On the notion of order in the stable module category

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On the notion of order in the stable module category*

Martin Langer

Abstract

The notion of order in triangulated categories, as introduced by Schwede, is investigated in the case of the stable category of kG-modules, where k is a field of characteristic p and G is a finite group. For Tate cohomology classes ζ of even degree, we obtain bounds on the ζ-order which are amazingly similar to corresponding results on the p-order in the stable homotopy category.

On our way we introduce a power operation $\mathcal{P}_1$ on Tate cohomology which serves as an obstruction for the ζ-order to be larger than its minimal possible value. Furthermore, it enables us to compute certain higher Massey products explicitly.

Contents

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Introduction

Let G be a finite group and k be a field of prime characteristic $p > 0$, and let us denote by $\mathfrak{mod}\text{-}kG$ the category of finitely generated, right kG-modules. In this category, the classes of injective and projective modules coincide, and this allows us to form its stable category $\underline{\mathfrak{mod}}\text{-}kG$, whose objects are the same as in $\mathfrak{mod}\text{-}kG$, and the morphisms are given by morphisms in $\mathfrak{mod}\text{-}kG$ modulo the subgroup of those morphisms factoring through a projective module. On $\underline{\mathfrak{mod}}\text{-}kG$ we have the translation functor Σ which can be defined as follows: Choose an inclusion $i : k \hookrightarrow P$ of the trivial kG-module k into a projective module P. For every object X define ΣX to be the cokernel of the (injective) map $i \otimes \mathrm{id}_X : X \to P \otimes X$. This functor serves as a translation functor of a triangulated structure on $\underline{\mathfrak{mod}}\text{-}kG$ (where the exact triangles are 'up to projectives' those coming from short exact sequences in $\mathfrak{mod}\text{-}kG$).

Let Ω denote the inverse of the shift functor Σ. Then we have

$$\underline{\mathrm{Hom}}_{kG}(\Omega^n X, Y) = \underline{\mathfrak{mod}}\text{-}kG(\Omega^n X, Y) \cong \widehat{\mathrm{Ext}}^n_{kG}(X, Y),$$

where $\widehat{\mathrm{Ext}}$ denotes Tate Ext-groups. Suppose we are given a non-zero Tate cohomology class $[\zeta] \in \hat{H}^n(G) = \widehat{\mathrm{Ext}}^n(k, k)$ represented by an unstable (surjective) map $\zeta : \Omega^n k \to k$. Denote by L_ζ the kernel of the map ζ; then we get an exact triangle

$$\cdots \to L_\zeta \to \Omega^n k \to k \to \Omega^{-1} L_\zeta =: k/\zeta \to \ldots$$

On k/ζ, we still have a multiplication by ζ, and the following theorem will be the starting point of our discussion.

Theorem (Carlson, [2]). *If p is odd and n is even, then multiplication by ζ on k/ζ vanishes; i.e.,*

$$\Omega^n k \otimes k/\zeta \xrightarrow{\zeta \otimes \mathrm{id}} k/\zeta$$

is stably zero.

If $p = 2$ then this does not need to be true. For instance, one can take $G = \mathbb{Z}/2 \times \mathbb{Z}/2$, then there is some non-zero $[\zeta] \in \hat{H}^{-2}(G)$ such that multiplication by ζ does not vanish on k/ζ. But why is the prime 2 special here?

In topology, we have a similar phenomenon. Suppose we are given a triangulated category $\mathcal{C}$, some object $X \in \mathcal{C}$ and a natural number m. On X, we have the 'multiplication by m', i.e., $m \cdot \mathrm{Id}_X : X \to X$; denote by X/m some choice of cone of this map. On this cone, we also have a multiplication by m. If $\mathcal{C}$ is the stable homotopy category, and p is a prime, then the mod-p Moore spectrum S/p (where S denotes the sphere spectrum) has a multiplication by p, which is zero if p is odd, but non-zero if $p = 2$.

Motivated by his proof of the Rigidity Theorem [24], Schwede introduced the notion of m-order (see [25]), which measures 'how strongly zero multiplication by m on some object is'. The m-order $m\text{-ord}(X)$ of an object $X \in \mathcal{C}$ is an element of $\{0, 1, 2, \ldots, \infty\}$, defined inductively by the following condition: $m\text{-ord}(X) \geq k$ if and only if for all objects K in $\mathcal{C}$ and all morphisms $f : K \to X$ there is an *extension* $\hat{f} : K/m \to X$ such that for some (and hence any) cone $C_{\hat{f}}$ of $\hat{f}$, m-ord$(C_{\hat{f}}) \geq k-1$. Here, extension means that the following diagram commutes:

$$\begin{tikzcd} \cdots \arrow[r] & K \arrow[r, "m\cdot"] & K \arrow[r] \arrow[d, "f"'] & K/m \arrow[r] \arrow[dl, dashed, "\hat{f}"] & \cdots \\ & & Y & & \end{tikzcd}$$

For instance, m-ord$(X) \geq 1$ if and only if multiplication by m vanishes on X, which is what you would expect from a reasonable definition of order. It can be compared with, say, multiplicities of zeroes of real polynomials.

In topology, we have the following results:

Theorem (Schwede, [24] and [25]). *Let p be a prime number and $\mathcal{C}$ be a topological triangulated category, i.e., the full subcategory of the homotopy category of a stable Quillen model category.*

(t1) For any object X of $\mathcal{C}$, the object X/p has p-order at least $p-2$.

(t2) In the stable homotopy category $\mathcal{SHC}$, the mod-p Moore spectrum S/p has p-order exactly $p-2$.

(t3) We can also go one step further. If the morphism $\alpha_1 \wedge X : \Sigma^{2p-3}X \to X$ is divisible by p, then X/p has p-order at least $p-1$. Here $\alpha_1 : \Sigma^{2p-3}S \to S$ is a generator of the p-torsion in the stable homotopy groups of spheres in the $(2p-3)$-stem.

To a certain extent, statement (t2) explains the phenomenon described above: multiplication by p on S/p vanishes if and only if the p-order of S/p is at least 1, and this is the case exactly if $p \geq 3$.

Let us turn back to the case $\mathcal{C} = \underline{\mathfrak{mod}}\text{-}kG$ and see what we get from the notion of order. It is certainly not very interesting to consider multiplication by m in our algebraic situation, so we extend the definition of order to elements in the graded center of $\mathcal{C}$. In degree n, the graded center of the triangulated category $\mathcal{C}$ consists of all natural transformations ζ from the identity functor to the functor Σ^n which commute with the functor Σ up to the sign $(-1)^n$. The notion of order as defined above can be modified to work for arbitrary elements ζ in the graded center, so we obtain a number ζ-ord(X) for every object X in $\mathcal{C}$. Now suppose we are given a cohomology class in $\hat{H}^n(G)$, represented by some unstable map $\zeta : \Omega^n k \to k$, which in turn induces an element in the graded center, also denoted by ζ:

$$\begin{tikzcd} \Omega^n X \arrow[r, "\zeta"] \arrow[d, "\wr"'] & X \arrow[d, "\wr"'] \\ \Omega^n k \otimes X \arrow[r, "\zeta\otimes\mathrm{id}"] & k \otimes X \end{tikzcd}$$

This should be thought of as multiplication by the class $[\zeta]$. With all this language at hand, Carlson's theorem above reads ζ-ord$(k/\zeta) \geq 1$ for all primes $p \geq 3$. This will be generalized by the following theorem whose proof is the main objective of this thesis.

Main Theorem. *Suppose that k is a field of prime characteristic p and G is a finite group.*

(a1) Let $\zeta \in \hat{H}^n(G)$ be a Tate cohomology class of even degree n. For any object X in $\underline{\mathfrak{mod}}$-$kG$, the ζ-order of X/ζ is at least $p-2$.

(a2) For every prime p and every field k of characteristic p, there exists a group G and a cohomology class $\zeta \in \hat{H}^*(G)$ of even degree such that the ζ-order of k/ζ is exactly $p-2$.

(a3) Suppose that $n > 0$. Recall that the Steenrod reduced powers act on group cohomology $H^*(G,k)$. The first non-trivial Steenrod operation gives an obstruction for the ζ-order of X/ζ to be at least $p-1$. More precisely, if $P_1\zeta \otimes X$ is divisible by ζ then ζ-ord$(X/\zeta) \geq p-1$.

Here, $\mathrm{P}_1\,\zeta$ is the first non-trivial Steenrod power operation, that is,

$$\mathrm{P}_1\,\zeta = \begin{cases} \mathrm{Sq}_1\,\zeta = \mathrm{Sq}^{n-1}\,\zeta & \text{if } p = 2, \\ \beta P^{\frac{n}{2}-1}\zeta & \text{if } p \text{ is odd.} \end{cases}$$

The plan of this thesis is as follows. As a first step, we recall several known facts about the objects we are going to work with. In the second section we prove the lower bound (a1) of our Main Theorem. In §3 we introduce a new power operation $\mathcal{P}_1$ on Tate cohomology which extends the Steenrod operation P_1 above to negative degrees, at the price of introducing a certain indeterminacy. Basic properties of the new operation are shown, and the operation is computed for elementary abelian p-groups. In the fourth section we show the obstruction statement (a3) of the Main Theorem. In §5 we show that certain higher Massey products give upper bounds on the order, and we will use the new power operation to compute such Massey products and thereby find an example for (a2). The last section is devoted to the question what happens with the statement of (a1) if we allow ζ to be any element of even degree in the graded center of $\underline{\mathfrak{mod}}$-$kG$.

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1 Prerequisites

1.1 The stable module category

We begin with a brief introduction to the stable module category; a more detailed exposition can be found in [3]. Throughout this thesis, we will work with a fixed finite group G and a field k of characteristic $p > 0$. Consider the category $\mathfrak{mod}\text{-}kG$ of finitely generated right modules over the group algebra kG. On this category, we have a symmetric monoidal tensor product $X \otimes_k Y$ defined for any two objects X, Y as follows. As a k-vector space, we take the usual tensor product $X \otimes_k Y$; the G-module structure is then given by the rule $(x \otimes y) \cdot g = (x \cdot g) \otimes (y \cdot g)$ for group elements $g \in G$. We will always drop the k from the notation and simply write $\otimes$ for the tensor product. Taking tensor products with a fixed module is an exact functor, since this is true for k-vector spaces and exactness does not depend on the G-module structure. The one-dimensional vector space k carries the structure of a trivial G-module by setting $\lambda \cdot g = \lambda$ for all $\lambda \in k$ and $g \in G$. This trivial module serves as a unit object for the symmetric monoidal product $\otimes$ in the sense that there are natural isomorphisms $X \otimes k \cong k \otimes X \cong X$ of kG-modules, given by $x \otimes 1 \mapsto 1 \otimes x \mapsto x$ for all $x \in X$. Whenever X and Y are kG-modules, we can equip $\mathrm{Hom}_k(X, Y)$ with a G-module structure by setting $(g \cdot f)(x) = f(x \cdot g^{-1}) \cdot g$ for all $g \in G$, $f \in \mathrm{Hom}_k(X, Y)$. We will simply write $\mathrm{Hom}(X, Y)$ for this G-module and write $X^\sharp = \mathrm{Hom}(X, k)$ where k is the trivial G-module. The algebra kG is self-injective, and therefore in the category $\mathfrak{mod}\text{-}kG$ the classes of projective and injective modules coincide. This allows us to form the stable module category $\underline{\mathfrak{mod}}\text{-}kG$, defined as follows. The objects of $\underline{\mathfrak{mod}}\text{-}kG$ are the same as in $\mathfrak{mod}\text{-}kG$; for any two objects A, B, the group of morphisms is given by $\underline{\mathrm{Hom}}_{kG}(A, B) = \mathrm{Hom}_{kG}(A, B)/\mathrm{PHom}(A, B)$, where $\mathrm{PHom}(A, B)$ denotes the set of all morphisms $A \to B$ which factor through a projective module. It is easily verified that the set $\mathrm{PHom}(A, B)$ is actually a subgroup of $\mathrm{Hom}_{kG}(A, B)$ and that the construction of $\underline{\mathrm{Hom}}$ is compatible with composition, so one indeed obtains a category $\underline{\mathfrak{mod}}\text{-}kG$. We refer to $\mathfrak{mod}\text{-}kG$ and $\underline{\mathfrak{mod}}\text{-}kG$ as the *unstable* and *stable* categories, respectively. There is a canonical functor from the unstable to the stable category which allows us to consider unstable maps as morphisms in the stable world. An unstable morphism will be called *stable isomorphism* if it maps to an isomorphism under the canonical functor. For instance, the projection $X \oplus P \to X$ and the inclusion $X \to X \oplus P$ are stable isomorphisms for projective modules P. We will sometimes denote stable isomorphisms by $\cong_{\mathrm{st}}$.

The stable category carries the structure of a triangulated category. We will only sketch the construction here and leave the technical details to the textbooks. Let us begin with the shift functor Σ. For every module X, choose a short exact sequence $X \hookrightarrow I(X) \twoheadrightarrow \Sigma X$ with an injective module $I(X)$. Any map $X \to Y$ can be lifted to a map $I(X) \to I(Y)$ which in turn induces a map $\Sigma X \to \Sigma Y$, the stable class of which only depends on the class of the map we started with. This implies that Σ is a functor on the stable category. In general, Σ will be a self-equivalence of $\underline{\mathfrak{mod}}\text{-}kG$, but if we are careful enough when choosing the $I(X)$ we can achieve that Σ is an automorphism of $\underline{\mathfrak{mod}}\text{-}kG$ (see, e.g., [7], §2). We will denote by Ω the inverse functor of Σ.

If Σ' is a self-equivalence arising from a construction as above, then Σ and Σ' are isomorphic as functors ([7], Remark on page 13). Let us apply this fact in a particular case. Whenever P is a projective module in $\mathfrak{mod}\text{-}kG$ and X is an arbitrary kG-module, then $P \otimes X$ is also projective (see [11], Theorem 3.2). This implies that the tensor product on $\mathfrak{mod}\text{-}kG$ descends to a symmetric monoidal tensor product on $\underline{\mathfrak{mod}}\text{-}kG$ which we also denote by $\otimes$. Now the exact sequence $k \hookrightarrow I(k) \twoheadrightarrow \Sigma k$ yields an exact sequence $X \hookrightarrow I(k) \otimes X \twoheadrightarrow \Sigma k \otimes X$. By the remarks above, the functors Σ and $\Sigma k \otimes -$ are isomorphic.

Next we are going to define the exact triangles. Let $f : X \to Y$ be a morphism in $\mathfrak{mod}\text{-}kG$. Then we obtain a diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & I(X) \arrow[r] \arrow[d] & \Sigma X \arrow[d, equal] \\ Y \arrow[r, "g"'] & C \arrow[r, "h"'] & \Sigma X \end{tikzcd}$$

where C is the pushout of f and i. The triangles $X \xrightarrow{f} Y \xrightarrow{g} C \xrightarrow{h} \Sigma X$ in $\underline{\mathfrak{mod}}\text{-}kG$ arising this way will be called standard triangles. A triangle in $\underline{\mathfrak{mod}}\text{-}kG$ is called exact if and only if it is isomorphic to a standard triangle. Now one has to prove that this structure indeed satisfies the axioms of a triangulated category; the interested reader may find this in [6], Theorem 9.4. In fact, the closed symmetric monoidal structure on $\underline{\mathfrak{mod}}\text{-}kG$ (given by $\otimes$ and Hom) is compatible with the triangulation in the sense of [10], appendix A.

We claim that short exact sequences $A \hookrightarrow B \twoheadrightarrow C$ in $\mathfrak{mod}\text{-}kG$ induce exact triangles in $\underline{\mathfrak{mod}}\text{-}kG$. We can lift the identity map of A to a commutative diagram as follows:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "a"] \arrow[d, equal] & B \arrow[r, "b"] \arrow[d] & C \arrow[r] \arrow[d, "w"'] & 0 \\ 0 \arrow[r] & A \arrow[r] & I(A) \arrow[r] & \Sigma A \arrow[r] & 0 \end{tikzcd}$$

Then, as is shown in Remark 2 in the proof of Theorem 9.4 in [6], we get an exact triangle $A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{-w} \Sigma A$ in $\underline{\mathfrak{mod}}\text{-}kG$. By the rotation axiom, we also get an exact triangle $\Omega C \xrightarrow{\Omega w} A \xrightarrow{a} B \xrightarrow{b} C$.

Now let n be a fixed integer; then all the k-vector spaces $\underline{\mathrm{Hom}}_{kG}(\Omega^{n+m}X, \Omega^m Y)$ for integers m are naturally isomorphic. This defines the *Tate cohomology groups*

$$\widehat{\mathrm{Ext}}^n_{kG}(X, Y) = \underline{\mathrm{Hom}}_{kG}(\Omega^n X, Y).$$

For any two morphisms $f \in \underline{\mathrm{Hom}}_{kG}(\Omega^n X, Y)$ and $g \in \underline{\mathrm{Hom}}_{kG}(\Omega^m Y, Z)$ we get a composition $g \circ \Omega^m f \in \underline{\mathrm{Hom}}_{kG}(\Omega^{n+m} X, Z)$. This describes the *composition product*

$$\widehat{\mathrm{Ext}}^m_{kG}(Y, Z) \otimes \widehat{\mathrm{Ext}}^n_{kG}(X, Y) \to \widehat{\mathrm{Ext}}^{m+n}_{kG}(X, Z).$$

Let us write $\hat{H}^*(G, M) = \widehat{\mathrm{Ext}}^*_{kG}(k, M)$ for short; then $\hat{H}^*(G, k)$ is a graded algebra, which we often denote by $\hat{H}^*(G)$. As in the case of ordinary cohomology, $\hat{H}^*(G)$ is graded commutative.

Note that for positive integers n, Tate cohomology $\widehat{\operatorname{Ext}}^n_{kG}(X,Y)$ agrees with the usual Ext-groups $\operatorname{Ext}^n_{kG}(X,Y)$. For example, every short exact sequence $A \hookrightarrow B \twoheadrightarrow C$ represents an element in $\operatorname{Ext}^1_{kG}(C,A)$; under the isomorphism $\operatorname{Ext}^1_{kG}(C,A) \cong \underline{\operatorname{Hom}}_{kG}(\Omega C, A)$ this element corresponds to the stable map $\Omega C \to A$ in the exact triangle $\Omega C \to A \to B \to C$.

Remark 1.1. It is well-known that the group algebra kG is a cocommutative Hopf algebra, the comultiplication $\Delta : kG \to kG \otimes kG$ and the antipode $\varepsilon : kG \to kG$ being the k-linear maps given by

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = g^{-1}$$

for all $g \in G$. Let us note that throughout this thesis we can replace the group algebra kG by an arbitrary (ungraded) finite-dimensional cocommutative Hopf algebra A over k. By a theorem of Larson and Sweedler [17], A is a Frobenius algebra and we are able to construct the stable module category $\underline{\mathfrak{mod}}$-$A$. Cocommutativity provides a symmetric monoidal tensor product $\otimes$ and will turn up later again when we discuss Steenrod operations. The tensor product of a projective module and an arbitrary module still is projective (see the last paragraph of §3 in [11]), and therefore the tensor product descends to a symmetric monoidal product on the stable module category with unit object k, the trivial module which gets its A-module structure via the counit of A.

We should also remark that we can generalize all results to the category $\mathfrak{Mod}$-A of arbitrary (not necessarily finite dimensional) A-modules and its corresponding stable category $\underline{\mathfrak{Mod}}$-$A$.

1.2 The graded center

Suppose that $\mathcal{C}$ is a triangulated category with shift functor Σ. The *graded center* $Z(\mathcal{C})$ of $\mathcal{C}$ consists of all natural transformations $\mathrm{id}_{\mathcal{C}} \to \Sigma^n$ which commute with Σ up to the sign $(-1)^n$; that is, the degree n-part of $Z(\mathcal{C})$ is given by

$$Z^n(\mathcal{C}) = \{\zeta : \mathrm{id}_{\mathcal{C}} \to \Sigma^n \mid \Sigma\zeta = (-1)^n \zeta\Sigma\}.$$

There is an obvious pairing $Z^n(\mathcal{C}) \times Z^m(\mathcal{C}) \to Z^{n+m}(\mathcal{C})$ mapping (φ, ψ) to the composition $\Sigma^m\varphi \circ \psi$. We will ignore set theoretical issues and assume that $Z^*(\mathcal{C})$ is a set – this will be obvious in all the cases we are interested in. Then $Z^*(\mathcal{C})$ is a graded commutative ring (see, e.g., [18], §2).

Now let $\mathcal{C}$ be the stable module category $\underline{\mathfrak{mod}}$-$kG$. Suppose that we are given a cohomology class in $\hat{H}^n(G)$, represented by some map $\zeta : \Omega^n k \to k$. Then multiplication by ζ induces an element of $Z^n(\mathcal{C})$ also denoted by ζ:

$$\begin{tikzcd}
\Omega^n X \arrow[r, "\zeta_{\Omega^n X}"] \arrow[d, "\wr"'] & X \arrow[d, "\wr"'] \\
\Omega^n k \otimes X \arrow[r, "\zeta \otimes \mathrm{id}"] & k \otimes X
\end{tikzcd}$$

Thus, we obtain a morphism of graded rings $\hat{H}^*(G) \to Z^*(\mathcal{C})$. If we compose this with the evaluation at k we get a retraction

$$\hat{H}^*(G) \to Z^*(\underline{\mathfrak{mod}}\text{-}kG) \xrightarrow{\mathrm{ev}_k} \hat{H}^*(G).$$

The first map is an inclusion of a direct summand, which will not be surjective in general. For example, take any element g in the center of G. Then the action of g on a given module is a natural morphism in $\mathfrak{mod}\text{-}kG$ and induces an element in $Z^0(\mathcal{C})$. If g is not the unit element of G then g does not come from $\hat{H}^*(G)$ since $\mathrm{ev}_k(g) = 1 \in \hat{H}^0(G)$ but $g \neq 1 \in Z^0(\mathcal{C})$ if p divides the order of G.

More generally, let A be a symmetric algebra, i.e., a Frobenius algebra A whose corresponding nondegenerate bilinear form $A \times A \to k$ is symmetric (as it is in the case $A = kG$). Define A^{op} to be the opposite algebra of A, and put $B = A^e = A^{op} \otimes_k A$, the enveloping algebra of A. Then B is also a symmetric algebra, and we can pass from $\mathfrak{mod}\text{-}B$, whose objects are the same as A-A-bimodules, to the stable category $\underline{\mathfrak{mod}}\text{-}B$ with shift-functor Σ_B. Then the *Tate version of Hochschild cohomology* is given by $\widehat{HH}^*(A) = \underline{\mathrm{Hom}}_B(\Omega_B^* A, A)$. Given an element in $\widehat{HH}^n(A)$ represented by some map $f : \Omega_B^n A \to A$ in $\mathfrak{mod}\text{-}B$, we obtain a natural transformation in $\underline{\mathfrak{mod}}\text{-}A$

$$\Omega^n X \cong \Omega_B^n X \cong \Omega_B^n A \otimes_A X \xrightarrow{f \otimes_A \mathrm{id}} A \otimes_A X \cong X,$$

where the first isomorphism comes from the fact that a projective resolution of B-modules is a projective resolution of A-modules. We therefore have constructed a map $\widehat{HH}^*(A) \to Z^*(\underline{\mathfrak{mod}}\text{-}A)$. In our case $A = kG$ we have a factorization

$$\hat{H}^*(G) \to \widehat{HH}^*(kG) \to Z(\underline{\mathfrak{mod}}\text{-}kG),$$

so the second map is more likely to be surjective than the whole composition. In fact, the central element given above not coming from $\hat{H}^*(G)$ comes from $\widehat{HH}^*(kG)$. Nevertheless, the map $\widehat{HH}^*(kG) \to Z^*(\underline{\mathfrak{mod}}\text{-}kG)$ is not surjective in general. For example, in the case that G is a p-group of rank at least 2 and k is algebraically closed Linckelmann and Stancu [19] proved that $Z^*(\underline{\mathfrak{mod}}\text{-}kG)$ is infinite-dimensional in all degrees for $p = 2$ and in odd degrees for $p > 2$.

Remark 1.2. The map $\hat{H}^*(G) \to Z^*(\underline{\mathfrak{mod}}\text{-}kG)$ is a special case of a more general setting. Suppose that $\mathcal{C}$ is a triangulated category with a symmetric monoidal product $\wedge$ compatible with the triangulated structure. Let us denote by S the unit of the product. Then we have a canonical map from the graded ring $[S, S]^*$ of graded self-maps of S to the center $Z^*(\mathcal{C})$ by mapping $\varphi : S \to \Sigma^n S$ to the family of maps

$$X \cong X \wedge S \xrightarrow{\mathrm{id}_X \wedge \varphi} X \wedge \Sigma^n S \cong \Sigma^n X.$$

In the case of $\mathcal{C} = \underline{\mathfrak{mod}}\text{-}kG$ with product $\otimes$ and $S = k$ being the trivial kG-module, we get our map $\hat{H}^*(G) \to Z^*(\underline{\mathfrak{mod}}\text{-}kG)$. In the case of the stable homotopy category $\mathcal{C} = \mathcal{SHC}$ with smash product $\wedge$ and S being the sphere spectrum we get a map from the stable homotopy groups of spheres $\pi_{-*}^s \to Z^*(\mathcal{SHC})$.

1.3 The notion of order

Now we are going to define the notion of order. Let $\mathcal{C}$ be a triangulated category with shift functor Σ, and let $\zeta \in Z^n(\mathcal{C})$ be an element in the graded center of $\mathcal{C}$. For every object K we denote by K/ζ some choice of cone of the map $\zeta_{\Sigma^{-n}K} : \Sigma^{-n}K \to K$, so we have exact triangles

$$\cdots \to \Sigma^{-1}K/\zeta \to \Sigma^{-n}K \xrightarrow{\zeta} K \to K/\zeta \to \Sigma^{1-n}K \to \ldots$$

The ambiguity in choosing these cones will turn out to be irrelevant for our purposes. Now suppose that $f : K \to Y$ is some map in our category $\mathcal{C}$; then we say that a map $g : K/\zeta \to Y$ is an *extension* of f if the following diagram commutes:

$$\begin{tikzcd} \cdots \arrow[r] & \Sigma^{-n}K \arrow[r, "\zeta"] & K \arrow[r] \arrow[d, "f"'] & K/\zeta \arrow[r] \arrow[dl, "g"] & \cdots \\ & & Y & & \end{tikzcd}$$

We are ready to define the main object of this thesis. The following is a straightforward generalization of a definition made by Schwede in [25].

Definition 1.3. *For every object X in the category $\mathcal{C}$ we define the ζ-*order* of X, denoted ζ-ord(X), inductively by the following two conditions:*

(i) *For every X we have ζ-ord$(X) \in \{0, 1, 2, \ldots\} \cup \{\infty\}$.*

(ii) *For every positive integer j and every object X we have ζ-ord$(X) \geq j$ if and only if for every object K and every morphism $f : K \to X$ there is some extension g of f such that for some (and hence any) choice of cone C_g of g we have ζ-ord$(C_g) \geq j-1$.*

We also define the ζ-order of the category $\mathcal{C}$ to be the ζ-order of some zero-object.

Remark 1.4. For every integer m and every object X in $\mathcal{C}$ we have multiplication by m on X, defined by the m-fold sum $\mathrm{id} + \mathrm{id} + \cdots + \mathrm{id} : X \to X$. This is natural in X and commutes with the shift functor Σ, so we get an element $m \in Z^0(\mathcal{C})$ and the definition applies. This way we recover the definition of m-order given in [25].

Lemma 1.5 (compare [25]). *Let $\mathcal{C}$ and ζ be as above. The ζ-order enjoys the following properties:*

(i) *The ζ-order is invariant under isomorphisms and shifts.*

(ii) *An object X of $\mathcal{C}$ has ζ-order at least 1 if and only if ζ acts as zero on X.*

(iii) *The ζ-order of $\mathcal{C}$ equals 1 plus the minimum of the ζ-orders of all objects of the form X/ζ.*

(iv) *The ζ-order is invariant under equivalences of triangulated categories.*

(v) *If $\mathcal{D} \subset \mathcal{C}$ is a full triangulated subcategory then we can restrict ζ to $\mathcal{D}$ and obtain an element in $Z^*(\mathcal{D})$. The notions of order in these two categories are related by the inequality ζ-ord$^{\mathcal{D}}(X) \geq \zeta$-ord$^{\mathcal{C}}(X)$ for all objects X of $\mathcal{D}$.*

Proof. Part (i) follows by a straightforward induction on j. At the same time the proof of the 'isomorphism' statement shows that the notion of order does not depend on the choices of cones in its definition. For (ii) note that if ζ is zero on X then there is some map $h : X/\zeta \to X$ for which the composition $X \to X/\zeta \xrightarrow{h} X$ is the identity of X. Then for every morphism $f : K \to X$ we have an extension $K/\zeta \xrightarrow{f/\zeta} X/\zeta \xrightarrow{h} X$, which means that $\zeta\text{-ord}(X) \geq 1$. Conversely, if the order of X is at least 1, then we can apply the definition to the identity map $f : X \to X$ and obtain some extension $g : X/\zeta \to X$. Then ζ is given by the composite

$$\Sigma^{-n} X \xrightarrow{\zeta} X \to X/\zeta \xrightarrow{g} X,$$

but the composition of the first two maps is zero. Part (iii) and (iv) follow directly from the definition. For (v) it is enough to show that $\zeta\text{-ord}^{\mathcal{C}}(X) \geq j$ implies $\zeta\text{-ord}^{\mathcal{D}}(X) \geq j$ which we show by induction on j, the case $j = 0$ being an empty statement. Suppose this is true for $j - 1$ an assume that $\zeta\text{-ord}^{\mathcal{C}}(X) \geq j$. Let $f : K \to X$ be any map in $\mathcal{D}$. By assumption, there is some extension $g : K/\zeta \to X$ in $\mathcal{C}$ such that for some choice of cone C_g of g we have $\zeta\text{-ord}^{\mathcal{C}}(C_g) \geq j-1$. Since $\mathcal{C}$ is a full triangulated subcategory, g and C_g lie in $\mathcal{D}$, and by induction hypothesis $\zeta\text{-ord}^{\mathcal{D}}(C_g) \geq j - 1$. This implies $\zeta\text{-ord}^{\mathcal{D}}(X) \geq j$. $\square$

1.4 Tensor powers

Suppose that X and Y are kG-modules; then we have already defined a kG-module structure on the tensor product $X \otimes Y$, coming from the comultiplication of the Hopf algebra structure on kG. We can therefore consider the n-fold tensor product $X \otimes X \otimes \cdots \otimes X$ which we denote by $X^{\otimes n}$. On this module we have an obvious Σ_n-action, where Σ_n denotes the symmetric group on n letters. The fact that the comultiplication $\Delta : kG \to kG \otimes kG$ is cocommutative shows that Σ_n acts by homomorphisms of kG-modules. Whenever we have a module X, we denote by T the map

$$X^{\otimes p} \to X^{\otimes p}$$
$$x_1 \otimes x_2 \otimes \cdots \otimes x_p \mapsto x_p \otimes x_1 \otimes \cdots \otimes x_{p-1}.$$

Then $T^p = \mathrm{id}$ and $(1-T)^p = 0$. We will also write $N = (1-T)^{p-1} = 1+T+T^2+\cdots+T^{p-1}$. All these notions generalize naturally to chain complexes of kG-modules. It should be noted that the action of Σ_n then involves a certain sign depending on the degrees of the elements, which can be deduced from the Koszul sign rule. Let us give the precise sign here. Whenever X is a chain complex of kG-modules and $x \in X$ is a homogeneous element, then we denote by $|x|$ the degree of x. The action of $\sigma \in \Sigma_n$ on a tensor product $x_1 \otimes \cdots \otimes x_n$ of homogeneous elements involves the sign $(-1)^m$ with

$$m = \sum_{\substack{i<j,\\ \sigma(i)>\sigma(j)}} |x_i| \cdot |x_j|.$$

For every chain complex X with differential ∂ we denote by $X[m]$ the chain complex with modules $X[m]_n = X_{n-m}$ and differential $(-1)^m\partial$. When X is a kG-module, we can view X as a chain complex concentrated in degree 0; then $X[1]$ is a chain complex concentrated in degree 1, and $\sigma \in \Sigma_n$ acts on the tensor product $X[1]^{\otimes n} \cong X^{\otimes n}[n]$ via its signum $(-1)^\sigma$.

1.5 Symmetric and exterior powers

For a kG-module X, the module $X^{\otimes n}$ has two famous quotients; on the one hand, the symmetric power

$$S^n X = X^{\otimes n} / \left\langle x_1 \otimes \cdots \otimes x_n - x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)} \right\rangle_k ,$$

and on the other hand, the exterior power

$$\Lambda^n X = X^{\otimes n} / \langle x_1 \otimes \cdots \otimes x_n \mid x_i = x_j \text{ for some } i \neq j \rangle_k .$$

Here we denoted by $\langle M \rangle_k$ the k-vector space generated by the elements of the set M. As usual, we define $X^{\otimes 0} = S^0 X = \Lambda^0 X = k$, the trivial kG-module. The obvious multiplication maps

$$\mu_{i,j} : S^i X \otimes S^j X \to S^{i+j} X, \quad \mu_{i,j} : \Lambda^i X \otimes \Lambda^j X \to \Lambda^{i+j} X$$

turn $S^* X$ and $\Lambda^* X$ into graded algebras. There are also comultiplications $\Delta : S^* X \to S^* X \otimes S^* X$ and $\Delta : \Lambda^* X \to \Lambda^* X \otimes \Lambda^* X$ which make $S^* X$ and $\Lambda^* X$ into graded Hopf algebras. For example, the former map is given by

$$\begin{aligned} \Delta_{i,j} : S^{i+j} X &\longrightarrow S^i X \otimes S^j X \\ x_1 \cdots x_{i+j} &\mapsto \sum_\sigma x_{\sigma(1)} \cdots x_{\sigma(i)} \otimes x_{\sigma(i+1)} \cdots x_{\sigma(i+j)}, \end{aligned}$$

where σ runs through all (i,j)-shuffles (see, e.g., [22] §1.3).

For any map $f : X \to Y$ let us denote by κ_{ij}^f the map

$$\begin{aligned} \kappa_{ij}^f : \Lambda^i X \otimes S^j Y &\longrightarrow \Lambda^{i-1} X \otimes S^{j+1} Y \\ x_1 \wedge \cdots \wedge x_i \otimes y_1 \cdots y_j &\mapsto \sum_{t=1}^{i} (-1)^{t-1} x_1 \wedge \cdots \wedge \widehat{x_t} \wedge \cdots \wedge x_i \otimes f(x_t) y_1 \cdots y_j . \end{aligned}$$

Notice that this map equals the composition

$$\Lambda^i X \otimes S^j Y \xrightarrow{\Delta \otimes \mathrm{id}} \Lambda^{i-1} X \otimes X \otimes S^j Y \xrightarrow{\mathrm{id} \otimes f \otimes \mathrm{id}} \Lambda^{i-1} X \otimes Y \otimes S^j Y \xrightarrow{\mathrm{id} \otimes \mu} \Lambda^{i-1} X \otimes S^{1+j} Y.$$

We will often drop the j from the notation and simply write κ_i^f for that map.

Lemma 1.6. *Suppose that $0 \to X \xrightarrow{\iota} Y \xrightarrow{\pi} Z \to 0$ is an exact sequence of kG-modules, and let n be a positive integer. Then the sequences*

$$0 \to \Lambda^n X \xrightarrow{\kappa_n^\iota} \Lambda^{n-1} X \otimes S^1 Y \xrightarrow{\kappa_{n-1}^\iota} \ldots \xrightarrow{\kappa_2^\iota} \Lambda^1 X \otimes S^{n-1} Y \xrightarrow{\kappa_1^\iota} S^n Y \xrightarrow{S^n \pi} S^n Z \to 0$$

and

$$0 \to \Lambda^n X \xrightarrow{\Lambda^n \iota} \Lambda^n Y \xrightarrow{\kappa_n^\pi} \Lambda^{n-1} Y \otimes S^1 Z \xrightarrow{\kappa_{n-1}^\pi} \ldots \xrightarrow{\kappa_2^\pi} \Lambda^1 Y \otimes S^{n-1} Z \xrightarrow{\kappa_1^\pi} S^n Z \to 0$$

are exact.

Remark 1.7. For $\iota = \mathrm{id}_X : X \to X$ and $Z = 0$ this is a well-known fact (see, e.g., [22], Theorem 1.1); one obtains the so-called *Koszul complex*

$$0 \to \Lambda^n X \to \Lambda^{n-1} X \otimes X \to \Lambda^{n-2} X \otimes S^2 X \to \dots \\ \dots \to X \otimes S^{n-1} X \to S^n X \to 0. \quad (1.8)$$

It is always exact, also for infinite dimensional modules X.

Proof. We only need to show a statement about exactness; we can therefore ignore the G-action and treat all modules as k-vector spaces. In particular, we can assume that $Y = X \oplus Z$ and ι and π are the obvious maps. Using the exponential property of the symmetric algebra functor $S^*(X \oplus Z) \cong S^* X \otimes S^* Z$ (see 1.4.(2) in [22]) the first sequence decomposes into a direct sum of sequences

$$0 \to \Lambda^i X \otimes S^{n-i} Z \to \Lambda^{i-1} X \otimes S^1 X \otimes S^{n-i} Z \to \Lambda^{i-2} X \otimes S^2 X \otimes S^{n-i} Z \to \dots \\ \dots \to \Lambda^1 X \otimes S^{i-1} X \otimes S^{n-i} Z \to S^i X \otimes S^{n-i} Z \to 0$$

which are exact by Remark 1.7, and one additional exact sequence $0 \to S^n Z \xrightarrow{\mathrm{id}} S^n Z \to 0$. Exactness of the second sequence is proved similarly. $\square$

Using the fact that there are natural isomorphisms $\Lambda^n(X^\sharp) \cong (\Lambda^n X)^\sharp$ for all n, and $S^n(X^\sharp) \cong (S^n X)^\sharp$ for $n < p$ (see §1.5 in [22]), we obtain the following dual result:

Corollary 1.9. *Suppose that $0 \to X \xrightarrow{\iota} Y \xrightarrow{\pi} Z \to 0$ is an exact sequence of kG-modules, and let n be an integer with $n < p$. Then the sequence*

$$0 \to S^n X \xrightarrow{S^n \iota} S^n Y \to \Lambda^1 Z \otimes S^{n-1} Y \to \dots \to \Lambda^{n-1} Z \otimes S^1 Y \to \Lambda^n Z \to 0$$

is exact, where the maps $\Lambda^i Z \otimes S^{n-i} Y \to \Lambda^{i+1} Z \otimes S^{n-i-1} Y$ are given by

$$z_1 \wedge z_2 \wedge \dots \wedge z_i \otimes y_1 y_2 \dots y_{n-i} \mapsto \sum_{j=1}^{n-i} z_1 \wedge \dots \wedge z_i \wedge \pi(y_j) \otimes y_1 \dots \widehat{y_j} \dots y_{n-i}.$$

Remark 1.10. Note that in the special case $X = 0$, $Y = Z$ and $\pi = \mathrm{id}_Y$ we get a sequence

$$0 \to S^n X \to X \otimes S^{n-1} X \to \Lambda^2 X \otimes S^{n-2} X \to \dots \\ \dots \to \Lambda^{n-1} X \otimes X \to \Lambda^n X \to 0, \quad (1.11)$$

which we call the *de Rham complex.* It is exact for $n < p$ but not in general, compare with Lemma 1.2 of [22].

Another consequence is the following

Corollary 1.12. *Suppose that $0 \to X \xrightarrow{\iota} Y \xrightarrow{\pi} k \to 0$ is an exact sequence of kG-modules. Then the map $\kappa_n^\pi : \Lambda^n Y \to \Lambda^{n-1} Y \otimes S^1 k \cong \Lambda^{n-1} Y$ induces a map $\Lambda^n Y \to \Lambda^{n-1} X$ which we also denote by κ_n^π, and the sequence*

$$0 \to \Lambda^n X \xrightarrow{\Lambda^n \iota} \Lambda^n Y \xrightarrow{\kappa_n^\pi} \Lambda^{n-1} X \to 0 \quad (1.13)$$

is exact.

Proof. By Lemma 1.6 we have a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \Lambda^n X \arrow[r, "\Lambda^n\iota"] \arrow[d, dotted] & \Lambda^n Y \arrow[r] & \Lambda^{n-1}Y \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & Y \arrow[r] \arrow[d, equal] & k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Lambda^{n-1}X \arrow[rr] & & \Lambda^{n-1}Y \arrow[r] & \cdots \arrow[r] & Y \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

showing that the dotted arrow exists and that (1.13) is exact. □

Remark 1.14. In the situation of the preceding corollary, consider the graded algebra $\Lambda^* X$. Then $\Lambda^{1+*}X$ and $\Lambda^{1+*}Y$ are a graded right $\Lambda^* X$-modules, and the corollary (and an easy to verify commutative diagram) show that we have an exact sequence

$$0 \to \Lambda^{1+*}X \xrightarrow{\iota} \Lambda^{1+*}Y \xrightarrow{\kappa} \Lambda^* X \to 0$$

of graded right $\Lambda^* X$-modules.

We end this section with a construction that will be helpful later for checking commutativity of certain diagrams. Suppose that Y is any kG-module, M^* is a graded right $\Lambda^* Y$-module and N^* is a graded left $S^* Y$-comodule; so we have maps

$$\mu : M^* \otimes \Lambda^* Y \to M^*, \quad \nu : N^* \to S^* Y \otimes N^*.$$

Then we have maps $M^i \otimes N^j \xrightarrow{d} M^{i+1} \otimes N^{j-1}$ given by the composition

$$M^i \otimes N^j \xrightarrow{\mathrm{id}\otimes\nu} M^i \otimes Y \otimes N^{j-1} \xrightarrow{\mu\otimes\mathrm{id}} M^{i+1} \otimes N^{j-1}.$$

We claim that $d^2 = 0$ as we can read off from the following commutative diagram:

$$\begin{tikzcd}
M^i \otimes N^j \arrow[r] \arrow[d] \arrow[dr, phantom, "N \text{ is a } S^*Y\text{-comodule}"] & M^i \otimes Y \otimes N^{j-1} \arrow[r, "\mu\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\nu"] & M^{i+1} \otimes N^{j-1} \arrow[d, "\mathrm{id}\otimes\nu"] \\
M^i \otimes S^2Y \otimes N^{j-2} \arrow[r] \arrow[dr, "0"'] & M^i \otimes Y \otimes Y \otimes N^{j-2} \arrow[r, "\mu\otimes\mathrm{id}"] \arrow[d] \arrow[dr, phantom, "M \text{ is a } \Lambda^*Y\text{-module}"] & M^{i+1} \otimes Y \otimes N^{j-2} \arrow[d] \\
& M^i \otimes \Lambda^2Y \otimes N^{j-2} \arrow[r] & M^{i+2} \otimes N^{j-2}
\end{tikzcd}$$

Definition 1.15. *We call*

$$\ldots \xrightarrow{d} M^{i-1} \otimes N^{j+1} \xrightarrow{d} M^i \otimes N^j \xrightarrow{d} M^{i+1} \otimes N^{j-1} \xrightarrow{d} \ldots$$

the chain complex associated to (M, N).

This is a functorial construction in the sense that maps of graded right $\Lambda^* Y$-modules and maps of graded left $S^* Y$-comodules induce maps of chain complexes. The same construction works for $N^* \otimes M^*$ when N is a graded *right* $S^* Y$-comodule and M a graded *left* $\Lambda^* Y$-module.

Example 1.16. We have already seen instances of such chain complexes: suppose that $\pi : Y \to Z$ is a map of kG-modules, and let $N = S^*Y$. Then $M = \Lambda^*Z$ becomes a Λ^*Y-module via the map π. The chain complex associated to (Λ^*Z, S^*Y) is part of the exact sequence of Corollary 1.9.

We can also swap the roles of S^* and Λ^* and consider graded left S^*Y-modules M and graded right Λ^*Y-comodules; then we also get an associated sequence in a similar way.

Example 1.17. Suppose again that $\pi : Y \to Z$ is a map of kG-modules, and let $M = \Lambda^*Y$. Similar to the example above, $N = S^*Z$ becomes an S^*Y-module via the map π. Since $S^*\pi : S^*Y \to S^*Z$ is a map of S^*Y-modules, we obtain a morphism of the associated chain complexes:

$$\begin{tikzcd}
\cdots & \Lambda^j Y \otimes S^i Z \arrow[l] & \Lambda^{j+1} Y \otimes S^{i-1} Z \arrow[l] & \cdots \arrow[l] \\
\cdots & \Lambda^j Y \otimes S^i Y \arrow[l] \arrow[u, "\mathrm{id} \otimes S^i\pi"] & \Lambda^{j+1} Y \otimes S^{i-1} Y \arrow[l] \arrow[u, "\mathrm{id} \otimes S^{i-1}\pi"'] & \cdots \arrow[l]
\end{tikzcd}$$

1.6 Symmetric and exterior powers in the stable category

For every kG-module X and $i < p$, there are natural maps

$$S^iX \to X^{\otimes i} : \qquad x_1 \cdots x_i \mapsto \frac{1}{i!} \sum_{\sigma \in \Sigma_i} x_{\sigma_1} \otimes \cdots \otimes x_{\sigma_i}, \tag{1.18}$$

$$\Lambda^iX \to X^{\otimes i} : \qquad x_1 \wedge \cdots \wedge x_i \mapsto \frac{1}{i!} \sum_{\sigma \in \Sigma_i} (-1)^\sigma x_{\sigma_1} \otimes \cdots \otimes x_{\sigma_i}.$$

Up to the scalar constants they are the i-fold iterated comultiplications of the graded Hopf algebra structures on S^*X and Λ^*X. We will refer to these maps as *the canonical maps* $S^iX \to X^{\otimes i}$ and $\Lambda^iX \to X^{\otimes i}$. The compositions $S^iX \to X^{\otimes i} \to S^iX$ and $\Lambda^iX \to X^{\otimes i} \to \Lambda^iX$ with the projections are the identity maps. Therefore, S^iX and Λ^iX both are direct summands of $X^{\otimes i}$. In particular, we have shown the following lemma which ensures that for $i < p$ the functors S^i and Λ^i induce functors on the stable category.

Lemma 1.19. *Suppose that $i < p$. If P is a projective kG-module, then so are S^iP and Λ^iP.*

Remark 1.20. Note that the conclusion of the lemma fails in general if $i \geq p$. For example, if G is the cyclic group of order p, then $\dim_k kG = p$ and hence $\dim_k \Lambda^p(kG) = 1$. One can check that $\Lambda^p(kG) \cong k$ as kG-modules, which is not projective. Also, $S^p(kG) \cong k \oplus (\text{free})$.

Lemma 1.21. *Suppose that $i < p$ and X is any kG-module. Then, in the stable module category, $S^i\Omega X \cong \Omega^i\Lambda^iX$ and $\Lambda^i\Omega X \cong \Omega^iS^iX$.*

Proof. Consider a short exact sequence $0 \to \Omega X \to P \to X \to 0$ in $\mathfrak{mod}\text{-}kG$ with a projective module P. By Corollary 1.9 there is an exact sequence

$$0 \to S^i\Omega X \to S^iP \to \Lambda^1X \otimes S^{i-1}P \to \cdots \to \Lambda^{i-1}X \otimes S^1P \to \Lambda^iX \to 0.$$

But the S^jP's are projective by Lemma 1.19, so $S^i\Omega X$ is stably isomorphic to $\Omega^i\Lambda^i X$. The other isomorphism is shown similarly using Lemma 1.6. $\square$

By induction, we conclude that

$$\Lambda^i\Omega^j X \cong \begin{cases} \Omega^{ij}S^iX & \text{if } j \text{ is odd,} \\ \Omega^{ij}\Lambda^iX & \text{if } j \text{ is even} \end{cases}$$

in the stable category. Since $\Omega^m X = 0$ in $\underline{\mathfrak{mod}}\text{-}kG$ implies that X is projective, we get the following corollary:

Corollary 1.22. *If n is even, then $\Lambda^i\Omega^n k$ is projective for $1 < i < p$.*

2 The lower bound

From now on, let $[\zeta] \in \hat{H}^n(G)$ be a non-zero Tate cohomology class of even degree n, represented by a map $\zeta : \Omega^n k \to k$ in $\mathfrak{mod}\text{-}kG$. Since ζ is non-zero, it must be surjective and we get an exact sequence $0 \to L_\zeta \xrightarrow{\iota} \Omega^n k \xrightarrow{\zeta} k \to 0$. This sequence yields an exact triangle

$$\Omega k \to L_\zeta \to \Omega^n k \to k$$

We denote by η an unstable representative of the stable map $\Omega k \to L_\zeta$.

Definition 2.1. *Let s be a non-negative integer. An s-coherent module X is a sequence $X_0, X_1, X_2, \dots, X_s$ of modules together with maps*

$$\mu_{i,j} : \Lambda^i L_\zeta \otimes X_j \to X_{i+j}$$

for all $i \geq 1$ and $j \geq 0$ with $i + j \leq s$, satisfying the following conditions:

- *(Unitality) For every $1 \leq i < s$, the composite*

$$\Omega k \otimes X_i \xrightarrow{\eta \otimes \mathrm{id}} L_\zeta \otimes X_i \xrightarrow{\mu_{1,i}} X_{1+i}$$

  is a stable isomorphism.

- *(Associativity) The square*

$$\begin{tikzcd} \Lambda^i L_\zeta \otimes \Lambda^j L_\zeta \otimes X_m \arrow[r, "\mathrm{id} \otimes \mu_{j,m}"] \arrow[d, "\mu_{i,j} \otimes \mathrm{id}"] & \Lambda^i L_\zeta \otimes X_{j+m} \arrow[d, "\mu_{i,j+m}"] \\ \Lambda^{i+j} L_\zeta \otimes X_m \arrow[r, "\mu_{i+j,m}"] & X_{i+j+m} \end{tikzcd}$$

  commutes for all $0 \leq i, j, m$ with $i + j + m \leq s$.

- *(Exactness) The sequence*

$$0 \to S^i L_\zeta \otimes X_0 \to S^{i-1} L_\zeta \otimes X_1 \to \cdots \to S^1 L_\zeta \otimes X_{i-1} \to X_i \to 0$$

is exact for all $i \leq s$. Here the map $S^j L_\zeta \otimes X_m \to S^{j-1} L_\zeta \otimes X_{m+1}$ is given by the composition

$$S^j L_\zeta \otimes X_m \xrightarrow{\Delta_{j-1,1}\otimes} S^{j-1} L_\zeta \otimes L_\zeta \otimes X_m \xrightarrow{\mathrm{id}\otimes\mu_{1,m}} S^{j-1} L_\zeta \otimes X_{m+1}.$$

We call X_1 the *underlying object* of X. A *map of s-coherent modules* $f : X \to Y$ is a family of maps $f_i : X_i \to Y_i$ compatible with the respective structure maps μ in the obvious sense. Such a map will be called injective (surjective) if each of these f_i's is injective (surjective).

Remark 2.2. We have chosen a rather lengthy and very explicit way of defining s-coherent modules in order to stress the similarity to the definition of k-coherent M-modules of [24], Definition 2.1. Notice that we could equally well have defined s-coherent modules to be graded left $\Lambda^* L_\zeta$-modules X_* with $X_i = 0$ for i outside $\{0, 1, \ldots, s\}$, satisfying the unitality and exactness conditions. The sequence of the exactness condition is the chain complex associated to the right $S^* L_\zeta$-comodule $S^* L_\zeta$ and the left $\Lambda^* L_\zeta$-module X_* (in the sense of Definition 1.15).

Example 2.3. For every $s < p$, there is the *tautological* s-coherent module, defined as follows. Put $X_i = \Lambda^i L_\zeta$, and define $\mu_{i,j} : \Lambda^i L_\zeta \otimes \Lambda^j L_\zeta \to \Lambda^{i+j} L_\zeta$ to be the natural maps. Then we have associativity, and exactness holds due to Corollary 1.9. To prove unitality, note that we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & L_\zeta \otimes \Lambda^i L_\zeta \arrow[r] \arrow[d] & \Omega^n k \otimes \Lambda^i L_\zeta \arrow[r] \arrow[d] & \Lambda^i L_\zeta \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Lambda^{1+i} L_\zeta \arrow[r] & \Lambda^{1+i} \Omega^n k \arrow[r] & \Lambda^i L_\zeta \arrow[r] & 0
\end{tikzcd}$$

where the upper row is obtained by tensoring the exact sequence $0 \to L_\zeta \to \Omega^n k \to k \to 0$ with $\Lambda^i L_\zeta$; it is therefore exact. The bottom row is the exact sequence constructed in Corollary 1.12, and the middle vertical map is the composite $\Omega^n k \otimes \Lambda^i L_\zeta \xrightarrow{\mathrm{id}\otimes\Lambda^i\iota} \Omega^n k \otimes \Lambda^i \Omega^n k \xrightarrow{\cdot\wedge\cdot} \Lambda^{i+1} \Omega^n k$. Passing to the stable category, we obtain a map of triangles:

$$\begin{tikzcd}
\Omega k \otimes \Lambda^i L_\zeta \arrow[r] \arrow[d, "\cong"] \arrow[dr, phantom, "(*)"] & L_\zeta \otimes \Lambda^i L_\zeta \arrow[r] \arrow[d] & \Omega^n k \otimes \Lambda^i L_\zeta \arrow[r] \arrow[d] & \Lambda^i L_\zeta \arrow[d, equal] \\
\Omega \Lambda^i L_\zeta \arrow[r] & \Lambda^{1+i} L_\zeta \arrow[r] & \Lambda^{1+i} \Omega^n k \arrow[r] & \Lambda^i L_\zeta
\end{tikzcd}$$

Since $0 < i < p-1$, we know (by Corollary 1.22) that $\Lambda^{1+i}\Omega^n k$ is projective and therefore isomorphic to 0 in the stable category. Hence, the first map of the bottom row is an isomorphism. The commutative diagram $(*)$ then shows the desired unitality condition.

More generally, if K is any module, we denote by $F(K)$ the tautological s-coherent module over $L_\zeta \otimes K$, given by $F(K)_j = \Lambda^j L_\zeta \otimes K$. The maps and properties are obtained from the previous example by tensoring everything with K from the right. This is also a functorial construction in the sense that given any map $f : K \to L$, we get an induced map $F(K) \xrightarrow{F(f)} F(L)$, which is surjective if f is.

Lemma 2.4. *Suppose that $f : X \to Y$ is a surjective map of s-coherent modules. Then the kernel of f is an s-coherent module in the natural way.*

Proof. The kernel of f is defined levelwise by $C_i = \ker f_i : X_i \to Y_i$. The multiplication maps are defined by the following commutative diagram:

$$\begin{tikzcd}
\Lambda^i L_\zeta \otimes C_j \arrow[r] \arrow[d, "\mu_{i,j}"] & \Lambda^i L_\zeta \otimes X_j \arrow[r, "\mathrm{id} \otimes f_j"] \arrow[d, "\mu_{i,j}"] & \Lambda^i L_\zeta \otimes Y_j \arrow[d, "\mu_{i,j}"] \\
C_{i+j} \arrow[r] & X_{i+j} \arrow[r, "f_{i+j}"] & Y_{i+j}
\end{tikzcd}$$

Associativity follows immediately from associativity on X and Y. To prove unitality, consider the following commutative diagram:

$$\begin{tikzcd}
\Omega k \otimes C_j \arrow[r] \arrow[d] & \Omega k \otimes X_j \arrow[r] \arrow[d] & \Omega k \otimes Y_j \arrow[d] \\
C_{1+j} \arrow[r] & X_{1+j} \arrow[r] & Y_{1+j}
\end{tikzcd}$$

When we pass to the stable category, the exact rows turn into exact triangles, and the second and third vertical map turn into isomorphisms. By the five lemma in triangulated categories, the first map must be an isomorphism as well.

We are left with the exactness condition. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & S^i L_\zeta \otimes C_0 \arrow[r] \arrow[d] & S^{i-1} L_\zeta \otimes C_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & C_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S^i L_\zeta \otimes X_0 \arrow[r] \arrow[d] & S^{i-1} L_\zeta \otimes X_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S^i L_\zeta \otimes Y_0 \arrow[r] & S^{i-1} L_\zeta \otimes Y_1 \arrow[r] & \cdots \arrow[r] & Y_i \arrow[r] & 0
\end{tikzcd}$$

Since the maps $C_i \to X_i$ are inclusions, we can deduce that the top row is a complex. Regarding the rows as chain complexes, the vertical maps form a short exact sequence of chain complexes, thus inducing a long exact sequence in homology. Together with the hypothesis that the second and the third row have zero homology, we obtain that the top row is an exact sequence. □

Lemma 2.5. *If X is an s-coherent module, then the multiplication maps $\mu_{i,j} : \Lambda^i L_\zeta \otimes X_j \to X_{i+j}$ are surjective.*

Proof. Let us prove this by induction on i. For $i = 1$, the map $L_\zeta \otimes X_j \to X_{1+j}$ is the same as the last non-trivial map in the exact sequence of the exactness condition; it is therefore surjective. For the inductive step, consider the following commutative diagram:

$$\begin{tikzcd}
L_\zeta \otimes \Lambda^i L_\zeta \otimes X_j \arrow[r, two heads] \arrow[d] & L_\zeta \otimes X_{i+j} \arrow[d, two heads] \\
\Lambda^{1+i} L_\zeta \otimes X_j \arrow[r] & X_{1+i+j}
\end{tikzcd}$$

Since the upper-right composition is surjective, so is the lower-left one. This implies that the bottom map is onto. $\square$

If X is an s-coherent module, let $X[1]$ be given by $X[1]_i = X_{1+i}$ for $i < s$. Then $X[1]$ is an $(s-1)$-coherent module in the natural way. The multiplication maps of X induce a map of $(s-1)$-coherent modules $F(X_1) \to X[1]$ which is surjective by the previous lemma.

Proposition 2.6. *If X is an s-coherent module with underlying object X_1, then we have that ζ-ord$(X_1) \geq s - 1$.*

Proof. We do this by induction on s, the case $s = 1$ being an empty statement. Now suppose that $s \geq 2$. We start with a map $K \to X_1$ in the stable category and assume that it is given by a surjective map $f : K \to X_1$ in $\mathfrak{mod}$-kG. Consider the composition of maps of $(s-1)$-coherent modules

$$F(K) \xrightarrow{F(f)} F(X_1) \longrightarrow X[1].$$

By Lemma 2.5, the maps are surjective. Therefore, by Lemma 2.4, the kernel C of this map is an $(s-1)$-coherent module. By induction hypothesis,

$$\zeta\text{-ord}(C_1) \geq s - 2. \tag{2.7}$$

Now consider the following commutative diagram in $\underline{\mathfrak{mod}}$-$kG$:

$$\begin{tikzcd}
\Omega k \otimes K \arrow[r, "\eta \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes f"'] & L_\zeta \otimes K \arrow[d, "\mathrm{id} \otimes f"] \arrow[ddl, dashed] \\
\Omega k \otimes X_1 \arrow[r, "\eta \otimes \mathrm{id}"] \arrow[d] & L_\zeta \otimes X_1 \arrow[dl, "\mu_{1,1}"] \\
X_2 &
\end{tikzcd}$$

By the unitality condition, the map $\Omega k \otimes X_1 \to X_2$ is a stable isomorphism. Therefore the dashed arrow stably lifts to a map $\hat{f} : L_\zeta \otimes K \to \Omega k \otimes X_1$. This is (up to a shift) an extension of f; we will show that some cone $C_{\hat{f}}$ of $\hat{f}$ satisfies ζ-ord$(C_{\hat{f}}) \geq s - 2$. It is enough to show that some cone of the dashed arrow has this property. Note that this map (in the unstable category) is surjective (because f and $\mu_{1,1}$ are). Therefore, up to a shift, the kernel of the composition

$$L_\zeta \otimes K \xrightarrow{\mathrm{id}\otimes f} L_\zeta \otimes X_1 \xrightarrow{\mu_{1,1}} X_2$$

is a possible choice of cone of the dashed arrow. Since this kernel is the underlying object of C, we get the inductive step from (2.7). $\square$

Corollary 2.8. *If X is any object in $\underline{\mathfrak{mod}}$-$kG$, then ζ-ord$(X/\zeta) \geq p-2$. In particular, the ζ-order of $\underline{\mathfrak{mod}}$-$kG$ is at least $p-1$.*

Proof. Since $L_\zeta \cong \Omega\big(k/\zeta\big)$ in $\underline{\mathfrak{mod}}$-$kG$, we get $\zeta\text{-ord}(X/\zeta) = \zeta\text{-ord}(L_\zeta \otimes X)$. But $L_\zeta \otimes X$ is the underlying object of the $(p-1)$-coherent module $F(X)$, so we get the result from Proposition 2.6. $\square$

Remark 2.9. It should be noted that, for the proof of Proposition 2.6, the exactness condition can be dropped from the definition of s-coherent modules. The reason why we introduced it was that we needed the surjectivity of the composite $L_\zeta \otimes K \to L_\zeta \otimes X_1 \to X_2$, because then the kernel of this map serves as a fiber in the stable category. We could equally well have used a functorial mapping cone construction on $\mathfrak{mod}$-kG as follows: for an unstable map $f : X \to Y$ define the mapping cone $C(f)$ to be the cokernel of the injective map

$$X \cong X \otimes k \xrightarrow{(\mathrm{id}\otimes i, f)} X \otimes kG \oplus Y.$$

Here, $i : k \to kG$ denotes the usual inclusion. The construction is very similar to the mapping cone for maps $f : X \to Y$ of simplicial sets, given as $C(f) = \Delta[1] \wedge X \cup_{1\times X} Y$ where $\Delta[1]$ is the standard 1-simplex, pointed by the 0-vertex.

The mapping cone is sufficiently well-behaved for our purposes: we have natural isomorphisms $C(\mathrm{id}_Z \otimes f) \cong Z \otimes C(f)$, and for every commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow[f']{} & Y' \end{array}$$

we get an induced map $C(f) \to C(f')$ making the obvious diagrams commute. Using these facts, it is straightforward to formulate an alternative proof of Proposition 2.6 not using the exactness condition of s-coherent modules.

Nevertheless, the way we have done it turns out to be useful in the proof of the more general statement in §4.

Remark 2.10. Let us note that Corollary 2.8 is not true for arbitrary non-zero even-degree elements ζ of the graded center. We will work out an example in the case $p = 3$ with $\zeta\text{-ord}(k/\zeta) = 0$ in §6.

3 The power operation

In this section we define a power operation $\mathcal{P}_1$ on the Tate cohomology $\hat{H}^*(G)$. The operation will serve as an obstruction for the ζ-order of k/ζ to be at least $p-1$. At the same time, it enables us to compute certain Massey products explicitly, and we will make use of this fact later on in order to prove $\zeta\text{-ord}(k/\zeta) = p-2$ for certain groups G and certain classes ζ. Let us gather several interesting properties of $\mathcal{P}_1$ in one big theorem.

Theorem 3.1. *There is a power operation $\mathcal{P}_1$ on Tate cohomology $\hat{H}^*(G)$ satisfying the following properties:*

(i) *For every Tate cohomology class $\zeta \in \hat{H}^n(G)$ (with n even if p is odd) the operation $\mathcal{P}_1(\zeta)$ is a coset of $\zeta^p \cdot \hat{H}^{-(2p-3)}(G)$ in $\hat{H}^{pn-(2p-3)}(G)$.*

(ii) *The operation is* linear *in the sense that $\mathcal{P}_1(\zeta+\varphi) \subseteq \mathcal{P}_1(\zeta)+\mathcal{P}_1(\varphi)$ and $\mathcal{P}_1(c\cdot\zeta) = c^p \cdot \mathcal{P}_1(\zeta)$ for all $\zeta, \varphi \in \hat{H}^n(G)$ and $c \in k$.*

(iii) *If $p = 2$ then for every ordinary cohomology class $\zeta \in H^n(G)$ we have that the Steenrod square $\mathrm{Sq}^{n-1}(\zeta)$ is an element of $\mathcal{P}_1(\zeta)$. If p is odd then for every ordinary cohomology class $\zeta \in H^n(G)$ of even degree we get that $\beta P^{\frac{n}{2}-1}(\zeta) \in \mathcal{P}_1(\zeta)$.*

(iv) *The operation is* natural *with respect to injective group homomorphisms $i : G \hookrightarrow H$: for every $\zeta \in \hat{H}^*(H)$ (of even degree if p is odd) we get $i^*\mathcal{P}_1(\zeta) \subseteq \mathcal{P}_1(i^*\zeta)$.*

(v) *For any $\zeta, \varphi \in \hat{H}^*(G)$ the* Cartan formula *$\mathcal{P}_1(\zeta\varphi) \subseteq \mathcal{P}_1(\zeta)\varphi^p + \zeta^p\mathcal{P}_1(\varphi)$ holds.*

Example 3.2. Property (iii) says that our power operation extends the Steenrod operations Sq_1 and P_1 that we have on ordinary cohomology to Tate cohomology, at the price of getting a certain indeterminacy. Let us demonstrate the effect of the indeterminacy for two examples of groups. Let p be an odd prime. We begin with $G = \mathbb{Z}/p\mathbb{Z}$, the cyclic group of order p. The structure of $\hat{H}^*(G)$ is known (see [4], §XII.7) to be $\hat{H}^*(G) \cong k[u, v^{\pm 1}]$ where u is an exterior class of degree 1 and v is a Laurent polynomial class of degree 2. For every non-zero ζ of even degree we get that $\zeta^p \cdot \hat{H}^{-(2p-3)}(G) = \hat{H}^{pn-(2p-3)}(G)$, the indeterminacy is the whole cohomology group, so that $\mathcal{P}_1$ does not store any information at all. As a second example let us study $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. The structure of $\hat{H}^*(G)$ is known and we will recall it in §3.6. In particular we have that $\zeta\varphi = 0$ for all classes ζ, φ whose degrees satisfy $|\zeta| < 0, |\varphi| < 0$ or $|\zeta| < 0, |\zeta\varphi| \geq 0$. This implies that $\mathcal{P}_1(\zeta)$ has *zero* indeterminacy for all ζ (except in the case $|\zeta| = 0$).

Example 3.3. Let us do an example of a non-commutative group which can be worked out completely using Theorem 3.1 only. Let $p = 2$ and $G = Q_8$, the quaternion group with 8 elements. The Tate cohomology ring is known (see, e.g., [4], XII.11, and [1], Lemma IV.2.10) to be

$$\hat{H}^*(G) \cong k[x, y, s^{\pm 1}]/(x^2 + xy + y^2, x^3)$$

with degrees $|x| = |y| = 1$ and $|s| = 4$. In order to describe $\mathcal{P}_1$ we first have a look at the indeterminacy. The vector space $\hat{H}^{-1}(G)$ is generated by xy^2s^{-1}, and for every homogeneous $z \in \hat{H}^*(G)$ we have $xy^2s^{-1}z^2 = 0$ unless $|z|$ is divisible by 4. Therefore, $\mathcal{P}_1(s^n) = \hat{H}^{8n-1}(G)$ for all integers n, and in all other degrees $\mathcal{P}_1$ does not have any indeterminacy. We know by part (iii) of the theorem that $\mathcal{P}_1(x) = \mathrm{Sq}_1(x) = \mathrm{Sq}^0(x) = x$ and $\mathcal{P}_1(y) = y$. The Cartan formula implies $\mathcal{P}_1(x^2) = 0$, $\mathcal{P}_1(y^2) = 0$, and $\mathcal{P}_1(x^2y) = 0$, and finally

$$\begin{aligned} \mathcal{P}_1(s^n x) &= \underbrace{\mathcal{P}_1(s^n)x^2}_{0} + s^{2n}\mathcal{P}_1(x) = s^{2n}x, & \mathcal{P}_1(s^n x^2) &= 0, \\ & & \mathcal{P}_1(s^n y^2) &= 0, \\ \mathcal{P}_1(s^n y) &= s^{2n}y, & \mathcal{P}_1(s^n x^2 y) &= 0. \end{aligned}$$

The plan of this section is as follows. The first two subsections have a preparative character. In the first one we introduce a new description of negative Ext-groups via complexes of projectives, whereas in the second one we prove the existence of a cochain map between certain cochain complexes. The definition of our power operation and proofs of the most obvious properties are given in the third subsection. In §3.4 we prove the Cartan formula, and in §3.5 we show that the new operation contains the first non-trivial Steenrod operation in positive degrees. In the last two subsections we compute $\mathcal{P}_1$ in the case of elementary abelian p-groups.

3.1 Negative Ext-groups

Let $n > 0$. It is well-known that $\mathrm{Ext}^n_{kG}(A,B) = \underline{\mathrm{Hom}}_{kG}(\Omega^n A, B)$ admits a description via extensions of A by B. We will now give a similar description of $\widehat{\mathrm{Ext}}^{-n}_{kG}(A,B) \cong \underline{\mathrm{Hom}}_{kG}(A, \Omega^n B)$. Let us define a category $\mathcal{K}_n(A,B)$, whose objects are all the chain complexes

$$C: \quad A \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow B$$

with projective modules $P_1, P_2, \ldots, P_n$, and a morphism of two such complexes is a commutative diagram as follows:

$$\begin{array}{ccccccccccccc} C & & A & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_1 & \longrightarrow & B \\ \downarrow & & \| & & \downarrow & & \downarrow & & & & \downarrow & & \| \\ C' & & A & \longrightarrow & P'_n & \longrightarrow & P'_{n-1} & \longrightarrow & \cdots & \longrightarrow & P'_1 & \longrightarrow & B \end{array}$$

For objects C and C', let us write $C \approx C'$ if there is a morphism $C \to C'$ in $\mathcal{K}_n(A,B)$. Define the relation $\sim$ on $\mathcal{K}_n(A,B)$ to be the equivalence relation generated by $\approx$, and put $K_n(A,B) = \mathcal{K}_n(A,B)/\sim$, the connected components of $\mathcal{K}_n(A,B)$. We will sometimes write $K_n^{kG}(A,B)$ if we want to emphasize that we are working over the algebra kG.

Let us fix a projective resolution of B:

$$P: \qquad 0 \longrightarrow \Omega^n B \xrightarrow{i} P_n \longrightarrow P_{n-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow B \longrightarrow 0 \qquad (3.4)$$

Theorem 3.5. *The map $\Phi : \mathrm{Hom}_{kG}(A, \Omega^n B) \to \mathcal{K}_n(A,B)$ which associates to each map $f : A \to \Omega^n B$ the complex $A \xrightarrow{i \circ f} P_n \to P_{n-1} \to \cdots \to P_1 \to B$ induces a bijection $\underline{\mathrm{Hom}}_{kG}(A, \Omega^n B) \overset{1:1}{\longleftrightarrow} K_n(A,B)$.*

To prove this, we need the following lemma.

Lemma 3.6. *Suppose we are given two finite chain complexes $A = (0 \to A_{n+1} \to \cdots \to A_0 \to 0)$ and $B = (0 \to B_{n+1} \to \cdots \to B_0 \to 0)$, where A_i is projective for $i = 1, 2, \ldots, n$, and B is exact. Let $f, g : A \to B$ be chain maps satisfying $f_0 = g_0 : A_0 \to B_0$. Then the classes of f_{n+1} and g_{n+1} in $\underline{\mathrm{Hom}}_{kG}(A_{n+1}, B_{n+1})$ are the same.*

Proof. This is a standard fact from homological algebra. We can assume that $g = 0$. Then $f_0 = 0$, hence $d_B\, f_1 = 0$. Since A_1 is projective and B is exact, there exists some $h_1 : A_1 \to B_2$ such that $d_B\, h_1 = f_1$. Inductively, one can find $h_j : A_j \to B_{j+1}$ satisfying $d_B\, h_j + h_{j-1}\, d_A = f_j$ for all $j = 2, 3, \ldots, n$. But then

$$d_B\, h_n\, d_A = (f_n - h_{n-1}\, d_A)\, d_A = f_n\, d_A = d_B\, f_{n+1},$$

and since $d_B : B_{n+1} \to B_n$ is injective, we get $h_n\, d_A = f_{n+1}$. Therefore, f_{n+1} factors through a projective module (namely A_n). □

Proof of Theorem 3.5: As a first step, we show that Φ induces a map $\underline{\mathrm{Hom}}_{kG}(A, \Omega^n B) \to K_n(A, B)$. Suppose we are given $f' \in \mathrm{Hom}_{kG}(A, \Omega^n B)$ such that $f' - f$ factors through some projective module R:

$$f' - f : \qquad A \xrightarrow{u} R \xrightarrow{w} \Omega^n B$$

Then the complexes $\Phi(f)$ and $\Phi(f')$ differ in their first map only; let us denote these by $\alpha, \alpha' : A \to P_n$, respectively. From the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, equal] & P_n \arrow[r] & P_{n-1} \arrow[r] \arrow[d, equal] & P_{n-2} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & B \arrow[d, equal] \\
A \arrow[r, "(\alpha\ u)"] \arrow[d, equal] & P_n \oplus R \arrow[u, "\binom{\mathrm{id}}{0}"'] \arrow[d, "\binom{\mathrm{id}}{d o w}"] \arrow[r, "\binom{d}{0}"] & P_{n-1} \arrow[r] \arrow[d, equal] & P_{n-2} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & B \arrow[d, equal] \\
A \arrow[r, "\alpha'"] & P_n \arrow[r] & P_{n-1} \arrow[r] & P_{n-2} \arrow[r] & \cdots \arrow[r] & B
\end{tikzcd}$$

we get that $\Phi(f) \sim \Phi(f')$. Therefore, we obtain a map $\underline{\mathrm{Hom}}_{kG}(A, \Omega^n B) \to K_n(A, B)$ which we also denote by Φ.

To construct an inverse for Φ, start with some object $C = (A \to Q_* \to B) \in \mathcal{K}_n(A, B)$. Since the Q_i's are projective and (3.4) is exact, we can lift the identity on B to a map of chain complexes $f : C \to P$:

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & Q_n \arrow[r] \arrow[d] & Q_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & Q_1 \arrow[r] \arrow[d] & B \arrow[d, equal] \\
\Omega^n B \arrow[r] & P_n \arrow[r] & P_{n-1} \arrow[r] & \cdots \arrow[r] & P_1 \arrow[r] & B
\end{tikzcd}$$

By Lemma 3.6, the stable class of the resulting map $f_{n+1} : A \to \Omega^n B$ is independent of the choice of the lift; let us write $\Psi(C) = f_{n+1} \in \underline{\mathrm{Hom}}_{kG}(A, B)$. Suppose we are given a morphism $g : C' \to C$ in $\mathcal{K}_n(A, B)$. Then $f \circ g$ is a lift of the identity on B to a map of chain complexes $C' \to P$. Since $g_{n+1} = \mathrm{id}_A$, we have $\Psi(C') = (f \circ g)_{n+1} = f_{n+1} = \Psi(C)$. Therefore, we have constructed a map $\Psi : K_n(A, B) \to \underline{\mathrm{Hom}}_{kG}(A, \Omega^n B)$. The proofs of $\Psi \circ \Phi = \mathrm{id}$ and $\Phi \circ \Psi = \mathrm{id}$ are immediate. □

Let us investigate the additive structure more closely.

Lemma 3.7. *Suppose that $A \xrightarrow{i} P_n \to \cdots \to P_1 \xrightarrow{f} B$ and $A \xrightarrow{j} Q_n \to \cdots \to Q_1 \xrightarrow{g} B$ are two complexes representing classes $\kappa, \lambda \in \widehat{\mathrm{Ext}}^{-n}_{kG}(A, B)$, respectively. Then the complex*

$$A \xrightarrow{\binom{i}{j}} P_n \oplus Q_n \to P_{n-1} \oplus Q_{n-1} \to \cdots \to P_1 \oplus Q_1 \xrightarrow{(f\ g)} B$$

represents the class $\kappa + \lambda$.

Corollary 3.8. *Suppose that $A \xrightarrow{i} P_* \xrightarrow{f} B$ and $A \xrightarrow{i} P_* \xrightarrow{g} B$ are complexes representing classes $\kappa, \lambda \in \widehat{\mathrm{Ext}}^{-n}_{kG}(A, B)$. Then $A \xrightarrow{i} P_* \xrightarrow{f+g} B$ is a complex representing $\kappa + \lambda$.*

We omit the straightforward proof of the lemma; the corollary is deduced by using the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, equal] & P_* \arrow[r, "f+g"] \arrow[d, "\Delta"] & B \arrow[d, equal] \\
A \arrow[r, "\binom{i}{i}"] & P_* \oplus P_* \arrow[r, "(f\ g)"] & B
\end{tikzcd}$$

where Δ is the diagonal map.

Proposition 3.9. *Suppose we have a commutative diagram*

$$\begin{tikzcd}
 & A \arrow[r] \arrow[d, "g"'] & P_n \arrow[r] \arrow[d] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d] & B \arrow[d, "f"] & \\
0 \arrow[r] & D \arrow[r] & E_n \arrow[r] & \cdots \arrow[r] & E_1 \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

in $\mathfrak{mod}$-kG. Assume further that the P_i's are projective, so that the upper row represents some element $\alpha \in \underline{\mathrm{Hom}}_{kG}(A, \Omega^n B)$, and assume that the lower row is exact, therefore representing some element $\beta \in \underline{\mathrm{Hom}}_{kG}(\Omega^n C, D)$. Then the diagram

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "g"'] & \Omega^n B \arrow[d, "\Omega^n f"] \\
D & \Omega^n C \arrow[l, "\beta"]
\end{tikzcd}$$

commutes stably.

Proof. Choose projective resolutions $\Omega^n B \to Q_* \to B$ and $\Omega^n C \to R_* \to C$. By the usual 'projective to acyclic'-argument, we get a diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\bar{\alpha}"'] & P_* \arrow[r] \arrow[d] & B \arrow[d, equal] \\
\Omega^n B \arrow[r] \arrow[d, "\Omega^n f"'] & Q_* \arrow[r] \arrow[d] & B \arrow[d, "f"] \\
\Omega^n C \arrow[r] \arrow[d, "\bar{\beta}"'] & R_* \arrow[r] \arrow[d] & C \arrow[d, equal] \\
D \arrow[r] & E_* \arrow[r] & C
\end{tikzcd}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are unstable representatives of α and β, respectively. The result follows from Lemma 3.6. $\square$

Remark 3.10. Suppose we have an exact sequence $A \hookrightarrow P_n \to \cdots \to P_1 \twoheadrightarrow B$ with projective modules $P_1, \ldots, P_n$. Then we can view this as an extension representing some stable isomorphism $\Omega^n B \to A$; but we can also consider this as an element of $\mathcal{K}_n(A, B)$, representing some stable isomorphism $A \to \Omega^n B$; by the previous proposition, the two maps are stable inverses of each other.

We have a composition product $\mathcal{K}_n(B, C) \times \mathcal{K}_m(A, B) \to \mathcal{K}_{n+m}(A, C)$ similar to the Yoneda splice: given $E : A \to P_* \to B$ and $E' : B \to Q_* \to C$ we define $E' \circ E$ to be the complex

$$E' \circ E : \quad \begin{tikzcd}
A \arrow[r] & P_* \arrow[rr] \arrow[dr] & & Q_* \arrow[r] & C. \\
& & B \arrow[ur] & &
\end{tikzcd}$$

This product is compatible with the equivalence relation $\sim$ and therefore induces a product

$$K_n(B, C) \times K_m(A, B) \to K_{n+m}(A, C).$$

Lemma 3.11. *The composition products on K_* and $\widehat{\operatorname{Ext}}_{kG}^{-*}$ coincide under the bijection of Theorem 3.5.*

Proof. Let us start with complexes $A \to P_* \to B$ and $B \to Q_* \to C$ representing stable maps $\alpha : A \to \Omega^m B$ and $\beta : B \to \Omega^n C$, respectively. Choose projective resolutions $\Omega^n C \to R_* \to C$ and $\Omega^{n+m} C \to T_* \to \Omega^n C$. Then we can lift the identity map on C to commutative diagrams as follows:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\bar{\gamma}"'] & P_* \arrow[r] \arrow[d] & B \arrow[d, "\bar{\beta}"] \\
\Omega^{n+m} C \arrow[r] & T_* \arrow[r] & \Omega^n C
\end{tikzcd} \qquad \begin{tikzcd}
B \arrow[r] \arrow[d, "\bar{\beta}"'] & Q_* \arrow[r] \arrow[d] & C \arrow[d, equal] \\
\Omega^n C \arrow[r] & R_* \arrow[r] & C
\end{tikzcd}$$

Here, $\bar{\beta}$ and $\bar{\gamma}$ are unstable representatives of β and some γ. Note that the extension $\Omega^{n+m} C \to T_* \to \Omega^n C$ represents the identity map $\mathrm{id} \in \underline{\mathrm{Hom}}_{kG}(\Omega^m \Omega^n C, \Omega^{n+m} C)$. By Proposition 3.9, the left diagram shows that $\gamma = \beta\alpha$. After splicing the two diagrams the result follows from Lemma 3.6. $\square$

There is also a way of composing an element $x \in \widehat{\operatorname{Ext}}^{-n}_{kG}(A, B)$ given as a complex $A \to P_* \to B$ with an element of $y \in \widehat{\operatorname{Ext}}^{m}_{kG}(B, C)$ (with $m > 0$) given as an extension $C \hookrightarrow M_* \twoheadrightarrow B$:

Lemma 3.12. *Suppose $m < n$. The identity map of B can be lifted to a diagram*

$$\begin{tikzcd}
A \arrow[r] & P_n \arrow[r] & \cdots \arrow[r] & P_{m+1} \arrow[r] \arrow[d] & P_m \arrow[r] \arrow[d] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d] & B \arrow[d, equal] \\
 & & & C \arrow[r] & M_m \arrow[r] & \cdots \arrow[r] & M_1 \arrow[r] & B
\end{tikzcd}$$

and for any such lifting, the complex $A \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_{m+1} \longrightarrow C$ represents the composition $y \cdot x \in \widehat{\operatorname{Ext}}^{m-n}_{kG}(A, C)$.

Proof. Existence of the lifting is common homological algebra. For the second statement choose a projective resolution $\Omega^{n-m}C \to R_* \to C$; then we have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[dd, "\bar{\gamma}"] & P_n \arrow[r] \arrow[dd] & \cdots \arrow[r] & P_{m+1} \arrow[rr] \arrow[dd] \arrow[dr] & & P_m \arrow[r] \arrow[dd] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[dd] & B \arrow[dd, equal] \\
 & & & & C \arrow[dr] & & & & \\
\Omega^{n-m}C \arrow[r] & R_{n-m} \arrow[r] & \cdots \arrow[r] & R_1 \arrow[rr] \arrow[ur] & & M_m \arrow[r] & \cdots \arrow[r] & M_1 \arrow[r] & B
\end{tikzcd}$$

The complex in question represents the stable class of the map $\bar{\gamma}$. The bottom row represents $y \in \underline{\operatorname{Hom}}_{kG}(\Omega^n B, \Omega^{n-m}C)$, the upper row represents $x \in \underline{\operatorname{Hom}}_{kG}(A, \Omega^n B)$. The result follows from Proposition 3.9. $\square$

Remark 3.13. We can also compose an element $x \in \widehat{\operatorname{Ext}}^{0}_{kG}(A, B) = \underline{\operatorname{Hom}}_{kG}(A, B)$ given by an unstable map $f : A \to B$ with an element $y \in \widehat{\operatorname{Ext}}^{-n}_{kG}(B, C)$ given by a complex $B \xrightarrow{i} P_* \xrightarrow{\pi} C$; the composition $y \cdot x \in \widehat{\operatorname{Ext}}^{-n}_{kG}(A, C)$ is then given by the complex $A \xrightarrow{i \circ f} P_* \xrightarrow{\pi} C$. Similarly, for any $z \in \widehat{\operatorname{Ext}}^{0}_{kG}(C, D)$ represented by some unstable map $g : C \to D$, the complex $B \xrightarrow{i} P_* \xrightarrow{g \circ \pi} D$ represents the product $z \cdot y \in \widehat{\operatorname{Ext}}^{-n}_{kG}(B, D)$. In fact, these statements are special cases of the following Proposition.

Proposition 3.14. *Suppose that we have a commutative diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & P_n \arrow[r] \arrow[d] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d] & B \arrow[d, "g"] \\
A' \arrow[r] & Q_n \arrow[r] & \cdots \arrow[r] & Q_1 \arrow[r] & B'
\end{tikzcd}$$

with projective modules P_i, Q_i for $i = 1, 2, \ldots, n$. Then the rows represent maps $x : A \to \Omega^n B$ and $y : A' \to \Omega^n B'$, respectively, and $y \circ f = \Omega^n(g) \circ x$ in $\underline{\mathfrak{mod}}$-$kG$.

Proof. Choose a projective resolution $\Omega^n B' \to R_* \to B'$. By usual homological algebra, we get a diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f'"'] & P_* \arrow[r] \arrow[d] & B \arrow[d, "g"] \\
A' \arrow[r] \arrow[d, "y"'] & Q_* \arrow[r] \arrow[d] & B' \arrow[d, equal] \\
\Omega^n B' \arrow[r] & R_* \arrow[r] & B'
\end{tikzcd}$$

and then the result follows from Proposition 3.9. □

Remark 3.15. There is a similar statement for extensions. Suppose that we have a diagram as in Proposition 3.14, but this time with exact rows and the P_i's and Q_i's are not necessarily projective. Then the rows represent maps $x : \Omega^n B \to A$ and $y : \Omega^n B' \to A'$, and $f \circ x = y \circ \Omega^n(g)$ in $\underline{\mathfrak{mod}}$-$kG$.

We end this part by mentioning that the bijection of Theorem 3.5 is natural with respect to injective group homomorphisms. Suppose that H is another group and $i : H \hookrightarrow G$ is an injective group homomorphism. Then the functor $i^* : \mathfrak{mod}\text{-}kG \to \mathfrak{mod}\text{-}kH$ is exact and maps projective kG-modules to projective kH-modules. This implies that $i^*(\Omega^n_G X) \cong \Omega^n_H(i^* X)$ in $\underline{\mathfrak{mod}}$-$kH$ for every kG-module X. Also, we get induced maps $K_n^{kG}(A, B) \to K_n^{kH}(i^*A, i^*B)$ and $\underline{\mathrm{Hom}}_{kG}(A, \Omega^n B) \to \underline{\mathrm{Hom}}_{kH}(i^*A, \Omega^n(i^*B))$. Naturality of the map Φ then shows the following lemma.

Lemma 3.16. *The diagram*

$$\begin{tikzcd}
K_n^{kG}(A, B) \arrow[r, leftrightarrow, "1:1"] \arrow[d] & \underline{\mathrm{Hom}}_{kG}(A, \Omega^n B) \arrow[d] \\
K_n^{kH}(i^*A, i^*B) \arrow[r, leftrightarrow, "1:1"'] & \underline{\mathrm{Hom}}_{kH}(i^*A, \Omega^n(i^*B))
\end{tikzcd}$$

commutes.

Example 3.17. Suppose that p divides the order of the group G. Then it is known that $\hat{H}^{-1}(G) \cong \underline{\mathrm{Hom}}_{kG}(k, \Omega k)$ is isomorphic to k. Under the bijection of Theorem 3.5, a canonical generator of that vector space is given by the complex

$$k \xrightarrow{\sum_{g \in G} g} kG \xrightarrow{\epsilon} k$$

where ϵ is the augmentation of kG. The previous lemma shows that this complex is invariant under every group automorphism of G.

Remark 3.18. The previous lemma generalizes to those morphisms $C \to C'$ of Hopf algebras for which C' becomes a projective C-module. In the case when $C = kH$ and $C' = kG$ are group algebras and the morphism comes from a morphism $f : H \to G$ of groups, it is a nice exercise to show that kG is projective as a kH-module if and only if p does not divide the order of the kernel of f.

3.2 A map of cochain complexes

Let X be any kG-module. Consider the complex

$$\mathcal{S}: \qquad X^{\otimes p} \xrightarrow{1-T} X^{\otimes p} \xrightarrow{N} X^{\otimes p} \xrightarrow{1-T} X^{\otimes p} \xrightarrow{N} \dots \xrightarrow{N} X^{\otimes p} \xrightarrow{1-T} X^{\otimes p} \tag{3.19}$$

consisting of $2p-2$ objects and $2p-3$ morphisms, which we refer to as the *Steenrod complex*. On the other hand, we can splice the complexes (1.11) and (1.8) to obtain the *de Rham-Koszul* complex

$$\begin{tikzcd}[column sep=small] \mathcal{RK}: & X \otimes S^{p-1}X \arrow[r] & \cdots \arrow[r] & \Lambda^{p-1}X \otimes X \arrow[r] \arrow[dr, dotted] & \Lambda^{p-1}X \otimes X \arrow[r] & \cdots \arrow[r] & X \otimes S^{p-1}X \\ & & & & \Lambda^p X \arrow[u, dotted] & & \end{tikzcd}$$

also consisting of $2p-2$ objects and $2p-3$ morphisms. We consider both complexes $\mathcal{S}$ and $\mathcal{RK}$ as cochain complexes.

Proposition 3.20. *There is a natural map of cochain complexes $\beta : \mathcal{S} \to \mathcal{RK}$ such that*

(i) *the last map $\beta_{2p-2} : X^{\otimes p} \to X \otimes S^{p-1}X$ is given by $x_1 \otimes x_2 \otimes \cdots \otimes x_p \mapsto x_1 \otimes x_2x_3 \cdots x_p$, and*

(ii) *the first map $\beta_1 : X^{\otimes p} \to X \otimes S^{p-1}X$ is given by $\beta_1 = -\beta_{2p-2}$.*

Proof. The statement is obvious for $p = 2$, so let us assume $p \geq 3$ from now on. Because of naturality, we can forget the G-module structures and consider this as a statement on k-vector spaces X. Let us recall some standard notions. Suppose that X, Y are k-vector spaces, and let H be a group acting on X from the right and on Y from the left. Then we have the usual definitions

$$\begin{aligned} X \otimes_H Y &= X \otimes Y / \langle (x \cdot h) \otimes y - x \otimes (h \cdot y) \mid x \in X, y \in Y, h \in H \rangle_k, \\ X/H &= X / \langle x \cdot h - x \mid x \in X, h \in H \rangle_k \cong X \otimes_H k, \end{aligned}$$

where k denotes the trivial H-module. Whenever X is a right H-module and $H_1 \subset H$ is a subgroup, then X is a right H_1-module and there is a natural map $X/H_1 \to X/H$. For every vector space X, the symmetric group Σ_n acts on $X^{\otimes n}$ from the right by permuting the factors. The cyclic subgroup $C_n = \mathbb{Z}/n\mathbb{Z} \subset \Sigma_n$ acts by cyclic permutation of the factors, and we define the *cyclic product* to be the quotient $C^n(X) = X^{\otimes n}/C_n$. Then we have a natural map $C^n(X) = X^{\otimes n}/C_n \to X^{\otimes n}/\Sigma_n = S^n(X)$.

All these notions generalize naturally to (co)chain complexes C, where one has to note that the action of Σ_n on $C^{\otimes n}$ involves signs depending on the degrees of the permuted elements. For instance, if $C = X[1]$ is the chain complex with X concentrated in degree 1, then $S^n(C) = (\Lambda^n X)[n]$ (here we use that p is odd), and $C^n(C) = (C^n(X))[n]$, so there is also a natural map $C^n(X) \to \Lambda^n X$.

We will use the following result of Swan. Let m be any integer, and define M to be the cochain complex over k generated by two elements x, y of degrees $m+1$ and m, respectively, subject to the condition $\partial y = x$. That is, M is isomorphic to $\cdots \to 0 \to k \xrightarrow{\mathrm{id}} k \to 0 \to \dots$.

Proposition 3.21 (Swan [27], Lemma 22.2 and Remark 22.1)**.** *Let $\mathcal{X}$ be the kC_p-cochain complex given by the sequence*

$$0 \to k \xrightarrow{N} kC_p \xrightarrow{1-T} kC_p \xrightarrow{N} \dots \xrightarrow{1-T} kC_p \xrightarrow{\varepsilon} k \to 0,$$

where T is a generator of C_p and $N = 1 + T + \dots + T^{p-1}$, and the k's are in degrees mp and $mp+p$. Then there is a C_p-equivariant homotopy equivalence $\psi : \mathcal{X} \to M^{\otimes p}$ such that $\psi_{mp+p}(1) = x^{\otimes p}$ and $\psi_{mp}(1) = c \cdot y^{\otimes p}$ with the constant $c = (-1)^{m \cdot \frac{p-1}{2}} \left(\frac{p-1}{2}\right)!$.

It follows from the proposition's proof that we can impose one of the following two extra conditions on ψ: either $\psi_{mp+p-1}(1) = y \otimes x^{\otimes(p-1)}$ or $\psi_{mp+1}(1) = c \cdot x \otimes y^{\otimes(p-1)}$.

Now let X be any k-vector space. Since $\psi : \mathcal{X} \to M^{\otimes p}$ is C_p-equivariant, we can form the composite

$$\mathcal{X} \otimes_{C_p} X^{\otimes p} \xrightarrow{\psi \otimes_{C_p} \mathrm{id}} M^{\otimes p} \otimes_{C_p} X^{\otimes p} \to M^{\otimes p} \otimes_{\Sigma_p} X^{\otimes p} \cong (M \otimes X)^{\otimes p}/\Sigma_p \cong S^p(M \otimes X).$$

If we now put $m = 0$, then this composite yields a map of cochain complexes

$$\begin{tikzcd}
C^pX \arrow[r, "N"] \arrow[d] & X^{\otimes p} \arrow[r, "1-T"] \arrow[d] & X^{\otimes p} \arrow[r, "N"] \arrow[d] & \cdots \arrow[r, "1-T"] & X^{\otimes p} \arrow[r] \arrow[d] & C^pX \arrow[d] \\
\Lambda^pX \arrow[r] & \Lambda^{p-1}X \otimes X \arrow[r] & \Lambda^{p-2}X \otimes S^2X \arrow[r] & \cdots \arrow[r] & X \otimes S^{p-1}X \arrow[r] & S^pX
\end{tikzcd} \tag{3.22}$$

where we can assume that $X^{\otimes p} \to X \otimes S^{p-1}X$ is the map β_{2p-2}. On the other hand we can put $m = 1$ to get

$$\begin{tikzcd}
C^pX \arrow[r, "N"] \arrow[d] & X^{\otimes p} \arrow[r, "1-T"] \arrow[d] & X^{\otimes p} \arrow[r, "N"] \arrow[d] & \cdots \arrow[r, "1-T"] & X^{\otimes p} \arrow[r] \arrow[d] & C^pX \arrow[d] \\
S^pX \arrow[r] & X \otimes S^{p-1}X \arrow[r] & \Lambda^2X \otimes S^{p-2}X \arrow[r] & \cdots \arrow[r] & \Lambda^{p-1}X \otimes X \arrow[r] & \Lambda^pX
\end{tikzcd} \tag{3.23}$$

with the map $X^{\otimes p} \to X \otimes S^{p-1}X$ being equal to β_1. By splicing the two complexes (3.22) and (3.23) after multiplying with suitable non-zero constants we obtain a proof of Proposition 3.20. □

Remark 3.24. To determine the constant c in Proposition 3.21 it seems to be necessary to write down a suitable cochain map ψ explicitly. This is not done in Swan [27], and we will also not do it here. In an earlier proof of Proposition 3.20, the author of this thesis constructed the cochain map β explicitly, thereby showing that the multiplicative constants are indeed as given. However, the proof is rather lengthy and not very enlightening, which is why we omit it here. For the main results of this thesis it is actually enough to know that β_1 is a *non-zero* multiple of β_{2p-2}, which follows from Swan's proof. Then, certain diagrams will only commute up to non-zero scalars, but this is enough for our purposes.

3.3 Definitions and basic properties

Now we give the definition of the power operation we are interested in. Let n be an integer. We assume that n is even if p is odd. Let $[\zeta] \in \hat{H}^n(G)$ be a Tate cohomology class; as a first step, we will define a subset $\mathcal{D}_i([\zeta])$ of $\hat{H}^{pn-i}(G)$ for all positive integers $i = 1, 2, \ldots$

To do so, choose an unstable map $\zeta : \Omega^n k \to k$ representing the class $[\zeta]$, and consider all commutative diagrams of the form

$$\begin{tikzcd}
(\Omega^n k)^{\otimes p} \arrow[r] \arrow[d, equal] & P_1 \arrow[r] \arrow[d] & P_2 \arrow[r] \arrow[d] & \cdots \arrow[r] & P_i \arrow[r] \arrow[d] & k \arrow[d, equal] \\
(\Omega^n k)^{\otimes p} \arrow[r, "1-T"'] & (\Omega^n k)^{\otimes p} \arrow[r, "N"'] & (\Omega^n k)^{\otimes p} \arrow[r, "1-T"'] & \cdots \arrow[r, "(*)"'] & (\Omega^n k)^{\otimes p} \arrow[r, "\zeta^{\otimes p}"'] & k
\end{tikzcd} \tag{3.25}$$

in which the upper row is a complex with projective modules P_j. It therefore defines an element of $\widehat{\operatorname{Ext}}^{-i}_{kG}\big((\Omega^n k)^{\otimes p}, k\big) \cong \hat{H}^{pn-i}(G)$. The set of all elements obtained this way is denoted by $\mathcal{D}_i(\zeta)$. Note that the map $(*)$ in the diagram is either N or $1 - T$, depending on the parity of i.

Lemma 3.26. *The set $\mathcal{D}_i(\zeta)$ defined above does not depend on the chosen representative ζ for the cohomology class $[\zeta]$. We can therefore write $\mathcal{D}_i([\zeta]) = \mathcal{D}_i(\zeta)$.*

Proof. If we assume that $\Omega^n k$ arises from a minimal resolution $\cdots \to P_n \to \ldots$, then we know that $\operatorname{PHom}_{kG}(\Omega^n k, k) = 0$. To see this, note that every morphism $\Omega^n k \to k$ which factors through a projective module also factors as $\Omega^n k \hookrightarrow P_{n-1} \to k$. Since we have chosen a minimal resolution, the differential of the complex $\operatorname{Hom}_{kG}(P_*, k)$ vanishes and therefore the composition $P_n \to P_{n-1} \to k$ is zero. From the surjectivity of $P_n \to \Omega^n k$ we then get that our original map must be zero. Therefore, $\operatorname{PHom}_{kG}(\Omega^n k, k) = 0$ and there is no ambiguity in choosing ζ.

Now suppose we have another version of $\Omega^n k$ and call it $\widetilde{\Omega^n} k$, and let $\tilde{\zeta} : \widetilde{\Omega^n} k \to k$ be a map representing the same cohomology class as ζ. Then we have that $\widetilde{\Omega^n} k \cong \Omega^n k \oplus R$ for some projective module R. We have retraction maps $\Omega^n k \xrightarrow{\iota} \widetilde{\Omega^n} k \xrightarrow{r} \Omega^n k$ and we know that $\tilde{\zeta}\iota = \zeta$. We want to show that the sets $\mathcal{D}_i(\zeta)$ and $\mathcal{D}_i(\tilde{\zeta})$ agree under the isomorphisms

$$\widehat{\operatorname{Ext}}^{-i}_{kG}((\Omega^n k)^{\otimes p}, k) \underset{(\iota^{\otimes p})^*}{\overset{(r^{\otimes p})^*}{\rightleftarrows}} \widehat{\operatorname{Ext}}^{-i}_{kG}((\widetilde{\Omega^n} k)^{\otimes p}, k) .$$

Let us suppress these isomorphisms from the notation and simply prove $\mathcal{D}_i(\tilde{\zeta}) = \mathcal{D}_i(\zeta)$.

Note that $(\widetilde{\Omega^n} k)^{\otimes p} \cong (\Omega^n k)^{\otimes p} \oplus S$ for some projective module S, and under this isomorphism, the map T decomposes as the direct sum of the map T on $(\Omega^n k)^{\otimes p}$ and some map on S. This implies that there is an isomorphism of complexes

$$\begin{tikzcd}
(\widetilde{\Omega^n} k)^{\otimes p} \arrow[r] \arrow[d] & (\widetilde{\Omega^n} k)^{\otimes p} \arrow[r] \arrow[d] & \cdots \arrow[r] & (\widetilde{\Omega^n} k)^{\otimes p} \arrow[r, "\tilde{\zeta}^{\otimes p}"] \arrow[d] & k \arrow[d, equal] \\
(\Omega^n k)^{\otimes p} \oplus S \arrow[r] & (\Omega^n k)^{\otimes p} \oplus S \arrow[r] & \cdots \arrow[r] & (\Omega^n k)^{\otimes p} \oplus S \arrow[r, "(\zeta^{\otimes p}, \alpha)"'] & k
\end{tikzcd}$$

where the lower row (without the last map) is a direct sum of two complexes.

Now suppose we have a diagram

$$\begin{tikzcd}
(\widetilde{\Omega^n k})^{\otimes p} \arrow[r, "d_0"] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d, "{(f_1,g_1)}"'] & \cdots \arrow[r, "d_{i-1}"] & P_i \arrow[r, "d_i"] \arrow[d, "{(f_i,g_i)}"'] & k \arrow[d, equal] \\
(\Omega^n k)^{\otimes p} \oplus S \arrow[r] & (\Omega^n k)^{\otimes p} \oplus S \arrow[r] & \cdots \arrow[r] & (\Omega^n k)^{\otimes p} \oplus S \arrow[r] & k
\end{tikzcd}$$

defining some element in $\mathcal{D}_i(\tilde{\zeta})$. Then we obtain a diagram

$$\begin{tikzcd}
(\Omega^n k)^{\otimes p} \arrow[r, "d_0 \circ \iota^{\otimes p}"] \arrow[d] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1"'] & \cdots \arrow[r, "d_{i-1}"] & P_i \arrow[r, "d_i - \alpha g_i"] \arrow[d, "f_i"'] & k \arrow[d, equal] \\
(\Omega^n k)^{\otimes p} \arrow[r] & (\Omega^n k)^{\otimes p} \arrow[r] & \cdots \arrow[r] & (\Omega^n k)^{\otimes p} \arrow[r, "\zeta"'] & k
\end{tikzcd}$$

defining some element in $\mathcal{D}_i(\zeta)$. By Corollary 3.8 the difference of that element and the element we started with is the cohomology class represented by the upper row of the diagram

$$\begin{tikzcd}
(\Omega^n k)^{\otimes p} \arrow[r] \arrow[d, "0"'] & P_1 \arrow[r] \arrow[d, "g_1"'] & \cdots \arrow[r] & P_i \arrow[r, "\alpha g_i"] \arrow[d, "g_i"'] & k \arrow[d, equal] \\
S \arrow[r] & S \arrow[r] & \cdots \arrow[r] & S \arrow[r, "\alpha"'] & k
\end{tikzcd}$$

and since the left-most vertical map is zero, we get that this difference vanishes by Proposition 3.14.

Conversely, let us start with some diagram

$$\begin{tikzcd}
(\Omega^n k)^{\otimes p} \arrow[r] \arrow[d, equal] & P_1 \arrow[r] \arrow[d, "f_1"'] & \cdots \arrow[r] & P_i \arrow[r, "g"] \arrow[d, "f_i"'] & k \arrow[d, equal] \\
(\Omega^n k)^{\otimes p} \arrow[r] & (\Omega^n k)^{\otimes p} \arrow[r] & \cdots \arrow[r] & (\Omega^n k)^{\otimes p} \arrow[r, "\zeta"'] & k
\end{tikzcd}$$

defining some element of $\mathcal{D}_i(\zeta)$; then we obtain a diagram

$$\begin{tikzcd}
(\widetilde{\Omega^n k})^{\otimes p} \cong (\Omega^n k)^{\otimes p} \oplus S \arrow[r] \arrow[d, equal] & P_1 \oplus S \arrow[r] \arrow[d, "f_1 \oplus \mathrm{id}"'] & \cdots \arrow[r] & P_i \oplus S \arrow[r, "{(g,\alpha)}"] \arrow[d, "f_i \oplus \mathrm{id}"'] & k \arrow[d, equal] \\
(\widetilde{\Omega^n k})^{\otimes p} \cong (\Omega^n k)^{\otimes p} \oplus S \arrow[r] & (\Omega^n k)^{\otimes p} \oplus S \arrow[r] & \cdots \arrow[r] & (\Omega^n k)^{\otimes p} \oplus S \arrow[r] & k
\end{tikzcd}$$

defining an element in $\mathcal{D}_i(\tilde{\zeta})$ which is the same as we started with because of the following commutative diagram:

$$\begin{tikzcd}
(\Omega^n k)^{\otimes p} \arrow[r] \arrow[d, "\iota^{\otimes p}"] & P_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & P_i \arrow[r, "g"] \arrow[d] & k \arrow[d, equal] \\
(\widetilde{\Omega^n k})^{\otimes p} \cong (\Omega^n k)^{\otimes p} \oplus S \arrow[r] & P_1 \oplus S \arrow[r] & \cdots \arrow[r] & P_i \oplus S \arrow[r, "{(g,\alpha)}"'] & k
\end{tikzcd}$$

$\square$

We could have equally well started with a slightly modified definition:

Lemma 3.27. *Every class in $\mathcal{D}_i(\zeta)$ comes from a diagram of the form* (3.25) *in which the sequence* $0 \to (\Omega^n k)^{\otimes p} \to P_1 \to \cdots \to P_i$ *is exact.*

Proof. Choose an injective resolution $(\Omega^n k)^{\otimes p} \hookrightarrow Q_1 \to \cdots \to Q_i$. Given any diagram of the form (3.25), we can lift the identity map on $(\Omega^n k)^{\otimes p}$ to a map of complexes $Q_* \to P_*$ (because the P_j's are injective modules and the resolution is an exact sequence). Finally define $Q_i \to k$ to be the composite $Q_i \to P_i \to k$. □

An immediate consequence is the following lower bound on the indeterminacy of $\mathcal{D}_i(\zeta)$.

Corollary 3.28. *If $a \in \mathcal{D}_i(\zeta)$, then $a + \zeta^p \cdot \hat{H}^{-i}(G) \subseteq \mathcal{D}_i(\zeta)$.*

Proof. We assume that a is given by a diagram

$$\begin{tikzcd}
(\Omega^n k)^{\otimes p} \arrow[r] \arrow[d, equal] & P_1 \arrow[r] \arrow[d] & \cdots \arrow[r, "d_{i-1}"] & P_i \arrow[r, "g"] \arrow[d, "f_i"'] & k \arrow[d, equal] \\
(\Omega^n k)^{\otimes p} \arrow[r] & (\Omega^n k)^{\otimes p} \arrow[r] & \cdots \arrow[r] & (\Omega^n k)^{\otimes p} \arrow[r, "\zeta^{\otimes p}"'] & k
\end{tikzcd}$$

in which the upper row is as in Lemma 3.27. Then the cokernel of d_{i-1} is a choice of $\Sigma^i(\Omega^n k)^{\otimes p}$. If we are given some class $b \in \hat{H}^{-i}(G)$ represented by some unstable map $h : \operatorname{coker} d_{i-1} \to (\Omega^n k)^{\otimes p}$ we can form the composition $w : P_i \to \operatorname{coker} d_{i-1} \xrightarrow{h} (\Omega^n k)^{\otimes p}$. Replace f_i by $f_i + w$ and g by $g + \zeta^{\otimes p} \circ w$, then we obtain a new diagram showing that $a + \zeta^p b \in \mathcal{D}_i(\zeta)$. □

Let us fix some non-zero map $\zeta : \Omega^n k \to k$. Let $A_1 \to A_2 \to \cdots \to A_{2p-2}$ be the de Rham-Koszul complex introduced in §3.2, so that $A_j = A_{2p-1-j} = \Lambda^j(\Omega^n k) \otimes S^{p-j}(\Omega^n k)$ for all $j = 1, 2, \ldots, p-1$. Furthermore, let K be the kernel of the surjective map $\zeta^{\otimes p} : (\Omega^n k)^{\otimes p} \to k$. Then we get an exact triangle

$$\Omega k \to K \to (\Omega^n k)^{\otimes p} \to k.$$

Let us denote the stable map $\Omega k \to K$ by υ. We also get a map $\lambda : A_{2p-3} \to K$ from the commutative diagram in $\mathfrak{mod}$-kG

$$\begin{tikzcd}
A_{2p-3} \arrow[r] \arrow[d, dashed, "\lambda"'] & A_{2p-2} = \Omega^n k \otimes S^{p-1}(\Omega^n k) \arrow[r] \arrow[d] & S^p(\Omega^n k) \arrow[d, "S^p\zeta"] \\
K \arrow[r] & (\Omega^n k)^{\otimes p} \arrow[r, "\zeta^{\otimes p}"] & k
\end{tikzcd}$$

in which the middle vertical map is the tensor product of the identity with the canoncial inclusion (1.18).

Lemma 3.29. *If $i < 2p-3$ then the composition $(\Omega^n k)^{\otimes p} \xrightarrow{a} \Omega^i k \xrightarrow{\Omega^{i-1}\upsilon} \Omega^{i-1} K$ vanishes for every $a \in \mathcal{D}_i(\zeta)$. For $i = 2p-3$, the composition equals the composition of the canonical map $(\Omega^n k)^{\otimes p} \to \Omega^n k \otimes S^{p-1}(\Omega^n k) = A_1$, multiplied by a minus sign, with the class represented by the complex $A_1 \to A_2 \to \cdots \to A_{2p-3} \xrightarrow{\lambda} K$.*

Corollary 3.30. *Suppose that $i \leq 2p-3$, and let us assume that $\mathcal{D}_i(\zeta) \neq \emptyset$. Then $\mathcal{D}_i(\zeta)$ is a coset of $\zeta^p \cdot \hat{H}^{-i}(G)$ in $\hat{H}^{pn-i}(G)$, and $0 \in \mathcal{D}_i(\zeta)$ unless $i = 2p-3$.*

Proof of the Corollary. From the lemma we know that the composition

$$(\Omega^n k)^{\otimes p} \xrightarrow{a} \Omega^i k \xrightarrow{\Omega^{i-1}\upsilon} \Omega^i K$$

does not depend on the choice of $a \in \mathcal{D}_i(\zeta)$. Therefore, if we are given $a, a' \in \mathcal{D}_i(\zeta)$, then $a - a'$ vanishes when postcomposed with $\Omega^{i-1}\upsilon$. Thus, $a - a'$ factors as $(\Omega^n k)^{\otimes p} \to \Omega^i(\Omega^n k)^{\otimes p} \xrightarrow{\zeta^{\otimes p}} \Omega^i k$, which proves $a - a' \in \zeta^p \cdot \hat{H}^{-i}(G)$. Together with Corollary 3.28 we get the first part. If $i < 2p-3$ then the composite $(\Omega^n k)^{\otimes p} \xrightarrow{a} \Omega^i k \xrightarrow{\Omega^{i-1}\upsilon} \Omega^i K$ vanishes for $a \in \mathcal{D}_i(\zeta)$; thus a is divisible by ζ^p which implies $0 \in \mathcal{D}_i(\zeta)$. $\square$

We will show in Proposition 3.47 that $\mathcal{D}_i(\zeta)$ is non-empty for all positive integers i.

Proof of Lemma 3.29. Let $a \in \mathcal{D}_i(\zeta)$ be defined by the first two rows of the following diagram:

$$\begin{tikzcd}
(\Omega^n k)^{\otimes p} \arrow[r] \arrow[d, equal] & P_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & P_{i-1} \arrow[r] \arrow[d] & P_i \arrow[r] \arrow[d] & k \arrow[d, equal] \\
(\Omega^n k)^{\otimes p} \arrow[r] \arrow[d, "\gamma"'] & (\Omega^n k)^{\otimes p} \arrow[r] \arrow[d] & \cdots \arrow[r] & (\Omega^n k)^{\otimes p} \arrow[r] \arrow[d] & (\Omega^n k)^{\otimes p} \arrow[r] \arrow[d, "\beta"'] & k \arrow[dd, equal] \\
A_{2p-2-i} \arrow[r] & A_{2p-1-i} \arrow[r] & \cdots \arrow[r] & A_{2p-3} \arrow[r] \arrow[d, "\lambda"'] & A_{2p-2} \arrow[d] \arrow[r, phantom, "(*)"] & \\
 & & & K \arrow[r] & (\Omega^n k)^{\otimes p} \arrow[r] & k
\end{tikzcd} \tag{3.31}$$

The connection between the second and the third row is given by the cochain map constructed in Proposition 3.20. In particular, β is the canonical map, and if $i = 2p-3$ then also $-\gamma$ is the canonical map. The diagram $(*)$ commutes due to the computation

$$u \otimes u_1 \otimes \cdots \otimes u_{p-1} \mapsto u \otimes u_1 u_2 \ldots u_{p-1} \mapsto \frac{1}{(p-1)!} \sum_{\sigma \in \Sigma_{p-1}} u \otimes u_{\sigma_1} \otimes \cdots \otimes u_{\sigma_{p-1}}$$
$$\mapsto \zeta(u)\zeta(u_1)\ldots\zeta(u_{p-1}).$$

Therefore, the diagram (3.31) commutes. Since the bottom row represents υ, Lemma 3.12 says that the composition $(\Omega^n k)^{\otimes p} \xrightarrow{a} \Omega^i k \xrightarrow{\Omega^{i-1}\upsilon} \Omega^{i-1} K$ is represented by the complex

$$(\Omega^n k)^{\otimes p} \to P_1 \to \cdots \to P_{i-1} \to K,$$

which in turn by Proposition 3.14 is the same as the composition of γ with the class in $\widehat{\mathrm{Ext}}_{kG}^{-i+1}(A_{2p-2-i}, K)$ represented by the complex

$$A_{2p-2-i} \to A_{2p-1-i} \to \cdots \to A_{2p-3} \to K.$$

If $i < 2p-3$ the latter class vanishes because A_{2p-2-i} is projective. $\square$

We conclude this section with the proof of two nice properties of the $\mathcal{D}_i$'s.

Lemma 3.32. *For all ζ and all $c \in k$ we have that $\mathcal{D}_i(c \cdot \zeta) = c^p \cdot \mathcal{D}_i(\zeta)$. Let us assume that $\mathcal{D}_{i+1}(\xi) \neq \emptyset$ for some ξ. Then the operation $\mathcal{D}_i$ is additive in the sense that $\mathcal{D}_i(\zeta + \varphi) \subseteq \mathcal{D}_i(\zeta) + \mathcal{D}_i(\varphi)$ for all ζ, φ of the same degree.*

Proof. Note that the statements given fit with the indeterminacy of the $\mathcal{D}_j$'s. The first statement is an immediate consequence of the definition. We will prove the second statement for odd numbers i only; the proof for even numbers i is similar. Take any diagram of the form

$$\begin{tikzcd}
(\Omega^n k)^{\otimes p} \arrow[r] \arrow[d, equal] & P_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & P_i \arrow[r] \arrow[d] & P_{i+1} \arrow[d] \\
(\Omega^n k)^{\otimes p} \arrow[r] & (\Omega^n k)^{\otimes p} \arrow[r] & \cdots \arrow[r] & (\Omega^n k)^{\otimes p} \arrow[r] & (\Omega^n k)^{\otimes p}
\end{tikzcd}$$

which exists due to our assumption. It is enough to show that the upper row of the diagram

$$\begin{tikzcd}
(\Omega^n k)^{\otimes p} \arrow[r] \arrow[d, equal] & P_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & P_i \arrow[r] \arrow[d] & k \arrow[d, equal] \\
(\Omega^n k)^{\otimes p} \arrow[r] & (\Omega^n k)^{\otimes p} \arrow[r] & \cdots \arrow[r, "1-T"'] & (\Omega^n k)^{\otimes p} \arrow[r, "\lambda"'] & k
\end{tikzcd}$$

represents the zero map, where $\lambda = (\zeta + \varphi)^{\otimes p} - \zeta^{\otimes p} - \varphi^{\otimes p}$. We claim that λ factors as $(\Omega^n k)^{\otimes p} \xrightarrow{N} (\Omega^n k)^{\otimes p} \to k$. To see this, note that λ is the sum over all p-fold tensor products of ζ's and φ's except $\zeta^{\otimes p}$ and $\varphi^{\otimes p}$, and the $\mathbb{Z}/p\mathbb{Z}$-action (by cyclic permutation) on these tensor products is free. Therefore, we get a diagram as follows:

$$\begin{tikzcd}
(\Omega^n k)^{\otimes p} \arrow[r] \arrow[dd, equal] & P_1 \arrow[r] \arrow[dd] & \cdots \arrow[r] & P_i \arrow[rr] \arrow[dd] \arrow[dr] & & k \arrow[dd, equal] \\
& & & & P_{i+1} \arrow[ur] \arrow[dd] & \\
(\Omega^n k)^{\otimes p} \arrow[r] & (\Omega^n k)^{\otimes p} \arrow[r] & \cdots \arrow[r, "1-T"'] & (\Omega^n k)^{\otimes p} \arrow[rr, "\lambda"'] \arrow[dr, "N"'] & & k \\
& & & & (\Omega^n k)^{\otimes p} \arrow[ur] &
\end{tikzcd}$$

But now the upper row represents the composition of a map $\Sigma^i(\Omega^n k)^{\otimes p} \to P_{i+1}$ with a map $P_{i+1} \to k$, which is stably trivial since P_{i+1} is projective. $\square$

Lemma 3.33. *Suppose that $i \leq 2p - 3$. The $\mathcal{D}_i$'s are natural in the following sense: whenever we are given an injective group homomorphism $f : H \to G$ and a cohomology class $[\zeta] \in \hat{H}^n(G)$, we have $f^*\mathcal{D}_i([\zeta]) \subseteq \mathcal{D}_i(f^*[\zeta])$ as subsets of $\hat{H}^{pn-i}(H)$.*

Proof. The functor $f^* : \mathfrak{mod}\text{-}kG \to \mathfrak{mod}\text{-}kH$ is exact and maps projective modules to projective modules (here we use the injectivity of f). Also, it maps the trivial G-module k to the trivial H-module k. This implies that $f^*(\Omega^n_G k) \cong \Omega^n_H k$ in $\underline{\mathfrak{mod}}\text{-}kH$. Now we start with a diagram of the form (3.25) defining some element a in $\mathcal{D}_i(\zeta)$ and apply the functor f^*. Then we obtain a diagram showing that $f^* a \in \mathcal{D}_i(f^*\zeta)$. $\square$

Remark 3.34. For the general case of a Hopf algebra, see Remark 3.18 which applies verbatim to the previous lemma.

Let us finally define our power operation.

Definition 3.35. *For all Tate cohomology classes ζ (of even degree if p is odd) we put $\mathcal{P}_1(\zeta) = -\mathcal{D}_{2p-3}(\zeta)$.*

For us the minus sign occurring in the definition is mainly motivated by the minus sign in the statement of Lemma 3.29. It is the same sign that shows up in any construction of the Steenrod operations, compare with (5.2) in [21].

3.4 The Cartan formula

Put $\mathcal{D}_0(\zeta) = \zeta^p$ (as a set with exactly one element). We are now going to prove the Cartan formula in the following form.

Proposition 3.36. *Let s be a positive integer with $s \leq 2p-3$, and let ζ, φ be cohomology classes (of even degree if p is odd). Then*

$$\mathcal{D}_s(\zeta \cdot \varphi) \subseteq \sum_{\substack{i+j=s \\ i,j \geq 0}} \mathcal{D}_i(\zeta) \cdot \mathcal{D}_j(\varphi).$$

We start with some auxiliary constructions and lemmas. As before let C_p be the cyclic group of order p, and let A be the algebra $k(C_p \times C_p)$ with augmentation $\epsilon : A \to k$ and augmentation ideal $I = \ker \epsilon$.

Let g be a generator of C_p, then we have a well-known exact sequence of free kC_p-modules

$$kC_p \xrightarrow{1-g} kC_p \xrightarrow{(1-g)^{p-1}} kC_p \xrightarrow{1-g} kC_p \xrightarrow{(1-g)^{p-1}} \dots$$

which we consider as a cochain complex with the first kC_p sitting in degree 0. When we tensor two such complexes we obtain an exact sequence $\mathcal{A}$ of free $kC_p \otimes kC_p = A$-modules

$$\mathcal{A}: \quad A \to A^{\oplus 2} \to A^{\oplus 3} \to A^{\oplus 4} \to \dots$$

where the i-th summand of $A^{\oplus s}$ comes from the tensor product of the two kC_p's sitting in degree $s-i$ and $i-1$, respectively. Let us denote the generators of the two factors of $C_p \times C_p$ by g and h; then we also have a cochain complex

$$\mathcal{B} : A \xrightarrow{1-g\otimes h} A \xrightarrow{(1-g\otimes h)^{p-1}} A \xrightarrow{1-g\otimes h} A \xrightarrow{(1-g\otimes h)^{p-1}} \dots,$$

which is actually exact, but we do not need this.

Lemma 3.37. *There is a map of cochain complexes of A-modules $\gamma : \mathcal{A} \to \mathcal{B}$ lifting the identity of A in degree 0 with the following property: when considered as a matrix, in degree s the map $\gamma_s : A^{\oplus(s+1)} \to A$ is of the form*

- $\begin{pmatrix} 1 & 1+I & 1 & \dots & 1 & 1+I \end{pmatrix}$ *if s is odd,*

- $\begin{pmatrix} 1 & 1+I & 1 & \dots & 1+I & 1 \end{pmatrix}$ *if s is even and $p = 2$,*
- $\begin{pmatrix} 1 & I & 1 & \dots & I & 1 \end{pmatrix}$ *if s is even and p is odd.*

Proof. We use the identification $A = k(C_p \times C_p) = k[x, y]/(x^p, y^p)$ given by $x = 1 - g$ and $y = 1 - h$. The augmentation ideal I is the ideal generated by x and y, and $1 - gh$ corresponds to $w = x + y - xy$. Let us define $z = (1 - x)y$; then $w = x + z$. The upper row in the diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge]
A \arrow[r, "\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)"] \arrow[d, equal] & A^2 \arrow[r, "\left(\begin{smallmatrix} x^{p-1} & \\ -y & x \\ & y^{p-1} \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 & \\ & 1-x \end{smallmatrix}\right)"] & A^3 \arrow[r, "\left(\begin{smallmatrix} x & & \\ y & x^{p-1} & \\ & -y^{p-1} & x \\ & & y \end{smallmatrix}\right)"] \arrow[d, "\left(\begin{smallmatrix} 1 & & \\ & 1-x & \\ & & 1 \end{smallmatrix}\right)"] & A^4 \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 1 & & & \\ & 1-x & & \\ & & 1 & \\ & & & 1-x \end{smallmatrix}\right)"] & \cdots \\
A \arrow[r, "\left(\begin{smallmatrix} x \\ z \end{smallmatrix}\right)"'] & A^2 \arrow[r, "\left(\begin{smallmatrix} x^{p-1} & \\ -z & x \\ & z^{p-1} \end{smallmatrix}\right)"'] & A^3 \arrow[r, "\left(\begin{smallmatrix} x & & \\ z & x^{p-1} & \\ & -z^{p-1} & x \\ & & z \end{smallmatrix}\right)"'] & A^4 \arrow[r] & \cdots
\end{tikzcd}$$

equals $\mathcal{A}$, and the diagram commutes because of the identities $x^{p-1}(1 - x) = x^{p-1}$ and $(1 - x)z^{p-1} = (1 - x)^p y^{p-1} = y^{p-1}$. Let us put $u = x^{p-2} - x^{p-3}z + - \cdots - z^{p-2} \in I$ for $p \geq 3$ and $u = 1$ if $p = 2$. Then the diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge]
A \arrow[r, "\left(\begin{smallmatrix} x \\ z \end{smallmatrix}\right)"] \arrow[d, equal] & A^2 \arrow[r, "\left(\begin{smallmatrix} x^{p-1} & \\ -z & x \\ & z^{p-1} \end{smallmatrix}\right)"] \arrow[d, "(1\ 1)"] & A^3 \arrow[r, "\left(\begin{smallmatrix} x & & \\ z & x^{p-1} & \\ & -z^{p-1} & x \\ & & z \end{smallmatrix}\right)"] \arrow[d, "(1\ u\ 1)"] & A^4 \arrow[r] \arrow[d, "(1\ 1\ 1\ 1)"] & \cdots \\
A \arrow[r, "w"'] & A \arrow[r, "w^{p-1}"'] & A \arrow[r, "w"'] & A \arrow[r] & \cdots
\end{tikzcd}$$

commutes because of the equalities

$$\begin{aligned} w^{p-1} &= x^{p-1} - zu = xu + z^{p-1}, \\ uw &= x^{p-1} - z^{p-1}. \end{aligned}$$

If we put the two diagrams together we get the desired cochain map, which in degree s is given by the matrix $\left(\begin{smallmatrix} 1 & 1-x & \dots & 1 & 1-x \end{smallmatrix}\right)$ if s is odd and by the matrix $\left(\begin{smallmatrix} 1 & (1-x)u & 1 & \dots & (1-x)u & 1 \end{smallmatrix}\right)$ if s is even. $\square$

Suppose we are given cochain complexes $X \to P_1 \to P_2 \to \cdots \to P_{i+j}$ and $Y \to Q_1 \to Q_2 \to \cdots \to Q_{i+j}$ with projective modules P_s and Q_s, and X and Y are sitting in degree 0. The tensor product of these two complexes is a complex $X \otimes Y \to R_1 \to R_2 \to \dots$ with projective modules R_s. Let $\alpha : P_i \to X'$ and $\beta : Q_j \to Y'$ be maps such that the sequences $X \to P_1 \to \cdots \to P_i \to X'$ and $Y \to Q_1 \to \cdots \to Q_j \to Y'$ are complexes, thus representing some classes $a \in \widehat{\operatorname{Ext}}_{kG}^{-i}(X, X')$ and $b \in \widehat{\operatorname{Ext}}_{kG}^{-j}(Y, Y')$, respectively.

Lemma 3.38. *In the situation above, the sequence* $X\otimes Y \to R_1 \to \cdots \to R_{i+j} \to X'\otimes Y'$, *in which the last map is the composite* $R_{i+j} \twoheadrightarrow P_i \otimes Q_j \xrightarrow{\alpha\otimes\beta} X'\otimes Y'$, *is a complex and represents* $a \otimes b \in \widehat{\operatorname{Ext}}_{kG}^{-(i+j)}(X\otimes Y, X'\otimes Y')$.

Proof. We know that $a\otimes b = (a\otimes \mathrm{id}_{Y'})(\mathrm{id}_X \otimes b)$ is represented by the complex

$$\begin{tikzcd}[column sep=small] X\otimes Y \arrow[r] & X\otimes Q_1 \arrow[r] & \cdots \arrow[r] & X\otimes Q_j \arrow[rr] \arrow[dr] & & P_1\otimes Y' \arrow[r] & \cdots \arrow[r] & P_i\otimes Y' \arrow[r] & X'\otimes Y'. \\ & & & & X\otimes Y' \arrow[ur] & & & & \end{tikzcd}$$

On the other hand we have a commutative diagram

$$\begin{tikzcd}[column sep=small] X\otimes Y \arrow[r] \arrow[d, equal] & R_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & R_j \arrow[r] \arrow[d] & R_{j+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & R_{i+j} \arrow[r] \arrow[d] & X'\otimes Y' \arrow[d, equal] \\ X\otimes Y \arrow[r] & X\otimes Q_1 \arrow[r] & \cdots \arrow[r] & X\otimes Q_j \arrow[r] & P_1\otimes Y' \arrow[r] & \cdots \arrow[r] & P_i\otimes Y' \arrow[r] & X'\otimes Y' \end{tikzcd}$$

given by the projection maps $R_s \to X\otimes Q_s$ for all $s = 1,2,\dots,j$ and by the compositions $R_{s+j} \to P_s \otimes Q_j \to P_s \otimes Y'$ for $s = 1,2,\dots,i$. The diagram also shows that its upper row is indeed a complex. $\square$

Proof of Proposition 3.36. The statement is trivial unless $s = 2p-3$, so we restrict attention to that case. Let $\zeta : \Omega^n k \to k$ and $\varphi : \Omega^m k \to k$ be given; then we will be interested in the complex

$$(\Omega^n k \otimes \Omega^m k)^{\otimes p} \xrightarrow{1-T} (\Omega^n k \otimes \Omega^m k)^{\otimes p} \xrightarrow{(1-T)^{p-1}} (\Omega^n k \otimes \Omega^m k)^{\otimes p} \xrightarrow{1-T} \dots,$$

which is isomorphic to the complex

$$(\Omega^n k)^{\otimes p} \otimes (\Omega^m k)^{\otimes p} \xrightarrow{1-T\otimes T} (\Omega^n k)^{\otimes p} \otimes (\Omega^m k)^{\otimes p} \xrightarrow{(1-T\otimes T)^{p-1}} \dots$$

via the obvious permutation maps. Let us put $X = (\Omega^n k)^{\otimes p}$ and $Y = (\Omega^m k)^{\otimes p}$. Then C_p acts on X and Y via the endomorphism T given by cyclic permutation. Let commutative diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & P_1 \arrow[r] \arrow[d, "f_1"'] & \cdots \arrow[r] & P_s \arrow[d, "f_s"'] \\ X \arrow[r, "1-T"'] & X \arrow[r, "N"'] & \cdots \arrow[r, "1-T"'] & X \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r] \arrow[d, equal] & Q_1 \arrow[r] \arrow[d, "g_1"'] & \cdots \arrow[r] & Q_s \arrow[d, "g_s"'] \\ Y \arrow[r, "1-T"'] & Y \arrow[r, "N"'] & \cdots \arrow[r, "1-T"'] & Y \end{tikzcd}$$

be given. Let us denote the cochain complexes in the bottom row by $\mathcal{X}$ and $\mathcal{Y}$, respectively. We write $X\otimes Y \to R_1 \to \cdots \to R_s \to \dots$ for the tensor product $(X \to P_*)\otimes(Y\to Q_*)$. Then we obtain a cochain map $(X\otimes Y \to R_*) \xrightarrow{f\otimes g} \mathcal{X}\otimes\mathcal{Y}$. Furthermore we have that

$$\mathcal{X}\otimes\mathcal{Y} \cong X \otimes_{C_p} \mathcal{A} \otimes_{C_p} Y. \tag{3.39}$$

This is easy to see, but the right hand side needs some explanation. The objects of $\mathcal{A}$ are free $k(C_p\times C_p)$-modules. On these we have a left C_p action given by the inclusion

$C_p \to C_p \times C_p$ of the first factor, and we have a right action given by the inclusion of the second factor, and the actions commute. On X and Y we have the C_p-actions given by the endomorphism T. When we endow the modules of $\mathcal{A}$ with the trivial G-action, (3.39) is an isomorphism of cochain complexes of kG-modules. Now Lemma 3.37 tells us that there is a certain map of cochain complexes

$$X \otimes_{C_p} \gamma \otimes_{C_p} Y : X \otimes_{C_p} \mathcal{A} \otimes_{C_p} Y \to X \otimes_{C_p} \mathcal{B} \otimes_{C_p} Y,$$

where the latter complex is isomorphic to

$$X \otimes Y \xrightarrow{1-T\otimes T} X \otimes Y \xrightarrow{(1-T\otimes T)^{p-1}} X \otimes Y \xrightarrow{1-T\otimes T} \dots$$

Putting things together we get the following diagram:

$$\begin{tikzcd}
X \otimes Y \arrow[r] \arrow[d, equal] & R_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & R_s \arrow[r] \arrow[d] & k \arrow[d, equal] \\
X \otimes Y \arrow[r] & X \otimes Y \arrow[r] & \cdots \arrow[r] & X \otimes Y \arrow[r, "\zeta^{\otimes p} \otimes \varphi^{\otimes p}"'] & k
\end{tikzcd}$$

In order to say which map is represented by the complex in the upper row, we need to investigate the map $R_s \to k$. Take a direct summand $P_i \otimes Q_j$ of R_s (with $i + j = s$) and consider the composite $P_i \otimes Q_j \hookrightarrow R_s \to X \otimes Y$. The description of γ in Lemma 3.37 says that this map is given as a composite

$$P_i \otimes Q_j \xrightarrow{f_i \otimes g_j} X \otimes Y \xrightarrow{\mathrm{id} + X \otimes_{C_p} \theta \otimes_{C_p} Y} X \otimes Y$$

where $\theta : A \to A$ is some map having its image in the augmentation ideal I of A. In particular, the composition

$$X \otimes Y \xrightarrow{X \otimes_{C_p} \theta \otimes_{C_p} Y} X \otimes Y \xrightarrow{\zeta^{\otimes p} \otimes \varphi^{\otimes p}} k$$

vanishes and the map $P_i \otimes Q_j \to k$ equals $(\zeta^{\otimes p} \circ f_i) \otimes (\varphi^{\otimes p} \circ g_j)$. Therefore, if we replace the last map of

$$X \otimes Y \to R_1 \to \dots \to R_s \to k$$

by its restriction to the direct summand $P_i \otimes Q_j$, Lemma 3.38 tells us that we obtain a complex representing some element of $\mathcal{D}_i(\zeta) \otimes \mathcal{D}_j(\varphi)$. Furthermore, if we replace the last map $R_s \to k$ by its restriction to the direct summand $X \otimes Q_s$, the resulting complex represents some element of $\zeta^{\otimes p} \otimes \mathcal{D}_s(\varphi)$, and similarly for $P_s \otimes Y$. Now the Cartan formula follows from Corollary 3.8. $\square$

3.5 Comparison with Steenrod operations

We are now going to show that our power operation extends certain Steenrod operations in the following sense: for every $\zeta \in H^n(G)$ of positive degree n we have that $\mathrm{Sq}^{n-1} \zeta \in \mathcal{P}_1(\zeta)$ if $p = 2$ and $\beta \mathrm{P}^{\frac{n}{2}-1} \zeta \in \mathcal{P}_1(\zeta)$ if p is odd and n is even. To do so, let us recall the definition of Steenrod operations on $H^*(G)$. There are several ways of constructing these operations which lead to the same result. One can use the isomorphism $H^*(G) \cong H^*(BG;k)$ where BG denotes the classifying space of G, and then work in the topological setting where we have Steenrod operations on the cohomology of a space. On the other hand, we can use the fact that kG is a cocommutative Hopf algebra, and there is a general construction of Steenrod operations on the cohomology of cocommutative Hopf algebras (see [21], §11). In either case, one usually constructs operations $D_i : H^n(G) \to H^{pn-i}(G)$ for all non-negative integers i, then proves that some of these operations vanish, and finally defines the Steenrod operations P^* and $\beta\mathrm{P}^*$ to be the non-vanishing ones. Let us go through this process more precisely; we will take the topological path, even though in the case of a cocommutative Hopf algebra the purely algebraic way is more appropriate.

Consider the simplicial set EG which can be obtained by forming the nerve of the category whose objects are the elements of G, and there is exactly one morphism for every ordered pair of objects. Then EG is a contractible simplicial set with a free G-action. We denote by P_* the corresponding chain complex over k, i.e., P_n is the k-vector space over the set $(EG)_n$ of n-simplices, and we have the differential $\partial = \sum_i \partial_i : P_n \to P_{n-1}$. The right G-action on EG gives a right G-action on P_n turning this into a kG-module. Naturality of the construction shows that ∂ is a kG-module map. The P_n's are in fact free kG-modules, and $k \longleftarrow P_*$ is a projective resolution of the trivial kG-module k. In what follows, we choose $\Omega^n k$ to be the cokernel of the map $\partial : P_{n+1} \to P_n$, and write $p_n : P_n \to \Omega^n k$ for the obvious map. Then p_n extends to a map $\pi : P_* \to (\Omega^n k)[n]$ of chain complexes.

We also have a complex Q_* of kG-modules defined by $Q_n = P_n^{\otimes p}$, the differential given by $\sum_i (\partial_i)^{\otimes p}$. The diagonal of EG induces a map $D : P_n \to Q_n$ of kG-chain complexes. Furthermore, we have the Alexander-Whitney map $\xi : Q_* \to (P^{\otimes p})_*$. Putting things together, we get a composition like this:

$$P_* \xrightarrow{D} (P_*)^{\otimes p} \xrightarrow{\xi} (P^{\otimes p})_* \xrightarrow{\pi^{\otimes p}} (\Omega^n k)[n]^{\otimes p} = (\Omega^n k)^{\otimes p}[pn]$$

Restricting these chain maps to degree pn, we obtain the following diagram:

$$\begin{tikzcd} P_{pn} \arrow[d, "p_{pn}"'] \arrow[dr, "\pi^{\otimes p}\,\xi\, D"] & \\ \Omega^{pn} k \arrow[r] & (\Omega^n k)^{\otimes p} \end{tikzcd} \tag{3.40}$$

The lower row is a stable isomorphism; in what follows, we will always mean this map whenever we write $\Omega^{pn} k \cong (\Omega^n k)^{\otimes p}$.

Recall that $T : (P^{\otimes p})_* \to (P^{\otimes p})_*$ is the chain map given by

$$T(a_1 \otimes a_2 \otimes \cdots \otimes a_p) = \pm\, a_p \otimes a_1 \otimes \cdots \otimes a_{p-1}, \tag{3.41}$$

with the usual sign convention; in particular, the sign is $+$ if the degrees of all the a_i's are the same. Let us write $\Delta_0 = \xi\, D$. From the general theory we know that $(1-T)\Delta_0$ is null-homotopic (in fact, naturally) via some homotopy Δ_1, i.e.,

$$(1-T)\Delta_0 = \Delta_1 \partial + \partial \Delta_1.$$

Proceeding inductively, we find (natural) maps Δ_i of degree i satisfying

$$(1-T)\Delta_i = \Delta_{i+1}\partial + \partial\Delta_{i+1} \qquad \text{if } i \text{ is even,} \tag{3.42}$$

$$(1+T+\cdots+T^{p-1})\Delta_i = \Delta_{i+1}\partial - \partial\Delta_{i+1} \qquad \text{if } i \text{ is odd.} \tag{3.43}$$

Naturality of the Δ_i's implies that they are kG-linear. We assume that these Δ_i are chosen as in the construction of Steenrod operations in [21], §7 (or, for the case of a general cocommutative Hopf algebra, §11 in the same paper). From now on, we assume that $p \cdot n$ is even.

Lemma 3.44. *Define $\alpha_i = \pi^{\otimes p}\Delta_i : P_{pn-i} \to (\Omega^n k)^{\otimes p}$. Then α is a cochain map:*

$$\begin{array}{ccccccccc}
P_{pn} & \longrightarrow & P_{pn-1} & \longrightarrow & P_{pn-2} & \longrightarrow & P_{pn-3} & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \\
(\Omega^n k)^{\otimes p} & \xrightarrow{1-T} & (\Omega^n k)^{\otimes p} & \xrightarrow{N} & (\Omega^n k)^{\otimes p} & \xrightarrow{1-T} & (\Omega^n k)^{\otimes p} & \xrightarrow{N} & \cdots
\end{array}$$

Proof. Note that the following diagram commutes:

$$\begin{array}{ccc}
(P^{\otimes p})_{pn} & \xrightarrow{T} & (P^{\otimes p})_{pn} \\
\downarrow{\scriptstyle \pi^{\otimes p}} & & \downarrow{\scriptstyle \pi^{\otimes p}} \\
(\Omega^n k)^{\otimes p} & \xrightarrow{T} & (\Omega^n k)^{\otimes p}
\end{array}$$

To see this, start with some element $a = a_1 \otimes \cdots \otimes a_p$ in $(P^{\otimes p})_{pn}$. If it does not belong to $(P_n)^{\otimes p} \subset (P^{\otimes p})_{pn}$, both a and Ta will be mapped to zero under $\pi^{\otimes p}$. If a lies in $(P_n)^{\otimes p}$, the sign occurring in (3.41) is $+$, because the degrees of all the a_j's are equal.

Using (3.42), we obtain

$$\begin{aligned}
(1-T)\alpha_i &= (1-T)\pi^{\otimes p}\Delta_i = \pi^{\otimes p}(1-T)\Delta_i \\
&= \pi^{\otimes p}\Delta_{i+1}\partial + \underbrace{\pi^{\otimes p}\partial}_{0}\Delta_{i+1} = \alpha_{i+1}\partial
\end{aligned}$$

for i even. A similar argument using (3.43) for odd i completes the proof of the lemma. $\square$

Let $[\zeta] \in H^n(G)$ be given by a map $\zeta : \Omega^n k \to k$, and let i be any positive integer. Define λ to be the composition $P_{pn-i} \xrightarrow{\alpha_i} (\Omega^n k)^{\otimes p} \xrightarrow{\zeta^{\otimes p}} k$. Then

$$\lambda\,\partial = \zeta^{\otimes p}\,\alpha_i\partial = \begin{cases} \underbrace{\zeta^{\otimes p}\,(1-T)}_{0}\,\alpha_{i-1} = 0 & \text{if } i \text{ is odd,} \\ \underbrace{\zeta^{\otimes p}\,N}_{0}\,\alpha_{i-1} = 0 & \text{if } i \text{ is even.} \end{cases}$$

Therefore, λ induces a map $\Omega^{pn-i}k \to k$ which represents $D_i([\zeta])$, where D_i is defined as in [21] (proof of Theorem 11.8, together with Definitions 2.2).

Lemma 3.45. *Viewed as an element in* $\underline{\mathrm{Hom}}_{kG}(\Omega^{pn}k, \Omega^i k)$*, the upper row of the commutative diagram*

$$\begin{tikzcd}
(\Omega^n k)^{\otimes p} \arrow[d, equal] & \Omega^{pn}k \arrow[l, "\cong_{st}"'] \arrow[r, hook] & P_{pn-1} \arrow[r] \arrow[d] & P_{pn-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_{pn-i} \arrow[r, "\lambda"] \arrow[d] & k \arrow[d, equal] \\
(\Omega^n k)^{\otimes p} \arrow[rr] & & (\Omega^n k)^{\otimes p} \arrow[r] & (\Omega^n k)^{\otimes p} \arrow[r] & \cdots \arrow[r] & (\Omega^n k)^{\otimes p} \arrow[r, "\zeta^{\otimes p}"'] & k
\end{tikzcd}$$

represents $\Omega^i D_i([\zeta])$*. In particular,* $D_i([\zeta]) \in \mathcal{D}_i(\zeta)$*.*

Proof. Commutativity of the diagram follows from the diagram (3.40), Lemma 3.44, and the definition of λ. The upper row is the composition of the complex

$$\Omega^{pn}k \hookrightarrow P_{pn-1} \longrightarrow P_{pn-2} \longrightarrow \cdots \longrightarrow P_{pn-i} \twoheadrightarrow \Omega^{pn-i}k$$

representing the identity in $\underline{\mathrm{Hom}}_{kG}(\Omega^{pn}k, \Omega^i\Omega^{pn-i}k)$ with $D_i([\zeta]) : \Omega^{pn-i}k \to k$. The result follows from Remark 3.13. □

Corollary 3.46. *We have that* $\mathrm{P}_1(\zeta) \in \mathcal{P}_1(\zeta)$*.*

Proof. Checking the signs in the formulas (1) and (2) of §5 in [21], we get that indeed $\mathrm{P}_1(\zeta) = -D_{2p-3}(\zeta)$. □

Proposition 3.47. *For all non-negative integers i and all Tate cohomology classes ζ (of even degree if p is odd) the set $\mathcal{D}_i(\zeta)$ is non-empty.*

Proof. We have already done this for ordinary cohomology classes ζ, and now we need to extend this result to negative degrees. To do so, we have to extend our projective resolution P_* and the maps Δ_i to the negative range. Recall that Δ_i was defined as a map (not a chain map in general) $P_* \to (P^{\otimes p})[-i]_*$.

The first step is easy: we simply extend P_* to a complete projective resolution of k. To define Δ_i, it turns out that we need to modify the notion of tensor product for our purposes. The usual tensor product of two chain complexes X_* and Y_* is defined by $(X \otimes Y)_n = \bigoplus_{i+j=n} X_i \otimes Y_j$ with a certain differential, but this is not what we want. For every complex X, let us write X^+ and X^- for the truncated complexes

$$X_n^+ = \begin{cases} X_n & \text{if } n \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$X_n^- = \begin{cases} X_n & \text{if } n < 0, \\ 0 & \text{otherwise,} \end{cases}$$

with the obvious differentials. Now we introduce a new product $\boxtimes$ for two given chain complexes X and X' as follows. Define K to be the kernel of the differential $X_{-1} \to X_{-2}$; then we obtain a complex $X^- \leftarrow K$ with K sitting in degree 0, and we also get an augmented complex $K \leftarrow X^+$. Similarly, we get the complexes $X'^- \leftarrow K'$ and $K' \leftarrow X'^+$. We can now form the tensor product of complexes $(X^- \leftarrow K) \otimes (X'^- \leftarrow K')$ and the tensor product of augmented complexes $K \otimes K' \leftarrow X^+ \otimes X'^+$. Connecting these two at their common object $K \otimes K'$ we obtain a new complex which we denote by $X \boxtimes X'$:

$$\cdots \leftarrow X_{-1} \otimes K' \oplus K \otimes X_{-1} \leftarrow X_0 \otimes X'_0 \leftarrow X_1 \otimes X'_0 \oplus X_0 \otimes X'_1 \leftarrow \ldots$$

One should bear in mind that the elements of K and K' are considered to be of degree 0; then the degree of an element of the form $a \otimes b$ equals $|a| + |b|$. The operation $\boxtimes$ is a symmetric monoidal product on the category of (unbounded) chain complexes. If X and Y are complete projective resolutions of the modules K and L, respectively, then by the Künneth theorem $X \boxtimes Y$ is a complete projective resolution of $K \otimes L$. Furthermore, it is an extension of the usual tensor product of non-negatively graded chain complexes in the sense that $(X \boxtimes Y)^+ = X^+ \otimes Y^+$. Also notice that $(X_n)^{\otimes p}$ is a direct summand of $(X^{\boxtimes p})_{pn}$ for every integer n.

We can extend T to a chain map $(P^{\boxtimes p})_* \to (P^{\boxtimes p})_*$ by putting $T(a_1 \otimes a_2 \otimes \cdots \otimes a_p) = \pm a_p \otimes a_1 \otimes \cdots \otimes a_{p-1}$ with the same sign rule as in §1.4. For every non-negative integer i, let us extend the maps $\Delta_i : P_*^+ \to ((P^+)^{\otimes p})_{*+i}$ to maps $\Delta_i : P_* \to (P^{\boxtimes p})_{*+i}$ satisfying the following properties:

$$\begin{aligned} 0 &= \Delta_0 \partial - \partial \Delta_0, && \\ (1-T)\Delta_i &= \Delta_{i+1}\partial + \partial \Delta_{i+1} && \text{if } i \text{ is even,} \\ (1 + T + \cdots + T^{p-1})\Delta_i &= \Delta_{i+1}\partial - \partial\Delta_{i+1} && \text{if } i \text{ is odd.} \end{aligned}$$

For $i = 0$, we need to extend a chain transformation Δ_0 which is defined on a large (in fact, infinite) range to the whole projective resolutions. It is a standard fact from homological algebra that this is possible. To do the inductive step, note that the term on the left-hand side is already defined everywhere and is a chain transformation $P \to (P^{\boxtimes p})[-i]$. Furthermore, Δ_{i+1} is a partially defined null-homotopy for that chain map, and again usual homological algebra tells us that we can find a suitable extension of Δ_{i+1} to the whole projective resolution. Let $p_n : P_n \to \Omega^n k$ be the obvious map for every integer n. For any fixed integer n, this map extends to a chain map $\pi : P_* \to (\Omega^n k)[n]$. We obtain a commutative diagram

$$\begin{tikzcd} P_{pn} \arrow[d, "p_{pn}"'] \arrow[dr, "\pi^{\boxtimes p}\Delta_0"] & \\ \Omega^{pn} k \arrow[r] & (\Omega^n k)^{\otimes p} \end{tikzcd}$$

in which the vertical arrow is a stable isomorphism which we refer to as 'the' isomorphism $\Omega^{pn} k \cong (\Omega^n k)^{\otimes p}$. If we put $\alpha_i = \pi^{\boxtimes p}\Delta_i : P_{pn-i} \to (\Omega^n k)^{\otimes p}$ we get a cochain map α:

$$\begin{tikzcd} P_{pn} \arrow[r] \arrow[d, "\alpha_0"] & P_{pn-1} \arrow[r] \arrow[d, "\alpha_1"] & P_{pn-2} \arrow[r] \arrow[d, "\alpha_2"] & P_{pn-3} \arrow[r] \arrow[d, "\alpha_3"] & \cdots \\ (\Omega^n k)^{\otimes p} \arrow[r, "1-T"] & (\Omega^n k)^{\otimes p} \arrow[r, "N"] & (\Omega^n k)^{\otimes p} \arrow[r, "1-T"] & (\Omega^n k)^{\otimes p} \arrow[r, "N"] & \cdots \end{tikzcd}$$

The proof of this fact is exactly the same as the one of Lemma 3.44. For every $\zeta : \Omega^n k \to k$ we get a commutative diagram

$$\begin{tikzcd}
(\Omega^n k)^{\otimes p} \arrow[d, equal] & \Omega^{pn} k \arrow[l, "\cong_{\mathrm{st}}"'] \arrow[r, hook] & P_{pn-1} \arrow[r] \arrow[d] & P_{pn-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_{pn-i} \arrow[r] \arrow[d] & k \arrow[d, equal] \\
(\Omega^n k)^{\otimes p} \arrow[rr] & & (\Omega^n k)^{\otimes p} \arrow[r] & (\Omega^n k)^{\otimes p} \arrow[r] & \cdots \arrow[r] & (\Omega^n k)^{\otimes p} \arrow[r, "\zeta^{\otimes p}"'] & k
\end{tikzcd}$$

proving that $\mathcal{D}_i(\zeta) \neq \emptyset$. $\square$

Proof of Theorem 3.1. This is essentially a table of contents for the previous results. Part (i) follows from Proposition 3.47 and Corollary 3.30. Part (ii) is Lemma 3.32, Part (iii) follows from Corollary 3.46, Part (iv) is Lemma 3.33. The Cartan formula (v) is shown in Proposition 3.36. $\square$

3.6 The power operation for $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$

In this section we will describe the power operation $\mathcal{P}_1$ on the Tate cohomology of the group $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Let us start with the case $p = 2$. It is well-known that the ordinary cohomology ring $H^*(G)$ is the graded algebra generated by two elements u_1, u_2 of degree 1. Recall from Example 3.17 that $\hat{H}^{-1}(G) \cong k$ has a certain canonical generator which is invariant under automorphisms of G. Now we use Tate duality (see, e.g., §4 of [28]) saying that the natural bilinear form

$$\hat{H}^{-1-n}(G) \otimes \hat{H}^n(G) \to \hat{H}^{-1}(G) \cong k$$

given by multiplication in the Tate cohomology is non-degenerate and therefore induces an isomorphism $(\hat{H}^{-1-n}(G))^* \cong \hat{H}^n(G)$. For a fixed non-negative degree n, the monomials $u_1^i u_2^j$ of degree n form a k-linear basis of $\hat{H}^n(G)$, and we denote the dual basis (which is a basis of $\hat{H}^{-1-n}(G)$) by $\varphi_{i,j}$. From [16], Proposition 4.21, we know that $\hat{H}^*(G)$ is the graded commutative algebra generated by the elements $u_1, u_2, \varphi_{i,j}$ (with $i, j \geq 0$) subject to the relations

$$\begin{aligned}
\varphi_{i,j} u_1 &= \begin{cases} \varphi_{i-1,j} & \text{if } i > 0, \\ 0 & \text{otherwise,} \end{cases} \\
\varphi_{i,j} u_2 &= \begin{cases} \varphi_{i,j-1} & \text{if } j > 0, \\ 0 & \text{otherwise,} \end{cases} \\
\varphi_{i,j} \varphi_{i',j'} &= 0.
\end{aligned}$$

Now let p be an odd prime; then the ordinary cohomology ring $H^*(G)$ is the graded commutative algebra generated by two exterior classes u_1, u_2 of degree 1 and two polynomial classes v_1, v_2 of degree 2. For a fixed non-negative degree n, we can consider all monomials of the form $v_1^a v_2^b u_1^c u_2^d$ of degree n with $a, b, c, d \geq 0$ and $c, d \leq 1$; these form a basis of $\hat{H}^n(G)$, and we denote the dual basis elements by $\varphi_{2a+c,2b+d}$. Again from [16], Proposition 4.21, we get that $\hat{H}^*(G)$ is the graded algebra generated by elements u_1, u_2, v_1, v_2

and $\varphi_{i,j}$ for all $i,j \geq 0$, of degrees $|u_1| = |u_2| = 1$, $|v_1| = |v_2| = 2$ and $|\varphi_{i,j}| = -1-(i+j)$, subject to the relations

$$\begin{aligned}
\varphi_{i,j}u_1 &= \begin{cases} \varphi_{i-1,j} & \text{if } i \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases} \\
\varphi_{i,j}u_2 &= \begin{cases} (-1)^i \varphi_{i,j-1} & \text{if } j \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases} \\
\varphi_{i,j}v_1 &= \begin{cases} \varphi_{i-2,j} & \text{if } i \geq 2, \\ 0 & \text{otherwise,} \end{cases} \\
\varphi_{i,j}v_2 &= \begin{cases} \varphi_{i,j-2} & \text{if } j \geq 2, \\ 0 & \text{otherwise,} \end{cases} \\
\varphi_{i,j}\varphi_{i',j'} &= 0.
\end{aligned}$$

(in fact, [16] obtains different signs due to a slightly different chosen basis in negative degrees). As an immediate consequence of the multiplicative structure of $\hat{H}^*(G)$ we get the following:

Lemma 3.48. *Outside degree* 0*, the operation* $\mathcal{P}_1$ *on* $\hat{H}^*(G)$ *has zero indeterminacy.*

We therefore obtain a map $\mathcal{P}_1 : \hat{H}^n(G) \to \hat{H}^{pn-(2p-3)}(G)$ for every non-zero integer n (with n even unless $p=2$), and our main objective will be to describe this map for all negative values of n. The following proposition does the first step.

Proposition 3.49. *Let* p *be an odd prime. Then there is some constant* $t \in k$ *with* $\mathcal{P}_1(\varphi_{1,0}) = t \cdot \varphi_{2p-1,2p-3}$ *and* $\mathcal{P}_1(\varphi_{0,1}) = -t \cdot \varphi_{2p-3,2p-1}$*.*

Proof. The result will follow from the naturality of $\mathcal{P}_1$ with respect to automorphisms of G, as established in Lemma 3.33. Suppose we are given an automorphism ψ of G which we think of as a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in the general linear group $\mathrm{GL}_2(\mathbb{F}_p)$. We have a natural isomorphism $H^1(G) \cong \mathrm{Hom}_{\mathrm{groups}}(G,k) \cong k^2$ which implies that ψ^* is given on $H^1(G) \cong k\{u_1,u_2\}$ by the transposed matrix, i.e.,

$$\psi^*(u_1) = au_1 + cu_2, \quad \psi^*(u_2) = bu_1 + du_2.$$

Furthermore, we know that the Bockstein homomorphism β is natural and maps u_i to v_i; hence $\psi^*(v_1) = av_1 + cv_2$ and $\psi^*(v_2) = bv_1 + dv_2$. This determines the morphism ψ^* of graded algebras uniquely on ordinary cohomology $H^*(G)$. Together with naturality of Tate duality we get that ψ^* is determined on Tate cohomology $\hat{H}^*(G)$. Now we will exploit this fact for several morphisms ψ.

Let us begin with a diagonal matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ with $a,b \in \mathbb{F}_p^*$. The action on $\hat{H}^1(G) \cong k\{u_1,u_2\}$ is given by the same matrix, and therefore the action on $\hat{H}^{-2}(G) \cong k\{\varphi_{1,0},\varphi_{0,1}\}$ is given by its transpose, which is again the same matrix, so that $\psi^*(\varphi_{1,0}) = a\varphi_{1,0}$ and similarly for $\varphi_{0,1}$. We are interested in the action of ψ^* on $\mathcal{P}_1(\varphi_{0,1})$ which lives in degree

$-4p+3$, so let us determine ψ^* on the dual space $H^{4p-4}(G)$. The latter space has the basis elements $v_1^i v_2^j$ with $i+j=2p-2$ and $v_1^i v_2^j u_1 u_2$ with $i+j=2p-3$. The action of ψ^* is given by the formula

$$\begin{aligned} v_1^i v_2^j &\mapsto a^i b^j v_1^i v_2^j, \\ v_1^i v_2^j u_1 u_2 &\mapsto a^{i+1} b^{j+1} v_1^i v_2^j u_1 u_2, \end{aligned}$$

which is a diagonal matrix. Therefore, the action of ψ^* on $\hat{H}^{-4p+3}(G)$ is given by

$$\begin{aligned} \varphi_{2i,2j} &\mapsto a^i b^j \varphi_{2i,2j}, \\ \varphi_{2i+1,2j+1} &\mapsto a^{i+1} b^{j+1} \varphi_{2i+1,2j+1}. \end{aligned}$$

On the other hand we know that $\mathcal{P}_1$ is a natural construction, i.e.,

$$\psi^* \mathcal{P}_1(\varphi_{1,0}) = \mathcal{P}_1(\psi^*(\varphi_{1,0})) = \mathcal{P}_1(a\varphi_{1,0}) = a^p \mathcal{P}_1(\varphi_{1,0}) = a\mathcal{P}_1(\varphi_{1,0}).$$

Hence $\mathcal{P}_1(\varphi_{1,0})$ is a linear combination of $\varphi_{2i,2j}$'s with $a^i b^j = a$ for all $a, b \in \mathbb{F}_p^*$ and $\varphi_{2i+1,2j+1}$'s with $a^{i+1} b^{j+1} = a$ for all $a, b \in \mathbb{F}_p^*$. Since $\mathbb{F}_p^*$ is the cyclic group of order $p-1$, we get from $a^i b^j = a$ that j is divisible by $p-1$; from $i+j=2p-2$ we get that i is also divisible by $p-1$, but then $a^i b^j = 1 \neq a$ in general. From $a^{i+1} b^{j+1} = a$ we can deduce that $j+1$ is divisible by $p-1$, so that $j \in \{p-2, 2p-3\}$. We have therefore shown that

$$\mathcal{P}_1(\varphi_{1,0}) = s \cdot \varphi_{1,4p-5} + t \cdot \varphi_{2p-1,2p-3} \tag{3.50}$$

for some constants $s, t \in k$.

The next matrix we consider is $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$; again, denote by ψ the corresponding automorphism of G. Using the same arguments as before we get:

$$\begin{aligned} u_1 &\mapsto u_1 + u_2 & v_1 &\mapsto v_1 + v_2 & \varphi_{1,0} &\mapsto \varphi_{1,0} \\ u_2 &\mapsto u_2 & v_2 &\mapsto v_2 & \varphi_{0,1} &\mapsto \varphi_{1,0} + \varphi_{0,1} \end{aligned}$$

Therefore, ψ acts on $H^{4p-4}(G)$ via

$$v_1^i v_2^j \mapsto \sum_{m=0}^{i} \binom{i}{m} v_1^m v_2^{i+j-m}, \qquad v_1^i v_2^j u_1 u_2 \mapsto \sum_{m=0}^{i} \binom{i}{m} v_1^m v_2^{i+j-m} u_1 u_2.$$

Passing to the dual space $\hat{H}^{-4p+3}(G)$ we get

$$\varphi_{2i+1,2j+1} \mapsto \sum_{m=i}^{2p-3} \binom{m}{i} \varphi_{2m+1,2(i+j-m)+1}.$$

We know that $\mathcal{P}_1(\varphi_{1,0})$ must be ψ^*-invariant. The element $\varphi_{2p-1,2p-3}$ is ψ^*-invariant because

$$\psi^*(\varphi_{2p-1,2p-3}) = \sum_{m=p-1}^{2p-3} \binom{m}{p-1} \varphi_{2m+1,4p-5-2m}$$

and $\binom{m}{p-1}$ is non-zero (mod p) if and only if $m \equiv -1 \pmod p$, which is only true for $m = p-1$. On the other hand, $\varphi_{1,4p-5}$ is not ψ^*-invariant because $\psi^*(\varphi_{1,4p-5}) = \sum_{m=0}^{2p-3} \varphi_{2m+1,4p-5-2m}$. Hence $s = 0$ in (3.50).

As a last step we use the automorphism ψ given by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ to find $\mathcal{P}_1(\varphi_{0,1})$. We know that ψ acts as follows:

$$u_1 \leftrightarrow u_2 \qquad\qquad v_1 \leftrightarrow v_2 \qquad\qquad \varphi_{1,0} \leftrightarrow \varphi_{0,1}$$

The action on $H^{4p-4}(G)$ is then given by $v_1^i v_2^j \mapsto v_1^j v_2^i$ and $v_1^i v_2^j u_1 u_2 \mapsto v_2^i v_1^j u_2 u_1 = -v_1^j v_2^i u_1 u_2$. From this we deduce that ψ acts on $\hat{H}^{-4p+3}(G)$ via $\varphi_{2i,2j} \mapsto \varphi_{2j,2i}$ and $\varphi_{2i+1,2j+1} \mapsto -\varphi_{2j+1,2i+1}$. Together with $\mathcal{P}_1(\varphi_{1,0}) = t \cdot \varphi_{2p-1,2p-3}$ we finally obtain $\mathcal{P}_1(\varphi_{0,1}) = -t \cdot \varphi_{2p-3,2p-1}$. □

In the case $p = 2$ we can also determine $\mathcal{P}_1(\varphi_{0,0})$ using similar methods.

Proposition 3.51. *If $p = 2$ then $\mathcal{P}_1(\varphi_{0,0}) = t \cdot \varphi_{1,1}$ for some constant $t \in k$.*

Proof. As before, start with some matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{GL}_2(\mathbb{F}_2)$. Written as a matrix with respect to the monomial basis u_1^2, u_1u_2, u_2^2, the action on $H^2(G)$ is given by the first of the following two matrices:

$$\begin{pmatrix} a^2 & ac & c^2 \\ 0 & ad+bc & 0 \\ b^2 & bd & d^2 \end{pmatrix} \qquad\qquad \begin{pmatrix} a^2 & 0 & b^2 \\ ac & 1 & bd \\ c^2 & 0 & d^2 \end{pmatrix} \tag{3.52}$$

Note that $ad+bc = ad-bc = 1$. Therefore, with respect to the basis $\varphi_{2,0}, \varphi_{1,1}, \varphi_{0,2}$ we get that ψ^* acts as the right hand matrix of (3.52) on $\hat{H}^{-3}(G)$. Note that $\varphi_{0,0}$ is ψ^*-invariant for every ψ. Therefore, $\mathcal{P}_1(\varphi_{0,0})$ must be ψ^*-invariant for all ψ. Using these facts for the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ one gets that $\mathcal{P}_1(\varphi_{0,0}) = t \cdot \varphi_{1,1}$ for some constant t. □

Together with the Cartan formula we are able to determine the power operation completely, up to the scalar constant. For this we need a technical lemma.

Lemma 3.53. *Let $x, y \in \hat{H}^n(G)$ be elements of negative degree n. If $p = 2$ then assume that $n \leq -4$, $xu_1^2 = yu_1^2$ and $xu_2^2 = yu_2^2$, and if p is odd then assume that $n \leq -4p$, $xv_1^p = yv_1^p$ and $xv_2^p = yv_2^p$. Then it follows that $x = y$.*

Proof. We may assume that $y = 0$. Let us start with $p = 2$. Then $xu_1^2 = 0$ implies $x \in \langle \varphi_{0,-n-1}, \varphi_{1,-n-2} \rangle_k$ and $xu_2^2 = 0$ implies $x \in \langle \varphi_{-n-2,1}, \varphi_{-n-1,0} \rangle_k$. Since $n \leq -4$ we get $x = 0$.

Now let p be an odd prime. Then

$$\begin{array}{lll} xv_1^p = 0 & \text{implies} & x \in \langle \varphi_{0,-n-1}, \varphi_{1,-n-2}, \dots, \varphi_{2p-1,-n-2p} \rangle_k \quad \text{and} \\ xv_2^p = 0 & \text{implies} & x \in \langle \varphi_{-n-2p,2p-1}, \varphi_{-n-2p+1,2p-2}, \dots, \varphi_{-n-1,0} \rangle_k . \end{array}$$

From $n \leq -4p$ we can deduce that $x = 0$. □

Proposition 3.54. *If $p = 2$ then there is some constant $t \in k$ such that $\mathcal{P}_1(\varphi_{i,j}) = t \cdot \varphi_{2i+1,2j+1}$ for all $i, j \geq 0$. If p is odd, then there is some constant $t \in k$ such that*

$$\begin{aligned}\mathcal{P}_1(\varphi_{2i+1,2j}) &= t \cdot \varphi_{2pi+2p-1,2pj+2p-3},\\ \mathcal{P}_1(\varphi_{2i,2j+1}) &= -t \cdot \varphi_{2pi+2p-3,2pj+2p-1}.\end{aligned}$$

Proof. Let us start with $p = 2$ and proceed by induction on $i+j$, the case $i = j = 0$ being covered by Proposition 3.49. The Cartan formula Theorem 3.1.(v) yields

$$\mathcal{P}_1(\varphi_{i,j})u_1^2 + \underbrace{\varphi_{i,j}^2}_{0}\mathcal{P}_1(u_1) = \mathcal{P}_1(\varphi_{i,j}u_1) = \begin{cases} t \cdot \varphi_{2i-1,2j+1} & \text{if } i \geq 1,\\ 0 & \text{otherwise,}\end{cases}$$

$$\mathcal{P}_1(\varphi_{i,j})u_2^2 + \underbrace{\varphi_{i,j}^2}_{0}\mathcal{P}_1(u_2) = \mathcal{P}_1(\varphi_{i,j}u_2) = \begin{cases} t \cdot \varphi_{2i+1,2j-1} & \text{if } j \geq 1,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, $t \cdot \varphi_{2i+1,2j+1}$ and $\mathcal{P}_1(\varphi_{i,j})$ satisfy the conditions of Lemma 3.53 and hence are equal.

The case of an odd prime p runs similarly. Let us prove the first equation only. Again we use induction on $i + j$ with $i = j = 0$ being already done. Using the Cartan formula we get

$$\begin{aligned}\mathcal{P}_1(\varphi_{2i+1,2j})v_1^p + \underbrace{\varphi_{2i+1,2j}^p}_{0}\mathcal{P}_1(v_1) &= \mathcal{P}_1(\varphi_{2i+1,2j}v_1)\\ &= \begin{cases} t \cdot \varphi_{2p(i-1)+2p-1,2pj+2p-3} & \text{if } i \geq 1,\\ 0 & \text{otherwise,}\end{cases}\\ \mathcal{P}_1(\varphi_{2i+1,2j})v_2^p + \underbrace{\varphi_{2i+1,2j}^p}_{0}\mathcal{P}_1(v_2) &= \mathcal{P}_1(\varphi_{2i+1,2j}v_2)\\ &= \begin{cases} t \cdot \varphi_{2pi+2p-1,2p(j-1)+2p-3} & \text{if } j \geq 1,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

As before, $t \cdot \varphi_{2pi+2p-1,2pj+2p-3}$ and $\mathcal{P}_1(\varphi_{2i+1,2j})$ satisfy the conditions of Lemma 3.53, and we are done. ☐

Remark 3.55. The previous proposition shows that the formulae for $\mathcal{P}_1$ given for odd primes p are actually also true for $p = 2$.

It remains to find the constants t in Proposition 3.54.

Proposition 3.56. *The constants in Proposition 3.54 equal $t = 1$.*

Proof. Let us denote by g and h generators of the two cyclic factors of the group G. Recall that, as an augmented algebra, $kG = k[x, y]/(x^p, y^p) = L$, where $x = 1 - g$ and $y = 1 - h$. The action of x on a tensor product of kG-modules is given by the formula

$$(a \otimes b) \cdot x = (ax) \otimes b + a \otimes (bx) - (ax) \otimes (bx), \tag{3.57}$$

and similarly for y. Let us define $\Omega^i k$ for small values of $|i|$ via the following (partially given) complete resolution:

$$\begin{tikzcd}[column sep=small]
\cdots & & L^2 \arrow[ll] \arrow[dl] & & L \arrow[ll, "\binom{x}{y}"'] \arrow[dl] & & L \arrow[ll, "x^{p-1}y^{p-1}"'] \arrow[dl] & & L^2 \arrow[ll, "(x\ y)"'] \arrow[dl] & & \cdots \arrow[ll] \\
& \Omega^{-2}k \arrow[ul] & & \Omega^{-1}k \arrow[ul] & & k \arrow[ul] & & \Omega k \arrow[ul] & & \Omega^2 k \arrow[ul] &
\end{tikzcd}$$

We consider Ωk as the submodule of L generated by x and y, $\Omega^{-1}k$ as the quotient $L/x^{p-1}y^{p-1}L$, and the leftmost L^2 has the basis e_1, e_2. We also view $\Omega^{-2}k$ as the quotient $L^2/(xe_1+ye_2)L$. With these notions, the map $\Omega^{-1}k \to L^2$ sends the class $[1]$ to xe_1+ye_2. The cohomology class u_1 is represented by the L-linear map $\Omega k \to k$ that sends x to 1 and y to 0. Similarly u_2 can be described by mapping x to 0 and y to 1. This implies that $\varphi_{1,0}$ is represented by the map $\Omega^{-1}k \to \Omega k$ sending $[1] \in L/x^{p-1}y^{p-1}L = \Omega^{-1}k$ to x. This map lifts to a diagram

$$\begin{tikzcd}
\Omega^{-1}k \arrow[r, "{[1]\mapsto xe_1+ye_2}"] \arrow[d, "{[1]\mapsto x}"'] & L^2 \arrow[r] \arrow[d, "(1\ 0)"'] & \Omega^{-2}k \arrow[d, "{[e_1]\mapsto 1, [e_2]\mapsto 0}"] \\
\Omega k \arrow[r] & L \arrow[r] & k
\end{tikzcd}$$

in which the rightmost vertical map represents $\varphi_{1,0}$ and will be denoted by φ. Now we are interested in $\mathcal{P}_1(\varphi)$. Define the module $M = \Omega^{-2}k/([e_2]\cdot L)$, that is, we divide out the submodule generated by $[e_2]$. Then φ factors uniquely as $\Omega^{-2}k \to M \to k$, and we denote the latter map by $\tilde{\varphi}$. Starting with some element of $-\mathcal{P}_1(\varphi) = \mathcal{D}_{2p-3}(\varphi)$ we obtain the following commutative diagram:

$$\begin{tikzcd}
(\Omega^{-2}k)^{\otimes p} \arrow[r] \arrow[d, equal] & P_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & P_{2p-3} \arrow[r] \arrow[d] & k \arrow[d, equal] \\
(\Omega^{-2}k)^{\otimes p} \arrow[r] \arrow[d] & (\Omega^{-2}k)^{\otimes p} \arrow[r] \arrow[d] & \cdots \arrow[r] & (\Omega^{-2}k)^{\otimes p} \arrow[r, "\varphi^{\otimes p}"'] \arrow[d] & k \arrow[d, equal] \\
M^{\otimes p} \arrow[r, "1-T"'] & M^{\otimes p} \arrow[r, "N"'] & \cdots \arrow[r, "1-T"'] & M^{\otimes p} \arrow[r, "\tilde{\varphi}^{\otimes p}"'] & k
\end{tikzcd} \tag{3.58}$$

Let us denote the bottom row by $\mathcal{M}$. In the following, whenever we write C_p we implicitly mean the second of the two factors of G, which is generated by the element h. Then we can consider the kG-modules as kC_p-modules. As such M is free of rank 1, generated by the class of $[e_1]$, and we will often omit this class and simply write 1 for that generator of M. We can construct another complex

$$0 \to k \to M \to \Lambda^2 M \to \cdots \to \Lambda^{p-1}M \to \Lambda^p M \to 0, \tag{3.59}$$

where the maps are given by $m \mapsto \sum_{r=0}^{p-1} m \wedge h^r$. Let K be the cokernel of the first map $k \to M$, and let us splice the complex with the Koszul complex associated to the map $\tilde{\varphi} : M \to k$ (see Lemma 1.6)

$$0 \to \Lambda^p M \xrightarrow{\kappa_p^{\tilde{\varphi}}} \Lambda^{p-1}M \to \cdots \to \Lambda^2 M \xrightarrow{\kappa_2^{\tilde{\varphi}}} M \xrightarrow{\tilde{\varphi}} k \to 0. \tag{3.60}$$

Then we get a complex

$$0 \to K \to \Lambda^2 M \to \cdots \to \Lambda^{p-1} M \to \Lambda^{p-1} M \to \cdots \to \Lambda^2 M \to M \to k \to 0,$$

which we denote by $\mathcal{L}$. Finally, we write σ for the composition $M^{\otimes p} \xrightarrow{\tau} M \to K$, where the former map is given by

$$\tau : h^{i_1} \otimes h^{i_2} \otimes \cdots \otimes h^{i_p} \mapsto \begin{cases} h^{i_1} & \text{if } \{i_1, \dots, i_p\} = \{0, \dots, p-1\} \pmod p, \\ 0 & \text{otherwise.} \end{cases} \tag{3.61}$$

Lemma 3.62. *The complex $\mathcal{L}$ is exact, and the identity map of k lifts to a map of cochain complexes $\mathcal{M} \to \mathcal{L}$ such that the first map equals $-\sigma : M^{\otimes p} \to K$.*

Proof. We take a closer look at the de Rham-complex $S^p M \to M \otimes S^{p-1} M \to \Lambda^2 M \otimes S^{p-2} M \to \cdots \to \Lambda^p M \to 0$. Let us denote by X^i_j the k-linear subspace of $\Lambda^i M \otimes S^{p-i} M$ generated by all elements of the form $h_1 \wedge h_2 \wedge \cdots \wedge h_i \otimes h_{i+1} \dots h_p$ where $h_s \in C_p$ for all $s = 1, 2, \dots, p$ and the set $\{h_1, h_2, \dots, h_p\}$ has size j. Then X^i_j is in fact a kG-module, and we have $\Lambda^i M \otimes S^{p-i} M \cong \bigoplus_{j=0}^p X^i_j$ as kG-modules. Furthermore, the differential of the de Rham-complex respects this decomposition, i.e., we have a direct sum of complexes $0 \to X^0_j \to X^1_j \to X^2_j \to \cdots \to X^p_j \to 0$. In the case $j = p$ we get the complex (3.59) which is therefore exact, except possibly at the two leftmost entries, where exactness is easily checked (in fact, non-exactness of the de Rham complex completely takes place in the $(j = 1)$-part of the decomposition above). The Koszul complex (3.60) is also exact by the second sequence of Lemma 1.6, because the dimension of M as a k-vector space equals p, so that $\Lambda^{p+1} M = 0$. Altogether we obtain that $\mathcal{L}$ is exact. Furthermore, we have a map of cochain complexes from the de Rham-Koszul complex $\mathcal{RK}$ of M to $\mathcal{L}$, where the first $p-1$ maps $\Lambda^i M \otimes S^{p-i} M \to \Lambda^i M$ are projections onto direct summands, and the last $p-1$ maps $\Lambda^i M \otimes S^{p-i} M \to \Lambda^i M$ equal $\mathrm{id} \otimes S^{p-i} \tilde{\varphi}$. Together with Proposition 3.20 we get the claim. $\square$

The exact sequence $\mathcal{L}$ represents an element γ of $\operatorname{Ext}^{2p-3}_{kG}(k, K)$, and Lemma 3.62 combined with diagram (3.58) and Proposition 3.9 says that the product of any element of $\mathcal{P}_1(\varphi)$ with γ stably equals the composition

$$\gamma \mathcal{P}_1(\varphi) = \alpha : (\Omega^{-2} k)^{\otimes p} \to M^{\otimes p} \xrightarrow{\sigma} K. \tag{3.63}$$

As a next step, we want to describe γ as a map $\Omega^{2p-3} k \to K$ more explicitly. Observe that we have a commutative diagram

$$\begin{tikzcd} L^{2p-2} \arrow[r] \arrow[d] & L^{2p-3} \arrow[r] \arrow[d] & \cdots \arrow[r] & L^3 \arrow[r] \arrow[d] & L^2 \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & k \arrow[d, equal] \\ M \arrow[r, "y"'] & M \arrow[r, "y^{p-1}"'] & \cdots \arrow[r] & M \arrow[r, "y^{p-1}"'] & M \arrow[r, "y"'] & M \arrow[r] & k \end{tikzcd} \tag{3.64}$$

where the upper row is a continuation of our partially chosen projective resolution, given by matrices of the form

$$\begin{pmatrix} x & y & & & & & \\ & x^{p-1} & -y^{p-1} & & & & \\ & & x & y & & & \\ & & & \ddots & & & \\ & & & & x^{p-1} & -y^{p-1} & \\ & & & & & x & y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x^{p-1} & -y & & & & & \\ & x & y^{p-1} & & & & \\ & & x^{p-1} & -y & & & \\ & & & \ddots & & & \\ & & & & x^{p-1} & -y & \\ & & & & & x & y^{p-1} \end{pmatrix}.$$

Let us define $\Omega^i k$ by this projective resolution. The vertical maps are given by projection onto the last factor $L^j \to L$ followed by the canonical projection map $L \to L/xL \cong M$.

Furthermore, there is a map of cochain complexes

$$\begin{tikzcd} M \arrow[r, "1-h"] \arrow[d] & M \arrow[r, "(1-h)^{p-1}"] \arrow[d] & \cdots \arrow[r, "(1-h)^{p-1}"] & M \arrow[r, "1-h"] \arrow[d] & M \arrow[r] \arrow[d] & k \arrow[d, equal] \\ M^{\otimes p} \arrow[r, "1-T"'] & M^{\otimes p} \arrow[r, "N"'] & \cdots \arrow[r, "N"'] & M^{\otimes p} \arrow[r, "1-T"'] & M^{\otimes p} \arrow[r, "\tilde{\varphi}^{\otimes p}"'] & k \end{tikzcd} \tag{3.65}$$

where each vertical arrow is the kC_p-linear map given by sending $1 \in M$ to $1 \otimes h^{p-1} \otimes h^{p-2} \otimes \cdots \otimes h \in M^{\otimes p}$; this is in fact a kG-linear map. Together with diagram (3.64) and Lemma 3.62 we obtain a diagram as follows:

$$\begin{tikzcd} L^{2p-2} \arrow[r] \arrow[d] & L^{2p-3} \arrow[r] \arrow[d] & \cdots \arrow[r] & L^2 \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & k \arrow[d, equal] \\ K \arrow[r] & \Lambda^2 M \arrow[r] & \cdots \arrow[r] & \Lambda^2 M \arrow[r] & M \arrow[r] & k \end{tikzcd}$$

The leftmost vertical map factors as $L^{2p-2} \twoheadrightarrow \Omega^{2p-3}k \to K$ where the second map represents $\gamma \in \operatorname{Ext}^{2p-3}_{kG}(k, K)$. We have therefore shown:

Lemma 3.66. *Consider $\Omega^{2p-3}k$ as a quotient of L^{2p-2}, and denote the basis elements of the latter by $f_1, f_2, \ldots, f_{2p-2}$. Then $\gamma : \Omega^{2p-3}k \to K$ is represented by the following unstable map:*

$$\begin{aligned} [f_{2p-2}] &\mapsto -\sigma(1 \otimes h^{p-1} \otimes h^{p-2} \otimes \cdots \otimes h), \\ [f_j] &\mapsto 0 \qquad\qquad \textit{for } j \neq 2p-2. \end{aligned}$$

At the end of our proof we are going to use the identity

$$\varphi_{2p-1,2p-3} \cdot v_1^{p-1} u_1 = \varphi_{0,2p-3},$$

so let us now construct unstable representatives of $v_1^{p-1}u_1$ and $\varphi_{0,2p-3}$. Let us define $b = [e_1] \cdot y^{p-1} \in (\Omega^{-2}k)^{\otimes p}$.

Lemma 3.67. *The class $v_1^{p-1}u_1 \in \hat{H}^*(G)$ is represented by the map $\Omega^{-1}k \to (\Omega^{-2}k)^{\otimes p}$ which sends $[1] \in L/x^{p-1}y^{p-1}L = \Omega^{-1}k$ to $[e_1] \otimes b^{\otimes(p-1)} \in \Omega^{-2}k$. Furthermore, the class $\varphi_{0,2p-3}$ is represented by the map $\phi : \Omega^{-1}k \to \Omega^{2p-3}k$ which sends $[1] \in L/x^{p-1}y^{p-1}L$ to $[f_{2p-2}] \in \Omega^{2p-3}k$.*

Proof. We know that v_1 is represented by the following chain map:

$$\begin{tikzcd}
\cdots & L \arrow[l] \arrow[d, "\begin{pmatrix} y^{p-1} \\ 0 \end{pmatrix}"'] & L^2 \arrow[l, "(x\ y)"'] \arrow[d, "(y^{p-1}\ 0)"] & L^3 \arrow[l, "\begin{pmatrix} x^{p-1} & -y & 0 \\ 0 & x & y^{p-1} \end{pmatrix}"'] \arrow[d, "(1\ 0\ 0)"] & \cdots \arrow[l] \\
\cdots & L^2 \arrow[l] & L \arrow[l, "\begin{pmatrix} x \\ y \end{pmatrix}"] & L \arrow[l, "x^{p-1}y^{p-1}"] & \cdots \arrow[l]
\end{tikzcd}$$

In particular, the map sending $1 \in k$ to $b \in \Omega^{-2}k$ represents v_1. Secondly, u_1 is represented by the chain map

$$\begin{tikzcd}
\cdots & L \arrow[l] \arrow[d, "\begin{pmatrix} 1 \\ 0 \end{pmatrix}"'] & L \arrow[l, "x^{p-1}y^{p-1}"'] \arrow[d, "x^{p-2}y^{p-1}"] & L^2 \arrow[l, "(x\ y)"'] \arrow[d, "(1\ 0)"] & \cdots \arrow[l] \\
\cdots & L^2 \arrow[l] & L \arrow[l, "\begin{pmatrix} x \\ y \end{pmatrix}"] & L \arrow[l, "x^{p-1}y^{p-1}"] & \cdots \arrow[l]
\end{tikzcd}$$

and therefore the map which sends $[1] \in L/x^{p-1}y^{p-1}L \cong \Omega^{-1}k$ to $[e_1] \in \Omega^{-2}k$ represents u_1. By forming a suitable tensor product of these two maps we see that $v_1^{p-1}u_1$ is represented by the map $\Omega^{-1}k \otimes k^{\otimes(p-1)} \to (\Omega^{-2}k)^{\otimes p}$ as stated.

For the second part it is enough to show that $(v_2^{p-2}u_2) \cdot \phi = \varphi_{0,0}$ and $m \cdot \phi m = 0$ for any other monomial m of degree $2p-3$. This follows immediately from the observation that the compositions $L^{2p-2} \to \Omega^{2p-3}k \to k$ (where the second map runs through all monomials of degree $2p-3$) are exactly the projection maps onto the single factors L followed by the augmentation $L \to k$, and $v_2^{p-2}u_2$ corresponds to projection onto the last factor. $\square$

Lemma 3.68. *The diagram*

$$\begin{tikzcd}
\Omega^{-1}k \arrow[r, "\varphi_{0,2p-3}"] \arrow[d, "v_1^{p-1}u_1"'] & \Omega^{2p-3}k \arrow[d, "\gamma"] \\
(\Omega^{-2}k)^{\otimes p} \arrow[r, "\alpha"'] & K
\end{tikzcd}$$

commutes, and both compositions are stably non-trivial.

Proof. By Lemmas 3.66 and 3.67, the upper-right composition sends $[1] \in \Omega^{-1}k$ to $-\sigma(1 \otimes h^{p-1} \otimes \cdots \otimes h) \in K$. For the other composition note that the quotient map $\Omega^{-2}k \to M$ sends b to the norm element $N = \sum_{j=0}^{p-1} h^j \in M$. By Lemma 3.67, the lower-left composition sends $[1] \in \Omega^{-1}k$ to $\sigma(1 \otimes N^{\otimes(p-1)})$. By definition of σ, it is therefore enough to show that

$$-\tau(1 \otimes h^{p-1} \otimes \cdots \otimes h) = \tau(1 \otimes N^{\otimes(p-1)}) \in M,$$

where τ was constructed in (3.61). But both sides equal $-1 = (p-1)! \in M$.

If the compositions were stably trivial, they would factor over the inclusion $\Omega^{-1}k \to L^2$. In particular, the image w of $[1] \in \Omega^{-1}k$ would lie in $K \cdot I$, where $I = \langle x, y\rangle_L$ is the augmentation ideal of L. But K is isomorphic to $k[y]/y^{p-1}$ (with trivial x-action) and w is a generator of K, so $w \notin K \cdot I$. $\square$

We are ready to complete the proof of Proposition 3.56. We know that

$$\begin{aligned}
\gamma\,\varphi_{0,2p-3} &= \alpha\, v_1^{p-1} u_1 && \text{by Lemma 3.68,}\\
&= \gamma\,\mathcal{P}_1(\varphi)\, v_1^{p-1} u_1 && \text{by (3.63),}\\
&= t \cdot \gamma\,\varphi_{0,2p-3} && \text{by Proposition 3.54,}
\end{aligned}$$

and all these maps in $\underline{\mathrm{Hom}}_{kG}(\Omega^{-1}k, K)$ are non-trivial by Lemma 3.68. This implies that $t = 1$. $\square$

3.7 Further examples

Let us also consider another family of examples. We take $G = (\mathbb{Z}/p\mathbb{Z})^r$, the direct product of r cyclic factors with $r \geq 3$. Then it will turn out that $\mathcal{P}_1$ vanishes on elements of negative degree. Let us briefly recall the multiplicative structure of the graded commutative algebra $\hat{H}^*(G)$, see also Proposition 4.12 in [16]. For $p = 2$ it is generated by elements $u_1, \ldots, u_r$ of degree 1 and elements φ_α of degree $-1 - |\alpha|$ for all multi-indices $\alpha \in \mathbb{N}^r$ subject to the relations

$$\begin{aligned}
\varphi_\alpha \varphi_\beta &= 0,\\
\varphi_\alpha u_i &= \begin{cases} \varphi_{\alpha-\epsilon_i} & \text{if } \alpha_i > 0,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Here ϵ_i denotes the multi-index $(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 sitting in the i-th position. For odd primes p we know that $\hat{H}^*(G)$ is generated by exterior classes $u_1, \ldots, u_r$ of degree 1, polynomial classes $v_1, \ldots, v_r$ of degree 2, and elements φ_α of degree $-1 - |\alpha|$ for all multi-indices $\alpha \in \mathbb{N}^r$, subject to the relations

$$\begin{aligned}
\varphi_\alpha \varphi_\beta &= 0,\\
\varphi_\alpha u_i &= \begin{cases} \pm\varphi_{\alpha-\epsilon_i} & \text{if } \alpha_i \text{ is odd},\\ 0 & \text{otherwise,}\end{cases}\\
\varphi_\alpha v_i &= \begin{cases} \pm\varphi_{\alpha-2\epsilon_i} & \text{if } \alpha_i \geq 2,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Let us write w^α for the monomial $v_1^{c_1} \cdots v_r^{c_r} u_1^{e_1} \cdots u_r^{e_r}$ with $\alpha_i = 2c_i + e_i$ and $e_i \in \{0, 1\}$. When α runs through all multi-indices with fixed norm $|\alpha|$, we obtain a k-linear basis of $H^{|\alpha|}(G)$, and we assume that its dual basis is the set of φ_α's in $\hat{H}^{-|\alpha|-1}(G)$.

Proposition 3.69. *For $G = (\mathbb{Z}/p\mathbb{Z})^r$ with $r \geq 3$ the power operation $\mathcal{P}_1$ vanishes on classes of negative degree.*

Proof. Let us do the proof for odd primes p, as it is the much more difficult one. As in Lemma 3.48 one sees that $\mathcal{P}_1$ does not have any indeterminacy. Using the same method as in the proof of Proposition 3.54 we see that it is enough to show that $\mathcal{P}_1(\varphi_\gamma) = 0$ for all φ_γ of even degree with $\gamma_i \in \{0, 1\}$ for all i. Suppose that two of the indices, say γ_1 and γ_2, equal zero. Then we have

$$\begin{aligned}
\pm\mathcal{P}_1(\varphi_\gamma) &= \mathcal{P}_1(\varphi_{\gamma+\epsilon_1+\epsilon_2} \cdot u_1u_2) \\
&= \mathcal{P}_1(\varphi_{\gamma+\epsilon_1+\epsilon_2}) \cdot \underbrace{(u_1u_2)^p}_{0} + \underbrace{\varphi^p_{\gamma+\epsilon_1+\epsilon_2}}_{0} \cdot \mathcal{P}_1(u_1u_2) = 0.
\end{aligned}$$

It is therefore enough to consider the case $\gamma = (1, 1, 1, \dots, 1, 1)$ for r odd and $\gamma = (1, 1, 1, \dots, 1, 0)$ for r even. Let us write $s = r$ for r odd and $s = r - 1$ for r even; then $s \geq 3$ is odd, $\gamma_i = 1$ for all $i \leq s$, and $\gamma_i = 0$ for $i > s$. The degree of φ_γ equals $-s - 1$. For the proof of $\mathcal{P}_1(\varphi_\gamma) = 0$ we will now use the naturality condition on $\mathcal{P}_1$ with respect to several automorphisms of G.

Let ψ be the automorphism of G represented by a diagonal matrix with entries $a_1, a_2, \dots, a_r \in \mathbb{F}_p^*$. As in the proof of Proposition 3.49 we get that $\psi^*(u_i) = a_iu_i$, $\psi^*(v_i) = a_iv_i$, and therefore $\psi^*(w^\alpha) = a^{c+e}w^\alpha$, where $\alpha = 2c + e$ for some multi-indices c and e with $e_i \in \{0, 1\}$ for all i. Here we use the notation $a^\beta = a_1^{\beta_1} \dots a_r^{\beta_r}$. By duality we obtain $\psi^*(\varphi_\alpha) = a^{c+e}w^\alpha$. From the naturality of $\mathcal{P}_1$ we get that

$$\psi^*(\mathcal{P}_1(\varphi_\alpha)) = \mathcal{P}_1(\psi^*(\varphi_\alpha)) = \mathcal{P}_1(a^{c+e}\varphi_\alpha) = a^{c+e}\mathcal{P}_1(\varphi_\alpha).$$

Therefore, $\mathcal{P}_1(\varphi_\gamma)$ is a k-linear combination of φ_α's with $a^\gamma = a^{c+e}$ for all $a = (a_1, \dots, a_r)$. This can only be true if $c_i + e_i \equiv 1 \pmod{p-1}$ which implies that

$$\alpha_i = 1 \text{ or } \alpha_i = 2 \text{ or } \alpha_i \geq 2p - 1, \text{ for all } i = 1, 2, \dots, s. \tag{3.70}$$

Now we take another automorphism as follows: choose indices $n, m \leq s$ with $n \neq m$ and let ψ be defined by

$$\psi^*(u_m) = u_n + u_m, \qquad \psi^*(u_i) = u_i \quad \text{for all } i \neq m.$$

Then ψ^* acts on ordinary cohomology $H^*(G)$ according to the formula

$$\psi^*(w^\alpha) = \begin{cases} \sum_{j=0}^{c_m} \binom{c_m}{j} \left(w^{\alpha-2j(\epsilon_m-\epsilon_n)} + w^{\alpha-(2j+1)(\epsilon_m-\epsilon_n)}\right) & \text{if } \alpha_m \text{ is odd and } \alpha_n \text{ is even,} \\ \sum_{j=0}^{c_m} \binom{c_m}{j} w^{\alpha-2j(\epsilon_m-\epsilon_n)} & \text{otherwise.} \end{cases}$$

for $\alpha = 2c + e$. By duality we get

$$\psi^*(\varphi_\alpha) = \begin{cases} \sum_{j\geq 0} \binom{c_m+j}{j} \left(\varphi_{\alpha+2j(\epsilon_m-\epsilon_n)} + \varphi_{\alpha+(2j+1)(\epsilon_m-\epsilon_n)}\right) & \text{if } \alpha_m \text{ is even and } \alpha_n \text{ is odd,} \\ \sum_{j\geq 0} \binom{c_m+j}{j} \varphi_{\alpha+2j(\epsilon_m-\epsilon_n)} & \text{otherwise.} \end{cases}$$

Here we used the convention that $\varphi_\beta = 0$ if $\beta_i < 0$ for some i. Since φ_γ is ψ-invariant, we know that $\mathcal{P}_1(\varphi_\gamma)$ is in the kernel of $\psi^* - \mathrm{id}$. Let us write $A = \mathcal{P}_1(\varphi_\gamma) = \sum_\alpha f_\alpha \cdot \varphi_\alpha$

with $f_\alpha \in k$, and let β be some multi-index with $\beta_m = 4$ such that $|\varphi_\beta| = |A|$. How often does φ_β occur in $(\psi^* - \mathrm{id})A$? The formulae above imply that it occurs with factor $\binom{2}{1} f_{\beta-2(\epsilon_m-\epsilon_n)} + \binom{2}{2} f_{\beta-4(\epsilon_m-\epsilon_n)}$. By (3.70) the second summand vanishes, because the m-th entry of the multi-index equals 0. Since $(\psi^* - \mathrm{id})A = 0$ we get that $f_{\beta-2(\epsilon_m-\epsilon_n)} = 0$. Note that the m-th entry of the multi-index equals 2, and every multi-index α of the right degree with $\alpha_m = 2$ is of that form for a suitable n. Hence A is a linear combination of φ_α's satisfying

$$\alpha_i = 1 \text{ or } \alpha_i \geq 2p-1, \text{ for all } i = 1, 2, \ldots, s. \tag{3.71}$$

Now let β be some multi-index with $\beta_m = 3$ such that $|\varphi_\beta| = |A|$. As before we see that φ_β occurs in $(\Psi^* - \mathrm{id})A$ with factor

$$\begin{cases} \binom{1}{0} f_{\beta-(\epsilon_m-\epsilon_n)} + \binom{1}{1} f_{\beta-2(\epsilon_m-\epsilon_n)} + \binom{1}{1} f_{\beta-3(\epsilon_m-\epsilon_n)} & \text{if } \beta_n \text{ is even,} \\ \binom{1}{1} f_{\beta-2(\epsilon_m-\epsilon_n)} & \text{otherwise.} \end{cases}$$

In the first case, the first and the last summand vanish due to (3.71). Therefore we get $f_{\beta-2(\epsilon_m-\epsilon_n)} = 0$ in both cases. The m-th entry of the multi-index equals 1 and every multi-index α of the right degree with $\alpha_m = 1$ is of that form. Together with (3.71) we obtain that A is a linear combination of φ_α's satisfying $\alpha_i \geq 2p-1$ for all $i = 1, 2, \ldots, s$. In particular, the degree satisfies $|\varphi_\alpha| \leq -s(2p-1)-1$. But we know that $A = \mathcal{P}_1(\varphi_\gamma)$ lives in degree $(-s-1)p-(2p-3) = -(s+3)p+3$. So we deduce that $-(s+3)p+3 \leq -s(2p-1)-1$, which is equivalent to $(s-3)(p-1) \leq -1$, a contradiction.

This completes the proof for odd primes p. For $p = 2$ one only needs to check that $\mathcal{P}_1(\varphi_{0,0,\ldots,0}) = 0$. We know that $\varphi_{0,0,\ldots,0}$ is invariant under all automorphisms ψ. By applying this fact to permutations of the r factors we get that

$$\mathcal{P}_1(\varphi_{0,0,\ldots,0}) = a \cdot \sum_i \varphi_{2\epsilon_i} + b \cdot \sum_{i<j} \varphi_{\epsilon_i+\epsilon_j}$$

for some $a, b \in k$. Then using an automorphism of the form

$$\begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

one easily deduces that $a = b = 0$. □

Remark 3.72. The following diagram might be conceptually helpful. It gathers information about the action of power operations on Tate cohomology classes of negative degree for elementary abelian p-groups.

	$\mathbb{Z}/p\mathbb{Z}$	$(Z/p\mathbb{Z})^{\times 2}$	$(Z/p\mathbb{Z})^{\times 3}$	$(Z/p\mathbb{Z})^{\times 4}$	...
$\mathcal{P}_0$	non-trivial	trivial	trivial	trivial	...
$\mathcal{P}_1$	undetermined	non-trivial	trivial	trivial	...

One might expect that this diagram can be extended downwards by defining higher order power operations, but we do not have any evidence for this.

4 An obstruction for higher order

We are now able to prove that our power operation serves as an obstruction for the order to be one larger. Recall that for every kG-module M we have a graded algebra $\widehat{\operatorname{Ext}}^*_{kG}(M,M)$ (which is not graded commutative in general) and there is a natural morphism of graded algebras $\hat{H}^*(G) \to \widehat{\operatorname{Ext}}^*_{kG}(M,M)$ given by tensoring with M. We can view $\widehat{\operatorname{Ext}}^*_{kG}(M,M)$ as an $\hat{H}^*(G)$-module via that map.

Theorem 4.1. *Let $\zeta \in \hat{H}^n(G)$ be a Tate cohomology class, of even degree if p is odd. Assume for some (hence any) element a of $\mathcal{P}_1(\zeta)$ that its image in $\widehat{\operatorname{Ext}}^*_{kG}(M,M)$ is divisible by ζ, that is, $a \otimes M \in \zeta \cdot \widehat{\operatorname{Ext}}^*_{kG}(M,M)$. Then ζ-ord$(M/\zeta) \geq p-1$.*

Corollary 4.2. *Let $\zeta \in \hat{H}^n(G)$ be a Tate cohomology class, of even degree if p is odd. Assume further that some (hence any) element of $\mathcal{P}_1(\zeta)$ is divisible by ζ, that is, $\mathcal{P}_1(\zeta)$ is contained in $\zeta \cdot \hat{H}^*(G)$. Then ζ-ord$(M/\zeta) \geq p-1$ for every kG-module M, and therefore ζ-ord$(\underline{\mathfrak{mod}}\text{-}kG) \geq p$.*

By Corollary 3.46 we also have the following consequence.

Corollary 4.3. *Suppose that $\zeta \in H^n(G)$ is an ordinary cohomology class, of even degree if p is odd. Assume further that the Steenrod power $\beta P^{\frac{n}{2}-1}\zeta$ (resp. $\mathrm{Sq}^{n-1}\zeta$ if $p=2$) is divisible by ζ, i.e., it is an element of $\zeta \cdot H^*(G)$. Then ζ-ord$(M/\zeta) \geq p-1$ for every kG-module M.*

In what follows, we will work with the same notation as in §2, that is, the cohomology class $[\zeta]$ is represented by a surjective unstable map $\zeta : \Omega^n k \to k$, we have a short exact sequence $0 \to L_\zeta \xrightarrow{\iota} \Omega^n k \xrightarrow{\zeta} k \to 0$ and an unstable map $\eta : \Omega k \to L_\zeta$ such that the triangle $\Omega k \xrightarrow{\eta} L_\zeta \xrightarrow{\iota} \Omega^n k \xrightarrow{\zeta} k$ is exact.

4.1 A commutative square

As a first step in the proof of Theorem 4.1, we will show that there is a commutative diagram in $\underline{\mathfrak{mod}}$-$kG$

$$\begin{tikzcd}
(\Omega^n k)^{\otimes p} \arrow[r, "P_1\zeta"] \arrow[d, "\wr"'] & \Omega^{2p-3}k \arrow[r, "\Omega^{2p-4}\eta"] & \Omega^{2p-4}L_\zeta \\
\Omega^n k \otimes S^{p-1}\Omega^n k \arrow[r, "\gamma"'] & \Omega^{p-2}\Lambda^p\Omega^n k \arrow[r, "\Omega^{p-2}\kappa_p"'] & \Omega^{p-2}\Lambda^{p-1}L_\zeta \arrow[u, "\wr"', "\Omega^{p-2}\omega_{p-1}"]
\end{tikzcd} \tag{4.4}$$

for every element $P_1\zeta \in \mathcal{P}_1(\zeta)$. The upper row vanishes if and only if $P_1\zeta$ is divisible by ζ. In a second step we will prove that vanishing of the bottom row implies ζ-ord$(X/\zeta) \geq p-1$.

Let us define the maps in (4.4). The vertical map on the left-hand side is the canonical morphism $(\Omega^n k)^{\otimes p} \to \Omega^n k \otimes S^{p-1}\Omega^n k$, we will show in Remark 4.11 that this map is a stable isomorphism. For the other vertical map note that by Corollary 1.22 the modules $\Lambda^i(\Omega^n k)$ are projective for $i = 2, 3, \dots, p-1$, and by Corollary 1.12 the sequence

$$0 \to \Lambda^j L_\zeta \hookrightarrow \Lambda^j(\Omega^n k) \xrightarrow{\kappa_j} \Lambda^{j-1}(\Omega^n k) \xrightarrow{\kappa_{j-1}} \dots \xrightarrow{\kappa_3} \Lambda^2(\Omega^n k) \xrightarrow{\kappa_2} L_\zeta \to 0 \tag{4.5}$$

is exact for every $j = 1, 2, \dots, p-1$. It therefore represents a stable isomorphism ω_j in $\underline{\mathrm{Hom}}_{kG}(\Lambda^j L_\zeta, \Omega^{j-1} L_\zeta)$ whose inverse is given by the same complex viewed as an extension (see Remark 3.10). Note that ω_1 is simply the identity map of L_ζ. Let us also consider the complex

$$\Omega^n k \otimes S^{p-1}\Omega^n k \to \Lambda^2\Omega^n k \otimes S^{p-2}\Omega^n k \to \cdots \to \Lambda^{p-1}\Omega^n k \otimes \Omega^n k \to \Lambda^p \Omega^n k;$$

again we use that the modules $\Lambda^i\Omega^n k$ are projective for $i = 2, 3, \dots, p-1$, so we get a class γ in $\underline{\mathrm{Hom}}_{kG}(\Omega^n k \otimes S^{p-1}\Omega^n k, \Omega^{p-2}\Lambda^p\Omega^n k)$.

Remark 4.6. Before we start proving that the diagram commutes, let us consider the case $p = 2$ and draw some analogies to the topological world. The diagram takes the following form:

$$\begin{tikzcd}
\Omega^n k \otimes \Omega^n k \arrow[r, "P_1\zeta"] \arrow[d, equal] & \Omega k \arrow[r, "\eta"] & L_\zeta \arrow[d, equal] \\
\Omega^n k \otimes \Omega^n k \arrow[r, "\cdot\wedge\cdot"] & \Lambda^2\Omega^n k \arrow[r, "\kappa_2"] & L_\zeta
\end{tikzcd}$$

This enables us to prove Theorem 4.1 in the case when $X = k$ by considering the commutative diagram on the left-hand side:

$$\begin{tikzcd}
\Omega^n k \otimes L_\zeta \arrow[rr, "\zeta\otimes\mathrm{id}"] \arrow[d] & & L_\zeta \\
& \Lambda^2\Omega^n k \arrow[ur, "\kappa_2"'] & \\
\Omega^n k \otimes \Omega^n k \arrow[ur, "\cdot\wedge\cdot"] \arrow[rr, "\mathrm{Sq}_1\,\zeta"'] & & k \arrow[uu, "\eta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
S/2 \arrow[r, "2\cdot"] \arrow[d, "\mathrm{pinch}"'] & S/2 \\
\Sigma S \arrow[r, "\boldsymbol{\eta}=\mathrm{Sq}_1(2)"'] & S \arrow[u, "\mathrm{incl}"']
\end{tikzcd}$$

Note the similarity to the topological situation on the right-hand side, where $\boldsymbol{\eta}$ denotes the Hopf map (compare §5 in [23]).

Proposition 4.7. *The diagram* (4.4) *commutes in* $\underline{\mathfrak{mod}}$*-kG.*

Proof. We postpone the case $p = 2$ to the end of the proof and assume $p \geq 3$. As in Lemma 3.29 denote by K the kernel of the surjective map $\zeta^{\otimes p} : (\Omega^n k)^{\otimes p} \to k$. Then we get the dashed arrows making the following diagram commute in $\mathfrak{mod}$-kG:

$$\begin{tikzcd}
\Lambda^2\Omega^n k \otimes S^{p-2}\Omega^n k \arrow[r] \arrow[d, dashed, "\lambda"'] & \Omega^n k \otimes S^{p-1}\Omega^n k \arrow[r] \arrow[d] & S^p\Omega^n k \arrow[d, "S^p\zeta"] \\
K \arrow[r] \arrow[d, dashed, "\rho"'] & (\Omega^n k)^{\otimes p} \arrow[r, "\zeta^{\otimes p}"] \arrow[d, "\mathrm{id}\otimes\zeta^{\otimes(p-1)}"] & k \arrow[d, equal] \\
L_\zeta \arrow[r, "\iota"'] & \Omega^n k \arrow[r, "\zeta"'] & k
\end{tikzcd} \tag{4.8}$$

On the other hand the diagram

$$\begin{tikzcd}
\Lambda^2\Omega^n k \otimes S^{p-2}\Omega^n k \arrow[r] \arrow[d, "\kappa_2 \otimes S^{p-2}\zeta"'] & \Omega^n k \otimes S^{p-1}\Omega^n k \arrow[d, "\mathrm{id} \otimes S^{p-1}\zeta"] \\
L_\zeta \arrow[r, "\iota"'] & \Omega^n k
\end{tikzcd}$$

commutes because both compositions are given by

$$x_1 \wedge x_2 \otimes x_3 \dots x_p \mapsto \Big(\zeta(x_1)x_2 - \zeta(x_2)x_1\Big)\zeta(x_3)\dots\zeta(x_p).$$

From the two diagrams we learn that $\rho\lambda = \kappa_2 \otimes S^{p-2}\zeta$. Therefore, the leftmost square in the diagram

$$\begin{tikzcd}
L_\zeta \arrow[d, equal] & \Lambda^2\Omega^n k \arrow[l, "\kappa_2"'] & \Lambda^3\Omega^n k \arrow[l, "\kappa_3"'] & \cdots \arrow[l, "\kappa_4"'] & \Lambda^{p-1}\Omega^n k \arrow[l, "\kappa_{p-1}"'] & \Lambda^p\Omega^n k \arrow[l, "\kappa_p"'] \arrow[d, equal] \\
L_\zeta & \Lambda^2\Omega^n k \otimes S^{p-2}\Omega^n k \arrow[l, "\rho\lambda"'] \arrow[u, "\mathrm{id} \otimes S^{p-2}\zeta"'] & \Lambda^3\Omega^n k \otimes S^{p-3}\Omega^n k \arrow[l] \arrow[u, "\mathrm{id} \otimes S^{p-3}\zeta"'] & \cdots \arrow[l] & \Lambda^{p-1}\Omega^n k \otimes \Omega^n k \arrow[l] \arrow[u, "\mathrm{id} \otimes \zeta"'] & \Lambda^p\Omega^n k \arrow[l]
\end{tikzcd}$$

commutes. When we apply Example 1.17 to the map $\zeta : \Omega^n k \to k$ we get that the remaining squares commute. The upper row (and therefore also the lower row) represents the composition $\Lambda^p\Omega^n k \xrightarrow{\kappa_p} \Lambda^{p-1}L_\zeta \xrightarrow{\omega_{p-1}} \Omega^{p-2}L_\zeta$ due to Remark 3.13.

We can now splice the lower row of the previous diagram with the complex representing γ and obtain a complex

$$\Omega^n k \otimes S^{p-1}\Omega^n k \to \Lambda^2\Omega^n k \otimes S^{p-2}\Omega^n k \to \cdots \to \Lambda^2\Omega^n k \otimes S^{p-2}\Omega^n k \to L_\zeta, \tag{4.9}$$

representing the composition

$$\Omega^n k \otimes S^{p-1}\Omega^n k \xrightarrow{\gamma} \Omega^{p-2}\Lambda^p\Omega^n k \xrightarrow{\Omega^{p-2}\kappa_p} \Omega^{p-2}\Lambda^{p-1}L_\zeta \xrightarrow{\Omega^{p-2}\omega_{p-1}} \Omega^{2p-4}L_\zeta.$$

On the other hand, the complex (4.9) represents the composition

$$\Omega^n k \otimes S^{p-1}\Omega^n k \cong (\Omega^n k)^{\otimes p} \xrightarrow{P_1\zeta} \Omega^{2p-3}k \xrightarrow{\Omega^{2p-4}\upsilon} \Omega^{2p-4}K \xrightarrow{\Omega^{2p-4}\rho} \Omega^{2p-4}L_\zeta$$

in $\underline{\mathfrak{mod}}$-$kG$ by Lemma 3.29, so we will be done as soon as we show $\rho \circ \upsilon = \eta$. But η and υ are represented by the bottom and middle row of (4.8), respectively. By Remark 3.15 the diagram shows $\rho\upsilon = \eta$.

We are left with the slightly different case $p = 2$. We get the dashed arrows making the following diagram commute in $\mathfrak{mod}$-kG:

$$\begin{tikzcd}
\Omega^n k \otimes \Omega^n k \arrow[r, "1+T"] \arrow[d, dashed, "\lambda"'] & \Omega^n k \otimes \Omega^n k \arrow[r] \arrow[d, equal] & S^2\Omega^n k \arrow[d, "S^2\zeta"] \\
K \arrow[r] \arrow[d, dashed, "\rho"'] & (\Omega^n k)^{\otimes 2} \arrow[r, "\zeta^{\otimes 2}"] \arrow[d, "\mathrm{id} \otimes \zeta"] & k \arrow[d, equal] \\
L_\zeta \arrow[r, "\iota"'] & \Omega^n k \arrow[r, "\zeta"'] & k
\end{tikzcd} \tag{4.10}$$

As before we get $\rho\upsilon = \eta$ in the stable category. Recall that $\gamma : \Omega^n k \otimes \Omega^n k \to \Lambda^2 \Omega^n k$ is the canonical projection map; therefore, the diagram

$$\begin{tikzcd} \Omega^n k \otimes \Omega^n k \arrow[r, "1+T"] \arrow[d, "\kappa_2 \circ \gamma"'] & \Omega^n k \otimes \Omega^n k \arrow[d, "\mathrm{id} \otimes \zeta"] \\ L_\zeta \arrow[r, "\iota"'] & \Omega^n k \end{tikzcd}$$

commutes. Together with (4.10) we get that $\rho\lambda = \kappa_2\gamma$. By Lemma 3.29, $\lambda = \upsilon \circ P_1\zeta$ for every element $P_1\zeta \in \mathcal{P}_1(\zeta)$. Gathering all the results, we have

$$\eta P_1 \zeta = \rho \upsilon P_1 \zeta = \rho\lambda = \kappa_2 \gamma. \qquad \square$$

Remark 4.11. Let us prove that the canonical map $(\Omega^n k)^{\otimes j} \to S^j \Omega^n k$ is a stable isomorphism for $j < p$. To do so, it is enough to check that the map $\Omega^n k \otimes S^{j-1} \Omega^n k \to S^j \Omega^n k$ is a stable isomorphism; but this map sits in an exact sequence

$$0 \to \Lambda^j \Omega^n k \to \Lambda^{j-1} \Omega^n k \otimes \Omega^n k \to \cdots \to \Omega^n k \otimes S^{j-1} \Omega^n k \to S^j \Omega^n k \to 0$$

in which all the $\Lambda^i \Omega^n k$ are projective for $i = 2, \ldots, j$.

4.2 Completion of the proof

Now we are ready to complete the proof of Theorem 4.1. We need a new kind of objects which store more information than s-coherent modules do.

Definition 4.12. *An (s,t)-coherent module consists of*

- *an s-coherent module X,*
- *a graded left $S^* L_\zeta$-comodule Y_* which vanishes outside degrees $0, 1, \ldots, t$, and*
- *a morphism $\sigma : X \to F(Y_t)$ of s-coherent modules*

such that the sequence

$$0 \to X_i \xrightarrow{\sigma_i} \Lambda^i L_\zeta \otimes Y_t \to \Lambda^{i+1} L_\zeta \otimes Y_{t-1} \to \cdots \to \Lambda^{i+t} L_\zeta \otimes Y_0 \to 0 \tag{4.13}$$

is exact for all $i = 0, 1, \ldots, s$, the maps being the same as in the chain complex associated to $(\Lambda^* L_\zeta, Y_*)$.

We call X_1 the underlying object of the (s,t)-coherent module.

Lemma 4.14. *Suppose that (X, Y) is an (s,t)-coherent module. Then the stable isomorphisms $\Omega k \otimes X_i \xrightarrow{\eta \otimes \mathrm{id}} L_\zeta \otimes X_i \to X_{1+i}$ and*

$$\Omega k \otimes \Lambda^i L_\zeta \otimes Y_0 \xrightarrow{\eta \otimes \mathrm{id}} L_\zeta \otimes \Lambda^i L_\zeta \otimes Y_0 \to \Lambda^{1+i} L_\zeta \otimes Y_0$$

induce isomorphisms of the groups $\widehat{\operatorname{Ext}}^t_{kG}(\Lambda^{i+t}L_\zeta \otimes Y_0, X_i)$ *for all* $i = 1, 2, \dots, s$. *Up to these isomorphisms, the exact sequences*

$$0 \to X_i \xrightarrow{\sigma_i} \Lambda^i L_\zeta \otimes Y_t \to \Lambda^{i+1}L_\zeta \otimes Y_{t-1} \to \cdots \to \Lambda^{i+t}L_\zeta \otimes Y_0 \to 0$$

represent the same element for all $i = 1, 2, \dots, s$. *In particular, the coherent module induces an element in* $\widehat{\operatorname{Ext}}^t_{kG}(\Lambda^{t+1}L_\zeta \otimes Y_0, X_1) \cong \underline{\operatorname{Hom}}_{kG}(\Omega^t\Lambda^{t+1}L_\zeta, X_1)$.

Proof. We have a commutative diagram as follows:

$$\begin{tikzcd}
0 \arrow[r] & \Omega k \otimes X_i \arrow[r] \arrow[d, "\eta\otimes\mathrm{id}"'] & \Omega k \otimes \Lambda^i L_\zeta \otimes Y_t \arrow[r] \arrow[d] & \cdots \arrow[r] & \Omega k \otimes \Lambda^{i+t}L_\zeta \otimes Y_0 \arrow[r] \arrow[d, "\eta\otimes\mathrm{id}"] & 0 \\
0 \arrow[r] & L_\zeta \otimes X_i \arrow[r] \arrow[d] \arrow[dr, phantom, "(a)"] & L_\zeta \otimes \Lambda^i L_\zeta \otimes Y_t \arrow[r] \arrow[d] \arrow[drr, phantom, "(b)"] & \cdots \arrow[r] & L_\zeta \otimes \Lambda^{i+t}L_\zeta \otimes Y_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_{1+i} \arrow[r] & \Lambda^{1+i}L_\zeta \otimes Y_t \arrow[r] & \cdots \arrow[r] & \Lambda^{1+i+t}L_\zeta \otimes Y_0 \arrow[r] & 0
\end{tikzcd}$$

Here (a) commutes because $\sigma : X \to F(Y_t)$ is a map of s-coherent modules, and (b) commutes because it is the map of complexes associated to the map of Λ^*L_ζ-modules $L_\zeta \otimes \Lambda^*L_\zeta \to \Lambda^{1+*}L_\zeta$ and the S^*L_ζ-comodule Y_*. □

Definition 4.15. *We denote by* $\Phi(X, Y)$ *the stable map* $\Omega^t\Lambda^{t+1}L_\zeta \otimes Y_0 \to X_1$ *constructed in Lemma 4.14.*

The next lemma allows us to construct coherent modules inductively.

Lemma 4.16. *Suppose that* (X, Y) *is an* (s,t)*-coherent module with* $t < p-1$, *and let* $\varphi : K \to X_1$ *be any map of* kG*-modules. Let us write* ν *for the* S^*L_ζ*-coaction map of* Y. *Then there exists a unique graded left* S^*L_ζ*-comodule* Z *with coaction map* ν' *satisfying the following conditions:*

(i) $Z_u = Y_u$ *for all* $u = 0, 1, 2, \dots, t$, $Z_{t+1} = K$, *and* $Z_u = 0$ *for* u *outside* $\{0, 1, \dots, t+1\}$,

(ii) $\nu'_{r,u} = \nu_{r,u}$ *for all* $0 \le r \le r+u \le t$, *and*

(iii) $\nu'_{1,t} : Z_{t+1} \to L_\zeta \otimes Z_t$ *agrees with the composition* $K \xrightarrow{\varphi} X_1 \xrightarrow{\sigma_1} L_\zeta \otimes Y_t$.

Proof. The first step is to define the remaining coaction maps $\nu'_{r,t+1-r}$. Consider the following diagram:

$$\begin{tikzcd}
 & Z_{t+1} \arrow[r, "\nu'_{1,t}"] \arrow[d, dashed] & L_\zeta \otimes Z_t \arrow[r] \arrow[d] & \Lambda^2 L_\zeta \otimes Z_{t-1} \arrow[d] \\
0 \arrow[r] & S^r L_\zeta \otimes Z_{t+1-r} \arrow[r] & L_\zeta \otimes S^{r-1}L_\zeta \otimes Z_{t+1-r} \arrow[r] & \Lambda^2 L_\zeta \otimes S^{r-2}L_\zeta \otimes Z_{t+1-r}
\end{tikzcd}$$

The upper composition is zero because the sequence (4.13) is a complex (for $i = 1$). Now one has to show that the square on the right hand side commutes — either by an easy verification, or by using the fact that $\nu : Y_* \to S^*L_\zeta \otimes Y_{t+1-r}$ is a map of comodules and passing to suitable associated chain complexes. Since the bottom row is exact by Corollary 1.9, we get the existence of the dashed arrow which we call $\nu'_{r,t+1-r}$. To check that the maps ν' define a comodule structure on Z_*, consider the diagram

$$\begin{tikzcd}
Z_{1+i+j+l} \arrow[rrr, "\nu'_{1+i,j+l}"] \arrow[dr] \arrow[ddd, "\nu'_{1+i+j,l}"'] & & & S^{1+i}L_\zeta \otimes Z_{j+l} \arrow[dl] \arrow[ddd] \\
& L_\zeta \otimes Z_{i+j+l} \arrow[r] \arrow[d] & L_\zeta \otimes S^i L_\zeta \otimes Z_{j+l} \arrow[d] & \\
& L_\zeta \otimes S^{i+j} L_\zeta \otimes Z_l \arrow[r] & L_\zeta \otimes S^i L_\zeta \otimes S^j L_\zeta \otimes Z_l & \\
S^{1+i+j} L_\zeta \otimes Z_l \arrow[ur] \arrow[rrr] & & & S^{1+i} L_\zeta \otimes S^j L_\zeta \otimes Z_l \arrow[ul, "*"']
\end{tikzcd}$$

for $i, j, l \geq 0$ with $i + j + l = t$. We claim that the exterior square commutes. The square (a) commutes because Y_* is a comodule, (b) commutes because S^*L_ζ is a coalgebra, and the squares (c) commute by definition of ν'. Injectivity of the map $*$ proves the claim. $\square$

Proposition 4.17. *Let X_1 be the underlying object of some (s,t)-coherent module (X, Y) with $t < p-1$, and let $f : K \to X_1$ be a surjective map. Then the kernel of the composition*

$$L_\zeta \otimes K \xrightarrow{\mathrm{id} \otimes f} L_\zeta \otimes X_1 \xrightarrow{\mu_{1,1}} X_2$$

is the underlying object of some $(s-1, t+1)$-coherent module.

Proof. Let (X, Y, σ) be an (s,t)-coherent module and denote by $\nu : Y_* \to S^*L_\zeta \otimes Y_*$ the structure map of Y_*. We want to define an $(s-1, t+1)$-coherent module (C, Z, σ') with $\nu' : Z_* \to S^*L_\zeta \otimes Z_*$ being the structure map of Z_*. As in the proof of Proposition 2.6, define C to be the kernel of the composite

$$F(K) \xrightarrow{F(f)} F(X_1) \to X[1].$$

Then C is an $(s-1)$-coherent module by Lemma 2.4. Furthermore, define the S^*L_ζ-comodule Z_* to be the one defined in Lemma 4.16 out of Y_* and $\varphi = f : K \to X_1$. The inclusion of C into $F(K)$ then defines the injective map $\sigma' : C \to F(Z_{t+1})$. Now we have defined all the data - it remains to show that

$$0 \to C_{i-1} \xrightarrow{\sigma'_{i-1}} \Lambda^{i-1} L_\zeta \otimes Z_{t+1} \to \Lambda^i L_\zeta \otimes Z_t \to \cdots \to \Lambda^{i+t} L_\zeta \otimes Z_0 \to 0 \tag{4.18}$$

is an exact sequence for all $i = 1, \dots s$. By definition, the sequence

$$\begin{tikzcd}
0 \arrow[r] & C_{i-1} \arrow[r, "\sigma'_{i-1}"] & \Lambda^{i-1} L_\zeta \otimes Z_{t+1} \arrow[rr] \arrow[dr, "\mathrm{id} \otimes f"'] & & X_i \arrow[r] & 0 \\
& & & \Lambda^{i-1} L_\zeta \otimes X_1 \arrow[ur, "\mu_{i-1,1}"'] & &
\end{tikzcd}$$

is exact. The Yoneda splice of this with (4.13) gives an exact sequence, which agrees with (4.18) due to the following commutative diagram:

$$\begin{tikzcd}
\Lambda^{i-1}L_\zeta \otimes Z_{t+1} \arrow[r, "\mathrm{id}\otimes\nu'_{1,t}"] \arrow[dr, "\mathrm{id}\otimes f"'] & \Lambda^{i-1}L_\zeta \otimes L_\zeta \otimes Z_t \arrow[r, "\mu_{i-1,1}\otimes\mathrm{id}"] & \Lambda^i L_\zeta \otimes Z_t \\
& \Lambda^{i-1}L_\zeta \otimes X_1 \arrow[u, "\mathrm{id}\otimes\sigma_1"'] \arrow[r, "\mu_{i-1,1}"'] & X_i \arrow[u, "\sigma_i"']
\end{tikzcd}$$

□

Suppose that (X, Y) is an (s, t)-coherent module with $t < p - 1$. By applying Lemma 4.16 to Y_* and the identity map $\varphi = \mathrm{id} : X_1 \to X_1$, we can view $(Y_0, Y_1, \ldots, Y_t, X_1)$ as an S^*L_ζ-comodule.

Definition 4.19. *We denote by $\lambda(X, Y) : X_1 \to S^{t+1}\Omega^n k \otimes Y_0$ the composition of the coaction map $X_1 \to S^{t+1}L_\zeta \otimes Y_0$ and the canonical inclusion $S^{t+1}L_\zeta \hookrightarrow S^{t+1}\Omega^n k$ tensored with Y_0.*

By Remark 1.14 the exact sequence $0 \to L_\zeta \to \Omega^n k \to k \to 0$ induces a map of graded right Λ^*L_ζ-modules $\kappa : \Lambda^{*+1}\Omega^n k \to \Lambda^* L_\zeta$; we therefore get a morphism of the associated chain complexes:

$$\begin{tikzcd}
& \Omega^n k \otimes X_1 \arrow[r] \arrow[d, "\zeta\otimes\mathrm{id}"] & \Lambda^2\Omega^n k \otimes Y_t \arrow[r] \arrow[d] & \Lambda^3\Omega^n k \otimes Y_{t-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \Lambda^{2+t}\Omega^n k \otimes Y_0 \arrow[d] \\
0 \arrow[r] & X_1 \arrow[r] & L_\zeta \otimes Y_t \arrow[r] & \Lambda^2 L_\zeta \otimes Y_{t-1} \arrow[r] & \cdots \arrow[r] & \Lambda^{1+t}L_\zeta \otimes Y_0 \arrow[r] & 0
\end{tikzcd} \tag{4.20}$$

On the other hand, the coaction map is a map of comodules $Y_* \xrightarrow{\nu} S^*L_\zeta \otimes Y_0$ and therefore induces a morphism of the associated chain complexes:

$$\begin{tikzcd}
\Omega^n k \otimes S^{t+1}L_\zeta \otimes Y_0 \arrow[r] & \Lambda^2\Omega^n k \otimes S^t L_\zeta \otimes Y_0 \arrow[r] & \cdots \arrow[r] & \Lambda^{2+t}\Omega^n k \otimes Y_0 \\
\Omega^n k \otimes X_1 \arrow[r] \arrow[u] & \Lambda^2\Omega^n k \otimes Y_t \arrow[r] \arrow[u] & \cdots \arrow[r] & \Lambda^{2+t}\Omega^n k \otimes Y_0 \arrow[u, equal]
\end{tikzcd} \tag{4.21}$$

Lemma 4.22. *Suppose that X_1 is the underlying object of some $(1, p-2)$-coherent module (X, Y). Then $\zeta_{X_1} : \Omega^n k \otimes X_1 \to X_1$ stably factors as the composition*

$$\begin{aligned}
\Omega^n k \otimes X_1 &\xrightarrow{\mathrm{id}\otimes\lambda(X,Y)} \Omega^n k \otimes S^{p-1}\Omega^n k \otimes Y_0 \\
&\xrightarrow{\gamma\otimes\mathrm{id}} \Omega^{p-2}\Lambda^p\Omega^n k \otimes Y_0 \xrightarrow{\Omega^{p-2}\kappa_p\otimes\mathrm{id}} \Omega^{p-2}\Lambda^{p-1}L_\zeta \otimes Y_0 \xrightarrow{\Phi(X,Y)} X_1.
\end{aligned} \tag{4.23}$$

Proof. As in (4.20) we get a diagram as follows:

$$\begin{tikzcd}
& \Omega^n k \otimes X_1 \arrow[r] \arrow[d, "\zeta\otimes\mathrm{id}"] & \Lambda^2\Omega^n k \otimes Y_{p-2} \arrow[r] \arrow[d] & \Lambda^3\Omega^n k \otimes Y_{p-3} \arrow[r] \arrow[d] & \cdots \arrow[r] & \Lambda^p\Omega^n k \otimes Y_0 \arrow[d, "\kappa_p\otimes\mathrm{id}"'] \\
0 \arrow[r] & X_1 \arrow[r] & L_\zeta \otimes Y_{p-2} \arrow[r] & \Lambda^2 L_\zeta \otimes Y_{p-3} \arrow[r] & \cdots \arrow[r] & \Lambda^{p-1}L_\zeta \otimes Y_0 \arrow[r] & 0
\end{tikzcd}$$

The exact lower row represents the element $\varphi = \Phi(X,Y)$ in $\widehat{\operatorname{Ext}}_{kG}^{p-2}(\Lambda^{p-1}L_\zeta \otimes Y_0, X_1)$. Since the modules $\Lambda^i\Omega^n k$ are projective for $i = 2,\dots,p-1$, the upper row represents some element $\psi \in \widehat{\operatorname{Ext}}_{kG}^{-p+2}(\Omega^n k \otimes X_1, \Lambda^p\Omega^n k \otimes Y_0)$. Commutativity of the diagram tells us that $\zeta_{X_1} = \varphi\kappa_p\psi$ by Proposition 3.9. Using diagram (4.21) and the inclusions $S^iL_\zeta \subset S^i\Omega^n k$, we get a diagram

$$\begin{tikzcd}
\Omega^n k \otimes S^{p-1}\Omega^n k \otimes Y_0 \arrow[r] & \Lambda^2\Omega^n k \otimes S^{p-2}\Omega^n k \otimes Y_0 \arrow[r] & \cdots \arrow[r] & \Lambda^p\Omega^n k \otimes Y_0 \\
\Omega^n k \otimes X_1 \arrow[r] \arrow[u, "\lambda = \mathrm{id}\otimes\lambda(X,Y)"] & \Lambda^2\Omega^n k \otimes Y_{p-2} \arrow[r] \arrow[u] & \cdots \arrow[r] & \Lambda^p\Omega^n k \otimes Y_0 \arrow[u, equal]
\end{tikzcd}$$

where the upper row represents $\gamma \otimes \mathrm{id}_{Y_0}$ and the lower row represents ψ. Hence, $\psi = \gamma\lambda$. Therefore, $\zeta_{X_1} = \varphi\kappa_p\psi = \varphi\kappa_p\gamma\lambda$. $\square$

Proposition 4.24. *Let X_1 be the underlying object of some (s,t)-coherent module (X,Y) with $s+t = p-1$. If the composition*

$$\Omega^n k \otimes S^{p-1}\Omega^n k \xrightarrow{\gamma} \Omega^{p-2}\Lambda^p\Omega^n k \xrightarrow{\Omega^{p-2}\kappa_p} \Omega^{p-2}\Lambda^{p-1}L_\zeta$$

is stably zero when we tensor it with Y_0, then ζ-ord$(X_1) \geq s$.

Proof. We do this by induction on s, starting with $s = 1$. By Lemma 4.22 and our assumptions, $\zeta_{X_1} = 0$ or equivalently ζ-ord$(X_1) \geq 1 = s$. The inductive step is done exactly the same way as in the proof of Proposition 2.6 by using Proposition 4.17. Notice that the object Y_0 does not change during the induction. $\square$

Proof of Theorem 4.1. Notice that $L_\zeta \otimes M$ is the underlying object of some $(p-1,0)$-coherent module as follows: the $(p-1)$-coherent module is $X = F(M)$, and $Y_0 = M$. The morphism $\sigma : X \to F(M)$ is simply the identity map, and the sequence (4.13) reduces to $0 \to X_i \xrightarrow{\sigma_i} \Lambda^i L_\zeta \otimes M \to 0$.

Our assumptions together with Proposition 4.7 tensored by M imply that the conditions of Proposition 4.24 are satisfied, so we get ζ-ord$(M/\zeta) = \zeta$-ord$(M \otimes L_\zeta) \geq p-1$. $\square$

5 Toda brackets

So far we only have considered lower bounds on the order. In this section, we will introduce techniques which enable us to prove upper bounds on the order. The main ingredient will be $(2p-1)$-fold Massey products. As a first step, we will recall the definition of these higher products and then show the connection to Toda brackets, an analog which can be defined in any triangulated category. In a second step we show that certain Toda brackets can be used to build up a criterion for the ζ-order to be bounded from above. More explicitly, we show that if certain $(2s-1)$-fold Toda brackets of the form $\langle\zeta, \alpha_1, \zeta, \dots, \alpha_s, \zeta\rangle$ do not contain zero, then ζ-ord$(k/\zeta) \leq s-1$. We will then try to find examples of such Toda brackets, and to do so, we will have to compute single elements of them, which is usually a difficult task. In our case we can use the relation $\alpha_1\alpha_2\dots\alpha_{p-1}\mathcal{P}_1(\zeta) \subseteq \langle\zeta, \alpha_1, \zeta, \dots, \alpha_{p-1}, \zeta\rangle$, which we will prove in the third subsection using a connection between Toda brackets and coherent modules. Finally we are able to give our explicit example in the fourth part of this section.

5.1 Toda brackets and Massey products

We are now going to prove that Massey products in $\hat{H}^*(G)$ agree with certain Toda brackets in $\underline{\mathfrak{mod}}$-$kG$.

Proposition 5.1. *Suppose we are given cohomology classes $a_1, a_2, \ldots, a_n \in \hat{H}^*(G)$. We will also view these elements as stable morphisms $\Omega^{|a_j|}k \to k$. The Massey product $\langle a_1, a_2, \ldots, a_n\rangle$ and the Toda bracket $\langle a_1, a_2, \ldots, a_n\rangle$ associated to the sequence of maps $\Omega^? k \xrightarrow{a_n} \Omega^? k \xrightarrow{a_{n-1}} \ldots \xrightarrow{a_1} k$ (where the $\Omega^? k$ are suitable shifts of k) define the same subset of $\hat{H}^{|a_1|+\cdots+|a_n|-(n-2)}(G)$.*

Let us recall the definitions first. Fix a complete projective resolution P of the trivial kG-module k, and let ∂ be the differential of P. Denote by $A = \mathrm{Hom}^*_{kG}(P_*, P_*)$ the endomorphism dga associated to P. More explicitly, A is the differential graded algebra whose degree n-part is given by

$$A^{(n)} = \prod_{j \in \mathbb{Z}} \mathrm{Hom}_{kG}(P_{n+j}, P_j),$$

and the differential is $df = \partial f - (-1)^n f\partial$. Then degree n-cocycles in A are chain transformations $P \to P[n]$, and two of those differ by a coboundary if and only if the chain transformations are homotopic. Therefore, $H^*A \cong \hat{H}^*(G)$.

An *n-fold defining system* is a collection $\{b_{ij}\}$ of elements of A for all pairs (i,j) with $1 \le i \le j \le n$ and $(i,j) \neq (1,n)$ satisfying

$$(-1)^{g_{i-1}} db_{ij} = \sum_{r=i}^{j-1} b_{i,r} b_{r+1,j}$$

for all $1 \le i \le j \le n$ with $(i,j) \neq (1,n)$, where $g_j = 1 + \sum_{r=1}^{j}(|b_{rr}| - 1)$. In particular, the b_{ii}'s are cocycles. Every defining system gives rise to a cocycle

$$c = \sum_{r=1}^{n-1} b_{1,r} b_{r+1,n}.$$

For fixed cocycles $b_{11}, \ldots, b_{nn}$, the set of cohomology classes represented by all cocycles arising from defining systems will be denoted by $\langle b_{11}, \ldots, b_{nn}\rangle$ and is called *n-fold Massey product*. It is well-known that this set only depends on the cohomology classes represented by $b_{11}, \ldots, b_{nn}$, and we will also write $\langle b_{11}, \ldots, b_{nn}\rangle = \langle c_1, \ldots, c_n\rangle$ where c_i is the cohomology class represented by b_{ii}.

Remark 5.2. It is worth noting that there are several different definitions of Massey products in the literature (see [12], [13], [20]), which differ by certain signs. A quick check of signs asserts that our definition leads to $(-1)^n$ times the set given in [12]. We have chosen this one because, as we will prove now, it agrees with the definition via certain Toda brackets which does not involve any choice of signs. Also it has the nice property that the two-fold Massey product is just the usual product (which in the other definitions is not always the case).

Remark 5.3. We also use a slightly different language than previous definitions of Massey products. Usually one says that a Massey product is *defined* if there is a defining system for it, and then the product is the set of cocycles associated to its defining systems. In the following, we will adopt the notion from Toda brackets and simply say that every Massey product is defined but might be the empty set.

For the definition of Toda brackets we take the definition of filtered objects from [26], Definition A.1. Let us work in a triangulated category $\mathcal{T}$ with shift functor [1]. Suppose that $X_{n-1} \xrightarrow{\lambda_{n-1}} X_{n-2} \xrightarrow{\lambda_{n-2}} \dots \xrightarrow{\lambda_2} X_1$ is a sequence of $n-2$ composable maps in $\mathcal{T}$. An $(n-1)$-*filtered object* $X \in \{\lambda_2, \dots, \lambda_{n-1}\}$ is a sequence of maps $* = F_0X \xrightarrow{i_0} F_1X \xrightarrow{i_1} \dots \xrightarrow{i_{n-1}} F_{n-1}X = X$ together with choices of exact triangles

$$F_jX \xrightarrow{i_j} F_{j+1}X \xrightarrow{p_{j+1}} X_{j+1}[j] \xrightarrow{d_j} F_jX[1]$$

such that $p_j[1] \circ d_j = \lambda_{j+1}[j]$. The maps $X_1 \cong F_1X \to F_{n-1}X$ and $F_{n-1}X \to X_{n-1}[n-2]$ are denoted by σ_X and σ'_X, respectively. The filtered object can be visualized as follows:

$$\begin{tikzcd}[column sep=small]
* = F_0X \arrow[rr] & & F_1X \arrow[rr] \arrow[dl] & & F_2X \arrow[rr] \arrow[dl] & & \cdots \arrow[rr] \arrow[dl] & & F_{n-2}X \arrow[rr] \arrow[dl] & & F_{n-1}X \arrow[dl] \\
& X_1 \arrow[ul, "\circ"] & & X_2[1] \arrow[ul, "\circ"] \arrow[ll, "\circ"] & & X_3[2] \arrow[ul, "\circ"] \arrow[ll, "\circ"] & & \cdots \arrow[ll, "\circ"] & & X_{n-1}[n-2] \arrow[ul, "\circ"] \arrow[ll, "\circ"]
\end{tikzcd}$$

Here $X \overset{\circ}{\longrightarrow} Y$ denotes a map of degree 1, that is, a map $X \to Y[1]$. Also, the diagram is *commutative-exact* in the following sense:

Definition 5.4. *A diagram is called* commutative-exact *if every triangle with exactly one map* $\overset{\circ}{\longrightarrow}$ *of degree one is an exact triangle, and all other triangles in the diagram are commutative.*

To be more precise, the first condition means that every triangle of the form

$$\begin{tikzcd}
X \arrow[rr, "a"] & & Y \arrow[dl, "b"] \\
& Z \arrow[ul, "c\ \circ"] &
\end{tikzcd}$$

represents an exact triangle $X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{c} X[1]$.

In the definition of a filtered object, suppose that the given set of composable maps $\lambda_{n-1}, \dots, \lambda_2$ lives in a strictly full triangulated subcategory $\mathcal{S}$. Then every $(n-1)$-filtered object X and its filtrations $F_0X, F_1X, \dots, F_{n-1}X$ also lie in $\mathcal{S}$. Recall that $\mathcal{S}$ is a *strictly* full subcategory if it is a full subcategory and for every isomorphism $B \cong B'$ in $\mathcal{T}$ for which B belongs to $\mathcal{S}$ we also have that B' belongs to $\mathcal{S}$.

Definition 5.5 ([26], Definition A.2)**.** *Suppose that* $X_n \xrightarrow{\lambda_n} X_{n-1} \xrightarrow{\lambda_{n-1}} \dots \xrightarrow{\lambda_1} X_0$ *is a sequence of* n *composable maps in a triangulated category* $\mathcal{T}$*. We say that a map*

$\gamma \in \mathcal{T}(X_n[n-2], X_0)$ *belongs to the n-fold* Toda bracket $\langle \lambda_1, \ldots, \lambda_n \rangle$ *if and only if there is an $(n-1)$-filtered object $X \in \{\lambda_2, \ldots, \lambda_{n-1}\}$ and a commutative diagram*

$$\begin{tikzcd} & X_1 \arrow[d, "\sigma_X"'] \arrow[dr, "\lambda_1"] & \\ X_n[n-2] \arrow[r, "\gamma_n"] \arrow[dr, "\lambda_n[n-2]"'] & X \arrow[r, "\gamma_0"'] \arrow[d, "\sigma'_X"] & X_0 \\ & X_{n-1}[n-2] & \end{tikzcd} \tag{5.6}$$

such that $\gamma = \gamma_0 \gamma_n$.

Let us note here that everything takes place in $\mathcal{S}$ if the n maps we started with belong to $\mathcal{S}$. It also follows from the definition that Toda brackets are compatible with exact equivalences of triangulated categories.

We will apply this to the case when $\mathcal{T} = \underline{\mathfrak{mod}}\text{-}kG$ and $\mathcal{S}$ is the strictly full triangulated subcategory generated by the trivial kG-module k. There is an equivalence of triangulated categories (see, e.g., [14], proof of Proposition 7.13 and Example 7.16)

$$Z : K_{\mathrm{ac}}(\mathrm{inj}{-}kG) \xrightarrow{\cong} \underline{\mathfrak{mod}}\text{-}kG \tag{5.7}$$

where $K_{\mathrm{ac}}(\mathrm{inj}{-}kG)$ denotes the homotopy category of (unbounded) acyclic chain complexes of (finitely generated) injective kG-modules. Let $\mathcal{M}$ be the strictly full triangulated subcategory generated by P in $K_{\mathrm{ac}}(\mathrm{inj}{-}kG)$. Since $Z(P) \cong k$ we get that $\mathcal{M}$ agrees to $\mathcal{S}$ under the equivalence Z, so we are going to describe $\mathcal{M}$ more explicitly.

Consider the set $\mathcal{C}$ of all chain complexes which arise from the following construction. Choose a non-negative integer r and integers $n_1, n_2, \ldots, n_r$; then consider the complex $(n_1, n_2, \ldots, n_r, D)$ whose modules are the same as in $P[n_1] \oplus P[n_2] \oplus \cdots \oplus P[n_r]$, but the differential is given by a matrix of the form

$$D = \begin{pmatrix} (-1)^{n_1}\partial & a_{11} & a_{12} & a_{13} & \cdots & a_{1,r} \\ & (-1)^{n_2}\partial & a_{22} & a_{23} & \cdots & a_{2,r} \\ & & (-1)^{n_3}\partial & a_{33} & \cdots & a_{3,r} \\ & & & \ddots & \ddots & \vdots \\ & & & & & (-1)^{n_r}\partial \end{pmatrix}.$$

Here ∂ is the differential of P, so that $(-1)^{n_i}\partial$ is the differential of $P[n_i]$. The a_{ij}'s are elements of the endomorphism dga A, and we assume that they are of suitable degrees and satisfy certain relations so that $D^2 = 0$. We will sometimes write $[D]$ for this complex.

Lemma 5.8. *The elements of $\mathcal{C}$ belong to $\mathcal{M}$, and every object in $\mathcal{M}$ is isomorphic (in $\mathcal{M}$) to an element of $\mathcal{C}$.*

Proof. Recall that one possibility of constructing the mapping cone of a map of chain complexes $f : D \to E$ is given as follows: take the same modules as in $E \oplus D[1]$, but with differential $\left(\begin{smallmatrix}\partial_E & f\\ & -\partial_D\end{smallmatrix}\right)$. If we have two objects $(n_1,\dots,n_r,D)$ and $(m_1,\dots,m_s,E)$ in $\mathcal{C}$ then a morphism $[D]\to[E]$ is a matrix F with entries in A satisfying $EF=FD$. Therefore, its mapping cone $\left(m_1,\dots,m_s,n_1+1,\dots,n_r+1,\left(\begin{smallmatrix}E & F\\ & -D\end{smallmatrix}\right)\right)$ is again an object in $\mathcal{C}$.

Let us prove by induction on r that $(n_1,\dots,n_r,D)\in\mathcal{C}$ belongs to $\mathcal{M}$. For $r=0$ this says that the trivial complex is in $\mathcal{M}$; for $r=1$ we need to show that shifts of P lie in $\mathcal{M}$, which is true by construction. Suppose $r\geq 2$, then we know that the object in question is the mapping cone of a morphism between two objects of $\mathcal{C}$ with smaller r. By induction hypothesis, this morphism belongs to $\mathcal{M}$, and so does its cone.

Now consider the strictly full subcategory $\mathcal{N}$ of $\mathcal{M}$ consisting of all objects which are isomorphic to objects in $\mathcal{C}$. Then $\mathcal{N}$ is closed under shifts and mapping cones (because the set $\mathcal{C}$ is), and P is contained in $\mathcal{N}$. This proves the second statement. $\square$

The following simple fact is a useful tool for lifting homotopy-commutative diagrams to strictly-commuting ones.

Lemma 5.9. *Suppose that $h:P\to Q$ is a chain map between acyclic complexes of injective modules, and that R is also an acyclic complex of injectives.*

(i) *Assume further that $g:P\to R$ is a chain map and $f:Q\to R$ is a morphism such that $fh=g$ in $K_{\mathrm{ac}}(\mathrm{inj}{-}kG)$. If h is levelwise an inclusion then there is a lift of f to a chain map $\hat{f}$ such that $\hat{f}h=g$ as chain maps, not only in $K_{\mathrm{ac}}(\mathrm{inj}{-}kG)$.*

(ii) *Assume that $g:R\to Q$ is a chain map and $f:R\to P$ is a morphism such that $hf=g$ in $K_{\mathrm{ac}}(\mathrm{inj}{-}kG)$. If h is levelwise a surjection then there is a lift of f to a chain map $\hat{f}$ such that $h\hat{f}=g$ as chain maps, not only in $K_{\mathrm{ac}}(\mathrm{inj}{-}kG)$.*

(i)
$$\begin{tikzcd} P \arrow[r, "h"] \arrow[d, "g"'] & Q \arrow[dl, dashed, "f"] \\ R & \end{tikzcd}$$

(ii)
$$\begin{tikzcd} P \arrow[r, "h"] & Q \\ & R \arrow[u, "g"'] \arrow[ul, dashed, "f"] \end{tikzcd}$$

Proof. Let us prove (i) only. We know that $fh=g$ up to some homotopy $H:P\to R[-1]$, that is, $fh-g=\partial_R H+H\partial_P$. Since h is levelwise an inclusion of a direct summand, we can lift H to a map $H':Q\to R[-1]$ satisfying $H=H'h$. Then $\hat{f}=f-\partial_R H'-H'\partial_Q$ satisfies $\hat{f}h=g$. $\square$

Proposition 5.10. *Suppose we are given a sequence of composable maps*

$$P[-m_1]=X_1\xleftarrow{a_2}X_2\xleftarrow{a_3}X_3\leftarrow\dots\xleftarrow{a_{n-1}}X_{n-1}$$

in $K_{\mathrm{ac}}(\mathrm{inj}{-}kG)$, where each a_i is a cocycle in A of degree m_i, m_1 is some integer, and the X_i's are suitable shifts of the complex P. Let us write $g_i=1+\sum_{j=1}^{i}(m_i-1)$. Then every $(n-1)$-filtered object in $\{a_2,\dots,a_{n-1}\}$ is isomorphic to an $(n-1)$-filtered object of *the following form:*

$$F_jX = (-g_1, -g_2, \dots, -g_j, \begin{pmatrix} (-1)^{g_1}\partial & a_{22} & a_{23} & \cdots & a_{2,j} \\ & (-1)^{g_2}\partial & a_{33} & \cdots & a_{3,j} \\ & & \ddots & \ddots & \vdots \\ & & & (-1)^{g_{j-1}}\partial & a_{j,j} \\ & & & & (-1)^{g_j}\partial \end{pmatrix})$$

and $i_j : F_jX \to F_{j+1}X$ is the inclusion into the first summands, $p_{j+1} : F_{j+1}X \to X_{j+1}[j]$ is the projection onto the last summand, $d_j : X_{j+1}[j] \to F_jX[1]$ is the map $-\begin{pmatrix} a_{2,j+1} & a_{3,j+1} & \dots & a_{j+1,j+1}\end{pmatrix}^T$, and $a_{jj} = -a_j$ for all j.

Here we used the notation M^T for the transpose of the matrix M.

Proof. As a first step we show that the given object is indeed an $(n-1)$-filtered object, i.e., $X_{j+1}[j-1] \xrightarrow{-d_j[-1]} F_jX \xrightarrow{i_j} F_{j+1}X \xrightarrow{p_{j+1}} X_{j+1}[j]$ is an exact triangle and $p_j[1] \circ d_j = -a_{j+1,j+1} = a_{j+1}$. The latter actually holds strictly (not only up to homotopy), and the former follows from the fact that $F_{j+1}X$ is the mapping cone (as constructed in the proof of Lemma 5.8) of $-d_j[-1]$, with i_j and p_{j+1} being the associated maps.

Now we argue by induction and consider a diagram

$$\begin{tikzcd}
{*} = F_0X \arrow[rr] & & F_1X \arrow[rr] \arrow[dl] & & F_2X \arrow[rr] \arrow[dl] & & \cdots \arrow[rr] \arrow[dl] & & F_{j-1}X \arrow[rr] \arrow[dl] \arrow[dr, phantom, "(*)"] & & F'_jX \arrow[dl] \\
& X_1 \arrow[ul, "o"] & & X_2[1] \arrow[ll, "o"'] \arrow[ul, "o"] & & X_3[2] \arrow[ll, "o"'] \arrow[ul, "o"] & & \cdots \arrow[ll, "o"'] & & X_j[j-1] \arrow[ll, "o"'] \arrow[ul, "o"] &
\end{tikzcd}$$

in which we assume that the boxed region is already of the form we want it to be. Since $F_{j-1}X \to X_{j-1}[j-2]$ is levelwise a surjection, we can assume by Lemma 5.9 that the map $d_{j-1} : X_j[j-1] \to F_{j-1}X$ is chosen in such a way that the triangle $(*)$ commutes strictly. Then d_{j-1} is of the form $-\begin{pmatrix} a_{2j} & a_{3j} & \dots & a_{j,j}\end{pmatrix}^T$ with $a_{j,j} = -a_j$. Now let F_jX be constructed as in the statement of the proposition; then the triangles

$$X_j[j-2] \xrightarrow{-d_j[-1]} F_{j-1}X \to F'_jX \to X_j[j-1]$$

and

$$X_j[j-2] \xrightarrow{-d_j[-1]} F_{j-1}X \to F_jX \to X_j[j-1]$$

are both exact and hence isomorphic. $\square$

Proof of Proposition 5.1. Due to the equivalence (5.7) we can work with Toda brackets in $\mathcal{M}$ instead of $\underline{\mathfrak{mod}}$-$kG$, so we have a sequence of composable maps $P = X_0 \xleftarrow{a_1} X_1 \xleftarrow{a_2} \dots \xleftarrow{a_{n-1}} X_{n-1} \xleftarrow{a_n} X_n$ with $a_i \in A$ and the X_i's are suitable shifts of P. Let us start with an $(n-1)$-filtered object X and maps γ_n, γ_0 such that $\gamma_0\gamma_n$ defines an element of the Toda bracket as in Definition 5.5. We can assume that X is of the form described in Proposition 5.10, where m_1 is the degree of a_1. Then $\sigma_X : P[-m_1] \to X$ is the inclusion of the first summand, and $\sigma'_X : X \to P[-g_{n-1}]$ is the projection onto the last summand. Using Lemma 5.9 we can assume that the diagram of chain complexes and chain maps

$$\begin{tikzcd}
 & X_1 \arrow[d, "\sigma_X"'] \arrow[dr, "a_1"] & \\
X_n[n-2] \arrow[r, "\gamma_n"] \arrow[dr, "a_n[n-2]"'] & X \arrow[r, "\gamma_0"'] \arrow[d, "\sigma'_X"] & X_0 \\
 & X_{n-1}[n-2] &
\end{tikzcd}$$

commutes strictly, which means that γ_n is of the form $-\begin{pmatrix} a_{2,n} & \dots & a_{n-1,n} & a_{n,n}\end{pmatrix}^T$ with $a_{nn} = -a_n$, and γ_0 is of the form $-\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1,n-1}\end{pmatrix}$ with $a_{11} = -a_1$. The fact that γ_0 and γ_n are chain maps can be expressed by saying that

$$\begin{pmatrix}
(-1)^{g_0}\partial & a_{11} & a_{12} & a_{13} & \cdots & a_{1,n-1} & 0 \\
 & (-1)^{g_1}\partial & a_{22} & a_{23} & \cdots & a_{2,n-1} & -a_{2,n} \\
 & & (-1)^{g_2}\partial & a_{33} & \cdots & a_{3,n-1} & -a_{3,n} \\
 & & & (-1)^{g_3}\partial & \cdots & a_{4,n-1} & -a_{4,n} \\
 & & & & \ddots & \vdots & \vdots \\
 & & & & & (-1)^{g_{n-1}}\partial & -a_{n,n} \\
 & & & & & & (-1)^{g_n}\partial
\end{pmatrix}^2 = \begin{pmatrix}
0 & 0 & \dots & -c \\
 & 0 & \ddots & 0 \\
 & & 0 & 0 \\
 & & & 0
\end{pmatrix},$$

where c is some cocycle representing the map $\gamma = \gamma_0\gamma_n$. By putting $b_{ij} = -a_{ij}$ (and in particular $b_{ii} = -a_{ii} = a_i$) we have therefore found a defining system whose associated cocycle is the element of the Toda bracket we started with. Going backwards through the arguments we get that every defining system yields an element of the Toda bracket. □

In view of Proposition 5.1 we will often not distinguish between Massey products and its corresponding Toda brackets. Still, one has to be careful with the signs: whenever we write a Toda bracket $\langle a_1, \dots, a_n\rangle$ with $a_i \in \hat{H}^*(G)$, we always mean the Toda bracket associated to the sequence of composable maps

$$\Omega^? k \xrightarrow{a_n} \Omega^? k \xrightarrow{a_{n-1}} \dots \xrightarrow{a_2} \Omega^? k \xrightarrow{a_1} k,$$

where the last object is k (not some shift of k).

Definition 5.11. *We say that a Toda bracket $\langle\lambda_1, \lambda_2, \dots, \lambda_n\rangle$ is* strictly defined *if for each pair $1 \le i < j \le n$ with $(i, j) \ne (1, n)$ we have $\langle\lambda_i, \dots, \lambda_j\rangle = \{0\}$.*

Corollary 5.12. *Let $c_1, c_2, \dots, c_n \in \hat{H}^*(G)$. The Toda bracket $\langle c_1, \dots, c_n\rangle$ in $\underline{\mathfrak{mod}}$-$kG$ is strictly defined if and only if the corresponding Massey product is (in the sense of May, [20]).*

Strictly defined Toda brackets (and Massey products) are much easier to deal with than arbitrary ones, mainly because every partially defined filtered object (and defining system) can be extended to a completely defined one. We will use this fact in the following version later.

Lemma 5.13. *Suppose that $X_n \xrightarrow{\lambda_n} X_{n-1} \xrightarrow{\lambda_{n-1}} \dots \xrightarrow{\lambda_1} X_0$ is a sequence of composable maps such that the Toda bracket $\langle \lambda_1, \dots, \lambda_n \rangle$ is strictly defined. Then for every $(n-1)$-filtered object $X \in \{\lambda_2, \dots, \lambda_{n-1}\}$ there are maps γ_n, γ_0 such that $\lambda_n[n-2] = \sigma'_X \gamma_n$ and $\lambda_1 = \gamma_0 \sigma_X$, that is, the diagram* (5.6) *commutes.*

Proof. Let us show the existence of γ_n only. For every pair $0 \leq i < j < n$ choose exact triangles $F_iX \to F_jX \to F_j/F_iX$. Notice that the exact triangles $F_jX \to F_{j+1}X \to X_{j+1}[j]$ induce exact triangles $F_j/F_iX \to F_{j+1}/F_iX \to X_{j+1}[j]$. Now we claim that for every $i = 0, 1, \dots, n-2$ there is a map $\beta_i : X_n[n-2] \to F_{n-1}/F_iX$ such that the composition $X_n[n-2] \xrightarrow{\beta_i} F_{n-1}/F_iX \to X_{n-1}[n-2]$ equals $\lambda_n[n-2]$. For $i = n-2$ we know that the second map in the latter composition is an isomorphism. Assume we have constructed β_i and let us construct β_{i-1}. We have an $(n-1-i)$-filtered object $* = F_i/F_iX \to F_{i+1}/F_iX \to \dots \to F_{n-1}/F_iX$ which lies in $\{\lambda_{i+2}, \dots, \lambda_{n-1}\}$, up to shifts. There is a commutative diagram

$$
\begin{tikzcd}
& F_{i+1}/F_iX \cong X_{i+1}[i] \arrow[d] \arrow[dr, "{\lambda_{i+1}[i]}"] & \\
X_n[n-2] \arrow[r, "\beta_i"] \arrow[dr, "{\lambda_n[n-2]}"'] & F_{n-1}/F_iX \arrow[r, "\tau"'] \arrow[d] & F_i/F_{i-1}X[1] \cong X_i[i] \\
& X_{n-1}[n-2] &
\end{tikzcd}
$$

where the map τ is the composite $F_{n-1}/F_iX \to F_iX[1] \to F_i/F_{i-1}X[1]$. Since $\tau\beta_i$ is an element of the Toda bracket $\langle \lambda_{i+1}, \dots, \lambda_n \rangle = \{0\}$ we know that $\tau\beta_i = 0$, which means that β_i can be factorized as $X_n[n-2] \xrightarrow{\beta_{i-1}} F_{n-1}/F_{i-1}X \to F_{n-1}/F_iX$. This completes the inductive step, and putting $\gamma_n = \beta_0$ we get the result. $\square$

We end this section with two technical lemmas. In the first one we basically recall the octahedral axiom in the form we will use it later, and we draw an immediate conclusion.

Lemma 5.14. *Suppose we are given the left-hand side diagram*

$$
\begin{tikzcd}
D \arrow[rr, "v"', "\circ"] \arrow[rrrr, dashed, bend left, "b", "\circ"'] & & B \arrow[dl, "\alpha"] \arrow[rr, "x"'] & & F \arrow[dl, "w", "\circ"'] \arrow[ddll, dashed, bend left, "a"] \\
& C \arrow[ul, "z"] \arrow[dr, "u"'] & & A \arrow[ul, "\gamma"'] \arrow[ll, "\beta"] & \\
& & E \arrow[ur, "y"', "\circ"] \arrow[uull, dashed, bend left, "c"] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
E[-1] \arrow[r, "{y[-1]}"] \arrow[d, "{c[-1]}"'] & A \arrow[d, "\gamma"] \\
D[-1] \arrow[r, "{v[-1]}"] & B
\end{tikzcd}
$$

in a triangulated category $\mathcal{T}$, without the dashed arrows, and assume further that it is commutative-exact. Then the dashed arrows exist in such a way that the diagram is still commutative-exact and $\gamma[1]y = vc$ and $u\alpha = ax$. Furthermore, $E[-1]$ is a weak pullback in the right-hand side diagram.

Proof. The first statement is the octahedral axiom. For the second statement let $T \xrightarrow{\lambda} A$ and $T \xrightarrow{\kappa} D[-1]$ be any two maps with $\gamma\lambda = v[-1] \circ \kappa$. Then $\beta\lambda = \alpha\gamma\lambda = \alpha \circ v[-1] \circ \kappa = 0$, so that λ factors as $T \xrightarrow{e} E[-1] \xrightarrow{y[-1]} A$ for some map e. Then $v[-1] \circ (c[-1] \circ e - \kappa) = \gamma \circ y[-1] \circ e - \gamma\lambda = 0$. Therefore, $c[-1] \circ e - \kappa$ factors as $T \xrightarrow{\tau} C[-1] \xrightarrow{z[-1]} D[-1]$. Define $f = e - u[-1]\tau : T \to E[-1]$. Then $c[-1]f = \kappa$ and $y[-1]f = \lambda$, as desired. $\square$

Lemma 5.15. *Suppose that $X_n \xrightarrow{\lambda_n} X_{n-1} \xrightarrow{\lambda_{n-1}} \dots \xrightarrow{\lambda_2} X_1$ is a sequence of $n-1$ composable maps in a triangulated category $\mathcal{T}$, and let $X \in \{\lambda_2, \dots, \lambda_{n-1}\}$ be an $(n-1)$-filtered object. Assume that the composition $X_n[n-3] \xrightarrow{\lambda_n[n-3]} X_{n-1}[n-3] \xrightarrow{d_{n-2}} F_{n-2}X$ can also be written as a composite $X_n[n-3] \xrightarrow{\phi} X_1 \cong F_1X \to F_{n-2}X$. Then $(-1)^{n-1}\phi$ is an element of the $(n-1)$-fold Toda bracket $\langle \lambda_2, \dots, \lambda_n \rangle$.*

Proof. Choose exact triangles $F_jX[-1] \xrightarrow{\pi_j} G_jX \xrightarrow{w_j} F_1X \to F_jX$ for all $j = 1, \dots, n-1$. For all $j = 1, 2, \dots, n-2$ we obtain a diagram

$$
\begin{tikzcd}
X_{j+1}[j-1] \arrow[rr, "-d_j{[-1]}", "\circ" marking] \arrow[rrrr, dashed, bend left, "\tilde{d}_j", "\circ" marking] & & F_jX[-1] \arrow[rr, "-\pi_j"] \arrow[dl, "i_j{[-1]}"] & & G_jX \arrow[dl, "w_j", "\circ" marking] \arrow[ddll, dashed, bend left, "-\tilde{i}_j"] \\
& F_{j+1}X[-1] \arrow[ul, "p_{j+1}{[-1]}"] \arrow[dr, "\pi_{j+1}"'] & & F_1X[-1] \arrow[ll] \arrow[ul] & \\
& & G_{j+1}X \arrow[ur, "-w_{j+1}"', "\circ" marking] \arrow[uull, dashed, bend left, "\tilde{p}_{j+1}"] & &
\end{tikzcd}
$$

(the signs coming from shifting of triangles), and the octahedral axiom assures that the dashed maps exist and form an exact triangle. Also, $\tilde{p}_j[1] \circ \tilde{d}_j = \tilde{p}_j[1]\pi_j[1]d_j[-1] = p_j \circ d_j[-1] = \lambda_{j+1}[j-1]$. Therefore $* \cong G_1X \xrightarrow{-\tilde{i}_1} G_2X \xrightarrow{-\tilde{i}_2} \dots \xrightarrow{-\tilde{i}_{n-2}} G_{n-1}X$ is an $(n-2)$-filtered object in $\{\lambda_3, \dots, \lambda_{n-1}\}$. Its associated maps are $\tilde{\sigma}'_X = \tilde{p}_{n-1} : G_{n-1}X \to X_{n-1}[n-3]$ and $\tilde{\sigma}_X = (-1)^{n-1}\tilde{i}_{n-2} \dots \tilde{i}_2\tilde{p}_2^{-1} : X_2 \to G_{n-1}X$. Our goal is to choose maps γ_n, γ_1 such that the diagram

$$
\begin{tikzcd}
& X_2 \arrow[d, "\tilde{\sigma}_X"'] \arrow[dr, "\lambda_2"] \arrow[dr, phantom, "(a)", shift right=4ex] & \\
X_n[n-3] \arrow[r, "\gamma_n"] \arrow[dr, "{\lambda_n[n-3]}"'] \arrow[dr, phantom, "(b)", shift left=4ex] & G_{n-1}X \arrow[r, "\gamma_1"'] \arrow[d, "\tilde{\sigma}'_X"] & X_1 \\
& X_{n-1}[n-3] &
\end{tikzcd}
$$

commutes and $\gamma_1\gamma_n = (-1)^{n-1}\phi$. We choose $\gamma_1 = (-1)^{n-1}p_1 w_{n-1}$, then (a) commutes due to the relations $w_j = w_{j+1}\tilde{i}_j$ and $\lambda_2\tilde{p}_2 = p_1 \circ d_1[-1] \circ \tilde{p}_2 = p_1 w_2$.

Let us write $i : F_1X \to F_{n-2}X$. By Lemma 5.14 and our assumptions, we have a weak pullback diagram

$$\begin{tikzcd}
X_n[n-3] \arrow[drr, bend left, "-p_1^{-1}\phi"] \arrow[dr, dashed, "\gamma_n"] \arrow[ddr, bend right, "\lambda_n[n-3]"'] & & \\
& G_{n-1}X \arrow[r, "-w_{n-1}"] \arrow[d, "\tilde{p}_{n-1}"] & F_1X \arrow[d, "i"] \\
& X_{n-1}[n-3] \arrow[r, "-d_{n-2}"'] & F_{n-2}X
\end{tikzcd}$$

which implies the existence of the dashed arrow γ_n. Then (b) commutes by definition, and $\gamma_1\gamma_n = (-1)^{n-1}\phi$ is an element of the Toda bracket. □

5.2 Toda brackets and order

From now on, we fix a Tate cohomology class ζ of even degree n.

Lemma 5.16. *Suppose that $t < p-1$ and $\alpha_1, \dots, \alpha_{t+1} \in \hat{H}^*(G)$ are Tate cohomology classes of even degree such that the $(2t+3)$-fold Toda bracket $\langle \zeta, \alpha_1, \zeta, \dots, \zeta, \alpha_{t+1}, \zeta \rangle$ is strictly defined. Then there is a $(2t+2)$-filtered object $* = F_0Z \to F_1Z \to \dots \to F_{2t+2}Z$ in $\{\zeta, \alpha_1, \zeta, \dots, \alpha_t, \zeta\}$, associated to the maps $k = Z_1 \xleftarrow{\zeta} Z_2 \xleftarrow{\alpha_1} Z_3 \leftarrow \dots \xleftarrow{\zeta} Z_{2t+2}$ (where the Z_i's are suitable shifts of k), with the following properties:*

(i) *ζ-ord$(F_{2t+2}Z) \geq \zeta$-ord$(k/\zeta) - t$.*

(ii) *If we write $Z_{2t+3} = \Omega^{|\alpha_{t+1}|}Z_{2t+2}$, then the map $\alpha_{t+1} : Z_{2t+3} \to Z_{2t+2}$ factors as $Z_{2t+3} \xrightarrow{\widehat{\alpha}_{t+1}} \Omega^{2t+1}F_{2t+2}Z \xrightarrow{\Omega^{2t+1}\sigma'_Z} Z_{2t+2}$.*

Proof. Recall that we have the following exact triangle:

$$\Omega k \xrightarrow{\eta} L_\zeta \xrightarrow{\iota} \Omega^n k \xrightarrow{\zeta} k$$

We prove the proposition by induction on t, beginning with $t = 0$. Define the filtered object to be as follows:

$$\begin{tikzcd}
* \arrow[rr] & & k \arrow[rr, "-\eta[1]"] \arrow[dl, equal] & & L_\zeta[1] \arrow[dl, "\iota[1]"] \\
& k \arrow[ul, "\circ"] & & \Omega^n k[1] \arrow[ll, "\circ", "\zeta[1]"'] \arrow[ul, "\circ", "\zeta[1]"'] &
\end{tikzcd}$$

Then $F_{2t+2}Z = L_\zeta[1] = k/\zeta$ so that (i) holds. The map $\widehat{\alpha}_1$ exists due to $\zeta\alpha_1 = 0$.

Now let us assume the statement is true for $t-1$ and prove it for t. By induction we know that there is a map $\widehat{\alpha}_t[2t] : Z_{2t+1}[2t] \to F_{2t}Z[1]$. Since

$$\zeta\text{-ord}(F_{2t}Z) \geq \zeta\text{-ord}(k/\zeta) - t + 1 \geq 1$$

by induction hypothesis and Corollary 2.8, we know that there exists some extension $F : k/\zeta \otimes Z_{2t+1}[2t] \to F_{2t}Z[1]$ of $-\widehat{\alpha}_t[2t]$ in such a way that there is an exact triangle

$$F_{2t}Z \to F_{2t+2}Z \to k/\zeta \otimes Z_{2t+1}[2t] \xrightarrow{F} F_{2t}Z[1]$$

with some object $F_{2t+2}Z$ satisfying $\zeta\text{-ord}(F_{2t+2}) \geq \zeta\text{-ord}(k/\zeta) - t$. We get the following diagram without the dashed and dotted arrows:

$$\begin{tikzcd}
F_{2t}Z \arrow[rr, dashed] \arrow[rrrr, bend left] & & F_{2t+1}Z \arrow[dl, dashed] \arrow[rr, dotted] & & F_{2t+2}Z \arrow[dl] \arrow[dll, bend left] \\
& Z_{2t+1}[2t] \arrow[ul, "\circ", "{\widehat{\alpha}_t[2t]}"'] \arrow[dr, "{-\eta[1]\otimes\mathrm{id}}"'] & & Z_{2t+2}[2t+1] \arrow[ll, "\circ", "\zeta"'] \arrow[ul, dotted, "\circ"] & \\
& & k/\zeta \otimes Z_{2t+1}[2t] \arrow[ur, "{\iota[1]}"'] \arrow[uull, bend left, "\circ", "F"'] & &
\end{tikzcd} \tag{5.17}$$

If we choose $F_{2t+1}Z$ to be some choice of cone of $\widehat{\alpha}_t[2t]$, we get the dashed arrows and the octahedral axiom guarantees the existence of the dotted arrows such that the resulting diagram is commutative-exact. We have therefore extended our $(2t)$-filtered object to a $(2t+2)$-filtered object, and (i) is also satisfied.

Now put $Z_{2t+3} = \Omega^{|\alpha_{t+1}|}Z_{2t+2}$. For the existence of $\widehat{\alpha}_{t+1}$, let us choose exact triangles $F_1Z \to F_iZ \to F_i/F_1Z \xrightarrow{\pi_i} F_1Z[1]$ for all $i = 1, 2, \ldots, 2t+2$. Then the $(2t+1)$-filtered object $* = F_1/F_1Z \to F_2/F_1Z \to \cdots \to F_{2t+2}/F_1Z$ is (up to shifts) an element of $\{\alpha_1, \zeta, \ldots, \alpha_t, \zeta\}$. By Lemma 5.13 there is a map ω yielding a commutative diagram as follows:

$$\begin{tikzcd}
& Z_2[1] \arrow[d] \arrow[dr, "\zeta"] & \\
Z_{2t+3}[2t+1] \arrow[r, dotted, "\omega"] \arrow[dr, "\alpha_{t+1}"'] & F_{2t+2}/F_1Z \arrow[r, "\pi_{2t+2}"'] \arrow[d] & F_1Z[1] \cong Z_1[1] \\
& Z_{2t+2}[2t+1] &
\end{tikzcd}$$

Then $\pi_{2t+2}\,\omega$ lies in the Toda bracket $\langle \zeta, \alpha_1, \ldots, \zeta, \alpha_{t+1} \rangle = \{0\}$. Therefore, $\Omega^{2t+1}\omega$ lifts to a map $\widehat{\alpha}_{t+1} : Z_{2t+3} \to \Omega^{2t+1}F_{2t+2}Z$, proving (ii). $\square$

Corollary 5.18. *Suppose that $s \leq p-1$, and let $\alpha_1, \ldots, \alpha_s \in \hat{H}^*(G)$ be Tate cohomology classes of even degree in such a way that the $(2s+1)$-fold Massey product $\langle \zeta, \alpha_1, \zeta, \ldots, \zeta, \alpha_s, \zeta \rangle$ is strictly defined. If ζ-ord$(k/\zeta) \geq s$, then 0 is an element of the Massey product.*

Proof. We put $t = s-1$ in Lemma 5.16 and get a $2s$-filtered object

$$\begin{tikzcd}[column sep=small]
* = F_0Z \arrow[rr] & & F_1Z \arrow[rr] \arrow[dl] & & F_2Z \arrow[rr] \arrow[dl] & & \cdots \arrow[rr] & & F_{2s}Z \arrow[dl] & \\
& Z_1 \arrow[ul, "\circ"] & & Z_2[1] \arrow[ll, "\circ"] \arrow[ul, "\circ"] & & \cdots \arrow[ll, "\circ"] \arrow[ul, "\circ"] & & Z_{2s}[2s-1] \arrow[ll, "\circ"] & & Z_{2s+1}[2s] \arrow[ll, "\circ", "{\alpha_s[2s]}"'] \arrow[ul, dotted, "\circ", "{\widehat{\alpha}_s[2s]}"']
\end{tikzcd}$$

together with a map $\widehat{\alpha}_s$ (the dotted arrow) making the diagram commutative-exact, and we know that ζ-ord$(F_{2s}Z) \geq 1$. By choosing a cone $F_{2s+1}Z$ of $\widehat{\alpha}_s[2s-1]$ we can view the diagram as a $(2s+1)$-filtered object. Put $Z_{2s+2} = \Omega^n Z_{2s+1}$; then the diagram

$$\begin{tikzcd}
Z_{2s+2}[2s-1] \arrow[r, "\zeta"] \arrow[d, "\Omega^n\widehat{\alpha}_s[2s-1]"'] & Z_{2s+1}[2s-1] \arrow[d, "\widehat{\alpha}_s[2s-1]"] \\
\Omega^n F_{2s}Z \arrow[r, "\zeta"] & F_{2s}Z
\end{tikzcd}$$

commutes. Since ζ-ord$(F_{2s}Z) \geq 1$, the bottom vertical map vanishes. Hence $\widehat{\alpha}_s\zeta = 0$, and therefore $0 \in \langle \zeta, \alpha_1, \zeta, \ldots, \alpha_s, \zeta\rangle$ due to Lemma 5.15. $\square$

5.3 Toda brackets and coherent modules

Proposition 5.19. *Suppose that $t < p-1$ and $\alpha_1, \ldots, \alpha_{t+1} \in \hat{H}^*(G)$ are Tate cohomology classes of even degree such that the $(2t+3)$-fold Toda bracket $\langle \zeta, \alpha_1, \zeta, \ldots, \zeta, \alpha_{t+1}, \zeta\rangle$ is strictly defined. Then there is a $(p-1-t, t)$-coherent module (X, Y) with $Y_0 = k$ and a $(2t+2)$-filtered object $* = F_0Z \to F_1Z \to \cdots \to F_{2t+2}Z$ in $\{\zeta, \alpha_1, \zeta, \ldots, \alpha_t, \zeta\}$, associated to the maps $k = Z_1 \xleftarrow{\zeta} Z_2 \xleftarrow{\alpha_1} Z_3 \leftarrow \ldots \xleftarrow{\zeta} Z_{2t+2}$ (where the Z_i's are suitable shifts of k), with the following properties in $\underline{\mathfrak{mod}}$-$kG$:*

(i) *$F_{2t+2}Z$ is isomorphic to $X_1[2t+1]$.*

(ii) *The map $\sigma_Z : k \to F_{2t+2}Z$ equals the composition*

$$k \xrightarrow{-\eta[1]} L_\zeta[1] \xrightarrow{\Omega^{-t-1}\omega_{t+1}^{-1}} \Omega^t\Lambda^{t+1}L_\zeta[2t+1] \xrightarrow{\Phi(X,Y)[2t+1]} X_1[2t+1],$$

where ω_{t+1} and $\Phi(X, Y)$ are constructed in (4.5) and Definition 4.15, respectively.

(iii) *The composition*

$$\Omega^{2t+1}F_{2t+2}Z \xrightarrow{\Omega^{2t+1}\sigma'_Z} Z_{2t+2} = \Omega^{|\alpha_1|+\cdots+|\alpha_t|+(t+1)n}k \xrightarrow{(-1)^t\alpha_1\ldots\alpha_t} \Omega^{(t+1)n}k$$

equals $\Omega^{2t+1}F_{2t+2}Z \cong X_1 \xrightarrow{\lambda(X,Y)} S^{1+t}\Omega^n k \cong (\Omega^n k)^{\otimes(1+t)}$, where $\lambda(X, Y)$ is defined in Definition 4.19.

(iv) *Let us write $Z_{2t+3} = \Omega^{|\alpha_{t+1}|}Z_{2t+2}$. The map $\alpha_{t+1} : Z_{2t+3} \to Z_{2t+2}$ factors as a composite $Z_{2t+3} \xrightarrow{\widehat{\alpha}_{t+1}} X_1 \cong \Omega^{2t+1}F_{2t+2}Z \xrightarrow{\Omega^{2t+1}\sigma'_Z} Z_{2t+2}$.*

(v) *By* (iii) *and* (iv)*, the composition*

$$Z_{2t+3} \xrightarrow{\widehat{\alpha}_{t+1}} X_1 \xrightarrow{\lambda(X,Y)} S^{1+t}(\Omega^n k) \cong (\Omega^n k)^{\otimes(1+t)}$$

equals $(-1)^t\alpha_1\alpha_2\ldots\alpha_{t+1} : Z_{2t+3} = \Omega^{|\alpha_1|+\cdots+|\alpha_{t+1}|+(t+1)n}k \to \Omega^{(t+1)n}k$.

We will draw a picture visualizing the situation in Remark 5.23, after the proof.

Proof. We do this by induction on t, beginning with $t = 0$. We have to define a $(p-1,0)$-coherent module, and we take $X_* = \Lambda^* L_\zeta = F(k)$, with $\sigma : X \to F(Y_0)$ being the identity map. Define the filtered object to be the following:

$$\begin{tikzcd}
* \arrow[rr] & & k \arrow[rr, "-\eta[1]"] & & L_\zeta[1] \arrow[dl, "\iota[1]"] \\
& k \arrow[ul, "o"] \arrow[ur, equal] & & \Omega^n k[1] \arrow[ll, "\zeta[1]", "o"'] \arrow[ul, "o\ \zeta[1]"'] &
\end{tikzcd}$$

Then $X_1[1] = L_\zeta[1] = F_{2t+2}Z$ so that (i) holds. The map $\sigma_Z : k \to L_\zeta[1]$ equals $-\eta[1]$ which proves (ii), because $\Phi(X,Y)[1] \in \underline{\mathrm{Hom}}_{kG}(L_\zeta[1], X_1[1])$ and $\omega_1^{-1} : L_\zeta[1] \to L_\zeta[1]$ are the identity maps. The map $\lambda(X,Y) : X_1 = L_\zeta \to \Omega^n k$ equals ι, which proves (iii). The map $\widehat{\alpha}_1$ exists due to $\zeta\alpha_1 = 0$, thus proving (iv).

Now let us assume the statement is true for $t-1$ and prove it for t. By induction we know that there is a map $\widehat{\alpha}_t : Z_{2t+1} \to X_1$, and we let $-f$ be a surjective lift of that stable map to the unstable world. As in Proposition 4.17 we get from f and the $(p-t, t-1)$-coherent module (X,Y) a $(p-1-t, t)$-coherent module (X', Y') whose underlying object X_1' is the kernel of the composition

$$F : L_\zeta \otimes Z_{2t+1} \xrightarrow{\mathrm{id} \otimes f} L_\zeta \otimes X_1 \to X_2.$$

We get the following diagram in $\underline{\mathfrak{mod}}$-$kG$ (without the dashed and dotted arrows):

$$\begin{tikzcd}
X_2[-1] \cong X_1[-2] \arrow[rr, dashed, "*"] \arrow[rrrr, bend left, "\mu"] & & C \arrow[rr, dotted, "*"] \arrow[dl, dashed] & & X_1' \arrow[dl] \arrow[ddll, bend left, "*"] \\
& Z_{2t+1}[-1] \arrow[ul, "-f[-1]", "o"'] \arrow[dr, "-\eta[1]\otimes\mathrm{id}"'] & & Z_{2t+2} \arrow[ll, "\zeta", "o"'] \arrow[ul, dotted, "o"'] & \\
& & L_\zeta \otimes Z_{2t+1} \arrow[uull, bend left, "F", "o"'] \arrow[ur, "*", "\iota"'] \arrow[ur, phantom, shift left=6, "(\natural)"] & &
\end{tikzcd} \tag{5.20}$$

We define C to be some fiber of $-f$ and obtain the dashed arrows. The octahedral axiom then tells us that the dotted arrows exist in such a way that the diagram is commutative-exact. Apply the shift functor $[2t+1]$ to the diagram, and at the same time multiply the $*$-marked arrows by (-1). This assures that the previously exact triangles are still exact. Then we can concatenate the result with the $(2t)$-filtered object we already have and obtain a $(2t+2)$-filtered object with $F_{2t+1}Z = C[2t+1]$ and $F_{2t+2}Z = X_1'[2t+1]$ which proves (i). To prove (ii), notice that the exact sequence

$$0 \to X_i' \to \Lambda^i L_\zeta \otimes Y_t' \to \Lambda^{i+1} L_\zeta \otimes Y_{t-1}' \to \cdots \to \Lambda^{i+t} L_\zeta \to 0$$

is the Yoneda splice of the exact sequences

$$0 \to X_{1+i} \to \Lambda^{1+i} L_\zeta \otimes Y_{t-1} \to \cdots \to \Lambda^{i+t} L_\zeta \to 0$$

and

$$0 \to X'_i \to \Lambda^i L_\zeta \otimes Y'_t \to X_{1+i} \to 0,$$

as in the proof of Proposition 4.17. Via the isomorphism

$$\operatorname{Ext}^1(X_{1+i}, X'_i) \cong \underline{\operatorname{Hom}}_{kG}(X_2[-1], X'_1)$$

the latter represents the map μ of (5.20), and (ii) follows.

In order to prove (iii), we set up two commutative diagrams. The first one uses the induction hypothesis on (v) as follows:

$$\begin{tikzcd}
X'_1 \arrow[r, "\Omega^{2t+1}\sigma'_Z"] \arrow[d] \arrow[dr, phantom, "(a)"] & Z_{2t+2} \arrow[r, "(-1)^t\alpha_1\cdots\alpha_t"] \arrow[d, "\cong"] \arrow[dr, phantom, "(v)"] & \Omega^n k \otimes S^t\Omega^n k \cong (\Omega^n k)^{\otimes(t+1)} \\
L_\zeta \otimes Z_{2t+1} \arrow[r, "\iota\otimes\mathrm{id}"'] & \Omega^n k \otimes Z_{2t+1} \arrow[r, "\mathrm{id}\otimes f"'] & \Omega^n k \otimes X_1 \arrow[u, "\mathrm{id}\otimes\lambda(X,Y)"']
\end{tikzcd} \tag{5.21}$$

The square (a) equals the triangle ($\natural$) in (5.20), so the diagram commutes. The second diagram takes the following form:

$$\begin{tikzcd}
X'_1 \arrow[rr, "\lambda(X',Y')"] \arrow[d] \arrow[drr, phantom, "(b)"] & & S^{1+t}(\Omega^n k) \arrow[r, "\cong"] \arrow[d] & (\Omega^n k)^{\otimes(1+t)} \arrow[d, equal] \\
L_\zeta \otimes Z_{2t+1} \arrow[r, "1\otimes f"'] & L_\zeta \otimes X_1 \arrow[r, "\iota\otimes\lambda(X,Y)"'] & \Omega^n k \otimes S^t(\Omega^n k) \arrow[r, "\cong"'] & (\Omega^n k)^{\otimes(1+t)}
\end{tikzcd} \tag{5.22}$$

Here, (b) commutes by the definition of the comodule structure on Y'. Putting the diagrams (5.21) and (5.22) together, we get a proof of (iii).

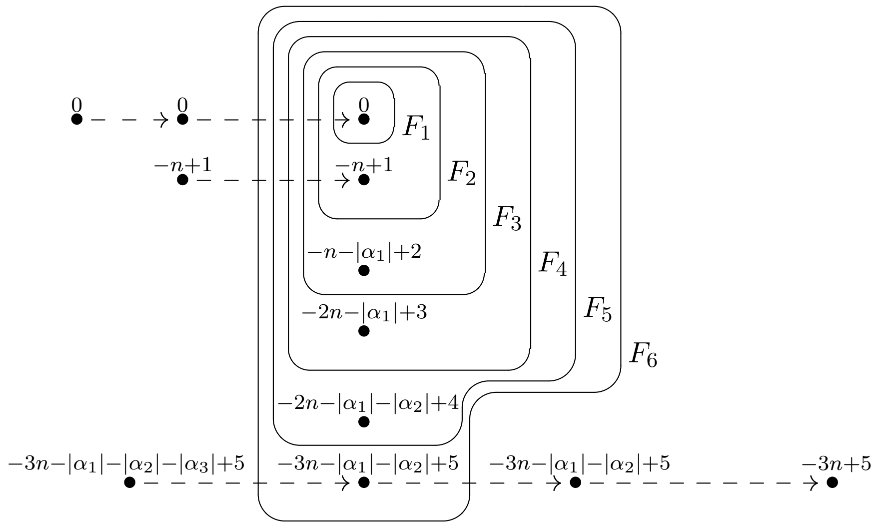
The existence of $\widehat{\alpha}_{t+1}$ in (iv) follows exactly as in the proof of Lemma 5.16. The statement (v) is an immediate consequence of (iii) and (iv), so we are done. $\square$

Remark 5.23. The statement of the previous proposition might become clearer by considering the following diagrammatic picture. In the topological world of the stable homotopy category $\mathcal{SHC}$, our trivial kG-module k corresponds to the sphere spectrum S, which can be viewed as a stable 0-cell. So let us consider k as a 0-cell, then $\Sigma^n k$ is an n-cell, $\Omega^n k$ is a $-n$-cell, and L_ζ has one -1-cell and one $-n$-cell. We draw pictures of these objects as follows:

$$\begin{tikzcd}
\overset{-1}{\bullet} \arrow[r] & \overset{-1}{\bullet} & \\
 & \overset{-n}{\bullet} \arrow[r] & \overset{-n}{\bullet} \\
\Omega k \arrow[r, "\eta"'] & L_\zeta \arrow[r, "\iota"'] & \Omega^n k
\end{tikzcd}$$

We draw a dot • for every cell, and indicate its dimension on top of it. Note that this does *not* mean that L_ζ decomposes as a direct sum of Ωk and $\Omega^n k$, the two cells are 'linked' by the map ζ in an appropriate sense, but we do not include this information into the figure. Now we draw a picture visualizing the situation of Proposition 5.19 in the special case $t = 2$ as follows:

$$\begin{tikzcd} k \arrow[r, "-\eta[1]"] \arrow[rr, bend left, "\sigma_Z"] & L_\zeta[1] \arrow[r, "\Phi(X,Y)"] & X_1[2t+1] \end{tikzcd}$$



$$\begin{tikzcd} Z_{2t+3}[2t+1] \arrow[r, "\widehat{\alpha}_{t+1}"'] & X_1[2t+1] \arrow[r, "\sigma'_Z"'] \arrow[rr, bend right, "\lambda(X,Y)"'] & Z_{2t+2}[2t+1] \arrow[r, "\alpha_1\alpha_2"'] & (\Omega^n k)^{\otimes(1+t)}[2t+1] \end{tikzcd}$$

Proposition 5.24. *Suppose that $\alpha_1, \ldots, \alpha_{p-1} \in \hat{H}^*(G)$ are Tate cohomology classes of even degree such that the $(2p-1)$-fold Massey product $\langle \zeta, \alpha_1, \zeta, \ldots, \alpha_{p-1}, \zeta \rangle$ is strictly defined. Then $-\alpha_1 \ldots \alpha_{p-1} \mathcal{P}_1(\zeta) \subseteq \langle \zeta, \alpha_1, \zeta, \ldots, \alpha_{p-1}, \zeta \rangle$.*

Remark 5.25. Notice that the sign occurring in the statement depends on the choice of the definition of Massey products.

Proof of Proposition 5.24. We put $t = p - 2$ in Proposition 5.19 and see what we get. There is a $(1, p-2)$-coherent module (X, Y) and a $(2p-2)$-filtered object

$$\begin{tikzcd}[column sep=small] * = F_0Z \arrow[rr] & & F_1Z \arrow[rr] \arrow[dl] & & F_2Z \arrow[rr] \arrow[dl] & & \cdots \arrow[rr] & & F_{2p-2}Z \arrow[dl] & \\ & Z_1 \arrow[ul, "o"] & & Z_2[1] \arrow[ll, "o"] \arrow[ul, "o"] & & \cdots \arrow[ll, "o"] \arrow[ul, "o"] & & Z_{2p-2}[2p-3] \arrow[ll, "o"] & & Z_{2p-1}[2p-2] \arrow[ll, "o", "{\alpha_{p-1}[2p-2]}"'] \arrow[ul, dotted, "o\,\widehat{\alpha}_{p-1}[2p-2]"'] \end{tikzcd}$$

together with a map $\widehat{\alpha}_{p-1}$ (the dotted arrow) making the diagram commutative-exact, and we know that $F_{2p-2}Z \cong X_1[2p-3]$, and $\sigma_Z : k = Z_1 \to F_{2p-2}Z$ equals the composition

$k \xrightarrow{-\eta[1]} L_\zeta[1] \xrightarrow{\Omega^{-(p-1)}\omega_{p-1}^{-1}} \Omega^{p-2}\Lambda^{p-1}L_\zeta[2p-3] \xrightarrow{\Phi(X,Y)[2p-3]} X_1[2p-3]$. By defining $F_{2p-1}Z$ to be some choice of fibre of $\widehat{\alpha}_{p-1}[2p-2]$ we can also consider the diagram above as a $(2p-1)$-filtered object.

Consider the composite $\Omega^n Z_{2p-1} \xrightarrow{\zeta} Z_{2p-1} \xrightarrow{\widehat{\alpha}_{p-1}} \Omega^{2p-3}F_{2p-2}Z \cong X_1$. By naturality of ζ, this equals

$$\Omega^n Z_{2p-1} \xrightarrow{\Omega^n\widehat{\alpha}_{p-1}} \Omega^n\Omega^{2p-3}F_{2p-2}Z \cong \Omega^n X_1 \xrightarrow{\zeta} X_1.$$

Let us choose any element $P_1 \in \mathcal{P}_1(\zeta)$ and consider the following diagram:

$$\begin{tikzcd}
\Omega^n Z_{2p-1} \arrow[rrr, "\Omega^n\widehat{\alpha}_{p-1}"] \arrow[d, "(-1)^{p-2}\alpha_1\ldots\alpha_{p-1}"'] \arrow[drr, phantom, "(a)"] & & & \Omega^n k \otimes X_1 \arrow[dl, "\mathrm{id}\otimes\lambda(X,Y)"'] \arrow[dddd, "\zeta\otimes\mathrm{id}"] \\
(\Omega^n k)^{\otimes p} \arrow[ddd, "P_1"'] \arrow[ddr, phantom, "(c)"] & & \Omega^n k \otimes S^{p-1}\Omega^n k \arrow[ll, "\cong"'] \arrow[d, "\gamma"] \arrow[ddr, phantom, "(b)"] & \\
& & \Omega^{p-2}\Lambda^p\Omega^n k \arrow[d, "\Omega^{p-2}\kappa_p"] & \\
& \Omega^{2p-4}L_\zeta \arrow[r, "\Omega^{p-2}\omega_{p-1}^{-1}"] \arrow[d, phantom, "(d)"] & \Omega^{p-2}\Lambda^{p-1}L_\zeta \arrow[dr, "\Phi(X,Y)"] & \\
\Omega^{2p-3}k \arrow[ur, "\Omega^{2p-4}\eta"'] \arrow[rrr, "-\Omega^{2p-3}\sigma_Z"'] & & & X_1
\end{tikzcd} \tag{5.26}$$

Here, (d) and (a) commute due to Proposition 5.19 (ii) and (v), (b) commutes because of Lemma 4.22, and (c) is Proposition 4.7.

Now we put $Z_{2p} = \Omega^n Z_{2p-1}$. Then we are in the situation of Lemma 5.15 (where the n in that statement corresponds to $2p$ in our case), the composable maps being $Z_{2p} \xrightarrow{\zeta} Z_{2p-1} \xrightarrow{\alpha_{p-1}} \ldots \xrightarrow{\zeta} Z_1 = k$. The filtered object is $* = F_0Z \to F_1Z \to \ldots F_{2p-2}Z \to F_{2p-1}Z$, and the map $Z_1 \cong F_1Z \to F_{2p-2}Z$ equals our σ_Z. Furthermore, (5.26) tells us that the diagram

$$\begin{tikzcd}
Z_{2p}[2p-3] \arrow[r, "\zeta"] \arrow[d, "\alpha_1\ldots\alpha_{p-1}P_1"'] & Z_{2p-1}[2p-3] \arrow[d, "\widehat{\alpha}_{p-1}[2p-3]"] \\
Z_1 = k \arrow[r, "\sigma_Z"'] & F_{2p-2}Z
\end{tikzcd}$$

commutes, and by Lemma 5.15 we get that $-\alpha_1\ldots\alpha_{p-1}P_1 \in \langle\zeta,\alpha_1,\ldots,\alpha_{p-1},\zeta\rangle$. □

Remark 5.27. In the case $p = 2$ we get the result $\alpha\mathcal{P}_1(\zeta) \subseteq \langle\zeta,\alpha,\zeta\rangle$, whenever the Massey product is defined. In the case of ordinary cohomology, this is a well-known fact. It is an immediate consequence of a formula relating the $\cup_1$- and the $\cup$-product in the singular cochain complex of a topological space; see [8]. For $G = \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ the formula enables us to compute all Massey products of the form $\langle\zeta,\alpha,\zeta\rangle$, and it is consistent with the computation of all Massey products done in [15], §5.

5.4 Example for equality

Now we can prove that Corollary 2.8 is a strong inequality in the sense that for every prime p and every field of characteristic p, there is a group G and a Tate cohomology class ζ of even degree such that $\zeta\text{-ord}(k/\zeta) = p - 2$. Consider the group

$$\mathrm{Aff}_1 = \{x \mapsto ax + b \,:\, a \in \mathbb{F}_p^*, b \in \mathbb{F}_p\} \cong \mathbb{F}_p \rtimes \mathbb{F}_p^*$$

of affine linear transformations of $\mathbb{F}_p$. The group operation is composition of such maps. This group has the normal subgroup of translations

$$T = \{x \mapsto x + b \,:\, b \in \mathbb{F}_p\},$$

and the quotient Aff_1/T is isomorphic to $\mathbb{F}_p^*$. Furthermore, T is the p-Sylow subgroup of Aff_1.

Our example will be the group $G = \mathrm{Aff}_1 \times \mathrm{Aff}_1$. By the considerations above, G has the normal p-Sylow subgroup $N = T \times T$. It is well-known that (see [4], Theorem 10.1 in Chapter XII) the inclusion $i : N \to G$ induces an isomorphism $\hat{H}^*(G) \to \hat{H}^*(N)^{G/N}$ from the Tate cohomology of G to the fixed points in Tate cohomology of N under the action of G/N. We will often suppress this isomorphism from notation. Let us now try to describe the Tate cohomology of G. To do so, we use the description of the Tate cohomology of N given in §3.6. Recall that for a fixed non-negative degree n we have the monomial basis of $\hat{H}^n(N)$ given by elements $v_1^{a_1}v_2^{a_2}u_1^{\epsilon_1}u_2^{\epsilon_2}$ with $2a_1 + 2a_2 + \epsilon_1 + \epsilon_2 = n$ and $\epsilon_1, \epsilon_2 \leq 1$. Using Tate duality, we denoted the dual basis by $\varphi_{2a_1+\epsilon_1, 2a_2+\epsilon_2}$. Let us write $D : \hat{H}^n(N) \to \hat{H}^{-1-n}(N)$ for the k-linear map which sends each basis element $v_1^{a_1}v_2^{a_2}u_1^{\epsilon_1}u_2^{\epsilon_2}$ to its dual $\varphi_{2a_1+\epsilon_1,2a_2+\epsilon_2}$. Also, let us write $q = 2p - 2$.

Lemma 5.28. *Suppose that $p \geq 3$. The ordinary group cohomology of G is given by $H^*(G) = k[b_1, b_2, e_1, e_2]$, the graded commutative algebra generated by two exterior classes e_1, e_2 of degree $q - 1$ and two polynomial classes b_1, b_2 of degree q. The elements are given by $e_i = v_i^{p-2}u_i$ and $b_i = v_i^{p-1}$. The structure of the Tate cohomology of G is obtained from the ordinary cohomology by using the fact that D restricts to an isomorphism $D : \hat{H}^n(G) \to \hat{H}^{-1-n}(G)$.*

Proof. The computation of $H^*(\mathrm{Aff}_1)$ is well-known, and $H^*(G)$ is deduced from this using the Künneth theorem. For the last statement it is enough to observe that the elements of $G/N \cong \mathbb{F}_p^* \times \mathbb{F}_p^*$ act as diagonal matrices with respect to our chosen basis: the element (c_1, c_2) acts as

$$(c_1, c_2) \cdot v_1^{a_1}v_2^{a_2}u_1^{\epsilon_1}u_2^{\epsilon_2} = c_1^{2a_1+\epsilon_1}c_2^{2a_2+\epsilon_2}v_1^{a_1}v_2^{a_2}u_1^{\epsilon_1}u_2^{\epsilon_2}.$$

Therefore, the elements of G/N act as the same diagonal matrices on the dual space $\hat{H}^{-1-n}(N)$ with respect to the dual basis $\varphi_?$. □

Remark 5.29. In the case $p = 2$ the classes u_1 and u_2 (and therefore also e_1 and e_2) are polynomial classes with $u_i^2 = v_i$. With this tiny difference in mind, the following goes through even in the case $p = 2$, because we will never use the fact that the u_i's are exterior classes.

As an immediate consequence of the preceeding lemma we can describe the structure of $\hat{H}^*(G)$ more explicitly in the range we will be interested in.

Corollary 5.30. *The following table describes the k-vector spaces $\hat{H}^i(G)$ in the range $-2q+1 \leq i \leq 2q-2$:*

deg	$-2q+1$	$-q-1$	$-q$	-1	0	$q-1$	q	$2q-2$
basis	$\varphi_{2p-3,2p-3}$	$\varphi_{2p-2,0}$ $\varphi_{0,2p-2}$	$\varphi_{2p-3,0}$ $\varphi_{0,2p-3}$	φ_{00}	1	$v_1^{p-2}u_1$ $v_2^{p-2}u_2$	v_1^{p-1} v_2^{p-1}	$v_1^{p-2}v_2^{p-2}u_1u_2$

For all other values of i in the given range we have $\hat{H}^i(G) = 0$.

Theorem 5.31. *Let $\zeta = \varphi_{2p-3,0} \in \hat{H}^{-q}(G)$ and $\alpha = v_1^{p-1} \in \hat{H}^q(G)$. Then we have the $(2p-1)$-fold Massey product $\langle \zeta, \alpha, \zeta, \ldots, \alpha, \zeta \rangle = -\varphi_{2p-3,2p-3}$ without indeterminacy. In particular, ζ-ord$(k/\zeta) = p-2$ by Corollaries 2.8 and 5.18.*

One step in the proof is easy: we can compute $-\alpha^{p-1}\mathcal{P}_1(\zeta)$, which is one element of the Massey product due to Proposition 5.24. We know $\mathcal{P}_1(\zeta) = \varphi_{2p^2-2p-1,2p-3}$ from Propositions 3.54 and 3.56, and hence

$$\alpha^{p-1}\mathcal{P}_1(\zeta) = v_1^{p^2-2p+1}\varphi_{2p^2-2p-1,2p-3} = \varphi_{2p-3,2p-3}.$$

The statement about the indeterminacy is slightly more complicated. Let us recall some facts we will frequently use in the proof.

Lemma 5.32 (May, [20])**.** *Suppose that the Massey product $\langle a_1, \ldots, a_n \rangle$ is strictly defined.*

(i) *For every $i = 1, 2, \ldots, n-1$ and every $x_i \in \hat{H}^*(G)$ of degree $|a_i|+|a_{i+1}|-1$ the Massey product $\langle a_1, \ldots, a_{i-1}, x_i, a_{i+2}, \ldots, a_n \rangle$ is strictly defined ([20], Proposition 2.4.(i)).*

(ii) *If every Massey product as in* (i) *consists only of the zero element, then $\langle a_1, \ldots, a_n \rangle$ has no indeterminacy ([20], Proposition 2.4.(ii)).*

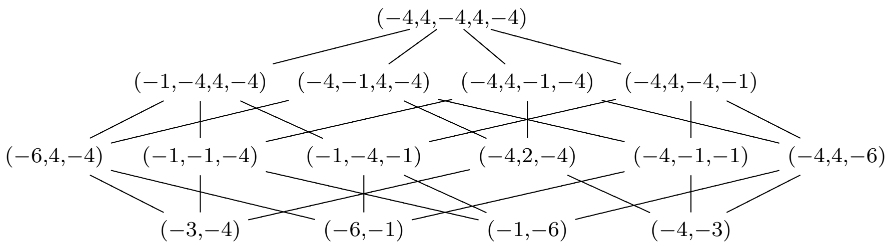
(iii) *If $a_0 \cdot \langle a_1, \ldots, a_n \rangle$ contains 0, then so does $\langle a_0a_1, \ldots, a_n \rangle$ (follows from part (ii) of Corollary 3.2 in [20]).*

Also, for arbitrary elements $a_1, \ldots, a_n$:

(iv) *If $\langle a_1, \ldots, a_{i-1}, a_ia_{i+1}, a_{i+2}, \ldots, a_n \rangle = \{0\}$ and $\langle a_1, \ldots, a_i, a_{i+1}a_{i+2}, a_{i+3}, \ldots, a_n \rangle$ is defined, then the latter contains 0 (this follows from Corollary 3.4.(iii) in [20]).*

Remark 5.33. Notice that May uses a different sign convention, so we formulated the facts in a sign-independent form we need later.

Let us now take a look at the sequences of the degrees of elements coming up in the Massey products we are interested in. Define S to be the smallest set of integer sequences of finite length with the following properties: the sequence $(-q, q, -q, \ldots, q, -q)$ of length $2p-1$ belongs to S, and whenever a sequence $(d_1, \ldots, d_n)$ of length $n \geq 3$ lies in S then so does $(d_1, \ldots, d_{i-1}, d_i + d_{i+1} - 1, d_{i+2}, \ldots, d_n)$ for every $i = 1, \ldots, n-1$. This notion is clearly motivated by Part (i) of Lemma 5.32. For example, in the case $p = 3$ the set S will consist of the following elements:



The occurring numbers are

$$q, q-2, q-4, \ldots, 2, -1, -3, \ldots, -q+1, -q, -q-2, \ldots, -2q+2. \tag{5.34}$$

Let us describe without proof some of the sequences in S more explicitly.

Lemma 5.35. *A sequence $(d_1, \ldots, d_n)$ with $2 \leq n \leq 2p-1$ and $d_i \in \{q, -1, -q\}$ for all i belongs to S if and only if it is of the form*

$$(\overline{-1}, -q, \overline{-1}, q, \overline{-1}, -q, \ldots, -q, \overline{-1})$$

where each $\overline{-1}$ stands for an arbitrary long (possibly empty) sequence of (-1)'s, and the total number of (-1)'s is $2p-1-n$.

We call a Massey product $\langle a_1, \ldots, a_n \rangle$ in $\hat{H}^*(G)$ *admissible* if the sequence of the degrees of its elements $(|a_1|, \ldots, |a_n|)$ belongs to S and for every $i = 1, \ldots, n$, $a_i = \zeta$ if $|a_i| = -q$ and $a_i = \alpha$ if $|a_i| = q$.

Lemma 5.36. *Every admissible Massey product is strictly defined.*

Proof. By Lemma 5.32.(i) it is sufficient to show that the Massey product $\langle a_1, \ldots, a_{2p-1} \rangle$ with

$$a_i = \begin{cases} \zeta & \text{if } i \text{ is odd,} \\ \alpha & \text{if } i \text{ is even} \end{cases}$$

is strictly defined. The two-fold products $\langle a_i, a_{i+1} \rangle = a_i a_{i+1}$ vanish due to the multiplicative relations in $\hat{H}^*(\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z})$. Now we can assume that $p \geq 3$. The three-fold products $\langle \zeta, \alpha, \zeta \rangle$ and $\langle \alpha, \zeta, \alpha \rangle$ are strictly defined and have no indeterminacy, and by Corollary 5.18 they contain zero.

Suppose we have some product $\langle a_i, \ldots, a_j \rangle$ of length $n \geq 4$. If n is even, then the product is of degree $-n+2$ with $4 \leq n \leq 2p-2 = q$; but then $\hat{H}^{-n+2}(G) = 0$ by Corollary 5.30. If n is odd, then the degree of the product is $\pm q - n + 2$ with $5 \leq n \leq 2p-3$, but then $\hat{H}^{\pm q-n+2}(G) = 0$. $\square$

Proposition 5.37. *Suppose that $\langle a_1, a_2, \ldots, a_n \rangle$ is an admissible Massey product. Then it does not have any indeterminacy. If $n < 2p-1$, then $\langle a_1, \ldots, a_n \rangle = \{0\}$.*

Proof. Let n be the smallest number for which any of the two statement fails. Suppose the first statement fails for that n. Then $n \geq 3$ and by minimality of n we know that all products of the form $\langle a_1, \ldots, a_{i-1}, x_i, a_{i+2}, \ldots, a_n \rangle$ with x_i of degree $|a_i| + |a_{i+1}| - 1$ equal $\{0\}$, because they are admissible (note here that x_i is not of degree $\pm q$). By Lemma 5.32.(ii) we know that $\langle a_1, \ldots, a_n \rangle$ does not have any indeterminacy.

We can therefore assume that for the minimal n the second statement fails, whereas the first one is true. Suppose that $\underline{a} = \langle a_1, \ldots, a_n \rangle$ is admissible; we need to show that $0 \in \underline{a}$. If there is some index i with $|a_i| \notin \{q, -1, -q\}$ then we can deduce from the list (5.34) and the table in Corollary 5.30 that $\hat{H}^{|a_i|}(G) = 0$ and therefore $a_i = 0$. Since $\underline{a}$ is strictly defined, it must contain 0 in this case. We can therefore assume that all Massey products $\underline{a}$ of length n with $\underline{a} \neq \{0\}$ satisfy $a_i \in \{q, -1, -q\}$ for all i. For every such Massey product there is a smallest index i with $|a_i| = -1$; denote that index by $d(\underline{a})$. Let us take one such product with minimal $d(\underline{a}) = i$.

Case 1: $i > 1$ and $|a_{i-1}| = -q$. Then $a_{i-1} = \zeta$. By Tate duality we know that there is some $b \in \hat{H}^{q-1}(G)$ with $a_i = b\zeta$. Then, by the description in Lemma 5.35, the Massey product $\langle a_1, \ldots, a_{i-2}, a_{i-1}b, \zeta, a_{i+1}, \ldots, a_n \rangle$ is admissible, and it equals $\{0\}$ by minimality of $d(\underline{a})$. By Lemma 5.32.(iii) $\underline{a}$ contains 0.

Case 2: $i > 1$ and $|a_{i-1}| = q$. Then $a_{i-1} = \alpha$. Again by Tate duality there is some element $b \in \hat{H}^{-q-1}(G)$ with $a_i = b\alpha$. The product $\langle a_1, \ldots, a_{i-2}, a_{i-1}b, \alpha, a_{i+1}, \ldots, a_n \rangle$ is admissible and equals $\{0\}$. As before we can deduce that $\underline{a}$ contains 0.

Case 3: $i = 1$. Then we know by Lemma 5.35 that the product $\langle \zeta, \alpha, a_2, a_3, \ldots, a_n \rangle$ is admissible and hence strictly defined. In particular, $\langle \alpha, a_2, \ldots, a_n \rangle = \{0\}$. By Tate duality we can write $a_1 = b\alpha$ for some class $b \in \hat{H}^{-q-1}(G)$. Then $b \cdot \langle \alpha, a_2, \ldots, a_n \rangle$ contains 0, and by Lemma 5.32.(iii) we get that $0 \in \underline{a}$. □

6 A counterexample

We shall now show that the lower bound of our Main Theorem (as given in Corollary 2.8) is no longer true in general if we allow ζ to be an arbitrary element of even degree in the graded center of $\underline{\mathfrak{mod}}$-$kG$. Let $p = 3$ and $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Let k be a field of characteristic 3; then $kG \cong A = k[x, y]/(x^3, y^3)$, and from now on we will work with A instead of kG. Since A is commutative, the enveloping algebra of A is simply given by $A^e = A \otimes A$, which we also denote by B. Define the elements $\bar{x} = 1 \otimes x - x \otimes 1$ and $\bar{y} = 1 \otimes y - y \otimes 1$ in B.

Lemma 6.1. *The beginning of a projective resolution of A as a right B-module is given by*

$$A \xleftarrow{\epsilon} B \xleftarrow{(\bar{x}\ \bar{y})} B^{\oplus 2} \xleftarrow{\left(\begin{smallmatrix} \bar{x}^2 & \bar{y} & \\ & -\bar{x} & \bar{y}^2 \end{smallmatrix}\right)} B^{\oplus 3} \leftarrow \ldots$$

where ϵ is the multiplication of A. In particular, we can choose $\Omega^2_B A$ to be the submodule of $B^{\oplus 2}$ generated by the three elements $\left(\begin{smallmatrix} \bar{x}^2 \\ 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \bar{y} \\ -\bar{x} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ \bar{y}^2 \end{smallmatrix}\right)$.

Proof. Consider the algebra $L = k[z]/(z^3)$ and let $\bar{z} = 1 \otimes z - z \otimes 1 \in L^e$. Simple combinatorial arguments show that

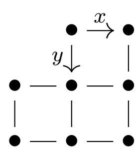
$$L \xleftarrow{\epsilon} L^e \xleftarrow{\bar{z}} L^e \xleftarrow{\bar{z}^2} L^e \xleftarrow{\bar{z}} L^e \xleftarrow{\bar{z}^2} \dots,$$

with ϵ being the multiplication of L, is a free resolution of L by L^e-modules. Tensoring two such complexes and using the Künneth theorem, we get the desired result. $\square$

If M is any A-module, we can apply the functor $M \otimes_A -$ to the projective resolution of the proposition and obtain a projective resolution of M. In particular, when we put $M = k$ the trivial A-module, we get the resolution

$$k \xleftarrow{\epsilon} A \xleftarrow{(x\ y)} A^{\oplus 2} \xleftarrow{\begin{pmatrix} x^2 & y & \\ & -x & y^2 \end{pmatrix}} A^{\oplus 3} \leftarrow \dots$$

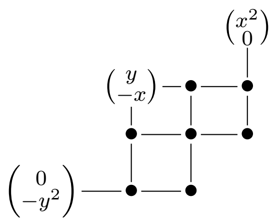
Let this resolution define the modules $\Omega^n k$ for $n \geq 0$. For a better understanding, let us draw pictures of these modules as follows: We take a suitable basis of the module as a k-vector space (drawn as $\bullet$) and draw vertical and horizontal lines to indicate multiplication by x and y, respectively:



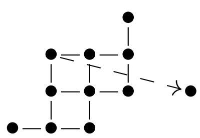
This picture represents $\Omega k = \langle x, y \rangle_A \subset A$. Whenever there is no horizontal line from a $\bullet$ to the right, we mean that the corresponding element vanishes under multiplication by x (and similarly downwards for y). Note that there are A-modules not representable in this form, but all the modules we are interested in will be. Let us draw two more examples:

k as a trivial module: $\bullet$

$\Omega^2 k$, considered as a submodule of $A^{\oplus 2}$:



Now let $\xi : \Omega^2 k \to k$ be the following map:



That is, $\left(\begin{smallmatrix} y \\ -x \end{smallmatrix}\right) \mapsto 1 \in k$ and all other drawn k-basis elements map to 0. Let L_ξ be the kernel of ξ, which can be drawn like this:

$$
\begin{tikzcd}[row sep=small, column sep=small]
 & & & b_4 \arrow[d, dash] \\
 & & b_3 \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[d, dash] \\
 & b_2 \arrow[r, dash] \arrow[d, dash] & \bullet \arrow[r, dash] \arrow[d, dash] & \bullet \\
b_1 \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet &
\end{tikzcd}
$$

It is the A-submodule of $A^{\oplus 2}$ spanned by the four vectors $b_1 = \left(\begin{smallmatrix} 0 \\ -y^2 \end{smallmatrix}\right)$, $b_2 = \left(\begin{smallmatrix} y^2 \\ -xy \end{smallmatrix}\right)$, $b_3 = \left(\begin{smallmatrix} xy \\ -x^2 \end{smallmatrix}\right)$ and $b_4 = \left(\begin{smallmatrix} x^2 \\ 0 \end{smallmatrix}\right)$. The map ξ induces an element $- \otimes \xi$ of degree 2 in the graded center $Z(\underline{\mathfrak{mod}}\text{-}A)$.

Let us turn back to the projective resolution of A as a B-module given in Lemma 6.1. Note that the elements $\bar{x}$ und $\bar{y}$ belong to the augmentation ideal; we therefore have constructed the beginning of a minimal resolution, and applying the functor $\mathrm{Hom}_B(-, A)$ to the resolution yields a complex $A \xrightarrow{0} A^{\oplus 2} \xrightarrow{0} A^{\oplus 3} \xrightarrow{0} \dots$; in particular $\widehat{HH}^2(A) \cong A^{\oplus 3}$. More explicitly, in order to specify a B-linear map $\varphi : \Omega^2_B A \to A$ we can freely choose the values of φ under the three generators $\left(\begin{smallmatrix} \bar{x}^2 \\ 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \bar{y} \\ -\bar{x} \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ \bar{y}^2 \end{smallmatrix}\right)$. Let us put

$$
\begin{aligned}
\varphi : \Omega^2_B A &\to A \\
\left(\begin{smallmatrix} \bar{x}^2 \\ 0 \end{smallmatrix}\right) &\mapsto x, \\
\left(\begin{smallmatrix} \bar{y} \\ -\bar{x} \end{smallmatrix}\right) &\mapsto 0, \\
\left(\begin{smallmatrix} 0 \\ \bar{y}^2 \end{smallmatrix}\right) &\mapsto 0.
\end{aligned}
$$

This map induces an element $- \otimes_A \varphi$ of degree 2 in the graded center $Z(\underline{\mathfrak{mod}}\text{-}A)$.

Proposition 6.2. *The element $\zeta = \varphi + \xi \in Z^2(\underline{\mathfrak{mod}}\text{-}A)$ satisfies ζ-ord$(k/\zeta) = 0$.*

Proof. Since x acts trivially on k, we get that $\varphi_k = 0$, so $\zeta_k = \xi_k$. Hence $k/\zeta \cong k/\xi \cong \Omega^{-1} L_\xi$. It is enough to show that $\zeta_{L_\xi} \neq 0$. Since ξ comes from a Tate cohomology class, we know that $\xi_{L_\xi} = 0$ by Corollary 2.8, so it remains to prove that $\varphi_{L_\xi} \neq 0$.

For this we need two easy but powerful lemmas. Suppose that M is a right A-module and $m \in M$. We say that m is divisible by x if there is some element $m' \in M$ with $m = m' \cdot x$, and similarly for y. Define I to be the ideal of A generated by the elements x^2, xy and y^2.

Lemma 6.3. *Let F be a free A-module. Suppose that $m \in F$ has the property that $m \cdot x$ is divisible by y and $m \cdot y$ is divisible by x. Then $m \in F \cdot I \subset F$.*

Proof. One can assume that F is free of rank 1. The set of elements m with $m \cdot x$ being divisible by y is the ideal generated by x^2 and y; the set of elements m with $m \cdot y$ being divisible by x is the ideal generated by x and y^2. The intersection of these two ideals is I. □

Lemma 6.4. *Suppose that $f : M \to N$ is a map of right A-modules which factors through a projective module. Assume further that $m \in M$ is such that $m \cdot x$ is divisible by y and $m \cdot y$ is divisible by x. Then $f(m) \in N \cdot I \subset N$.*

Proof. Because A is the group algebra of a 3-group, all projective modules are free. Write f as a composition $M \xrightarrow{g} F \xrightarrow{h} N$ with a free module F. Then $g(m)$ satisfies the conditions of Lemma 6.3; therefore $g(m) \in F \cdot I$ and hence $f(m) = h(g(m)) \in N \cdot I$. □

We want to apply this to φ acting on L_ξ, i.e., the map

$$L_\xi \otimes_A \Omega_B^2 A \to L_\xi \otimes_A A \cong L_\xi.$$

Let $e_1, e_2 \in B^{\oplus 2}$ be given by $e_1 = \binom{1}{0}$ and $e_2 = \binom{0}{1}$. Define the elements $f_i = b_{i-1} \otimes e_1 - b_i \otimes e_2 \in L_\xi \otimes_A B^{\oplus 2}$ for $i = 1, \dots, 5$ (where by convention $b_0 = 0$ and $b_5 = 0$). When we apply the differential $d : L_\xi \otimes_A B^{\oplus 2} \to L_\xi \otimes_A B$ to f_i we get

$$\begin{aligned} df_i &= b_{i-1} \otimes \bar{x} - b_i \otimes \bar{y} \\ &= b_{i-1} \otimes (1 \otimes x) - b_{i-1}x \otimes (1 \otimes 1) - b_i \otimes (1 \otimes y) + b_i y \otimes (1 \otimes 1) \\ &= b_{i-1} \otimes (1 \otimes x) - b_i \otimes (1 \otimes y); \end{aligned}$$

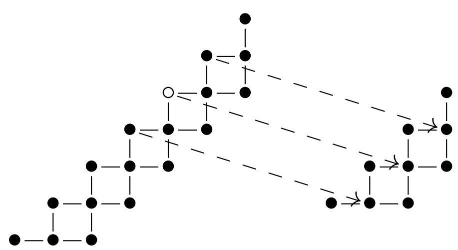
here we used $b_{i-1}x = b_i y$.

Now consider $b'_i = f_{i+1}x^2 + f_{i+2}xy + f_{i+3}y^2$ (where $f_i = 0$ for $i > 5$). From the computation above we get $db'_i = 0$, so $b'_i \in L_\xi \otimes_A \Omega_B^2 A$, and we have $b'_2 y = b'_1 x$ and $b'_2 x = b'_3 y$. Therefore, b'_2 satisfies the conditions of Lemma 6.4, and it remains to show that the image of b'_2 does not lie inside $L_\xi \cdot I$. A rather lengthy computation shows that

$$b'_2 = b_4 \otimes \binom{\bar{y}}{-\bar{x}} y + b_3 \otimes \binom{\bar{y}}{-\bar{x}} x + b_3 x \otimes \binom{\bar{y}}{-\bar{x}} + b_2 \otimes \binom{\bar{x}^2}{0}.$$

Therefore, under φ, b'_2 is mapped to $b_2 \cdot x \in L_\xi$ which is not inside $L_\xi \cdot I$. □

Remark 6.5. The last step of the proof might become clearer if we draw the map φ_{L_ξ} using the diagrams introduced above. It turns out that, up to a stable isomorphism in the source, the map $\varphi : L_\xi \otimes_A \Omega_B^2 A \to L_\xi$ can be viewed as follows:



The point marked $\circ$ satisfies the conditions of Lemma 6.4, thus showing that the indicated map is non-trivial in the stable category.

Remark 6.6. The element ζ of the graded center comes from the Hochschild cohomology of $A \cong kG$. Since G is abelian, we know that $HH^*(G) \cong H^*(G) \otimes A$ (see [9], or [5] for a more general result). Using the description of H^*G given in §3.6, the element ζ comes from $u_1 u_2 + v_1 \cdot x \in HH^*(G)$.

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List of Symbols

Greek letters

γ	in §4 and §5 a certain map $\Omega^n k \otimes S^{p-1}\Omega^n k \to \Omega^{p-2}\Lambda^p\Omega^n k$, page 56
Δ	may refer to
	comultiplication maps of the coalgebras S^*X and Λ^*X, page 12
	maps Δ_i arising in the construction of Steenrod powers, page 39
ζ	may refer to
	an element in the graded center of a triangulated category, page 9
	an unstable map $\Omega^n k \to k$, page 16
η	an unstable representative of the stable map $\Omega k \to L_\zeta$, page 16
ι	the inclusion $L_\zeta \to \Omega^n k$, page 16
κ_i^f	the i-th map in the Koszul complex associated to the map f, page 12
$\Lambda^n X$	the exterior power of a module X, page 12
$\lambda(X,Y)$	a certain map associated to a coherent module (X,Y), page 61
μ	multiplication maps of the algebras S^*X and Λ^*X, page 12
Σ	shift functor in $\underline{\mathfrak{mod}}$-$kG$, page 6
Σ_n	the symmetric group on n letters, page 11
σ_X, σ'_X	maps associated to a filtered object X, page 64
$\Phi(X,Y)$	a certain map associated to a coherent module (X,Y), page 59
Ω	inverse of the shift functor of $\underline{\mathfrak{mod}}$-$kG$, page 6
ω_j	a certain stable isomorphism $\Lambda^j L_\zeta \to \Omega^{j-1} L_\zeta$, page 56

Other expressions

$\mathcal{D}_i$	certain predecessor of the power operation, page 30
$\widehat{\mathrm{Ext}}$	Tate cohomology groups, page 7
$F(K)$	the tautological coherent module over $L_\zeta \otimes K$, page 18
G	a fixed finite group, page 6
$\widehat{HH}^*$	Tate version of Hochschild cohomology, page 9
$\underline{\mathrm{Hom}}$	morphisms in the stable category, page 6
$\hat{H}^*(G)$	Tate cohomology algebra, page 7
k	may refer to
	the ground field, of characteristic p, page 6
	the trivial kG-module k, unit of $\otimes$, page 6

Further symbols

Zusammenfassung

Einführung

Motiviert durch seinen Beweis des Starrheitssatzes hat Schwede den Begriff der *Ordnung* in triangulierten Kategorien eingeführt. Sei dazu $\mathcal{C}$ eine triangulierte Kategorie mit Translations-Funktor Σ. Das *graduierte Zentrum* von $\mathcal{C}$ ist der graduierte Ring, dessen Elemente im Grad n gegeben sind durch diejenigen natürlichen Transformationen vom Identitätsfunktor zu Σ^n, welche mit Σ bis auf ein Vorzeichen kommutieren. Gegeben solch ein Element ζ, bezeichne man mit X/ζ einen Kegel über der Abbildung ζ_X. Die ζ-Ordnung eines Objektes X von $\mathcal{C}$ wird induktiv definiert: Die Ordnung $\zeta\text{-ord}(X)$ ist ein Element von $\{0, 1, 2, \ldots, \infty\}$, und $\zeta\text{-ord}(X) \geq k$ für positives k gilt genau dann, wenn für jeden Morphismus $f : K \to X$ eine *Erweiterung* $\hat{f} : K/\zeta \to X$ existiert so, dass die Komposition $K \to K/\zeta \xrightarrow{\hat{f}} X$ gleich f ist, und für einen (und daher alle) Kegel $C_{\hat{f}}$ über $\hat{f}$ gilt $\zeta\text{-ord}(C_{\hat{f}}) \geq k-1$. Die ζ-Ordnung misst gewissermaßen, „wie sehr“ ζ auf X Null ist. Beispielsweise gilt $\zeta\text{-ord}(X) \geq 1$ genau dann, wenn $\zeta_X = 0$. Für jede natürliche Zahl m ist die Multiplikation mit m, $X \xrightarrow{m\cdot} X$, ein Element im Zentrum vom Grad 0. Insbesondere wurde hiermit also die m-Ordnung eines Objektes definiert.
Für den Fall von sogenannten *topologischen* triangulierten Kategorien hat Schwede folgendes bewiesen:

(t1) Sei $\mathcal{C}$ eine topologische triangulierte Kategorie und p eine Primzahl. Für jedes Objekt $X \in \mathcal{C}$ ist die p-Ordnung von X/p mindestens gleich $p-2$.

(t2) In der stabilen Homotopiekategorie $\mathcal{SHC}$ ist die p-Ordnung des mod-p-Moore Spektrums S/p *genau* gleich $p-2$.

(t3) Falls für ein Objekt $X \in \mathcal{C}$ die Abbildung $\alpha_1 \wedge X : \Sigma^{2p-3}X \to X$ durch p teilbar ist, so ist die p-Ordnung von X/p mindestens gleich $p-1$.

Die Aufgabe war nun, den Begriff der Ordnung im Fall der stabilen Modulkategorie $\mathcal{C} = \underline{\mathfrak{mod}}\text{-}kG$ zu untersuchen, wobei k ein Körper der Charakteristik $p > 0$ und G eine endliche Gruppe ist. Hat man eine Tate Kohomologie-Klasse $[\zeta] \in \widehat{\mathrm{Ext}}^n(k,k) = \hat{H}^n(G,k)$ von *geradem* Grad n gegeben, so wird diese von einer Abbildung $\zeta : \Omega^n k \to k$ in $\underline{\mathfrak{mod}}\text{-}kG$ repräsentiert, welche wiederum ein Element des Zentrums induziert, welches auch mit ζ bezeichnet wird:

$$\zeta : \Omega^n X \cong \Omega^n k \otimes X \xrightarrow{\zeta \otimes X} k \otimes X \cong X.$$

Es war bereits bekannt (Carlson 1987), dass für ungerades p dieses ζ auf $L_\zeta = \ker \zeta$, also auf $\Omega(k/\zeta)$ verschwindet. Dies entspricht der Aussage $\zeta\text{-ord}(k/\zeta) \geq 1$. Für $p = 2$ ist dies im allgemeinen falsch.

Resultate

In der vorliegenden Arbeit werden folgende Ergebnisse beweisen, die als Analoga zu (t1) bis (t3) angesehen werden können:

(a1) Für jedes Objekt $X \in \underline{\mathfrak{mod}}\text{-}kG$ gilt $\zeta\text{-ord}(X/\zeta) \geq p-2$.

Dies verallgemeinert Carlsons Resultat.

(a2) Für jede Primzahl p und jeden Körper k gibt es eine endliche Gruppe G und eine Kohomologieklasse $[\zeta] \in \hat{H}^*(G)$ von geradem Grad mit $\zeta\text{-ord}(k/\zeta) = p-2$.

Bekanntlich ist die gewöhnliche Gruppenkohomologie $H^*(G)$ die Kohomologie des klassifizierenden Raumes BG. Auf dieser hat man Steenrod Operationen. Wir definieren für Kohomologieklassen ζ vom Grad n (mit geradem n falls p ungerade ist):

$$P_1\zeta = \begin{cases} \mathrm{Sq}_1\zeta = \mathrm{Sq}^{n-1}\zeta & \text{falls } p = 2, \\ \beta \mathrm{P}^{\frac{n}{2}-1}\zeta & \text{falls } p \geq 3. \end{cases}$$

Dann gilt:

(a3) Sei X ein kG-Modul. Falls $P_1\zeta \otimes X$ durch ζ teilbar ist, dann gilt $\zeta\text{-ord}(k/\zeta) \geq p-1$.

Es stellt sich heraus, dass (a2) der schwierigste der drei Punkte ist. Für dessen Nachweis wird zunächst die Steenrod Operation P_1 auf die Tate Kohomologie erweitert; die neue Operation, genannt $\mathcal{P}_1$, bildet dann eine Kohomologieklasse ζ vom Grad n nicht mehr auf ein einzelnes Element, sondern auf eine Nebenklasse von $\zeta^p \cdot \hat{H}^*(G)$ in $\hat{H}^{pn-(2p-3)}(G)$ ab. Die Konstruktion dieser neuen Operation, der Nachweis elementarer Eigenschaften (Natürlichkeit, Cartan-Formel) sowie die konkrete Berechnung der Operation im Fall der Gruppen $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ sind zentraler Bestandteil dieser Arbeit. Abschließend wird über $(2p-1)$-fache Massey Produkte der Form $\langle \zeta, \alpha_1, \zeta, \ldots, \zeta, \alpha_{p-1}, \zeta \rangle$ ein Zusammenhang zum Ordnungsbegriff hergestellt, wodurch ein Beispiel für (a2) gefunden werden kann.