Twisted conjugation braidings and link invariants

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Abstract

This work is about link invariants arising from enhanced Yang-Baxter operators. For each enhanced Yang-Baxter operator $\mathcal{R} = (R, D, \lambda, \beta)$ and any braid $Br(n)$ Turaev defined a link invariant $T_{\mathcal{R}}(\xi) = \lambda^{-\omega(\xi)}\beta^{-n}\mathrm{trace}(b_R(\xi) \circ D^{\otimes n})$, where $\omega : Br(n) \to \mathbb{Z}$ is a homomorphism and b_R is the representation of the Artin braid group $Br(n)$ arising from the solution of the Yang-Baxter equation R. Therefore, we first introduce new solutions of the Yang-Baxter equation $B^{\varphi} : V^{\otimes 2} \to V^{\otimes 2}$, $B^{\varphi}(a \otimes b) = ab\varphi(a)^{-1} \otimes \varphi(a)$, for $V = \mathbb{K}[G]$, $\varphi \in Aut(G)$, where G is any group. We call these solutions *twisted conjugation braidings*.
Then we give sufficient and necessary conditions for a map D to decide whether the quadruple $(B^{\varphi}, D, \lambda, \beta)$ is an EYB-operator. Moreover, we prove that the twisted conjugation braidings B^{φ} can be enhanced using character theory. These enhancements are called *character enhancements*. It turns out that for every character enhancement D of the twisted conjugation brading B^{φ} the link invariant is constantly 1, i.e., $T_{\mathcal{B}}(\xi) = 1$ for all $\xi \in Br(n)$. In general, we prove that the link invariant for all $\xi \in Br(n)$ and for every enhancement D of the twisted conjugation braiding B^{φ} is a map $T_{\mathcal{B}}(\xi) = \beta^{-n}\mathrm{trace}(b_{\mathcal{B}^{\varphi}}) \circ D^{\otimes n}$.
Our main result is the following theorem.
Let γ be a fixed invertible element of $\mathbb{K}$ and let D denote a linear map. Asumme that $D \otimes D$ commutes with the twisted conjugation braiding B^{φ}. Then

1. $Sp_2((B^{\varphi})^{\pm 1} \circ (D \otimes D)) = \gamma D \implies D^2 = \gamma D$
2. $Sp_2(B^{\varphi} \circ (D \otimes D)) = \gamma D \iff Sp_2((B^{\varphi})^{-1} \circ (D \otimes D)) = \gamma D$

In the last part of this work, we prove that for finite groups G the twisted conjugation braiding B^{φ} satisfies $(B^{\varphi})^l(a \otimes b) = a \otimes b$, with $l = 2 \cdot \mathrm{lcm}(ord(a), ord(b))$. From this follows that the link invariant is $T_{\mathcal{B}}(\xi) = \left(\frac{m_1}{\beta}\right)^n$, for braids ξ in $Br(n)$, with $\xi = \sigma_{\sigma_{i_1}}^{\epsilon_1} \dots \sigma_{i_l}^{\epsilon_l}$, and with $\epsilon_1, \dots, \epsilon_l \equiv\ 0 \bmod l$, where $m_1 = \mathrm{trace}(D)$. We call such braids *mod-l* braids. Furthermore, it follows that the link invariant is $T_{\mathcal{B}} = \left(\frac{m_1}{\beta}\right)^{n-1}$ for braids $\xi \in Br(n)$ such that $\xi = \sigma_i^{\epsilon}$, with $\epsilon \equiv\ 0 \bmod l$. We call these braids *single-power* braids. Moreover, we wrote a program in JAVA programming language which computes the link invariants for the enhancement $D = \gamma I$, $(\gamma \in \mathbb{K}^*)$ for braids $\xi \in Br(p)$, (p prime) with $\xi = (\sigma_1\sigma_2 \dots \sigma_{p-1})^q$, and with $(p, q) = 1$ for the cases $G = \Sigma_n$ and $G = \mathbb{Z}/n\mathbb{Z}$. In the cases were we have computed the link invariants $T_{\mathcal{B}}$ "the polynomial is constant," i.e., $T_{\mathcal{B}} \in \mathbb{K}$, since the only braidings we consider are permutations of the basis $\mathbb{K}[G]^{\otimes 2}$.

Contents

Introduction

In the 1988's [14] Turaev defined a criteria called an enhancement. If satisfied, would produce a Markov trace and hence lead to a link invariant. To describe his criteria let $\mathbb{K}$ be a commutative ring with 1 and let V be a $\mathbb{K}-$ free module of finite rank $m \geq 0$. A solution of the Yang-Baxter equation R is an invertible linear map $R : V \otimes V \to R \otimes R$ which satisfies the equation $(R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R)$ in $Aut(V^{\otimes 3})$. This equation first has appeared in independent papers of C. N. Yang and R. J. Baxter in the late 1960's and early 1970's, respectively. This equation and its solutions play a fundamental role in statistical mechanics ([18]) and in knot theory ([7], [9], [10]). For example, a relationship between the Yang-Baxter equation and polynomial invariants of links can be found in [6]. In this paper, Jones introduced his famous polynomial of links via the study of certain finite dimensional von Neumann algebras. A remark of D. Evans mentioned in [6] points out that these algebras were earlier discovered by physicists who used them to study the Potts model of statistical mechanics.

For describing Turaev's criteria we need to recall as well his definition of an enhanced Yang-Baxter operator. An enhanced Yang-Baxter operator (EYB) is a quadruple $\mathcal{R} = (R, D : V \to V, \lambda \in \mathbb{K}^*, \beta \in \mathbb{K}^*)$, where R is a solution of the Yang-Baxter equation and D is an endomorphism of V which satisfies

(T1) $D \otimes D$ commutes with R,

(T2a) $Sp_2(R \circ (D \otimes D)) = \lambda^{\pm 1} \beta D$,

(T2b) $Sp_2(R^{-1} \circ (D \otimes D)) = \lambda^{\pm 1} \beta D$, where $Sp_2 : V \to V$ denotes the partial trace on the second factor. For the definition and properties of partial trace we refer the reader to Definition 2.1.1, Lemma 2.1.2 and Lemma 2.1.3.

In chapter 1 we use group rings $V = \mathbb{K}[G]$ and automorphisms of the group G to introduce new solutions of the Yang-Baxter equation $B^{\varphi} : V^{\otimes 2} \to V^{\otimes 2}$. We define $B^{\varphi}(a \otimes b) = ab\varphi(a)^{-1} \otimes \varphi(a)$, for any group G and for $V = \mathbb{K}[G]$, and $\varphi \in Aut(G)$. Throughout this work B^{φ} will be called *twisted conjugation braiding* and by a link we will understand a finite family of disjoint, smooth oriented or unoriented, closed curves in $\mathbb{R}^3$, or equivalently S^3. An example of a solution B^{φ} is the following. Set G to be an abelian group. Then the twisted conjugation braiding $B^{\varphi}(a \otimes b) = aba^{-1} \otimes a$. Moreover, observe that if G is commutative then B^{φ} is the twist map.

In Theorem 2.2.6 we completely characterize EYB-operators by a set of three equations. This allows us to show that the twisted conjugation braiding B^{φ} is an enhanced Yang-Baxter operator. (We refer the reader to Theorem 2.2.6 for a precise formulation).

As a corollary of Theorem 2.2.6, we have:

Corollary 2.2.7 *Let G be any finite group, $V = \mathbb{K}[G]$, and $D = qId$, where q is an invertible element of $\mathbb{K}$. Then, $B' = (B^{\varphi}, D, \lambda = 1, \beta = q)$ is an EYB-operator.*

Moreover, in Chapter 3 we prove in terms of characters of the group $G \times G$ that the twisted conjugation braiding B^{φ} is an enhanced Yang-Baxter operator. Indeed we have

Theorem 3.2.1 *Let χ be a character defined from $G \times G$ into $\mathbb{K}^*$. Define the $\mathbb{K}$-linear map $D : \mathbb{K}[G] \to \mathbb{K}[G]$, via its action on the basis elements $a \in G$,*

$$D(a) = \sum_{c \in G} \chi(a, c)c,$$

then the following three conditions are satisfied:

1. *The quadruple $\mathcal{B} = (B^{\varphi}, D, \lambda = 1, \beta = trace(D))$ is an EYB-operator,*
2. *$B^{\varphi} \circ (D \otimes D) = D \otimes D$,*
3. *$Sp_2(B^{\varphi} \circ (D \otimes D)) = trace(D)\ D$*

Coming back to the description of Turaev's criteria. For each EYB operator $\mathcal{R}$, Turaev defines in [14] a map $T_{\mathcal{R}} : \coprod Br(n) \to \mathbb{K}$, as follows.
For a braid $\xi \in Br(n)$,

$$T_{\mathcal{R}}(\xi) = \lambda^{-\omega(\xi)} \beta^{-n} \operatorname{trace}(b_R(\xi) \circ D^{\otimes n}),$$

where ω is the homomorphism from $Br(n)$ to the additive group of integers $\mathbb{Z}$ which sends $\sigma_1, \ldots, \sigma_{n-1}$ into 1, and b_R is the representation of the Artin braid group $Br(n)$, arising from the Yang-Baxter solution $R : V^{\otimes 2} \to V^{\otimes 2}$. Namely, b_R sends σ_i into $id^{\otimes (i-1)} \otimes R \otimes id^{(n-i-1)}$.

The most important properties of the map $T_{\mathcal{R}}$ are given by the following theorem.

Theorem ((3.1.2), [14]) *For any $\xi, \eta, \in Br(n)$*

$$T_{\mathcal{R}}(\eta^{-1}\xi\eta) = T_{\mathcal{R}}(\xi\sigma_n) = T_{\mathcal{R}}(\xi\sigma_n^{-1}) = T_{\mathcal{R}}(\xi).$$

Due to a theorem of J.W. Alexander (first part) and A. A. Markov, any oriented link is isotopic to the closure of some braid. The closures of two braids are isotopic (in the category of oriented links) if and only if these braids are equivalent with respect to the equivalence relation in $\coprod_n Br(n)$ generated by the Markov moves $\xi \mapsto \eta^{-1}\xi\eta$, $\xi \mapsto \xi\sigma_n^{\pm 1}$, where $\xi, \eta \in Br(n)$. Turaev's theorem

(Theorem 2.3.1) shows that for any enhanced Yang-Baxter operator $\mathcal{R} = (R, D, \lambda, \beta)$, the mapping $T_{\mathcal{R}} : \coprod_n Br(n) \to \mathbb{K}$ induces a mapping of the set of oriented isotopy classes of links into $\mathbb{K}$.

Motivated by Turaev's work (mentioned above), we prove in Chapter 2 (Corollary 2.5.3) that the link invariant $T_{\mathcal{B}}$ of any EYB-operator $\mathcal{B} = (B^{\varphi}, D, \lambda, \beta)$ of the twisted conjugation braiding B^{φ} is given by the formula

$$T_{\mathcal{B}}(\xi) = \beta^{-n} \text{ trace}(b_{B^{\varphi}}(\xi) \circ D^{\otimes n})$$

for any braid $\xi \in Br(n)$.

Moreover, in Chapter 3 we prove that the link invariant associated to any character enhancement D_{χ} of the twisted conjugation braiding B^{φ} is constantly 1, i.e., $T_{\mathcal{B}}(\xi) = 1 \quad \text{for all} \quad \xi \in Br(n)$. (We refer the reader to Theorem 3.3.2 for a precise formulation).

Remark Theorem 3.3.2 shows that new link invariants will only arise from enhancements D of the twisted conjugation braiding B^{φ} that do not arise from a character $\chi : G \times G \to \mathbb{K}$.

The main result in this work is that any enhancement D of the twisted conjugation braiding B^{φ} is idempotent. Indeed we have the following theorem.

Theorem 4.1.1 (Idempotence) *Let γ be fixed invertible element of $\mathbb{K}$, and let D denote a linear map. Assume that $D \otimes D$ commutes with the twisted conjugation braiding B^{φ}.*

1. *If $Sp_2(B^{\varphi} \circ (D \otimes D)) = \gamma \cdot D$, then $D^2 = \gamma D$.*
2. *If $Sp_2((B^{\varphi})^{-1} \circ (D \otimes D)) = \gamma \cdot D$, then $D^2 = \gamma D$.*
3. *The following two statements are equivalent.*
   (a) $Sp_2(B^{\varphi} \circ (D \otimes D)) = \gamma D$,
   (b) $Sp_2((B^{\varphi})^{-1} \circ (D \otimes D)) = \gamma D$.

Other important properties of the map $T_{\mathcal{R}}$ are given by the following result of Turaev (see [14]).

For the trivial knot $\bigcirc$ we have

$$T_{\mathcal{R}}(\bigcirc) = \beta^{-1}\text{trace}(D).$$

If a link $L = L_1 \sqcup L_2$ is the disjoint union of two links L_1 and L_2 then

$$T_{\mathcal{R}}(L) = T_{\mathcal{R}}(L_1)\, T_{\mathcal{R}}(L_2),$$

i.e., the map $T_{\mathcal{R}}$ is multiplicative.

In particular, if L is the trivial n-component link, then

$$T_{\mathcal{R}}(L) = \beta^{-n}\text{trace}(D)^n.$$

In this work, we compute the link invariants $T_{\mathcal{B}}$ for enhancements of the twisted conjugation braiding B^{φ}, for braids ξ in $Br(n)$, with $\xi = \sigma_{i_1}^{\epsilon_1} \dots \sigma_{i_l}^{\epsilon_l}$, and with $\epsilon_1, \dots, \epsilon_l \equiv 0 \bmod l$. Such braids are called mod-l braids. We also compute the link invariants $T_{\mathcal{B}}$ for enhancements of the twisted conjugation braiding B^{φ} for braids $\xi \in Br(n)$ such that $\xi = \sigma_i^{\epsilon}$, with $\epsilon \equiv 1 \bmod l$. We call these braids *single-power* braids. In Chapter 6, by using the program "Bhi_orders" we compute the link invariants for the enhancement $D = \gamma I$, ($\gamma \in K^*$) for braids $\xi \in Br(p)$, (p prime) with $\xi = (\sigma_1 \sigma_2 \dots \sigma_{p-1})^q$, and with $(p, q) = 1$.

Our results are the following.

Remark In the cases were we have computed the link invariants $T_{\mathcal{B}}$, "the polynomial is constant", i.e, $T_{\mathcal{B}} \in \mathbb{K}$ as we see in the following table (Table 6.13), since the only braidings we consider are permutations of the basis of $\mathbb{K}[G]^{\otimes 2}$.

Table 1: Link invariants for $G = \Sigma_5$, $\varphi(s) = s_2 s s_2^{-1}$ and $D = \gamma I$

Knot	Name	(p, q)	$T_{\mathcal{B}}$
	Hop link	$(2, 2)$	840
3_1	Trefoil knot	(2, 3)	600
5_1	Solomon's seal knot	(2, 5)	720
7_1	7 crossing torus knot	(2, 7)	120
8_{19}	8 crossing torus knot	(3, 4)	1200
9_1	9 crossing torus knot	(2, 9)	600
10_{124}	10 crossing torus knot	(3, 5)	600
	11 crossing torus knot	(2, 11)	120

Proposition 5.1.1 *Asumme that D is an enhancement of the twisted conjugation braiding B^{φ}. Moreover, assume that $(B^{\varphi})^l \circ (D \otimes D) = D \otimes D$ for some $l \in \mathbb{N}$. Then*

1. $T_{\mathcal{B}}(\xi) = \left(\frac{m_1}{\beta}\right)^n$, *for all mod-$l$braids $\xi \in Br(n)$, where $m_1 = rank\ (D)$.*
2. $T_{\mathcal{B}}(\xi) = \left(\frac{m_1}{\beta}\right)^{n-1}$, *for all single-power braids $\xi = \sigma_i^{\epsilon} \in Br(n)$, with $\epsilon \equiv 1 \bmod l$.*

In particular, for the enhancement $D = qI$, with $q \in \mathbb{K}$ (invertible)

1. $T_{\mathcal{B}}(\xi) = |d|^n$, *for all mod-$l$ braids $\xi \in Br(n)$, where, $d = |G|$*
2. $T_{\mathcal{B}}(\xi) = |d|^{n-1}$, *for all single-power braids $\xi = \sigma_i^{\epsilon} \in Br(n)$, with $\epsilon \equiv 1 \bmod l$.*

Examples of enhancements D of the twisted conjugation braiding B^{φ}, satisfying the hypothesis $(B^{\varphi})^l \circ (D \otimes D) = D \otimes D$, occur for example in the following situations.

Examples

1. Let G be commutative group and set $\varphi = id$. Then the twisted conjugation braiding B^φ is the twist map, i.e. $B^\varphi(a \otimes b) = b \otimes a$. Therefore, $(B^\varphi)^2 = id$ (see Proposition 5.1.3).

   Let $G = \mathbb{Z}/3\mathbb{Z} = \{1, x, x^2\}$, with $x^3 = 1$ and assume that φ is the automorphims which sends $x \mapsto x^2, x^2 \mapsto x$. Then, $(B^\varphi)^3 = id$ (see Proposition 5.1.5).

Another example of enhancements D of the twisted conjugation braiding B^φ, satisfying the condition $(B^\varphi)^l \circ (D \otimes D) = D \otimes D$ of previous Lemma is given by the following theorem.

Theorem 5.1.9 *Let $D : \mathbb{K}[G] \to \mathbb{K}[G]$, defined as $D(a) = \sum \Delta_{c\in}(a, c)c$. Assume that $(D \otimes D)$ commutes with the twisted conjugation braiding B^φ. Moreover, assume that there is no pair of elements a and $c \in G$ such that $\Delta(a, c)$ and $\Delta(\varphi(a), \varphi(c))$ vanish at the same time. Then*

$$B^\varphi \circ (D \otimes D) \circ B^\varphi = D \otimes D$$

In particular,

$$(B^\varphi)^2 \otimes (D \otimes D) = D \otimes D = (D \otimes D) \circ (B^\varphi)^2.$$

Our work is organized as follows:

In **Chapter 1**, we introduce the twisted conjugation braiding (solution of the Yang-Baxter equation) B^φ. Moreover, motivated by the work of Sarah Schardt, (see [11]), we define an action of the Braid group $Br(n)$ on $\mathbb{K}[G]^{\otimes n}$. With the help of this action, we give a slight generalization of Schardt's Hopf algebra $\mathcal{H}(G)$. Namely, we define two Hopf algebra structures, $(\mu_R^\varphi, \Delta, \epsilon, \eta)$ and $(\mu_R^\varphi, \Delta, \epsilon, \eta)$, on the tensor algebra $\mathcal{H}^\varphi := \oplus_{n\geq 0} V^{\otimes n}$, compare with [11] Moreover, we prove that these Hopf algebras have invertible antipode maps S_L^φ and S_R^φ, respectively.

In **Chapter 2,** we recall the definition of the partial trace (Definition 2.1.1, Definition 2.1.4, see [3, 8]), and we prove that the partial trace does not depend on the choice of the basis (Lemma 2.1.2). Moreover, we recall Turaev's work (see [14]) and we give the proof of Theorem 2.2.6 and Corollary 2.2.7.

In **Chapter 3,** we prove in terms of characters of the group $G \times G$ that the twisted conjugation braiding B^φ is an enhanced Yang-Baxter operator. Namely, we prove that if the map $D : \mathbb{K}[G] \to \mathbb{K}[G]$ is defined as $D(a) = \sum_{c\in G} \chi(a, c)c$, for all $a \in G$, with χ a character from $G \times G$ into a field $\mathbb{K}$. Then D is an enhancement of the twisted braiding B^φ. Such enhancements will be called *character enhancements* and will be denoted by D_χ. Moreover, we prove that character enhancements D_χ of the twisted conjugation braiding B^φ satisfy the property

$$B^\varphi \circ (D \otimes D) = D \otimes D.$$

At the end of this chapter we give the proof of Theorem 3.3.2.

In **Chapter 4,** we prove that any enhancement D of the twisted conjugation braiding B^{φ} satisifies $D^2 = \gamma \cdot D$, where γ is a fixed invertible element in $\mathbb{K}$. In particular, if D is invertible then $D = \gamma I$, i.e. we recover the enhancement D given by Corollary 2.2.7.

In **Chapter 5,** we give the proof of Proposition 5.1.1 and give some examples of enhancements D of the twisted conjugation braiding B^{φ}, satisfying the hypothesis $(B^{\varphi})^l \circ (D \otimes D) = D \otimes D$. At the end of this chapter we give the proof of Theorem 5.1.9.

In **Chapter 6,** we prove that $ord(B^{\varphi}) = ord(B^{id})$ for all $\varphi \in Inn(G)$. Moreover, we prove that if the least common mutiple m of the order of all elements $a \in G$ exists, then the order of the twisted conjugation braiding B^{φ} is smaller than or equal to $2m$. With the help of he computer program "Bphi_orders," which is written in JAVA programming language, we compute at the end of this chapter the link invariants $T_{\mathcal{B}}$ for the enhancement $D = \gamma I$ ($\gamma \in \mathbb{K}^*$) for braids $\xi \in Br(p)$ (p prime) with $\xi = (\sigma_1 \ldots \sigma_{p-1})^q$, and with $(p, q) = 1$ for the cases $G = \Sigma_n$ and $G = \mathbb{Z}/n\mathbb{Z}$.

In **Appendix A,** we prove that the Hopf algebras $(\mathcal{H}^{\varphi}(G), \mu_L^{\varphi}, \Delta, \eta, \epsilon, S_L^{\varphi})$ and $(\mathcal{H}^{\varphi}(G), \mu_R^{\varphi}, \Delta, \eta, \epsilon, S_R^{\varphi})$ are neither quasi-commutative nor quasi-cocommutative, therefore they are not quantum groups.

In **Appendix B,** using Whitehouse and Worocnicz's (see [15] and [17]) solutions of the YB-equation, we prove that the Hopf algebras $(\mathcal{H}^{\varphi}(G), \mu_L^{\varphi}, \Delta, \eta, \epsilon, S_L^{\varphi})$ and $(\mathcal{H}^{\varphi}(G), mu_R^{\varphi}, \Delta, \eta, \epsilon, S_R^{\varphi})$ are not braided Hopf algebras.

In **Appendix C,** we recall the main properties of the tensor product of matrices.

In **Appendix D,** we explain how to use the program "Bphi_orders" which is written in JAVA programming language.

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Chapter 1

The twisted shuffle Hopf algebra of a group

In the first section of this chapter we recall Schardt's Hopf algebra $\mathcal{H}(G), (see[11])$. In the second section, we define the *twisted conjugation braiding* B^{φ} (solution of the Yang-Baxter equation), which will play an important role throughout this work, since it will help us to describe some link invariants for some finite groups, as we will see in the next chapter of this thesis. In section 3, we give a slight generalization of Schardt's Hopf algebra $\mathcal{H}G$. The main part of this chapter is based on her work. We define two Hopf algebra structures on the tensor algebra $\mathcal{H}^{\varphi}(G)$. First, we define the two products μ_L^{φ} and μ_R^{φ}, respectively. We then define the twist maps tw_L^{φ} and tw_R^{φ}, respectively, and a coproduct Δ. Secondly, we prove that the coproduct Δ is compatible with both products, and finally we show that the Hopf algebras $(\mathcal{H}^{\varphi}(G), \mu_L^{\varphi}, \Delta, \eta, \epsilon)$ and $(\mathcal{H}^{\varphi}(G), \mu_R^{\varphi}, \Delta, \eta, \epsilon)$ have antipode maps S_L^{φ} and S_R^{φ}, respectively. Moreover, in Apendix A and Appendix B, we prove that these Hopf algebras are neither quasi-commutative nor quasi-cocommutative; therefore they are not quantum groups. We will show as well using Whitehouse and Woroniwicz's solutions of the YBE Ψ, Ψ'; respectively Φ, Φ'. (See [15], [17]), that they are not braided Hopf algebras.

1.1 Schardt's Hopf algebra $\mathcal{H}(G)$

In this section, we recall Schardt's Hopf algebra, which has been introduced in [11], for two reasons. First, because the main part of this chapter is based on her work and second, because it is an example of the Hopf algebra $\mathcal{H}^{\varphi}(G)$, which will be introduced later in this chapter. Thus, using her definition of the shuffle product on $\mathcal{H}(G)$, we compute the shuffle-products, coproduct and antipode maps, when we set G to be the trivial group.

In [11], Schardt introduced the Hopf algebra $\mathcal{H}(G)$, associated to a group as follows: Let $\mathbb{K}$ be any commutative ring with unit 1, and denote $V = \mathbb{K}[G]$ the ring group of G. Set $\mathcal{H}(G) = \bigoplus_{n \geq 0} V^{\otimes n}$.

If we use the usual *concatenation product*

$$(x_1 \otimes \cdots \otimes x_n)(y_1 \otimes \cdots \otimes y_m) = (x_1 \otimes \cdots \otimes x_n \otimes y_1 \otimes \cdots \otimes y_m)$$

on $\mathcal{H}(G)$ we called it the tensor algebra, but Schardt defined a *shuffle-product* μ, as:

$$(x_1 \otimes \cdots \otimes x_l).(x_{l+1} \otimes \cdots \otimes x_n) = \sum_{\sigma \in (l,n-l)-\text{shuffle}} \operatorname{sgn}(\sigma)(x_1^\sigma \otimes \cdots \otimes x_n^\sigma)$$

with

$$x_j^\sigma = \begin{cases} x_{\sigma^{-1}(j)} & \text{if } \sigma^{-1}(j), \in \{l+1, \dots, n\} \\ (x_{\sigma^{-1}(j)})_{x_{l+1} \dots x_{l+r}} & \text{if } \sigma^{-1}(j), \in \{1, \dots, l\} \\ & \text{and } \sigma(l+r) < j < \sigma(l+r+1) \end{cases}$$

and $x_y = y^{-1}xy$.

Moreover, she defined a coproduct Δ and an antipode map S, which are given as:

$$\begin{aligned} \Delta(x_1 \otimes \cdots \otimes x_n) &= \textstyle\sum_{l=0}^n (x_1, \dots, x_l) \otimes (x_{l+1}, \dots, x_n) \\ S(x_1 \otimes \dots x_n) &= (-1)^{\lceil \frac{n}{2} \rceil} n(x_n, (x_{n-1})_{x_n}, \dots, (x_2)_{x_3 \dots x_n}, (x_1)_{x_2 \dots x_n}) \end{aligned} \tag{1.1.1}$$

Furthermore, she proved that $\mathcal{H}(G)$

1. is a graded differential algebra with the differential given by

$$\partial = \sum_{i=1}^{n-1} \partial_i$$

   with

$$\partial_i(x_1 \otimes \cdots \otimes x_n) = (x_1, \dots, x_i x_{i+1}, \dots, x_n).$$

2. S has finite order if the order of all elements of the group G have finite smallest common multiple. In particular, S is invertible for all finite groups.

3. $\mathcal{H}$ is neither commutative nor cocommutative.

Example

Set $G = \{e\}$. Recall that $\mathbb{K}[G] \cong \mathbb{K}$ and that $Aut(G) \cong \{id\}$.
Denote by $\epsilon_k = 1 \otimes \cdots \otimes 1 \quad (k \text{ times})$ and $\epsilon_l = 1 \otimes \cdots \otimes 1 \quad (l\text{-times})$ the generators of $\mathcal{H}_k = \mathbb{K}[G]^k$ and $\mathcal{H}_l = \mathbb{K}[G]^l$, respectively. If $k = l = 1$ the shuffle product $\epsilon_1 \bullet \epsilon_1 = \epsilon_2 - \epsilon_2$ vanishes. For any k and $l = 1$, the shuffle product is given by:

$$\epsilon_k \bullet \epsilon_1 = \epsilon_{k+1} - \epsilon_{k+1} + \cdots + (-1)^k \epsilon_{k+1} = \begin{cases} 0 & \text{for } k \text{ odd} \\ \epsilon_{k+1} & \text{for } k \text{ even} \end{cases} = \left(\frac{1 + (-1)^k}{2} \right) \epsilon_{k+1}$$

Recursively, one can deduce that the shuffle product of ϵ_k and ϵ_l is given by:

$$\epsilon_k \bullet \epsilon_l = \sum_{\sigma \in \mathrm{Sh}(k,l)} \mathrm{sgn}(\sigma)\ \epsilon_{k+l} := C_{k,l}\ .\ \epsilon_{k+l} = \begin{cases} \epsilon_{k+l} & \text{if } k = 0 \text{ or } l = 0 \\ 0 & \text{otherwise} \end{cases}$$

where

$$C_{k,l} = \left(\frac{1 + (-1)^{k+l-1}}{2} \right) C_{k,l-1} = \prod_{i=1}^{l} \left(\frac{1 + (-1)^{k+i-1}}{2} \right) C_{k,0} \tag{1.1.2}$$

$$C_{k,0} = 1, \quad C_{0,l} = 1, \quad C_{0,0} = 1 \quad \text{and } \frac{1+(-1)^{k+i-1}}{2} = \begin{cases} 0 & \text{for } k+i \text{ even} \\ 1 & \text{for } k+i \text{ odd} \end{cases}$$

The antipode and the coproduct maps are given by:

$$\begin{array}{rcl} \Delta(\epsilon_k) & = & \sum_{i+j=k} \epsilon_i \otimes \epsilon_j \\ & = & \epsilon_0 \otimes \epsilon_k + \epsilon_1 \otimes \epsilon_{k-1} + \cdots + \epsilon_k \otimes \epsilon_0 \end{array}$$

where by convention we set $\epsilon_0 \in (\mathbb{K}[G])^{\otimes 0} = \mathbb{K}$, $\epsilon_0 = 1$ in $\mathbb{K}$

$$S(\epsilon_k) = (-1)^{\lceil \frac{k}{2} \rceil} \epsilon_k$$

1.2 The twisted conjugation braiding B^{φ}

In this section, we give a slight generalization of Schardt's conjugation braiding, which has been introduced in [11]. More precisely, for a a group G (not necessarily commutative) she defines $B : \mathbb{K}[G]^{\otimes 2} \to \mathbb{K}[G]^{\otimes 2}$ as $a \otimes b \mapsto aba^{-1} \otimes a$.

Before we give the generalization of Schardt's conjugation braiding B, we need to recall the following definition.

Definition 1.2.1. A solution of the Yang-Baxter equation is a linear map $R : V^{\otimes 2} \to V^{\otimes 2}$ which satisfies

$$(R \otimes id_V)(id_V \otimes R)(R \otimes id_V) = (id_V \otimes R)(R \otimes id_V)(id_V \otimes R)$$

in $Aut(V^{\otimes 3})$, where V is a finitely generated $\mathbb{K}$-module of rank $m \geq 0$.

Definition 1.2.2. *Let G be a group, and let $\varphi : G \to G$ be an automorphism. Define the twisted conjugation braidng $B^{\varphi} : V^{\otimes 2} \to V^{\otimes 2}$, where $V = \mathbb{K}[G]$ by:*

$$B^{\varphi}(a \otimes b) := ab\varphi(a)^{-1} \otimes \varphi(a).$$

It is easy to see that B^{φ} is invertible. Its inverse $(B^{\varphi})^{-1} : \mathbb{K}[G]^{\otimes 2} \to \mathbb{K}[G]^{\otimes 2}$ is given by

$$a \otimes b \longmapsto \varphi^{-1}(b) \otimes \varphi^{-1}(b)^{-1}ab$$

for all $a \otimes b$ generator of $\mathbb{K}[G]^{\otimes 2}$. Figure 1.1 gives a graphic representation of the twisted conjugation brading B^{φ}.

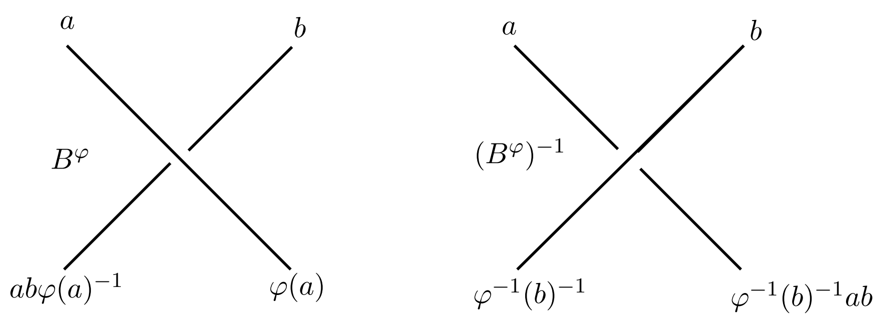


Figure 1.1: The braiding B^{φ} and its inverse $(B^{\varphi})^{-1}$.

Proposition 1.2.3. *B^{φ} satisfies the braiding equation in $Aut(V^{\otimes 3})$, i.e.,*

$$B_{12}B_{23}B_{12} = B_{23}B_{12}B_{23},$$

where $B_{12} = B^{\varphi} \otimes 1$ and $B_{23} = 1 \otimes B^{\varphi}$.

Proof Let $a \otimes b \otimes c$ be a generator of $V^{\otimes 3}$ then:

$$B_{12}(a \otimes b \otimes c) = ab\varphi(a)^{-1} \otimes \varphi(a) \otimes c$$

and

$$B_{23}(a \otimes b \otimes c) = a \otimes bc\varphi(b)^{-1} \otimes \varphi(b).$$

Therefore,

$$\begin{aligned} B_{12}B_{23}B_{12}(a \otimes b \otimes c) &= B_{12}B_{23}(ab\varphi(a)^{-1} \otimes \varphi(a) \otimes c) \\ &= B_{12}(ab\varphi(a)^{-1} \otimes \varphi(a)c\varphi^2(a)^{-1} \otimes \varphi^2(a)) \\ &= abc\varphi(ab)^{-1} \otimes \varphi(ab)\varphi^2(a)^{-1} \otimes \varphi^2(a) \\ &= B_{23}(abc\varphi(ab)^{-1} \otimes \varphi(a) \otimes \varphi(b)) \\ &= B_{12}B_{23}(a \otimes bc\varphi(b)^{-1} \otimes \varphi(b)) \\ &= B_{23}B_{12}B_{23}(a \otimes b \otimes c) \end{aligned}$$

From this follows that B^{φ} satisfies the braid equation.

∎

Remark 1.2.4.

1. *Here, unless mentioned otherwise, we will understand by a braiding a solution of the Yang-Baxter equation.*

2. *Let $\psi, \varphi : G \to G$ be homomorphism of the group G. Define $B^{\psi}, B^{\psi} : \mathbb{K}[G]^{\otimes 2} \to \mathbb{K}[G]^{\otimes 2}$ as above. Consider $B = B^{\psi} \circ B^{\varphi}$, which is*

$$a \otimes b \longmapsto ab\ \psi\varphi(a)\ \psi(ab)^{-1} \otimes \psi(ab)\ \psi\varphi(a)^{-1}.$$

It is easy to see that B does not satisfy the Yang Baxter equation. But, up to an isomorphism C it is

$$C_{\psi(ab)}\ (B^{\psi\varphi}(a \otimes b)) = B^{\psi}(B^{\varphi}(a \otimes b))$$

with $C_x(a \otimes b) := ax^{-1} \otimes xb$.

Therefore, in general composition of the Yang-Baxter equation is not a solution of the Yang-Baxter equation.

Lemma 1.2.5. *Let $V = \mathbb{K}[G]^{\otimes l}$, let $\varphi = \varphi_1 \times \varphi_2 \times \cdots \times \varphi_l$, with φ_i $inAut(G)$ for all $i \in \{1, \ldots, l\}$. Define $B : V \otimes V \to V \otimes V$ as:*

$$a \otimes b \mapsto a_1 b_1 \varphi_1(a_1)^{-1} \otimes a_2 b_2 \varphi_2(a_2)^{-1} \otimes \cdots \otimes a_l b_l \varphi_l(a_l)^{-1} \otimes \varphi_1(a_1) \otimes \varphi_2(a_2) \otimes \cdots \otimes \varphi_l(a_l),$$

for $a \otimes b$ generator of $V \otimes V$. $(a = (a_1, \ldots, a_l), b = (b_1, b_2, \ldots, b_l))$. Then B is a braiding on V.

Proof It is similar to the proof of Proposition 1.2.3.

∎

1.3 Action of the braid group $Br(k)$ on $T_k G$

In this section, we define two actions of the braid group on $\mathbb{K}[G]^{\otimes k}$.

Let G denote a group G (not necessarily commutative). Let φ be an automorphism of the group G. The following proposition gives two actions of the braid group $Br(k)$ on $T_k G$, where $T_k G = \mathbb{K}[G]^{\otimes k}$. In the next section, we will use these actions to describe the two algebras and coalgebras structures on the tensor algebra $\mathcal{H}^{\varphi}(G) = \bigoplus_{k \geq 0} T_k G$. Moreover, with the help of these actions we define twists maps and the antipode maps of the corresponding Hopf algebras.

Definition 1.3.1. *For each $k \geqslant 0$ the braid group $Br(k)$ is defined as:*

$$\begin{aligned} Br(k) \;\; = \;\; & < b_1, \ldots, b_{k-1} \;\mid\; \forall\, 1 \leq i, j \leq k-1: \; b_i b_j = b_j b_i \;\; \textit{for} \;\; |i-j| > 1 \\ & \textit{and} \;\; b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} > \end{aligned}$$

Proposition 1.3.2. *For all $k \geq 0$, the braid group $Br(k)$ acts on T_kG, this action is given by:*

$$b_i \cdot (g_1, \ldots, g_i, g_{i+1}, \ldots, g_k) := (g_1, \ldots, g_i g_{i+1} \varphi(g_i)^{-1}, \varphi(g_i), \ldots, g_k)$$

and

$$(g_1, \ldots, g_i, g_{i+1}, \ldots, g_k) \cdot b_i := (g_1, \ldots, \varphi^{-1}(g_{i+1}), \varphi^{-1}(g_{i+1})^{-1} g_i g_{i+1}, \varphi(g_i), \ldots, g_k)$$

for all tuple $(g_1, g_2, \ldots, g_k) \in T_kG$ and each generator b_i of $Br(k)$.

Proof The action of $b_i \in Br(k)$ is an automorphism of T_kG; an inverse is given by:

$$\begin{array}{ccc} T_kG & \longrightarrow & T_kG \\ (g_1, \ldots, g_i, g_{i+1}, \ldots, g_k) & \longmapsto & (g_1, \ldots, \varphi^{-1}(g_{i+1}), \varphi^{-1}(g_{i+1})^{-1} g_i g_{i+1}, \ldots, g_k). \end{array}$$

Now, it remains to prove the compatibility with the relations on the braid group.

Let $b_i, b_j \in Br_k$ with $i < j, \; |i-j| > 1$. Then:

$b_i b_j \cdot (g_1 \ldots, g_i, g_{i+1}, \ldots, g_j, g_{j+1}, \ldots, g_k)$

$= (g_1, \ldots, g_i g_{i+1} \varphi(g_i)^{-1}, \varphi(g_i), \ldots, g_j g_{j+1} \varphi(g_j)^{-1}, \varphi(g_j), \ldots, g_k)$

$= b_j b_i \cdot (g_1, \ldots, g_i, g_{i+1}, \ldots, g_j, g_{j+1}, \ldots, g_k)$

Now, if $i < j, \; |i-j| = 1$ and $j = i+1$, then

$$\begin{aligned} b_i b_{i+1} b_i \cdot (g_1, \ldots, g_k) \;\; &= \;\; b_i b_{i+1} \cdot (g_1, \ldots, g_i g_{i+1} \varphi(g_i)^{-1}, \varphi(g_i), \ldots, g_k) \\ &= \;\; b_i \cdot (g_1, \ldots, g_i g_{i+1} \varphi(g_i)^{-1}, \varphi(g_i) g_{i+2} \varphi^2(g_i)^{-1}, \varphi(g_i), \ldots, g_k) \\ &= \;\; (g_1, g_2, \ldots, g_i g_{i+1} g_{i+2} \varphi(g_i g_{i+1})^{-1}, \varphi(g_i g_{i+1} \varphi^2(g_i)^{-1}, \varphi^2(g_i), \ldots, g_k) \end{aligned}$$

On the other hand:

$$\begin{aligned} b_{i+1} b_i b_{i+1} \cdot (g_1, \ldots, g_k) \;\; &= \;\; b_{i+1} b_i \cdot (g_1, \ldots, g_i, g_{i+1} g_{i+2} \varphi(g_{i+1})^{-1}, \varphi(g_{i+1}, \ldots, g_k) \\ &= \;\; b_{i+1} \cdot (g_1, \ldots, g_i g_{i+1} g_{i+2} \varphi(g_i g_{i+1})^{-1}, \varphi(g_i), \varphi(g_{i+1}, \ldots, g_k) \end{aligned}$$

From this follows that $\quad b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}.$

∎

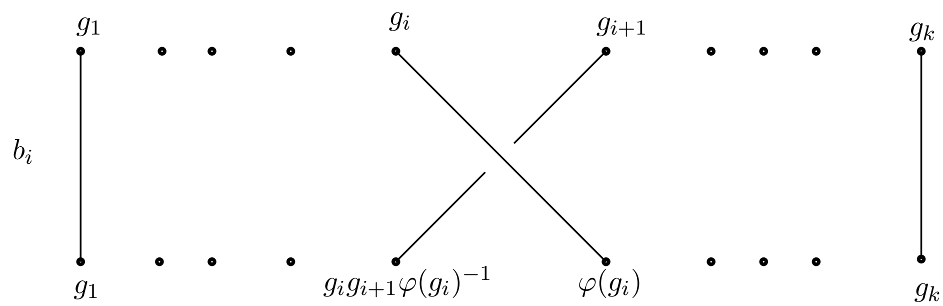


Figure 1.2: Left braid action.

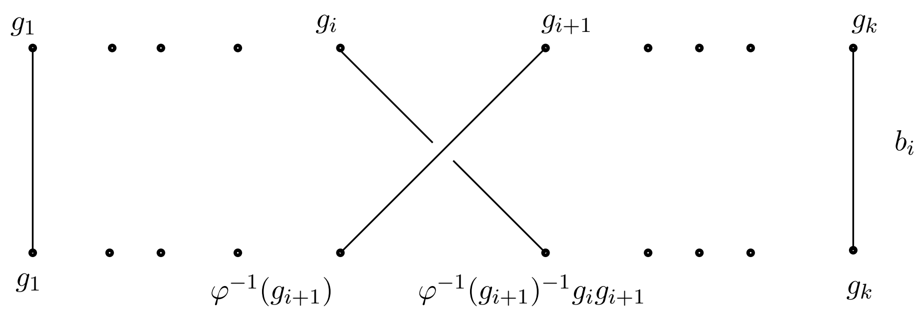


Figure 1.3: Right braid action.

1.4 Algebra structure on $\mathcal{H}^{\varphi}(G)$.

With the help of proposition 1.3.2 we define in this section two algebra structures μ_L^{φ} respectively μ_R^{φ} on $\mathcal{H}^{\varphi}(G)$.

Definition 1.4.1. (Left Product) *We define a left product:* $\mu_L^{\varphi} : T_l G \otimes T_{k-l} G \to T_k G$*:*

$$\mu_L^{\varphi}(a \otimes b) := \sum_{\substack{\sigma \in (l,k-l) \\ -shuffle}} \underbrace{sgn(\sigma)(b_{\sigma(k)} \dots b_{k-2} b_{k-1}) \dots (b_{\sigma(l+2)} \dots b_l b_{l+1}) \cdot (b_{\sigma(l+1)} \dots b_{l-1} b_l) \cdot (a, b)}_{=: S_L^{\varphi}(a,b;\sigma)_{l,k-l}}$$

for $a \in T_l G$ *and for* $b \in T_{k-l} G$.

We define a unit:

$$\eta : \mathbb{K} \longrightarrow \mathcal{H}^{\varphi}(G)$$

$$1 \longmapsto 1 \in T_0 G = \mathbb{K}$$

Remark In view of the definition of the action of the braid group $Br(n)$ on $T_n V$ (see Proposition 1.3.2), we can describe μ_L^{φ} as in Figure 1.4.

Definition 1.4.2. (Right product) *We define a right product:* $\mu_R^{\varphi} : T_l G \otimes T_{k-l} G \to T_k G$

$$\mu_R^{\varphi} : (a \otimes b) := \sum_{\sigma \in (l,k-l)-shuffle} \underbrace{sgn(\sigma)(a \otimes b) \cdot (b_l b_{l+1} \ldots b_{\sigma(l-1)} \cdot (b_{l-1} b_l \ldots b_{\sigma(l-1)-1}) \ldots (b_1 b_2 \ldots b_{\sigma(1)-1})}_{=:S_R^{\varphi}(a,b;\sigma)_{l,k-l}}$$

for $a \in T_l G$ *and for* $b \in T_{k-l} G$.

We define a unit:

$$\eta : \mathbb{K} \to \mathcal{H}^{\varphi}(G)$$

$$1 \longmapsto 1 \in T_0 G.$$

Note, that each of these products together with the unit η give a structure of graded algebra to $\mathcal{H}^{\varphi}(G)$.

Remark The algebra $\mathcal{H}^{\varphi}(G)$ is not commutative. Indeed we have that he following diagram

$$\begin{tikzcd} T_l G \otimes T_{k-l} G \arrow[rr, "\mu_L^{\varphi}"] \arrow[dr, "T"'] & & T_k G \\ & T_{k-l} G \otimes T_l G \arrow[ur, "\mu_L^{\varphi}"'] & \end{tikzcd}$$

does not commute in general, where T denotes the twist map, $T_k(a \otimes b) = (-1)^{pq} b \otimes a$ for $a \in T_p G$ and $b \in T_q G$ and $p + q = k$.

Notation Let $a = (g_1, \ldots, g_l) \in T_l G$ and let $b = (g_{l+1}, \ldots, g_k) \in T_{k-l} G$. Denote by $S_{L,\sigma}^{\varphi}(a, b) := S_L^{\varphi}(a, b, \sigma)_{l,k-l}$.

1.5 Coalgebra structure on $\mathcal{H}^{\varphi}(G)$

In this section, we describe a coalgebra structure on $(\mathcal{H}^{\varphi}(G), \mu_L^{\varphi}, \eta)$, and on $(\mathcal{H}^{\varphi}(G), \mu_r^{\varphi}, \eta)$, respectively. Moreover, we define right and twist maps tw_R^{φ}, tw_L^{φ} and we prove that the coproduct is compatible with both products.

Definition 1.5.1. *We define*

$$\Delta : T_kG \to (TG \otimes TG)_k = \bigoplus_{l=0}^{k} (T_lG \otimes T_{k-l}G)$$

$$\Delta(g_1, \ldots, g_k) := \sum_{l=0}^{k} \underbrace{(g_1, \ldots, g_l) \otimes (g_{l+1}, \ldots, g_k)}_{=:\Delta_l(g_1,\ldots,g_k)}$$

Define a counit $\epsilon : \mathcal{H}^{\varphi}(G) \to \mathbb{K}$ as $\quad T_0G \ni 1 \longmapsto 1$ $(g_1, \ldots, g_k) \longmapsto 0$ for all $k > 0$.

The above definition of Δ together with the definition of the counit ϵ give a graded coalgebra structure to $\mathcal{H}^{\varphi}(G)$.

Remark $\mathcal{H}^{\varphi}(G)$ is not cocommutative. Indeed we have that the following diagram

$$\begin{tikzcd}
\bigoplus_{l=0}^{k}(T_lG \otimes T_{k-l}G) \arrow[dr, "T"'] & T_kG \arrow[l, "\Delta"'] \arrow[d, "\Delta"] \\
& \bigoplus_{l=0}^{k}(T_{k-l}G \otimes T_lG)
\end{tikzcd}$$

does not commute in general, where T denotes the twist map.

Definition 1.5.2. (Right twist map) *Let* $a = (g_1, \ldots, g_l) \in T_lG$ *and let* $b = (g_{\in}T_{k-l}G$. *We define the rigth twist map:*

$$tw_R^{\varphi} : T_lG \otimes T_{k-l}G \to T_{k-l}G \otimes T_lG$$

$$tw_R^{\varphi}(a \otimes b) \quad := \quad (-1)^{l(k-l)} \underbrace{\Delta_{k-l}((a,b) \cdot (b_l b_{l+1} \ldots b_{k-1}) \cdot (b_{l-1} b_l \ldots b_{k-2}) \ldots (b_1 b_2 \ldots b_{k-l}))}_{t_R^{\varphi}(a,b)_{l,k-l}}$$

Definition 1.5.3. (Left Twist map) *Let* $a = (g_1, \ldots, g_l) \in T_lG$ *and let* $b = (g_{l+1}, \ldots, g_k) \in T_{k-l}G$*. We define the left twist map:*

$$tw_L^{\varphi} : T_lG \otimes T_{k-l}G \to T_{k-l}G \otimes T_lG$$

$$tw_L^{\varphi}(a \otimes b) : \quad = \quad (-1)^{l(k-l)} \underbrace{\Delta_{k-l}((b_{k-l} \ldots b_{k-2}b_{k-1}) \ldots (b_2 \ldots b_l b_{l+1}) \cdot (b_1 \ldots b_{l-1}b_l) \cdot (a, b))}_{t_L^{\varphi}(a,b)_{l,k-l}}$$

Using the action of the braid group $Br(k)$ on T_kG, we see that the left twist map and the right twist map respectively, can be defined as

$$tw_L^{\varphi}(a \otimes b) = (-1)^{l(k-l)}(ag_{l+1}\varphi(a)^{-1}, \varphi(a)g_{l+2}\varphi^2(a)^{-1}, \ldots, \varphi^{k-l-2}(g_1 \ldots g_{l-1})g_k)$$

$$\otimes\, (\varphi^{l-2}(g_1), \ldots, \varphi^{k-l-2}(g_{l-1}), \varphi^{k-l-1}(g_l))$$

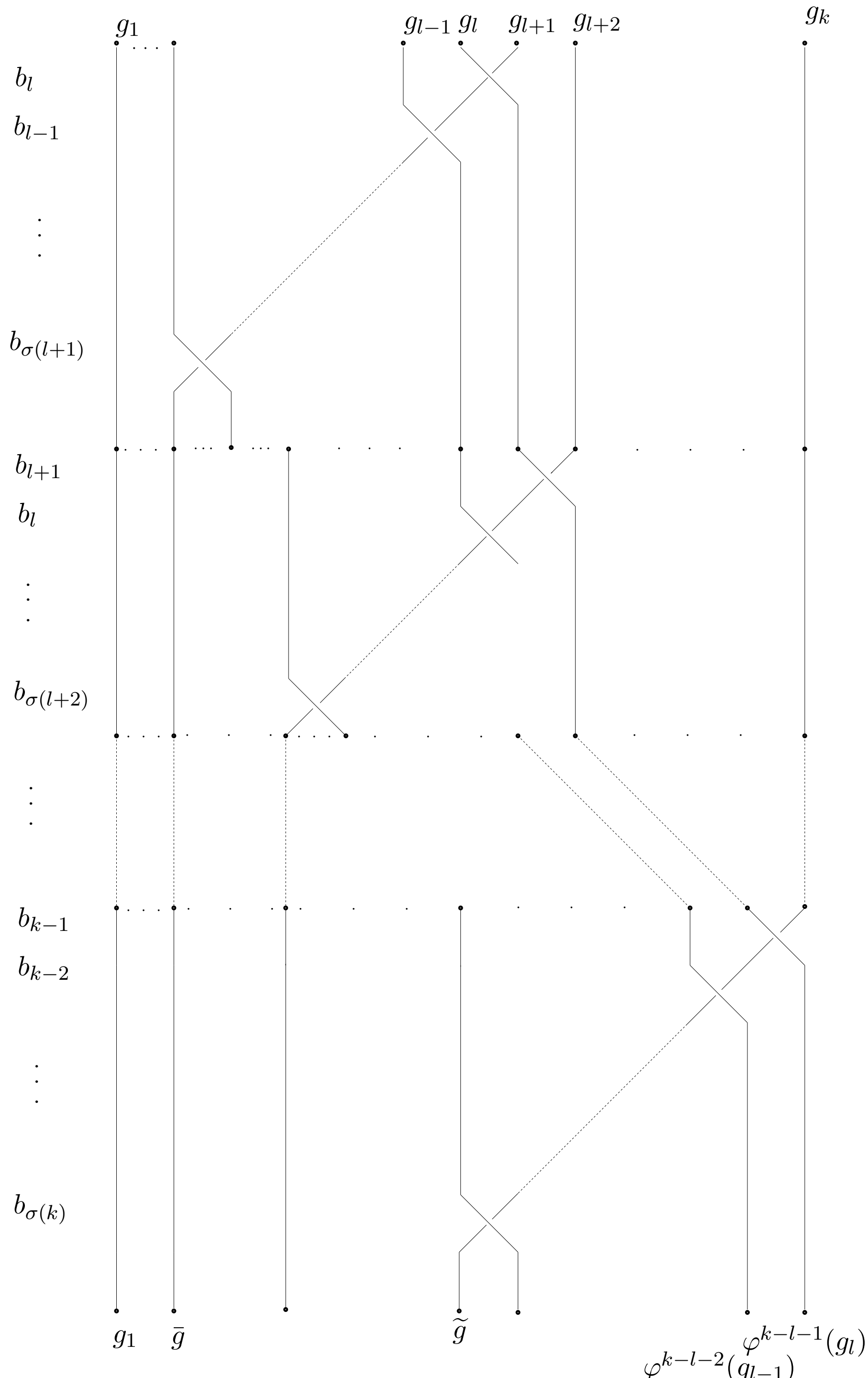
This is graphically represented in Figure 1.5.

$$tw_R^{\varphi}(a \otimes b) = (-1)^{l(k-l)}(\varphi^{-(l+2)}(g_{l+1}), \varphi^{(l-1)}(g_{l+2}), \ldots, \varphi^{-(k-l-1)}(g_k))$$

$$\otimes\, (\varphi^{(-k-l-2)}(g_1, \ldots,), \ldots, \varphi^{-2}(g_k \cdots g_{l+1})^{-1}g_{l-1}\varphi^{-1}(g_k \ldots, g_{l+1}g_l), \varphi^{-1}(g_l \cdot b))$$

This is graphically represented in Figure 1.6.

Remark 1.5.4. $tw_R^{\varphi} \circ tw_L^{\varphi} = tw_L^{\varphi} \circ tw_R^{\varphi} = id.$



$\widetilde{g} = \varphi^{k-r-2}(g_{r_k}) \dots \varphi^{k-l-2}(g_1)\, g_k \varphi^{k-r-1}(g_{r_k})^{-1}$

$\bar{g} = g_{r_{l+1}} \dots g_{l+1}\, \varphi(g_{r_{l+1}} \dots g_l)^{-1}$

Figure 1.4: Graphic representation of the left-shuffle product.

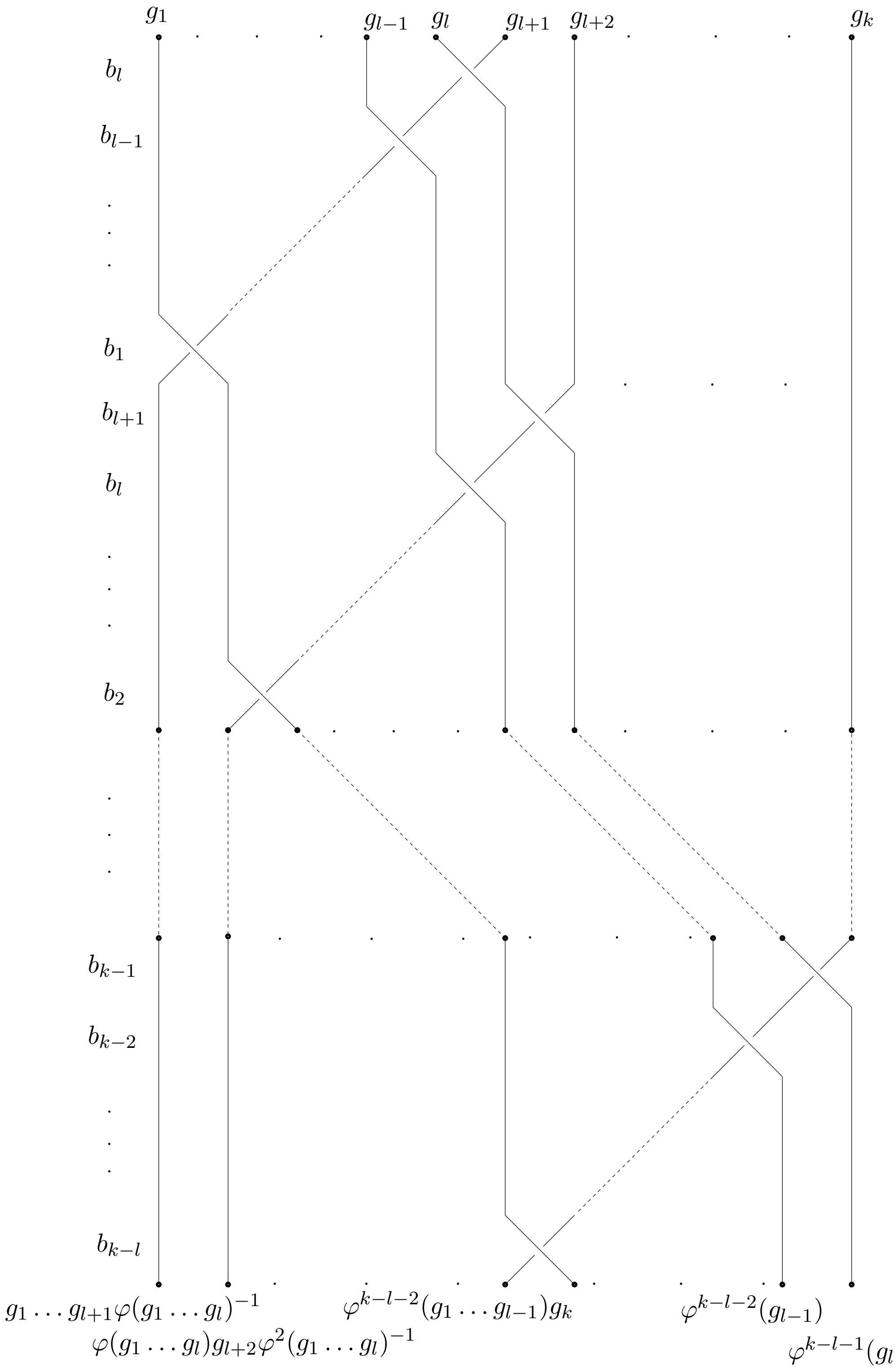


Figure 1.5: Graphic representation of the left twist map.

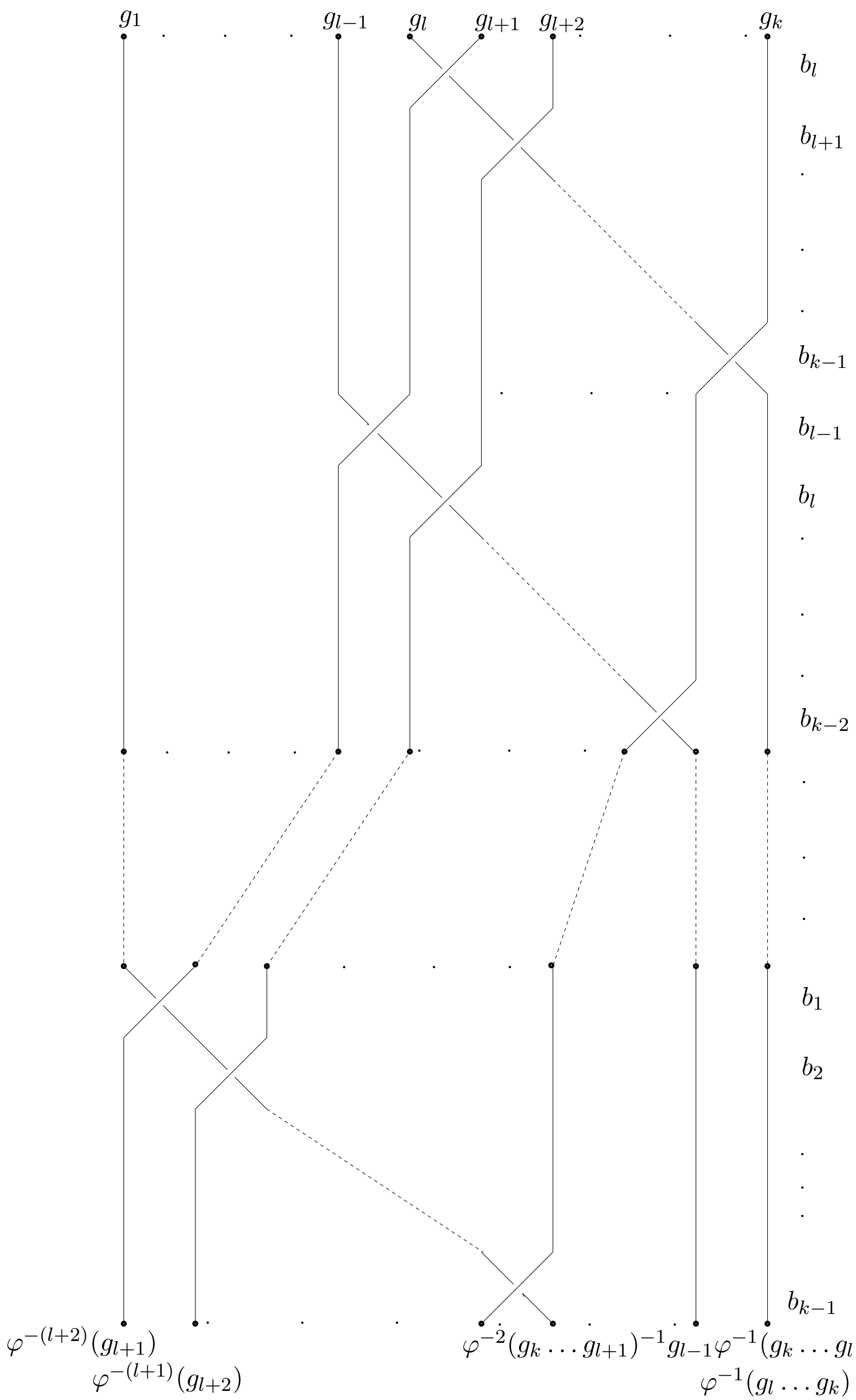


Figure 1.6: Graphic representation of the right twist map.

Proposition 1.5.5. *Δ is an algebra homomorphism for μ^φ_R and for μ^φ_L; i.e*

$$\Delta \circ \mu^\varphi_L \quad = \quad (\mu^\varphi_L \otimes \mu^\varphi_L) \circ (id \otimes tw^\varphi_L \otimes id) \circ (\Delta \otimes \Delta)$$

$$\Delta \circ \mu^\varphi_R \quad = \quad (\mu^\varphi_R \otimes \mu^\varphi_R) \circ (id \otimes tw^\varphi_R \otimes id) \circ (\Delta \otimes \Delta),$$

respectively.

Proof We only will prove the first equality, because the proof for the second equality is similar. Let $a = (a_1, \dots, a_s) \in T_s G$ and $b = (b_1, \dots, b_t) \in T_t G$. Let $s' \in \{0, \dots s\}$ and $t' \in \{0, \dots, t\}$. Let σ_1 and σ_2 denote a fixed (s', t') and $(s - s', t - t')$- shuffles respectively. We have:

$$((\mu^\varphi_L \otimes \mu^\varphi_L) \circ (id \otimes tw^\varphi_L \otimes id) \circ (\Delta \otimes \Delta)(a \otimes b))_{s',t',\sigma_1,\sigma_2:} =$$

$$= (S^\varphi_(L, \sigma_1) \otimes S^\varphi_(L, \sigma_2)) \circ (id \otimes tw^\varphi_L \otimes id)(\Delta_s(a_1, \dots, a_s) \otimes \Delta_t(b_1, \dots, b_t))$$

$$= (S^\varphi_{L,\sigma_1} \otimes S^\varphi_{L,\sigma_2})((-1)^{(s-s')t'}((a_1, \dots, a_{s'}) \otimes t^\varphi_L(a, b)_{s-s',t'}) \otimes (b_{t'+1}, \dots, b_t)$$

$$= (-1)^{(s-s')t'} S^\varphi_L(a, b, \sigma_1)_{s',t'} \otimes S^\varphi_L(a, b, \sigma_2)_{s-s',t-t'}$$

Now, consider the permutation $\sigma_0 \in \Sigma_{s+t}$ which is given by:

$$\{1, \dots, s + t\} \longrightarrow \{1, \dots, s + t\}$$

$$i \longmapsto \begin{cases} i & \text{if } 1 \leq i \leq s' \\ i + t' & \text{if } s' + 1 \leq i \leq s \\ i - (s - s') & \text{if } s + 1 \leq i \leq s + t' \\ i & \text{if } s + t' + 1 \leq i \leq s + t \end{cases}$$

Clearly, $\operatorname{sgn}(\sigma_0) = (-1)^{(s-s')t'}$. On the other hand, let $\sigma'_1 \in \Sigma_{s+t}$ denote the permutation that coincides with σ_1 in the first $k + l$ positions, and the identity in the remained positions. Let $\sigma'_2 \in \Sigma_{s+t}$ denote the permutation that coincides with σ_2 in the last $s + t - (k + l)$ positions, and the identity in the remained positions. It is not difficult to see that $\sigma' := \sigma'_1 \ . \ \sigma'_2 \ . \ \sigma_0$ is a (s, t)-shuffle.

We have:

$$\begin{aligned} (\Delta \circ \mu^\varphi_L)_{s'+t',\sigma'}(a \otimes b) &:= \Delta_{s'+t'} \circ S^\varphi_L(a, b, \sigma') \\ &= (-)^{(s-s')t'} \Delta_{s'+t'}(S^\varphi_L(a, b, \sigma'_1 \sigma'_2)) \\ &= (-1)^{(s-s')t'}(S^\varphi_L(a, b, \sigma_1) \otimes S^\varphi_L(a, b, \sigma_2)), \end{aligned}$$

$supp(\sigma'_1) \subseteq \{1, \dots, s' + t'\}$ and $supp(\sigma'_2) \subseteq \{s' + t' + 1, \dots, s + t\}$.

∎

1.6 The antipode maps S_L^φ and S_R^φ.

Before we define the antipode maps, we need to recall the definition of a *convolution product.*

Definition 1.6.1. *Given an algebra (A, μ, η) and a coalgebra (C, Δ, ϵ), and given $f, g \in Hom(C, A)$, then its convolution product $\star$, is defined by the following commutative diagram:*

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[drr, "f\star g"'] & C \otimes C \arrow[r, "f\otimes g"] & A \otimes A \arrow[d, "\mu"] \\ & & A \end{tikzcd}$$

Definition 1.6.2. *Let $(H, \mu, \eta, \Delta, \epsilon)$ be a Hopf algebra. An endomorphism S of H is called an antipode for H if*

$$S \star id_H = id_H \star S = \eta \circ \epsilon.$$

Therefore, to define an antipode S_L^φ for $(\mathcal{H}^\varphi(G), \mu_L^\varphi, \Delta, \eta, \epsilon)$ we must have the following equalities:

$$\mu_L^\varphi \circ (id \otimes S_L^\varphi) \circ \Delta = \eta \circ \epsilon = \mu_L^\varphi \circ (S_L^\varphi \otimes id) \circ \Delta,$$

and for defining an antipode for $(\mathcal{H}^\varphi(G), \mu_R^\varphi, \Delta, \eta, \epsilon)$ we have to have the following equalities:

$$\mu_R^\varphi \circ (id \otimes S_R^\varphi) \circ \Delta = \eta \circ \epsilon = \mu_R^\varphi \circ (S_R^\varphi \otimes id) \circ \Delta.$$

Theorem 1.6.3. *For $(\mathcal{H}^\varphi G, \mu_L^\varphi, \Delta, \eta, \epsilon)$ and $(\mathcal{H}^\varphi G, \mu_R^\varphi, \Delta, \eta, \epsilon)$ there are unique antipodes*

$$S_L^\varphi : T_k G \to T_k G \quad \textit{and} \quad S_R^\varphi : T_k G \to T_k G$$

defined as:

$$S_L^\varphi(g_1, \dots, g_k) = (-1)^{\lceil \frac{k}{2} \rceil}\, b_{k-1} \,.\, (b_{k-2} b_{k-1}) \dots (b_1 \dots b_{k-2} b_{k-1}) \,.\, (g_1, \dots, g_k),$$

$$S_R^\varphi(g_1, \dots, g_k) = (-1)^{\lceil \frac{k}{2} \rceil}(g_1, \dots, g_k) \,.\, (b_1 \dots b_{k-1}) \dots (b_1 b_2) \,.$$

These antipodes are given by using de definition of the action of the braid group (Lemma 1.3.2), as: (see Figure 1.7 and 1.8).

$$\begin{aligned}
S_L^{\varphi}(g_1,\dots,g_k) &= (-1)^{\lceil \frac{k}{2}\rceil}(g_1\dots g_k\varphi(g_1\dots g_{k-1})^{-1},\varphi(g_1\dots g_{k-1})\varphi^2(g_1\dots g_{k-2})^{-1},\dots,\\
&\quad \varphi^{k-2}(g_1)\varphi^{k-2}(g_2)\varphi^{k-1}(g_1)^{-1},\varphi^{k-1}(g_1))
\end{aligned}$$

and

$$\begin{aligned}
S_R^{\varphi}(g_1,\dots,g_k) &= (-1)^{\lceil \frac{k}{2}\rceil}(\varphi^{-(k-1)}(g_k),\varphi^{-(k-1)}(g_{k-1})^{-1}\varphi^{-(k-2)}(g_kg_{k-1}),\dots,\\
&\quad \varphi^{-2}(g_3\dots g_k)\varphi^{-1}(g_2\dots g_k),\varphi^{-1}(g_2\dots g_k)g_1\dots g_k))
\end{aligned}$$

Proof of Theorem 1.6.3 We only do the proof the theorem for S_L^{φ}, because the proof for S_R^{φ} is similar.

Induction on the lenght k.

For $k=0$

$$\begin{aligned}
\mu_L^{\varphi}\circ(id\otimes S_L^{\varphi})\circ\Delta(1) &= \mu_L^{\varphi}\circ(S_L^{\varphi}\otimes id)\circ\Delta(1)\\
&= S_L^{\varphi}(1)=1=\eta\circ\epsilon(1)
\end{aligned}$$

For $k=1$,

$$\begin{aligned}
\mu_L^{\varphi}\circ(id\otimes S_L^{\varphi})\circ\Delta(g_1,g_2) &= \mu_L^{\varphi}\circ(id\otimes S_L^{\varphi})[1\otimes(g_1,g_2)+g_1\otimes g_2+(g_1,g_2)\otimes 1]\\
&= \mu_L^{\varphi}[1\otimes(g_1,g_2)-g_1\otimes g_2-(g_1g_2\varphi(g_1)^{-1},\varphi(g_1))\otimes 1]\\
&= (g_1,g_2)-(g_1,g_2)+(g_1g_2\varphi(g_1)^{-1},\varphi(g_1))-(g_1g_2\varphi(g_1),\varphi(g_1))\\
&= 0
\end{aligned}$$

The last equality follows by definition $\eta\circ\epsilon=0$ for all $k>0$.

Induction step:

Let $(g_1,\dots,g_k)\in T_kG$. Then

$\mu_L^{\varphi}\circ(id\otimes S_L^{\varphi})\circ\Delta(g_1,\dots,g_k)=$

$=\mu_L^{\varphi}\circ(id\otimes S_L^{\varphi})(\sum_{l=0}^{k}(g_1,\dots,g_l)\otimes(g_{l+1},\dots,g_k))$

$=\mu_L^{\varphi}(\sum_{l=0}^{k}(g_1,\dots,g_l)\otimes S_L^{\varphi}(g_{l+1},\dots,g_k))$

$=S_L^{\varphi}(g_1,\dots,g_k)+\mu_L^{\varphi}(\sum_{l=1}^{k-1}(g_1,\dots,g_l)\otimes S_L^{\varphi}(g_{l+1},\dots,g_k))+(g_1,\dots,g_k)$

$=S_L^{\varphi}(g_1,\dots,g_k)+\mu_L^{\varphi}(\sum_{l=1}^{k-1}(g_1,\dots,g_l)\otimes(-1)^{\lceil\frac{k-l}{2}\rceil}(g_{l+1}\dots g_k\varphi(g_l\dots g_k)^{-1},\varphi(g_{l+1}\dots g_k)\varphi^2(g_l\dots g_k)^{-1},$

$, \ldots, \varphi^{k-2}(g_l)\varphi^{k-2}(g_{l+1}\varphi^{k-1}(g_l)^{-1}, \varphi^{k-1}(g_l)) + (g_1, \ldots, g_k)$

By induction step and analyzing the above formula one can easily see that the last shuffle product on the sum will cancel the element $(g_1, \ldots, g_k)$ and that all the other elements will cancel each other up to the tuple

$$(-1)^{k-1}(-1)^{\lceil \frac{k-1}{2} \rceil}(g_1 \ldots g_{k-1}\varphi(g_1 \ldots g_k)^{-1}, \ldots, \varphi^{k-1}(g_1)).$$

Therefore, we must have

$$S_L^{\varphi}(g_1, \ldots, g_k) + (-1)^{k-1}(-1)^{\lceil \frac{k-1}{2} \rceil}(g_1 \ldots g_{k-1}\varphi(g_1 \ldots g_k)^{-1}, \ldots, \varphi^{k-1}(g_1)) = 0.$$

From this follows $\mu_L^{\varphi} \circ (id \otimes S_L^{\varphi}) \circ \Delta = \eta \circ \epsilon$.

Now S_L^{φ} is unique, because if there is another $\tilde{S}_L^{\varphi}$ such that

$$\tilde{S}_L^{\varphi} \star id = id \star \tilde{S}_L^{\varphi} = \eta \circ \epsilon$$

then

$$S^{\varphi} = S^{\varphi} \star (\eta\epsilon) = S^{\varphi} \star (id \star \tilde{S}^{\varphi}) = (S^{\varphi} \star id) * \tilde{S}^{\varphi} = (\eta\epsilon) \star \tilde{S}^{\varphi} = \tilde{S}^{\varphi}.$$

A similar argument will prove that S_R^{φ} is an antipode for $(\mathcal{H}^{\varphi}(G), \mu_R^{\varphi}, \Delta, \eta, \epsilon)$.

∎

The above theorem proves that $(\mathcal{H}^{\varphi}(G), \mu_L^{\varphi}, \Delta, S_L^{\varphi}, \eta, \epsilon)$ and $(\mathcal{H}^{\varphi}(G), \mu_R^{\varphi}, \Delta, S_R^{\varphi}, \eta, \epsilon)$ are graded Hopf algebras with an antipode map.

Lemma 1.6.4. *The antipode maps S_L^{φ} and S_R^{φ} are invertible. Namely we have*

$$S_L^{\varphi} \circ S_R^{\varphi} = S_R^{\varphi} \circ S_L^{\varphi} = id.$$

Proof Induction on the lenght k.

For $k = 0$

$(S_L^{\varphi} \circ S_R^{\varphi})(1) = S_R^{\varphi}(1) = 1$

For $k = 2$, let $(g_1, g_2) \in T_2G$ then

$$(S_L^{\varphi} \circ S_R^{\varphi})(g_1, g_2) = -S_L^{\varphi}\varphi(g_2)^{-1}, \varphi^{-1}(g_2g_1)g_1g_2)) = (g_1, g_2)$$

Now assume the result for $k - 1$, i.e. for all $(g_1, \ldots, g_{k-1}) \in T_{k-1}G$ we have

$$S_L^{\varphi} \circ S_R^{\varphi}(g_1, \ldots, g_{k-1}) = (g_1, \ldots, g_{k-1}).$$

So, let $(g_1, \dots, g_k) \in T_kG$ then:

$$
\begin{aligned}
(S_L^\varphi \circ S_R^\varphi)(g_1, \dots, g_k) &= S_L^\varphi((-1)^{\lceil \frac{k}{2} \rceil}(\varphi^{-(k-1)}(g_k), \dots, \varphi^{-1}(g_2, \dots, g_k)g_1 \dots g_k) \\
&= (g_1, \dots, g_k)
\end{aligned}
$$

The last equality follows by the induction step $k-1$.

∎

Proposition 1.6.5. *Let μ^φ, S^φ, Δ be defined as before. Then*

$$
\mu^\varphi \circ (\mu^\varphi \otimes 1) \circ (1 \otimes S^\varphi \otimes 1) \circ (\Delta \otimes 1) \circ \Delta = 1.
$$

Proof By definition of S^φ we have:

$$
\mu^\varphi \circ (S^\varphi \otimes 1) \circ \Delta = \mu^\varphi \circ (1 \otimes S^\varphi) \circ \Delta = \eta \circ \epsilon \tag{1.6.1}
$$

From it follows that:

$$
\begin{aligned}
\mu^\varphi \circ (\mu^\varphi \bar{\otimes} 1) \circ (1 \bar{\otimes} S^\varphi \bar{\otimes} 1) \circ (\Delta \bar{\otimes} 1) \circ \Delta &= \mu^\varphi \otimes [(\eta \circ \epsilon) \bar{\otimes} 1] \circ \Delta \\
&= \mu^\varphi \circ [(\eta \bar{\otimes} 1) \circ (\epsilon \bar{\otimes} 1)] \circ \Delta
\end{aligned}
$$

Let $(x_1, \dots, x_k) =: x$ be generator of T_kG, then:

case 1: If $x = 1$, then:

$$
\begin{aligned}
\mu^\varphi \circ (\mu^\varphi \bar{\otimes} 1) \circ (1 \bar{\otimes} S^\varphi \bar{\otimes} 1) \circ (\Delta \bar{\otimes} 1) \circ \Delta(1) &= \mu^\varphi \otimes [(\eta \circ \epsilon) \bar{\otimes} 1](1 \bar{\otimes} 1) \\
&= \mu^\varphi \circ [(\eta \bar{\otimes} 1) \circ (1 \bar{\otimes} 1)] = 1
\end{aligned}
$$

Case 2: If $|x| > 0$, then:

$$
\begin{aligned}
(\epsilon \bar{\otimes} 1) \circ \Delta(x_1, \dots, x_k) &= (\epsilon \bar{\otimes} 1)[\textstyle\sum_{l=0}^k (x_1 \otimes \cdots \otimes x_l) \bar{\otimes} (x_{l+1} \otimes \cdots \otimes x_k)] \\
&= \textstyle\sum_{l=0}^k \epsilon(x_1 \otimes \cdots \otimes x_l) \bar{\otimes} (x_{l+1} \otimes \cdots \otimes x_k) \\
&= 1 \bar{\otimes} (x_1 \otimes \cdots \otimes x_k)
\end{aligned}
$$

The last equality holds by definition $\epsilon(x_1 \otimes \cdots \otimes x_l) = 0$ for all $l > 0$. Therefore, we get:

$$
\begin{aligned}
\mu^\varphi \circ (\eta \bar{\otimes} 1)(1 \bar{\otimes} (x_1 \otimes \cdots \otimes x_k) &= \mu^\varphi (1 \bar{\otimes} (x_1 \bar{\otimes} (x_1 \otimes \cdots \otimes x_k) \\
&= (x_1 \otimes \cdots \otimes x_k)
\end{aligned}
$$

∎

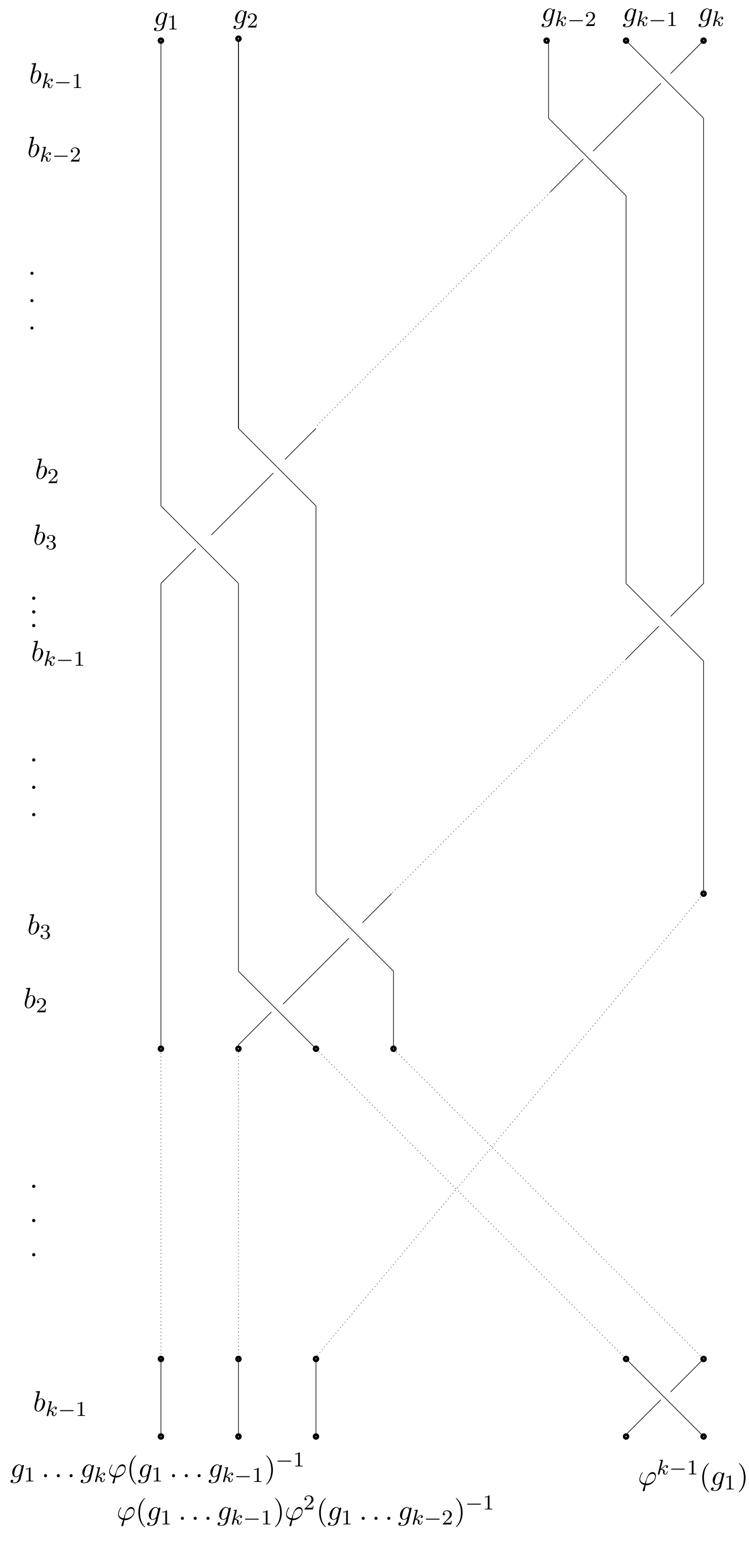


Figure 1.7: Graphic respresentation of the left antipode map.

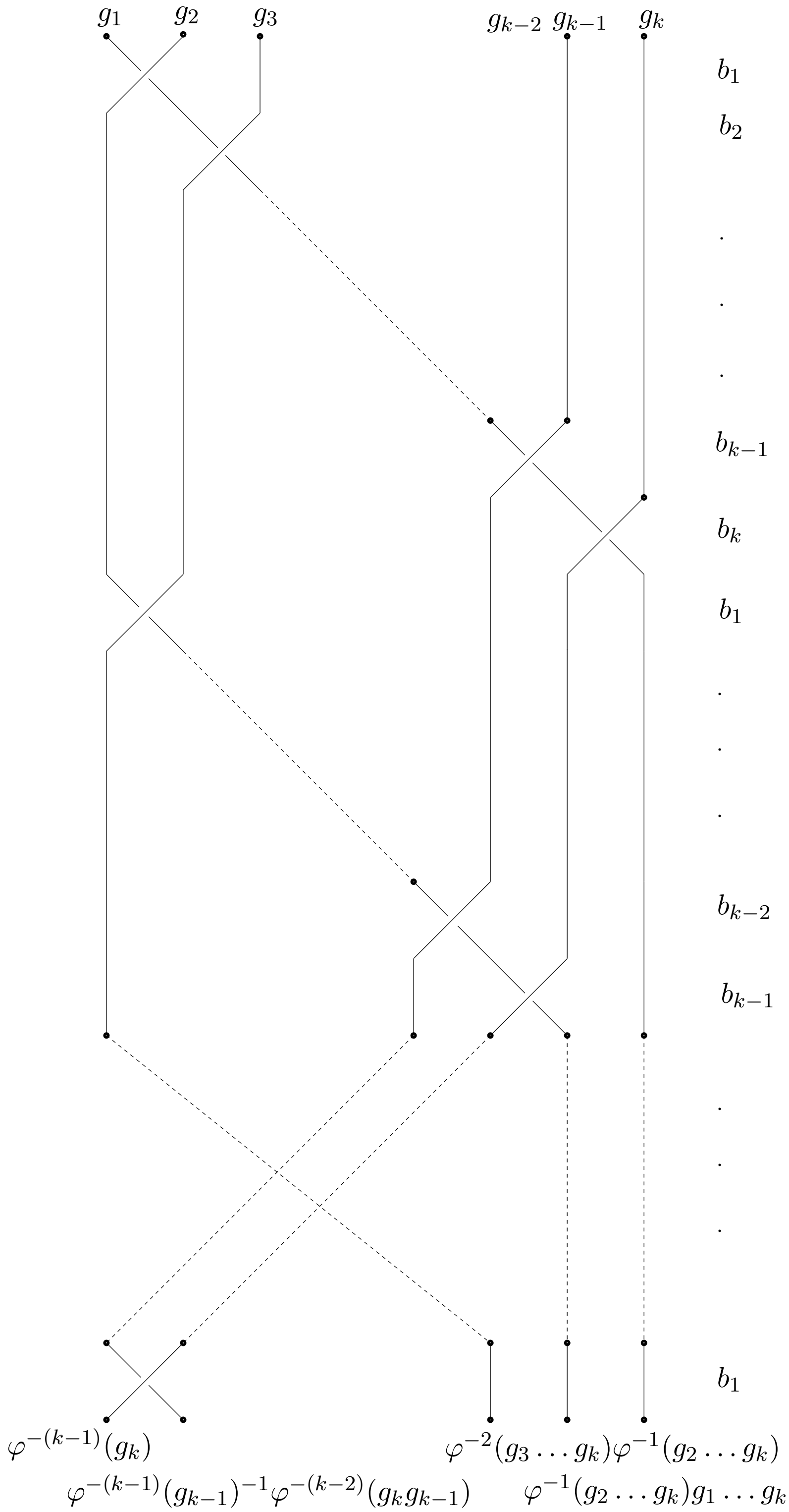


Figure 1.8: Graphic representation of the right antipode map.

Chapter 2

The Yang-Baxter Equation and knot invariants

In this chapter we recall the definition of enhanced Yang Baxter operator introduced in [14]. Just like in the case when we have solutions of the Yang-Baxter equation, we give a lemma that allows to construct new enhancements from old ones. We give a survey about Turaev's work ([14], Thm. 2.3.1, Thm. 3.1.2). Based on Turaev's work, we prove that the twisted conjugation braiding B^{φ} introduced in chapter 2, is an enhanced Yang-Baxter operator for any finite group G. In the last section, we prove that for the twisted conjugation braiding B^{φ}, the link invariant $T_{\mathcal{B}}$ is

$$T_{\mathcal{B}}(\xi) = \beta^{-n}\ \mathrm{trace}(b_{B^{\varphi}}(\xi) \circ D^{\otimes n}), \quad \text{for all braid } \ \xi \in Br(n).$$

2.1 Traces and partial traces

In this section, we recall the definition of trace of a homomorphism $f : V \to V$. Moreover, we recall the definition of partial trace and its properties.

Notation and agreements

Here $\mathbb{K}$ denotes a fixed commutative ring with 1, and V denotes a fixed finitely generated free $\mathbb{K}$-module of rank $m \geq 1$. For $n \geq 0$ the n-fold tensor product $V \otimes_{\mathbb{K}} V \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} V$ is denoted by $V^{\otimes n}$. Each basis $v_1, \ldots, v_m$ in V gives rise to a basis in $V^{\otimes n}$ which consists of vectors $v_{i_1} \otimes \cdots \otimes v_{i_n}$, with $i_1, \ldots, i_n \in \{1, 2, \ldots, m\}$. In this basis, each endomorphism f of $V^{\otimes n}$ determines the multindexed matrix $(f^{j_1,\ldots,j_n}_{i_1,\ldots,i_n})$, $1 \leq i_1, j_1, \ldots, i_n, j_n \leq m$ defined by the equation

$$f(v_{i_1} \otimes \cdots \otimes v_{i_n}) = \sum_{1 \leq j_1,\ldots,j_n \leq m} f^{j_1,\ldots,j_n}_{i_1,\ldots,i_n} v_{j_1} \otimes \cdots \otimes v_{j_n}.$$

Definition 2.1.1. *For each homomorphism $f : V^{\otimes n} \to V^{\otimes n}$ its partial trace (on the k-th factor) $Sp_k(f)$ is the homomorphism $V^{\otimes(n-1)} \to V^{\otimes(n-1)}$, where $k \in \{1, \ldots, n\}$ is given as follows.*

For any $i_1, \ldots, i_{n-1} \in \{1, 2, \ldots, m\}$

$$Sp_k(f)(v_{i_1} \otimes \cdots \otimes v_{i_{n-1}}) = \sum_{1 \leq j_1, \ldots, \widehat{j}_k, \ldots, j_n \leq m} (\sum_{j_k=1}^{m} f_{i_1, \ldots, j_k, \ldots, i_{n-1}}^{j_1, \ldots, j_k, \ldots, j_n}) v_{j_1} \otimes \cdots \otimes \widehat{v}_{j_k} \otimes \cdots \otimes v_{j_n},$$

Lemma 2.1.2. *The partial trace $Sp_k(f)$ does not depend on the choice of a basis of V.*

Proof We do the proof for $n = 2$, i.e. when $f : V^{\otimes 2} \to V^{\otimes 2}$. A similar argument will prove the result in the case when we consider homomorphisms $f : V^{\otimes n} \to V^{\otimes n}$.

We have to prove

$$Sp_2((A \otimes A) \circ f \circ (A \otimes A)^{-1}) = A \circ Sp_2(f) \circ A^{-1},$$

where $A = [a_{ij}]$

Notation

1. Fix a basis $\mathcal{B} = \{v_1, \ldots, v_m\}$ of V then we get a basis $\mathcal{B}' = \{v_1 \otimes v_1, v_1 \otimes v_2, \ldots, v_m \otimes v_m\}$. Notice that this basis comes with a given order, namely the lexicographic order.
2. On the basis $\mathcal{B}$, the homomorphism f has the following matrix representation (by blocks),

$$[f(i,j)] := \begin{pmatrix} f(1,1) & \ldots & f(1,m) \\ \ldots & \ldots & \ldots \\ f(m,1) & \ldots & f(m,m) \end{pmatrix}$$

   where each $f(i,j)$ is a square $m \times m$ matrix .Notice that $[f(i,j)]$ is a $m^2 \times m^2$ matrix composed by m^2 matrices of size $m \times m$.
3. On the basis $\mathcal{B}'$, the partial trace $Sp_2(f) : V \to V$ has an associated $m \times m$ matrix $S = [S_{i,j}]$, with the entry $D_{i,j} = \text{trace } (f(i,j))$.
4. If $A = [a_{ij}]$, then $A \otimes A = [a_{ij} A]$
5. If A is invertible, then $(A \otimes A)^{-1} = A^{-1} \otimes A^{-1}$

Parts (2) and (3) of previous Remark imply that

$$\begin{aligned} A \circ Sp_2(f) \circ A^{-1} &= [a_{ij}] \, [\text{trace } (f(j,k))] \, [b_{kl}] \\ &= [\textstyle\sum_{j,k} a_{ij} \, \text{trace } (f(j,k)) \, b_{kl}] \end{aligned}$$

$$
\begin{aligned}
(A \otimes A) \circ f \circ (A \otimes A)^{-1} &= [a_{ij} A]\, [f(j,k)]\, [b_{kl}\, A^{-1}] \\
&= [\textstyle\sum_{j,k} (a_{ij} A\ f(j,k)\ A^{-1})\ b_{kl}]
\end{aligned}
$$

Using part (4) and (5) of previous remark we get :

$$
\begin{aligned}
Sp_2((A \otimes A) \circ f \circ (A \otimes A)^{-1}) &= [\mathrm{trace}(\textstyle\sum_{j,k} (a_{ij} A\ f(j,k)\ A^{-1})\ b_{kl})] \\
&= [\textstyle\sum_{j,k} a_{ij} \mathrm{trace} A\ f(j,k)\ A^{-1}]\ b_{kl}] \\
&= [\textstyle\sum_{j,k} a_{ij} \mathrm{trace}\ (f(j,k))\ b_{kl}] \\
&= A \circ Sp_2(f) \circ A^{-1}
\end{aligned}
$$

The last equality follows from the fact that trace is invariant under change of basis.

∎

Remark In general, we have

$$Sp_1(f) \neq Sp_2(f)$$

Because, let $dim V = 2$, then with the above notations we have:

$$
f(1,1) := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad f(1,2) := \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}
$$

$$
f(2,1) := \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \quad f(2,2) := \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}
$$

From it follows that the $f : V^{\otimes 2} \to V^{\otimes 2}$, has the following matrix representation:

$$
A(f) = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ c_{11} & c_{12} & d_{11} & d_{12} \\ c_{21} & c_{22} & d_{21} & d_{22} \end{pmatrix}
$$

Moreover, we have:

$$
Sp_1(f) = \begin{pmatrix} a_{11} + d_{11} & a_{12} + d_{12} \\ a_{21} + d_{21} & a_{22} + d_{22} \end{pmatrix} \neq \begin{pmatrix} a_{11} + a_{22} & b_{11} + b_{22} \\ c_{11} + c_{22} & d_{11} + d_{22} \end{pmatrix} = Sp_2(f)
$$

Lemma 2.1.3. *If f, g, h are endomorphisms of $V^{\otimes(n+1)}, V^{\otimes n}, V^{\otimes k}$ $(n \geq k)$ respectively then,*

1. *$trace(Sp_k(g)) = trace(g)$, where trace is the ordinary trace of a homomorphism.*

2. *$Sp_{n+1}((g \otimes Id_V) \circ f)) = g \circ Sp_{n+1}(f)$,*

3. $Sp_{n+1}(f \circ (g \otimes Id_V)) = Sp_{n+1}(f) \circ g$,

4. $Sp_{n+1}(Id_V^{\otimes(n-k)} \otimes h) = Id_V^{\otimes(n-k)} \otimes Sp_{n-k}(h)$

Proof We do the proof for (2), since (3) and (4) will hold by a similar argument. First of all, we have that:

$$\begin{aligned}((f \otimes Id) \circ g) &= \sum_{j_1,\ldots,j_{n+1}} f \otimes Id(g^{i_1,\ldots,i_{n+1}}_{j_1,\ldots,j_{n+1}} v_{j_1,\ldots,j_{n+1}}) \\ &= \sum_{k_1,\ldots,k_n} \sum_{j_1,\ldots,j_{n+1}} f^{j_1,\ldots,j_n}_{k_1,\ldots,k_n} g^{i_1,\ldots,i_{n+1}}_{j_1,\ldots,j_{n+1}} v_{k_1,\ldots,k_n,j_{n+1}}.\end{aligned}$$

Notice that all summands in above equation range from 1 to n. Now, by the definition of partial trace on the $n+1$ factor, we get:

$$\begin{aligned}Sp_{n+1}((f \otimes id) \circ g)(v_{i_1}, \ldots, v_{i_n}) &= \sum_{k_1,\ldots,k_n} \sum_{k_{n+1}} ((f \otimes id) \circ g)_{k_1}, \ldots, k_n, k_{n+1}{}^{i_1,\ldots,i_n,k_{n+1}} v_{k_1,\ldots,k_n} \\ &= \sum_{k_1,\ldots,k_n} \sum_{k_{n+1}} \sum_{j_1,\ldots,j_n} f^{j_1,\ldots,j_n}_{k_1,\ldots,k_n} g^{i_1,\ldots,i_n,k_{n+1}}_{j_1,\ldots,j_n,k_{n+1}} v_{k_1,\ldots,k_n} \\ &\quad \sum_{j_1,\ldots,j_n} \sum_{k_{n+1}} f(g^{i_1,\ldots,i_n,k_{n+1}}_{j_1,\ldots,j_n,k_{n+1}} v_{j_1,\ldots,j_n}) \\ &= (f \otimes Sp_{n+1})(v_{i_1,\ldots,i_n})\end{aligned}$$

∎

There is an equivalent definition of partial trace Sp_k on the k-th factor, for an endomorphism $f : V^{\otimes n} \to V^{\otimes n},\ k \in \{1, \ldots, n\}$, which sometimes will be useful for avoiding nasty computations.

Recall that $End(V^{\otimes n}) \cong End(V)^{\otimes k-1} \otimes End(V) \otimes End(V)^{\otimes(n-k)}$, where $End(V)$ denotes the group of endomorphisms of V. Denote this isomorphism by $\bar{\lambda}$.

Definition 2.1.4. *The partial trace on the $k-th$-factor Sp_k, is defined by the following commutative diagram*

$$\begin{tikzcd} End(V^{\otimes n}) \arrow[r, "Sp_k"] \arrow[d, "\bar{\lambda}"'] & End(V^{\otimes(n-1)}) \\ End(V)^{\otimes(k-1)} \otimes End(V) \otimes End(V)^{\otimes(n-k)} \arrow[r, "\tilde{\Phi}"'] & End(V)^{\otimes(k-1)} \otimes \mathbb{K} \otimes End(V)^{(n-k)} \arrow[u, "\cong"'] \end{tikzcd}$$

with $\tilde{\Phi} := id^{\otimes(k-1)} \otimes trace \otimes id^{\otimes(n-k)}$.

As an example we have the following:

Example If $f(v_i \otimes v_j) = \sum_{k,l=1}^{m} f_{i,j}^{k,l} v_k \otimes v_l$, then

$$Sp_2(f)(v_i) = \sum_{j,k=1}^{m} f_{i,j}^{k,j} v_k \text{ and } Sp_1(f)(v_j) = \sum_{i,l=1}^{m} f_{i,j}^{i,l} v_l.$$

Moreover, $Sp_1(Sp_1(f)) = Sp_1(Sp_2(f)) = \mathrm{trace}(f)$.

2.2 Enhanced Yang-Baxter operator

In this section, we recall the notion (due to Turaev, [14]) of an Enhanced Yang-Baxter operator. For simplicity, we will write EYB-operator. Moreover, we give some examples of EYB-operators and a lemma which allows to construct new EYB-operators from old ones, just like in the case when we have a solution of the Yang-Baxter equation. At the end of this section, we recall a theorem due to Turaev ([14], Thm.), which restates conditions $(T1), (T2)$ of the definition of a EYB-operator such that a solution of the Yang-Baxter equation $R : V^{\otimes 2} \to V^{\otimes 2}$ is a EYB-operator, when the map $D : V \to V$, is defined as $D(v_i) = a_i v_i$, with v_i element of the basis of the vector space V and $a_i \in \mathbb{K}^*$, for all $i \in \{1, \ldots, m\}$.

Definition 2.2.1. *Let V be a free module of finite rank over a commutative ring $\mathbb{K}$. An enhanced (quantum) Yang-Baxter Operator on $V \otimes V$ is a quadruple (R, D, λ, β) consisting of an invertible solution $R \in End(V \otimes V)$ of the Yang-Baxter equation and a map $D \in End(V)$, such that*

(T1) $(D \otimes D) \circ R = R \circ (D \otimes D)$

(T2a) $Sp_2(R \circ (D \otimes D)) = \lambda\beta\ D$, *and*

(T2b) $Sp_2(R^{-1} \circ (D \otimes D)) = \lambda^{-1}\beta\ D$

where λ, β are invertible elements of the ring $\mathbb{K}$

Remark 2.2.2. *1. If D is an invertible map, condition $(T2a)$, and $(T2b)$ of Definition 2.2.1 are equivalent to*

$$Sp_2(R^{\pm 1} \circ (1 \otimes D)) = \lambda^{\pm 1} \beta Id_V,$$

because we can write $(D \otimes D) = (1 \otimes D)\ (D \otimes 1)$, and thus, the claim follows from Lemma 2.1.3.

2. *It is not loss of generality to assume that $\lambda, \beta = 1$ in the above definition, for (R, D, λ, μ) is an enhanced Yang-Baxter operator, $(\lambda^{-1}R, \beta^{-1}D, 1, 1)$ is one too. However, it is not always covenient to make this normalization.*

Example 1. 1. *Let V be a vector space of dimension 1. For each solution $R = (\alpha)$, $\alpha \in \mathbb{K}^*$ of the Yang-Baxter equation and $D = (\gamma)$, $\gamma \in \mathbb{K}^*$. Then, the quadruple $\mathcal{R} = (R, D, \lambda, \beta)$ is an enhanced Yang-Baxter operator.*

2. *Consider the following solution of the Yang-Baxter equation*

$$R = \begin{pmatrix} 1 & . & . & . \\ . & . & q & . \\ . & q & 1-q^2 & . \\ . & . & . & 1 \end{pmatrix}$$

with $q \in \mathbb{C}$ an invertible element.

The quadruple $\mathcal{R} = (R, D, \lambda = \pm 1, \beta = \pm 1)$, is a EYB-operator, where D is given as follows:

$$\begin{pmatrix} 1 & 0 \\ 0 & q^2 \end{pmatrix}$$

The following lemma gives a method how to construct new enhanced Yang-Baxter operators from old ones.

Lemma 2.2.3.

1. *Let $\mathcal{R} = (R, D, \lambda, \beta)$ be an enhanced Yang-Baxter operator. Then $\tilde{\mathcal{R}} = (pR, qD, p\lambda, q\beta)$ with $p, q \in \mathbb{K}^*$, is an enhanced Yang-Baxter operator.*

2. *If $\mathcal{R} = (R, D, \lambda, \beta)$ is an enhanced Yang-Baxter operator. Then the quadruples $(R^t, D^t, \lambda, \beta)$ and $(R^{-1}, D, \lambda^{-1}, \beta)$ are enhanced Yang-Baxter operators.*

3. *If $\mathcal{R} = (R, D, \lambda, \beta)$ is an enhanced Yang-Baxter operator and if $A \in Aut(V)$. Then (R', D', λ, β), where*
$$R' = (A \otimes A) \circ R \circ (A \otimes A)^{-1}, \; D' = A \circ D \circ A^{-1},$$
is an enhanced Yang-Baxter operator.

Proof Notice that $D' \otimes D' = (A \otimes A) \circ (D \otimes D) \circ (A \otimes A)^{-1}$. Therefore, last part of the theorem follows by Lemma 2.1.2 i.e by the invariance of the partial trace Sp_2 on the second factor under conjugation.

∎

Now, if we consider the case when D is an isomorphism presented by a diagonal matrix with respect to some basis of V. The following theorem restates conditions (T1), (T2a) and (T2b) in the Definition of a EYB-operator (Definition 2.2.1) such that a solution $R : V^{\otimes 2} \to V^{\otimes 2}$ is an EYB-operator.

Theorem 2.2.4. (Turaev, [14]) *Let $R : V^{\otimes 2} \to V^{\otimes 2}$ be a solution of the YBE. Let $v_1, \ldots, v_m$ be basis of the m-dimensional vector space V and D be an isomorphism $V \to V$ given by*

$$D(v_i) = a_i v_i$$

with $a_1, \ldots, a_m \in \mathbb{K}^*$. The collection $(R, D, \lambda \in \mathbb{K}^*, \beta \in \mathbb{K}^*)$ is an EYB-operator if and only if the following two conditions are satisfied:

1. *For any $i, j, k, l \in \{1, \ldots, m\}$*

$$(a_i a_j - a_k a_l) R_{i,j}^{k,l} = 0.$$

2. *For any $i, k \in \{1, 2, \ldots, m\}$*

$$\sum_{j=1}^{m} R_{i,j}^{k,j} a_j = \lambda \beta \delta_i^k, \quad \sum_{j=1}^{m} (R^{-1})_{i,j}^{k,j} a_j = \lambda^{-1} \beta \delta_i^k$$

(here δ_i^k denotes the Kronecker symbol.)

Proof Under the conditions of Theorem we have that, for all $i, i_1, i_2 \in \{1, \ldots, m\}$

$R(v_{i_1} \otimes v_{i_2}) = \sum_{1 \leq j_1, j_2 \leq m} R_{i_1,i_2}^{j_1,j_2} v_{j_1} \otimes v_{j_2}, \qquad D(v_i) = a_i v_i, \qquad$ and the tensor product $D \otimes D$ is

$$(D \otimes D)(v_i \otimes v_j) = a_i a_j \ (v_i \otimes v_j)$$

Now, it is easy to see that

$$(R \circ (D \otimes D))(v_i \otimes v_j) = \sum_{k,l} a_i a_j R_{i,j}^{k,l} \ (v_k \otimes v_l) \tag{2.2.1}$$

and that

$$((D \otimes D) \circ R)(v_i \otimes v_j) = \sum_{k,l} a_k a_l R_{i,j}^{k,l} \ (v_k \otimes v_l) \tag{2.2.2}$$

Moreover, we have that

$$(R^{-1} \circ (D \otimes D))(v_i \otimes v_j) = \sum_{k,l} a_i a_j (R^{-1})^{k,l}_{i,j} \ (v_k \otimes v_l) \tag{2.2.3}$$

Therefore, R commutes with $D \otimes D$ if and only if $a_i a_j R^{k,l}_{i,j} = a_k a_l R^{k,l}_{i,j}$. Moreover, from equations 2.2.1 and 2.2.3 we can compute $Sp_2(R^{\pm 1} \circ (D \circ D))$, by summing over all the terms with $j = l$; i.e

$$\begin{array}{rcl} Sp_2(R \circ (D \otimes D))(v_i) & = & \sum_{j=1}^{m} R^{k,j}_{i,j} a_j; \\ Sp_2(R^{-1} \circ (D \otimes D))(v_i) & = & \sum_{j=1}^{m} (R^{-1})^{k,j}_{i,j} a_j \end{array} \tag{2.2.4}$$

From equation 2.2.4, we get then that $Sp_2(R^{\pm -1} \circ (D \otimes D)) = \lambda^{\pm 1} \beta D$ if and only

$$\sum_{j=1}^{m} R^{k,j}_{i,j} a_j = \delta^k_i \lambda \beta$$

and

$$\sum_{j=1}^{m} (R^{\pm 1})^{k,j}_{i,j} a_j = \lambda^{-1} \beta \delta^k_i.$$

∎

Remarks Clearly, $D \otimes D$ commutes with R if and only if $D \otimes D$ commutes with R^{-1}. Therefore, any of the conditions (1), (2) in Theorem 2.2.4 implies that for arbitrary i, j, k, l

$$(a_i a_j - a_k a_l)(R^{-1})^{k,l}_{i,j} = 0.$$

The condition (2) of Theorem 2.2.4 implies that

$$\tilde{\mathcal{R}} a = \begin{pmatrix} \lambda\beta \\ \vdots \\ \lambda\beta \end{pmatrix},$$

$\tilde{\mathcal{R}}$ is the $m \times m$-matrix $\tilde{\mathcal{R}}$, with $\tilde{R}_{ij} = R^{i,j}_{i,j}$ and $a = (a_1, \ldots, a_m)$.

The same is true for the matrix $\tilde{\mathcal{R}}^{-1}$ if we replace λ by λ^{-1}. Therefore, if at least one of the matrices $\tilde{\mathcal{R}}$ or $\tilde{\mathcal{R}}^{-1}$ is invertible over $\mathbb{K}$ then there exist at most one sequence $a_1, \ldots, a_m$ which satisfy (2) for given λ, β.

In the general case $a_1, \ldots, a_m$ (if exist) are not uniquely determined by R, λ, β. Because of the following Lemma.

Lemma 2.2.5. *For any homomorphism $D : V \to V$ the collection $(Id_{V^{\otimes 2}}, D, \lambda = 1, \beta = Sp(D))$ is a EYB-operator.*

Proof Denote by $R = Id_{V^{\otimes 2}}$. Then, on the respectively basis of V and $V \otimes V$, we have.

$$R(v_i \otimes v_j) \quad = \quad \sum_{k,l} R_{i,j}^{k,l}\, v_k \otimes v_l$$

with

$$R_{i,j}^{k,l} = \begin{cases} 1 & \text{if } \ i = k, \text{ and } j = l \\ 0 & \text{otherwise} \end{cases}$$

Moreover, we have

$$\begin{array}{rcl} D(v_i) & = & \sum_k D_{i,k}\, v_k \\ \mathrm{trace}(D(v_i)) & = & \sum_i D_{i,i}\, v_i \\ (D \otimes D)(v_i \otimes v_j) = \sum_{k,l} D_{i,k}\, D_{j,l}\, v_k \otimes v_l & & \end{array}$$

From it follows:

$$(R \circ (D \otimes D))(v_i \otimes v_j) \quad = \quad \sum_{k,l,s,t} D_{i,k}\, D_{j,l}\, R_{k,l}^{s,t}\, v_s \otimes v_t$$

with

$$R_{k,l}^{s,t} = \begin{cases} 1 & \text{if } \ k = s, \text{ and } l = t \\ 0 & \text{otherwise} \end{cases}$$

Hence, $(R \circ (D \otimes D))(v_i \otimes v_j) = \sum_{s,t} D_{i,s}\, D_{j,t}\, v_s \otimes v_t$.

On the other hand,

$$((D \otimes D) \circ R)(v_i \otimes v_j) = \sum_{k,l,s,t} D_{i,s}\, D_{j,t}\, R_{s,t}^{k,l}\, v_s \otimes v_t$$

with

$$R_{s,t}^{k,l} = \begin{cases} 1 & \text{if } \ s = k, \text{ and } t = l \\ 0 & \text{otherwise} \end{cases}$$

Hence,

$$((D \otimes D) \circ R)(v_i \otimes v_j) = \sum_{s,t} D_{i,s}\, D_{j,t}\, v_s \otimes v_t. \tag{2.2.5}$$

Thus, $(T1)$, holds. To finish the proof, rest to prove, conditions $(T2a)$ and $(T2b)$ of the Definition 2.2.1. To do it, we need to compute Sp_2 of $(R^{\pm 1} \circ (D \otimes D)$. But, it can be computed, from equation 2.2.5, just by summing over the terms which satisfy $j = t$; i.e

$$Sp_2(R^{\pm 1} \circ (D \otimes D)) = \sum_j D_{i,s}\, D_{j,j}\, v_j$$

∎

Notation Here G will denote a finite group and unless that is is specified $\mathbb{K}$ will denote a commutative ring with 1.

Consider the twisted conjugation braiding $B^{\varphi} : \mathbb{K}[G]^{\otimes 2} \to \mathbb{K}[G]^{\otimes 2}$.(See Definition 1.1). Can the twisted conjugation braiding B^{φ} be enhanced ? i.e., is there a homomorphism $D : \mathbb{K}[G] \to K[G]$, such that satisfies Turaev's conditions (T1), (T2a) and (T2b) of Definition 2.2.1.

In this part we give an answer to question A. Moreover we give some explicit examples for such a D.

It is very well known that a basis for $\mathbb{K}$ is given by the elements of G; which here we will denote by $g_1, g_2, \ldots, g_{|G|}$. So, we get a basis $\{g_i \otimes g_k\}$ with $i, j \in \{1, 2, \ldots, |G|\}$, for $\mathbb{K}[G] \otimes \mathbb{K}[G]$. On the basis of $\mathbb{K}[G]$, the map D is given as

$$D(g_i) = \sum_{j=1}^{|G|} D_{i,j}\; g_j$$

Moreover, on the basis for $\mathbb{K}[G]^{\otimes 2}$, we have:

$$(D \otimes D)(g_j \otimes g_k) = \sum_{m,n=1}^{|G|} D_{m,j} D_{n,k} g_m \otimes g_n$$

and

$$B^{\varphi}(g_m \otimes g_n) = \theta(m, n) \otimes \varphi(g_m)$$

with $\theta(m, n) = g_m g_n \varphi(g_m)^{-1}$.

Notice that $\varphi(g_j) \in G$, so there is an index $\Phi(j)$ [1] such that $\varphi(g_j) = g_{\Phi(j)}$. for the same reason, there is an index $\Theta(j, k)$ [2] such that $\theta(j, k) = g_{\Theta(j,k)}$. From it follows that B^{φ} has the following matrix representation.

$$[B^{\varphi}]_{p,q;m,n} = \begin{cases} 1 & \text{if } p = \Theta(m, n) \text{ and } q = \Phi(n) \\ 0 & \text{otherwise} \end{cases}$$

Since the twisted conjugation braiding B^{φ} is invertible (Remark 1.2.4, (4)) , we have that for every indexes (p, q) there is a second pair (m, n) such that $p = \Theta(m, n)$ and $q = \Phi(m)$. Thus the commutativity of $D \otimes D$ and B^{φ} is granted under the following condition:

$$D_{m,j}\; D_{n,k} = D_{\Theta(m,n),\Theta(j,k)}\; D_{\Phi(m),\Phi(j)} \tag{2.2.6}$$

Moreover, $Sp_2((D \otimes D) \circ B^{\varphi})$ can be calculated from 2.2.6 by adding all the terms whose (second) indices satisfy $k = \Phi(m)$. In order to simplify notation, we fix $p = \Theta(m, n)$.

$$\lambda\beta\; D_{p,j} \;\;=\;\; \sum_{k=1}^{|G|} D_{p,\Theta(j,k)}\; D_{k,\Phi(j)} \tag{2.2.7}$$

[1] Notice that $\Phi \in \Sigma_{|G|}$

[2] Notice that $\Theta : |G| \times |G| \to |G|$.

Therefore, now, the inverse $(B^{\varphi})^{-1}$ is given by:

$$(B^{\varphi})^{-1}(g_{\Theta(p,q)} \otimes g_{\Phi(p)}) = g_p \otimes g_q.$$

So, we have that $(B^{\varphi})^{-1}$ has the following matrix representation.

$$[B^{\varphi}]^{-1}_{p,q;m,n} = \begin{cases} 1 & \text{if } \ m = \Theta(p,q) \text{ and } n = \Phi(p) \\ 0 & \text{otherwise} \end{cases}$$

Notice that $D \otimes D$ commutes with the inverse of the twisted conjugation braiding $(B^{\varphi})^{-1}$, because it does with the twisted conjugation braiding B^{φ}. Again, we can compute $Sp_2(B^{\varphi})^{-1} \otimes (D \otimes D))$, by summing over all terms whose (second) indices satisfy $k = q$,

$$\lambda^{-1}\beta \, D_{p,j} = \sum_{q=1}^{|G|} D_{\Theta(p,q),j} \, D_{\Phi(p),q}. \tag{2.2.8}$$

Hence, we have proved the following Theorem.

Theorem 2.2.6. *The collection $\mathcal{B} = (B^{\varphi}, D, \lambda, \beta)$ is a EYB-operator if and only if the conditions (2.2.6), (2.2.7), (2.2.8) are satisfied.*

The following Corollaries are an easy consequence of Theorem 2.2.6.

Corollary 2.2.7. *Define $D : \mathbb{K}G \to \mathbb{K}G$ as $g_i \mapsto q g_i$, for all $i \in \{1, \ldots, |G]\}$, $q \in \mathbb{K}^*$ i.e its matrix representation is:*

$$[D] = \begin{pmatrix} q & 0 & \ldots & 0 \\ 0 & q & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & q \end{pmatrix}$$

The quadruple $\mathcal{B} = (B^{\varphi}, D, \lambda = 1, \beta = q)$ is an enhanced Yang- Baxter operator.

Proof we have that

$$D(g_i) = \sum_{m=1}^{|G|} D_{m,i} \, g_m,$$

with

$$D_{m,i} = \begin{cases} q & \text{if } \ m = i \\ 0 & \text{otherwise} \end{cases}$$

Its tensor product is

$$(D \otimes D)(g_i \otimes g_j) = \sum_{i,j=1}^{|G|} D_{m,i} \, D_{n,j} \, g_m \otimes g_n$$

with

$$D_{m,i}\, D_{n,j} = \begin{cases} q^2 & \text{if } \ m = i \text{ and } n = j \\ 0 & \text{otherwise} \end{cases}$$

From, it follows

$$(D \otimes D)(g_i \otimes g_j) = \sum_{i,j=1}^{|G|} q^2\ g_i \otimes g_j. \tag{2.2.9}$$

Now,

$$(B^{\varphi} \circ (D \otimes D))(g_i \otimes g_j) = \sum_{i,j=1}^{|G|} q^2\ (g_i g_j \varphi(g_i)^{-1} \otimes \varphi(g_i))$$

On the other hand:

$$\begin{array}{rcl} ((D \otimes D) \circ B^{\varphi})(g_i \otimes g_j) & = & (D \otimes D)(g_i g_j \varphi(g_i)^{-1} \otimes \varphi(g_i)) \\ & = & \sum_{m,n=1}^{|G|} D_{\Theta(m,n),s}\ D_{\Phi(m),t}\ g_s \otimes g_t \end{array}$$

with

$$D_{\Theta(m,n),s} D_{\Phi(m),t} = \begin{cases} q^2 & \text{if } \ \Theta(m,n) = s \text{ and } \Phi(m) = t \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$((D \otimes D) \circ B^{\varphi})(g_i \otimes g_j) = \sum_{m,n} q^2\ g_{\Theta(m,n)} \otimes g_{\Phi(m)} \tag{2.2.10}$$

Thus, (T1), follows from equations (2.2.9), and (2.2.10) and Theorem 2.2.6.

Thus, it remains to prove that the equations (2.2.7) and (2.2.8) are satisifed. But, they follows from (2.2.9) and (2.2.10), just by summing over the terms $n = j$ and $\Phi(m) = t$.

∎

Corollary 2.2.8. *Define D as $D(g) = qN$, for all $g \in G$, and with $N = g_1 + \cdots + g_{|G|}$, the norm element in $V = \mathbb{K}[G]$; i.e.*

$$[D] = \begin{pmatrix} q & q & \cdots & q \\ q & q & \cdots & q \\ \vdots & \vdots & \ddots & \vdots \\ q & q & \cdots & q \end{pmatrix}$$

The collection $(B^{\varphi}, D, \lambda = 1, \beta = traceD)$ is an EYB-operator.

2.3 Invariants of braids and links

Recall that every Yang-Baxter operator $R : V^{\otimes} \to V^{\otimes 2}$ gives rise to a finite dimensional representation of Artin's braid group

$$\begin{array}{rcl} Br(n) & = & < \sigma_1, \ldots, \sigma_{k-1} \mid \forall\, 1 \leq i, j \leq k-1 : \ \sigma_i\sigma_j = \sigma_j\sigma_i \ \text{ for } \ |i-j| > 1 \\ & & \text{and } \ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} > \end{array}$$

(here $n \geq 1$, See [2]). Namely, put $R_i = R_i(n) : V^{\otimes n} \to V^{\otimes n}$ and notice that $R_iR_j = R_jR_i$ for $|i-j| \geq 2$ and (in view of the Yang Baxter equality) $R_iR_{i+1}R_i = R_{i+1}R_iR_{i+1}$ for $i = 1, \ldots, n-1$. Therefore there is a unique homomorphism $Br(n) \to Aut(V^{\otimes n})$ which transforms σ_i into R_i for all i. Denote this homomorphism by b_R. We shall also use the homomorphism ω from $Br(n)$ to the additive group of integers which sends $\sigma_1, \ldots, \sigma_{n-1}$ into 1.

Every EYB-operator $\mathcal{R} = (R, D, \lambda, \beta)$ determines a mapping $T_{\mathcal{R}} : \coprod_{n\geq 1} Br(n) \to \mathbb{K}$ as follows. For $n \geq 1$ denote the homomorphism $D \otimes \cdots \otimes D : V^{\otimes n} \to V^{\otimes n}$ by $D^{\otimes n}$. For a braid $\xi \in Br(n)$ put

$$T_{\mathcal{R}}(\xi) = \lambda^{-\omega(\xi)}\ \beta^{-n}\ \mathrm{trace}(b_R(\xi) \circ D^{\otimes n} : V^{\otimes n} \to V^{\otimes n}). \tag{2.3.1}$$

The most important porperties of $T_{\mathcal{R}}$ are given by the following theorem.

Theorem 2.3.1. (Turaev, [14]) *For any $\xi, \eta \in Br(n)$*

$$T_{\mathcal{R}}(\eta^{-1}\xi\eta) = T_{\mathcal{R}}(\xi\sigma_n) = T_{\mathcal{R}}(\xi\sigma_n^{-1}) = T_{\mathcal{R}}(\xi).$$

Proof of Theorem 2.3.1 It follows from the definition of EYB-operator that

$$(D^{\otimes n} \circ b(\eta)) = (b(\eta) \circ D^{\otimes n}), \quad \text{for any } \ \eta \ \in \ Br(n), \text{where } b = b_R : Br(n) \to Aut(V^{\otimes n}).$$

Thus

$$\begin{array}{rcl} \mathrm{trace}(b(\eta^{-1}\xi\eta) \circ D^{\otimes n} & = & \mathrm{trace}(b(\eta^{-1})b(\xi)b(\eta) \circ D^{\otimes n}) \\ & = & \mathrm{trace}(b(\eta^{-1})b(\xi) \circ D^{\otimes n}b(\eta)) \end{array}$$

By properties of the usual trace (Lemma C.2.1), the last equality is equal to:

$$\mathrm{trace}(b(\xi) \circ D^{\otimes n})$$

Also,

$$\omega(\eta^{-1}\xi\eta) = \omega(\xi), \quad \text{since } \ \omega : Br(n) \to \mathbb{Z} \text{ is a homomorphism.}$$

Therefore

$$\begin{array}{rcl} T_{\mathcal{R}}(\eta^{-1}\ \xi\ \eta) & = & \lambda^{-\omega(\eta^{-1}\xi\eta)}\ \beta^{-n}\ \mathrm{trace}[b_R(\eta^{-1}\xi\eta) \circ D^{\otimes n)} : V^{\otimes n} \to V^{\otimes n}] \\ & = & \lambda^{-\omega(\xi)}\ \beta^{-n}\ \mathrm{trace}[b_R(\xi) \circ D^{\otimes n} : V^{\otimes n} \to V^{\otimes n}] \\ & = & T_{\mathcal{R}}(\xi) \end{array}$$

Now, we want to prove that

$$T_{\mathcal{R}}(\xi\sigma_n) = T_{\mathcal{R}}(\xi)$$

Notice,

$$b(\xi\sigma_n) = (b(\xi) \otimes Id_V) \circ R_n : V^{\otimes(n+1)} \to V^{\otimes(n+1)}.$$

Thus,

$$\begin{array}{rcl}
\text{trace}[b(\xi\sigma_n) \circ D^{\otimes(n+1)}) & = & \text{trace}[(b(\xi) \otimes Id_V) \circ R_n) \circ D^{\otimes(n+1)}] \\
& = & \text{trace}[(b(\xi) \otimes Id_V) \circ R_n \circ (Id_V^{\otimes(n-1)} \otimes D \otimes D) \circ (D^{\otimes(n-1)} \otimes Id_V^{\otimes 2})] \\
& = & \text{trace}(\ (b(\xi) \otimes Id_V) \circ (Id_V^{\otimes(n-1)} \otimes R) \circ (Id_V^{\otimes(n-1)} \otimes D \otimes D) \\
& \circ & (D^{\otimes(n-1)} \otimes Id_V^{\otimes 2})\ ) \\
& = & \text{trace}\{(b(\xi) \otimes Id_V) \otimes (Id_v^{\otimes(n-1)} \otimes (R \circ (D \otimes D)\}) \circ (D^{\otimes(n-1)} \otimes Id_V^{\otimes 2})) \\
& = & \text{trace}\{Sp_{n+1}(\ (b(\xi) \otimes Id_V) \circ (Id_V^{\otimes(n-1)} \otimes (R \circ (D \otimes D)\ ) \\
& \circ & (D^{\otimes(n-1)} \otimes Id_V^{\otimes 2})]\}
\end{array}$$

By properties of the partial trace (Lemma 2.1.3) , the last equality is equal to

$$b(\xi) \circ \{Id^{\otimes(n-1)} \otimes Sp_2(R \circ (D \otimes D))\} \circ (D^{\otimes(n-1)} \circ Id_V).$$

thus, by definition of EYB-operator (Definition 2.2.1), this is equal to $\lambda\, \beta\, (b(\xi) \otimes D^{\otimes n})$. Hence

$$\text{trace}(b(\xi\sigma_n) \circ D^{\otimes(n+1)}) = \lambda\, \beta\, \text{trace}(b(\xi) \circ D^{\otimes n}).$$

Clearly, $\omega(\xi\sigma_n) = \omega(\xi) + 1$. These equalities imply that

$$T_{\mathcal{R}}(\xi\sigma_n) = T_{\mathcal{R}}(\xi).$$

to finish the proof, one notice that the equality

$$T_{\mathcal{R}}(\xi\sigma^{-1}) = T_{\mathcal{R}}(\xi)$$

is proved similarly.

∎

Remark Due to a theorem of J. Alexander (first part) and A. Markov [2]. Any oriented link is isotopic to the clousure of some braid (Figure 2.1). The closures of two braids are isotopic (in the category of oriented links) if and only if these braids are equivalent with respect to the equivalence relation in $\coprod_n Br(n)$ generated by Markov moves $\xi \mapsto \eta^{-1}\xi\eta,\ \xi \mapsto \xi\sigma_n^{\pm 1}$, where $\xi, \eta, \in Br(n)$. Tuaev's theorem (Theorem 2.3.1) shows that for any EYB-operator $\mathcal{R} = (R, D, \lambda, \beta)$ the mapping $T_{\mathcal{R}} : \coprod_n Br(n) \to \mathbb{K}$ induces a mapping of the set of oriented isotopy classes of links into $\mathbb{K}$.

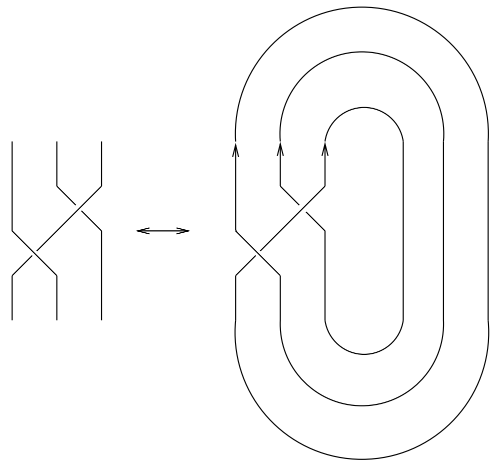


Figure 2.1: Closure of a braid.

2.4 Elementary properties of $T_{\mathcal{R}}$.

In this section, we recall the properties of the link invariant $T_{\mathcal{R}}$.

Theorem 2.4.1. *1. For the trivial knot $\bigcirc$ we have*

$$T_{\mathcal{R}}(\bigcirc) = \beta^{-1} Sp(D)$$

2. $T_{\mathcal{R}}$ is multiplicative, i.e. if $L = L_1 \sqcup L_2$ is the disjoint union of two links L_1 and L_2 then

$$T_{\mathcal{R}}(L) = T_{\mathcal{R}}(L_1) \ \cdot \ T_{\mathcal{R}}(L_2).$$

Corollary 2.4.2. *If L is the trivial n-component link then*

$$T_{\mathcal{R}}(L) = \beta^{-n} Sp(D)^n.$$

Proof of Theorem 2.4.1

1. Consider the generator σ_1 of $Br(2)$, then

$$\begin{aligned} T_{\mathcal{R}}(\bigcirc) &= \lambda^{-1}\,\beta^{-2}\,\mathrm{trace}(R \circ (D \otimes D) \\ &= \lambda^{-1}\,\beta^{-2}\,\mathrm{trace}(Sp_2(R \circ (D \otimes D)) \\ &= \beta^{-1}\,\mathrm{trace}(D) \end{aligned}$$

2. Let $\sigma_1 \in Br(n_1)$ and $\beta_2 \in Br(n_2)$ two braids which closures are the links L_1 and L_2 respectively. It follows from Figure 2.2 that $\beta_1 \sqcup \beta_2 \in Br_(n_1 + n_2)$ is a braid which clousure is $L_1 \sqcup L_2$. Therefore

$$\begin{aligned} b_{R,n_1+n_2}(\beta_1\beta_2) &= b_{R,n_1+n_2}(\beta_1) \circ b_{R,n_1+n_2}(\beta_2) \\ &= b_{R,n_1}(\beta_1) \otimes b_{R,n_2}(\beta_2) \end{aligned}$$

Now, by part (6) of Lemma C.2.1

$$\begin{aligned} T_{\mathcal{R}}(b_{R,n_1+n_2}(\beta_1\beta_2)) &= \operatorname{trace}(b_{R,n_1}(\beta_1) \otimes b_{R,n_2}(\beta_2)) \\ &= \operatorname{trace}(b_{R,n_1}(\beta_1)).\ \operatorname{trace}(b_{R,n_2}(\beta_2)) \end{aligned}$$

From it follows:

$$\begin{aligned} T_{\mathcal{R}}(L_1 \sqcup L_2) &= \lambda^{-\omega(\beta_1+\beta_2)}\beta^{-(n_1+n_2)Sp(b_{R,n_1+n_2}(\beta_1\beta_2))} \\ &= \lambda^{-\omega(\beta_1)}\ \lambda^{-\beta_2}\ \beta^{-n_1}\ \beta^{-n_2} \operatorname{trace}(b_{R,n_1}(\beta_1))\ .\ \operatorname{trace}(b_{R,n_2}(\beta_2)) \end{aligned}$$

∎

Proof of Corollary 2.4.2 It follows from Theorem 2.4.1, that for the trivial link $\bigcirc$,

$$T_{\mathcal{R}}(\bigcirc) = \beta^{-1} \operatorname{trace}(D),$$

and that $T_{\mathcal{R}}$ is multiplicative. Thus,

$$T_{\mathcal{R}}(\bigcirc^n) = \beta^{-n} \operatorname{trace}(D)^n$$

∎

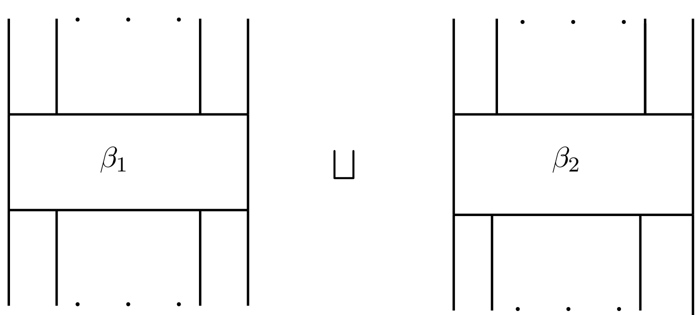


Figure 2.2:

The following proposition, gives the link invariants associated to the new EYB-operators constructed on Lemma 2.2.3.

Proposition 2.4.3. *Let $T_{\mathcal{R}}$ be the associated link invariant to an enhanced Yang-Baxter operator $\mathcal{R} = (R, D, \lambda, \beta)$. Then:*

1. *If $\mathcal{R}' = (pR, qD, p\lambda, q\beta)$, then $T_{\mathcal{R}'} = q\, T_{\mathcal{R}}$,*
2. *If $\widetilde{\mathcal{R}} = (R^t, D^t, \lambda, \beta)$, then $T_{\mathcal{R}} = T_{\widetilde{\mathcal{R}}}$,*
3. *If $A \in Aut(V)$, and $\widetilde{R} = (A \otimes A) \circ R \circ (A \otimes A)^{-1}$ and $\widetilde{D} = A \otimes D \otimes A^{-1}$, then $T_{\mathcal{R}} = T_{\mathcal{R}'}$, where R' is the enhanced Yang-Baxter operator $(\widetilde{R}, \widetilde{D}, \beta, \lambda)$.*

Proof First of all notice that, by properties of the tensor product (See [5]).

$$b_R(\sigma_i)^t = (id^{\otimes(i-1)} \otimes R \otimes id^{\otimes(n-i-1)})^t = id^{\otimes(i-1)} \otimes R^t \otimes id^{\otimes(n-i-1)}.$$

Thus, the second part of proposition holds.

For the third part, we have

$$(\tilde{D})^{\otimes n} = A^{\otimes n} \circ D^{\otimes n} \circ A^{\otimes -n}.$$

Moreover, we have: $id^{\otimes(i-1)} \otimes \tilde{R} \otimes id^{\otimes(n-i-1)} = A^{\otimes n} \circ (id^{\otimes(i-1} \otimes R \otimes id^{\otimes(n-i-1)}) \circ A^{\otimes -n}$.

Therefore, $T_{\mathcal{R}} = T_{\tilde{\mathcal{R}}}$, holds from the invariance of the trace (Lemma C.2.1).

∎

2.5 The link invariants for the twisted conjugation braiding

Notation Here G denotes a finite group, $\varphi \in Aut(G), \mathbb{K}$ denotes a fixed commutative ring with 1; unless it is mentioned $\mathbb{K}$ will denote the field of complex numbers $\mathbb{C}$. Recall that given a basis $\{g\}_{g\in G}$ for $\mathbb{K}[G]$, we get a basis $\{a \otimes b\}_{a,b\in G}$ for $\mathbb{K}[G]^{\otimes 2}$.

In this section we prove that for any enhancement D of the twisted conjugation braiding B^{φ}, $\lambda = 1$; i.e. the link invariant $T_{\mathcal{B}}$ associated to any enhancement of the twisted conjugation braiding B^{φ} is given by

$$T_{\mathcal{B}}(\xi) = \beta^{-n}\, \mathrm{trace}(b_{B^{\varphi}} \circ D^{\otimes n}),$$

for any braid $\xi \in Br(n)$ and $\beta \in \mathbb{K}^*$.

Notice that for any EYB-operator $\mathcal{R} = (R, D, \lambda, \beta)$, it is not always true that $\lambda = 1$.

Remark 2.5.1. *Since the twisted conjugation braiding B^φ is invertible, we have that for every basis element $c \otimes d \in \mathbb{K}[G]$ there is a second basis element $a \otimes b \in K[G]^{\otimes 2}$ sucht that $c = ab\varphi(a)^{-1}$ and $d = \varphi(a)$.*

Now, let D be the linear map from $\mathbb{K}[G]$ into $\mathbb{K}[G]$. We may characterise D via its action on the basic elements $a \in G$. Thus, we have a collection of coefficients $\Delta(a, c) \in \mathbb{K}$ such that

$$D(a) = \sum_{c \in G} \Delta(a, c)\ c \tag{2.5.1}$$

$$D(\sum_{a \in G} \beta_a a) = \sum_{a \in G} \beta_a\ D(a) = \sum_{a,c \in G} \beta_a\ \Delta(a, c)\ c.$$

the tensor product $D \otimes D$ is also defined via its action on the basis elements $a \otimes b$ of $\mathbb{K}[G]^{\otimes 2}$

$$(D \otimes D)(a \otimes b) = \sum_{c,d \in G} \Delta(a, c)\Delta(b, d)c \otimes d \tag{2.5.2}$$

Using the definition of the twisted conjugation braiding B^φ (Definition 1.1), and equation (2.5.2), is easy to see that:

$$\begin{aligned} (B^\varphi \circ (D \otimes D))(a \otimes b) &= \textstyle\sum_{c,d \in G} \Delta(a, c)\ \Delta(b, d)\ D(c \otimes d) \\ &= \textstyle\sum_{c,d \in G} \Delta(a, c)\ \Delta(b, d)\ (cd\ \varphi(c)^{-1} \otimes \varphi(c)) \end{aligned} \tag{2.5.3}$$

and that:

$$\begin{aligned} ((D \otimes D) \circ B^\varphi)(a \otimes b) &= (D \otimes D)(ab\ \varphi(a)^{-1} \otimes \varphi(a)) \\ &= \textstyle\sum_{s,t \in G} \Delta(ab\ \varphi(a)^{-1}, s)\ \Delta(\varphi(a), t)s \otimes t \end{aligned} \tag{2.5.4}$$

Now, by Remark 2.5.1 , for every basic element $s \otimes t$ there is a second basis element $c \otimes d$ such that $s = cd\ \varphi(c)^{-1}$ and $t = \varphi(c)$. Therefore $D \otimes D$ commutes with the twisted onjugation braiding B^φ if and only if

$$\Delta(a, c)\Delta(b, d) = \Delta(ab\ \varphi(a)^{-1}, cd\ \varphi(c)^{-1})\ \Delta(\varphi(a), \varphi(c)) \tag{2.5.5}$$

In particular,

$$\begin{aligned} \Delta(a, c)\Delta(b, \varphi(c)) &= \Delta(ab\varphi(a)^{-1}, c)\Delta(\varphi(a), \varphi(c)) \\ \Delta(a, c)\Delta(\varphi(a), d) &= \Delta(a, cd\varphi(c)^{-1})\Delta(\varphi(a), \varphi(c)) \end{aligned} \tag{2.5.6}$$

Theorem 2.5.2. *Assume that D is a non-zero linear map. Moreover, assume that $D \otimes D$ commutes with B^{φ} and that there exist a pair of elements $\beta_1, \beta_2 \in \mathbb{K}^*$ (invertible elements) such that:*

$$Sp_2(B^{\varphi} \circ (D \otimes D)) = \beta_1 \cdot D \quad \textit{and} \quad Sp_2((B^{\varphi})^{-1} \circ (D \otimes D)) = \beta_2 \cdot D \tag{2.5.7}$$

Then,

$$\beta_1 = \beta_2.$$

Proof First of all notice that by Remark 2.5.1, the equation

$$(D \otimes D)(a \otimes b) = \sum_{c,d \in G} \Delta(a,c)\ \Delta(b,d)\ c \otimes d$$

is equivalent to the following equation:

$$(D \otimes D)(a \otimes b) = \sum_{c,d \in G} \Delta(a, cd\ \varphi(c)^{-1})\ \Delta(b, \varphi(c)\ (cd\ \varphi(c)^{-1} \otimes \varphi(c) \tag{2.5.8}$$

Now, it is very easy to see that:

$$(B^{\varphi} \circ (D \otimes D))(a \otimes b) = \sum_{c,d \in G} \Delta(ab\ \varphi(a)^{-1}, c)\ \Delta(\varphi(a), d)\ c \otimes d \tag{2.5.9}$$

and

$$((B^{\varphi})^{-1} \circ (D \otimes D))(a \otimes b) = \sum_{c,d \in G} \Delta(a, cd\ \varphi(c)^{-1})\Delta(b, \varphi(c))\ c \otimes d \tag{2.5.10}$$

Thus from the equations (2.5.9) and (2.5.10), we can calculate the partial traces on the second factor of $(B^{\varphi} \circ (D \otimes D))$ and of $(B^{\varphi})^{-1} \circ (D \otimes D)$ respectively.

$$Sp_2(B^{\varphi} \circ (D \otimes D))(a) = \sum_{c \in G} (\sum_{d \in G} \Delta(ad\ \varphi(a)^{-1}, c)\ \Delta(\varphi(a), d)\ )\ c \tag{2.5.11}$$

and

$$Sp_2((B^{\varphi})^{-1} \circ (D \otimes D))(a) = \sum_{c \in G} (\sum_{d \in G} \Delta(a, cd\ \varphi(c)^{-1})\ \Delta(d, \varphi(c)\ )\ )\ c \tag{2.5.12}$$

A direct application of equations (2.5.11) and (2.5.12) yields that equations (2.5.7) holds if and only if the following equations are satisfied:

$$\beta_1\ .\ \Delta(a,c) = \sum_{d \in G} \Delta(ad\ \varphi(a)^{-1}, c)\ \Delta(\varphi(a), d) \tag{2.5.13}$$

and

$$\beta_2\ .\ \Delta(a,c) = \sum_{d \in G} \Delta(a, cd\ \varphi(c)^{-1})\ \Delta(d, \varphi(c)) \tag{2.5.14}$$

Multiplying equations (2.5.13) and (4.1.6) by $\Delta(a,c)$ respectively and using equations 2.5.6, we get the desired result $\beta_1 = \beta_2$, because there is at least one entry $\Delta(a,c) \neq 0$. i.e

$$\begin{aligned} \beta_1 \cdot \Delta(a,c)^2 &= \sum_{d\in G} \Delta(ad\,\varphi(a)^{-1}, c)\Delta(a, cd\,\varphi(c)^{-1})\Delta(\varphi(a), \varphi(c)) \\ &= \beta_2 \cdot \Delta(a,c)^2. \end{aligned}$$

∎

Corollary 2.5.3. *If $\mathcal{B} = (B^{\varphi}, D, \lambda, \beta)$ is an enhanced Yang-Baxter operator of the twisted conjugation braiding B^{φ}. Then its associated link invariant $T_{\mathcal{R}}$ is*

$$T_{\mathcal{B}}(\xi) = \beta^{-n}\ trace(b_{B^{\varphi}}(\xi) \circ D^{\otimes n}).$$

Proof By definition $\mathcal{B} = (B^{\varphi}, D, \lambda, \beta)$ is an EYB-operator, i.e.

(T1) $D \otimes D$ commutes with B^{φ},

(T2a) $Sp_2((B^{\varphi}) \circ (D \otimes D)) = \lambda\, \beta D$, and

(T2b) $Sp_2((B^{\varphi})^{-1} \circ (D \otimes D)) = \lambda^{-1}\, \beta D$.

Thus, Theorem 2.5.2 implies $\lambda\beta = \lambda^{-1}\beta$, the last equality implies that $\lambda = 1$, because $\lambda, \beta \in \mathbb{K}^*$ are invertible elements. Thus by the definition of $T_{\mathcal{B}}$ we get that

$$T_{\mathcal{B}}(\xi) = \beta^{-n} Sp(b_{B^{\varphi}}(\xi) \otimes D^{\otimes n}).$$

∎

Chapter 3

Character enhancements

In this chapter, we prove in terms of characters of the group $G \times G$ that the twisted conjugation braiding B^{φ} is an enhanced Yang-Baxter operator. Indeed we prove that if $D : \mathbb{K}[G] \to \mathbb{K}[G]$ is defined as $D(a) = \sum_{c \in G} \chi(a, c)c$, with χ a character from $G \times G$ into a field $\mathbb{K}$, then D is an enhancement of the twisted conjugation braiding B^{φ}; such enhancements will be called *character enhancements* and will be denoted by D_{χ}. Moreover, we prove that character enhancements satisfy the following property $B^{\varphi} \circ (D \otimes D) = D \otimes D$. This condition implies that the link invariant $T_{\mathcal{B}}(\xi) = 1$ for all braid $\xi \in Br(n)$.

3.1 Character χ

In this section, we recall the definition of character and give some examples.

Definition 3.1.1. *If G is group and $\mathbb{K}$ is a field. A character is a group homomorphism χ from G into $\mathbb{K}^*$. See [1].*

If G is an abelian group, then the set $Ch(G)$ of these characters forms a group under the operation

$$(\chi_1 \cdot \chi_2)(a) = \chi_1(a) \cdot \chi_2(a).$$

It is called, the character group. Sometimes only unitary characteres are considered (so that the image is in the unit circle); other such homomorphisms are then called quasi-characteres.

3.2 Character enhancements D_χ

In this section we prove in terms of character theory of $G\times G$ that the twisted conjugation braiding B^φ is an enhanced Yang-Baxter operator.

Given a character $\chi : G \times G \to \mathbb{K}^*$. Define the $\mathbb{K}$- linear map $D : \mathbb{K}[G] \to \mathbb{K}[G]$ via its action on the basis elements $a \in G$,

$$D(a) = \sum_{c\in G} \chi(a,c)c, \tag{3.2.1}$$

therefore,

$$D(\sum_{a\in G} \beta_a a) = \sum_{c\in G} \chi(a,c)\beta_a c.$$

The tensor product $D\otimes D$ is also defined via its action on the basic elements $a\otimes b$ of $\mathbb{K}[G]^{\otimes 2}$,

$$\begin{array}{rcl}(D\otimes D)(a\otimes b) & = & \sum_{c\otimes d} \chi(a,c)\ \chi(b,d)\ c\otimes d \\ & = & \sum_{c\otimes d}\ \chi(ab,cd)\ (c\otimes d).\end{array}$$

We have that:

$$\begin{array}{rcl}(B^\varphi \circ (D\otimes D))(a\otimes b) & = & B^\varphi(\sum_{c\otimes d}\ \chi(a,c)\ c\chi(b,d)\ c\otimes d) \\ & = & \sum_{c\otimes d} \chi(ab,cd)\ (cd\ \varphi(c)^{-1}\otimes \varphi(c))\end{array}$$

On the other hand,

$$\begin{array}{rcl}(D\otimes D)\circ(B^\varphi)(a\otimes b) & = & (D\otimes D)\ (ab\varphi(a)^{-1}\otimes\varphi(a)) \\ & = & \sum_{c\otimes d} \chi(ab\varphi(a)^{-1}\ \varphi(a),cd)\ c\otimes d \\ & = & \sum_{c\otimes d} \chi(ab,cd)\ (c\otimes d)\end{array}$$

Since B^φ is invertible for each basis element $c\otimes d$, there exists a basis element $s\otimes t$ such that $s = cd\ \varphi(c)^{-1}$ and $t = \varphi(c)$. Hence, commutativity of $D\otimes D$ with B^φ holds if and only if the following equation holds for all $a, b, c, d \in G$,

$$\chi(ab, cd\ \varphi(c)^{-1}\ \varphi(c)) = \chi(ab, cd).$$

From it follows, then:

$$(D\otimes D) = B^\varphi\circ(D\otimes D) = (D\otimes D)\circ B^\varphi = (B^\varphi)^{-1}\circ(D\otimes D) = (D\otimes D)\circ(B^\varphi)^{-1}. \tag{3.2.2}$$

It implies:

$$Sp_2(B^\varphi\circ(D\otimes D)) = Sp_2((B^\varphi)^{-1}\circ(D\otimes D)) = Sp_2(D\otimes D) = \text{trace}(D)\ .\ D$$

Hence, we have proved the following theorem.

Theorem 3.2.1. *Define $D : \mathbb{K}G \to \mathbb{K}G$ as in 3.2.1. Then, the following three statements hold:*

1. *$\mathcal{B} = (B^{\varphi}, D, \lambda = 1, \beta = trace(D)$ is an enhanced Yang-Baxter operator.*
2. *$B^{\varphi}(D \otimes D) = (D \otimes D) = (D \otimes D) \circ B^{\varphi} = (D \otimes D)(B^{\varphi})^{-1}$.*
3. *$Sp_2(B^{\varphi} \circ (D \otimes D)) = trace(D) \,.\, D$*

Remark 3.2.2. 1. *The definition of D does not depend on the automorphism φ of G.*

2. *Enhancements D of the twisted braiding B^{φ}, arising from a character χ will be denoted by D_{χ} and will be called character enhancements.*

Given a character $\chi : G \times G \to \mathbb{K}^{*}$. Define $D : \mathbb{K}[G] \to \mathbb{K}[G]$, as

$$D(a) = \sum_{c} \chi(a, c)\ \bar{c} \tag{3.2.3}$$

with $\bar{c} = \psi(c)$, with ψ a homomorphism from G into G. The tensor product $D \otimes D$, is given as

$$(D \otimes D)(a \otimes b) = \sum_{c \otimes d} \chi(a, c)\ \chi(b, d)\ (\bar{c} \otimes \bar{d})$$

Thus,

$$\begin{aligned} (B^{\varphi} \circ (D \otimes D))(a \otimes b) &= B^{\varphi}(\textstyle\sum_{c \otimes d} \chi(a, c)\ \chi(b, d)\ (\bar{c} \otimes \bar{d}) \\ &= \textstyle\sum_{c \otimes d} \chi(a, c)\ \chi(c, d)\ (\bar{c}\bar{d}\ \varphi(\bar{c})^{-1} \otimes \varphi(\bar{c})) \\ &= \textstyle\sum \chi(a, c)\ \chi(b, d)\ (\psi(c)\psi(d)\varphi(\psi(c)^{-1}) \otimes \varphi(\psi(c))) \end{aligned} \tag{3.2.4}$$

On the other hand,

$$\begin{aligned} ((D \otimes D) \circ B^{\varphi}))(a \otimes b) &= (D \otimes D)(ab\ \varphi(a)^{-1} \otimes \varphi(a)) \\ &= \textstyle\sum_{s \otimes t} \chi(ab\varphi(a)^{-1}, s)\ \chi(\varphi(a), t)(\bar{s} \otimes \bar{t}) \\ &= \textstyle\sum_{s \otimes t} \chi(ab\varphi(a)^{-1}, s)\ \chi(\varphi(a), t)\ (\psi(s) \otimes \psi(t)) \end{aligned} \tag{3.2.5}$$

Notice, that $\psi(s), \psi(t) \in G$. Hence, by the invertibiliy of B^{φ}, for each basis element $\bar{c} \otimes \bar{d}$, there exist a basis element $\psi(s) \otimes \psi(t)$, such that, $\psi(s) = \psi(cd)\varphi(\psi(c))^{-1}$, and $\psi(t) = \varphi(\psi(c))$. From it follows, that $D \otimes D$ commutes with B^{φ} if and only if the following equation holds

$$\chi(a, c)\chi(b, d) = \chi(ab\varphi(a)^{-1}, s)\chi(\varphi(a), t) \tag{3.2.6}$$

for all $a, b \in G$. However, if we assume ψ to be an automorphism, then equation 3.2.6 is equivalent to have:

$$\chi(a, c)\chi(b, d) = \chi(ab\varphi(a)^{-1}, cd\varphi(c)^{-1})\chi(\varphi(a), \varphi(c)) \tag{3.2.7}$$

and it will imply that

$$(D \otimes D) \circ B^{\varphi} = D \otimes D.$$

3.3 Constancy of the link invariant $T_{\mathcal{B}}(\xi)$

We have seen that $\mathcal{B} = (B^{\varphi}, D, \lambda, \beta)$, where $D(a) = \sum_{c \in G} \chi(a,c)c$, is an enhanced Yang-Baxter operator, (Theorem 3.2.1). Moreover we have proved that

$$(D \otimes D) = (D \otimes D) \circ B^{\varphi} = (D \otimes D)(B^{\varphi})^{-1}.$$

In this section we prove that all for any group G, all the link invariants arising from these enhancements are constantly 1.

Recall from chapter Yang-Baxter solution R gives rise to a representation b_R of the braid group $Br(n)$ on n-strands.

Remark 3.3.1. *Let D_{χ} denote any character enhnacement of the twisted conjugation braiding B^{φ}. For $n \geq 0$,*

$$(B^{\varphi})^n \circ (D \otimes D) = D \otimes D, \ \ (B^{\varphi})^{\otimes(n-1)} \circ D^{\otimes n} = D^{\otimes n}.$$

The above remark, follows from equation (3.2.2).

Theorem 3.3.2. *Let $\mathcal{B}$ be the enhanced Yang-Baxter operator of Theorem 3.2.1. Let $\xi \in Br(n)$ denote a braid, and b_B the corresponding braid representation of B^{φ}. Then, the link invariant associated to any character enhancement D_{χ} is trivial; i.e., for all $\xi \in Br(n)$*

$$T_{\mathcal{B}}(\xi) = 1$$

Proof

Let $\xi \in Br(n)$. Then β can be written as the product of the $\sigma_i's$ and their inverses, i.e., $\xi = \sigma_{i_1}^{\epsilon_1} \dots \sigma_{i_k}^{\epsilon_k}$ with $1 \leq i_1, \dots, i_k \leq n-1$, $\epsilon_i \in \{\pm 1\}$ for $1 \leq i \leq k$, $k \in \mathbb{N}$.

Furthemore, notice that

$$\begin{aligned} B_i^m &:= (id^{\otimes i-1} \otimes B^{\varphi} \otimes id^{\otimes n-i-1})^m = id^{\otimes i-1} \otimes (B^{\varphi})^m \otimes id^{\otimes n-i-1} \\ B_i^m \circ D^{\otimes n} &= D^{\otimes i-1} \otimes B^{\varphi} \otimes D^{\otimes n-i-1} \end{aligned} \tag{3.3.1}$$

Moreover, it follows from Remark 3.3.1 that

$$B_i^m \circ D^{\otimes n} = D^{\otimes n}$$

Thus,

$$
\begin{array}{rcl}
T(\xi) & = & \beta^{-n}\ \mathrm{trace}(b_{B^\varphi}(\xi) \circ D^{\otimes n}) \\
& = & \beta^{-n}\ \mathrm{trace}(b_{B^\varphi}(\sigma_{i_1}^{\epsilon_1} \ldots \sigma_{i_k}^{\epsilon_k}) \circ D^{\otimes n}) \\
& = & \beta^{-n}\ \mathrm{trace}((B_{i_1}^{\epsilon_1} \ldots B_{i_k}^{\epsilon_k}) \circ D^{\otimes n}) \\
& \stackrel{(3.3.1)}{=} & \beta^{-n}\ \mathrm{trace}((B_{i_1}^{\epsilon_1} \ldots B_{i_{k-1}}^{\epsilon_{k-1}}) \circ (D^{\otimes i-1} \otimes (B^\varphi)^k \otimes D^{\otimes n-i-1})) \\
& \stackrel{(3.3.1)}{=} & \beta^{-n}\ \mathrm{trace}((B_{i_1}^{\epsilon_1} \ldots B_{i_{k-1}}^{\epsilon_{k-1}}) \circ D^{\otimes n}) \\
& \vdots & \\
& = & \beta^{-n}\ \mathrm{trace}(B_{i_1}^{\epsilon_1} \otimes D^{\otimes n}) \\
& = & \beta^{-n}\ \mathrm{trace}(D^{\otimes n}) \\
& \stackrel{(3.2.1)}{=} & 1
\end{array}
$$

∎

Example 3.3.3. *1. Consider the character $\chi : G \to \mathbb{K}^*$. Defined as $\chi_a \equiv 1$, Let q be a fixed element in $\mathbb{K}^*$. Then, $q\chi$, is the map D of Corollary 2.2.8.*

2. The following function

$$\chi(a \otimes b) = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$

is not a character. But, it is invariant under B^φ; i.e.

$$\chi(a, b)\chi(c, d) = \chi(ac\varphi(a)^{-1}, b)\chi(\varphi(a), d)$$

and it is the type of invariance that one needs to prove the three conditions of theorem 3.2.1.

3. If we replace the linear map D in Theorem 3.2.1, by the following map

$$D(a) = \chi(\bar{a}, c)\ c,$$

where χ is a character and $\bar{c} = \psi(c)$, with $\psi : G \to G$ a homomorphism. Then, we do not change the invariant $T_{\mathcal{B}}$, of Theorem 3.3.2, because it is easy to see that

$$D \otimes D = B^\varphi \circ (D \otimes D).$$

4. Given $\mathbb{Z}/3\mathbb{Z} = \{1, x, x^2\}$ and $\rho = \exp(2\pi i3)$, the character $\chi(x^j \otimes x^k) = \rho^{j-k}$ yields the matrix

$$[D] = \begin{pmatrix} 1 & \rho^2 & \rho \\ \rho & 1 & \rho^2 \\ \rho^2 & \rho & 1 \end{pmatrix}$$

Chapter 4

The projection enhancements

In this chapter, we prove that any enhancement D of the twisted conjugation braiding B^φ satisfies $D^2 = \gamma\ D$, with γ a fixed invertible element in $\mathbb{K}$. In particular, if D is invertible then, $D = \gamma\ I$, i.e., we recover the enhancement D of Corollary 2.2.7.

4.1 The idempotence Theorem

Theorem 4.1.1. (Idempotence)

Let γ be a fixed invertible element in $\mathbb{K}$. Let D denote an endomorphism of $\mathbb{K}[G]$. Assume that $D \otimes D$ commutes with the twisted conjugation braiding B^φ.

1. *If $Sp_2(B^\varphi \circ (D \otimes D)) = \gamma\ D$, then $D^2 = \gamma\ D$.*
2. *If $Sp_2((B^\varphi)^{-1} \circ (D \otimes D)) = \gamma\ D$, then $D^2 = \gamma\ D$.*
3. *The following two statements are equivalent.*
   - *(a) $Sp_2(B^\varphi \circ (D \otimes D)) = \gamma\ D$,*
   - *(b) $Sp_2((B^\varphi)^{-1} \circ (D \otimes D)) = \gamma\ D$.*

Proof It is not loss of generality to assume that D is a non zero, because the above three statements are obviously equivalent if D the zero map.

Let $D(a) = \sum c \in G \Delta(a,c)c$ for all $a \in G$. First of all, notice that

$$(B^\varphi \circ (D \otimes D))(a \otimes b) = \sum_{c,d \in G} \Delta(ab\ \varphi(a)^{-1}, c)\ \Delta(\varphi(a), d)\ c \otimes d \tag{4.1.1}$$

and

$$((B^{\varphi})^{-1}\circ(D\otimes D))(a\otimes b)=\sum_{c,d\in G}\Delta(a,cd\,\varphi(c)^{-1})\Delta(b,\varphi(c))\ c\otimes d \tag{4.1.2}$$

Thus from 4.1.1 and 4.1.2, we can calculate the partial traces on the second factor of $(B^{\varphi}\circ(D\otimes D))$ and of $(B^{\varphi})^{-1}\circ(D\otimes D)$:

$$Sp_2(B^{\varphi}\circ(D\otimes D))(a)=\sum_{c\in G}(\sum_{d\in G}\Delta(ad\,\varphi(a)^{-1},c)\ \Delta(\varphi(a),d)\ )\ c \tag{4.1.3}$$

and

$$Sp_2((B^{\varphi})^{-1}\circ(D\otimes D))(a)=\sum_{c\in G}(\sum_{d\in G}\Delta(a,cd\,\varphi(c)^{-1})\ \Delta(d,\varphi(c)\ )\ )\ c \tag{4.1.4}$$

A direct application of equations 4.1.3 and 4.1.4 yields that equations

$$Sp_2(B^{\varphi}\circ(D\otimes D))=\gamma\ D\ \text{ and }\ Sp_2((B^{\varphi})^{-1}\circ(D\otimes D))=\gamma\ D$$

holds if and only if the following equations are satisfied:

$$\gamma\ \cdot\ \Delta(a,c)=\sum_{d\in G}\Delta(ad\,\varphi(a)^{-1},c)\ \Delta(\varphi(a),d) \tag{4.1.5}$$

and

$$\gamma\ \cdot\ \Delta(a,c)=\sum_{d\in G}\Delta(a,cd\,\varphi(c)^{-1})\ \Delta(d,\varphi(c)) \tag{4.1.6}$$

for all $a,c\in G$. Multiplying, both sides of the last two equations by $\Delta(\varphi(a),\varphi(c)$, we get

$$\begin{aligned}\gamma\ .\ \Delta(\varphi(a),\varphi(c))\ \Delta(a,c) &= \textstyle\sum_{d\in G}\Delta(ad\varphi(a)^{-1},c)\ \Delta(\varphi(a),d)\ \Delta(\varphi(a),\varphi(c))\\ \gamma\ .\ \Delta(a,c)\ \Delta(\varphi(a),\varphi(c)) &= \textstyle\sum_{d\in G}\Delta(a,cd\varphi(c)^{-1})\ \Delta(d,\varphi(c))\ \Delta(\varphi(a),\varphi(c))\end{aligned} \tag{4.1.7}$$

But, by hypothesis there is a least one entry $\Delta(a,c)\neq 0$,thus $Sp_2(B^{\varphi}\circ(D\otimes D))=\gamma\ D$ and $Sp_2((B^{\varphi})^{-1}\circ(D\otimes D))=\gamma\ D$, if and only if the following equations holds

$$\begin{aligned}\gamma\ .\ \Delta(\varphi(a),\varphi(c)) &= \textstyle\sum_{d\in G}\frac{\Delta(ad\varphi(a)^{-1},c)\ \Delta(\varphi(a),d)\ \Delta(\varphi(a),\varphi(c))}{\Delta(a,c)}\\ &\overset{(2.5.6)}{=} \textstyle\sum_{d\in G}\Delta(\varphi(a),d)\Delta(d,\varphi(c))\end{aligned} \tag{4.1.8}$$

and

$$\begin{aligned}\gamma\ .\ \Delta(\varphi(a),\varphi(c)) &= \textstyle\sum_{d\in G}\frac{\Delta(a,cd\varphi(c)^{-1})\ \Delta(d,\varphi(c))\ \Delta(\varphi(a),\varphi(c))}{\Delta(a,c)}\\ &\overset{(2.5.6)}{=} \textstyle\sum_{d\in G}\Delta(\varphi(a),d)\ \Delta(d,\varphi(c))\end{aligned} \tag{4.1.9}$$

for some $a,c\in G$.

Now, is easy to deduce that $Sp_2((B^{\varphi})^{\pm1}\circ(D\otimes D))=\gamma\ D$ implies $D^2=\gamma\ D$, because φ is bijective and the last part of equations 4.1.8 and 4.1.9 are indeed

$$(D\circ D)(a)\ =\ \textstyle\sum_{d\in G}\Delta(a,c)\ \Delta(c,d)\ d \tag{4.1.10}$$

Two finish the proof of theorem rest to prove the equivalence between statements (a) and (b), but it holds from a direct application of Theorem 2.5.2.

∎

The following corollary is an easy consequence of Theorem 4.1.1.

Corollary 4.1.2. *Let γ be a fixed invertible element in $\mathbb{K}$. Then, any enhancement D, of the twisted conjugation braiding B^{φ} satisifies $D^2 = \gamma D$.*
In particular, if D is invertible then $D = \gamma I$, with $\gamma \in \mathbb{K}^*$

Remark 4.1.3. Asumme that D in an enhancement of the twisted braiding B^{φ}, and let $\gamma \in \mathbb{K}$ (fixed and invertible). Set $D' = \frac{1}{\gamma}D$; it is easy to see that D' is an idempotent. Moreover,

$$\frac{1}{\gamma^2} Sp_2((B^{\varphi})^{-1} \circ (D \otimes D)) = \frac{1}{\gamma} D'$$

and by Lemma C.2.1,

$$(D' \otimes D')^2 = (D' \otimes D');$$

i.e., its tensor product is an idempotent, too.

Remark 4.1.4. Let D be an enhancement of the twisted conjugation braiding B^{φ}. Then, $V = \mathbb{K}[G] = V_1 \oplus V_2$, with $V_1 = Im(D)$ and $V_2 = Ker(D)$. The map D has the matrix representation

$$\begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix},$$

where $D_1 : V_1 \to V_1$ and $D_2 : V_2 \to V_2$. Notice that in some basis of V the matrix representation of D is given as follows

$$\begin{pmatrix} \gamma I & 0 \\ 0 & 0 \end{pmatrix}$$

where γ is the fixed invertible element in $\mathbb{K}$ of the idempotent Theorem 4.1.3. Moreover, we have that trace$(D) = \gamma \dim Im(D)$.

Indeed we have, that for all $(o, v_2) \in Ker(D)$,

$$0 = D(0, v_2)^t = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} 0 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ D_2\, v_2 \end{pmatrix}$$

The last equation implies that $D_2\, v_2 = 0$ for all $v_2 \in V_2$, thus $D_2 = 0$. Now, it follows from the idempotent theorem (Theorem 4.1.3) that $D_1^2 = \gamma\, D_1$, for a fixed invertible element in $\mathbb{K}$. Moreover, since D_1 acts on $Im(D)$, then $Ker(D_1) = 0$, i.e., D_1 is invertible. Therefore $D_1 = \gamma\ I$, where I is the $m_1 \times m_1$ matrix, with $m_1 = \dim Im(D)$.

The last part of the remark follows, because $D^2 = \gamma\ D$ (Theorem 4.1.3), and it says that there exist an invertible $d \times d$ matrix P such that

$$P \circ D \circ P^{-1} = \begin{pmatrix} \gamma I & 0 \\ 0 & 0 \end{pmatrix}$$

where I is the $m_1 \times m_1$ identity matrix. From it follows,

$$\mathrm{trace}(D) = \mathrm{trace}(P \circ D \circ P^{-1}) = \gamma \dim Im(D).$$

Lemma 4.1.5. *1. Let B, C denote matrices and let A denote a non-singular matrix. The matrix $A^{-1}BA$ commutes with the matrix C if and only if the matrix B commutes with the matrix ACA^{-1}.*

2. Let D be an enhancement of the twisted conjugation braiding B^{φ}. The tensor product $D \otimes D$ commutes with the twisted conjugation braiding B^{φ} if and only its Jordan form $\widetilde{J}$ commutes with $A^{\otimes 2} B^{\varphi} A^{\otimes(-2)}$, ($A \circ \widetilde{J} \circ A^{-1} = D$.)

3. For the twisted conjugation braiding the following holds

$$\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} = (A \otimes A) \circ B^{\varphi} \circ (A \otimes A)^{-1}$$

where B_1 is a diagonal matrix $m_1^2 \times m_1^2$.

4.2 Examples of projection enhancements

Consider the twisted conjugation braiding B^{φ}, and define $D : \mathbb{K}[G] \to \mathbb{K}[G]$ as $D(g) = \Psi(g)$, with $\Psi = \mathbb{K}(\psi)$, with $\psi \in End(G)$. Assume that φ and ψ commute.

Claim The map $D \otimes D$ commutes with the twisted conjugation braiding B^{φ}.

Proof Let $a \otimes b$ denote a basis element of $\mathbb{K}[G]$. Then,

$$(B^{\varphi} \circ (D \otimes D))(a \otimes b) = B^{\varphi}(\psi(a) \otimes \psi(b)) = \psi(ab)\varphi(\psi(a))^{-1} \otimes \varphi(\psi(a) \qquad (4.2.1)$$

On the other hand

$$((D \otimes D) \circ B^{\varphi})(a \otimes b) = (D \otimes D)(ab\varphi(a)^{-1} \otimes \varphi(a) = \psi(ab)\psi(\varphi(a)^{-1} \otimes \psi(\varphi(a)) \tag{4.2.2}$$

Now, claim follows from equations (4.2.1), (4.2.2) and the commutativity of ψ and φ.

∎

Set $F := B^{\varphi} \circ (D \otimes D)$, lets compute the partial trace Sp_2 of F. First of all, we observe that $F(a \otimes b) = \underbrace{\psi(ab)\psi(\varphi(a))^{-1}}_{:=c} \otimes \underbrace{\psi(\varphi(a))}_{:=d}$.

Notice that c is a function which depends on a and on b, while d is a function which depends on a. Now, write $F(a \otimes b) = \sum_{c,d \in G} f_{a,b}^{c,d}\, c \otimes d$, where

$$f_{a,b}^{c,d} = \begin{cases} 1 & \text{if } \; c = \psi(ab)\psi(\varphi(a)^{-1} \text{ and } d = \psi(\varphi(a)) \\ 0 & \text{else} \end{cases}$$

Observe that for each $a \otimes b$ there is exactly one $c \otimes D$ such that $f_{a,b}^{c,d} \neq 0$, and that from equation (4.2.1) we can compute the partial trace Sp_2 of F, by summing over all terms with the property $b = d$, i.e.,

$$Sp_2(F)(a) = \sum_{b \in G} f_{a,b}^{c,b}\, c \tag{4.2.3}$$

But, now notice that for a given a, there is for each b exactly one $f_{a,b}^{c,b} \neq 0$. Namely

$$\sum f_{a,\psi(\varphi(a)}^{\psi(ab)\psi(\varphi(a)^{-1},\psi(\varphi(a))}.$$

Therefore last equation is equal to have the following equation

$$Sp_2(F)(a) = \sum f_{a,\psi(\varphi(a)}^{\psi(ab)\psi(\varphi(a)^{-1},\psi(\varphi(a))}\ \psi(ab)\psi(\varphi(a)^{-1} = \psi(a)\psi(\psi(\varphi(a)))\psi(\varphi(a))^{-1} \tag{4.2.4}$$

Hence, the condition (T2a) of the definition of EYB-operator (Definition 2.2.1) holds if and only if

$$\psi(a)\psi(\psi(\varphi(a)))\psi(\varphi(a))^{-1} = \beta\, \psi(a)$$

Properties

1. If $\Psi = id$, then $Sp_2(F) = id$.
2. If $\psi\varphi(g^{-1})\varphi(g) \in Ker(\psi)$ for all g. Then, $\psi^2 = \psi$

Remark 4.2.1. Note that the last property (Property (2) 2), shows that $D = [\psi]$ is an enhancement of the twisted conjugation braiding if and only if $\psi\varphi(g^{-1})\varphi(g) \in Ker(\psi)$ for all g.

Chapter 5

Link invariants for EYB-operators of the twisted conjugation braiding

In this chapter we compute the associated link invariants $T_{\mathcal{R}}$, for any EYB-operator $\mathcal{B}$ of the twisted conjugation braiding B^{φ}, for the case when we assume that $(B^{\varphi})^l \circ (D \otimes D) = D \otimes D$, and for all braids $\xi \in Br(n)$, with $\xi = \sigma_{i_1}^{\epsilon_1} \dots \sigma_{i_l}^{\epsilon_l}$ with $\epsilon_1, \dots, \epsilon_l \equiv 0 \bmod l$ and for all braids $\xi = \sigma_i^{\epsilon}$, with $\epsilon \equiv 1 \bmod l$. As a particular case, we get the link invariants for the case when we consider commutative groups G and we set φ to be the identity automorphism. In particular, we get the associated link invariants of the EYB-operator given by Corollary 2.2.7; i.e. when $D = qId$, $q \in \mathbb{K}^*$. At the end of this section, we compute $T_{\mathcal{B}}$, for any EYB-operator $\mathcal{B}$ of the twisted braiding B^{φ}, for the cyclic group $\mathbb{Z}/3\mathbb{Z}$, when we consider $\varphi \neq id$. In particular, the invariants for the EYB-operator given by Corollary 2.2.7.

5.1 Computations of link invariants for some braids $\xi \in Br(n)$

First of all we fix our notation.

1. A braid $\xi \in Br(n)$, with $\xi = \sigma_{i_1}^{\epsilon_1} \dots \sigma_{i_l}^{\epsilon_l}$, and $\epsilon_1, \dots, \epsilon_l \equiv 0 \bmod l$, will be called a *mod-l braid.*
2. A braid $\xi \in Br(n)$, with $\xi = \sigma_i^{\epsilon}$ (for some $i = 1, \dots, n-1$), will be called *single-power braid.*

Proposition 5.1.1. *Asumme that D is an enhancement of the twisted conjugation braiding B^{φ}. Moreover, assume that $(B^{\varphi})^l \circ (D \otimes D) = D \otimes D$ for some $l \in \mathbb{N}$. Then*

1. $T_{\mathcal{B}}(\xi) = \left(\frac{m_1}{\beta}\right)^n$, *for all mod-l braids $\xi \in Br(n)$, where $m_1 = trace(D) = dim\ Im(D)$,*
2. $T_{\mathcal{B}}(\xi) = \left(\frac{m_1}{\beta}\right)^{n-1}$, *for all single-power braids $\xi = \sigma_i^{\epsilon} \in Br(n)$, with $\epsilon \equiv 1 \bmod l$.*

Proof

1. First of all, notice that

$$\begin{aligned} B_i^m \quad &:= \quad (id^{\otimes(i-1)} \otimes B^\varphi \otimes id^{\otimes(n-i-1)} = id^{\otimes(i-1)} \otimes (B^\varphi)^m \otimes id^{\otimes(n-i-1)} \\ B_i^m \circ D^{\otimes n} \quad &= \quad D^{\otimes(i-1)} \otimes (B^\varphi)^m \otimes D^{\otimes(n-i-1)} \end{aligned} \tag{5.1.1}$$

   for all $m \geq 0$ and for all $i \in \{1, \dots, n-1\}$.

   Now, observe that the hypothesis $(B^\varphi)^l \circ (D \otimes D) = D \otimes D$, together with the last equation imply that $B_i^m \circ D^{\otimes n} = D^{\otimes n}$, for all $m \equiv 0 \mod l$.

   Therefore, if ξ is a mod-l braid in $Br(n)$, then

$$\begin{aligned} \text{trace}(b_{B^\varphi}(\xi) \circ D^{\otimes n}) \quad &= \quad \text{trace}(b_{B^\varphi}(\sigma_{i_1}^{\epsilon_1} \dots \sigma_{i_l}^{\epsilon_l}) \circ D^{\otimes n}) \\ &= \quad \text{trace}((B_{i_1}^{\epsilon_1} \dots B_{i_l}^{\epsilon_l}) \circ D^{\otimes n}) \\ &\overset{(5.1.1)}{=} \quad \text{trace}((B_{i_1}^{\epsilon_1} \dots B_{i_{l-1}}^{\epsilon_{l-1}}) \circ D^{\otimes(i_l-1)} \otimes (B^\varphi)^{\epsilon_l} \otimes D^{\otimes(n-i_l-1)}) \\ &= \quad \text{trace}((B_{i_1}^{\epsilon_1} \dots B_{i_{l-1}}^{\epsilon_{l-1}}) \circ D^{\otimes n}) \\ &\vdots \\ &= \quad \text{trace}(B_{i_1}^{\epsilon_1} \circ D^{\otimes n}) \\ &= \quad \text{trace}(D^{\otimes n}) \\ &\overset{(C.2.1)}{=} \quad \text{trace}(D)^n \\ &\overset{(4.1.4)}{=} \quad m_1^n \end{aligned}$$

   Now, proof of part (1) of Lemma, follows by the definition of $T_{\mathcal{B}}$.

2. Part (2) follows from equation 5.1.1, the fact that $(B^\varphi)^\epsilon = B^\varphi$, properties of the partial trace $\text{trace}(B^\varphi \circ (D \otimes D)) = \text{trace}(Sp_2(B^\varphi \circ (D \otimes D))$ and part (T2a) of the Definition of an enhanced Yang-Baxter operator.

∎

Corollary 5.1.2. *Assume that the twisted conjugation braiding B^φ satisifies the following equation $(B^\varphi)^l \circ (D \otimes D) = D \otimes D$, for some $l \geq 0$. If $D = qD$, with $q \in \mathbb{K}$ (invertible)*

1. $T_{\mathcal{B}}(\xi) = |G|^n$, *for all mod-l braids* $\xi \in Br(n)$, *where,*

2. $T_{\mathcal{B}}(\xi) = |G|^{n-1}$, *for all single-power braids* $\xi = \sigma_i^\epsilon \in Br(n)$, *with* $\epsilon \equiv 1 \mod l$.

Example Consider the braid ξ given as in Figure 5.1, which closure is the trefoil knot.

We have $\xi = b_1^3$ and that the braid representation $\rho : Br(2) \to Aut(\mathbb{K}[G]^{\otimes 2})$ associated to b_1, where b_1 denotes the generator of the braid group in 2 strings $Br(2)$ is B^φ. Hence,

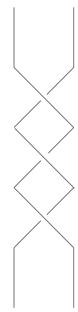


Figure 5.1: Braid with 3 crossings.

$$
\begin{aligned}
T_{\mathcal{B}}(b_1^3) &= q^{-1}\,\mathrm{trace}(b(\sigma_1^3) \circ D^{\otimes 2}) \\
&= 2
\end{aligned}
$$

Examples of enhancements D of the twisted conjugation braiding B^{φ}, satisfying the hypothesis $(B^{\varphi})^l \circ (D \otimes D) = D \otimes D$, of Lemma 5.1.1 occur for example in the following situations.

Proposition 5.1.3. *Let $\mathcal{B}$ denote a EYB-operator of the twisted conjugation braiding B^{φ}. Assume that G is commutative and that φ is the identity automorphism. Then,*

1. $T_{\mathcal{B}}(\xi) = \left(\frac{d}{\beta}\right)^n$, *for all mod-$l$ braids $\xi \in Br(n)$, where $d = trace(D)$,*
2. $T_{\mathcal{B}}(\xi) = \left(\frac{d}{\beta}\right)^{n-1}$, *for all single-power braids $\xi = \sigma_i^{\epsilon} \in Br(n)$, with $\epsilon \equiv 1 \bmod l$.*

Proof of Proposition 5.1.3 Since, G is assumed to be a commutative, B^{φ} is the twist map, i.e.,

$$B^{\varphi}(a \otimes b) = b \otimes a.$$

Therefore, $(B^{\varphi})^2 = id$. Hence, proof follows by Lemma 5.1.1

∎

Remark 5.1.4. Observe, that if in Lemma 5.1.1 we assume that the EYB-operator $\mathcal{B}$ of the twisted conjugation braiding B^{φ} is given as $\mathcal{B} = (B^{\varphi}, D, \lambda = 1, \beta = \mathrm{trace}D)$. Then, $T_{\mathcal{B}}(\xi) = 1$ for all mod-l braids $\xi \in Br(n)$ and for all single-power braids $\xi = \sigma_i^{\epsilon}$, with $\epsilon \equiv 1$ mod l.

Proposition 5.1.5. *Let $\mathcal{B}$ denote a EYB-operator of the twisted conjugation braiding B^{φ}. Set $G = \mathbb{Z}/3\mathbb{Z} =< x >$. Let φ denote the automorphism which sends $x \mapsto x^2,\ x^2 \mapsto x$.*

1. $T_{\mathcal{B}}(\xi) = \left(\frac{d}{\beta}\right)^n$, *for all mod-3 braids $\xi \in Br(n)$, where $d = trace(D)$,*

2. $T_{\mathcal{B}}(\xi) = \left(\frac{d}{\beta}\right)^{n-1}$, *for all single-power braids $\xi = \sigma_i^{\epsilon} \in Br(n)$, with $\epsilon \equiv 1 \bmod 3$.*

3. $T_{\mathcal{B}}(\xi) = \beta^{n-1}\, \widetilde{d}$ *for all single-power braids $\xi = \sigma_i^{\epsilon} \in Br(n)$, with $\epsilon \equiv 2 \bmod 3$, where*

   $\widetilde{d} = trace((B^{\varphi})^2 \circ (D \otimes D))$.

Proof Notice, that a basis for $\mathbb{K}[G]^{\otimes 2}$ is:

$$\mathcal{C} = \{1 \otimes 1, 1 \otimes x, 1 \otimes x^2, x \otimes 1, x \otimes x, x \otimes x^2, x^2 \otimes 1, x^2 \otimes x, x^2 \otimes x^2\}.$$

On the basis $\mathcal{C}$ the braiding B^{φ} has the following matrix representation:

$$\begin{pmatrix} 1 & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & 1 \\ . & . & . & . & 1 & . & . & . & . \\ . & 1 & . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 & . & . \\ . & . & . & . & . & 1 & . & . & . \\ . & . & 1 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & 1 & . \\ . & . & . & 1 & . & . & . & . & . \end{pmatrix}$$

Now, it is not difficult to prove that for all $m \geq 0$,

$$(B^{\varphi})^m = \begin{cases} Id & \text{if } m = 3k,\ k \in \mathbb{N} \\ B^{\varphi} & \text{if } m = 3k+1, k = 0, 1, \dots \\ (B^{\varphi})^2 & \text{if } m = 3k+2,\ k = 0, 1, \dots \end{cases}$$

Hence, proof of proposition follows by Lemma 5.1.1.

∎

As a consequence of previous Proposition, we get the following Corollary.

Corollary 5.1.6. *Consider $\mathcal{B}$ to be the EYB-operator given by Corollary 2.2.7; i.e. $D = qI$, with $q \in \mathbb{K}$ invertible.*

1. $T_{\mathcal{B}}(\xi) = 1$, *for all mod-3 braids $\xi \in Br(n)$, where $d = trace(D)$,*

2. $T_{\mathcal{B}}(\xi) = 3$, *for all single-power braids $\xi = \sigma_i^{\epsilon} \in Br(n)$, with $\epsilon \equiv 1 \bmod 3$.*

3. $T_{\mathcal{B}}(\xi) = 2$ *for all single-power braids* $\xi = \sigma_i^{\epsilon} \in Br(n)$, *with* $\epsilon \equiv 2 \mod 3$,

Another examples where the hypothesis $(B^{\varphi})^l \circ (D \otimes D) = D \otimes D$ of Lemma 5.1.1 is satisified are given by the following theorems.

Definition 5.1.7. *Consider the* $\mathbb{K}$*-linear* D*, given as in 2.5.1. We say that* D *satisfies the* **weak hypothesis** *with respect to* φ *if and only if* $\Delta(a, c) \neq 0$ *whenever* $\Delta(\varphi(a), \varphi(c)) = 0$.

Theorem 5.1.8. *Assume that the twisted conjugation braiding* B^{φ} *commutes with* $D \otimes D$*, and that* D *satisfies the weak hypothesis with respect to* φ*. Then*

$$(D^{-1} \otimes D^{-1}) \circ B^{\varphi} \circ (D \otimes D) = (B^{\varphi})^{-1} \tag{5.1.2}$$

In particular,

$$(B^{\varphi})^2 \circ (D \otimes D) = D \otimes D.$$

Proof Using the definition of the twisted conjugation braiding B^{φ} (Definition 1.1), and formula 2.5.3, we get

$$(B^{\varphi} \circ (D \circ D) \circ B^{\varphi})(a \otimes b) = \sum_{c,d \in G} \Delta(ab\varphi(a)^{-1}, c)\Delta(\varphi(a), d)(cd\varphi(c)^{-1} \otimes \varphi(c))$$

On the other hand we have seen that $D \otimes D$ is given by the formula

$$(D \otimes D)(a \otimes b) = \sum_{s,t \in G} \Delta(a, s)\Delta(b, t) s \otimes t$$

Therefore, using again the fact that for every basis element $s \otimes t$, there is a second element $c \otimes d$ such that $s = cd\varphi(c)^{-1}$ and $t = \varphi(c)$, equality 5.1.2 will hold if and only if

$$\Delta(ab\, \varphi(a)^{-1}, c)\Delta(\varphi(a), d) = \Delta(a, cd\, \varphi(c)^{-1})\Delta(b, \varphi(c)).$$

Now, assume that $\Delta(\varphi(a), \varphi(c)) \neq 0$, then equation 2.5.6 implies

$$\Delta(ab\varphi(a)^{-1}, c)\Delta(\varphi(a), d) = \frac{\Delta(a, b)\Delta(b, \varphi(c))\Delta(\varphi(a), d)}{\Delta(\varphi(a), \varphi(c)} = \Delta(a, cd\varphi(c)^{-1})\Delta(b, \varphi(c))$$

On the other hand, if $\Delta(\varphi(a), \varphi(\varphi(c)) \neq 0$, then by the given hypothesis $\Delta(a, b) \neq 0$. So, equation 2.5.6 implies that $\Delta(b, \varphi(c))$ and $\Delta(\varphi(a), d)$ both will vanish and therefore

$$\Delta(ab\varphi(a)^{-1}, c)\Delta(\varphi(a), d) = 0 = \Delta(a, cd\varphi(c)^{-1})\Delta(b, \varphi(c)).$$

∎

Remark We can write Theorem 5.1.2 a little bit more general as follows:

Theorem 5.1.9. *Suppose that $D, D \otimes D$ and B^φ are defined as in Theorem 5.1.2. Moreover, assume that $(D \otimes D)$ and B^φ commute and that there is no pair of elements a and $c \in G$ such that $\Delta(a, c)$ and $\Delta(\varphi(a), \varphi(c))$ vanish at the same time. Then*

$$B^\varphi \circ (D \otimes D) \circ B^\varphi = D \otimes D \tag{5.1.3}$$

In particular

$$(B^\varphi)^2 \circ (D \otimes D) = D \otimes D = (D \otimes D) \circ (B^\varphi)^2.$$

Proof It is similar to proof of Theorem 5.1.2. Because equation 5.1.3 holds if and only if

$$\Delta(ab\ \varphi(a)^{-1}, c)\Delta(\varphi(a), d) = \Delta(a, cd\ \varphi(c)^{-1})\Delta(b, \varphi(c))$$

Now, if $\Delta(a, c) \neq 0$, equation 2.5.6 implies

$$\Delta(ab\ \varphi(a)^{-1}, c)\Delta(\varphi(a), d) = \frac{\Delta(ab\ \varphi(a)^{-1}, c)\ \ \Delta(\varphi(a), \varphi(c))\ \ \Delta(a, cd\ \varphi(c)^{-1})}{\Delta(a, c)} = \Delta(a, cd\ \varphi(c)^{-1})\Delta(b, \varphi(c))$$

On the other hand, if $\Delta(\varphi(a), \varphi(c)) \neq 0$, equation 2.5.6 implies

$$\Delta(ab\varphi(a)^{-1}, c)\Delta(\varphi(a), d) = \frac{\Delta(b, \varphi(c))\ \Delta(a, c)\ \Delta(\varphi(a), d)}{\Delta(\varphi(a), \varphi(c))} = \Delta(a, cd\varphi(c)^{-1})\Delta(b, \varphi(c)).$$

∎

Chapter 6

Specific computations

In the first section of this chapter, we prove that $ord(B^{\varphi}) = ord(B^{id})$ for all $\varphi \in Inn(G)$. Moreover, we prove that for finite groups G the twisted conjugation braiding B^{φ} satisifies $(B^{\varphi})^{l}(a \otimes b) = a \otimes b)$, for $l = 2 \cdot lcm(ord(a), ord(b))$. From this and Proposition 5.1.1 follows that the link invariant is $T_{\mathcal{B}}(\xi) = \left(\frac{m_1}{\beta}\right)^{n}$, for all mod-l braids $\xi \in Br(n)$, where $m_1 = \text{trace}(D)$ and $T_{\mathcal{B}}(\xi) = \left(\frac{m_1}{\beta}\right)^{n-1}$ for all single power braids $\xi = \sigma_i^{\epsilon} \in Br(n)$, with $\epsilon \equiv 1 \bmod l$. With the help of he computer program "Bphi_orders," which is written in JAVA programming language, we compute at the end of this chapter the link invariants $T_{\mathcal{B}}$ for the enhancement $D = \gamma I$ ($\gamma \in \mathbb{K}^{*}$) for braids $\xi \in Br(p)$ (p prime) with $\xi = (\sigma_1 \dots \sigma_{p-1})^{q}$, and with $(p, q) = 1$ for the cases $G = \Sigma_n$ and $G = \mathbb{Z}/n\mathbb{Z}$.

6.1 Orders of B^{φ} for symmetric groups

In this section, we prove that $ord(B^{\varphi_c}) = ord(B^{id})$, where $\varphi(c) = cgc^{-1}$. Moreover, we prove that for finite groups G the twisted conjugation braiding B^{φ} satisfies $(B^{\varphi})^{l}(a \otimes b) = a \otimes b$ for $l = 2 \cdot lcm(ord(a), ord(b))$. We give a table of the orders of the twisted conjugation braiding B^{φ}, for the case when we consider the symmetric group Σ_n, with $n = 3, 4, 5, 7$. For the case we consider G to be the symmetric group Σ_6, we compute the orders of the twisted conjugation braiding only for the case when the automorphism φ is an inner automorphism.

Proposition 6.1.1. *Let G be any group and let $\varphi(g) := cgc^{-1}$ be an inner automorphism of G. There exists an invertible map $\Gamma : \mathbb{K}[G]^{\otimes 2} \to \mathbb{K}[G]^{\otimes 2}$, such that $B^{\varphi} = \Gamma \circ B^{id} \circ \Gamma^{-1}$.*

In particular, $ord(B^{\varphi}) = ord(B^{id})$ for all $\varphi \in Inn(G)$.

Proof Define the map $\Gamma : \mathbb{K}[G]^{\otimes 2} \to \mathbb{K}[G]^{\otimes 2}$ as:

$$\Gamma(a \otimes b) = (L_c)^{-1}(a) \otimes (R_c)^{-1}(b),$$

where $(L_c)^{-1}$ and $(R_c)^{-1}$ denote the inverse maps of the left and right translation maps, respectively. It is easy to see that Γ is invertible; an inverse is:

$$(\Gamma)^{-1}(a \otimes b) = L_c(a) \otimes R_c(b).$$

Now, it is left to prove that $B^{\varphi} = \Gamma \circ B^{id} \circ \Gamma^{-1}$.

On the one hand we have:

$$\Gamma \circ B^{id}(a \otimes b) \quad = \quad \Gamma(aba^{-1} \otimes a) = c^{-1}(aba^{-1}) \otimes ac^{-1}$$

On the other hand:

$$\begin{aligned} B^{\varphi} \circ \Gamma(a \otimes B) &= B^{\varphi}(c^{-1}a \otimes bc^{-1}) = (c^{-1}abc^{-1}\varphi_c(c^{-1}a)^{-1} \otimes \varphi_c(c^{-1}a) \\ &= c^{-1}abc^{-1}(cc^{-1}ac^{-1})^{-1} \otimes cc^{-1}ac^{-1} \\ &= c^{-1}aba^{-1} \otimes ac^{-1} \end{aligned}$$

Now, it follows from the bijectivity of Γ that:

$$B^{\varphi} = \Gamma \circ B^{id} \circ \Gamma^{-1}.$$

In particular, $ord(B^{\varphi}) = ord(B^{id})$ for all $\varphi \in Inn(G)$.

∎

Remark 6.1.2.
1. For the symmetric group Σ_n $(n \neq 6)$ we have $ord(B^{\varphi}) = ord(B^{id})$, for all $\varphi \in Aut(\Sigma_n)$.
2. If Σ_6, then $ord(B^{\varphi}) = ord(B^{id})$, for all $\varphi \in Inn(\Sigma_6)$.
3. trace$((B^{\varphi})^m)$ = trace$((B^{id})^m)$ for all $\varphi \in Inn(G)$.

Notation Let $a, b \in G$. Denote by $b_a := aba^{-1}$

Lemma 6.1.3. *Let G be any group and $a, b \in G$, and let $k \in \mathbb{N}$.*

(a) If $k = 2l+1$ is odd, then $(B^{id})^{2l+1}(a \otimes b) = (b_a)_{(ab)^l} \otimes a_{(ab)^l}$

(b) If $k = 2l$ is even, then $(B^{id})^{2l}(a \otimes b) = a_{(ab)^l} \otimes b_{(ab)^l}$

Proof Follows by an easy computation. ∎

Proposition 6.1.4. *Let G denote a finite group and let $a, b \in G$. Assume that $\varphi \in Inn(G)$ then*

$$(B^{\varphi})^{2\cdot lcm(ord(a),ord(b))}(a \otimes b) = a \otimes b$$

Proof Note that, since G is finite there exists $\mathrm{lcm}(ord(a), ord(b))$. From Lemma 6.1.1, we saw that there exists an invertible map $\Gamma : \mathbb{K}[G]^{\otimes 2} \to \mathbb{K}[G]^{\otimes 2}$ such that $B^{\varphi} = \Gamma B^{id} \Gamma^{-1}$ for all $\varphi \in Inn(G)$. Thus, it is enough to prove the proposition for B^{id}.

From Lemma 6.1.3 follows that:

$$(B^{id})^{2\cdot \mathrm{lcm}(ord(a),ord(b))}(a \otimes b) = a_{(ab)^{\mathrm{lcm}(ord(a),ord(b))}} \otimes b_{(ab)^{\mathrm{lcm}(ord(a),ord(b))}} = a \otimes b$$

∎

Remark The above proposition shows that if the least common mutiple m of the order of all elements $a \in G$ exists, then the order of the twisted conjugation braiding B^{φ} is smaller than or equal to $2m$. From Proposition 5.1.1 and the above proposition follows that the link invariant is $T_{\mathcal{B}}(\xi) = \left(\frac{m_1}{\beta}\right)^n$ for all mod-l braids $\xi \in Br(n)$, where $m_1 = \mathrm{trace}(D)$ and $T_{\mathcal{B}}(\xi) = \left(\frac{m_1}{\beta}\right)^{n-1}$ for all single power braids $\xi = \sigma_i^{\epsilon} \in Br(n)$, with $\epsilon \equiv\ 1 \bmod l$.

Proposition 6.1.5. *Consider the symmetric group Σ_n. Let $\varphi \in Inn(G)$, and let $a, b \in G$. Then*

$$(B^{id})^{2\cdot lcm(1,2,\ldots,n)}(a \otimes b) = a \otimes b$$

Moreover, the order l' of the twisted conjugation braiding B^{φ} is equal to $2 \cdot lcm(1, 2, \ldots, n)$.

Proof It is enough to prove the proposition for the case $\varphi = id$, because according to Proposition 6.1.1 there exists an invertible map $\Gamma : \mathbb{K}[G]^{\otimes 2} \to \mathbb{K}[G]^{\otimes 2}$ such that $B^{\varphi} = \Gamma B^{id} \Gamma^{-1}$, for all $\varphi \in Inn(G)$.

Now, from Lemma 6.1.3 follows that

$$(B^{id})^{2\cdot \mathrm{lcm}(1,2,\ldots,n)}(a \otimes b) = a_{(ab)^{\mathrm{lcm}(1,2,\ldots,n)}} \otimes b_{(ab)^{\mathrm{lcm}(1,2,\ldots,n)}}$$

Note that the permutation ab decomposes into a product of disjoint cycles $c_1, \ldots, c_m$ of length $l_1, \ldots, l_m$, with $\sum_{i=1}^{m} l_i = n$.

We have $ord(c_i) = l_i$ for all $i = 1, \ldots, m$. Thus, $ord(ab) = ord(c_1 \ldots c_m) = \mathrm{lcm}(l_1, \ldots, l_m)$.

Observe that $\mathrm{lcm}(l_1, \ldots, l_m) \mid \mathrm{lcm}(1, 2, \ldots, n)$. Therefore, $(B^{id})^{2\cdot \mathrm{lcm}(1,2,\ldots,n)}(a \otimes b) = a \otimes b$

Now, it is left to prove that the order l' of the twisted conjugation braiding B^{φ} is equal to $2 \cdot \text{lcm}(1, 2, \ldots, n)$. To prove it, we have to show that for all $m \in \{1, 2, \ldots, n\}$ there exist $g \in \mathbb{K}[\Sigma_n]^{\otimes 2}$ such that $(B^{\varphi})^{2m}(g) = g$ and $(B^{\varphi})^{r}(g) \neq g$ for all $r \leq 2m$.

Choose $m \in \{1, 2, \ldots, n\}$ and define $g := (1,2)(2,3) \cdots (s-1, s) \otimes (s, s+1) \cdots (m-1, m)$ Observe that $((1,2)(2,3) \cdots (m-1,m))^m = 1 = (23 \ldots m1)^m$. Therefore, $(B^{\varphi})^{2m}(g) = g$.
For all $s \in \{1, \ldots, m-1\}$ it holds:

$$(s)_{(23 \ldots m1)^k} \otimes (s+1)_{(23 \ldots m1)^k} = s + k \ (\text{mod } m) \otimes s + 1 + k \ (\text{mod } m) \neq s \otimes (s+1) \text{ for } k < m$$

Therefore, $(B^{\varphi})^{2k}(g) \neq g$ for $k < m$.

Moreover, it holds:

$$(s)_{(12)(s-1,s)} \otimes (s+1)_{(12)(s-1,s)} = (s)_{(23 \ldots s1)} \otimes (s+1)_{(23 \ldots s1)} = 1 \otimes (s+1)$$

and

$$(1)_{(23 \ldots m1)^k} \otimes (s+1)_{(23 \ldots m1)^k} = 1 + k \ (\text{mod } m) \otimes r + 1 \ (\text{mod } m) \neq s \otimes (s+1) \text{ for } k < m$$

Therefore, $(B^{\varphi})^{2k+1}(g) \neq g$ for $k < 2m$. Thus, for all $r < 2m$ the twisted conjugation braiding B^{φ} satisifies $(B^{\varphi}(g))^r \neq g$.

∎

Examples

The table below (Table (6.1)) shows the order of the twisted conjugation braiding for the symmetric groups Σ_n (for $n = 3, 4, 5, 6, 7$).

Table 6.1: Orders of B^{φ} for Σ_n

Automorphism φ	Group Σ_n	Order of the B^{φ}
$\varphi(s) = s_2 s s_2^{-1}$	Σ_3	12
$\varphi(s) = s_2 s s_2^{-1}$	Σ_4	24
$\varphi(s) = s_2 s s_2^{-1}$	Σ_5	120
$\varphi(s) = s_2 s s_2^{-1}$	Σ_6	120
$\varphi(s) = s_2 s s_2^{-1}$	Σ_7	840

Remark For Σ_6 we consider only inner automorphisms.

6.2 Orders of B^φ for cyclic groups C_p

In this section, we give tables of the values of the orders of the twisted conjugation braiding B^φ for 7 cyclic groups $\mathbb{Z}/n\mathbb{Z} =< x, x^n = 1 >$. To compute the orders of the twisted conjugation braiding B^φ, we used the fact that if $\varphi \in Aut(\mathbb{Z}/n\mathbb{Z})$, then $\varphi(x) = x^l$ for some l with $gcd(n, l) = 1$.

Proposition 6.2.1. *Let $G = \mathbb{Z}/n\mathbb{Z} =< x, x^n = 1 >$. Let $\varphi \in Aut(G)$, i.e $\varphi(x) = x^l$ for some $l \in \mathbb{Z}$ with $gcd(n, l) = 1$. For the twisted conjugation braiding B^φ it holds:*
$(B^\varphi)^k(a \otimes b) = a \otimes b$,

$$k = \begin{cases} \frac{p-1}{gcd(k_1, p-1)} & if \quad n = p \\ lcm\left(\frac{p_i^{\alpha_i}(p_i-1)}{gcd(l_i, p_i^{\alpha_i-1}(p_i-1))}\right) & if \quad n = p_1^{\alpha_1} \dots p_r^{\alpha_r} \end{cases}$$

where $k_1 \in \mathbb{Z}$ with $(-l) \equiv a^{k_1} \bmod p$ and with a a primitive root of unity mod p. And where $l_i \in \mathbb{Z}$ for all $i = 1, \dots, r$ with $(-l) \equiv a_i^{l_i} \bmod p_i^{\alpha_i}$, and with a_i a primitive root of unity mod $p_i^{\alpha_i}$ for all $i = 1, \dots, r$.

Proof For every generator $a \otimes b \in \mathbb{K}[G]^{\otimes 2}$ we write the twisted conjugation braiding B^φ additively:

$$B^\varphi(a, b) = (a + b - la, la).$$

As a matrix it is:

$$\begin{pmatrix} 1-l & 1 \\ l & 0 \end{pmatrix}$$

It is not difficult to see that the above matrix is similar to the following matrix:

$$\begin{pmatrix} 1 & 0 \\ 1 & -l \end{pmatrix}$$

Moreover, it is not difficult to see that :

$$\begin{pmatrix} 1 & 0 \\ 1 & -l \end{pmatrix}^k = \begin{pmatrix} 1 & 0 \\ c_{k-1}(-l) & (-l)^k \end{pmatrix}$$

where $c_{k-1}(-l) = 1 - l + l^2 + \dots + (-1)^{k-1}l^{k-1}$.

To finish the proof, we have to find the minimun k such that the following congruences hold:

(i) $C_{k-1}(-l) \equiv 2 \cdot \bmod n$ and

(ii) $(-l)^k - 1 \equiv 2 \cdot \bmod n$

Case 1 $l = -1$, then $k = n$

Case 2 $l \neq -l$, then $k = ord(-l)$ in $(\mathbb{Z}/n\mathbb{Z})^*$

If $n = p$, (p prime) then $ord(-l) = \frac{p-1}{\gcd(k_1,\, p-1)}$, where $k_1 \in \mathbb{Z}$, with $(-l) \equiv a^{k_1} \bmod p$, and with a a primitive root of the unity mod p.

For $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$, it is known that:

1. $(\mathbb{Z}/n\mathbb{Z})^* \cong (\mathbb{Z}/p_1^{\alpha_1})^* \times \cdots \times (\mathbb{Z}/p_r^{\alpha_r})^*$
2. $(\mathbb{Z}/2^\alpha\mathbb{Z})^* \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{\alpha-2}$ for $\alpha \geq 2$
3. $(\mathbb{Z}/p^\alpha\mathbb{Z})^* \cong \mathbb{Z}/p^{\alpha-1}(p-1)$

From this follows, that $ord(-l) = \operatorname{lcm}\left(\frac{p_i^{\alpha_i}(p_i-1)}{\gcd(l_i,\, p_i^{\alpha_i-1}(p_i-1))}\right)$, where $(-l) \equiv a_i^{l_i} \bmod p_i^{\alpha_i}$, and with a_i a primitive root of unity mod $p_i^{\alpha_i}$ for all $i = 1, \dots, r$.

∎

Examples

The following tables contain the orders of the twisted conjugation braiding B^φ for the case when we consider $G = \mathbb{Z}/n\mathbb{Z}$, where $n = 3, 5, 7, 8, 10, 11, 13, 17$ and all its automorphisms.

Table 6.2: Orders of the B^φ for C_3

Automorphism φ	Order of the B^φ
$\varphi(x) = x$	2
$\varphi(x) = x^2$	3

Table 6.3: Orders of the B^φ for C_5

Automorphism φ	Order of the B^φ
$\varphi(x) = x$	2
$\varphi(x) = x^2$	4
$\varphi(x) = x^3$	4
$\varphi(x) = x^4$	5

Table 6.4: Orders of the B^{φ} for C_7

Automorphism φ	Order of the B^{φ}
$\varphi(x) = x$	2
$\varphi(x) = x^2$	6
$\varphi(x) = x^3$	3
$\varphi(x) = x^4$	6
$\varphi(x) = x^5$	3
$\varphi(x) = x^6$	7

Table 6.5: Orders of the B^{φ} for C_{11}

Automorphism φ	Order of the B^{φ}
$\varphi(x) = x$	2
$\varphi(x) = x^2$	5
$\varphi(x) = x^3$	10
$\varphi(x) = x^4$	10
$\varphi(x) = x^5$	10
$\varphi(x) = x^6$	5
$\varphi(x) = x^7$	5
$\varphi(x) = x^8$	5
$\varphi(x) = x^9$	10
$\varphi(x) = x^{10}$	11

Table 6.6: Orders of the B^{φ} for C_{13}

Automorphism φ	Order of the B^{φ}
$\varphi(x) = x$	2
$\varphi(x) = x^2$	12
$\varphi(x) = x^3$	6
$\varphi(x) = x^4$	3
$\varphi(x) = x^5$	4
$\varphi(x) = x^6$	12
$\varphi(x) = x^7$	12
$\varphi(x) = x^8$	4
$\varphi(x) = x^9$	6
$\varphi(x) = x^{10}$	3
$\varphi(x) = x^{11}$	12
$\varphi(x) = x^{12}$	13

Table 6.7: Orders of the B^{φ} for C_{17}

Automorphism φ	Order of the B^{φ}
$\varphi(x) = x$	2
$\varphi(x) = x^2$	8
$\varphi(x) = x^3$	16
$\varphi(x) = x^4$	4
$\varphi(x) = x^5$	16
$\varphi(x) = x^6$	16
$\varphi(x) = x^7$	16
$\varphi(x) = x^8$	8
$\varphi(x) = x^9$	8
$\varphi(x) = x^{10}$	16
$\varphi(x) = x^{11}$	16
$\varphi(x) = x^{12}$	16
$\varphi(x) = x^{13}$	4
$\varphi(x) = x^{14}$	16
$\varphi(x) = x^{15}$	8
$\varphi(x) = x^{16}$	17

The following tables contain the orders of the twisted conjugation braiding B^{φ} for the cyclic group $\mathbb{Z}/8\mathbb{Z}$ and for the cyclic group $\mathbb{Z}/10\mathbb{Z}$.

Table 6.8: Orders of the B^{φ} for $\mathbb{Z}/8\mathbb{Z}$

Automorphism φ	Order of the B^{φ}
$\varphi(x) = x$	2
$\varphi(x) = x^3$	8
$\varphi(x) = x^5$	4
$\varphi(x) = x^7$	8

Table 6.9: Orders of the B^{φ} for $\mathbb{Z}/10\mathbb{Z}$

Automorphism φ	Order of the B^{φ}
$\varphi(x) = x$	2
$\varphi(x) = x^3$	4
$\varphi(x) = x^7$	4
$\varphi(x) = x^9$	10

Remark All orders of the twisted conjugation braiding B^{φ} were computed using the program "Bphi_orders".

6.3 Consideration of the matrix sizes

The size of the matrices B_i of the tensor product $id^{\otimes(i-1)} \otimes B^\varphi \otimes id^{\otimes(p-i-1)}$ is $d^p \times d^p$, where $d = |G|$. Therefore, to compute the traces for the word braid $(\sigma_1\sigma_2 \ldots \sigma_{p-1})^q$, when we consider the enhancement $D = \gamma I$, where $\gamma \in \mathbb{K}^*$ of the twisted conjugation braiding B^φ, turns out to be a very complicated computation by using the program "Bphi_orders" , as we can see in the following tables for the case when we set the group G to be either the symmetric group Σ_n (n=3, ..., 7) or to be the cyclic group.

Notation Denote by $a = d^p$, where $d = |G|$ and p as above. The following tables show the values of a for the case when we consider the symmetric group Σ_n and the cyclic group $\mathbb{Z}/n\mathbb{Z}$.

Table 6.10: Symmetric group and the values of $a = (n!)^p$

	p:=2	3	4	5	6	7	8	9	10
n:=2	4	8	16	32	64	128	256	512	1024
3	36	216	1296	7776	46656	279936	1679616	10077696	60466176
4	576	13824	331776	7962624	19×10^7	45×10^8	11×10^{10}	26×10^{11}	63×10^{12}
5	14400	1728000	207×10^6	24×10^9	29×10^{11}	35×10^{13}	42×10^{15}		
6	518400	373×10^6	26×10^{11}	19×10^{13}					
7	25×10^6	12×10^{10}	64×10^{13}						

Table 6.11: Cyclic group and the values of $a = n^p$

	p:=2	3	4	5	6	7	8	9	10
n:=2	4	8	16	32	64	128	256	512	1024
3	9	27	81	243	729	2187	6561	19683	59049
4	16	64	256	1024	4096	16384	65536	262144	1048576
5	25	125	625	3125	15625	78125	390625	1953125	9765625
6	36	216	1296	7776	46656	279936	1679616	10077696	

Remark The program computes the trace of $(\sigma_1 \ldots \sigma_{p-1})^q$ for bigger cyclic groups, but nevertheless

Table 6.12: Cyclic group a x a matrix

	p=6	7	8
n=11	1771561	19×10^6	21×10^6
12	2925924	35×10^6	42×10^7
13	4226209	62×10^6	752×10^7

Remark By using the above tables (Tables (6.9), (6.10) and (6.11) we can compute the amount of required RAM memory for the computation: multiply the value of a with 4-bytes and then divide it by 1 GB (Giga-byte). For example, if $a = (n!)$, then $a \times 4/(1024)^3$ = number of GB you need for computing the trace of the map $b_{B^\varphi}(\xi)$, where $\xi = (\sigma_1 \ldots \sigma_{p-1})^q$.

6.4 Link invariants of torus knots

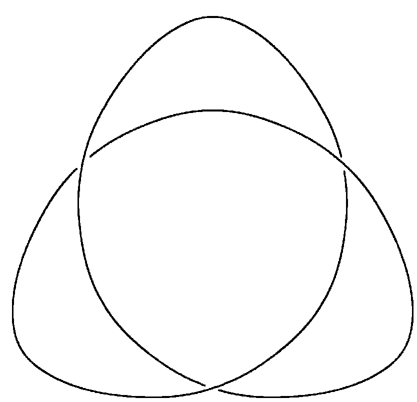
Notation Here, $\mathbb{K}$ denotes the field of the complex numbers $\mathbb{C}$. Let D denote the enhancement $D = qI$, where q is an invertible element of the field $\mathbb{C}$. Let (p, q) denote a pair of coprime integers.

In this section, we give a table of the values of the link invariants of a (p, q)-torus knot, for the cases when we consider the enhancement $D = \gamma\ ; I$, where $q \in \mathbb{K}^*$ of the twisted conjugation braiding B^{φ}. These values have been calculated by using the computer program “Bphi_orders”.

Recall that any (p, q)-torus knot can be made from a closed braid with p strands. The appropiate braid word is

$$(\sigma_1\sigma_2 \ldots \sigma_{p-1})^q$$

Notice that torus knots are trivial if and only if either p or q is equal to 1. The simplest nontrivial example is the $(2, 3)$-torus knot, also known as the trefoil knot (see following Figure).



Remark 6.4.1. If D is an invertible enhancement of the twisted conjugation braiding B^{φ}, then

$$T_{\mathcal{B}}(\xi) = \operatorname{trace}(b(\xi)),$$

for any braid $\xi \in Br(n)$. Thus, if $\xi \in Br(p)$, with $\xi = (\sigma_1\sigma_2 \ldots \sigma_{p-1})^q$, and with $(p, q) = 1$, then

$$T_{\mathcal{B}}(\xi) = \operatorname{trace}(b(\sigma_1 \ldots \sigma_{p-1})^q).$$

Indeed we proved that the link invariant $T_{\mathcal{B}}$ associated to any enhancement of twisted conjugation braiding B^{φ} is given by the following formula

$$T_{\mathcal{B}}(\xi) = \beta^{-n}\text{trace}\,(b(\xi) \circ D^{\otimes n})$$

for any braid $\xi \in Br(n)$. (See Corollary 2.5.3). Moreover, we proved that any enhancement D of the twisted conjugation braiding B^{φ} satisfies $D^2 = \gamma\ \cdot D$. (Idempotent Theorem 4.1.3).

Notation In the programm we used the following notation for the elements of the symmetric group $s_0 = 1, s_1, \ldots, s_{n!-1}$.

Now, recall that if G is a finite group, then trace$((B^{\varphi})^m)$ = trace$((B^{id})^m)$ for all $\varphi \in Inn(G)$ (see Remark 6.1.2). By using the program "Bphi_orders" we get the following values for the link invariants $T_{\mathcal{B}}$ of the torus knot for the case when we set G to be the symmetric group Σ_5, $\varphi(s) = s_2 s s_2^{-1}$, for all $s \in \Sigma_5$ and with $s_2 \in \Sigma_5$, and for the case that we consider the enhancement $D = \gamma I$ of the twisted conjugation braiding B^{φ}.

Table 6.13: Link invariants for $G = \Sigma_5$, $\varphi(s) = s_2 s s_2^{-1}$ and $D = \gamma I$

Knot	Name	(p, q)	$T_{\mathcal{B}}$
	Hopf link	$(2, 2)$	840
3_1	Trefoil knot	(2, 3)	600
5_1	Solomon's seal knot	(2, 5)	720
7_1	7 crossing torus knot	(2, 7)	120
8_{19}	8 crossing torus knot	(3, 4)	1200
9_1	9 crossing torus knot	(2, 9)	600
10_{124}	10 crossing torus knot	(3, 5)	600
	11 crossing torus knot	(2, 11)	120

Remark From the previous table (Table (6.13)) we can see that the trefoil knot σ_1^3, the 9 crossing torus knot and the 10 crossing knot have the same link invariant $T_{\mathcal{B}}$ associated to the enhancement $D = \gamma I$ (γ invertible).

By using the program Bphi_orders we get the following link invariants $T_{\mathcal{B}}$ (see Table 6.16) of the enhancement $D = \gamma I$ of the twisted conjugation braiding B^{φ}. For the case that we consider torus knots and for the case that we set the group G to be the symmetric group Σ_4. We set the automorphism to be $\varphi(s) = s_3 s s_3^{-1}$ for all $s \in \Sigma_4$.

Table 6.14: Link invariants for $G = \Sigma_4, \varphi(s) = s_3 s s_3^{-1}$ and $D = \gamma I$

Knot	Name	(p, q)	$T_{\mathcal{B}}$
	Hopf link	$(2, 2)$	120
3_1	Trefoil knot	(2, 3)	96
5_1	Solomon's seal knot	(2, 5)	24
7_1	7 crossing torus knot	(2, 7)	24
8_{19}	8 crossing torus knot	(3, 4)	144
9_1	9 crossing torus knot	(2, 9)	96
10_{124}	10 crossing torus knot	(3, 5)	24
	11 crossing torus knot	(2, 11)	24

By using the program "Bphi_orders" we get the following link invariants $T_{\mathcal{B}}$ (see Table 6.15) of the enhancement $D = \gamma I$ of the twisted conjugation braiding B^{φ}, for the case when we consider torus knots.

Table 6.15: Link invariants for $G = \Sigma_7, \varphi(s) = s_2 s s_2^{-1}$ and $D = \gamma I$

Knot	Name	(p, q)	$T_{\mathcal{B}}$
	Hopf link	$(2, 2)$	7920
3_1	Trefoil knot	(2, 3)	6480
5_1	Solomon's seal knot	(2, 5)	11520
7_1	7 crossing torus knot	(2, 7)	720
9_1	9 crossing torus knot	(2, 9)	6480
	11 crossing torus knot	(2, 11)	720

Remark By looking at the above tables (see Tables 6.13, 6.16 and 6.15), we can see that our results are almost of the kind "the polynomial is constant," i.e., $T_{\mathcal{B}} \in \mathbb{K}$. Since the only braidings we consider are permutations of the basis of $\mathbb{K}[G]^{\otimes 2}$.

Table 6.16: Link invariants for $G = \mathbb{Z}/10\mathbb{Z}, \varphi(x) = x^9$ and $D = \gamma I$

Knot	Name	(p, q)	$T_{\mathcal{B}}$
	Hopf link	$(2, 2)$	20
3_1	Trefoil knot	(2, 3)	10
5_1	Solomon's seal knot	(2, 5)	50
7_1	7 crossing torus knot	(2, 7)	10
8_{19}	8 crossing torus knot	(3, 4)	10
9_1	9 crossing torus knot	(2, 9)	10
10_{124}	10 crossing torus knot	(3, 5)	10
	11 crossing torus knot	(2, 11)	10

Table 6.17: Link invariants for $G = \mathbb{Z}/20\mathbb{Z}, \varphi(x) = x^7$ and $D = \gamma I$

Knot	Name	(p, q)	$T_{\mathcal{B}}$
	Hopf link	$(2, 2)$	40
3_1	Trefoil knot	(2, 3)	20
5_1	Solomon's seal knot	(2, 5)	20
7_1	7 crossing torus knot	(2, 7)	20
8_{19}	8 crossing torus knot	(3, 4)	20
9_1	9 crossing torus knot	(2, 9)	20
10_{124}	10 crossing torus knot	(3, 5)	20
	11 crossing torus knot	(2, 11)	20

Appendix A

A.1 Connection to quasi-cocommutative Hopf algebras.

In Chapter 1, we defined the Hopf algebras $(H^{\varphi}(G), \mu_L^{\varphi}, \eta, \Delta, \epsilon, S_L^{\varphi})$, and$(H^{\varphi}(G), \mu_R^{\varphi}, \eta, \Delta, \epsilon, S_R^{\varphi})$. Moreover, we saw that these are neither commutative nor cocommutative Hopf algebras. So, our next task is to prove whether these Hopf algebras are quasi-cocommutative or quasi-commutative. The answer is given by the following lemma:

Lemma A.1.1. *The Hopf algebras* $(H^{\varphi}, \mu_L^{\varphi}, \eta, \Delta, \epsilon, S_L^{\varphi})$*,* $(H^{\varphi}, \mu_R^{\varphi}, \eta, \Delta, \epsilon, S_R^{\varphi})$ *are neither quasi-cocommutative nor quasicocommutative. Therefore they are not quantum groups.*

Before proving the previous lemma, we recall the definition and some properties about quasi-cocommutative and quasi-commutative Hopf algebras.

Definition A.1.2. *A bialgebra* $(H, \mu, \eta, \Delta, \epsilon, S)$ *is called quasi-cocommutative if there exists an invertible element* $R \in H \otimes H$ *such that:* $\forall\ x \in H: \quad \tau_{H,H} \circ \Delta(x) = R\Delta R^{-1}$*, where* $\tau_{H,H}$ *is the twist map on* H*.*

An element R with above property is called Universal $R-$ matrix.

Definition A.1.3. *A Hopf algebra* $(H, \mu, \eta, \Delta, \epsilon, S)$ *is called quasi-triangular or quantum group if is is a quasi-cocommutative and the* R *satisifies the following two properties*

1. $(\Delta \otimes id)(R) = R_{1,3}R_{23}$
2. $(id \otimes \Delta)(R) = R_{13}R_{12}$

Theorem A.1.4. *Let $(H, \mu, \eta, \Delta, \epsilon, S, S^{-1}, R)$ be a quasi-cocommutative Hopf algebra with an invertible antipode. Then there exists an element $u \in H$ such that:*

$$\forall \;\; x \in H : \quad S^2(x) = uxu^{-1}$$

Proof See [8].

∎

Definition A.1.5. *A bialgebra $(H, \mu, \eta, \Delta, \epsilon, S)$ is called quasi-commutative if there exists a linear form r in $H \otimes H$ such that*

1. *There is a linear form $\bar{r}$ in $H \otimes H$, such that $r \star \bar{r} = \bar{r} \star r = \epsilon$,*
2. *$\mu \otimes \tau_{H,H} = r \star \mu \star \bar{r}$*

where $\tau_{H,H}$ is the twist map in $H \otimes H$ and $\star$ is the convolution product. (See Definition 1.6.1.) An element r with these properties is called universal $R-$form.

Definition A.1.6. *A quasi-commutative Hopf algebra $(H, \mu, \eta, \Delta, \epsilon, S, r)$ is called quasi-triangular or quantum group if r satisfies the following property:*

$$r(\mu \otimes id_H) = r_{13} \star r_{23} \text{ and } r(id_H \otimes \mu) = r_{13} \star r_{12}$$

Theorem A.1.7. *Let $(H, \mu, \eta, \Delta, \epsilon, S, S^{-1}, r)$ be a quasi-commutative Hopf algebra with an invertible antipode. Then there is an invertible element $u \in H^*$ such that*

$$S^2 = u \star id_H \star \bar{u}.$$

Proof See [8]. ∎

Proof of lemma A.1.1 : Every invertible element u in $\mathcal{H}^{\varphi}(G)$ has to be of the form

$$u = 1_{\mathcal{H}^{\varphi}(G)} + \underbrace{\ldots}_{\text{degree}>0} .$$

And every invetible linear form in $\mathcal{H}^{\varphi}(G)^*$ has to send $1_{\mathcal{H}^{\varphi}}$ to $1_{\mathbb{Z}}$.

Assume that $(\mathcal{H}^\varphi G, \mu_L^\varphi, \eta, \Delta, \epsilon, S_L^\varphi)$ is a quasi-cocommutative Hopf algebra. Then, by Theorem A.1.4, there exists an invertible element $u \in \mathcal{H}^\varphi(G)$ such that

$$(S_L^\varphi)^2(g) = u(g_1, \ldots, g_k)u^{-1}(g_1, \ldots, g_k) + \underbrace{\ldots}_{\text{degree}>0}$$

$\forall \;\; g = (g_1, \ldots, g_k) \in \mathcal{H}^\varphi G$

The last equation does not hold in general. Indeed set $\varphi = id$, then we get Schardt's Hopf algebra $\mathcal{H}(G)$. And it has been proved in [11] that it is not a quasi-cocommutative Hopf algebra.

By a similar argument, we can porve that $\mathcal{H}^\varphi(G)$ is not a quasi-commutative Hopf algebra.

∎

Appendix B

B.1 Connection to braided Hopf algebras

In the previous section, we saw that the Hopf algebras $(H^{\varphi}G, \mu_L^{\varphi}, \Delta, \epsilon, \eta, S_L^{\varphi})$, $(H^{\varphi}G, \mu_R^{\varphi}, \Delta, \epsilon, \eta, S_R^{\varphi})$ are neither quasi-commutative nor quasi-cocommutative Hopf algebras with an invertible antipode $S_L^{\varphi}, S_R^{\varphi}$ respectively . Therefore by Whitehouse's work [15] we have solutions of the YBE Ψ, Ψ' respectively. In the same way, by Worocnocz's work [17] we have that there exist solutions of the Yang Baxter equation Φ, Φ'.
Hence, the next question to be asked is whether they are braided Hopf algebras. The answer is given by the following proposition:

Proposition B.1.1. *The Hopf algebras $(H^{\varphi}G, \mu_L^{\varphi}, \Delta, \epsilon, \eta, S_L^{\varphi}, S_L^{\varphi})$, $(H^{\varphi}G, \mu_R^{\varphi}, \Delta, \epsilon, \eta, S_R^{\varphi}), S_R^{\varphi})$ are not braided algebras with respect to Whitehouse's solutions of the YB equation Ψ, Ψ' respectively with the Woronocwicz solutions of the Yang-Baxter equation Φ, Φ'.*

To prove Lemma B.1.1 we first need to recall Whitehouse and Woronowicz's work. Moreover we need to recall the definition of braided Hopf algebra.

B.1.1 Whitehouse's solutions of the Yang-Baxter-equation

In this section, we briefly recall Whitehouse's work on the Yang-Baxter equation. See [15].

In [15], Whitehouse described two different actions of the braid group $Br_(n)$ on $\mathcal{H}^{\otimes n}$, where $\mathcal{H}$ is a Hopf algebra with multiplication μ, diagonal Δ and an invertible antipode S. Namely, she proved the following:

Theorem B.1.2. ([15], Theo 2.1] *Let $\mathcal{H}$ be as above. Then $\Psi : \mathcal{H}^{\otimes 2} \to \mathcal{H}^{\otimes 2}$ defined by*

$$\Psi = (\mu \otimes 1) \circ (\mu \otimes 1 \otimes 1) \circ (1 \otimes S \otimes 1 \otimes 1) \circ (243) \circ (\Delta \otimes 1 \otimes 1) \circ (\Delta \circ 1)$$

is a solution of the Yang-Baxter equation. Since S is invertible its inverse is given by:

$$\Psi^{-1} = (1 \otimes \mu) \circ (1 \otimes 1 \otimes \mu) \circ (1423) \circ (1 \otimes S^{-1} \otimes 1 \otimes 1) \circ (1 \otimes 1 \otimes \Delta) \circ (1 \otimes \Delta).$$

Proof: Under Sweedler's notations, we have that the map Ψ is given by

$$\Psi(x \otimes y) = \sum x^{(1)} S(x^3) y \otimes x^2.$$

Now, we would like to prove that the following equation

$$(\Psi \otimes 1)(1 \otimes \Psi)(\Psi \otimes 1) = (1 \otimes \Psi)(\Psi \otimes 1)(1 \otimes \Psi)$$

holds in $Aut(\mathcal{H}^{\otimes 3})$.

It is easy to compute that the left hand side of the formula is given by

$$\sum x^1 S(x^5) y^1 S(y^3) z \otimes x^2 S(x^4) y^2 \otimes x^3$$

To obtain the same formula for the right hand side of equation, use first coassociativity repeatedly, that the comultiplication is an algebra map, that S is an anti-algebra homomorphism; that S is an anti-coalgebra homomorphism (twice), the formula $\mu \circ (S \otimes 1) \circ \Delta = \eta\epsilon$ and unit (counit) properties.

∎

We recall that the dual $\mathcal{H}^* = Hom(\mathcal{H}, \mathbb{K})$ of a finite dimensional Hopf algebra is also a Hopf algebra, $\mathcal{H}^* = (\mathcal{H}^*, \Delta^*, \epsilon^*, \mu^*, \eta^*, S^*)$. The following Yang-Baxter solution is dual to that of theorem B.1.2.

Theorem B.1.3. ([15], Theo 2.2] *Let $(\mathcal{H}, \mu, \eta, \Delta, \epsilon, S)$ be a Hopf algebra over $\mathbb{K}$. Define $\Psi' : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ by*

$$\Psi' = (\mu \otimes 1) \circ (\mu \otimes 1 \otimes 1) \circ (234) \circ (1 \otimes S \otimes 1 \otimes 1) \circ (\Delta \otimes 1 \otimes 1) \circ (\Delta \otimes 1)$$

Then Ψ' is a solution of the Yang-Baxter equation. Moreover if the antipode S of $\mathcal{H}$ is invertible then Ψ' is invertible.

Lemma B.1.4. *1. If $\mathcal{H}$ is cocommutative then Ψ is the twist map.*

2. If $\mathcal{H}$ is commutative, then Ψ' is the twist map.

Proof: Use cossasociative, then associativity. The fact that $\mathcal{H}$ is a cocommutative (respectively commutative) Hopf algebra. Again, use coassociativity and associativity and the formula

$$\mu \circ (1 \otimes S) \circ \Delta = \eta \circ \epsilon.$$

$\blacksquare$

Proposition B.1.5. *Let H be a Hopf algebra and consider Whitehouse's solutions Ψ, Ψ' respectively of the Yang-Baxter equation. If D is an isomorphism of the Hopf algebra H, then $D \otimes D$ commutes with Ψ and Ψ', i.e.*

1. $(D \otimes D) \circ \Psi = \Psi \circ (D \otimes D)$,
2. $(D \otimes D) \circ \Psi' = \Psi' \circ (D \otimes D)$.

Proof of Proposition B.1.5: The proof follows by the commutativity of the following diagram:

$$\begin{tikzcd}
H \otimes H \arrow[r, "D\otimes D"] \arrow[d, "\Delta\otimes 1"'] & H \otimes H \arrow[d, "\Delta\otimes 1"] \\
H \otimes H \otimes H \arrow[r] \arrow[d, "\Delta\otimes 1\otimes 1"'] & H \otimes H \otimes H \arrow[d, "\Delta\otimes 1\otimes 1"] \\
H \otimes H \otimes H \otimes H \arrow[r] \arrow[d, "(234)"'] & H \otimes H \otimes H \otimes H \arrow[d, "(234)"] \\
H \otimes H \otimes H \otimes H \arrow[r] \arrow[d, "1\otimes S\otimes 1\otimes 1"'] & H \otimes H \otimes H \otimes H \arrow[d, "1\otimes S\otimes 1\otimes 1"] \\
H \otimes H \otimes H \otimes H \arrow[r] \arrow[d, "\mu\otimes 1\otimes 1"'] & H \otimes H \otimes H \otimes H \arrow[d, "\mu\otimes 1\otimes 1"] \\
H \otimes H \otimes H \arrow[r] \arrow[d, "\mu\otimes 1"'] & H \otimes H \otimes H \arrow[d, "\mu\otimes 1"] \\
H \otimes H \arrow[r, "D\otimes D"'] & H \otimes H
\end{tikzcd}$$

Note, that the composition of the maps on the left and right vertical arrows is Ψ'. A similar commutative diagram will prove the proposition for Ψ.

$\blacksquare$

B.1.2 Woronowicz's solutions of the Yang-Baxter equation

In this section, the two Woronowicz solutions Φ, Φ' of the Yang-Baxter equation defined in [17] are recalled.

Let H be a Hopf algebra with multiplication μ, comultiplication Δ, counit η, unit ϵ and an invertible antipode S. Here, we use the notation of Sweedler [13].

$\Delta(b) = \sum_{(b)} b^{(1)} \otimes b^{(2)}$,

$(1 \otimes \Delta)\Delta(b) = (\Delta \otimes 1)\Delta(b) = \sum_{(b)} b^{(1)} \otimes b^{(2)} \otimes b^{(3)}$.

Theorem B.1.6. [17] *Let Φ, Φ' be linear operators acting on $H \otimes H$ introduced by the formula*

$$\Phi(a \otimes b) = \sum_{(b)} b^{(2)} \otimes aS(b^{(1)})b^{(3)},$$

$$\Phi'(a \otimes b) = \sum_{(b)} b^{(1)} \otimes S(b^{(2)})ab^{(3)},$$

for any $a, b \in H$. But, S is invertible thus both maps are invertible with inverses given by:

$$\Phi^{-1}(a \otimes b) = \sum_{(b)} bS^{-1}(a^{(3)})a^{(1)} \otimes a^{(2)},$$

$$\Phi'^{-1}(a \otimes b) = \sum_{(b)} a^{(3)}bS^{-1}(a^{(2)}) \otimes a^{(1)},$$

for any $a, b \in H$. These operators satisfy the Yang-Baxter equation.

Remark B.1.7. *1.* $\Phi((a \otimes 1)\Delta(b)) = (1 \otimes a)\Delta(b)$

2. If H is either cocommutative or commutative , then Φ is the twist map.

Proof of Remark B.1.7. First of all, note that since H is cocommutative, we have

$$\sum_{(b)} b^{(1)} \otimes b^{(2)} = \sum_{(b)} b^{(2)} \otimes b^{(1)}$$

and

$$\Phi = (a \otimes b) = \sum_{(b)} b^{(1)} \otimes aS(b^{(2)})b^{(3)} = b \otimes a.$$

Similarly, if H is commutative, then

$$\Phi(a \otimes b) = \sum_{(b)} b^{(1)} \otimes aS(b^{(2)})b^{(3)} = b \otimes a.$$

∎

Proof of Theorem B.1.6: For any $b \in H$ set $ad(a) = \sum_{(b)} b^{(2)} \otimes S(b^{(1)})b^{(3)}$. Is easy to verify that

$$\Phi(a \otimes b) = (1 \otimes a)ad(b), \tag{B.1.1}$$

$$(ad \otimes 1)ad(b) = (1 \otimes \Delta)ad(b), \tag{B.1.2}$$

$$\Phi((a \otimes b)\Delta(c)) = (1 \otimes a)\ ad(b)\ \Delta(c). \tag{B.1.3}$$

Using equation B.1.2, we get

$$(\Phi \otimes 1)\ (1 \otimes \Phi)\ (q \otimes c) = (1 \otimes q)\, (1 \otimes \Delta)\ ad(c). \tag{B.1.4}$$

For any $a, b, c \in H$ and $q \in H \otimes H$. Let $a, b, c \in H.$ Using B.1.4 and B.1.3, we get

$$(1 \otimes \Phi)\ (\Phi \otimes 1)\ (1 \otimes \Phi)\, (a \otimes b \otimes c) = (1 \otimes 1 \otimes a)\ (1 \otimes ad(b))\ (1 \otimes \Delta)\ ad(c).$$

On the other hand, using B.1.2 and B.1.4, we obtain

$$(\Phi \otimes 1)\ (1 \otimes \Phi)\ (\Phi \otimes 1)\ (a \otimes b \otimes c) = (1 \otimes 1 \otimes a)\ (1 \otimes ad(b))\ (1 \otimes \Delta)\ ad(c).$$

From these equations follows that Φ is a solution of the Yang-Baxter equation.

The second proof follows by duality.

∎

In analogy to Proposition B.1.5, we get the following Proposition, when we consider Worocnicz's solutions of the Yang-Baxter equation.

Proposition B.1.8. *Let Φ, Φ' denote the Woronocwiz solutions of the Yang-Baxter equation. Let D and H be given as in Proposition B.1.5. Then, $D \otimes D$ commutes with Φ and Φ', i.e.*

1. $(D \otimes D) \circ \Phi = \Phi \circ (D \otimes D)$,
2. $(D \otimes D) \circ \Phi' = \Phi' \circ (D \otimes D)$

Proof: We will not prove this Proposition, since it follows by a similar commutative diagram used in the proof of Proposition B.1.5.

B.1.3 Braided Hopf algebras

Let $\mathcal{C}$ be a monoidal category, for instance, the category of vector spaces over a field $\mathbb{K}$. We write $\otimes$ and I for the tensor product and the unit of $\mathcal{C}$, respectively. Let V, W be objects in $\mathcal{C}$ and let $c : V \otimes W \to W \otimes V$ be a morphism in $\mathcal{C}$. The following definition was taken from [4].

Definition B.1.9. *A braided bialgebra in $\mathcal{C}$ is an object H of $\mathcal{C}$ endowed with an algebra structure, a coalgebra structure and a solution of the Yang Baxter equation c_H such that:*

1. *c_H is compatible with the algebra and coalgebra structures of H; i.e*

   (a) $c_H \circ (\eta \otimes 1) = 1 \otimes \eta$ *and* $c_H \circ (\mu \otimes 1) = (1 \otimes \mu) \circ (c_H \otimes 1) \circ (1 \otimes c)$.

   (b) $(1 \otimes \epsilon) \circ c_H = \epsilon \otimes 1$ *and* $(1 \otimes \Delta) \circ c_H = (c_H \otimes 1) \circ (1 \otimes c_H) \circ (\Delta \otimes 1)$.

2. *η is a coalgebra morphism and ϵ is an algebra morphism and*

3. $\Delta \circ \mu = (\mu \otimes \mu) \circ (1 \otimes c_H \otimes 1) \circ (\Delta \otimes \Delta)$.

Moreover, if the antipode S of H is invertible we say that H is a braided Hopf algebra. To read more about braided Hopf algebras and its connection to knot invariants see [12].

Before giving the proof of Proposition B.1.1, we need the following Lemma.

Let $V(r, n) = \bigoplus_{n_1+\cdots+n_r=r} \mathcal{H}_{n_1} \otimes \cdots \otimes \mathcal{H}_{n_r}$ be the finite dimensional subspaces of $(\mathcal{H}^{\varphi}(G))^{\otimes n}$, where $\mathcal{H}_m = \mathbb{K}[G]^{\otimes m}$.

Lemma B.1.10. *Let Ψ, Ψ' be Whitehouse's solutions of the Yang-Baxter equation. The finite dimensional subspaces $V(r, n)$ of $(\mathcal{H}^{\varphi}(G))^{\otimes n}$ are invariant under Ψ, Ψ'.*

Proof: We do the proof for Ψ, because by a similar argument the proof will hold for Ψ'. Consider $a, b \in \mathcal{H}^{\varphi}(G)$, with $a = (a_1 \otimes \cdots \otimes a_m)$ and $b = (b_1 \otimes \cdots \otimes b_n)$. Let $m' \in \{1, \ldots m\}, n' \in \{1, \ldots, n\}$. Let $S_{(a)_n} := S_L^{\varphi}(a)$ denote the left antipode map. Fix σ_1 to be the (m', n') shuffle, and σ_2 to be the $(n - n', n' + m')$-shuffle. Then,

$$\begin{aligned}
(\Psi(a \otimes b))_{m',n',\sigma_1,\sigma_2} &:= (\mu \otimes 1) \circ (\mu \otimes 1 \otimes 1) \circ (1 \otimes S \otimes 1 \otimes 1) \circ (243) \circ (\Delta \otimes 1 \otimes 1) \\
&\circ \quad (\Delta \circ 1))_{m',n',\sigma_2,\sigma_1}(a \otimes b) \\
&:= (S^{\varphi}_{L,\sigma_2} \otimes 1) \circ (S^{\varphi}_{L,\sigma_1} \otimes 1 \otimes 1) \circ (1 \otimes S \otimes 1 \otimes 1)(\Delta'_m \otimes \Delta'_n)(a \otimes b)
\end{aligned}$$

$$= \quad (S^{\varphi}_{L,\sigma_2} \otimes 1) \circ (S^{\varphi}_{L,\sigma_1} \otimes 1 \otimes 1)((a_1, \ldots, a'_m) \otimes (b_1, \ldots, b'_n)$$

$$\otimes \quad (b_{n'+1}, \ldots, b_n) \otimes (a_{m'+1}, \ldots, a_m))$$

$$= \quad (S^{\varphi}_{L,\sigma_2} \otimes 1)(S^{\varphi}_{L,\sigma_1} \otimes 1 \otimes 1)((a_1, \ldots, a_{m'}) \otimes S_{(b)_{n'}}$$

$$\otimes \quad (b_{n'+1}, \ldots, b_n) \otimes (a_{m'+1}, \ldots, a_m))$$

$$= \quad (S^{\varphi}_{L,\sigma_2} \otimes 1)(S^{\varphi}_{L}(a, S_{(b)_{n'}}, \sigma_1)_{m',n'} \otimes (b_{n'+1}, \ldots, b_n) \otimes (a_{m'+1}, \ldots, a_m))$$

$$= \quad S^{\varphi}_{L}(S^{\varphi}_{L,\sigma_1}(aS_{(b)_{n'}}), b, \sigma_2)_{n-n',n'+m'} \otimes (a_{m'+1}, \ldots, a_m)$$

We observe that $(a_{m'+1}, \ldots, a_m) \in \mathcal{H}_{m-m'}$ and we observe that $S_{(b)_{n'}} \in \mathcal{H}_{n'}$, because for each $g \in G$ $\varphi(g) \in G$. Now, it is not difficult to see that $S^{\varphi}_{L}(S^{\varphi}_{L,\sigma_1}(a, S_{(b)_{n'}}), b, \sigma_2)_{n-n',n'+m'} \in \mathcal{H}_{n+m'}$. Thus, $(\Psi(a \otimes b))_{m',n',\sigma_1,\sigma_2} \subseteq V(r,n)$

∎

Lemma B.1.11. *Let Ψ, Ψ' be Whitehouse's solutions of the Yang Baxter equation. Let G be any commutative group. Asumme that, $\varphi = id$, and let a_1, a_2 be generators of $\mathcal{H}_1$. Then:*

1. $\Psi(a_1 \bar{\otimes} a_2) = a_2 \bar{\otimes} a_1$

   $\Psi'(a_1 \bar{\otimes} a_2) = a_2 \bar{\otimes} a_1 - 2(a_2 \otimes a_1)\bar{\otimes} 1 + 2(a_1 \otimes a_2)\bar{\otimes} 1$

2. $\Psi(a_1 \bar{\otimes} 1) = 1 \bar{\otimes} a_1$

   $\Psi'(a_1 \bar{\otimes} 1) = 1 \bar{\otimes} a_1 - 2(1 \otimes a_1)\bar{\otimes} 1 + 2(a_1 \otimes 1)\bar{\otimes} 1$

3. $\Psi(1 \bar{\otimes} 1) = \Psi'(1 \bar{\otimes} 1) = 1 \bar{\otimes} 1$

4. $\Psi'(1 \bar{\otimes} a_2) = a_2 \bar{\otimes} 1 - 2(a_2 \otimes 1)\bar{\otimes} 1 + 2(1 \otimes a_2)\bar{\otimes} 1$

5. *Let $a_1, (a_2 \otimes a_3)$ be generators of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then,*

   $\Psi(a_1 \bar{\otimes}(a_2 \otimes a_3)) = (a_2 \otimes a_3)\bar{\otimes} a_1$

   $\Psi'(a_1 \bar{\otimes}(a_2 \otimes a_3)) = (a_2 \otimes a_3)\bar{\otimes} a_1 - (a_2 \otimes a_3 \otimes a_1)\bar{\otimes} 1 + (a_1 \otimes a_2 \otimes a_3)\bar{\otimes} 1$

Proof: Follows by the definition of $\Psi, \Delta, \mu^{\varphi}_{L}$ and S^{φ}_{L}.

∎

In analogy to Lemmas B.1.10 and B.1.11, we get the following Lemmas.

Lemma B.1.12. *Woronowicz solutions of the Yang-Baxter equation Φ, Φ', leave invariant the finite dimensional subspaces $V(r,n)$.*

Lemma B.1.13. *Let Φ, Φ', denote Worocnowicz's solutions of the Yang-Baxter equation. Assume that G is commutative. Moreover, assume that φ is the identity automorphism. Let a_1, a_2 be generators of $\mathcal{H}_1$, then:*

1. $\Phi(a_1 \bar{\otimes} a_2) = a_2 \bar{\otimes} a_1$

   $\Phi'(a_1 \bar{\otimes} a_2) = a_2 \bar{\otimes} a_1 + 2(1 \bar{\otimes} (a_1 \otimes a_2)) - 2(1 \bar{\otimes} (a_2 \otimes a_1)$

2. $\Phi(1 \bar{\otimes} 1) = 1 \bar{\otimes} 1 = \Phi'(1 \bar{\otimes} 1)$

3. $\Phi(a_1 \bar{\otimes} 1) = -1 \bar{\otimes} a_1 + a_1 \bar{\otimes} 1$

   $\Phi'(a_1 \bar{\otimes} 1) = 1 \bar{\otimes} a_1$

4. *Let $a_1, (a_2 \otimes a_3)$ be generators of $\mathcal{H}_1$ and $\mathcal{H}_2$, repectively. Then*

$$\begin{aligned}
\Phi(a_1 \bar{\otimes} (a_2 \otimes a_3)) \;&=\; a_2 \bar{\otimes} (a_1 \otimes a_3) - a_2 \bar{\otimes} (a_3 \otimes a_1) + (a_2 \otimes a_3) \bar{\otimes} a_1 - a_3 \bar{\otimes} (a_1 \otimes a_2) \\
&+\; a_3 \bar{\otimes} (a_2 \otimes a_1) + 2(1 \bar{\otimes} (a_2 \otimes a_1 \otimes a_3)) - 2(1 \bar{\otimes} (a_3 \otimes a_1 \otimes a_2)) \\
&+\; 1 \bar{\otimes} (a_1 \otimes a_3 \otimes a_2) - 1 \bar{\otimes} (a_1 \otimes a_2 \otimes a_3)
\end{aligned}$$

$$\begin{aligned}
\Phi'(a_1 \bar{\otimes} (a_2 \otimes a_3)) \;&=\; (a_2 \otimes a_3) \bar{\otimes} a_1 - 2(a_2 \bar{\otimes} (a_3 \otimes a_1)) + 2(a_2 \bar{\otimes} (a_1 \otimes a_3)) \\
&-\; 1 \bar{\otimes} (a_3 \otimes a_2 \otimes a_1) + 1 \bar{\otimes} (a_3 \otimes a_1 \otimes a_2) - 2(1 \bar{\otimes} (a_2 \otimes a_1 \otimes a_3)) \\
&+\; 2(1 \bar{\otimes} (a_1 \otimes a_2 \otimes a_3))
\end{aligned}$$

Lemma B.1.14. *Let G be a group and $V = \mathbb{K}[G]$. Define the coproduct structure by $\Delta(g) = 1 \otimes g + g \otimes 1$ and the coproduct structure μ given by the product on G and antipode map S given by $S(g) = g^{-1}$, for all $g \in G$.*

1. *Consider Whithouse's solution of the Yang-Baxter equation Ψ. Then*

$$(\mu \otimes \mu) \circ (1 \otimes \Psi \otimes 1) \circ (\Delta \otimes \Delta) = \Delta \circ \mu + id + \Psi.$$

2. *If we consider the solution B^{φ} of the Yang-Baxter equation, then it is compatible with the algebra and coalgebra structure of V, but*

$$\Delta \circ \mu \neq (\mu \otimes \mu) \circ (1 \otimes B^{\varphi} \otimes 1) \circ (\Delta \otimes \Delta).$$

Proof: Let v, w generators of V, then by the definition of the coproduct ,product and the antipode map, we have:

$$\begin{aligned}
(\Delta \otimes \Delta)(v \otimes w) &= 1 \otimes v \otimes 1 \otimes w + 1 \otimes v \otimes w \otimes 1 + v \otimes 1 \otimes 1 \otimes w + v \otimes 1 \otimes w \otimes 1 \\
\Psi(v \otimes w) &= v^{-1}w \otimes 1 + w \otimes v + vw \otimes 1 \\
(\Delta \circ \mu)(v \otimes w) &= 1 \otimes vw + vw \otimes 1 \\
\Psi(1 \otimes w) &= w \otimes 1 \\
\Psi(v \otimes 1) &= 1 \otimes v
\end{aligned}$$

Thus,

$$\begin{aligned}
(\mu \otimes \mu) \circ (1 \otimes \Psi \otimes 1) \circ (\Delta \otimes \Delta)(v \otimes w) &= (\mu \otimes \mu) \circ (1 \otimes \Psi \otimes 1)(1 \otimes v \otimes 1 \otimes w + 1 \otimes v \otimes w \otimes 1 \\
&+ v \otimes 1 \otimes 1 \otimes w + v \otimes 1 \otimes w \otimes 1) \\
&= (\mu \otimes \mu)(1 \otimes \Psi(v \otimes 1) \otimes w + 1 \otimes \Psi(v \otimes w) \otimes 1 \\
&+ v \otimes \Psi(1 \otimes 1) \otimes w + v \otimes \Psi(1 \otimes w) \otimes 1) \\
&= 1 \otimes vw + vw \otimes 1 + v \otimes w + vw \otimes 1 + v^{-1}w \otimes 1 \\
&+ w \otimes v \\
&= (\Delta \circ \mu)(v \otimes w) + id(v \otimes w) + \Psi(v \otimes w)
\end{aligned}$$

∎

Using Lemma B.1.11 and B.1.13, respectively, we get the following remarks.

Remark B.1.15. 1. *The above lemma implies that $\mathbb{K}[G]$ is not a braided Hopf algebra, neither with respect to Whitehouse's solution of the Yang-Baxter equation Ψ nor with respect to the solution of the Yang-Baxter solution B^{φ}.*

2. *Consider Whitehouse's solutions of the Yang-Baxter equation Ψ, Ψ' respectively. For any group G.*

(a) Ψ, Ψ' *are not compatible with the algebra and coalgebra structures of* $\mathcal{H}^{\varphi}(G)$. *Moreover, we have*

(b) $(\mu \otimes \mu) \circ (1 \otimes \Psi \otimes 1) \circ (\Delta \otimes \Delta) \neq \Delta \circ \mu$

(c) $(\mu \otimes \mu) \circ (1 \otimes \Psi' \otimes 1) \circ (\Delta \otimes \Delta) \neq \Delta \circ \mu$

3. Consider Woronowicz's solutions of the Yang-Baxter equation Φ, Φ' *respectively. For any group* G.

(a') In general, is not true that Φ, Φ' *are compatible with the algebra and coalgebra structures of* $\mathcal{H}^{\varphi}(G)$. *Moreover, we have*

(b') $(\mu \otimes \mu) \circ (1 \otimes \Phi \otimes 1) \circ (\Delta \otimes \Delta) \neq \Delta \circ \mu$

(c') $(\mu \otimes \mu) \circ (1 \otimes \Phi' \otimes 1) \circ (\Delta \otimes \Delta) \neq \Delta \circ \mu$

Proof of Proposition B.1.1. Follows from Remark B.1.15.

∎

Appendix C

C.1 Tensor product of matrices

In this appendix we recalled the tensor product of matrices. In analysis or linear algebra it is named Kronecker product after Leopold Kronecker, even though there is a little evidence that he was the first to define and use it. Indeed, in the past the tensor product of matrcies was sometimes called the Zehfuss matrix, after Johann Georg Zehfuss. All the material of this Appendix has been taken from the book of Horn, (see [5]).

Definition C.1.1. *If A is an $n \times n$ matrix and B is a $p \times q$ matrix, then the tensor product $A \otimes B$ is the $mp \otimes nq$ block matrix.*

$$A \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{pmatrix}$$

More explicity, we have

$$A \otimes B = \begin{pmatrix}
a_{11}b_{11} & a_{11}b_{12} & \dots & a_{11}b_{1q} & \dots & \dots & a_{1n}b_{1n} & a_{1n}b_{12} & \dots & a_{1n}b_{1q} \\
a_{n1}b_{21} & a_{11}b_{12} & \dots & a_{11}b_{1q} & \dots & \dots & a_{1n}b_{21} & a_{1n}b_{22} & \dots & a_{1n}b_{2q} \\
\vdots & \vdots & & \vdots & \ddots & & & \vdots & \dots & \vdots \\
\vdots & \vdots & & \vdots & & \ddots & & \vdots & \dots & \vdots \\
a_{11}b_{p1} & a_{11}b_{p2} & \dots & a_{11}b_{pq} & \dots & \dots & a_{1n}b_{p1} & a_{1n}b_{p2} & \dots & a_{1n}b_{pq} \\
\vdots & \vdots & \ddots & & & & \vdots & \vdots & \ddots & \vdots \\
a_{m1}b_{11} & a_{m1}b_{12} & \dots & a_{m1}b_{1q} & \dots & \dots & a_{mn}b_{11} & a_{mn}b_{12} & \dots & a_{mn}b_{pq} \\
a_{m1}b_{21} & a_{m1}b_{12} & \dots & a_{m1}b_{1q} & \dots & \dots & a_{mn}b_{11} & a_{mn}b_{12} & \dots & a_{mn}b_{pq} \\
\vdots & \vdots & \ddots & & & & \vdots & \vdots & \ddots & \vdots \\
a_{m1}b_{p1} & a_{m1}b_{p2} & \dots & a_{m1}b_{pq} & \dots & \dots & a_{mn}b_{p1} & a_{mn}b_{p2} & \dots & a_{mn}b_{pq}
\end{pmatrix}$$

Remark C.1.2. *The tensor product of matrices, corresponds to the tensor product of linear maps. Specifically, if the vector spaces,* V, W, X *and* Y *have bases* $\{v_1, \ldots, v_m\}$, $\{w_1, \ldots, w_n\}$, $\{x_1, \ldots, x_d\}$ *and* $\{y_1, \ldots, y_l\}$, *respectively, and if the matrices* A *and* B *represent the linear transfromations* $S : V \to X$ *and* $T : W \to Y$, *respectively in the corresponding bases, then the matrix* $A \otimes B$ *represents the tensor product of the two maps* $S \otimes T : V \otimes W \to X \otimes Y$ *with respect to the basis* $\{v_1 \otimes w_1, v_1 \otimes w_2, \ldots, v_2 \otimes w_1, \ldots, v_m \otimes w_n\}$ *of* $V \otimes W$ *and the similarly basis of* $X \otimes Y$.

C.2 Properties

In the following is assumed that A, B, C and D take values in a field $\mathbb{K}$, and that $\alpha \in \mathbb{K}$. Some identities only hold for appropriately dimensional matrices.

Lemma C.2.1. 1. *The tensor product of matrices is bilinear:*

$$\begin{aligned} A \otimes (\alpha B) &= \alpha(A \otimes B) \\ (\alpha A) \otimes B &= \alpha(A \otimes B). \end{aligned}$$

2. *It distributes over addition:*

$$\begin{aligned} (A + B) \otimes C &= (A \otimes C) + (B \otimes C) \\ A \otimes (B + C) &= (A \otimes B) + (A \otimes C) \end{aligned}$$

3. *It is associtive, and in general it is not commutative:*

$$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$

4.
$$(A \otimes B)(C \otimes D) = AC \otimes BD,$$

this property is called the mixed-product property, because it mixes the ordinary matrix product and the tensor product of matrices. It follows that $A \otimes B$ *is invertible if and only if* A *and* B *are invertible, in which case the inverse is given by*

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

5.
$$det(A_{n\times n} \otimes B_{m\times m}) = det(A)^m \;.\; det(B)^n$$

6.
$$trace(A \otimes B) = trace(A)\ .\ trace(B).$$

Proof We only prove part (6) of the Lemma. Because the other proofs are similar.
It follows from Remark C.1.2, that the tensor product of matrices corresponds to the tensor product of linear maps. Therefore, it is enough to prove that, if U, V are finite dimensional vector spaces, and if f (respectively g) is a an endomorphism of U (respectively V.) Then

$$\mathrm{trace}(f \otimes g) = \mathrm{trace}(f)\ \mathrm{trace}(g).$$

Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_m$ be basis of U resp. V. Then:

$$f(u_i) = \sum_{j=1}^{n} f_j^i u_j \text{ and}$$

$$g(v_i) = \sum_{j=1}^{m} g_j^i v_j$$

for the map $f \otimes g$ then we have:

$$(f \otimes g)(u_{i_1} \otimes v_{i_2}) = \sum_{j_1}^{n} \sum_{j_2=1}^{m} f_{j_1}^{i_1} g_{j_2}^{i_2} u_{j_1} \otimes v_{j_2}$$

From it we get:

$$\begin{aligned} \mathrm{trace}(f \otimes g) &= \textstyle\sum_{i_1}^{n} \sum_{i_2}^{m} f_{i_1}^{i_1} g_{i_2}^{i_2} \\ &= \textstyle(\sum_{i_1=1}^{n} f_{i_1}^{i_1})(\sum_{i_2}^{i_2} g_{i_2}^{i_2}) \\ &= \mathrm{trace}(f)\ \mathrm{trace}(g) \end{aligned}$$

∎

Appendix D

D.1 The computer program

In this Appendix, we explain how to use the program "Bhi_orders" which has been written in Java programming language.

This program calculates the orders of the twisted conjugation braiding B^{φ} introduced in Chapter 1 of this thesis. (see 1.2.3). It also computes the trace of the following composition of maps $b_{B^{\varphi}} \circ D^{\otimes p}$ for the case when we consider the enhancement $D = \gamma\, I$ ($\gamma \in \mathbb{C}^*$) of the twisted conjugation braiding B^{φ}, and when we consider braids $\xi \in Br(p)$, with $\xi = (\sigma_1 \sigma_2 \dots \sigma_{p-1})^q$, and with $(p, q) = 1$.

1. Compute the order of the twisted conjugation braiding B^{φ} for the symmetric group Σ_n (n=3, ..., 7) and for the cyclic group $\mathbb{Z}/n\mathbb{Z}$.

   Input:

   java Bphi_orders <arg1> <arg2>

   <arg1> of type String declares which group will be considered.

   "sym" for symmetric group

   "cyc" for cyclic group

   <arg2> of type int defines the level of the chosen group ($G = \Sigma_n$ or $G = \mathbb{Z}/n\mathbb{Z}$)

2. Compute the trace of the link invariant $T_{\mathcal{B}}$

   Input:

java Bphi_orders <arg0> <arg1> <arg2> <arg3> <arg4>

<arg0> of type String, declares the trace of the group which will be considered

"trsym" for computing the trace of the composition of the torus knot $(\sigma_1\sigma_2\ldots\sigma_{p-1})^q$.

<arg1> of type int defines the level of the chosen group $G = \Sigma_n$

<arg2> of type int defines the inner automorphism

<arg3> of type int defines the value of p

<arg4> of type int defines the value of q

Remark In case of the computation of the trace the user should give integers p and q sucht that $(p, q) = 1$ as an input.

Output:

- Bphi_orders calculates the order of the twisted conjugation braiding B^φ, where G is either the symmetric group Σ_n, with $n = 3, 4, 5, 7$ or G is the cyclic group $\mathbb{Z}/n\mathbb{Z}$.

- If $G = \Sigma_6$, then it computes the orders of the twisted conjugation braiding B^φ for the inner automorphims, i.e., $\varphi \in Inn(G)$.

- If $G = \Sigma_n$ (n=3, 4, 5, 6, 7), (or $G = \mathbb{Z}/n\mathbb{Z}$) then it calculates the trace of the map $b_{B^\varphi}(\xi) \circ D^{\otimes p}$, for the case that we consider the enhancement $D = \gamma I$ (γ invertible) of the twisted congation braiding B^φ. For braids $\xi \in Br(p)$, with $\xi = (\sigma_1 \ldots \sigma_{p-1})^q$ with p and q integers sucht that $(p, q) = 1$.

Using Bphi_orders:

To run the program, the folder Bphi_orders should contain the following three classes:

1. Bphi_orders.class
2. CyclicGroup.class
3. SymmetricGroup.class

Set the path of the shell command line to the directory "../Bphi_orders". For the symmetric group use the command line:

java Bphi_orders sym 4

For the cyclic group use the command line:

java Bphi_orders cyc 11

For computing the trace use the command line:

java Bphi_orders trsym 5 2 3 4

or the command line: java Bphi_orders trcyc 5 2 3 7

Compiling:

Set the path of the shell to the folder where the source code "Bphi_orders.java" is located (here "../Bphi_orders").

Compile with the command

javac Bphi_orders.java

The compiler generates the classes into the same folder of the source code file "Bphi_orders.java".

After compiling, the folder will contain the following files:

1. Bphi_orders.java
2. Bphi_orders.class
3. CyclicGroup.class
4. SymmetricGroup.class

Now the folder contains the executable classes.

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