Cardinals as Ultrapowers

A Canonical Measure Analysis
under the Axiom of Determinacy

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Preface

> The game is up.
>
> William Shakespeare (1564 - 1616)
> "Cymbeline", Act 3 scene 3

My interest in logic and set theory was first raised when I realized that mathematics is not just about calculations with numbers but about formal systems, about the consequences that follow from applying specific rules to formal statements, so that the whole of mathematics can be concluded from axioms and rules of deduction. After reading about Gödels theorems I was fascinated. This was when I was still in high school and my first years of studying mathematics were more concerned with topics like functional analysis and algebraic topology.

Then I had to decide what the topic of my Master's thesis should be. I remembered that I always wanted to know more about set theory. So I went to Professor Peter Koepke and asked him if he would be my supervisor. That was when I really started to learn about logic and set theory. The set theory lecture course lead to seminars about models, large cardinals and determinacy. My Master's thesis was about supercompact cardinals under the Axiom of Determinacy and would not have happened without the support of Benedikt Löwe.

I started my PhD studies in Bonn under the supervision of Benedikt Löwe who soon after moved to Amsterdam. In Bonn, I was first a teaching assistant and then hired in the bilateral Amsterdam-Bonn project "Determiniertheitsaxiome, Infinitäre Kombinatorik und ihre Wechselwirkungen" (DFG-NWO Bilateral Cooperation Project KO1353/3-1/DN 61-532). As part of the project research, I went to Denton, Texas for a year in order to learn from and work with Steve Jackson. I spent my time in Denton by understanding his computation of the projective ordinals under AD and working as a teaching assistant.

After returning to Europe, I continued my project work in Amsterdam at the Institute for Logic, Language and Computation (ILLC). I had known before that logic was not restricted to mathematics, but at the ILLC I saw a truly interdisciplinary interaction between mathematics, philosophy, linguistics, and computer science. In January 2007, I returned to Bonn to finish writing my thesis.

But it is not only the mathematics and travelling to other countries that makes studying set theory so exiting and fun. Even before finishing my Master's thesis I helped out at the conference "Foundations of the Formal Sciences II" (FotFS II, Bonn 2000). Later I was a helper at the "Logic Colloquium 2002" in Münster and at the conference FotFS IV (Bonn 2003). I was part of the Organizing Committee of FotFS V (Bonn 2004) and of "Computability in Europe 2005" in Amsterdam. In 2007 I helped with the "International Conference On Logic, Navya-Nyaya & Applications" in Kolkata. My largest event was the "European Summer School in Logic, Language and Information 2008" in Hamburg, where I was responsible for catering and coordination. Planning and running a conference is sometimes exhausting but when all is over, the participants were happy, and everything ran (more or less) as planned, that makes it all worthwhile.

Such events must be advertised of course, so designing posters, printing shirts and bags, and writing small pamphlets with technical and local information is also part of the job. If a conference was a scientific success, a proceedings volume might be published, and so an organizer becomes an editorial assistant for a scientific publication. All together, you learn to be a mathematician, an event manager, a designer, and an editor.

So this is what I did in my seven years as a PhD student: writing this thesis was only a small fragment of my work in mathematical logic. When I started studying mathematics I would have never believed how many different things I would learn and do. But all of this would not have happened without the help of a lot of people.

I want to thank Peter Koepke for bringing me to set theory and keeping me there. This thesis is based on Steve Jackson's work on the projective ordinals under AD and would not have been possible without him helping me understanding his results. My supervisor Benedikt Löwe was always there for me. His response time sometimes seemed to contradict the laws of physics and he kept me going till the finish line. I really cannot thank him enough.

There are too many fellow PhD students I worked and had fun with to thank them all. So I restrict myself to two: my office-mate Ross Bryant from the University of North Texas, Denton, who made me feel at home in Texas, and my houseboat-mate Tikitu de Jager from the ILLC, who, among many other things, is the cause of me needing more space for books.

Last but definitively not least I want to thank Eva Bischoff. Without her support (and telling me to get behind the desk again) this thesis might still not be finished.

Cologne November 10, 2009 Stefan Bold

Contents

Introduction

Among the extensions of Zermelo-Fraenkel Set Theory (ZF) that contradict the Axiom of Choice (AC), the **Axiom of Determinacy** is one of the most interesting. The Axiom of Determinacy (AD) is a game-theoretic statement expressing that all infinite two-player perfect information games with a countable set of possible moves are determined, *i.e.*, admit a winning strategy for one of the players. The restriction to countable sets of possible moves makes AD essentially a statement about real numbers and sets of real numbers, and as a consequence it may come as a surprise that AD has strikingly peculiar consequences for the combinatorics on uncountable cardinals. Before we[1] go into more detail concerning those consequences let us give one reason why AD could have an impact on cardinals that seem far removed from the reals. If we let

$$\Theta := \sup\{\alpha \in \mathrm{On}\,;\ \text{there is a surjection from } \mathbb{R} \text{ onto } \alpha\},$$

then it is a consequence of Moschovakis' Coding Lemma (observed by H. Friedman and R. Solovay, for details, *cf.* [Ka94, Exercises 28.16 & 28.17]) that under AD we have $\Theta = \aleph_\Theta$, so Θ is a limit cardinal much larger than, for example, $\aleph_{\omega^\omega}$.

Since Θ is the supremum of the range of surjections from the set of real numbers onto an ordinal, part of the combinatorial theory of cardinals $\kappa < \Theta$ is affected by the theory of the reals. For example, the Axiom of Determinacy contradicts the full Axiom of Choice, but it implies countable choice for subsets of reals, and we can use surjections to get countable choice for subsets of κ^κ if κ is less than Θ.

Let us look at some of the remarkable combinatorial consequences of AD. Many properties that under full AC cannot hold or define large cardinals can be proven to hold under AD. An example from the early investigations of AD for the latter would be the existence of a normal measure on ω_1 which was proven by Solovay in 1967 [Ke78a], *cf.* [Ka94, Theorem 28.2], making the first uncountable

[1] I will use the first-person plural "we" throughout the thesis as it is common in most mathematical texts, we hope this will also enable the reader to feel more involved.

cardinal ω_1 a measurable cardinal. In order to present a combinatorial property that has witnesses under AD and cannot hold under AC we need to give some definitions first. We write $\kappa \to (\kappa)^\alpha$ to denote the fact that for every partition P of $[\kappa]^\alpha$, the set of increasing functions from κ to κ, into two sets there is a subset H of κ of size κ such that the partition P is constant on the set $[H]^\alpha$. If κ fulfils $\kappa \to (\kappa)^\kappa$ we say that κ has the **strong partition property**. Under AC no partition property with an infinite exponent can hold by a result of Erdős and Rado [ErRa52], *cf.* [Ka94, Proposition 7.1], but under AD many infinite partition properties are realized, an example for this would be the strong partition property of ω_1 which was shown by Martin in 1973, *cf.* [Ka94, Theorem 28.12].

Kleinberg [Kl77], *cf.* [Ka94, Theorem 28.14], proved that a normal measure μ on a strong partition cardinal κ generates a sequence $\langle \kappa_n^\mu\,;\, n < \omega\rangle$ of Jónsson cardinals (called a **Kleinberg sequence**) and computed the sequence derived from Martin's result about the measurability of ω_1 under AD: Let $\mathcal{C}$ be the normal measure that witnesses the measurability of ω_1, then

$$\kappa_n^{\mathcal{C}} = \aleph_n.$$

Nowadays, we know much more about infinitary combinatorics under AD, and it was mainly the work of Steve Jackson [Ja88, Ja99] that gave us many more strong partition cardinals and normal measures below $\aleph_{\boldsymbol{\varepsilon}_0}$. He computed the values of definable analogues of the cardinal Θ, the so-called **projective ordinals**

$$\boldsymbol{\delta}_n^1 := \sup\{\xi\ ;\ \xi \text{ is the length of a prewellordering of } \omega^\omega \text{ in } \boldsymbol{\Delta}_n^1\},$$

thus solving the fifth Victoria Delfino problem.[2] Furthermore, his computation showed that all projective ordinals with odd index have the strong partition property. A key part of this analysis was the concept of **descriptions**, finitary objects that "described" how to build ordinals less than a projective ordinal.

By the term "measure analysis" we shall understand informally the following procedure: given a strong partition cardinal κ and some cardinal $\lambda > \kappa$, we assign a measure μ on κ to λ such that $\kappa^\kappa/\mu = \lambda$. A central tool for measure analyses is Martin's Theorem on measures on strong partition cardinals (*cf.* [Ja99, Theorem 7.1]), which states that the ultrapower κ^κ/μ is a cardinal if μ is a measure on a strong partition cardinal κ.

By a **canonical measure analysis** we mean that there is a measure assignment for cardinals larger than a strong partition cardinal κ and a binary operation $\oplus$ on the measures of this assignment that corresponds to the iterated successor operation on cardinals, *i.e.*, if the ultrapower κ^κ/μ_1 is the αth successor of κ and the ultrapower κ^κ/μ_2 is the βth successor of κ, then the ultrapower $\kappa^\kappa/\mu_1 \oplus \mu_2$ is the $(\alpha+\beta)$th successor of κ. We will formalize these notions in Chapter 3.

[2]The first five of the Victoria Delfino problems can be found in [KeMo78, p. 279], problems six to twelve in [KeMaSt88, p. 221].

In 1990 Jackson and Khafizov [JaKh∞] provided a full analysis for cardinals less than $\boldsymbol{\delta}^1_5 = \aleph_{\omega^{\omega^\omega}+1}$, using the description theory developed by Jackson. This analysis was used by Benedikt Löwe [Lö02] to compute more Kleinberg sequences, corresponding to the normal measures on $\boldsymbol{\delta}^1_3$.

However, a uniform analysis of cardinals in terms of measures was still a *desideratum* since this analysis could not easily be generalized to larger projective ordinals. In 2004 Benedikt Löwe and the author developed a simple inductive argument for a measure analysis with just two measures that reaches the first ω^2 cardinals after a projective ordinal [BoLö07]. The argument consists of an abstract combinatorial induction and the concrete computation of certain ultrapowers, thus not needing the full description theory of Jackson. The combinatorial induction was then generalized to arbitrary sums of measures in [BoLö06]. But the computation of the ultrapowers needed in order to apply the combinatorial induction was at that time still missing.

In [JaLö06], Löwe and Jackson presented a general introduction to measure analyses under the Axiom of Determinacy and gave some algorithmic applications of the existence of an canonical measure assignment. This included the computation of the cofinalities of all cardinals in the scope of the measure assignment and the Kleinberg sequences associated to the normal ultrafilters on projective ordinals.

In 2005, Steve Jackson, Benedikt Löwe, and the author were working on material related to this thesis, when Steve Jackson managed to prove a general theorem about proving Jónssoness from a canonical measure assignment. During GLLC 12, the 12th workshop "Games in Logic, Language and Computation" at the Amsterdamer ILLC (Institute for Logic, Language and Computation) in 2006, Steve Jackson gave a talk in which he presented this result. The measure analysis developed in this thesis is similar to the one used by Jackson but differs in certain key ingredients. In Section 7.3 we will use Jackson's argument, slightly adapted to work with our measure assignment. This presentation is based on the slides of Jackson's talk at GLLC 12.

With the algebraic foundation of measure analysis developed by Jackson and Löwe in [JaLö06] and the combinatorial argument from [BoLö06] the way to a canonical measure analysis under AD was clear. What was needed was a way to derive additive ordinal algebras from ordinal algebras with multiplication so that the combinatorial argument could be used. And then compute the value of specific ultrapowers to prove the canonicity of the measure analysis with that argument. This thesis presents a solution to the first problem and also a computation of the first ω many ultrapowers needed.

In [JaLö06] ordinal algebras as an algebraic foundation for the measure analysis were introduced, in this thesis we develop the related notion of additive ordinal algebra and show that in the case of measure assignments from order measures canonicity of the measure assignment follows from the canonicity of the induced measure assignment for the additive ordinal algebra, see Lemma 3.3.3.

We also prove with Corollary 5.2.3 a generalization of the combinatorial Theorem 24 from [BoLö07] to arbitrary sums of order measures. This result allows us to reduce the question of canonicity for measure assignments for additive ordinal algebras essentially to the computation of certain ultrapowers. At this time this method is the best tool for an inductive proof of the canonicity of a measure assignment.

In [Lö02, p. 75] $\aleph_{\omega\cdot 2+2}$ was named as "the first infinite cardinal of which we do not know whether it has any large cardinal properties under AD." In [BoLö07] it was shown that $\aleph_{\omega\cdot 2+2}$ is Jónsson and $\aleph_{\omega\cdot 2+3}$ became the first such cardinal. In the last chapter of the thesis we show that all cardinals that are ultrapowers with respect to certain basic order measures are Jónsson cardinals. This allows us to enlarge the number of cardinals under AD for which we can prove that they are Jónsson. With the amount of canonicity proven in this thesis we can state that, if κ is an odd projective ordinal, $\kappa^{(n)}$, $\kappa^{(\omega\cdot n+1)}$, and $\kappa^{(\omega^n+1)}$, for $n<\omega$, are Jónsson under AD, see Theorem 7.3.9.

Naturally this leads to the question whether this is true for all ultrapowers of our measure assignment. Using the analysis of cardinals below $\boldsymbol{\delta}^1_5$, Steve Jackson showed 2005 that all successor cardinals below $\boldsymbol{\delta}^1_5$ are Jónsson, see Theorem 7.3.2. It would be enough to show the analogue of Lemma 7.3.7 for arbitrary order measures to prove that all successor cardinals in the scope of the canonical measure assignment are Jónsson.

Due to the results of this thesis we now have canonicity of the measure assignment up to the ω^ωth cardinal after an odd projective cardinal. In order to enlarge the scope of our measure analysis it will be necessary to inductively compute the values of larger and larger ultrapowers, corresponding to the variables in the additive ordinal algebra. This entails the use of Martin Trees that give upper bounds for ordinals with higher cofinalities. The first step after the results in this thesis would be to compute the ultrapower with respect to the ω_2-cofinal measure and products of it.

In Chapter 1 we will set up the mathematical foundations. We will define key notions like measures, club sets, and ultrapowers and present necessary results concerning those objects. Furthermore, we will introduce partition properties and types of cardinals with special partition properties, like the strong and weak partition property, as well as Jónsson cardinals.

After that we present a Theorem by Kleinberg stating that the iterated ultrapowers of a normal measure on a strong partition cardinal are Jónsson cardinals. The rest of Chapter 1 is about special types of functions, more precisely functions that are increasing, of uniform cofinality ω, and either continuous or discontinuous on all limit ordinals. We call them functions of continuous or discontinuous type, respectively. We show that, restricted to those functions, the homogeneous sets we get from the weak or strong partition property are in fact club sets.

In Chapter 2 we introduce the aforementioned Axiom of Determinacy. We give a formal definition and present some of its consequences. Most important for this thesis are of course the projective ordinals under AD, their values and properties, especially the strong partition property for odd projective ordinals. We will also often use that the odd projective ordinals are closed under ultrapowers. Furthermore we state the existence of Kunen trees and Martin trees on the odd projective ordinals.

A Kunen tree is a tree that gives us an upper bound for a function f for all ordinals in the domain of f with cofinality ω. A Martin tree is a generalization of this idea to ordinals of higher cofinality, we will work with Martin trees that give upper bounds for ordinals with cofinality ω_1. These trees will be used in the computation of ultrapowers in Chapter 6.

In Chapter 3 we will formalize our notion of measure analysis. First we introduce the algebraic foundation, the ordinal algebras. Then we define tree representations of terms in those algebras. Using those, we now can connect terms and measures and define the notion of measure assignments. If a measure assignment behaves in a way such that it respects the ordinal structure of terms then we call it canonical. The concepts in this chapter are purely algebraic and most general.

Which leads to Chapter 4, where we define order measures. Order measures are measures on a cardinal that arise from the weak or strong partition property of that cardinal; they are defined by lifting a measure on a smaller ordinal, using functions of continuous type. A similar procedure, called the strong lift, enables us to lift measures on a strong partition cardinal to measures on the respective ultrapower. This construction uses functions of discontinuous type.

We show how to define measure assignments from order measures and prove that some special measures, like the ω- and ω_1-cofinal normal measure on an odd projective ordinal, are order measures. At the end of Chapter 4 we finally define the measure assignment that we want to prove to be canonical.

In Chapter 5 we begin to prove the canonicity of our measure assignment. Theorem 5.2.2, which we call The Really Helpful Theorem (RHT), reduces the problem of showing the canonicity of our measure assignment essentially to the computation of certain ultrapowers corresponding to the variables in the additive ordinal algebra. We show that the first step in our measure analysis is canonical.

In Chapter 6 we compute the ultrapowers with respect to products of the ω_1-cofinal measure on an odd projective ordinal. The proof is by induction, we need the exact values of smaller ultrapowers in order to compute the next one. The results enables us to state that our measure assignment is canonical for the first ω^ω many cardinals after an odd projective ordinal.

Applications of the canonical measure analysis are given in Chapter 7. We show how it enable us to compute the cofinality of all cardinals in its scope and prove that some of the thus analyzed and computed cardinals are Jónsson cardinals. With these results we conclude the thesis.

Chapter 1

Mathematical Background

In this chapter we introduce basic definitions and results of Set Theory necessary for the concepts and proofs in later chapters. Mainly this chapter is meant as a reminder on material that is covered by standard set theory textbooks like [Ka94] or [Je02]. The last two sections of this chapter deal with certain types of functions and partition properties for sets of those functions. The results presented there are essential for the concept of order measures, see Chapter 4, and will be used often in our proofs.

1.1 The Basics

Our basic theory is Zermelo-Fraenkel Set Theory ZF, all additional assumptions (the Axiom of Determinacy (AD) mostly) will be explicitly stated. We write On for the class of ordinals, Lim for the class of limit ordinals and Card for the class of cardinals. As usual we will use α, β, γ, and so on, to denote ordinals. Unless otherwise noted κ will be a cardinal, and λ is our canonical choice for limit ordinals. We write ω for the first infinite cardinal, use ω_1 and $\aleph_1$ interchangeably for the first uncountable cardinal, and the same goes for ω_2 and $\aleph_2$. All larger cardinals will be denoted with $\aleph_\alpha$. For the product of two sets we write $X \times Y$ and $\alpha \cdot \beta$ for the product of two ordinals α and β in the sense of ordinal arithmetic. For cardinals κ the function $\ulcorner\cdot,\cdot\urcorner$ is the Gödel pairing function, *i.e.*, a definable bijection between $\kappa \times \kappa$ and κ. Similarly, $\ulcorner\cdot\urcorner : \kappa^n \to \kappa$ is a definable bijection between κ^n and κ. We define the **iterated successor operation** on cardinals κ by transfinite recursion:

- $\kappa^{(0)} = \kappa$,
- $\kappa^{(\alpha+1)} = (\kappa^{(\alpha)})^+$ for all ordinals α, and
- $\kappa^{(\lambda)} = \sup_{\alpha<\lambda} \kappa^{(\alpha)}$ for limit ordinals λ.

If X and Y are sets, then Y^X is the set of functions from X to Y. For ordinals it will be clear from the context whether by α^β we mean the set of functions or the ordinal derived by ordinal exponentiation. We write $X^{<\alpha}$ for the set of functions with range in X and domain less than α. If $x = \langle x_i\,;\, i < \alpha\rangle \in X^\alpha$ is a X-sequence of length α and β an ordinal less than α, then we write $x{\restriction}\beta$ for its restriction $\langle x_i\,;\, i < \beta\rangle$ to length β. The length α of a sequence $x \in X^\alpha$ will be denoted by $\mathbf{lh}(x)$. Quite often we will write $\vec{x}$ to denote a sequence $\langle x_i\,;\, i < n\rangle$, the length of which will be clear from the context. If $M \subset \mathrm{On}$ is a set of ordinals that has cardinality at least κ, then we denote the set of κ-sized subsets of M by $[M]^\kappa$ and identify it with the set of strictly increasing M-sequences of length κ and the set of strictly increasing functions from κ to M. If $F : X \to Y$ is a function and A a subset of X, then we write $F"(A)$ for the set $\{F(x)\,;\, x \in A\}$, the image of A under F. If α is an ordinal, C a subset of α and β an ordinal less than α, then we denote the set $C\backslash(\beta + 1)$ of ordinals in C that are greater than β by $C_{>\beta}$.

An **order** $\prec$ on a set X is an irreflexive, transitive relation $\prec\, \subseteq X \times X$ on X. The order $<_T$ on X-sequences is defined by: $t <_T s$ iff s is a proper initial segment of t, *i.e.*, $s = t{\restriction}\alpha$ for some $\alpha < \mathbf{lh}(t)$. The **lexicographic order** $<_{\mathrm{lex}}$ is the following order on $\mathrm{On}^{<\omega}$: $\vec{\alpha} <_{\mathrm{lex}} \vec{\beta}$ iff $\alpha_i < \beta_i$ holds for the least i such that $\alpha_i \neq \beta_i$, or $\vec{\alpha}$ is a proper initial segment of $\vec{\beta}$. If we reverse the ordering with respect to initial segments, then we get the **Kleene-Brouwer order** $<_{\mathrm{KB}}$ on $\mathrm{On}^{<\omega}$: $\vec{\alpha} <_{\mathrm{KB}} \vec{\beta}$ iff $\alpha_i < \beta_i$ holds for the least i such that $\alpha_i \neq \beta_i$, or $\vec{\beta}$ is a proper initial segment of $\vec{\alpha}$. And if we order sequences from right to left instead from left to right we get the **reverse lexicographic order** $<_{\mathrm{rlex}}$ on $\mathrm{On}^{<\omega}$: $\vec{\alpha} <_{\mathrm{rlex}} \vec{\beta}$ iff $\alpha_i < \beta_i$ holds for the largest i such that $\alpha_i \neq \beta_i$, or $\vec{\alpha}$ is a proper initial segment of $\vec{\beta}$. An order $\prec$ on a set A is **wellfounded** if every subset $B \subseteq A$ has a $\prec$-minimal element. If α and β are ordinals, then the $<_{\mathrm{rlex}}$-order type of the set $\alpha \times \beta$ is the ordinal $\alpha \cdot \beta$, and the function $\langle \gamma, \delta\rangle \mapsto \alpha \cdot \delta + \gamma$ is an isomorphism between $\langle \alpha \times \beta, <_{\mathrm{rlex}}\rangle$ and $\langle \alpha \cdot \beta, <\rangle$.

A **tree** T on a set X is a subset of $X^{<\omega}$ that is closed under initial segments. We call the elements of a tree also **nodes**. A node that has no $<_T$-predecessor is called a **root** of T, most of the time we will work with trees that have one root. Conversely, a node that has no $<_T$-successor is a **terminal node**, or **leaf**, of T. A sequence of immediate $<_T$-successors is called a **branch** in T. Mostly we will talk about branches that start with the root and in this case identify the branch with its terminal node, if such exists. An **infinite branch** of T is an element $t \in X^\omega$ such that $t{\restriction}n$ is an element of T for all $n < \omega$. A tree $T \subseteq \alpha^{<\omega}$ is called **wellfounded** iff the order $<_T$ on T is wellfounded. For trees on ordinals this is equivalent to the non-existence of infinite branches:

1.1.1. Lemma. *Let $T \subseteq \alpha^{<\omega}$ be a tree on an ordinal α. Then the following are equivalent:*

1. *The tree T is wellfounded.*

2. *The tree T has no infinite branches.*

Proof. 1. $\Rightarrow$ 2. Assume T has an infinite branch t, then the set $\{t \restriction n\,;\, n < \omega\} \subseteq T$ has no $<_T$-minimal element, contradicting the wellfoundedness of T.

2. $\Rightarrow$ 1. Let B be an arbitrary subset of T. If t is an element of B then adding the initial segments of t to B will not change the existence or non-existence of a $<_T$-minimal element in B, so we can assume that B is closed under initial segments. Let s be the leftmost branch in B, *i.e.*,

$$s_n := \min\{\beta\,;\, \beta = t_n \text{ for some } t \in B \text{ such that } t \restriction n = \langle s_i\,;\, i < n\rangle\}.$$

Then by assumption s has finite length and no extensions in B, so it is a $<_T$-minimal element in B. q.e.d.

A tree T on an ordinal α is linearly ordered by the Kleene-Brouwer order $<_{\mathrm{KB}}$ and if T is wellfounded this is a wellorder. If T is a wellfounded tree on some ordinal α then we denote the **rank** of T in the Kleene-Brouwer ordering $<_{\mathrm{KB}}$ by $|T|$. If $t \in T$, then $|T|(t)$ denotes the rank of t in the Kleene-Brouwer ordering on T. For infinite ordinals α we fix bijections $p : \alpha \to \alpha^{<\omega}$ between α and $\alpha^{<\omega}$ and write $|T|(\beta)$ for $|T|(p(\beta))$, if that exists, otherwise it is undefined. If δ is an ordinal we write $T \restriction \delta$ for the tree that is the restriction $T \cap \delta^{<\omega}$ of T to δ.

1.2 Filters and Measures

A **filter** $\mathcal{F}$ on a set M is a nonempty subset of the powerset $\mathcal{P}(M)$ that is closed under finite intersections, supersets, and does not include the empty set. A filter is **σ-closed** if it is closed under countable intersections. Let κ be a cardinal, a filter $\mathcal{F}$ is **κ-closed** if it is closed under unions of length less than κ, so σ-closed is the same as ω_1-closed. A filter $\mathcal{F}$ on a set S is **non-principal** if its intersection is empty and it is an **ultrafilter** if for every subset S of M either S or $M \backslash S$ is in $\mathcal{F}$. A ultrafilter on an ordinal κ is said to **contain end segments** if for all $\gamma < \kappa$ the set $\{\alpha < \kappa\,;\, \gamma < \alpha\}$ is an element of the ultrafilter. As usual we call a σ-complete ultrafilter a **measure**. To denote measures we often use the Greek letters μ and ν. We write $\mu_{\{\alpha\}}$ for the principal measure that concentrates on $\{\alpha\}$.

In this section we introduce basic properties of measures that will be used throughout this thesis. Especially the measures derived from club sets on cardinals, see Definition 1.3.1, will play an important role.

1.2.1. Definition. If $\mathcal{F}$ is a filter on a set X and $F : X \to Y$ a function we define the **image filter** $\mathcal{F}_F$ on Y by

$$A \in \mathcal{F}_F \quad :\Longleftrightarrow \quad \text{There is a set } B \in \mathcal{F} \text{ such that } F\text{"}(B) \subseteq A.$$

1.2.2. LEMMA. *Let $\mathcal{F}$ be a filter on a set X, $F : X \to Y$ a function, and κ a cardinal. Then*

1. *$\mathcal{F}_F$ is a filter,*
2. *if $\mathcal{F}$ is an ultrafilter then so is $\mathcal{F}_F$, and*
3. *if $\mathcal{F}$ is κ-complete then so is $\mathcal{F}_F$.*

Proof. 1. Since $F"(B)$ is nonempty for all nonempty sets $B \subseteq X$ we have $\emptyset \notin \mathcal{F}_F$, and from $F"(X) \subseteq Y$ follows $Y \in \mathcal{F}_F$. If $A \subseteq Y$ is in $\mathcal{F}_F$ then there is a set $B \in \mathcal{F}$ such that $F"(B) \subseteq A$. But for all sets $D \subseteq Y$ with $A \subseteq D$ we have also $F"(B) \subseteq D$, so $\mathcal{F}$ is closed under supersets. If $A \subseteq Y$ and $C \subseteq Y$ are in $\mathcal{F}_F$ then there are sets $B, D \in \mathcal{F}$ such that $F"(B) \subseteq A$ and $F"(D) \subseteq C$. Since $\mathcal{F}$ is a filter the set $B \cap D$ is in $\mathcal{F}$, and since $F"(B \cap D) \subseteq A \cap C$ that means $\mathcal{F}_F$ is closed under intersections. So $\mathcal{F}_F$ is a filter.

2. Let A be an arbitrary subset of Y and $B = \{x \in X\,;\, F(x) \in A\}$ its pre-image under F. Since $\mathcal{F}$ is an ultrafilter we have either $B \in \mathcal{F}$ or $X \backslash B \in \mathcal{F}$. In the first case we get $A \in \mathcal{F}_F$. Otherwise, since $F"(X \backslash B) \subseteq Y \backslash A$ and $\mathcal{F}_F$ is closed under supersets, we get $Y \backslash A \in \mathcal{F}_F$. So $\mathcal{F}_F$ is an ultrafilter if $\mathcal{F}$ is one.

3. Let γ be an ordinal less than κ and $\langle A_\alpha\,;\, \alpha < \gamma\rangle$ a sequence of subsets of Y with $A_\alpha \in \mathcal{F}_F$ for all $\alpha < \gamma$. By definition of $\mathcal{F}_F$ there is a sequence $\langle B_\alpha\,;\, \alpha < \gamma\rangle$ of subsets of X with $B_\alpha \in \mathcal{F}$ and $F"(B_\alpha) \subseteq A_\alpha$ for all $\alpha < \gamma$. Since $\mathcal{F}$ is κ-complete we have $\bigcap_{\alpha<\gamma} B_\alpha \in \mathcal{F}$ and from $F"(\bigcap_{\alpha<\gamma} B_\alpha) \subseteq \bigcap_{\alpha<\gamma} F"(B_\alpha) \subseteq \bigcap_{\alpha<\gamma} A_\alpha$ follows the κ-completeness of $\mathcal{F}_F$. q.e.d.

The following equivalences to being κ-complete for an ultrafilter allow us different approaches in proving the completeness of measures and will be used later:

1.2.3. LEMMA. *Let M be a set, $\mathcal{U} \subseteq \mathcal{P}(M)$ an ultrafilter on M and $\kappa \leq \mathrm{Card}(M)$ a cardinal. Then the following are equivalent:*

1. *$\mathcal{U}$ is a κ-complete ultrafilter on M.*
2. *For all $\gamma < \kappa$ and sequences $\langle X_\alpha\,;\, \alpha < \gamma\rangle$ of subsets of M, if $\bigcup_{\alpha<\gamma} X_\beta$ is an element of $\mathcal{U}$ then there is an $\beta < \gamma$ such that $X_\beta \in \mathcal{U}$.*
3. *For all $\gamma < \kappa$ and γ-partitions $\langle X_\alpha\,;\, \alpha < \gamma\rangle$ of M there is an $\beta < \gamma$ such that $X_\beta \in \mathcal{U}$.*

Proof. 1. $\Rightarrow$ 2. If no X_β is an element of U, then all $\mathcal{P}(M)\backslash X_\beta$ are elements of U and κ-completeness implies $\bigcap_{\alpha<\gamma} \mathcal{P}(M)\backslash X_\alpha = \mathcal{P}(M)\backslash \bigcup_{\alpha<\gamma} X_\alpha \in U$, a contradiction to $\bigcup_{\alpha<\gamma} X_\alpha \in U$.

2. $\Rightarrow$ 3. The union of all X_β is the set $\mathcal{P}(M)$, an element of U, so by 2. there has to be a β such that $X_\beta \in U$.

3. $\Rightarrow$ 1. We define $X'_\alpha := X_\alpha \backslash \bigcup_{\beta\neq\alpha} X_\beta$, $X'_\gamma := \bigcap_{\beta<\gamma} X_\beta$, and $X'_{\gamma+1} := \mathcal{P}(M)\backslash \bigcup_{\beta<\gamma} X_\beta$. Since U is an ultrafilter only X'_γ can be an element of U, and by 3. it has to be. q.e.d.

If a κ-complete measure on a cardinal κ exists then this cardinal is called **measurable**. Under AC such a cardinal is a strong limit, *i.e.*, a large cardinal. Without AC we can at least prove the regularity of the cardinal:

1.2.4. LEMMA. *Let κ be cardinal and μ a κ-complete ultrafilter on κ that contains end segments. Then κ is regular.*

Proof. Assume κ is singular, then there is an ordinal $\lambda < \kappa$ and a sequence $\langle \alpha_\gamma\,;\, \gamma < \lambda\rangle \in \kappa^\lambda$ with $\kappa = \sup_{\gamma<\lambda} \alpha_\gamma$. Define $U_\gamma := \{\alpha \in \kappa\,;\, \alpha > \gamma\}$, since μ contains end segments the set U_γ is an element of μ for all $\gamma < \lambda$ and because μ is κ-complete the intersection $\bigcap_{\gamma<\lambda} U_\gamma$ is also in μ. But since $\kappa = \sup_{\gamma<\lambda} \alpha_\gamma$ we have $\bigcap_{\gamma<\lambda} U_\gamma = \emptyset$ which contradicts the fact that μ is a filter. q.e.d.

A function that is bounded below κ on a measure one set is in fact constant on a measure one set if the measure is κ-complete:

1.2.5. LEMMA. *Let M be a set, κ a cardinal and $\mathcal{U} \subseteq \mathcal{P}(M)$ a κ-complete ultrafilter on M. If $F : M \to \mathrm{On}$ is a function and there is an ordinal $\alpha < \kappa$ such that for $\mathcal{U}$-almost all x we have $F(x) < \alpha$, i.e.,*

$$\exists A \in \mathcal{U}\ \forall x \in A\ F(x) < \alpha,$$

then F is $\mathcal{U}$-almost constant:

$$\exists \beta < \kappa\ \exists B \in \mathcal{U}\ \forall x \in B\ F(x) = \beta.$$

Proof. We define a sequence $\langle X_\gamma\,;\, \gamma < \alpha\rangle$ of subsets of M by

$$X_\gamma := \{x \in M\,;\, F(x) = \gamma\}.$$

Then A is a subset of $\bigcup_{\gamma<\alpha} X_\gamma$, so $\bigcup_{\gamma<\alpha} X_\gamma$ is an element of the filter $\mathcal{U}$. And since $\mathcal{U}$ is a κ-complete ultrafilter by Lemma 1.2.3 that means there is a $\beta < \alpha$ such that X_β is in $\mathcal{U}$, so by definition of X_β the proof is finished. q.e.d.

Let $\vec{X} = \langle X_\alpha\,;\, \alpha < \kappa\rangle$ be a sequence of subsets of κ, then the **diagonal intersection** $\triangle_{\alpha<\kappa} X_\alpha$ of $\vec{X}$ is defined by

$$\triangle_{\alpha\in\kappa} X_\alpha := \{\xi < \kappa\,;\, \xi \in \bigcap_{\alpha<\xi} X_\alpha\}.$$

We call a filter F on a cardinal κ **normal** if it is closed under diagonal intersections, *i.e.*, if $\triangle_{\alpha<\kappa} X_\alpha \in F$ for all $\vec{X} \in F^\kappa$. We include the requirement of being non-principal in our definition of normal since the only normal measures that concern us fulfill this requirement.

Although normal measures on a cardinal κ are often considered to be κ-complete this is only true if the measure in question contains end segments, an assumption that is often included in the definition of an ultrafilter.

1.2.6. LEMMA. *Let κ be a cardinal and μ a normal ultrafilter on κ that contains end segments. Then μ is a κ-complete ultrafilter.*

Proof. Let $\lambda < \kappa$ be an ordinal and $\langle U_\gamma\,;\, \gamma < \lambda\rangle$ a sequence of subsets of κ that are in μ. For $\lambda \leq \gamma < \kappa$ define $U_\gamma := \{\alpha \in \kappa\,;\, \alpha > \gamma\}$, then those U_γ are also elements of μ since μ contains end segments. So the diagonal intersection $\triangle_{\gamma<\kappa} U_\gamma$ is an element of μ and thus also the set $S := \triangle_{\gamma<\kappa} U_\gamma \cap U_\lambda$. An ordinal $\alpha > \lambda$ is an element of S if $\alpha \in \bigcap_{\gamma<\alpha} U_\gamma$ holds. Since by definition of the U_γ we have $\alpha \in U_\gamma$ for $\lambda < \gamma < \alpha$ that means $\alpha \in \bigcap_{\gamma<\lambda} U_\gamma$ for all $\alpha \in S$. But μ is closed under supersets, so $\bigcap_{\gamma<\lambda} U_\gamma$ is an element of μ and thus μ is κ-complete. q.e.d.

Let $f : \alpha \to \alpha$ be a function on some ordinal α and X be a subset of α. If $f(\beta) < \beta$ holds for all β in X then f is **regressive** on X.

1.2.7. LEMMA. *Let $\kappa > \omega$ be a cardinal. For a measure μ on κ are equivalent:*

1. *μ is normal.*
2. *For every function $f : \kappa \to \kappa$ that is regressive on some $X \subseteq \kappa$ in μ there is a set $Y \subseteq \kappa$ in μ such that f is constant on Y, i.e., there is an ordinal $\alpha < \kappa$ such that $f(\beta) = \alpha$ for all β in Y.*

Proof. 1. $\Rightarrow$ 2. Let $f : \kappa \to \kappa$ be a function that is regressive on the set $X \in \mu$. If there is no set $Y \subseteq \kappa$ in μ such that f is constant on Y then the set $Y_\alpha := \{\beta < \kappa\,;\, f(\beta) \neq \alpha\}$ is in μ for all $\alpha < \kappa$. Since μ is normal the set $X \cap \triangle_{\beta<\kappa} Y_\beta$ is also an element of μ. By definition of the diagonal intersection α is in $\triangle_{\beta<\kappa} Y_\beta$ if $f(\alpha) \neq \beta$ holds for all $\beta < \alpha$, *i.e.*, if $f(\alpha) \geq \alpha$. So the intersection of X and $\triangle_{\beta<\kappa} Y_\beta$ is empty, a contradiction. So there is some set $Y \subseteq \kappa$ in μ such that f is constant on Y.

2. $\Rightarrow$ 1. Let $\langle Y_\alpha\,;\, \alpha < \kappa\rangle$ be a sequence of elements of μ. If $\triangle_{\beta<\kappa} Y_\beta$ is not an element of μ then the set $X := \{\alpha \in \kappa\,;\, \alpha \notin \bigcap_{\beta<\alpha} Y_\beta\}$ is. We define a function $f : \kappa \to \kappa$ by

$$f(\alpha) := \begin{cases} 0 & \text{if } \alpha \notin X \text{ and} \\ \min\{\beta < \alpha\,;\, \alpha \notin Y_\beta\} & \text{if } \alpha \in X. \end{cases}$$

Then f is regressive on X by definition and so by 2. for some $\gamma < \kappa$ the set $Z := \{\beta \in \kappa\,;\, f(\beta) = \gamma\}$ is in μ. Since μ is a measure we can assume without loss of generality that Z is a subset of X. So for all α in Z we have $\alpha \notin Y_\gamma$ and thus the intersection of Z and Y_γ is empty, a contradiction. So $\triangle_{\beta<\kappa} Y_\beta$ is an element of μ. q.e.d.

1.2.8. Lemma. *If μ is a normal measure on a cardinal κ then the set* $\mathrm{Lim} \cap \kappa$ *of limit ordinals less than κ is an element of μ.*

Proof. Assume otherwise, since μ is an ultrafilter that means the set $\mathrm{Succ} \cap \kappa$ of successor ordinals less than κ is an element of μ. Define $f(\alpha)$ to be the predecessor of α for successor ordinals α and 0 otherwise. Then on $\mathrm{Succ} \cap \kappa$ the function f is regressive, so by Lemma 1.2.7 there is an ordinal $\beta < \kappa$ and a set $B \in \mu$ with $f(\alpha) = \beta$ for all $\alpha \in B$. We can assume that B is a subset of Succ, so $f(\alpha) = \beta$ for all $\alpha \in B$ means there are successor ordinals that are not equal but have the same predecessor, a contradiction. So the set $\mathrm{Lim} \cap \kappa$ must be an element of μ.
q.e.d.

Let μ be a measure on a set X and ν a measure on a set Y. The **product measure** $\mu \times \nu$ on $X \times Y$ is defined as follows: For $C \subseteq X \times Y$

$$C \in \mu \times \nu \quad :\Longleftrightarrow \quad \{y \in Y\,;\, \{x \in C\,;\, \langle x, y\rangle \in C\} \in \mu\} \in \nu.$$

1.2.9. Lemma. *Let μ, ν, and η be measures on sets X, Y, and Z, respectively. Let κ be a cardinal.*

1. *The product measure $\mu \times \nu$ is indeed a measure, i.e., an σ-complete ultrafilter.*
2. *If μ and ν are κ-complete then so is $\mu \times \nu$.*
3. *The operation $\times$ on measures is associative, i.e., $(\mu \times \nu) \times \eta = \mu \times (\nu \times \eta)$. We often write μ^n for the n-folded product of a measure μ.*

Proof. We start with a bit of notational convenience: For subsets C of $X \times Y$ and elements y of Y let us define $C(y) := \{x \in X\,;\, \langle x, y\rangle \in C\}$. So $\emptyset(y) = \emptyset$, $(X \times Y)(y) = X$ and $X \backslash C(y) = (X \times Y \backslash C)(y)$ for all $y \in Y$.

1. By definition of $\mu \times \nu$ we have $\emptyset \notin \mu \times \nu$ and $X \times Y \in \mu \times \nu$. If $C \subseteq X \times Y$ is in $\mu \times \nu$ and $D \subseteq X \times Y$ is a superset of C we have $C(y) \subseteq D(y)$ for all $y \in Y$. If $C(y)$ is in μ then so is $D(y)$ since μ is closed under supersets. Which means $\{y \in Y\,;\, C(y) \in \mu\}$ is a subset of $\{y \in Y\,;\, D(y) \in \mu\}$ and since ν is also closed under supersets this set is in ν. So $\mu \times \nu$ is closed under supersets. Let γ be an ordinal less than ω_1 and $\langle C_\alpha\,;\, \alpha < \gamma\rangle$ a sequence of subsets of $X \times Y$ with $C_\alpha \in \mu \times \nu$ for all $\alpha < \gamma$. By definition of $\mu \times \nu$ the set $\{y \in Y\,;\, C_\alpha(y) \in \mu\}$ is in ν for all $\alpha < \gamma$ and since ν is σ-complete the set

$$\bigcap_{\alpha<\gamma} \{y \in Y\,;\, C_\alpha(y) \in \mu\} = \{y \in Y\,;\, C_\alpha(y) \in \mu \text{ for all } \alpha < \gamma\}$$

is also in ν. If $C_\alpha(y)$ is in μ for all $\alpha < \gamma$ then $\bigcap_{\alpha<\gamma} C_\alpha(y)$ is also in μ since μ is σ-complete. But $\bigcap_{\alpha<\gamma} C_\alpha(y) = (\bigcap_{\alpha<\gamma} C_\alpha)(y)$, so the set of $y \in Y$ such that $(\bigcap_{\alpha<\gamma} C_\alpha)(y)$ is in μ is in ν. That means $\bigcap_{\alpha<\gamma} C_\alpha$ is in $\mu \times \nu$ and $\mu \times \nu$ is thus also σ-complete. At last we have to prove the ultrafilter property of $\mu \times \nu$. Let C be an arbitrary subset of $X \times Y$. Either C is in $\mu \times \nu$ and we are done or C is not in $\mu \times \nu$. Then the set of $y \in Y$ such that $C(y)$ is in μ is not in ν. Since ν is an ultrafilter that means the set of $y \in Y$ such that $C(y)$ is not in μ is in ν. If $C(y)$ is not in μ then $(X \times Y \backslash C)(y) = X \backslash C(y)$ is in μ since μ is an ultrafilter. So the set $\{y \in Y\,;\, (X \times Y \backslash C)(y) \in \mu\}$ is in ν, which means $X \times Y \backslash C$ is in $\mu \times \nu$ and thus $\mu \times \nu$ is an ultrafilter. This finishes the proof that $\mu \times \nu$ is a measure.

2. Here we can use the same argument as in part 1. when we proved the σ-completeness of $\mu \times \nu$ from the σ-completeness of μ and ν.

3. A set $D \subseteq X \times Y \times Z$ is in $(\mu \times \nu) \times \eta$ if the set $\{z \in Z\,;\, D(z) \in \mu \times \nu\}$ is in η. The set $D(z)$ is in $\mu \times \nu$ if the set $\{y \in Y\,;\, (D(z))(y) \in \mu\}$ is in ν. So D is in $(\mu \times \nu) \times \eta$ if

$$\{z \in Z\,;\, \{y \in Y\,;\, \{x \in X\,;\, \langle x, y, z\rangle \in D\} \in \mu\} \in \nu\} \in \eta$$

holds. On the other hand D is in $\mu \times (\nu \times \eta)$ if the set $\{\langle y, z\rangle \in Y \times Z\,;\, D(\langle y, z\rangle) \in \mu\}$ is in $\nu \times \eta$. The set $\{\langle y, z\rangle \in Y \times Z\,;\, D(\langle y, z\rangle) \in \mu\}$ is in $\nu \times \eta$ if the set

$$\{z \in Z\,;\, (\{\langle y, z\rangle \in Y \times Z\,;\, D(\langle y, z\rangle) \in \mu\})(z) \in \nu\}$$

is in η. We have

$$(\{\langle y, z\rangle \in Y \times Z\,;\, D(\langle y, z\rangle) \in \mu\})(z) =$$

$$(\{\langle y, z\rangle \in Y \times Z\,;\, \{x \in X\,;\, \langle x, y, z\rangle \in D\} \in \mu\})(z) =$$
$$\{y \in Y\,;\, \{x \in X\,;\, \langle x, y, z\rangle \in D\} \in \mu\}.$$

So D is in $(\mu \times \nu) \times \eta$ if $\{z \in Z\,;\, \{y \in Y\,;\, \{x \in X\,;\, \langle x, y, z\rangle \in D\} \in \mu\} \in \nu\} \in \eta$ holds, which means the measures $(\mu \times \nu) \times \eta$ and $\mu \times (\nu \times \eta)$ are identical. q.e.d.

Let us take a closer look at products of measures on ordinals. Let α and β be ordinals. The wellorders $\langle \alpha \times \beta, <_{\text{rlex}}\rangle$ and $\langle \alpha \cdot \beta, <\rangle$ are isomorphic with the bijection $\pi(\langle \gamma, \delta\rangle) = \alpha \cdot \delta + \gamma$. This leads to another notion of product measure: Let μ be a measure on an ordinal α and ν a measure on an ordinal β, we denote the image filter induced by π with $\mu \times' \nu$. So for $C \subseteq \alpha \cdot \beta$ we have

$$C \in \mu \times' \nu \quad :\Leftrightarrow \quad \{y \in \beta\,;\, \{x \in \alpha\,;\, \alpha \cdot y + x \in C\} \in \mu\} \in \nu.$$

The measure $\mu \times \nu$ lives on the set $\alpha \times \beta$ and the filter $\mu \times' \nu$ on the ordinal $\alpha \cdot \beta$, but essentially they are the same measure:

1.2.10. LEMMA. *Let μ and ν be measures on ordinals α and β, respectively. Let π be the bijection between $\alpha \times \beta$ and $\alpha \cdot \beta$ defined by $\pi(\langle \gamma, \delta\rangle) = \alpha \cdot \delta + \gamma$.*

1. *If $A \in \mu \times \nu$ then $\pi``(A) \in \mu \times' \nu$ and if $B \in \mu \times' \nu$ then $\pi^{-1}``(B) \in \mu \times \nu$.*

2. *The filter $\mu \times' \nu$ is a measure and if μ and ν are κ-complete then so is $\mu \times' \nu$.*

3. *The operation $\times'$ on measures is associative, i.e., $(\mu \times' \nu) \times' \eta = \mu \times' (\nu \times' \eta)$.*

4. *If μ and ν contain end segments then so does $\mu \times' \nu$.*

Proof. Since $\mu \times' \nu$ is the image filter of $\mu \times \nu$ under the function π and π is a bijection part 1. of the lemma follows directly from the definition of an image filter. Part 2. follows from Lemma 1.2.2 and part 3. is a direct consequence of the associativity of $\times$ and the bijectivity of π. So we only need to prove part 4. Let ξ be an element of $\alpha \cdot \beta$, there are $\gamma \in \alpha$ and $\delta \in \beta$ such that $\xi = \alpha \cdot \delta + \gamma$. We have to show that the set $C := \{\zeta \in \alpha \cdot \beta\,;\, \zeta \geq \xi\}$ is in $\mu \times' \nu$. For all $\gamma \leq \sigma < \alpha$ and $\delta \leq \tau < \beta$ we have $\alpha \cdot \tau + \sigma \in C$. Since μ contains end segments the set $\{\sigma \in \alpha\,;\, \alpha \cdot \tau + \sigma \in C\}$ is an element of μ for all $\delta \leq \tau < \beta$. And ν also contains end segments so the set $\{\tau \in \beta\,;\, \{\sigma \in \alpha\,;\, \alpha \cdot \tau + \sigma \in C\} \in \mu\}$ is in ν. We conclude that C is an element of $\mu \times' \nu$, *i.e.*, that this product measure contains end segments. q.e.d.

1.3 Club Sets

We now come to the notion of club sets. Club subsets of cardinals and the filters that can be derived from the set of club subsets of a cardinal are essential tools in our work toward analyzing cardinals as ultrapowers.

1.3.1. DEFINITION. Let α be an ordinal and $C \subseteq \alpha$ a subset of α. We call C **unbounded** in α if for all β in α exists a δ in C such that $\delta \geq \beta$. If C includes every ordinal λ such that $\lambda \cap C = \lambda$ we call C **closed**. A subset $C \subseteq \alpha$ is **club** if it is unbounded in α and closed. Obviously the ordinal α itself is a club subset of α.

1.3.2. LEMMA. *Let α be a limit ordinal with cofinality $\mathrm{cf}(\alpha)$ greater ω.*

1. *The set of club subsets of α is closed under intersections of less than $\mathrm{cf}(\alpha)$-many club subsets.*

2. *If α is a regular limit ordinal then the set of club subsets of α is also closed under diagonal intersections.*

3. *For all club subsets $C \subseteq \alpha$ and ordinals $\beta < \alpha$ the set $C_{>\beta} := C \backslash (\beta + 1)$ of elements in C greater than β is a club subset of α.*

4. *The set $\mathrm{Lim}(\alpha)$ of limit ordinals less than α is a club set.*

Proof. 1. Let δ be an ordinal less than $\mathrm{cf}(\alpha)$ and $\langle C_\beta\,;\,\beta<\delta\rangle$ a sequence of club subsets of α. We have to show that $C := \bigcap_{\beta<\delta} C_\beta$ is club. Assume we have $\lambda \cap C = \lambda$ for some $\lambda < \alpha$. Since C is a subset of C_β for all $\beta < \delta$ we also have $\lambda \cap C_\beta = \lambda$ for all $\beta < \delta$. The C_β are club so λ is an element of all C_β, which means λ is also an element of C. This shows that C is closed.

Let ζ be an element of α. For each $\beta < \delta$ we build an increasing ω-sequence $\langle c^\beta_n\,;\,n<\omega\rangle$ in C_β that dominates ζ: For all $\beta<\delta$ let $c^\beta_0 :=$ the least element of C_β greater ζ and for $n<\omega$ let $c^\beta_{n+1} :=$ the least element of C_β greater $\sup_{\xi<\delta} c^\xi_n$. This is welldefined since δ is less than the cofinality of α, which also is greater ω. From this also follows that $\sup_{n<\omega} c^\beta_n$ is an element of α for all $\beta<\delta$. The C_β are closed so $\sup_{n<\omega} c^\beta_n$ as a limit of elements from C_β is an element of C_β. From the definition of the c^β_n we get $\sup_{n<\omega} c^\beta_n = \sup_{n<\omega} c^\eta_n$ for all $\beta,\eta\in\delta$. So $\sup_{n<\omega} c^0_n$ is an element of $\bigcup_{\beta<\delta} C_\beta = C$ that is greater ζ, which proves that C is unbounded.

2. Let $\langle C_\beta\,;\,\beta<\alpha\rangle$ be a sequence of club subsets of α and $C := \{\xi\in\kappa\,;\,\xi\in\bigcap_{\beta<\xi} C_\beta\}$ the diagonal intersection of this sequence. First we show that C is closed. Let $\lambda<\kappa$ be a ordinal with $C\cap\lambda=\lambda$. We have to prove $\lambda\in C$, which by definition of the diagonal intersection is equal to $\lambda\in\bigcap_{\beta<\lambda} C_\beta$. Fix a $\beta<\lambda$. Since $\lambda=\lambda\cap C$ there is an strictly increasing sequence $\langle c_\xi\,;\,\xi<\delta\rangle$ with supremum λ and length $\delta\le\lambda$ in C. Let γ be the least ordinal less than δ such that c_γ is larger than β. For all ξ between γ and δ we have $c_\xi\in\bigcup_{\zeta<\xi} C_\zeta$ and thus $c_\xi\in C_\beta$. The set C_β is club which means that $\sup_{\beta<\xi<\delta} c_\xi$ is an element of C_β. But $\sup_{\beta<\xi<\delta} c_\xi$ is equal to $\sup_{\xi<\delta} c_\xi=\lambda$ and so λ is an element of C_β. This is true for all $\beta<\lambda$, so λ is in $\bigcap_{\beta<\lambda} C_\beta$ and thereby C is closed.

Now we prove the unboundedness of C. Let β be an ordinal less than α. We construct an increasing sequence $\langle c_n\,;\,n<\omega\rangle$ in α by $c_0:=\beta$ and

$$c_{n+1} := \min\{\delta\in\alpha\,;\,\delta>c_n \text{ and } \delta\in\bigcap_{\xi<c_n} C_\xi\}.$$

Then c_0 is an element of α and if c_n is an element of α then the set $\bigcap_{\xi<c_n} C_\xi$ is a club subset of α since α is regular. So c_{n+1} is also an element of α. That means $\langle c_n\,;\,n<\omega\rangle$ is a welldefined sequence in α. The cofinality of α is greater than ω, so $\gamma:=\sup_{n<\omega} c_n$ is an element of α. Let δ be an ordinal less than γ. There is a natural number n_δ such that c_{n_δ} is larger than δ. By definition of the c_n we have $c_n\in C_\delta$ for all $n>n_\delta$. So $\gamma=\sup_{n<\omega} c_n=\sup_{n_\delta<n<\omega} c_n$ is an element of C_δ since C_δ is club. The ordinal $\delta<\gamma$ was arbitrary which means γ is an element of $\bigcap_{\delta<\gamma} C_\delta$ and thus γ is an element of the diagonal intersection C. By its definition γ is larger than β and so we have shown that C is unbounded in α.

3. Since α is a limit ordinal the set $D:=\{\delta\in\alpha\,;\,\delta>\beta\}$ is closed and unbounded in α. So by 1. the set $C_{>\beta}=C\cap D$ is also club.

4. The set $\mathrm{Lim}(\alpha)$ is obviously closed and since α is a limit ordinal greater ω it is also unbounded in α. q.e.d.

1.3.3. Corollary. *Let α be a limit ordinal with* $\mathrm{cf}(\alpha) > \omega$*. The set*

$$\mathcal{C}_\alpha := \{A \subseteq \alpha\,;\ \textit{There is a club subset } C \subseteq \alpha \textit{ such that } C \subseteq A\}$$

is a non-principal $\mathrm{cf}(\alpha)$*-complete filter that is closed under end segments. If α is regular then $\mathcal{C}_\alpha$ is also normal. We call $\mathcal{C}_\alpha$ the **club filter** on α.*

Proof. By its definition $\mathcal{C}_\alpha$ contains α, does not contain $\emptyset$, and is closed under supersets. From Lemma 1.3.2 follows that $\mathcal{C}_\alpha$ is closed under intersections of less than $\mathrm{cf}(\alpha)$-many sets, contains end segments, and is closed under diagonal intersections if α is regular. That $\mathcal{C}_\alpha$ is non-principal follows from α being a limit ordinal and $\mathcal{C}_\alpha$ containing end segments: Assume $\mathcal{C}_\alpha$ is principal, *i.e.*, there is a set $A \subseteq \alpha$ that is a subset of every element of $\mathcal{C}_\alpha$. Let δ be the minimum of A, since α is a limit ordinal the ordinal $\delta + 1$ is still smaller than α. Since $\mathcal{C}_\alpha$ contains end segments the set $D := \{\beta < \alpha\,;\ \delta < \beta\}$ is an element of $\mathcal{C}_\alpha$. But δ is no element of D, so A is no subset of D and $\mathcal{C}_\alpha$ cannot be principal. q.e.d.

1.3.4. Definition. Let $\lambda < \alpha$ be ordinals, the **λ-cofinal filter** $\mathcal{C}^\lambda_\alpha$ is defined as the filter generated by the λ-closed unbounded sets in α, *i.e.*,

$$A \in \mathcal{C}^\lambda_\alpha :\Longleftrightarrow \text{ there is a club set } C \subseteq \alpha \text{ such that } \{\beta \in C\,;\ \mathrm{cf}(\beta) = \lambda\} \subseteq A.$$

For example, the filter $\mathcal{C}^\omega_{\omega_1}$ is clearly the ordinary club filter on ω_1.

1.3.5. Corollary. *Let α be a limit ordinal with* $\mathrm{cf}(\alpha) > \omega$ *and λ an ordinal less than α. Then $\mathcal{C}^\lambda_\alpha$ is a non-principal* $\mathrm{cf}(\alpha)$*-complete filter. And if α is regular then $\mathcal{C}^\lambda_\alpha$ is also normal.*

Proof. As in the proof of the Corollary 1.3.3, this follows rather directly from the definition of $\mathcal{C}^\lambda_\alpha$ and Lemma 1.3.2. q.e.d.

1.3.6. Definition. Let κ be a cardinal and μ a measure on κ. If μ contains all club subsets of κ, then we call μ a **semi-normal measure on κ**.

Using the Gödel pairing function we can get a measure on a cardinal κ as the image measure from a product measure on κ^n. The measure we get this way from the n-fold product of the ω-cofinal filter on a cardinal will be used several times later in our analysis.

1.3.7. Definition. Let $\kappa > \omega$ be a cardinal and $n < \omega$. The measure $\mathcal{W}^n_\kappa$ on κ is defined by

$$A \in \mathcal{W}^n_\kappa :\Longleftrightarrow \quad \text{there is a set } B \in (\mathcal{C}^\omega_\kappa)^n \text{ such that } \ulcorner\vec{\alpha}\urcorner \in A \text{ for all } \vec{\alpha} \in B.$$

So $\mathcal{W}^n_\kappa$ is the image filter of the n-fold product measure $(\mathcal{C}^\omega_\kappa)^n$ under the Gödel pairing function. As such $\mathcal{W}^n_\kappa$ is a measure and κ-complete if $(\mathcal{C}^\omega_\kappa)^n$ has those properties.

One of the most important properties of club sets is that there are several thinning procedures such that the resulting set retains the club properties: For example, we can intersect club sets, we can take the end segments of a club set, and the result will again be a club set. In the following we will introduce other operations that create club sets from club sets :

1.3.8. LEMMA. *Let $\kappa > \omega$ be a regular cardinal and $C \subseteq \kappa$ a club subset of κ. Then exist a club subset $C' \subseteq C$ of κ such that every element in C' is the supremum of an increasing C-sequence. We call this club set the **set of C-limits**.*

Proof. We define an increasing sequence $\langle c_\alpha\,;\,\alpha < \kappa\rangle$ in C of length κ by $c_0 :=$ the ωth element of C, $c_{\alpha+1} :=$ the ωth element of C greater than c_α for $\alpha < \kappa$ and $c_\lambda := \sup_{\alpha<\lambda} c_\alpha$ for limit ordinals $\lambda < \kappa$. Then $C' := \bigcup_{\alpha<\kappa}\{c_\alpha\}$ is a subset of C and by definition unbounded and closed under κ, *i.e.*, a club subset of κ. q.e.d.

1.3.9. LEMMA. *Let κ be a regular cardinal, $C \subseteq \kappa$ a club subset of κ and $f : \kappa \to \kappa$ a function. Then there is a club subset $C' \subseteq C$ of κ such that $f(\beta) < \alpha$ holds for all $\alpha, \beta \in C'$ with $\beta < \alpha$. We call such a club set **closed under** f.*

Proof. We define an increasing sequence $\langle c_\alpha\,;\,\alpha < \kappa\rangle$ in C of length κ by $c_0 := \min C$, $c_{\alpha+1} := \min\{\beta \in C\ :\ \beta > \max\{c_\alpha, \sup_{\gamma\leq\alpha} f(c_\gamma)\}\}$ for $\alpha < \kappa$ and $c_\lambda := \sup_{\alpha<\lambda} f(c_\alpha)$ for limit ordinals $\lambda < \kappa$. Since κ is regular and C a club subset of κ this is welldefined. Let $C' := \bigcup_{\alpha<\kappa}\{c_\alpha\}$. Then by definition of $\langle c_\alpha\,;\,\alpha < \kappa\rangle$ we have $f(\beta) < \alpha$ for all $\alpha, \beta \in C'$ with $\beta < \alpha$ and also $C' \subseteq C$. An increasing sequence of length κ in κ is unbounded in κ since κ is regular and C' is by definition closed, so C' is club. q.e.d.

1.3.10. LEMMA. *Let κ be a regular cardinal and $C \subseteq \kappa$ a club subset of κ. Then there is a club subset $C' \subseteq C$ of κ such that for all $\alpha \in C'$ the αth element of C is α. So C' contains only **closure points** of C.*

Proof. Let $f : \kappa \to C$ be an enumeration of C. By Lemma 1.3.9 exist a club set $C' \subseteq C$ that is closed under f, by Lemma 1.3.2 we can assume that C' contains only limit ordinals. Then for all α in C' we have that for all $\beta < \alpha$ the βth element of C is smaller than α. Since α is a limit and both C and C' are club we get that $\sup_{\beta<\alpha} f(\beta) =$ the αth element of C is less or equal α. But of course the αth element of C is greater or equal α so for all α in C' the αth element of C is α. q.e.d.

Slightly weaker than being a club set is the the notion of a stationary set. Note that since the intersection of two club sets is again a club set all club sets are stationary, whereas the converse is generally not true.

1.3.11. Definition. Let λ be a limit ordinal. A subset $S \subseteq \lambda$ such that for any club subset $C \subseteq \lambda$ the intersection $S \cap C$ is non-empty is called **stationary** in λ. So if the cofinality of λ is greater ω then every club subset of λ is stationary in λ.

1.3.12. Lemma (Fodor 1956). *Let $\lambda > \omega$ be a regular cardinal and S a stationary subset of λ. Assume $f : S \to \lambda$ is a regressive function on S, i.e., $f(\alpha) < \alpha$ holds for all $\alpha \in S$. Then there is an ordinal $\gamma < \lambda$ and a subset $S' \subseteq S$ that is stationary in λ such that $f(\alpha) = \gamma$ holds for all $\alpha \in S'$.*

Proof. Assume this is not the case. Then for all $\gamma < \lambda$ the set

$$U_\gamma := \{\alpha \in \lambda\,;\, f(\alpha) = \gamma\}$$

is not stationary in λ. That means there is a club set whose intersection with U_γ is empty, let

$$C_\gamma := \bigcup \{C \subseteq \lambda\,;\, C \text{ is club and } C \cap U_\gamma = \emptyset\},$$

so C_γ is club and $U_\gamma \cap C_\gamma = \emptyset$. The diagonal intersection $C_\triangle := \triangle_{\gamma<\lambda} C_\gamma$ of the C_γ is itself a club subset of λ, since λ is regular. But then the intersection of $C_\triangle$ and S is nonempty, let $\alpha \in C_\triangle \cap S$ be a witness. Since α is an element of $C_\triangle$ we have $\alpha \in C_\gamma$ for $\gamma < \alpha$, *i.e.*, $f(\alpha) \neq \gamma$ for $\gamma < \alpha$ by definition of the C_γ. On the other hand α is an element of S, so $f(\alpha) < \alpha$, a contradiction that concludes the proof. q.e.d.

1.4 Ultrapowers

For a measure μ on a cardinal κ and an ordinal α we denote the corresponding **ultrapower** of α with respect to the measure μ by α^κ/μ. The ultrapower is ordered by E_μ with $[f]_\mu E_\mu [g]_\mu$ iff $\{\alpha \in \kappa\,;\, f(\alpha) < g(\alpha)\} \in \mu$. Since measures are closed under finite intersections this order is welldefined. If an ultrapower α^κ/μ is wellfounded with respect to E_μ we identify the ultrapower with its Mostowski collapse, so in this case α^κ/μ is an ordinal. If the ultrapower κ^κ/μ is wellfounded we also call the measure **wellfounded**.

In Lemma 1.1.1 we showed that a tree is wellfounded if it has no infinite branch, *i.e.*, no infinite descending sequence in the $<_T$-order. For ultrapowers we can show a similar result if we assume the **Principle of Dependent Choices** DC.

1.4.1. Definition. Let X be a nonempty set, then $\mathsf{DC}(X)$ is the statement:

> If R is a subset of $X \times X$ and for all x in X exists an y in X such that $\langle y, x\rangle$ is in R then there is a function $f : \omega \to X$ such that $\langle f(n+1), f(n)\rangle$ is in R for all n in ω.

The principle DC is of course the assumption that $\mathsf{DC}(X)$ holds for all sets X.

1.4.2. LEMMA. *Assume* $\mathsf{DC}(\kappa^\kappa)$. *Let* μ *be a measure on a cardinal* κ *and* λ *an ordinal. The ultrapower* λ^κ/μ *is wellfounded if and only if it has no infinite descending sequences.*

Proof. Let E_μ be the ordering of the ultrapower. If $\langle [f_i]_\mu\,; i<\omega\rangle$ is a descending sequence in λ^κ/μ then the set $\{[f_i]_\mu\,; i<\omega\}$ obviously has no minimal element and the ultrapower is illfounded. Now assume there is a subset A of the ultrapower that has no minimal element. We define a relation R on $\kappa^\kappa \times \kappa^\kappa$ by

$$\langle y, x\rangle \in R \quad :\Leftrightarrow \quad [y]_\mu \in A \text{ and } ([x]_\mu \notin A \text{ or } ([x]_\mu \in A \text{ and } [y]_\mu E_\mu [x]_\mu)).$$

Then for all $x \in \kappa^\kappa$ exists an $y \in \kappa^\kappa$ with $\langle y, x\rangle \in R$ and we can use $\mathsf{DC}(\kappa^\kappa)$ to get a sequence $\langle f_i\,; i<\omega\rangle$ with $\langle f_{i+1}, f_i\rangle \in R$ for all $i<\omega$. But then by definition of R the sequence $\langle [f_{i+1}]_\mu\,; i<\omega\rangle$ is infinite and descending in the E_μ-order. q.e.d.

An **embedding** π from the ultrapower α^κ/μ into the ultrapower β^ξ/ν is an order-preserving function $\pi : \alpha^\kappa/\mu \to \beta^\xi/\nu$. If the ultrapowers in question are wellfounded then this means order-preserving with respect to $\in$, since in this case we identify the ultrapower with its Mostowski collapse as we mentioned before. Otherwise we mean order-preserving with respect to the orders E_μ and E_ν. Most of the time we will define an embedding in terms of representatives of the equivalence classes: If π is a function from α^κ to β^ξ we denote (slightly abusing notation) by π also the the function $\pi : [f]_\mu \mapsto [\pi(f)]_\nu$. Of course in this case we have to consider the following:

1.4.3. REMARK. In order to show that $f \mapsto \pi(f)$ induces an embedding, we have to show two properties:

1. the function is welldefined, *i.e.*, if $[f]_\mu = [g]_\mu$, then $[\pi(f)]_\nu = [\pi(g)]_\nu$, and
2. the function is order-preserving, *i.e.*, if $[f]_\mu < [g]_\mu$, then $[\pi(f)]_\nu < [\pi(g)]_\nu$.

Obviously, the proofs of these two statements are typically parallel, and if they are we reduce them in most places of this thesis to one proof where we show the implication for "$\substack{=\\<}$". This is meant to indicate that the proof works with both $<$ and $=$ and thus proves 1. and 2.

1.4.4. DEFINITION. Let μ and ν be wellfounded measures on sets X and Y, respectively. We say that μ and ν are **equivalent** ($\mu \simeq \nu$) if their ultrapowers are the same, *i.e.*, if $X^X/\mu = Y^Y/\nu$.

We mentioned in Lemma 1.2.10 that for measures μ and ν the product measures $\mu \times \nu$ and $\mu \times' \nu$ are essentially the same, so the corresponding ultrapowers should be equivalent:

1.4.5. Lemma. *Let μ and ν be measures on ordinals α and β, respectively. Let λ be an ordinal and assume the ultrapower $\lambda^{\alpha\cdot\beta}/\mu \times \nu$ is wellfounded. The bijection π between $\alpha \times \beta$ and $\alpha \cdot \beta$ is as usual defined by $\pi(\langle\gamma, \delta\rangle) = \alpha \cdot \delta + \gamma$.*

We have $[f]_{\mu\times\nu} = [f \circ \pi^{-1}]_{\mu\times'\nu}$ and $[g]_{\mu\times'\nu} = [g \circ \pi]_{\mu\times\nu}$ for all functions f, g with range λ and domains $\alpha \times \beta$ and $\alpha \cdot \beta$, respectively. So the measures $\mu \times \nu$ and $\mu \times' \nu$ are equivalent.

Proof. First we show that $f \mapsto f \circ \pi^{-1}$ induces an embedding from $\lambda^{\alpha\times\beta}/\mu \times \nu$ into $\lambda^{\alpha\cdot\beta}/\mu\times'\nu$, using our convention from Remark 1.4.3: Let h and f be functions in $\lambda^{\alpha\times\beta}$, if $[h]_{\mu\times\nu} \mathrel{\substack{=\\<}} [f]_{\mu\times\nu}$ then the set $\{x \in \alpha \times \beta\,;\, h(x) \mathrel{\substack{=\\<}} f(x)\}$ is in $\mu \times \nu$. From the definition of $\mu \times' \nu$ as the image filter of $\mu \times \nu$ under π follows that the set $\{y \in \alpha \cdot \beta\,;\, h(\pi^{-1}(x)) \mathrel{\substack{=\\<}} f(\pi^{-1}(x))\}$ is in $\mu \times' \nu$ which is equivalent to $[h \circ \pi^{-1}]_{\mu\times'\nu} \mathrel{\substack{=\\<}} [f \circ \pi^{-1}]_{\mu\times'\nu}$, so we have indeed an embedding.

We also have an embedding from $\lambda^{\alpha\times\beta}/\mu \times' \nu$ into $\lambda^{\alpha\cdot\beta}/\mu \times \nu$ that is induced by $g \mapsto g \circ \pi$, the proof is analogous to that for $f \mapsto f \circ \pi^{-1}$. It follows that we have an order preserving bijection between the two ultrapowers. Since we identify an wellfounded ultrapower with its Mostowski this means the two ultrapowers are in fact the same ordinal. q.e.d.

1.4.6. Remark. When working with functions on products of ordinals, product measures on ordinals, and equivalence classes with respect to these product measures we will often implicitly use Lemma 1.4.5 and identify the functions f and $f \circ \pi$ as well as the equivalence classes $[f]_{\mu\times\nu}$ and $[f \circ \pi]_{\mu\times'\nu}$, just writing $[f]_{\mu\times\nu}$.

Let α^κ/μ and β^ξ/ν be two wellfounded ultrapowers. If π is an embedding from α^κ/μ into β^ξ/ν then we have $\alpha^\kappa/\mu \leq \beta^\xi/\nu$, since π is order-preserving.

1.4.7. Lemma. *Let μ be a measure on a cardinal κ and $\alpha < \beta$ be ordinals such that the ultrapowers α^κ/μ and β^κ/μ are wellfounded. Then $\alpha^\kappa/\mu \leq \beta^\kappa/\mu$.*

Proof. The identity function $\mathrm{id} : f \mapsto f$ for functions $f \in \alpha^\kappa$ defines obviously an embedding from α^κ/μ into β^κ/μ. q.e.d.

This result of course generalizes if we look at sequences of ordinals and their supremum.

1.4.8. Lemma. *Let μ be a measure on a cardinal κ, α an ordinal, and $\langle\gamma_i\,;\, i < \alpha\rangle$ a sequence of ordinals. Assume that the ultrapowers ${\gamma_i}^\kappa/\mu$ for $i < \alpha$ and $(\sup_{i<\alpha}\gamma_i)^\kappa/\mu$ are wellfounded. Then* $\sup_{i<\alpha}({\gamma_i}^\kappa/\mu) \leq (\sup_{i<\alpha}\gamma_i)^\kappa/\mu$.

Proof. If $\sup_{i<\alpha}\gamma_i = \gamma_\beta$ for some $\beta < \alpha$ then $\sup_{i<\alpha}({\gamma_i}^\kappa/\mu) = {\gamma_\beta}^\kappa/\mu = (\sup_{i<\alpha}\gamma_i)^\kappa/\mu$ by Lemma 1.4.7 and we are finished. So let us assume that $\gamma_\beta < \sup_{i<\alpha}\gamma_i$ holds for all $\beta < \alpha$. By Lemma 1.4.7 we have ${\gamma_\beta}^\kappa/\mu \leq (\sup_{i<\alpha}\gamma_i)^\kappa/\mu$ for all $\beta < \alpha$ and $\sup_{i<\alpha}({\gamma_i}^\kappa/\mu) \leq (\sup_{i<\alpha}\gamma_i)^\kappa/\mu$ follows. q.e.d.

Can we compute the cofinality of an ultrapower of an ordinal from that ordinal? In certain cases the answer to this question is simple:

1.4.9. LEMMA. *Let $\kappa < \lambda$ be cardinals, μ a measure on κ and $\mathrm{cf}(\lambda) > \kappa$. If λ^κ/μ is wellfounded then $\mathrm{cf}(\lambda^\kappa/\mu) = \mathrm{cf}(\lambda)$.*

Proof. "$\leq$": For $\alpha < \lambda$, let $c_\alpha : \kappa \to \lambda$ be the constant function $c_\alpha(\xi) = \alpha$. We shall show that $\{[c_\alpha]_\mu \,;\, \alpha \in \lambda\}$ is cofinal in λ^κ/μ:

Let $f \in \lambda^\kappa$ be arbitrary. Since $\mathrm{cf}(\lambda) > \kappa$, the range of the function f is bounded in λ, *i.e.*, there is an $\alpha' \in \lambda$ such that $\{f(\xi)\,;\, \xi \in \kappa\} \subseteq \alpha'$. Then $[f]_\mu < [c_{\alpha'}]_\mu$.

"$\geq$": Now let $X \subseteq \lambda^\kappa/\mu$ be a cofinal subset. If $\xi \in X$, there is some $\alpha \in \lambda$ such that $\xi \leq [c_\alpha]_\mu$ by the above argument. Let α_ξ be the least such ordinal. We claim that $A := \{\alpha_\xi\,;\, \xi \in X\}$ is a cofinal subset of λ: Let $\gamma \in \lambda$ be arbitrary. Since X was cofinal, pick some $\xi_\gamma \in X$ such that $\xi_\gamma > [c_\gamma]_\mu$. But then, $\alpha_{\xi_\gamma} \in A$ with $\alpha_{\xi_\gamma} > \gamma$. So, A is cofinal in λ. But $\mathrm{Card}(A) \leq \mathrm{Card}(X)$, so $\mathrm{cf}(\lambda) \leq \mathrm{cf}(\lambda^\kappa/\mu)$. q.e.d.

The first κ-many elements of a wellfounded ultrapower have canonical representatives:

1.4.10. LEMMA. *Let κ be a regular cardinal and μ a κ-complete measure on κ such that the ultrapower κ^κ/μ is wellfounded. Then for all $\gamma < \kappa$ we have $[c_\gamma]_\mu = \gamma$, where $c_\gamma \in \kappa^\kappa$ is the function with constant value γ.*

Proof. We prove this result by induction over $\gamma < \kappa$:

1. The smallest element of the ultrapower is the equivalence class of c_0, so $[c_0]_\mu = 0$.

2. Now assume $[c_\beta]_\mu = \beta$ for all $\beta < \gamma$. Obviously $\beta = [c_\beta]_\mu < [c_\gamma]_\mu$ for all $\beta < \gamma$, *i.e.*, $\gamma \leq [c_\gamma]_\mu$. If $\gamma = [c_\gamma]_\mu$ we are done, so assume towards a contradiction that there is $f \in \kappa^\kappa$ with $[f]_\mu < [c_\gamma]_\mu$ and $[f]_\mu = \gamma$. From $[f]_\mu < [c_\gamma]_\mu$ follows that there is a set $A \in \mu$ with $f(\alpha) < \gamma$ for all $\alpha \in A$. Define
$$A_\beta := \{\alpha \in A\,;\, f(\alpha) = \beta\}.$$
Then $\bigcup_{\beta<\gamma} A_\beta = A \in \mu$, so by part 3. of Lemma 1.2.3 there is a $\beta < \gamma$ such that $A_\beta \in \mu$. But this means $[f]_\mu = [c_\beta]_\mu = \beta$, which contradicts our assumption $[f]_\mu = \gamma$.

Due to Lemma 1.2.3 Step 2. works for all $\gamma < \kappa$. q.e.d.

1.4.11. LEMMA. *Let κ be a regular cardinal and μ a κ-complete measure on κ such that the ultrapower κ^κ/μ is wellfounded. Then a function $f \in \kappa^\kappa$ is either μ-almost constant or there is a set $A \in \mu$ such that f restricted to A is unbounded in κ.*

Proof. Assume that for all $A \in \mu$ the function f restricted to A is bounded in κ. Pick $A \in \mu$ and $\gamma < \kappa$ such that $f(\alpha) < \gamma$ for all $\alpha \in A$. As in the Lemma 1.4.10 let

$$A_\beta := \{\alpha \in A\, ;\, f(\alpha) = \beta\}.$$

Then $\bigcup_{\beta<\gamma} A_\beta = A \in \mu$, so again by part 3. of Lemma 1.2.3 there is a $\beta < \gamma$ such that $A_\beta \in \mu$. But this means f is μ-almost constant (and thus by the Lemma 1.4.10 we have $[f]_\mu = \beta$ for some $\beta < \kappa$). q.e.d.

1.5 Partition Properties

In this section we will introduce the notation for partition properties of cardinals.

1.5.1. Definition. Let γ, λ, and κ be cardinals with $\kappa \geq \gamma, \lambda > 1$. We denote by $\kappa \to (\kappa)^\lambda_\gamma$ the following partition property: For every partition $P : [\kappa]^\lambda \to \gamma$ of $[\kappa]^\lambda$ into γ-many blocks there is a set $H \subseteq \kappa$ of size κ such that P is constant on $[H]^\lambda$. We call such a set H homogeneous for the partition P.

If γ is equal to 2 we omit it in the notation, *i.e.*, we write $\kappa \to (\kappa)^\lambda$ instead of $\kappa \to (\kappa)^\lambda_2$. Please note that we omitted the trivial partition properties, if we write $\kappa \to (\kappa)^\lambda_\gamma$ then λ and γ are both greater than 1.

Even the partition property $\kappa \to (\kappa)^2$ tells us something about the cardinal κ, since it has the following non-trivial consequence:

1.5.2. Lemma. *Let κ be a cardinal and assume $\kappa \to (\kappa)^2$ holds. Then κ is regular.*

Proof. Assume κ is not regular, then there is a cardinal $\gamma < \kappa$ such that $\kappa = \bigcup_{\xi<\gamma} X_\beta$ for a γ-partition $\langle X_\xi\, ;\, \xi < \gamma\rangle$ of κ with $\mathrm{Card}(X_\xi) < \kappa$ for all $\xi < \gamma$. We define a partition $P : [\kappa]^2 \to 2$ by $P(\langle \alpha, \beta\rangle) := 1$ iff $\alpha, \beta \in X_\xi$ for some $\xi < \gamma$. Then there can be no homogeneous set of size κ for the partition P, so by contradiction κ has to be regular. q.e.d.

1.5.3. Corollary. *The partition property $\kappa \to (\kappa)^\lambda_\gamma$ implies $\kappa \to (\kappa)^{\lambda'}_{\gamma'}$ for λ' less than λ and γ' less than γ.*

Proof. Let $P : [\kappa]^{\lambda'} \to \gamma'$ be a γ'-partition of $[\kappa]^{\lambda'}$. For $f \in [\kappa]^\lambda$ we let $f{\restriction}\lambda'$ be the restriction of f to λ', *i.e.*, $f{\restriction}\lambda' \in [\kappa]^{\lambda'}$ and $f{\restriction}\lambda'(\alpha) := f(\alpha)$ for all $\alpha < \lambda'$. Then $Q : [\kappa]^\lambda \to \gamma$ defined by $Q(f) := P(f{\restriction}\lambda')$ is a γ-partition of $[\kappa]^\lambda$, so by assumption there is a homogeneous set $H \subseteq \kappa$ of size κ for this partition. Let $g \in [H]^{\lambda'}$, since κ is regular we can define an extension $f \in [H]^\lambda$ of g by

$$f(\alpha) := \left\{ \begin{array}{l} g(\alpha) \text{ if } \alpha < \lambda' \\ \min\{\xi \in H\, ;\, \xi > \sup_{\beta<\alpha} f(\beta)\} \end{array} \right. .$$

Then $f \restriction \lambda' = g$, so $P(g) = Q(f)$, which means that H is also homogeneous for the partition P. q.e.d.

So if we have a partition property for a certain set of functions we can always switch to coarser partitions. What about the converse? Lemma 1.5.4 will show that we get partition properties for finer partitions if we restrict the set of functions that we want to partition:

1.5.4. Lemma (Kleinberg 1970). *Let $\lambda < \kappa$ be cardinals and assume $\kappa \to (\kappa)^{\lambda+\lambda}$ holds. Then the partition property $\kappa \to (\kappa)^{\lambda}_{\gamma}$ holds for arbitrary $\gamma < \kappa$.*

Proof. For the original proof see [Kl70, Lemma 1.2]. If x is an element of $[\kappa]^{\lambda+\lambda}$ we denote its first block of length λ by x_1 and the second by x_2, *i.e.*, we have $x_1 := x \restriction \lambda$ and $x_2(\alpha) := x(\lambda + \alpha)$ for $\alpha < \lambda$. Now fix an ordinal $\gamma < \kappa$ and a γ-partition P of $[\kappa]^{\lambda}$. From P we define a 2-partition P' of $[\kappa]^{\lambda+\lambda}$:

$$P'(x) = 0 \quad :\Leftrightarrow \quad P(x_1) = P(x_2).$$

By assumption exist a homogeneous set H for the partition P'. Since H has cardinality κ we can partition it in κ many subsets $h_\alpha \subseteq H$ of size λ:

Let h_α be the set of the first λ-many elements of $H \setminus \bigcup_{\beta<\alpha} h_\beta$, for $\alpha < \lambda$.

And since γ is less than κ there are $\alpha < \beta < \kappa$ such that $P(h_\alpha) = P(h_\beta)$, *i.e.*, $P'(h_\alpha{}^\frown h_\beta) = 0$. The set H is homogeneous for P', so this implies $P(x_1) = P(x_2)$ for all $x \in [H]^{\lambda+\lambda}$. Now we show that H is also homogeneous for P: Let x and y be arbitrary elements of $[H]^{\lambda}$. The cardinal κ is regular, so neither x nor y is cofinal in H and there exists $z \in [H]^{\lambda}$ such that $\inf z > \max\{\sup x, \sup y\}$. Then the homogeneity of H for P' implies

$$P(x) = P((x^\frown z)_1) = P((x^\frown z)_2) = P(z) \text{ and}$$

$$P(y) = P((y^\frown z)_1) = P((y^\frown z)_2) = P(z),$$

so $P(x)$ is constant for all $x \in [H]^{\lambda}$. q.e.d.

In Section 1.9 we will present the consequences of these partition properties for partitions of sets of certain special functions, and later we will mainly use those consequences in our proofs.

1.6 Partition Cardinals

By a result of Erdős and Rado (See [Ka94, Proposition 7.1]) any partition relation $\kappa \to (\kappa)^{\lambda}_{\gamma}$ with infinite exponent λ violates the Axiom of Choice AC.

We say that κ has the **strong partition property** if the partition relation $\kappa \to (\kappa)^{\kappa}$ holds, and that κ has the **weak partition property** if the partition relation $\kappa \to (\kappa)^{\alpha}$ holds for all $\alpha < \kappa$.

As mentioned in the introduction, a measure analysis essentially an assignment that assigns a measure μ_γ on a cardinal κ to a cardinal $\gamma > \kappa$ such that κ^κ/μ_γ is γ.Thanks to the following theorem by Martin we know that on cardinals with the strong partition property all wellfounded measures generate cardinals. So for those cardinals a measure analysis is at least a possibility.

1.6.1. Theorem (Martin). *Let κ be a strong partition cardinal and let μ be a measure on κ. If the ultrapower κ^κ/μ is wellfounded then it is a cardinal.*

Proof. See [Ja99, Theorem 7.1]. q.e.d.

If a cardinal κ has the strong partition property it implies the existence of many concrete measures on κ, as the following theorem of Kleinberg shows:

1.6.2. Theorem (Kleinberg 1970). *Let κ be a cardinal with the weak partition property and $\lambda < \kappa$ a regular cardinal. Then $\mathcal{C}^\lambda_\kappa$ is a normal measure. In addition, if κ is not weakly Mahlo, then these are the only normal measures on κ.*

Proof. See [Kl70, Theorem 2.1] and [Kl77, Theorem 3.5], also [Ka94, Theorem 28.10 & Exercise 28.11]. q.e.d.

In other words, the strong partition property of κ not only implies the mere existence of measures, but in our case (our cardinals will be below $\aleph_{\varepsilon_0}$ and thus not weakly Mahlo) also a structured pattern of all of the normal measures on κ (indexed by the regular cardinals below κ).

The strong partition property also connects to other combinatorial properties that are well known from usual (AC) combinatorial set theory: A cardinal κ is called a **Jónsson cardinal** if the partition relation $\kappa \to [\kappa]^{<\omega}_\kappa$ holds, *i.e.*, for every partition of $[\kappa]^{<\omega}$ into κ blocks there is a set H of order type κ with the property that $[H]^{<\omega}$ doesn't meet all blocks.[1]

A cardinal κ is called a **Rowbottom cardinal** if for all $\lambda < \kappa$ the partition relation $\kappa \to [\kappa]^{<\omega}_{\lambda,<\omega_1}$ holds, *i.e.*, for every partition of $[\kappa]^{<\omega}$ into λ blocks there is a set H of order type κ with the property that $[H]^{<\omega}$ meets only countably many blocks.

As an application for the measure analysis that is the topic of this thesis we will prove in section 7.3 the existence of some (new) Jónsson cardinals under AD.

1.7 Kleinberg Sequences

The following notion of Kleinberg sequences shows us that, under the right conditions, every normal measure leads to a sequence of Jónsson cardinals.

[1]This is equivalent to saying that every algebra (in the sense of universal algebra) on κ has a proper subalgebra of size κ. [Co96] is a nice survey of the algebraic side of the Jónsson property.

1.7.1. DEFINITION. Let κ be a cardinal and μ a normal measure on κ. We then define a sequence of structures $\langle \kappa_n^\mu\ ;\ n < \omega \rangle$ as follows:

- $\kappa_0^\mu := \kappa$,
- $\kappa_{n+1}^\mu := (\kappa_n^\mu)^\kappa / \mu$.

If this is a sequence of wellordered structures we also define

- $\kappa_\omega^\mu := \sup\{\kappa_n^\mu\ ;\ n \in \omega\}$.

This sequence is called the **Kleinberg sequence** derived from the measure μ.

1.7.2. THEOREM (KLEINBERG 1977). *Let κ be a strong partition cardinal and μ be a normal measure on κ. If the Kleinberg sequence $\langle \kappa_i^\mu\,;\, i \leq \omega \rangle$ is a sequence of wellordered structures then*

1. *κ_0^μ and κ_1^μ are measurable,*
2. *for all $n \geq 1$, $\mathrm{cf}(\kappa_n^\mu) = \kappa_1^\mu$,*
3. *κ_n^μ is a Jónsson cardinal, and*
4. *κ_ω^μ is a Rowbottom cardinal.*

Moreover, if $\kappa_1^\mu = \kappa^+$, then $\kappa_{n+1}^\mu = (\kappa_n^\mu)^+$ for all $n \in \omega$, and $\kappa_1^\mu \to (\kappa_1^\mu)^\alpha$ for all $\alpha < \kappa_1^\mu$.

Proof. The proofs for parts 1. to 3. are Theorem 5.1 [Kl77], and part 4. is Theorem 6.4. q.e.d.

The following theorem by Benedikt Löwe is an elaboration of the proof of the “moreover” part in Theorem 1.7.2. We will use it extensively in the proof of Theorem 5.2.2, our Really Helpful Theorem (RHT).

1.7.3. THEOREM (ULTRAPOWER SHIFTING LEMMA). *Let β and γ be ordinals and let μ be a σ-complete ultrafilter on κ with $\kappa^\kappa/\mu = \kappa^{(\gamma)}$. If the ultrapower $(\kappa^{(\beta)})^\kappa/\mu$ is wellfounded and for all cardinals $\kappa < \nu \leq \kappa^{(\beta)}$*

- *either ν is a successor and $\mathrm{cf}(\nu) > \kappa$,*
- *or ν is a limit and $\mathrm{cf}(\nu) = \omega$,*

then $(\kappa^{(\beta)})^\kappa/\mu \leq \kappa^{(\gamma+\beta)}$.

Proof. See [Lö02, Lemma 2.7]. q.e.d.

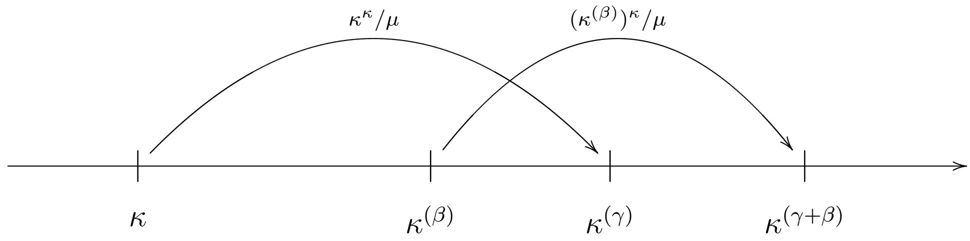

Figure 1.1: Ultrapower Shifting Lemma

1.8 Functions of Various Types

Let $\alpha \leq \kappa$ be ordinals and A a subset of κ. A function $f : \alpha \to \kappa$ is **continuous**[2] if and only if for all limit ordinals $\lambda < \alpha$

$$f(\lambda) = \sup\{f(\xi)\,;\, \xi < \lambda\}.$$

Conversely a function $f : \alpha \to \kappa$ is **discontinuous** if and only if for all limit ordinals $\lambda < \alpha$,

$$f(\lambda) > \sup\{f(\xi)\,;\, \xi < \lambda\}.$$

The function f has **uniform cofinality** ω if there is a function $h : \omega \times \alpha \to \kappa$, which is increasing in the first argument, such that for $\gamma < \alpha$, we have

$$f(\gamma) \quad = \quad \sup\{h(n, \gamma)\,;\, n \in \omega\}.$$

We say that a function $f : \alpha \to \kappa$ has **discontinuous type** α if it is increasing, discontinuous and has uniform cofinality ω, and it has **continuous type** α if it is increasing, continuous and has uniform cofinality ω at successor ordinals and 0. If $X \subseteq \kappa$, we write $\mathfrak{C}^\alpha_X$ for the set of functions from α to X that are of continuous type α and $\mathfrak{D}^\alpha_X$ for those of discontinuous type. If $\vec{\alpha} = \langle \alpha_i\,;\, i < n\rangle$ is a sequence of ordinals and $\lambda = \sum_{i<n} \alpha_i$, we write $\mathfrak{C}^{\vec{\alpha}}_X$ for the set of functions from λ to X that are piecewise of continuous type.[3] So $f \in \mathfrak{C}^{\vec{\alpha}}_X$ if there are functions $f_i \in \mathfrak{C}^{\alpha_i}_X$ such that $\sup f_i < \inf f_j$ for $i < j < n$ and f is induced by $\langle f_i\,;\, i < n\rangle$:

$$f\left(\sum_{i<j} \alpha_i + \gamma\right) = f_j(\gamma) \qquad \text{for } j < n.$$

We say functions $f \in \mathfrak{C}^{\vec{\alpha}}_X$ have **continuous type** $\vec{\alpha}$. For a sequence $\vec{f} = \langle f_i\,;\, i < n\rangle$ of functions $f_i : \alpha_i \to X$ we write $\vec{f} \in \mathfrak{C}^{\vec{\alpha}}_X$ if that sequence induces a function $f \in \mathfrak{C}^{\vec{\alpha}}_X$, *i.e.*, if $f_i \in \mathfrak{C}^{\alpha_i}_X$ and $\sup f_i < \inf f_j$ for $i < j < n$. If $C \subseteq \kappa$ is club and

[2]For increasing functions, this is the ordinary notion of continuity for the order topology on ordinals.

[3]We could define $\mathfrak{D}^{\vec{\alpha}}_X$ in a similar manner, but that would be superfluous, since easily with $\lambda = \sum_{i<n} \alpha_i$ we have $\mathfrak{D}^{\vec{\alpha}}_X = \mathfrak{D}^\lambda_X$.

and α is an element of κ, we write as before $C_{>\alpha}$ for the club set of elements in C that are greater than α.

There is a simple correspondence between the sets of functions of continuous type and discontinuous type if we restrict the ranges to club sets:

1.8.1. LEMMA. *Let κ be a regular cardinal, $\vec{\alpha} = \langle \alpha_i\,;\, i < n\rangle$ a sequence of ordinals less than κ and $\lambda = \sum_{i<n} \alpha_i$. Then there is a bijection π between $\mathfrak{D}^\lambda_\kappa$ and $\mathfrak{C}^{\vec{\alpha}}_\kappa$ such that $\mathfrak{C}^{\vec{\alpha}}_C$ is the π-image of $\mathfrak{D}^\lambda_C$ if $C \subseteq \kappa$ is a club set.*

Proof. Let $f : \lambda \to \kappa$ be a function of discontinuous type λ, we define $\pi(f) : \lambda \to \kappa$ by:

- Let $\pi(f)(0)$ be $f(0)$,
- let $\pi(f)(\alpha)$ be the least element of $\mathrm{ran}(f) > \sup_{\beta<\alpha} \pi(f)(\beta)$ if α is a limit and $\alpha = \sum_{i<j} \alpha_i$ for some $j < n$,
- let $\pi(f)(\alpha)$ be $\sup_{\beta<\alpha} \pi(f)(\beta)$ if α is a limit and $\sum_{i<j} \alpha_i < \alpha < \sum_{i<j+1} \alpha_i$ for some $j < n$, and
- let $\pi(f)(\alpha + 1)$ be the least element of $\mathrm{ran}(f) > \pi(f)(\alpha)$ for $\alpha < \lambda$.

Since club sets are closed we have $\mathrm{ran}(\pi(f)) \subseteq C$ if $\mathrm{ran}(f) \subseteq C$ for some club set $C \subseteq \kappa$. Now let $g : \lambda \to \kappa$ be a function of continuous type $\vec{\alpha}$, we define $\tau(g) : \lambda \to \kappa$ by:

- Let $\tau(g)(0)$ be $g(0)$,
- let $\tau(g)(\alpha)$ be the least element of $\mathrm{ran}(g) > \sup_{\beta<\alpha} \tau(g)(\beta)$ if α is a limit, and
- let $\tau(g)(\alpha + 1)$ be the least element of $\mathrm{ran}(g) > \tau(g)(\alpha)$ for $\alpha < \lambda$.

Again we have $\mathrm{ran}(\tau(g)) \subseteq C$ if $\mathrm{ran}(g) \subseteq C$ for some club set $C \subseteq \kappa$. Then τ is the inverse of π and as shown this bijection respects the range if it is club. q.e.d.

1.8.2. DEFINITION. Let $\alpha > \omega$ be an ordinal. If a function $f : \alpha \to \mathrm{On}$ is increasing and of continuous type α except for finitely many initial values, *i.e.*, if for some $n < \omega$ the function $g(\beta) := f(n + \beta)$ is of continuous type α, we say that f is **nearly of continuous type** α.

1.8.3. LEMMA. *Let $\alpha > \omega$ be an ordinal and μ a measure on α that contains end segments. Let $f : \alpha \to \mathrm{On}$ be a function that is nearly of continuous type α and let $n < \omega$ be such that the function $g(\beta) := f(n + \beta)$ is of continuous type α. Then f and g have the same same supremum, the range of g is a subset of the range of f and $[f]_\mu = [g]_\mu$.*

Proof. That f and g have the same same supremum and that the range of g is a subset of the range of f follows from the definition of g. Let $C := \alpha\backslash\omega$, then C as an end segment is an element of μ and we have $f(\beta) = g(\beta)$ for all $\beta \in C$, so $[f]_\mu = [g]_\mu$. q.e.d.

1.8.4. LEMMA. *Let $\kappa > \omega$ be a regular cardinal and μ a measure on κ that contains all club sets of κ. Let $f, g : \kappa \to \kappa$ be functions of discontinuous type κ such that $[f]_\mu < [g]_\mu$ holds. Then exist functions $f', g' : \kappa \to \kappa$ of discontinuous type κ with $f'(\alpha) < g'(\alpha) < f'(\alpha + 1)$ for all $\alpha < \kappa$, $\mathrm{ran}(f') \subseteq \mathrm{ran}(f)$, $\mathrm{ran}(g') \subseteq \mathrm{ran}(g)$, and $[f]_\mu = [f']_\mu$, $[g]_\mu = [g']_\mu$.*

Proof. We define f' and g' recursively by $f'(\alpha) :=$ the least element in the range of f greater $\sup_{\beta<\alpha} g'(\beta)$ (so $f'(0) := f(0)$) and $g'(\alpha) :=$ the least element in the range of g greater $f'(\alpha)$. Since κ is regular and f and g are of discontinuous type the functions f' and g' are welldefined. By definition we have $\mathrm{ran}(f') \subseteq \mathrm{ran}(f)$, $\mathrm{ran}(g') \subseteq \mathrm{ran}(g)$, $f'(\alpha) < g'(\alpha) < f'(\alpha + 1)$ for all $\alpha < \kappa$ and the functions f' and g' are of discontinuous type κ, so we only have to prove $[f]_\mu = [f']_\mu$ and $[g]_\mu = [g']_\mu$. From Lemma 1.3.9 we get a club set $C \subseteq \kappa$ that is closed under g'. Since f is increasing and discontinuous we have $\alpha < f(\alpha)$ for all $\alpha < \kappa$ and if $\alpha < \kappa$ is an element of C we have $\sup_{\beta<\alpha} g'(\beta) \leq \alpha$. So for all α in C the the least element in the range of f greater $\sup_{\beta<\alpha} g'(\beta)$ is $f(\alpha)$, *i.e.*, $f(\alpha) = f'(\alpha)$. The measure μ contains all club sets and $[f]_\mu = [f']_\mu$ follows. We assumed $[f]_\mu < [g]_\mu$ so there is a set $A \subseteq \kappa$ in μ with $f(\alpha) < g(\alpha)$ for all $\alpha \in A$. By definition of g' we have $g(\alpha) \leq g'(\alpha)$ for all $\alpha < \kappa$, so for all $\alpha \in A \cap C$ the least element in the range of g greater $f'(\alpha) = f(\alpha)$ is $g(\alpha)$, *i.e.*, $g(\alpha) = g'(\alpha)$ and $[g]_\mu = [g']_\mu$ follows. q.e.d.

Now we have the notational equipment to formulate some variations of partition properties.

1.9 More about Partition Properties

The homogeneous set we get from a partition property of some κ has not necessary any other properties except from being homogeneous and of size κ. If we restrict our partitions to sets of functions of continuous or discontinuous type we can show that the corresponding variations of the partition property give us homogeneous club sets. In this thesis we will mainly work with those.

1.9.1. LEMMA. *Let κ be a cardinal and λ, γ ordinals. Assume $\kappa \to (\kappa)^{\omega\cdot\lambda}_\gamma$ holds. Then for any partition P of the set $\mathfrak{D}^\lambda_\kappa$ into γ-many parts exists a homogeneous club set C, i.e., there is an ordinal $\alpha \in \gamma$ and a club set $C \subseteq \kappa$ such that $P(f) = \alpha$ for all $f \in \mathfrak{D}^\lambda_C$.*

Proof. Let P' be a γ-partition of $\mathfrak{D}^{\lambda}_{\kappa}$. For $f \in [\kappa]^{\omega\cdot\lambda}$ we define g_f by

$$g_f(\alpha) := \sup_{\beta<\omega\cdot(\alpha+1)} f(\beta).$$

Then $g_f : \lambda \to \kappa$ is a function of discontinuous type λ, so $P(f) := P'(g_f)$ defines a γ-partition of $[\kappa]^{\omega\cdot\lambda}$. The partition property $\kappa \to (\kappa)^{\omega\cdot\lambda}_{\gamma}$ gives us a homogeneous set H for P, *i.e.*, a set $H \subseteq \kappa$ of size κ such that $P(f)$ is constant for all $f \in [H]^{\omega\cdot\lambda}$. Let C be the set of limit points of H

$$C := \{\alpha \in \kappa\,;\, \sup(\alpha \cap H) = \alpha\}.$$

Then C is closed by definition and since H has size κ and κ is regular C is also unbounded, *i.e.*, C is a club subset of κ. Let $g : \lambda \to C$ be a function of discontinuous type λ. Since it has uniform cofinality ω there is an increasing function $f : \omega\cdot\lambda \to \kappa$ such that g is induced by f, $g = g_f$. We define a function $f' : \omega\cdot\lambda \to H$ by

$$f'(n\cdot\alpha) := \min\{\beta \in H\,;\, \beta \geq f(n\cdot\alpha)\}.$$

Since C consists of the limit points of H we get $\sup_{n<\omega} f'(n\cdot\alpha) = \sup_{n<\omega} f(n\cdot\alpha)$ for all $\alpha < \kappa$, so g is also induced by f'. By homogeneity of H we now know that $P'(g) = P'(g_{f'}) = P(f')$ is constant for all $g \in \mathfrak{D}^{\lambda}_{C}$. q.e.d.

1.9.2. COROLLARY. *Let κ be a cardinal, λ, γ ordinals and $\vec{\alpha} = \langle \alpha_i\,;\, i < n\rangle$ a sequence of ordinals with $\lambda = \sum_{i<n} \alpha_i$. Assume $\kappa \to (\kappa)^{\omega\cdot\lambda}_{\gamma}$ holds. Then for any partition P of the set $\mathfrak{C}^{\vec{\alpha}}_{\kappa}$ into γ-many parts exists a homogeneous club set C, i.e., there is an ordinal $\alpha \in \gamma$ and a club set $C \subseteq \kappa$ such that $P(f) = \alpha$ for all $f \in \mathfrak{C}^{\vec{\alpha}}_{C}$.*

Proof. Let P be a γ-partition of $\mathfrak{C}^{\vec{\alpha}}_{\kappa}$. We use the bijection from Lemma 1.8.1 to define a γ-partition P' of $\mathfrak{D}^{\lambda}_{\kappa}$: $P'(f) := P(\pi(f))$. By Lemma 1.9.1 exists a homogeneous club set $C \subseteq \kappa$ for this partition and since $\mathfrak{D}^{\lambda}_{C}$ is the π-image of $\mathfrak{C}^{\vec{\alpha}}_{C}$ the club set C is also homogeneous for the partition P: We have $P(f) := P'(\pi^{-1}(f))$ constant for all $f \in \mathfrak{C}^{\vec{\alpha}}_{C}$ q.e.d.

If $\kappa > \omega$ is a cardinal with the strong or weak partition property we have by Lemma 1.5.4 the partition property $\kappa \to (\kappa)^{\lambda}_{\gamma}$ for all $\lambda, \gamma < \kappa$. As said before, most of our partition arguments will use the above versions of the partition property that give us homogeneous club sets for the sets of functions of continuous or discontinuous type. Often we will invoke a partition property to get homogeneous club sets for partitions of sets of tuples of those functions. Here are the details:

1.9.3. LEMMA. *Let κ be a weak partition cardinal, $\gamma < \kappa$ an ordinal and $\lambda < \kappa$ a limit ordinal. Let S be the set of of tuples $\langle f, g\rangle$, where $f, g \in \mathfrak{D}^{\lambda}_{\kappa}$ are functions of discontinuous type λ such that for all $\alpha < \lambda$*

$$f(\alpha) < g(\alpha) < f(\alpha + 1).$$

Then for any partition P of the set S into γ-many parts exists a homogeneous club set C, i.e., there is an ordinal $\gamma_c \in \gamma$ and a club set $C \subseteq \kappa$ such that $P(\langle f, g\rangle) = \gamma_c$ for all $f, g \in \mathfrak{D}^\lambda_C$ with $f(\alpha) < g(\alpha) < f(\alpha + 1)$ for all $\alpha < \lambda$.

Proof. Let P be a γ-partition of the set S. Similar to Lemma 1.8.1 we have a bijection between S and $\mathfrak{D}^\lambda_\kappa$ that respects the range. If $\langle f, g\rangle$ is an element of the set S, then the function $h : \lambda \to \kappa$ defined by

- Let $h(\alpha)$ be $f(\alpha)$ if α is a limit ordinal or 0,
- let $h(\alpha + 1)$ be the least element of $\mathrm{ran}(g) > h(\alpha)$ if $h(\alpha) \in \mathrm{ran}(f)$, and
- let $h(\alpha + 1)$ be the least element of $\mathrm{ran}(f) > h(\alpha)$ if $h(\alpha) \in \mathrm{ran}(g)$.

is a function of discontinuous type λ. If the ranges of f and g are subsets of some $A \subseteq \kappa$, then the range of h is also a subset of A. Conversely, if $h : \lambda \to \kappa$ is a function of discontinuous type λ, we can define functions $f, g : \lambda \to \kappa$ by

- Let $f(\alpha)$ be $h(\alpha)$ if α is a limit ordinal or 0,
- let $g(\alpha)$ be the least element of $\mathrm{ran}(h) > f(\alpha)$ for $\alpha < \lambda$, and
- let $f(\alpha + 1)$ be the least element of $\mathrm{ran}(h) > g(\alpha)$ for $\alpha < \lambda$.

that are of discontinuous type λ and the ranges of f and g are subsets of the range of h. Those two operations are obviously inverses of each other, so the homogeneous club set C for the partition $P'(h) := P(\langle f, g\rangle)$ that we get from Lemma 1.9.1 is also a homogeneous club set for P. q.e.d.

1.9.4. Corollary. *Let κ be a strong partition cardinal, $\gamma < \kappa$ an ordinal and $\lambda < \kappa$ a limit ordinal. Let S be the set of of tuples $\langle f, g\rangle$, where $f : \lambda \to \kappa$ is a function of continuous type $\vec{\alpha}$ with $\lambda = \sum_{i<n} \alpha_i$ and $g : \lambda \to \kappa$ is a function of discontinuous type λ such that for all $\alpha < \lambda$*

$$f(\alpha) < g(\alpha) < f(\alpha + 1).$$

Then for any partition P of the set S into γ-many parts exists a homogeneous club C, i.e., there is an ordinal $\gamma_c \in \gamma$ and a club set $C \subseteq \kappa$ such that $P(\langle f, g\rangle) = \gamma_c$ for all $f \in \mathfrak{C}^{\vec{\alpha}}_C$ and $g \in \mathfrak{D}^\lambda_C$ with $f(\alpha) < g(\alpha) < f(\alpha + 1)$ for all $\alpha < \lambda$.

Proof. This is nearly the same proof as in Lemma 1.9.3, the main difference is that if $\langle f, g\rangle$ is an element of S and the ranges of f and g are in some club set C, then h is a function of continuous type $\vec{\alpha}$ with range C and vice versa. So we have to use Corollary 1.9.2 instead of Lemma 1.9.1. q.e.d.

Chapter 2

The Axiom of Determinacy

In this chapter we will define the Axiom of Determinacy and present some properties of the mathematical universe we get if we replace the Axiom of Choice with it.

2.1 Definition of AD

Let X be a non-empty set. For $A \subseteq X^\omega$ we denote with $\mathrm{G}_X(A)$ the **perfect information two player game** of length ω on X with **payoff** A: There are two **players**, I and II, that alternately play elements of X. First player I chooses an $x(0) \in X$; then player II chooses an $x(1) \in X$; then player I chooses an $x(2)$; and so on. Each choice is called a **move**, and a player can use the knowledge of all previous moves to choose his next move. The game ends after ω-many moves and the resulting sequence $x = \langle x(i)\,;\, i < \omega\rangle \in X^\omega$ is called a **play** in the game $\mathrm{G}_X(A)$.

We write x_{I} to denote the sequence $\langle x(2i)\,;\, i < \omega\rangle$ of moves of player I in the play x, analogous x_{II} for the sequence $\langle x(2i+1)\,;\, i < \omega\rangle$ of moves of player II. If y and z are elements of X^ω we write $y * z$ for the play that results from player I playing the sequence y and player II the sequence z. That means $y * z$ is defined by $y * z(2i) := y(i)$ and $y * z(2i+1) := z(i)$ for all $i < \omega$. So $x = x_{\mathrm{I}} * x_{\mathrm{II}}$ for all plays $x \in X^\omega$.

$$\left.\begin{array}{c|ccccccc} \mathrm{I} & x(0) & & x(2) & & x(4) & \cdots & = x_{\mathrm{I}} \\ \hline \mathrm{II} & & x(1) & & x(3) & & \cdots & = x_{\mathrm{II}} \end{array}\right\} x_{\mathrm{I}} * x_{\mathrm{II}} = x$$

Figure 2.1: The game $\mathrm{G}_X(A)$

A **strategy** $\sigma : \bigcup_{j<\omega} X^j \to X$ is a function that tells a player what moves to make during a play, *i.e.*, if $x(2n) = \sigma(x{\upharpoonright}2n)$ for all $n < \omega$ then we say that player I plays in the play x according to the strategy σ, analogous for player II, if

$x(2n+1) = \sigma(x{\restriction}2n+1)$ for all $n < \omega$ then player II plays in the play x according to the strategy σ. We write $\sigma * z$ for the play that results from player I following the strategy σ and player II playing z. So the play $\sigma * z$ goes like this:

$$\begin{array}{c|ccccc}
\mathrm{I} & \sigma(\emptyset) & & \sigma(\langle\sigma(\emptyset), z(0)\rangle) & & \sigma(\langle\sigma(\emptyset), z(0), \sigma(\langle\sigma(\emptyset), z(0)\rangle), z(1)\rangle) \quad \cdots \\ \hline
\mathrm{II} & & z(0) & & z(1) & \cdots
\end{array}$$

Similarly we write $y * \tau$ for the play that results from player I playing y and player II following the strategy τ.

$$\begin{array}{c|cccccc}
\mathrm{I} & y(0) & & y(1) & & y(2) & \cdots \\ \hline
\mathrm{II} & & \tau(\langle y(0)\rangle) & & \tau(\langle y(0), \tau(\langle y(0)\rangle), y(1)\rangle) & & \cdots
\end{array}$$

Finally $\sigma * \tau$ denotes the play we get when player I follows strategy σ and player II strategy τ:

$$\begin{array}{c|cccc}
\mathrm{I} & \sigma(\emptyset) & & \sigma(\langle\sigma(\emptyset), \tau(\langle\sigma(\emptyset)\rangle)\rangle) & \cdots \\ \hline
\mathrm{II} & & \tau(\langle\sigma(\emptyset)\rangle) & & \cdots
\end{array}$$

Player I **wins** the play x in the game $\mathrm{G}(A)$ if x is an element of the payoff A, otherwise player II wins. A strategy σ is a **winning strategy** for player I in the game $\mathrm{G}_X(A)$ if player I wins all plays where he plays according to σ, *i.e.*, if $\{\sigma * z\,;\, z \in X^\omega\} \subseteq A$. Analogous a strategy τ is a winning strategy for player II in the game $\mathrm{G}_X(A)$ if player II wins all plays where he plays according to τ, *i.e.*, if $\{y * \tau\,;\, y \in X^\omega\} \cap A = \emptyset$. We say that the game $\mathrm{G}_X(A)$ is **determined** if one of the players has a winning strategy and call a set $A \subseteq X^\omega$ determined if the game $\mathrm{G}_X(A)$ is determined.

With those definitions, notations, and conventions we now can formulate the **Axiom of Determinacy** AD as the following statement:

(AD) All subsets of ω^ω are determined.

The Axiom of Determinacy (then called Axiom of Determinateness) was first proposed by the Polish mathematicians Mycielski and Steinhaus in 1962. We refer to Kanamori [Ka94] for more on the history of this notion.

2.2 The Universe under AD

In this section we will describe the mathematical world under AD a little and emphasize its differences to the world under AC. The most fundamental is that AD and AC cannot hold both to their full extent:

2.2.1. Theorem (Gale-Stewart 1953). *Assume* AC. *There is a subset of* ω^ω *that is not determined.*

Proof. The original proof can be found in [GaSt53]. We include it in this thesis since it is an important result and also serves as an example of how to work with infinite games.

Let $\langle \sigma_\alpha\,;\,\alpha < 2^\omega \rangle$ be an enumeration of the set of strategies for player I and $\langle \tau_\alpha\,;\,\alpha < 2^\omega \rangle$ an enumeration of the set of strategies for player II. We define recursively sets

$$A := \{a_\alpha\,;\,\alpha < 2^\omega\}$$
$$B := \{b_\alpha\,;\,\alpha < 2^\omega\}$$

as follows:

- Choose $a_0 \neq b_0$ such that $a_0 = x * \tau_0$ and $b_0 = \sigma_0 * y$ for some $x, y \in \omega^\omega$.
- Assuming we have defined a_β, b_β for $\beta < \alpha$, we choose $a_\alpha \notin \{b_\beta\,;\,\beta < \alpha\}$ such that $a_\alpha = x * \tau_\alpha$ for some $x \in \omega^\omega$. This is possible since the cardinality of the set $\{x * \tau_\alpha\,;\,x \in \omega^\omega\}$ is 2^ω and the cardinality of the set $\{b_\beta\,;\,\beta < \alpha\}$ is less than 2^ω.
- We choose $b_\alpha \notin \{a_\beta\,;\,\beta \leq \alpha\}$ such that $b_\alpha = \sigma_\alpha * y$ for some $y \in \omega^\omega$. This is possible since the cardinality of the set $\{\sigma_\alpha * y\,;\,y \in \omega^\omega\}$ is 2^ω and the cardinality of the set $\{a_\beta\,;\,\beta \leq \alpha\}$ is less than 2^ω.

Then A and B are by disjoint sets, this follows from their definition by a trivial induction. Now we can show that the set A is not determined. One one hand, if player I plays according to a strategy σ_α then by definition of B exists $y \in \omega^\omega$ such that $\sigma_\alpha * y = b_\alpha \notin A$. So player I has no winning strategy. On the other hand, if player II plays according to a strategy τ_α then by definition of A exits $x \in \omega^\omega$ such that $x * \tau_\alpha = a_\alpha \in A$. That means player II also has no winning strategy, *i.e.*, the set A is not determined. q.e.d.

Maybe the main incentive behind the formulation of AD were its topological consequences:

2.2.2. THEOREM. *Assume* AD. *Then*

1. *All subsets of the reals are Lebesgue-measurable.*
2. *All subsets of the reals have the Baire property.*
3. *All subsets of the reals have the perfect set property.*

Proof. The Lebesgue-measurability is due to Mycielski and Swierczkowski [MySt64]. Stefan Banach used an idea by Stanisław Mazur to prove the Baire property in 1935, but the first published proof is due to Oxtoby [Ox57]. The third result is due to Davis [Da64, Theorem 2.1]. q.e.d.

Another surprising result was that under AD there are witnesses for properties that define large cardinals under AC:

2.2.3. Theorem (Solovay). *Assume* AD*. The first uncountable cardinal ω_1 is measurable, i.e., there exists a σ-complete ultrafilter on ω_1.*

Proof. See [Ka94, S. 384, Theorem 28.2] for a proof and also an historical play-by-play of the result and its consequences. q.e.d.

Later we will see that under AD there are many more measurable cardinals and other large cardinals like Jónsson and Rowbottom cardinals. Even λ-supercompact cardinals κ with λ greater κ exists under the Axiom of Determinacy.[1]

Although by Theorem 2.2.1 full choice is not an option under AD we can use a typical game argument to get a useful fragment of AC directly via a determinacy.

2.2.4. Lemma (Swierczkowski, Mycielski 1964). *Assume* AD*. Then the statement $\mathsf{AC}_\omega(\omega^\omega)$ holds, i.e., every countable family of nonempty subsets of ω^ω has a choice function.*

Proof. Suppose that for all $i < \omega$ the set $X_i \subseteq \omega^\omega$ is nonempty. Then let $\mathrm{G}_{\langle X_i\,;\,i<\omega\rangle}$ be the following game:

Player I plays a natural number n, Player II then plays in ω moves a sequence $y \in \omega^\omega$. Player II wins $\mathrm{G}_{\langle X_i\,;\,i<\omega\rangle}$ if $y \in X_n$. Clearly, player I cannot have a winning strategy, so by AD, player II has one. But a winning strategy for player II is a choice function for the family $\langle X_i\,;\, i < \omega\rangle$. q.e.d.

The axiom AD has consequences for the combinatorial structure of ω^ω, which in turn has consequences for other mathematical properties. For example:

2.2.5. Lemma. *Assume* AD*, then there are no non-principal ultrafilters over ω. Hence, every ultrafilter is ω_1-complete.*

Proof. See [Ka94, Proposition 28.1]. q.e.d.

By an argument of Bernstein, see [Ka94, Proposition 11.4], if all subsets of reals have the perfect set property then there is no order preserving injection from ω_1 into the reals. If there is no order preserving injection from ω_1 into the reals then no uncountable wellorderable set of reals exists. Since by part 3. of Theorem 2.2.2 under AD all subsets of reals have the perfect set property this means that the powerset of ω is not wellorderable. Furthermore there are only order preserving injections from countable cardinals into the reals. Consideration of the possibilities for surjections from the reals onto ordinals lead to the definition of the cardinal Θ:

$$\Theta := \sup\{\alpha\,;\ \text{there is a surjection from } \omega^\omega \text{ onto } \alpha\}.$$

Fundamental for results concerning Θ and its definable analoges was the following coding lemma by Moschovakis.

[1] See the articles [Be81a], [Be81b] by Howard Becker and the joint article [BeJa01] by Becker and Jackson. The master thesis [Bo02] of the author discusses some of these results.

2.2.6. THEOREM (MOSCHOVAKIS' CODING LEMMA). *Assume* AD. *If there is a surjection from* ω^ω *onto an ordinal* α *then there is a surjection from* ω^ω *onto* $\mathcal{P}(\alpha)$.

Proof. See [Ka94, Theorem 28.15]. q.e.d.

With a little basic set theory we can derive from Moschovakis' Coding Lemma the following corollary:

2.2.7. COROLLARY. *Assume* AD. *If there is a surjection from* ω^ω *onto an ordinal* α *then there is a surjection from* ω^ω *onto* α^α.

Proof. Let $h : \omega^\omega \to \mathcal{P}(\alpha)$ be the surjection that Moschovakis' Coding Lemma gives us. Let $g : \mathcal{P}(\alpha) \to \mathcal{P}(\alpha \times \alpha)$ be the bijection induced by the Gödel Pairing function. We define a function $f : \mathcal{P}(\alpha \times \alpha) \to \alpha^\alpha$ by

$$f(X)(\beta) := \min\{\gamma \in \alpha\,;\, \langle \beta, \gamma \rangle \in X\}.$$

Then f, g, and h are surjections, so $f \circ g \circ h : \omega^\omega \to \alpha^\alpha$ is also a surjection. q.e.d.

For all cardinals below Θ we now immediately get countable choice from Lemma 2.2.4 and Corollary 2.2.7.

2.2.8. LEMMA. *Assume* AD *and let* $\alpha < \Theta$ *be an ordinal. Then* $\mathsf{AC}_\omega(\alpha^\alpha)$ *holds, i.e., every countable family of nonempty subsets of* α^α *has a choice function.*

Proof. Since $\alpha < \Theta$, Corollary 2.2.7 yields a surjection $\pi : \omega^\omega \to \alpha^\alpha$. Suppose that for all $i < \omega$ the set $X_i \subseteq \alpha^\alpha$ is nonempty. Then let $\mathrm{G}_{\langle X_i\,;\,i<\omega\rangle}$ be the following game:

Player I plays a natural number n, Player II then plays in ω moves a sequence $y \in \omega^\omega$. Player II wins $\mathrm{G}_{\langle X_i\,;\,i<\omega\rangle}$ if $\pi(y) \in X_n$. Clearly, player I cannot have a winning strategy, so by AD, player II has one. But a winning strategy for player II is a choice function for the family $\langle X_i\,;\, i < \omega\rangle$. q.e.d.

This is nearly enough to show that all the ultrapowers we are interested in are wellfounded, as the following lemma shows.

2.2.9. LEMMA. *Assume* AD. *Let* $\gamma \leq \kappa < \Theta$ *be ordinals and* μ *a measure on* γ *then the ultrapower* κ^γ/μ *has no infinite descending sequences.*

Proof. Let E_μ be the induced ordering on the ultrapower, *i.e.*, for all $[f]_\mu, [g]_\mu \in \kappa^\gamma/\mu$ holds

$$[f]_\mu E_\mu [g]_\mu \quad :\Longleftrightarrow \quad \{\alpha \in \gamma\,;\, f(\alpha) < g(\alpha)\} \in \mu.$$

Assume toward a contradiction that there is a sequence $\langle x_i\,;\, i < \omega\rangle \in (\kappa^\gamma/\mu)^\omega$ such that $x_i E_\mu x_j$ holds for all $i < j < \omega$. Every x_i is a subset of κ^γ, so by Lemma

2.2.8 we can choose a sequence $\langle f_i\,;\,i<\omega\rangle \in (\kappa^\gamma)^\omega$ such that $[f_i]_\mu = x_i$ for all $i<\omega$. All measures are ω_1-complete under AD, so the set

$$\bigcap_{i<\omega}\{\alpha\in\gamma\,;\,f_{i+1}<f_i\}$$

is an element of μ and thus nonempty. But any element of this set leads to an infinite decreasing $<$-chain of ordinals, which is a contradiction. q.e.d.

Having no infinite sequences is not the same as being wellfounded, but if we include DC in our assumption we get:

2.2.10. Lemma. *Assume* $\mathsf{AD}+\mathsf{DC}$. *Let $\gamma\le\kappa<\Theta$ be ordinals and μ a measure on γ then the ultrapower κ^γ/μ is wellfounded.*

Proof. This is a direct corollary of Lemmas 2.2.9 and 1.4.2. q.e.d.

From now on in our statements about ultrapowers we will either assume their wellfoundedness or include $\mathsf{AD}+\mathsf{DC}$ in our assumptions, thus guaranteeing their wellfoundedness. Note that if the relation R from the proof of Lemma 1.4.2 can be shown to be Suslin then DC is not needed. This is due to the fact that in this case a function f with $\langle f(n+1), f(n)\rangle \in R$ for all $n<\omega$ can be constructed via a leftmost branch argument. So a thorough analysis of all the concrete measures used in terms of descriptive complexity could, and should, enable us to forgo all uses of DC in this thesis. But such an analysis is not our main goal and thus we join the majority of set theorists working under AD and assume DC, if only for convenience.

Another consequence of the choice available under AD is that for measures on ordinals greater ω the cofinality of the corresponding ultrapower is greater ω:

2.2.11. Lemma. *Assume* AD. *Let $\omega<\alpha<\Theta$ be an ordinal with $\mathrm{cf}(\alpha)>\omega$ and μ a measure on α. If α^α/μ is wellfounded then its cofinality is greater ω.*

Proof. Let $\langle\alpha_i\,;\,i<\omega\rangle$ be a sequence in α^α/μ. Lemma 2.2.8 allows us to pick a sequence $\langle f_i\,;\,i<\omega\rangle\in(\alpha^\alpha)^\omega$ such that $[f_i]_\mu=\alpha_i$ for $i<\omega$. For $\xi\in\alpha$ let $g(\xi):=\sup_{i\in\omega} f_i(\xi)$. Since $\mathrm{cf}(\alpha)>\omega$, we have $g\in\alpha^\alpha$ and by definition of g we have $[f_i]_\mu\le[g]_\mu\in\alpha^\alpha/\mu$ for all $i<\omega$, so the sequence $\langle\alpha_i\,;\,i<\omega\rangle$ cannot be cofinal in α^α/μ. q.e.d.

Let us connect the theory of strong partition cardinals to the Axiom of Determinacy AD. In descriptive set theory, definable analogues of the cardinal Θ have been investigated, the so-called **projective ordinals**

$$\boldsymbol{\delta}^1_n := \sup\{\xi\;;\;\xi \text{ is the length of a prewellordering of } \omega^\omega \text{ in } \boldsymbol{\Delta}^1_n\}.$$

We will call the projective ordinals with odd index **odd projective ordinals** and those with even index **even projective ordinals**. In the early 1980s, a lot of combinatorial consequences of AD for the projective ordinals were known, among them the following:

2.2.12. THEOREM. *Assume* AD *and let n be a natural number. Then:*

1. *(Kunen, Martin 1971)* $\boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$,
2. *(Kechris 1974)* $\boldsymbol{\delta}^1_{2n+1}$ *is the cardinal successor of a cardinal of cofinality* ω,
3. *(Martin, Kunen 1971) all* $\boldsymbol{\delta}^1_n$ *are measurable and distinct,*
4. *(Martin, Kunen 1971)* $\boldsymbol{\delta}^1_1 = \aleph_1$, $\boldsymbol{\delta}^1_2 = \aleph_2$, $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$, *and* $\boldsymbol{\delta}^1_4 = \aleph_{\omega+2}$,
5. *(Martin, Paris 1971)* $\boldsymbol{\delta}^1_1 \to (\boldsymbol{\delta}^1_1)^{\boldsymbol{\delta}^1_1}$, *and for all* $\alpha < \boldsymbol{\delta}^1_2$, *the relation* $\boldsymbol{\delta}^1_2 \to (\boldsymbol{\delta}^1_2)^\alpha$ *holds,*
6. *(Martin 1971) for all* $\alpha < \omega_1$ *the partition relation* $\boldsymbol{\delta}^1_{2n+1} \to (\boldsymbol{\delta}^1_{2n+1})^\alpha$ *holds,*
7. *(Kunen 1971) the* ω*-cofinal measure* $\mathcal{C}^\omega_{\boldsymbol{\delta}^1_{2n+1}}$ *is a normal measure on* $\boldsymbol{\delta}^1_{2n+1}$ *with* ${\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}}/\mathcal{C}^\omega_{\boldsymbol{\delta}^1_{2n+1}} = \boldsymbol{\delta}^1_{2n+2} = (\boldsymbol{\delta}^1_{2n+1})^+$.

Proof. Proofs of all parts can be found in [Ke78a]. Fact 2.2.12 comprises of Theorem 3.12, Theorem 3.10, Theorem 5.1, §6, Theorem 12.1, Corollary 13.4, Theorem 11.2, and Theorem 14.3 of [Ke78a]. q.e.d.

Theorem 2.2.12 gives an indication of how representing cardinals as ultrapowers of measures helps in computing their value. Let us make this more explicit in one example: Suppose we are working in AD, we know that $\boldsymbol{\delta}^1_2 = \aleph_2$ and we know that for each $n > 2$ there is some measure μ on $\aleph_1$ such that ${\boldsymbol{\delta}^1_2}^{\omega_1}/\mu = \aleph_n$. Suppose furthermore that it is our goal to compute $\boldsymbol{\delta}^1_3$ (so, we are trying to prove part 4. of Theorem 2.2.12). Tony Martin showed that $\boldsymbol{\delta}^1_3 \leq \aleph_{\omega+1}$ (just from the existence of sharps; see [Ka94, p. 428]); by Theorem 2.2.12 (3), we know that $\boldsymbol{\delta}^1_3 > \boldsymbol{\delta}^1_2 = \aleph_2$ is a regular cardinal, so there are only two options left: either $\boldsymbol{\delta}^1_3 = \aleph_n$ for some $n > 2$ or $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$.

As a consequence of Lemma 1.4.9, we get from our assumptions that $\mathrm{cf}(\aleph_n) = \aleph_2$ for $n > 2$, and thus these cardinals are singular. This leaves $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$ as the last remaining possibility and the computation of $\boldsymbol{\delta}^1_3$ is finished.

By carrying information about the cofinality of the cardinals, the representation of the $\aleph_n$ as ultrapowers allowed us to exclude them from the list of candidates for being $\boldsymbol{\delta}^1_3$. This idea will be used again in the proof of the Really Helpful Theorem 5.2.2. The whole argument is paradigmatic for measure analyses; ideas like this were fully exploited in the work of Steve Jackson when he computed all of the projective ordinals under the assumption of AD.

2.2.13. THEOREM (JACKSON). *Assume* AD. *Let* $\mathbf{e}_0 := 0$ *and* $\mathbf{e}_{n+1} := \omega^{(\omega^{\mathbf{e}_n})}$ *(i.e.,* $\mathbf{e}_n$ *is a exponential* ω*-tower of height* $2n-1$*). Then for every* $n < \omega$,

$$\boldsymbol{\delta}^1_{2n+1} = \aleph_{\mathbf{e}_n+1},$$

and all odd projective ordinals have the strong partition property.

Proof. The original paper is [Ja88]; a more accessible (but still very involved) proof of the case $n = 2$ can be found in [Ja99]. q.e.d.

The proof of Theorem 2.2.13 is outside of the scope of this thesis, but deeply connected with the techniques we are using.

We call a cardinal κ **closed under ultrapowers** if for all $\gamma, \delta < \kappa$ and wellfounded measures μ on γ we have $\gamma^\delta/\mu < \kappa$.

2.2.14. LEMMA. *Let κ be a regular cardinal that is closed under ultrapowers and μ a measure on $\gamma < \kappa$. If the ultrapower κ^γ/μ is wellfounded then $\kappa^\gamma/\mu = \kappa$.*

Proof. For $\alpha < \kappa$ let $c_\alpha : \gamma \to \{\alpha\}$ be the function with constant value α. Then $[c_\alpha]_\mu < [c_\beta]_\mu$ for $\alpha < \beta < \kappa$, so $\kappa \leq \kappa^\gamma/\mu$. Fix $f \in \kappa^\gamma$, since κ is regular we have $\sup f < \kappa$. If $[g]_\mu < [f]_\mu$ then there is a function $g' : \gamma \to \sup f$ such that $[g]_\mu = [g']_\mu$, so

$$\begin{aligned} [f]_\mu &= \{[g]_\mu \in \kappa^\gamma/\mu\,;\, [g]_\mu < [f]_\mu\} \\ &= \{[g]_\mu \in (\sup f)^\gamma/\mu\,;\, [g]_\mu < [f]_\mu\} \\ &\leq (\sup f)^\gamma/\mu. \end{aligned}$$

But κ is closed under ultrapowers, which means $[f]_\mu \leq (\sup f)^\gamma/\mu < \kappa$, and so $\kappa^\gamma/\mu = \sup\{[f]_\mu\,;\, f \in \kappa^\gamma\} \leq \kappa$. q.e.d.

If a cardinal κ is closed under ultrapowers, then club sets can be thinned out to have another special property: A club set $C \subseteq \kappa$ such that for all $\beta, \alpha \in C$ with $\beta < \alpha$ we have

$$[f]_\mu \leq \alpha \text{ for all } f : \gamma \to \beta$$

is called **closed under** μ.

2.2.15. LEMMA. *Let κ be a regular cardinal that is closed under ultrapowers and μ a measure on $\gamma < \kappa$ such that the ultrapower κ^γ/μ is wellfounded. If $C \subseteq \kappa$ is a club set then exist a club set $C' \subseteq C$ that is closed under μ.*

Proof. Let $c : \kappa \to C$ be an increasing enumeration of C. We will define C' by an increasing enumeration $c' : \kappa \to \kappa$.

- Let $c'(0)$ be the least element of C,
- let $c'(\alpha + 1)$ be the least element β of C such that $[f]_\mu \leq \beta$ holds for all $f : \gamma \to c'(\alpha)$, and
- let $c'(\lambda)$ be $\sup_{\alpha<\lambda} c'(\alpha)$ for limits λ.

We only have to show that $c'(\alpha+1)$ exist. We have that $[f]_\mu < [\xi \mapsto c'(\alpha)]_\mu$ holds for all $f : \gamma \to c'(\alpha)$ and since κ is closed under ultrapowers by Lemma 2.2.14 we get $[\xi \mapsto c'(\alpha)]_\mu < \kappa$. So there exists a $\beta \in C$ such that $[f]_\mu < [\xi \mapsto c'(\alpha)]_\mu \leq \beta$ holds for all $f : \gamma \to c'(\alpha)$, which shows that c' is welldefined. By definition the set $C' := \{c'(\alpha)\,;\, \alpha < \kappa\}$ is nonempty, closed and unbounded in κ, since the sequence $\langle c'(\alpha)\,;\, \alpha < \kappa\rangle$ is strictly increasing and κ is regular. So we found our club subset $C' \subseteq C$. q.e.d.

We will now show that, in the case of normal measures, the equivalence class of a function of continuous type is determined by its supremum. This result will allow us sometimes to ignore the specific function that generates an equivalence class, thus enabling us to substitute it with another function with the same supremum.

2.2.16. Lemma. *Let κ be a regular cardinal that is closed under ultrapowers. If μ is a normal measure on an infinite cardinal $\gamma < \kappa$, the ultrapower κ^γ/μ is wellfounded, and $C \subseteq \kappa$ is a club set that is closed under μ, then for all functions $f \in \mathfrak{C}_C^\gamma$ of continuous type γ we have*

$$[f]_\mu = \sup f.$$

Proof. Since f is an increasing function and γ a limit ordinal, for each $\alpha < \sup f$ there is an $\beta < \gamma$ such that $\alpha < f(\beta)$. Then for all $\delta < \gamma$ with $\delta > \beta$ we have $\alpha < f(\delta)$, so $[\xi \mapsto \alpha]_\mu < [f]_\mu$ holds since μ contains end segments. Now $\sup f \leq [f]_\mu$ follows from the fact that $[\xi \mapsto \alpha]_\mu < [\xi \mapsto \beta]_\mu$ is true for all $\alpha < \beta < \sup f$. So we only have to show $[f]_\mu \leq \sup f$.

First we show that $[f]_\mu$ must be a limit ordinal. If $[g]_\mu < [f]_\mu$ for some $g : \gamma \to \kappa$ then there is a set $A \in \mu$ such that for all $\alpha \in A$ we have $g(\alpha) < f(\alpha)$. Without loss of generality we can assume that C contains only limit ordinals, so $f(\alpha)$ is a limit and we can define a function $g' : \gamma \to \kappa$ with $[g]_\mu < [g']_\mu < [f]_\mu$ by $g'(\alpha) := g(\alpha) + 1$. So $[f]_\mu$ cannot be a successor.

Now take an arbitrary $g : \gamma \to \kappa$ with $[g]_\mu < [f]_\mu$. There is a set $A \in \mu$ with $g(\alpha) < f(\alpha)$ for all $\alpha \in A$ and with Lemma 1.2.8 we can assume that A contains only limit ordinals. That means $f(\alpha)$ is a limit for all $\alpha \in A$, so there is an ordinal $\beta < \alpha$ such that $g(\alpha) < f(\beta) < f(\alpha)$. The function $\alpha \mapsto \beta$ is regressive on $A \in \mu$, so there is a set $B \in \mu$ and an ordinal β' with $g(\alpha) < f(\beta') < \sup f$ for all $\alpha \in B$. The ordinals $f(\beta')$ and $\sup f$ are elements of C, which is closed under μ, so we get $[g]_\mu \leq \sup f$. The function g was arbitrary, which means we have $[g]_\mu \leq \sup f$ for all g with $[g]_\mu < [f]_\mu$ and since $[f]_\mu$ is a limit $[f]_\mu = \sup_{[g]_\mu < [f]_\mu} [g]_\mu \leq \sup f$ follows. q.e.d.

One result of Steve Jackson's inductive computation of the projective ordinals under AD was the following theorem:

2.2.17. THEOREM (JACKSON). *Assume* AD *and let κ be an odd projective ordinal, i.e., $\kappa = \boldsymbol{\delta}^1_{2n+1}$ for some natural number n. Then κ is closed under ultrapowers.*

Proof. For the projective ordinals up to $\boldsymbol{\delta}^1_5$ this follows from Theorem 5.2 and Theorem 7.2 in [Ja88], see also Remark 4.13 in [Ja∞]. The necessary details to prove the theorem for arbitrary $\boldsymbol{\delta}^1_{2n+1}$ can be found in [Ja88]. q.e.d.

So with Lemma 2.2.15 we often can assume that a club subset of an odd projective ordinal is closed under certain ultrapowers, which in the case of normal measures allows us to use Lemma 2.2.16 and identify the equivalence class of a function with its supremum.

Kunen trees were first used by Kunen in his analysis of measures on $\aleph_\omega$, see [So78b], Martin trees are generalizations of Kunen trees, see [Ke78b]. They arise from the analysis of functions on the projective ordinals via homogeneous trees. Originally restricted to ω_1 and $\boldsymbol{\delta}^1_3$, the generalization of these trees to arbitrary odd projective ordinals was also part of Jackson's inductive computation of the projective ordinals, see [Ja88, p. 213].

2.2.18. THEOREM (JACKSON). *Assume* AD. *Let κ be an odd projective ordinal and $g : \kappa \to \kappa$ a function from κ to κ. Then there is a tree T on κ such that $g(\alpha) < |T{\restriction}\alpha|$ holds for $\mathcal{C}^\omega_\kappa$-almost all α. Such a tree is called a **Kunen Tree**.*

Proof. Kunen's original construction for the case $\kappa = \omega_1$ can also be found as Lemma 3.1 in [Ja∞], the case $\kappa = \boldsymbol{\delta}^1_3$ is Theorem 2.1 in [Ja99]. The general case is discussed in Section 4.1 of [Ja∞]. q.e.d.

2.2.19. THEOREM (JACKSON). *Assume* AD + DC. *Let $\kappa > \omega_1$ be an odd projective ordinal and $g : \kappa \to \kappa$ a function from κ to κ. Then there is a tree T on κ such that $g(\alpha) < |T{\restriction}\sup_n \alpha^{\omega_1}/\mathcal{W}^n_{\omega_1}|$ holds for $\mathcal{C}^{\omega_1}_\kappa$-almost all α. A tree like this is called **Martin Tree**.*

Proof. As with the Kunen trees the case $\kappa = \boldsymbol{\delta}^1_3$ is Theorem 2.1 in [Ja99] and the general case is discussed in Section 4.1 of [Ja∞]. q.e.d.

Chapter 3

Measure Analysis

In this chapter we develop the algebraic basis for our notion of measure analysis. By the term "measure analysis", we understand (informally) the following procedure: given a cardinal κ and some cardinal $\lambda > \kappa$, we assign a measure μ on κ to λ such that $\kappa^\kappa/\mu = \lambda$.

This algebraic foundation, using ordinal algebras with operations that corresponded to ordinal multiplication and addition, first appeared in Jackson and Löwe's paper [JaLö06]. One of the main theorems in proving canonicity for the measure assignment defined in Section 4.5, Theorem 5.2.2, will allow us to work with sums of measures, which corresponds to the addition operation in ordinal algebras, but not with products. So it was necessary to develop a similar framework using only additive ordinal algebras and connect the two approaches. That means the definitions and results concerning ordinal algebras in this thesis are due to Jackson and Löwe, whereas those concerning additive ordinal algebras are due to the author.

3.1 Ordinal Algebras

Let $\mathfrak{V}$ be a nonempty set of variables and G a set of symbols. Then the **free algebra** $\mathbf{Free}_G(\mathfrak{V})$ is the smallest set $\mathfrak{A} \subseteq (\mathfrak{V} \cup G \cup \{(,)\})^{<\omega}$ closed under the following conditions:

1. If $v \in \mathfrak{V}$ is a variable then it is an element of $\mathfrak{A}$.
2. If t and s are in $\mathfrak{A}$ then so is the term (tgs) for all $g \in G$.

An **ordinal algebra** is a free algebra $\mathfrak{A} = \mathbf{Free}_{\oplus,\otimes}(\mathfrak{V})$ over a set of generators $\mathfrak{V} = \langle \mathsf{V}_\beta\,;\, \beta < \alpha\rangle$ using the binary operations $\oplus$ and $\otimes$. We write $\mathfrak{A}_\alpha$ for the ordinal algebra with α-many generators, so $\mathfrak{A}_\alpha \subseteq \mathfrak{A}_\beta$ for $\alpha < \beta$. For any ordinal algebra $\mathfrak{A}_\alpha$ we define a function $o_\alpha : \mathfrak{A}_\alpha \to \mathrm{ht}(\mathfrak{A}_\alpha)$ onto an ordinal $\mathrm{ht}(\mathfrak{A}_\alpha)$ that we call the **height** of $\mathfrak{A}_\alpha$. We begin with $o_1(\mathsf{V}_0) := 1$ and extend o to the whole algebra $\mathfrak{A}_1$ by $o_1(t_1 \oplus t_2) := o_1(t_1) + o_1(t_2)$ and $o_1(t_1 \otimes t_2) := o_1(t_1) \cdot o_\alpha(t_2)$ for all terms $t_1, t_2 \in \mathfrak{A}_1$, so $\mathrm{ht}(\mathfrak{A}_1) = \omega$. Assuming we have defined o_α on $\mathfrak{A}_\alpha$, we let $o_{\alpha+1}(t) := o_\alpha(t)$ for all terms $t \in \mathfrak{A}_\alpha$, set $o_{\alpha+1}(\mathsf{V}_\alpha) := \mathrm{ht}(\mathfrak{A}_\alpha) = \sup\{o_\alpha(t)\,;\, t \in \mathfrak{A}_\alpha\}$ and extend this to the whole of $\mathfrak{A}_{\alpha+1}$ by $o_{\alpha+1}(t_1 \oplus t_2) := o_\alpha(t_1) + o_\alpha(t_2)$ and $o_{\alpha+1}(t_1 \otimes t_2) := o_\alpha(t_1) \cdot o_\alpha(t_2)$. For limit ordinals λ, the function o_λ is just the union of the functions o_α for $\alpha < \lambda$. Since for $\alpha < \beta$ the ordinal algebra $\mathfrak{A}_\beta$ with function o_β extends the ordinal algebra $\mathfrak{A}_\alpha$ with function o_α we drop the index in our notation and just talk about the ordinal algebra $\mathfrak{A}$ and the function o.

We will also work with **additive ordinal algebras**. The additive ordinal algebra with α-many generators $\mathfrak{S} = \langle \mathsf{S}_\beta\,;\, \beta < \alpha\rangle$ is the free algebra $\mathbf{Free}_\oplus(\mathfrak{S})$ and will be denoted by $\mathfrak{A}^\oplus_\alpha$. As in the case of ordinal algebras we define a function $u : \mathfrak{A}^\oplus \to \mathrm{ht}(\mathfrak{A}^\oplus)$ for additive ordinal algebras $\mathfrak{A}^\oplus$ by $u(\mathsf{S}_0 \otimes n) := n$, $u(\mathsf{S}_\alpha) := \mathrm{ht}(\mathfrak{A}^\oplus_\alpha) = \sup\{u(t)\,;\, t \in \mathfrak{A}^\oplus_\alpha\}$ and extend this to $\mathfrak{A}^\oplus_{\alpha+1}$ by $u(t_1 \oplus t_2) := u(t_1) + u(t_2)$. At limit steps we again take the union.

Let us take the simplest ordinal algebras as an example. For this, we introduce some notations for finitely iterated sums and products:

$$\mathsf{V} \otimes n := \mathsf{V} \underbrace{\oplus \ldots \oplus}_{n} \mathsf{V},$$

$$\mathsf{V}^{\otimes n} := \mathsf{V} \underbrace{\otimes \ldots \otimes}_{n} \mathsf{V}, \text{ and}$$

$$\bigoplus_{i<n} \mathsf{V}_{\alpha_i} = \mathsf{V}_{\alpha_0} \oplus \ldots \otimes \mathsf{V}_{\alpha_{n-1}}.$$

$$\bigotimes_{i<n} \mathsf{V}_{\alpha_i} = \mathsf{V}_{\alpha_0} \otimes \ldots \otimes \mathsf{V}_{\alpha_{n-1}}.$$

Then the first two ordinal algebras look as follows:

- $(\alpha = 1)$. If $\mathfrak{V} = \{\mathsf{V}_0\}$, then $o(\mathsf{V}_0) = 1$, so $o(\mathsf{V}_0 \otimes n) = n$, meaning $\mathrm{ht}(\mathfrak{A}) = \omega$.
- $(\alpha = \omega)$. We have ω-many generators $\mathfrak{V} = \langle \mathsf{V}_0, \mathsf{V}_1, \mathsf{V}_2, \mathsf{V}_3, \ldots\rangle$, where $o(\mathsf{V}_0) = 1$, and $o(\mathsf{V}_1) = \omega$. So $o(\mathsf{V}_2) = \sup_{n<\omega} o(\mathsf{V}_2{}^{\otimes n}) = \omega^\omega$, the same way we get $o(\mathsf{V}_3) = \omega^{\omega^2}$, $o(\mathsf{V}_4) = \omega^{\omega^3}$, etc., and $\mathrm{ht}(\mathfrak{A}) = \omega^{\omega^\omega}$.

The o-value of a variable V_α in an ordinal algebra $\mathfrak{A}$ is the height $\mathrm{ht}(\mathfrak{A}_\alpha)$ of the ordinal algebra with α-many generators. And the height of an ordinal algebra is the supremum of the o-values of its terms. Which means we can easily compute those values:

3.1.1. PROPOSITION. *Let $\alpha > 0$ be an ordinal. We have:*

$$o(\mathsf{V}_\alpha) := \mathrm{ht}(\mathfrak{A}_\alpha) = \begin{cases} \omega^{\omega^{\alpha-1}} & 0 < \alpha < \omega \\ \omega^{\omega^\alpha} & \alpha \geq \omega. \end{cases}$$

Proof. We will prove the claim by induction on α. We already know $\mathrm{ht}(\mathfrak{A}_1) = \omega$. The first case to consider is $1 < \alpha < \omega$ with $\alpha = \beta + 1$. Then by induction hypothesis we have

$$\mathrm{ht}(\mathfrak{A}_\alpha) = \sup_{n<\omega} o(\mathsf{V}_\beta{}^{\otimes n}) = \omega^{\omega^{\beta-1+1}} = \omega^{\omega^{\alpha-1}}.$$

Furthermore by definition of ht we get

$$\mathrm{ht}(\mathfrak{A}_\omega) = \sup_{\alpha<\omega} o(\mathsf{V}_\alpha) = \omega^{\omega^\omega}.$$

Now we come to the case $\alpha > \omega$. The induction hypothesis states that $\mathrm{ht}(\mathfrak{A}_\beta) = \omega^{\omega^\beta}$ for $\omega \leq \beta < \alpha$. Then we get

$$\begin{aligned}\mathrm{ht}(\mathfrak{A}_\alpha) &= \sup\{o(\mathsf{V}_\beta{}^{\otimes n})\,;\, \omega \leq \beta < \alpha, n < \omega\} \\ &= \sup\{(\omega^{\omega^\beta})^n\,;\, \omega \leq \beta < \alpha, n < \omega\} \\ &= \sup\{\omega^{\omega^{\beta+1}}\,;\, \omega \leq \beta < \alpha\} \\ &= \omega^{\omega^\alpha}.\end{aligned}$$

q.e.d.

Analogous to Proposition 3.1.1 we get the following result about the height of additive ordinal algebras:

3.1.2. Proposition. *Let $\alpha > 0$ be an ordinal. We have $u(\mathsf{S}_\alpha) := \mathrm{ht}(\mathfrak{A}^\oplus_\alpha) = \omega^\alpha$.*

Proof. By induction on α. We have $u(\mathsf{S}_1) = \omega = \omega^1$, and if $u(\mathsf{S}_\beta) := \omega^\beta$ for $0 < \beta < \alpha$ we get $u(\mathsf{S}_\alpha) = \sup\{u(\mathsf{S}_\beta \otimes n)\,;\, 0 < \beta < \alpha \text{ and } n < \omega\} = \omega^\alpha$ for both successor and limit ordinals α. q.e.d.

By definition the function $o : \mathfrak{A} \to \mathrm{ht}(\mathfrak{A})$ is a homeomorphism between ordinal algebras and ordinals, and the function $u : \mathfrak{A} \to \mathrm{ht}(\mathfrak{A})$ is a homeomorphism between additive ordinal algebras and ordinals. An ordinal algebra $\mathfrak{A}$ is essentially a formal syntactic way of describing the ordinal $\mathrm{ht}(\mathfrak{A})$ while only using the operations of ordinal addition and multiplication, or just using ordinal addition in the case of an additive ordinal algebra $\mathfrak{A}^\oplus$.

We get natural embeddings π from an additive ordinal algebra $\mathfrak{A}^\oplus$ into any ordinal algebra $\mathfrak{A}$ with height greater than $\mathrm{ht}(\mathfrak{A}^\oplus)$ by assigning certain products of generators from $\mathfrak{A}$ to generators from $\mathfrak{A}^\oplus$.

3.1.3. Definition. Let $\mathfrak{A}_\alpha$ be an ordinal algebra and let $\mathfrak{P}_\alpha \subseteq \mathfrak{A}_\alpha$ be the following set of product terms from $\mathfrak{A}_\alpha$:

$$\mathfrak{P}_\alpha := \{\bigotimes_{i<n} \mathsf{V}_{\beta_i}\,;\, \vec{\beta} \in \alpha^n \text{ with } \beta_i \geq \beta_j > 0 \text{ for all } i < j < n \text{ and } n < \omega\} \cup \{\mathsf{V}_0\}.$$

We consider $\mathfrak{P}_\alpha$ ordered with the normal lexicographic order $<_{\text{lex}}$, *i.e.*, $t := \mathsf{V}_{\beta_0} \otimes \ldots \otimes \mathsf{V}_{\beta_{n-1}}$ is less than $s := \mathsf{V}_{\gamma_0} \otimes \ldots \otimes \mathsf{V}_{\gamma_{m-1}}$ if t is an initial segment of s or if $\beta_i < \gamma_i$ for the first i such that $\beta_i \neq \gamma_i$. Since for $\beta > \alpha$ the order $\langle \mathfrak{P}_\beta, <_{\text{lex}} \rangle$ extends the order $\langle \mathfrak{P}_\alpha, <_{\text{lex}} \rangle$ we will often simply write $\mathfrak{P}$ for the corresponding ordered set of finite products of an ordinal algebra $\mathfrak{A}$ and assume $\mathfrak{P} = \mathfrak{P}_\alpha$ for a large enough α.

If we have an ordinal algebra $\mathfrak{A}_\alpha$ we call $\mathfrak{A}^{\oplus}_{\mathfrak{P}_\alpha} := \mathbf{Free}_{\oplus}(\mathfrak{P}_\alpha)$ the **induced additive ordinal algebra**. If β is the length of $\mathfrak{P}_\alpha$ with respect to $<_{\text{lex}}$ then we have an obvious isomorphism between $\mathfrak{A}^{\oplus}_{\mathfrak{P}_\alpha}$ and $\mathfrak{A}^{\oplus}_{\beta}$, so we can identify them and see $\mathfrak{A}^{\oplus}_{\mathfrak{P}_\alpha}$ as a normal additive ordinal algebra. But since the generators of $\mathfrak{A}^{\oplus}_{\mathfrak{P}_\alpha}$ are terms of $\mathfrak{A}_\alpha$, a term in $\mathfrak{A}^{\oplus}_{\mathfrak{P}_\alpha}$ is also a term of $\mathfrak{A}_\alpha$.

3.1.4. Lemma. *Let $\mathfrak{A}_\alpha$ be an ordinal algebra and $\mathfrak{A}^{\oplus}_{\mathfrak{P}_\alpha}$ the induced additive ordinal algebra. If $\alpha > 0$ is finite then $\mathfrak{A}^{\oplus}_{\mathfrak{P}_\alpha}$ has $\omega^{\alpha-1}$-many generators, otherwise ω^α-many. So by Propositions 3.1.1 and 3.1.2 the algebras $\mathfrak{A}_\alpha$ and $\mathfrak{A}^{\oplus}_{\mathfrak{P}_\alpha}$ have the same height. Furthermore, for all terms t of $\mathfrak{A}^{\oplus}_{\mathfrak{P}_\alpha}$ we have $u(t) = o(t)$ (on the right side we interpret t as a term of $\mathfrak{A}_\alpha$).*

Proof. Fix an ordinal algebra $\mathfrak{A}_\alpha$ and let $\lambda := \omega^{\alpha-1}$ if α is finite, $\lambda := \omega^\alpha$ otherwise. We define the auxiliary function $h : \lambda \to \mathrm{On}$ by $h(\beta) := \beta + 1$ if β is finite and $h(\beta) := \beta$ otherwise. We define a sequence $\langle \mathsf{S}_\beta \,;\, \beta < \lambda \rangle$ in the following way: We start with $\mathsf{S}_0 = \mathsf{V}_0$. If β is greater 0 then we have $\beta = \sum_{i<n} \omega^{\beta_i} \cdot m_i > 0$, where $\beta_i \geq \beta_j$ if $i < j$. We let $\mathsf{S}_\beta := \bigotimes_{i<n} \mathsf{V}^{\otimes m_i}_{h(\beta_i)}$ in this case. Then $\langle \mathsf{S}_\beta \,;\, \beta < \lambda \rangle$ is an enumeration of $\mathfrak{P}_\alpha$, the set of generators of $\mathfrak{A}^{\oplus}_{\mathfrak{P}_\alpha}$. We prove this by induction over β. We have $\mathsf{S}_0 = \mathsf{V}_0$ and $\mathsf{S}_1 = \mathsf{V}_1$ so our hypothesis holds for $\beta < 2$. Let $\beta > 1$ and assume it holds for $\gamma < \beta$. If $\beta = \gamma + 1$ is a successor then $\mathsf{S}_\beta = \mathsf{S}_\gamma \otimes \mathsf{V}_1$, which is the $<_{\text{lex}}$-successor of S_γ. If β is a limit then it is of the form $\beta = \sum_{i<n} \omega^{\beta_i} \cdot m_i > 0$, where $\beta_i \geq \beta_j > 0$ if $i < j$. Assume without loss of generality that $m_{n-1} = 1$. We have $\beta = \sup_{\delta<\beta_{n-1}, m<\omega} \big((\sum_{i<n-1} \omega^{\beta_1} \cdot m_i) + \omega^\delta \cdot m \big)$ and by induction hypothesis we know that for all $\delta < \beta_{n-1}$ and $m < \omega$ we have $\mathsf{S}_\gamma = (\bigoplus_{i<n-1} \mathsf{V}^{\otimes m_i}_{h(\beta_i)}) \otimes \mathsf{V}^{\otimes m}_{\delta}$ with $\gamma = (\sum_{i<n-1} \omega^{\beta_1} \cdot m_i) + \omega^\delta \cdot m$. The $<_{\text{lex}}$-least element of $\mathfrak{P}_\alpha$ that is $<_{\text{lex}}$-larger than all those γs is $(\bigoplus_{i<n-1} \mathsf{V}^{\otimes m_i}_{h(\beta_i)}) \otimes \mathsf{V}_{\beta_{n-1}} = \mathsf{S}_\beta$, which proves the limit case.

Now let $t = \bigoplus_{i<n} \mathsf{S}_{\beta_i}$ be an arbitrary term in $\mathfrak{A}^{\oplus}_{\mathfrak{P}_\alpha}$. By definition of o and u we have $o(t) = \sum_{i<n} o(\mathsf{S}_{\beta_i})$ and $u(t) = \sum_{i<n} u(\mathsf{S}_{\beta_i})$. So to prove $o(t) = u(t)$ it suffices to show $o(\mathsf{S}) = u(\mathsf{S})$ for all S in $\mathfrak{A}^{\oplus}_{\mathfrak{P}_\alpha}$. Let S_β be the βth generator of $\mathfrak{A}^{\oplus}_{\mathfrak{P}_\alpha}$. We know that either $\beta = 0$, in which case we have $\mathsf{S}_0 = \mathsf{V}_0$ and $u(\mathsf{S}_0) = o(\mathsf{V}_0) = 1$ follows, or $\beta = \sum_{i<n} \omega^{\beta_i} \cdot m_i > 0$, where $\beta_i \geq \beta_j$ if $i < j$, and $\mathsf{S}_\beta = \bigotimes_{i<n} \mathsf{V}^{\otimes m_i}_{h(\beta_i)}$. With Proposition 3.1.1 we get

$$o(\bigotimes_{i<n} \mathsf{V}^{\otimes m_i}_{h(\beta_i)}) = \prod_{i<n} o(\mathsf{V}_{h(\beta_i)})^{\otimes m_i} = \prod_{i<n} \left(\omega^{\omega^{\beta_i}} \right)^{m_i},$$

and using Proposition 3.1.2 we get

$$u(S_\beta) = \omega^{\sum_{i<n} \omega^{\beta_i} \cdot m_i} = \prod_{i<n} \left(\omega^{\omega^{\beta_i}}\right)^{m_i}.$$

This proves $o(t) = u(t)$ for all terms t of $\mathfrak{A}^{\oplus}_{\mathfrak{P}_\alpha}$. q.e.d.

So if $\mathfrak{A}^{\oplus}_\beta$ is an additive ordinal algebra then we can embed $\mathfrak{A}^{\oplus}_\beta$ into any ordinal algebra $\mathfrak{A}_\alpha$ with $\mathrm{ht}(\mathfrak{A}_\alpha) \geq \mathrm{ht}(\mathfrak{A}^{\oplus}_\beta)$ by identifying its generators with the right $\otimes$-products of generators from $\mathfrak{A}_\lambda$. And on the other hand by Definition 3.1.3 and Lemma 3.1.4 for every ordinal algebra $\mathfrak{A}$ there is an additive ordinal algebra $\mathfrak{A}^{\oplus}$ of the same height, whose generators correspond to product terms from $\mathfrak{A}$.

3.2 Terms as Trees

In the following, we shall identify terms in an (additive) order algebra with finite labelled ordered trees $\langle T, \ell \rangle$, where the trees live on ω, *i.e.*, for each tree T there are $m, n < \omega$ such that $T \subseteq m^n$. All of our trees have a root $\bullet$ and the labelling function ℓ is a map from $T \backslash \{\bullet\}$ into $\mathfrak{V}$. When denoting elements of those trees we leave out the node $\bullet$. We assume that the order on the set of immediate successors of a node is reflected in the pictures of trees, they are ordered from left to right according to their values. For example, in Figure 3.1 the tree T_x has two branches, $\langle i_0, i_1 \rangle$ and $\langle j_0 \rangle$, and we have $i_0 < j_0$.

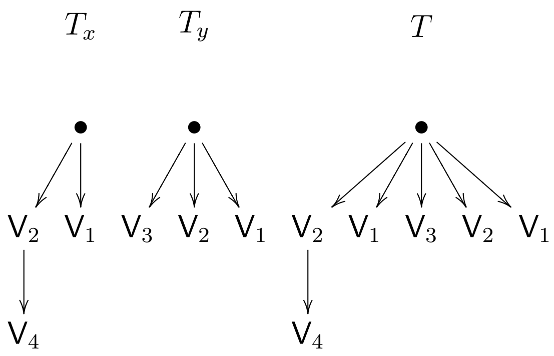

Figure 3.1: Adding the trees for $x = (\mathsf{V}_4 \otimes \mathsf{V}_2) \oplus \mathsf{V}_1$ and $y = \mathsf{V}_3 \oplus \mathsf{V}_2 \oplus \mathsf{V}_1$.

3.2.1. Definition. Let $\mathfrak{A}$ be an ordinal algebra and $\mathfrak{A}^{\oplus}$ an additive ordinal algebra. We recursively associate a labelled tree $\langle T_x, \ell_x \rangle$ to each term c in $\mathfrak{A}$:

1. We identify the variable v with the tree consisting of a root $\bullet$ and one immediate successor node v such that $\ell(v) := \mathsf{v}$.

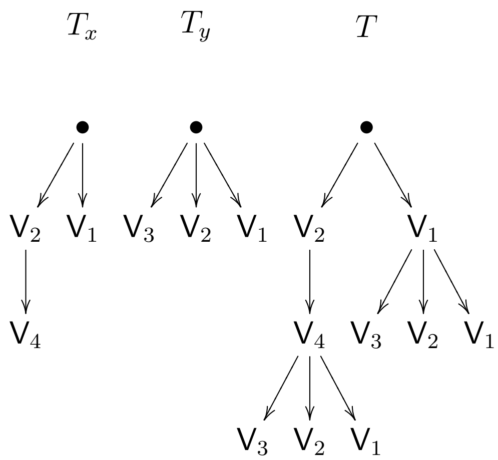

Figure 3.2: Multiplying the trees for $x = (\mathsf{V}_4 \otimes \mathsf{V}_2) \oplus \mathsf{V}_1$ and $y = \mathsf{V}_3 \oplus \mathsf{V}_2 \oplus \mathsf{V}_1$.

2. If $x, y \in \mathfrak{A}$ are represented by $\langle T_x, \ell_x \rangle$ and $\langle T_y, \ell_y \rangle$, respectively, then we represent $x \oplus y$ by defining a tree T as follows: we juxtapose T_x and T_y with a common root and take the union of the labelling functions. An example for $x = (\mathsf{V}_4 \otimes \mathsf{V}_2) \oplus \mathsf{V}_1$ and $y = \mathsf{V}_3 \oplus \mathsf{V}_2 \oplus \mathsf{V}_1$ can be seen in Figure 3.1.

3. If $x, y \in \mathfrak{A}$ are represented by $\langle T_x, \ell_x \rangle$ and $\langle T_y, \ell_y \rangle$, respectively, then we represent $x \otimes y$ by defining a tree T as follows: we start with T_y and glue a copy of T_x to each terminal node of T_x. An example for $x = (\mathsf{V}_4 \otimes \mathsf{V}_2) \oplus \mathsf{V}_1$ and $y = \mathsf{V}_3 \oplus \mathsf{V}_2 \oplus \mathsf{V}_1$ can be seen in Figure 3.2.

This corresponds directly to the representation of ordinal addition and multiplication by finite trees. Note that the order of the successors of a node in the tree is highly relevant, as ordinal addition and multiplication are not commutative. The tree itself describes the operations on the objects that are given by the labelling function.

We likewise recursively associate a labelled tree $\langle T_x, \ell_x \rangle$ to each term c in $\mathfrak{A}^{\oplus}$. This is done the same way as for an ordinal algebra, only part 3. of the definition is superfluous. Again this corresponds directly to the representation of ordinal addition by finite trees.

3.2.2. Definition. If we have a term x in an (additive) ordinal algebra with corresponding tree T_x and a function $v : T_x \backslash (\bullet) \to \mathrm{On}$ then the v-**evaluation** $\mathbf{val}(x, v)$ of x is the ordinal we get if we apply the ordinal operations derived from the tree to the ordinals given by the labelling function v. More formally we define $\mathbf{val}(x, v) : T_x \to \mathrm{On}$ by

1. If t is a terminal node of T_x, then $\mathbf{val}(x, v)(t) := v(t)$.

2. If t is not the root and has successor nodes $\langle s_i\,;\, i < n\rangle$, then $\mathbf{val}(x,v)(t) := (\mathbf{val}(x,v)(s_0) + \ldots + \mathbf{val}(x,v)(s_{n-1})) \cdot v(t)$.

3. If t is the root and has successor nodes $\langle s_i\,;\, i < n\rangle$, then $\mathbf{val}(x,v)(t) := \mathbf{val}(x,v)(s_0) + \ldots + \mathbf{val}(x,v)(s_{n-1})$.

For the evaluation $\mathbf{val}(x,v)(\bullet)$ we simply write $\mathbf{val}(x,v)$.

3.2.3. Lemma. *Let $\mathfrak{A}$ be an ordinal algebra and $\mathfrak{A}^{\oplus}$ an additive ordinal algebra.*

1. *If x is a term in an ordinal algebra with corresponding labelled tree $\langle T_x, \ell_x\rangle$ and v is the function that assigns to each node t the ordinal $o(\ell_x(t))$, then $o(x) = \mathbf{val}(x,v)$.*

2. *If x is a term in an additive ordinal algebra with corresponding labelled tree $\langle T_x, \ell_x\rangle$ and v is the function that assigns to each node t the ordinal $u(\ell_x(t))$, then $u(x) = \mathbf{val}(x,v)$.*

Proof. This follows directly from the definition of o and u, respectively. q.e.d.

We will use these tree representations of elements of an (additive) ordinal algebra in Chapter 4 to define binary operations for measures.

3.3 Measure Assignments

Now we can make the connection between ordinal algebras and measure analysis.

3.3.1. Definition. Let κ be a strong partition cardinal and $\mathfrak{A}$ an (additive) ordinal algebra. A κ-**measure assignment** for $\mathfrak{A}$ is a function $\mathbf{meas}_\kappa : \mathfrak{A} \to \mathcal{P}(\mathcal{P}(\kappa))$ that assigns to terms in $\mathfrak{A}$ wellfounded measures on κ.

If $\mathfrak{A}$ is an ordinal algebra, we call a measure assignment $\mathbf{meas}_\kappa$ for $\mathfrak{A}$ **canonical** up to height κ^λ if $\kappa^\kappa/\mathbf{meas}_\kappa(t) = \kappa^{(o(t))}$ for all terms $t \in \mathfrak{A}$ with $o(t) < \omega$ and $\kappa^\kappa/\mathbf{meas}_\kappa(t) = \kappa^{(o(t)+1)}$ for all terms $t \in \mathfrak{A}$ with $\omega \leq o(t) < \lambda$. If $\mathfrak{A}^{\oplus}$ is an additive ordinal algebra, we call a measure assignment $\mathbf{meas}_\kappa$ for $\mathfrak{A}^{\oplus}$ **canonical** up to height κ^λ if $\kappa^\kappa/\mathbf{meas}_\kappa(t) = \kappa^{(u(t))}$ for all terms $t \in \mathfrak{A}$ with $u(t) < \omega$ and $\kappa^\kappa/\mathbf{meas}_\kappa(t) = \kappa^{(u(t)+1)}$ for all terms $t \in \mathfrak{A}^{\oplus}$ with $\omega \leq u(t) < \lambda$.

Let $\mathbf{meas}_\kappa : \mathfrak{A}_\alpha \to \mathcal{P}(\mathcal{P}(\kappa))$ be a measure assignment for an ordinal algebra $\mathfrak{A}_\alpha$. Since the terms of the induced additive ordinal algebra $\mathfrak{A}^{\oplus}_{\mathfrak{P}_\alpha}$ are also terms of $\mathfrak{A}_\alpha$ we can see $\mathbf{meas}_\kappa$ as a κ-measure assignment $\mathbf{meas}_\kappa : \mathfrak{A}^{\oplus}_{\mathfrak{P}_\alpha} \to \mathcal{P}(\mathcal{P}(\kappa))$ for the additive ordinal algebra $\mathfrak{A}^{\oplus}_{\mathfrak{P}_\alpha}$. To avoid confusion we will write $\mathbf{meas}^{\oplus}_\kappa$ for the measure assignment on $\mathfrak{A}^{\oplus}_{\mathfrak{P}_\alpha}$ that we get this way from a measure assignment $\mathbf{meas}_\kappa$ on $\mathfrak{A}_\alpha$.

Note that canonicity implies a lot of non-obvious claims about the behavior of sums and products of measures: while $o(\mathsf{V}_1 \oplus \mathsf{V}_2) = \omega = o(\mathsf{V}_2)$, there is no *a priori* reason that the measures associated to these two terms should be similar. Canonicity of the measure assignment ensures that they are, in the sense that they give the same ultrapowers. If a measure assignment for an ordinal algebra allows us to change terms into terms that are induced by an additive ordinal algebra, *i.e.*, if we can transform them into sums of $\mathfrak{P}$-terms, then canonicity for the full ordinal algebra reduces to canonicity of the measure assignment for the induced additive ordinal algebra. Which naturally leads to the following definition:

3.3.2. Definition. Let $\mathbf{meas}_\kappa : \mathfrak{A} \to \mathcal{P}(\mathcal{P}(\kappa))$ be a κ-measure assignment for an ordinal algebra $\mathfrak{A}$. Let γ be an ordinal. We write $\mu \simeq \nu$ if the corresponding ultrapowers are the same. We call $\mathbf{meas}_\kappa$ **almost canonical** up to height $\kappa^{(\lambda)}$ if for terms r_1, r_2, t_1, t_2, t_3, where the terms r_1 and r_2 may be empty, and finite sequences of generators $\langle \mathsf{V}_{\alpha_i}\,;\, i < n\rangle$ in $\mathfrak{A}$ we have

1. associativity of $\oplus$ with respect to $\mathbf{meas}_\kappa$, so if $o(r_1)+o(t_1)+o(t_2)+o(t_3)+o(r_2) < \lambda$ then
$$\mathbf{meas}_\kappa((r_1 \oplus (t_1 \oplus (t_2 \oplus t_3))) \oplus r_2) \simeq \mathbf{meas}_\kappa((r_1 \oplus ((t_1 \oplus t_2) \oplus t_3)) \oplus r_2),$$
2. left-distributivity of $\otimes$ with respect to $\mathbf{meas}_\kappa$, so if $o(r_1) + o(t_1)\cdot(o(t_2)+o(t_3)) + o(r_2) < \lambda$ then
$$\mathbf{meas}_\kappa(r_1 \oplus t_1 \otimes (t_2 \oplus t_3) \oplus r_2) \simeq \mathbf{meas}_\kappa(r_1 \oplus t_1 \otimes t_2 \oplus t_1 \otimes t_3) \oplus r_2),$$
3. associativity of $\otimes$ with respect to $\mathbf{meas}_\kappa$, so if $o(r_1) + o(t_1)\cdot o(t_2)\cdot o(t_3) + o(r_2) < \lambda$ then
$$\mathbf{meas}_\kappa((r_1 \oplus (t_1 \otimes (t_2 \otimes t_3))) \oplus r_2) \simeq \mathbf{meas}_\kappa((r_1 \oplus ((t_1 \otimes t_2) \otimes t_3)) \oplus r_2),$$
4. the same behavior for applying $\otimes$ from the right as for ordinal multiplication, so if $(o(r_1) + (o(t_1)+o(t_2))\cdot o(t_3) + o(r_2)) < \lambda$ then
$$\mathbf{meas}_\kappa(r_1 \oplus (t_1 \oplus t_2) \otimes t_3 \oplus r_2) \simeq \mathbf{meas}_\kappa(r_1 \oplus (t_1 \oplus t_2) \oplus r_2),$$
if $o(t_3) = 1$ and with $t_i = t_1$ if $o(t_1) \geq o(t_2)$ and $t_i = t_2$ otherwise
$$\mathbf{meas}_\kappa(r_1 \oplus (t_1 \oplus t_2) \otimes t_3 \oplus r_2) \simeq \mathbf{meas}_\kappa(r_1 \oplus (t_i \otimes t_3) \oplus r_2)$$
if $o(t_3) > 1$ and t_3 is of the form $\bigotimes_{i<n} \mathsf{V}_{\alpha_i}$ with $\langle \mathsf{V}_{\alpha_i}\,;\, i<n\rangle$ a sequence of generators of $\mathfrak{A}$, and

5. the same behavior for n-folded $\otimes$-products as for ordinal multiplication, so if $o(r_1) + o(\mathsf{V}_{\alpha_0} \otimes \ldots \otimes \mathsf{V}_{\alpha_{n-1}}) + o(r_2) < \lambda$ then

$$\mathbf{meas}_\kappa(r_1 \oplus \bigotimes_{i<n} \mathsf{V}_{\alpha_i} \oplus r_2) \simeq \mathbf{meas}_\kappa(r_1 \oplus \bigotimes_{i<m} \mathsf{V}_{\beta_i} \oplus r_2),$$

where $\bigotimes_{i<m} \mathsf{V}_{\beta_i}$ is the $<_{\text{lex}}$-smallest element in $\mathfrak{P}$ with $o(\bigotimes_{i<n} \mathsf{V}_{\alpha_i}) = o(\bigotimes_{i<m} \mathsf{V}_{\beta_i})$.

3.3.3. Lemma. *Let* $\mathbf{meas}_\kappa : \mathfrak{A}_\alpha \to \mathcal{P}(\mathcal{P}(\kappa))$ *be a* κ*-measure assignment for an ordinal algebra* $\mathfrak{A}_\alpha$*. Let* γ *be an ordinal. If* $\mathbf{meas}_\kappa$ *is almost canonical up to height* $\kappa^{(\lambda)}$ *and the* κ*-measure assignment* $\mathbf{meas}^\oplus_\kappa$ *for the induced additive ordinal algebra* $\mathfrak{A}^\oplus_{\mathfrak{P}_\alpha}$ *we get from Definition 3.1.3 is canonical up to height* $\kappa^{(\lambda)}$ *then* $\mathbf{meas}_\kappa$ *is in fact canonical up to height* $\kappa^{(\lambda)}$*.*

Proof. We explicitly defined "almost canonical" such that for all terms $t \in \mathfrak{A}_\alpha$ we have $\mathbf{meas}_\kappa(t) \simeq \mathbf{meas}_\kappa(t')$, where $t' \in \mathfrak{A}_\alpha$ is an $\oplus$-sum of $\mathfrak{P}$-minimal $\otimes$-products with $o(t) = o(t')$. That means we can interpret t' as a term of $\mathfrak{A}^\oplus_{\mathfrak{P}_\alpha}$. If $\mathbf{meas}^\oplus_\kappa$ is canonical we get for terms t with $o(t) < \omega$

$$\kappa^\kappa/\mathbf{meas}_\kappa(t) = \kappa^\kappa/\mathbf{meas}_\kappa(t') = \kappa^\kappa/\mathbf{meas}^\oplus_\kappa(t') = \kappa^{(u(t'))} = \kappa^{(u(t))} = \kappa^{(o(t))}$$

and for terms t with $o(t) \geq \omega$

$$\kappa^\kappa/\mathbf{meas}_\kappa(t) = \kappa^\kappa/\mathbf{meas}_\kappa(t') = \kappa^\kappa/\mathbf{meas}^\oplus_\kappa(t') = \kappa^{(u(t')+1)} = \kappa^{(u(t)+1)} = \kappa^{(o(t)+1)}.$$

This is due to the canonicity of $\mathbf{meas}^\oplus_\kappa$ and the fact that by Lemma 3.1.4 we have $u(t) = o(t)$ for all terms t of $\mathfrak{A}^\oplus_{\mathfrak{P}_\alpha}$. q.e.d.

So to prove that an almost canonical κ-measure assignment for an ordinal algebra $\mathfrak{A}$ is in fact canonical up to κ^λ we need only to check canonicity up to κ^λ for the induced additive ordinal algebra:

3.3.4. Corollary. *Let* $\mathbf{meas}_\kappa : \mathfrak{A}_\alpha \to \mathcal{P}(\mathcal{P}(\kappa))$ *be a* κ*-measure assignment for an ordinal algebra* $\mathfrak{A}_\alpha$*. Let* γ *be an ordinal. Assume* $\mathbf{meas}_\kappa$ *is almost canonical up to height* $\kappa^{(\lambda)}$*. If for all terms* x *of* $\mathfrak{A}_\alpha$ *with* $o(x) < \kappa^\lambda$ *that are of the form* $x = \bigoplus_{i<n} x_i$*, where the* x_i *are products of generators, i.e.,* $x_i = \bigotimes_{j<n_i} \mathsf{v}_{i,j}$ *with* $\mathsf{v}_{i,j} \in \mathfrak{V}$*, we have*

1. $\kappa^\kappa/\mathbf{meas}_\kappa(x) = \kappa^{(n)}$ *if* $o(x) < \omega$*, and*
2. $\kappa^\kappa/\mathbf{meas}_\kappa(x) = \kappa^{(\sum_{i<n}\prod_{j<n_i} o(\mathsf{v}_{i,j})+1)}$ *if* $o(x) \geq \omega$*,*

then $\mathbf{meas}_\kappa$ *is canonical up to height* $\kappa^{(\lambda)}$*.*

Proof. This follows directly from Lemma 3.3.3 and the definition of canonicity for measure assignments for additive ordinal algebras. q.e.d.

Chapter 4

Order Measures

We now introduce the measures that will enable us to define a canonical measure assignment under AD.

4.1 Definition of Order Measures and the Weak Lift

4.1.1. DEFINITION. Let $\mathfrak{A}$ be an (additive) ordinal algebra. An **order type** is a function $\mathsf{ot} : T\backslash\{\bullet\} \to \mathrm{On}$ where T is a finite tree with root $\bullet$. A **germ** is a function germ defined on $T\backslash\{\bullet\}$ assigning a measure on some ordinal to each non-root node of the tree T. We say that a germ germ **lives on** an order type ot if for each non-root node v, $\mathsf{germ}(v)$ is a measure on $\mathsf{ot}(v)$ that contains end segments. A pair of functions $\mathrm{GA}_{\mathfrak{A}} = \langle \mathbf{germ}, \mathbf{ot}\rangle$ is a **germ assignment** on $\mathfrak{A}$ if **ot** and **germ** assign order types and germs to elements of $\mathfrak{A}$, respectively, such that the following conditions hold.

1. For $x \in \mathfrak{A}$ both germ $\mathbf{germ}(x)$ and $\mathbf{ot}(x)$ have domain $T_x\backslash\{\bullet\}$ and $\mathbf{germ}(x)$ lives on $\mathbf{ot}(x)$. Note that for a generator v, the order type $\mathbf{ot}(\mathsf{v})$ is essentially one ordinal, and the germ $\mathbf{germ}(\mathsf{v})$ is a measure on this ordinal. So in the case of generators we shall identify order type and germ with the ordinal and the measure, respectively.

2. For $x \in \mathfrak{A}$ with corresponding labelled tree $\langle T_x, \ell_x\rangle$ we have $\mathbf{germ}(x)(v) = \mathbf{germ}(\ell_x(v))$ and $\mathbf{ot}(x)(v) = \mathbf{ot}(\ell_x(v))$ for all non-root nodes v of T_x. So the order type assignment for the generators of $\mathfrak{A}$ defines all order types for terms in $\mathfrak{A}$ and the same goes for the germ assignment.

3. The order type of the first generator is 1, *i.e.*, $\mathbf{ot}(\mathsf{V}_0) = 1$, and the order types for all other generators are limit ordinals, *i.e.*, $\mathbf{ot}(\mathsf{V}_\alpha) \in \mathrm{Lim}$ for $\alpha > 0$. Furthermore, for generators V_α and V_β in $\mathfrak{A}$ with $\alpha < \beta$ we have

$\mathbf{ot}(\mathsf{V}_\alpha) < \mathbf{ot}(\mathsf{V}_\beta)$. This means the order type **ot** behaves like the evaluation function o for (additive) ordinal algebras.

Let κ be a cardinal, a germ assignment $\mathrm{GA}_{\mathfrak{A}} = \langle \mathbf{germ}, \mathbf{ot} \rangle$ is a κ-**germ assignment** if the range of **ot** is less than κ and

$$\kappa^{\mathbf{ot}(\mathsf{v}_0)\cdot\ldots\cdot\mathbf{ot}(\mathsf{v}_{n-1})}/\mathbf{germ}(\mathsf{v}_0) \times \ldots \times \mathbf{germ}(\mathsf{v}_{n-1}) = \kappa$$

holds for all finite sequences $\langle \mathsf{v}_i\,;\, i < n \rangle$ of generators of $\mathfrak{A}$.

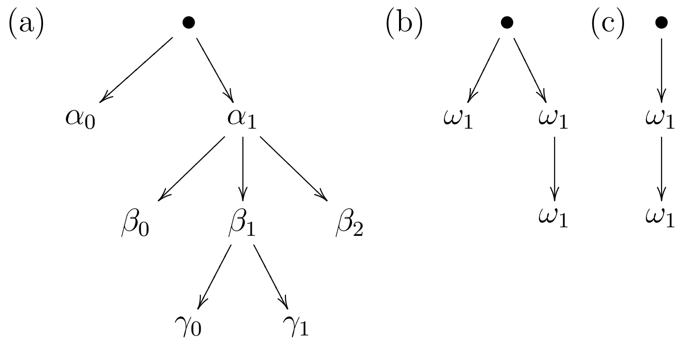

Figure 4.1: (a). A tree representation of $\alpha_0 + (\beta_0 + (\gamma_0 + \gamma_1) \cdot \beta_1 + \beta_2) \cdot \alpha_1$. (b) & (c). Two (different) tree representations of $\omega_1 + (\omega_1)^2 = (\omega_1)^2$.

As we mentioned before finite trees (with ordinals as nodes) can be seen as representing ordinal addition and multiplication, Definition 3.2.2 defined the function **val** accordingly. So the order type $\mathbf{ot}(x)$ can be evaluated as the single ordinal $\mathbf{val}(x, \mathbf{ot}(x))$, computed recursively from the values of $\mathbf{ot}(v)$ for the nodes v of T_x. Examples can be seen in Figure 4.1. In the following we shall often identify $\mathbf{ot}(x)$ with the ordinal $\mathbf{val}(x, \mathbf{ot}(x))$, it will be clear from the context which interpretation we mean in a specific situation. We say that the **range** of a germ assignment $\mathrm{GA}_{\mathfrak{A}}$ is the supremum of the ordinals $\mathbf{ot}(x)$ for all $x \in \mathfrak{A}$.

4.1.2. Remark. In a finite tree T_x we call the rightmost immediate successor v of the root the **trailing node**. If one considers the tree as representing an ordinal then the ordinal $\mathbf{ot}(\mathsf{v})$ corresponds to the rightmost term in the ordinal presentation of $\mathbf{ot}(x)$. By that we mean the following: Every term x is of the form $x = t \oplus r \cdot \mathsf{v}$, with t and r (possibly empty) terms. If we evaluate x according to $\mathbf{val}(x, \mathbf{ot}(x))$ then we we get an ordinal of the form $\mathbf{val}(t, \mathbf{ot}(t)) + \mathbf{val}(r, \mathbf{ot}(r)) \cdot \mathbf{val}(\mathsf{v}, \mathbf{ot}(\mathsf{v}))$. So if $\mathbf{val}(\mathsf{v}, \mathbf{ot}(\mathsf{v}))$ is a limit ordinal then the cofinality of the ordinal $\mathbf{val}(x, \mathbf{ot}(x))$ is the same as that of the ordinal $\mathbf{val}(\mathsf{v}, \mathbf{ot}(\mathsf{v}))$. Using our convention to see $\mathbf{ot}(x)$ as an ordinal we thus have $\mathrm{cf}(\mathbf{ot}(x)) = \mathrm{cf}(\mathbf{ot}(\mathsf{v}))$ in this case. We will later use this fact in Section 7.2 to compute the cofinality of cardinals in the reach of our canonical measure assignment.

4.1.3. Definition. Let $\mathfrak{A}$ be an (additive) ordinal algebra, $x \in \mathfrak{A}$ a term in $\mathfrak{A}$ with derived labelled tree $\langle T_x, \ell_x \rangle$, and $\mathrm{GA}_{\mathfrak{A}} = \langle \mathbf{germ}, \mathbf{ot} \rangle$ a germ assignment on $\mathfrak{A}$. We will now define the restriction of a function $f : \mathbf{ot}(x) \to \mathrm{On}$ to terminal nodes of T_x. This will be used in Definition 4.1.4.

Let A_x be the set of tuples $\langle j_k, \alpha_k, j_{k-1}, \alpha_{k-1}, \cdots, j_0, \alpha_0 \rangle$, where $\langle j_0, \cdots, j_k \rangle$ is a terminal node of T_x and $\alpha_i < \mathbf{ot}(\ell_x(\langle j_0, \cdots, j_i \rangle))$ for all $i < k$. Ordered with $<_{\mathrm{rlex}}$ the set A_x has ordertype $\mathbf{ot}(x)$, let $\pi_{<_{\mathrm{lex}}}(s)$ denote the $<_{\mathrm{lex}}$ rank of $s \in A_x$.

For each terminal node $t = \langle j_i \,; i < n \rangle$ in T_x we define

$$\lambda_t := \mathbf{ot}(\ell_x(\langle j_0, \ldots, j_{n-1} \rangle)) \times \ldots \times \mathbf{ot}(\ell_x(\langle j_0 \rangle)).$$

Using the isomorphism between $\mathbf{ot}(\ell_x(\langle j_0, \ldots, j_{n-1} \rangle)) \times \ldots \times \mathbf{ot}(\ell_x(\langle j_0 \rangle)))$ with order $<_{\mathrm{rlex}}$ and the ordinal $\mathbf{ot}(\ell_x(\langle j_0, \ldots, j_{n-1} \rangle)) \cdot \ldots \cdot \mathbf{ot}(\ell_x(\langle j_0 \rangle))$ we can identify λ_t with this ordinal.

Let $f : \mathbf{ot}(x) \to \mathrm{On}$ be a function of discontinuous type $\mathbf{ot}(x)$. Using π we can consider f to be a function on the set A_x and define its restriction $f_t : \lambda_t \to \mathrm{On}$ to the terminal node $t = \langle j_i \,; i < n \rangle$ by

$$f_t(\alpha_{n-1}, \ldots, \alpha_0) := f(\pi_{<_{\mathrm{lex}}}(\langle j_{n-1}, \alpha_{n-1}, \cdots, j_0, \alpha_0 \rangle)).$$

Then f_t is a function of discontinuous type λ_t. By Lemma 1.8.1 we have a bijection between the set of functions of discontinuous type and the set of functions of continuous type, let $\hat{f}_t$ be the image of f_t under this bijection. That means we have $\hat{f}_t(0) := f_t(0)$, $\hat{f}_t(\gamma) := \sup_{\alpha<\gamma} f_t(\alpha)$ for limit ordinals $\gamma < \lambda_t$ and $\hat{f}_t(\alpha+1) := f_t(\alpha)$ for successor ordinals $\alpha+1 < \lambda_t$. The function $\hat{f}_t : \lambda_t \to \mathrm{On}$ is by construction of continuous type λ_t and has the same supremum as the function f_t, *i.e.*, $\sup \hat{f}_t = \sup f_t$.

We also assign to each terminal node $t = \langle j_i \,; i < n \rangle$ a measure $\mathbf{germ}_t$ by

$$\mathbf{germ}_t := \mathbf{germ}(x)(\langle j_0, \ldots, j_{n-1} \rangle) \times \ldots \times \mathbf{germ}(x)(\langle j_0 \rangle),$$

where $\times$ denotes the normal product of measures. If, according to Remark 1.4.6, we interpret $\mathbf{germ}_t$ as a measure on the ordinal λ_t we of course use $\times'$ instead.

In order to keep our formulas to a reasonable length we introduce the following notation: For x an $\mathfrak{A}$-term, T_x the corresponding tree, $S \subseteq T_x$ a subtree of T_x, $\langle t_i \,; i < n \rangle$ the set of terminal nodes of S, and $f : \mathbf{ot}(x) \to \mathrm{On}$ a function we write

$$\overrightarrow{[f]_S} := \langle [f_{t_0}]_{\mathbf{germ}_{t_0}}, \ldots, [f_{t_n}]_{\mathbf{germ}_{t_n}} \rangle.$$

Let x be a term in an (additive) ordinal algebra $\mathfrak{A}$, $\mathrm{GA}_{\mathfrak{A}} = \langle \mathbf{germ}, \mathbf{ot} \rangle$ a κ-germ assignment, and $f : \mathbf{ot}(x) \to \mathrm{On}$ a function of continuous type $\mathbf{ot}(x)$. The properties of the function f_t depend strongly on the form of the term x and the position of the terminal node t in the tree T_x. Let us state some results that follow rather directly from the definition of f_t.

We have $\sup f_t = \sup f$ if t is the rightmost terminal node. If x is of the form $y_1 \oplus y_2$ then the functions corresponding to terminal nodes in T_{y_1} are smaller than those corresponding to terminal nodes in T_{y_2}, *i.e.*, we have $f_t(\alpha) < f_s(\beta)$ for all terminal nodes $t \in T_{y_1}$, $s \in T_{y_2}$ and $\alpha \in \lambda_t$, $\beta \in \lambda_s$.

If $x = r_1 \oplus t_1 \oplus r_2$ and $y = r_1 \oplus t_2 \oplus r_2$ are terms with $\mathbf{ot}(t_1) = \mathbf{ot}(t_2)$ then the functions corresponding to terminal nodes in the subtrees T_{r_1} and T_{r_1} are equal to their counterparts corresponding to terminal nodes in T_y. That means if t is a terminal node in T_x that is in the subtree of T_x defined by r_1 and t' is the analogous terminal node in T_y then, since A_{r_1} is an initial part of A_x and A_y, we have $f_t = f_{t'}$. And if t is a terminal node in T_x that is in the subtree of T_x defined by r_2 and t' is the analogous terminal node in T_y then, since the order type of $A_{r1\oplus t_1}$ is the same as that of $A_{r1\oplus t_2}$, we also have $f_t = f_{t'}$.

If $x = r_1 \oplus \bigotimes_{i<n} \mathsf{V}_{\alpha_i} \oplus r_2$ and $y = r_1 \oplus \bigotimes_{i<m} \mathsf{V}_{\beta_i} \oplus r_2$ are terms with $\langle \mathsf{V}_{\alpha_i}\,;\, i < n\rangle$ and $\langle \mathsf{V}_{\beta_i}\,;\, i < n\rangle$ being sequences of generators of $\mathfrak{A}$ with $\mathbf{ot}(\bigotimes_{i<n} \mathsf{V}_{\alpha_i}) = \mathbf{ot}(\bigotimes_{i<m} \mathsf{V}_{\beta_i})$ then $f_t = f_s$, where t is the terminal node in T_x corresponding to $\bigotimes_{i<n} \mathsf{V}_{\alpha_i}$ and s is the terminal node in T_y corresponding to $\bigotimes_{i<m} \mathsf{V}_{\beta_i}$.

If $x = r_1 \oplus (t \otimes s) \oplus r_2$, with $s = \bigotimes_{i<n} \mathsf{V}_{\alpha_i}$, order type $\mathbf{ot}(s) \neq 1$, and $\langle \mathsf{V}_{\alpha_i}\,;\, i < n\rangle$ being a sequence of generators of $\mathfrak{A}$, then the supremum of all functions f_t derived from terminal nodes in $T_{t\otimes s}$ is the same, *i.e.*, $\sup f_t = \sup f_s$ for all terminal nodes that are in the subtree $T_{t\otimes s}$ of T_x.

4.1.4. Definition. Let κ be a cardinal, $\mathfrak{A}$ an (additive) ordinal algebra, and $\mathrm{GA}_{\mathfrak{A}} = \langle \mathbf{germ}, \mathbf{ot}\rangle$ a κ-germ assignment. For terms x in $\mathfrak{A}$ we define the κ-**lift** of the germ $\mathbf{germ}(x)$ by

$$\begin{aligned}\mathbf{lift}_\kappa(\mathbf{germ}(x)) := \{A \subseteq \kappa;\ & \text{there is a club } C \subseteq \kappa \text{ such that for all } f \in \mathfrak{D}_C^{\mathbf{ot}(x)} \\ & \text{we have } \ulcorner [\hat{f}_{t_0}]_{\mathbf{germ}_{t_0}}, \ldots, [\hat{f}_{t_{n-1}}]_{\mathbf{germ}_{t_{n-1}}} \urcorner \in A, \\ & \text{where } \langle t_i\,;\, i < n\rangle \text{ are the terminal nodes of } T_x\}. \\ = \{A \subseteq \kappa;\ & \text{there is a club set } C \subseteq \kappa \text{ such that} \\ & \text{for all } f \in \mathfrak{D}_C^{\mathbf{ot}(x)} \text{ we have } \ulcorner \overrightarrow{[\hat{f}]_{T_x}} \urcorner \in A\}.\end{aligned}$$

4.1.5. Theorem (Lifting Theorem). *Let κ be a weak partition cardinal, $\mathfrak{A}$ an (additive) ordinal algebra, and $\mathrm{GA}_{\mathfrak{A}} = \langle \mathbf{germ}, \mathbf{ot}\rangle$ a κ-germ assignment on $\mathfrak{A}$. Let x be a term from $\mathfrak{A}$, then $\mathbf{lift}_\kappa(\mathbf{germ}(x))$ is a κ-complete measure on κ.*

Proof. From the definition of $\mathbf{lift}_\kappa(\mathbf{germ}(x))$ and the properties of club sets follows directly that $\emptyset \notin \mathbf{lift}_\kappa(\mathbf{germ}(x))$, $\kappa \in \mathbf{lift}_\kappa(\mathbf{germ}(x))$, and that $\mathbf{lift}_\kappa(\mathbf{germ}(x))$ is closed under finite unions and supersets, so $\mathbf{lift}_\kappa(\mathbf{germ}(x))$ is a filter.

Now fix an ordinal $\gamma < \kappa$ and a γ-partition $\{X_\alpha\,;\, \alpha < \gamma\}$ of κ. Since κ is a cardinal we have $\gamma + \gamma < \kappa$. Let $\langle t_i\,;\, i < n\rangle$ be the terminal nodes of T_x. We define a γ-partition P of $\mathfrak{D}_\kappa^{\mathbf{ot}(x)}$ by

$$P(f) = \alpha \quad \Leftrightarrow \quad \ulcorner [\hat{f}_{t_0}]_{\mathbf{germ}_{t_0}}, \ldots, [\hat{f}_{t_{n-1}}]_{\mathbf{germ}_{t_{n-1}}} \urcorner \in X_\alpha.$$

From Lemma 1.5.4 and Lemma 1.9.1 we know that there is a homogeneous club set $C \subseteq \kappa$ for this partition, *i.e.*, there exists a club set $C \subseteq \kappa$ such that for all $f \in \mathfrak{D}_C^{\mathbf{ot}(x)}$ we have $\ulcorner [\hat{f}_{t_0}]_{\mathbf{germ}_{t_0}}, \ldots, [\hat{f}_{t_{n-1}}]_{\mathbf{germ}_{t_{n-1}}} \urcorner \in X_\alpha$ for some $\alpha < \gamma$. But this means that X_α is in $\mathbf{lift}_\kappa(\mathbf{germ}(x))$, so by Lemma 1.2.3 the filter $\mathbf{lift}_\kappa(\mathbf{germ}(x))$ is κ-complete. The ultrafilter property of $\mathbf{lift}_\kappa(\mathbf{germ}(x))$ is just the case $\gamma = 2$ of this argument, since then either $A \in \mathbf{lift}_\kappa(\mathbf{germ}(x))$ or $\kappa \backslash A \in \mathbf{lift}_\kappa(\mathbf{germ}(x))$.

q.e.d.

Since the weak partition property of κ is used to prove that the lift $\mathbf{lift}_\kappa$ of a germ is a measure we call the operation $\mathbf{lift}_\kappa$ also the **weak lift**. In Section 4.2 we will deal with the corresponding result that we get from the strong partition property of a cardinal. With Theorem 4.1.5 we can now define the most important notion for our measure analysis under AD:

4.1.6. Definition. Let κ be a weak partition cardinal, $\mathfrak{A}$ an (additive) ordinal algebra, and $\mathrm{GA}_{\mathfrak{A}} = \langle \mathbf{germ}, \mathbf{ot} \rangle$ a κ-germ assignment. We call the κ-lift $\mathbf{lift}_\kappa(\mathbf{germ}(x))$ of a germ $\mathbf{germ}(x)$ an **order measure** on κ. The operations $\oplus$ and $\otimes$ from the ordinal algebra induce corresponding binary operations on order measures, we will denote them with the same symbols: Let $\mu = \mathbf{lift}_\kappa(\mathbf{germ}(x))$ and $\nu = \mathbf{lift}_\kappa(\mathbf{germ}(y))$ be order measures. Then the **sum of the order measures** μ **and** ν, written $\mu \oplus \nu$, is the order measure $\mathbf{lift}_\kappa(\mathbf{germ}(x \oplus y))$ and the **product of the order measures** μ **and** ν, written $\mu \otimes \nu$, is the order measure $\mathbf{lift}_\kappa(\mathbf{germ}(x \otimes y))$.

Let $\langle \mathsf{V}_{\alpha_i}\,;\, i < n \rangle$ be a sequence of $\mathfrak{A}$-generators. If x is of the form $\bigotimes_{i<n} \mathsf{V}_{\alpha_i}$, *i.e.*, the order measure $\mathbf{lift}_\kappa(\mathbf{germ}(x))$ is generated by a single product measure $\mathbf{germ}_t$ on some ordinal λ_t, we call $\mathbf{lift}_\kappa(\mathbf{germ}(x)$ a **basic order measure**. If $\mu = \mathbf{lift}_\kappa(\mathbf{germ}(x)$ is a basic order measure we often write $\mathbf{germ}_\mu$ for its germ $\mathbf{germ}(x)$, $\mathbf{ot}_\mu$ for its ordertype $\mathbf{ot}(x)$, and $\mathbf{lift}_\kappa(\mathbf{germ}_\mu)$ for $\mathbf{lift}_\kappa(\mathbf{germ}(x))$. For example, if v is a generator of $\mathfrak{A}$ and $\mathbf{germ}(\mathsf{v}) = \mathcal{C}_{\omega_1}^{\omega}$ we write $\mathbf{lift}_\kappa(\mathcal{C}_{\omega_1}^{\omega})$ instead of $\mathbf{lift}_\kappa(\mathbf{germ}(\mathsf{v}))$. If μ and ν are basic order measures then $\mu \otimes \nu$ is again a basic order measure with germ $\mathbf{germ}_{\mu \otimes \nu} = \mathbf{germ}_\mu \times \mathbf{germ}_\nu$ and order type $\mathbf{ot}_{\mu \otimes \nu} = \mathbf{ot}_\mu \cdot \mathbf{ot}_\nu$.

If $\alpha + 1$ is a successor ordinal, then the only measure on it that contains end segments is the measure $\mu_{\{\alpha\}}$. The measure $\mathbf{lift}_\kappa(\mu_{\{\alpha\}})$ on κ generated by $\mu_{\{\alpha\}}$ is the same for all $\alpha + 1 < \kappa$, see Lemma 4.4.1. This is one of the reasons we restricted the order types for generators to the set $\{1\} \cup \mathrm{Lim}$. Note also that there is no measure containing end segments on ω, so the next possible order type after 1 is ω_1.

Now we will show that the lift of a normal measure on a regular cardinal is itself a normal measure. In Section 7.1 we will use this fact when, under the assumption that our measure assignment is canonical, we compute inductively all regular cardinals below the supremum of the projective ordinals.

4.1.7. LEMMA. *Let κ be a weak partition cardinal that is closed under ultrapowers and μ a normal measure on a regular cardinal $\varrho < \kappa$ such that the ultrapower κ^ϱ/μ is wellfounded, then* $\mathbf{lift}_\kappa(\mu)$ *is the ϱ-cofinal normal measure on κ.*

Proof. By Lemma 2.2.15 we can assume that all club set we are working with are closed under μ. And by Lemma 2.2.16 we have $[f]_\mu = \sup f$ for all functions $f \in \mathfrak{C}^\varrho_C$ of continuous type ϱ. So, for any club set $C \subseteq \kappa$ and any limit point α of C of cofinality ϱ, there is a function $f : \varrho \to C$ of continuous type ϱ such that $\alpha = \sup f = [f]_\mu$. On the other hand, for any function $f : \varrho \to C$ of continuous type ϱ we have $[f]_\mu = \sup f$ and since ϱ is regular this means that $[f]_\mu$ is an element of C with cofinality ϱ. By definition of $\mathbf{lift}_\kappa(\mu, \varrho)$ we get that $A \subseteq \kappa$ is an element of $\mathbf{lift}_\kappa(\mu, \varrho)$ if and only if there is a club set $C \subseteq \kappa$ such that all limit point α of C of cofinality ϱ are elements of A, *i.e.*, $\mathbf{lift}_\kappa(\mu, \varrho)$ is the ϱ-cofinal normal measure on κ. q.e.d.

4.2 The Strong Lift

In a similar manner to Theorem 4.1.5 we can use the strong partition property of a cardinal κ to lift a measure on κ to its ultrapower. If κ has the strong partition property, μ is a measure on κ and the ultrapower κ^κ/μ is wellfounded then we define

$$\mathbf{slift}_\kappa(\mu) \;:=\; \{A \subseteq \kappa^\kappa/\mu \;;\; \text{there is a club set } C \subseteq \kappa \text{ such that for all } f \in \mathfrak{D}^\kappa_C \text{ we have } [f]_\mu \in A\}.$$

Note that we used functions of continuous type in the definition of the (weak) lift but functions of discontinuous type in that of the strong lift.

4.2.1. THEOREM (STRONG LIFTING THEOREM). *If κ is a strong partition cardinal and μ a measure on κ such that the ultrapower κ^κ/μ is wellfounded, then* $\mathbf{slift}_\kappa(\mu)$ *is a κ-complete measure on κ^κ/μ that contains end segments.*

Proof. From the definition of $\mathbf{slift}_\kappa(\mu)$ it is clear that $\emptyset \notin \mathbf{slift}_\kappa(\mu)$, $\kappa^\kappa/\mu \in \mathbf{slift}_\kappa(\mu)$, and that $\mathbf{slift}_\kappa(\mu)$ is closed under finite unions and supersets, so $\mathbf{slift}_\kappa(\mu)$ is a filter.

We will use Lemma 1.2.3 to show that $\mathbf{slift}_\kappa(\vec{\mu}, \vec{\varrho})$ is a κ-complete measure. Fix an ordinal $\gamma < \kappa$ and a γ-partition $\{X_\alpha \,;\, \alpha < \gamma\}$ of κ^κ/μ. Since κ is a cardinal we have $\gamma + \gamma < \kappa$. We define a γ-partition P of $\mathfrak{D}^\kappa_\kappa$ by

$$P(f) = \alpha \quad :\Leftrightarrow \quad [f]_\mu \in X_\alpha.$$

From Lemma 1.5.4 and Lemma 1.9.2 we know that there is a homogeneous club set $C \subseteq \kappa$ for this partition, *i.e.*, there exists a club set $C \subseteq \kappa$ such that for all $f \in \mathfrak{D}^\kappa_C$ we have $[f]_\mu \in X_\alpha$ for some $\alpha < \gamma$. But this means that X_α is in $\mathbf{slift}_\kappa(\mu)$, so the filter $\mathbf{slift}_\kappa(\mu)$ is κ-complete. The ultrafilter property of $\mathbf{slift}_\kappa(\mu)$ is just the case $\gamma = 2$ of this argument, since then either $A \in \mathbf{slift}_\kappa(\mu)$ or $\kappa^\kappa/\mu \backslash A \in \mathbf{slift}_\kappa(\mu)$.

Now we show that $\mathbf{slift}_\kappa(\mu)$ contains end segments. Towards a contradiction we assume that there exists a function $g : \kappa \to \kappa$ and a club set $C \subseteq \kappa$ such that $[f]_\mu < [g]_\mu$ for all functions $f \in \mathfrak{D}^\kappa_C$. We define a function $f : \kappa \to C$ by $f(0) :=$ ωth element of C larger than $g(0)$, $f(\alpha+1) :=$ ωth element of C that is larger than $\max\{g(\alpha+1), f(\alpha)\}$ for $\alpha < \kappa$, and $f(\lambda) :=$ ωth element of C that is larger than $\max\{\sup_{\alpha<\lambda} f(\alpha), g(\lambda)\}$ for limit ordinals $\lambda < \kappa$. Then f is of discontinuous type κ with range C and for all $\alpha < \kappa$ we have $g(\alpha) < f(\alpha)$. So $[g]_\mu < [f]_\mu$, which contradicts our assumption on g. That means for all $g : \kappa \to \kappa$ exists a club set $C \subseteq \kappa$ such that $[g]_\mu \leq [f]_\mu$ for all functions $f \in \mathfrak{D}^\kappa_C$ and thus $\mathbf{slift}_\kappa(\mu)$ contains end segments. q.e.d.

The measure assignment we are working toward will be defined in Section 4.5 by an iteration of applications of the weak lift and the strong lift. In Section 7.2 we will compute the cofinality of cardinals in the reach of this measure assignment. An important part of this computation is the identification of regular cardinals. If there is a normal measure that contains end segments on a cardinal then by lemmas 1.2.6 and 1.2.4 that cardinal is regular. So we need a way to identify the normal order measures. In Lemma 4.2.2 we showed that the weak lift of a normal measure is itself a normal measure, now we will show that also the strong lift of a normal measure is a normal measure. The following lemma is a reproduction of Lemma 15 from [JaLö06].

4.2.2. Lemma (Jackson-Löwe). *Let $\kappa > \omega_1$ be a strong partition cardinal and μ a semi-normal measure on κ. If the ultrapower κ^κ/μ is wellfounded then $\mathbf{slift}_\kappa(\mu)$ is a normal measure on $\gamma := \kappa^\kappa/\mu$.*

Proof. We will use Lemma 1.2.7 to show that the measure μ is normal. So let $F : \gamma \to \gamma$ be a function that is regressive on $X \in \mathbf{slift}_\kappa(\mu)$. By definition of the strong lift that means there is a club set $C \subseteq \kappa$ such that $F([f]_\mu) < [f]_\mu$ holds for all $f : \kappa \to C$ of discontinuous type κ. We can assume that C contains only limit ordinals greater ω_1. Let S be the set of tuples $\langle f, g\rangle$, where $f, g : \kappa \to \kappa$ are functions of discontinuous type κ such that $f(\alpha) < g(\alpha) < f(\alpha+1)$ is true for all $\alpha < \kappa$.

First we partition the set S according to whether

$$F([g]_\mu) < [f]_\mu$$

holds or not. By Lemma 1.9.3 exist a homogeneous club set for this partition, without loss of generality we can assume that C is this club set.

If C is homogeneous for the contrary side of this partition then for all $f, g \in \mathfrak{D}^\kappa_C$ with $\langle f, g\rangle$ in S we have $[f]_\mu \leq F([g]_\mu)$. Let $g : \kappa \to C_{\mathrm{lim}}$ be a function of discontinuous type κ into the set of closure points of C. We know $F([g]_\mu) < [g]_\mu$, let $f \in \mathfrak{D}^\kappa_C$ be a function of discontinuous type κ with $F([g]_\mu) < [f]_\mu < [g]_\mu$. From Lemma 1.8.4 we get $f', g' \in \mathfrak{D}^\kappa_C$ with $[f']_\mu = [f]_\mu$, $[g']_\mu = [g]_\mu$, and $\langle f', g' \rangle$ in S. So we get $F([g]_\mu) < [f]_\mu \leq F([g]_\mu)$, a contradiction. So C must be homogeneous for the stated side.

Now we partition the set S according to whether

$$F([f]_\mu) \leq F([g]_\mu)$$

holds or not. Again by Lemma 1.9.3 exist a homogeneous club set for this partition and we can assume that C is this club set. We will show that C has to be homogeneous for the stated side of the partition. We define a sequence $\langle f_i\,;\, i \in \omega \rangle$ of functions by

1. Let $f_0(0)$ be the ωth element of C.
2. For $\alpha > 0$ let $f_0(\alpha)$ be the ωth element of C greater than $\sup_{i<\omega} \sup_{\beta<\alpha} f_i(\beta)$.
3. For $i > 0$ and $\alpha \in \kappa$ let $f_i(\alpha)$ be the ωth element of C greater $\sup_{j<i} f_j(\alpha)$.

Since $\kappa > \omega_1$ is regular this sequence is welldefined and all f_i are of discontinuous type κ with range C. Furthermore if $j < i$ then $f_j(\alpha) < f_i(\alpha) < f_{j+1}(\alpha + 1)$ for all $\alpha < \kappa$, so the tuple $\langle f_j, f_i \rangle$ is an element of S. So if C would be homogeneous for the contrary side we would get an infinite descending sequence

$$F([f_0]_\mu) > F([f_1]_\mu) > \ldots > F([f_i]_\mu) > F([f_{i+1}]_\mu) > \ldots,$$

a contradiction to the wellfoundedness of the $<$-relation on the ordinals.

At last we partition the set S according to whether

$$F([f]_\mu) = F([g]_\mu)$$

holds or not, get a homogeneous club set for this partition by Lemma 1.9.3 and assume that C is this club set.

If C is homogeneous for the stated side of this partition, then we are done since then Lemma 1.8.4 implies that for any $f, g \in \mathfrak{D}^\kappa_C$ we have $F([f]_\mu) = F([g]_\mu)$. Meaning the function F is $\mathbf{slift}_\kappa(\mu)$-almost constant, which in turn by Lemma 1.2.7 implies that the measure $\mathbf{slift}_\kappa(\mu)$ is normal.

So suppose C is homogeneous for the contrary side of the partition. Let δ be an ordinal such that $\delta = [k]_\mu$ for some $k \in \mathfrak{D}^\kappa_C$ and fix $f \in \mathfrak{D}^\kappa_C$ with $[f]_\mu > \delta$.

Let $h\colon \{(\alpha, \beta)\colon \alpha < f(\beta)\} \to C$ be of uniform cofinality ω, discontinuous, and order-preserving with respect to reverse lexicographic ordering. Define a map $\pi\colon [f]_\mu \to \delta$ as follows: For $\gamma = [g]_\mu < [f]_\mu$ let $\pi(\gamma) := [g']_\mu$, where $g'(\beta) := h(g(\beta), \beta)$ if $g(\beta) < f(\beta)$, and $:= h(0, \beta)$ otherwise. It is now easy to check that π is a well-defined, order-preserving map from $[f]_\mu$ into δ, a contradiction since $[f]_\mu > \delta$. q.e.d.

4.3 Measure Assignments from Order Measures

With the lifting operation we now can define a κ-measure assignment from an κ-germ assignment.

4.3.1. Definition. Let κ be a weak partition cardinal, $\mathfrak{A}$ an ordinal algebra and $\mathrm{GA}_{\mathfrak{A}} = \langle \mathbf{germ}, \mathbf{ot} \rangle$ a κ-germ assignment on $\mathfrak{A}$. Let x be a term in $\mathfrak{A}$, then by Theorem 4.1.5 the order measure

$$\mathbf{meas}_\kappa(x) := \mathbf{lift}_\kappa(\mathbf{germ}(x))$$

is indeed a measure on κ and thus $\mathbf{meas}_\kappa$ is a κ-measure assignment for the algebra $\mathfrak{A}$. We call $\mathbf{meas}_\kappa$ a **natural κ-measure assignment**.

From now on we will consider only measure assignments that stem from germ assignments, *i.e.*, natural measure assignments. We can show the following:

4.3.2. Theorem. *Let κ be a weak partition cardinal that is closed under ultrapowers, $\mathfrak{A}$ an ordinal algebra, and $\mathrm{GA}_{\mathfrak{A}} = \langle \mathbf{germ}, \mathbf{ot} \rangle$ a κ-germ assignment. Then the natural κ-measure assignment $\mathbf{meas}_\kappa$ we get from from Definition 4.3.1 is almost canonical up to $\kappa^{(\mathrm{ht}(\mathfrak{A}))}$.*

Proof. We have to check the five conditions from Definition 3.3.2. Let t_1, t_2, t_3, r_1, and r_2 be terms in an (additive) ordinal algebra, r_1 and r_2 can be empty terms.

(1) If we compare at the trees T_x and T_y that correspond to terms $x = r_1 \oplus t_1 \oplus (t_2 \oplus t_3) \oplus r_2$ and $y = r_1 \oplus (t_1 \oplus t_2) \oplus t_3 \oplus r_2$ we see that they are identical. So by definition the measures $\mathbf{meas}_\kappa(x)$ and $\mathbf{meas}_\kappa(y)$ are also identical, which means the operation $\oplus$ is associative, *i.e.*, for order measures μ, ν, and η we have $\mu \oplus (\nu \oplus \eta) = (\mu \oplus \nu) \oplus \eta$. So as usual we will omit the brackets to simplify notation.

(2) We get left-distributivity also directly from the definition of order measures: The trees T_x and T_y corresponding to $x = r_1 \oplus t_1 \otimes (t_2 \oplus t_3) \oplus r_2$ and $y = r_1 \oplus t_1 \otimes t_2 \oplus t_1 \otimes t_3 \oplus r_2$ are identical, so by definition the order measures $\mathbf{meas}_\kappa(x)$ and $\mathbf{meas}_\kappa(y)$ are also identical.

(3) We can argue the same way as in (1) and (2): The trees T_x and T_y corresponding to $x = r_1 \oplus t_1 \otimes (t_2 \otimes t_3) \oplus r_2$ and $y = r_1 \oplus (t_1 \otimes t_2) \otimes t_3 \oplus r_2$ are identical, so by definition the order measures $\mathbf{meas}_\kappa(x)$ and $\mathbf{meas}_\kappa(y)$ are also identical and the operation $\otimes$ is associative. As in the $\oplus$ case we will omit unnecessary brackets.

(4) Let $x = r_1 \oplus (t_1 \oplus t_2) \otimes t_3 \oplus r_2$. By (2) we can assume that t_3 is of the form $\bigotimes_{i<n} \mathsf{V}_{\alpha_i}$ for some sequence $\langle \mathsf{V}_{\alpha_i}\,;\, i < n \rangle$ of $\mathfrak{A}$-generators. The subterm $(t_1 \oplus t_2) \otimes t_3$ of x corresponds to two subtrees $T_{t_1 \otimes t_3}$ and $T_{t_2 \otimes t_3}$ of T_x. Those two trees start with the same nodes, those that correspond to t_3, and are disjunct afterwards.

Let n_1, n_2, n_3, n_4 be the number of terminal nodes in the subtrees T_{r_1}, $T_{t_1 \oplus t_3}$, $T_{t_2 \oplus t_3}$, and T_{r_2} of T_x, respectively. Let $\langle p_i \,;\, i < n_2 \rangle$ be the sequence of terminal nodes of $T_{t_1 \otimes t_3}$ and $\langle q_i \,;\, i < n_3 \rangle$ the sequence of terminal nodes of $T_{t_2 \otimes t_3}$, we assume the nodes in the sequences are ordered from left to right.

If $o(t_3) = 1$ then t_3 is of the form $\bigotimes_{i<n} \mathsf{V}_0$ and we know $\mathbf{ot}(\mathsf{V}_0) = 1$. We define $y := r_1 \oplus (t_1 \oplus t_2) \oplus r_2$ and have to show $\mathbf{meas}_\kappa(x) \simeq \mathbf{meas}_\kappa(x)$. Let T_{r_1} and T_{r_2} be the subtrees of T_x that correspond to the subterms r_1 and r_2 of x and T'_{r_1} and T'_{r_2} the subtrees of T_y that correspond to the subterms r_1 and r_2 of y. Let $\langle p'_i \,;\, i < n_2 \rangle$ be the sequence of terminal nodes of the subtree T'_{t_1} of T_y that corresponds to t_1 and $\langle q_i \,;\, i < n_3 \rangle$ the sequence of terminal nodes of the subtree T'_{t_2} of T_y that corresponds to t_2, we again assume the nodes in the sequences are ordered from left to right.

By definition of the κ-lift we have that $A \subseteq \kappa$ is in $\mathbf{lift}_\kappa(\mathbf{germ}(x))$ if there is a club $C \subseteq \kappa$ such that for all $f \in \mathfrak{D}_C^{\mathbf{ot}(x)}$ we have

$$\ulcorner \overrightarrow{[\hat{f}]_{T_{r_1}}}, \overrightarrow{[\hat{f}]_{T_{t_1 \otimes t_3}}}, \overrightarrow{[\hat{f}]_{T_{t_2 \otimes t_3}}}, \overrightarrow{[\hat{f}]_{T_{r_2}}} \urcorner \in A.$$

As we mentioned after Definition 4.1.3, since $\mathbf{ot}((t_1 \oplus t_2) \otimes t_3) = \mathbf{ot}(t_2 1 \oplus t_2)$, we have $\overrightarrow{[\hat{f}]_{T_{r_1}}} = \overrightarrow{[\hat{f}]_{T'_{r_1}}}$ and $\overrightarrow{[\hat{f}]_{T_{r_1}}} = \overrightarrow{[\hat{f}]_{T'_{r_1}}}$. Since $\mathbf{ot}(t_3) = 1$ the sets $A_{(t_1 \oplus t_2) \otimes t_3}$ and $A_{t_1 \oplus t_2}$ ordered by $<_{\mathrm{rlex}}$ are equivalent, so we get $f_{p_i} = f_{p'_i}$ and $f_{q_j} = f_{q'_j}$ for $i < n_2$, $j < n_3$. Which means we have

$$[\hat{f}_{p_i}]_{\mathbf{germ}_{p_i}} = [\hat{f}_{p'_i}]_{\mathbf{germ}_{p'_i}} \text{ and } [\hat{f}_{q_j}]_{\mathbf{germ}_{q_j}} = [\hat{f}_{q'_j}]_{\mathbf{germ}_{q'_i}}$$

for all $i < n_2$, $j < n_3$. So there is a club $C \subseteq \kappa$ such that for all $f \in \mathfrak{D}_C^{\mathbf{ot}(y)}$ we have

$$\ulcorner \overrightarrow{[\hat{f}]_{T'_{r_1}}}, \overrightarrow{[\hat{f}]_{T'_{t_1}}}, \overrightarrow{[\hat{f}]_{T'_{t_2}}}, \overrightarrow{[\hat{f}]_{T_{r_2}}} \urcorner \in A,$$

which means A is an element of $\mathbf{lift}_\kappa(\mathbf{germ}(y))$. So the two measures are the same.

If $o(t_3) \neq 1$ then we know that $\mathbf{ot}(t_3)$ is a limit ordinal. Assume $\mathbf{ot}(t_1) \geq \mathbf{ot}(t_2)$. Then by ordinal arithmetic we have $\mathbf{ot}(x) = \mathbf{ot}(r_1 \oplus t_1 \otimes t_3 \oplus r_2)$. We define $y := r_1 \oplus t_1 \otimes t_3 \oplus r_2$ and have to show $\mathbf{meas}_\kappa(x) \simeq \mathbf{meas}_\kappa(x)$. Let T_{r_1} and T_{r_2} be the subtrees of T_x that correspond to the subterms r_1 and r_2 of x and T'_{r_1} and T'_{r_2} the subtrees of T_y that correspond to the subterms r_1 and r_2 of y. Let $\langle p'_i \,;\, i < n_2 \rangle$ be the sequence of terminal nodes of the subtree $T'_{t_1 \otimes t_3}$ of T_y that corresponds to t_1, we again assume the nodes in the sequences are ordered from left to right.

We get an embedding from $\kappa^\kappa/\mathbf{meas}_\kappa(y)$ into $\kappa^\kappa/\mathbf{meas}_\kappa(x)$: For $F : \kappa \to \kappa$ we define $F' : \kappa \to \kappa$ by

$$F'(\ulcorner \vec{\alpha}, \vec{\beta}, \vec{\gamma}, \vec{\delta} \urcorner) := F(\ulcorner \vec{\alpha}, \vec{\beta}, \vec{\gamma} \urcorner),$$

where $\vec{\alpha} \in \kappa^{n_1}$, $\vec{\beta} \in \kappa^{n_2}$, $\vec{\gamma} \in \kappa^{n_3}$, and $\vec{\delta} \in \kappa^{n_4}$. Then $F \mapsto F'$ induces an embedding, as we now will show. If we have $[F]_{\mathbf{meas}_\kappa(y)} \overset{=}{<} [G]_{\mathbf{meas}_\kappa(y)}$ then there is a club set $C \subseteq \kappa$ such that for all $f \in \mathfrak{D}_C^{\mathbf{ot}(x)}$ we have

$$F(\ulcorner \overrightarrow{[\hat{f}]_{T_{r_1}}}, \overrightarrow{[\hat{f}]_{T_{t_1 \otimes t_3}}}, \overrightarrow{[\hat{f}]_{T_{r_2}}} \urcorner) \overset{=}{<} G(\ulcorner \overrightarrow{[\hat{f}]_{T_{r_1}}}, \overrightarrow{[\hat{f}]_{T_{t_1 \otimes t_3}}}, \overrightarrow{[\hat{f}]_{T_{r_2}}} \urcorner).$$

By definition of F' and G' this means there is a club set $C \subseteq \kappa$ such that for all $f \in \mathfrak{D}_C^{\mathbf{ot}(x)}$ we have

$$F'(\ulcorner \overrightarrow{[\hat{f}_t]_{T_{r_1}}}, \overrightarrow{[\hat{f}]_{T_{t_1 \otimes t_3}}}, \overrightarrow{[\hat{f}]_{T_{t_2 \otimes t_3}}}, \overrightarrow{[\hat{f}]_{T_{r_2}}} \urcorner) \overset{=}{<} G'(\ulcorner \overrightarrow{[\hat{f}]_{T_{r_1}}}, \overrightarrow{[\hat{f}]_{T_{t_1 \otimes t_3}}}, \overrightarrow{[\hat{f}]_{T_{t_2 \otimes t_3}}}, \overrightarrow{[\hat{f}]_{T_{r_2}}} \urcorner),$$

which is equivalent to $[F']_{\mathbf{meas}_\kappa(x)} \overset{=}{<} [G']_{\mathbf{meas}_\kappa(x)}$.

We also get an embedding from $\kappa^\kappa/\mathbf{meas}_\kappa(x)$ into $\kappa^\kappa/\mathbf{meas}_\kappa(y)$: If p' is a terminal node of the subtree $T_{t_1 \otimes t_3}$ of T_y, p the corresponding terminal node of T_x and q_i a terminal node of the subtree $T_{t_2 \otimes t_3}$ of T_x then, since $\mathbf{ot}(t_1) \geq \mathbf{ot}(t_2)$ and the order type $\mathbf{ot}(t_3)$ is an limit ordinal, we can define $\hat{f}_{q_i}$ from $\hat{f}_p$. For $F : \kappa \to \kappa$ we define $F' : \kappa \to \kappa$ by

$$F'(\ulcorner \vec{\alpha}, \vec{\beta}, \vec{\delta} \urcorner) := F(\ulcorner \vec{\alpha}, \vec{\beta}, \vec{\gamma}, \vec{\delta} \urcorner),$$

where $\vec{\alpha} \in \kappa^{n_1}$, $\vec{\beta} \in \kappa^{n_2}$, $\vec{\delta} \in \kappa^{n_4}$, and $\vec{\gamma} \in \kappa^{n_3}$ is the sequence $\langle [\hat{f}_{q_i}]_{\mathbf{germ}_{q_i}} ; i < n_3 \rangle$ that we get from $\beta_0 = [\hat{f}_{p0}]_{\mathbf{germ}_{p_0}}$. Then $F \mapsto F'$ induces an embedding: If we have $[F]_{\mathbf{meas}_\kappa(x)} \overset{=}{<} [G]_{\mathbf{meas}_\kappa(x)}$ then there is a club set $C \subseteq \kappa$ such that for all $f \in \mathfrak{D}_C^{\mathbf{ot}(x)}$ we have

$$F(\ulcorner \overrightarrow{[\hat{f}]_{T_{r_1}}}, \overrightarrow{[\hat{f}]_{T_{t_1 \otimes t_3}}}, \overrightarrow{[\hat{f}]_{T_{t_2 \otimes t_3}}}, \overrightarrow{[\hat{f}]_{T_{r_1}}} \urcorner) \overset{=}{<} G(\ulcorner \overrightarrow{[\hat{f}]_{T_{r_1}}}, \overrightarrow{[\hat{f}]_{T_{t_1 \otimes t_3}}}, \overrightarrow{[\hat{f}]_{T_{t_2 \otimes t_3}}}, \overrightarrow{[\hat{f}]_{T_{r_1}}} \urcorner).$$

This means there is a club set $C \subseteq \kappa$ such that for all $f \in \mathfrak{D}_C^{\mathbf{ot}(y)}$ we have

$$F'(\ulcorner \overrightarrow{[\hat{f}]_{T_{r_1}}}, \overrightarrow{[\hat{f}]_{T_{t_1 \otimes t_3}}}, \overrightarrow{[\hat{f}]_{T_{r_1}}} \urcorner) \overset{=}{<} G'(\ulcorner \overrightarrow{[\hat{f}_t]_{T_{r_1}}}, \overrightarrow{[\hat{f}_t]_{T_{t_1 \otimes t_3}}}, \overrightarrow{[\hat{f}_t]_{T_{r_1}}} \urcorner),$$

which is equivalent to $[F']_{\mathbf{meas}_\kappa(y)} \overset{=}{<} [G']_{\mathbf{meas}_\kappa(y)}$. So we have embeddings in both directions which means the measures $\mathbf{meas}_\kappa(x)$ and $\mathbf{meas}_\kappa(y)$ are equivalent.

(5) Assume $x = r_1 \oplus \mathsf{V}_{\alpha_0} \otimes \ldots \otimes \mathsf{V}_{\alpha_{n-1}} \oplus r_2$, where $\langle \mathsf{V}_{\alpha_i} ; i < n \rangle$ is a sequence of generators of $\mathfrak{A}$. There is a smallest element $\bigotimes_{i<m} \mathsf{V}_{\beta_i}$ of $\mathfrak{P}$ that has the same $\mathbf{ot}$ value as $\bigotimes_{i<n} \mathsf{V}_{\alpha_i}$, we let $y = r_1 \oplus \mathsf{V}_{\beta_0} \otimes \ldots \otimes \mathsf{V}_{\beta_{n-1}} \oplus r_2$. We have $\mathbf{ot}(x) = \mathbf{ot}(y)$ and as we mentioned after Definition 4.1.3 we know $f_t = f_s$, where t is the terminal node of T_x that corresponds to $\bigotimes_{i<n} \mathsf{V}_{\alpha_i}$ and s the terminal node of T_y that corresponds to $\bigotimes_{i<m} \mathsf{V}_{\beta_i}$, and, since $\lambda := \mathbf{ot}(\mathsf{V}_{\alpha_0} \otimes \ldots \otimes \mathsf{V}_{\alpha_{n-1}}) = \mathbf{ot}(\mathsf{V}_{\beta_0} \otimes \ldots \otimes \mathsf{V}_{\beta_{m-1}})$, we have also $\overrightarrow{[\hat{f}]_{T_{r_1}}} = \overrightarrow{[\hat{f}]_{T'_{r_1}}}$ and $\overrightarrow{[\hat{f}]_{T_{r_2}}} = \overrightarrow{[\hat{f}]_{T'_{r_2}}}$. Let n_1 be the number of terminal nodes in T_{r_1} and n_2 the number of terminal nodes in T_{r_2}.

We get an embedding from $\kappa^\kappa/\mathbf{meas}_\kappa(y)$ into $\kappa^\kappa/\mathbf{meas}_\kappa(x)$: For $F:\kappa\to\kappa$ we define $F':\kappa\to\kappa$ by

$$F'(\ulcorner\vec{\alpha},[f]_{\mathbf{germ}_t},\vec{\gamma}\urcorner) := F(\ulcorner\vec{\alpha},[f]_{\mathbf{germ}_s},\vec{\gamma}\urcorner),$$

where $\vec{\alpha}\in\kappa^{n_1}$, $f\in\kappa^\lambda$, and $\vec{\gamma}\in\kappa^{n_2}$. Then $F\mapsto F'$ induces an embedding, as we now will show. If we have $[F]_{\mathbf{meas}_\kappa(y)} \overset{=}{\underset{<}{}} [G]_{\mathbf{meas}_\kappa(y)}$ then there is a club set $C\subseteq\kappa$ such that for all $f\in\mathfrak{D}_C^{\mathbf{ot}(y)}$ we have

$$F(\ulcorner\overrightarrow{[\hat{f}]_{T_{r_1}}},[\hat{f}_s]_{\mathbf{germ}_s},\overrightarrow{[\hat{f}]_{T_{r_2}}}\urcorner) \overset{=}{\underset{<}{}} G(\ulcorner\overrightarrow{[\hat{f}]_{T_{r_1}}},[\hat{f}_s]_{\mathbf{germ}_s},\overrightarrow{[\hat{f}]_{T_{r_2}}}\urcorner).$$

We know $\hat{f}_s=\hat{f}_t$ and by definition of F' and G' this means there is a club set $C\subseteq\kappa$ such that for all $f\in\mathfrak{D}_C^{\mathbf{ot}(x)}$ we have

$$F'(\ulcorner\overrightarrow{[\hat{f}]_{T_{r_1}}},[\hat{f}_t]_{\mathbf{germ}_t},\overrightarrow{[\hat{f}]_{T_{r_2}}}\urcorner) \overset{=}{\underset{<}{}} G'(\ulcorner\overrightarrow{[\hat{f}]_{T_{r_1}}},[\hat{f}_t]_{\mathbf{germ}_t},\overrightarrow{[\hat{f}]_{T_{r_2}}}\urcorner).$$

We get an embedding from $\kappa^\kappa/\mathbf{meas}_\kappa(x)$ into $\kappa^\kappa/\mathbf{meas}_\kappa(y)$ the same way. So we have embeddings in both directions which means the measures $\mathbf{meas}_\kappa(x)$ and $\mathbf{meas}_\kappa(y)$ are equivalent. q.e.d.

This means if κ is at least a weak partition cardinal that is closed under ultrapowers and $\mathrm{GA}_{\mathfrak{A}}$ is a κ-germ assignment then the κ-measure assignment derived from $\mathrm{GA}_{\mathfrak{A}}$ is almost canonical. In order to get full canonicity up to height κ^λ for the natural measure assignment it is enough to to prove canonicity up to height κ^λ for the induced measure assignment on the induced additive ordinal algebra. Which means dealing with terms x that are sums of products of generators and the germ assignments and the corresponding lifts for those terms have a simpler form than in the general case and we can simplify our notation accordingly.

4.3.3. Remark. Let $\mathfrak{A}$ be an ordinal algebra with generators $\mathfrak{V}$ and x a term in $\mathfrak{A}$ of the form $x=x_0\oplus\ldots\oplus x_{n-1}$, where $\langle x_i\,;\,i,n\rangle$ is a sequence of products of generators of $\mathfrak{A}$. That means the corresponding tree T_x consists just of the root $\bullet$ and a number of branches that do not branch themselves, so $\mathbf{germ}(x)$ is essentially a sequence $\langle\mu_i\,;\,i<n\rangle$ of (product) measures and $\mathbf{ot}(x)$ a sequence $\langle\varrho_i\,;\,i<n\rangle$ of ordinals. If we add two terms x and y of this form then $\mathbf{germ}(x\oplus y)$ is just the concatenation $\mathbf{germ}(x)^\frown\mathbf{germ}(y)$ and $\mathbf{ot}(x\oplus y)$ is the concatenation $\mathbf{ot}(x)^\frown\mathbf{ot}(y)$. Let $f\in\mathfrak{D}_\kappa^{\mathbf{ot}(x)}$ be a function of discontinuous type $\mathbf{ot}(x)$. If t is the terminal node in T_x that corresponds to x_i then $f_t:\mathbf{ot}(x_i)\to\kappa$ is of the form

$$f_t(\alpha)=f(\sum_{j<i}\mathbf{ot}(x_j)+\alpha).$$

4.3.4. LEMMA. *Let κ be a weak partition cardinal that is closed under ultrapowers, $\mathfrak{A}$ an ordinal algebra, and $\mathrm{GA}_{\mathfrak{A}} = \langle \mathbf{germ}, \mathbf{ot}\rangle$ a κ-germ assignment for this algebra. Let x be a sum of products of generators of $\mathfrak{A}$ with $\mathbf{germ}(x) = \langle \mu_i\,;\, i < n\rangle$ and $\mathbf{ot}(x) = \langle \varrho_i\,;\, i < n\rangle$. Using the notation from Remark 4.3.3 we have*

$$\begin{aligned}\mathbf{lift}_\kappa(\mathbf{germ}(x)) = \{A \subseteq \kappa;\ & \text{there is a club } C \subseteq \kappa \text{ such that for all } f \in \mathfrak{D}_C^{\mathbf{ot}(x)} \\ & \text{we have } \ulcorner[\hat{f}_{t_0}]_{\mathbf{germ}_{t_0}}, \ldots, [\hat{f}_{t_{n-1}}]_{\mathbf{germ}_{t_{n-1}}}\urcorner \in A, \\ & \text{where } \langle t_i\,;\, i < n\rangle \text{ are the terminal nodes of } T_x\}. \\ = \{A \subseteq \kappa;\ & \text{there is a club set } C \subseteq \kappa \text{ such that for all } \vec{g} \in \mathfrak{C}_C^{\vec{\varrho}} \\ & \text{we have } \ulcorner[g_0]_{\mu_0}, \ldots, [g_{n-1}]_{\mu_{n-1}}\urcorner \in A\}.\end{aligned}$$

Proof. Let $C \subseteq \kappa$ be a club set. If $f : \mathbf{ot}(x) \to C$ is a function of discontinuous type $\mathbf{ot}(x)$ then $f_{t_i} : \varrho_i \to C$ is a function of discontinuous type ϱ_i and we know that $\sup f_{t_i} < f_{t_{i+1}}(0)$ is true for all $i < n$. Since C is closed we get $\hat{f}_{t_o} \in \mathfrak{C}_C^{\varrho_0}$ and $\hat{f}_{t_{i+1}} \in \mathfrak{C}^{\varrho_{i+1}}_{C_{>\sup \hat{f}_i}}$ for $i < n$ from the construction of $\hat{f}_{ti}$. So for all $f \in \mathfrak{D}_C^{\mathbf{ot}(x)}$ we have $\vec{\hat{f}} \in \mathfrak{C}_C^{\mathbf{ot}(x)}$, which proves "$\subseteq$".

An the other hand, if $\vec{g} \in \mathfrak{C}_C^{\mathbf{ot}(x)}$ is a sequence of functions $g_i : \varrho_i \to C$ of continuous type ϱ_i with $\sup g_i < g_{i+1}(0)$ then we can construct a function $f \in \mathfrak{D}_C^{\mathbf{ot}(x)}$ by $f(0) := g_0(0)$, $f(\alpha+1) := g_i(\beta)$, where $i = \max\{j\,;\, \sum_{k<j} \varrho_k \leq \alpha + 1\}$ and $\beta = \min\{\gamma \in \mathrm{rang}_i\,;\, f(\alpha) < g_i(\gamma)\}$. From the construction of f follows that we have $\hat{f}_{t_i} = g_i$ for $i < n$, which proves "$\supseteq$". q.e.d.

We will use the new representation given by Lemma 4.3.4 when we deal with sums of basic order measures. In order to to facilitate working with this new representation we define some new notations and conventions.

4.3.5. DEFINITION. Let μ and ν be sums of basic order measures, *i.e.*, assume that there is a weak partition cardinal κ and κ-germ assignment for an ordinal algebra $\mathfrak{A}$ such that $\mu = \mathbf{lift}_\kappa(\mathbf{germ}(x))$ and $\nu = \mathbf{lift}_\kappa(\mathbf{germ}(x))$ for some $\mathfrak{A}$-terms x and y that are sums of products of generators. We call order measures of this form **simple order measures**. As we said in Definition 4.1.6, if x is a product of generators of $\mathfrak{A}$, we call μ a basic order measure and write $\mathbf{germ}_\mu$ for the germ $\mathbf{germ}(x)$ of μ and $\mathbf{ot}_\mu$ for its ordertype $\mathbf{ot}(x)$. We now extend this notation to simple order measures. If $\mathbf{germ}(x) = \langle \mu_i\,;\, i < n\rangle$ and $\mathbf{ot}(x) = \langle \varrho_i\,;\, i < n\rangle$ we write $\mathbf{germ}_\mu$ and $\mathbf{ot}_\mu$ for these sequences and $\mathbf{germ}_{\mu,i}$, $\mathbf{ot}_{\mu,i}$ for their elements μ_i, ϱ_i, respectively. If $\vec{f} = \langle f_i\,;\, i < n\rangle$ is a sequence of functions we abbreviate the sequence $\langle [f_0]_{\mathbf{germ}_{\mu,0}}, \ldots, [f_{n-1}]_{\mathbf{germ}_{\mu,n-1}}\rangle$ with $\overrightarrow{[f]}_{\mathbf{germ}_\mu}$. Furthermore we write $\sup \vec{f}$ for $\sup_{i<n} \sup f$.

With these notations we can write down properties of order measures much more compactly. For example, if μ and ν are simple order measures then we have

$$\begin{aligned}\mu \oplus \nu = \{A \subseteq \kappa;\ & \text{there is a club } C \subseteq \kappa \text{ such that for all } \vec{x} \in \mathfrak{C}_C^{\mathbf{ot}_\mu},\ \vec{y} \in \mathfrak{C}^{\mathbf{ot}_\nu}_{C_{>\sup \vec{x}}} \\ & \text{we have } \ulcorner\overrightarrow{[x]}_{\mathbf{germ}_\mu} {}^\frown \overrightarrow{[y]}_{\mathbf{germ}_\nu}\urcorner \in A\},\end{aligned}$$

and if they are basic order measures their order measure product is simply

$$\mu \otimes \nu = \{A \subseteq \kappa; \begin{array}{l} \text{there is a club } C \subseteq \kappa \text{ such that for all } x \in \mathfrak{C}_C^{\mathbf{ot}_\mu \cdot \mathbf{ot}_\nu} \\ \text{we have } [x]_{\mathbf{germ}_\mu \times \mathbf{germ}_\nu} \in A\}. \end{array}$$

4.4 Some Special Order Measures

We can show that some canonical measures are indeed order measure.

4.4.1. LEMMA. *Let κ be a weak partition cardinal and $\alpha+1$ an ordinal less than κ. Then $\mathcal{C}_\kappa^\omega = \mathbf{lift}_\kappa(\mu_{\{\alpha\}})$, with $\mu_{\{\alpha\}}$ being the principal measure on $\alpha+1$ that concentrates on α. We will most often use the fact that the order measure with germ $\mu_{\{0\}}$ and order type 1 is $\mathcal{C}_\kappa^\omega$, i.e., $\mathcal{C}_\kappa^\omega = \mathbf{lift}_\kappa(\mu_{\{0\}})$.*

Proof. A set $A \subseteq \kappa$ is an element of $\mathcal{C}_\kappa^\omega$ if there is a club set $C \subseteq \kappa$ such that all elements of C that have cofinality ω are elements of A. Since κ is greater than ω and regular this is equivalent to $y(\alpha)$ being an element of A for all functions $y : \alpha+1 \to C$ of continuous type $\alpha+1$, *i.e.*, the measures $\mathcal{C}_\kappa^\omega$ and $\mathbf{lift}_\kappa(\mu_{\{\alpha\}})$ are identical. q.e.d.

Let us take a closer look at the simple order measure $\mathcal{C}_{\omega_1}^\omega \otimes n$. It is a measure on ω_1 and using Lemma 4.4.1 we have

$$A \in \mathcal{C}_{\omega_1}^\omega \otimes n \quad \Leftrightarrow \quad \begin{array}{l} \text{there is a club set } C \subseteq \omega_1 \text{ such that} \\ \text{for all } x_1 \in \mathfrak{C}_C^1, \ldots, x_n \in \mathfrak{C}_{C > x_n(0)}^1 \text{ we have} \\ \ulcorner x_1(0), \ldots, x_n(0) \urcorner \in A. \end{array}$$

Since all limits in ω_1 have cofinality ω and we can without loss of generality assume that a club set contains only limits we get the following reformulation:

$$A \in \mathcal{C}_{\omega_1}^\omega \otimes n \quad \Leftrightarrow \quad \begin{array}{l} \text{there is a club set } C \subseteq \omega_1 \text{ such that} \\ \text{for all } \vec{\alpha} \in [C]^n \text{ we have } \ulcorner \vec{\alpha} \urcorner \in A. \end{array}$$

That means if we identify ω_1 and $[\omega_1]^n$ through the bijection $\ulcorner \cdot \urcorner : \omega_1 \to [\omega_1]^n$ then we can interpret a function $f : \omega_1 \to \mathrm{On}$ as a function on $[\omega_1]^n$:

4.4.2. REMARK. In the following we will often view equivalence classes $[f]_{\mathcal{C}_{\omega_1}^\omega \otimes n}$ as generated by functions $f : [\omega_1]^n \to \kappa$. We then write $[\vec{\alpha} \to f(\vec{\alpha})]_{\mathcal{C}_{\omega_1}^\omega \otimes n}$ to make clear that the domain of f is $[\omega_1]^n$. This is not to be confused with the use of $\vec{\alpha}$ in the case of product measures, for example, $[\vec{\alpha} \to f(\vec{\alpha})]_{{\mathcal{C}_{\omega_1}^\omega}^{\otimes n}}$, where the domain of f is $(\omega_1)^n$.

4.4.3. LEMMA. *Let κ be a weak partition cardinal and $0 < n < \omega$ a natural number. Then $\mathcal{W}_\kappa^n = \mathcal{C}_\kappa^\omega \otimes n$, i.e., the projection of the n-fold product of $\mathcal{C}_\kappa^\omega$ under* *the Gödel pairing function is equal to the n-fold order measure sum of $\mathcal{C}^\omega_\kappa$. From this we get directly an alternative way of describing the product measure $(\mathcal{C}^\omega_{\omega_1})^n$:*

$$A \in (\mathcal{C}^\omega_{\omega_1})^n \quad \Leftrightarrow \quad \begin{array}{l} \textit{there is a club set } C \subseteq \omega_1 \textit{ such that} \\ \textit{for all } \vec{\alpha} \in [C]^n \textit{ we have } \vec{\alpha} \in A. \end{array}$$

Proof. By Lemma 1.2.2 the filter $\mathcal{W}^n_\kappa$ is an measure since $(\mathcal{C}^\omega_\kappa)^n$ is one and by Theorem 4.1.5 $\mathcal{C}^\omega_\kappa \otimes n$ is also a measure. So we are finished if we can show that every element of $\mathcal{C}^\omega_\kappa \otimes n$ is also an element of $\mathcal{W}^n_\kappa$.

Let $A \subseteq \kappa$ be an element of $\mathcal{C}^\omega_\kappa \otimes n$, so there is a club set $C \subseteq \kappa$ such that $\ulcorner [x_0]_{\mu_{\{0\}}}, \ldots, [x_{n-1}]_{\mu_{\{0\}}} \urcorner \in A$ for all $\vec{x} \in \mathfrak{C}^n_C$. For $\beta < \kappa$ let D be the subset of C consisting of ordinals with cofinality ω. As always we let $D_{>\beta}$ be the subset of all ordinals in D that are greater than β. Then for all $\vec{\alpha} \in [D]^n$ we have $\ulcorner \vec{\alpha} \urcorner \in A$, so for all $\alpha_0 \in D$ we have that for all $\alpha_1 \in D_{>\alpha_0}$ ... we have that for all $\alpha_{n-1} \in D_{>\alpha_{n-2}}$ we have $\ulcorner \langle \alpha_0, \ldots, \alpha_{n-1} \rangle \urcorner \in A$. Since $D_{>\beta}$ is in $\mathcal{C}^\omega_\kappa$ for all $\beta < \kappa$ we know that the set

$$B := \{\vec{\alpha} \in \kappa^n \,;\, \alpha_0 \in D, \alpha_1 \in D_{>\alpha_0}, \ldots, \alpha_{n-1} \in D_{>\alpha_{n-2}}\}$$

is in $(\mathcal{C}^\omega_\kappa)^n$. So there is a set $B \in (\mathcal{C}^\omega_\kappa)^n$ such that $\ulcorner \vec{\alpha} \urcorner \in A$ for all $\vec{\alpha} \in B$, which means A is in $\mathcal{W}^n_\kappa$. q.e.d.

Using Lemma 4.1.7 we can identify the κ-lift of $\mathcal{C}^\omega_{\omega_1}$:

4.4.4. Corollary. *Assume* AD. *Let $\kappa > \omega_1$ be an odd projective ordinal. Then $\mathcal{C}^{\omega_1}_\kappa = \mathbf{lift}_\kappa(\mathcal{C}^\omega_{\omega_1})$.*

This is essentially the first step in the computation of normal measures on odd projective ordinals that we will undertake in Section 7.1.

4.5 The Natural Measure Assignment

We want a germ assignment for odd projective ordinals $\boldsymbol{\delta}^1_{2n+1}$ under AD that induces a canonical $\boldsymbol{\delta}^1_{2n+1}$-measure assignment up to the next odd projective ordinal. We will now define this germ assignment, it is essentially the same as in Section 5. of Jackson and Löwe's paper [JaLö06]. The difference lies in the fact that in [JaLö06] the ordinal algebras have a generator for 0, whereas our first generator has value 1.

4.5.1. Definition. Assume AD + DC. We define by recursion germ assignments $\mathrm{GA}_{\boldsymbol{\delta}^1_{2n+1}}$ for ordinal algebras with $1 + \mathbf{e}_n$-many generators. This will be done in a such a way that $\mathrm{GA}_{\boldsymbol{\delta}^1_{2n+3}}$ extends $\mathrm{GA}_{\boldsymbol{\delta}^1_{2n+1}}$, so we can denote the germ and order type functions uniformly with **germ** and **ot**. We know that by definition $\mathbf{e}_0 = 0$ and $\mathbf{e}_{n+1} = \omega^{\omega^{\mathbf{e}_n}}$.

We start with $n = 0$ where we only have one generator and assign $\mathbf{germ}(\mathsf{V}_0) := \mu_{\{0\}}$ and $\mathbf{ot}(\mathsf{V}_0) := 1$. Assume we have defined the germ assignment for $1 + \mathbf{e}_n$-many generators. We extend this to a germ assignment for $\mathbf{e}_{n+1}$-many generators in the following way: For $\alpha < 1 + \mathbf{e}_n$ we keep the assignment we have. We set $\mathbf{germ}(\mathsf{V}_{1+\mathbf{e}_n}) = \mathcal{C}^{\omega}_{\boldsymbol{\delta}^1_{2n+1}}$ and $\mathbf{ot}(\mathsf{V}_{1+\mathbf{e}_n}) = \boldsymbol{\delta}^1_{2n+1}$. If $1 + \mathbf{e}_n < \alpha < \mathbf{e}_{n+1}$ then $\alpha = 1+\mathbf{e}_n+\beta$ with $\beta < \mathbf{e}_{n+1}$. The height of $\mathfrak{A}_1$ is $\omega = \mathbf{e}_1$ and for $n > 0$ the height of $\mathfrak{A}_{\mathbf{e}_n}$ is equal to $\omega^{\omega^{e_n}} = e_{n+1}$ by Proposition 3.1.1. So in the ordinal algebra with $1 + \mathbf{e}_n$-many generators there is a unique term $\bigoplus_{i<n}(\bigotimes_{j<m} \mathsf{V}_{\gamma_{i,j}})$ with the following properties: it has o value β, we have for all $i < n$ that $\gamma_{i,\ell} \geq \gamma_{i,k}$ for $\ell < k < m$, and $o(\bigotimes_{j<m} \mathsf{V}_{\gamma_{\ell,j}}) \geq o(\bigotimes_{j<m} \mathsf{V}_{\gamma_{k,j}})$ for $\ell < k < n$. We name that term t_α and extend the germ assignment by setting

$$\mathbf{germ}(\mathsf{V}_\alpha) := \mathbf{slift}_{\boldsymbol{\delta}^1_{2n+1}}(\mathbf{lift}_{\boldsymbol{\delta}^1_{2n+1}}(\mathbf{germ}(t_\alpha))),$$

$$\mathbf{ot}(\mathsf{V}_\alpha) := {\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}} / \mathbf{lift}_{\boldsymbol{\delta}^1_{2n+1}}(\mathbf{germ}(t_\alpha).$$

Then $\mathrm{GA}_{\boldsymbol{\delta}^1_{2n+1}}$ is a $\boldsymbol{\delta}^1_{2n+1}$-germ assignment for all $n < \omega$: $\mathrm{GA}_{\boldsymbol{\delta}^1_1}$ is by definition a $\boldsymbol{\delta}^1_1$-germ assignment. Assume $\mathrm{GA}_{\boldsymbol{\delta}^1_{2n+1}}$ is a $\boldsymbol{\delta}^1_{2n+1}$-germ assignment. In order to show that $\mathrm{GA}_{\boldsymbol{\delta}^1_{2n+3}}$ is a $\boldsymbol{\delta}^1_{2n+3}$-germ assignment we only have to check for α with $1 + \mathbf{e}_n \leq \alpha < 1 + \mathbf{e}_{n+1}$ that $\mathbf{germ}(\mathsf{V}_\alpha)$ and $\mathbf{ot}(\mathsf{V}_\alpha)$ fulfill the requirements of Definition 4.1.1. By Theorem 4.2.1 $\mathbf{germ}(\mathsf{V}_\alpha)$ is a measure on $\mathbf{ot}(\mathsf{V}_\alpha)$ and since under AD the projective ordinals are closed under ultrapowers we know that $\mathbf{ot}(\mathsf{V}_\alpha)$ is less than $\boldsymbol{\delta}^1_{2n+3}$. So by Lemma 2.2.14 we have

$$(\boldsymbol{\delta}^1_{2n+3})^{\mathbf{ot}(\mathsf{v}_0)\cdot\ldots\cdot\mathbf{ot}(\mathsf{v}_{n-1})} / \mathbf{germ}(\mathsf{v}_0) \times \ldots \times \mathbf{germ}(\mathsf{v}_{n-1}) = \boldsymbol{\delta}^1_{2n+3}$$

for all finite sequences $\langle \mathsf{v}_i\,;\, i < n\rangle$ of generators of $\mathfrak{A}^{\oplus}_{1+\mathbf{e}_{2n+3}}$. Furthermore by Theorem 4.2.1 the measure $\mathbf{germ}(\mathsf{V}_\alpha)$ is closed under end segments and by Lemma 1.2.10 the same goes for products of those measures. So $\mathrm{GA}_{\boldsymbol{\delta}^1_{2n+3}}$ is a $\boldsymbol{\delta}^1_{2n+3}$-germ assignment.

From the $\boldsymbol{\delta}^1_{2n+1}$-germ assignment $\mathrm{GA}_{\boldsymbol{\delta}^1_{2n+1}}$ for the ordinal algebra with $1 + \mathbf{e}_n$-many generators we get a natural $\boldsymbol{\delta}^1_{2n+1}$-measure assignment for the same ordinal algebra and its induced additive ordinal algebra.

4.5.2. Definition. Assume $\mathsf{AD}+\mathsf{DC}$. We write $\mathrm{NMA}_{\boldsymbol{\delta}^1_{2n+1}}$ for the natural $\boldsymbol{\delta}^1_{2n+1}$-measure assignment we get by Definition 4.3.1 from the $\boldsymbol{\delta}^1_{2n+1}$-germ assignment $\mathrm{GA}_{\boldsymbol{\delta}^1_{2n+1}}$ defined in Definition 4.5.1.

Now we have to prove that the measure assignment $\mathrm{NMA}_{\boldsymbol{\delta}^1_{2n+1}}$ is in fact a canonical measure assignment up to a certain height. We will do this in the following chapters for the first ω^2-many generators of the additive ordinal algebra.

Let us conclude this section by making a table for the initial part of the germ assignment in order to better understand Definition 4.5.1. In this table we put the generators of the additive ordinal algebra with their corresponding germ, order type, and the measure this generates on a projective ordinal $\boldsymbol{\delta}^1_{2n+1}$. Remember that $\mathbf{lift}_{\boldsymbol{\delta}^1_{2n+1}}(\mu_{\{0\}}) = \mathcal{C}^{\omega}_{\boldsymbol{\delta}^1_{2n+1}}$ and $\mathbf{lift}_{\boldsymbol{\delta}^1_{2n+1}}(\mathcal{C}^{\omega}_{\omega_1}) = \mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}}$. At stage ω for $\mathsf{S}_\omega = \mathsf{V}_2$ we have $\mathbf{germ}(\mathsf{V}_2) = \mathbf{slift}_{\boldsymbol{\delta}^1_1}(\mathbf{lift}_{\boldsymbol{\delta}^1_1}(\mathbf{germ}(\mathsf{V}_0)))$. Since the measure $\mathbf{lift}_{\boldsymbol{\delta}^1_1}(\mathbf{germ}(\mathsf{V}_0)) = \mathcal{C}^{\omega}_{\boldsymbol{\delta}^1_1}$ contains all club subsets of $\boldsymbol{\delta}^1_1$ by Lemma 4.2.2 the measure $\mathbf{germ}(\mathsf{V}_2)$ is a normal measure on ${\boldsymbol{\delta}^1_1}^{\boldsymbol{\delta}^1_1}/\mathcal{C}^{\omega}_{\boldsymbol{\delta}^1_1} = \omega_2$. So ω_2 is regular and by Lemma 4.1.7 we now know that for $n > 0$ the measure $\mathbf{lift}_{\boldsymbol{\delta}^1_{2n+1}}(\mathbf{germ}(\mathsf{V}_2))$ is the ω_2-cofinal measure $\mathcal{C}^{\omega_2}_{\boldsymbol{\delta}^1_{2n+1}}$ on $\boldsymbol{\delta}^1_{2n+1}$. So the table for the first ω^2 generators of the additive ordinal algebra looks like this:

	germ	order type	measure on $\boldsymbol{\delta}^1_{2n+1}$
$\mathsf{S}_0 = \mathsf{V}_0$	$\mu_{\{0\}}$	1	$\mathcal{C}^{\omega}_{\boldsymbol{\delta}^1_{2n+1}}$
$\mathsf{S}_1 = \mathsf{V}_1$	$\mathcal{C}^{\omega}_{\omega_1}$	ω_1	$\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}}$
$\mathsf{S}_2 = \mathsf{V}_1^{\otimes 2}$	$(\mathcal{C}^{\omega}_{\omega_1})^2$	$(\omega_1)^2$	$(\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}})^{\otimes 2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathsf{S}_n = \mathsf{V}_1^{\otimes n}$	$(\mathcal{C}^{\omega}_{\omega_1})^n$	$(\omega_1)^n$	$(\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}})^{\otimes n}$
$\mathsf{S}_\omega = \mathsf{V}_2$	$\mathbf{slift}_{\omega_1}(\mathcal{C}^{\omega}_{\omega_1})$	ω_2	$\mathcal{C}^{\omega_2}_{\boldsymbol{\delta}^1_{2n+1}}$
$\mathsf{S}_{\omega+1}$	$\mathbf{slift}_{\omega_1}(\mathcal{C}^{\omega}_{\boldsymbol{\delta}^1_1}) \times \mathcal{C}^{\omega}_{\omega_1}$	$\omega_2 \cdot \omega_1$	$\mathcal{C}^{\omega_2}_{\boldsymbol{\delta}^1_{2n+1}} \otimes \mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathsf{S}_{\omega+n}$	$\mathbf{slift}_{\omega_1}(\mathcal{C}^{\omega}_{\boldsymbol{\delta}^1_1}) \times (\mathcal{C}^{\omega}_{\omega_1})^n$	$\omega_2 \cdot (\omega_1)^n$	$\mathcal{C}^{\omega_2}_{\boldsymbol{\delta}^1_{2n+1}} \otimes (\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}})^{\otimes n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathsf{S}_{\omega\cdot 2}$	$(\mathbf{slift}_{\omega_1}(\mathcal{C}^{\omega}_{\boldsymbol{\delta}^1_1}))^2$	$(\omega_2)^2$	$(\mathcal{C}^{\omega_2}_{\boldsymbol{\delta}^1_{2n+1}})^{\otimes 2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathsf{S}_{\omega\cdot m+n}$	$(\mathbf{slift}_{\omega_1}(\mathcal{C}^{\omega}_{\boldsymbol{\delta}^1_1}))^m \times (\mathcal{C}^{\omega}_{\omega_1})^n$	$(\omega_2)^m \cdot (\omega_1)^n$	$(\mathcal{C}^{\omega_2}_{\boldsymbol{\delta}^1_{2n+1}})^{\otimes m} \otimes (\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}})^{\otimes n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathsf{S}_{\omega^2} = \mathsf{V}_3$	$\mathbf{slift}_{\omega_1}(\mathcal{C}^{\omega}_{\omega_1} \oplus \mathcal{C}^{\omega}_{\omega_1})$	ω_3	

Chapter 5

Canonicity of the Natural Measure Assignment

In this chapter we will give conditions for the canonicity of the natural measure assignment we defined in Definition 4.5.2 and proceed to prove that these conditions are fulfilled under AD.

5.1 Embeddings between Ultrapowers of Order Measures

In this section, we will derive various embedding results for order measures. Kleinberg's Theorem 1.7.2 gives us partition properties for cardinals that can be represented as an iterated ultrapower with respect to a normal measure. So can we get a connection between iterated ultrapowers and order measures? This following result is a preliminary answer and will also be used in the proof of Theorem 5.2.2 where this question will be answered in detail:

5.1.1. Lemma. *Let κ be a weak partition cardinal and let μ and ν be simple order measures, both on κ. Let $\lambda \geq \kappa$ be a cardinal and assume the ultrapowers $\lambda^\kappa/(\mu \oplus \nu)$, λ^κ/ν, and $(\lambda^\kappa/\nu)^\kappa/\mu$ are wellfounded. Then*

$$\lambda^\kappa/(\mu \oplus \nu) \leq (\lambda^\kappa/\nu)^\kappa/\mu.$$

Proof. For $f : \kappa \to \lambda$ define $\hat{f} : \kappa \to \lambda^\kappa/\nu$ by $\hat{f}(\ulcorner\vec{\alpha}\urcorner) := [\ulcorner\vec{\beta}\urcorner \mapsto f(\ulcorner\vec{\alpha}^\frown\vec{\beta}\urcorner)]_\nu$. We shall show that $f \to \hat{f}$ induces an embedding from $\lambda^\kappa/(\mu\oplus\nu)$ into $(\lambda^\kappa/\nu)^\kappa/\mu$. We use our convention from Remark 1.4.3, *i.e.*, write "$\overset{=}{<}$" to denote that the proof works for "$=$" and "$<$". If $[f]_{\mu\oplus\nu} \overset{=}{<} [g]_{\mu\oplus\nu}$, then there is a club set $C \subseteq \kappa$ such that for all $\vec{x} \in \mathfrak{C}_C^{\mathbf{ot}_\mu}$ and all $\vec{y} \in \mathfrak{C}_{C_{>\sup\vec{x}}}^{\mathbf{ot}_\nu}$ we have

$$f(\ulcorner\overrightarrow{[x]}_{\mathbf{germ}_\mu}{}^\frown\overrightarrow{[y]}_{\mathbf{germ}_\nu}\urcorner) \overset{=}{<} g(\ulcorner\overrightarrow{[x]}_{\mathbf{germ}_\mu}{}^\frown\overrightarrow{[y]}_{\mathbf{germ}_\nu}\urcorner).$$

Momentarily fixing $\vec{x}$ in this statement and observing that $C_{>\sup \vec{x}}$ is club in κ, we get that there is a club set $C \subseteq \kappa$ such that for all $\vec{x} \in \mathfrak{C}_C^{\mathbf{ot}_\mu}$ we have

$$[\ulcorner\vec{\beta}\urcorner \mapsto f(\ulcorner\overrightarrow{[x]}_{\mathbf{germ}_\mu}{}^\frown\vec{\beta}\urcorner)]_\nu \underset{<}{=} [\ulcorner\vec{\beta}\urcorner \mapsto g(\ulcorner\overrightarrow{[x]}_{\mathbf{germ}_\mu}{}^\frown\vec{\beta}\urcorner)]_\nu,$$

which by definition of $f \mapsto \hat{f}$ translates to $[\hat{f}]_\mu \underset{<}{=} [\hat{g}]_\mu$. q.e.d.

We want to show that the measure assignment we define in Definition 4.5.1 is canonical. That means the operations $\oplus$ and $\otimes$ on the order measures have to correspond to $+$ and $\cdot$ in the sense of the iterated successor operation The next lemma will be used to prove Corollary 5.2.3 where we state the conditions that must be fulfilled to get canonicity for our measure assignment.

5.1.2. Lemma. *Assume κ is a weak partition cardinal and let μ, η and ν be simple order measures on κ. Then, assuming the ultrapowers are wellfounded,*

1. $\kappa^\kappa/\mu \leq \kappa^\kappa/\mu \oplus \nu$, $\kappa^\kappa/\nu \leq \kappa^\kappa/\mu \oplus \nu$, *and*
2. $\kappa^\kappa/\mu \oplus \nu \leq \kappa^\kappa/\mu \oplus \eta \oplus \nu$.

Proof. For $f : \kappa \to \kappa$, we define $f_0, f_1 : \kappa \to \kappa$ by $f_0(\ulcorner\vec{\alpha}{}^\frown\vec{\beta}\urcorner) := f(\ulcorner\vec{\alpha}\urcorner)$ and $f_1(\ulcorner\vec{\alpha}{}^\frown\vec{\beta}\urcorner) := f(\ulcorner\vec{\beta}\urcorner)$. Now $f \to f_0$ induces an embedding from κ^κ/μ into $\kappa^\kappa/\mu \oplus \nu$ and $f \to f_1$ induces an embedding from κ^κ/ν into $\kappa^\kappa/\mu \oplus \nu$. As the proofs for the two parts of 1. are identical we do only the one for the second part:

$$\begin{array}{rcl}
[f]_\nu \underset{<}{=} [g]_\nu & \Leftrightarrow & \text{there is a club set } C \subseteq \kappa \text{ such that} \\
& & \text{for all } \vec{y} \in \mathfrak{C}_C^{\mathbf{ot}_\nu} \text{ we have } f(\ulcorner\overrightarrow{[y]}_{\mathbf{germ}_\nu}\urcorner) \underset{<}{=} g(\ulcorner\overrightarrow{[y]}_{\mathbf{germ}_\nu}\urcorner). \\
& \Rightarrow & \text{there is a club set } C \subseteq \kappa \text{ such that} \\
& & \text{for all } \vec{x} \in \mathfrak{C}_C^{\mathbf{ot}_\mu},\ \vec{y} \in \mathfrak{C}_{C_{>\sup \vec{x}}}^{\mathbf{ot}_\nu} \text{ we have} \\
& & f_1(\ulcorner\overrightarrow{[x]}_{\mathbf{germ}_\mu}{}^\frown\overrightarrow{[y]}_{\mathbf{germ}_\nu}\urcorner) \underset{<}{=} g_1(\ulcorner\overrightarrow{[x]}_{\mathbf{germ}_\mu}{}^\frown\overrightarrow{[y]}_{\mathbf{germ}_\nu}\urcorner) \\
& \Leftrightarrow & [f_1]_{\mu\oplus\nu} \underset{<}{=} [g_1]_{\mu\oplus\nu}.
\end{array}$$

The proof of (3) is similar. For $f : \kappa \to \kappa$, we define $f_2 : \kappa \to \kappa$ by $f_2(\ulcorner\vec{\alpha}{}^\frown\vec{\beta}{}^\frown\vec{\gamma}\urcorner) := f(\ulcorner\vec{\alpha}{}^\frown\vec{\gamma}\urcorner)$. Now $f \to f_2$ induces an embedding from $\kappa^\kappa/\mu \oplus \nu$ into $\kappa^\kappa/\mu \oplus \eta \oplus \nu$:

$$\begin{array}{rcl}
[f]_{\mu\oplus\nu} \underset{<}{=} [g]_{\mu\oplus\nu} & \Leftrightarrow & \text{there is a club set } C \subseteq \kappa \text{ such that} \\
& & \text{for all } \vec{x} \in \mathfrak{C}_C^{\mathbf{ot}_\mu},\ \vec{y} \in \mathfrak{C}_{C_{>\sup \vec{x}}}^{\mathbf{ot}_\nu} \text{ we have} \\
& & f(\ulcorner\overrightarrow{[x]}_{\mathbf{germ}_\mu}{}^\frown\overrightarrow{[y]}_{\mathbf{germ}_\nu}\urcorner) \underset{<}{=} g(\ulcorner\overrightarrow{[x]}_{\mathbf{germ}_\mu}{}^\frown\overrightarrow{[y]}_{\mathbf{germ}_\nu}\urcorner) \\
& \Rightarrow & \text{there is a club set } C \subseteq \kappa \text{ such that} \\
& & \text{for all } \vec{x} \in \mathfrak{C}_C^{\mathbf{ot}_\mu},\ \vec{z} \in \mathfrak{C}_{C_{>\sup \vec{x}}}^{\mathbf{ot}_\eta},\ \vec{y} \in \mathfrak{C}_{C_{>\sup \vec{z}}}^{\mathbf{ot}_\nu} \text{ we have} \\
& & f(\ulcorner\overrightarrow{[x]}_{\mathbf{germ}_\mu}{}^\frown\overrightarrow{[y]}_{\mathbf{germ}_\nu}\urcorner) \underset{<}{=} g(\ulcorner\overrightarrow{[x]}_{\mathbf{germ}_\mu}{}^\frown\overrightarrow{[y]}_{\mathbf{germ}_\nu}\urcorner) \\
& \Leftrightarrow & [f_2]_{\mu\oplus\eta\oplus\nu} \underset{<}{=} [g_2]_{\mu\oplus\eta\oplus\nu}.
\end{array}$$

q.e.d.

Up to now we only got embedding results for sums of basic order measures. In order to deal also with products of basic order measures it is useful to define the notion of strong embeddings.

5.1.3. Definition. Let κ be a weak partition cardinal. Let μ and ν be basic order measures on κ. We say that μ **strongly embeds into** ν ($\mu \preccurlyeq \nu$) if there exists a measure η on an ordinal $\lambda < \kappa$ and a function $H : \lambda \mapsto {}^{\mathbf{ot}_\mu}\mathbf{ot}_\nu$ such that

- For η-almost all $\alpha \in \lambda$ the function $H(\alpha) : \mathbf{ot}_\mu \mapsto \mathbf{ot}_\nu$ is order preserving and continuous.
- If $A \subseteq \mathbf{ot}_\nu$ has $\mathbf{germ}_\nu$-measure 1, then for η-almost all $\alpha \in \lambda$ it is the case that for $\mathbf{germ}_\mu$-almost all $x \in \mathbf{ot}_\mu$ $H(\alpha)(x) \in A$.

If $\mu = \mu_1 \oplus \ldots \oplus \mu_n$ and $\nu = \nu_1 \oplus \ldots \oplus \nu_n$ are basic order measure sums of the same length, we say that μ **strongly embeds into** ν ($\mu \preccurlyeq \nu$) if $\mu_i \preccurlyeq \nu_i$ holds for all $i \leq n$.

5.1.4. Lemma. *Let κ be a cardinal, X a subset of κ and $\alpha, \beta < \kappa$ ordinals. If $h : \alpha \mapsto \beta$ is an order preserving, continuous function and $x \in \mathfrak{C}^\beta_X$, then $x \circ h \in \mathfrak{C}^\alpha_X$.*

Proof. Since both h and x are increasing and continuous, $x \circ h$ is also an increasing, continuous function. Let $g : \omega \times \beta \mapsto X$ witness the uniform cofinality ω of x, then $g' : \omega \times \alpha \mapsto X$, defined by $g'(n, \gamma) := g(n, h(\gamma))$, witnesses the uniform cofinality ω of $x \circ h$. q.e.d.

Now we can show that if a simple order measure μ strongly embeds into another simple order measure ν then there is in fact an embedding from the ultrapower with respect to μ into the ultrapower with respect to ν.

5.1.5. Lemma. *Let κ be a weak partition cardinal. Let $\mu = \mu_1 \oplus \ldots \oplus \mu_n$ and $\nu = \nu_1 \oplus \ldots \oplus \nu_n$ be sums of basic order measures on κ. If μ strongly embeds into ν, then there exists an embedding from κ^κ/μ into κ^κ/ν and thus $\kappa^\kappa/\mu \leq \kappa^\kappa/\nu$ if the ultrapowers are wellfounded.*

Proof. Let $i \leq n$ and the strong embedding between μ_i and ν_i be witnessed by the measure η on λ and the function $H : \lambda \mapsto {}^{\mathbf{ot}_{\mu_i}}\mathbf{ot}_{\nu_i}$. For $f : \kappa \to \kappa$ define $\hat{f} : \kappa \to \kappa$ by

$$\hat{f}(\ulcorner \alpha_1, \ldots, [y]_{\mathbf{germ}_{\nu_i}}, \ldots, \alpha_n \urcorner) := [\beta \mapsto f(\ulcorner \alpha_1, \ldots, [y \circ H(\beta)]_{\mathbf{germ}_{\mu_i}}, \ldots, \alpha_n \urcorner)]_\eta.$$

We will show that $f \mapsto \hat{f}$ defines an embedding from κ^κ/μ to $\kappa^\kappa/\mu_1 \oplus \ldots \oplus \nu_i \oplus \ldots \oplus \mu_n$. First we have to prove that $\hat{f}$ is welldefined. If $[y]_{\mathbf{germ}_{\nu_i}} = [y']_{\mathbf{germ}_{\nu_i}}$ there is a $\mathbf{germ}_{\nu_i}$-measure 1 set A such that $y(\gamma) = y'(\gamma)$ for all $\gamma \in A$. From part 2. of Definition 5.1.3 follows that for η-almost all $\alpha \in \lambda$ it is the case that for $\mathbf{germ}_{\mu_i}$-almost all $\delta \in \mathbf{ot}_{\mu_i}$ $H(\alpha)(\delta) \in A$. So for η-almost all $\alpha \in \lambda$ we have $[y \circ H(\alpha)]_{\mathbf{germ}_{\mu_i}} = [y' \circ H(\alpha)]_{\mathbf{germ}_{\mu_i}}$, *i.e.*

$$\hat{f}(\ulcorner \alpha_1, \ldots, [y]_{\mathbf{germ}_{\nu_i}}, \ldots, \alpha_n \urcorner) = [\beta \mapsto f(\ulcorner \alpha_1, \ldots, [y \circ H(\beta)]_{\mathbf{germ}_{\mu_i}}, \ldots, \alpha_n \urcorner)]_\eta =$$

$$[\beta \mapsto f(\ulcorner \alpha_1, \ldots, [y' \circ H(\beta)]_{\mathbf{germ}_{\mu_i}}, \ldots, \alpha_n \urcorner)]_\eta = \hat{f}(\ulcorner \alpha_1, \ldots, [y']_{\mathbf{germ}_{\nu_i}}, \ldots, \alpha_n \urcorner).$$

Now we can show that $f \to \hat{f}$ induces an embedding from κ^κ/μ to $\kappa^\kappa/\mu_1 \oplus \ldots \oplus \nu_i \oplus \ldots \oplus \mu_n$, again using "$\overset{=}{<}$" to mean both "=" and "<" as mentioned in Remark 1.4.3. Assume $[f]_\mu \overset{=}{<} [f']_\mu$, ie,

There exists a club set $C \subseteq \kappa$ such that for all $x_1 \in \mathfrak{C}_C^{\mathbf{ot}_{\mu_1}}, \ldots, x_n \in \mathfrak{C}_{C > \sup x_{n-1}}^{\mathbf{ot}_{\mu_n}}$

$$f(\ulcorner [x_1]_{\mathbf{germ}_{\mu_1}}, \ldots, [x_n]_{\mathbf{germ}_{\mu_n}} \urcorner) \overset{=}{<} f'(\ulcorner [x_1]_{\mathbf{germ}_{\mu_1}}, \ldots, [x_n]_{\mathbf{germ}_{\mu_n}} \urcorner).$$

Fix such a club set C. By part 1. of Definition 5.1.3 and Lemma 5.1.4 we know if $y \in \mathfrak{C}_C^{\mathbf{ot}_{\nu_i}}$ then $y \circ H(\alpha) \in \mathfrak{C}_C^{\mathbf{ot}_{\mu_i}}$ for η-almost all α. So by our choice of C for all $y \in \mathfrak{C}_C^{\mathbf{ot}_{\nu_i}}$ it is the case that for η-almost all α we have $f([y \circ H(\alpha)]_{\mathbf{germ}_\mu}) \overset{=}{<} f'([y \circ H(\alpha)]_{\mathbf{germ}_\mu})$. In other words, there exists a club set $C \subseteq \kappa$ such that for all $x_1 \in \mathfrak{C}_C^{\mathbf{ot}_{\mu_1}}, \ldots,\ y \in \mathfrak{C}_{C > \sup x_{i-1}}^{\mathbf{ot}_{\nu_i}}, \ldots, x_n \in \mathfrak{C}_{C > \sup x_{n-1}}^{\mathbf{ot}_{\mu_n}}$ we have

$$[\alpha \mapsto f(\ulcorner [x_1]_{\mathbf{germ}_{\mu_1}}, \ldots, [y \circ H(\beta)]_{\mathbf{germ}_{\mu_i}}, \ldots, [x_n]_{\mathbf{germ}_{\mu_n}} \urcorner)]_\eta \overset{=}{<}$$

$$[\alpha \mapsto f'(\ulcorner [x_1]_{\mathbf{germ}_{\mu_1}}, \ldots, [y \circ H(\beta)]_{\mathbf{germ}_{\mu_i}}, \ldots, [x_n]_{\mathbf{germ}_{\mu_n}} \urcorner)]_\eta.$$

By definition of $\hat{f}$ this is equivalent to

$$\hat{f}(\ulcorner [x_1]_{\mathbf{germ}_{\mu_1}}, \ldots, [y]_{\mathbf{germ}_{\nu_i}}, \ldots, [x_n]_{\mathbf{germ}_{\mu_n}} \urcorner) \overset{=}{<}$$

$$\hat{f}'(\ulcorner [x_1]_{\mathbf{germ}_{\mu_1}}, \ldots, [y]_{\mathbf{germ}_{\nu_i}}, \ldots, [x_n]_{\mathbf{germ}_{\mu_n}} \urcorner),$$

and we conclude $[\hat{f}]_{\mu_1 \oplus \ldots \oplus \nu_i \oplus \ldots \oplus \mu_n} \overset{=}{<} [\hat{f}']_{\mu_1 \oplus \ldots \oplus \nu_i \oplus \ldots \oplus \mu_n}$. So $f \mapsto \hat{f}$ indeed defines an embedding from κ^κ/μ to $\kappa^\kappa/\mu_1 \oplus \ldots \oplus \nu_i \oplus \ldots \oplus \mu_n$. All that is left to do is a simple induction on the length of μ, using this embedding:

$$\kappa^\kappa/\mu = \kappa^\kappa/\mu_1 \oplus \ldots \oplus \mu_n \leq \kappa^\kappa/\nu_1 \oplus \mu_2 \oplus \ldots \oplus \mu_n \leq \ldots \leq \kappa^\kappa/\nu_1 \oplus \ldots \oplus \nu_n = \kappa^\kappa/\nu.$$

q.e.d.

Let us state some basic properties concerning strong embeddings:

5.1.6. PROPOSITION. *Let κ be a weak partition cardinal and μ, ν and ν' simple order measures on κ.*

1. *A basic order measure μ on κ strongly embeds into itself with every measure η on any ordinal $\lambda < \kappa$ and the constant function $H(\alpha) = \mathrm{id}_{\mathbf{ot}_\mu}$, where $\mathrm{id}_{\mathbf{ot}_\mu}$ is the the identity function on $\mathbf{ot}_\mu$.*

2. *If ν strongly embeds into ν', then $\mu \oplus \nu$ strongly embeds into $\mu \oplus \nu'$ and $\nu \oplus \mu$ strongly embeds into $\nu' \oplus \mu$.*

Proof. 1. If $H(\alpha) = \mathrm{id}_{\mathbf{ot}_\mu}$ is the identity function on $\mathbf{ot}_\mu$, then $H(\alpha)$ is obviously order preserving and continuous for all $\alpha \in \lambda$, independent of η and λ. And since $H(\alpha)(x) = \mathrm{id}_{\mathbf{ot}_\mu}(x) = x$, we have $H(\alpha)(x) \in A$ $\mathbf{germ}_\mu$-almost always for any $\mathbf{germ}_\mu$-measure 1 set A and any $\alpha \in \lambda$.

2. This follows directly from 1. and the definition of strong embedding for sums of basic order measures. q.e.d.

Now we will show that the measure $\mathcal{C}^\omega_\kappa$ is a neutral element with respect to the operation $\otimes$ on the set of equivalence classes of basic order measures on κ.

5.1.7. Lemma. *Let κ be a weak partition cardinal and μ a basic order measure on κ . Then we get the following strong embedding results:*

1. *$\mathcal{C}^\omega_\kappa$ strongly embeds into μ.*

2. *μ strongly embeds into $\mathcal{C}^\omega_\kappa \otimes \mu$ and $\mu \otimes \mathcal{C}^\omega_\kappa$ and $\mu \otimes \mathcal{C}^\omega_\kappa$ and $\mathcal{C}^\omega_\kappa \otimes \mu$ strongly embed into μ.*

Proof. Let $\mu_{\{0\}}$ be the principal measure on the ordinal 1. We have $\mathcal{C}^\omega_\kappa = \mathbf{lift}_\kappa(\mu_{\{0\}})$,*i.e.*, $\mathbf{germ}_{\mathcal{C}^\omega_\kappa} = \mu_{\{0\}}$ and $\mathbf{ot}_{\mathcal{C}^\omega_\kappa} = 1$. So $\mathbf{germ}_{\mu\otimes\mathcal{C}^\omega_\kappa} = \mathbf{germ}_\mu \times \mu_{\{0\}}$ and $\mathbf{ot}_{\mu\otimes\mathcal{C}^\omega_\kappa} = 1 \cdot \mathbf{ot}_\mu = \mathbf{ot}_\mu$. The same way we get $\mathbf{germ}_{\mathcal{C}^\omega_\kappa\otimes\mu} = \mu_{\{0\}} \times \mathbf{germ}_\mu$ and $\mathbf{ot}_{\mathcal{C}^\omega_\kappa\otimes\mu} = \mathbf{ot}_\mu$.

1. For $\alpha \in \mathbf{ot}_\mu$ let $H(\alpha)(0) := \alpha$, then $H(\alpha)$ is trivially an increasing function from 1 to $\mathbf{ot}_\mu$. If A has $\mathbf{germ}_\mu$-measure one then for all $\alpha \in A$ we have $H(\alpha)(0) = \alpha \in A$ and so H and the measure $\mathbf{germ}_\mu$ on $\mathbf{ot}_\mu$ witness the strong embedding from $\mathcal{C}^\omega_\kappa$ into μ.

2. Define $H : 1 \mapsto {}^{\mathbf{ot}_\mu}\mathbf{ot}_\mu \cdot 1$ by $H(0)(\alpha) := \alpha$. Then the strong embedding $\mu \preccurlyeq \mu \otimes \mathcal{C}^\omega_\kappa$ is witnessed by the measure $\mu_{\{0\}}$ and the function H:

- Since $H(0)$ is the identity on $\mathbf{ot}_\mu$ it is clearly order preserving and continuous.

- If $A \subseteq \mathbf{ot}_\mu \cdot 1$ has $\mathbf{germ}_\mu \times \mu_{\{0\}}$ measure 1, it also has $\mathbf{germ}_\mu$ measure 1 and since $H(0)(\alpha) = \alpha$ we have $H(0)(\alpha) \in A$ for $\mathbf{germ}_\mu$-almost all $\alpha \in \mathbf{ot}_\mu$.

The proofs for the rest of 2. are nearly identical to this one with the same measure $\mathbf{germ}_\mu$ and function H as witnesses for the strong embedding. q.e.d.

By part 2. of Lemma 5.1.7 we have $\mu \preccurlyeq \mu \otimes \mathcal{C}^\omega_\kappa \preccurlyeq \mu$ and $\mu \preccurlyeq \mathcal{C}^\omega_\kappa \otimes \mu \preccurlyeq \mu$, so $\mathcal{C}^\omega_\kappa$ is neutral with respect to $\otimes$: From Lemma 5.1.5 follows directly

$$\kappa^\kappa/\mu \otimes \mathcal{C}^\omega_\kappa = \kappa^\kappa/\mathcal{C}^\omega_\kappa \otimes \mu = \kappa^\kappa/\mu.$$

We can also use the technique of strong embeddings to get an embedding from κ^κ/μ into $\kappa^\kappa/\mu \otimes \nu$.

5.1.8. Lemma. *Let κ be a weak partition cardinal and μ, ν basic order measures on κ. Then μ strongly embeds into $\mu \otimes \nu$ so $\kappa^\kappa/\mu \leq \kappa^\kappa/\mu \otimes \nu$ if those ultrapowers are wellfounded.*

Proof. We show that the measure $\mathbf{germ}_\nu$ on $\mathbf{ot}_\nu$ and the function $H(\alpha)(\beta) := \mathbf{ot}_\mu \cdot \alpha + \beta$ are witnesses for the strong embedding.

- For every $\alpha \in \mathbf{ot}_\nu$ the function $H(\alpha)$ is order preserving and continuous.
- If $C \in \mathbf{germ}_\mu \times \mathbf{germ}_\nu$, then there exist by definition sets $B \in \mathbf{germ}_\mu$ and $A \in \mathbf{germ}_\nu$ such that for all $\alpha \in A$ and $\beta \in B$ we have $\mathbf{ot}_\mu \cdot \alpha + \beta \in C$, *i.e.*, for $\mathbf{germ}_\nu$-almost all α it is the case that for $\mathbf{germ}_\mu$-almost all β we have $H(\alpha)(\beta) = \mathbf{ot}_\mu \cdot \alpha + \beta \in C$.

q.e.d.

For basic order measures μ and ν we have by Lemma 5.1.8 $\kappa^\kappa/\mu \leq \kappa^\kappa/\mu \otimes \nu$ and by Lemma 5.1.7 we know $\kappa^\kappa/\mu \leq \kappa^\kappa/\mathcal{C}^\omega_\kappa \otimes \mu$. To show the general case, *i.e.*, that there is also an embedding from κ^κ/ν into $\kappa^\kappa/\mu \otimes \nu$ we need some more assumptions and the following lemma:

5.1.9. Lemma. *Let κ be a weak partition cardinal that is closed under ultrapowers and μ, ν basic order measures on κ with order types that are limit ordinals and such that there is a cofinal sequence of length $\mathbf{ot}_\mu$ in $\mathbf{ot}_\nu$. Assume the ultrapowers κ^κ/μ and κ^κ/ν are wellfounded. Then there is a cofinal embedding from κ^κ/μ into κ^κ/ν, so $\kappa^\kappa/\mu \leq \kappa^\kappa/\nu$.*

Proof. Let $\langle \gamma_\alpha\,;\, \alpha < \mathbf{ot}_\mu \rangle$ be a cofinal sequence in $\mathbf{ot}_\nu$. For a function $y : \mathbf{ot}_\nu \to C$ we define the function $x_y : \mathbf{ot}_\mu \to C$ by $x_y(0) := y(0)$, $x_y(\alpha) := y(\gamma_\alpha + 1)$ for successor ordinals $\alpha < \mathbf{ot}_\mu$, and $x_y(\lambda) := \sup_{\alpha<\lambda} y(\gamma_\alpha + 1)$ for limit ordinals $\lambda < \mathbf{ot}_\mu$. If y is of continuous type $\mathbf{ot}_\nu$ then x_y is by definition of continuous type $\mathbf{ot}_\mu$ with range in C.

We will show that $[F]_\mu \mapsto [\hat{F}]_\nu$ with $\hat{F}([y]_{\mathbf{germ}_\nu}) := F([x_y]_{\mathbf{germ}_\mu})$ for functions $y \in \kappa^{\mathbf{ot}_\nu}$ is a welldefined embedding. The function $\hat{F}$ itself is welldefined, since we have $[x_y]_{\mathbf{germ}_\mu} = [x_{y'}]_{\mathbf{germ}_\mu}$ if $[y]_{\mathbf{germ}_\nu} = [y']_{\mathbf{germ}_\nu}$. Assume F and G are functions from κ to κ and $[F]_\mu \overset{=}{<} [G]_\mu$. Then there is a club set $C \subseteq \kappa$ such that

$F([x]_{\mathbf{germ}_\mu}) \stackrel{=}{<} G([x]_{\mathbf{germ}_\mu})$ holds for all functions $x : \mathbf{ot}_\mu \to C$ of continuous type $\mathbf{ot}_\mu$. That means for all functions $y \in \mathfrak{C}_C^{\mathbf{ot}_\nu}$ we get

$$\hat{F}([y]_{\mathbf{germ}_\nu}) = F([x_y]_{\mathbf{germ}_\mu}) \stackrel{=}{<} G([x_y]_{\mathbf{germ}_\mu}) = \hat{G}([y]_{\mathbf{germ}_\nu})$$

from our assumption on F and G. So $[F]_\mu \mapsto [\hat{F}]_\nu$ is a welldefined embedding from κ^κ/μ into κ^κ/ν.

Now we show that this embedding is cofinal in κ^κ/ν. Let $[G]_\nu$ be an element of κ^κ/ν. We need to find $F : \kappa \to \kappa$ such that $[G]_\nu \leq [\hat{F}]_\nu$. Define the function $F : \kappa \to \kappa$ by

$$F(\alpha) := \sup\{G([y]_{\mathbf{germ}_\nu})\,;\, y \in \mathfrak{C}_\kappa^{\mathbf{ot}_\nu} \text{ and } [x_y]_{\mathbf{germ}_\mu} \leq \alpha\}.$$

Since κ is closed under ultrapowers this is welldefined. For $y \in \mathfrak{C}_C^{\mathbf{ot}_\nu}$ we get

$$\hat{F}([y]_{\mathbf{germ}_\nu}) = F([x_y]_{\mathbf{germ}_\mu})$$

$$\begin{aligned} &= \sup\{G([y']_{\mathbf{germ}_\nu})\,;\, y' \in \mathfrak{C}_\kappa^{\mathbf{ot}_\nu} \text{ and } [x_{y'}]_{\mathbf{germ}_\mu} \leq [x_y]_{\mathbf{germ}_\nu}\} \\ &\geq G([y]_{\mathbf{germ}_\nu}) \end{aligned}$$

which proves $[G]_\nu \leq [\hat{F}]_\nu$. q.e.d.

5.1.10. Corollary. *Let κ be a weak partition cardinal that is closed under ultrapowers and μ, ν basic order measures on κ. Assume the ultrapowers κ^κ/ν and $\kappa^\kappa/\mu \otimes \nu$ are wellfounded. If ν is not the measure $\mathcal{C}_\kappa^\omega$ then there is a cofinal embedding from κ^κ/ν into $\kappa^\kappa/\mu \otimes \nu$, so $\kappa^\kappa/\nu \leq \kappa^\kappa/\mu \otimes \nu$.*

Proof. This follows directly from Lemma 5.1.9 and the fact that $\alpha \mapsto \mathbf{ot}_\mu \cdot \alpha$ defines a cofinal sequence in $\mathbf{ot}_\mu \cdot \mathbf{ot}_\nu$ of length $\mathbf{ot}_\nu$. q.e.d.

5.1.11. Corollary. *Let κ be a weak partition cardinal that is closed under ultrapowers and μ and ν basic order measures on κ. Assume the ultrapowers κ^κ/ν and $\kappa^\kappa/\mu \otimes \nu$ are wellfounded. If ν is not the measure $\mathcal{C}_\kappa^\omega$ then the cofinality of the ultrapower $\kappa^\kappa/\mu \otimes \nu$ is the same as that of the ultrapower κ^κ/ν.*

Proof. Let $\pi : \kappa^\kappa/\nu \to \kappa^\kappa/\mu \otimes \nu$ be the cofinal embedding from Lemma 5.1.10. If $\langle \gamma_\alpha\,;\, \alpha < \mathrm{cf}(\kappa^\kappa/\nu)\rangle$ is a cofinal sequence in κ^κ/ν then $\langle \pi(\gamma_\alpha)\,;\, \alpha < \mathrm{cf}(\kappa^\kappa/\nu)\rangle$ is a cofinal sequence in $\kappa^\kappa/\mu \otimes \nu$, so $\mathrm{cf}(\kappa^\kappa/\mu \otimes \nu) \leq \mathrm{cf}(\kappa^\kappa/\nu)$. If $\langle \beta_\alpha\,;\, \alpha < \delta\rangle$ is a sequence in $\kappa^\kappa/\mu \otimes \nu$ of length $\delta < \mathrm{cf}(\kappa^\kappa/\nu)$ we can define a sequence $\langle \gamma_\alpha\,;\, \alpha < \delta\rangle$ in κ^κ/ν by $\gamma_\alpha = \min\{\xi \in \kappa^\kappa/\mu \otimes \nu\,;\, \pi(\xi) > \beta_\alpha\}$. Then $\sup_{\alpha<\delta} \gamma_\alpha$ is an element of κ^κ/ν, so $\pi(\sup_{\alpha<\delta} \gamma_\alpha)$is an element of $\kappa^\kappa/\mu \otimes \nu$ and we have $\sup_{\alpha<\delta} \beta_\alpha \leq \pi(\sup_{\alpha<\delta} \gamma_\alpha)$. Which means that $\langle \beta_\alpha\,;\, \alpha < \delta\rangle$ is not cofinal in $\kappa^\kappa/\mu \otimes \nu$ and thus $\mathrm{cf}(\kappa^\kappa/\mu \otimes \nu) \geq \mathrm{cf}(\kappa^\kappa/\nu)$. q.e.d.

5.2 A Really Helpful Theorem

Kleinberg's theorem, Theorem 1.7.2, can be understood as analyzing the natural measure assignment for the ordinal algebra with one generator. If we have a measure μ such that $\kappa^\kappa/\mu = \kappa^+$, then the natural measure assignment corresponds to the ordinal algebra above κ (*i.e.*, the next ω cardinals).

In [BoLö06] Benedikt Löwe and the author do the same for the additive ordinal algebra of height ω^2 which has two generators: if μ and ν are measures, and $\kappa^\kappa/\mu = \kappa^+$ and $\kappa^\kappa/\nu = \kappa^{(\omega+1)}$, *i.e.*, the successor cardinal just after the height of the ordinal algebra with one generator, then the natural measure assignment is canonical.

These proofs use the ultrapower shifting lemma (UPSL) for the computation of the upper bounds of the ultrapowers and proceed by induction. The next step would bee to go to height κ^{ω^2+1}. This corresponds to taking products of measures, and everything would work perfectly if we had an analogue of the UPSL for products of measures, but regrettably we do not have such an analogue.

Our approach is to work with the induced additive ordinal algebras, *i.e.*, using all of the product terms as generators, assuming for the time being that these products behave as they should, *i.e.*, that the corresponding ultrapowers have the right value. This leads to the really helpful theorem (RHT) of this section, Theorem 5.2.2. The RHT is the generalization of Theorem 24 from [BoLö07] to arbitrary finite sums of measures and appeared in the preprint [BoLö06] as Theorem 7. It states that if all products behave correctly then the inductive proof idea from the preprint [BoLö06] can be pushed through and we can calculate all ultrapowers.

So in order to prove the proof of the canonicity of the natural measure assignment we only need to compute the values of the ultrapowers that are generated by the measures that correspond to the generators of the induced additive ordinal algebra. We will do this for ω^ω many generators in Chapter 6.

If $\xi < \varepsilon_0 (= \sup_{n<\omega} \mathbf{e}_n)$ is an ordinal then there is a unique representation of it that uses the u-values of the generators of the additive ordinal algebra. We call this relativized version of the Cantor normal form the $\oplus$-normal form of ξ and formally it is defined as follows:

Let $\langle \theta_\alpha\, ;\, \alpha \in \varepsilon_0 \rangle$ be the sequence of u-values of generators of the additive ordinal algebra, Proposition 3.1.2 tells us that we have $\theta_\alpha := \omega^\alpha$. We know $\theta_0 = 1$, $\theta_{\alpha+1} = \theta_\alpha \cdot \omega$ for $\alpha < \varepsilon_0$, and $\theta_\lambda = (\sup_{\alpha<\lambda} \theta_\alpha)$ for limit ordinals $\lambda < \varepsilon_0$.

For every ordinal $\xi < \varepsilon_0$ the **$\oplus$-normal form of ξ** is the decomposition of ξ into a finite sum of elements of $\langle \theta_\alpha\, ;\, \alpha \in \varepsilon_0 \rangle$, *i.e.*,

$$\xi = \theta_{\alpha_0} + \ldots + \theta_{\alpha_m},$$

where $m \in \omega$. It is defined by $\alpha_0 := \min\{\alpha \in \gamma\, ;\, \xi < \theta_{\alpha+1}\}$ and $\alpha_{i+1} := \min\{\alpha \in \gamma\, ;\, \xi < \theta_{\alpha_0} + \ldots + \theta_{\alpha_i} + \theta_{\alpha+1}\}$.

By wellfoundedness of $<$ the $\oplus$-normal form of ξ is welldefined and unique. In the case of infinite successor ordinals we will often write the $\oplus$-normal form in the form $\theta_{\alpha_0} + \ldots + \theta_{\alpha_m} + 1$.

We will work a lot with iterated ultrapowers in this section and in order to facilitate readability we define the following notation for iterated ultrapowers:

5.2.1. Definition. We assume the wellfoundedness of all ultrapowers in this definition. Let $\langle \mu_i\,;\, i < m\rangle$ be a finite sequence of measures on $\kappa < \Theta$. For $m = 0$ we define $\mathrm{iUlt}_\alpha(\emptyset) := \alpha$ and for $m > 0$ we write $\mathrm{iUlt}_\alpha(\mu_0, \ldots, \mu_{m-1})$ to denote the iterated ultrapower

$$\mathrm{iUlt}_\alpha(\mu_0, \ldots, \mu_{m-1}) := (\ldots(\alpha^\kappa/\mu_{m-1})^\kappa/\ldots)^\kappa/\mu_0.$$

We write $\mathrm{iUlt}(\mu_0, \ldots, \mu_{m-1})$ for $\mathrm{iUlt}_\kappa(\mu_0, \ldots, \mu_{m-1})$.

5.2.2. Theorem (A Really Helpful Theorem (RHT)). *Assume* $\mathsf{AD}+\mathsf{DC}$. *Let $\kappa < \Theta$ be a strong partition cardinal and $\gamma < \varepsilon_0$ an ordinal. Let $\langle \mu_\alpha\,;\, \alpha \in \gamma\rangle$ be a sequence of basic order measures on κ and $\langle \iota_\alpha\,;\, \alpha \in \gamma\rangle$ the sequences of cofinalities of the corresponding ultrapowers, i.e., $\iota_\alpha := \mathrm{cf}(\kappa^\kappa/\mu_\alpha)$. We assume that the following properties are true:*

i) $\kappa^\kappa/\mu_0 = \kappa^{(\theta_0)} = \kappa^+$ *and* $\kappa^\kappa/\mu_\alpha = \kappa^{(\theta_\alpha+1)}$ *for* $0 < \alpha < \gamma$,

ii) $(\kappa^\kappa/\nu)^{(\theta_0)} \leq \kappa^\kappa/\nu \oplus \mu_0$ *and* $(\kappa^\kappa/\nu)^{(\theta_\alpha+1)} \leq \kappa^\kappa/\nu \oplus \mu_\alpha$ *for order measure sums* $\nu = \bigoplus_{i<n} \mu_{\alpha_i}$ *and* $0 < \alpha < \gamma$, *and*

iii) $\iota_\alpha = \mathrm{cf}(\kappa^\kappa/\mu_\alpha) > \kappa$ *for* $\alpha < \gamma$.

Then for all $\xi < \sup_{\alpha<\gamma}(\theta_\alpha \cdot \omega)$ the following is true:

1. *If $\xi > 0$ is a limit ordinal and $\theta_{\alpha_0} + \ldots + \theta_{\alpha_m}$ its $\oplus$-normal form then*
$$\kappa^{(\xi)} = \mathrm{iUlt}_{\kappa^{(\theta_{\alpha_m})}}(\mu_{\alpha_0}, \ldots, \mu_{\alpha_{m-1}}) = (\kappa^{(\theta_{\alpha_m})})^\kappa/(\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_{m-1}}).$$

2. *If $\xi < \omega$ is a successor ordinal and $\theta_{\alpha_0} + \ldots + \theta_{\alpha_m} (= \sum_{i\leq m} 1)$ its $\oplus$-normal form then*
$$\kappa^{(\xi)} = \mathrm{iUlt}(\mu_{\alpha_0}, \ldots, \mu_{\alpha_m}) = \kappa^\kappa/(\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_m}).$$

3. *If $\xi > \omega$ is a successor ordinal and $\theta_{\alpha_0} + \ldots + \theta_{\alpha_m} + 1$ its $\oplus$-normal form then*
$$\kappa^{(\xi)} = \mathrm{iUlt}(\mu_{\alpha_0}, \ldots, \mu_{\alpha_m}) = \kappa^\kappa/(\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_m}).$$

4. $$\mathrm{cf}(\kappa^{(\xi)}) := \begin{cases} \kappa & \text{if } \xi = 0, \\ \omega & \text{if } \xi > 0 \text{ is a limit}, \\ \iota_0 & \text{if } \xi \text{ is finite}. \\ \iota_{\alpha_m} & \text{if } \xi = \theta_{\alpha_0} + \ldots + \theta_{\alpha_m} + 1 \text{ is a successor}. \end{cases}$$

Proof. By assumption κ is a strong partition cardinal, thus regular. Also, for all limit ordinals $\xi < \sup_{n<\omega} \mathbf{e}_n$, the cofinality of $\kappa^{(\xi)}$ is ω. So the first two parts of 4. are trivial.

We proceed by induction on $\xi > 0$, using the following induction hypothesis:

$$(\mathrm{IH}_\xi)\quad \left[\begin{array}{l} \text{For all } 0 < \beta \leq \xi \text{, the following three conditions hold:} \\ \text{1. If } \beta \text{ is a limit and } \beta = \theta_{\alpha_0} + \ldots + \theta_{\alpha_m}\text{, then} \\ \kappa^{(\beta)} = \mathrm{iUlt}_{\kappa^{(\theta_{\alpha_m})}}(\mu_{\alpha_0}, \ldots, \mu_{\alpha_{m-1}}) = (\kappa^{(\theta_{\alpha_m})})^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_{m-1}}. \\ \text{2. If } \beta < \omega \text{ is a successor and } \beta = \theta_{\alpha_0} + \ldots + \theta_{\alpha_m} \text{ then} \\ \kappa^{(\beta)} = \mathrm{iUlt}(\mu_{\alpha_0}, \ldots, \mu_{\alpha_m}) = \kappa^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_m} \\ \text{3. If } \beta > \omega \text{ is a successor and } \beta = \theta_{\alpha_0} + \ldots + \theta_{\alpha_m} + 1 \text{ then} \\ \kappa^{(\beta)} = \mathrm{iUlt}(\mu_{\alpha_0}, \ldots, \mu_{\alpha_m}) = \kappa^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_m} \\ \text{4. } \mathrm{cf}(\kappa^{(\beta)}) = \iota_0 \text{ if } \beta \text{ is a finite successor.} \\ \text{5. } \mathrm{cf}(\kappa^{(\beta)}) = \iota_{\alpha_m} \text{ if } \beta = \theta_{\alpha_0} + \ldots + \theta_{\alpha_m} + 1 \text{ is an infinite successor.} \end{array}\right.$$

Obviously, if all (IH_ξ) (for $\xi < \sup_{\alpha<\gamma}(\theta_\alpha \cdot \omega)$) hold, the theorem is proven.

By assumption we have $\kappa^\kappa/\mu_0 = \kappa^+$ and $\mathrm{cf}(\kappa^+) = \mathrm{cf}(\kappa^\kappa/\mu_0) = \iota_0$, so the base case (IH_1) holds.

For the successor step we assume that (IH_ξ) holds and prove $(\mathrm{IH}_{\xi+1})$. We have to distinguish between $\xi + 1$ being finite and $\xi + 1$ being infinite. We start with the finite successor case, so let $\theta_{\alpha_0} + \ldots + \theta_{\alpha_m}$ be the $\oplus$-normal form of $\xi + 1$. This means of course that we have $\theta_{\alpha_i} = \theta_0$ for all $i \leq m$. We prove part 2. of $(\mathrm{IH}_{\xi+1})$ as follows:

$$\begin{array}{rcll} \kappa^{(\xi+1)} & = & (\kappa^{(\theta_{\alpha_0}+\ldots+\theta_{\alpha_{m-1}})})^{(\theta_{\alpha_m})} & \\ & = & (\kappa^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_{m-1}})^{(\theta_{\alpha_m})} & \text{Induction Hypothesis} \\ & \leq & \kappa^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_m} & \text{Assumption ii)} \\ & \leq & (\kappa^\kappa/\mu_{\alpha_m})^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_{m-1}} & \text{Lemma 5.1.1} \\ & \vdots & & \\ & \leq & \mathrm{iUlt}(\mu_{\alpha_0}, \ldots, \mu_{\alpha_m}) & \text{Lemma 5.1.1} \\ & = & (\mathrm{iUlt}(\mu_{\alpha_1}, \ldots, \mu_{\alpha_m})^\kappa/\mu_{\alpha_0} & \\ & = & (\kappa^{(\theta_{\alpha_1}+\ldots+\theta_{\alpha_m})})^\kappa/\mu_{\alpha_0} & \text{Induction Hypothesis} \\ & \leq & \kappa^{(\theta_{\alpha_0}+\ldots+\theta_{\alpha_m})} & \text{UPSL (Theorem 1.7.3)} \\ & = & \kappa^{(\xi+1)}. & \end{array}$$

Using $\kappa^{(\xi+1)} = \mathrm{iUlt}(\mu_{\alpha_0}, \ldots, \mu_{\alpha_m})$ with $\theta_{\alpha_i} = \theta_0$ for all $i \leq m$ and Lemma 1.4.9 (repeatedly) we get

$$\iota_0 = \mathrm{cf}(\kappa^\kappa/\mu_{\alpha_m}) = \ldots = \mathrm{cf}((\mathrm{iUlt}(\mu_{\alpha_1}, \ldots, \mu_{\alpha_m}))^\kappa/\mu_{\alpha_0}) = \mathrm{cf}(\kappa^{(\xi+1)}),$$

which proves part 4. of $(\mathrm{IH}_{\xi+1})$ and thus the validity of $(\mathrm{IH}_{\xi+1})$ for finite successor ordinals $\xi + 1$.

We proceed with the infinite successor case, so let $\theta_{\alpha_0} + \ldots + \theta_{\alpha_m} + 1$ be the $\oplus$-normal form of $\xi + 1$. We can use nearly the same method as in the finite successor case but we have to take special care of the finite "tail" of $\theta_{\alpha_0} + \ldots + \theta_{\alpha_m} + 1$. If θ_{α_m} itself is infinite then either $m = 0$, *i.e.*, $\xi + 1 = \theta_\alpha + 1$ for some $0 < \alpha < \gamma$, so we have by assumption $\kappa^{(\xi+1)} = \kappa^\kappa/\mu_\alpha$ and $\mathrm{cf}(\kappa^{(\xi+1)}) = \mathrm{cf}(\kappa^\kappa/\mu_\alpha) = \iota_\alpha$ and thus $(\mathrm{IH}_{\xi+1})$ holds, or $m > 0$, *i.e.*, θ_{α_m} and $\theta_{\alpha_{m-1}}$ are infinite, and we can prove part 3. of $(\mathrm{IH}_{\xi+1})$ as follows:

$$\begin{array}{rcll}
\kappa^{(\xi+1)} & = & \left(\kappa^{(\theta_{\alpha_0}+\ldots+\theta_{\alpha_{m-1}}+1)}\right)^{(\theta_{\alpha_m}+1)} & \\
& = & (\kappa^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_{m-1}})^{(\theta_{\alpha_m}+1)} & \text{Induction Hypothesis} \\
& \leq & \kappa^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_m} & \text{Assumption ii)} \\
& \leq & (\kappa^\kappa/\mu_{\alpha_m})^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_{m-1}} & \text{Lemma 5.1.1} \\
& \vdots & & \\
& \leq & \mathrm{iUlt}(\mu_{\alpha_0}, \ldots, \mu_{\alpha_m}) & \text{Lemma 5.1.1} \\
& = & (\mathrm{iUlt}(\mu_{\alpha_1}, \ldots, \mu_{\alpha_m})^\kappa/\mu_{\alpha_0} & \\
& = & (\kappa^{(\theta_{\alpha_1}+\ldots+\theta_{\alpha_m}+1)})^\kappa/\mu_{\alpha_0} & \text{Induction Hypothesis} \\
& \leq & \kappa^{(\theta_{\alpha_0}+\ldots+\theta_{\alpha_m}+1)} & \text{UPSL (Theorem 1.7.3)} \\
& = & \kappa^{(\xi+1)}. &
\end{array}$$

If θ_{α_m} is not infinite then it is 1, *i.e.*, $\theta_{\alpha_m} = \theta_0$ and we can write the ordinal ξ as $\theta_{\alpha_0} + \ldots + \theta_{\alpha_{m-1}} + \theta_0 + 1$. In this case we prove part 3. of $(\mathrm{IH}_{\xi+1})$ as follows:

$$\begin{array}{rcll}
\kappa^{(\xi+1)} & = & \left(\kappa^{(\theta_{\alpha_0}+\ldots+\theta_{\alpha_{m-1}}+1)}\right)^{(\theta_0)} & \\
& = & (\kappa^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_{m-1}})^{(\theta_0)} & \text{Induction Hypothesis} \\
& \leq & \kappa^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_m} & \text{Assumption ii)} \\
& \leq & (\kappa^\kappa/\mu_{\alpha_m})^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_{m-1}} & \text{Lemma 5.1.1} \\
& \vdots & & \\
& \leq & \mathrm{iUlt}(\mu_{\alpha_0}, \ldots, \mu_{\alpha_m}) & \text{Lemma 5.1.1} \\
& = & (\mathrm{iUlt}(\mu_{\alpha_1}, \ldots, \mu_{\alpha_m})^\kappa/\mu_{\alpha_0} & \\
& = & (\kappa^{(\theta_{\alpha_1}+\ldots+\theta_{\alpha_m}+1)})^\kappa/\mu_{\alpha_0} & \text{Induction Hypothesis} \\
& \leq & \kappa^{(\theta_{\alpha_0}+\ldots+\theta_{\alpha_m}+1)} & \text{UPSL (Theorem 1.7.3)} \\
& = & \kappa^{(\xi+1)}. &
\end{array}$$

To prove the necessary cofinality result, *i.e.*, part 5. of $(\mathrm{IH}_{\xi+1})$ we again use $\kappa^{(\xi+1)} = \mathrm{iUlt}(\mu_{\alpha_0}, \ldots, \mu_{\alpha_m})$ and Lemma 1.4.9 (repeatedly) to get

$$\iota_{\alpha_m} = \mathrm{cf}(\kappa^\kappa/\mu_{\alpha_m}) = \ldots = \mathrm{cf}((\mathrm{iUlt}(\mu_{\alpha_1}, \ldots, \mu_{\alpha_m}))^\kappa/\mu_{\alpha_0}) = \mathrm{cf}(\kappa^{(\xi+1)}),$$

which proves the validity of $(\mathrm{IH}_{\xi+1})$ for infinite successor ordinals $\xi + 1$.

Now for the limit case in the induction. We assume that (IH_β) holds for all $\beta < \xi$ and will show (IH_ξ). If $\xi = \theta_\alpha$ for some $\alpha < \gamma$ we have nothing to prove since then part 1. of the induction hypothesis reduces to $\kappa^{(\xi)} = \mathrm{iUlt}_{\kappa^{(\xi)}}(\emptyset) = \kappa^{(\xi)}$ and thus IH_ξ holds trivially. So let $\theta_{\alpha_0} + \ldots + \theta_{\alpha_m}$ be the $\oplus$-normal form of ξ

with $m > 0$. If α_m is a successor we have $\theta_{\alpha_m} = \sup_{n\in\omega} \theta_{\alpha_m - 1} \cdot n$ and so we get

$$\begin{array}{rlll}
\kappa^{(\xi)} & = & \sup_{n\in\omega} \left(\kappa^{(\theta_{\alpha_0}+\ldots+\theta_{\alpha_{m-1}}+\theta_{\alpha_m-1}\cdot n)}\right) & \\
& = & \sup_{n\in\omega} \left(\kappa^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_{m-1}} \oplus \mu_{\alpha_m-1} \otimes n\right) & \text{Induction Hypothesis} \\
& \leq & \sup_{n\in\omega} \left((\kappa^\kappa/\mu_{\alpha_m-1} \otimes n)^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_{m-1}}\right) & \text{Lemma 5.1.1} \\
& = & \sup_{n\in\omega} \left((\kappa^{(\theta_{\alpha_m-1}\cdot n)})^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_{m-1}}\right) & \text{Induction Hypothesis} \\
& \leq & (\kappa^{(\theta_{\alpha_m})})^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_{m-1}} & \text{Lemma 1.4.8} \\
& \leq & \mathrm{iUlt}_{\kappa^{(\theta_{\alpha_m})}}(\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_{m-2}}, \mu_{\alpha_{m-1}}) & \text{Lemma 5.1.1} \\
& \vdots & & \\
& \leq & \mathrm{iUlt}_{\kappa^{(\theta_{\alpha_m})}}(\mu_{\alpha_0}, \ldots, \mu_{\alpha_{m-1}}) & \text{Lemma 5.1.1} \\
& \leq & \mathrm{iUlt}_{\kappa^{(\theta_{\alpha_{m-1}}+\theta_{\alpha_m})}}(\mu_{\alpha_0}, \ldots, \mu_{\alpha_{m-2}}) & \text{UPSL (Theorem 1.7.3)} \\
& \vdots & & \\
& \leq & \kappa^{(\theta_{\alpha_0}+\ldots+\theta_{\alpha_m})} & \text{UPSL (Theorem 1.7.3)} \\
& = & \kappa^{(\xi)}. &
\end{array}$$

On the other hand, if α_m is a limit we have $\theta_{\alpha_m} = \sup_{\beta\in\alpha_m} \theta_\beta$ and so we get

$$\begin{array}{rlll}
\kappa^{(\xi)} & = & \sup_{\beta\in\alpha_m} \left(\kappa^{(\theta_{\alpha_0}+\ldots+\theta_{\alpha_{m-1}}+\theta_\beta)}\right) & \\
& = & \sup_{\beta\in\alpha_m} \left(\kappa^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_{m-1}} \oplus \mu_\beta\right) & \text{Induction Hypothesis} \\
& \leq & \sup_{\beta\in\alpha_m} \left((\kappa^\kappa/\mu_\beta)^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_{m-1}}\right) & \text{Lemma 5.1.1} \\
& = & \sup_{\beta\in\alpha_m} \left((\kappa^{(\theta_\beta)})^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_{m-1}}\right) & \text{Induction Hypothesis} \\
& \leq & (\kappa^{(\theta_{\alpha_m})})^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_{m-1}} & \text{Lemma 1.4.8} \\
& \leq & \mathrm{iUlt}_{\kappa^{(\theta_{\alpha_m})}}(\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_{m-2}}, \mu_{\alpha_{m-1}}) & \text{Lemma 5.1.1} \\
& \vdots & & \\
& \leq & \mathrm{iUlt}_{\kappa^{(\theta_{\alpha_m})}}(\mu_{\alpha_0}, \ldots, \mu_{\alpha_{m-1}}) & \text{Lemma 5.1.1} \\
& \leq & \mathrm{iUlt}_{\kappa^{(\theta_{\alpha_{m-1}}+\theta_{\alpha_m})}}(\mu_{\alpha_0}, \ldots, \mu_{\alpha_{m-2}}) & \text{UPSL (Theorem 1.7.3)} \\
& \vdots & & \\
& \leq & \kappa^{(\theta_{\alpha_0}+\ldots+\theta_{\alpha_m})} & \text{UPSL (Theorem 1.7.3)} \\
& = & \kappa^{(\xi)}. &
\end{array}$$

In both cases we have shown

$$\kappa^{(\xi)} = (\kappa^{(\theta_{\alpha_m})})^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_{m-1}} = \mathrm{iUlt}_{\kappa^{(\theta_{\alpha_m})}}(\mu_{\alpha_0}, \ldots, \mu_{\alpha_{m-1}}),$$

which means that IH_ξ holds. We have dealt with successor and limit case, so the induction and thus the proof is finished. q.e.d.

5.2.3. COROLLARY. *Assume* $\mathsf{AD} + \mathsf{DC}$. *Let $\kappa < \Theta$ be a strong partition cardinal and $\gamma < \varepsilon_0$ an ordinal. If $\langle \mu_\alpha\,;\, \alpha < \gamma\rangle$ is a sequence of basic order measures on κ that fulfills the requirements of Theorem 5.2.2, then for all $\xi < \omega^\gamma$ and finite sequences $\langle \alpha_i\,;\, i \leq m\rangle \in \gamma^{m+1}$ we have*

$$\kappa^{(\theta_{\alpha_0}+\ldots+\theta_{\alpha_m}+\xi)} = \mathrm{iUlt}_{\kappa^{(\xi)}}(\mu_{\alpha_0}, \ldots, \mu_{\alpha_m}) = \left(\kappa^{(\xi)}\right)^\kappa/(\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_m}).$$

Proof. If $\theta_{\beta_0} + \ldots + \theta_{\beta_n}$ is the $\oplus$-normal form of $\theta_{\alpha_0} + \ldots + \theta_{\alpha_m} + \xi$, then the $\oplus$-normal form of ξ is an end segment of $\theta_{\beta_0} + \ldots + \theta_{\beta_n}$, *i.e.* there is a $k \geq 0$ such that $\theta_{\beta_k} + \ldots + \theta_{\beta_n}$ is the $\oplus$-normal form of ξ. And for all $i < k$ there is a $j \leq m$ such that $\theta_{\beta_i} = \theta_{\alpha_j}$. So using RHT (Theorem 5.2.2) we get

$$\kappa^{(\theta_{\alpha_0}+\ldots+\theta_{\alpha_m}+\xi)} = \kappa^{(\theta_{\beta_0}+\ldots+\theta_{\beta_n})} = \kappa^\kappa/\mu_{\beta_0} \oplus \ldots \oplus \mu_{\beta_n}.$$

Lemma 5.1.2 allows us to insert the missing elements of the sequence $\langle \mu_{\alpha_i}\,;\, i \leq m\rangle$ and then we can apply Lemma 5.1.1 in order to get

$$\begin{aligned} &\kappa^\kappa/\mu_{\beta_0} \oplus \ldots \oplus \mu_{\beta_n} \leq \kappa^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_m} \oplus \mu_{\beta_k} \oplus \ldots \oplus \mu_{\beta_n} \\ &\leq \left(\kappa^\kappa/\mu_{\beta_k} \oplus \ldots \oplus \mu_{\beta_n}\right)^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_m} = \left(\kappa^{(\xi)}\right)^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_m}. \end{aligned}$$

And finally we can use Lemma 5.1.1 and Theorem 1.7.3, both repeatedly as we did before in the proof of RHT (Theorem 5.2.2), to show that we have equality:

$$\left(\kappa^{(\xi)}\right)^\kappa/\mu_{\alpha_0} \oplus \ldots \oplus \mu_{\alpha_m} \leq \mathrm{iUlt}_{\kappa^{(\xi)}}(\mu_{\alpha_0}, \ldots, \mu_{\alpha_m}) \leq \kappa^{(\theta_{\alpha_0}+\ldots+\theta_{\alpha_m}+\xi)}.$$

q.e.d.

Lemma 3.3.3 together with Theorem 4.3.2 allowed us to reduce the question of canonicity for the natural measure assignment to whether the induced measure assignment on the additive ordinal algebra is canonical or not. The Really Helpful Theorem is called really helpful because we can combine it with these results to get simple conditions for the canonicity of natural measure assignments.

5.2.4. LEMMA. *Assume* $\mathsf{AD} + \mathsf{DC}$. *Let* $\kappa < \Theta$ *be a strong partition cardinal that is closed under ultrapowers. Let* $\mathrm{GA}_\kappa = \langle \mathbf{germ}, \mathbf{ot}\rangle$ *be a* κ*-germ assignment for the ordinal algebra* $\mathfrak{A}_\gamma$ *with* γ*-many generators* $\mathfrak{V}$. *As before let*

$$\mathfrak{P}_\gamma := \{\bigotimes_{i<n} \mathsf{V}_{\beta_i}\,;\, \vec\beta \in \gamma^n \text{ with } \beta_i \geq \beta_j > 0 \text{ for all } i < j < n \text{ and } n < \omega\} \cup \{\mathsf{V}_0\}.$$

We consider $\mathfrak{P}_\gamma$ *ordered with the normal lexicographic order* $<_{\mathrm{lex}}$ *and denote the* α*th element of* $\mathfrak{P}_\gamma$ *with* S_α. *Let* λ *be less or equal to the length of* $\mathfrak{P}_\gamma$. *If the following conditions* $\mathrm{I}(\kappa,\alpha)$, $\mathrm{II}(\kappa,\alpha)$, *and* $\mathrm{III}(\kappa,\alpha)$ *hold for* $\alpha < \lambda$ *then the natural* κ*-measure assignment* NMA_κ *derived from the germ assignment* GA_κ *is canonical up to* $\kappa^{(\omega^\lambda)}$.

$\mathrm{I}(\kappa,0)$:	$\kappa^\kappa/\mathbf{lift}_\kappa(\mathbf{germ}(\mathsf{S}_0)) = \kappa^+$,
$\mathrm{I}(\kappa,\alpha)$:	$\kappa^\kappa/\mathbf{lift}_\kappa(\mathbf{germ}(\mathsf{S}_\alpha)) = \kappa^{(\omega^\alpha+1)}$ *for* $0 < \alpha$,
$\mathrm{II}(\kappa,0)$:	$(\kappa^\kappa/\nu)^+ \leq \kappa^\kappa/\nu \oplus \mathbf{lift}_\kappa(\mathbf{germ}(\mathsf{S}_0))$ *for simple order measures* $\nu = \mathbf{lift}_\kappa(\mathbf{germ}(x))$ *with* $o(x) < \omega^\lambda$,
$\mathrm{II}(\kappa,\alpha)$:	$(\kappa^\kappa/\nu)^{(\omega^\alpha+1)} \leq \kappa^\kappa/\nu \oplus \mathbf{lift}_\kappa(\mathbf{germ}(\mathsf{S}_\alpha))$ *for simple order measures* $\nu = \mathbf{lift}_\kappa(\mathbf{germ}(x))$ *with* $o(x) < \omega^\lambda$ *for* $0 < \alpha$, *and*
$\mathrm{III}(\kappa,\alpha)$:	$\mathrm{cf}(\kappa^\kappa/\mathbf{lift}_\kappa(\mathbf{germ}(\mathsf{S}_\alpha))) > \kappa$.

Proof. From Theorem 4.3.2 we know that NMA_κ is almost canonical up to $\kappa^{(\mathrm{ht}(\mathfrak{A}_\gamma))}$. And if $\mathrm{I}_{\kappa,\alpha}$, $\mathrm{II}_{\kappa,\alpha}$, and $\mathrm{III}_{\kappa,\alpha}$ hold for $\alpha < \lambda$ then by Corollary 5.2.3 the measure assignment NMA_κ is canonical up to $\kappa^{(\mathrm{ht}(\mathfrak{A}_\lambda^\oplus))} = \kappa^{(\omega^\lambda)}$ for the induced additive ordinal algebra $\mathfrak{A}_\lambda^\oplus$, so by Lemma 3.3.3 the natural measure assignment NMA_κ is canonical up to $\kappa^{(\omega^\lambda))}$. q.e.d.

In view of Lemma 5.2.4 it is clear how we have to proceed. We want to show that the natural measure assignment we defined in Section 4.5 is canonical and Lemma 5.2.4 tells is that this task essentially boils down to the computation of the values of certain ultrapowers ($\mathrm{I}(\kappa, \alpha)$, $\mathrm{II}(\kappa, \alpha)$) and and their cofinalities ($\mathrm{III}(\kappa, \alpha)$). In the next section we will start an inductive proof to show that conditions $\mathrm{I}(\kappa, \alpha)$, $\mathrm{II}(\kappa, \alpha)$, and $\mathrm{III}(\kappa, \alpha)$ under AD hold for odd projective ordinals κ and for $\alpha < \omega^\omega$.

5.3 The First Step, the Order Measure $\mathcal{C}^\omega_{\boldsymbol{\delta}^1_{2n+1}}$

We want to prove that the measure assignment $\mathrm{NMA}_{\boldsymbol{\delta}^1_{2n+1}}$ is canonical, so we naturally start with the basic case, where we have the germ $\mathbf{germ}(\mathsf{S}_0) = \mathbf{germ}(\mathsf{V}_0) = \mu_{\{0\}}$ with order type $\mathbf{ot}(\mathsf{S}_0) = 1$. We have to check conditions $\mathrm{I}(\boldsymbol{\delta}^1_{2n+1}, 0)$, $\mathrm{II}(\boldsymbol{\delta}^1_{2n+1}, 0)$, and $\mathrm{III}(\boldsymbol{\delta}^1_{2n+1}, 0)$ of Lemma 5.2.4 for all odd projective ordinals $\boldsymbol{\delta}^1_{2n+1}$. From Lemma 4.4.1 we know that $\mathbf{lift}_{\boldsymbol{\delta}^1_{2n+1}}(\mu_{\{0\}})$ is the ω-cofinal measure $\mathcal{C}^\omega_{\boldsymbol{\delta}^1_{2n+1}}$ on $\boldsymbol{\delta}^1_{2n+1}$, condition $\mathrm{II}(\boldsymbol{\delta}^1_{2n+1}, 0)$ demands that we take a closer look at the properties of this order measure.

5.3.1. Proposition. *Assume* $\mathsf{AD}+\mathsf{DC}$. *Let* $\kappa < \Theta$ *be a strong partition cardinal and* μ *an simple order measure on* κ. *Then*

1. $\kappa^+ \leq \kappa^\kappa/\mathcal{C}^\omega_\kappa$, *and*
2. $(\kappa^\kappa/\mu)^+ \leq \kappa^\kappa/(\mu \oplus \mathcal{C}^\omega_\kappa)$.

Proof. Claim 1. is Theorem 2.2.12. So we need only to prove claim 2. For $f : \kappa \to \kappa$ define $\hat{f} : \kappa \to \kappa$ by $\hat{f}(\ulcorner\vec{\alpha}{}^\frown\beta\urcorner) := f(\ulcorner\vec{\alpha}\urcorner)$. By Lemma 5.1.2 this induces an embedding from κ^κ/μ into $\kappa^\kappa/(\mu \oplus \mathcal{C}^\omega_\kappa)$. We'll show that this embeds κ^κ/μ into a proper initial segment of $\kappa^\kappa/(\mu \oplus \mathcal{C}^\omega_\kappa)$ which is enough by Martin's Theorem 1.6.1. Let $f \in \kappa^\kappa$ be arbitrary and let π be defined by $\pi(\ulcorner\vec{\alpha}{}^\frown\beta\urcorner) := \beta$. We shall show that $[\hat{f}]_{\mu\oplus\mathcal{C}^\omega_\kappa} < [\pi]_{\mu\oplus\mathcal{C}^\omega_\kappa}$ holds:

Let S be the set of tuples (x, α), where $x : \varrho \to \kappa$ is a function of continuous type $\mathbf{ot}_\mu$ and $\alpha < \kappa$ an ordinal with cofinality ω such that $\alpha > \sup x$. Note that S is just the set $\mathfrak{C}_\kappa^{\mathbf{ot}_\mu{}^\frown 1}$ in a different notation. We partition this set according to whether

$$f(\ulcorner\overrightarrow{[x]}_{\mathbf{germ}_\mu}\urcorner) < \alpha$$

holds or not. By Lemma 1.9.2 exists a homogeneous club set $C \subseteq \kappa$ for this partition. Toward a contradiction we assume that C is homogeneous for the contrary side, *i.e.*, for all $x \in \mathfrak{C}_C^{\mathbf{ot}_\mu}$ and ordinals $\alpha \in C$ that are greater than $\sup x$ and have cofinality ω

$$f(\ulcorner \overrightarrow{[x]}_{\mathbf{germ}_\mu} \urcorner) \geq \alpha$$

holds. For $\beta < \kappa$ let $o(\beta)$ be the ωth element of C greater than $\max\{\beta, \sup x\}$. Then

$$\forall \beta < \kappa \quad f(\ulcorner \overrightarrow{[x]}_{\mathbf{germ}_\mu} \urcorner) \geq o(\beta) > \beta,$$

which means $f(\ulcorner \overrightarrow{[x]}_{\mathbf{germ}_\mu} \urcorner) \geq \kappa$, a contradiction.

So C is a homogeneous club set for the stated side, let us write down what that means: There is a club set $C \subseteq \kappa$ such that for all $x \in \mathfrak{C}_C^{\mathbf{ot}_\mu}$ and ordinals $\alpha > \sup x$ with cofinality ω, we have $f(\ulcorner \overrightarrow{[x]}_{\mathbf{germ}_\mu} \urcorner) < \alpha$. If we see α as a function $y : 1 \to C$ and use the definition of $\hat{f}$ and π this translates to: There is a club set $C \subseteq \kappa$ such that for all $x \in \mathfrak{C}_C^{\mathbf{ot}_\mu}$, for all $y \in \mathfrak{C}^1_{C_{>\sup x}}$ we have

$$\hat{f}(\ulcorner \overrightarrow{[x]}_{\mathbf{germ}_\mu} {}^\frown [y]_{\mu_{\{0\}}} \urcorner) < \pi(\ulcorner \overrightarrow{[x]}_{\mathbf{germ}_\mu} {}^\frown [y]_{\mu_{\{0\}}} \urcorner).$$

We know that $\mu_{\{0\}}$ is the germ of $\mathcal{C}^\omega_\kappa$, so this is exactly what we wanted to prove. q.e.d.

We now can prove the first step in our inductive proof of the canonicity of the natural measure assignment $\mathrm{NMA}_{\boldsymbol{\delta}^1_{2n+1}}$:

5.3.2. Corollary. *Assume* AD+DC. *The* $\boldsymbol{\delta}^1_{2n+1}$*-measure assignment* $\mathrm{NMA}_{\boldsymbol{\delta}^1_{2n+1}}$ *for* $\mathfrak{A}_1$ *derived from* $\mathrm{GA}_{\boldsymbol{\delta}^1_{2n+1}}$ *is canonical up to height* $(\boldsymbol{\delta}^1_{2n+1})^{(\omega)}$.

Proof. From part 7. of Theorem 2.2.12 we know that

$${\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}} / \mathcal{C}^\omega_{\boldsymbol{\delta}^1_{2n+1}} = (\boldsymbol{\delta}^1_{2n+1})^+$$

and from Theorem 1.7.2 follows

$$\mathrm{cf}({\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}} / \mathcal{C}^\omega_{\boldsymbol{\delta}^1_{2n+1}}) = (\boldsymbol{\delta}^1_{2n+1})^+,$$

so requirements $\mathrm{I}(\boldsymbol{\delta}^1_{2n+1}, 0)$ and $\mathrm{III}(\boldsymbol{\delta}^1_{2n+1}, 0)$ of Lemma 5.2.4 are met. And from Proposition 5.3.1 we get condition $\mathrm{II}(\boldsymbol{\delta}^1_{2n+1}, 0)$. So Lemma 5.2.4 proves this corollary. q.e.d.

The canonicity of the measure assignment $\mathrm{NMA}_{\boldsymbol{\delta}^1_{2n+1}}$ up to height $(\boldsymbol{\delta}^1_{2n+1})^{(\omega)}$ means that we can compute the values of ultrapowers with respect to $\mathcal{C}^\omega_{\boldsymbol{\delta}^1_{2n+1}}$-sums.

5.3.3. Corollary. *Assume* AD + DC. *Let* $\boldsymbol{\delta}^1_{2n+1}$ *be an odd projective ordinal and* $n \in \omega$. *Then* ${\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}} / \mathcal{C}^\omega_{\boldsymbol{\delta}^1_{2n+1}} \otimes n = (\boldsymbol{\delta}^1_{2n+1})^{(n)}$.

Proof. Let κ be a projective ordinal $\boldsymbol{\delta}^1_{2n+1}$. We know $\mathbf{lift}_\kappa(\mu_{\{0\}}) = \mathcal{C}^\omega_\kappa$ and from the canonicity of the measure assignment with $\mathbf{germ}(\mathsf{V}_0) = \mu\{0\}$ we get

$$\kappa^\kappa/\mathcal{C}^\omega_\kappa \otimes n = \kappa^\kappa/\mathbf{lift}_\kappa(\mathbf{germ}(\mathsf{V}_0 \otimes n)) = (\kappa)^{(o(\mathsf{V}_0 \otimes n))} = (\kappa)^{(n)}.$$

q.e.d.

Chapter 6

Computation of the Ultrapowers

In Section 5.3 we did the first step in our inductive proof of the canonicity of the measure assignment NMA_κ. By Lemma 5.2.4 we have to check the conditions

- $\mathrm{I}(\kappa,\alpha)$: $\kappa^\kappa/\mathbf{lift}_\kappa(\mathbf{germ}(\mathsf{S}_\alpha)) = \kappa^{(\omega^\alpha+1)}$ for $0 < \alpha$.
- $\mathrm{II}(\kappa,\alpha)$: $(\kappa^\kappa/\nu)^{(\omega^\alpha+1)} \leq \kappa^\kappa/\nu \oplus \mathbf{lift}_\kappa(\mathbf{germ}(\mathsf{S}_\alpha))$ for simple order measures $\nu = \mathbf{lift}_\kappa(\mathbf{germ}(x))$ with $o(x) < \omega^\lambda$ for $0 < \alpha$.
- $\mathrm{III}(\kappa,\alpha)$: $\mathrm{cf}(\kappa^\kappa/\mathbf{lift}_\kappa(\mathbf{germ}(\mathsf{S}_\alpha))) > \kappa$.

for all odd projective ordinals κ (and for increasing values of α) to continue with this proof. Condition $\mathrm{I}(\kappa,\alpha)$ means we have to compute certain ultrapowers, whereas $\mathrm{II}(\kappa,\alpha)$ only needs lower bounds for certain ultrapowers.

Our general technique for computing the cardinal value of a wellfounded ultrapower is to find upper and lower bounds for the ultrapower and show that they coincide. The lower bound is normally found by embedding sufficiently many ultrapowers into the ultrapower in question. For example, Lemma 6.1.1 will state that for odd projective ordinals κ we have for all $n < \omega$

$$\kappa^\kappa/\mathcal{C}^\omega_\kappa \otimes n \leq \kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa$$

and since by Lemma 2.2.11 no ultrapower on a regular cardinal has cofinality ω we can conclude

$$\kappa^{(\omega+1)} \leq \kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa,$$

which gives us a lower bound for $\kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa$. So if we can show that the lower bound is also an upper bound, *i.e.*,

$$\kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa \leq \kappa^{(\omega+1)},$$

then we have computed the value of $\kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa$ to be $\kappa^{(\omega+1)}$.

6.1 Lower Bounds

We now continue in our induction and examine the ultrapowers generated by the lift of $\mathbf{germ}(\mathsf{V}_1) = \mathcal{C}^{\omega}_{\omega_1}$, $\mathbf{germ}(\mathsf{V}_1 \otimes \mathsf{V}_1) = (\mathcal{C}^{\omega}_{\omega_1})^{\otimes 2}$, and so on. By Corollary 4.4.4 we know $\mathbf{lift}_{\boldsymbol{\delta}^1_{2n+1}}(\mathcal{C}^{\omega}_{\omega_1}) = \mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}}$ and we will prove that the ω_1-cofinal measure generates an ultrapower larger than any of the finite iterations of the ω-cofinal measure. This will give us a lower bound for the ultrapower and also prove condition II$(\kappa, 1)$. In a similar manner we get lower bounds for the ultrapowers $\kappa^\kappa/(\mathcal{C}^{\omega}_{\omega_1})^{\otimes n}$ and prove condition II(κ, n) for all $n < \omega$. First we show that we can embed enough ultrapowers into $\kappa^\kappa/(\mathcal{C}^{\omega_1}_{\kappa})^{\otimes n}$:

6.1.1. LEMMA. *Assume* $\mathsf{AD} + \mathsf{DC}$. *Let κ be an odd projective ordinal, μ a simple order measure on κ, ν a basic order measure on κ, and $n \in \omega$, then*

1. $\kappa^\kappa/\mathcal{C}^{\omega}_{\kappa} \otimes n \leq \kappa^\kappa/\mathcal{C}^{\omega_1}_{\kappa}$,
2. $\kappa^\kappa/\mu \oplus \mathcal{C}^{\omega}_{\kappa} \otimes n \leq \kappa^\kappa/\mu \oplus \mathcal{C}^{\omega_1}_{\kappa}$,
3. $\kappa^\kappa/\nu \otimes n \leq \kappa^\kappa/\nu \otimes \mathcal{C}^{\omega_1}_{\kappa}$, *and*
4. $\kappa^\kappa/\mu \oplus \nu \otimes n \leq \kappa^\kappa/\mu \oplus \nu \otimes \mathcal{C}^{\omega_1}_{\kappa}$.

Proof. 1. and 2. are just 3. and 4. with $\nu = \mathcal{C}^{\omega}_{\kappa}$, since from Lemma 5.1.7 we know that $\mathcal{C}^{\omega}_{\kappa} \otimes \mathcal{C}^{\omega_1}_{\kappa} = \mathcal{C}^{\omega_1}_{\kappa}$. The proof for 3. is just a simpler version of that for 4., so we will only present a proof for 4. Let η be the measure $\mathcal{C}^{\omega}_{\omega_1} \otimes n$ on ω_1, so

$$A \in \eta \;:\Leftrightarrow\; \text{there is a club } C \subseteq \omega_1 \forall \alpha_0 < \cdots < \alpha_{n-1} \in C \text{ such that } \ulcorner\vec{\alpha}\urcorner \in A.$$

Let $m := \mathbf{lh}_\mu$ and define for each $f : \kappa \to \kappa$ a function $\hat{f} : \kappa \to \kappa$ by

$$\hat{f}(\ulcorner\vec{\beta}^\frown[y]_{\mathbf{germ}_\nu \times \mathcal{C}^{\omega}_{\omega_1}}\urcorner) := [\vec{\alpha} \mapsto f(\ulcorner\vec{\beta}^\frown\overrightarrow{[\gamma \mapsto y(\mathbf{ot}_\nu \cdot \alpha + \gamma)]_{\mathbf{germ}_\nu}}\urcorner)]_\eta,$$

where $\vec{\beta} \in [\kappa]^m$ and $\overrightarrow{[\gamma \mapsto y(\mathbf{ot}_\nu \cdot \alpha + \gamma)]_{\mathbf{germ}_\nu}}$ stands for the sequence

$$\langle [\gamma \mapsto y(\mathbf{ot}_\nu \cdot \alpha_0 + \gamma)]_{\mathbf{germ}_\nu}, \ldots, [\gamma \mapsto y(\mathbf{ot}_\nu \cdot \alpha_{n-1} + \gamma)]_{\mathbf{germ}_\nu})\rangle.$$

First we have to prove that $\hat{f}$ is welldefined. If $[y]_{\mathbf{germ}_\nu \times \mathcal{C}^{\omega}_{\omega_1}} = [y']_{\mathbf{germ}_\nu \times \mathcal{C}^{\omega}_{\omega_1}}$ then there is a club set $C \subseteq \omega_1$ such that for all $\delta \in C$ the set

$$A_\delta := \{\gamma < \mathbf{ot}_\nu \,;\, y(\mathbf{ot}_\nu \cdot \delta + \gamma) = y'(\mathbf{ot}_\nu \cdot \delta + \gamma)\}$$

is in $\mathbf{germ}_\nu$. Which means that for all $\delta \in C$ we have

$$[\gamma \mapsto y(\mathbf{ot}_\nu \cdot \delta + \gamma)]_{\mathbf{germ}_\nu} = [\gamma \mapsto y'(\mathbf{ot}_\nu \cdot \delta + \gamma)]_{\mathbf{germ}_\nu}.$$

It follows immediately that for all $\vec{\alpha} \in [C]^n$

$$f(\ulcorner\vec{\beta}^\frown\overrightarrow{[\gamma \mapsto y(\mathbf{ot}_\nu \cdot \alpha + \gamma)]_{\mathbf{germ}_\nu}}\urcorner) = f(\ulcorner\vec{\beta}^\frown\overrightarrow{[\gamma \mapsto y'(\mathbf{ot}_\nu \cdot \alpha + \gamma)]_{\mathbf{germ}_\nu}}\urcorner),$$

so by definition of $\hat{f}$ we get $\hat{f}(\ulcorner\vec{\beta}^\frown[y]_{\mathbf{germ}_\nu \times \mathcal{C}^\omega_{\omega_1}}\urcorner) = \hat{f}(\ulcorner\vec{\beta}^\frown[y']_{\mathbf{germ}_\nu \times \mathcal{C}^\omega_{\omega_1}}\urcorner)$, which means $\hat{f}$ is welldefined with respect to $\mathbf{germ}_\nu \times \mathcal{C}^\omega_{\omega_1}$ equivalence classes.

Now we can show that $f \to \hat{f}$ induces a welldefined embedding from $\kappa^\kappa/\mu \oplus \nu \otimes n$ into $\kappa^\kappa/\mu \oplus \nu \otimes \mathcal{C}^{\omega_1}_\kappa$. Assume $[f]_{\mu\oplus\nu\otimes n} \overset{=}{<} [g]_{\mu\oplus\nu\otimes n}$, *i.e.*, there is a club $C \subseteq \kappa$ such that for all $\vec{z} \in \mathfrak{C}^{\mathbf{ot}_\mu}_C$ and all $\vec{x} \in \mathfrak{C}^{\mathbf{ot}_{\nu\otimes n}}_{C_{>\sup \vec{z}}}$ we have

$$f(\ulcorner\overrightarrow{[z]}_{\mathbf{germ}_\mu}{}^\frown\overrightarrow{[x]}_{\mathbf{germ}_{\nu\otimes n}}\urcorner) \overset{=}{<} g(\ulcorner\overrightarrow{[z]}_{\mathbf{germ}_\mu}{}^\frown\overrightarrow{[x]}_{\mathbf{germ}_{\nu\otimes n}}\urcorner).$$

Let $\vec{z} \in \mathfrak{C}^{\mathbf{ot}_\mu}_C$ and $y \in \mathfrak{C}^{\mathbf{ot}_\nu \cdot \mathbf{ot}_{\mathcal{C}^{\omega_1}_\kappa}}_{C_{>\sup \vec{z}}} = \mathfrak{C}^{\mathbf{ot}_\nu \cdot \omega_1}_{C_{>\sup \vec{z}}}$ be a function of continuous type $\mathbf{ot}_{\nu\otimes\mathcal{C}^{\omega_1}_\kappa}$. For $\vec{\alpha} \in (\omega_1)^n$ define functions $x_{\alpha_i} : \mathbf{ot}_\nu \to C_{>\sup \vec{z}}$ by $x_{\alpha_i}(\gamma) := y(\mathbf{ot}_\nu \cdot \alpha_i + \gamma)$. Since $y : \mathbf{ot}_\nu \cdot \omega_1 \to C_{>\sup \vec{z}}$ is a function of continuous type $\mathbf{ot}_\nu \cdot \omega_1$ we have $\langle x_{\alpha_0}, \ldots, x_{\alpha_{n-1}}\rangle \in \mathfrak{C}^{\mathbf{ot}_{\nu\otimes n}}_{C_{>\sup \vec{z}}}$ for all increasing sequences $\vec{\alpha} \in [\omega_1]^n$. By assumption on C that means for all increasing sequences $\vec{\alpha} \in [\omega_1]^n$ we get

$$\begin{aligned} & f(\ulcorner\overrightarrow{[z]}_{\mathbf{germ}_\mu}{}^\frown\langle[x_{\alpha_0}]_{\mathbf{germ}_\nu}, \ldots, [x_{\alpha_{n-1}}]_{\mathbf{germ}_\nu}\rangle\urcorner) \\ \overset{=}{<}\ & g(\ulcorner\overrightarrow{[z]}_{\mathbf{germ}_\mu}{}^\frown\langle[x_{\alpha_0}]_{\mathbf{germ}_\nu}, \ldots, [x_{\alpha_{n-1}}]_{\mathbf{germ}_\nu}\rangle\urcorner). \end{aligned}$$

Since ω_1 itself is a club subset of ω_1 we immediately get

$$\begin{aligned} & [\vec{\alpha} \mapsto f(\ulcorner\overrightarrow{[z]}_{\mathbf{germ}_\mu}{}^\frown\overrightarrow{[\gamma \mapsto y(\mathbf{ot}_\nu \cdot \alpha + \gamma)]_{\mathbf{germ}_\nu}}\urcorner)]_\eta \\ \overset{=}{<}\ & [\vec{\alpha} \mapsto g(\ulcorner\overrightarrow{[z]}_{\mathbf{germ}_\mu}{}^\frown\overrightarrow{[\gamma \mapsto y(\mathbf{ot}_\nu \cdot \alpha + \gamma)]_{\mathbf{germ}_\nu}}\urcorner)]_\eta, \end{aligned}$$

where $\overrightarrow{[\gamma \mapsto y(\mathbf{ot}_\nu \cdot \alpha + \gamma)]_{\mathbf{germ}_\nu}}$ again stands for the sequence

$$\langle[\gamma \mapsto y(\mathbf{ot}_\nu \cdot \alpha_0 + \gamma)]_{\mathbf{germ}_\nu}, \ldots, [\gamma \mapsto y(\mathbf{ot}_\nu \cdot \alpha_{n-1} + \gamma)]_{\mathbf{germ}_\nu})\rangle.$$

So for all $\vec{z} \in \mathfrak{C}^{\mathbf{ot}_\mu}_C$ and $y \in \mathfrak{C}^{\mathbf{ot}_\nu \cdot \omega_1}_{C_{>\sup \vec{z}}}$ we have

$$\hat{f}(\ulcorner\overrightarrow{[z]}_{\mathbf{germ}_\mu}{}^\frown[y]_{\mathbf{germ}_\nu \times \mathcal{C}^\omega_{\omega_1}}\urcorner) \overset{=}{<} \hat{g}(\ulcorner\overrightarrow{[z]}_{\mathbf{germ}_\mu}{}^\frown[y]_{\mathbf{germ}_\nu \times \mathcal{C}^\omega_{\omega_1}}\urcorner)$$

and this proves that $f \mapsto \hat{f}$ induces a welldefined embedding. q.e.d.

Now we can use Lemma 6.1.1 to show that condition $\mathrm{II}(\kappa, n)$ of Lemma 5.2.4 holds for the natural measure assignment NMA_κ for all odd projective ordinals κ and finite n. Lemma 6.1.1 also enables us to get lower bounds for the ultrapowers $\kappa^\kappa/(\mathcal{C}^{\omega_1}_\kappa)^{\otimes n}$:

6.1.2. PROPOSITION. *Assume* $\mathsf{AD}+\mathsf{DC}$. *Let* κ *be an odd projective ordinal and* μ *a simple order measure on* κ*, then for all natural numbers* $n > 0$ *we have*

1. $\kappa^{(\omega^n+1)} \leq \kappa^\kappa/(\mathcal{C}_\kappa^{\omega_1})^{\otimes n}$, *and*
2. $(\kappa^\kappa/\mu)^{(\omega^n+1)} \leq \kappa^\kappa/(\mu \oplus (\mathcal{C}_\kappa^{\omega_1})^{\otimes n})$.

Proof. We prove this by induction over n and start with $n = 1$. By repeatedly applying Proposition 5.3.1 we get that for any $m \in \omega$

$$\kappa^{(m)} \leq \kappa^\kappa/\mathcal{C}_\kappa^\omega \otimes m \text{ and}$$
$$(\kappa^\kappa/\mu)^{(m)} \leq \kappa^\kappa/(\mu \oplus \mathcal{C}_\kappa^\omega \otimes m).$$

By taking the supremum over n on both sides and using Lemma 6.1.1 this yields

$$\kappa^{(\omega)} \leq \sup_{m\in\omega}(\kappa^\kappa/\mathcal{C}_\kappa^\omega \otimes m) \leq \kappa^\kappa/\mathcal{C}_\kappa^{\omega_1} \text{ and}$$
$$(\kappa^\kappa/\mu)^{(\omega)} \leq \sup_{m\in\omega}(\kappa^\kappa/\mu \oplus \mathcal{C}_\kappa^\omega \otimes m) \leq \kappa^\kappa/\mu \oplus \mathcal{C}_\kappa^{\omega_1}.$$

But we know from Lemma 2.2.11 that both $\mathrm{cf}(\kappa^\kappa/\mathcal{C}_\kappa^{\omega_1})$ and $\mathrm{cf}(\kappa^\kappa/\mu \oplus \mathcal{C}_\kappa^{\omega_1})$ are greater ω, so

$$\kappa^{(\omega+1)} \leq \kappa^\kappa/\oplus\, \mathcal{C}_\kappa^{\omega_1} \text{ and}$$
$$(\kappa^\kappa/\mu)^{(\omega+1)} \leq \kappa^\kappa/(\mu \oplus \mathcal{C}_\kappa^{\omega_1}).$$

For the induction step assume $\kappa^{(\omega^n+1)} \leq \kappa^\kappa/(\mathcal{C}_\kappa^{\omega_1})^{\otimes n}$ and $(\kappa^\kappa/\mu')^{(\omega^n+1)} \leq \kappa^\kappa/(\mu' \oplus (\mathcal{C}_\kappa^{\omega_1})^{\otimes n}$ holds for n and all simple order measures μ'. Let $m > 0$ be a natural number, then we get by repeated application of this induction hypothesis

$$\kappa^{(\omega^n\cdot m+1)} \leq (\kappa^\kappa/(\mathcal{C}_\kappa^{\omega_1})^{\otimes n})^{(\omega^n\cdot(m-1)+1)} \leq \ldots \leq \kappa^\kappa/(\mathcal{C}_\kappa^{\omega_1})^{\otimes n} \otimes m \text{ and}$$
$$(\kappa^\kappa/\mu)^{(\omega^n\cdot m+1)} \leq (\kappa^\kappa/(\mu \oplus (\mathcal{C}_\kappa^{\omega_1})^{\otimes n})^{(\omega^n\cdot(m-1)+1)} \leq \ldots \leq \kappa^\kappa/(\mu \oplus (\mathcal{C}_\kappa^{\omega_1})^{\otimes n} \otimes m.$$

Now we use Lemma 6.1.1 with $\nu = (\mathcal{C}_\kappa^{\omega_1})^{\otimes n}$ in order to get

$$\kappa^\kappa/(\mathcal{C}_\kappa^{\omega_1})^{\otimes n} \otimes m \leq \kappa^\kappa/(\mathcal{C}_\kappa^{\omega_1})^{\otimes n} \otimes \mathcal{C}_\kappa^{\omega_1} = \kappa^\kappa/(\mathcal{C}_\kappa^{\omega_1})^{\otimes(n+1)} \text{ and}$$
$$\kappa^\kappa/\mu \oplus (\mathcal{C}_\kappa^{\omega_1})^{\otimes n} \otimes m \leq \kappa^\kappa/\mu \oplus (\mathcal{C}_\kappa^{\omega_1})^{\otimes n} \otimes \mathcal{C}_\kappa^{\omega_1} = \kappa^\kappa/\mu \oplus (\mathcal{C}_\kappa^{\omega_1})^{\otimes(n+1)}.$$

By taking the supremum over m we arrive at

$$\kappa^{(\omega^{n+1})} = \sup_{m\in\omega}\left(\kappa^{(\omega^n\cdot m+1)}\right) \leq \kappa^\kappa/(\mathcal{C}_\kappa^{\omega_1})^{\otimes(n+1)} \text{ and}$$
$$(\kappa^\kappa/\mu)^{(\omega^{n+1})} = \sup_{m\in\omega}\left((\kappa^\kappa/\mu)^{(\omega^n\cdot m+1)}\right) \leq \kappa^\kappa/\mu \oplus (\mathcal{C}_\kappa^{\omega_1})^{\otimes(n+1)}.$$

We know from Lemma 2.2.11 that the cofinality of the ultrapowers $\kappa^\kappa/(\mathcal{C}_\kappa^{\omega_1})^{\otimes(n+1)}$ and $\kappa^\kappa/\mu \oplus (\mathcal{C}_\kappa^{\omega_1})^{\otimes(n+1)}$ is greater ω, so

$$\kappa^{(\omega^{n+1})} < \kappa^\kappa/(\mathcal{C}_\kappa^{\omega_1})^{\otimes(n+1)} \text{ and}$$
$$(\kappa^\kappa/\mu)^{(\omega^{n+1})} < \kappa^\kappa/\mu \oplus (\mathcal{C}_\kappa^{\omega_1})^{\otimes(n+1)},$$

which proves the induction step, since by Martins Theorem 1.6.1 the two ultrapowers are cardinals. q.e.d.

6.2 How to Compute Upper Bounds

Our technique for computing the cardinal value of a wellfounded ultrapower is to find upper and lower bounds for the ultrapower and show they coincide. Proposition 6.1.2 provided us with lower bounds for ${\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}}/(\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}})^{\otimes n}$ and proved the validity of condition $\mathrm{II}(\boldsymbol{\delta}^1_{2n+1}, n)$ for $n < \omega$. Now we have to find a good upper bounds for the ultrapowers ${\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}}/(\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}})^{\otimes n}$.

In order to do this we reduce the problem of finding an upper bound to finding an upper bound for restricted versions of the ultrapower in question. For this we will need to be able to get dominating functions for the functions that generate equivalence classes:

6.2.1. LEMMA. *Assume $\kappa <$ is a cardinal with the weak partition property and μ a basic order measure on κ with order type $\varrho < \kappa$. If $G : \kappa \to \kappa$ represents the equivalence class $[G]_\mu \in \kappa^\kappa/\mu$ then there is a function $g : \kappa \to \kappa$ such that there is a club set $C \subseteq \kappa$ such that $G([f]_{\mathbf{germ}_\mu}) \leq g(\sup f)$ for all functions $f \in \mathcal{C}^\varrho_C$.*

Proof. Let S be the set of tuples (f, α), where $f : \varrho \to \kappa$ is a function of continuous type ϱ and α an ordinal with cofinality ω such that $\alpha > \sup f$. Note that S is just the set $\mathfrak{C}^{\langle\varrho,1\rangle}_\kappa$ in a different notation. We partition this set according to whether

$$G([f]_{\mathbf{germ}_\mu}) < \alpha$$

holds or not. By Lemma 1.9.2 exists a homogeneous club set $C \subseteq \kappa$ for this partition. Toward a contradiction we assume that C is homogeneous for the contrary side, *i.e.*, for all $f \in \mathfrak{C}^\varrho_C$ and ordinals $\alpha \in C$ that are greater than $\sup f$ and have cofinality ω

$$G([f]_{\mathbf{germ}_\mu}) \geq \alpha$$

holds. For $\beta < \kappa$ let $\sigma(\beta)$ be the ωth element of C greater than $\max\{\beta, \sup f\}$. Then

$$\forall \beta < \kappa \quad G([f]_{\mathbf{germ}_\mu}) \geq \sigma(\beta) > \beta,$$

which means $G([f]_{\mathbf{germ}_\mu}) \geq \kappa$, a contradiction.

So C is a homogeneous club set for the stated side. For $\alpha < \kappa$ we define

$$g(\alpha) := \delta, \text{ where } \delta \text{ is the } \omega\text{th element of } C \text{ greater than } \alpha,$$

this function does the job: If f is an element of $\mathfrak{C}^\varrho_C$ then $g(\sup f)$ is larger than $\sup f$, has cofinality ω and is an element of C, so $G([f]_{\mathbf{germ}_\mu}) < g(\sup f)$ holds. q.e.d.

Now we can use Martin trees to show that the successor of the supremum of certain restricted versions of the ultrapower is an upper bound for the ultrapower itself.

6.2.2. Lemma. *Assume* $\mathsf{AD}+\mathsf{DC}$, *let* $\kappa > \omega_1$ *be an odd projective ordinal and* μ *a basic order measure on* κ *with* $\mathrm{cf}(\mathbf{ot}_\mu) = \omega_1$. *Then*

$$\kappa^\kappa/\mu \leq (\sup_n [G_n]_\mu)^+,$$

where

$$G_n([f]_{\mathbf{germ}_\mu}) := (\sup f)^{\omega_1}/\mathcal{C}^\omega_{\omega_1} \otimes n.$$

Proof. Let $[G]_\mu$ be an element of κ^κ/μ, where $G : \kappa \to \kappa$. By Lemma 6.2.1 there is a function $g : \kappa \to \kappa$ and a club set $C \subseteq \kappa$ such that

$$\forall f : \mathbf{ot}_\mu \to C \quad G([f]_{\mathbf{germ}_\mu}) \leq g(\sup f).$$

Since κ is an odd projective ordinal we can apply Theorem 2.2.19 to the function g. So we get a Martin tree T on κ and a club set $C \subseteq \kappa$ such that for all $\alpha \in C$ with $\mathrm{cf}(\alpha) = \omega_1$ we have $g(\alpha) < |T \upharpoonright \sup_n \left(\alpha^{\omega_1}/\mathcal{W}^n_{\omega_1}\right)|$. For all functions $f \in \mathfrak{C}^{\mathbf{ot}_\mu}_C$ of continuous type $\mathbf{ot}_\mu$ we have that $\sup f$ is an element of C with cofinality $\mathbf{ot}_\mu = \omega_1$. Which means that for all functions $f \in \mathfrak{C}^{\mathbf{ot}_\mu}_C$ we have

$$G([f]_{\mathbf{germ}_\mu}) \leq g(\sup f) < |T \upharpoonright \sup_n \left((\sup f)^{\omega_1}/\mathcal{C}^\omega_{\omega_1} \otimes n\right)|.$$

Since no wellorder on an ordinal is longer than the cardinal successor of this ordinal this means

$$\forall f \in \mathfrak{C}^{\mathbf{ot}_\mu}_C \quad G([f]_{\mathbf{germ}_\mu}) < \left(\sup_n \left((\sup f)^{\omega_1}/\mathcal{C}^\omega_{\omega_1} \otimes n\right)\right)^+.$$

So for all $G : \kappa \to \kappa$ exists a club set $C \subseteq \kappa$ such that

$$\forall f \in \mathfrak{C}^{\mathbf{ot}_\mu}_C \quad G([f]_{\mathbf{germ}_\mu}) < \left(\sup_n G_n([f]_{\mathbf{germ}_\mu})\right)^+,$$

in other words $\kappa^\kappa/\mu \leq (\sup_n [G_n]_\mu)^+$. q.e.d.

Which means, if we can compute $[G_n]_{\mathcal{C}^{\omega_1}_\kappa}$ for all n we get an upper bound for $\kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa$, if we can compute $[G_n]_{\mathcal{C}^{\omega_1}_\kappa \otimes \mathcal{C}^{\omega_1}_\kappa}$ we get an upper bound for $\kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa \otimes \mathcal{C}^{\omega_1}_\kappa$, etc.

We will show that for all natural numbers $n > 0$ and basic order measures $\mu \neq \mathcal{C}^\omega_\kappa$ on κ there is an cofinal embedding $\pi_{\mu,n} : \kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa \to [G_n]_{\mu \otimes \mathcal{C}^{\omega_1}_\kappa}$.

6.2.3. Definition. Assume AD. Let $\kappa > \omega_1$ be an odd projective ordinal. Let $\mu \neq \mathcal{C}^\omega_\kappa$ be a basic order measure on κ and $0 < n < \omega$, we define $\pi_{\mu,n} : \kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa \to [G_n]_{\mu \otimes \mathcal{C}^{\omega_1}_\kappa}$ as follows: $\pi_{\mu,n}([F]_{\mathcal{C}^{\omega_1}_\kappa}) := [\pi_{\mu,n}(F)]_{\mu \otimes \mathcal{C}^{\omega_1}_\kappa}$, where for $g : \mathbf{ot}_\mu \cdot \omega_1 \to \kappa$ of continuous type $\mathbf{ot}_\mu \cdot \omega_1$ we have

$$\pi_{\mu,n}(F)([g]_{\mathbf{germ}_\mu \times \mathcal{C}^\omega_{\omega_1}}) := [\vec{\alpha} \to F(\sup_\beta g(\beta, \alpha_{n-1}))]_{\mathcal{C}^\omega_{\omega_1} \otimes n}.$$

Then $\pi_{\mu,n}(F)([g]_{\mathbf{germ}_\mu \times \mathcal{C}^\omega_{\omega_1}}$ is welldefined: Let $g, g' : \mathbf{ot}_\mu \cdot \omega_1 \to \kappa$ be functions of continuous type $\mathbf{ot}_\mu \cdot \omega_1$ and assume $[g]_{\mathbf{germ}_\mu \times \mathcal{C}^\omega_{\omega_1}} = [g']_{\mathbf{germ}_\mu \times \mathcal{C}^\omega_{\omega_1}}$ holds. In Remark 1.4.6 we mentioned that we can view g as a function on a product set instead of on a product ordinal. So there is a club set $C \subseteq \omega_1$ such that the set $D_\alpha := \{\delta < \mathbf{ot}_\mu\,;\, g(\delta,\alpha) = g'(\delta,\alpha)\}$ is in μ for all $\alpha \in C$, and since g and g' are of continuous type $\mathbf{ot}_\mu \cdot \omega_1$ and $\mathbf{germ}_\mu$ contains end segments we have for all $\alpha \in C$

$$\sup_\beta g(\beta,\alpha) = \sup_{\beta \in D_\alpha} g(\beta,\alpha) = \sup_{\beta \in D_\alpha} g'(\beta,\alpha) = \sup_\beta g(\beta,\alpha),$$

i.e., for all $\vec{\alpha} \in [C]^n$ we have $F(\sup_\beta g(\beta,\alpha_n)) = F(\sup_\beta g'(\beta,\alpha_n))$.

By Lemma 1.3.9 exists a club set $C \subset \kappa$ that is closed under F, so for all functions $g \in \mathfrak{C}_C^{\mathbf{ot}_\mu \cdot \omega_1}$ we have $F(\sup_\beta g(\beta,\alpha) < \sup g$ for all $\alpha \in \omega_1$. And since ω_1 is itself a club subset of ω_1 that means for all functions $g \in \mathfrak{C}_C^{\mathbf{ot}_\mu \cdot \omega_1}$ exits a club set $D \subseteq \omega_1$ such that

$$F(\sup_\beta g(\beta,\alpha_{n-1}) < \sup g \text{ for all } \vec{\alpha} \in [D]^n,$$

i.e., $[\vec{\alpha} \mapsto F(\sup_\beta g(\beta,\alpha_{n-1}))]_{\mathcal{C}^\omega_{\omega_1} \otimes n} \in (\sup g)^{\omega_1}/\mathcal{C}^\omega_{\omega_1} \otimes n$ for all functions $g \in \mathfrak{C}_C^{\mathbf{ot}_\mu \cdot \omega_1}$, and thus

$$[\pi_{\mu,n}(F)]_{\mu \otimes \mathcal{C}^{\omega_1}_\kappa} \in [G_n]_{\mu \otimes \mathcal{C}^{\omega_1}_\kappa}.$$

Now we have to prove that $\pi_{\mu,n}$ is a welldefined embedding. So let $F, G : \kappa \to \kappa$ be functions such that $[F]_{\mathcal{C}^{\omega_1}_\kappa} \overset{=}{\underset{<}{}} [G]_{\mathcal{C}^{\omega_1}_\kappa}$. That means there exists a club subset $C \subseteq \kappa$ such that for all functions $g \in \mathfrak{C}_C^{\omega_1}$ there exits a club set $D \subseteq \omega_1$ such that for all $\alpha \in D$

$$F(g(\alpha)) \overset{=}{\underset{<}{}} G(g(\alpha)).$$

Let $h \in \mathfrak{C}_C^{\mathbf{ot}_\mu \cdot \omega_1}$ be a function of continuous type $\mathbf{ot}_\mu \cdot \omega_1$ and define the function $g_h : \omega_1 \to C$ by $g_h(\alpha) := \sup_\beta h(\beta,\alpha)$, then g_h is a function of continuous type ω_1 with range C. So for all functions $h \in \mathfrak{C}_C^{\mathbf{ot}_\mu \cdot \omega_1}$ exits a club set $D \subseteq \omega_1$ such that for all $\alpha \in D$

$$F(\sup_\beta h(\beta,\alpha)) \overset{=}{\underset{<}{}} G(\sup_\beta h(\beta,\alpha)),$$

we can go from D to $[D]^n$ as we did before and get for all functions $h \in \mathfrak{C}_C^{\mathbf{ot}_\mu \cdot \omega_1}$

$$[\vec{\alpha} \mapsto F(\sup_\beta h(\beta,\alpha_{n-1}))]_{\mathcal{C}^\omega_{\omega_1} \otimes n} \overset{=}{\underset{<}{}} [\vec{\alpha} \mapsto G(\sup_\beta h(\beta,\alpha_{n-1}))]_{\mathcal{C}^\omega_{\omega_1} \otimes n}.$$

Which is equivalent to $[\pi_{\mu,n}(F)]_{\mu \otimes \mathcal{C}^{\omega_1}_\kappa} \overset{=}{\underset{<}{}} [\pi_{\mu,n}(G)]_{\mu \otimes \mathcal{C}^{\omega_1}_\kappa}$, so $\pi_{\mu,n}$ is indeed a welldefined embedding.

6.2.4. Lemma. *Let $\kappa > \omega$ be a regular cardinal. If $f : [\omega_1]^n \to \kappa$ is a function then there is $f' : \omega_1 \to \kappa$ such that $[f]_{\mathcal{C}^\omega_{\omega_1} \otimes n} \leq [\vec{\alpha} \mapsto f'(\alpha_{n-1})]_{\mathcal{C}^\omega_{\omega_1} \otimes n}$ and $\sup f = \sup f'$.*

Proof. Let $f'(\alpha) := \sup_{\alpha_0<\alpha_1<\ldots<\alpha_{n-2}<\alpha} f(\alpha_0, \ldots, \alpha_{n-2}, \alpha)$. Let C be a club subset of ω_1, then $f(\vec{\alpha}) \leq f'(\alpha_{n-1})$ holds for all $\vec{\alpha} \in [C]^n$, *i.e.*, $[f]_{\mathcal{C}^\omega_{\omega_1}\otimes n} \leq [\vec{\alpha} \mapsto f'(\alpha_{n-1})]_{\mathcal{C}^\omega_{\omega_1}\otimes n}$, and $\sup f = \sup_{\alpha_{n-1}} \sup_{\alpha_0<\alpha_1<\ldots<\alpha_{n-2}} f(\vec{\alpha}) = \sup f'$ q.e.d.

6.2.5. Lemma. *Assume* $\mathsf{AD}+\mathsf{DC}$. *Let* $\kappa > \boldsymbol{\delta}^1_1$ *be an odd projective ordinal. Let* $\mu \neq \mathcal{C}^\omega_\kappa$ *be a basic order measure on* κ *and* $0 < n < \omega$. *For every function* $G : \kappa \to \kappa$ *with* $[G]_{\mu\otimes\mathcal{C}^{\omega_1}} \in [G_n]_{\mu\otimes\mathcal{C}^{\omega_1}_\kappa}$ *exists a function* $F : \kappa \to \kappa$ *such that*

$$\pi_{\mu,n}([F]_{\mathcal{C}^{\omega_1}_\kappa}) > [G]_{\mu\otimes\mathcal{C}^{\omega_1}_\kappa},$$

i.e., $\pi_{\mu,n}$ *is cofinal into* $[G_n]_{\mu\otimes\mathcal{C}^{\omega_1}_\kappa}$.

Proof. Since $\mu \neq \mathcal{C}^\omega_\kappa$ the order type $\mathbf{ot}_\mu$ is a limit ordinal greater ω. Let P be the set of pairs (g, f), where $g : \mathbf{ot}_\mu \cdot \omega_1 \to \kappa$ is of continuous type $\mathbf{ot}_\mu \cdot \omega_1$, $f : \omega_1 \to \kappa$ is of discontinuous type ω_1, and

$$g(0, \alpha+1) \left(= \sup_{\beta<\mathbf{ot}_\mu} g(\beta, \alpha) \right) < f(\alpha) < g(1, \alpha+1,)$$

holds for all $\alpha < \omega_1$. We partition this set according to whether

$$G([g]_{\mathbf{germ}_\mu\times\mathcal{C}^\omega_{\omega_1}}) < [\vec{\alpha} \mapsto f(\alpha_{n-1})]_{\mathcal{C}^\omega_{\omega_1}\otimes n}$$

holds or not.

Toward a contradiction, suppose $C \subseteq \kappa$ is homogeneous for the contrary side. Since $[G]_{\mu\otimes\mathcal{C}^{\omega_1}_\kappa} \in [G_n]_{\mu\otimes\mathcal{C}^{\omega_1}_\kappa}$ we can also assume that for all functions $g : \mathbf{ot}_\mu\cdot\omega_1 \to C$ of continuous type $\mathbf{ot}_\mu \cdot \omega_1$ we have

$$G([g]_{\mathbf{germ}_\mu\times\mathcal{C}^\omega_{\omega_1}}) < G_n([g]_{\mathbf{germ}_\mu\times\mathcal{C}^\omega_{\omega_1}}) = (\sup g)^{\omega_1}/\mathcal{C}^\omega_{\omega_1} \otimes n.$$

Let $[f]_{\mathcal{C}^\omega_{\omega_1}\otimes n}$ be an element of $(\sup g)^{\omega_1}/\mathcal{C}^\omega_{\omega_1} \otimes n$ that dominates $G([g]_{\mathbf{germ}_\mu\times\mathcal{C}^\omega_{\omega_1}})$, we can assume $\sup f = \sup g$ and by Lemma 6.2.4 we can also assume that f only depends on α_{n-1}, *i.e.*, is a function from ω_1 to κ:

$$G([g]_{\mathbf{germ}_\mu\times\mathcal{C}^\omega_{\omega_1}}) < [\vec{\alpha} \mapsto f(\alpha_{n-1})]_{\mathcal{C}^\omega_{\omega_1}\otimes n}.$$

We do a shifting argument and define functions $g' : \mathbf{ot}_\mu \cdot \omega_1 \to C$ and $f' : \omega_1 \to C$ as follows:

- Let $g'(\beta, 0)$ be $g(\beta, 0)$, for all $\beta < \mathbf{ot}_\mu$,
- let $g'(0, \alpha+1)$ be $\sup_{\beta<\mathbf{ot}_\mu} g'(\beta, \alpha)$, for all $\alpha < \omega_1$,
- let $g'(\beta, \lambda,)$ be $g(\beta, \lambda')$, where λ' is such that $g(0, \lambda') = \sup_{\alpha<\lambda,\beta<\mathbf{ot}_\mu} g'(\beta, \alpha)$, for all $\beta < \mathbf{ot}_\mu$ and limits $\lambda < \omega_1$,

- let $f'(\alpha)$ be the ωth element of C greater than the maximum of $g'(0, \alpha+1)$ and $f(\alpha)$, for all $\alpha < \omega_1$,
- let $g'(\beta, \alpha+1)$ be $g(\beta, \alpha')$, where α' is least such that $g(1, \alpha') > f'(\alpha)$, for all $0 < \beta < \mathbf{ot}_\mu$ and $\alpha < \omega_1$.

Then $g' : \mathbf{ot}_\mu \cdot \omega_1 \to C$ is of continuous type $\mathbf{ot}_\mu \cdot \omega_1$, $f' : \omega_1 \to C$ is of discontinuous type ω_1, $[g']_{\mathbf{germ}_\mu \times \mathcal{C}^\omega_{\omega_1}} = [g]_{\mathbf{germ}_\mu \times \mathcal{C}^\omega_{\omega_1}}$, $[\vec{\alpha} \mapsto f'(\alpha_n)]_{\mathcal{C}^\omega_{\omega_1} \otimes n} \geq [\vec{\alpha} \mapsto f(\alpha_n)]_{\mathcal{C}^\omega_{\omega_1} \otimes n}$, $\sup g' = \sup f' = \sup f = \sup g$, and f' and g' are ordered as in P. But since $G([g]_{\mathbf{germ}_\mu \times \mathcal{C}^\omega_{\omega_1}})$ depends only on the $\mathbf{germ}_\mu \times \mathcal{C}^\omega_{\omega_1}$-equivalence class of g this contradicts the homogeneity of C for the contrary side, so C has to be homogeneous for the stated side.

That means we have $G([g]_{\mathbf{germ}_\mu \times \mathcal{C}^\omega_{\omega_1}}) < [\vec{\alpha} \mapsto f(\alpha_n)]_{\mathcal{C}^\omega_{\omega_1} \otimes n}$ for all pairs (g, f) in P such that f and g have range in C. We use the club set C to define a function $F : \kappa \to C$:

Let $F(\alpha)$ be the ωth element of C greater than α.

This function F will do the job, we have to show that $[\pi(F)]_{\mu \otimes \mathcal{C}^{\omega_1}_\kappa} > [G]_{\mu \otimes \mathcal{C}^{\omega_1}_\kappa}$ holds. Let $g : \mathbf{ot}_\mu \cdot \omega_1 \to C$ be a function of continuous type $\mathbf{ot}_\mu \cdot \omega_1$, we define functions $g' : \mathbf{ot}_\mu \cdot \omega_1 \to C$ and $f_g : \omega_1 \to C$ as follows:

- Let $g'(\beta, 0)$ be $g(\beta, 0)$, for all $\beta < \mathbf{ot}_\mu$,
- let $g'(0, \alpha+1)$ be $g(0, \alpha+1)$,*i.e.*, $\sup_{\beta < \mathbf{ot}_\mu} g(\beta, \alpha)$, for all $\alpha < \omega_1$,
- let $g'(\beta, \lambda)$ be $g(\beta, \lambda)$, for all $\beta < \mathbf{ot}_\mu$ and limits $\lambda < \omega_1$,
- let $f_g(\alpha)$ be the ωth element of C greater than $g(0, \alpha+1)$, for all $\alpha < \omega_1$,
- let $g'(\beta, \alpha+1)$ be $g(\eta+\beta, \alpha)$, where η is least such that $g(\eta, \alpha) > f_g(\alpha)$, for all $0 < \beta < \mathbf{ot}_\mu$ and $\alpha < \omega_1$.

Then $g' : \mathbf{ot}_\mu \cdot \omega_1 \to C$ is of continuous type $\mathbf{ot}_\mu \cdot \omega_1$, $[g']_{\mathbf{germ}_\mu \times \mathcal{C}^\omega_{\omega_1}} = [g]_{\mathbf{germ}_\mu \times \mathcal{C}^\omega_{\omega_1}}$, $f_g : \omega_1 \to C$ is of discontinuous type ω_1, the functions g' and f_g are ordered as in P and $\sup_{\beta < \mathbf{ot}_\mu} g(\beta, \alpha) = \sup_{\beta < \mathbf{ot}_\mu} g'(\beta, \alpha)$ for all limit ordinals $\alpha < \omega_1$. Since C was homogeneous for the stated side of the partition we get

$$G([g]_{\mathbf{germ}_\mu \times \mathcal{C}^\omega_{\omega_1}}) = G([g']_{\mathbf{germ}_\mu \times \mathcal{C}^\omega_{\omega_1}}) < [\vec{\alpha} \mapsto f_g(\alpha_{n-1})]_{\mathcal{C}^\omega_{\omega_1} \otimes n} \Leftrightarrow$$

$$G([g]_{\mathbf{germ}_\mu \times \mathcal{C}^\omega_{\omega_1}}) < [\vec{\alpha} \mapsto \text{ the } \omega\text{th element of } C \text{ greater } g'(0, \alpha_{n-1}+1)]_{\mathcal{C}^\omega_{\omega_1} \otimes n}$$

$$\Leftrightarrow G([g]_{\mathbf{germ}_\mu \times \mathcal{C}^\omega_{\omega_1}}) < [\vec{\alpha} \mapsto F(\sup_{\beta < \mathbf{ot}_\mu} g'(\beta, \alpha_{n-1})]_{\mathcal{C}^\omega_{\omega_1} \otimes n}$$

$$\Leftrightarrow G([g]_{\mathbf{germ}_\mu \times \mathcal{C}^\omega_{\omega_1}}) < [\vec{\alpha} \mapsto F(\sup_{\beta < \mathbf{ot}_\mu} g(\beta, \alpha_{n-1})]_{\mathcal{C}^\omega_{\omega_1} \otimes n}$$

$$\Leftrightarrow G([g]_{\mathbf{germ}_\mu \times \mathcal{C}^\omega_{\omega_1}}) < \pi_{\mu,n}(F)([g]_{\mathbf{germ}_\mu \times \mathcal{C}^\omega_{\omega_1}}).$$

q.e.d.

6.3 Computation of $(\boldsymbol{\delta}^1_{2n+1})^{\boldsymbol{\delta}^1_{2n+1}}/\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}}$

We proceed with the computation of the ultrapowers $\kappa^\kappa/(\mathcal{C}^{\omega_1}_\kappa)^{\otimes n}$ for odd projective ordinals κ. In this section we will deal with the case $n = 1$, *i.e.*, we show that the ultrapower $\kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa$ is $\kappa^{(\omega+1)}$. This is not a new result, Martin showed ${\boldsymbol{\delta}^1_3}^{\boldsymbol{\delta}^1_3}/\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_3}$ from AD before 1981, see the appendix of [KeMo78]. The proof of Theorem 6.4.4 is similar but more complicated than the proof of this result, so that the method can be seen more clearly in this simpler case. Theorem 6.3.4 is also needed as induction basis for Theorem 6.4.4.

From Proposition 6.1.2 we have a lower bound for the ultrapower $\kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa$ and by Lemma 6.2.2 it is enough to get upper bounds for the restricted versions of the ultrapower to get an upper bound for $\kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa$. In the proof of Lemma 6.2.2 we used Martin trees to get upper bounds for functions and we will use a similar idea in the proofs of Theorem 6.3.4 and Theorem 6.4.4. In this section we need only the result for Kunen trees:

6.3.1. LEMMA. *Assume* AD + DC *and let κ be an odd projective ordinal. Let $\vartheta : \kappa \to \kappa$ be a function, C a club subset of κ and K a tree on κ such that for all $g : \omega_1 \to C$ of continuous type ω_1 we have*

$$\vartheta([g]_{\mathcal{C}^\omega_{\omega_1}}) < [\vec{\alpha} \mapsto |K{\restriction}g(\alpha_{n-1})|]_{\mathcal{C}^\omega_{\omega_1}\otimes n}.$$

Then there is a function $\vartheta^1 : \kappa \times [\omega_1]^n \to \kappa$ such that for all functions $g : \omega_1 \to C$ of continuous type ω_1 we have

$$\vartheta([g]_{\mathcal{C}^\omega_{\omega_1}}) = [\vec{\alpha} \mapsto |K{\restriction}g(\alpha_{n-1})|(\vartheta^1([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}))]_{\mathcal{C}^\omega_{\omega_1}\otimes n} \text{ and}$$

$$\vartheta^1([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}) < g(\alpha_{n-1}) \text{ for all } \vec{\alpha} \in [\omega_1]^n.$$

Proof. Let $g : \omega_1 \to C$ be a function of continuous type ω_1. Let $t : [\omega_1]^n \to \kappa$ represent the ordinal $\vartheta([g]_{\mathcal{C}^\omega_{\omega_1}})$ with respect to $\mathcal{C}^\omega_{\omega_1} \otimes n$. We define the function $\vartheta^1([g]_{\mathcal{C}^\omega_{\omega_1}}) : [\omega_1]^n \to g(\alpha_{n-1})$ by

$$\vartheta^1([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}) := \min\{\beta < \kappa\,;\, t(\vec{\alpha}) = |K{\restriction}g(\alpha_{n-1})|(\beta)\}.$$

So if $g, g' : \omega_1 \to C''$ are functions of continuous type ω_1 with $[g]_{\mathcal{C}^\omega_{\omega_1}} = [g]_{\mathcal{C}^\omega_{\omega_1}}$ and $t, t' : [\omega_1]^n \to \kappa$ are functions such that $\vartheta([g]_{\mathcal{C}^\omega_{\omega_1}}) = [t]_{\mathcal{C}^\omega_{\omega_1}\otimes n} = [t']_{\mathcal{C}^\omega_{\omega_1}\otimes n}$ then for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$ we have

$$\min\{\beta < \kappa\,;\, t(\vec{\alpha}) = |K{\restriction}g(\alpha_{n-1})|(\beta)\} = \min\{\beta < \kappa\,;\, t'(\vec{\alpha}) = |K{\restriction}g'(\alpha_{n-1})|(\beta)\}.$$

That means ϑ^1 is welldefined on a $\mathcal{C}^\omega_{\omega_1} \otimes n$-measure 1 set and by its definition we have for all $g : \omega_1 \to C$ of continuous type ω_1

$$\vartheta([g]_{\mathcal{C}^\omega_{\omega_1}}) = [\vec{\alpha} \mapsto |K{\restriction}g(\alpha_{n-1})|(\vartheta^1([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}))]_{\mathcal{C}^\omega_{\omega_1}\otimes n}$$

with $\vartheta^1([g]_{\mathcal{C}^{\omega}_{\omega_1}})(\vec{\alpha}) < g(\alpha_{n-1})$ for all $\vec{\alpha} \in [\omega_1]^n$. q.e.d.

So we analyze elements $[\vartheta]_{\mathcal{C}^{\omega_1}_\kappa}$ of the ultrapower $(\boldsymbol{\delta}^1_{2n+1})^\kappa/\mathcal{C}^{\omega_1}_\kappa$ using Kunen trees. We want to do this until we have a complete representation of $[\vartheta]_{\mathcal{C}^{\omega_1}_\kappa}$ for which we then can compute its ordinal value. This will require the following technical lemmas:

6.3.2. Lemma. *We assume the weak partition property of ω_1. Let $\kappa > \omega$ be a regular cardinal, n a natural number, $H : [\omega_1]^{n+1} \to \kappa$ a function, and $g : \omega_1 \to \kappa$ an increasing continuous function. If there is a club set $D \subseteq \omega_1$ such that $H(\vec{\alpha}) < g(\alpha_n)$ holds for all $\vec{\alpha} \in [D]^{n+1}$ then there is a club set $C \subseteq \omega_1$ such that*

- *there is a function $\ell_g : \omega_1 \to \omega_1$ such that $H(\vec{\alpha}) < g(\ell_g(\alpha_{n-1}))$ holds for all $\vec{\alpha} \in [C]^{n+1}$ if $n > 0$ and*
- *there is an ordinal $\gamma_g \in \omega_1$ such that $H(\vec{\alpha}) < g(\gamma_g)$ holds for all $\vec{\alpha} \in [C]^{n+1}$ if $n = 0$.*

Proof. Since the proof for the second case is a simpler version of that for the first case we will do only the case $n > 0$. We can assume without loss of generality that D contains only limit ordinals. Fix an increasing sequence $\langle \alpha_0, \ldots, \alpha_{n-1} \rangle \in [D]^n$. Then for $\alpha \in D$ greater α_{n-1} we have $H(\langle \alpha_0, \ldots, \alpha_{n-1}, \alpha \rangle) < g(\alpha)$. Since g is a continuous function and α a limit ordinal there is a $\beta < \alpha$ such that $H(\langle \alpha_0, \ldots, \alpha_{n-1}, \alpha \rangle) < g(\beta)$. Let $r(\alpha)$ be the minimum of such β, then $r : D_{>\alpha_{n-1}} \to \omega_1$ is a regressive function on the club set $D_{>\alpha_{n-1}}$, the set of ordinals in D that are greater than α_{n-1}. By Lemma 1.3.12 that means that there is an ordinal $\gamma < \omega_1$ and a stationary subset $S \subseteq D_{>\alpha_{n-1}}$ such that $r(\alpha) = \gamma$ is true for all $\alpha \in S$. Let $\ell(\langle \alpha_0, \ldots, \alpha_{n-1} \rangle)$ be the smallest such γ, we now can define the function $\ell_g : \omega_1 \to \omega_1$ by

$$\ell_g(\alpha) := \sup\{\ell(\langle \alpha_0, \ldots, \alpha_{n-2}, \alpha \rangle)\,;\, \langle \alpha_0, \ldots, \alpha_{n-2}, \alpha \rangle \in [D]^n\}.$$

Now we partition the set $\mathfrak{C}^{n+1}_{\omega_1}$ of continuous functions x of type $n+1$ according to whether

$$H(\langle x(0), \ldots, x(n) \rangle) < g(\ell_g(x(n-1)))$$

is true or not. By the weak partition property of ω_1 there is a homogeneous club set C for this partition. Assume C is homogeneous for the contrary side, *i.e.*, $H(\langle x(0), \ldots, x(n) \rangle) \geq g(\ell_g(x(n-1)))$ holds for all $x \in \mathfrak{C}^{n+1}_C$. Let $y(0) < \ldots < y(n-1)$ be elements of $D \cap C$, by definition of ℓ_g we have $\ell_g(y(n-1)) \geq \ell(\langle y(0), \ldots, y(n-1) \rangle)$ and there is a stationary set $S \subseteq D_{>y(n-1)}$ such that

$$H(\langle y(0), \ldots, y(n-1), \alpha \rangle) < g(\ell(\langle y(0), \ldots, y(n-1) \rangle))$$

holds for all $\alpha \in S$. But the intersection $S \cap C$ is non-empty and with $y(n) \in S \cap C$ we have an element y of $\mathfrak{C}^{n+1}_C$ such that $H(\langle y(0), \ldots, y(n) \rangle) < g(\ell_g(y(n-1)))$, which contradicts our assumption on C.

So C must be homogeneous for the stated side, *i.e.*, for all $x \in \mathfrak{C}_C^{n+1}$ we have $H(\langle x(0),\dots,x(n)\rangle) < g(\ell_g(x(n-1)))$. Since all ordinals in C are countable limits that means for all $\vec{\alpha} \in [C]^{n+1}$ we have $H(\vec{\alpha}) < g(\ell_g(\alpha_{n-1}))$, which finishes the proof. q.e.d.

6.3.3. Lemma. *Assume κ is a strong partition cardinal, n a natural number, and $F : \kappa \to \kappa$ a function. If the ultrapower $\kappa^{\omega_1^n}/\mathcal{C}_\kappa^{\omega_1 \otimes n}$ is wellfounded and there is a club set $C \subseteq \kappa$ such that for all $g : \omega_1^n \to C$ of continuous type ω_1^n we have*

$$F([g]_{(\mathcal{C}_{\omega_1}^{\omega})^n}) < \sup g,$$

then F is $\mathcal{C}_\kappa^{\omega_1 \otimes n}$-almost constant.

Proof. Let S be the set of tuples (α, g), where $g : \omega_1^n \to \kappa$ is a function of continuous type ω_1^n and α an ordinal with cofinality ω such that $\alpha < \inf g$. Note that S is just the set $\mathfrak{C}_\kappa^{\langle 1,\omega_1^n\rangle}$ in a different notation. We partition this set according to whether

$$F([g]_{(\mathcal{C}_{\omega_1}^{\omega})^n}) < \alpha$$

holds or not. By Lemma 1.9.2 exists a homogeneous club set $C' \subseteq \kappa$ for this partition which without loss of generality we can assume to be equal to C. Toward a contradiction we take C to be homogeneous for the contrary side. By our assumption on C we have $F([g]_{(\mathcal{C}_{\omega_1}^{\omega})^n}) < \sup g$ for all $g \in \mathfrak{C}_C^{\omega_1^n}$, fix such a g. Then there is an $\alpha' \in \operatorname{ran}(g)$ such that $F([g]_{(\mathcal{C}_{\omega_1}^{\omega})^n}) < \alpha' < \sup g$ and if α is the ωth element of $\operatorname{ran}(g)$ greater than α' we still have

$$F([g]_{(\mathcal{C}_{\omega_1}^{\omega})^n}) < \alpha < \sup g.$$

Now we shift the function g by α to get a function g':

- Let $g'(0)$ be the ωth element of $\operatorname{ran}(g) > \alpha$,
- let $g'(\alpha+1)$ be the least element of $\operatorname{ran}(g) > g'(\alpha)$, and
- let $g'(\alpha)$ be $\sup_{\beta<\alpha} g'(\beta)$ if α is a limit ordinal.

Then $[g']_{(\mathcal{C}_{\omega_1}^{\omega})^n} = [g]_{(\mathcal{C}_{\omega_1}^{\omega})^n}$ holds, so

$$F([g']_{(\mathcal{C}_{\omega_1}^{\omega})^n}) = F([g]_{(\mathcal{C}_{\omega_1}^{\omega})^n}) < \alpha,$$

but since $g' : \omega_1^n \to C$ is a function of continuous type ω_1^n and $\alpha < \inf g'$ is an element of C with cofinality ω this contradicts our assumption on C.

So let $C \subseteq \kappa$ be a homogeneous club set for the stated side. Define α to be the ωth element of C and remember that C_α is the club set of elements in $C > \alpha$, then for all $g : \omega_1^n \to C_\alpha$ of continuous type ω_1^n we have $F([g]_{(\mathcal{C}_{\omega_1}^{\omega})^n}) < \alpha$. So for $\mathcal{C}_\kappa^{\omega_1 \otimes n}$-almost all β we have $F(\beta) < \alpha$, which by Lemma 1.2.5 means that F is $\mathcal{C}_\kappa^{\omega_1 \otimes n}$-almost constant. q.e.d.

Now we come to the second step in our inductive computation of ultrapowers.

6.3.4. THEOREM. *Assume* $\mathsf{AD}+\mathsf{DC}$ *and let* $\kappa > \omega_1$ *be a projective ordinal with odd index. Then* $\kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa = \kappa^{(\omega+1)}$.

Proof. By Proposition 6.1.2 we already have the lower bound $\kappa^{(\omega+1)} \le \kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa$, so it suffices to show $\kappa^\kappa/\mathcal{C}^{\omega_1}_\kappa \le \kappa^{(\omega+1)}$, which by Lemma 6.2.2 reduces to proving $(\sup_n [G_n]_{\mathcal{C}^{\omega_1}_\kappa})^+ \le \kappa^{(\omega+1)}$ for $G_n([f]_{\mathcal{C}^\omega_{\omega_1}}) := (\sup f)^{\omega_1}/\mathcal{C}^\omega_\kappa \otimes n$. In other words, if we can show $[G_n]_{\mathcal{C}^{\omega_1}_\kappa} = \kappa^{(n)}$ we are done.

So fix an n and let $[\vartheta]_{\mathcal{C}^{\omega_1}_\kappa}$ be an element of $[G_n]_{\mathcal{C}^{\omega_1}_\kappa}$. That means there is a club set $C \subseteq \kappa$ that contains only limit ordinals such that for all functions $g : \omega_1 \to C$ of continuous type ω_1 we have

$$\vartheta([g]_{\mathcal{C}^\omega_{\omega_1}}) < G_n([g]_{\mathcal{C}^\omega_{\omega_1}}) = (\sup g)^{\omega_1}/\mathcal{C}^\omega_{\omega_1} \otimes n.$$

We will now use a partition argument to get a function $F : \kappa \to \kappa$ such that for all functions $g \in \mathfrak{C}^{\omega_1}_C$ we have

$$\vartheta([g]_{\mathcal{C}^\omega_{\omega_1}}) < [\vec{\alpha} \mapsto F(g(\alpha_{n-1}))]_{\mathcal{C}^\omega_{\omega_1}\otimes n}.$$

So, let P be the set of pairs (g, f), where $g : \omega_1 \to \kappa$ is of continuous type ω_1, $f : \omega_1 \to \kappa$ is of discontinuous type ω_1, and $g(\alpha) < f(\alpha) < g(\alpha+1)$ holds for all $\alpha < \omega_1$. We partition this set according to whether

$$\vartheta([g]_{\mathcal{C}^\omega_{\omega_1}}) < [\vec{\alpha} \mapsto f(\alpha_{n-1})]_{\mathcal{C}^\omega_{\omega_1}\otimes n}$$

holds or not. By Lemma 1.9.4 there exists a club set that is homogeneous for this partition and we can assume without loss of generality that C is this club set.

Suppose C is homogeneous for the contrary side and $g : \omega_1 \to C$ is a function of continuous type ω_1. Let $[f]_{\mathcal{C}^\omega_{\omega_1}\otimes n}$ be an element of $(\sup g)^{\omega_1}/\mathcal{C}^\omega_{\omega_1} \otimes n$ that dominates $\vartheta([g]_{\mathcal{C}^\omega_{\omega_1}})$, we can assume $\sup f = \sup g$ and by Lemma 6.2.4 we can also assume that f only depends on α_{n-1}, *i.e.*, f is a function from ω_1 to κ such that

$$\vartheta([g]_{\mathcal{C}^\omega_{\omega_1}}) < [\vec{\alpha} \mapsto f(\alpha_{n-1})]_{\mathcal{C}^\omega_{\omega_1}\otimes n}.$$

Next we do a shifting argument, another technique that will appear repeatedly from now on. We define functions $g' : \omega_1 \to C$ and $f' : \omega_1 \to C$ as follows:

- Let $g'(0)$ be $g(0)$,
- let $f'(\alpha)$ be the ωth element of C greater than $\max\{g'(\alpha), f(\alpha)\}$,
- let $g'(\alpha+1)$ be $g(\alpha')$, where α' is least such that $g(\alpha') > f'(\alpha)$, and
- let $g'(\lambda)$ be $\sup_{\alpha<\lambda} g'(\alpha)$, for limits λ.

Then g' is of continuous type ω_1, f' is of discontinuous type ω_1, $[g']_{\mathcal{C}^\omega_{\omega_1}} = [g]_{\mathcal{C}^\omega_{\omega_1}}$, $[\vec{\alpha} \mapsto f'(\alpha_{n-1})]_{\mathcal{C}^\omega_{\omega_1} \otimes n} \geq [\vec{\alpha} \mapsto f(\alpha_{n-1})]_{\mathcal{C}^\omega_{\omega_1} \otimes n}$, $\sup g' = \sup f' = \sup f = \sup g$, both functions have range C, and f' and g' are ordered as in P. But this contradicts the homogeneity of C for the contrary side, so C has to be homogeneous for the stated side. Now we can define our function $F : \kappa \to \kappa$:

Let $F(\alpha)$ be the ωth element of C greater than α.

We have to show that this F will do its job, *i.e.*, that $\vartheta([g]_{\mathcal{C}^\omega_{\omega_1}}) < [\vec{\alpha} \mapsto F(g(\alpha_{n-1})]_{\mathcal{C}^\omega_{\omega_1} \otimes n}$ holds. Let $g : \omega_1 \to C$ be a function of continuous type ω_1, we will have to do some shifting again to define functions $g' : \omega_1 \to C$ and $f_g : \omega_1 \to C$:

- Let $g'(0)$ be $g(0)$,
- let $f_g(\alpha)$ be the ωth element of C greater than $g'(\alpha)$,
- let $g'(\alpha + 1)$ be $g(\alpha')$, where α' is least such that $g(\alpha') > f_g(\alpha)$, and
- let $g'(\lambda)$ be $\sup_{\alpha<\lambda} g'(\alpha)$, for limits λ.

Then g' is of continuous type ω_1, f_g is of discontinuous type ω_1, g' and f_g are ordered as in P and $[g']_{\mathcal{C}^\omega_{\omega_1}} = [g]_{\mathcal{C}^\omega_{\omega_1}}$, *i.e.*, for $\mathcal{C}^\omega_{\omega_1}$-almost all α we have $g(\alpha) = g'(\alpha)$. So by our partition we have

$$\vartheta([g]_{\mathcal{C}^\omega_{\omega_1}}) = \vartheta([g']_{\mathcal{C}^\omega_{\omega_1}}) < [\vec{\alpha} \mapsto f_g(\alpha_{n-1})]_{\mathcal{C}^\omega_{\omega_1} \otimes n} \;\Leftrightarrow$$
$$\vartheta([g]_{\mathcal{C}^\omega_{\omega_1}}) < [\vec{\alpha} \mapsto \text{ the } \omega\text{th element of } C \text{ greater than } g'(\alpha_{n-1})]_{\mathcal{C}^\omega_{\omega_1} \otimes n} \Leftrightarrow$$
$$\vartheta([g]_{\mathcal{C}^\omega_{\omega_1}}) < [\vec{\alpha} \mapsto \text{ the } \omega\text{th element of } C \text{ greater than } g(\alpha_{n-1})]_{\mathcal{C}^\omega_{\omega_1} \otimes n}$$
$$\Leftrightarrow \;\vartheta([g]_{\mathcal{C}^\omega_{\omega_1}}) < [\vec{\alpha} \mapsto F(g(\alpha_{n-1})]_{\mathcal{C}^\omega_{\omega_1} \otimes n}.$$

Since $g(\alpha_{n-1})$ is an ordinal of cofinality ω for $\mathcal{C}^\omega_{\omega_1}$-almost all α_{n-1} we can use Lemma 2.2.18 to get a Kunen tree K^1 on κ such that for all $g : \omega_1 \to C$ of continuous type ω_1 we have

$$\vartheta([g]_{\mathcal{C}^\omega_{\omega_1}}) < [\vec{\alpha} \mapsto |K^1 \restriction g(\alpha_{n-1})|]_{\mathcal{C}^\omega_{\omega_1} \otimes n}.$$

By Lemma 6.3.1 there is a function $\vartheta : \kappa \times [\omega]^n \to \kappa$ such that for all $g : \omega_1 \to C$ of continuous type ω_1 we have

$$\vartheta([g]_{\mathcal{C}^\omega_{\omega_1}}) = [\vec{\alpha} \mapsto |K^1 \restriction g(\alpha_{n-1})|(\vartheta^1([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}))]_{\mathcal{C}^\omega_{\omega_1} \otimes n}$$

and $\vartheta^1([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}) < g(\alpha_{n-1})$ for all $\vec{\alpha} \in [\omega_1]^n$. There are the two cases $n - 1 > 0$ and $n - 1 = 0$ and we prove them separately. Let $m = n - 1$.

Case $m > 0$:

If $\vartheta^{n-m}([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}) < g(\alpha_m)$ for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$ then by Lemma 6.3.2 there is a function $\ell_g : \omega_1 \to \omega_1$ such that $\vartheta^1([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}) < g(\ell_g(\alpha_{m-1})$ for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$. The next step is again a partition argument. Let P be the set of pairs (g, f), where $g : \omega_1 \to \kappa$ is of continuous type ω_1, $f : \omega_1 \to \kappa$ of discontinuous type ω_1 and $g(\alpha) < f(\alpha) < g(\alpha+1)$ holds for all $\alpha < \omega_1$. We partition this set according to whether

$$\vartheta^{n-m}([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}) \leq f((\alpha_{m-1})$$

holds for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$ or not.

By Lemma 1.9.4 there exists a club set that is homogeneous for this partition and we can assume without loss of generality that C is this club set. Suppose C is homogeneous for the contrary side. Fix a function $g : \omega_1 \to C$ of continuous type ω_1, then $\vartheta^{n-m}([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}) < g(\ell_g(\alpha_{m-1})$ for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$.

We shift g to get functions $g' : \omega_1 \to \sup g$ and $f : \omega_1 \to \sup g$:

- Let $g'(0)$ be $g(0)$,
- let $f(\alpha)$ be the ωth element of $C \cap C'$ greater than $\max\{g'(\alpha), g(\ell_g(\alpha))\}$,
- let $g'(\alpha+1)$ be the least element of $\operatorname{ran}(g)$ greater than $f(\alpha)$, and
- let $g'(\lambda)$ be $\sup_{\alpha<\lambda} g'(\alpha)$, for limits λ.

Then g' is a function of continuous type ω_1, f a function of discontinuous type ω_1, they have range C, $[g']_{\mathcal{C}^\omega_{\omega_1}} = [g]_{\mathcal{C}^\omega_{\omega_1}}$, and for all $\alpha \in \omega_1$ we have $g'(\alpha) < f(\alpha) < g'(\alpha+1)$ and $g(\ell_g(\alpha)) < f(\alpha)$. So (g', f) is in P and since $\vartheta^{n-m}([g]_{\mathcal{C}^\omega_{\omega_1}})$ depends only on the equivalence class of g, which is the same as that of g', we get that for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$

$$\vartheta^{n-m}([g']_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}) = \vartheta^{n-m}([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}) < g(\ell_g(\alpha_{m-1})) < f(\alpha_{m-1}).$$

But this contradicts the homogeneity of C for the contrary side, so C has to be homogeneous for the stated side. Now we can define a function $H : \kappa \to C$ by:

Let $H(\alpha)$ be the ωth element of C greater than α.

For a function $g : \omega_1 \to C$ of continuous type ω_1 we define $g' : \omega_1 \to C$ and $f_g : \omega_1 \to C$ as follows:

- Let $g'(0)$ be $g(0)$,
- let $f_g(\alpha)$ be the ωth element of C greater than $g'(\alpha)$,
- let $g'(\alpha+1)$ be the least element of $\operatorname{ran}(g)$ greater than $f_g(\alpha)$, and
- let $g'(\lambda)$ be $\sup_{\alpha<\lambda} g'(\alpha)$, for limits λ.

Then again (g', f_g) is in P, $\operatorname{ran}(g'), \operatorname{ran}(f) \subseteq C$, and $[g']_{\mathcal{C}^\omega_{\omega_1}} = [g]_{\mathcal{C}^\omega_{\omega_1}}$, so by the homogeneity of C for the right side of our partition we have that for all $g : \omega_1 \to C$ of continuous type ω_1 and for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$ holds

$$\vartheta^{n-m}([g]_{\mathcal{C}^\omega_{\omega_1}}) = \vartheta^{n-m}([g']_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}) \leq f_g(\alpha_{m-1}) = H(g(\alpha_{m-1})).$$

Now we are nearly in the same position as in the beginning of this proof, only we dropped from α_m to α_{m-1}. So we use again Lemma 2.2.18 to get a Kunen tree $K^{n-(m-1)}$ on κ such that for all $g : \omega_1 \to C$ of continuous type ω_1 and for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$ we have

$$\vartheta^{n-m}([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}) < |K^{n-(m-1)} \restriction g(\alpha_{m-1})|.$$

Lemma 6.3.1 gives us again a function $\vartheta^{n-(m-1)} : \kappa \times [\omega_1]^n \to \sup g$ such that for all $g : \omega_1 \to C$ of continuous type ω_1 we have

$$\vartheta^{n-m}([g]_{\mathcal{C}^\omega_{\omega_1}}) = [\vec{\alpha} \mapsto |K^{n-(m-1)} \restriction g(\alpha_{m-1})|(\vartheta^{n-(m-1)}([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}))]_{\mathcal{C}^\omega_{\omega_1} \otimes n}$$

and $\vartheta^{n-(m-1)}([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}) < g(\alpha_{m-1})$ for all $\vec{\alpha} \in [\omega_1]^n$. Now either $m-1 = 0$ and we jump to Case $m = 0$, or we continue with the Case $m - 1 > 0$.

End of Case $m > 0$.

After finitely many iterations of Case $m > 0$ we have the following situation: For all $g : \omega_1 \to C$ of continuous type ω_1

$$\vartheta([g]_{\mathcal{C}^\omega_{\omega_1}}) = [\vec{\alpha} \mapsto |K^1 \restriction g(\alpha_{n-1})|(\vartheta^1([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}))]_{\mathcal{C}^\omega_{\omega_1} \otimes n},$$

$$[\vec{\alpha} \mapsto \vartheta^1([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha})]_{\mathcal{C}^\omega_{\omega_1} \otimes n} = [\vec{\alpha} \mapsto |K^2 \restriction g(\alpha_{n-2})|(\vartheta^2([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}))]_{\mathcal{C}^\omega_{\omega_1} \otimes n},$$

$$\vdots$$

$$[\vec{\alpha} \mapsto \vartheta^{n-(m+1)}([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha})]_{\mathcal{C}^\omega_{\omega_1} \otimes n} = [\vec{\alpha} \mapsto |K^{n-m} \restriction g(\alpha_m)|(\vartheta^{n-m}([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}))]_{\mathcal{C}^\omega_{\omega_1} \otimes n},$$

$$\vdots$$

$$[\vec{\alpha} \mapsto \vartheta^{n-1}([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha})]_{\mathcal{C}^\omega_{\omega_1} \otimes n} = [\vec{\alpha} \mapsto |K^n \restriction g(\alpha_0)|(\vartheta^n([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}))]_{\mathcal{C}^\omega_{\omega_1} \otimes n} \text{ and}$$

$$[\vec{\alpha} \mapsto \vartheta^n([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha})]_{\mathcal{C}^\omega_{\omega_1} \otimes n} < [\vec{\alpha} \mapsto g(\alpha_0)]_{\mathcal{C}^\omega_{\omega_1} \otimes n}.$$

Case $m = 0$:

If $\vartheta^n([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}) < g(\alpha_0)$ for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha} \in [\omega_1]^n$ then by the second part of Lemma 6.3.2 for all $g : \omega_1 \to C$ of continuous type ω_1 there is an ordinal $\gamma_g \in \omega_1$ such that

$$\vartheta^n([g]_{\mathcal{C}^\omega_{\omega_1}})(\vec{\alpha}) < g(\gamma_g) \text{ for } \mathcal{C}^\omega_{\omega_1} \otimes n\text{-almost all } \vec{\alpha}.$$

With Lemma 2.2.15 we can assume without loss of generality that C is closed under $\mathcal{C}^\omega_{\omega_1} \otimes n$, *i.e.*,

$$[\vartheta^n([g]_{\mathcal{C}^\omega_{\omega_1}})]_{\mathcal{C}^\omega_{\omega_1}\otimes n} \leq g(\gamma_g + 1) < \sup g.$$

Now we can use Lemma 6.3.3 and get that $[\vartheta^n([g]_{\mathcal{C}^\omega_{\omega_1}})]_{\mathcal{C}^\omega_{\omega_1}\otimes n}$ is constant for all $g : \omega_1 \to C$ of continuous type ω_1, let $c_\vartheta : [\omega_1]^n \to \kappa$ represent that constant with respect to $\mathcal{C}^\omega_{\omega_1} \otimes n$. End of Case 0.

To summarize, we started with an element $[\vartheta]_{\mathcal{C}^{\omega_1}_\kappa}$ of $[G_n]_{\mathcal{C}^{\omega_1}_\kappa}$ and our analysis enables us to derive Kunen trees $K^1, \ldots, K^n$ and a constant c_ϑ such that there is a club set $C \subseteq \kappa$ such that for all $g : \omega_1 \to C$ of continuous type ω_1

$$\vartheta([g]_{\mathcal{C}^\omega_{\omega_1}}) = [\vec{\alpha} \mapsto |K^1{\restriction}g(\alpha_{n-1})|(|K^2{\restriction}g(\alpha_{n-2})|(\ldots|K^n{\restriction}g(\alpha_0)|(c_\vartheta(\vec{\alpha}))\ldots))]_{\mathcal{C}^\omega_{\omega_1}\otimes n}.$$

We are now at the point where we can define a surjective embedding π from $\kappa^\kappa/\mathcal{C}^\omega_\kappa \otimes n$ to $[G_n]_{\mathcal{C}^{\omega_1}_\kappa}$: For $F : \kappa \to \kappa$ we define $\pi([F]_{\mathcal{C}^\omega_\kappa\otimes n}) := [\pi(F)]_{\mathcal{C}^{\omega_1}_\kappa}$, where

$$\pi(F)([g]_{\mathcal{C}^\omega_{\omega_1}}) := [\vec{\alpha} \mapsto F(\ulcorner g(\alpha_0), \ldots, g(\alpha_{n-1})\urcorner)]_{\mathcal{C}^\omega_{\omega_1}\otimes n}$$

for functions $g : \omega_1 \to \kappa$ of continuous type ω_1. First we have to show of course that $\pi(F)$ is welldefined with respect to $\mathcal{C}^\omega_{\omega_1}$-equivalence classes:

If $[g]_{\mathcal{C}^\omega_{\omega_1}} = [g']_{\mathcal{C}^\omega_{\omega_1}}$ then there is a club set $C \subseteq \omega_1$ such that $g(\alpha) = g'(\alpha)$ for all $\alpha \in C$, *i.e.*, for all $\vec{\alpha} \in [C]^n$ we have

$$F(\ulcorner g(\alpha_0), \ldots, g(\alpha_{n-1})\urcorner) = F(\ulcorner g'(\alpha_0), \ldots, g'(\alpha_{n-1})\urcorner).$$

So $\pi(F)([g]_{\mathcal{C}^\omega_{\omega_1}})$ is welldefined. And it is an welldefined embedding: Let D be the set of limit ordinals less than ω_1, note that all its elements have cofinality ω. Assume $[F]_{\mathcal{C}^\omega_\kappa\otimes n} \overset{=}{<} [F']_{\mathcal{C}^\omega_\kappa\otimes n}$, this means there is a club set $C \subseteq \kappa$ such that for all $f \in \mathfrak{C}^n_C$ we have

$$F(\ulcorner f(0), \ldots, f(n-1)\urcorner) \overset{=}{<} F'(\ulcorner f(0), \ldots, f(n-1)\urcorner).$$

But this is equivalent to the statement that for all $g \in \mathfrak{C}^{\omega_1}_C$ and all $\vec{\alpha} \in [D]^n$ we have $F(\ulcorner g(\alpha_0), \ldots, g(\alpha_{n-1})\urcorner) \overset{=}{<} F'(\ulcorner g(\alpha_0), \ldots, g(\alpha_{n-1})\urcorner)$. Since $D \subseteq \omega_1$ is a club set we get

$$[\vec{\alpha} \mapsto F(\ulcorner g(\alpha_0), \ldots, g(\alpha_{n-1})\urcorner)]_{\mathcal{C}^\omega_{\omega_1}\otimes n} \overset{=}{<} [\vec{\alpha} \mapsto F'(\ulcorner g(\alpha_0), \ldots, g(\alpha_{n-1})\urcorner)]_{\mathcal{C}^\omega_{\omega_1}\otimes n}$$

for all $g \in \mathfrak{C}^{\omega_1}_C$, *i.e.*, $[\pi(F)]_{\mathcal{C}^{\omega_1}_\kappa} \overset{=}{<} [\pi(F')]_{\mathcal{C}^{\omega_1}_\kappa}$, which means π is a welldefined embedding.

The only thing left to show is the surjectivity of π. Let $[\vartheta]_{\mathcal{C}^{\omega_1}_\kappa}$ be an element of $[G_n]_{\mathcal{C}^{\omega_1}_\kappa}$, we get Kunen trees $K^1, \ldots, K^n$ and a constant c_ϑ from our analysis of $[\vartheta]_{\mathcal{C}^{\omega_1}_\kappa}$. Let

$$F_\vartheta(\ulcorner \alpha_0, \ldots, \alpha_{n-1}\urcorner) := |K^1{\restriction}\alpha_{n-1}|(|\ldots|K^n{\restriction}\alpha_0|(c_\vartheta(\vec{\alpha}))\ldots),$$

then there is a club set $C \subseteq \kappa$ such that for all $g : \omega_1 \to C$ of continuous type ω_1

$$\vartheta([g]_{\mathcal{C}^{\omega}_{\omega_1}}) = [\vec{\alpha} \mapsto |K^1 \restriction g(\alpha_{n-1})|(\ldots |K^n \restriction g(\alpha_0)|(c_\vartheta(\vec{\alpha}))\ldots)]_{\mathcal{C}^{\omega}_{\omega_1} \otimes n} = \pi(F)([g]_{\mathcal{C}^{\omega}_{\omega_1}}).$$

So $[\vartheta]_{\mathcal{C}^{\omega_1}_{\kappa}} = [\pi(F)]_{\mathcal{C}^{\omega_1}_{\kappa}}$, π is bijective, and using Corollary 5.3.3 we get

$$[G_n]_{\mathcal{C}^{\omega_1}_{\kappa}} = \kappa^\kappa / \mathcal{C}^{\omega}_{\kappa} \otimes n = \kappa^{(n)}.$$

q.e.d.

Now that we have computed the value of the ultrapower $\kappa^\kappa/\mathcal{C}^{\omega_1}_{\kappa}$ we can prove the validity of the conditions from Lemma 5.2.4 for the second generator of the ordinal algebra and conclude:

6.3.5. Corollary. *Assume* $\mathsf{AD} + \mathsf{DC}$. *The* $\boldsymbol{\delta}^1_{2n+1}$*-measure assignment for* $\mathfrak{A}_2$ *derived from* $\mathrm{GA}_{\boldsymbol{\delta}^1_{2n+1}}$ *is canonical up to height* $(\boldsymbol{\delta}^1_{2n+1})^{(\omega^2)}$.

Proof. We apply Lemma 5.2.4. From Corollary 5.3.2 we have the necessary properties for $\mathbf{germ}(\mathsf{V}_0)$. By Theorem 6.3.4 we know that ${\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}}/\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}} = (\boldsymbol{\delta}^1_{2n+1})^{(\omega+1)}$ so we have $\mathrm{I}(\boldsymbol{\delta}^1_{2n+1}, 1)$. Condition $\mathrm{II}(\boldsymbol{\delta}^1_{2n+1}, 1)$ follows from Proposition 5.3.1. From Lemma 4.2.2 we get a that $\mathbf{slift}_{\boldsymbol{\delta}^1_{2n+1}}(\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}})$ is a normal measure on $(\boldsymbol{\delta}^1_{2n+1})^{(\omega+1)}$, so by Lemma 1.2.4 the cardinal $(\boldsymbol{\delta}^1_{2n+1})^{(\omega+1)}$ is regular, which takes care of $\mathrm{III}(\boldsymbol{\delta}^1_{2n+1}, 1)$. q.e.d.

Corollary 6.3.5 is exactly Corollary 25 from [BoLö07]. There the lower bound for the ultrapower $\kappa^\kappa/\mathcal{C}^{\omega_1}_{\kappa}$ was computed in the same manner as in this thesis, but no proof for the upper bound was given. In this thesis we computed the upper bound explicitly in Theorem 6.3.4.

Since the natural measure assignment is canonical up to height $(\boldsymbol{\delta}^1_{2n+1})^{(\omega^2)}$ we now can compute the values of ultrapowers with respect to order measure sums of $\mathcal{C}^{\omega}_{\boldsymbol{\delta}^1_{2n+1}}$ and $\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}}$.

6.3.6. Corollary. *Assume* $\mathsf{AD} + \mathsf{DC}$. *Let* $\boldsymbol{\delta}^1_{2n+1} > \omega_1$ *be an odd projective ordinal and* $n \in \omega$. *Then* ${\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}}/\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}} \otimes n \oplus \mathcal{C}^{\omega}_{\boldsymbol{\delta}^1_{2n+1}} \otimes m = (\boldsymbol{\delta}^1_{2n+1})^{(\omega\cdot n+m)}$.

Proof. Let κ be a projective ordinal $\boldsymbol{\delta}^1_{2n+1} > \omega_1$. We know $\mathbf{lift}_\kappa(\mu_{\{0\}}) = \mathcal{C}^{\omega}_{\kappa}$, $\mathbf{lift}_\kappa(\mathcal{C}^{\omega}_{\omega_1}) = \mathcal{C}^{\omega_1}_{\kappa}$, and from the canonicity of the measure assignment with $\mathbf{germ}(\mathsf{V}_0) = \mu\{0\}$ and $\mathbf{germ}(\mathsf{V}_1) = \mathcal{C}^{\omega}_{\omega_1}$ we get

$$\begin{aligned} \kappa^\kappa/\mathcal{C}^{\omega_1}_{\kappa} \otimes n \oplus \mathcal{C}^{\omega}_{\kappa} \otimes m &= \kappa^\kappa/\mathbf{lift}_\kappa(\mathbf{germ}(\mathsf{V}_1 \otimes n \oplus \mathsf{V}_0 \otimes m)) \\ &= (\kappa)^{(o(\mathsf{V}_1 \otimes n \oplus \mathsf{V}_0 \otimes m))} = (\kappa)^{((\omega+1)\cdot n+m)}. \end{aligned}$$

q.e.d.

The results concerning canonicity of the measure assignment up to this point are also covered by [BoLö07], although we included some details that were left out in that paper. In the next section we continue the inductive proof of the canonicity of the natural measure assignment, thus extending the results from [BoLö07].

6.4 Computation of $(\boldsymbol{\delta}^1_{2n+1})^{\boldsymbol{\delta}^1_{2n+1}}/\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}}{}^{\otimes m}$

Let us remind the reader that by Corollary 5.1.11 from $\mathrm{cf}({\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}}/\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}}) = (\boldsymbol{\delta}^1_{2n+1})^{(\omega+1)}$ we can conclude the cofinality of ${\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}}/\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}}{}^{\otimes m}$ for any $m < \omega$:

6.4.1. Lemma. *Assume* $\mathsf{AD} + \mathsf{DC}$. *For all $m > 0$ we have*

$$\mathrm{cf}\left({\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}}/(\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}})^{\otimes m}\right) = {\boldsymbol{\delta}^1_{2n+1}}^{(\omega+1)}.$$

Proof. We know $\mathrm{cf}\left({\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}}/\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}}\right) = {\boldsymbol{\delta}^1_{2n+1}}^{(\omega+1)}$, see the proof of Corollary 6.3.5. So by Corollary 5.1.11 we have

$$\mathrm{cf}\left({\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}}/(\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}})^{\otimes m}\right) = \mathrm{cf}\left({\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}}/\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}}\right) = {\boldsymbol{\delta}^1_{2n+1}}^{(\omega+1)}.$$

q.e.d.

This takes care of condition $\mathrm{III}(\boldsymbol{\delta}^1_{2n+1}, m)$ for finite m, from Proposition 6.1.2 we already have $\mathrm{II}(\boldsymbol{\delta}^1_{2n+1}, m)$ for finite m. So once we have proven that the cardinal value of $(\boldsymbol{\delta}^1_{2n+1})^{\boldsymbol{\delta}^1_{2n+1}}/\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}}{}^{\otimes m}$ is $(\boldsymbol{\delta}^1_{2n+1})^{(\omega^m+1)}$ we know that the $\boldsymbol{\delta}^1_{2n+1}$-measure assignment for $\mathfrak{A}_2$ derived from $\mathrm{GA}_{\boldsymbol{\delta}^1_{2n+1}}$ is canonical up to height $(\boldsymbol{\delta}^1_{2n+1})^{(\omega^{m+1})}$. Which in turn means we can for all odd projective ordinals κ compute the value of $(\kappa)^\kappa/(\mathcal{C}^{\omega_1\otimes m}_\kappa) \otimes n$ to be $(\kappa)^{(\omega^m \cdot n+1)}$ for $0 < n < \omega$.

In Section 6.3 we used Kunen trees to get upper bounds for elements of the restricted ultrapower. Now we need the analogue of Lemma 6.3.1 for Martin trees.

6.4.2. Lemma. *Assume* AD, *let $m > 1$ be a natural number and $\kappa > \omega_1$ an odd projective ordinal. Let $\vartheta : \kappa \to \kappa$ be a function, C a club subset of κ and T a tree on κ such that for all $g : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$ we have*

$$\vartheta([g]_{(\mathcal{C}^\omega_{\omega_1})^m}) < [\vec{\alpha} \mapsto |T{\restriction}\sup_k(\sup_{\vec{\beta}} g(\vec{\beta}, \alpha_{n-1}))^{\omega_1}/\mathcal{C}^\omega_{\omega_1} \otimes k|]_{\mathcal{C}^\omega_{\omega_1}\otimes n}.$$

Then exists a function $\vartheta^1 : \kappa \times [\omega_1]^n \to \kappa$ such that for all functions $g : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$ we have

$$\vartheta([g]_{(\mathcal{C}^\omega_{\omega_1})^m}) = [\vec{\alpha} \mapsto |T{\restriction}\sup_k(\sup_{\vec{\beta}} g(\vec{\beta}, \alpha_{n-1}))^{\omega_1}/\mathcal{C}^\omega_{\omega_1} \otimes k|(\vartheta^1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}))]_{\mathcal{C}^\omega_{\omega_1}\otimes n} \text{ and}$$

$$\vartheta^1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) < \sup_k(\sup_{\vec{\beta}} g(\vec{\beta}, \alpha_{n-1}))^{\omega_1}/\mathcal{C}^\omega_{\omega_1} \otimes k \text{ for all } \vec{\alpha} \in [\omega_1]^n.$$

Proof. Let $g : (\omega_1)^m \to C$ be a function of continuous type $(\omega_1)^m$. Let $t : [\omega_1]^n \to \kappa$ represent the ordinal $\vartheta([g]_{(\mathcal{C}^\omega_{\omega_1})^m})$ with respect to $\mathcal{C}^\omega_{\omega_1} \otimes n$. We define the function $\vartheta^1([g]_{(\mathcal{C}^\omega_{\omega_1})^m}) : [\omega_1]^n \to \kappa$ by

$$\vartheta^1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) := \min\{\beta < \kappa\,;\, t(\vec{\alpha}) = |T{\restriction}\sup_k(\sup_{\vec{\beta}} g(\vec{\beta}, \alpha_{n-1}))^{\omega_1}/\mathcal{C}^\omega_{\omega_1} \otimes k|(\beta)\}.$$

Let $g, g' : (\omega_1)^m \to C$ be functions of continuous type $(\omega_1)^m$ with $[g]_{(\mathcal{C}^\omega_{\omega_1})^m} = [g']_{(\mathcal{C}^\omega_{\omega_1})^m}$ then by definition of $(\mathcal{C}^\omega_{\omega_1})^m$ there is a club set $D \subseteq \omega_1$ such that for all $\alpha \in D$ we have

$$\{\vec{\beta} \in (\omega_1)^{m-1}\,;\, g(\vec{\beta}, \alpha) = g'(\vec{\beta}, \alpha)\} \in (\mathcal{C}^\omega_{\omega_1})^{m-1},$$

which means for all $\alpha \in D$ we get $[\vec{\beta} \mapsto g(\vec{\beta}, \alpha)]_{(\mathcal{C}^\omega_{\omega_1})^{m-1}} = [\vec{\beta} \mapsto g'(\vec{\beta}, \alpha)]_{(\mathcal{C}^\omega_{\omega_1})^{m-1}}$. The functions $\vec{\beta} \mapsto g(\vec{\beta}, \alpha)$ and $\vec{\beta} \mapsto g'(\vec{\beta}, \alpha)$ are increasing and have the same values on a $(\mathcal{C}^\omega_{\omega_1})^{m-1}$-measure one set, which means their supremums are the same. So for all α in D we get $\sup_{\vec{\beta}} g(\vec{\beta}, \alpha) = \sup_{\vec{\beta}} g'(\vec{\beta}, \alpha)$. It follows that for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$ we have $\sup_{\vec{\beta}} g(\vec{\beta}, \alpha_{n-1}) = \sup_{\vec{\beta}} g'(\vec{\beta}, \alpha_{n-1})$ and if $t, t' : [\omega_1]^n \to \kappa$ are functions such that $\vartheta([g]_{(\mathcal{C}^\omega_{\omega_1})^m}) = [t]_{\mathcal{C}^\omega_{\omega_1} \otimes n} = [t']_{\mathcal{C}^\omega_{\omega_1} \otimes n}$ then for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$ we have

$$\min\{\beta < \kappa\,;\, t(\vec{\alpha}) = |T{\restriction}\sup_k(\sup_{\vec{\beta}} g(\vec{\beta}, \alpha_{n-1}))^{\omega_1}/\mathcal{C}^\omega_{\omega_1} \otimes k|(\beta)\} =$$

$$= \min\{\beta < \kappa\,;\, t'(\vec{\alpha}) = |T{\restriction}\sup_k(\sup_{\vec{\beta}} g'(\vec{\beta}, \alpha_{n-1}))^{\omega_1}/\mathcal{C}^\omega_{\omega_1} \otimes k|(\beta)\}.$$

That means ϑ^1 is welldefined on a $\mathcal{C}^\omega_{\omega_1} \otimes n$-measure 1 set and by its definition we have for all $g : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$

$$\vartheta([g]_{(\mathcal{C}^\omega_{\omega_1})^m}) = [\vec{\alpha} \mapsto |T{\restriction}\sup_k(\sup_{\vec{\beta}} g(\vec{\beta}, \alpha_{n-1}))^{\omega_1}/\mathcal{C}^\omega_{\omega_1} \otimes l|(\vartheta^1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}))]_{\mathcal{C}^\omega_{\omega_1} \otimes n}$$

with $\vartheta^1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) < \sup_k(\sup_{\vec{\beta}} g(\vec{\beta}, \alpha_{n-1}))^{\omega_1}/\mathcal{C}^\omega_{\omega_1} \otimes k$ for all $\vec{\alpha} \in [\omega_1]^n$. q.e.d.

6.4.3. Lemma. *Let $\kappa > \omega$ be a regular cardinal. Let $f : (\omega_1)^m \to \kappa$ be a function of continuous type $(\omega_1)^m$ and $\delta : [\omega_1]^n \to \sup f$ an arbitrary function. Then there is a club set $C \subseteq \omega_1$ and a function $\ell : \omega_1 \to \omega_1$ so that $\delta(\vec{\alpha}) < f(\vec{0}, \ell(\alpha_{n-1}))$ for all $\vec{\alpha} \in [C]^n$.*

Proof. By Lemma 6.2.4 there is a club set $C \subseteq \omega_1$ and a function $\delta' : \omega_1 \to \sup f$ such that for all $\vec{\alpha} \in [C]^n$ we have $\delta(\vec{\alpha}) \leq \delta'(\alpha_{n-1})$. If we define $\ell : \omega_1 \to \omega_1$ by

$$\ell(\alpha) := \min\{\beta \in \omega_1\,;\, \delta'(\alpha) < f(\vec{0}, \beta)\}$$

we get that for all $\vec{\alpha} \in [C]^n$ we have $\delta(\vec{\alpha}) \leq \delta'(\alpha_{n-1}) < f(\vec{0}, \ell(\alpha_{n-1}))$. q.e.d.

6.4.4. THEOREM. *Assume* $\mathsf{AD}+\mathsf{DC}$ *and let* $\kappa > \omega_1$ *be a projective ordinal with odd index. Then* $\kappa^\kappa/\mathcal{C}^{\omega_1\otimes m}_\kappa = \kappa^{(\omega^m+1)}$ *for all natural numbers* $m > 0$.

Proof. From Proposition 6.1.2 we already have the lower bounds, so we need only to prove $\kappa^\kappa/\mathcal{C}^{\omega_1\otimes m}_\kappa \leq \kappa^{(\omega^m+1)}$ for $m > 0$. We do this by induction over m. Theorem 6.3.4 takes care of the case $m = 1$, so for the induction step let m be greater 1 and assume $\kappa^\kappa/\mathcal{C}^{\omega_1\otimes(m-1)}_\kappa = \kappa^{(\omega^{m-1}+1)}$ is true. Fix a $0 < n < \omega$ and a $[\vartheta]_{\mathcal{C}^{\omega_1\otimes m}_\kappa} \in [G_n]_{\mathcal{C}^{\omega_1\otimes m}_\kappa}$, by Lemma 6.2.5 exists $F : \kappa \to \kappa$ such that $[\vartheta]_{\mathcal{C}^{\omega_1\otimes m}_\kappa} < [\pi_{\mathcal{C}^{\omega_1\otimes(m-1)}_\kappa,n}(F)]_{\mathcal{C}^{\omega_1\otimes m}_\kappa}$, *i.e.*, there is a club set $C \subseteq \kappa$ such that for all functions $g : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$

$$\vartheta([g]_{(\mathcal{C}^\omega_{\omega_1})^m}) < [\vec{\alpha} \mapsto F(\sup_{\vec{\beta}} g(\vec{\beta}, \alpha_{n-1}))]_{\mathcal{C}^\omega_{\omega_1}\otimes n}.$$

From Lemma 2.2.19 we get a tree T^1 on κ and a club set $C' \subseteq \kappa$ such that $F(\alpha) < |T^1 \upharpoonright \sup_k \alpha^{\omega_1}/\mathcal{C}^\omega_{\omega_1} \otimes k|$ for all $\alpha \in C'$ with cofinality ω_1, we can without loss of generality assume $C = C'$. If $g : (\omega_1)^m \to C$ is a strictly increasing[1] function then $\sup_{\vec{\beta}} g(\vec{\beta}, \alpha_{n-1})$ is an element of C with cofinality ω_1, so for all functions $g : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$ we get

$$\vartheta([g]_{(\mathcal{C}^\omega_{\omega_1})^m}) < [\vec{\alpha} \to |T^1 \upharpoonright \sup_k(\sup_{\vec{\beta}} g(\vec{\beta}, \alpha_{n-1}))^{\omega_1}/\mathcal{C}^\omega_{\omega_1} \otimes k|]_{\mathcal{C}^\omega_{\omega_1}\otimes n}.$$

To keep formulas slightly more understandable we introduce the following notation: $\delta(g, \alpha) := \sup_k(\sup_{\vec{\beta}} g(\vec{\beta}, \alpha))^{\omega_1}/\mathcal{C}^\omega_\kappa \otimes k$. Which means the statement above transforms to: For all functions $g : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$ we have

$$\vartheta([g]_{(\mathcal{C}^\omega_{\omega_1})^m}) < [\vec{\alpha} \to |T^1 \upharpoonright \delta(g, \alpha_{n-1})|]_{\mathcal{C}^\omega_{\omega_1}\otimes n}.$$

By Lemma 6.4.2 exists a function $\vartheta^1_1 : \kappa \times [\omega_1]^n \to \kappa$ such that for all functions $g : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$

$$\vartheta([g]_{(\mathcal{C}^\omega_{\omega_1})^m}) = [\vec{\alpha} \to |T^1 \upharpoonright \delta(g, \alpha_{n-1})|(\vartheta^1_1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}))]_{\mathcal{C}^\omega_{\omega_1}\otimes n}, \text{ and}$$

$$\vartheta^1_1([g])(\vec{\alpha}) < \sup_k(\sup_{\vec{\beta}} g(\vec{\beta}, \alpha_{n-1}))^{\omega_1}/\mathcal{C}^\omega_{\omega_1} \otimes k \text{ for all } \vec{\alpha} \in [\omega_1]^n.$$

Since $\operatorname{ran}(\vartheta^1_1([g])) \subseteq \sup_k(\sup_{\vec{\beta}} g(\vec{\beta}, \alpha_{n-1})^\kappa/\mathcal{C}^\omega_{\omega_1} \otimes k$ there is a least k and a club set $D \subseteq \omega_1$ such that for all $\vec{\alpha} \in [D]^n$ we have

$$\vartheta^1_1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) < (\sup_{\vec{\beta}} g(\vec{\beta}, \alpha_{n-1}))^{\omega_1}/\mathcal{C}^\omega_{\omega_1} \otimes k.$$

Now we use Lemma 6.4.3, so for $\vec{\alpha} \in [D]^n$ there is a function $\ell : \omega_1 \to \omega_1$ such that for $\mathcal{C}^\omega_{\omega_1} \otimes k$-almost all $\vec{\gamma} \in [\omega_1]^k$ we have $\vartheta^1_1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha})(\vec{\gamma}) < g(\vec{0}, \ell(\gamma_{k-1}), \alpha_{n-1}))$.

[1] With respect to the order $<_{\mathrm{rlex}}$.

If $\ell(\alpha) = \ell'(\alpha)$ for $\mathcal{C}^\omega_{\omega_1}$-almost all α, *i.e.*, $[\ell]_{\mathcal{C}^\omega_{\omega_1}} = [\ell']_{\mathcal{C}^\omega_{\omega_1}}$, then the ℓ' also does the job, so for all $\vec{\alpha} \in [D]^n$ there is a $[\ell]_{\mathcal{C}^\omega_{\omega_1}} \in \omega_2$ such that for $\mathcal{C}^\omega_{\omega_1} \otimes k$-almost all $\vec{\gamma} \in [\omega_1]^k$ we have $\vartheta^1_1([g])(\vec{\alpha})(\vec{\gamma}) < g(\vec{0}, \ell(\gamma_{k-1}), \alpha_{n-1})))$.

And since $\mathrm{cf}(\omega_2) > \mathrm{cf}(\omega_1)$ we can switch quantifiers, which means for all functions $g : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$ there is a function $\ell : \omega_1 \to \omega_1$ such that for all $\vec{\alpha} \in [D]^n$ we have

$$\vartheta^1_1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) < [\vec{\gamma} \mapsto g(\vec{0}, \ell(\gamma_{k-1}), \alpha_{n-1}))]_{\mathcal{C}^\omega_{\omega_1} \otimes k}.$$

The next step is again a partition argument. Let P be the set of pairs (g, f), where $g : (\omega_1)^m \to \kappa$ is of continuous type $(\omega_1)^m$, $f : (\omega_1)^m \to \kappa$ is of discontinuous type $(\omega_1)^m$ and $g(\vec{\delta}) < f(\vec{\delta}) < g(\vec{\zeta})$ holds for all $\vec{\delta} <_{\mathrm{rlex}} \vec{\zeta} \in (\omega_1)^m$. We partition this set according to whether

$$\vartheta^1_1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) \leq [\vec{\gamma} \mapsto f(\vec{0}, \gamma_{k-1}, \alpha_{n-1})]_{\mathcal{C}^\omega_{\omega_1} \otimes k}$$

holds for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$ or not.

There is a homogeneous club subset of κ for this partition, without loss of generality we can assume that C is this club set. Suppose C is homogeneous for the contrary side. Fix a function $g : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$ such that $\vartheta^1_1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})$ is defined, *i.e.*, $\vartheta([g]_{(\mathcal{C}^\omega_{\omega_1})^m}) < [\vec{\alpha} \to |T^1 \restriction \delta(g, \alpha_{n-1})|]_{\mathcal{C}^\omega_{\omega_1} \otimes n}$.

By a shifting argument we get functions $g', f : (\omega_1)^m \to C$ of continuous and discontinuous type $(\omega_1)^m$, respectively, such that $\mathrm{ran}(g'), \mathrm{ran}(f) \subseteq C$, $[g']_{(\mathcal{C}^\omega_{\omega_1})^m} = [g]_{(\mathcal{C}^\omega_{\omega_1})^m}$, $[\vec{\gamma} \mapsto f(\vec{0}, \gamma_{k-1}, \alpha_{n-1})]_{\mathcal{C}^\omega_{\omega_1} \otimes k} = [\vec{\gamma} \mapsto g(\vec{0}, \ell(\gamma_{k-1}), \alpha_{n-1}))]_{\mathcal{C}^\omega_{\omega_1} \otimes k}$, and g' an f are ordered as in P.

Now g' and f are functions of the required type and ordered as in P, but since $\vartheta^1_1([g])$ depends only on the equivalence class of g, which is the same as that of g', we get that for $(\mathcal{C}^\omega_{\omega_1})^n$-almost all $\vec{\alpha}$

$$\vartheta^1_1([g])(\vec{\alpha}) < [\vec{\gamma} \mapsto g(\vec{0}, \ell(\gamma_{k-1}), \alpha_{n-1}))]_{\mathcal{C}^\omega_{\omega_1} \otimes k} = [\vec{\gamma} \mapsto f(\vec{0}, \gamma_{k-1}, \alpha_{n-1})]_{\mathcal{C}^\omega_{\omega_1} \otimes k}.$$

But this contradicts the homogeneity of C, so $C \subseteq \kappa$ has to be homogeneous for the stated side.

Now that we have our club set, we can define a function $H : \kappa \to C$:

$$H(\alpha) := \text{ the } \omega^{\text{th}} \text{ element of } C \text{ greater than } \alpha.$$

For functions $g : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$ let $f : (\omega_1)^m \to C$ be defined by

$$f(\alpha) := \text{ the } \omega^{\text{th}} \text{ element of } C \text{ greater than } g(\alpha).$$

Then f is of discontinuous type $(\omega_1)^m$ and g and f are ordered as in P. So by the partition we have for all $g : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$, for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$

$$\vartheta^1_1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) \leq [\vec{\gamma} \mapsto f(\vec{0}, \gamma_{k-1}, \alpha_{n-1})]_{\mathcal{C}^\omega_{\omega_1} \otimes k} = [\vec{\gamma} \mapsto H(g(\vec{0}, \gamma_{k-1}, \alpha_{n-1}))]_{\mathcal{C}^\omega_{\omega_1} \otimes k}.$$

So there is a club set $C \subseteq \kappa$ such that for all $g : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$

$$\vartheta([g]_{(\mathcal{C}^\omega_{\omega_1})^m}) = [\vec{\alpha} \to |T^1 \restriction \delta(g, \alpha_{n-1})|(\vartheta^1_1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}))]_{\mathcal{C}^\omega_{\omega_1}\otimes n},$$

where for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$

$$\vartheta^1_1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) \leq [\vec{\gamma} \mapsto H(g(\vec{0}, \gamma_{k-1}, \alpha_{n-1}))]_{\mathcal{C}^\omega_{\omega_1}\otimes k}.$$

At the beginning of this proof we used a Martin tree to dominate the function F, now we are in a similar position with the function H, but we can use a Kunen tree instead of a Martin tree, since we are only interested in values of H at points of cofinality ω. So by Lemma 2.2.18 there is a tree K^1_1 on κ such that for all $g : (\omega_1)^m \to C$ of continuous type and for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$

$$\vartheta^1_1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) < [\vec{\gamma} \mapsto |K^1_1 \restriction g(\vec{0}, \gamma_{k-1}, \alpha_{n-1})|]_{\mathcal{C}^\omega_{\omega_1}\otimes k},$$

which means for all $g : (\omega_1)^m \to C$ of continuous type and for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$ there is a function $\vartheta^1_2([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) : [\omega_1]^k \to \kappa$ such that

$$\vartheta^1_1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) = [\vec{\gamma} \mapsto |K^1_1 \restriction g(\vec{0}, \gamma_{k-1}, \alpha_{n-1})|]_{\mathcal{C}^\omega_{\omega_1}\otimes k},$$

and $\vartheta^1_2([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha})(\vec{\gamma}) < g(\vec{0}, \gamma_{k-1}, \alpha_{n-1})$ for $\mathcal{C}^\omega_{\omega_1} \otimes k$-almost all $\vec{\gamma}$. We can assume that γ_{k-1} is a limit ordinal, and since g is continuous at limit points we get that for $\mathcal{C}^\omega_{\omega_1} \otimes k$-almost all $\vec{\gamma}$ there is a $\delta < \gamma_{k-1}$ such that

$$\vartheta^1_2([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha})(\vec{\gamma}) < g(\vec{0}, \delta, \alpha_{n-1}).$$

This δ depends only on the values of γ_0 to γ_{k-2}, in fact with Lemma 6.2.4 we can conclude that it depends only on γ_{k-2}. In other words, for $\mathcal{C}^\omega_{\omega_1} \otimes k$-almost all $\vec{\gamma}$ there is a function $\ell : \omega_1 \to \omega_1$ such that $\vartheta^1_2([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha})(\vec{\gamma}) < g(\vec{0}, \ell(\gamma_{k-2}), \alpha_{n-1}$. We call this behavior "g is pressing down on the $(k-1)$th variable".

This is nearly the same situation as we had for ϑ^1_1 before, only we dropped down from $k-1$ to $k-2$. So we can iterate this process and get:

There is a club set $C \subseteq \kappa$ such that for all $g : (\omega_1)^m \to C$ of continuous type

$$\vartheta([g]_{(\mathcal{C}^\omega_{\omega_1})^m}) = [\vec{\alpha} \mapsto |T^1 \restriction \delta(g, \alpha_{n-1})|(\vartheta^1_1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}))]_{\mathcal{C}^\omega_{\omega_1}\otimes n},$$

where for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$

$$\vartheta^1_1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) = [\vec{\gamma} \mapsto |K^1_1 \restriction g(\vec{0}, \gamma_{k-1}, \alpha_{n-1})|(\vartheta^1_2([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha})(\vec{\gamma}))]_{\mathcal{C}^\omega_{\omega_1}\otimes k},$$

with

$$\vartheta^1_2([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) = [\vec{\gamma} \mapsto |K^1_2 \restriction g(\vec{0}, \gamma_{k-2}, \alpha_{n-1})|(\vartheta^1_3([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha})(\vec{\gamma}))]_{\mathcal{C}^\omega_{\omega_1}\otimes k},$$

with

$$\vartheta^1_3([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) = [\vec{\gamma} \mapsto |K^1_3 \restriction g(\vec{0}, \gamma_{k-3}, \alpha_{n-1})|(\vartheta^1_4([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha})(\vec{\gamma}))]_{\mathcal{C}^\omega_{\omega_1} \otimes k},$$

$$\vdots$$

$$\vartheta^1_k([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) = [\vec{\gamma} \mapsto |K^1_k \restriction g(\vec{0}, \gamma_0, \alpha_{n-1})|(\vartheta^1_{k+1}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha})(\vec{\gamma}))]_{\mathcal{C}^\omega_{\omega_1} \otimes k},$$

with

$$\vartheta^1_{k+1}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) < [\vec{\gamma} \mapsto g(\vec{0}, \gamma_0, \alpha_{n-1})]_{\mathcal{C}^\omega_{\omega_1} \otimes k}.$$

So g is pressing down on γ_0, which means that there is a function $\eta : \omega_1 \to \kappa$ with $\eta(\alpha_{n-1}) < \sup_{\vec{\beta}} g(\vec{\beta}, \alpha_{n-1})$ such that $\vartheta^1_{k+1}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) < \eta(\alpha_{n-1})$.

Now comes again a partition argument. We partition the set of function pairs (g, f), where $g : (\omega_1)^m \to \kappa$ is of continuous type, $f : \omega_1 \to \kappa$ is of discontinuous type and $g(\vec{0}, \alpha) < f(\alpha) < g(1, \vec{0}, \alpha)$, according to whether $\vartheta^1_{k+1}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) < f(\alpha_{n-1})$ for $\mathcal{C}^\omega_\omega \otimes n$-almost all $\vec{\alpha}$ holds or not. Assume C is homogeneous for the contrary side and such that for all $g : (\omega_1)^m \to C$ of continuous type, for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$ there is a $\eta(\alpha_{n-1}) \in C$ such that $\vartheta^1_{k+1}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) < \eta(\alpha_{n-1})$, with $\eta(\alpha_{n-1}) < \sup_{\vec{\beta}} g(\vec{\beta}, \alpha_{n-1})$. Let $g : (\omega_1)^m \to C$ be such a function. Then a shifting argument gives us a pair of functions (g', f) of the right type and the right order with range in C such that $[g']_{(\mathcal{C}^\omega_{\omega_1})^m} = [g]_{(\mathcal{C}^\omega_{\omega_1})^m}$ and $f(\alpha) > \eta(\alpha)$ for $\mathcal{C}^\omega_{\omega_1}$-almost all α, which contradicts our assumption on C.

Now we repeat the argument that gives us the Kunen tree for, in this case, $\vartheta^1_{k+1}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha})$. So let C be homogeneous for the stated side, define

$$F(\alpha) := \text{ the } \omega\text{th element greater than } \alpha,$$

and for $g : (\omega_1)^m \to C$ let $f_g(\alpha) :=$ the ωth element greater than $g(\vec{0}, \alpha)$. The functions g and f_g are then ordered in the right way, so by our partition we get for $\mathcal{C}^\omega_\omega \otimes n$-almost all $\vec{\alpha}$

$$\vartheta^1_{k+1}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) < f_g(\alpha_{n-1}) = F(g(\vec{0}, \alpha_{n-1}).$$

Since $g(\vec{0}, \alpha)$ has cofinality ω for $\mathcal{C}^\omega_{\omega_1}$-almost all α we can use Lemma 2.2.18 to get a tree K^1_{k+1} on κ such that such that for all $g : (\omega_1)^m \to C$ of continuous type, for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$

$$\vartheta^1_{k+1}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) < |K^1_{k+1} \restriction g(\vec{0}, \alpha_{n-1})|.$$

Which means we can introduce a new $\vartheta^1_{k+2}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})$ such that for all $g : (\omega_1)^m \to C$ of continuous type, for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$

$$\vartheta^1_{k+1}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) = |K^1_{k+1} \restriction g(\vec{0}, \alpha_{n-1})|(\vartheta^1_{k+2}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha})),$$

with $\vartheta^1_{k+2}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) < g(\vec{0},\alpha_{n-1})$ for $\mathcal{C}^\omega_{\omega_1}\otimes n$-almost all $\vec{\alpha}$. Since $g(\vec{0},\alpha_{n-1}) = \sup_{\gamma<\alpha_{n-1},\vec{\beta}} g(\vec{\beta},\gamma)$, for $\mathcal{C}^\omega_{\omega_1}\otimes n$-almost all $\vec{\alpha}$ there is an ordinal $\gamma < \alpha_{n-1}$ such that $\vartheta^1_{k+2}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) < \sup_{\vec{\beta}} g(\vec{\beta},\gamma)$, *i.e.*, there is a function $\ell : \omega_1 \to \omega_1$ such that $\vartheta^1_{k+2}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) < \sup_{\vec{\beta}} g(\vec{\beta},\ell(\alpha_{n-2}))$ for $\mathcal{C}^\omega_{\omega_1}\otimes n$-almost all $\vec{\alpha}$.

We use this ℓ in a partition argument similar to the one in Lemma 6.2.5: Let P be the set of pairs (g,f), where $g : (\omega_1)^m \to \kappa$ is of continuous type, $f : \omega \to \kappa$ is of discontinuous type, and $g(\vec{0},\alpha) < f(\alpha) < g(1,\vec{0},\alpha)$ holds for all $\alpha < \omega_1$. We partition this set according to whether

$$\vartheta^1_{k+2}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) < f(\alpha_{n-2})$$

for $\mathcal{C}^\omega_{\omega_1}\otimes n$-almost all $\vec{\alpha}$ holds or not. As before, there is a homogeneous club set $C \subseteq \kappa$ for the stated side and we use this C to construct a function $F(\alpha) :=$ the ωth element greater than α, so we get: For all $g : (\omega_1)^m \to C$ of continuous type, for $\mathcal{C}^\omega_{\omega_1}\otimes n$-almost all $\vec{\alpha}$

$$\vartheta^1_{k+2}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) < F(\sup_\beta g(\vec{\beta},\alpha_{n-2})).$$

Now we are in nearly the same situation as at the beginning of our proof, but we dropped from α_{n-1} to α_{n-2}. The next step is again finding a Martin tree, T^2, and then a finite number of Kunen trees, K^2_1, etc., till we can drop down to α_{n-3}, and so on. If we denote the variable k from step i by k_i, we arrive at: There are Martin trees $T^1,\ldots,T^n$ and Kunen trees $K^1_1,\ldots,K^1_{k_1+1},K^2_1,\ldots,K^n_{k_n+1}$ such that:

There is a club set $C \subseteq \kappa$ such that for all $g : (\omega_1)^m \to C$ of continuous type

$$\vartheta([g]_{(\mathcal{C}^\omega_{\omega_1})^m}) = [\vec{\alpha} \mapsto |T^1 \restriction \delta(g,\alpha_{n-1})|(\vartheta^1_1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}))]_{\mathcal{C}^\omega_{\omega_1}\otimes n},$$

where for $\mathcal{C}^\omega_{\omega_1}\otimes n$-almost all $\vec{\alpha}$

$$\vartheta^1_1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) = [\vec{\gamma} \mapsto |K^1_1 \restriction g(\vec{0},\gamma_{k_1-1},\alpha_{n-1})|(\vartheta^1_2([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha})(\vec{\gamma}))]_{\mathcal{C}^\omega_{\omega_1}\otimes k_1},$$

where for $\mathcal{C}^\omega_{\omega_1}\otimes n$-almost all $\vec{\alpha}$

$$\vartheta^1_2([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) = [\vec{\gamma} \mapsto |K^1_2 \restriction g(\vec{0},\gamma_{k_1-2},\alpha_{n-1})|(\vartheta^1_3([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha})(\vec{\gamma}))]_{\mathcal{C}^\omega_{\omega_1}\otimes k_1},$$

$$\vdots$$

where for $\mathcal{C}^\omega_{\omega_1}\otimes n$-almost all $\vec{\alpha}$

$$\vartheta^1_{k_1}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) = [\vec{\gamma} \mapsto |K^1_{k_1} \restriction g(\vec{0},\gamma_0,\alpha_{n-1})|(\vartheta^1_{k_1+1}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha})(\vec{\gamma}))]_{\mathcal{C}^\omega_{\omega_1}\otimes k_1},$$

where for $\mathcal{C}^\omega_{\omega_1}\otimes n$-almost all $\vec{\alpha}$

$$\vartheta^1_{k_1+1}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) = |K^1_{k_1+1} \restriction g(\vec{0},\alpha_{n-1})|(\vartheta^1_{k_1+2}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha})),$$

where for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$

$$\vartheta^1_{k_1+2}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) = |T^2 \restriction \delta(g, \alpha_{n-2})|(\vartheta^2_1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha})),$$

where for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$

$$\vartheta^2_1([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) = [\vec{\gamma} \mapsto |K^2_1 \restriction g(\vec{0}, \gamma_{k_2-1}, \alpha_{n-2})|(\vartheta^2_2([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha})(\vec{\beta}))]_{\mathcal{C}^\omega_{\omega_1} \otimes k_2},$$

$$\vdots$$

where for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$

$$\vartheta^n_{k_n}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) = [\vec{\gamma} \mapsto |K^n_{k_n} \restriction g(\vec{0}, \gamma_0, \alpha_0)|(\vartheta^n_{k_n+1}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha})(\vec{\beta}))]_{\mathcal{C}^\omega_{\omega_1} \otimes k_n},$$

where for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$

$$\vartheta^n_{k_n+1}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) = |K^n_{k_n+1} \restriction g(\vec{0}, \alpha_0)|(\vartheta^n_{k_n+2}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha})),$$

where for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$

$$\vartheta^n_{k_n+2}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) < g(\vec{0}, \alpha_0) = \sup_{\gamma < \alpha_0, \vec{\beta}} g(\vec{\beta}, \gamma).$$

For all functions $g : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$ the function $\alpha \mapsto \sup_{\vec{\beta}} g(\vec{\beta}, \alpha)$ is increasing and continuous, so by the second part of Lemma 6.3.2 there is an ordinal $\gamma_g \in \omega_1$ such that for $\mathcal{C}^\omega_{\omega_1} \otimes n$-almost all $\vec{\alpha}$

$$\vartheta^n_{k_n+2}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})(\vec{\alpha}) < \sup_{\vec{\beta}} g(\vec{\beta}, \gamma_g).$$

With Lemma 2.2.15 we can assume without loss of generality that C is closed under $\mathcal{C}^\omega_{\omega_1} \otimes n$, *i.e.*,

$$[\vartheta^n_{k_n+2}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})]_{\mathcal{C}^\omega_{\omega_1} \otimes n} \leq \sup_{\vec{\beta}} g(\vec{\beta}, \gamma_g + 1) < \sup g.$$

Now we can use Lemma 6.3.3 and get that $[\vartheta^n_{k_n+2}([g]_{(\mathcal{C}^\omega_{\omega_1})^m})]_{\mathcal{C}^\omega_{\omega_1} \otimes n}$ is constant for all $g : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$, let $c_\vartheta : [\omega_1]^n \to \kappa$ represent that constant with respect to $\mathcal{C}^\omega_{\omega_1} \otimes n$

To summarize, we started with an element $[\vartheta]_{\mathcal{C}^{\omega_1}_\kappa \otimes m}$ of $[G_n]_{\mathcal{C}^{\omega_1}_\kappa \otimes m}$ and our analysis enables us to derive Martin trees $T^1, \ldots, T^n$, Kunen trees $K^1_1, \ldots, K^1_{m_1+1}, K^2_1, \ldots, K^n_{m_n+1}$ and a constant c_ϑ such that there is a club set $C \subseteq \kappa$ such that for all functions $g : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$

$$\vartheta([g]_{(\mathcal{C}^\omega_{\omega_1})^m} = [\vec{\alpha} \mapsto |T^1 \restriction \sup_k (\sup_{\vec{\beta}} g(\vec{\beta}, \alpha_{n-1}))^\kappa / (\mathcal{C}^\omega_\kappa)^k|(t^1_1)]_{\mathcal{C}^\omega_{\omega_1} \otimes n}, \text{ where}$$

$$t^1_1 = [\vec{\gamma} \mapsto |K^1_1 \restriction g(\vec{0}, \gamma_{k_1-1}, \alpha_{n-1})|(t^1_2)]_{\mathcal{C}^\omega_{\omega_1} \otimes k_1}, \text{ where}$$
$$t^1_2 = [\vec{\gamma} \mapsto |K^1_2 \restriction g(\vec{0}, \gamma_{k_1-2}, \alpha_{n-1})|(t^1_3)]_{\mathcal{C}^\omega_{\omega_1} \otimes k_1}, \text{ where}$$
$$\vdots$$
$$t^1_{k_1} = [\vec{\gamma} \mapsto |K^1_{k_1} \restriction g(\vec{0}, \gamma_0, \alpha_{n-1})|(t^1_{k_1+1})]_{\mathcal{C}^\omega_{\omega_1} \otimes k_1}, \text{ where}$$
$$t^1_{k_1+1} = |K^1_{k_1+1} \restriction g(\vec{0}, \alpha_{n-1})|(t^1_{k_1+2}), \text{ where}$$
$$t^1_{k_1+2} = |T^2 \restriction \sup_k (\sup_{\vec{\beta}} g(\vec{\beta}, \alpha_{n-2})^\kappa / (\mathcal{C}^\omega_\kappa)^m|(t^2_1), \text{ where}$$
$$t^2_1 = [\vec{\gamma} \mapsto |K^2_1 \restriction g(\vec{0}, \gamma_{k_2-1}, \alpha_{n-2})|(t^2_2)]_{\mathcal{C}^\omega_{\omega_1} \otimes k_2}, \text{ where}$$
$$\vdots$$
$$t^n_{k_n} = |K^n_{k_n} \restriction g(\vec{0}, \gamma_0, \alpha_0)|(t^n_{k_n+1}), \text{ where}$$
$$t^n_{k_n+1} = |K^n_{k_n+1} \restriction g(\vec{0}, \alpha_0)|(t^n_{k_n+2}), \text{ where}$$
$$t^n_{k_n+2} = c_\vartheta.$$

We are now at the point where we can define a surjective embedding π from $\kappa^\kappa/(\mathcal{C}^\omega_\kappa \oplus \mathcal{C}^{\omega_1}_\kappa{}^{\otimes m-1}) \otimes n$ to $[G_n]_{\mathcal{C}^{\omega_1}_\kappa{}^{\otimes m}}$: For $F : \kappa \to \kappa$ we define $\pi([F]_{(\mathcal{C}^\omega_\kappa \oplus \mathcal{C}^\omega_\kappa{}^{\otimes m-1}) \otimes n})$ to be $[\pi(F)]_{\mathcal{C}^{\omega_1}_\kappa{}^{\otimes m}}$, with

$$\pi(F)([g]_{(\mathcal{C}^\omega_{\omega_1})^m}) :=$$
$$[\vec{\alpha} \mapsto F(\ulcorner g(\vec{0}, \alpha_0), \sup_{\vec{\beta}} g(\vec{\beta}, \alpha_0), \ldots, g(\vec{0}, \alpha_{n-1}), \sup_{\vec{\beta}} g(\vec{\beta}, \alpha_{n-1})\urcorner)]_{\mathcal{C}^\omega_{\omega_1} \otimes n}$$

for functions $g : (\omega_1)^m \to \kappa$ of continuous type $(\omega_1)^m$. First we have to show of course that $\pi(F)$ is welldefined with respect to $(\mathcal{C}^\omega_{\omega_1})^m$-equivalence classes:

If $g, g' : (\omega_1)^m \to \kappa$ are functions of continuous type $(\omega_1)^m$ with $[g]_{(\mathcal{C}^\omega_{\omega_1})^m} = [g']_{(\mathcal{C}^\omega_{\omega_1})^m}$ then there is a club set $D \subseteq \omega_1$ such that for all $\alpha \in D$ the set

$$\{\vec{\beta}\,;\, g(\vec{\beta}, \alpha) = g'(\vec{\beta}, \alpha)\}$$

is in $(\mathcal{C}^\omega_{\omega_1})^{m-1}$. Let $C \subseteq \kappa$ be a club set of κ. If $f : (\omega_1)^m \to C$ is a function of continuous type $(\omega_1)^m$ and $\alpha \in \omega_1$ then the function $f_\alpha : (\omega_1)^{m-1} \to C$ defined by $f_\alpha(\vec{\beta}) := f(\vec{\beta}, \alpha)$ is of continuous type $(\omega_1)^{m-1}$. So by the above argument for functions $g, g' : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$ with $[g]_{(\mathcal{C}^\omega_{\omega_1})^m} = [g']_{(\mathcal{C}^\omega_{\omega_1})^m}$ there is a club set $D \subseteq \omega_1$ such that $[g_\alpha]_{(\mathcal{C}^\omega_{\omega_1})^{m-1}} = [g'_\alpha]_{(\mathcal{C}^\omega_{\omega_1})^{m-1}}$ for all $\alpha \in D$. Since the functions g_α and g'_α are increasing we get $\sup_{\vec{\beta}} g(\vec{\beta}, \alpha) = \sup_{\vec{\beta}} g'(\vec{\beta}, \alpha)$, *i.e.*, for all $\vec{\alpha} \in [D]^n$ and $i < n$ we have

$$g(\vec{0}, \alpha_i) = g'(\vec{0}, \alpha_i) \text{ and } \sup_{\vec{\beta}} g(\vec{\beta}, \alpha_i = \sup_{\vec{\beta}} g'(\vec{\beta}, \alpha_i).$$

So $\pi(F)([g]_{(\mathcal{C}^{\omega}_{\omega_1})^m})$ is welldefined.

And it is an welldefined embedding: Assume

$$[F]_{(\mathcal{C}^{\omega}_{\kappa}\oplus\mathcal{C}^{\omega}_{\kappa}{}^{\otimes m-1})^{\otimes n}} \mathrel{\substack{=\\<}} [F']_{(\mathcal{C}^{\omega}_{\kappa}\oplus\mathcal{C}^{\omega}_{\kappa}{}^{\otimes m-1})^{\otimes n}}.$$

This means there is a club set $C \subseteq \kappa$ such that we have

$$F(\ulcorner[f_0]_{\mathcal{C}^{\omega}_{\omega}}, \dots, [f_{2n-1}]_{(\mathcal{C}^{\omega}_{\omega_1})^{m-1}}\urcorner) \mathrel{\substack{=\\<}} F'(\ulcorner[f_0]_{\mathcal{C}^{\omega}_{\omega}}, \dots, [f_{2n-1}]_{(\mathcal{C}^{\omega}_{\omega_1})^{m-1}}\urcorner)$$

for all $f_0 \in \mathfrak{C}^{\omega}_{C}$, ..., $f_{2n-1} \in \mathfrak{C}^{(\omega_1)^{m-1}}_{C_{\sup f_{2n-2}}}$

Let $g : (\omega_1)^m \to C$ be a function of continuous type $(\omega_1)^m$, as mentioned before for any $\alpha \in \omega_1$ the function $g_\alpha : (\omega_1)^{m-1} \to C$ defined by $g_\alpha(\vec{\beta}) := g(\vec{\beta}, \alpha)$ is of continuous type $(\omega_1)^{m-1}$. Furthermore, since g is increasing in the $<_{\text{rlex}}$-order, if $\alpha < \alpha'$ are limit ordinals then $\sup g_\alpha < g_{\alpha'}$.

Let D be the set of limit ordinals less than ω_1, note that all its elements have cofinality ω. So for all functions $g : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$ we get that for all $\vec{\alpha} \in [D]^n$ we have $g_{\alpha_0} \in \mathfrak{C}^{(\omega_1)^{m-1}}_{C}$, ..., $g_{\alpha_{n-1}} \in \mathfrak{C}^{(\omega_1)^{m-1}}_{C_{\sup g_{\alpha_{n-2}}}}$.

If $g \in \mathfrak{C}^{(\omega_1)^m}_{C}$ then there is a $g' \in \mathfrak{C}^{(\omega_1)^m}_{C}$ such that $\sup_{\vec{\beta}} g(\vec{\beta}, \alpha_i) = [g'_{\alpha_i}]_{(\mathcal{C}^{\omega}_{\omega_1})^{m-1}}$, which means we get the statement that for all $g \in \mathfrak{C}^{(\omega_1)^m}_{C}$ and all $\vec{\alpha} \in [D]^n$ we have

$$F(\ulcorner g(\vec{0}, \alpha_0), \dots, \sup_{\vec{\beta}} g(\vec{\beta}, \alpha_{n-1})\urcorner) \mathrel{\substack{=\\<}} F'(\ulcorner g(\vec{0}, \alpha_0), \dots, \sup_{\vec{\beta}} g(\vec{\beta}, \alpha_{n-1})\urcorner).$$

But since $D \subseteq \omega_1$ is a club set that means $[\pi(F)]_{\mathcal{C}^{\omega_1 \otimes m}_{\kappa}} \mathrel{\substack{=\\<}} [\pi(F')]_{\mathcal{C}^{\omega_1 \otimes m}_{\kappa}}$.

The only thing left to show is the surjectivity of π. Let $[\vartheta]_{\mathcal{C}^{\omega_1 \otimes m}_{\kappa}}$ be an element of $[G_n]_{\mathcal{C}^{\omega_1 \otimes m}_{\kappa}}$, we get Martin trees $T^1, \dots, T^n$, Kunen trees K^1_1, ..., $K^1_{m_1+1}$, K^2_1, ..., $K^n_{m_n+1}$ and a constant c_ϑ from our analysis of $[\vartheta]_{\mathcal{C}^{\omega_1 \otimes m}_{\kappa}}$. Let

$$F_\vartheta(\ulcorner\beta_0, \alpha_0, \dots, \alpha_{n-1}\urcorner) := |T^1 {\restriction} \alpha_{n-1}|(| \dots |K^n_{k_n+1} \restriction \beta_0|(c_\vartheta) \dots),$$

then there is a club set $C \subseteq \kappa$ such that for all $g : \omega_1 \to C$ of continuous type ω_1

$$\vartheta([g]_{(\mathcal{C}^{\omega}_{\omega_1})^m}) =$$

$$[\vec{\alpha} \mapsto |T^1 {\restriction} \sup_{\vec{\beta}} g(, \vec{\beta}, \alpha_{n-1})|(\dots |K^n_{k_n+1} \restriction g(\vec{0}, \alpha_0)|(c_\vartheta) \dots)]_{\mathcal{C}^{\omega}_{\omega_1}{}^{\otimes n}}$$

$$= \pi(F)([g]_{(\mathcal{C}^{\omega}_{\omega_1})^m}).$$

So $[\vartheta]_{\mathcal{C}^{\omega_1 \otimes m}_{\kappa}} = [\pi(F)]_{\mathcal{C}^{\omega_1 \otimes m}_{\kappa}}$ and thus π is bijective. From our induction assumption $\kappa^\kappa/\mathcal{C}^{\omega_1 \otimes (m-1)}_{\kappa} = \kappa^{(\omega^{m-1}+1)}$ follows, as we remarked at the beginning of this section, the canonicity of our measure assignment up to height $\kappa^{(\omega^{m-1}+1)}$. So we have $\kappa^\kappa/(\mathcal{C}^\omega_\kappa \oplus \mathcal{C}^{\omega_1 \otimes m-1}_\kappa) \otimes n = \kappa^{(\omega^{m-1}\cdot n+1)}$, which means

$$[G_n]_{\mathcal{C}^{\omega_1 \otimes m}_\kappa} = \kappa^\kappa/(\mathcal{C}^\omega_\kappa \oplus \mathcal{C}^{\omega_1 \otimes m-1}_\kappa) \otimes n = \kappa^{(\omega^{m-1}\cdot n+1)}.$$

We now can conclude the proof of the induction step with

$$\kappa^\kappa/\mathcal{C}^{\omega_1 \otimes m}_\kappa = (\sup_{n<\omega}[G_n]_{\mathcal{C}^{\omega_1 \otimes m}_\kappa})^+ = (\sup_{n<\omega}\kappa^{(\omega^{m-1}\cdot n+1)})^+ = \kappa^{(\omega^m+1)}.$$

q.e.d.

From Theorem 6.4.4 follows directly that the next ω many necessary conditions for the canonicity of our measure assignment are proven. Analogous to Corollary 6.3.6 we can conclude:

6.4.5. COROLLARY. *Assume* $\mathsf{AD}+\mathsf{DC}$*. The* $\boldsymbol{\delta}^1_{2n+1}$*-measure assignment for* $\mathfrak{A}_2$ *derived from* $\mathrm{GA}_{\boldsymbol{\delta}^1_{2n+1}}$ *is canonical up to height* $(\boldsymbol{\delta}^1_{2n+1})^{(\omega^\omega)}$*.*

Proof. This is again an application of Lemma 5.2.4. From the Corollary 5.3.2 we have the necessary properties for $\mathbf{germ}(\mathsf{S}_0)$ and $\mathbf{germ}(\mathsf{S}_1)$. Let $n>0$, by Theorem 6.4.4 we know that $\boldsymbol{\delta}^1_{2n+1}{}^{\boldsymbol{\delta}^1_{2n+1}}/(\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}})^{\otimes n} = (\boldsymbol{\delta}^1_{2n+1})^{(\omega^n+1)}$ so we have $\mathrm{I}(\boldsymbol{\delta}^1_{2n+1}, n)$. Condition $\mathrm{II}(\boldsymbol{\delta}^1_{2n+1}, n)$ follows from Proposition 6.1.2 and condition $\mathrm{III}(\boldsymbol{\delta}^1_{2n+1}, n)$ from Lemma 6.4.1. q.e.d.

This result extends the measure analysis from [BoLö07], where canonicity was shown only up to height $(\boldsymbol{\delta}^1_{2n+1})^{(\omega^2)}$. We now can compute the values of ultrapowers with respect to order measure sums of $\mathcal{C}^\omega_{\boldsymbol{\delta}^1_{2n+1}}$ and $(\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}})^{\otimes m}$.

6.4.6. COROLLARY. *Assume* $\mathsf{AD}+\mathsf{DC}$*. Let* $\boldsymbol{\delta}^1_{2n+1} > \omega_1$ *be an projective ordinal,* $k >$ *a natural number and* $\langle\langle n_i, m_i\rangle\,;\, i<\ell\rangle$ *a finite sequence of pairs of positive natural numbers. Then*

$$\boldsymbol{\delta}^1_{2n+1}{}^{\boldsymbol{\delta}^1_{2n+1}}/(\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}}{}^{\otimes m_0}) \otimes n_0 \oplus \ldots \oplus (\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}}{}^{\otimes m_{\ell-1}}) \otimes n_{\ell-1} \oplus \mathcal{C}^\omega_{\boldsymbol{\delta}^1_{2n+1}} \otimes k$$

$$= (\boldsymbol{\delta}^1_{2n+1})^{(\omega^{m_0}\cdot n_0+\ldots+\omega^{m_{\ell-1}}\cdot n_{\ell-1}+k+1)},$$

and of course

$$\boldsymbol{\delta}^1_{2n+1}{}^{\boldsymbol{\delta}^1_{2n+1}}/(\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}}{}^{\otimes m_0}) \otimes n_0 \oplus \ldots \oplus (\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}}{}^{\otimes m_{\ell-1}}) \otimes n_{\ell-1}$$

$$= (\boldsymbol{\delta}^1_{2n+1})^{(\omega^{m_0}\cdot n_0+\ldots+\omega^{m_{\ell-1}}\cdot n_{\ell-1}+1)}.$$

Proof. As in Corollaries 5.3.3 and 6.3.6 this is a simple application of the canonicity of our measure assignment. q.e.d.

In order to proceed with the inductive proof of the canonicity of NMA_κ we would have next to compute the value of the ultrapower with respect to the ω_2-cofinal measure. This entails the use of Martin Trees that give upper bounds for ordinals with higher cofinalities like ω_2, *i.e.*, analogues of Theorem 2.2.19. Note that if $\kappa^\kappa/\mathcal{C}_\kappa^{\omega_2}$ is computed to be $\kappa^{(\omega^\omega+1)}$ we could use a simple derivation of Theorem 6.4.4 to compute the ultrapowers with respect to measures of the form $\mathcal{C}_\kappa^{\omega_2} \otimes \mathcal{C}_\kappa^{\omega_1 \otimes n}$.

Chapter 7

Applications of the Canonical Measure Analysis

If the natural measure assignment defined by Definition 4.5.1 is canonical up to ${\boldsymbol{\delta}^1_{2n+1}}^{(\gamma)}$ then we can compute the cardinal value of ultrapowers of measures from the measure assignment up to ${\boldsymbol{\delta}^1_{2n+1}}^{(\gamma)}$ and get ultrapower representations for successor cardinals between $\boldsymbol{\delta}^1_{2n+1}$ and ${\boldsymbol{\delta}^1_{2n+1}}^{(\gamma)}$. We have proven in Corollary 6.4.5 that the natural measure assignment is canonical at least up to height ${\boldsymbol{\delta}^1_{2n+1}}^{(\omega^\omega)}$.

In Sections 7.1 and 7.2 we present results from the paper [JaLö06] about the cofinality of cardinals in the reach of a canonical measure assignment. We will use these results in Section 7.3 to show that our canonicity result allows us to enlarge the number of cardinals under AD for which we can show that they are Jónsson.

7.1 Regular Cardinals

In this chapter we will work again with ordinal algebras, not with additive ordinal algebras. We will compute regular cardinals by identifying special variables in the set $\mathfrak{V}$ of generators of an ordinal algebra. We call these variables **normal**, as they will be the ones that are assigned normal measures by our recursive assignment.

7.1.1. Definition. We say that V_0 and V_1 are normal. In each of the iteration steps from $\mathfrak{A}_{2+\mathbf{e}_n}$ to $\mathfrak{A}_{\mathbf{e}_{n+1}}$ we identify those new variables as normal that correspond to normal variables in $\mathfrak{A}_{2+\mathbf{e}_n}$:

For $\xi < 2 + \mathbf{e}_n$, the variable V_ξ is normal if and only if it was designated as normal in $\mathfrak{A}_{2+\mathbf{e}_n}$. For $\xi = 2 + \mathbf{e}_n + \eta$ for some $\eta < \mathbf{e}_{n+1}$, the variable V_ξ is normal if and only if $\eta = o(\mathsf{v})$ for some normal variable v.

By Proposition 3.1.1, for infinite ordinals ξ, the function o is just $\xi \mapsto \omega^{\omega^\xi}$. Therefore we can easily compute the indices of the normal variables by the following algorithm: write down the 2^{n+1} normal variables for $\mathfrak{A}_{\mathbf{e}_n}$, write down the values of o for these variables underneath the variables, then compute the 2^{n+1} new normal variables as $\mathbf{e}_{n+1} + o(\mathsf{v})$ for the values of o in your list. You can see the first three steps of the algorithm in the following table:

$\mathfrak{A}_2$		0	1		
	o	0	1		
$\mathfrak{A}_\omega$		$2 = 2+0$	$3 = 2+1$		
	o	ω	ω^ω		
$\mathfrak{A}_{\mathbf{e}_2}$		$\omega = \omega + 0$	$\omega + 1$	$\omega \cdot 2 = \omega + \omega$	$\omega^\omega = \omega + \omega^\omega$
	o	$\mathbf{e}_3$	$\omega^{\omega^{\omega+1}}$	$\omega^{\omega^{\omega \cdot 2}}$	$\omega^{\omega^{\omega^\omega}}$

It is easy to show that the normal variables give rise to normal measures:

7.1.2. LEMMA. *Let κ be a strong partition cardinal closed under ultrapowers. If μ is a normal measure on $\varrho < \kappa$ and the ultrapowers κ^ϱ/μ and $\kappa^\kappa/\mathbf{lift}_\kappa(\mu)$ are wellfounded then both $\mathbf{lift}_\kappa(\mu)$ and $\mathbf{slift}(\mathbf{lift}_\kappa(\mu))$ are normal measures.*

Proof. This is just a summary of Lemmas 4.1.7 and 4.2.2. q.e.d.

7.1.3. THEOREM (JACKSON-LÖWE). *Assume $\mathsf{AD} + \mathsf{DC}$. The canonical measure assignment from Definition 4.5.2 assigns normal measures to all normal variables. Consequently, ${\boldsymbol{\delta}^1_{2n+1}}^{\boldsymbol{\delta}^1_{2n+1}}/\mathbf{meas}_{\boldsymbol{\delta}^1_{2n+1}}(\mathsf{v})$ is a regular cardinal for all normal variables v.*

Proof. The first part of the claim follows immediately by induction from Lemma 7.1.2. The second part follows from the fact that, if there is a normal ultrafilter an a cardinal, then that cardinal is regular, see Lemmas 1.2.4 and 1.2.6. q.e.d.

Assuming that the measure assignment is canonical, we can now compute these regular cardinals easily from the table given above by looking at the row containing the o-values. In the following table, we list the first 32 of such cardinals (up to $\boldsymbol{\delta}^1_9$):

$\aleph_0$	$\boldsymbol{\delta}^1_1 = \aleph_1$	$\boldsymbol{\delta}^1_2 = \aleph_2$	$\boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$
$\boldsymbol{\delta}^1_4 = \aleph_{\omega+2}$	$\aleph_{\omega\cdot 2+1}$	$\aleph_{\omega^\omega+1}$	$\boldsymbol{\delta}^1_5 = \aleph_{\omega^{\omega^\omega}+1}$
$\boldsymbol{\delta}^1_6 = \aleph_{\omega^{\omega^\omega}+2}$	$\aleph_{\omega^{\omega^\omega}+\omega+1}$	$\aleph_{\omega^{\omega^\omega}+\omega^\omega+1}$	$\aleph_{\omega^{\omega^\omega}\cdot 2+1}$
$\aleph_{\omega^{\omega^{\omega+1}}+1}$	$\aleph_{\omega^{\omega^{\omega\cdot 2}}+1}$	$\aleph_{\omega^{\omega^{\omega^\omega}}+1}$	$\boldsymbol{\delta}^1_7 = \aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}+1}$
$\boldsymbol{\delta}^1_8 = \aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}+2}$	$\aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}+\omega+1}$	$\aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}+\omega^\omega+1}$	$\aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}+\omega^{\omega^\omega}+1}$
$\aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}+\omega^{\omega^{\omega+1}}+1}$	$\aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}+\omega^{\omega^{\omega\cdot 2}}+1}$	$\aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}+\omega^{\omega^{\omega^\omega}}+1}$	$\aleph_{\omega^{\omega^{\omega^{\omega^\omega}}}\cdot 2+1}$
$\aleph_{\omega^{\omega^{(\omega^{\omega^\omega}+1)}}+1}$	$\aleph_{\omega^{\omega^{(\omega^{\omega^\omega}+\omega)}}+1}$	$\aleph_{\omega^{\omega^{(\omega^{\omega^\omega}+\omega^\omega)}}+1}$	$\aleph_{\omega^{\omega^{(\omega^{\omega^\omega}\cdot 2)}}+1}$
$\aleph_{\omega^{\omega^{(\omega^{\omega^{\omega+1}})}}+1}$	$\aleph_{\omega^{\omega^{(\omega^{\omega^{\omega\cdot 2}})}}+1}$	$\aleph_{\omega^{\omega^{\omega^{\omega^{\omega^\omega}}}}+1}$	$\boldsymbol{\delta}^1_9 = \aleph_{\omega^{\omega^{\omega^{\omega^{\omega^{\omega^\omega}}}}}+1}$

Note that we have not yet proved that these are the only regular cardinals. This will be done in Section 7.2.

7.2 Cofinalities

In this section we will show that the canonical measure assignment enables us to compute the cofinality of all cardinals that are in the reach of the measure assignment. We will do this by giving a concrete algorithm to compute those cofinalities.

7.2.1. LEMMA. *Let κ be a weak partition cardinal closed under ultrapowers. Let μ be a measure containing end segments on $\varrho < \kappa$ with $\mathrm{cf}(\varrho) = \delta$. Then there is a cofinal embedding from $\kappa^\kappa/\mathcal{C}^\delta_\kappa$ into $\kappa^\kappa/\mathbf{lift}_\kappa(\mu)$, assuming those ultrapowers are wellfounded.*

Proof. We define a function $\pi : \kappa^\kappa/\mathcal{C}^\delta_\kappa \to \kappa^\kappa/\mathbf{lift}_\kappa(\mu)$ as follows: If $F \in \kappa^\kappa$, let $G \in \kappa^\kappa$ be defined as:

1. If $\alpha = [g]_\mu$ for some $g : \varrho \to \kappa$ of continuous type ϱ, let $G(\alpha) := F(\sup(g))$.
2. For all other $\alpha < \kappa$ let $G(\alpha) := 0$.

We set $\pi([F]_{\mathcal{C}^\delta_\kappa}) := [G]_{\mathbf{lift}_\kappa(\mu)}$. This is well-defined, for if $[g_1]_\mu = [g_2]_\mu$ and g_1, g_2 are both increasing, then $\sup(g_1) = \sup(g_2)$ (since any μ measure one set is cofinal in ϱ).

We have to show that π is indeed a cofinal embedding. Suppose $[F_1]_{\mathcal{C}^\delta_\kappa} = [F_2]_{\mathcal{C}^\delta_\kappa}$. Then there is a club set $C \subseteq \kappa$ such that for all $\alpha \in C$ of cofinality δ we have $F_1(\alpha) = F_2(\alpha)$. So for any $\beta < \kappa$ represented by a function $g : \varrho \to C$ of continuous type ϱ we have $G_1(\beta) = F_1(\sup(g)) = F_2(\sup(g)) = G_2(\beta)$, since $\sup g$ is an element of C with cofinality ϱ. This means $[G_1]_{\mathbf{lift}_\kappa(\mu)} = [G_2]_{\mathbf{lift}_\kappa(\mu)}$ and π is an embedding from $\kappa^\kappa/\mathcal{C}^\delta_\kappa$ into $\kappa^\kappa/\mathbf{lift}_\kappa(\mu)$.

To see that the embedding π is cofinal we take an arbitrary $H \in \kappa^\kappa$. Define the function $F \in \kappa^\kappa$ by

$$F(\alpha) := \sup\{H([g]_\mu)\,;\, g \in \mathfrak{C}^\varrho_\kappa \text{ with } \sup(g) = \alpha\}$$

for $\alpha < \kappa$ of cofinality δ, $F(\alpha) := 0$ otherwise.

This is well-defined as κ is regular and closed under ultrapowers. Then we have for all functions $g \in \mathfrak{C}^\varrho_\kappa$ of continuous type ϱ

$$G([g]_\mu) = F(\sup(g)) = \sup\{H([g']_\mu)\,;\, \sup(g') = \sup(g)\} \geq H([g]_\mu).$$

So by definition of π we get $\pi([F]_{\mathcal{C}^\delta_\kappa}) \geq [H]_{\mathbf{lift}_\kappa(\mu)}$, which means π is cofinal. q.e.d.

As an immediate consequence, we can reduce the computation of the cofinality of $\kappa^\kappa/\mathbf{lift}_\kappa(\mathbf{germ}(x))$ for an arbitrary term $x \in \mathfrak{A}_{\varepsilon_0}$ to the cofinalities of the basic variables:

7.2.2. Corollary. *Let $x \in \mathfrak{A}_{\varepsilon_0}$ be a term with trailing node v such that $\ell_x(v) = \mathsf{v}$. If $o(x) < \mathbf{e}_{n+1}$, write $\kappa := \boldsymbol{\delta}^1_{2n+1}$ and $\lambda := \kappa^\kappa/\mathbf{meas}_\kappa(\mathsf{v})$. Then*

$$\mathrm{cf}(\kappa^\kappa/\mathbf{meas}_\kappa(x)) = \mathrm{cf}(\lambda).$$

Proof. This is immediate from Lemma 7.2.1, keeping in mind that $\mathrm{cf}(\mathbf{ot}(x)) = \mathrm{cf}(\ell_x(v))$, as mentioned in Remark 4.1.2. q.e.d.

We shall now recursively define a function nor : $\mathfrak{V} \to \mathfrak{V}$ assigning normal variables to arbitrary generators in the algebra. Our recursion will go along the tower of algebras $\mathfrak{A}_{2+\mathbf{e}_n}$ as in the definition of the measure assignment in Definition 4.5.2.

In $\mathfrak{A}_2$, all basic variables are normal, so the function nor will just be the identity. Suppose that we have defined the function nor on $\mathfrak{A}_{2+\mathbf{e}_n}$ and want to extend it to $\mathfrak{A}_{\mathbf{e}_{n+1}}$. Each of the generators V_α of $\mathfrak{A}_{\mathbf{e}_{n+1}}$ was either already in $\mathfrak{A}_{2+\mathbf{e}_n}$, in which case we already have defined its nor value, or is of the form $\mathsf{V}_{2+\mathbf{e}_n+\xi}$ for some $\xi < \mathrm{ht}(\mathfrak{A}_{2+\mathbf{e}_n})$. By the recursive measure assignment from Definition 4.5.2, this variable $\mathsf{V}_\alpha = \mathsf{V}_{2+\mathbf{e}_n+\xi}$ is linked to terms $x \in \mathfrak{A}_{2+\mathbf{e}_n}$ such that $o(x) = \xi$. Let x be such a term with representing tree $\langle T_x, \ell_x\rangle$ and trailing node $v \in T_x$.

Then $\ell_x(v)$ is a generator of $\mathfrak{A}_{2+\mathbf{e}_n}$. We can now define for such α

$$\mathrm{nor}(\mathsf{V}_\alpha) := \mathsf{V}_{2+\mathbf{e}_n+o(\mathrm{nor}(\ell_x(v)))}.$$

7.2.3. Theorem (Jackson-Löwe). *Assume* $\mathsf{AD}+\mathsf{DC}$. *For each generator* v *of $\mathfrak{A}_{\varepsilon_0}$ and every odd projective ordinal $\kappa = \boldsymbol{\delta}^1_{2n+1}$ such that $\mathbf{ot}(v) < \kappa$, we have that*

$$\mathrm{cf}(\kappa^\kappa/\mathbf{meas}_\kappa(\mathsf{v}) = \mathrm{cf}(\kappa^\kappa/\mathbf{meas}_\kappa(\mathrm{nor}(\mathsf{v}))).$$

Proof. The claim is proved by induction on n. Recall that the generators v with $\mathbf{ot}(v) < \boldsymbol{\delta}^1_{2n+1}$ are precisely those in $\mathfrak{A}_{2+\mathbf{e}_n}$. The case $n = 0$ is trivial as nor is the identity on the generators in $\mathfrak{A}_2$ (*i.e.*, V_0, V_1). Assume the theorem holds for n, that is for $\boldsymbol{\delta}^1_{2n+1}$ and $\mathfrak{A}_{2+\mathbf{e}_n}$, and we show it holds for $n+1$, that is, for $\boldsymbol{\delta}^1_{2n+3}$ and $\mathfrak{A}_{\mathbf{e}_{n+1}}$.

Let v be a generator in $\mathfrak{A}_{\mathbf{e}_{n+1}}$, so $\mathsf{v} = \mathsf{V}_{2+\mathbf{e}_n+\xi}$ for some $\xi < \mathbf{e}_{n+1} = \mathrm{ht}(\mathfrak{A}_{2+\mathbf{e}_n})$. Fix $x \in \mathfrak{A}_{2+\mathbf{e}_n}$ such that $o(x) = \xi$, let v be the trailing term of $\langle T_x, \ell_x\rangle$, and $\mathsf{v}^* := \ell_x(v)$. By definition of nor, we have $\mathrm{nor}(v) = \mathsf{V}_{2+\mathbf{e}_n+o(\mathrm{nor}(\mathsf{v}^*))}$. Let $\lambda := \boldsymbol{\delta}^1_{2n+1}$. By Corollary 7.2.2 and the induction hypothesis, we have that

$$\mathrm{cf}(\lambda^\lambda/\mathbf{meas}_\lambda(x)) = \mathrm{cf}(\lambda^\lambda/\mathbf{meas}_\lambda(\mathsf{v}^*)).$$

But $\mathrm{cf}(\lambda^\lambda/\mathbf{meas}_\lambda(x)) = \mathbf{ot}(v)$ and $\mathrm{cf}(\lambda^\lambda/\mathbf{meas}_\lambda(\mathsf{v}^*)) = \mathbf{ot}(\mathrm{nor}(\mathsf{v}))$. Now we can apply Lemma 7.2.1 (with the κ there being $\boldsymbol{\delta}^1_{2n+3}$) to finish the claim. q.e.d.

7.2.4. COROLLARY. *Assume* $\mathsf{AD}+\mathsf{DC}$ *and let* $\boldsymbol{\delta}^1_{2n+1}$ *be an odd projective ordinal. All regular cardinals between* $\boldsymbol{\delta}^1_{2n+1}$ *and* $\boldsymbol{\delta}^1_{2n+3}$ *are of the form*

$$\boldsymbol{\delta}^1_{2n+1}{}^{\boldsymbol{\delta}^1_{2n+1}}/\mathbf{meas}_{\boldsymbol{\delta}^1_{2n+1}}(\mathsf{v}),$$

where v *is a normal variable. Furthermore, every normal measure on* $\boldsymbol{\delta}^1_{2n+1}$ *is of the form* $\mathbf{meas}_{\boldsymbol{\delta}^1_{2n+1}}(\mathsf{v})$, *with* v *being a normal variable.*

Proof. The first part follows directly from Theorems 7.1.3 and 7.2.3. The second from the recursive definition of normal variable and Theorem 1.6.2. q.e.d.

Using Corollary 7.2.2 and Theorem 7.2.3, we can now describe the algorithm to compute the value of $\mathrm{cof}_\kappa(x) := \mathrm{cf}(\kappa^\kappa/\mathbf{meas}_\kappa(x))$ recursively for arbitrary x. Suppose that we have already computed $\mathrm{cof}_\kappa{\restriction}\mathfrak{A}_{2+\mathbf{e}_n}$ for all odd projective ordinals $\kappa \leq \boldsymbol{\delta}^1_{2n+1}$. We shall give an algorithm to compute $\mathrm{cof}_\kappa{\restriction}\mathfrak{A}_{\mathbf{e}_{n+1}}$ for all $\kappa \leq \boldsymbol{\delta}^1_{2n+3}$.

Given a term $x \in \mathfrak{A}_{\mathbf{e}_{n+1}}$, ask whether $x \in \mathfrak{A}_{2+\mathbf{e}_n}$ or not.
Case 1. $x \in \mathfrak{A}_{2+\mathbf{e}_n}$.
Then $\mathrm{cof}_\kappa(x)$ has already been determined.
Case 2. $x \notin \mathfrak{A}_{2+\mathbf{e}_n}$.
Find the trailing term v of $\langle T_x, \ell_x\rangle$.
Set $\mathsf{v} := \ell_x(v)$.
Compute $\mathrm{nor}(\mathsf{v})$.
Then $\mathrm{cof}_\kappa(x) = \kappa^\kappa/\mathbf{meas}_\kappa(\mathrm{nor}(\mathsf{v}))$.

Figure 7.1: The algorithm to compute cofinalities.

7.2.5. COROLLARY. *The algorithm described in Figure 7.1 correctly computes the cofinality of* $\kappa^\kappa/\mathbf{meas}_\kappa(x)$.

Proof. Obvious from Corollary 7.2.2 and Theorem 7.2.3. q.e.d.

Corollary 7.2.5 and the canonicity assumption give an algorithm for computing the cofinality of any successor cardinal $\aleph_{\alpha+1}$ for $\alpha < \varepsilon_0$. Namely, first find the n such that $\mathbf{e}_n \leq \alpha < \mathbf{e}_{n+1}$. Let α' be such that $\alpha = \mathbf{e}_n + \alpha'$. Let $x \in \mathfrak{A}_{\mathbf{e}_n}$ be a term with $o(x) = \alpha'$. Let $\mathsf{v} = \mathrm{nor}(x)$. Then $\mathrm{cf}(\aleph_{\alpha+1} = (\boldsymbol{\delta}^1_{2n-1})^{\boldsymbol{\delta}^1_{2n-1}}/\mathbf{meas}_{\boldsymbol{\delta}^1_{2n-1}}(\mathsf{v}) = \aleph_{\mathbf{e}_n+o(\mathsf{v})+1}$.

7.3 Jónsson Cardinals

In Section 1.6 we introduced Jónsson cardinals. Kleinbergs Theorem 1.7.2 states that if μ is a normal measure on a strong partition cardinal κ then all elements of the corresponding Kleinberg sequence are Jónsson. Under AD the projective ordinals have the strong partition property and we know from Theorem 7.1.3 that the normal measures on $\boldsymbol{\delta}^1_{2n+1}$ are order measures that are included in our canonical measure analysis. Which means that we can compute the cardinal values of the elements of the corresponding Kleinberg sequence. So for every normal measure on $\boldsymbol{\delta}^1_{2n+1}$ we get a sequence of Jónsson cardinals with computable values.

7.3.1. LEMMA. *Assume* $\mathsf{AD} + \mathsf{DC}$ *and that the measure assignment from Definition 4.5.2 is canonical. If* $\mathcal{C}^\lambda_\kappa$ *is a normal ultrafilter on* $\kappa := \boldsymbol{\delta}^1_{2n+1}$*, then the Kleinberg sequence on* κ *derived from* $\mathcal{C}^\lambda_\kappa$ *is given by* $\kappa_0^{\mathcal{C}^\lambda_\kappa} = \kappa$ *and, for* $0 < n < \omega$*,*

$$\kappa_n^{\mathcal{C}^\lambda_\kappa} := \kappa^\kappa / \mathcal{C}^\lambda_\kappa \otimes n.$$

Proof. By Theorem 5.2.2 we know that taking iterated ultrapowers as in the definition of the Kleinberg sequence corresponds to taking order measure sums.

q.e.d.

By Corollary 7.2.4 we know that the normal measures on an odd projective ordinal are all generated by normal variables, and by Corollary 7.2.5 we have a simple algorithm to compute the o-values of those normal variables. Therefore, we can read off the values of the Kleinberg sequences as $o(\mathsf{v}) \cdot n$ for a normal variable v. As an example, we can read off the Kleinberg sequences on $\boldsymbol{\delta}^1_5$ as follows:

$$\kappa_n^{\mathcal{C}^\omega} = \aleph_{\omega^{\omega^\omega}+n+1},\ \kappa_n^{\mathcal{C}^{\omega_1}} = \aleph_{\omega^{\omega^\omega}+\omega\cdot n+1},\ \kappa_n^{\mathcal{C}^{\omega_2}} = \aleph_{\omega^{\omega^\omega}+\omega^\omega\cdot n+1},$$

$$\kappa_n^{\mathcal{C}^{\aleph_{\omega+1}}} = \aleph_{\omega^{\omega^\omega}\cdot n+1},\ \kappa_n^{\mathcal{C}^{\aleph_{\omega+2}}} = \aleph_{\omega^{\omega^{\omega+1}}\cdot n+1},\ \kappa_n^{\mathcal{C}^{\aleph_{\omega\cdot 2+1}}} = \aleph_{\omega^{\omega^{\omega\cdot 2}}\cdot n+1},$$

$$\text{and } \kappa_n^{\mathcal{C}^{\aleph_{\omega^\omega+1}}} = \aleph_{\omega^{\omega^{\omega^\omega}}\cdot n+1}.$$

But our measure analysis allows us not only to compute the values of Kleinberg sequences. In section 1.6 we defined Jónsson cardinals, a cardinal κ is a Jónsson cardinal if for every function $F : [\kappa]^{<\omega} \to \kappa$ there is a set H of order type κ and an element $\beta \in \kappa$ such that $F(\vec{\alpha}) \neq \beta$ holds for all $\vec{\alpha} \in [H]^{<\omega}$.

In 2005 Steve Jackson proved that all successor cardinals in the scope of a special measure assignment are Jónsson if that measure assignment is canonical. He presented this result at the workshop GLLC12 ("Games in Logic, Language and Computation") held in Amsterdam on June 14 2006 in a talk titled "Canonical Measure Assignments and Jónsson Cardinals". The measure assignment he used in his argument is similar to our measure assignment from Definition 4.5.2, Jackson's measure assignment is defined using a lifting operation called **wlift** that is nearly the same as the **lift** operation from Definition 4.1.4, the difference being that it uses functions of discontinuous type instead of functions of continuous type.

7.3.2. THEOREM (JACKSON). *Assume* $\mathsf{AD}+\mathsf{DC}$ *and that the measure assignment derived from the* **wlift** *operation is canonical up to* $\boldsymbol{\delta}^1_5$. *Then all infinite successor cardinals below* $\boldsymbol{\delta}^1_5$ *are Jónsson cardinals.*

The following is an adaptation of Steve Jackson's arguments, based on the slides of his talk at GLLC12. Since Steve Jackson's measure assignment is defined using a different lifting operation, the Theorem 7.3.2 does not directly apply to our measure assignment, for which we have shown a certain degree of canonicity by Corollary 6.4.5. In the following we will focus on cardinals in the reach of our canonical measure analysis. We will show that cardinals that can be represented as ultrapowers with respect to measures of the form $(\mathcal{C}^{\omega_1}_{\boldsymbol{\delta}^1_{2n+1}})^{\otimes m}$ are in fact Jónsson cardinals.

First we show that for odd projective ordinals κ the set

$$D := \{[f]_\mu\,;\, f : \kappa \to C \text{ is a function of discontinuous type } \kappa\}$$

has order type κ^κ/μ. We will need this result in the proof of Theorem 7.3.8 where we present our Jónsson results.

7.3.3. LEMMA. *Assume* $\mathsf{AD}+\mathsf{DC}$, *let* $\kappa = \boldsymbol{\delta}^1_{2n+1}$ *be an odd projective ordinal and* μ *a basic order measure on* κ, *so* $\mu = \mathbf{lift}_\kappa(\nu)$ *with* ν *a measure on some limit ordinal* $\lambda < \kappa$. *If for a function* $f \in \kappa^\kappa$ *there is a set* $A \in \mu$ *such that* f *restricted to* A *is unbounded then* f *is* μ*-almost everywhere increasing, i.e., there is a set* $B \in \mu$ *such that* $f(\alpha) < f(\beta)$ *for all* $\alpha, \beta \in B$ *with* $\alpha < \beta$.

Proof. We partition the set of pairs $\langle x, y\rangle \in \mathfrak{C}^\lambda_\kappa \times \mathfrak{C}^\lambda_\kappa$ of functions with $\sup x < \min y$ according to whether $f([x]_\nu) < f([y]_\nu)$ or not. Using Corollary 1.9.2 we get a homogeneous club set $C \subseteq \kappa$ for this partition. Toward a contradiction, assume $f([y]_\nu) \leq f([x]_\nu)$ for all $x, y \in \mathfrak{C}^\lambda_C$ with $\sup x < \min y$. We know from the unboundedness of f on A that there is a club set $C' \subseteq \kappa$ such that for all $\alpha < \kappa$ exists a $z \in \mathfrak{C}^\lambda_{C'}$ such that $\alpha < f([z]_\nu)$. Fix a function $x_0 \in \mathfrak{C}^\lambda_{C'\cap C}$. Then for all $\alpha < \kappa$ there is a $y \in \mathfrak{C}^\lambda_{C'\cap C}$ with $\sup x_0 < \min y$ such that

$$\alpha < f([y]_\nu) \text{ and } f([y]_\nu) \leq f([x_0]_\nu).$$

By taking the supremum over $\alpha < \kappa$ follows that $f([x]_\nu)$ is equal to κ, a contradiction.

We now partition the set of pairs $\langle x, y\rangle \in \mathfrak{C}^\lambda_\kappa \times \mathfrak{C}^\lambda_\kappa$ of functions with $x(\alpha) < y(\alpha) < x(\alpha+1)$ according to whether $f([x]_\nu) < f([y]_\nu)$ or not. Like above we get homogeneous club set for the stated side and can assume without loss of generality that C is that club set.

Let x and y be continuous functions of type λ into C with $[x]_\nu$ less than $[y]_\nu$. We either have $\sup x < \sup y$ or $\sup x = \sup y$. In the first case, since λ is a limit ordinal there is some $\alpha_0 < \lambda$ with $\sup x < y(\alpha_0)$. If we define

$y' : \lambda \to C$ by $y'(\alpha) := y(\alpha_0 + \alpha)$ then y' is of continuous type λ and we have $\sup x < \min y'$, so we know $f([x]_\nu) < f([y']_\nu)$. But since y' is a simple shift of y we have $[y']_\nu = [y]_\nu$ and thus $f([x]_\nu) < f([y]_\nu)$. In the second case we get in the usual way functions x' and y' with range in C and $[x']_\nu = [x]_\nu$, $[y']_\nu = [y]_\nu$ and also conclude $f([x]_\nu) < f([y]_\nu)$, which proves that f is μ-almost everywhere increasing.
q.e.d.

7.3.4. Lemma. *Assume* $\mathsf{AD}+\mathsf{DC}$*, let* $\kappa = \boldsymbol{\delta}^1_{2n+1}$ *be an projective ordinal and* μ *a basic order measure on* κ*, so* $\mu = \mathbf{lift}_\kappa(\nu)$ *with* ν *a measure on some limit ordinal* $\lambda < \kappa$*. Let* $f \in \kappa^\kappa$ *be a function that is* μ*-almost everywhere increasing, i.e., there is a set* $B \in \mu$ *such that* $f(\alpha) < f(\beta)$ *for all* $\alpha, \beta \in B$ *with* $\alpha < \beta$*. Then there is a function* $f'' \in \kappa^\kappa$ *with* $[f'']_\mu = [f]_\mu$ *that is everywhere increasing.*

Proof. We know that there is a club set $C \subseteq \kappa$ such that for all $x, y \in \mathfrak{C}^\lambda_C$ with $[x]_\nu < [y]_\nu$ we have $f([x]_\nu) < f([y]_\nu)$. We can assume that C is closed under ν. For functions $x \in \kappa^\lambda$ we define $x' \in C^\lambda$ by $x'(\alpha) :=$ the $x(\alpha)$th element of C. Now let $f' \in \kappa^\kappa$ be defined by $f'(\alpha) := f([x']_\nu)$ when $\alpha < \kappa$ is represented by $[x]_\nu$ and $f'(\alpha) := 0$ otherwise. Since $[x']_\nu = [y']_\nu$ follows directly from $[x]_\nu = [y]_\nu$ this is welldefined.

Let $C' \subseteq C$ be the set of closure points of C we get from Lemma 1.3.10. So for $x \in \mathfrak{C}^\lambda_{C'}$ we have that $x(\alpha)$ is equal to the $x(\alpha)$th element of C, *i.e.*, $x'(\alpha)$, for all $\alpha < \lambda$. That means $f'([x]_\nu) = f([x']_\nu) = f([x]_\nu)$ and so $[f']_\mu = [f]_\mu$.

If $x : \lambda \to \kappa$ is a function of continuous type λ then x' is also a function of continuous type λ and with range in C. Let x, y be functions in $\mathfrak{C}^\lambda_\kappa$. If $[x]_\nu < [y]_\nu$ then $[x']_\nu < [y']_\nu$. Since x' and y' are in $\mathfrak{C}^\lambda_C$ we get $f([x']_\nu) < f([y']_\nu)$ and so $f'([x]_\nu) < f'([y]_\nu)$. That means on the set

$$H := \{[x]_\nu \,;\, x \in \mathfrak{C}^\lambda_C\}$$

the function f' is increasing.

Now let $\pi : \kappa \to H$ be the increasing enumeration of H and define the function $f'' : \kappa \to \kappa$ by $f''(\alpha) := f'(\pi(\alpha))$. Then the function f'' is everywhere increasing. Let $C'' \subseteq \kappa$ be a club set that is closed under π, so $\sup_{\beta<[x]_\nu} \pi(\beta) \le [x]_\nu$. Since we have $\alpha \le \pi(\alpha)$ for all $\alpha < \kappa$ this means $\pi([x]_\nu) = [x]_\nu$. It follows that $f''([x]_\nu) = f'(\pi([x]_\nu)) = f'([x]_\nu)$ is true for all $x \in \mathfrak{C}^\lambda_{C''}$, *i.e.*, $[f'']_\mu = [f']_\mu = [f]_\mu$.
q.e.d.

7.3.5. Lemma. *Assume* $\mathsf{AD}+\mathsf{DC}$*, let* $\kappa = \boldsymbol{\delta}^1_{2n+1}$ *be an odd projective ordinal,* μ *a basic order measure on* κ*, and* $C \subseteq \kappa$ *a club set. Then the set*

$$D := \{[f]_\mu \,;\, f : \kappa \to C \text{ is a function of discontinuous type } \kappa\}$$

has order type κ^κ/μ.

Proof. From Lemmas 1.4.10 and 1.4.11 follows that κ^κ/μ is the disjunct union of κ and and the set

$$U := \{[f]_\mu\,;\, f \in \kappa^\kappa \text{and } f"(A) \text{ is unbounded in } \kappa \text{ for some } A \in \mu\}.$$

So the set U has ordertype κ^κ/μ. From Lemmas 7.3.3 and 7.3.4 we get a different description of the set U,

$$U = \{[f]_\mu\,;\, f \in \kappa^\kappa \text{and } f \text{ is increasing}\}.$$

If we can show that there is an order preserving injection π from the set U into the set D we are finished with the proof, since then D must have ordertype κ^κ/μ.

We define the function $\pi : U \to \kappa^\kappa/\mu$ as follows: For $f \in \kappa^\kappa$ let $\pi([f]_\mu) := [\hat{f}]_\mu$, with $\hat{f}(\alpha) :=$ the $(\omega \cdot f(\alpha) + \omega)$th element of C. This is obviously welldefined with respect to μ, *i.e.*, if $[f]_\mu = [g]_\mu$ then $[\hat{f}]_\mu = [\hat{g}]_\mu$. If $[f]_\mu < [g]_\mu$ then there is a set $A \in \mu$ such that $f(\alpha) < g(\alpha)$ for all $\alpha \in A$. It follows that we have $\omega \cdot f(\alpha) + \omega < \omega \cdot g(\alpha) + \omega$ for all $\alpha \in A$, *i.e.*, $[\hat{f}]_\mu < [\hat{g}]_\mu$, meaning π is orderpreserving.

We are finished if we can show that the range of π is a subset of D, *i.e.*, if $\hat{f}$ is in $\mathfrak{D}^\kappa_C$ for all f in U. This is done in the following four steps:

1. Let $f \in \kappa^\kappa$ be an increasing function. By definition of $\hat{f}$ for all $\alpha < \kappa$ we have $\hat{f}(\alpha) \in C$, so ran$\hat{f} \subseteq C$.

2. If $\alpha < \beta < \kappa$ then $f(\alpha) < f(\beta)$, since f is increasing. It follows that $\hat{f}(\alpha) = \omega \cdot f(\alpha) + \omega < \omega \cdot f(\beta) + \omega = \hat{g}(\alpha)$, so $\hat{f}$ is an increasing function.

3. Define $\ell : \kappa \times \omega \to \kappa$ by $\ell(\alpha, n) :=$ the $(\omega \cdot f((\alpha) + n)$th element of C. Since C as a club set is closed under supremums we have $\sup_{n<\omega} \ell(\alpha, n) =$ the $(\omega \cdot f(\alpha) + \omega)$th element of C, which is the definition of $\hat{f}(\alpha)$. So ℓ witnesses the uniform cofinality ω of $\hat{f}$.

4. Let $\lambda < \kappa$ be a limit ordinal. Since f is increasing we have

$$\sup_{\alpha<\omega}(\omega \cdot f(\alpha) + \omega) = \sup_{\alpha<\omega} \omega \cdot f(\alpha) \leq \omega \cdot f(\lambda) < \omega \cdot f(\lambda) + \omega,$$

so $\sup_{\alpha<\lambda} \hat{f}(\alpha) < \hat{f}(\lambda)$, meaning $\hat{f}$ is discontinuous.

q.e.d.

Now we prepare ourself to prove that certain cardinals that can be presented as ultrapowers are Jónsson. For that proof we will need to analyze the functions that define the ultrapowers. Let $\kappa > \omega_1$ be an odd projective ordinal and $f : (\omega_1)^m \to \kappa$ a function of continuous type $(\omega_1)^m$. For $i \leq m$ the ith **invariant** $\text{inv}_i(f)$ of f is the function from $(\omega_1)^i$ to κ defined by

$$\text{inv}_i(f)(\vec{\alpha}) := \sup_{\vec{\beta} \in (\omega)^{m-i}} f(\vec{\beta}, \vec{\alpha}).$$

So the mth invariant of f is equal to f and the 0th invariant of f is $\sup f$. For $\alpha < \kappa$ represented by $f : (\omega_1)^m \to \kappa$ of continuous type $(\omega_1)^m$, let also $\mathrm{inv}_i(\alpha) = [\mathrm{inv}_i(f)]_{(\mathcal{C}^\omega_{\omega_1})^m}$. This is easily well-defined. For a function $F : \kappa \to \kappa$ of discontinuous type κ we define the ith **invariant** of F by

$$\mathrm{inv}_i(F)([h]_{(\mathcal{C}^\omega_{\omega_1})^m}) := \sup\{F([h']_{(\mathcal{C}^\omega_{\omega_1})^m})\,;\, h' \in \mathfrak{C}_\kappa^{(\omega_1)^m} \text{ and } \mathrm{inv}_i(h') = \mathrm{inv}_i(h)\}.$$

7.3.6. Lemma. *Assume* $\mathsf{AD} + \mathsf{DC}$*, let* $\kappa = \boldsymbol{\delta}^1_{2n+1}$ *be an odd projective ordinal,* $m > 0$ *a natural number, and suppose that* $f, g : (\omega_1)^m \to \kappa$ *are functions of continuous type* $(\omega_1)^m$ *with* $[f]_{(\mathcal{C}^\omega_{\omega_1})^m} < [g]_{(\mathcal{C}^\omega_{\omega_1})^m}$.

1. *Either* $\mathrm{inv}_0(f) = \sup(f) < \sup(g) = \mathrm{inv}_0(g)$*. In this case there are functions* f', $g' : (\omega_1)^m \to \kappa$ *with* $[f']_{(\mathcal{C}^\omega_{\omega_1})^m} = [f]_{(\mathcal{C}^\omega_{\omega_1})^m}$, $[g']_{(\mathcal{C}^\omega_{\omega_1})^m} = [g]_{(\mathcal{C}^\omega_{\omega_1})^m}$, $\mathrm{ran}(f') \subseteq \mathrm{ran}(f)$, $\mathrm{ran}(g') \subseteq \mathrm{ran}(g)$*, and* $f'(\vec\alpha) < g'(\vec\beta)$ *for all* $\vec\alpha, \vec\beta \in (\omega_1)^m$.

2. *Or there is an* $0 < i \leq m$ *such that* $[\mathrm{inv}_{i-1}(f)]_{(\mathcal{C}^\omega_{\omega_1})^{i-1}} = [\mathrm{inv}_{i-1}(g)]_{(\mathcal{C}^\omega_{\omega_1})^{i-1}}$*, and* $[\mathrm{inv}_i(f)]_{(\mathcal{C}^\omega_{\omega_1})^i} < [\mathrm{inv}_i(g)]_{(\mathcal{C}^\omega_{\omega_1})^i}$*. Then there are* f', $g' : (\omega_1)^m \to \kappa$ *with* $[f']_{(\mathcal{C}^\omega_{\omega_1})^m} = [f]_{(\mathcal{C}^\omega_{\omega_1})^m}$, $[g']_{(\mathcal{C}^\omega_{\omega_1})^m} = [g]_{(\mathcal{C}^\omega_{\omega_1})^m}$, $\mathrm{ran}(f') \subseteq \mathrm{ran}(f)$, $\mathrm{ran}(g') \subseteq \mathrm{ran}(g)$*, and for all* $\vec\alpha$, $\vec\beta$,

$$f(\vec\alpha) < g(\vec\beta) \textit{ iff } (\alpha_i, \ldots, \alpha_{m-1}) \leq_{\mathrm{rlex}} (\beta_{m-i}, \ldots, \beta_{m-1}).$$

Proof. In case 1., define $f' = f$ and $g'(\alpha) = g(\alpha_0 + \alpha)$ where α_0 is least such that $g(\alpha_0) > \sup(f)$. Then obviously $[f']_{(\mathcal{C}^\omega_{\omega_1})^m} = [f]_{(\mathcal{C}^\omega_{\omega_1})^m}$, $\mathrm{ran}(f') \subseteq \mathrm{ran}(f)$, $\mathrm{ran}(g') \subseteq \mathrm{ran}(g)$, and $f'(\vec\alpha) < g'(\vec\beta)$ for all $\vec\alpha, \vec\beta \in (\omega_1)^m$. Since the measure $(\mathcal{C}^\omega_{\omega_1})^m$ contains end segments we also have $[g']_{(\mathcal{C}^\omega_{\omega_1})^m} = [g]_{(\mathcal{C}^\omega_{\omega_1})^m}$.

In case 2, we first deal with $i < m$. Then there is a club set $C \subseteq \omega_1$ such that for all $\vec\alpha \in (C)^i$ we have

$$\sup_{\vec\beta \in (\omega)^{m-i+1}} f(\vec\beta, \alpha_1, \ldots, \alpha_{m-1}) = \sup_{\vec\beta \in (\omega)^{m-i+1}} g(\vec\beta, \alpha_1, \ldots, \alpha_{m-1})$$

and

$$\sup_{\vec\beta \in (\omega)^{m-i}} f(\vec\beta, \alpha_0, \ldots, \alpha_{m-1}) < \sup_{\vec\beta \in (\omega)^{m-i}} g(\vec\beta, \alpha_0, \ldots, \alpha_{m-1}).$$

For $\vec\alpha \in (C)^i$ let $\vec\beta_{\vec\alpha}$ be the $<_{\mathrm{rlex}}$-least element of $(\omega)^{m-i}$ such that

$$\sup_{\vec\beta \in (\omega)^{m-i}} f(\vec\beta, \alpha_0, \ldots, \alpha_{m-1}) < g(\vec\beta_{\vec\alpha}, \alpha_0, \ldots, \alpha_{m-1})$$

and define $\hat g(\vec\beta, \vec\alpha) := g(\vec\beta_{\vec\alpha} + \vec\beta, \vec\alpha)$. We now have for $\vec\alpha, \vec\beta \in (C)^m$

$$f(\vec\alpha) < \hat g(\vec\beta) \text{ iff } (\alpha_{m-i}, \ldots, \alpha_{m-1}) \leq_{\mathrm{rlex}} (\beta_{m-i}, \ldots, \beta_{m-1}).$$

All that is left is to extend this property from almost everywhere to everywhere. We do this in the usual way and define

$$\begin{aligned} f'(\alpha_0,\ldots,\alpha_{m-1}) &:= f(\alpha_0\text{th element of } C,\ldots,\alpha_{m-1}\text{th element of } C)\\ g'(\alpha_0,\ldots,\alpha_{m-1}) &:= \hat{g}(\alpha_0\text{th element of } C,\ldots,\alpha_{m-1}\text{th element of } C)\end{aligned}$$

We get $\mathrm{ran}(f') \subseteq \mathrm{ran}(f)$, $\mathrm{ran}(g') \subseteq \mathrm{ran}(g)$ directly from the definition and if C' denotes the the club set of closure points of C that we get from Lemma 1.3.10 then the functions f' and f, g' and g, respectively, coincide on $(C')^m$ and so $[f']_{(\mathcal{C}^{\omega}_{\omega_1})^m} = [f]_{(\mathcal{C}^{\omega}_{\omega_1})^m}$ and $[g']_{(\mathcal{C}^{\omega}_{\omega_1})^m} = [g]_{(\mathcal{C}^{\omega}_{\omega_1})^m}$.

The argument for $i = m$ is a simpler version of the argument for $i < m$, we have from the assumption $f(\vec{\alpha}) < g(\vec{\alpha})$ for all $\vec{\alpha} \in (C)^m$ and can go directly to the definition of f' and g'.

q.e.d.

7.3.7. Lemma. *Assume* $\mathsf{AD}+\mathsf{DC}$, *let* $\kappa = \boldsymbol{\delta}^1_{2n+1}$ *be an odd projective ordinal and* $m > 0$ *a natural number. Let* $F_1,\ldots,F_n : \kappa \to \kappa$ *be functions of discontinuous type with* $[F_i]_{(\mathcal{C}^{\omega_1}_{\kappa})^{\otimes m}} < [F_j]_{(\mathcal{C}^{\omega_1}_{\kappa})^{\otimes m}}$ *for* $0 < i < j < n$. *Then there are* $F'_1,\ldots,F'_n$ *with* $[F'_i]_{(\mathcal{C}^{\omega_1}_{\kappa})^{\otimes m}} = [F_i]_{(\mathcal{C}^{\omega_1}_{\kappa})^{\otimes m}}$, $\mathrm{ran}(F'_i) \subseteq \mathrm{ran}(F_i)$, *and such that for all* $1 \le i < j \le n$, *there is a* p *such that for all* $\alpha,\beta \in \kappa$ *we have*

$$F'_i(\alpha) < F'_j(\beta)$$

iff

$$\mathrm{inv}_p(\alpha) \le \mathrm{inv}_p(\beta).$$

Proof. The general result follows directly if we can prove it for the case of two functions F and G, with $[F]_{(\mathcal{C}^{\omega_1}_{\kappa})^{\otimes m}} < [G]_{(\mathcal{C}^{\omega_1}_{\kappa})^{\otimes m}}$.

From $[F]_{(\mathcal{C}^{\omega_1}_{\kappa})^{\otimes m}} < [G]_{(\mathcal{C}^{\omega_1}_{\kappa})^{\otimes m}}$ we know that there is a club set $C \subseteq \kappa$ such that $F([f]_{(\mathcal{C}^{\omega}_{\omega_1})^m}) < G([f]_{(\mathcal{C}^{\omega}_{\omega_1})^m})$ for all functions $f : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$. Let i be least such that $[\mathrm{inv}_i(F)]_{(\mathcal{C}^{\omega_1}_{\kappa})^{\otimes m}} < [\mathrm{inv}_i(G)]_{(\mathcal{C}^{\omega_1}_{\kappa})^{\otimes m}}$. We want to show that $F([f]_{(\mathcal{C}^{\omega}_{\omega_1})^m}) < G([g]_{(\mathcal{C}^{\omega}_{\omega_1})^m})$ is true if and only if we have $\mathrm{inv}_i(f) \le \mathrm{inv}_i(g)$ $(\mathcal{C}^{\omega}_{\omega_1})^i$-almost everywhere.

To prove our desired result we need to analyze pairs of functions $f, g : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$. If $[f]_{(\mathcal{C}^{\omega}_{\omega_1})^m} \le [g]_{(\mathcal{C}^{\omega}_{\omega_1})^m}$ then we have $\mathrm{inv}_i(f) \le \mathrm{inv}_i(g)$ $(\mathcal{C}^{\omega}_{\omega_1})^i$-almost everywhere and immediately get

$$F([f]_{(\mathcal{C}^{\omega}_{\omega_1})^m}) < G([f]_{(\mathcal{C}^{\omega}_{\omega_1})^m}) < G([g]_{(\mathcal{C}^{\omega}_{\omega_1})^m})$$

from our assumption on C and the fact that G is increasing. If $[g]_{(\mathcal{C}^{\omega}_{\omega_1})^m} < [f]_{(\mathcal{C}^{\omega}_{\omega_1})^m}$ then we have to consider the different ways that f and g can be interleaved, we get those from Lemma 7.3.6. If the largest k such that $\mathrm{inv}_k(g) = \mathrm{inv}_k(f)$ almost everywhere is smaller than i then we get

$$F([g]_{(\mathcal{C}^{\omega}_{\omega_1})^m}) < G([g]_{(\mathcal{C}^{\omega}_{\omega_1})^m}) < G([f]_{(\mathcal{C}^{\omega}_{\omega_1})^m})$$

again from our assumption on C and the fact that G is increasing. That leaves pairs of functions $f, g : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$ with $\mathrm{inv}_j(g) < \mathrm{inv}_j(f)$ almost everywhere for some $j > i$ and $\mathrm{inv}_k(g) = \mathrm{inv}_k(f)$ almost everywhere for $k < j$.

We partition pairs $f, g : (\omega_1)^m \to \kappa$ of functions of continuous type $(\omega_1)^m$, with $g(\vec{\alpha}) < f(\vec{\beta})$ iff $(\alpha_{m-j}, \ldots, \alpha_m) \leq_{\mathrm{rlex}} (\beta_{m-j}, \ldots, \beta_m)$ according to whether

$$F([f]_{(\mathcal{C}^{\omega}_{\omega_1})^m}) < G([g]_{(\mathcal{C}^{\omega}_{\omega_1})^m}).$$

From Lemma 1.9.1 follows that there is a homogeneous club set $C' \subseteq \kappa$ for this partition, we can assume $C = C'$. Assume towards a contradiction that C is homogeneous for the contrary side, *i.e.*, that for all functions $f, g : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$ with

$$g(\vec{\alpha}) < f(\vec{\beta}) \text{ iff } (\alpha_{m-j}, \ldots, \alpha_{m-1}) \leq_{\mathrm{rlex}} (\beta_{m-j}, \ldots, \beta_{m-1})$$

we have

$$F([f]_{(\mathcal{C}^{\omega}_{\omega_1})^m}) \geq G([g]_{(\mathcal{C}^{\omega}_{\omega_1})^m}).$$

From our assumption $[\mathrm{inv}_i(F)]_{(\mathcal{C}^{\omega_1}_{\kappa})^{\otimes m}} < [\mathrm{inv}_i(G)]_{(\mathcal{C}^{\omega_1}_{\kappa})^{\otimes m}}$ we get a function $h : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$ with $\mathrm{inv}_i(F)([h]_{(\mathcal{C}^{\omega}_{\omega_1})^m}) < \mathrm{inv}_i(G)([h]_{(\mathcal{C}^{\omega}_{\omega_1})^m})$, *i.e.*,

$$\sup_{\mathrm{inv}_i(h')=\mathrm{inv}_i(h)} F([h']_{(\mathcal{C}^{\omega}_{\omega_1})^m}) < \sup_{\mathrm{inv}_i(h')=\mathrm{inv}_i(h)} G([h']_{(\mathcal{C}^{\omega}_{\omega_1})^m}).$$

That means there is a function $g' : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$ with $\mathrm{inv}_i(g') = \mathrm{inv}_i(h)$ and $\sup_{\mathrm{inv}_i(h')=\mathrm{inv}_i(h)} F([h']_{(\mathcal{C}^{\omega}_{\omega_1})^m}) < G([g']_{(\mathcal{C}^{\omega}_{\omega_1})^m})$. Now take a function $f' : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$ with $\mathrm{inv}_i(f') = \mathrm{inv}_i(h)$ and $\mathrm{inv}_j(f') > \mathrm{inv}_j(g')$ $(\mathcal{C}^{\omega}_{\omega_1})^{i+1}$-almost everywhere. Using Lemma 7.3.6 we get functions $g'', g'' : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$ with $g''(\vec{\alpha}) < f''(\vec{\beta})$ iff $(\alpha_{m-j}, \ldots, \alpha_{m-1}) \leq_{\mathrm{rlex}} (\beta_{m-j}, \ldots, \beta_{m-1})$ and $[f'']_{(\mathcal{C}^{\omega}_{\omega_1})^m} = [f']_{(\mathcal{C}^{\omega}_{\omega_1})^m}$, $[g'']_{(\mathcal{C}^{\omega}_{\omega_1})^m} = [g']_{(\mathcal{C}^{\omega}_{\omega_1})^m}$, *i.e.*, $F([f'']_{(\mathcal{C}^{\omega}_{\omega_1})^m}) < G([g'']_{(\mathcal{C}^{\omega}_{\omega_1})^m})$. From this contradiction follows that C must be homogeneous for the stated side of the partition.

So if $f, g : (\omega_1)^m \to C$ are functions of continuous type $(\omega_1)^m$ with $\mathrm{inv}_j(g) < \mathrm{inv}_j(f)$ almost everywhere and $\mathrm{inv}_k(g) = \mathrm{inv}_k(f)$ almost everywhere for $k < j$ then from Lemma 7.3.6 we get functions $\hat{f}$ and $\hat{g}$ with $[\hat{f}]_{(\mathcal{C}^{\omega}_{\omega_1})^m} = [f]_{(\mathcal{C}^{\omega}_{\omega_1})^m}$ and $[\hat{g}]_{(\mathcal{C}^{\omega}_{\omega_1})^m} = [g]_{(\mathcal{C}^{\omega}_{\omega_1})^m}$ so that by our partition we have

$$F([f]_{(\mathcal{C}^{\omega}_{\omega_1})^m}) = F([\hat{f}]_{(\mathcal{C}^{\omega}_{\omega_1})^m}) < G([\hat{g}]_{(\mathcal{C}^{\omega}_{\omega_1})^m}) = G([g]_{(\mathcal{C}^{\omega}_{\omega_1})^m}).$$

Since we now have considered all possibilities we know that there is a club set $C \subseteq \kappa$ such that for all functions $f, g : (\omega_1)^m \to C$ of continuous type $(\omega_1)^m$ we have $F([f]_{(\mathcal{C}^{\omega}_{\omega_1})^m}) < G([g]_{(\mathcal{C}^{\omega}_{\omega_1})^m})$ if and only if $\mathrm{inv}_i(f) \leq \mathrm{inv}_i(g)$ $(\mathcal{C}^{\omega}_{\omega_1})^i$-almost everywhere.

To finish the proof we define $F'([f]_{(\mathcal{C}^{\omega}_{\omega_1})^m}) = F([f']_{(\mathcal{C}^{\omega}_{\omega_1})^m})$ and $G'([f]_{(\mathcal{C}^{\omega}_{\omega_1})^m}) = G([f']_{(\mathcal{C}^{\omega}_{\omega_1})^m})$, where $f'(\vec{\alpha}) = f(\vec{\alpha})^{\text{th}}$ element of C. Then $\operatorname{ran}(F') \subseteq \operatorname{ran}(F)$, $\operatorname{ran}(G') \subseteq \operatorname{ran}(G)$, using the set of closure points of C we can show as usual $[F']_{(\mathcal{C}^{\omega_1}_{\kappa})^{\otimes m}} = [F]_{(\mathcal{C}^{\omega_1}_{\kappa})^{\otimes m}}$, $[G']_{(\mathcal{C}^{\omega_1}_{\kappa})^{\otimes m}} = [G]_{(\mathcal{C}^{\omega_1}_{\kappa})^{\otimes m}}$, and for all $\alpha, \beta \in \kappa$ we have $F'(\alpha) < G'(\beta)$ iff $\operatorname{inv}_p(\alpha) \leq \operatorname{inv}_p(\beta)$.

q.e.d.

If $\langle F_0, \ldots, F_{n-1} \rangle$ is a sequence of functions from κ to κ of discontinuous type with $[F_i]_{(\mathcal{C}^{\omega_1}_{\kappa})^{\otimes m}} < [F_j]_{(\mathcal{C}^{\omega_1}_{\kappa})^{\otimes m}}$ for $0 < i < j < n$ and such that for all $i < j < n$ there is a $p_{i,j}$ such that for all $\alpha, \beta \in \kappa$ we have $F_i(\alpha) < F_j(\beta)$ iff $\operatorname{inv}_{p_{i,j}}(\alpha) \leq \operatorname{inv}_{p_{i,j}}(\beta)$, then we say that this sequence of functions has **type** $\pi := \langle p_{i,j}\,;\, i < j < n \rangle$. The type determines how the functions interleave and that means we can use a variation of Lemma 1.9.3 for sequences of functions that have such a type.

With Lemmas 7.3.3 to 7.3.7 we now can apply our main result Corollary 6.4.5 to our measures assignment based on functions of continuous type and determine new Jónsson cardinals, going beyond the results of [Lö02] and [BoLö07].

7.3.8. THEOREM. *Assume* $\mathsf{AD} + \mathsf{DC}$. *Let* $\kappa = \boldsymbol{\delta}^1_{2n+1}$ *be an odd projective ordinal. Then for all* $0 < n < \omega$ *the ultrapower* $\lambda := \kappa^\kappa / (\mathcal{C}^{\omega_1}_{\kappa})^{\otimes n}$ *is a Jónsson cardinal.*

Proof. Let $\mu = (\mathcal{C}^{\omega_1}_{\kappa})^{\otimes m}$. Define $S_{n,\pi}$ to be the set of n-tuples $\langle F_j\,;\, j < n \rangle$ of functions $F_j : \kappa \to \kappa$ of discontinuous type κ such that the sequence is of type π. Now let $J : [\lambda]^n \to \lambda$ be an arbitrary function. For $n > 0$ and $\ell < n$ let $P_{n,\pi,\ell}$ be the partition of $S_{n,\pi}$ according to whether or not

$$J([F_0]_\mu, \ldots, [F_{\ell-1}]_\mu, [F_{\ell+1}]_\mu, \ldots, [F_{n-1}]_\mu) \neq [F_\ell]_\mu$$

is true. By Lemma 1.9.3 exist a homogeneous club set $C_{n,\pi,\ell}$ for this partition. We can easily construct a sequence $\langle F_j\,;\, j < n+1 \rangle$ of functions $F_j : \kappa \to C_{n,\pi,\ell}$ of discontinuous type κ such that both sequences $\langle F_j\,;\, j < n+1, j \neq \ell+1 \rangle$ and $\langle F_j\,;\, j < n+1, j \neq \ell \rangle$ are of type π. Since we cannot have both

$$J([F_0]_\mu, \ldots, [F_{\ell-1}]_\mu, [F_{\ell+2}]_\mu, \ldots, [F_n]_\mu) = [F_\ell]_\mu$$

and

$$J([F_0]_\mu, \ldots, [F_{\ell-1}]_\mu, [F_{\ell+2}]_\mu, \ldots, [F_n]_\mu) = [F_{\ell+1}]_\mu$$

the club set $C_{n,\pi,\ell}$ must be homogeneous for the stated side of partition $P_{n,\pi,\ell}$.

Now let C be the intersection $\bigcap_{0<n<\omega} \bigcap_{\ell \leq n} \bigcap_{\pi} C_{n,\pi\ell}$, since club sets are closed under countable intersections the set C is also a club subset of κ. Let $D \subset C$ be the club set of closure points of C we get from Lemma 1.3.10. By Lemma 7.3.5 the set

$$H := \{[F]_\mu\,;\, F : \kappa \to D \text{ is a function of discontinuous type } \kappa\}$$

has order type λ. Let $G : \kappa \to C \backslash D$ be a function of discontinuous type κ, it follows that $[G]_\mu \neq [F]_\mu$ holds for all functions $F : \kappa \to D$ of discontinuous type κ. Let $\langle F_j \,;\, j \leq n \rangle$ be a tuple of functions $F_j : \kappa \to D$ of discontinuous type κ with $[F_i]_\mu < [F_j]_\mu$ for $i < j < nj$. Then we can "stick" $[G]_\mu$ in this increasing sequence, *i.e.*, we can define a tuple $\langle F'_j \,;\, j \leq n+1 \rangle$ such that $[F'_i]_\mu < [F'_j]_\mu$ holds for $i < j \leq n+1$ and $F'_\ell = G$. By Lemma 7.3.7 we can assume that the sequence $\langle F'_j \,;\, j \leq n+1 \rangle$ has some type π. Since C was homogeneous for the partition $P_{n+1,\pi,\ell}$ we get (since by definition of the F'_i we have $[F_i]_\mu = [F'_i]_\mu$ for $i < \ell$ and $[F_i]_\mu = [F'_{i+1}]_\mu$ for $i > \ell$)

$$J([F_0]_\mu, \ldots, [F_n]_\mu) \neq [G]_\mu.$$

That means there is no increasing n-tuple with elements in H whose J-image is $[G]_\mu$, in other words the function $J : [H]^{<\omega} \to \lambda$ is not surjective and thus λ is a Jónsson cardinal. q.e.d.

7.3.9. Theorem. *Assume* $\mathsf{AD} + \mathsf{DC}$. *Let* $\kappa = \boldsymbol{\delta}^1_{2n+1}$ *be an odd projektive ordinal. Then the following cardinals are all Jónsson cardinals:*

$$\kappa^{(n)},\ \kappa^{(\omega \cdot n+1)},\ \kappa^{(\omega^n+1)},\ \textit{for } n < \omega.$$

Proof. The first two sequences of cardinals are just the Kleinberg sequences we get from $\mathcal{C}^\omega_\kappa$ and $\mathcal{C}^{\omega_1}_\kappa$, respectively. From Lemma 7.3.1 we know that the elements of the Kleinberg sequences are ultrapowers with respect to order measure sums, and Corollary 6.4.6 allows us to compute their values. The third sequence is derived using Theorem 7.3.8, the values are again computed using Corollary 6.4.6. q.e.d.

Now we know that under AD the following cardinals are Jónsson (for $n \geq 1$): the sequence on $\boldsymbol{\delta}^1_1$, $\aleph_n$; the three sequences on $\boldsymbol{\delta}^1_3$, $\aleph_{\omega+n}$, $\aleph_{\omega \cdot n+1}$, and $\aleph_{\omega^n+1}$; the three sequences on $\boldsymbol{\delta}^1_5$, $\aleph_{\omega^{\omega^\omega}+n}$, $\aleph_{\omega^{\omega^\omega}+\omega \cdot n+1}$, and $\aleph_{\omega^{\omega^\omega}+\omega^n+1}$; and so on. That means even the first small steps of the canonical measure assignment enable us get more Jónsson cardinals than before. If we look at the first of the newly proven to be Jónsson cardinals we get:

7.3.10. Theorem. *Assume* $\mathsf{AD} + \mathsf{DC}$. *Then* $\aleph_{\omega^n+1}$ *is Jónsson (for* $n < \omega$*).*

Proof. Special case of Theorem 7.3.9, with $\kappa = \boldsymbol{\delta}^1_3$. q.e.d.

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