{pmatrix} -q \\ p \\ 0 \end{pmatrix}, \quad \boldsymbol{v} := \begin{pmatrix} 0 \\ r \\ -q \end{pmatrix}.$$

Remark 2.2.1. *If $q = 0$ and $p \neq 0$, then*

$$\boldsymbol{u}^* := \begin{pmatrix} -q \\ p \\ 0 \end{pmatrix}, \quad \boldsymbol{v}^* := \begin{pmatrix} r \\ 0 \\ -p \end{pmatrix}$$

are linearly independent elements of $\ker(p,q,r)$ and everything we do in the following with $\boldsymbol{u}$ and $\boldsymbol{v}$ can analoguesly be done with $\boldsymbol{u}^*$ and $\boldsymbol{v}^*$.

Let $\boldsymbol{w} \in \mathbb{Z}^3$ and $\boldsymbol{D}$ be the matrix which we have introduced on p. 15. From now on we denote in this section $\boldsymbol{D}\boldsymbol{w} \in \mathbb{Z}^3$ by $\boldsymbol{w}'$.

Since S is an abelian group the multiplication map

$$\mu : S \times S \to S, \quad (s_1, s_2) \mapsto s_1 s_2$$

is a homomorphism. Let $g_{\boldsymbol{w}'}$ and $g_{\boldsymbol{v}'}$ be the homomorphisms from $U(1)$ to G which are induced by $\boldsymbol{u}'$ resp. $\boldsymbol{v}'$ in the sense of Definition 2.1.5. We conclude from the previous lemmas that

$$\begin{aligned} M^{pqr} &= \frac{\frac{SU(3)}{SU(2)} \times SU(2) \times U(1)}{\mu \circ (g_{\boldsymbol{u}'} \times g_{\boldsymbol{v}'})(U(1) \times U(1))} \\ &= \frac{\dfrac{\frac{SU(3)}{SU(2)} \times SU(2)}{g_{\boldsymbol{u}'}(U(1))} \times U(1)}{g_{\boldsymbol{v}'}(U(1))}. \end{aligned}$$

Lemma 2.2.2. $\dfrac{\frac{SU(3)}{SU(2)} \times SU(2)}{g_{\boldsymbol{u}'}(U(1))}$ *is diffeomorphic to* $M^{\frac{p}{gcd(p,q)} \frac{q}{gcd(p,q)} 0}$.

Proof. Let

$$\bar{\boldsymbol{u}} := \begin{pmatrix} -\frac{q}{\gcd(p,q)} \\ \frac{p}{\gcd(p,q)} \\ 0 \end{pmatrix}, \quad \boldsymbol{v}_1 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

As we have defined the maps $g_{\boldsymbol{u}'}, g_{\boldsymbol{v}'}$ for $\boldsymbol{u}'$ and $\boldsymbol{v}'$ we analoguesly define the maps $g_{\bar{\boldsymbol{u}}'}$ and $g_{\boldsymbol{v}'_1}$ for $\bar{\boldsymbol{u}}'$ and $\boldsymbol{v}'_1$ resp. Then

$$\begin{aligned} M^{\frac{p}{\gcd(p,q)}\,\frac{q}{\gcd(p,q)}\,0} &= \frac{\frac{SU(3)}{SU(2)} \times SU(2) \times U(1)}{\mu \circ (g_{\bar{\boldsymbol{u}}'} \times g_{\boldsymbol{v}'_1})(U(1) \times U(1))} \\ &\cong \frac{\frac{SU(3)}{SU(2)} \times SU(2)}{g_{\bar{\boldsymbol{u}}'}}. \end{aligned}$$

But $g_{\boldsymbol{u}'}$ and $g_{\bar{\boldsymbol{u}}'}$ have the same images which finishes the proof. ■

In the proof of the last lemma we introduced the map $g_{\bar{\boldsymbol{u}}'}$. By dividing out the kernel of $g_{\bar{\boldsymbol{u}}'}$ we obtain an embedding of $U(1)$ into $\dfrac{SU(3)}{SU(2)} \times SU(2)$ and a free and smooth (right) $U(1)$-action on $\dfrac{SU(3)}{SU(2)} \times SU(2)$ which gives rise to a fibre bundle

$$U(1) \to \frac{SU(3)}{SU(2)} \times SU(2) \to M^{\frac{p}{\gcd(p,q)}\,\frac{q}{\gcd(p,q)}\,0}.$$

The long exact sequence of homotopy groups for this fibration,

$$\cdots \to \pi_2(\tfrac{SU(3)}{SU(2)} \times SU(2)) \to \pi_2(M^{\frac{p}{\gcd(p,q)}\,\frac{q}{\gcd(p,q)}\,0}) \to$$

$$\to \pi_1(U(1)) \to \pi_1(\tfrac{SU(3)}{SU(2)} \times SU(2)) \to \pi_1(M^{\frac{p}{\gcd(p,q)}\,\frac{q}{\gcd(p,q)}\,0}) \to \pi_0(U(1)) \to \cdots,$$

implies

Lemma 2.2.3. *The Witten space $M^{\frac{p}{\gcd(p,q)}\,\frac{q}{\gcd(p,q)}\,0}$ is simply-connected and $\pi_2(M^{\frac{p}{\gcd(p,q)}\,\frac{q}{\gcd(p,q)}\,0})$ is isomorphic to $\mathbb{Z}$.*

The diffeomorphism

$$\Delta_1 \times \Delta_2 : \frac{SU(3)}{SU(2)} \times SU(2) \xrightarrow{\sim} S^5 \times S^3$$

and $g_{\bar{\boldsymbol{u}}'}$ induce the following action on $S^5 \times S^3$:

$$\begin{aligned} E_{\bar{\boldsymbol{u}}'} : U(1) \times (S^5 \times S^3) &\to S^5 \times S^3, \\ (z, ((z_1, z_2, z_3), (z_4, z_5))) &\mapsto (z^{-\frac{2q}{\gcd(p,q)}}(z_1, z_2, z_3), z^{\frac{2p}{\gcd(p,q)}}(z_4, z_5)). \end{aligned}$$

The action given by $E_{\bar{\boldsymbol{u}}'}$ induces the following free $U(1)$-action

$$\begin{aligned} G_{\bar{\boldsymbol{u}}'} : U(1) \times (S^5 \times S^3) &\to S^5 \times S^3, \\ (z, ((z_1, z_2, z_3), (z_4, z_5))) &\mapsto (z^{-\frac{q}{\gcd(p,q)}}(z_1, z_2, z_3), z^{\frac{p}{\gcd(p,q)}}(z_4, z_5)). \end{aligned}$$

We denote the orbit space of this free $U(1)$-action by $\dfrac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}}$ and it's clear that

$$M^{\frac{p}{\gcd(p,q)}\frac{q}{\gcd(p,q)}0} \cong \frac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}}.$$

Let's further denote the vector $\frac{1}{2\gcd(q,r)}\boldsymbol{v}'$ by $\bar{\boldsymbol{v}}'$ and $g_{\bar{\boldsymbol{v}}'}$ is the homomorphism from $U(1)$ to G which is induced by $\bar{\boldsymbol{v}}'$ (in the sense of Definition 2.1.5.). We notice that $g_{\bar{\boldsymbol{v}}'}$ is an embedding of $U(1)$ into G.
Thus $g_{\bar{\boldsymbol{v}}'}$ gives rise to a smooth (right) $U(1)$-action on $M^{\frac{p}{\gcd(p,q)}\frac{q}{\gcd(p,q)}0} \times U(1)$ and hence on $\dfrac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}} \times U(1)$:

$$\begin{aligned} F_{\bar{\boldsymbol{v}}'} : U(1) \times \left(\frac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}}} \times U(1)\right) &\longrightarrow \frac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}}} \times U(1), \\ (z, ([(z_1,z_2,z_3),(z_4,z_5)], z_6)) &\longmapsto ([(z_1,z_2,z_3), z^{\frac{r}{\gcd(q,r)}}(z_4,z_5)], z^{-\frac{q}{\gcd(q,r)}} z_6). \end{aligned}$$

The orbit space of this $U(1)$-action is denoted by $\dfrac{\frac{S^5\times S^3}{\sim_{\bar{\boldsymbol{u}}'}} \times U(1)}{\sim_{\bar{\boldsymbol{v}}'}}$ and the above considerations make clear that $\dfrac{\frac{S^5\times S^3}{\sim_{\bar{\boldsymbol{u}}'}} \times U(1)}{\sim_{\bar{\boldsymbol{v}}'}}$ and M^{pqr} are diffeomorphic.
Elements of $M^{\frac{p}{\gcd(p,q)}\frac{q}{\gcd(p,q)}0}$ are equivalence classes of matrices, whereas elements in $\dfrac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}}$ are equivalence classes of points in $S^5 \times S^3$ which themselves can be written as complex vectors. In order to keep notations as simple as possible we prefer to study the action $F_{\bar{\boldsymbol{v}}'}$ on $\dfrac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}} \times U(1)$ in order to understand M^{pqr} better.
Let $x := ([(z_1,z_2,z_3),(z_4,z_5)], z_6) \in \dfrac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}} \times U(1)$. We denote the $U(1)$-orbit of x by $x \cdot U(1)$. Each $U(1)$-orbit in $\dfrac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}} \times U(1)$ intersects $\dfrac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}} \times \{1\}$. We identify $\dfrac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}} \times \{1\}$ with $\dfrac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}}$ and it is clear what we mean when we speak about

$$\textit{the intersection points of } x \cdot U(1) \textit{ with } \frac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}}.$$

The figure on the next page illustrates the last lines in the case of an $U(1)$-action on $U(1) \times U(1)$ which comes from a Lie group homomorphism from $U(1)$ to $U(1) \times U(1)$ with finite kernel.

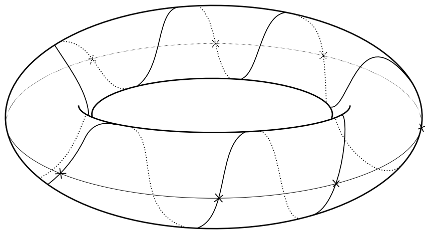


Figure 2.1: The image of an orbit of an $U(1)$-action on the 2-torus $U(1) \times U(1)$. The meridian is identified with $U(1) \times \{1\}$ and the little crosses mark the intersection points of the orbit with $U(1) \times \{1\}$.

We say that x_1 and $x_2 \in \frac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}}$ are equivalent if there exists a $y \in \frac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}} \times U(1)$ s.t.

$$x_1, x_2 \in y \cdot U(1) \cap \frac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}}.$$

Let's denote the set of intersection points of $x \cdot U(1)$ and $\frac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}}$ by $\mathcal{I}_x$.

By multiplying x with $z_6^{\frac{\gcd(q,r)}{q}}$ (in the sense of the defined action) we obtain

$$([(z_1, z_2, z_3), z_6^{\frac{\gcd(q,r)}{q}\frac{r}{\gcd(q,r)}}(z_4, z_5)], 1).$$

Thus an element $\bar{x} \in \frac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}} \times U(1)$ lies in the $U(1)$-orbit of x if and only if there exists a $z \in U(1)$ s. t.

$$\bar{x} = ([(z_1, z_2, z_3), z^{\frac{r}{\gcd(q,r)}} z_6^{\frac{\gcd(q,r)}{q}\frac{r}{\gcd(q,r)}}(z_4, z_5)], z^{\frac{q}{\gcd(q,r)}}).$$

Hence

$$\mathcal{I}_x = \left\{ [(z_1, z_2, z_3), \underbrace{e^{2\pi i \frac{\gcd(q,r)}{q} \frac{r}{\gcd(q,r)} m}}_{= e^{2\pi i \frac{r}{q} m}} z_6^{\frac{r}{q}}(z_4, z_5)] | 0 \leq m < \frac{q}{\gcd(q,r)} \right\}. \tag{2.3}$$

The elements of

$$\left\{ e^{2\pi i \frac{r}{q} m} | 0 \leq m < \frac{q}{\gcd(q,r)} \right\}$$

form a cyclic group of order $\frac{q}{\gcd(q,r)}$ with a generator given by $e^{2\pi i \frac{r}{q}}$. It's clear that the order of this cyclic group doesn't depend on the choice of x.

The equivalence relation which we have defined on $\dfrac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}}$ induces a smooth $\mathbb{Z}/\frac{q}{\gcd(q,r)}$-action on $\dfrac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}}$ (compare with Fig. 2.1):

$$\begin{array}{rcl} \mathbb{Z}/\dfrac{q}{\gcd(q,r)} \times \dfrac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}} & \to & \dfrac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}}, \\ (m, [(z_1, z_2, z_3), (z_4, z_5)]) & \mapsto & [(z_1, z_2, z_3), e^{2\pi i \frac{r}{q} m}(z_4, z_5)]. \end{array}$$

We denote the quotient space under this action by $\dfrac{\frac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}}}{\mathbb{Z}/\frac{q}{\gcd(q,r)}}$. By the definition of the smooth $\mathbb{Z}/\frac{q}{\gcd(q,r)}$-action there is a canonical diffeomorphism between

$$\frac{\frac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}} \times U(1)}{\sim_{\bar{\boldsymbol{v}}'}} \text{ and } \frac{\frac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}}}{\mathbb{Z}/\frac{q}{\gcd(q,r)}}.$$

Thus M^{pqr} and $\dfrac{\frac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}}}{\mathbb{Z}/\frac{q}{\gcd(q,r)}}$ are diffeomorphic.

If $q = 0$ and $p \neq 0$, then Remark 2.2.1. indicates that by going through a similar procedure as above one deduces the following smooth $\mathbb{Z}/\frac{p}{\gcd(p,r)}$-action on $\dfrac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}}$:

$$\begin{array}{rcl} \mathbb{Z}/\dfrac{p}{\gcd(p,r)} \times \dfrac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}} & \to & \dfrac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}}, \\ (m, [(z_1, z_2, z_3), (z_4, z_5)]) & \mapsto & [e^{2\pi i \frac{r}{p} m}(z_1, z_2, z_3), (z_4, z_5)]. \end{array}$$

We denote the quotient space by $\dfrac{\frac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}}}{\mathbb{Z}/\frac{p}{\gcd(p,r)}}$ and again by the definition of the $\mathbb{Z}/\frac{p}{\gcd(p,r)}$-action there is a canonical diffeomorphism between $\dfrac{\frac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}} \times U(1)}{\sim \bar{\boldsymbol{v}}^{*\prime}}$ and $\dfrac{\frac{S^5 \times S^3}{\sim_{\bar{\boldsymbol{u}}'}}}{\mathbb{Z}/\frac{p}{\gcd(p,r)}}$ which shows that M^{pqr} and $\dfrac{\frac{S^5 \times S^3}{\sim_{\bar{u}'}}}{\mathbb{Z}/\frac{p}{\gcd(p,r)}}$ are diffeomorphic.

Proposition 2.2.4. *Let $p, q \in \mathbb{Z}$ s.t. $p \neq 0$ or $q \neq 0$. Then M^{pqr} is diffeomorphic to the orbit space of a smooth and free $\mathbb{Z}/gcd(p,q)$-action on $M^{\frac{p}{gcd(p,q)} \frac{q}{gcd(p,q)} 0}$.*

Proof. Let $s := \gcd(p,q)$, $k := \gcd(q,r)$, $k' := \gcd(p,r)$ and $[(z_1, z_2, z_3), (z_4, z_5)]$ be a point in $\dfrac{S^5 \times S^3}{\sim_{\bar{u}'}}$.

If $q \neq 0$ and $p = 0$, then $s = q$ and $k = 1$. We analyze which elements of $\mathbb{Z}/\frac{q}{k}$ fix $[(z_1, z_2, z_3), (z_4, z_5)]$:

$$\begin{array}{rcl} [(z_1, z_2, z_3), e^{2\pi i m \frac{r}{q}}(z_4, z_5)] & = & [(z_1, z_2, z_3), (z_4, z_5)] \\ & \Longleftrightarrow & \\ \exists z \in U(1) \text{ s.t. } ((z_1, z_2, z_3), e^{2\pi i m \frac{r}{q}}(z_4, z_5)) & = & (z^{-\frac{q}{s}}(z_1, z_2, z_3), (z_4, z_5)). \end{array}$$

The last line implies that only the trivial element of $\mathbb{Z}/\frac{q}{k}$ fixes x.

If $p \neq 0$ and $q = 0$, then $s = p$ and $k' = 1$. We conclude by similar arguments that again only the trivial element of $\mathbb{Z}/\frac{p}{k'}$ fixes x.

Thus in the first two cases we have free $\mathbb{Z}/s$-actions on $\dfrac{S^5 \times S^3}{\sim_{\bar{u}'}}$.

If $p \neq 0$ and $q \neq 0$, then we study which elements of $\mathbb{Z}/\frac{q}{k}$ fix $[(z_1, z_2, z_3), (z_4, z_5)]$:

$$\begin{array}{rcl} [(z_1, z_2, z_3), e^{2\pi i m \frac{r}{q}}(z_4, z_5)] & = & [(z_1, z_2, z_3), (z_4, z_5)] \\ & \Longleftrightarrow & \\ \exists z \in U(1) \text{ s.t. } ((z_1, z_2, z_3), e^{2\pi i m \frac{r}{q}}(z_4, z_5)) & = & (z^{-\frac{q}{s}}(z_1, z_2, z_3), z^{\frac{p}{s}}(z_4, z_5)), \\ & \Longleftrightarrow & \\ \exists z \in \left\{ e^{2\pi i f \frac{s}{q}} | f \in \mathbb{Z} \right\} & \text{s. t.} & z \in \left\{ e^{2\pi i m \frac{r}{q}\frac{s}{p}} e^{2\pi i \frac{s}{p} n} | n \in \mathbb{Z} \right\}, \\ & \Longleftrightarrow & \\ \exists f, n \in \mathbb{Z} \text{ s.t. } e^{2\pi i f \frac{s}{q}} & = & e^{2\pi i (m \frac{r}{q}\frac{s}{p} + \frac{s}{p} n)}, \\ & \Longleftrightarrow & \\ \exists f, n \in \mathbb{Z} & \text{s.t.} & \dfrac{s}{q} f - \dfrac{r}{q}\dfrac{s}{p} m - \dfrac{s}{p} n \in \mathbb{Z}. \qquad (\star) \end{array}$$

We define the following map

$$\begin{array}{rcl} \alpha : I_{\pm} & \to & \mathbb{N}, \\ (p, q, r) & \mapsto & \begin{cases} |\frac{q}{k}| & , \text{ if } p = 0 \\ |\frac{p}{k'}| & , \text{ if } q = 0 \\ \min\{m | m \in \mathbb{N} \text{ s.t. } (\star) \text{ is fulfilled}\} & , \text{ if } p \neq 0 \text{ and } q \neq 0. \end{cases} \end{array}$$

We notice that M^{pqr} is diffeomorphic to the orbit space of a free $\mathbb{Z}/\alpha(p, q, r)$-action on $\dfrac{S^5 \times S^3}{\sim_{\bar{u}'}}$.

It remains to show that

$$\alpha(p, q, r) = s. \tag{2.4}$$

As we have already seen if $p = 0$ or $q = 0$ the equality (2.4) trivially holds.

But (2.4) also holds if $p \neq 0$ and $q \neq 0$:

Let's denote by $\mathcal{O}_{pqr}$ the set of positive integers m which fulfill ($\star$). We show first that s lies in $\mathcal{O}_{pqr}$. We notice that there exist coprime integers t and w s.t. $p = ts$ and $q = ws$. It's clear that s is an element of $\mathcal{O}_{pqr}$ if and only if there exist $n, f \in \mathbb{Z}$ s.t. $\frac{r+nw-ft}{tw} \in \mathbb{Z}$. Since t, w are coprime there exist $n, f \in \mathbb{Z}$ s.t. $nw - ft = tw - r$ which proves that $s \in \mathcal{O}_{pqr}$.
The last step is to show that s is the minimum of $\mathcal{O}_{pqr}$. For $s = 1$ it's clear. Otherwise we assume that the minimum $\bar{m}$ of $\mathcal{O}_{pqr}$ is smaller than s. But then there exist $n, f \in \mathbb{Z}$ s.t.

$$\frac{r\frac{\bar{m}}{s} + nw - ft}{tw} \in \mathbb{Z}. \qquad (\star\star)$$

Since s and r are coprime we conclude $r\frac{\bar{m}}{s} \notin \mathbb{Z}$ which leads to a contradiction to ($\star\star$). The fact that $M^{\frac{p}{\gcd(p,q)}\frac{q}{\gcd(p,q)}0}$ and $\dfrac{S^5 \times S^3}{\sim_{\bar{u}'}}$ are canonically diffeomorphic (compare p. 18) finishes the proof. ∎

Corollary 2.2.5. *i) The universal covering space of the Witten space M^{pqr} is $M^{\frac{p}{gcd(p,q)}\frac{q}{gcd(p,q)}0}$ if $p \neq 0$ or $q \neq 0$ and $\mathbb{C}P^2 \times S^2 \times \mathbb{R}$ if $p = q = 0$.*

ii) The fundamental group of a Witten space:

$$\pi_1(M^{pqr}) \cong \begin{cases} \mathbb{Z}/gcd(p,q) & , \text{ if } \quad p \neq 0 \text{ or } q \neq 0 \\ \mathbb{Z} & , \text{ if } \quad p = 0 = q. \end{cases}$$

Thus each cyclic group is realized as the fundamental group of a Witten space.

iii) The Witten spaces M^{pqr} and $M^{pqr'}$ are diffeomorphic.

Proof. If $p \neq 0$ or $q \neq 0$, then Lemma 2.2.3. and Proposition 2.2.4. imply that $M^{\frac{p}{\gcd(p,q)}\frac{q}{\gcd(p,q)}0}$ is the universal covering space of M^{pqr}. If $p = q = 0$, then M^{pqr} is diffeomorphic to $\mathbb{C}P^2 \times \mathbb{C}P^1 \times S^1$ and its universal covering space is $\mathbb{C}P^2 \times \mathbb{C}P^1 \times \mathbb{R}$. This proves i).

Let $l \in \mathbb{N}$ then $\pi_1(M^{0l1}) \cong \mathbb{Z}/l$. If $p = q = 0$, then $\pi_1(M^{001}) \cong \mathbb{Z}$ which proves ii).

If $p = q = 0$, then iii) is clearly true. If $p, q \in \mathbb{Z}$ s.t. $p \neq 0$ or $q \neq 0$, then iii) follows from the definition of the $\mathbb{Z}/\gcd(p,q)$-action on $M^{\frac{p}{\gcd(p,q)}\frac{q}{\gcd(p,q)}0}$ and the fact that finite cyclic subgroups of $U(1)$ are determined by their order. ∎

2.3 An equivariant diffeomorphism classification

Let $\Phi, \Phi' \in \mathcal{F}$ and $\frac{G}{\Phi(H)}$ and $\frac{G}{\Phi'(H)}$ be Witten spaces as we have defined them at the beginning of this chapter. In this section we consider a Witten space as a homogeneous space together with the obvious left G-action which comes from the construction. If β is a Lie group automorphism of G, then we define the β-twisted G-action on $\frac{G}{\Phi(H)}$ which is given by

$$(g, x) \mapsto \beta(g) \cdot x,$$

where "$\cdot$" denotes the obvious multiplication of an element of G with a point in $\dfrac{G}{\Phi(H)}$. If we equip $\dfrac{G}{\Phi(H)}$ with the β-twisted G-action we write $(\dfrac{G}{\Phi(H)}, G_\beta)$ and $(\dfrac{G}{\Phi(H)}, G_{id})$ is simply denoted by $\dfrac{G}{\Phi(H)}$.

Definition/Lemma 2.3.1. *i) Let $A(G; \Phi(H), \Phi'(H))$ be the set of Lie group automorphisms of G which map $\Phi(H)$ bijectively to $\Phi'(H)$.*
ii) Assume that $A(G; \Phi(H), \Phi'(H))$ is not empty then for all $\beta \in A(G; \Phi(H), \Phi'(H))$ the map

$$\begin{array}{ccc} \dfrac{G}{\Phi(H)} & \to & (\dfrac{G}{\Phi'(H)}, G_\beta), \\ g\Phi(H) & \mapsto & \beta(g)\Phi'(H) \end{array}$$

is an equivariant diffeomorphism which sends $\Phi(H)$ to $\Phi'(H)$.

iii) There is the following equivalence relation on the set of Witten spaces. We say that $\dfrac{G}{\Phi(H)}$ and $\dfrac{G}{\Phi'(H)}$ are equivalent, $\dfrac{G}{\Phi(H)} \sim_G \dfrac{G}{\Phi'(H)}$, if $A(G; \Phi(H), \Phi'(H))$ is not empty.

Proof. Statement ii) is easily verified. The relation in iii) is clearly an equivalence relation. ■

Proposition 2.3.2. *(An equivariant diffeomorphism classification of the Witten spaces) Let M^{pqr}, $M^{p'q'r'} \in \mathcal{W}$. Then $M^{pqr} \sim_G M^{p'q'r'}$ if and only if $|p| = |p'|$ and $|q| = |q'|$ and $|r| = |r'|$.*

Proof. We recall how the Witten space M^{pqr} was defined: Let

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

be a basis of $\boldsymbol{D}\ker(p, q, r)$ then M^{pqr} is the quotient of $SU(3) \times SU(2) \times U(1)$ divided by the image of the homomorphism from H to G which sends

$$(A, z_1, z_2) \text{ to } \left(\begin{pmatrix} Az_1^{a_1} z_2^{b_1} & 0 \\ 0 & z_1^{-2a_1} z_2^{-2b_1} \end{pmatrix}, \begin{pmatrix} z_1^{2a_2} z_2^{2b_2} & 0 \\ 0 & z_1^{-2a_2} z_2^{-2b_2} \end{pmatrix}, z_1^{2a_3} z_2^{2b_3} \right).$$

The image of such embeddings lies in

$$\left\{\left(\left(\begin{array}{cc} B & 0 \\ 0 & 1 \end{array}\right)\left(\begin{array}{ccc} w & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & w^{-1}x^{-1} \end{array}\right), \left(\begin{array}{cc} y & 0 \\ 0 & y^{-1} \end{array}\right), z\right) | B \in SU(2);\ \begin{array}{c} w, x \\ y, z \end{array} \in U(1)\right\}.$$

We also know from Corollary 2.1.6. that the image of such a homomorphism only depends on p, q, r and not on the choice of two linear independent elements in $\boldsymbol{D}\ker(p, q, r)$. This consideration justifies to denote the image of the map above by H_{pqr}. In the following we denote the set

$$\{H_{abc} | a, b, c \text{ are coprime}\}$$

by C.

By definition $M^{pqr} \sim_G M^{p'q'r'}$ if and only if $A(G; H_{pqr}, H_{p'q'r'})$ is nonempty.

Let $\mathrm{Aut}(G)$ be the group of Lie group automorphisms of G which operates on the set $\mathcal{S}$ of all Lie subgroups of G in the obvious way. Furthermore we denote by $\mathrm{Aut}(G) \cdot H_{pqr}$ the $\mathrm{Aut}(G)$-orbit of H_{pqr} in $\mathcal{S}$ and by C_{pqr} we denote the intersection between $\mathrm{Aut}(G) \cdot H_{pqr}$ and C.

It's clear that $M^{pqr} \sim_G M^{p'q'r'}$ if and only if $H_{p'q'r'} \in C_{pqr}$. The next aim is to analyze C_{pqr}.

We claim that

$$\mathrm{Aut}(G) = \mathrm{Aut}(SU(3)) \times \mathrm{Aut}(SU(2)) \times \mathrm{Aut}(U(1)). \qquad (\star)$$

It's clear that $\mathrm{Aut}(G) \supseteq \mathrm{Aut}(SU(3)) \times \mathrm{Aut}(SU(2)) \times \mathrm{Aut}(U(1))$.
"$\subseteq$": Each factor of G is a normal subgroup of G and an element of $\mathrm{Aut}(G)$ sends a normal subgroup to a normal subgroup. Assume that there is an automorphism α which is not an element of the right hand side of $(\star)$. Then at least the image of one of the three factors of G contains an element with nontrivial coordinates in one of the other factors which we denote by G_i. We further denote by pr_i the projection onto G_i. Smoothness of α implies that $\mathrm{im}(\mathrm{pr}_i \circ \alpha) =: I_i$ is a non-discrete Lie subgroup of G_i. But I_i has to be a normal subgroup of G_i which is a contradiction to the fact that each of the three factors of G is a simple Lie group (see for expl. [He, p. 451]).

We denote by T^4 the following maximal torus of G

$$\left\{\left(\left(\begin{array}{ccc} w & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & w^{-1}x^{-1} \end{array}\right), \left(\begin{array}{cc} y & 0 \\ 0 & y^{-1} \end{array}\right), z\right) | w, x, y, z \in U(1)\right\}.$$

There is the following observation:

Assume that Ψ is a Lie group automorphism of G s.t. $\Psi(H_{pqr}) \in C_{pqr}$. Then

$\Psi(T^4) = T^4$.

If Ψ is an inner automorphism, then we know that $\Psi(\cdot) = g \cdot g^{-1}$ for some $g \in N(T^4)$, where $N(T^4)$ is the normalisator of T^4 in G. We are interested in how $N(T^4)$ acts on T^4. But since T^4 is abelian it's enough to know how $\frac{N(T^4)}{T^4}$ acts on T^4 and hence on elements in $\mathcal{S}$ that belong to C. The group $\frac{N(T^4)}{T^4}$ which we denote by $\Omega(T^4)$, is called the Weyl group of T^4.
The Weyl group of T^4 is isomorphic to the product of the symmetric group $S(3)$ and the symmetric group $S(2)$ ([Br-tD, p. 171]) and it acts on T^4 by permuting the elements factorwise. But the only nontrivial element in $\Omega(T^4)$ which sends H_{pqr} to C is the nontrivial element in $S(2)$.

Our next aim is to find all outer Lie group automorphisms of G which send H_{pqr} to C. From [Mc, p. 6] we know that

- $SU(3)$ has exactly one outer automorphism, namely complex conjugation,
- $SU(2)$ has no outer automorphism (complex conjugation is an inner automorphism).

It's well known that complex conjugation on $U(1)$ is the only outer automorphism of $U(1)$.

Let c_3 and c_1 denote the complex conjugation maps in $SU(3)$ respectively in $U(1)$. From the definition of H_{pqr} it's clear that $c_3 \times id_{SU(2)} \times id_{U(1)}$ and $id_{SU(3)} \times id_{SU(2)} \times c_1$ map H_{pqr} to C.

From $(\star)$ it follows that all outer automorphisms of G leave C invariant.

We show that $c_3 \times id_{SU(2)} \times id_{U(1)}$ maps H_{pqr} to $H_{(-p)qr}$: The map $c_3 \times id_{SU(2)} \times id_{U(1)}$ sends

$$\left(\begin{pmatrix} Az_1^{a_1} z_2^{b_1} & 0 \\ 0 & z_1^{-2a_1} z_2^{-2b_1} \end{pmatrix}, \begin{pmatrix} z_1^{2a_2} z_2^{2b_2} & 0 \\ 0 & z_1^{-2a_2} z_2^{-2b_2} \end{pmatrix}, z_1^{2a_3} z_2^{2b_3} \right) \in H_{pqr}$$

to

$$\left(\begin{pmatrix} \bar{A}z_1^{-a_1} z_2^{-b_1} & 0 \\ 0 & z_1^{2a_1} z_2^{2b_1} \end{pmatrix}, \begin{pmatrix} z_1^{2a_2} z_2^{2b_2} & 0 \\ 0 & z_1^{-2a_2} z_2^{-2b_2} \end{pmatrix}, z_1^{2a_3} z_2^{2b_3} \right).$$

Since $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ form a basis of $\boldsymbol{D}\ker(p,q,r)$ the vectors $\begin{pmatrix} -a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\begin{pmatrix} -b_1 \\ b_2 \\ b_3 \end{pmatrix}$ form a basis of $\boldsymbol{D}\ker(-p,q,r)$ and we realize that $c_3 \times id_{SU(2)} \times id_{U(1)}$ maps

H_{pqr} bijectively to $H_{(-p)qr}$.

Studying the other Lie group automorphisms which leave C invariant yields that C_{pqr} consists only of those H_{abc}'s with $|a| = |p|$, $|b| = |q|$ and $|c| = |r|$. ■

Remark 2.3.3. *Let C_{pqr} be the set which we have introduced in the proof of the last proposition. The only element of C_{pqr} which is nontrivially conjugated to H_{pqr} is $H_{p(-q)r}$:*

$$H_{p-qr} = c_2 H_{pqr} c_2^{-1}, \text{ with } c_2 := \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 1 \right).$$

Corollary 2.3.4. *There are manifolds in $\mathcal{W}$ which are diffeomorphic but not equivariantly diffeomorphic in the sense of Definition 2.3.1.ii).*

Proof. By Corollary 2.2.5.iii) M^{123} and M^{125} are diffeomorphic but by Proposition 2.3.2. $M^{123} \nsim_G M^{125}$. ■

There is the following problem:

Let M and M' be two Witten spaces. Are there metrics m, m' which are homogeneous w.r.t. G on M resp. M' s.t. (M, m) and (M', m') are isometric?

We don't solve this problem but we present some results which are related to it and which will play a role in the next chapter.

Let E be a compact Lie group and K, K' be isomorphic compact Lie subgroups of E.

Proposition 2.3.5. *If $\frac{E}{K} \sim_E \frac{E}{K'}$, then there exist metrics m, m' on $\frac{E}{K}$ and $\frac{E}{K'}$ which are homogeneous w.r.t. E s.t. $(\frac{E}{K}, m)$ and $(\frac{E}{K'}, m')$ are isometric.*

Proof. Let
$$F : \frac{E}{K} \to (\frac{E}{K'}, E_\phi)$$
be an equivariant diffeomorphism for a $\phi \in A(E; K, K')$. Furthermore let m be a metric on $\frac{E}{K}$ which is homogeneous w.r.t. E and $m' := F^* m$ be the pushforward metric on $\frac{E}{K'}$. By the construction of the metrics it's clear that F is an isometry and it is straightforward to show that the metric m' is homogeneous w.r.t. E. ■

We will apply Proposition 2.3.5. in the next chapter where we prove that M^{pq1} and $M^{p'q'1}$ are diffeomorphic if and only if there exist homogeneous metrics m, m' on M^{pq1}

and $M^{p'q'1}$ resp. s.t. (M^{pq1}, m) and $(M^{p'q'1}, m')$ are isometric.

Furthermore we obtain the following

Proposition 2.3.6. *Assume that there exist metrics m, m' on $\frac{E}{K}$ resp. $\frac{E}{K'}$ which are homogeneous w.r.t. E s.t. $Isom(\frac{E}{K}, m) = Isom(\frac{E}{K'}, m') = E$ and $(\frac{E}{K}, m)$ and $(\frac{E}{K'}, m')$ are isometric. Then $\frac{E}{K} \sim_E \frac{E}{K'}$.*

Proof. Let $\frac{E}{K}$ and $\frac{E}{K'}$ be equipped with homogeneous metrics w.r.t. E s.t. there exists an isometry $f : \frac{E}{K} \to \frac{E}{K'}$. W.l.o.g. we may assume $f(K) = K'$, otherwise we can multiply an appropriate element $h \in E$ to $f(K)$ s.t. $h \cdot f(K) = K'$. Let $g \in E$.

$$\begin{array}{ccc} & f & \\ \frac{E}{K} & \rightarrow & \frac{E}{K'} \\ g \uparrow & \circlearrowleft & \uparrow l \\ \frac{E}{K} & \leftarrow & \frac{E}{K'} \\ & f^{-1} & \end{array}$$

Since $\text{Isom}(\frac{E}{K}, m) = \text{Isom}(\frac{E}{K'}, m') = E$ the commutative diagram above implies that $fgf^{-1} \in E$. Hence we get the following Lie group automorphism of E :

$$\phi_f : E \to E, \quad g \mapsto fgf^{-1}.$$

Since $f(K) = K'$ and by the definition of ϕ_f it is easy to verify that $\phi_f(K) = K'$. Thus we obtain an equivariant diffeomorphism Φ_f from $\frac{E}{K}$ to $(\frac{E}{K'}, E_{\phi_f})$ with $\Phi_f(K) = K'$.
■

Corollary 2.3.7. *Assume $Isom(\frac{E}{K}, g) = Isom(\frac{E}{K'}, g') = E$ for each choice of homogeneous metrics g, g' w.r.t. E. Then there exist metrics m, m' on $\frac{E}{K}$ resp. $\frac{E}{K'}$ which are homogeneous w.r.t. E s.t. $(\frac{E}{K}, m)$ and $(\frac{E}{K'}, m')$ are isometric if and only if $\frac{E}{K} \sim_E \frac{E}{K'}$.*

Remark 2.3.8. *Let $n \in \mathbb{N}$. We denote by (S^{2n+1}, h) the $(2n+1)$-dimensional sphere regarded as the Riemannian submanifold of the euclidean space $\mathbb{R}^{2(n+1)}$, obtained as*

the collection of all unit vectors and equipped with the metric h which is the induced metric of $\mathbb{R}^{2(n+1)}$. *This metric on* S^{2n+1} *makes* (S^{2n+1}, h) *to a homogeneous space with isometry group* $O(2(n+1))$. *We know that lens spaces are orbit spaces of isometric actions of finite cyclic groups on* (S^{2n+1}, h). *Thus lens spaces are homogeneous spaces with respect to* $O(2(n+1))$. *The homeomorphism and diffeomorphism classification of all lens spaces ([L, p. 45]) couldn't reveal any exotic differentiable structures on non-simply-connected lens spaces.*
In [M.1] J. Milnor proves the existence of an "exotic" involution on S^7, *i.e. the existence of a free* $\mathbb{Z}/2$*-action on* S^7 *s.t. the orbit space is homeomorphic but not diffeomorphic to* $\mathbb{R}P^7$. *But it's not known whether this action respects any geometric structure.*

Thus it seems to be an interesting question whether there exists a homogeneous space or Riemannian manifold with non-trivial cyclic isometric group actions s.t. the resulting orbit spaces are homeomorphic but not diffeomorphic.

2.4 A topological characterisation

The following proposition is of great significance for the main theorem in the next chapter where we classify the non-simply-connected Witten spaces upto diffeomorphism.

Proposition 2.4.1. *A Witten space* M^{pqr} *is diffeomorphic to the total space of the principal* $U(1)$*-bundle over* $\mathbb{C}P^2 \times \mathbb{C}P^1$,

$$U(1) \to TS(E_{pq}) \stackrel{S(E_{pq})}{\to} \mathbb{C}P^2 \times \mathbb{C}P^1,$$

which is given by the first Chern class

$$c_1(S(E_{pq})) = px + qy \in H^2(\mathbb{C}P^2 \times \mathbb{C}P^1; \mathbb{Z}),$$

where $x \in H^2(\mathbb{C}P^2; \mathbb{Z})$ *and* $y \in H^2(\mathbb{C}P^1; \mathbb{Z})$ *are the standard generators of the corresponding cohomology groups.*

Proof. From Corollary 2.2.5. we know that M^{pqr} and M^{pq1} are diffeomorphic. Thus w.l.o.g. we prove the proposition for M^{pq1}. The proof is subdivided into the following steps:

1) First we define the projection map $\Pi : M^{pq1} \to \mathbb{C}P^2 \times \mathbb{C}P^1 \times \mathbb{C}P^0$ and show that the fibre is $U(1)$. Then we prove that there is a smooth and free $U(1)$-action on M^{pq1} which preserves the fibre.

2) We prove that there is a bundle isomorphism between Π and the sphere bundle $S(E_{pq})$ of the following complex line bundle:

$$E_{pq} : (\mathrm{pr}_1^* \gamma_2^p) \otimes (\mathrm{pr}_2^* \gamma_1^q) \otimes \mathrm{pr}_0^* \gamma_0 \to \mathbb{C}P^2 \times \mathbb{C}P^1 \times \mathbb{C}P^0,$$

where γ_2 resp. γ_1 denotes the tautological bundle over $\mathbb{C}P^2$ resp. $\mathbb{C}P^1$, γ_0 is the trivial complex line bundle over $\mathbb{C}P^0$ and $pr_i : \mathbb{C}P^1 \to \mathbb{C}P^2 \times \mathbb{C}P^1 \times \mathbb{C}P^0$ is the projection map onto the i-th factor ($i \in \{0, 1, 2\}$).

1) Let $\boldsymbol{a} := (a_1, a_2, a_3)$, $\boldsymbol{b} := (b_1, b_2, b_3) \in \mathbb{Z}^3$ be two linearly independent elements in $\boldsymbol{D}\ker(p, q, 1)$. We recall that

$$M^{pq1} \quad = \quad \frac{SU(3) \times SU(2) \times U(1)}{SU(2) \times (U(1) \times U(1))_{pq1}},$$

where $(U(1) \times U(1))_{pq1}$ is the image of the following homomorphism:

$F_{\boldsymbol{a},\boldsymbol{b}} : U(1) \times U(1) \to SU(3) \times SU(2) \times U(1)$, where $F_{\boldsymbol{a},\boldsymbol{b}}(z_1, z_2)$ equals

$$\left(\begin{pmatrix} z_1^{-a_1} z_2^{-b_1} & 0 & 0 \\ 0 & z_1^{-a_1} z_2^{-b_1} & 0 \\ 0 & 0 & z_1^{2a_1} z_2^{2b_1} \end{pmatrix}, \begin{pmatrix} z_1^{2a_2} z_2^{2b_2} & 0 \\ 0 & z_1^{-2a_2} z_2^{-2b_2} \end{pmatrix}, z_1^{2a_3} z_2^{2b_3} \right).$$

Let's denote by $\dfrac{S^5 \times S^3 \times S^1}{\sim_{\boldsymbol{ab}}}$ the quotient of the following smooth $U(1) \times U(1)$-action on $S^5 \times S^3 \times S^1$:

$$\begin{aligned} G_{\boldsymbol{a},\boldsymbol{b}} : U(1) \times U(1) \times (S^5 \times S^3 \times S^1) \quad &\to \quad S^5 \times S^3 \times S^1, \\ (z_1, z_2, ((x_1, x_2, x_3), (x_4, x_5), x_6)) \quad &\mapsto \quad (z_1^{a_1} z_2^{b_1}(x_1, x_2, x_3), z_1^{a_2} z_2^{b_2}(x_4, x_5), z_1^{a_3} z_2^{b_3} x_6). \end{aligned}$$

The diffeomorphisms $\Delta_1 : \frac{SU(3)}{SU(2)} \xrightarrow{\sim} S^5$ and $\Delta_2 : SU(2) \xrightarrow{\sim} S^3$ which we gave on p. 14 induce a diffeomorphism

$$\alpha : M^{pq1} \overset{\cong}{\to} \frac{S^5 \times S^3 \times S^1}{\sim_{\boldsymbol{ab}}}$$

(compare with p. 18).
Let $\tilde{\Pi}$ be the map from $\dfrac{S^5 \times S^3 \times S^1}{\sim_{\boldsymbol{ab}}}$ to $\mathbb{C}P^2 \times \mathbb{C}P^1 \times \mathbb{C}P^0$ which maps $[(x_1, x_2, x_3), (x_4, x_5), x_6]$ to $([x_1 : x_2 : x_3], [x_4 : x_5], [x_6])$. We define Π to be $\tilde{\Pi} \circ \alpha$.

Let $d, e, f \in \mathbb{Z}$ s.t. $dp + eq + f = 1$. We have the following split short exact sequence

$$1 \to U(1) \times U(1) \xrightarrow{\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}_\star} U(1) \times U(1) \times U(1) \underset{(d,e,f)_\star}{\overset{(p,q,1)_\star}{\rightleftarrows}} U(1) \to 1, \tag{2.5}$$

where $(\cdot)_\star$ denotes the induced homomorphisms in the sense of Definition 2.1.5.
From (2.5) we conclude that $\tilde{\Pi}$ is a $U(1)$-fibre bundle, where the $U(1)$-action is given by

$$[(x_1, x_2, x_3), (x_4, x_5), x_6] * z = [(x_1, x_2, x_3) \cdot z^d, (x_4, x_5) \cdot z^e, x_6 \cdot z^f].$$

Hence Π is a principal $U(1)$-fibre bundle.

2) We denote the total space of $S(E_{pq})$ by $TS(E_{pq})$. The following diagram commutes:

$$\begin{array}{ccc} \dfrac{S^5 \times S^3 \times S^1}{\sim_{ab}} & \xrightarrow{\psi} & TS(E_{pq}) \\ & \tilde{\Pi} \searrow \quad \swarrow S(E_{pq}) & \\ & \mathbb{C}P^2 \times \mathbb{C}P^1 \times \mathbb{C}P^0 & , \end{array}$$

where

$$\psi([(x_1, x_2, x_3), (x_4, x_5), x_6])$$

equals

$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \otimes ... \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{p \text{ copies}} \otimes \underbrace{\begin{pmatrix} x_4 \\ x_5 \end{pmatrix} \otimes ... \otimes \begin{pmatrix} x_4 \\ x_5 \end{pmatrix}}_{q \text{ copies}} \otimes x_6$$

and

$$S(E_{pq}) \left(\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \otimes ... \otimes \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{p \text{ copies}} \otimes \underbrace{\begin{pmatrix} x_4 \\ x_5 \end{pmatrix} \otimes ... \otimes \begin{pmatrix} x_4 \\ x_5 \end{pmatrix}}_{q \text{ copies}} \otimes x_6 \right)$$

equals

$$([x_1 : x_2 : x_3], [x_4 : x_5], [x_6]).$$

From the exactness of (2.5) it follows that ψ is well defined and bijective and from the relation $dp+eq+f = 1$ we derive that ψ is an $U(1)$-equivariant map. The multiplicative properties of the (total) Chern classes imply

$$c_1(S(E_{pq})) = px + py,$$

where $x = c_1(\gamma_2) \in H^2(\mathbb{C}P^2; \mathbb{Z})$ and $y = c_1(\gamma_1) \in H^2(\mathbb{C}P^1; \mathbb{Z})$ are the standard generators of the corresponding cohomology groups. The fact that $\mathbb{C}P^0$ is a point finishes the proof. ■

Remark 2.4.2. *The reason why we have chosen the parametrisation of the set of Witten spaces as we have done it on p. 15 is the following: Given a Witten space M^{abc} we can according to Proposition 2.4.1. immediately find without any further calculations a fibre bundle with total space diffeomorphic to M^{abc}.*

Corollary 2.4.3. *Let $s \in \mathbb{N}$ and M^{pqr} a Witten space with $\pi_1(M^{pqr}) \cong \mathbb{Z}/s$. Its universal covering space $M^{\frac{p}{s}\frac{q}{s}0}$ is diffeomorphic to the total space of the $U(1)$-bundle over $\mathbb{C}P^2 \times \mathbb{C}P^1$ with $\frac{p}{s}x + \frac{q}{s}y$ as its first Chern class.*
Homotopically $\pi_1(M^{pqr})$ acts trivially on $M^{\frac{p}{s}\frac{q}{s}0}$.

Proof. This follows from the description of the deck transformation on p. 20 and in the proof of Proposition 2.2.4. and the description of the fibre bundle structure in the proof of the last proposition. There we see that the action of $\pi_1(M^{pqr})$ on $M^{\frac{p}{s}\frac{q}{s}0}$ via deck transformation is the restriction of the $U(1)$-action which comes from the $U(1)$-fibre bundle structure of $M^{\frac{p}{s}\frac{q}{s}0}$ restricted to the $\gcd(p,q)$-th roots. Thus homotopically $\pi_1(M^{pqr})$ acts trivially on $M^{\frac{p}{s}\frac{q}{s}0}$. ■

2.5 Invariants

From Corollary 2.2.5. we know that a Witten space M^{abc} is diffeomorphic to M^{ab1}. We simplify the notation and write M^{ab} instead of M^{ab1}.
In this section we are interested in homotopy and diffeomorphism invariants thus instead of studying the whole set of Witten spaces we may only concentrate on

$$\{M^{ab} | a, b \in \mathbb{Z}\} \subset \mathcal{W}.$$

If not otherwise stated we identify M^{ab} with the total space $ST(E_{ab})$ of the $U(1)$-fibre bundle which was given in Proposition 2.4.1.
From Corollary 2.2.5. we know that the only Witten space M^{ab} with infinite cyclic fundamental group is M^{00} which is diffeomorphic to the well known manifold $\mathbb{C}P^2 \times \mathbb{C}P^1 \times S^1$.

In this section we reveal some basic informations concerning the cohomological structure, compute certain characteristic classes and determine the so called normal 2-type of each Witten space with finite cyclic fundamental group. All these invariants are important ingredients for the proof of the classification theorem (Theorem 2.7.9.) and for the last chapter.

2.5.1 The integral cohomology ring

Let $M^{pq} \in \mathcal{W}$ with finite cyclic fundamental group, i.e. $(p,q) \neq (0,0)$. From the fact that $H_1(M^{pq};\mathbb{Z})$ is finite cyclic it immediately follows that $H^1(M^{pq};\mathbb{Z}) \cong 0$. The main tool for computing the integral cohomology ring of M^{pq} is the Gysin sequence associated to the fibre bundle

$$S^1 \to M^{pq} \xrightarrow{\Pi} \mathbb{C}P^2 \times \mathbb{C}P^1 \text{ with } c_1(\Pi) = px + qy,$$

where x and y are the standard generators of $H^2(\mathbb{C}P^2;\mathbb{Z})$ resp. $H^2(\mathbb{C}P^1;\mathbb{Z})$. Part of this sequence is

$$\begin{array}{lll} \cdots & \to H^{i-1}(M^{pq};\mathbb{Z}) & \to H^{i-2}(\mathbb{C}P^2 \times \mathbb{C}P^1;\mathbb{Z}) \stackrel{\cup c_1(\Pi)}{\to} H^i(\mathbb{C}P^2 \times \mathbb{C}P^1;\mathbb{Z}) \stackrel{\Pi^*}{\to} \\ & H^i(M^{pq};\mathbb{Z}) & \to H^{i-1}(\mathbb{C}P^2 \times \mathbb{C}P^1;\mathbb{Z}) \to \cdots . \end{array}$$

If i is even, we observe that

$$H^i(M^{pq};\mathbb{Z}) \cong \frac{H^i(\mathbb{C}P^2 \times \mathbb{C}P^1;\mathbb{Z})}{\operatorname{im}(\cup c_1(\Pi))}.$$

This means in more explicite terms that

$$\begin{aligned} H^2(M^{pq};\mathbb{Z}) &= \frac{< \Pi^*(x), \Pi^*(y) >}{< p\Pi^*(x) + q\Pi^*(y) >}, \\ H^4(M^{pq};\mathbb{Z}) &= \frac{< \Pi^*(x^2), \Pi^*(xy) >}{< p\Pi^*(x^2) + q\Pi^*(xy), p\Pi^*(xy) >}, \\ H^6(M^{pq};\mathbb{Z}) &= \frac{< \Pi^*(x^2y) >}{< p\Pi^*(x^2y), q\Pi^*(x^2y) >}. \end{aligned}$$

The Gysin sequence also implies that $H^1(M^{pq};\mathbb{Z})$ and $H^3(M^{pq};\mathbb{Z})$ are trivial and $H^5(M^{pq};\mathbb{Z})$ is isomorphic to $\mathbb{Z}$.
In order to understand the dependence of $H^{2k}(M^{pq};\mathbb{Z})$ on p, q better we want to know to what abstract groups they are isomorphic:

By Poincaré duality $H^6(M^{pq};\mathbb{Z})$ is isomorphic to $\mathbb{Z}/\gcd(p,q)$.

We claim that $H^2(M^{pq};\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/\gcd(p,q)$:
First we choose $m, n \in \mathbb{Z}$ s.t. $m\frac{q}{\gcd(p,q)} + n\frac{p}{\gcd(p,q)} = 1$ which is possible since $\frac{p}{\gcd(p,q)}$ and $\frac{q}{\gcd(p,q)}$ are coprime integers. Then it's clear that $\frac{p}{\gcd(p,q)}\Pi^*(x) + \frac{q}{\gcd(p,q)}\Pi^*(y)$ and $m\Pi^*(x) - n\Pi^*(y)$ generate $H^2(M^{pq};\mathbb{Z})$, where $m\Pi^*(x) - n\Pi^*(y)$ generates a $\mathbb{Z}-$summand and $\frac{p}{\gcd(p,q)}\Pi^*(x) + \frac{q}{\gcd(p,q)}\Pi^*(y)$ generates the torsion part.

Lemma 2.5.1. *Let $M^{pq} \in \mathcal{W}$. If $p = 0$, then $H^4(M^{pq};\mathbb{Z}) \cong 0$ and if $p \neq 0$, then $H^4(M^{pq};\mathbb{Z}) \cong \mathbb{Z}/\frac{p^2}{gcd(p,q)} \oplus \mathbb{Z}/gcd(p,q)$.*

Proof. Let $C := \begin{pmatrix} p & 0 \\ q & p \end{pmatrix}$. We denote

$$\operatorname{coker}\Big((H^2(\mathbb{C}P^2 \times \mathbb{C}P^1;\mathbb{Z}), \Pi^*(x), \Pi^*(y)) \xrightarrow{C} (H^4(\mathbb{C}P^2 \times \mathbb{C}P^1;\mathbb{Z}), \Pi^*(x^2), \Pi^*(xy))\Big)$$

by Co. The order of Co has to be $\det(C) = p^2$. Thus if $p = 0$, then $H^4(M^{pq};\mathbb{Z}) \cong 0$. Now we assume that $p \neq 0$. We prove that Co is isomorphic to $\mathbb{Z}/\frac{p^2}{\gcd(p,q)} \oplus \mathbb{Z}/\gcd(p,q)$. It's enough to show that there are two linearly independent elements in Co of order $\frac{p^2}{\gcd(p,q)}$ and $\gcd(p,q)$ respectively.

Let $m, n \in \mathbb{Z}$ s.t. $m\frac{q}{\gcd(p,q)} + n\frac{p}{\gcd(p,q)} = 1$. The statement that $[(m, -n)] \in$Co is a nontrivial element of order $\frac{p^2}{\gcd(p,q)}$ is equivalent to the statement that the smallest positive integer k s.t.

$$kC^{-1}\begin{pmatrix} m \\ -n \end{pmatrix} \in \mathbb{Z}^2 \tag{2.6}$$

is $\frac{p^2}{\gcd(p,q)}$. But this is clearly the case since

$$\begin{aligned} kC^{-1}\begin{pmatrix} m \\ -n \end{pmatrix} &= k\begin{pmatrix} p & 0 \\ q & p \end{pmatrix}^{-1}\begin{pmatrix} m \\ -n \end{pmatrix} = k\frac{1}{p^2}\begin{pmatrix} p & 0 \\ -q & p \end{pmatrix}\begin{pmatrix} m \\ -n \end{pmatrix} \\ &= k\begin{pmatrix} \frac{m}{p} \\ -\underbrace{(\frac{p}{\gcd(p,q)}n + \frac{q}{\gcd(p,q)}m)}_{=1}\frac{\gcd(p,q)}{p^2} \end{pmatrix}. \end{aligned}$$

We denote $[(m, -n)]$ by g_1 and $[(\frac{p}{\gcd(p,q)}, \frac{q}{\gcd(p,q)})]$ by g_2 and realize that they are not multiples of each other which follows from the fact that $(m, -n)$ and $(\frac{p}{\gcd(p,q)}, \frac{q}{\gcd(p,q)})$ form a basis of $\mathbb{Z}^2$. It's clear that g_2 is an element of order $\gcd(p,q)$. We claim that g_1 and g_2 generate direct summands in Co. Thus if $\gcd(p,q) = 1$, then the claim is trivially true since $g_2 = 0 \in$ Co and g_1 generates A. Now we assume that $\gcd(p,q)$ is greater than 1. If the claim wasn't true, then there would exist $h \in \{1, ..., \frac{p^2}{\gcd(p,q)} - 1\}$ and $j \in \{1, ..., \gcd(p,q) - 1\}$ s.t.

$$hg_1 = jg_2.$$

This is equivalent to

$$j\begin{pmatrix} \frac{p}{\gcd(p,q)} \\ \frac{q}{\gcd(p,q)} \end{pmatrix} - h\frac{p^2}{\gcd(p,q)}\begin{pmatrix} m \\ -n \end{pmatrix} \in \operatorname{im}(C).$$

We know that

$$\begin{aligned} C\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} p \\ q \end{pmatrix}, \\ C\begin{pmatrix} -pm \\ \underbrace{qm + pn}_{\gcd(p,q)} \end{pmatrix} &= -p^2\begin{pmatrix} m \\ -n \end{pmatrix}. \end{aligned}$$

Then the preimage of $j\begin{pmatrix}\frac{p}{\gcd(p,q)}\\ \frac{q}{\gcd(p,q)}\end{pmatrix} - h\frac{p^2}{\gcd(p,q)}\begin{pmatrix}m\\ -n\end{pmatrix}$ under C which should be

$$\frac{j}{\gcd(p,q)}\begin{pmatrix}1\\ 0\end{pmatrix} - \begin{pmatrix}\frac{h}{\gcd(p,q)}pm\\ h\end{pmatrix}$$

has to be an element of $\mathbb{Z}^2$. But since $\gcd(p,q)$ was assumed to be greater than 1 and $\frac{h}{\gcd(p,q)}pm \in \mathbb{Z}$ this can't be the case. ■

The next proposition summarizes the cohomological properties of Witten spaces with finite cyclic fundamental groups we have found so far.

Proposition 2.5.2. *Let $M^{pq} \in \mathcal{W}$ and $m, n \in \mathbb{Z}$ s.t. $m\frac{q}{gcd(p,q)} + n\frac{p}{gcd(p,q)} = 1$. We further denote by x, y the standard generators of $H^2(\mathbb{C}P^2;\mathbb{Z})$ respectively $H^2(\mathbb{C}P^1;\mathbb{Z})$. Then:*

- $H^1(M^{pq};\mathbb{Z}) \cong 0$.
- $H^2(M^{pq};\mathbb{Z}) \cong \mathbb{Z}/gcd(p,q) \oplus \mathbb{Z}$. *The elements*

$$\frac{p}{gcd(p,q)}\Pi^*(x) + \frac{q}{gcd(p,q)}\Pi^*(y) \text{ and } m\Pi^*(x) - n\Pi^*(y)$$

  form a basis of $H^2(M^{pq};\mathbb{Z})$.
- $H^3(M^{pq};\mathbb{Z}) \cong 0$.
- *If $p = 0$, then $H^4(M^{pq};\mathbb{Z}) \cong 0$. If $p \neq 0$, then $H^4(M^{pq};\mathbb{Z}) \cong \mathbb{Z}/gcd(p,q) \oplus \mathbb{Z}/\dfrac{p^2}{gcd(p,q)}$ in this case the elements*

$$\frac{p}{gcd(p,q)}\Pi^*(x^2) + \frac{q}{gcd(p,q)}\Pi^*(xy) \text{ and } m\Pi^*(x^2) - n\Pi^*(xy)$$

  form a basis of $H^4(M^{pq};\mathbb{Z})$.
- $H^5(M^{pq};\mathbb{Z}) \cong \mathbb{Z}$.
- $H^6(M^{pq};\mathbb{Z}) \cong \mathbb{Z}/gcd(p,q)$. *The element $\Pi^*(x^2y)$ generates $H^6(M^{pq};\mathbb{Z})$.*

Remark 2.5.3. *i) The cup product of classes of even degree is obvious from the Gysin sequence. The only nontrivial cup product pairing involving odd degree cohomology classes is $\cup : H^2(M^{pq};\mathbb{Z}) \times H^5(M^{pq};\mathbb{Z}) \to H^7(M^{pq};\mathbb{Z})$. But this we understand with the help of Poincaré duality.*

ii) Since $H^3(M^{pq};\mathbb{Z}) \cong 0$ it follows that all Witten spaces are spin^c manifolds. In section 2.5.3 we give a criterion to decide whether a Witten space admits a spin structure or not.

2.5.2 The mod -($|\pi_1(M^{pq})|$)-cohomology ring

From now on we always assume that M^{pq} is an element of $\mathcal{W}$, where $\pi_1(M^{pq})$ is finite cyclic, i.e. $(p,q) \neq (0,0)$.

The computations in the last section and the Universal Coefficient Theorem (UCT) enable us to calculate the $\mathbb{Z}/\mathrm{gcd}(p,q)$-cohomology groups of M^{pq}:

$$H^*(M^{pq};\mathbb{Z}/\mathrm{gcd}(p,q)) \cong \begin{cases} \mathbb{Z}/\mathrm{gcd}(p,q), & \text{if } * = 0,1,6,7 \\ (\mathbb{Z}/\mathrm{gcd}(p,q))^2, & \text{if } * = 2,3,4,5 \end{cases}.$$

This information determines the differentials in the following E_2-term of the Leray-Serre spectral sequence, associated to the fibration $S^1 \xrightarrow{i} M^{pq} \xrightarrow{\Pi} \mathbb{C}P^2 \times \mathbb{C}P^1$ which converges to the $\mathbb{Z}/\mathrm{gcd}(p,q)-$cohomology of M^{pq}:

$$\begin{array}{cccccc} 1 & \mathbb{Z}/\mathrm{gcd}(p,q) & 0 & (\mathbb{Z}/\mathrm{gcd}(p,q))^2 & 0 & \\ & & \searrow d_2 & & \searrow d_2 & \\ 0 & \mathbb{Z}/\mathrm{gcd}(p,q) & 0 & (\mathbb{Z}/\mathrm{gcd}(p,q))^2 & 0 & \\ & 0 & 1 & 2 & 3... & \end{array}$$

All the d_2-differentials are trivial especially $d_2 : E_2^{01} \to E_2^{20}$ which has the following consequence: The multiplicative structure of this spectral sequence implies that there are "global classes" $a \in H^1(M^{pq};\mathbb{Z}/\mathrm{gcd}(p,q))$ with the property that $i^*(a)$ is a generator of $H^1(S^1;\mathbb{Z}/\mathrm{gcd}(p,q))$. Thus there are elements in $H^*(M^{p,q};\mathbb{Z}/\mathrm{gcd}(p,q))$ s.t. their images under i^* form a basis of $H^*(S^1;\mathbb{Z}/\mathrm{gcd}(p,q))$. In such a situation we can apply the Leray-Hirsch theorem ([Ha, p.432]):

Lemma 2.5.4. $\Pi^* : H^*(\mathbb{C}P^2 \times \mathbb{C}P^1;\mathbb{Z}/gcd(p,q)) \to H^*(M^{pq};\mathbb{Z}/gcd(p,q))$ *is an injective ring homomorphism and further*

$$\begin{aligned} H^*(M^{pq};\mathbb{Z}/gcd(p,q)) &= \Lambda[a] \otimes \Pi^*(\ H^*(\mathbb{C}P^2 \times \mathbb{C}P^1;\mathbb{Z}/gcd(p,q))\ ) \\ &= \Lambda[a] \otimes \frac{\mathbb{Z}/gcd(p,q)\,[\Pi^*(x),\Pi^*(y)]}{\Pi^*(x^3),\Pi^*(y^2)}. \end{aligned}$$

2.5.3 Characteristic classes

In this section we want to give explicite formulas for the first and second Stiefel-Whitney class $\omega_{1,2}(M^{pq}) := \omega_{1,2}(\tau_{M^{pq}})$ and the first Pontrjagin class $p_1(M^{pq}) := p_1(\tau_{M^{pq}})$ of the tangent bundle of a Witten space M^{pq}.

Let $D^{pq} \xrightarrow{pr} \mathbb{C}P^2 \times \mathbb{C}P^1$ be the disc bundle over $\mathbb{C}P^2 \times \mathbb{C}P^1$ associated to the S^1-bundle $M^{pq} \xrightarrow{\Pi} \mathbb{C}P^2 \times \mathbb{C}P^1$ and $j : M^{pq} \hookrightarrow D^{pq}$ the inclusion. It's clear that $pr \circ j = \Pi$ and it's true that $\tau_{D^{pq}} \cong pr^*(\tau_{\mathbb{C}P^2\times\mathbb{C}P^1}) \oplus \underline{\epsilon}_1^2$, where $\underline{\epsilon}_1^2$ denotes the real 2-dimensional trivial bundle over D^{pq}. Hence

$$\tau_{M^{pq}} \oplus \underline{\epsilon}_1 \cong j^*(\tau_{D^{pq}}) \cong j^*(pr^*(\tau_{\mathbb{C}P^2\times\mathbb{C}P^1}) \oplus \underline{\epsilon}_1^2) \cong \Pi^*(\tau_{\mathbb{C}P^2\times\mathbb{C}P^1}) \oplus \underline{\epsilon}_1^2.$$

Let $\rho_2 : H^2(M^{pq};\mathbb{Z}) \to H^2(M^{pq};\mathbb{Z}/2)$ be the mod-2-reduction map. Since the characteristic classes we are interested in are stable characteristic classes and the characteristic classes of $\mathbb{C}P^1$ are trivial we get the following equalities:

$$\begin{aligned}
\omega_1(M^{pq}) &= \omega_1(\Pi^*(\tau_{\mathbb{C}P^2})) = \Pi^*(\omega_1(\mathbb{C}P^2)) = \Pi^*(\omega_1(3\cdot\overline{\gamma})) = 0, && (2.7)\\
\omega_2(M^{pq}) &= \omega_2(\Pi^*(\tau_{\mathbb{C}P^2})) = \Pi^*(\omega_2(\mathbb{C}P^2)) = \Pi^*(\omega_2(3\cdot\overline{\gamma})) = 3\Pi^*\rho_2 c_1(\overline{\gamma}), && (2.8)\\
p_1(M^{pq}) &= p_1(\Pi^*(\tau_{\mathbb{C}P^2})) = \Pi^*(p_1(\mathbb{C}P^2)) = \Pi^*(3(\overline{\gamma})^2) = 3\Pi^*(x^2), && (2.9)
\end{aligned}$$

where $\overline{\gamma}$ is the dual of the tautological bundle over $\mathbb{C}P^2$ (see [M-S, p. 169]). Formula (2.7) tells us that all Witten spaces are orientable.

Lemma 2.5.5. *Let M^{pq} be a Witten manifold. Then M^{pq} is a spin manifold if and only if gcd(p,q) is odd and $\frac{q}{gcd(p,q)}$ is even.*

Proof."$\Leftarrow$": We recall that $\frac{p}{\gcd(p,q)}\Pi^*(x) + \frac{q}{\gcd(p,q)}\Pi^*(y)$ generates the torsion and $m\Pi^*(x) - n\Pi^*(y)$ a $\mathbb{Z}$-summand of $H^4(M^{pq};\mathbb{Z})$. We may write $3c_1(\overline{\gamma}) = 3\Pi^*(x)$ as

$$\begin{aligned}
&3\{n(\frac{p}{\gcd(p,q)}\Pi^*(x) + \frac{q}{\gcd(p,q)}\Pi^*(y))\\
&+\frac{q}{\gcd(p,q)}(m\Pi^*(x) - n\Pi^*(y))\}. \qquad (\star)
\end{aligned}$$

The part of the Bockstein sequence for M^{pq} associated to $0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$ which is of interest for us is

$$\cdots H^2(M^{pq};\mathbb{Z}) \to H^2(M^{pq};\mathbb{Z}) \xrightarrow{\rho_2} H^2(M^{pq};\mathbb{Z}/2) \to 0 \cdots .$$

From this we observe together with the UCT that the map ρ_2 factorizes over $\frac{H^2(M^{pq};\mathbb{Z})}{\text{torsion}} (\cong \mathbb{Z})$, i.e. ρ_2 sends the torsion part of $H^2(M^{pq};\mathbb{Z})$ to zero. This together with $(\star)$ and (2.8) implies that

$$\omega_2(M^{pq}) = \frac{q}{\gcd(p,q)}(m\Pi^*(\bar{x}) - n\Pi^*(\bar{y})),$$

where $\bar{x}$ and $\bar{y}$ are the mod-2-reductions of x and y resp. Since $\frac{q}{\gcd(p,q)}$ is even the first part is proven.

"$\Rightarrow$": If M^{pq} admits a spin structure, then $\omega_2(M^{pq}) = 0$ and $(\star)$ implies that $\frac{q}{\gcd(p,q)}$ is even. This follows from the fact that image of $\frac{p}{\gcd(p,q)}\Pi^*(x) + \frac{q}{\gcd(p,q)}\Pi^*(y)$ and $m\Pi^*(x) - n\Pi^*(y)$ under ρ_2 generate direct summands of $H^2(M^{pq};\mathbb{Z}/2)$. Now we assume that $\gcd(p,q)$ is even which means that $\gcd(p,q)$ and $\frac{q}{\gcd(p,q)}$ are not coprime. With the Bockstein sequence mentioned above we observe that the image of the map ρ_2 restricted to the torsion part of $H^2(M^{pq};\mathbb{Z})$ is nonzero in $H^2(M^{pq};\mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$. This means that the n appearing in $(\star)$ in front of the first term has to be even either. But $m\frac{q}{\gcd(p,q)} + n\frac{p}{\gcd(p,q)} = 1$ implies that n and $\frac{q}{\gcd(p,q)}$ have to be coprime. ■

Corollary 2.5.6. *Let M^{pq} be a Witten space.*
(i) There are no Witten spaces with fundamental group of even order that are spin manifolds.
(ii) The existence of a spin structure on the universal covering space $\widetilde{M^{pq}}$ of M^{pq} doesn't imply that M^{pq} itself admits a spin structure.
(iii) If a Witten manifold admits a spin structure, then the spin structure is unique.

Proof. The statements (i) and (ii) follow directly from the last lemma. The universal covering space of M^{28} is M^{14} and we see that M^{14} admits a spin structure where M^{28} doesn't admit one. Statement (iii) is a consequence of the fact that $H_1(M^{pq}; \mathbb{Z}/2)$ is 0 for M^{pq} a spin manifold. ■

2.5.4 The normal 2-type

Matthias Kreck introduced the concept of the normal k-type of a smooth manifold in the framework of his *modified surgery theory* [Kr], where $k \in \mathbb{N}$. In the next section we give a very brief outline of the *classification program* based on Kreck's surgery and present a successful application to the classification of the set of simply-connected Witten spaces.

We start this section with a formal definition of the normal k-type of a smooth manifold and finish it with the determination of the normal 2-type of a Witten space which will play a central role in chapter four.

Let M be a manifold and $\nu : M \to BO$ its stable normal Gauss map. Assume that the following diagram commutes:

$$\begin{array}{ccc} & & B \\ & \overset{\bar{\nu}}{\nearrow} & \downarrow \xi \\ M & \xrightarrow{\nu} & BO \end{array}$$

with ξ a fibration.

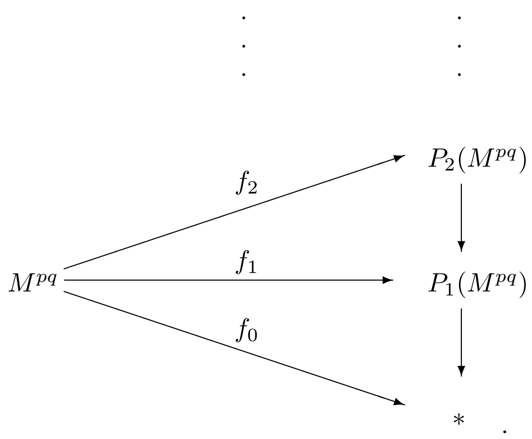
Definition 2.5.7. *Let $k \in \mathbb{N}$.*
*i) We call the lift $\bar{\nu}$ over ξ of the normal Gauss map ν a **normal k-smoothing** of M in (M, ξ) if $\bar{\nu}$ is a $(k+1)$-equivalence.*

*ii) If the homotopy groups of the fibre of ξ vanish in the degrees greater than or equal to $k+1$, then we call B **k-universal**.*

The theory of Moore-Postnikov decompositions of maps implies that there is a fibration $B_k \to BO$ with B_k is k-universal admitting a normal k-smoothing of M. And if B and B' are both k-universal and admitting a k-smoothing of M, then obstruction theory implies that the fibrations $B, B' \to BO$ are fibre homotopy equivalent. For all these homotopy theoretic facts we refer to [Kr.1, p. 14-15] and [Bau, p. 306-311].
Thus the fibre homotopy type of the fibration B_k over BO is an invariant of M and we call it the *normal k-type* of M.

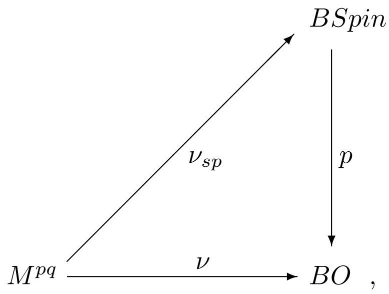
Determination of the normal 2-type of a Witten space

Let $M^{pq} \in \mathcal{W}$ be a Witten space and $i \in \mathbb{N}$. We denote the space in the i-th level of the Postnikov decomposition of M^{pq} by $P_i(M^{pq})$.



In the following we only use the properties of the map $f_2 : M^{pq} \to P_2(M^{pq})$ but in the last part of this section we are concerned with the explicite shape of $P_2(M^{pq})$.

Now we assume that M^{pq} is a spin manifold. In this case there is the following commutative diagram

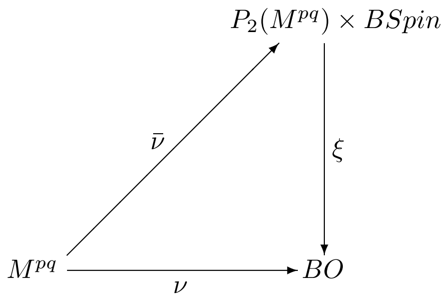


,

where p is the 3-connected cover of BO, $\nu : M^{pq} \to BO$ is the classifying map of the stable normal bundle $\nu_{M^{pq}}$ of M^{pq} and ν_{sp} is a lift of ν. We define the following maps:

$$\begin{aligned} \overline{\nu} &:= f_2 \times v_{sp} : M^{pq} \to P_2(M^{pq}) \times BSpin, \\ \xi &:= p \circ pr_2 : P_2(M^{pq}) \times BSpin \to BO, \end{aligned}$$

where pr_2 is the projection map onto the second factor. One can easily check that the following diagram commutes:

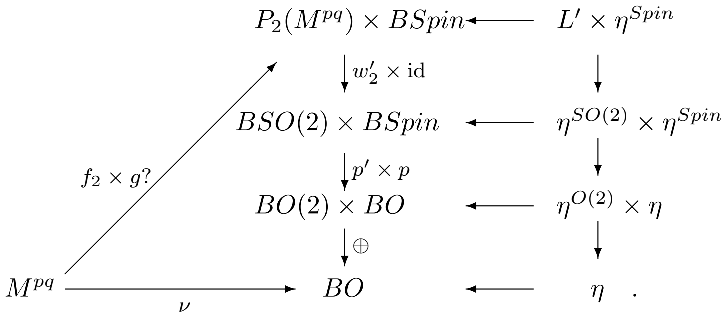


Since $p : BSpin \to BO$ is the 3-connected cover of BO it follows that $\overline{\nu}$ induces isomorphisms on π_1 and π_2 and the fibre of ξ has vanishing homotopy groups π_n for n greater than or equal to 3.

But what is the normal 2-type of a Witten Space that is non-spin?

From Remark 2.5.3.ii) we know that M^{pq} ia a spin^c manifold, this means that there exists a class $\tilde{w}_2$ in $H^2(M^{pq}; \mathbb{Z})$ with the property that its mod-2 reduction is the second Stiefel-Whitney class $w_2(M^{pq})$ of the tangent bundle of M^{pq}. Since the map $f_2 : M^{pq} \to P_2(M^{pq})$ is a 3-equivalence there exists a class w_2' in $H^2(P_2(M^{pq}); \mathbb{Z})$ that corresponds to $\tilde{w}_2$ under f_2^*. It's well known that $H^2(P_2(M^{pq}); \mathbb{Z})$ is isomorphic to

$[P_2(M^{pq}), K(\mathbb{Z}, 2)](= [P_2(M^{pq}), BSO(2)])$ (see [G-M, p. 73]). Now let $L' \to P_2(M^{pq})$ be a complex line bundle with first Chern class w_2'.
We wonder whether there exists a map $g : M^{pq} \to BSpin$ s.t. the following diagram commutes up to homotopy:



The η's are the corresponding "tautological bundles" and $\oplus$ is the classifying map for $\eta^{O(2)} \times \eta \to BO(2) \times BO$. If there was such a g, then $f_2 \times g$ would be a normal 2-smoothing of M^{pq} and

$$(P_2(M^{pq}) \times BSpin, \oplus \circ (p' \times p) \circ (w_2' \times id))$$

the normal 2-type of M^{pq}.

Let's denote the pullback fibration of L' under f_2 by L. If $\nu_{M^{pq}} \oplus L^{-1}$ was spin, then we could define g as a classifying map of $\nu_{M^{pq}} \oplus L^{-1}$.
We assert that $\nu_{M^{pq}} \oplus L^{-1}$ is spin. To prove this we use the fact that M^{pq} is orientable. The total Stiefel Whitney class $w(\nu_{M^{pq}})$ of $\nu_{M^{pq}}$ starts as

$$1 + 0 + \overline{w_2}(M^{pq}) + \cdots$$

and the total Stiefel Whitney class $w(L)$ of L equals

$$1 + 0 + w_2(M^{pq}).$$

This implies that $w(L^{-1}) = 1 + 0 + \overline{w_2}(M^{pq})$ and thus

$$w(\nu_{M^{pq}} \oplus L^{-1}) = 1 + 0 + 0 + \cdots ,$$

which shows that $\nu_{M^{pq}} \oplus L^{-1}$ is spin. This leads to the following

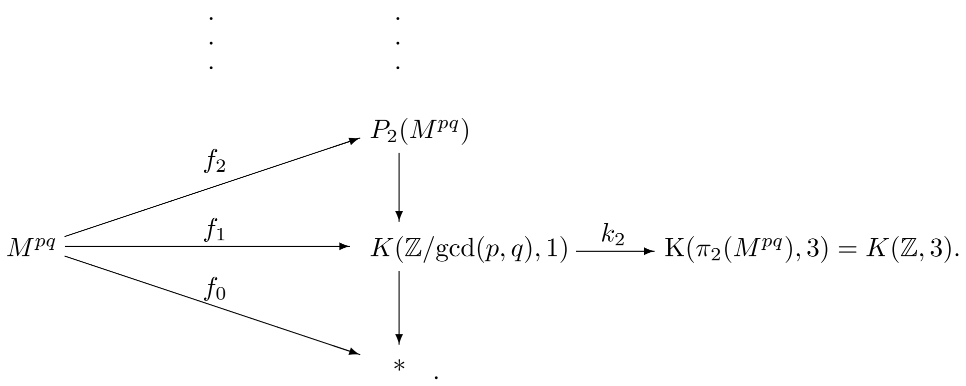
Proposition 2.5.8. *Let M^{pq} be a Witten space. The normal 2-type of M^{pq} is*

$$(P_2(M^{pq}) \times BSpin, \xi_{M^{pq}}),$$

where $P_2(M^{pq})$ is the space in the second stage of the Postnikov tower of M^{pq} and depending on $\omega_2(M^{pq})$ the map $\xi_{M^{pq}} : P_2(M^{pq}) \times BSpin \to BO$ is one of the fibrations which we have defined above.

What is $P_2(M^{pq})$?

From Corollary 2.4.3. we know that homotopically $\pi_1(M^{pq})$ acts trivially on $\widetilde{M^{pq}}$, the universal covering space of M^{pq} and Lemma 2.2.3. implies that $\pi_2(M^{pq}) \cong \mathbb{Z}$. Thus we can construct the Postnikov tower of M^{pq} with the help of k-invariants. The Postnikov tower of M^{pq} begins as follows:



Thus $P_2(M^{pq})$ is the total space of the pullback fibration induced by k_2. Let L_r^∞ denote the infinite dimensional lens space with fundamental group isomorphic to $\mathbb{Z}/r$ and we identify $K(\mathbb{Z}/\text{gcd}(p,q),1)$ with $L^\infty_{\text{gcd}(p,q)}$ and $K(\mathbb{Z},2)$ with the infinite dimensional complex projective space $\mathbb{C}P^\infty$.
By obstruction theory one can see that the homotopy class $[k_2]$ of k_2 can be identified with an element in $H^3(L^\infty_{\text{gcd}(p,q)};\mathbb{Z})$. But by the UCT $H^3(L^\infty_{\text{gcd}(p,q)};\mathbb{Z})$ is trivial. Hence k_2 is nullhomotopic and this implies that $P_2(M^{pq})$ is fibre homotopy equivalent to

$$L^\infty_{\text{gcd}(p,q)} \times \mathbb{C}P^\infty.$$

Thus if M^{pq} is a Witten space with fundamental group isomorphic to $\mathbb{Z}/\text{gcd}(p,q)$, then the normal 2-type of M^{pq} is

$$(L^\infty_{\text{gcd}(p,q)} \times \mathbb{C}P^\infty \times BSpin,\ \xi_{M^{pq}}),$$

where $\xi_{M^{pq}}$ depends only on $\omega_2(M^{pq})$.

2.6 A diffeomorphism classification: The simply-connected case

This and the next section deal with the classification of the following set of manifolds:

$$\{M^{ab} := M^{ab1} | a, b \in \mathbb{Z}\} \subset \mathcal{W}.$$

With a classification theorem of these manifolds in the hands, we easily obtain via Corollary 2.2.5. a classification of the set of all Witten spaces.

Modified surgery

The main reason why mathematicians like Browder, Milnor, Wall and others developed the so called classical surgery theory was given by the following problem:

Find a way to distinguish the diffeomorphism and homeomorphism types within a given homotopy type (in dimension ≥ 5*).*

Classical surgery is a very sophisticated mathematical theory which led to many deep insights, as for example

the calculation of the groups of diffeomorphism classes of oriented homotopy spheres in dimension greater than or equal to 5 [K-M].

However if one encounters the problem to compare the diffeomorphism type between two given closed smooth manifolds of dimension greater than or equal to 5, classical surgery theory implies to decide first whether the given manifolds share the same (simple) homotopy type. But even if one has ensured the existence of a (simple) homotopy equivalence between manifolds one needs to know the homotopy equivalence fairly explicitely in order to continue the *surgery program*, i.e. in order to tackle the problem of whether the homotopy equivalence can be covered by bundle maps of the normal bundles and whether they are normally bordant. However the problem to prove the existence of a homotopy equivalence between two smooth manifolds is in general a very hard one.
Thus classical surgery seems to be not that efficient in *practical life*.
This was maybe one of the reasons why Matthias Kreck developed an extension of classical surgery which is called *modified Surgery* [Kr]. Let $k, n \in \mathbb{N}$ s.t. n is greater than or equal to 5 and k is greater than or equal to $\left[\frac{n}{2}\right]$. Sometimes modified surgery enables to classify n-dimensional smooth closed manifolds up to diffeomorphism (or homeomorphism) although roughly speaking only the k-skeleton is known.

In the last section we introduced the normal k-type and normal k-smoothings of a smooth n-manifold. These mathematical objects enable us to formulate the surgery program which Theorem 3. and Theorem 4. in [Kr] suggest:

1) First we check whether the manifolds under consideration have the same normal 2-type (B, ξ).

2) Then we try to decide whether two normal $([\frac{n}{2}] - 1)$-smoothings are normally bordant in (B, ξ).

3) The last step is to analyze the obstruction for a B-bordism to be transformed by surgery into a s-cobordism.

This strategy for classifying manifolds led to the following abstract classification theorem for the simply-connected Witten spaces.

Theorem 2.6.1. *(A special case of [Kr, Theorem 6])*
Two simply-connected Witten spaces M and M' are diffeomorphic if and only if they

- *have the same normal 2-type B,*

- *admit normal 2-smoothings $f : M \to B$ and $f' : M' \to B$ and a B-bordism (W, F) between (M, f) and (M', f') s.t.*

$$\begin{array}{ll} i) & sign(W) = 0 \text{ and} \\ ii) & \langle F^*(u) \cup F^*(v), [W, \partial W] \rangle = 0 \text{ for all } u, v \in H^4(B; \mathbb{Q}). \end{array}$$

The condition ii) is to be understood in the following way: From the cohomological properties of the Witten spaces we conclude that $H^3(\partial W; \mathbb{Q}) \cong 0 \cong H^4(\partial W; \mathbb{Q})$. Hence there is an isomorphism $H^4(W, \partial W; \mathbb{Q}) \xrightarrow{\cong} H^4(W; \mathbb{Q})$. We identify $F^*(u)$ and $F^*(v)$ as elements in $H^4(W, \partial W; \mathbb{Q})$ before taking the cup product.

In [Kr-St] Matthias Kreck and Stephan Stolz succeeded in both the translation of Theorem 2.6.1. in terms of congruences between integral functions which depend on the parametrisation of the set of simply-connected Witten spaces and giving a homeomorphism classification as well in terms of integral congruences.

Theorem 2.6.2. *([Kr-St, Theorem B])*
Suppose M^{pq} and $M^{p'q'}$ are elements of $\mathcal{W}$ which are simply-connected. Then M^{pq} is homeomorphic to $M^{p'q'}$ if and only if $p' = \pm p$ and $q' \equiv q \mod a_p p^2$ where

$$a_p = \begin{cases} 1, & \text{if } 2|p \text{ and } 4 \nmid p \\ 2, & \text{otherwise.} \end{cases}$$

And M^{pq} is diffeomorphic to $M^{p'q'}$ if and only if $p' = \pm p$ and $q' \equiv q \mod 2^{\lambda_2(q)} 7^{\lambda_7(q)} q^2$, *where*

$$\lambda_2(p) = \begin{cases} 0, & \text{if } p \equiv 2,6 \quad mod\ 8 \\ 1, & \text{if } p \equiv 1,7 \quad mod\ 8 \\ 2, & \text{if } p \equiv 3,5 \quad mod\ 8 \\ 3, & \text{if } p \equiv 0,4 \quad mod\ 8 \end{cases}$$

$$\lambda_7(p) = \begin{cases} 0, & \text{if } p \equiv 1,2,5,6 \quad mod\ 7 \\ 1, & \text{if } p \equiv 0,3,4 \quad mod\ 7. \end{cases}$$

Remark 2.6.3. *i) With the help of the last theorem Matthias Kreck and Stephan Stolz discovered the following remarkable phenomenon :*

Within the category of homogeneous spaces there are homeomorphic but non-diffeomorphic manifolds.

ii) Other examples of successful application of modified surgery are:
The classification of the simply-connected Aloff-Wallach spaces [Kr-St.1] and the classification of complete intersections [Tr].

2.7 A diffeomorphism classification: The non-simply-connected case

2.7.1 The equivariant signature

Throughout this section we denote by G a nontrivial finite cyclic group. The main reference for the two subsequent sections is [A-S].

Definition 2.7.1. *Let $k \in \mathbb{N}$ and W be a $2k$-dimensional compact smooth oriented manifold equipped with a smooth and orientation preserving G-action. The boundary ∂W of W inherits an orientation and has the induced G-action. We call W a **G-manifold**.*

In the last definition the case $\partial W = \emptyset$ is not excluded.

We know that there is the following $(-1)^k$-hermitian form called the intersection form of W:

$$\lambda \ : \ H^k(W, \partial W; \mathbb{Z}) \times H^k(W, \partial W; \mathbb{Z}) \to \mathbb{Z}.$$

This form is G-invariant, i.e. $\lambda(ga, gb) = \lambda(a, b)$. This follows from the assumption that G acts on W in an orientation preserving way.
The radical $\mathrm{rad}(\lambda)$ of λ equals $ker(i^* : H^k(W, \partial W; \mathbb{Z}) \to H^k(W, \mathbb{Z}))$. We easily see that the form

$$\bar{\lambda} \ : \ \frac{H^k(W, \partial W; \mathbb{Z})}{\mathrm{rad}(\lambda)} \times \frac{H^k(W, \partial W; \mathbb{Z})}{\mathrm{rad}(\lambda)} \to \mathbb{Z},$$

given by

$$\bar{\lambda}([a],[b]) := \lambda(a,b),$$

is a non-degenerate G-invariant $(-1)^k$-hermitian form.
In the following we denote $\frac{H^k(W,\partial W;\mathbb{Z})}{\mathrm{rad}(\lambda)}$ by $\hat{H}(W)$. Tensoring $\hat{H}(W)$ with $\mathbb{C}$ over $\mathbb{Z}$ yields a complex vector space which we denote by $\hat{H}(W)^{\mathbb{C}}$. We extend $\bar{\lambda}$ to $\hat{H}(W)^{\mathbb{C}}$ in the following way:

$$\bar{\lambda}_{\mathbb{C}} \;:\; \hat{H}(W)^{\mathbb{C}} \times \hat{H}(W)^{\mathbb{C}} \to \mathbb{C},\;(x \otimes z_1, y \otimes z_2) \mapsto \bar{\lambda}(x,y)\cdot(z_1 \cdot z_2).$$

This complex valued quadratic form is a $(-1)^k$-hermitian G-invariant unimodular form.

Now we construct a hermitian form on $\hat{H}(W)^{\mathbb{C}}$ which is positive definite and G-invariant:

Let $(\tilde{z}_1, ..., \tilde{z}_l)$ be a complex basis for $\hat{H}(W)^{\mathbb{C}}$ and let $\langle \cdot, \cdot \rangle$ be the oriented standard hermitian form on $\hat{H}(W)^{\mathbb{C}}$, i.e. $\langle \tilde{z}_i, \tilde{z}_j \rangle = \delta_{ij}$. We define the following G-invariant hermitian product:

$$\langle \cdot, \cdot \rangle_G := \sum_{g \in G} \langle g(\cdot), g(\cdot) \rangle.$$

We see that $(\hat{H}(W)^{\mathbb{C}}, \langle \cdot, \cdot \rangle_G)$ is a unitary G-representation.
In order to define the G-signature of W we need the complex linear map $\Phi : \hat{H}(W)^{\mathbb{C}} \to \hat{H}(W)^{\mathbb{C}}$ which is characterized by the following equation:

$$\bar{\lambda}_{\mathbb{C}}(a,b) = \langle a, \Phi(b) \rangle_G \;\; \forall\; a, b \in \hat{H}(W)^{\mathbb{C}}.$$

Let $(z_1, ..., z_l)$ be an orthonormal basis with respect to $\langle \cdot, \cdot \rangle_G$ then we have the following matrix representation A of Φ (w.r.t. this basis):

$$(\bar{\lambda}_{\mathbb{C}}(z_i, z_j)) =: (a_{ij}).$$

The properties of $\bar{\lambda}_{\mathbb{C}}$ which we have mentioned above imply:

$$A^* = (-1)^k A, \qquad (1)$$
$$A(gx) = g(Ax) \;\forall g \in G,\; x \in \hat{H}(W)^{\mathbb{C}}, \qquad (2)$$

where A^* denotes the adjoint of A.

From the unimodularity of $\bar{\lambda}_{\mathbb{C}}$ we conclude that the eigenvalues of A are non-zero and if k is even then (1) implies that the eigenvalues of A are real and

$$\hat{H}(W)^{\mathbb{C}} = H_+ \oplus H_-,$$

where H_+ is the eigenspace of the positive eigenvalues and H_- is the eigenspace of the negative eigenvalues. And if k is odd, then (1) implies that the eigenvalues of A are purely imaginary and

$$\hat{H}(W)^{\mathbb{C}} = H_i \oplus H_{-i},$$

where H_i is the eigenspace of the eigenvalues which are positive multiples of i and H_{-i} is the eigenspace of the eigenvalues which are negative multiples of i. From (2) we derive that H_+, H_-, H_i and H_{-i} are all G-invariant. This means that H_+ and H_- are orthogonal G-representations hence we get two real valued characters which we denote by $\rho_\pm : G \to \mathbb{R}$. And H_i and H_{-i} are unitary G-representations thus we obtain two complex valued characters $\rho_{\pm i} : G \to \mathbb{C}$.

Definition 2.7.2. *Let $RO(G)$, $RU(G)$ be the real, complex representation ring of G resp. The G-**signature** $sign(G, W)$ of W is defined as*

$$\rho_+ - \rho_- \in RO(G) \text{ or } \rho_i - \rho_{-i} \in RU(G)$$

depending on whether k is even or odd.
Let $g \in G$ then

$$sign(g, W) = \begin{cases} \rho_+(g) - \rho_-(g) & \in \mathbb{R} \text{ f } k \text{ is even} \\ \rho_i(g) - \rho_{-i}(g) & \in i\mathbb{R} \text{ if } k \text{ is odd} \end{cases}$$

The G-signature is well defined. This follows from the following three facts:

a) The characters depend continuously on the inner product.

b) The space of all G-invariant hermitian products on $\hat{H}(W)^{\mathbb{C}}$ is connected.

c) The characters of a compact group are discrete.

We immediately observe that $\mathrm{sign}(1, W)$ equals the *classical* signature $\mathrm{sign}(W)$.

The G-signature has the following very important property which was proven by Novikov.

Proposition 2.7.3. *([A-S, Prop. (7.1)]) Let Y and Y' be $2k$-dimensional G-manifolds with $\partial Y = X$ and $\partial Y' = -X$. Let further $Z := Y \cup_X Y'$ be the closed G-manifold that one obtains when one glues Y and Y' together along the boundary X via the identity. Then*

$$sign(G, Z) = \mathrm{sign}(G, Y) + sign(G, Y').$$

The fixed point set

Let W be a $2k$-dimensional G-manifold and $g \in G$. By W^g we denote the fixed point set of g in W. In general W^g is not connected. So we write W^g as $\dot{\bigcup}_{j \in I} W_j^g$, the disjoint union of all its connected components. Let x be a point of W_j^g, T_xW the tangent space of W at x and $(dg)_x$ the differential of the diffeomorphism which is associated to g at the point x. By standard techniques we can find a metric on W

s.t. g acts as an isometry. Since g respects the orientation of W there exists a basis of T_xW s.t. the representation matrix of $(dg)_x$ is

$$\begin{pmatrix} 1 & 0 & \cdots & & & \cdots & 0 & \\ 0 & \ddots & & & & & \vdots & \\ \vdots & & 1 & & & & & \\ & & & \cos(\theta_1) & \sin(\theta_1) & & \vdots & \\ & & & -\sin(\theta_1) & \cos(\theta_1) & & & \\ \vdots & & & & & \ddots & 0 & \\ 0 & \cdots & & \cdots & & 0 & \begin{matrix} \cos(\theta_k) & \sin(\theta_k) \\ -\sin(\theta_k) & \cos(\theta_k) \end{matrix} \end{pmatrix},$$

where θ_l is greater than 0 and less than or equal to π for $l \in \{1, \ldots, k\}$. Of course the multipicity of the eigenvalue 1 equals the dimension of W_j^g. Since W is even dimensional we conclude that for all $j \in I$ the manifold W_j^g has to be even dimensional.

Let $\mathcal{N}_j^g$ denote the normal bundle of W_j^g in W. Now we look at the action of G on $\mathcal{N}_j^g$.
Since G is cyclic its irreducible real representations are of two types:

(i) one-dimensional with $g \mapsto \pm 1$ and

(ii) two-dimensional with $g \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

In (ii) the representations given by θ and $-\theta$ are equivalent and we may therefore restrict to the case $0 < \theta < \pi$. Such a two-dimensional real G-module has then a (canonical) complex structure in which g acts as the complex scalar $e^{i\theta}$.

As above let x be a point in W_i^g and $\mathcal{N}_x^g$ be the restriction of the normal bundle $\mathcal{N}_i^g$ to x. We have the following decomposition of $\mathcal{N}_x^g$:

$$\mathcal{N}_x^g = \mathcal{N}_x^g(-1) \oplus \sum_{0<\theta<\pi} \mathcal{N}_x^g(\theta),$$

where $\mathcal{N}_x^g(-1)$ is the subvectorspace of $\mathcal{N}_x^g$ on which g acts by -1 and $\mathcal{N}_x^g(\theta)$ is the complex subvectorspace of $\mathcal{N}_x^g$ where g acts by $e^{i\theta}$ for some $\theta \in (0, \pi)$.
The decomposition of $\mathcal{N}_x^g$ extends to a decomposition of $\mathcal{N}_i^g$:

$$\mathcal{N}_i^g = \mathcal{N}_i^g(-1) \oplus \sum_{0<\theta<\pi} \mathcal{N}_i^g(\theta),$$

where $\mathcal{N}_i^g(-1)$ denotes a real and $\mathcal{N}_i^g(\theta)$ a complex vector bundle over W_i^g (see [A-S, p. 560] or [A, Thm. 1.6.2.]).

If we have a complex structure on the normal bundle over W_i^g, then we orient W_i^g (if orientable) in the following way:
We first choose the orientation of the normal bundle which comes from the complex structure then we orient the tangent bundle of W_i^g in such a way that we obtain the orientation of the tangent bundle of W restricted to W_i^g.

Assume that $\mathcal{N}_i^g(\theta)$ is decomposable into the direct sum of r complex line bundles over W_i^g. We associate to $\mathcal{N}_i^g(\theta)$ the formal power series in r variables which is implicitely given by

$$\mathcal{M}^\theta := \prod_{0<l\leq r} \frac{\tanh(i\frac{\theta}{2})}{\tanh(\frac{x_l+i\theta}{2})}. \tag{2.10}$$

This means that the formal power series is just the Taylor expansion of the right handside of (2.10) around zero.
Applying the so called *Hirzebruch formalism* ([Hi, Ch. 1, §1] or [M-S, §19])) one can derive certain *characteristic polynomials* $\{\mathcal{M}_k^\theta(\mathcal{N}_i^g(\theta))\}_{k\in\mathbb{N}_0}$ associated to $\mathcal{M}^\theta$ which depend on the Chern classes of $\mathcal{N}_i^g(\theta)$. Let $\mathcal{M}^\theta(\mathcal{N}_i^g(\theta))$ be the formal sum of all $\mathcal{M}_k^\theta(\mathcal{N}_i^g(\theta))$'s, i.e.

$$\mathcal{M}^\theta(\mathcal{N}_i^g(\theta)) := \sum_{j=0}^{\infty} \mathcal{M}_j^\theta(\mathcal{N}_i^g(\theta)).$$

Assume that $\mathcal{N}_i^g(-1)$ is decomposable into the direct sum of s real line bundles over W_i^g. Then we associate to $\mathcal{N}_i^g(-1)$ the formal power series in s variables which is implicitely given by

$$\mathcal{L}^{-1} := \prod_{0<l\leq s} \frac{\tanh(\frac{x_l}{2})}{\frac{x_l}{2}}.$$

As above we obtain characteristic polynomials

$$\{\mathcal{L}_j^{-1}(\mathcal{N}_i^g(-1))\}_{j\in\mathbb{N}_0},$$

but here they depend on the Pontrjagin classes of $\mathcal{N}_i^g(-1)$. We'll make use of the following notation:

$$\mathcal{L}^{-1}(\mathcal{N}_i^g(-1)) := \sum_{k=0}^{\infty} \mathcal{L}_k^{-1}(\mathcal{N}_i^g(-1)).$$

Let $\mathcal{L}_k(W_i^g)$ denotes the k-th *Hirzebruch $\mathcal{L}$-polynomial* depending on the Pontrijagin classes of W_i^g (see for expl. [M-S, p. 224]). Let's denote $\sum_{k=0}^{\infty}\mathcal{L}_k(W_i^g)$ by $\mathcal{L}(W_i^g)$.

Theorem 2.7.4. *([A-S, The G-Signature Theorem, Thm. 6.12])*
Let W be a closed $2k$-dimensional G-manifold, $g \in G$ and $W^g = \dot{\bigcup}_n W_n^g$ the fixed point set of g in W, $\mathcal{N}^g = \bigcup \mathcal{N}_n^g$ the normal bundle of W_g in W and

$$\mathcal{N}^g = \bigcup_n \left(\mathcal{N}_n^g(-1) \oplus \sum_{0<\theta<\pi} \mathcal{N}_n^g(\theta) \right)$$

the decomposition of $\mathcal{N}^g$ determined by the eigenvalue of g. Then $\mathcal{N}_j^g(-1)$ is a real vector bundle of dimension $2r_j$, $\mathcal{N}_j^g(\theta)$ is a complex vector bundle of complex dimension $s_j(\theta)$ and by $2t_j$ we denote the dimension of W_j^g. Then the following holds:

$$sign(g,W) = \sum_j (2^{-r_i} \prod_{0<\theta<\pi} (i\tan(\frac{\theta}{2}))^{s_j(\theta)} \mathcal{L}(W_j^g)\mathcal{L}^{-1}(\mathcal{N}_j^g(-1))e(\mathcal{N}_j^g(-1)) \prod_{0<\theta<\pi} \mathcal{M}^\theta(\mathcal{N}_j^g(\theta))[W_j^g]),$$

where $e(\mathcal{N}_j^g(-1))$ denotes the "twisted" Euler class of $\mathcal{N}_j^g(-1)$ and $[W_j^g]$ is the "twisted" fundamental class of W_j^g, both twistings being defined by the local coefficient system of orientations of W_j^g.

For a precise explanation of what "twisted" Euler class and fundamental class really means and further immediate corollaries of the last theorem we refer to [A-S, pp. 581-603].

2.7.2 h-cobordism invariants

We call a G-manifold where the action of G is free a **free G-manifold**. In this section we define an invariant for free G-manifolds. Let W be such a manifold then the invariant is going to be a map

$$\sigma(\cdot, W) : G \setminus \{0\} \to \mathbb{C}$$

s.t. two free G-manifolds V, W which differ by a G-equivariant diffeomorphism have the same invariant.

Let X be a G-manifold with $\partial X = W$.

Definition/Lemma 2.7.5. *Let $g \in G \setminus \{0\}$ then we define $\sigma(g,W)$ to be*

$$L(g,X) - sign(g,X),$$

where $L(g,X)$ is the expression which appears on the right hand side of the equivariant signature formula.

Proof. We have to show that $\sigma(g,W)$ just depends on W and not on the choice of the G-bordism X. Let X' be another G-manifold with $\partial X' = W$. We denote $L(g,X) - \mathrm{sign}(g,X)$ by $\sigma_X(g,W)$ and $L(g,X') - \mathrm{sign}(g,X')$ by $\sigma_{X'}(g,W)$. Then we glue X and $-X'$ together along W with the identity map and the result

$$Z := X \cup_W X'$$

is a closed G-manifold. By the additivity property of the equivariant signature (Proposition 2.7.3.) we get

$$\mathrm{sign}(g, Z) = \mathrm{sign}(g, X) - \mathrm{sign}(g, X') \tag{2.11}$$

and since g has no fixed points in W the fixed point set Z^g is the disjoint union of X^g and X'^g. This implies

$$L(g, Z) = L(g, X) - L(g, X'). \tag{2.12}$$

And we obtain:

$$\begin{aligned} (2.12) - (2.11) \quad &= \quad \{L(g, X) - \mathrm{sign}(g, X)\} - \{L(g, X') - \mathrm{sign}(g, X')\} \\ &= \quad \sigma_X(g, W) - \sigma'_X(g, W) \\ &= \quad L(g, Z) - \mathrm{sign}(g, Z) \\ &\overset{Thm.2.7.4.}{=} \quad 0. \end{aligned}$$

■

Proposition 2.7.6. *Let V, W be closed $(2k-1)$-dimensional free G-manifolds and X a G-manifold with $\partial X = V - W$. If $X^g = \emptyset$ and $im(i^* : H^n(X, \partial X; \mathbb{R}) \to H^n(X; \mathbb{R})) = 0$, then*

$$\sigma(g, V) = \sigma(g, W), \quad \forall g \in G \setminus \{0\}.$$

Proof. Let Y and Y' be G-manifolds with $\partial Y = V$ and $\partial Y' = W$. By glueing X, Y and $-Y'$ together in the obvious way we obtain

$$Z := Y \cup_V X \cup_W Y'.$$

The assumptions of the proposition and the additivity property of the equivariant signature imply

$$L(g, Z) = L(g, Y) - L(g, Y')$$

and

$$\mathrm{sign}(g, Z) = \mathrm{sign}(g, Y) - \mathrm{sign}(g, Y').$$

By Theorem 2.7.4. it follows that

$$\begin{aligned} 0 \quad &= \quad L(g, Z) - \mathrm{sign}(g, Z) \\ &= \quad \{L(g, Y) - \mathrm{sign}(g, Y)\} - \{L(g, Y') - \mathrm{sign}(g, Y')\} \\ &= \quad \sigma(g, V) - \sigma(g, W). \end{aligned}$$

■

Definition/Proposition 2.7.7. *Let M be a closed oriented $(2k-1)$-dimensional manifold with nontrivial finite cyclic fundamental group G. Assume that the universal covering space $\widetilde{M}$ of M bounds as an oriented G-manifold another G-manifold, then*

$$\sigma(\cdot, \widetilde{M}) : G \setminus \{0\} \to \mathbb{C}$$

is an h-cobordism invariant of M.

Proof. Let W be an h-cobordism between M and M' then $\widetilde{W}$ is an h-cobordism between $\widetilde{M}$ and $\widetilde{M}'$ and let $i : M \hookrightarrow W$ and $j : M' \hookrightarrow W$ be the inclusions.
It's clear that $\widetilde{W}^g = \emptyset$ and since $j^* : H^k(\widetilde{W}; \mathbb{R}) \to H^k(\partial\widetilde{W}; \mathbb{R})$ is injective it follows from the long exact sequence in cohomology for $(\widetilde{W}, \partial\widetilde{W})$ that $\mathrm{im}(i^* : H^k(\widetilde{W}, \partial\widetilde{W}; \mathbb{R}) \to H^k(\widetilde{W}; \mathbb{R})) = 0$ and the assumptions of the last proposition are fulfilled. Thus $\sigma(g, \widetilde{M}) = \sigma(g, \widetilde{M}')$, where we have identified $\pi_1(M')$ with G via $(i^*)^{-1}j^*$. ■

Lemma 2.7.8. *Let M be a closed oriented $(2k-1)$-dimensional manifold with G as its fundamental group and $g \in G \setminus \{0\}$. If $k \equiv 3$ mod 4, then $\sigma(g, \widetilde{M}) \in \mathbb{R}$ and if $k \equiv 1$mod 4, then $\sigma(g, \widetilde{M}) \in i\mathbb{R}$.*

Proof. The σ-invariant is additive w.r.t. the disjoint union. On the other hand it follows from the free cobordism theory of Conner and Floyd [C-F, Chapter 3 and 7] that for a $(2k-1)$-dimensional free G-manifold X there exists a natural number l s.t. $\dot{\bigcup}_{i=1}^{l} X_i$ bounds an oriented free G-manifold, where X_i is a copy of X. Putting these facts together leads to the following: Let $g \in G \setminus \{0\}$, then

$$\sigma(g, \dot{\bigcup}_{i=1}^{l} X_i) = l\sigma(g, X).$$

But this implies that

$$\sigma(g, X) = -\frac{\mathrm{sign}(g, X)}{l}.$$

If $\dim(X) \equiv 3$ mod4, then $\mathrm{sign}(g, X) \in \mathbb{R}$ and if $\dim(X) \equiv 1$ mod 4, then $\mathrm{sign}(g, X) \in i\mathbb{R}$ which finishes the proof.

■

2.7.3 The main result

Theorem 2.7.9. *Let s be a natural number greater than 1 and M^{pq}, $M^{p'q'}$ be two Witten spaces with $\pi_1(M^{pq}) \cong \pi_1(M^{p'q'}) \cong \mathbb{Z}/s$. Then the following statements are equivalent:*

1) *There exist homogeneous metrics m_1 and m_2 on M^{pq} and $M^{p'q'}$ w.r.t. $SU(3) \times SU(2) \times U(1)$ s.t. (M^{pq}, m_1) and (M^{pq}, m_2) are isometric.*

2) *M^{pq} and $M^{p'q'}$ are diffeomorphic.*

3) *$|p| = |p'|$ and $|q| = |q'|$.*

Proof. Statement 1) obviously implies statement 2).

"3) $\Rightarrow$ 1)": In section 5 of this chapter we defined M^{pq} and $M^{p'q'}$ to be M^{pq1} and $M^{p'q'1}$ respectively. Proposition 2.3.2. implies that the algebraic condition in 3) implies the existence of an equivariant diffeomorphism between the corresponding Witten spaces in the sense of Definition 2.3.1. But the existence of such a diffeomorphism and Proposition 2.3.5. imply 1).

The hard part of the proof is to show that 2) implies 3). We show this by doing the following steps:

i) Computation of the σ-invariants for Witten spaces with fundamental groups which have order greater than or equal to three.

ii) Proof of "2) $\Rightarrow$ 3)" for these manifolds and for those with $2||\pi_1(\cdot)|$ but $|\pi_1(\cdot)| > 4$.

iii) Computation of the σ-invariants for Witten spaces with fundamental group isomorphic to $\mathbb{Z}/2$ or $\mathbb{Z}/4$.

iv) Proof of "2) $\Rightarrow$ 3)" for these manifolds.

i) Before we begin with the calculation of the invariants we equip the (orientable) Witten spaces with an orientation. Therefore we identify them with the total space of the S^1-fibre bundle over $\mathbb{C}P^2 \times \mathbb{C}P^1$,

$$S^1 \to M^{ab} \overset{\Pi_{ab}}{\to} \mathbb{C}P^2 \times \mathbb{C}P^1,$$

with $c_1(\Pi_{ab}) = ax + by$, where $x \in H^2(\mathbb{C}P^2;\mathbb{Z})$ and $y \in H^2(\mathbb{C}P^1;\mathbb{Z})$ are the standard generators.
If we speak of M^{ab} being oriented, then M^{ab} is equipped with the orientation which we obtain in the following way:
First we equip the base $\mathbb{C}P^2 \times \mathbb{C}P^1$ with the standard orientation and then we choose the orientation of the fibre S^1 which is compatible with the orientation of the fibre $\mathbb{C}$ of the complex line bundle

$$\mathbb{C} \to E^{ab} \overset{pr_{ab}}{\to} \mathbb{C}P^2 \times \mathbb{C}P^1$$

with $c_1(pr_{ab}) = ax + by$.
An oriented Witten space M^{ab} induces an orientation on its universal covering space which coincides with the orientation of $\widetilde{M}^{ab}$ considered as an oriented Witten space.

Let s be an odd natural number greater than 3 and M^{pq} be an oriented Witten space with $|\pi_1(M^{pq})| = s$ and we identify $\widetilde{M}^{pq}$ with $M^{\frac{p}{s}\frac{q}{s}}$. As we have already mentioned at the beginning of this section we regard $M^{\frac{p}{s}\frac{q}{s}}$ as the total space of a certain fibre bundle.
In order to compute the σ-invariant for $M^{\frac{p}{s}\frac{q}{s}}$ one needs an oriented $\pi_1(M^{pq})$-manifold

which has $M^{\frac{p}{s}\frac{q}{s}}$ as its boundary s.t. the $\pi_1(M^{pq})$-action on the boundary coincides with the given one on $M^{\frac{p}{s}\frac{q}{s}}$. An obvious choice of such a bordism is the (oriented) disc bundle $D^{\frac{p}{s}\frac{q}{s}}$ associated to $M^{\frac{p}{s}\frac{q}{s}}$, where the orientation on $D^{\frac{p}{s}\frac{q}{s}}$ is chosen in such a way that it is compatible with the orientation of $M^{\frac{p}{s}\frac{q}{s}}$. From Corollary 2.4.3. we know that the deck transformation on $M^{\frac{p}{s}\frac{q}{s}}$ preserves the fibre thus we may smoothly extend the $\pi_1(M^{pq})$-action on the boundary to the disc bundle in the most obvious way ("by smoothly decreasing the radius"). Thus the fixed point set of any nontrivial element in $\pi_1(M^{pq})$ is the base space $\mathbb{C}P^2 \times \mathbb{C}P^1$. Let x and y be the standard generators of $H^2(\mathbb{C}P^2;\mathbb{Z})$ and $H^2(\mathbb{C}P^1;\mathbb{Z})$ resp. The normal bundle of the base space in $D^{\frac{p}{s}\frac{q}{s}}$ is the complex line bundle over $\mathbb{C}P^2 \times \mathbb{C}P^1$ which is given by $\pm(\frac{p}{s}x + \frac{q}{s}y) \in H^2(\mathbb{C}P^2 \times \mathbb{C}P^1;\mathbb{Z})$ as its first Chern class (the sign depends on the complex structure of the normal bundle). Let $g \in \pi_1(M^{pq})$ be a nontrivial element s.t. g acts on the normal bundle by fibrewise multiplication with $e^{i\theta_g}$, where

$$\theta_g \in \{\frac{2\pi}{s} j | 0 < j \leq \left[\frac{r}{2}\right]\} =: A_s \subset (0, \pi).$$

The orientation of $D^{\frac{p}{s}\frac{q}{s}}$ and the complex structure of the normal bundle which is determined by g induce an orientation of the fixed point set.

Let's denote the (oriented) normal bundle over $\mathbb{C}P^2 \times \mathbb{C}P^1$ together with the multiplication with g by $\mathcal{N}_{\theta_g}$. The σ-invariant associated on g is:

$$\begin{aligned} \sigma(g, M^{\frac{p}{s}\frac{q}{s}}) &= (i \tan \frac{\theta_g}{2})^{-1} \sum_{i=0} \mathcal{L}_i(\mathbb{C}P^2 \times \mathbb{C}P^1) \sum_{j=0} \mathcal{M}_j^{\theta_g}(\mathcal{N}_{\theta_g}) \left[\mathbb{C}P^2 \times \mathbb{C}P^1\right]_\pm \\ & \quad - \mathrm{sign}(g, D^{\frac{p}{s}\frac{q}{s}}), \end{aligned}$$

where $[\mathbb{C}P^2 \times \mathbb{C}P^1]_\pm$ denotes $+1$ or -1 times the standard fundamental class of $\mathbb{C}P^2 \times \mathbb{C}P^1$, where the sign depends on how g acts on the normal bundle (compare with p.48, the remark on how we orient the fixed point set).

By the construction of the $\pi_1(M^{pq})$-action on $D^{\frac{p}{s}\frac{q}{s}}$ we see that homotopically it operates trivially on $D^{\frac{p}{s}\frac{q}{s}}$. Thus

$$\mathrm{sign}(g, D^{\frac{p}{s}\frac{q}{s}}) = \mathrm{sign}(D^{\frac{p}{s}\frac{q}{s}}) \ \ \forall g \in \pi_1(M^{pq}).$$

We show that $\mathrm{sign}(D^{\frac{p}{s}\frac{q}{s}}) = 0$:

The compution of the integral cohomology of M^{pq} has shown that $H^3(M^{pq};\mathbb{Z})$ and $H^4(M^{pq};\mathbb{Z})$ consist only of torsion (compare with Proposition 2.5.2.). This implies that the cup product pairing

$$\cup : H^4(D^{\frac{p}{s}\frac{q}{s}}, M^{\frac{p}{s}\frac{q}{s}};\mathbb{R}) \times H^4(D^{\frac{p}{s}\frac{q}{s}}, M^{\frac{p}{s}\frac{q}{s}};\mathbb{R}) \to \mathbb{R}$$

is unimodular. By Poincaré-Lefschetz duality $H^4(D^{\frac{p}{s}\frac{q}{s}}, M^{\frac{p}{s}\frac{q}{s}};\mathbb{R})$ is isomorphic to $H_4(D^{\frac{p}{s}\frac{q}{s}};\mathbb{R})$. Since $\mathbb{C}P^2 \times \mathbb{C}P^1$ is a deformation retract of $D^{\frac{p}{s}\frac{q}{s}}$ we conclude that

$H_4(D^{\frac{p}{s}\frac{q}{s}};\mathbb{R})$ is isomorphic to $\mathbb{R}^2$. The homology class $[\mathbb{C}P^2]$ which is representable by $\mathbb{C}P^2(\cong \mathbb{C}P^2 \times pt \subset \mathbb{C}P^2 \times \mathbb{C}P^1)$, considered as a submanifold of $D^{\frac{p}{s}\frac{q}{s}}$, is a generator of $H_4(D^{\frac{p}{s}\frac{q}{s}};\mathbb{R})$. Taking the cup product of the Poincaré-Lefschetz dual $[\mathbb{C}P^2]^*$ of $[\mathbb{C}P^2]$ with itself and evaluate it on the relative fundamental class of $(D^{\frac{p}{s}\frac{q}{s}}, M^{\frac{p}{s}\frac{q}{s}})$ is the same as the geometric intersection of $\mathbb{C}P^2$ with itself. For this we write: $[\mathbb{C}P^2]\cdot[\mathbb{C}P^2]$. But we immediately see that we can move $\mathbb{C}P^2$ within the base in the direction of $\mathbb{C}P^1$. Thus the "moved $\mathbb{C}P^2$" and the "original $\mathbb{C}P^2$" intersect trivially which means that $[\mathbb{C}P^2]\cdot[\mathbb{C}P^2] = 0$.
Let's choose another generator Z of $H^4(D^{\frac{p}{s}\frac{q}{s}}, M^{\frac{p}{s}\frac{q}{s}};\mathbb{R})$ s.t. $[\mathbb{C}P^2]^*$ and Z form a basis of $H^4(D^{\frac{p}{s}\frac{q}{s}}, M^{\frac{p}{s}\frac{q}{s}};\mathbb{R})$. The representation matrix of the adjoint of the cup product pairing on $H^4(D^{\frac{p}{s}\frac{q}{s}}, M^{\frac{p}{s}\frac{q}{s}};\mathbb{R})$ w.r.t. the chosen basis is given by

$$\begin{pmatrix} 0 & k \\ k & f \end{pmatrix},$$

where $k \in \mathbb{R}_{>0}$ and $f \in \mathbb{R}$. One eigenvalue of the adjoint is positive and the other is negative hence $\operatorname{sign}(D^{\frac{p}{s}\frac{q}{s}}) = 0$. Thus

$$\sigma(g, M^{\frac{p}{s}\frac{q}{s}}) = (i \tan\frac{\theta_g}{2})^{-1} \sum_{i=0} \mathcal{L}_i(\mathbb{C}P^2 \times \mathbb{C}P^1) \sum_{j=0} \mathcal{M}_j^{\theta_g}(\mathcal{N}_{\theta_g}) \left[\mathbb{C}P^2 \times \mathbb{C}P^1\right]_{\pm}.$$

Since the fixed point set is 6-dimensional we are only interested in $\mathcal{M}_r^{\theta}(\mathcal{N}_\theta)$ for $0 < r \leq 3$. We know from the section 2.7.1 that the polynomials $\mathcal{M}_j^{\theta}$ are implicitely given by the formal power series that one obtains from the Taylor expansion of

$$\mathcal{M}^{\theta}(x) := \frac{\tanh(i\frac{\theta}{2})}{\tanh(\frac{x+i\theta}{2})} = \coth(\frac{x+i\theta}{2}) \cdot \tanh(i\frac{\theta}{2})$$

at 0.

Let $\mathcal{M}^{\theta} = 1 + \lambda_1 + \lambda_2 + \lambda_3 + \cdots$ be the beginning of the Taylor expansion. We prove that the following is true:

$$\lambda_1 = \frac{i}{\sin\theta}, \quad \lambda_2 = -\frac{1}{4\sin^2(\frac{\theta}{2})}, \quad \lambda_3 = -i\left(\frac{1}{12\sin\theta} + \frac{\cos\frac{\theta}{2}}{8\sin^3\frac{\theta}{2}}\right). \tag{2.13}$$

By $\mathcal{M}^{\theta,k}(0)$ we denote the k-th derivative of $\mathcal{M}^{\theta}(x)$ at $x = 0$. Then around 0 we have

$\mathcal{M}^\theta(x) = \sum_{k=0}^{\infty} \frac{\mathcal{M}^{\theta,k}(0)}{k!} x^k$, where

$$\begin{aligned}
\mathcal{M}^{\theta,1}(0) &= \coth(\frac{x+i\theta}{2})'|_{x=0} \cdot \tanh(i\frac{\theta}{2}) \\
&= -\frac{\tanh(i\frac{\theta}{2})}{2\sinh^2(i\frac{\theta}{2})} \\
&= \frac{i}{\sin\theta}, \\
\mathcal{M}^{\theta,2}(0) &= \coth(\frac{x+i\theta}{2})''|_{x=0} \cdot \tanh(i\frac{\theta}{2}) \\
&= \frac{\cosh(i\frac{\theta}{2})}{2\sinh^3(i\frac{\theta}{2})}\tanh(i\frac{\theta}{2}) \\
&= -\frac{1}{-4\sin^2\frac{\theta}{2}}, \\
\mathcal{M}^{\theta,3}(0) &= \coth(\frac{x+i\theta}{2})'''|_{x=0} \cdot \tanh(i\frac{\theta}{2}) \\
&= \frac{1}{\sinh i\frac{\theta}{2}\cosh i\frac{\theta}{2}} - \frac{3\cosh i\frac{\theta}{2}}{4\sinh^3 i\frac{\theta}{2}} \\
&= \frac{-i}{2\sin\theta} - \frac{3i\cos\frac{\theta}{2}}{4\sin^3\frac{\theta}{2}}.
\end{aligned}$$

Thus we have proved assertion (2.13). But actually we are interested in

$$\mathcal{M}_j^{\theta_g}(\mathcal{N}_{\theta_g}) = \mathcal{M}_r^{\theta_g}(c_1(\mathcal{N}_{\theta_g}), \ldots\ c_r(\mathcal{N}_{\theta_g})).$$

and since $c_{i\geq 2}(\mathcal{N}_{\theta_g}) = 0$ the Hirzebruch formalism (see e.g. [M-S, §19]) implies

$$\begin{aligned}
\mathcal{M}_0^{\theta_g} &= 1, \\
\mathcal{M}_1^{\theta_g} &= \lambda_1 c_1(\mathcal{N}_{\theta_g}) \\
&= \frac{i}{\sin\theta_g} c_1(\mathcal{N}_{\theta_g}), \\
\mathcal{M}_2^{\theta_g} &= \lambda_2 c_1^2(\mathcal{N}_{\theta_g}) \\
&= -\frac{1}{4\sin^2\frac{\theta_g}{2}} c_1^2(\mathcal{N}_{\theta_g}), \\
\mathcal{M}_3^{\theta_g} &= \lambda_3 c_1^3(\mathcal{N}_{\theta_g}) \\
&= -i\left(\frac{1}{12\sin\theta_g} + \frac{\cos\frac{\theta_g}{2}}{8\sin^3\frac{\theta_g}{2}}\right) c_1^3(\mathcal{N}_{\theta_g}).
\end{aligned}$$

The relevant Hirzebruch $\mathcal{L}$-polynomials for the fixed point set are:

$$\begin{aligned}
\mathcal{L}_0(\mathbb{C}P^2 \times \mathbb{C}P^1) &= 1, \\
\mathcal{L}_1(\mathbb{C}P^2 \times \mathbb{C}P^1) &= \frac{1}{3} p_1(\mathbb{C}P^2 \times \mathbb{C}P^1) = x^2.
\end{aligned}$$

We denote $\frac{p}{s}$, $\frac{p'}{s}$ by $\bar{p}$,$\bar{p}'$ and $\frac{q}{s}$, $\frac{q'}{s}$ by $\bar{q}$, $\bar{q}'$ resp. Let's assume that the rotation by θ_g respects the orientation of the normal bundle which is given by the first Chern class $\bar{p}x+\bar{q}y$, where the orientation of the normal bundle comes from the complex structure. Then

$$\begin{aligned}\sigma(g, M^{\frac{p}{s}\frac{q}{s}}) &= (i\tan\frac{\theta_g}{2})^{-1}(1+x^2)\\ &\quad \left(1+i\frac{\bar{p}x+\bar{q}y}{\sin\theta_g}-\frac{\bar{p}^2x^2+2\bar{p}\bar{q}xy}{4\sin^2\frac{\theta_g}{2}}+i\left(\frac{1}{12\sin\theta_g}+\frac{\cos\frac{\theta_g}{2}}{8\sin^3\frac{\theta_g}{2}}\right)3\bar{p}^2\bar{q}x^2y\right)\\ &\quad \left[\mathbb{C}P^2\times\mathbb{C}P^1\right]_+\\ &= -(\tan\frac{\theta_g}{2})^{-1}\left(3\left(\frac{1}{12\sin\theta_g}+\frac{\cos\frac{\theta_g}{2}}{8\sin^3\frac{\theta_g}{2}}\right)\bar{p}^2\bar{q}+\frac{\bar{q}}{\sin\theta_g}\right).\end{aligned}$$

Let's denote

$$-(\tan\frac{\theta_g}{2})^{-1}\left(3\left(\frac{1}{12\sin\theta_g}+\frac{\cos\frac{\theta_g}{2}}{8\sin^3\frac{\theta_g}{2}}\right)\bar{p}^2\bar{q}+\frac{\bar{q}}{\sin\theta_g}\right)$$

by $f_{\bar{p}^2}(\theta_g)$. Then we have

$$\sigma(g, M^{\frac{p}{s}\frac{q}{s}}) = \bar{q}f_{\bar{p}^2}(\theta_g).$$

ii) Let s be a natural number greater than or equal to 3. If we show that for any $\bar{p}\geq 1$ there exists a $\theta\in A_s\cap(0,\pi)$ s.t. $f_{\bar{p}^2}(\theta)\neq 0$, then there exists a nontrivial maximum of the set

$$\{|f_{\bar{p}^2}(\frac{2\pi}{s}j)||0<j\leq\left[\frac{s}{2}\right]\}$$

which we call $m_{\bar{p},s}$.

Let's assume that M^{pq} and $M^{p'q'}$ are diffeomorphic. Then since $|H^4(M^{pq};\mathbb{Z})| = p^2$ (see Proposition 2.5.2.) we conclude that $|p| = |p'|$. And since M^{pq} and $M^{p'q'}$ are assumed to be diffeomorphic the following value sets of σ-invariants have to coincide

$$\{\bar{q}f_{\bar{p}^2}(\frac{2\pi}{s}j)|0<j\leq\left[\frac{s}{2}\right]\} = \{\bar{q}'f_{\bar{p}^2}(\frac{2\pi}{s}j)|0<j\leq\left[\frac{s}{2}\right]\}.$$

But this implies that $|\bar{q}m_{\bar{p},s}| = |\bar{q}'m_{\bar{p},s}|$, thus $|\bar{q}| = |\bar{q}'|$.

Now we show that for any $\bar{p}\geq 1$ there exists a $\theta\in A_s\cap(0,\pi)$ s.t. $f_{\bar{p}^2}(\theta)\neq 0$ or equivalently

$$\underbrace{-3\left(\frac{1}{12\sin\theta}+\frac{\cos\frac{\theta}{2}}{8\sin^3\frac{\theta}{2}}\right)\bar{p}^2+\frac{1}{\sin\theta}}_{=:\tilde{f}_{\bar{p}^2}(\theta)}\neq 0.$$

First case: $|\bar{p}| = 1$:

$$\tilde{f}_1(\theta) = \frac{3}{8}\frac{1-2\cos^2\frac{\theta}{2}}{\sin^3\frac{\theta}{2}\cos\frac{\theta}{2}}.$$

Assume that $\tilde{f}_1(\theta) = 0$ for some $\theta \in (0, \pi)$ then $1 - 2\cos^2 \frac{\theta}{2} = 0$ which is equivalent to $\cos^2 \frac{\theta}{2} = \frac{1}{2}$ and thus $\cos \frac{\theta}{2} = \pm\sqrt{\frac{1}{2}}$. But this implies that $\theta = \frac{\pi}{2}$ or $\frac{3}{2}\pi$. Since s is either odd or even but greater than 4, we see that $\frac{2\pi}{s} \in A_s \cap (0, \pi)$ is neither $\frac{\pi}{2}$ nor $\frac{3}{2}\pi$.

Second case: $|\bar{p}| > 1 \Leftrightarrow \bar{p}^2 \geq 4$.
Assume $\tilde{f}_{\bar{p}^2}(\theta) = 0$ for some $\theta \in A_s \cap (0, \pi)$ which is equivalent to

$$\bar{p}^2 = \underbrace{\frac{1}{3}\left(\frac{1}{\frac{1}{12} + \frac{\sin\theta\cos\frac{\theta}{2}}{8\sin^3\frac{\theta}{2}}}\right)}_{=:h(\theta)} \geq 4.$$

But this is equivalent to

$$\frac{\sin\theta\cos\frac{\theta}{2}}{8\sin^3\frac{\theta}{2}} \leq 0$$

which is clearly impossible for $\theta \in (0, \pi)$.

iii) Suppose M^{pq} is a Witten space with fundamental group isomorphic either to $\mathbb{Z}/2$ or $\mathbb{Z}/4$. In this case the formula for the σ-invariant associated to the only element in $\pi_1(M^{pq})$ of order 2 is given by

$$\frac{1}{4}\mathcal{L}(\mathbb{C}P^2 \times \mathbb{C}P^1)\mathcal{L}^{-1}(\mathcal{N}(-1))e(\mathcal{N}(-1))\left[\mathbb{C}P^2 \times \mathbb{C}P^1\right], \tag{2.14}$$

where $\mathcal{N}(-1)$ denotes the normal bundle of $\mathbb{C}P^2 \times \mathbb{C}P^1$ in D^{pq}. It's clear that $e(\mathcal{N}(-1))$ is $\bar{p}x + \bar{q}y$ Since $\mathcal{N}(-1)$ is a 2-dimensional real orientable vector bundle we only have to know what $p_1(\mathcal{N}(-1))$ is:

$$p_1(\mathcal{N}(-1)) = e^2(\mathcal{N}(-1)) = \bar{p}x^2 - 2\bar{p}\bar{q}xy.$$

Let B_j denotes the j-th Bernoulli number. Applying the Hirzebruch formalism to the formal power series which is given by

$$\begin{aligned} \frac{\tanh\frac{x}{2}}{\frac{x}{2}} &= \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)}{(2n)!} B_{2n} \left(\frac{x}{2}\right)^{2n-2} \\ &= 1 - \frac{1}{12}x^2 + \frac{1}{120}x^4 \cdots \end{aligned}$$

shows that $\mathcal{L}^{-1}(\mathcal{N}(-1)) = 1 - \frac{1}{12}p_1(\mathcal{N}(-1))$. Plugging all this data into the formula (2.14) yields that σ-invariant for the element of order 2 is

$$\underbrace{\bar{q}\,\frac{1}{4}\left(1 + \frac{1}{12}\bar{p}^2\right)}_{=:h_{\bar{p}^2}}.$$

iv) Since $h_{\bar{p}^2}$ is not 0 for any $\bar{p}$ the proof of "1) $\Rightarrow$ 3)" goes analoguesly as the proof in ii). ■

Remark 2.7.10. *The last theorem resembles much the classification theorem of the lens spaces which one can find for example in [M.1, p. 406] or in [L, p. 45].*
For a proof of the classification of lens spaces which doesn't use the Reidemeister-Franz torsion we refer to [A-S.1].

From Corollary 2.2.5. and the last theorem we conclude

Corollary 2.7.11. *Let* $M^{pqr}, M^{p'q'r'} \in \mathcal{W}$. *Then* M^{pqr} *and* $M^{p'q'r'}$ *are diffeomorphic if and only if* $|p| = |p'|$ *and* $|q| = |q'|$.

3 On a family of homogeneous 5-manifolds with cyclic fundamental group

3.1 A definition and invariants

In this section we introduce another class of homogeneous spaces which in many ways resembles the class of Witten spaces. Let's denote by A the Lie group $SU(2)\times SU(2)\times U(1)$, by T^2 the torus $U(1)\times U(1)$ and let

$$\phi : T^2 \to A$$

be a Lie group homomorphism with finite kernel. We define $\frac{A}{\phi(T^2)}$ to be the left coset space A divided by the image of B under ϕ. If not otherwise stated we just consider $\frac{A}{\phi(T^2)}$ as a smooth manifold.

Notation 3.1.1. *We define $\mathcal{L}$ to be the set of all smooth 5-manifolds obtained in the way as described above.*

All the properties of these manifolds which we'll mention in this chapter are without proof since the methods of the proofs can be copied from the corresponding results in the last chapter. We will just refer to the analogues statements in chapter two.
We can parametrise $\mathcal{L}$ by the set of coprime triples: Let p, q, r be coprime integers and $\boldsymbol{a},\boldsymbol{b} \in \mathbb{Z}^3$ be linearly independent elements of $\ker((p,q,r) : \mathbb{Z}^3 \to \mathbb{Z})$ then we define N^{pqr} to be $\frac{A}{\phi(T^2)}$, where ϕ is the homomorphism from T^2 to A which is induced by $(\boldsymbol{a},\boldsymbol{b})$ in the sense of Definition 2.1.5. Thus

$$\mathcal{L} = \{N^{pqr} | p, q, r \in \mathbb{Z} \text{ being coprime}\}.$$

Proposition 3.1.2. *Let $N^{pqr}, N^{p'q'r'} \in \mathcal{L}$.*

i) Then N^{pqr} is diffeomorphic to the orbit space of a smooth and free $\mathbb{Z}/gcd(p,q)$-action on $N^{\frac{p}{gcd(p,q)}\frac{q}{gcd(p,q)}0}$.
ii) If we consider N^{pqr} and $N^{p'q'r'}$ as homogeneous spaces with the A-action coming from the construction, then $N^{pqr} \sim_A N^{p'q'r'}$ (in the sense of Definition 2.3.1.) if and only if $|p| = |p'|, |q| = |q'|, |r| = |r'|$.

Proof. For i) see Proposition 2.2.4. and for ii) see Proposition 2.3.2. ∎

Corollary 3.1.3. *i) The fundamental group of N^{pqr} is isomorphic to $\mathbb{Z}/gcd(p,q)$ and each finite cyclic group is realized as the fundamental group of a manifold in $\mathcal{L}$.*

ii) Let p,q,r and $r' \in \mathbb{Z}$ s.t. the triples p,q,r and p,q,r' are coprime. Then N^{pqr} and $N^{pqr'}$ are diffeomorphic.

Proof. See Corollary 2.2.5. ■

From now on we denote N^{pq1} by N^{pq}.

Proposition 3.1.4. *i) Let $N^{pq} \in \mathcal{L}$. Then N^{pq} is diffeomorphic to the total space of the principal $U(1)$-bundle over $\mathbb{C}P^1 \times \mathbb{C}P^1$ which is given by the first Chern class*

$$c_1(S^1 \to N^{pq} \to \mathbb{C}P^1 \times \mathbb{C}P^1) = px + qy \in H^2(N^{pq};\mathbb{Z}),$$

where $x, y \in H^2(\mathbb{C}P^1 \times \mathbb{C}P^1;\mathbb{Z})$ are the standard generators.

ii) Homotopically the fundamental group of N^{pq} operates trivially on $\widetilde{N}^{pq}$ which is diffeomorphic to $N^{\frac{p}{gcd(p,q)}\frac{q}{gcd(p,q)}}$.

Proof. See Proposition 2.4.1. and Corollary 2.4.3. ■

Remark 3.1.5. *The following follows immediately from the definition of the manifolds in $\mathcal{L}$.*

- $N^{pq} \cong N^{qp}$.
- $N^{01} \cong S^2 \times S^3$.
- $N^{0q>0} \cong S^2 \times L(q,1,1)$*, where* $L(q,1,1)$ *is the standard 3-dimensional lens space with fundamental group isomorphic to* $\mathbb{Z}/q$.

Proposition 3.1.6. *Let $N^{pq} \in \mathcal{L}$ then*

- $H^1(N^{pq};\mathbb{Z}) \cong 0$.
- $H^2(N^{pq};\mathbb{Z}) \cong \mathbb{Z}/gcd(p,q) \oplus \mathbb{Z}$.
- $H^3(N^{pq};\mathbb{Z}) \cong \mathbb{Z}$.
- $H^4(N^{pq};\mathbb{Z}) \cong \mathbb{Z}/gcd(p,q)$.

Proof. Apply the Gysin sequence as in the proof of Proposition 2.5.2. ■

Lemma 3.1.7. *All manifolds in $\mathcal{L}$ are string manifolds.*

Proof. This follows from the fact that the tangent bundle of S^2 is stably trivial and from the considerations we have done on p. 35. ■

Let L_r^∞ denote the infinite dimensional lens space with $\pi_1(L_r^\infty) \cong \mathbb{Z}/r$.

Proposition 3.1.8. *Let $N^{pq} \in \mathcal{L}$ then*

$$(\underbrace{L^{\infty}_{gcd(p,q)} \times \mathbb{C}P^{\infty}}_{=:B_{gcd(p,q)}} \times BSpin, \xi)$$

is the normal 2-type of N^{pq}, where $\xi : B_{gcd(p,q)} \times BSpin \to BO$ is the fibration which we explained in section 2.5.4.

Proof. See Proposition 2.5.8. ■

3.2 A bordism classification of normal 2-smoothings

Let $N^{ab} \in \mathcal{L}$. From now on we identify N^{ab} with the total space of the fibre bundle which we have given in Proposition 3.1.4.

Let r be an odd natural number and let $N^{pq} \in \mathcal{L}$ be oriented. A normal 2-smoothing $f \times \nu^{pq}_{sp} : N^{pq} \to L^{\infty}_r \times \mathbb{C}P^{\infty} \times BSpin$ represents an element in $\Omega_5(L^{\infty}_r \times \mathbb{C}P^{\infty} \times BSpin, \xi)$. For a detailed description of this (generalized) bordism group we refer to [St, Ch. 2].

Definition/Lemma 3.2.1. *Let L be a n-dimensional smooth closed spin manifold and $h \times \nu^{L}_{sp} : L \to P \times BSpin$ a map, where $\nu^{L}_{sp} : L \to BSpin$ is the classifying map of a spin bundle over L and P is a CW-complex. Then $(L, h \times \nu^{L}_{sp})$ represents the zero-element in $\Omega_n(P \times BSpin, \xi)$ if and only if (L, h) represents the zero-element in $\Omega^{Spin}_n(P)$, where N is equipped with the chosen spin structure. Let*

$$(P_2(L) \times BSpin, \xi)$$

be the normal 2-type of L, where $P_2(L)$ denotes the second level of its Postnikov decomposition and ξ is the fibration which we have described above.
If $h \times \nu^{L}_{sp} : L \to P_2(L) \times BSpin$ is a normal 2-smoothing, then we call h a ***2-smoothing*** *of L.*

Proposition 3.2.2. *Let $r \in \mathbb{N}$ s.t. $gcd(r, 6) = 1$ and N, N' be smooth oriented closed 5-manifolds equipped with their unique spin structures. Furthermore let $f : N \to L^{\infty}_r \times \mathbb{C}P^{\infty}$ and $f' : N' \to L^{\infty}_r \times \mathbb{C}P^{\infty}$ be maps. The bordism group $\Omega^{Spin}_5(L^{\infty}_r \times \mathbb{C}P^{\infty})$ is isomorphic to $(\mathbb{Z}/r)^4$ and (N, f), (N', f') represent the same element in $\Omega^{Spin}_5(L^{\infty}_r \times \mathbb{C}P^{\infty})$ if and only if*

$$\begin{aligned} \langle \rho_r(p_1(N)) f^*(v_1), [N]_{\mathbb{Z}/r} \rangle &\equiv \langle \rho_r(p_1(N')) f'^*(v_1), [N']_{\mathbb{Z}/r} \rangle \; mod \; r, \\ f^*([N]) &= f'^*([N']), \end{aligned}$$

where v_1 is a generator of $H^1(L^{\infty}_r \times \mathbb{C}P^{\infty}; \mathbb{Z}/r)$, ρ_r denotes the mod-r-reduction in cohomology.

Proof. We know that

$$H_j(L_r^\infty;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & j=0 \\ \mathbb{Z}/r, & j \text{ is odd} \\ 0, & j \text{ else.} \end{cases}$$

And this implies that

$$H_k(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k \text{ is even} \\ (\mathbb{Z}/r)^j, & 2j-1=k. \end{cases}$$

The entries of the E_2-term of the AHSS for $\Omega_{a+b}^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ are $H_a(L_r^\infty \times \mathbb{C}P^\infty; \Omega_b^{Spin}(pt.))$ and since $r \equiv 1 \bmod 2$ it looks for $a+b \leq 6$ as follows:

b								
$\vdots$								
6	0							
5	0	0						
4	$\mathbb{Z}$	$\mathbb{Z}/r$	$\mathbb{Z}$					
3	0	0	0	0				
2	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$			
1	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0		
0	$\mathbb{Z}$	$\mathbb{Z}/r$	$\mathbb{Z}$	$(\mathbb{Z}/r)^2$	$\mathbb{Z}$	$(\mathbb{Z}/r)^3$	$\mathbb{Z}$	
	0	1	2	3	4	5	6	$\dots a$.

What are the ∞-terms in the fifth diagonal, i.e. what is $E_{a,b}^\infty(L_r^\infty \times \mathbb{C}P^\infty)$ for $a+b=5$? There is an exterior product

$$m_r : E_{a,b}^r(L_r^\infty) \otimes E_{s,t}^r(\mathbb{C}P^\infty) \to E_{a+s,b+t}^\infty(L_r^\infty \times \mathbb{C}P^\infty).$$

And the differentials d_r behave as derivations w.r.t. these exterior products, i.e.

$$d_r(m_r(x \otimes y)) = m_r(d_r x \otimes y) + (-1)^{|x|} m_r(x \otimes d_r y). \tag{3.1}$$

From [T, p. 7] we know what the differentials d_2 in $E_{a,b}^2(L_r^\infty)$ resp. $E_{a,b}^2(\mathbb{C}P^\infty)$ from the first to the second row and from the second to the third row are. In the first case they are just the dual of the Steenrod square $Sq^2 : H^*(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}/2) \to H^{*+2}(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}/2)$ precomposed with the reduction map:

$$\text{red}_2 : H_*(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}) \to H_*(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}/2).$$

In the second case they are the dual of Sq^2. The Steenrod square for lens spaces and projective spaces is given by:

$$Sq^2(\alpha^n) = \binom{n}{2} \alpha^{n+2} \text{ for } \alpha \in H^1(\cdot;\mathbb{Z}/2) \text{ (see [Ha, p. 490]).}$$

The relevant part of $E^5_{ab}(L^\infty_r \times \mathbb{C}P^\infty)$ for computing the ∞-terms in the range $a+b = 1, 3, 5$ is given here:

5	0						
4		$\mathbb{Z}/r$					
3	0		0				
2		0		0			
1	0		0		0		
0		$\mathbb{Z}/r$		$(\mathbb{Z}/r)^2$		$(\mathbb{Z}/r)^3$	$\mathbb{Z}$
	0	1	2	3	4	5	6 .

From the facts that $d_5 : E^5_{60}(L^\infty_r) \to E^5_{14}(L^\infty_r)$ and $d_5 : E^5_{60}(\mathbb{C}P^\infty) \to E^5_{14}(\mathbb{C}P^\infty)$ are trivial it follows from the Leibniz rule (3.1) that

$$d_5 : E^5_{60}(L^\infty_r \times \mathbb{C}P^\infty) \to E^5_{14}(L^\infty_r \times \mathbb{C}P^\infty)$$

also has to be trivial. Thus for $a+b = 1, 3, 5$ the ∞-term of the AHSS equals $E^5_{a,b}(L^\infty_r \times \mathbb{C}P^\infty)$ and we realize that

$$\begin{array}{rcl} h_1 : \Omega_1^{Spin}(L^\infty_r \times \mathbb{C}P^\infty) & \to & H_1(L^\infty_r \times \mathbb{C}P^\infty; \mathbb{Z}), \\ [(S,g)] & \mapsto & g_*([S]) \end{array}$$

is an isomorphism.
Let K be a Kummer surface equipped with its ususal orientation. We know from [M.2] that K generates $\Omega_4^{Spin}(pt.)$. The construction of the AHSS and its infinity term imply the following extension problem:

$$0 \to \Omega_1^{Spin}(L^\infty_r \times \mathbb{C}P^\infty) \overset{\mu_K}{\to} \Omega_5^{Spin}(L^\infty_r \times \mathbb{C}P^\infty) \overset{h_5}{\to} H_5(L^\infty_r \times \mathbb{C}P^\infty; \mathbb{Z}) \to 0, \quad (3.2)$$

where

$$\begin{array}{rcl} \mu_K : \Omega_1^{Spin}(L^\infty_r \times \mathbb{C}P^\infty) & \to & \Omega_5^{Spin}(L^\infty_r \times \mathbb{C}P^\infty), \\ [(S,g)] & \mapsto & [(K \times S, g \circ \mathrm{pr}_2)], \end{array}$$

pr_2 is the projection onto the second factor and

$$\begin{array}{rcl} h_5 : \Omega_5^{Spin}(L^\infty_r \times \mathbb{C}P^\infty) & \to & H_5(L^\infty_r \times \mathbb{C}P^\infty; \mathbb{Z}), \\ [(N,f)] & \mapsto & f_*([N]). \end{array}$$

Let $S^1 (\subset \mathbb{C})$ be equipped with the standard orientation and $i : S^1 \to L^\infty_r$ be the inclusion of S^1 as the 1-skeleton of L^∞_r. The fact that h_1 is an isomorphism implies that (S^1, i) represents a generator of $\Omega_1^{Spin}(L^\infty_r \times \mathbb{C}P^\infty)$.

Constructing a splitting

First we define a homomorphism n_1 from $\Omega_5^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ to $\mathbb{Z}/r$ s.t. $n_1 \circ \mu_K([S^1, i])$ is a unit in $\mathbb{Z}/r$:

Let v_1 be a generator of $H^1(L_r^\infty \times \mathbb{C}P^\infty; \mathbb{Z}/r)$ then

$$\begin{aligned} n_1 : \Omega_5^{Spin}(L_r^\infty \times \mathbb{C}P^\infty) &\rightarrow \mathbb{Z}/r, \\ [(N, f)] &\mapsto \left\langle \rho_r(p_1(N)) f^*(v_1), [N]_{\mathbb{Z}/r} \right\rangle, \end{aligned}$$

where ρ_r is the mod-r-reduction in cohomology and $p_1(N)$ the first Pontrijagin class of the tangent bundle of N. We assert that the map n_1 is a well defined homomorphism:

Assume (N, f) bounds (W, F), where W is oriented and let $i : N \hookrightarrow W$ be the inclusion. Then it's enough to show that $n_1(N, f) = 0$. We know that $[N] = \partial[W, N]$, where ∂ is the boundary homomorphism in the homology long exact sequence and $[W, N] \in H_6(W, N; \mathbb{Z})$ is the relative fundamental class of the pair (W, N). Furthermore there is the following property of the Kronecker product: Let $x \in H^5(M; \mathbb{Z}/r)$ then

$$\left\langle x, [M]_{\mathbb{Z}/r} \right\rangle = \left\langle x, \partial[W, M]_{\mathbb{Z}/r} \right\rangle = \langle \delta x, [W, M] \rangle,$$

where δ is the coboundary map. In order to prove that $n_1(N, f) = 0$ it's enough to show that $p_1(N) f^*(v_1)$ lies in the kernel of δ or equivalently in the image of i^* which is the case: Clearly

$$f^*(v_1) = i^*(F^*(v_1))$$

and since $\tau_N \oplus \mathbb{R} \cong i^*(\tau_W)$

$$p_1(N) = i^*(p_1(W)).$$

We claim that $n_1 \circ \mu_K([S^1, i])$ lies in $(\mathbb{Z}/r)^*$:

The Künneth theorem implies

$$\begin{aligned} n_1 \circ \mu_K([S^1, i]) &= \left\langle \rho_r(p_1(K \times S^1))(i \circ \mathrm{pr}_2)^*(v_1), [K \times S^1]_{\mathbb{Z}/r} \right\rangle \\ &= \left\langle \rho_r(p_1(K)), [K]_{\mathbb{Z}/r} \right\rangle \left\langle i^*(v_1), [S^1]_{\mathbb{Z}/r} \right\rangle \end{aligned}$$

It's clear that $\left\langle i^*(v_1), [S^1]_{\mathbb{Z}/r} \right\rangle$ is a generator of $\mathbb{Z}/r$.

But what's $\left\langle \rho_r(p_1(K)), [K]_{\mathbb{Z}/r} \right\rangle$?

By the Hirzebruch signature theorem it's known that

$$\left\langle \frac{p_1}{3}(K), [K] \right\rangle = \mathrm{sign}(K) = -16.$$

Thus $\left\langle \rho_r(p_1(K)), [K]_{\mathbb{Z}/r} \right\rangle \equiv -48 \bmod r$ is a generator of $\mathbb{Z}/r$ if and only if $\gcd(r, 48) = 1$. But $\gcd(r, 48) = 1$ if and only if $\gcd(r, 6) = 1$ which is the case by assumption.

Thus $n_1 \circ \mu_K([S^1, i]) \in (\mathbb{Z}/r)^*$ and we can compose n_1 with an appropriate isomorphism α from $\mathbb{Z}/r$ to $\Omega_5^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ s.t. $\alpha \circ n_1$ is a splitting of the short exact sequence (3.2).

Thus $\Omega_5^{Spin}(L_r^\infty \times \mathbb{C}P^\infty) \cong (\mathbb{Z}/r)^4$ and (N^{pq}, f), $(N^{p'q'}, f')$ represent the same element in $\Omega_5^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ if and only if

$$n_1(N^{pq}, f) = n_1(N^{p'q'}, f') \text{ and } h_5(N^{pq}, f) = h_5(N^{p'q'}, f').$$

■

An orientation convention

We know that N^{pq} is orientable. We choose the orientation on N^{pq} in the following way:

First we orient the fibre $U(1)$ s.t. it is compatible with the orientation of the corresponding complex line bundle and then we orient the base $\mathbb{C}P^1 \times \mathbb{C}P^1$ by the standard orientation.

A parametrisation of 2-smoothings

Let $N^{pq} \in \mathcal{L}$ with $\pi_1(N^{pq}) \cong \mathbb{Z}/r$, $f : N^{pq} \to L_r^\infty \times \mathbb{C}P^\infty$ be a 2-smoothing and Π denotes the projection map of the corresponding fibre bundle. Furthermore we denote the standard generators of $H^2(\mathbb{C}P^1 \times \mathbb{C}P^1; \mathbb{Z})$ by x, y, i.e. $xy \in H^4(\mathbb{C}P^1 \times \mathbb{C}P^1; \mathbb{Z})$ is the Kronecker dual of the (standard) fundamental class of the base. We regard $U(1) \subset \mathbb{C}$ as the set of complex numbers with norm 1 and we orient $U(1)$ anticlockwise. We denote the mod-r-reduction of the chosen fundamental class of $U(1)$ by $[U(1)]_{\mathbb{Z}/r}$. Let $i : U(1) \hookrightarrow N^{pq}$ be the inclusion of $U(1)$ as the fibre which preserves the chosen orientation of the fibre (see above). Let further $m, n \in \mathbb{Z}$ s.t. $m\frac{q}{r} + n\frac{p}{r} = 1$ then $H^2(N^{pq}; \mathbb{Z}) \cong \mathbb{Z}/r \oplus \mathbb{Z}$, where $\frac{p}{r}\Pi^*(x) + \frac{q}{r}\Pi^*(y)$ is a generator of the torsion part and $m\Pi^*(x) - n\Pi^*(y)$ is a generator of a $\mathbb{Z}$-summand (compare with the proof of Prop. 3.1.6. resp. Prop. 2.5.2.). By a we denote the generator of $H^1(N^{pq}; \mathbb{Z}/r)$ with the property that

$$\langle i^*(a), [U(1)]_{\mathbb{Z}/r} \rangle = 1.$$

Let v_1, z be the standard generators of $H^1(L_r^\infty; \mathbb{Z}/r)$, $H^2(\mathbb{C}P^\infty; \mathbb{Z})$ resp. and $f := f_1 \times f_2 : N^{pq} \to L_r^\infty \times \mathbb{C}P^\infty$ be a 2-smoothing. The map f is up to homotopy uniquely determined by

$$f^*(v_1) = sa,$$

for $s \in (\mathbb{Z}/r)^*$ a unit in $\mathbb{Z}/r$ and

$$f^*(z) = \underbrace{\epsilon(f)}_{\in\{\pm 1\}} (m\Pi^*(x) - n\Pi^*(y)) + \underbrace{k(f, m, n)}_{\in \mathbb{Z}/r}(\frac{p}{r}\Pi^*(x) + \frac{q}{r}\Pi^*(y)).$$

Lemma 3.2.3. *Let $N^{pq} \in \mathcal{L}$ be oriented with $\pi_1(N^{pq}) \cong \mathbb{Z}/r$. Fixing a choice of $m, n \in \mathbb{Z}$ s.t. $m\frac{q}{r} + n\frac{p}{r} = 1$ then there is a 1-1 correspondence between the set S of homotopy classes of 2-smoothings of N^{pq} and the set of triples $\{(\epsilon, s, k) | \epsilon \in \{\pm 1\}, s \in (\mathbb{Z}/r)^*, k \in \mathbb{Z}/r\} =: T$, where the bijection is given as follows:*

$$\begin{aligned} \mathcal{C}: \quad S &\rightarrow \quad T, \\ [f] &\mapsto \quad (\epsilon(f), s(f), k(f, m, n)). \end{aligned}$$

Proof. It's clear that $\mathcal{C}$ is injective. We claim that $\mathcal{C}$ is also surjective, i.e. for fixed $m, n \in \mathbb{Z}$ as above any triple $(\epsilon, s, k) \in T$ has a preimage under $\mathcal{C}$. We write f as $f_1 \times f_2$ and the homotopy class $[f_1]$ of f_1 can be seen as an element in $H^1(N^{pq}; \mathbb{Z}/r) \cong \mathbb{Z}/r$. Any automorphism of $\mathbb{Z}/r$ is given by a unit s of $\mathbb{Z}/r$, $(1 \mapsto s)$. Furthermore there is a 1-1 correspondence beween automorphisms of $\pi_1(L_r^\infty)(\cong \mathbb{Z}/r)$ and homotopy classes of self-maps of L_r^∞. Thus the homotopy classes of self-maps of L_r^∞ correspond bijectively to automorphisms of $H_1(L_r^\infty; \mathbb{Z})$. Now let g be a self-map of L_r^∞ then "naturality" of the UCT implies that

$$\begin{aligned} g^* : H^1(L_r^\infty; \mathbb{Z}) \cong \mathrm{Hom}(H_1(L_r^\infty; \mathbb{Z}), \mathbb{Z}/r) \quad &\rightarrow \quad \mathrm{Hom}(H_1(L_r^\infty; \mathbb{Z}), \mathbb{Z}/r), \\ h \quad &\mapsto \quad g^*(h) = h \circ g_*. \end{aligned}$$

This means that $g^* : H^1(L_r^\infty; \mathbb{Z}) \rightarrow H^1(L_r^\infty; \mathbb{Z})$ is an automorphism if and only if $g_* : H_1(L_r^\infty; \mathbb{Z}) \rightarrow H_1(L_r^\infty; \mathbb{Z})$ is an automorphism. Thus the set of self-maps of L_r^∞ that induce automorphism on $\pi_1(L_r^\infty)$ is in 1-1 correspondence to $(\mathbb{Z}/r)^*$ which itself corresponds bijectively to automorphisms of $H^1(L_r^\infty; \mathbb{Z})$. We conclude via Whitehead's theorem that the set of self-maps of L_r^∞ that induce automorphism on $\pi_1(L_r^\infty)$ corresponds bijectively to the homotopy classes of self-homotopy equivalences of L_r^∞. Hence by precomposing the f_1 in $f = f_1 \times f_2 : N^{pq} \rightarrow L_r^\infty \times \mathbb{C}P^\infty$ by a suitable self-homotopy equivalence one can realize any s in the above sense.

Now we show that the homotopy class of a map $h : N^{pq} \rightarrow \mathbb{C}P^\infty$ that realizes (ϵ, k) induces an isomorphism on π_2. Therefore we gather some facts:

a) Proposition 3.1.4. ii) justifies the application of the cohomology version of the Leray-Serre spectral sequence for the fibration $\widetilde{N}^{pq} \xrightarrow{pr} N^{pq} \rightarrow L_r^\infty$. We obtain the following:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_{20}^\infty & \longrightarrow & H^2(N^{pq}; \mathbb{Z}) & \xrightarrow{u} & E_{02}^\infty & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & H^2(L_r^\infty; \mathbb{Z}) & \longrightarrow & H^2(N^{pq}; \mathbb{Z}) & \xrightarrow{u} & H^2(\widetilde{N}^{pq}; \mathbb{Z}) & \longrightarrow & 0 \end{array},$$

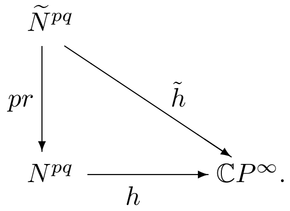
where $u = pr^*$ (see [McCl, Thm. 5.9.]). Hence $pr^* : H^2(N^{pq}; \mathbb{Z}) \rightarrow H^2(\widetilde{N}^{pq}; \mathbb{Z})$ is surjective with kernel isomorphic to $\mathbb{Z}/r$.

b) Again "naturality" of the UCT implies that the set of homotopy classes of maps from $\widetilde{N}^{pq}$ to $\mathbb{C}P^\infty$ that induce isomorphism on $H^2(\cdot;\mathbb{Z})$ equals the set of homotopy classes of maps that induce isomorphism on $H_2(\cdot;\mathbb{Z})$.

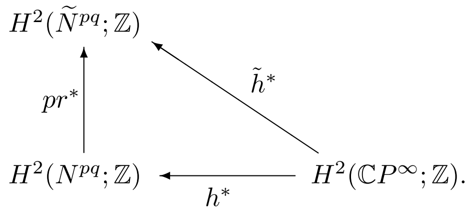
c) With the Hurewicz theorem one sees that a map between simply-connected CW-complexes that induces isomorphism on $H_2(\cdot;\mathbb{Z})$ also induces isomorphism on $\pi_2(\cdot)$.

With the facts a)-c) we finish the proof:

There exists the following commutative diagram:



Applying the $\mathbb{Z}$-cohomology functor $H^2(\cdot;\mathbb{Z})$, we get the following commutative diagram:



Since $H^2(\widetilde{N}^{pq};\mathbb{Z}) \cong \mathbb{Z}$, $H^2(N^{pq};\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/r$ and $H^2(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}$ a) implies that $\tilde{h}^*$ is an isomorphism.

By b) h induces an isomorphism on $H_2(\cdot;\mathbb{Z})$ and thus by c) h induces an isomorphism on $\pi_2(\cdot)$. This proves that $\mathcal{C}$ is surjective. ∎

Proposition 3.2.4. *Let $r \in \mathbb{N}$ s.t. $gcd(r,6) = 1$ and N^{pq}, $N^{p'q'} \in \mathcal{L}$ be oriented with $\pi_1(N^{pq}) \cong \pi_1(N^{p'q'}) \cong \mathbb{Z}/r$ and (m,n), $(m',n') \in \mathbb{Z}^2$ s.t. $m\frac{q}{r} + n\frac{p}{r} = 1 =$*

$m'\frac{q'}{r} + n'\frac{p'}{r}$. There exist normal 2-smoothings $g : N^{pq} \to L_r^\infty \times \mathbb{C}P^\infty \times BSpin$ and $g' : N^{p'q'} \to L_r^\infty \times \mathbb{C}P^\infty \times BSpin$ s.t. (N^{pq}, g) and $(N^{p'q'}, g')$ represent the same element in $\Omega_5(L_r^\infty \times \mathbb{C}P^\infty \times BSpin, \xi)$ if and only if there exist triples (s, ϵ, k) and (s', ϵ', k') in T s.t.

$$(1) \qquad s(\epsilon m + k\frac{p}{r})(\epsilon n - k\frac{q}{r}) \equiv s'(\epsilon' m' + k'\frac{p'}{r})(\epsilon' n' - k'\frac{q'}{r}) mod\ r,$$

$$(2) \qquad s^2(\frac{q}{r}(\epsilon m + k\frac{p}{r}) - \frac{p}{r}(\epsilon n - k\frac{q}{r})) \equiv s'^2(\frac{q'}{r}(\epsilon' m' + k'\frac{p'}{r}) - \frac{p'}{r}(\epsilon n' - k'\frac{q'}{r}))mod\ r,$$

$$(3) \qquad s^3\frac{pq}{r^2} \equiv s'^3\frac{p'q'}{r^2} mod\ r.$$

Proof. By Lemma 3.2.1. we know that $(N^{pq}, g = f \times \nu_{sp})$ and $(N^{p'q'}, g' = f' \times \nu_{sp'})$ represent the same element in $\Omega_5(L_r^\infty \times \mathbb{C}P^\infty \times BSpin, \xi)$ if and only if (N^{pq}, f) and $(N^{p'q'}, f')$ represent the same element in $\Omega_5^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$. Thus we are going to classify 2-smoothings up to bordism.

Proposition 3.2.2. tells us that (N^{pq}, f) and $(N^{p'q'}, f')$ represent the same element in $\Omega_5^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ if and only if

$$\left\langle \rho_r(p_1(N^{pq}))f^*(v_1), [N^{pq}]_{\mathbb{Z}/r} \right\rangle \equiv \left\langle \rho_r(p_1(N^{p'q'}))f'^*(v_1), [N^{p'q'}]_{\mathbb{Z}/r} \right\rangle \text{ mod } r$$
$$\text{and}$$
$$f^*([N^{pq}]) = f'^*([N^{p'q'}]).$$

But since $p_1(N^{ab})$ is zero for each $N^{ab} \in \mathcal{L}$ (Lemma 3.1.7.) we conclude that (N^{pq}, f) and $(N^{p'q'}, f')$ represent the same element in $\Omega_5^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ if and only if

$$f^*([N^{pq}]) = f'^*([N^{p'q'}]).$$

We observe that

$$H_5(L_r^\infty \times \mathbb{C}P^\infty; \mathbb{Z}) \cong (\mathbb{Z}/r)^3 \cong H_5(L_r^\infty \times \mathbb{C}P^\infty; \mathbb{Z}/r)$$

and

$$H^5(L_r^\infty \times \mathbb{C}P^\infty; \mathbb{Z}/r) \cong (\mathbb{Z}/r)^3.$$

A basis of $H^5(L_r^\infty \times \mathbb{C}P^\infty; \mathbb{Z}/r)$ is given by

$$v_1 z_r^2, \quad v_1(\beta_r(v_1))z_r, \quad v_1(\beta_r(v_1))^2,$$

where v_1 is a generator of $H^1(L_r^\infty \times \mathbb{C}P^\infty; \mathbb{Z}/r)$, z_r is a generator of $H^2(L_r^\infty \times \mathbb{C}P^\infty; \mathbb{Z}/r)$ which comes from the mod-r-reduction of the standard generator z of $H^2(\mathbb{C}P^\infty; \mathbb{Z})$.

Thus

$$\begin{aligned} h_5(N^{pq}, f) = h_5(N^{p'q'}, f') \quad &\Leftrightarrow \quad f_*([N^{pq}]) = f'_*([N^{p'q'}]), \\ &\Leftrightarrow \quad f_*([N^{pq}])_{\mathbb{Z}/r} = f'_*([N^{p'q'}])_{\mathbb{Z}/r}, \\ &\Leftrightarrow \quad \left\langle b, f_*[N^{pq}]_{\mathbb{Z}/r} \right\rangle = \left\langle b, f'_*[N^{p'q'}]_{\mathbb{Z}/r} \right\rangle, \\ &\qquad \forall b \in H^5(L_r^\infty \times \mathbb{C}P^\infty; \mathbb{Z}/r), \\ &\Leftrightarrow \quad \left\langle f^*(b), [N^{pq}]_{\mathbb{Z}/r} \right\rangle = \left\langle f'^*(b), [N^{p'q'}]_{\mathbb{Z}/r} \right\rangle, \\ &\qquad \forall b \in H^5(L_r^\infty \times \mathbb{C}P^\infty; \mathbb{Z}/r). \end{aligned}$$

But this is equivalent to the following equations:

$$\begin{aligned} (1') \quad & \left\langle f^*(v_1 z^2), [N^{pq}]_{\mathbb{Z}/r} \right\rangle = \left\langle f'^*(v_1 z^2), [N^{p'q'}]_{\mathbb{Z}/r} \right\rangle, \\ (2') \quad & \left\langle f^*(v_1(\beta_r(v_1))z), [N^{pq}]_{\mathbb{Z}/r} \right\rangle = \left\langle f'^*(v_1(\beta_r(v_1))z), [N^{p'q'}]_{\mathbb{Z}/r} \right\rangle, \\ (3') \quad & \left\langle f^*(v_1(\beta_r(v_1))^2), [N^{pq}]_{\mathbb{Z}/r} \right\rangle = \left\langle f'^*(v_1(\beta_r(v_1))^2), [N^{p'q'}]_{\mathbb{Z}/r} \right\rangle. \end{aligned}$$

Notation: By $x_a(y_a) \in H^2(\mathbb{C}P^1; \mathbb{Z}/a)$ we denote the element of $H^2(\mathbb{C}P^1 \times \mathbb{C}P^1; \mathbb{Z}/a)$ which is the mod-a-reduction of the standard generator of $H^2(\cdot; \mathbb{Z})$ of the first factor (second factor) of $\mathbb{C}P^1 \times \mathbb{C}P^1$.

We know that

$$f^*(\beta_r(v_1)) = \beta_r(f^*(v_1)) = s\beta_r(a).$$

Hence in order to compute the Kronecker products above we have to understand what $\beta_r(a)$ is in terms of $\Pi^*(x)$ and $\Pi^*(y)$, i.e.

$$\beta_r(a) = b_1 \Pi^*(x_r) + b_2 \Pi^*(y_r)$$

for some $b_1, b_2 \in \mathbb{Z}/r$ The cohomological structure of $L_r^\infty \times \mathbb{C}P^\infty$ implies that $\beta_r(a)$ lies in the image of ρ_r restricted to the torsion part of $H^2(N^{pq}; \mathbb{Z})$, i.e.

$$\beta_r(a) = t(\frac{p}{r}\Pi^*(x_r) + \frac{q}{r}\Pi^*(y_r)) \tag{3.3}$$

for some $t \in (\mathbb{Z}/r)^*$. We claim that modulo r b_1 equals $u_r \frac{p}{r}$ and b_2 equals $u_r \frac{q}{r}$ and thus $t = u_r$ for some (universal) $u_r \in (\mathbb{Z}/r)^*$.

Proof of the last claim. An idea to obtain information about the $\Pi^*(x_r)$-component of $\beta_r(a)$ is to analyze the "restricted bundles"

$$\begin{aligned} U(1) \xrightarrow{i} N^{pq}|_{\mathbb{C}P_1^1} \quad &\xrightarrow{\bar{\Pi}} \quad \mathbb{C}P^1, \\ U(1) \xrightarrow{j} N^{pq}|_{\mathbb{C}P_2^1} \quad &\xrightarrow{\bar{\Pi}} \quad \mathbb{C}P^1, \end{aligned}$$

where the first fibre bundle is the restriction of the fibre bundle associated to N^{pq} to the first $\mathbb{C}P^1$-factor. Its first Chern class is px hence $N^{pq}|_{\mathbb{C}P^1_1}$ is the familiar lens space $L^3(p;1,1) =: L^3_p$. The second bundle is the restriction of the fibre bundle associated to N^{pq} to the second $\mathbb{C}P^1$-factor thus its first Chern class is qx and hence $N^{pq}|_{\mathbb{C}P^1_2}$ is the lens space $L^3(q;1,1) =: L^3_q$. Let $a_p \in H^1(L^3_p;\mathbb{Z}/p)$ s.t. $\langle i^*(a_p), [U(1)]_{\mathbb{Z}/p}\rangle = 1$ and $a_q \in H^1(L^3_q;\mathbb{Z}/q)$ s.t. $\langle j^*(a_q), [U(1)]_{\mathbb{Z}/q}\rangle = 1$. Let i_p and i_q be the obvious inclusion of L^3_p resp. L^3_q in N^{pq} then it's clear that $i_p^*(a) = \bar{a}_p$ and $i_q^*(a) = \bar{a}_q$, where $\bar{a}_p \in H^1(L^3_p;\mathbb{Z}/r)$ and $\bar{a}_q \in H^1(L^3_q;\mathbb{Z}/r)$ are the images of a_p resp. a_q under the corresponding coefficient homomorphism.
On the other hand we have the following facts: $i_p^*(\Pi^*(x)) = \bar{\Pi}^*(x)$, $i_q^*(\Pi^*(x)) = \tilde{\Pi}^*(x)$ and $i_p^*(\beta_r(a)) = \beta_r(i_p^*(a)) = \beta_r(\bar{a})$, $i_q^*(\beta_r(a)) = \beta_r(i_q^*(a)) = \beta_r(\tilde{a})$. Furthermore we conclude from the construction of the maps together with the long exact sequence in $\mathbb{Z}/r$-cohomology for the pairs (N^{pq}, L^3_p) and (N^{pq}, L^3_q) that $i_p^*(\Pi^*(x))$ and $i_q^*(\Pi^*(x))$ vanish. Summarizing the last considerations leads to the following:

$$\begin{aligned} i_p^*(\beta_r(a)) &= i_p^*(t(\frac{p}{r}\Pi^*(x_r) + \frac{q}{r}\Pi^*(y_r))) = t\frac{p}{r}\bar{\Pi}^*(x_r) = \beta_r(\bar{a}_p), \\ i_q^*(\beta_r(a)) &= i_q^*(t(\frac{p}{r}\Pi^*(x_r) + \frac{q}{r}\Pi^*(y_r))) = t\frac{p}{r}\tilde{\Pi}^*(y_r) = \beta_r(\bar{a}_q). \end{aligned}$$

Thus if we knew $\beta_r(\bar{a}_p)$ and $\beta_r(\bar{a}_q)$ in terms of $\bar{\Pi}^*(x_r)$ resp. $\tilde{\Pi}^*(y_r)$, then we would know what $\beta_r(a)$ is.

Assume $\beta_p(a_p) = u_p\bar{\Pi}^*(x_p)$ for some $u_p \in (\mathbb{Z}/p)^*$. We compare the short exact sequences associated to β_r and β_p:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}/r & \longrightarrow & \mathbb{Z}/r^2 & \xrightarrow{\tilde{\pi}} & \mathbb{Z}/r & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \mathrm{red}_{p,r}} & & \uparrow{\scriptstyle \mathrm{red}_{p^2,r}} & & \uparrow{\scriptstyle \mathrm{red}_{p,r}} & & \\ 0 & \longrightarrow & \mathbb{Z}/p & \longrightarrow & \mathbb{Z}/p^2 & \xrightarrow{P} & \mathbb{Z}/p & \longrightarrow & 0, \end{array}$$

where $\mathrm{red}_{\cdot,\cdot}$ denotes the reduction homomorphism. The maps in the squares above commute, hence we get the following commutative diagram:

$$\begin{array}{ccc} H^1(L^3(p);\mathbb{Z}/r) & \xrightarrow{\beta_r} & H^2(L^3(p);\mathbb{Z}/r) \\ \uparrow{\scriptstyle \rho_{p,r}} & & \uparrow{\scriptstyle \frac{p}{r}\rho_{p,r}} \\ H^1(L^3(p);\mathbb{Z}/p) & \xrightarrow{\beta_p} & H^2(L^3(p);\mathbb{Z}/p), \end{array}$$

where $\rho_{\cdot,\cdot}$ is the "change of coefficient-homomorphism", i.e

$$\beta_r \circ \rho_{p,r} = \frac{p}{r}\rho_{p,r} \circ \beta_p.$$

It's clear that $\rho_{p,r}(\tilde{a})$ is a cohomological fundamental class of $U(1)$ in $\mathbb{Z}/r$-cohomology. Thus $\beta_r(\bar{a}_p) = u_{p,r}\frac{p}{r}\tilde{\Pi}^*(x_r)$, where $u_{p,r} \in \mathbb{Z}/r$ is the mod-r-reduction of u_p. It's clear that $u_{p,r}$ is a unit in $\mathbb{Z}/r$.
If $\beta_q(\bar{a}_q) = u_q\tilde{\Pi}^*(x_q)$ for some $u_q \in (\mathbb{Z}/q)^*$, then in the same way we obtain:

$$\beta_r(\bar{a}_q) = u_{q,r}\frac{q}{r}\tilde{\Pi}^*(x_r),$$

where $u_{q,r}$ is the mod-r-reduction of u_q. By (3.3) the following has to be true:

$$\begin{pmatrix} u_{p,r} & 0 \\ 0 & u_{q,r} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}(\frac{p}{r}\Pi^*(x_r), \frac{q}{r}\Pi^*(y_r)) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}(\frac{p}{r}\Pi^*(x_r), \frac{q}{r}\Pi^*(y_r)).$$

This implies that $u_{p,r} = u_{q,r} =: u_r$ and thus $\beta_r(a) = u_r(\frac{p}{r}\Pi^*(x_r) + \frac{q}{r}\Pi^*(y_r))$.

By definition we have

$$\begin{aligned} f^*(v_1) &= sa, \\ f^*(\beta_r(v_1)) &= su_r(\frac{p}{r}\Pi^*(x) + \frac{q}{r}\Pi^*(y)), \\ f^*(z) &= \epsilon(m\Pi^*(x) - n\Pi^*(y)) + k(\frac{p}{r}\Pi^*(x) + \frac{q}{r}\Pi^*(y)). \end{aligned}$$

Thus

$$\begin{aligned} f^*(v_1 z^2) &= sa((\epsilon m + k\frac{p}{r})\Pi^*(x) - (\epsilon n - k\frac{q}{r})\Pi^*(y))^2 \\ &= -2s(\epsilon m + k\frac{p}{r})(\epsilon n - k\frac{q}{r})a\Pi^*(xy), \\ f^*(v_1(\beta_r v_1)z) &= u_r sa((\epsilon m + k\frac{p}{r})\Pi^*(x) - (\epsilon n - k\frac{q}{r})\Pi^*(y))(\frac{p}{r}\Pi^*(x) + \frac{q}{r}\Pi^*(y)) \\ &= u_r s((\epsilon m + k\frac{p}{r})\frac{q}{r} - (\epsilon n - k\frac{q}{r})\frac{p}{r})a\Pi^*(xy), \\ f^*(v_1(\beta_r v_1)^2) &= au_r^2 s^3(\frac{p}{r}\Pi^*(x) + \frac{q}{r}\Pi^*(y))^2 \\ &= \frac{u_r^2 s^3 pq}{r^2}a\Pi^*(xy). \end{aligned}$$

It's true that

$$\langle a\Pi^*(xy), [N^{pq}]_{\mathbb{Z}/r}\rangle = \langle a, \Pi^*(xy) \cap [N^{pq}]_{\mathbb{Z}/r}\rangle$$

and by the choice of the orientation of N^{pq} it follows from Propostion 2 in [G] and its proof that $\langle a\Pi^*(xy), [N^{pq}]_{\mathbb{Z}/r}\rangle = 1 \text{mod } r$. Hence since 2 and u_r are units of $\mathbb{Z}/r$ the equations (1')-(3') (p. 69) translate into the following congruences:

(1) $\quad s(\epsilon m + k\frac{p}{r})(\epsilon n - k\frac{q}{r}) \equiv s'(\epsilon' m' + k'\frac{p'}{r})(\epsilon' n' - k'\frac{q'}{r}) \text{mod } r,$

(2) $\quad s^2(\frac{q}{r}(\epsilon m + k\frac{p}{r}) - \frac{p}{r}(\epsilon n - k\frac{q}{r})) \equiv s'^2(\frac{q'}{r}(\epsilon' m' + k'\frac{p'}{r}) - \frac{p'}{r}(\epsilon n' - k'\frac{q'}{r}))\text{mod } r,$

(3) $\quad s^3\frac{pq}{r^2} \equiv s'^3\frac{p'q'}{r^2}\text{mod } r.$ ∎

3.3 A diffeomorphism classification: The simply-connected case

We have the following fundamental result by Stephan Smale concerning the differential topology of smooth simply-connected closed spin 5-manifolds.

Theorem 3.3.1. *([S]) There is a 1-1 correspondence between the set $\mathcal{D}^5$ of diffeomorphism classes of smooth simply-connected closed 5-manifolds with vanishing second Stiefel-Whitney class and the set Ab of isomorphism classes of finitely generated abelian groups. The correspondence is given by:*

$$\begin{array}{rcl} \phi : \mathcal{D}^5 & \rightarrow & Ab, \\ M & \mapsto & F \oplus \frac{1}{2}T, \end{array}$$

where $H_2(M;\mathbb{Z}) = F \oplus T$ is a direct sum decomposition of $H_2(M;\mathbb{Z})$ into a free part and a torsion part and $\frac{1}{2}T \oplus \frac{1}{2}T$ is a direct sum decomposition of T.

Corollary 3.3.2. *Let $N^{pq}, N^{p'q'} \in \mathcal{L}$ be simply-connected then N^{pq} and $N^{p'q'}$ are diffeomorphic.*

Proof. Since N^{pq} and $N^{p'q'}$ are simply-connected they fulfill the assumptions of Smale's theorem. From Propostion 3.1.6. we know that $H_2(N^{pq};\mathbb{Z}) \cong H_2(N^{p'q'};\mathbb{Z}) \cong \mathbb{Z}$ thus Theorem 3.3.1. implies that $N^{pq} \cong S^2 \times S^3 \cong N^{p'q'}$. ■

3.4 A diffeomorphism classification: The non-simply-connected case

From the first part of this chapter we know that if $\pi_1(N^{ab}) \cong \mathbb{Z}$, then $a = 0 = b$. Hence the classification of the N^{pq}'s with infinite cyclic fundamental group up to diffeomorphism (or homotopy) is trivial. In this section we concentrate on the diffeomorphism classification of the non-simply-connected manifolds in $\mathcal{L}$, where the order of the fundamental groups is coprime to 6.

In section 2.6 we gave a rather philosophical description of modified surgery. The purpose of this section is to introduce the main mathematical objects that enable us to formulate an abstract version of the classification theorem in the context of the 5-manifolds we are interested in. Then we classify the non-simply-connected manifolds in $\{N^{ab} \in \mathcal{L} | \gcd(|\pi_1(N^{ab})|, 6) = 1\}$.

3.4.1 The surgery obstruction

This section deals with the following question:

Let $m \in \mathbb{N}$ greater than or equal to 3. Does there exist an "algebraic measure" *which helps to decide whether a $2m$-dimensional B-bordism (W, F) between two normal $(m-1)$-smoothings $(N, f), (N', f')$ can be transformed relative boundary into a s-cobordism?*

A very detailed exposition of the treatment of this problem can be found in [Kr.1].

The Surgery Obstruction Groups

The basic ingredients of the definition of the even dimensional surgery obstruction groups are quadratic forms. Let $\mathbb{Z}[\pi]$ be a group ring, $\omega : \pi \to \mathbb{Z}/2$ a homomorphism and $^{-} : \mathbb{Z}[\pi] \to \mathbb{Z}[\pi]$ be the involution which sends $g \in \pi$ to $\omega(g)g^{-1} =: \bar{g}$.
From now on we denote $\mathbb{Z}[\pi]$ by Λ.

Definition 3.4.1. *(Form parameter) Let $\epsilon \in \{\pm 1\}$ be fixed. A subgroup $S_\epsilon \subset \Lambda$ is called a **form parameter** if it fulfills the following properties:*

- *If $a \in S_\epsilon$, then $a + \epsilon\bar{a} = 0$.*
- *If $a \in S_\epsilon$, then $ba\bar{b} \in S_\epsilon$, $\forall b \in \Lambda$.*
- *S_ϵ contains the subgroup $\{a - \epsilon\bar{a} | a \in \Lambda\}$.*

These properties ensure that for $[a] \in \frac{\Lambda}{S_\epsilon}$, $a + \epsilon\bar{a}$ is a well defined element in Λ and that $ba\bar{b} \in \frac{\Lambda}{S_\epsilon}$ is well defined for any $b \in \Lambda$. If $S_\epsilon = \{a - \epsilon\bar{a} | a \in \Lambda\}$ we denote $\frac{\Lambda}{S_\epsilon}$ by Q_ϵ.

Definition 3.4.2. *(Quadratic forms) Let $\epsilon \in \{\pm 1\}$ be fixed. An **ϵ-quadratic form** over (Λ, S_ϵ) consists of a triple (M, λ, μ), where*

- *M is a left Λ-module,*
- *$\lambda : M \times M \to \Lambda$ is an ϵ-hermitian form and*
- *$\mu : M \to \frac{\Lambda}{S_\epsilon}$ is a quadratic refinement of λ.*

That means λ and μ have to fulfill the following properties:

- *For $y \in M$ fixed the map $M \to \Lambda$, $x \mapsto \lambda(x, y)$ is a Λ-homomorphism.*
- *$\lambda(x, y) = \epsilon\overline{\lambda(y, x)}$ $\forall x, y \in M$.*
- *$\lambda(x, x) = \mu(x) + \epsilon\overline{\mu(x)} \in \Lambda$ $\forall x \in M$.*
- *$\mu(x + y) = \mu(x) + \mu(y) + [\lambda(x, y)]$ $\forall x, y \in M$.*
- *$\mu(ax) = a\mu(x)\bar{a}$ $\forall x \in M$ and $\forall a \in \Lambda$.*

For $S_\epsilon = \{a - \epsilon\bar{a} | a \in \Lambda\}$ such forms were introduced in [Wa, §5].

An important special case are the ϵ-hyperbolic forms:

$$H^r_\epsilon(\Lambda) := \underbrace{H_\epsilon(\Lambda) \oplus \cdots \oplus H_\epsilon(\Lambda)}_{r \text{ times}},$$

where $H_\epsilon(\Lambda) = \left(\Lambda \oplus \Lambda, \lambda = \begin{pmatrix} 0 & 1 \\ \epsilon & 0 \end{pmatrix}, \mu(e) = \mu(f) = 0\right)$
((e, f) is the canonical basis of $\Lambda \oplus \Lambda$).

Definition 3.4.3. *(M, λ, μ) is called*

- ***weakly based*** *if (M, λ, μ) is equipped with an equivalence class of bases, where two bases are equivalent if the matrix of the base change has vanishing Whitehead torsion in $Wh(\pi)$,*
- ***non-singular*** *if the adjoint of λ is an isomophism,*
- ***based*** *if it is weakly based, non-singular and the adjoint of λ is a simple isomorphism.*

We always assume that $H^r_\epsilon(\Lambda)$ is based by e_i, f_i.

Definition 3.4.4. *Two quadratic forms (M, λ, μ) and (M', λ', μ') are called **stably equivalent** if for some r and $s \in \mathbb{N}$, $(M, \lambda, \mu) \oplus H^r_\epsilon(\Lambda)$ is isomorphic to $(M', \lambda', \mu') \oplus H^s_\epsilon(\Lambda)$. If M and M' are weakly based we require that the isomorphism is simple.*

Definition/Lemma 3.4.5.

- *The s-**decorated** L-group $L^s_{2m}(\pi, \omega)$ is the set of stable equivalence classes of based non-singular $(-1)^m$-quadratic forms over Λ with group structure given by orthogonal sum.*
- *Let $S_{(-1)^m}$ be a form parameter then we define $L^s_{2m}(\pi, \omega, S_{(-1)^m})$ to be the set of stable equivalence classes of based non-singular $(-1)^m$-quadratic forms over $(\Lambda, S_{(-1)^m})$.*
- *We denote the set of stable equivalence classes of weakly based non-singular $(-1)^m$-quadratic forms over $(\Lambda, S_{(-1)^m})$ by $L^{s,\tau}_{2m}(\pi, \omega, S_{(-1)^m})$. If $S_{(-1)^m} = \{a - (-1)^m a | a \in \Lambda\}$, then we denote $L^{s,\tau}_{2m}(\pi, \omega, S_{(-1)^m})$ by $L^{s,\tau}_{2m}(\pi, \omega)$. Orthogonal sum defines a group structure on $L^{s,\tau}_{2m}(\pi, \omega, S_{(-1)^m})$.*

If $\omega \equiv 1$, then we don't mention the ω in the notation of the various L-groups.

Proof. To show that $L^{s,\tau}_{2m}(\pi, \omega, S_{(-1)^m})$ is a group it is enough to show that $L^{s,\tau}_{2m}(\pi, \omega)$ is a group. There is the following exact sequence:

$$0 \to L^s_{2m}(\pi, \omega) \to L^{s,\tau}_{2m}(\pi, \omega) \xrightarrow{\tau} Wh(\pi),$$

where $\tau(M, \lambda, \mu)$ equals the torsion of the adjoint w.r.t. to the given equivalence class of basis. If $\mathrm{im}(\tau) \subset Wh(\pi)$ is a group, then $L^{s,\tau}_{2m}(\pi, \omega)$ is a group. This is shown in [Kr.1, p. 36]. ■

The surgery obstruction

Let $m \in \mathbb{N}$ greater than or equal to 3 and (W, F) be a $(2m)$-dimensional B-bordism between two normal 2-smoothings (N_0, f_0) and (N_1, f_1) s.t. F is a m-equivalence (see [Kr, Prop. 4]). Let's further denote $\ker(F_* : \pi_m(W) \to \pi_m(B))$ by $K\pi_m(W)$ and the image of $\ker(f_{i*} : \pi_m(N_i) \to \pi_m(B))$ under $i_* : \pi_m(N_i) \to \pi_m(W)$ by $\mathrm{im}K\pi_m(N_i)$, where $i = 0, 1$. On $K\pi_m(W)$ there we have the intersection form λ. In [Kr.1, p.54] it is shown that

- $\mathrm{im}K\pi_m(N_0) = \mathrm{im}K\pi_m(N_1)$,
- the radical of λ equals $\mathrm{im}K\pi_m(N_0)$.

Thus the induced quadratic form

$$\bar{\lambda} : \frac{K\pi_m(W)}{\mathrm{im}K\pi_m(N_0)} \times \frac{K\pi_m(W)}{\mathrm{im}K\pi_m(N_0)} \to \Lambda$$

is a non-singular $(-1)^m$ hermitian form.

Let's denote by $\xi : B \to BO$ the normal $(m-1)$-type of N_0 and by γ the stable tautological bundle over BO. If $m \neq 3, 7$ or $m = 3, 7$ and $\omega_{m+1}(\alpha^*(\xi^*\gamma)) = 0$ for all $\alpha \in \pi_{m+1}(B)$, then there is a quadratic refinement μ of λ defined on $K\pi_m(W)$ and μ restricted on $\mathrm{im}K\pi_m(N_0)$ is a homomorphism (compare [Kr.1, p. 54]). We denote the subgroup of Λ which projects to $\mathrm{im}\mu|_{\mathrm{im}K\pi_m(N_0)} \subset Q_{(-1)^m}$ by $S(W)$. If $m = 3, 7$ and there exists an $\alpha \in \pi_{m+1}(B)$ s.t. $\omega_{m+1}(\alpha^*(\xi^*\gamma)) \neq 0$, we have to work with a quadratic refinement $\bar{\mu}$ which takes values in $\frac{\Lambda}{S(W) \oplus \mathbb{Z}}$. If we equip $\frac{K\pi_m(W)}{\mathrm{im}K\pi_m(N_0)}$ with a preferred basis (see [Kr.1, p. 58]), then $(\bar{\lambda}, \mu)$ (resp. $(\bar{\lambda}, \bar{\mu})$) represents an element in $L^{s,\tau}_{2m}(\pi_1(B), \omega_1(B), S(W))$ (resp. $L^{s,\tau}_{2m}(\pi_1(B), \omega_1(B), S(W) \oplus \mathbb{Z})$) and we denote it by $\Theta(W, F)$.

The following theorem is a reformulation of a part of Theorem 5.2. in [Kr.1].

Theorem 3.4.6. *Let (W, F) be a B-bordism between two normal 2-smoothings (N, f) and (N', f'), where $N, N' \in \mathcal{L}$. Then $\Theta(W, F) \in L^{s,\tau}_{2m}(\pi_1(B), S(W)$ or $S(W) \oplus \mathbb{Z})$ is a bordism invariant rel. boundary. Furthermore (W, F) is bordant rel. boundary to an s-cobordism if and only if $\Theta(W, F) = 0$.*

The multisignature

The *multisignature* is an important tool for distinguishing elements in $L^s_{2n}(\pi)$. If π is the trivial group one knows that

$$\text{sign} : L^s_{4k}(\{1\}) \to 8\mathbb{Z}$$

is an isomorphism. It is also true that the so called Arf-invariant gives an isomorphism between $L^s_{4k+2}(\{1\})$ and $\mathbb{Z}/2$. A very useful application of the multisignature is the following

Theorem 3.4.7. *([Wa, Thm. 13A.4.]) Let π be cyclic group of odd order. There is a decomposition*

$$L^s_{2k}(\pi) = L^s_{2k}(\{1\}) \oplus \widetilde{L}^s_{2k}(\pi) \quad \text{([Wa, p. 179])},$$

where the multisignature maps $\widetilde{L}^s_{2k}(\pi)$ injectively to the characters (real or imaginary as appropriate) trivial on 1 and divisible by 4.

For an extensive study of surgery obstruction groups for finite groups we refer to [H-T].

Definition of the multisignature

The so called multisignature extends the notion of the ordinary signature of a quadratic form over the integers in a certain sense (see below). The most general definition of the multisignature which one can find in [Wa.1] or [Wa, p. 174] is applicable to unimodular forms over group rings of finite groups with either trivial or nontrivial involution. Since we deal with forms over group rings with trivial involution ("all manifolds are orientable") we give a definition of the multisignature which is equivalent to the general definition restricted to the orientable case [Wa, p. 175].

Let H be a free Λ-module and $\lambda : H \times H \to \Lambda$ a non-degenerate ϵ-hermitian form, where ϵ lies in $\{\pm 1\}$. Let further denote by p_c the map from Λ to $\mathbb{Z}$ which sends $a_0 + \sum_{g \in \pi \setminus \{0\}} a_g$ to a_0. The composition $p_c \circ \lambda$ is an ϵ-hermitian $\mathbb{Z}$-valued non-degenerate form. We extend $p_c \circ \lambda$ to $H^{\mathbb{C}} := H \otimes_{\mathbb{Z}} \mathbb{C}$ in the obvious way which yields an ϵ-hermitian unimodular $\mathbb{C}$-valued form ϕ, i.e.

$$\phi(x, y) = \epsilon\overline{\phi(y, x)} \quad \forall x, y \in H^{\mathbb{C}}.$$

It is clear that $H^{\mathbb{C}}$ inherits a π-action from H and we easily realize that

$$\phi(xg, yg) = \phi(x, y) \quad \forall x, y \in H^{\mathbb{C}} \text{ and } g \in \pi.$$

Now we choose a positive definite π-invariant hermitian form $\langle \cdot, \cdot \rangle$ on $H^{\mathbb{C}}$ (see p. 45). There is the following linear map A of $H^{\mathbb{C}}$ to itself defined by

$$\phi(x,y) = \langle x, Ay\rangle \quad \forall x,y \in H^{\mathbb{C}}.$$

It follows that A is an ϵ-hermitian π-equivariant automorphism of $H^{\mathbb{C}}$. If A is $(+1)$-hermitian, then all eigenvalues of A are real and nonzero. If A is (-1)-hermitian, then the eigenvalues of A are purely imaginary and nonzero. Thus

$$H^{\mathbb{C}} = H^{\mathbb{C}}_{+} \oplus H^{\mathbb{C}}_{-},$$

where $H^{\mathbb{C}}_{\pm}$ is the sum of the eigenspaces corresponding to positive multiples of ± 1 or $\pm i$. Since the eigenspaces $H^{\mathbb{C}}_{+}$ and $H^{\mathbb{C}}_{-}$ are π-invariant they define complex π-representations. We denote the characters of these π-representations by ρ_+ and ρ_- resp.

Definition 3.4.8. *1) The multisignature of λ which we denote by $MS(\lambda)$ is the element of the complex representation ring $RU(\pi)$ given by*

$$\rho_+ - \rho_-.$$

2) Let g be an element of π then $MS(g,\lambda) := \rho_+(g) - \rho_-(g)$.

The vanishing of the surgery obstruction

Let (W,F) be a bordism between normal 2-smoothings (N,f) and (N',f'), where $N, N' \in \mathcal{L}$ with fundamental groups isomorphic to $\mathbb{Z}/r$ and F a 3-equivalence. Let $\xi : BSpin \to BO$ be the 3-connected cover of BO thus $\xi_* : \pi_4(BSpin) \to \pi_4(BO)(\cong \mathbb{Z})$ is an isomorphism. From these considerations we conclude that there exists some $\alpha \in \pi_4(L^\infty_r \times \mathbb{C}P^\infty \times BSpin)$ s.t. $\omega_4(\alpha^*(\xi^*\gamma)) \neq 0$ which proves that $\Theta(W,F)$ is an element of $L^{s,\tau}_6(\mathbb{Z}/r, S(W) \oplus \mathbb{Z})$. Thus $\Theta(W,F) = (\bar\lambda : \Lambda^d \times \Lambda^d \to \Lambda, \bar\mu)$ (see p. 75), where Λ^d is equipped with some preferred basis.

We have already seen on p. 76 that there is the following decomposition of $L^s_6(\mathbb{Z}/r)$:

$$L^s_6(\mathbb{Z}/r) = L^s_6(\{1\}) \oplus \widetilde{L}^s_6(\mathbb{Z}/r).$$

Let $[(\Lambda^d, \lambda, \mu)] =: B$ be an element in $L^s_6(\mathbb{Z}/r)$. The first coordinate of B w.r.t. the above decomposition is detected by the Arf-invariant in the following sense:

To an element $[(\Lambda^d, \lambda, \mu)]$ in $L^s_6(\mathbb{Z}/r)$ one can assign an element in $L^s_6(\{1\})$:
As we identify Λ with $\mathbb{Z}^r$ in a canonical way we regard Λ^d as a $\mathbb{Z}$-module. Let $\epsilon : \Lambda \to \mathbb{Z}$ be the augmentation map which is a ring homomorphism and let $\tilde\epsilon$ be the projection of Q_{-1} to $\frac{\epsilon(\Lambda)}{\epsilon(\{2a|a\in\Lambda\})} \cong \mathbb{Z}/2$.
We compose λ with ϵ and compose the quadratic refinement μ with $\tilde\epsilon$ then we obtain an integral non-degenerate (-1)-hermitian quadratic form which represents an element b in $L^s_6(\{1\})$. We define $\mathrm{Arf}(B)$ to be the *classical Arf-invariant* of b (see for expl. [L, p. 95]).

The second coordinate of B w.r.t. the decomposition above is according to Theorem 3.4.7. detected by the multisignature.

The difference between $L_6^s(\mathbb{Z}/r)$ and $L_6^s(\mathbb{Z}/r, S(W) \oplus \mathbb{Z})$ comes from the different choices of form parameters. In order to understand $L_6^s(\mathbb{Z}/r, S(W) \oplus \mathbb{Z})$ we do the same considerations as for $L_6^s(\mathbb{Z}/r)$. We observe that because of the $\mathbb{Z}$ in $S(W) \oplus \mathbb{Z}$ elements in $L_6^s(\mathbb{Z}/r, S(W) \oplus \mathbb{Z})$ are uniquely determined by the multisignature.

If the multisignature of $\Theta(W, F)$ is zero, then this implies that there is a base $\mathcal{B}$ of Λ^d s.t. the quadratic form $(\bar{\lambda} : \Lambda^d \times \Lambda^d \to \Lambda, \bar{\mu})$, where Λ^d is equipped with the basis $\mathcal{B}$, represents the zero-element in $L_6^s(\mathbb{Z}/r, S(W) \oplus \mathbb{Z})$ and hence in $L_6^{s,\tau}(\mathbb{Z}/r, S(W) \oplus \mathbb{Z})$. We call this element $\Theta(W, F)_{\mathcal{B}}$ and we denote a based (-1)-hyperbolic form over $(\Lambda, S(W) \oplus \mathbb{Z})$ again by $H_{-1}^n(\Lambda)$ (for some $n \in \mathbb{N}$).
If $\Theta(W, F)_{\mathcal{B}} = 0$, then there is a $r \in \mathbb{N}$ s.t. $H_{-1}^r(\Lambda) \oplus \Theta(W, F)_{\mathcal{B}}$ is isomorphic to $H_{-1}^s(\Lambda)$ for some $s \in \mathbb{N}$, where the isomorphism is simple. Let A be the matrix of base change w.r.t. the base in $H_{-1}^r(\Lambda) \oplus \Theta(W, F)$ and $H_{-1}^r(\Lambda) \oplus \Theta(W, F)_{\mathcal{B}}$. The element in $\mathrm{Wh}(\mathbb{Z}/r)$ that is represented by A is denoted by $\tau(A)$.

Proposition 3.4.9. *Let r be an odd natural number and $\pi_1(W) \cong \mathbb{Z}/r$. Then*

i) $\Theta(W, F) \in L_6^{s,\tau}(\mathbb{Z}/r, S(W) \oplus \mathbb{Z})$ and

ii) $\Theta(W, F) = 0$ if and only if $MS(\Theta(W, F))$ and $\tau(A)$ are trivial.

The proof of Theorem 5.2. (p. 58) in [Kr.1] implies

Corollary 3.4.10. *W is rel. boundary bordant to an h-cobordism $(W_h; N, N')$ if and only if $MS(\Theta(W, F))$ is trivial. Thus the vanishing of the algebraic torsion $\tau(A)$ is equivalent to the vanishing of the so called Whitehead torsion of the inclusion $N \hookrightarrow W_h$ (see next section).*

3.4.2 The Whitehead torsion

An elabotate and detailed treatment of the following can be found in [M]. The purpose of this section is rather to give a flavour of the concept of *torsion* in differential topology than to state precise and technical definitions.

Let R be an associative ring with unit, $GL(A)$ the infinite general linear group which is the obvious colimit construction of the finite general linear groups $GL_n(R)$. $GL(R)$ divided by its commutator subgroup yields a group which we call $K_1(R)$. The matrix (-1) defines an element in $K_1(R)$. The quotient $K_1(R)/\{0, -1\}$ is denoted by $\bar{K}_1(R)$ and is called the *reduced Whitehead group.* Whenever there is a based finite acyclic chain complex over a ring R with unit, i.e. a based chain complex with finitely many generators and vanishing homology groups, one can associate to it its *torsion* ([M, §3, §7]) which lies in $\bar{K}_1(R)$ of R. We have the following topological application of this algebraic concept:

Let (K, L) be a pair consisting of a finite, connected CW-complex K and a subcomplex L which is a deformation retract of K. We denote the fundamental group of K simply by π. For (K, L) there is the chain complex $C_*(K, L)$ defined by setting

$$C_p(K, L) := H_p(|K^p \cup L|, |K^{p-1} \cup L|),$$

where H_* denotes singular homology with integer coefficients, and $|K^p|$ denotes the underlying topological space of the p-skeleton of K. This p-th chain group is free abelian with one generator for each p-cell of $K - L$. We denote the homology group $H_pC_*(K, L)$ by $H_p(K, L)$. We consider the universal covering complexes $\widetilde{L} \subset \widetilde{K}$ of K and L, where π operates on $\widetilde{K}$ via deck transformation. From the chain complex $C_*(K, L)$ we obtain a chain complex $C_*(\widetilde{K}, \widetilde{L})$ of the pair $(\widetilde{K}, \widetilde{L})$, as follows: For a generator e_p of $C_p(K, L)$ there are as many cells in $C_p(\widetilde{K}, \widetilde{L})$ as π has elements. The resulting chain module is $\mathbb{Z}[\pi]$-free with one generator for each p-cell in $K - L$. Since K is finite it follows that $C_*(\widetilde{K}, \widetilde{L})$ is finitely generated over $\mathbb{Z}[\pi]$. It's clear that the homology groups $H_j(\widetilde{K}, \widetilde{L})$ vanish, since $|\widetilde{L}|$ is a deformation retract of $|\widetilde{K}|$. A $\mathbb{Z}[\pi]$-basis c_p for $C_p(\widetilde{K}, \widetilde{L})$ is given by a choice of lifts of a basis $\{e_1, ..., e_s\}$ of $C_p(K, L)$, i.e. $c_p := \{\tilde{e}_1, ..., \tilde{e}_s\}$. Having chosen such a basis the *torsion* of the acyclic chain complex $C_*(\widetilde{K}, \widetilde{L})$ is defined and an element of $\bar{K}_1(\mathbb{Z}[\pi])$. In order to get something which doesn't depend on the choice of a lift we map this torsion to the

$$\bar{K}_1(\mathbb{Z}[\pi])/\pi =: Wh(\pi)$$

the *Whitehead group* of π ([M, p. 377ff]). Thus we have associated to (K, L) a well defined element $\tau(K, L)$ in Wh(π) which we call the *Whitehead torsion* of (K, L).

There is the following useful result:

Theorem 3.4.11. *([M, Thm. 6.4]) If π is finite abelian, then Wh(π) is a free abelian group.*

Let $f : X \to Y$ be a cellular homotopy equivalence between finite CW-complexes and M_f denotes its mapping cylinder which is equipped with an obvious cell structure (compare [M, p. 381]). The torsion $\tau(f)$ of f is defined to be that element of $Wh(\pi_1(Y))$ which corresponds to $\tau(M_f, X) \in Wh(\pi_1(Y))$ under the isomorphism between $Wh(\pi_1(M_f))$ and $Wh(\pi_1(Y))$ which is induced by the inclusion of Y into M_f. We call f *simple* if $\tau(f) = 0$. There are the following properties (compare [M, p. 382]):

Lemma 3.4.12. *i) If $f : X \to Y$ and $g : y \to Z$ are cellular homotopy equivalences, then*

$$\tau(g \circ f) = \tau(g) + g_*(\tau(f)).$$

ii) Let (K, L) be as above and $i : L \hookrightarrow K$ be the inclusion then $\tau(i) = \tau(K, L)$.

Let W be a smooth compact and connected manifold with boundary, where $\partial W = N_0 \cup N_1$ is the disjoint union of closed submanifolds N_0 and N_1. If N_0 and N_1 are deformation retracts of W, then we call $(W; N_0, N_1)$ an h-cobordism. By choosing a C^1-triangulation of (W, N) we can associate a Whitehead torsion $\tau(W, N)$.

There is the following deep theorem by T.A. Chapman (see [Ch]):

Theorem 3.4.13. *Let $f : X \to Y$ be a homeomorphism of finite CW-complexes then $\tau(f) = 0$.*

This implies that the Whitehead torsions $\tau(K, L)$ doesn't depend on the choice of the CW-structure and thus is a well defined invariant which only depends on the inclusion $L \hookrightarrow K$.

The Whitehead Torsion of Special Complexes

Definition 3.4.14. *A finite complex is called **special** if its fundamental group is finite abelian and operates trivially on the rational homology groups of the universal covering space.*

Examples of special complexes are the lens spaces, since the universal covering spaces of them are spheres. The Witten spaces and the 5-dimensional manifolds which we consider in this chapter are special, since the action of the fundamental groups via deck transformation on the universal covering spaces is homotopically trivial (see Prop. 3.1.4.ii)).

Let X be a special complex and $\pi_1(X)$ be denoted by π. In the following we define an invariant of X which lives in the rational group ring $\mathbb{Q}[\pi]$ and furthermore we present a relation between this invariant and the Whitehead torsion $\tau(W, N_0)$ associated to an h-cobordism $(W; N_0, N_1)$ between special manifolds of odd dimension.

Let $\mathcal{K}$ denote the kernel of the augmentation map

$$\mathbb{Q}[\pi] \to \mathbb{Q}, \ \sum_{g \in \pi} a_g g \mapsto \sum_{g \in \pi} a_g$$

and let Σ denote the sum of all the group elements. Then one can easily check that $\mathbb{Q}[\pi]$ splits as the direct sum

$$\mathcal{K} \oplus \mathbb{Q}[\Sigma].$$

The subring $\mathcal{K}$ can be thought of as an algebra which is isomorphic to $\mathbb{Q}[\pi]/\mathbb{Q}[\Sigma]$.
The direct sum decomposition of $\mathbb{Q}[\pi]$ gives rise to a decompositions of the chain complex

$$C(\widetilde{X}; \mathbb{Q}) = \mathcal{K}C(\widetilde{X}; \mathbb{Q}) \oplus \Sigma C(\widetilde{X}; \mathbb{Q})$$

and the homology modules

$$H_*(\widetilde{X};\mathbb{Q}) = \mathcal{K}H_*(\widetilde{X};\mathbb{Q}) \oplus \Sigma H_*(\widetilde{X};\mathbb{Q}).$$

Since X is a special complex π operates trivially on the rational homology groups of $\widetilde{X}$ and thus

$$\mathcal{K}H_*(\widetilde{X};\mathbb{Q}) = 0,$$

i.e. $\mathcal{K}C(\widetilde{X};\mathbb{Q})$ is acyclic. Furthermore $\mathcal{K}C(\widetilde{X};\mathbb{Q})$ inherits a preferred basis from $C(\widetilde{X};\mathbb{Q})$ via the natural homomorphism

$$C(\widetilde{X};\mathbb{Q}) \to C(\widetilde{X};\mathbb{Q})/\Sigma C(\widetilde{X};\mathbb{Q}) = \mathcal{K}C(\widetilde{X};\mathbb{Q}).$$

From this acyclic complex we obtain the torsion

$$\tau(\mathcal{K}C(\widetilde{X};\mathbb{Q})) \in \bar{K}_1\mathcal{K}/im(\pi), \quad \text{see [M, p. 405].}$$

Since X is a special complex π is a finite abelian group and $\mathcal{K}$ is isomorphic to a cartesian product of cyclotomic fields, and hence K_1 can be identified with the group of units $U(\mathcal{K})$ of $\mathcal{K}$. There is an element $\Delta(X) \in U(\mathcal{K})$ which projects on $\tau(\mathcal{K}C(\widetilde{X};\mathbb{Q}))$. We can conclude the following:
To each special complex X we can associate the *R-torsion*

$$\Delta(X) \in U(\mathcal{K}) \subset \mathcal{K} \subset \mathbb{Q}[\pi]$$

which is well defined up to multiplication by plus or minus a group element. We introduce the following notation: For two elements Δ and Δ' in $U(\mathcal{K})$ we write $\Delta \sim \Delta'$ if they differ by plus or minus a group element.

There is the following

Lemma 3.4.15. *([M, Lemma 12.5.]) A homotopy equivalence $f : X \to Y$ between special complexes is a simple homotopy equivalence if and only if $f_*(\Delta(X)) \sim \Delta(Y)$.*

That the R-torsion of a special complex X is up to multiplication with a group element an invariant of the topological space X, i.e. doesn't depend on the CW-structure, follows from the last lemma, Theorem 3.4.10. and Lemma 3.4.11. Just choose Y to be X and $f = g = id_X$ but it also follows immediately from Chapman's result Thm. 3.4.12.

If a compact manifold is a special complex we call it a *special manifold*. There is the following connection to the Whitehead torsion:

Let $(W; N_0, N_1)$ be an h-cobordism between two odd dimensional special manifolds N_0 and N_1. We identify all three fundamental groups with π.

Theorem 3.4.16. *([M, Thm. 12.8]) $\Delta(N_1) \sim u^2\Delta(N_0)$, where u is a unit in $\mathbb{Z}[\pi]$ and $u^2 \sim 1$ if and only if $\tau(W, N_0) = 0$.*

3.4.3 The main results

Let $r \in \mathbb{N}$ and greater than 1 and $\mathcal{D}_r^5$ be the set of diffeomorphism classes of smooth closed oriented non-simply-connected 5-manifolds with fundamental group isomorphic to $\mathbb{Z}/r$ and $\pi_2(\cdot)$ isomorphic to $\mathbb{Z}$.
Let N, N' be smooth 5-manifolds s.t. they represent elements in $\mathcal{D}_r^5$. It follows from Theorem 3.3.1. that the universal covering space of N is diffeomorphic to $S^2 \times S^3$ and since r is odd the fundamental group acts trivially on the homology of $S^2 \times S^3$. The Leray-Serre SS for computing integral cohomology of N (associated to $\widetilde{N} \to N \to L_r^\infty$) together with the UCT implies that $H^2(N;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/r$.

In order to state the first theorem of this section in an uncomplicated way we introduce the following equivalence relation on $\mathcal{D}_r^5$. Let $u \in H^1(N;\mathbb{Z}/r)$ be a generator then we say that N, N' are equivalent,

$$N \sim N',$$

if and only if there exist

- an isomorphism $$\alpha : H^1(N;\mathbb{Z}/r) \xrightarrow{\sim} H^1(N';\mathbb{Z}/r)$$
- and $z \in H^2(N;\mathbb{Z})$ and $z' \in H^2(N';\mathbb{Z})$ which project to a generator of $\frac{H^2(N;\mathbb{Z})}{\text{torsion}}$ and $\frac{H^2(N';\mathbb{Z})}{\text{torsion}}$ resp. s.t.

  1) $\left\langle u(\beta_r(u))^2, [N]_{\mathbb{Z}/r} \right\rangle \equiv \left\langle \alpha(u)(\beta_r(\alpha(u)))^2, [N']_{\mathbb{Z}/r} \right\rangle \bmod r$;
  2) $\left\langle u\beta_r(u)z, [N]_{\mathbb{Z}/r} \right\rangle \equiv \left\langle \alpha(u)\beta_r(\alpha(u))z', [N']_{\mathbb{Z}/r} \right\rangle \bmod r$;
  3) $\left\langle uz^2, [N]_{\mathbb{Z}/r} \right\rangle \equiv \left\langle \alpha(u)z'^2, [N']_{\mathbb{Z}/r} \right\rangle \bmod r$;
  4) $\left\langle \rho_r(p_1(N))u, [N]_{\mathbb{Z}/r} \right\rangle \equiv \left\langle \rho_r(p_1(N'))\alpha(u), [N']_{\mathbb{Z}/r} \right\rangle \bmod r$, where ρ_r is the mod-r-reduction in cohomolgy;
  5) $\sigma(g \in \pi_1(N'), \widetilde{N'}) = \sigma(\tilde{\alpha}(g), \widetilde{N})$ for all $g \in \pi_1(N') \setminus \{0\}$, where $\tilde{\alpha} : \pi_1(N') \xrightarrow{\sim} \pi_1(N)$ is Kronecker dual to α.

This relation is indeed an equivalence relation on $\mathcal{D}_r^5$: Symmetry and reflexivity are obvious. Transitivity: If $N \sim N'$ and $N' \sim N''$, then clearly there exists an isomorphism $\alpha'' : H^1(N;\mathbb{Z}/r) \xrightarrow{\sim} H^1(N'';\mathbb{Z}/r)$ s.t. the conditions 1),4) and 5) are fulfilled. Since $N' \sim N''$ there exist y' and z'' s.t.

$$\left\langle \alpha(u)\beta_r(\alpha(u))y', [N']_{\mathbb{Z}/r} \right\rangle \equiv \left\langle \alpha''(u)\beta_r(\alpha''(u))z'', [N'']_{\mathbb{Z}/r} \right\rangle \bmod r$$
$$\left\langle \alpha(u)y'^2, [N']_{\mathbb{Z}/r} \right\rangle \equiv \left\langle \alpha''(u)z''^2, [N'']_{\mathbb{Z}/r} \right\rangle \bmod r.$$

The Postnikov decomposition of manifolds in $\mathcal{D}_r^5$ (compare with Proposition 3.1.8.) implies that y' has to be $z' + k\beta_r(u')$ for some $k \in \mathbb{Z}/r$. Let's denote $z + k\beta_r(u)$ by y. It's straightforward to show that the conditions 2) and 3) are fulfilled if we choose y

and z'' as the generators of $H^2(N;\mathbb{Z})$ and $H^2(N'';\mathbb{Z})$ resp.

For manifolds in $\mathcal{D}_r^5$ it's obviously true that the fundamental group always acts trivially on the cohomology of the universal covering space. Thus in these cases it makes sense to speak of the R-torsion of such manifolds, as we have defined it on p. 81.

Theorem 3.4.17. *Let $r \in \mathbb{N}$ greater than 1 s.t. gcd$(r, 6) = 1$ and N, N' be oriented smooth closed 5-dimensional spin manifolds with $\pi_1(N) \cong \pi_1(N') \cong \mathbb{Z}/r$ and $\pi_2(N) \cong \pi_2(N') \cong \mathbb{Z}$. Then N and N' represent the same element in $\mathcal{D}_r^5$, i.e. are diffeomorphic if and only if*

i) $N \sim N'$,

ii) *The R-torsions are equivalent, i.e.* $\Delta(N) \sim \Delta(N')$.

Proof. "$\Rightarrow$": Is obviously clear.

"$\Leftarrow$": If we have proven that condition i) is equivalent to N and N' being h-cobordant, then the proof follows from Theorem 3.4.16.
From the proof of Lemma 3.2.3. it follows that the second stage of the Postnikov decomposition of such manifolds is $L_r^\infty \times \mathbb{C}P^\infty$. This also follows from the fact that we can build the Postnikov tower up to the second stage with the help of k-invariants since the fundamental group acts trivially on the higher homotopy groups in this range. This leads to the same result (compare with p. 41).

Let's denote $L_r^\infty \times \mathbb{C}P^\infty \times BSpin$ by B_r.

From Corollary 3.4.10. we know that N and N' are h-cobordant if and only if there exist normal 2-smoothings $f \times \nu_{sp} : N \to B_r$, $f' \times \nu'_{sp} : N' \to B_r$ and a bordism (W, F) beween $(N, f \times \nu_{sp})$ and $(N', f' \times v'_{sp})$, where F is a 3-equivalence s.t. the multisiganture of the surgery obstruction is trivial. From Lemma 3.2.1., Proposition 3.2.2. and the calculations in the proof of Proposition 3.2.4. it follows that there exist bordant normal 2-smoothings if and only if there exist generators $u \in H^1(N;\mathbb{Z}/r)$, $u' \in H^1(N;\mathbb{Z}/r)$ and $z \in H^2(N;\mathbb{Z})$ and $z' \in H^2(N';\mathbb{Z})$ s.t. they project to generators of $\frac{H^2(N;\mathbb{Z})}{\text{torsion}}$ and $\frac{H^2(N';\mathbb{Z})}{\text{torsion}}$ with the property that

$$\begin{aligned}
&\langle u(\beta_r(u))^2, [N]_{\mathbb{Z}/r}\rangle \equiv \langle u'(\beta_r(u'))^2, [N']_{\mathbb{Z}/r}\rangle \bmod r;\\
&\langle u\beta_r(u)z, [N]_{\mathbb{Z}/r}\rangle \equiv \langle u'\beta_r(u')z', [N']_{\mathbb{Z}/r}\rangle \bmod r;\\
&\langle uz^2, [N]_{\mathbb{Z}/r}\rangle \equiv \langle u'z'^2, [N']_{\mathbb{Z}/r}\rangle \bmod r;\\
&\langle \rho_r(p_1(N))u, [N]_{\mathbb{Z}/r}\rangle \equiv \langle \rho_r(p_1(N'))u', [N']_{\mathbb{Z}/r}\rangle \bmod r.
\end{aligned}$$

We have made use of the facts that a map g from a CW-complex into a product of Eilenberg Maclane spaces can be identified with certain cohomology classes.
If these conditions are fulfilled, then there exist normal 2-smoothings $(N, f \times \nu_{sp})$ and $(N', f' \times \nu'_{sp})$ which are bordant. We may choose the bordism (W, F) in such a way that F is a 3-equivalence (see [Kr, Prop. 4]).

We denote $\pi_1(B_r)$ by π. There is a π-action on $\widetilde{W}$, the universal covering space of W, and on $\widetilde{N}$, $\widetilde{N}'$, which comes from the identification of the fundamental groups via the maps into the normal 2-type B_r. In the following we relate the multisignature of $\Theta(W,F)$ to the π-equivariant signature of $\widetilde{W}$.

We equip W and $\widetilde{W}$ with basepoints b and $\tilde{b}$ resp. s.t. $\tilde{b}$ is a lift of b under the universal covering map. We want to study the unimodular skew-hermitian form

$$\bar{\lambda} : \frac{K\pi_3(W)}{imK\pi_3(N)} \times \frac{K\pi_3(W)}{imK\pi_3(N)} \to \Lambda,$$

which comes from the intersection pairing defined on $K\pi_3(W)$. We recall that

$$K\pi_3(W) := ker(F_\star : \pi_3(W) \to \pi_3(B)) \overset{\pi_3(B)=0}{=} \pi_3(W).$$

On the other hand there is the following composition of maps

$$\begin{aligned} \pi_3(\widetilde{W}) \overset{\mathcal{H}}{\to} H_3(\widetilde{W};\mathbb{Z}) \quad &\overset{(\cap[\widetilde{W},\partial\widetilde{W}])^{-1}}{\to} \quad H^3(\widetilde{W},\partial\widetilde{W};\mathbb{Z}) \\ &\to \quad \underbrace{\frac{H^3(\widetilde{W},\partial\widetilde{W};\mathbb{Z})}{ker(i^* : H^3(\widetilde{W},\partial\widetilde{W};\mathbb{Z}) \to H^3(\widetilde{W};\mathbb{Z}))}}_{=:\hat{H}^3(\widetilde{W})} \end{aligned}$$

which we call Φ. The map Φ is π-equivariant and since $\widetilde{W}$ is 1-connected we know that the Hurewicz map $\mathcal{H}$ is surjective.
We also know from section 2.7.1. that the π-signature for $\widetilde{W}$ is defined for the non-singular pairing

$$\gamma : \hat{H}^3(\widetilde{W}) \times \hat{H}^3(\widetilde{W}) \to \mathbb{Z}$$

which comes from the cup-product-pairing on $H^3(\widetilde{W},\partial\widetilde{W};\mathbb{Z})$ (see pp. 44-45). We denote the intersection pairing on $K\pi_3(W) = \pi_3(W)$ by λ. Let's recall how λ was defined. For a detailed exposition of the following we refer to [Wa, Ch.5].

Elements of $K\pi_3(W)$ are (regular) homotopy classes of immersions $f : S^3 \to W$. Now let α_1 and $\alpha_2 \in K\pi_3(W)$ then we find representatives (f_1, w_1) and (f_2, w_2) s.t. the images of these maps intersect (transversally) only in finitely many points. Let's call the set of intersection points D. From α_1 and α_2 we obtain a welldefined element in Λ, namely

$$\lambda(\alpha_0, \alpha_1) := \sum_{d \in D} \epsilon(d) g(d),$$

where $\epsilon(d) \in \{\pm 1\}$ and $g(d) \in \pi_1(W)$ ([Wa, p. 45]). There is a unique lift $\tilde{f}_i$ of f_i to $\widetilde{W}$ determined by $\tilde{b}$. Let

$$\lambda_{\mathbb{Z}}([\tilde{f}_0], [\tilde{f}_1])$$

be the $\mathbb{Z}$-valued algebraic intersection number of the homology classes $[\tilde{f}_0]$ and $[\tilde{f}_1]$ then we alternatively can write $\lambda(\alpha_0, \alpha_1)$ as

$$\sum_{g \in \pi} \lambda_{\mathbb{Z}}([\tilde{f}_0], [l_{g^{-1}} \circ \tilde{f}_1])g,$$

where $l_{g^{-1}}$ denotes the left multiplication by g^{-1}. This means that

$$p_c \circ \lambda(\alpha_0, \alpha_1) = \lambda_{\mathbb{Z}}([\tilde{f}_0], [\tilde{f}_1]).$$

But

$$\lambda_{\mathbb{Z}}([\tilde{f}_0], [\tilde{f}_1]) = [\tilde{f}_0]^* \cup [\tilde{f}_1]^* \in H^3(\widetilde{W}, \partial\widetilde{W}; \mathbb{Z}) \cong \mathbb{Z},$$

where $[\tilde{f}_i]^*$ denotes the Poincaré-Lefschetz dual of $[\tilde{f}_i]$.
All in all we obtain

$$p_c \circ \lambda = \gamma(\Phi(\cdot), \Phi(\cdot)).$$

Thus the radicals of $p_c \circ \lambda$, λ and $\gamma(\Phi(\cdot), \Phi(\cdot))$ are the same and equal to $\ker\Phi$. But on the other hand they equal $\mathrm{im} K\pi_3(N^{pq})$ (see p.75). From the definition of the multisignature p. 76 and the equivariant signature in section 2.7.1. we conclude:

Computing the multisignature of $\bar{\lambda}$ is the same as computing the π-signature of $\widetilde{W}$.

Thus

$$MS(g, \Theta(W, F)) = \mathrm{sign}(g, \widetilde{W}) \ \ \forall g \in \pi.$$

In the same way we have done it on p. 84 we identify the fundamental groups of the bordant manifolds and the bordism with the fundamental group of B_r. Let $g \in \pi \setminus \{0\}$ then from the above formula and Novikov's additivity formula for the equivariant signature it follows that

$$MS(g, \Theta(W, F)) \quad = \quad \sigma(g, \widetilde{N}) - \sigma(g, \widetilde{N}').$$

Since $\widetilde{W}$ is 6-dimensional it follows that $\mathrm{sign}(\widetilde{W})$ thus

$$\begin{array}{rcl} MS(g, \Theta(W, F)) & = & 0 \ \ \forall g \in \pi \\ & \Leftrightarrow & \\ \sigma(h \in \pi_1(N) \setminus \{0\}, \widetilde{N}) & = & \sigma((f_*)^{-1} \circ f'_*(h), \widetilde{N}') \ \ \forall h \in \pi_1(N) \setminus \{0\}. \end{array}$$

The map α is $f'^* \circ (f^*)^{-1} : H^1(N; \mathbb{Z}/r) \to H^1(N'; \mathbb{Z}/r)$ and $\tilde{\alpha}$ is $(f_*)^{-1} \circ f'_* : \pi_1(N') \to \pi_1(N)$.

All in all we see that if the conditions 1)-5) on p. 82 are fulfilled for an isomorphism $\alpha : H^1(N; \mathbb{Z}/r) \to H^1(N'; \mathbb{Z}/r)$ and $z \in H^2(N; \mathbb{Z})$ and $z' \in H^2(N'; \mathbb{Z})$ that project to generators of $\frac{H^2(N;\mathbb{Z})}{\text{torsion}}$ and $\frac{H^2(N';\mathbb{Z})}{\text{torsion}}$ resp. then there exist bordant 2-smoothings of N and N' s.t. one can choose a bordism (W, F) with the following properties:

- F is 3-connected.
- $\widetilde{W}$ has trivial π-signature.

It follows that W is bordant rel. boundary to an h-cobordism and we may apply Theorem 3.4.16. which finishes the proof. ∎

An application of the last theorem is the following

Theorem 3.4.18. *Let r be a natural number greater than 1 s.t. $gcd(r, 6) = 1$ and N^{pq}, $N^{p'q'} \in \mathcal{L}$ be oriented (see the convention on p.65) with $\pi_1(N^{pq}) \cong \pi_1(N^{p'q'}) \cong \mathbb{Z}/r$. Let further (m, n), (m', n') be pairs of integral numbers s.t. $m\frac{q}{r}+n\frac{p}{r} = 1 = m'\frac{q'}{r}+n'\frac{p'}{r}$. Then N^{pq} and $N^{p'q'}$ are diffeomorphic if and only if there exist $\epsilon, \epsilon', \delta \in \{\pm 1\}$ and $k, k' \in \mathbb{Z}/r$ s.t.*

$$\begin{aligned} pq &= \delta p'q' \\ (\epsilon m + k\frac{p}{r})(\epsilon n - k\frac{q}{r}) &\equiv \delta(\epsilon' m' + k'\frac{p'}{r})(\epsilon' n' - k'\frac{q'}{r}) mod\ r, \\ \frac{q}{r}(\epsilon m + k\frac{p}{r}) - \frac{p}{r}(\epsilon n - k\frac{q}{r}) &\equiv \frac{q'}{r}(\epsilon' m' + k'\frac{p'}{r}) - \frac{p'}{r}(\epsilon n' - k'\frac{q'}{r}) mod\ r. \end{aligned}$$

Remark 3.4.19. *The examples which are given in the following remarks were produced by a computer program:*

i) The first arithmetic condition in the previous theorem doesn't imply the two other ones. Example: $(p, q) = (66, 385)$ and $(p', q') = (165, 154)$, where $r = gcd(p, q) = gcd(p', q') = 11$ and $pq = 25410 = p'q'$.

ii) That N^{pq} and $N^{p'q'}$ are diffeomorphic doesn't imply that they are equivariantly diffeomorphic in the sense of Definition 2.3.1. Example: $N^{5,30}$ and $N^{10,15}$, where $r = 5$.

iii) Let N be a smooth manifold and $\phi : U(1) \times N \to N$ a smooth and free $U(1)$-action on N. Let $r \in \mathbb{N}$ then ϕ induces a free $\mathbb{Z}/r$-action on N, namely by restricting on the r'th roots of unity. We call this action ϕ_r. Furthermore we denote the orbit space w.r.t. ϕ_r by $\frac{N}{\phi_r}$.

There are smooth and free $U(1)$-actions γ, ψ on $S^2 \times S^3$ which are not equivalent, i.e. they "don't differ" by an $U(1)$-equivariant self-diffeomorphism of $S^2 \times S^3$ s.t. $\frac{S^2\times S^3}{\gamma_r}$ and $\frac{S^2\times S^3}{\psi_r}$ are diffeomorphic for some $r \in \mathbb{N}$. Example: same as in ii).

Proof. Following Theorem 3.4.16. N^{pq} and $N^{p'q'}$ are diffeomorphic if and only if $N^{pq} \sim N^{p'q'}$ and $\Delta(N^{pq}) \sim \Delta(N^{p'q'})$. The proof of this theorem is organized as follows:

A) We first compute the σ-invariants for an $N^{ab} \in \mathcal{L}$ fulfilling the assumptions of the theorem.

B) Then we show that the conditions 1)-5) (p. 82) are fulfilled if and only if there exists congruences as stated in the theorem.

C) In the last part of the proof we show that the R-torsion of an N^{pq} doesn't depend on p and q, i.e. $\Delta(N^{pq}) \sim \Delta(N^{p'q'})$.

A) We know that $\widetilde{N}^{ab} \cong N^{\frac{a}{r}\frac{b}{r}}$ is the total space of the $U(1)$-bundle over $S^2 \times S^2$,

$$S^1 \to N^{\frac{a}{r}\frac{b}{r}} \xrightarrow{\Pi} S^2 \times S^2,$$

with $c_1(\Pi) = \frac{a}{r}x + \frac{b}{r}y$. The content of the following passage resmbles much what we have done in the proof of Theorem 2.7.9., where all the technical details were explained.

Let $E^{\frac{a}{r}\frac{b}{r}}$ denote the disc bundle which is associated to Π. Since the $\pi_1(N^{ab})$-action on $N^{\frac{a}{r}\frac{b}{r}}$ preserves the fibre there is an obvious $\pi_1(N^{ab})$-action on $E^{\frac{a}{r}\frac{b}{r}}$ which is compatible with the $\pi_1(N^{ab})$-action on the boundary $N^{\frac{a}{r}\frac{b}{r}}$. And for any nontrivial element g of $\pi_1(N^{ab})$ the fixed point set $(E^{\frac{a}{r}\frac{b}{r}})^g$ is the base space of the disc bundle, namely $S^2 \times S^2$. The normal bundle $\mathcal{N}_{ab}$ over $S^2 \times S^2$ is oriented in such a way that it is compatible with the g-action, thus $c_1(\mathcal{N}_{ab}) = \pm(\frac{a}{r}x + \frac{b}{r}y)$. Let g be a nontrivial element of $\pi_1(N^{ab})$ then Θ_g denotes the rotation angle between 0 and π and furthermore we denote by $(\mathcal{L}_j)_{j\in\mathbb{N}_0}$ and $(\mathcal{M}_k^\theta)_{k\in\mathbb{N}_0}$ the characteristic polynomials which we have explained on p. 48. According to Definition 2.7.5. the σ-invariant of $N^{\frac{a}{r}\frac{b}{r}}$ for a nontrivial element g of $\pi_1(N^{ab})$ is

$$L(g, E^{\frac{a}{r}\frac{b}{r}}) - \operatorname{sign}(g, E^{\frac{a}{r}\frac{b}{r}}),$$

where

$$L(g, E^{\frac{a}{r}\frac{b}{r}}) = (i \tan\frac{\theta_g}{2})^{-1} \sum_{j=0} \mathcal{L}_j(S^2 \times S^2) \sum_r \mathcal{M}_r^{\theta_g}(\mathcal{N}_{\theta_g}) \left[S^2 \times S^2\right]_\pm . \quad (3.4)$$

The signs in $\left[S^2 \times S^2\right]_\pm$ depend on how g operates on $N^{\frac{a}{r}\frac{b}{r}}$ (compare with the remark on how we orient the fixed point set, p.48). We have already computed some $\mathcal{L}$- and $\mathcal{M}_\theta$-polynomials on pp. 54 - 55. The $\mathcal{L}$-polynomial applied to $S^2 \times S^2$ which is relevant for us is:

$$\mathcal{L}_0(S^2 \times S^2) = 1.$$

And the $\mathcal{M}_\theta$-polynomials applied to $\mathcal{N}_{\theta_g}$ which are relevant for us are the following:

$$\begin{aligned}
\mathcal{M}_0^{\theta_g}(\mathcal{N}_{\theta_g}) &= 1, \\
\mathcal{M}_1^{\theta_g}(\mathcal{N}_{\theta_g}) &= \frac{i}{\sin\theta_g} c_1(\mathcal{N}_{\theta_g}), \\
\mathcal{M}_2^{\theta_g}(\mathcal{N}_{\theta_g}) &= -\frac{1}{4\sin^2(\frac{\theta_g}{2})} c_1(\mathcal{N}_{\theta_g})^2.
\end{aligned}$$

Now we show that $\operatorname{sign}(g, E^{\frac{a}{r}\frac{b}{r}}) = 0$ for all $g \in \pi_1(N^{\frac{a}{r}\frac{b}{r}})$:
Since homotopically $\pi_1(N^{ab})$ acts trivially on $E^{\frac{a}{r}\frac{b}{r}}$ it follows that $\operatorname{sign}(g, E^{\frac{a}{r}\frac{b}{r}}) = \operatorname{sign}(E^{\frac{a}{r}\frac{b}{r}})$ for all $g \in \pi_1(N^{ab})$. But $\operatorname{sign}(E^{\frac{a}{r}\frac{b}{r}}) = 0$ as $\dim(E^{\frac{a}{r}\frac{b}{r}})$ is not divisible by 4. This also follows from the fact that

$$H^3(E^{\frac{a}{r}\frac{b}{r}}; \mathbb{Z}) \cong H^3(S^2 \times S^2; \mathbb{Z}) \cong 0.$$

All in all we conclude the following: If the rotation by θ_g respects the orientation of the fiber, then

$$\begin{aligned}
\sigma(g, N^{\frac{a}{r}\frac{b}{r}}) &\overset{(3.4)}{=} (i\tan\frac{\theta_g}{2})^{-1} \sum_{j=0} \mathcal{L}_j(S^2 \times S^2) \sum_r \mathcal{M}_r^{\theta_g}(\mathcal{N}_{\theta_g}) \left[S^2 \times S^2\right] \\
&= \underbrace{-i\frac{\cos(\frac{\theta_g}{2})}{2r^2\sin^3(\frac{\theta_g}{2})}}_{=:h(\theta_g)} ab\,. \qquad (3.5)
\end{aligned}$$

If $ab = 0$, then it follows that $N^{ab} \cong N^{0\pm r} \cong N^{\pm r0}$. Thus in this case the theorem is trivially true. In the following we study the case where $ab \neq 0$.

Therefore we gather some observations:

Let $g \in \pi_1(N^{ab}) \setminus \{0\}$ then

$$\sigma(-g, N^{\frac{a}{r}\frac{b}{r}}) = -\sigma(g, N^{\frac{a}{r}\frac{b}{r}}). \qquad (3.6)$$

This is a consequence of the sign change in the above formula which comes from the change of the orientation of the normal bundle $\mathcal{N}_{\frac{a}{r}\frac{b}{r}}$.
It is also true that the map

$$\sigma(\cdot, N^{\frac{a}{r}\frac{b}{r}}) : \pi_1(N^{ab}) \setminus \{0\} \to i\mathbb{R} \qquad (3.7)$$

is injective. This follows from the injectivity of the map

$$h : (0, \pi) \quad \to \quad i\mathbb{R}.$$

From (3.5) and (3.7) we conclude that if the value sets of the σ-invariants of $N^{\frac{p}{r}\frac{q}{r}}$ and $N^{\frac{p'}{r}\frac{q'}{r}}$ have to be equal, then $|pq|$ and $|p'q'|$ have to coincide.

Proposition 3.2.4. and its proof imply that the conditions 1) to 4)(p. 82) are fulfilled if and only if there exist $s, s' \in (\mathbb{Z}/r)^*$, $k, k' \in \mathbb{Z}/r$ and $\epsilon, \epsilon'\{\pm 1\}$ s.t. the following congruences hold:

$$
\begin{aligned}
& s^3\frac{pq}{r^2} \equiv s'^3\frac{p'q'}{r^2} \bmod r, \\
& s(\epsilon m + k\frac{p}{r})(\epsilon n - k\frac{q}{r}) \equiv s'(\epsilon' m' + k'\frac{p'}{r})(\epsilon' n' - k'\frac{q'}{r}) \bmod r, \\
& s^2(\frac{q}{r}(\epsilon m + k\frac{p}{r}) - \frac{p}{r}(\epsilon n - k\frac{q}{r})) \equiv s'^2(\frac{q'}{r}(\epsilon' m' + k'\frac{p'}{r}) - \frac{p'}{r}(\epsilon n' - k'\frac{q'}{r})) \bmod r.
\end{aligned}
$$

(Recall that $p_1(N^{ab})$=0 thus condition 4) is always fulfilled.)

B) If we further require that condition 5) also has to hold, then (3.6) and (3.7) imply that $s' = \delta s$ if $pq = \delta p'q'$, where $\delta \in \{\pm 1\}$. Hence $N^{pq} \sim N^{p'q'}$ if and only if

$$
\begin{aligned}
& \frac{pq}{r^2} = \delta\frac{p'q'}{r^2}, \\
& (\epsilon m + k\frac{p}{r})(\epsilon n - k\frac{q}{r}) \equiv \delta(\epsilon' m' + k'\frac{p'}{r})(\epsilon' n' - k'\frac{q'}{r}) \bmod r, \\
& \frac{q}{r}(\epsilon m + k\frac{p}{r}) - \frac{p}{r}(\epsilon n - k\frac{q}{r}) \equiv \underbrace{\delta^2}_{=1}(\frac{q'}{r}(\epsilon' m' + k'\frac{p'}{r}) - \frac{p'}{r}(\epsilon n' - k'\frac{q'}{r})) \bmod r.
\end{aligned}
$$

Reflecting Corollary 3.4.10., what we have gained up to now is an h-cobordism classification of the non-simply-connected oriented manifolds in $\mathcal{L}$ with $\gcd(|\pi_1(\cdot)|, 6) = 1$.

C) In the following we compute the R-torsion $\Delta(N^{pq})$ for N^{pq}.

Let's denote $\pi_1(N^{pq})$ by π_1. In order to compute $\Delta(N^{pq})$ we need a π_1-equivariant CW-structure on $\widetilde{N}^{pq} = N^{\frac{p}{r}\frac{q}{r}}$, the universal covering space of N^{pq}. We finish the proof by doing the following steps:

i) We find a CW-structure on N^{pq} ,

ii) lift this CW-structure to a π_1-equivariant CW-structure on $\widetilde{N}^{pq} = N^{\frac{p}{r}\frac{q}{r}}$ and then

iii) compute $\Delta(N^{pq})$.

i) For the purpose to detect a CW-structure on N^{pq} we use the fact that N^{pq} is diffeomorphic to the total space of the $U(1)$-fibre bundle over $S^2 \times S^2$ with first Chern class $px + qy$. There is the following CW-decomposition of the base $S^2 \times S^2$:

$$S^2 \times S^2 = S^2 \vee S^2 \cup_f e^4$$

which is the product CW-structure where S^2 is constructed by attaching a 2-cell to a 0-cell. This implies the following decomposition of N^{pq}: Let's denote by L^3_p and L^3_q the standard 3-dimensional lens spaces with fundamental group isomorphic to $\mathbb{Z}/p$, $\mathbb{Z}/q$ resp. which are decomposed in the following way (see [F, §8], [Ha, pp. 144-146]):

$$L^3_p = e^0_p \cup e^1_p \cup e^2_p \cup e^3_p,\ d_i(e^i_p) = \begin{cases} 0, & \text{if } i \text{ is odd} \\ pe^{i-1}_p, & \text{else} \end{cases} \tag{3.8}$$

and

$$L^3_q = e^0_q \cup e^1_q \cup e^2_q \cup e^3_q,\ d_i(e^i_q) = \begin{cases} 0 & , \quad \text{if } i \text{ is odd} \\ qe^{i-1}_q & , \quad \text{else.} \end{cases} \quad . \tag{3.9}$$

With these CW-decompositions of lens spaces in mind we obtain

$$N^{pq} = \underbrace{L^3_p \cup_{S^1} L^3_q}_{=:X^3_{pq}} \cup_{f_{pq}} (e^4 \times S^1),$$

where X^3_{pq} is the result of glueing L^3_p and L^3_q together along their 1-cells via the identity map (respecting the orientation of the 1-cells) and f_{pq} is the glueing map which is induced by f.
Furthermore we decompose $e^4 \times S^1$ in the following way:

$$e^4 \times S^1 = \partial(e^4 \times S^1) \cup e^4 \cup e^5.$$

Thus we obtain the following cellular decomposition of N^{pq}:

$$N^{pq} = e^0 \cup e^1 \cup e^2_p \cup e^2_q \cup e^3_p \cup e^3_q \cup e^4 \cup e^5.$$

The chain complex associated to this decomposition looks as:

$$\mathbb{Z} \overset{d_5}{\to} \mathbb{Z} \overset{d_4}{\to} (\mathbb{Z})^2 \overset{d_3}{\to} (\mathbb{Z})^2 \overset{d_2}{\to} \mathbb{Z} \overset{d_1}{\to} \mathbb{Z}.$$

All the differentials except d_4 can easily be deduced from the decomposition: d_5 and d_1 have to be 0, d_3 and d_2 come from the differentials in (3.8), (3.9), i.e.

$$d_3(e^3_p) = 0,\quad d_3(e^3_q) = 0,\quad d_2(e^2_p) = pe^1,\quad d_2(e^2_q) = qe^1.$$

What's d_4?

Let g_{pq} be the attaching map of e^4 to X^3_{pq}. It's known that $d_4(e^4) = \alpha_{1,p}e^3_p + \alpha_{2,q}e^3_q$, where

$$\begin{aligned} \alpha_{1,p} &:= \deg(S^3 \overset{f_{pq}}{\to} X^3_{pq} \to X^3_{pq}/(X^3_{pq} - e^3_p)) \\ \alpha_{2,q} &:= \deg(S^3 \overset{f_{pq}}{\to} X^3_{pq} \to X^3_{pq}/(X^3_{pq} - e^3_q)), \end{aligned}$$

where the last maps in the compositions are the corresponding quotient maps from X^3_{pq} to $X^3_{pq}/(X^3_{pq} - e^3_p)$, $X^3_{pq}/(X^3_{pq} - e^3_q)$ resp.

The differential d_4 for N^{00}:

We know that $N^{00} \cong S^1 \times S^2 \times S^2 = S^1 \times S^2 \cup_{S^1} S^1 \times S^2 \cup_{f_{00}} e^4 \cup e^5$. Since $H^4(N^{00};\mathbb{Z}) \cong \mathbb{Z}$ the differential d_4 has to be trivial, thus $\alpha_{1,0} = 0 = \alpha_{2,0}$.

The differential d_4 for N^{10}:

We know that $N^{10} \cong S^3 \times S^2 = S^3 \cup_{S^1} S^1 \times S^2 \cup_{f_{10}} e^4 \cup e^5$ thus $H_4(N^{10};\mathbb{Z})$ is trivial which means that d_4 can not be trivial. Since

$$\begin{aligned} \alpha_{2,0} &= \deg(S^3 \stackrel{f_{10}}{\to} X^3_{10} \to X^3_{10}/(X^3_{10} - e^3_1)) \\ &= \deg(S^3 \stackrel{f_{00}}{\to} X^3_{00} \to X^3_{00}/(X^3_{00} - e^3_0)) \\ &= 0 \end{aligned}$$

we conclude that $\alpha_{1,1}$ has to be ± 1. If $|\alpha_{1,1}| > 1$, then $H_3(N^{10};\mathbb{Z})$ would have nontrivial torsion which is a contradiction to the fact that $H_3(N^{10};\mathbb{Z}) \cong \mathbb{Z}$. W.l.o.g. we orient e^4 s.t. $\alpha_{1,1} = 1$.
In the same way we obtain for general N^{pq} the following:

$$\alpha_{1,p} = p \text{ and } \alpha_{1,q} = q.$$

ii) From i) we obtain for the universal covering space a (non-equivariant) CW-structure, i.e.

$$N^{\frac{p}{r}\frac{q}{r}} = L^3_{\frac{p}{r}} \cup_{S^1} L^3_{\frac{q}{r}} \cup_{f_{\frac{p}{r}\frac{q}{r}}} (e^4 \times S^1) (= e^0 \cup e^1 \cup e^2_p \cup e^2_q \cup e^3_p \cup e^3_q \cup e^4 \cup e^5)$$

and we know the covering map:

$$pr : L^3_{\frac{p}{r}} \cup_{S^1} L^3_{\frac{q}{r}} \cup_{f_{\frac{p}{r}\frac{q}{r}}} (e^4 \times S^1) \to L^3_p \cup_{S^1} L^3_q \cup_{f_{pq}} (e^4 \times S^1).$$

We choose a basepoint x in N^{pq} and lift it to a point $\tilde{x}$ in $N^{\frac{p}{r}\frac{q}{r}}$. Furthermore we lift the cells of N^{pq} w.r.t. $\tilde{x}$: The lifted cells are denoted by $\hat{e}^0$, $\hat{e}^1$, $\hat{e}^2_p$, $\hat{e}^2_q$, $\hat{e}^3_p$, $\hat{e}^3_q$, $\hat{e}^4$ and $\hat{e}^5$. We choose the preferred generator t of π_1 to be the one which corresponds to the (canonical) generator $a \in H^1(N^{pq};\mathbb{Z})$ (see p.65). The π_1-equivariant chain complex that we obtain is of the following form:

$$\mathbb{Z}[\pi_1] \stackrel{\tilde{d}_5}{\to} \mathbb{Z}[\pi_1] \stackrel{\tilde{d}_4}{\to} (\mathbb{Z}[\pi_1])^2 \stackrel{\tilde{d}_3}{\to} (\mathbb{Z}[\pi_1])^2 \stackrel{\tilde{d}_2}{\to} \mathbb{Z}[\pi_1] \stackrel{\tilde{d}_1}{\to} \mathbb{Z}[\pi_1].$$

The equivariant cell structure of the universal covering spaces of lens spaces which one can find in [M, p. 404] together with the knowledge of the π_1-action on $e^4 \times S^1 \subset N^{\frac{p}{r}\frac{q}{r}}$

implies:

$$\begin{aligned}
\tilde{d}_1(\hat{e}^1) &= (t-1)\hat{e}^0,\\
\tilde{d}_2(\hat{e}^2_p) &= \frac{p}{r}(1+t+\cdots+t^{r-1})\hat{e}^1,\\
\tilde{d}_2(\hat{e}^2_q) &= \frac{q}{r}(1+t+\cdots+t^{r-1})\hat{e}^1,\\
\tilde{d}_3(\hat{e}^3_p) &= (t-1)\hat{e}^2_p,\\
\tilde{d}_3(\hat{e}^3_q) &= (t-1)\hat{e}^2_q,\\
\tilde{d}_4(\hat{e}^4) &= \frac{p}{r}(1+t+\cdots+t^{r-1})\hat{e}^3_p+\frac{q}{r}(1+t+\cdots+t^{r-1})\hat{e}^3_q,\\
\tilde{d}_5(\hat{e}^5) &= (t-1)\hat{e}^4.
\end{aligned}$$

iii) Let $\widetilde{C}^{odd}$, $\widetilde{C}^{even}$ be the nontrivial chain-modules of the equivariant CW-complex associated to $N^{\frac{p}{r}\frac{q}{r}}$ of odd resp. even degrees. According to [M, pp. 405-406] the torsion is defined to be

$$\det : (d_*^{odd} : \widetilde{C}^{odd} \to \widetilde{C}^{even}).$$

We choose $\hat{e}^5$, $\hat{e}^3_p$, $\hat{e}^3_q$, $\hat{e}^1$ to be a $\mathbb{Z}[\pi_1]$-basis of $\widetilde{C}^{odd}$ and $\hat{e}^4$, $\hat{e}^2_p$, $\hat{e}^2_q$, $\hat{e}^0$ to be a basis of $\widetilde{C}^{even}$ then the representation matrix of d_*^{odd} is

$$\begin{pmatrix} t-1 & 0 & 0 & 0\\ 0 & t-1 & 0 & 0\\ 0 & 0 & t-1 & 0\\ 0 & 0 & 0 & t-1 \end{pmatrix}$$

thus $\Delta(N^{pq}) \sim (t-1)^4$ which finishes the proof. ∎

Let $r \in \mathbb{N}$ with $\gcd(r,6)=1$. The non-simply-connected manifolds in $\mathcal{L}$ with fundamental group isomorphic to $\mathbb{Z}/r$ don't represent all diffeomorphism classes of smooth, closed spin 5-manifolds with second Betti number equal to 1:

Let S^3 be the 3-sphere and let $q_1, q_2 \in \mathbb{Z}$ with $\gcd(q_1,r)=1=\gcd(q_2,r)$ and $(q_1q_2)^2 \not\equiv \pm 1 \bmod r$. By $L(r;q_1,q_2)$ we denote the 3-dimensional oriented lens space which is the orbit space of the following smooth free $\mathbb{Z}/r$-action on S^3:

$$\begin{aligned}
\mathbb{Z}/r \times S^3 &\to S^3,\\
(t,(z_1,z_2)) &\mapsto (e^{2\pi t\frac{q_1}{r}}z_1, e^{2\pi t\frac{q_2}{r}}z_2).
\end{aligned}$$

Since r is odd it follows that $H^2(L(r;q_1,q_2);\mathbb{Z}/2)$ is trivial thus $L(r;q_1,q_2)$ admits a spin structure. Let S^2 be the 2-sphere then

$$S^2 \times L(r;q_1,q_2) =: S(r;q_1,q_2)$$

is an orientable smooth and closed 5-manifold with fundamental group isomorphic to $\mathbb{Z}/r$ which admits a unique spin structure. We claim that $S(r;q_1,q_2)$ is not diffeomorphic to any manifold in $\mathcal{L}$.

Proof of the claim: If we show that $\Delta(S(r;q_1,q_2) \nsim \Delta(N^{pq})$ for all $p,q \in \mathbb{Z}$, we are finished. For $L(r;q_1,q_2)$ there is a CW-decomposition ([F, §8]) of the following form:

$$L(r;q_1,q_2) \cong e_0 \cup e_1 \cup e_3 \cup e_3,$$

with differentials

$$d_3(e_3) = 0, \quad d_2(e_2) = re_1, \quad d_1(e_1) = 0.$$

There is a preferred generator t of $\pi_1(L(r;q_1,q_2))$ (see [L, p. 39]). Let $b \in L(r;q_1,q_2)$ be the zero-cell and $\tilde{b} \in \widetilde{L}(r;q_1,q_2) = S^3$ a lift of b. The choice of $\tilde{b}$ determines unique lifts of the cells of $L(r;q_1,q_2)$. Thus we obtain a $\pi_1(L(r;q_1,q_2))$-equivariant CW-structure on S^3.

In order to compute the R-torsion of $S(r;q_1,q_2)$ we first choose the product CW-structure, where $S^2 = s^0 \cup s^2$, i.e.

$$S(r;q_1,q_2) = e^0 \cup e^1 \cup (e_1^2 \cup e_2^2) \cup (e_1^3 \cup e_2^3) \cup e_4 \cup e_5,$$

where $e_1^2 = s^0 \times e^2$, $e_2^2 = s^2 \times e^0$, $e_1^3 = s^0 \times e^3$ and $e_2^3 = s^2 \times e^1$.
Then we choose the zero-cell as the base point p for $S(r;q_1,q_2)$ and a lift $\tilde{p}$. As above the choice of a lift of p determines a unique lift of all the other cells. We obtain a $\pi_1(S(r;q_1,q_2))$-equivariant CW-decomposition of $S^2 \times S^3$:

$$\mathbb{Z}[\pi_1] \overset{\tilde{d}_5}{\to} \mathbb{Z}[\pi_1] \overset{\tilde{d}_4}{\to} (\mathbb{Z}[\pi_1])^2 \overset{\tilde{d}_3}{\to} (\mathbb{Z}[\pi_1])^2 \overset{\tilde{d}_2}{\to} \mathbb{Z}[\pi_1] \overset{\tilde{d}_1}{\to} \mathbb{Z}[\pi_1]$$

and again as in the proof of the previous theorem it follows from [F, §9] that

$$\begin{aligned}
\tilde{d}_1(\hat{e}^1) &= (t^{q_1}-1)\hat{e}^0,\\
\tilde{d}_2(\hat{e}_1^2) &= (1+t+\cdots+t^{r-1})\hat{e}^1,\\
\tilde{d}_2(\hat{e}_2^2) &= 0,\\
\tilde{d}_3(\hat{e}_1^3) &= (t^{q_1}-1)\hat{e}_1^2,\\
\tilde{d}_3(\hat{e}_2^3) &= (t^{q_2}-1)\hat{e}_2^2,\\
\tilde{d}_4(\hat{e}^4) &= (1+t+\cdots+t^{r-1})\hat{e}_2^3,\\
\tilde{d}_5(\hat{e}^5) &= (t^{q_2}-1)\hat{e}^4.
\end{aligned}$$

Thus $\Delta(S(r;q_1,q_2)) \sim (t^{q_1}-1)^2(t^{q_2}-1)^2$. If $(t^{q_1}-1)^2(t^{q_2}-1)^2$ and $(t-1)^4$ were equivalent in the sense of Milnor, then there should exist a unit u in $\mathbb{Z}[\mathbb{Z}/r]$ s.t. $u^2(t-1)^4 = (t^{q_1}-1)^2(t^{q_2}-1)^2$. We show that this equation is impossible by applying the following

Lemma 3.4.20. *([M, Lemma 12.10.]) There exists a unit d of $\mathbb{Z}[\mathbb{Z}/r]$ satisfying the equation*

$$(t^{r_1}-1)\cdots(t^{r_n}-1) = d(t^{s_1}-1)\cdots(t^{s_p}-1)$$

if and only if $n = p$ and

$$r_1 \cdots r_n \equiv \pm s_1 \cdots s_n \, mod \, r.$$

By the choice of q_1 and q_2 the claim follows from Lemma 3.4.20. and Theorem 3.4.17. An example of such a manifold is $S(7;1,2)$.

The author is attracted by the problem to classify more general families of lens space bundles over S^2 up to diffeomorphism. Therefore Theorem 3.4.17. could be very useful.

Remark 3.4.21. *From Corollary 3.1.3.ii) and Theorem 3.4.17. we easily obtain a diffeomorphism classification of* $\{N^{abc} \in \mathcal{L} | gcd(gcd\ (a,b),6) = 1\}$.

4 A bordism classification of normal 2-smoothings of certain 7-manifolds

The non-simply-connected Witten spaces are interesting examples of 7-manifolds with cyclic fundamental group and second homotopy group isomorphic to $\mathbb{Z}$. If one is interested in classifying such manifolds, then one could try to apply the modified surgery. The first step is the determination of the so called *normal 2-type* which in the spin^c case is

$$(K(\mathbb{Z}/r, 1) \times K(\mathbb{Z}, 2) \times BSpin, \xi),$$

where $\xi : K(\mathbb{Z}/r, 1) \times K(\mathbb{Z}, 2) \times BSpin \to BO$ is a fibration depending on the second Stiefel-Whitney class of the tangent bundle of the manifold. The next step is to decide whether two normal 2-smoothings of manifolds under consideration are normally bordant. This of course requires a study of the corresponding bordism group. We will carry out the last step for spin manifolds of the prescribed type where the order of the fundamental group is coprime to 6.

We recall the following fact: Let M be a smooth manifold as above and $f \times \nu_{sp} : M \to L_r^\infty \times \mathbb{C}P^\infty \times BSpin$ be a normal 2-smoothing, where ν_{sp} is the classifying map of a chosen spin structure on M. The map $f \times \nu_{sp}$ represents an element in $\Omega_7(L_r^\infty \times \mathbb{C}P^\infty \times BSpin, \xi)$. As we have already stated in Lemma 3.2.1. the normal 2-smoothing $f \times \nu_{sp}^{pq}$ represents the zero-element in $\Omega_7(L_r^\infty \times \mathbb{C}P^\infty \times BSpin, \xi)$ if and only if (M, f) represents the trivial element in $\Omega_7^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$, where M is equipped with the chosen spin structure.

Although we apply the same methods for computing $\Omega_7^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ as for the calculation of $\Omega_5^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ in the proof of Proposition 3.2.2. we mention the main techniques again.

4.1 $\Omega_7^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$

Throughout this chapter we assume that r is an odd natural number. As we have already noticed in chapter three.

$$H_k(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & k \text{ is even} \\ (\mathbb{Z}/r)^j, & 2j-1=k. \end{cases}$$

The entries of the E_2-page of the AHSS for $\Omega_*^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ is given by: $E^2_{ab} = H_a(L_r^\infty \times \mathbb{C}P^\infty);\Omega_b^{Spin}(pt))$. Since $r \equiv 1 \bmod 2$ the E^2-term in the range $a+b \leq 8$ looks like

b										
$\vdots$										
8	$\mathbb{Z}^2$									
7	0	0								
6	0	0	0							
5	0	0	0	0						
4	$\mathbb{Z}$	$\mathbb{Z}/r$	$\mathbb{Z}$	$(\mathbb{Z}/r)^2$	$\mathbb{Z}$					
3	0	0	0	0	0	0	0			
2	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$			
1	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0		
0	$\mathbb{Z}$	$\mathbb{Z}/r$	$\mathbb{Z}$	$(\mathbb{Z}/r)^2$	$\mathbb{Z}$	$(\mathbb{Z}/r)^3$	$\mathbb{Z}$	$(\mathbb{Z}/r)^4$	$\mathbb{Z}$	
	0	1	2	3	4	5	6	7	8	$\dots$ a .

What are the ∞-terms in the seventh diagonal, i.e. what is $E^\infty_{a,b}$ for $a+b=7$?

The differentials d_2 from the 0-row to the 1-row and from the 1-row to the 2-row are given as follows (see [T, p. 7]):

In the first case they are just the dual of the Steenrod square

$$Sq^2 : H^*(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}/2) \to H^{*+2}(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}/2)$$

precomposed with the reduction map:

$$\rho_2 : H_*(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}) \to H_*(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}/2).$$

In the second case they are dual to Sq^2.

The Steenrod square properties for lens spaces and projective spaces yield the following result for $E^\infty_{a,b}$ for $a+b=1,3,5,7$ (compare pp. 62-63):

7	0							
6		0						
5	0		0					
4		0		$(\mathbb{Z}/r)^2$				
3	0		0		0			
2		0		0		0		
1	0		0		0		0	
0		$\mathbb{Z}/r$		$(\mathbb{Z}/r)^2$		$(\mathbb{Z}/r)^3$		$(\mathbb{Z}/r)^4$
	0	1	2	3	4	5	6	7 .

For $\Omega_7^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ we get the following extension problem:

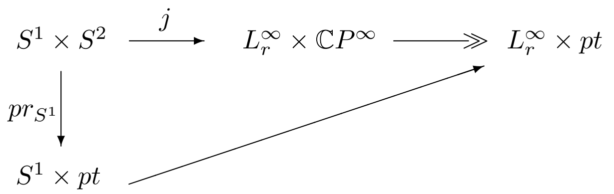
$$0 \to H_3(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}) \stackrel{\mu_k}{\to} \Omega_7^{Spin}(L_r^\infty \times \mathbb{C}P^\infty) \stackrel{h_7}{\to} H_7(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}) \to 0. \quad (4.1)$$

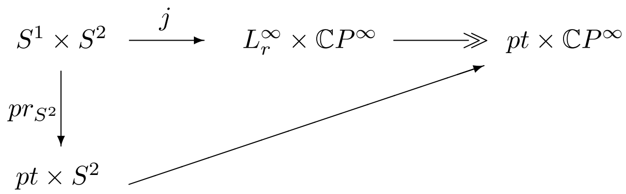
The map h_7 is a Hurewicz homomorphism, it sends $[(M,g)] \in \Omega_7^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ to $g_*([M])$. But what is μ_k?

Before we give an answer to that we recall some facts:
The E^∞-term of the AHSS above implies that the Hurewicz homomorphism

$$\begin{aligned} h_3 : \Omega_3^{Spin}(L_r^\infty \times \mathbb{C}P^\infty) &\rightarrow H_3(L_r^\infty \times \mathbb{C}P^\infty), \\ [N,f] &\mapsto f_*([N]) \end{aligned}$$

is an isomorphism. Thus we may identify $(\mathbb{Z}/r)^2$ with $\Omega_3^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$. Let L_r^3 be the standard 3-dimensional oriented lens space where the orientation comes from its universal covering space S^3 which is oriented in the standard way. Furthermore we equip $S^1 \times S^2$ with the orientation which we obtain by first orientating S^1 and then S^2 in the usual way. Generators of $\Omega_3^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ are (L_r^3, i) and $(S^1 \times S^2, j)$ where i is the composition of the following obvious maps: $L_r^3 \to L_r^\infty \to L_r^\infty \times pt \to L_r^\infty \times \mathbb{C}P^\infty$ and j is given as follows: we identify S^1 with the 1-skeleton in L_r^∞ and S^2 with the 2-skeleton in $\mathbb{C}P^\infty$ then we define j by the following commuting triangles:





That these two elements of $\Omega_3^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ are generators can easily be seen by using h_3 and the fact that L_r^3 and $S^1 \times S^2$ represent generators of $H_3(L_r^\infty;\mathbb{Z})$ resp. $H_1(L_r^\infty;\mathbb{Z}) \otimes H_2(\mathbb{C}P^\infty;\mathbb{Z})$, i.e.

$$\begin{aligned} h_3[(L_r^3, i)] &= i_*([L_r^3]) \in H_3(L_r^\infty, \mathbb{Z}) \subset H_3(L_r^\infty;\mathbb{Z}) \oplus H_1(L_r^\infty;\mathbb{Z}) \otimes H_2(\mathbb{C}P^\infty;\mathbb{Z}) \\ &\qquad = H_3(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}), \\ h_3[(S^1 \times S^2, j)] &= j_*([S^1 \times S^2]) \in H_1(L_r^\infty;\mathbb{Z}) \otimes H_2(\mathbb{C}P^\infty;\mathbb{Z}) \subset H_3(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}) \end{aligned}$$

form a basis of $H_3(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z})$.
Furthermore we know that a Kummer surface K generates $\Omega_4^{Spin}(pt.)$ ([M.2]) and we introduce the following homomorphism:

$$\tilde{\mu}_K : \Omega_3^{Spin}(L_r^\infty \times \mathbb{C}P^\infty) \to \Omega_7^{Spin}(L_r^\infty \times \mathbb{C}P^\infty), \quad [(N, g)] \mapsto [(N \times K, g \circ pr_N)],$$

where pr_N is the projection onto the first factor. The multiplicative structure of the AHSS implies that μ_K is given as

$$\begin{array}{ccc} H_3(L_r^\infty \times \mathbb{C}P^\infty) & \xrightarrow{\mu_K} & \Omega_7^{Spin}(L_r^\infty \times \mathbb{C}P^\infty) \\ h_3 \uparrow & \nearrow \tilde{\mu}_K & \\ \Omega_3^{Spin}(L_r^\infty \times \mathbb{C}P^\infty) & & \end{array}$$

If we could show that there is a homomorphism s from $\Omega_7^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ to $\Omega_3^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ s.t. $s \circ \mu_K = id_{\Omega_3^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)}$, then $\Omega_7^{Spin}(L_r^\infty \times \mathbb{C}P^\infty) \cong (\mathbb{Z}/r)^6$ and the maps s and h_7 would decide whether two 7 dimensional spin manifolds as described at the beginning of this chapter equipped with maps into $L_r^\infty \times \mathbb{C}P^\infty$ represent the same element in $\Omega_7^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ or not.

Constructing a splitting

An idea for constructing a map from $\Omega_7^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ to $\Omega_3^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ is to look at certain characteristic numbers.

We want to study two characteristic numbers n_1 and n_2 that take values in $\mathbb{Z}/r$ s.t. the evaluation of $n_1 \times n_2$ on $\mu_K[(L_r^3, i)] = [(L_r^3 \times K, i \circ pr)]$ and $\mu_K[(S^1 \times S^2, j)] = [(S^1 \times S^2 \times K, j \circ pr)]$ yields a basis of $(\mathbb{Z}/r)^2$. Before we introduce two candidates of interesting characteristic numbers, we should understand the $\mathbb{Z}/r$-cohomology of $L_r^\infty \times \mathbb{C}P^\infty$. We know that $H^\star(L_r^\infty;\mathbb{Z}/r) = \Lambda[v_1] \otimes \mathbb{Z}/r[\beta_r(v_1)]$, where v_1 is a generator of $H^1(L_r^\infty;\mathbb{Z}/r)$ and $\beta_r(v_1)$ is the mod-r-Bockstein homomorphism evaluated on v_1. We also know that $H^\star(\mathbb{C}P^\infty;\mathbb{Z}/r) = \mathbb{Z}/r[z_r]$, where z_r is the mod-r-reduction of the standard generator of $H^2(\mathbb{C}P^\infty;\mathbb{Z})$. Let ρ_r be the mod-r-reduction then the candidates are:

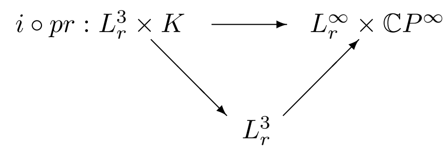
$$\begin{array}{rcl} n_1 : \Omega_7^{Spin}(L_r^\infty \times \mathbb{C}P^\infty) & \rightarrow & \mathbb{Z}/r, \\ [(M,f)] & \mapsto & \left\langle \rho_r(p_1(M)) f^*(v_1 z_r), [M]_{\mathbb{Z}/r} \right\rangle \end{array}$$

and

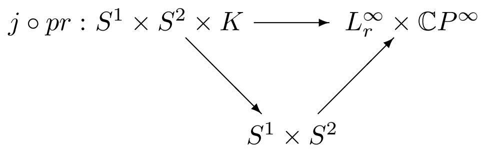
$$\begin{array}{rcl} n_2 : \Omega_7^{Spin}(L_r^\infty \times \mathbb{C}P^\infty) & \rightarrow & \mathbb{Z}/r, \\ [(M,f)] & \mapsto & \left\langle \rho_r(p_1(M)) f^*(v_1 \beta_r(v_1)), [M]_{\mathbb{Z}/r} \right\rangle, \end{array}$$

where $p_1(M)$ is the first Pontrijagin class of the tangent bundle of M and $[M]_{\mathbb{Z}/r}$ denotes the mod-r-reduction of the integral fundamental class of M.
In order to show that $n_i : \Omega_7^{Spin}(L_r^\infty \times \mathbb{C}P^\infty) \rightarrow \mathbb{Z}/r$ is well defined for $i = 1, 2$ one has to show that n_i is a bordism invariant. But for this look on p.64.

Now we show that $n_1 \times n_2(L_r^3 \times K, i \circ pr)$ and $n_1 \times n_2(S^1 \times S^2 \times K, j \circ pr)$ form a basis of $(\mathbb{Z}/r)^2$. First we have a look at



and observe that $i^*(z_r) = 0$ which implies that $(i \circ pr)^*(v_1 z) = 0$ and $i^*(v_1 \beta_r(v_1))$ generates $H^3(L_r^\infty; \mathbb{Z}/r)$. By looking at



we observe that $(j \circ pr)^*(v_1 \beta_r(v_1)) = 0$. Furthermore via the Künneth formula and the fact that $S^1 \subset L_r^\infty$ and $S^2 \subset \mathbb{C}P^\infty$ represent the Kronecker duals of the classes v_1 and z resp. it follows that $j^*(v_1 z)$ is a generator of $H^3(S^1 \times S^2; \mathbb{Z}/r)$. The product formula for Pontrjagin classes implies that for any closed oriented 3-manifold L $p_1(L \times K) = p_1(K) \in H^4(L \times K; \mathbb{Z})$. In the following we denote $(S^1 \times S^2 \times K, j \circ pr)$

by G_1 and $(L_r^3 \times K, j \circ pr)$ by G_2. The Künneth theorem implies that

$$n_1(G_1) = \left\langle \rho_r(p_1(K)), [K]_{\mathbb{Z}/r} \right\rangle \underbrace{\left\langle (j \circ pr)^*(v_1 z_r), [S^1 \times S^2]_{\mathbb{Z}/r} \right\rangle}_{=:A_{G_1}} \tag{4.2}$$

$$n_1(G_2) = \left\langle \rho_r(p_1(K)), [K]_{\mathbb{Z}/r} \right\rangle \underbrace{\left\langle (i \circ pr)^*(v_1 z_r), [L_p^3]_{\mathbb{Z}/r} \right\rangle}_{=:A_{G_2}} \tag{4.3}$$

$$n_2(G_1) = \left\langle \rho_r(p_1(K)), [K]_{\mathbb{Z}/r} \right\rangle \underbrace{\left\langle (j \circ pr)^*(v_1 \beta_r(v_1)); [S^1 \times S^2]_{\mathbb{Z}/r} \right\rangle}_{=:B_{G_1}} \tag{4.4}$$

$$n_2(G_2) = \left\langle \rho_r(p_1(K)), [K]_{\mathbb{Z}/r} \right\rangle \underbrace{\left\langle (i \circ pr)^*(v_1 \beta_r(v_1)), [L_p^3]_{\mathbb{Z}/r} \right\rangle}_{=:B_{G_2}}. \tag{4.5}$$

Lemma 4.1.1. *A_{G_1} generates $\mathbb{Z}/r$, $A_{G_2} = 0$, $B_{G_1} = 0$, B_{G_2} generates $\mathbb{Z}/r$.*

Proof. That $n_1(G_2) = 0 = n_2(G_1)$ is easily verified from one of the above considerations. The fact that $j^\star(v_1 z_r)$ is a generator of $H^3(S^1 \times S^2; \mathbb{Z}/r)$ implies that A_{G_2} is a generator of $\mathbb{Z}/r$. We know that

$$H^\star(L_r^3; \mathbb{Z}/r) \cong \frac{\mathbb{Z}/r[a,b]}{a^2, b^2}$$

with $|a| = 1, |b| = 2$ and the fact that $(i \circ pr)^\star(v_1)$ is a generator of $H^1(L_r^3; \mathbb{Z}/r)$ and $(i \circ pr)^\star(v_2)$ is a generator of $H^2(L_r^3; \mathbb{Z}/r)$ shows that B_{G_2} is a generator of $\mathbb{Z}/r$. ■

Proposition 4.1.2. *Let $r \in \mathbb{N}$ s.t. $gcd(r,6) = 1$. There exists an isomorphism*

$$\alpha : (\mathbb{Z}/r)^2 \to \Omega_3^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$$

s.t.

$$\alpha \circ (n_1 \times n_2) : \Omega_7^{Spin}(L_r^\infty \times \mathbb{C}P^\infty) \to \Omega_3^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$$

is a splitting of

$$0 \to \Omega_3^{Spin}(L_r^\infty \times \mathbb{C}P^\infty) \overset{\tilde{\mu}_k}{\to} \Omega_7^{Spin}(L_r^\infty \times \mathbb{C}P^\infty) \overset{h_7}{\to} H_7(L_r^\infty \times \mathbb{C}P^\infty) \to 0.$$

Thus

$$\Omega_7^{Spin}(L_r^\infty \times \mathbb{C}P^\infty) \cong (\mathbb{Z}/r)^6.$$

Proof. By the Hirzebruch signature theorem it's known that

$$\left\langle \frac{p_1}{3}(K), [K] \right\rangle = \mathrm{sign}(K) = -16.$$

Thus $\left\langle \rho_r(p_1(K)), [K]_{\mathbb{Z}/r} \right\rangle \equiv -48 \bmod r$ is a generator of $\mathbb{Z}/r$ if and only if $\gcd(r, 48) = 1$. But $\gcd(r, 48) = 1$ if and only if $\gcd(r, 6) = 1$. In this situation the assertion follows from the last lemma and the equations (4.2)-(4.5). ■

Now we are prepared to prove the following

Theorem 4.1.3. *Let $r \in \mathbb{N}$ greater than 1 s.t. $gcd(r,6) = 1$ and M, M' be closed smooth oriented spin 7-manifolds equipped with their unique spin structures and with normal 2-type*

$$(\underbrace{L_r^\infty \times \mathbb{C}P^\infty \times BSpin}_{=:B_r}, \xi),$$

where $\xi : B_r \to BO$ is a certain fibration (see p.40). Furthermore let $\nu_{sp} : M \to BSpin$, $\nu'_{sp} : M' \to BSpin$ be the classifying maps of the corresponding Spin(7)-bundles. Let $g := f \times \nu_{sp} : M \to B_r$ and $g' := f' \times \nu_{sp} : M' \to B_r$ be maps then (M, g) and (M', g') represent the same element in $\Omega_7(B_r, \xi)$ if and only if

$$\begin{aligned}
\left\langle \rho_r(p_1(M)) f^*(v_1 z_r), [M]_{\mathbb{Z}/r} \right\rangle &\equiv \left\langle \rho_r(p_1(M')) f'^*(v_1 z_r), [M']_{\mathbb{Z}/r} \right\rangle \, mod \ r, \\
\left\langle \rho_r(p_1(M)) \beta_r f^*(v_1) f^*(v_1), [M]_{\mathbb{Z}/r} \right\rangle &\equiv \left\langle \rho_r(p_1(M')) \beta_r f'^*(v_1) f'^*(v_1), [M']_{\mathbb{Z}/r} \right\rangle \\
&\qquad mod \ r, \\
f^*([M]) &= f'^*([M']),
\end{aligned}$$

where ρ_r is the mod-r-reduction in cohomology, v_1 is a generator of $H^1(B_r; \mathbb{Z}/r)$ and z_r is the mod-r-reduction of the standard generator of $H^2(\mathbb{C}P^\infty; \mathbb{Z}) \subset H^2(B_r; \mathbb{Z})$.

Proof. From the short exact sequence (4.1) and Proposition 4.1.2. it follows that (M, f) and (M, f') represent the same element in $\Omega_7(B_r, \xi)$ if and only if $n_1 \times n_2((M, f)) = n_1 \times n_2((M', f'))$ and $f^*([M]) = f'^*([M'])$. ■

4.2 Normal 2-smoothings of non-simply-connected spin Witten spaces

For the purpose of this section we identify a Witten space M^{ab} with the total space of the principal $U(1)$-bundle over $\mathbb{C}P^2 \times \mathbb{C}P^1$ given by the first Chern class $px + qy$, where x, y are the standard generators of $H^2(\mathbb{C}P^2; \mathbb{Z})$ and $H^2(\mathbb{C}P^1; \mathbb{Z})$ resp. (see Proposition 2.4.1.). In order to classify normal 2-smoothings of Witten spaces up to bordism we have to take orientations into account.

We orient the Witten spaces as we have done it on p.53.

Let i be the inclusion of the fibre $U(1)$ into M^{pq}. Furthermore we denote by u the element in $H^1(M^{pq}; \mathbb{Z}/r)$ with the property that $\left\langle i^*(u), [U(1)]_{\mathbb{Z}/r} \right\rangle = 1$, where $[U(1)]_{\mathbb{Z}/r}$ is the mod-r-reduction of the integral fundamental class of $U(1)$.
Let $m, n \in \mathbb{Z}$ s.t. $m\frac{q}{r} + n\frac{p}{r} = 1$ then we know from Proposition 2.5.2. that

$$H^2(M^{pq}; \mathbb{Z}) = \underbrace{\langle m\Pi^*(x) - n\Pi^*(y) \rangle}_{\cong \mathbb{Z}} \oplus \underbrace{\left\langle \frac{p}{r}\Pi^*(x) + \frac{q}{r}\Pi^*(y) \right\rangle}_{\cong \mathbb{Z}/r}.$$

We also know that

$$H^*(L_r^\infty \times \mathbb{C}P^\infty; \mathbb{Z}/r) \cong \Lambda[v_1] \otimes \mathbb{Z}/r[\beta_r(v_1), z_r],$$

where z_r is the mod-r-reduction of the standard generator of $H^2(\mathbb{C}P^\infty;\mathbb{Z})$, v_1 a generator of $H^1(L_r^\infty;\mathbb{Z}/r)$ and β_r the mod-r-Bockstein homomorphism.

Let $f = f_1 \times f_2 : M^{pq} \to L_r^\infty \times \mathbb{C}P^\infty$ be a 2-smoothing. Then the map f is up to homotopy uniquely determined by

$$f^*(v_1) = sa,$$

for $s \in (\mathbb{Z}/r)^*$ a unit in $\mathbb{Z}/r$ and

$$f^*(z) = \underbrace{\epsilon(f)}_{\in\{\pm 1\}}(m\Pi^*(x) - n\Pi^*(y)) + \underbrace{k(f,m,n)}_{\in \mathbb{Z}/r}(\frac{p}{r}\Pi^*(x) + \frac{q}{r}\Pi^*(y)).$$

Lemma 4.2.1. *Let M^{pq} be a non-simply-connected spin Witten space with $\pi_1(M^{pq}) \cong \mathbb{Z}/r$. Fixing a choice of $m, n \in \mathbb{Z}$ s.t. $m\frac{q}{r} + n\frac{p}{r} = 1$ then there is a 1-1 correspondence between the set S of homotopy classes of 2-smoothings of M^{pq} and the set of triples $\{(\epsilon, s, k) | \epsilon \in \{\pm 1\}, s \in (\mathbb{Z}/r)^*, k \in \mathbb{Z}/r\} =: T$. The bijection is given as follows:*

$$\begin{aligned} \mathcal{C}: \quad S &\to \quad T, \\ [f] &\mapsto \quad (\epsilon(f), s(f), k(f,m,n)). \end{aligned}$$

Proof. Same proof as of Lemma 3.2.3. ■

Theorem 4.2.2. *(A bordism classification of Witten spaces)*
Let r be a natural number with $gcd(r,6) = 1$ and M^{pq}, $M^{p'q'}$ be oriented spin Witten spaces with $\pi_1(M^{pq}) \cong \pi_1(M^{p'q'}) \cong \mathbb{Z}/r$ and (m,n), $(m',n') \in \mathbb{Z}^2$ s.t. $m\frac{q}{r} + n\frac{p}{r} = 1 = m'\frac{q'}{r} + n'\frac{p'}{r}$. There are normal 2-smoothings $f \times \nu_{sp} : M^{pq} \to B_r$ and $f' \times \nu'_{sp} : M^{p'q'} \to B_r$ s.t. $[(M^{pq}, f \times \nu_{sp})] = [(M^{p'q'}, f' \times \nu'_{sp})] \in \Omega_7(L_r^\infty \times \mathbb{C}P^\infty \times BSpin, \xi)$ if and only if there exist triples (s, ϵ, k) and (s', ϵ', k') in T s.t.

$$(1) \qquad s^2\frac{q}{r} \equiv s'^2\frac{q'}{r} \ mod\ r,$$

$$(2) \qquad s(k\frac{q}{r} - \epsilon n) \equiv s'(k'\frac{q'}{r} - \epsilon' n') \ mod\ r,$$

$$(3) \qquad s(\epsilon m + k\frac{p}{r})^2(k\frac{q}{r} - \epsilon n) \equiv s'(\epsilon' m' + k'\frac{p'}{r})^2(k'\frac{q'}{r} - \epsilon' n') \ mod\ r,$$

$$(4) \qquad s(\epsilon m + k\frac{p}{r})((\epsilon m + k\frac{p}{r})\frac{q}{r} - 2(\epsilon n - k\frac{q}{r})\frac{p}{r}) \equiv s' \cdot \\ \cdot(\epsilon' m' + k'\frac{p'}{r})((\epsilon' m' + k'\frac{p'}{r})\frac{q'}{r} - 2(\epsilon' n' - k'\frac{q'}{r})\frac{p'}{r}) \ mod\ r,$$

$$(5) \qquad \frac{s^3}{r^2}(2pq(\epsilon m + k\frac{p}{r}) - p^2(\epsilon n - k\frac{q}{r})) \equiv \frac{s'^3}{r^2}(2p'q'(\epsilon' m' + k'\frac{p'}{r}) \\ -p'^2(\epsilon' n' - k'\frac{q'}{r})) \ mod\ r,$$

$$(6) \qquad s^4\frac{p^2q}{r^3} \equiv s'^4\frac{p'^2q'}{r^3} \ mod\ r.$$

If we choose M^{pq} or $M^{p'q'}$ with the opposite orientation, then we have to multiply simultaneously -1 to the corresponding side of all congruences above.

The last theorem enables us to answer the following question:

Are there non-simply-connected oriented Witten spaces which do not admit a diffeomorphism of degree -1?

An answer is given by

Corollary 4.2.3. *There exists an infinite family of Witten spaces that do not admit an orientation reversing self-diffeomorphism (they are called **chiral**).*

For an extended treatment of the notion of *orientation reversal on manifolds* we refer to [Ml].

Proof of Corollary 4.2.3. Let $r \in \mathbb{N}$ be as in the previous Theorem. If there existed such a self-map, then the last part of the theorem and condition (1) imply that there should exist a unit s of $\mathbb{Z}/r$ s.t.

$$-1 \equiv s^2 \bmod r. \tag{4.6}$$

Let's denote by $\mathcal{N}_r$ the following set:

$$\{M^{pr} | \gcd(p, r) = r\}.$$

But the equation (4.6) can't hold for example for $r = 7$. By Proposition 2.5.2. we immediately realize that the Witten spaces in $\mathcal{N}_7$ represent pairwise different chiral diffeomorphism-(homotopy-)classes. ■

Proof of Theorem 4.2.2. The first part of the statement in the above theorem is by Lemma 3.2.1. equivalent to $[(M^{pq}, f)] = [(M^{p'q'}, f')] \in \Omega_7^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$. From Theorem 4.1.3. we know that two 2-smoothings $f : M^{pq} \to L_r^\infty \times \mathbb{C}P^\infty$ and $f' : M^{p'q'} \to L_r^\infty \times \mathbb{C}P^\infty$ define the same class in $\Omega_7^{Spin}(L_r^\infty \times \mathbb{C}P^\infty)$ if and only if

$$\left\langle \rho_r(p_1(M^{pq}))f^*(v_1 z_r), [M^{pq}]_{\mathbb{Z}/r} \right\rangle \equiv \left\langle \rho_r(p_1(M^{p'q'}))f'^*(v_1 z_r), [M^{p'q'}]_{\mathbb{Z}/r} \right\rangle \bmod r,$$

$$\left\langle \rho_r(p_1(M^{pq}))\beta_r(f^*(v_1))f^*(v_1), [M^{pq}]_{\mathbb{Z}/r} \right\rangle \equiv \left\langle \rho_r(p_1(M^{p'q'}))\beta_r(f'^*(v_1))f'^*(v_1), [M^{p'q'}]_{\mathbb{Z}/r} \right\rangle \bmod r,$$

$$f^*([M^{pq}]) = f'^*([M^{p'q'}]).$$

We introduce a notational convention: Let $l \in \mathbb{N}$ then β_l is the Bockstein homomorphism in $\mathbb{Z}/l$-cohomology, associated to $0 \to \mathbb{Z}/l \to \mathbb{Z}/l^2 \to \mathbb{Z}/l \to 0$ and $a \in H^1(M^{pq}; \mathbb{Z}/r)$ is the class with the property $\left\langle i^*(a), [U(1)]_{\mathbb{Z}/r} \right\rangle = 1$.

First we analyze what $\langle \rho_r(p_1(M))f^*(v_1\beta_r(v_1)), [M]_{\mathbb{Z}/r}\rangle$:

Let's denote by x_r, y_r the mod-r-reductions x, y resp. and by definition $f^*(v_1) = sa$ for $s \in (\mathbb{Z}/r)^*$. From formula (2.9) on p. 36 we know that $p_1(M^{pq}) = 3\Pi^*(x^2)$ hence $\langle \rho_r(p_1(M^{pq}))f^*(v_1\beta_r(v_1)), [M^{pq}]_{\mathbb{Z}/r}\rangle$ equals $3\langle \Pi^*(x_r^2)\cdot sa\cdot s\beta_r(a), [M^{pq}]_{\mathbb{Z}/r}\rangle$. In order to understand what this Kronecker product is we have to understand what $\beta_r(a)$ is in terms of $\Pi^*(x_r)$ and $\Pi^*(y_r)$, i.e.

$$\beta_r(a) = c_1\Pi^*(x_r) + c_2\Pi^*(y_r).$$

The fact that $L_r^\infty \times \mathbb{C}P^\infty$ is homotopy equivalent to the second stage of the Postnikov decomposition of M^{pq} implies that $\beta_r(a)$ comes from a torsion element of $H^2(M^{pq};\mathbb{Z})$ under the mod-r-reduction map. This is true since the analogues statement holds for $L_r^\infty \times \mathbb{C}P^\infty$. Hence

$$\beta_r(a) = t(\frac{p}{r}\Pi^*(x_r) + \frac{q}{r}\Pi^*(y_r))$$

for some $t \in (\mathbb{Z}/r)^*$. This means that the $\Pi^*(y_r)$-coordinate of $\beta_r(a)$ induces the $\Pi^*(x_r)$-coordinate of $\beta_r(a)$.
We claim that $c_2 = d_r\frac{q}{r}$ and hence $t = d_r$ for some (universal) $d_r \in (\mathbb{Z}/r)^*$. The proof of the last claim is analogues to the corresponding claim in the proof of Proposition 3.2.4. Thus $\beta_r(a) = d_r(\frac{p}{r}\Pi^*(\bar{x}) + \frac{q}{r}\Pi^*(\bar{y}))$. We know that

$$\langle a\Pi^*(x_r^2y_r), [M^{pq}]_{\mathbb{Z}/r}\rangle = \langle a, \Pi^*(x_r^2y_r)\cap[M^{pq}]_{\mathbb{Z}/r}\rangle.$$

And by Proposition 2 in [G] and its proof we obtain that

$$\langle a\Pi^*(x_r^2y_r), [M^{pq}]_{\mathbb{Z}/r}\rangle = 1.$$

Hence $\langle \rho_r(p_1(M^{pq}))\beta_r(f^*(v_1))f^*(v_1), [M^{pq}]_{\mathbb{Z}/r}\rangle = 3d_r\frac{q}{r} \bmod r$.
By definition we have

$$f^*(z_r) = \epsilon(f)(m\Pi^*(x_r) - n\Pi^*(y_r)) + k(f,m,n)(\frac{p}{r}\Pi^*(x_r) + \frac{q}{r}\Pi^*(y_r)).$$

Hence

$$\begin{aligned}\langle \rho_r(p_1(M^{pq}))f^*(v_1z), [M^{pq}]_{\mathbb{Z}/r}\rangle &= 3\langle \Pi^*(x_r^2)\cdot sa\cdot f^*(z_r), [M^{pq}]_{\mathbb{Z}/r}\rangle \\ &= 3s(k\frac{q}{r} - \epsilon n) \bmod r.\end{aligned}$$

The fact that $3, d_r$ are units in $\mathbb{Z}/r$ implies that the first 2 congruences at the beginning of the proof are equivalent to

$$(1) \qquad s^2\frac{q}{r} \equiv s'^2\frac{q'}{r} \bmod r,$$

$$(2) \qquad s(k\frac{q}{r} - \epsilon n) \equiv s'(k'\frac{q'}{r} - \epsilon' n') \bmod r.$$

Now we calculate $f_*([M^{pq}])$: First observe that

$$\mathrm{red}_r : H_7(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}) \to H_7(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}/r) \cong (\mathbb{Z}/r)^4$$

is an isomorphism and by the UCT

$$H^7(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}/r) \cong \mathrm{Hom}(H_7(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}),\mathbb{Z}/r) \cong (\mathbb{Z}/r)^4.$$

A basis of $H^7(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}/r)$ is given by

$$v_1 z_r^3,\quad v_1\beta_r(v_1)z_r^2,\quad v_1(\beta_r(v_1))^2 z_r,\quad v_1(\beta_r(v_1))^3.$$

Thus

$$\begin{aligned}
f_*([M^{pq}]) = f'_*([M^{p'q'}]) \quad &\Leftrightarrow \quad f_*[M^{pq}] = f'_*[M^{p'q'}]\\
&\Leftrightarrow \quad f_*[M^{pq}]_{\mathbb{Z}/r} = f'_*[M^{p'q'}]_{\mathbb{Z}/r}\\
&\Leftrightarrow \quad \left\langle b, f_*[M^{pq}]_{\mathbb{Z}/r}\right\rangle = \left\langle b, f'_*[M^{p'q'}]_{\mathbb{Z}/r}\right\rangle,\\
&\qquad \forall b \in H^7(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}/r)\\
&\Leftrightarrow \quad \left\langle f^*(b), [M^{pq}]_{\mathbb{Z}/r}\right\rangle = \left\langle f'^*(b), [M^{p'q'}]_{\mathbb{Z}/r}\right\rangle,\\
&\qquad \forall b \in H^7(L_r^\infty \times \mathbb{C}P^\infty;\mathbb{Z}/r).
\end{aligned}$$

But this is equivalent to the following equations:

$$\begin{aligned}
&(3') \quad \left\langle f^*(v_1 z_r^3), [M^{pq}]_{\mathbb{Z}/r}\right\rangle = \left\langle f'^*(v_1 z_r^3), [M^{p'q'}]_{\mathbb{Z}/r}\right\rangle\\
&(4') \quad \left\langle f^*(v_1(\beta_r(v_1))z_r^2), [M^{pq}]_{\mathbb{Z}/r}\right\rangle = \left\langle f'^*(v_1(\beta_r(v_1))z_r^2), [M^{p'q'}]_{\mathbb{Z}/r}\right\rangle\\
&(5') \quad \left\langle f^*(v_1(\beta_r(v_1))^2 z_r), [M^{pq}]_{\mathbb{Z}/r}\right\rangle = \left\langle f'^*(v_1(\beta_r(v_1))^2 z_r), [M^{p'q'}]_{\mathbb{Z}/r}\right\rangle\\
&(6') \quad \left\langle f^*(v_1(\beta_r(v_1))^3), [M^{pq}]_{\mathbb{Z}/r}\right\rangle = \left\langle f'^*(v_1(\beta_r(v_1))^3), [M^{p'q'}]_{\mathbb{Z}/r}\right\rangle.
\end{aligned}$$

Evaluations of f^*:

$$\begin{aligned}
f^*(v_1 z_r^3) &= sa((\epsilon m + k\frac{p}{r})\Pi^*(x) - (\epsilon n - k\frac{q}{r})\Pi^*(x))^3 \\
&= 3s(\epsilon m + k\frac{p}{r})^2(k\frac{q}{r} - \epsilon n)a\Pi^*(x^2y) \\
f^*(v_1(\beta_r(v_1))z_r^2) &= sd_r a((\epsilon m + k\frac{p}{r})\Pi^*(x) - (\epsilon n - k\frac{q}{r}))^2(\frac{p}{r}\Pi^*(x) + \frac{q}{r}\Pi^*(y)) \\
&= sd_r(\epsilon m + k\frac{p}{r})((\epsilon m + k\frac{p}{r})\frac{q}{r} - 2(\epsilon n - k\frac{q}{r})\frac{p}{r})a\Pi^*(x^2y) \\
f^*(v_1(\beta_r(v_1))^2 z_r) &= sas^2d_r^2(\frac{p}{r}\Pi^*(x) + \frac{q}{r}\Pi^*(y))^2((\epsilon m + k\frac{p}{r})\Pi^*(x) - (\epsilon n - k\frac{q}{r})) \\
&= \frac{s^3d_r^2}{r^2}(2pq(\epsilon m + k\frac{p}{r}) - p^2(\epsilon n - k\frac{q}{r}))a\Pi^*(x^2y) \\
f^*(v_1(\beta_r(v_1))^3) &= sas^3d_r^3(\frac{p}{r}\Pi^*(x) + \frac{q}{r}\Pi^*(y))^3 \\
&= 3s^4d_r^3\frac{p^2q}{r^3}a\Pi^*(x^2y).
\end{aligned}$$

We do the same computations for f'^* and since 3 and d_r are units in $\mathbb{Z}/r$ the conditions (3')-(6') are equivalent to the following congruences:

$$\begin{aligned}
(3') \quad &\Leftrightarrow \quad (3) \quad s(\epsilon m + k\frac{p}{r})^2(k\frac{q}{r} - 3\epsilon n) \equiv s'(\epsilon' m' + k'\frac{p'}{r})^2(k'\frac{q'}{r} - 3\epsilon' n') \bmod r, \\
(4') \quad &\Leftrightarrow \quad (4) \quad s(\epsilon m + k\frac{p}{r})((\epsilon m + k\frac{p}{r})\frac{q}{r} - 2(\epsilon n - k\frac{q}{r})\frac{p}{r}) \equiv \\
&\qquad s'(\epsilon' m' + k'\frac{p'}{r})((\epsilon' m' + k'\frac{p'}{r})\frac{q'}{r} - 2(\epsilon' n' - k'\frac{q'}{r})\frac{p'}{r}) \bmod r, \\
(5') \quad &\Leftrightarrow \quad (5) \quad \frac{s^3}{r^2}(2pq(\epsilon m + k\frac{p}{r}) - p^2(\epsilon n - k\frac{q}{r})) \equiv \\
&\qquad \frac{s'^3}{r^2}(2p'q'(\epsilon' m' + k'\frac{p'}{r}) - p'^2(\epsilon' n' - k'\frac{q'}{r})) \bmod r, \\
(6') \quad &\Leftrightarrow \quad (6) \quad s^4\frac{p^2q}{r^3} \equiv s'^4\frac{p'^2q'}{r^3} \bmod r,
\end{aligned}$$

which finishes the proof. ∎

Remark 4.2.4. *With the help of the previous theorem and the diffeomorphism classification of Witten spaces (Theorem 2.7.9.) one could try to answer the question whether there are bordant normal 2-smoothings of Witten spaces which aren't diffeomorphic but have diffeomorphic universal covering spaces. If this phenomenon occurs it can be useful for the study of the obstruction monoids $l_8(\mathbb{Z}/r)$. For a general definition of these monoids see [Kr, §5].*

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